Convergence Analysis of Robust and Sparse M-Estimation of DOA

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ITA 2024, San Diego, Feb. 19, 2024

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M-Estimation based on Complex Elliptically Symmetric (CES)-distribution

Sketch of the Proof

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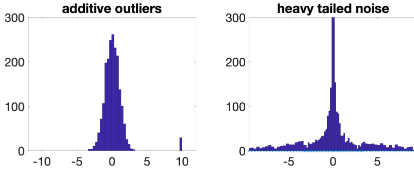
- In this talk, the CES distribution is mentioned a lot
- The CES distribution is defined by the form of its pdf

$$p(\boldsymbol{y}|0, \Sigma) = \det(\Sigma^{-1}) g(\boldsymbol{y}^{\mathrm{H}} \Sigma^{-1} \boldsymbol{y})$$

for a suitable density generator g.

Introduction 2/2

We are interested in array processing for data with
- additive outliers
- heavy-tailed noise



Array data in simulations is one of
- Complex Gaussian
- Multivariate Complex t_ν (MVT)
- ϵ-contaminated Complex Gaussian

M-Estimator for Direction of Arrival (DOA) derived for general loss function

Sparse Bayesian Learning (SBL) approach for this model

ICASSP 2022 contribution generalized IEEE SPL 2016 paper

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Array Data Model 1/2 (CES model)

- We observe narrowband waves on N sensors for L snapshots
  - Any array geometry works, but our results are for Uniform Linear Array (ULA) with half wavelength spacing
- $\boldsymbol{y}_\ell$ is the ℓth array data snapshot
- Array data $\boldsymbol{Y} = [\boldsymbol{y}_1 \ldots \boldsymbol{y}_L] \in \mathbb{C}^{N \times L}$ (given)
- Array data covariance matrix $\mathrm{E}(\boldsymbol{y}_\ell \boldsymbol{y}_\ell^{\mathsf{H}})$ exists

Array Data Model 2/2 (CES model)

- We assume $\boldsymbol{y}_\ell \sim$ CES-distributed with unknown scatter matrix Σ

$$p(\boldsymbol{Y}|0, \Sigma) = \prod_{\ell=1}^{L} \det(\Sigma^{-1}) g(\boldsymbol{y}_\ell^{\mathrm{H}} \Sigma^{-1} \boldsymbol{y}_\ell)$$

for a suitable density generator g.

- Σ is the so-called scatter matrix
- Σ equals the array data covariance matrix $\mathrm{E}(\boldsymbol{y}_\ell \boldsymbol{y}_\ell^H)$ for Gaussian data

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M-Estimation based on CES-distribution 1/7

- General approach based on loss functions:
- M-estimator of Σ is minimizer of

$$\mathcal{L}(\Sigma) = \frac{1}{Lb} \sum_{\ell=1}^{L} \rho(\boldsymbol{y}_\ell^{\mathrm{H}} \Sigma^{-1} \boldsymbol{y}_\ell) - \log \det(\Sigma^{-1})$$

  where $\rho(\cdot)$ is a chosen loss function

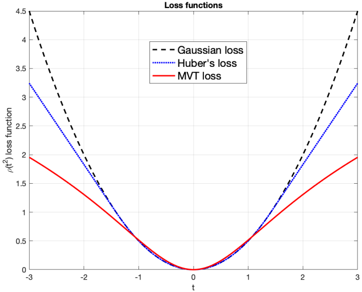
- By introducing the consistency factor b, the maximizer $\hat{\Sigma}$ of $\mathcal{L}(\Sigma)$ becomes consistent for the array data covariance matrix for Gaussian data
  - b enables identifying covariance with scatter matrix, $\mathrm{E}(\boldsymbol{y}_\ell \boldsymbol{y}_\ell^{\mathrm{H}}) = \Sigma$
  - b is evaluated a priori

- Example loss functions:

$$\text{Gaussian loss,} \quad \rho_{\mathrm{G}}(t) = t$$

$$\text{Huber's loss,} \quad \rho_{\mathrm{H}}(t) = \begin{cases} t & \text{for } t \leqslant c^2, \\ c^2\big(\log(t/c^2) + 1\big) & \text{for } t > c^2, \end{cases}$$

$$\text{MVT loss,} \quad \rho_{\mathrm{T}}(t) = \frac{\nu+2N}{2}\log(\nu + 2t)$$



the ρ differ in their weighting of outliers

M-Estimation based on CES-distribution 3/7 (Table)

loss name	data density generator $g(t)$	loss function $\rho(t)$	weight $u(t)$
	$g(t)$	$-\log g(t)$	$\rho'(t)$
Gauss	e^{-t}	t	1
MVT	$(1 + t/\nu)^{-(\nu+2N)/2}$	$\frac{\nu+2N}{2} \log(\nu + 2t)$	$\frac{\nu+2N}{\nu+2t}$
Huber	$\mathrm{e}^{-\rho_{\mathrm{Huber}}(t)}$	$= \begin{cases} t \text{ if } & t \leqslant c^2, \\ c^2\big(\log(t/c^2) + 1\big) & \text{else} \end{cases}$	$= \begin{cases} 1 \text{ if } t < c^2 \\ c^2/t \text{ else} \end{cases}$
Tyler	t^{-N}	$N \log t$	N/t

M-Estimation based on CES-distribution 4/7 (generalized Jaffer's condition)

- $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{\Gamma}\boldsymbol{A}^{\mathsf{H}} + \sigma^2 \boldsymbol{I}_N$ where $\boldsymbol{\Gamma} = \mathrm{diag}(\boldsymbol{\gamma})$ and $\boldsymbol{\gamma} = [\gamma_1 \ldots \gamma_M]^{\mathsf{T}}$
- First order condition: $\frac{\partial \mathcal{L}}{\partial \gamma_m} = 0$
- Generalizes Jaffer's condition (ICASSP 1988) on $\boldsymbol{\Sigma}$

$$\boxed{\boldsymbol{a}_m^{\mathsf{H}} \boldsymbol{\Sigma}^{-1} \boldsymbol{a}_m = \boldsymbol{a}_m^{\mathsf{H}} \boldsymbol{\Sigma}^{-1} \boldsymbol{R}_{\boldsymbol{Y}} \boldsymbol{\Sigma}^{-1} \boldsymbol{a}_m,}$$

where $\boldsymbol{R}_{\boldsymbol{Y}}$ is the weighted sample covariance matrix,

$$\boldsymbol{R}_{\boldsymbol{Y}} = \frac{1}{Lb} \sum_{\ell=1}^{L} u(\boldsymbol{y}_\ell^{\mathsf{H}} \boldsymbol{\Sigma}^{-1} \boldsymbol{y}_\ell) \boldsymbol{y}_\ell \boldsymbol{y}_\ell^{\mathsf{H}}$$

Weight function $u(\cdot) = \rho'(\cdot)$ depends on chosen loss $\rho(\cdot)$

M-Estimation based on CES-distribution 5/7 (Iterating $\boldsymbol{R_Y}$ and Σ)

$$\Gamma^{(j-1)} = \mathrm{diag}(\gamma^{(j-1)})$$

$$\Sigma^{(j-1)} = \boldsymbol{A}\Gamma^{(j-1)}\boldsymbol{A}^H + (\sigma^2)^{(j-1)}\boldsymbol{I}_N$$

$$\boldsymbol{R_Y}^{(j)} = \frac{1}{Lb}\sum_{\ell=1}^{L} u(\boldsymbol{y}_\ell^H(\Sigma^{(j-1)})^{-1}\boldsymbol{y}_\ell;\cdot)\boldsymbol{y}_\ell\boldsymbol{y}_\ell^H$$

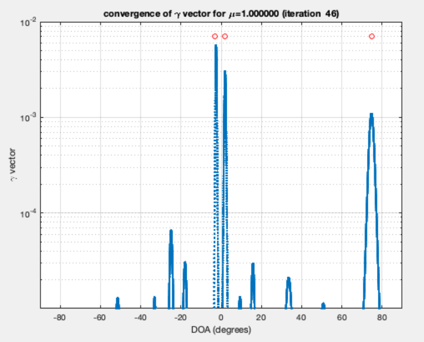
$$\boxed{\gamma_m^{(j)} = (1-\mu)\gamma_m^{(j-1)} + \mu\gamma_m^{(j-1)}\left(\frac{\boldsymbol{a}_m^H\Sigma^{-1}\boldsymbol{R_Y}\Sigma^{-1}\boldsymbol{a}_m}{\boldsymbol{a}_m^H\Sigma^{-1}\boldsymbol{a}_m}\right)}$$

$$j = j+1$$

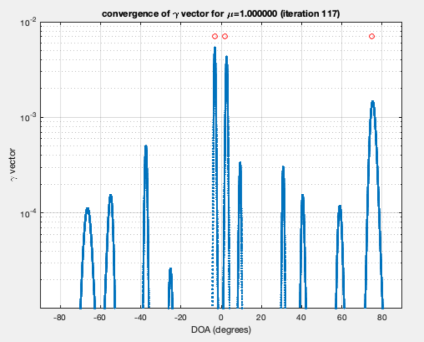
Show animation of iterations: γ vector convergence

Matlab-code

M-Estimation based on CES-distribution 6/7 (Animation for $\mu = 1$)

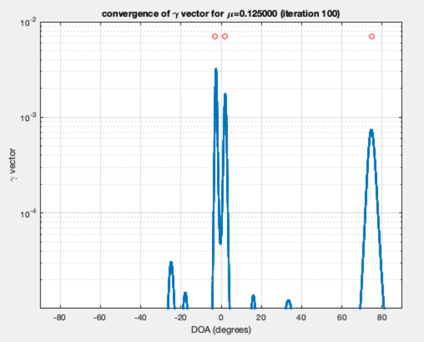


3 sources, $\mu = 1.000$, Gaussian data

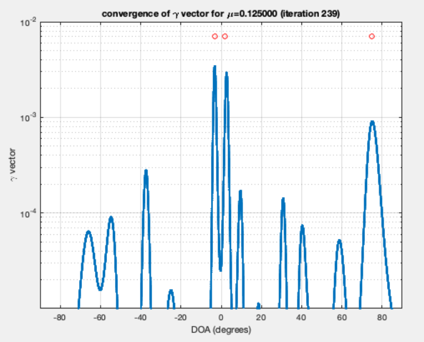


3 sources, $\mu = 1.000$, ϵ-contaminated data

M-Estimation based on CES-distribution 6/7 (Animation for $\mu = 0.125$)



3 sources, $\mu = 0.125$, Gaussian data



3 sources, $\mu = 0.125$, ϵ-contaminated data

$$\boxed{\gamma_m^{(j)} = (1-\mu)\gamma_m^{(j-1)} + \mu\gamma_m^{(j-1)} \left(\frac{\boldsymbol{a}_m^H \Sigma^{-1} \boldsymbol{R}_{\boldsymbol{Y}} \Sigma^{-1} \boldsymbol{a}_m}{\boldsymbol{a}_m^H \Sigma^{-1} \boldsymbol{a}_m} \right)}$$

Stepsize $\mu \in [0,1]$ determines γ convergence

Main result: Convergence $\lim_{j\to\infty} \gamma_m^{(j)}$ is guaranteed...

- for any $\mu \in [0,1]$ if $\gamma_m = 0$, i.e., $\forall m \notin \mathcal{M}$
- for some $\mu \in [0, \mu_{\max}]$ if $\gamma_m > 0$, i.e., $\forall m \in \mathcal{M}$, with $\mu_{\max} < 1$

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Sketch of the Proof (1/5)

The proof proceeds in two steps

1. The true solution is a fixed point of the iteration
2. The iteration update decreases the error

According to the update rule:

$$\gamma_m^{(j)} = (1-\mu)\gamma_m^{(j-1)} + \mu\gamma_m^{(j-1)} \left(\frac{\boldsymbol{a}_m^H \Sigma^{-1} \boldsymbol{R}_{\boldsymbol{Y}} \Sigma^{-1} \boldsymbol{a}_m}{\boldsymbol{a}_m^H \Sigma^{-1} \boldsymbol{a}_m} \right)$$

and assuming perfect estimate $\boldsymbol{R}_{\boldsymbol{Y}} = \Sigma$ we see that

$$\boxed{\boldsymbol{\gamma}^{(j)} = \boldsymbol{\gamma}^{\mathrm{true}} \quad \text{if} \quad \boldsymbol{\gamma}^{(j-1)} = \boldsymbol{\gamma}^{\mathrm{true}}}$$

where $\boldsymbol{\gamma}^{\mathrm{true}}$ is the true vector of source powers for any μ.

The estimate $\boldsymbol{R}_{\boldsymbol{Y}}$ is unbiased for Σ when $\boldsymbol{y}_\ell$ follow any of distributions in the Table (thanks to b).

For convergence:

the new error must be less than the previous error,

$$|\gamma_m^{\mathrm{true}} - \gamma_m^{(j)}| \quad < \quad |\gamma_m^{\mathrm{true}} - \gamma_m^{(j-1)}|.$$

This is ensured if

$$\left| 1 + \mu\gamma_m^{(j-1)} \left(\frac{\partial G_m(\gamma)}{\partial \gamma_m} \right)_{\gamma=\tilde{\gamma}} \right| < 1,$$

where

$$G_m(\gamma) = \frac{\boldsymbol{a}_m^H \Sigma^{-1} \boldsymbol{R}_{\boldsymbol{Y}} \Sigma^{-1} \boldsymbol{a}_m}{\boldsymbol{a}_m^H \Sigma^{-1} \boldsymbol{a}_m}$$

For inactive indices m, we have $\gamma_m^{\mathrm{true}} = 0$,

and it turns out that any μ with $0 < \mu < 1$ guarantees $\gamma_m^{(j)} \to 0$.

The active set is estimated for any μ

The active set determines DOAs

For active indices m, we have $\gamma_m^{\mathrm{true}} > 0$,

and it turns out that any μ with $0 < \mu < \mu_{\mathsf{max}}$ guarantees $\gamma_m^{(j)} \to \gamma_m^{\mathrm{true}}$.

Source power estimation requires smaller μ than DOA estimation

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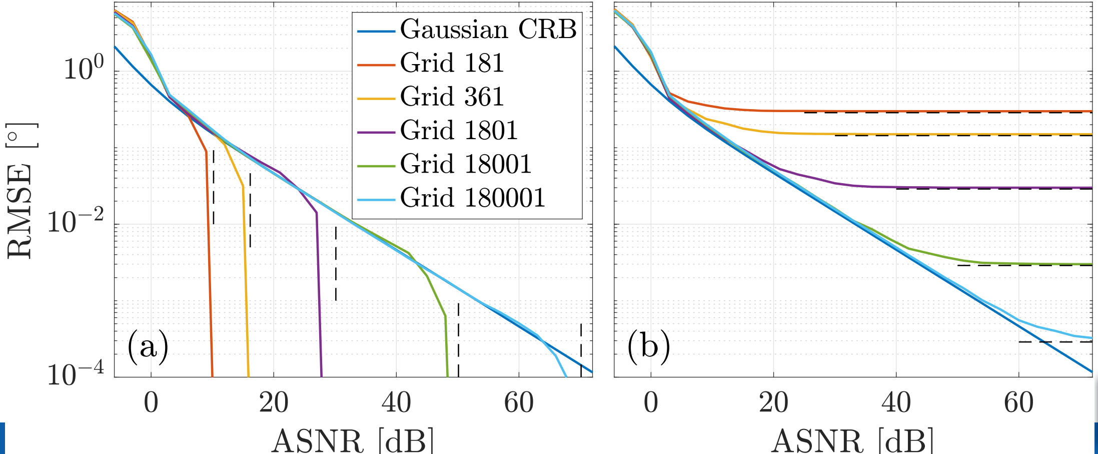
Simulations Results

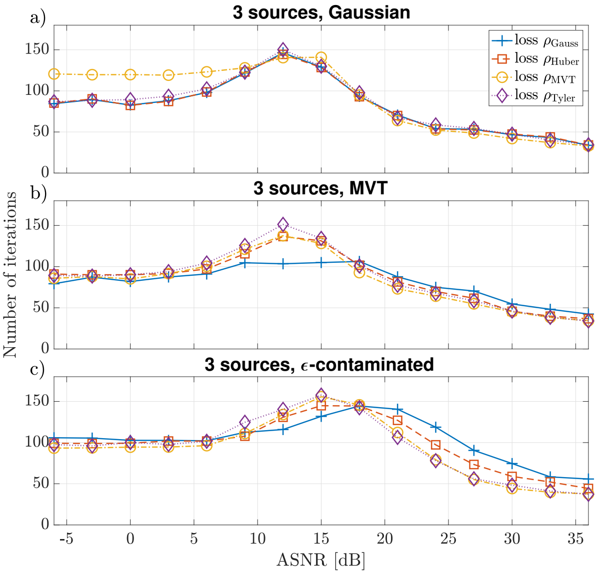
- Simulation with synthetic data for ULA,
  - $N = 20$ sensors,
  - $L = 25$ array data snapshots,
  - $K = 1$ a single active source at DOA $\theta = -45^\circ$
  - Two DOA grid sizes $M = 181$ (low res) and $M = 18001$ (high res).
- We evaluate Root Mean Square Error (RMSE) of DOA estimators vs. Array Signal to Noise Ratio (ASNR).
- Results averaged over $10^6/L = 4 \cdot 10^4$ realizations.
- Several noise models:
  - Complex multivariate Gaussian
  - Complex multivariate Student, $\nu = 2.1$
  - ϵ-contaminated complex multivariate Gaussian, $\epsilon = 0.05$, $\lambda = 10$

Simulation Results (Grid Resolution Effect on RMSE)

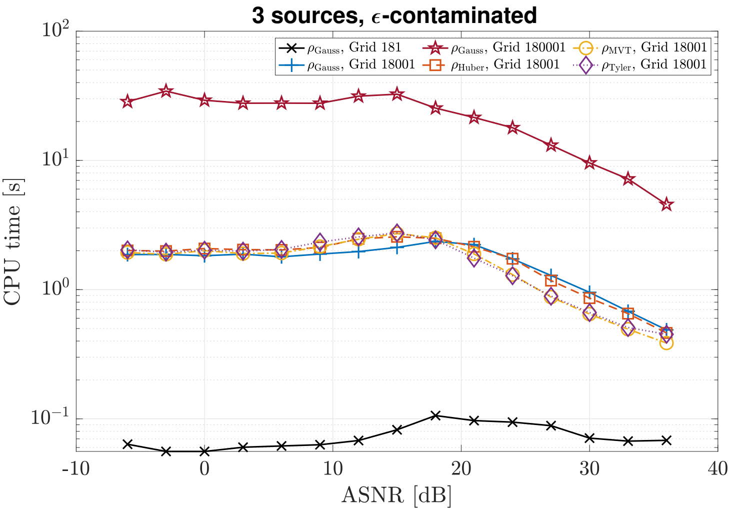
left: fixed DOA -10° on grid, right: uniformly distributed DOA $\sim -10^\circ + U(-\delta/2, \delta/2)$.



Simulation Results (Array Data Distribution Effect on Required Iteration Count)



Simulation Results (Grid Resolution Effect on Required Iteration Count)



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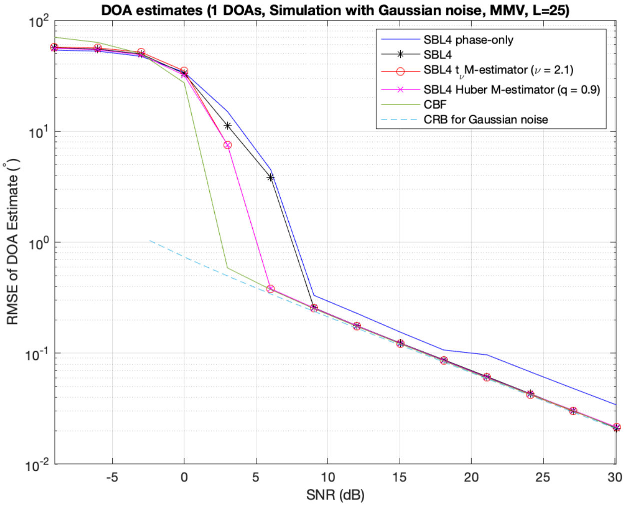
Summary and Conclusion

- Convergence of an M-Estimator for DOAs investigated for CES array data
- General approach based on loss functions
- Investigation of iteration convergence: Analytically and Numerically
-
- M-Estimator DOA performs well for all investigated noise models
- Matlab implementation on GitHub
  https://github.com/NoiseLabUCSD/RobustSBL

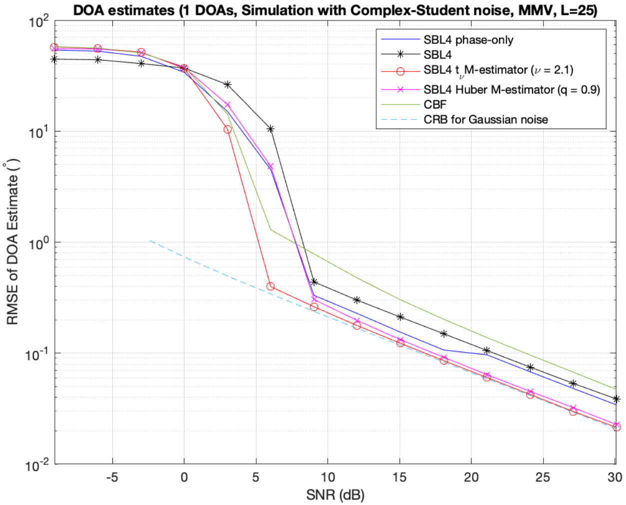
Backup: Simulation Results (Gaussian noise)

$M = 18001$, DOA grid spacing 0.01°



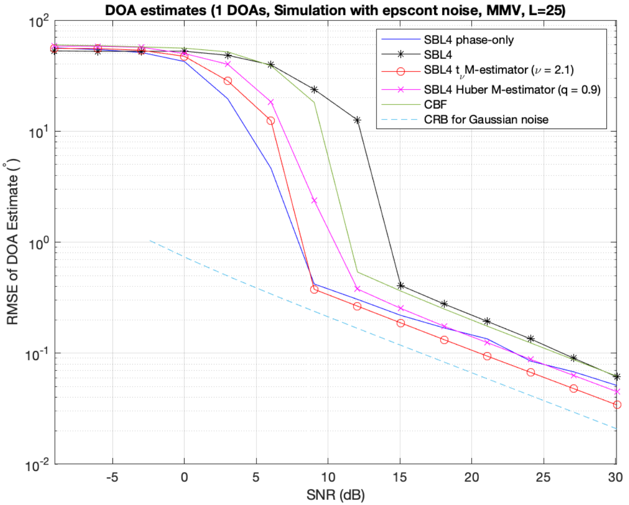
Backup: Simulation Results (Complex Student noise, $\nu = 2.1$)

$M = 18001$, DOA grid spacing 0.01°



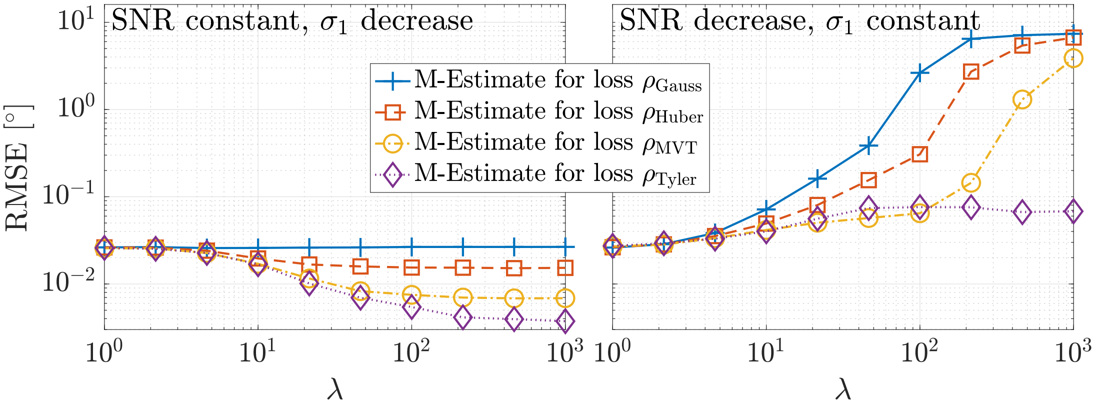
Backup: Simulation Results (ϵ-contaminated noise, $\epsilon = 0.05$, $\lambda = 10$)

$M = 18001$, DOA grid spacing 0.01°



Backup: Simulation Results (Outlier Strength Effect on RMSE)

ϵ-contaminated data $\epsilon = 0.05$ with one source, high SNR



Backup: DOA M-Estimation Algorithm

input $\boldsymbol{Y} \in \mathbb{C}^{N \times L}$ array data to be analyzed
choose the desired weight function $u(\cdot;\cdot)$ and loss parameter
constant $\boldsymbol{A} \in \mathbb{C}^{N \times M}$ dictionary matrix
constants ν, $K, j_{\max} = 1200$
initialize $\hat{\sigma}^2$, γ^{new}, $j = 0$
repeat
    $j = j + 1$, $\gamma^{\text{old}} = \gamma^{\text{new}}$, $\Gamma = \operatorname{diag}(\gamma^{\text{new}})$
    $\Sigma = \boldsymbol{A}\Gamma\boldsymbol{A}^H + \hat{\sigma}^2 \boldsymbol{I}_N$
    $\boldsymbol{R}_{\boldsymbol{Y}} = \frac{1}{L}\sum_{\ell=1}^{L} u(\boldsymbol{y}_\ell^H \Sigma^{-1} \boldsymbol{y}_\ell; \cdot)\boldsymbol{y}_\ell \boldsymbol{y}_\ell^H$
    $\gamma_m^{\text{new}} = (1 - \mu)\gamma_m^{\text{old}} + \mu\gamma_m^{\text{old}} \left(\frac{\boldsymbol{a}_m^H \Sigma^{-1} \boldsymbol{R}_{\boldsymbol{Y}} \Sigma^{-1} \boldsymbol{a}_m}{\boldsymbol{a}_m^H \Sigma^{-1} \boldsymbol{a}_m} \right)$
    $\mathcal{M} = \{m \in \mathbb{N} \mid K \text{ largest peaks in } \gamma^{\text{new}}\}$ active set
    $\boldsymbol{A}_{\mathcal{M}} = [\boldsymbol{a}_{m_1}, \ldots, \boldsymbol{a}_{m_K}]$
    $\hat{\sigma}^2 = \hat{\sigma}_R^2 = \frac{\operatorname{tr}\left[(\boldsymbol{I}_N - \boldsymbol{A}_{\mathcal{M}}\boldsymbol{A}_{\mathcal{M}}^{+})\boldsymbol{R}_{\boldsymbol{Y}}\right]}{N-K}$
until convergence or $j > j_{\max}$
output $\mathcal{M}$, γ^{new}, $\hat{\sigma}^2$

Table: DOA M-Estimation using Sparse Bayesian Learning

- The CRB is in [21, Eq. (8.106)].

$$C_{CR}(\boldsymbol{\theta}) = \frac{\sigma^2}{2L}\left\{\mathrm{Re}\left\{\left[\Gamma_{\mathcal{M}}\left[\left(\boldsymbol{I}_K + \boldsymbol{A}_{\mathcal{M}}^H \boldsymbol{A}_{\mathcal{M}} \frac{\Gamma_{\mathcal{M}}}{\sigma^2}\right)^{-1}\left(\boldsymbol{A}_{\mathcal{M}}^H \boldsymbol{A}_{\mathcal{M}} \frac{\Gamma_{\mathcal{M}}}{\sigma^2}\right)\right]\right] \odot \boldsymbol{H}^T\right\}\right\}^{-1}$$

with

$$\begin{aligned}
\Gamma_{\mathcal{M}} &= \mathrm{diag}(\boldsymbol{\gamma}_{\mathcal{M}}),\\
\boldsymbol{H} &= \boldsymbol{D}^H\left(\boldsymbol{I}_N - \boldsymbol{A}_{\mathcal{M}}\boldsymbol{A}_{\mathcal{M}}^{+}\right)\boldsymbol{D},\\
\boldsymbol{D} &= \left[\left(\frac{\partial \boldsymbol{a}(\theta)}{\partial\theta}\right)_{\theta=\theta_1} \cdots \left(\frac{\partial \boldsymbol{a}(\theta)}{\partial\theta}\right)_{\theta=\theta_K}\right].
\end{aligned}$$

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ITA 2024, San Diego, Feb. 19, 2024

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