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Undecidability of the universal support problem for weighted automata over zero-sum-free commutative semirings ☆

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ABSTRACT

We show that there is an effectively given zero-sum-free commutative semiring S, contained as the subsemiring of nonnegative elements in an effectively given commutative ordered ring, for which there are no procedures deciding, given a weighted finite automaton over S, whether its support is the language of all words or whether its support is infinite. In particular, by a result of D. Kirsten (2011), since S is zero-sum-free and commutative, the support is recognizable by a classical finite automaton, but such an automaton or even just a pushdown automaton for its support cannot be constructed effectively.

1. Introduction

Weighted finite automata, introduced by Schützenberger [17], can be used to model quantitative properties of systems. Their behaviors associate to each word a value reflecting the cost for executing the word in the system. The computations are usually performed in a semiring. This model contains classical finite automata which are obtained by taking the two-valued Boolean semiring as structure of values. This connection led to a rich theory of weighted finite automata, as witnessed by the books [7,16,11,1,14,6] and by the surveys [15,12,5].

A fundamental question in the theory of weighted finite automata is when their support, i.e., the language of all words with non-zero value, is a classical recognizable language. As easy examples show, even over the ring of integers as weight structure the supports might not be context-free. However, if the semiring is finite, or just locally finite, or positive (i.e., zero-sum-free and zero-divisor-free), the support is easily seen to be recognizable. This includes important examples like all Boolean algebras, the semiring of natural numbers and the max-plus-semiring and min-plus-semiring of nonnegative real numbers. If the semiring is just zero-sum-free and commutative, Kirsten [10] showed that the support of any weighted finite automaton is recognizable; moreover, he gave a necessary and sufficient condition for the semiring so that an automaton for the support can be constructed effectively from the given weighted finite automaton. However, it remained open whether there are effectively given semirings violating this condition.

The goal of this paper is to construct zero-sum-free commutative semirings for which there are no procedures deciding, given a weighted finite automaton, whether its support is the language of all words or whether its support is infinite. In particular, by Kirsten's result, the support is recognizable by a classical finite automaton, but such an automaton or even just a pushdown automaton for

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its support cannot be constructed effectively. The semirings will be constructed in an effective way, i.e., with effectively given representations of the elements and their operations and with an effective test whether two representations give the same element. Moreover, they can be taken to be contained, as subsemirings, in commutative ordered rings, which could be viewed as supporting that the semirings arise as natural examples.

In the literature, several authors have achieved related undecidability results for reachability problems, like the mortality problem or the zero-in-the corner problems, for products of matrices over the ring of integers or the max-plus-semiring of integers, cf. [3,8,2]. These results easily translate into undecidability results whether weighted finite automata over these rings resp. semirings generate 0 as value; the latter is equivalent to the support not containing all words. However, these results use positive and negative integers for the constructions; if restricted to non-negative integer values, the semiring considered is positive and hence the universality of the support problem is decidable, in contrast to our results. In [9], it was shown that it is undecidable whether an arbitrary weighted automaton over the semiring $\mathbb{N}$ of natural numbers generates *all* natural numbers as values.

Our construction yields a zero-sum-free commutative semiring contained as the set of nonnegative elements in a commutative ordered ring. Our proof idea rests on a construction presented in [13, Ex. 1], but requires the construction of a suitable semiring. The method was also used recently in [4] for a construction of an effectively given idempotent commutative semiring such that for given weighted finite automata it is undecidable whether the image of their behavior is infinite. These semirings, being idempotent, are not contained as subsemirings in commutative rings.

2. Background

We denote by $\mathbb{Z}$ the set of all integers and by $\mathbb{N}$ the subset of nonnegative integers. Let $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$.

A *semiring* is an algebra $(S, +, \times, 0, 1)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \times, 1)$ is a monoid, multiplication is distributive over addition from both sides and $0 \times s = s \times 0 = 0$ for all $s \in S$. We say that the semiring S is

- *commutative*, if the multiplication $\times$ is commutative,
- *zero-sum-free*, if $s + s' = 0$ implies $s = s' = 0$, for all $s, s' \in S$,
- a *ring*, if $(S, +, 0)$ is a group.

Let Σ be an alphabet and $(S, +, \times, 0, 1)$ a semiring.

A *weighted finite automaton* over Σ and S is a quadruple $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ such that Q is a non-empty finite set, $\mathrm{in} : Q \to S$ is the initial weight mapping, $\mathrm{wt} : Q \times \Sigma \times Q \to S$ is the transition weight mapping, and $\mathrm{out} : Q \to S$ is the final weight mapping.

Let $n \in \mathbb{N}$ and $w = \sigma_1 \cdots \sigma_n \in \Sigma^*$ with $\sigma_1, \ldots, \sigma_n \in \Sigma$. A *run* of $\mathcal{A}$ on w is any sequence $\rho = (q_0, q_1, \ldots, q_n)$ of states $q_0, q_1, \ldots, q_n \in Q$. The *weight* of ρ on w is $\mathrm{wt}(w, \rho) = \mathrm{wt}(q_0, \sigma_1, q_1) \times \cdots \times \mathrm{wt}(q_{n-1}, \sigma_n, q_n)$. Here, if $n = 0$, we let $\rho = q_0$ and $\mathrm{wt}(w, \rho) = 1$. We also let $\mathrm{in}(\rho) = \mathrm{in}(q_0)$ and $\mathrm{out}(\rho) = \mathrm{out}(q_n)$. Let $R(w)$ be the set of all runs ρ of $\mathcal{A}$ on w.

The *behavior, series*, or *weighted language* of $\mathcal{A}$ is the function $||\mathcal{A}|| : \Sigma^* \to S$ defined, for each $w \in \Sigma^*$, by

$$||\mathcal{A}||(w) = \sum_{\rho \in R(w)} \mathrm{in}(\rho) \times \mathrm{wt}(w, \rho) \times \mathrm{out}(\rho).$$

In particular, $||\mathcal{A}||(\epsilon) = \sum_{q \in Q} \mathrm{in}(q) \times \mathrm{out}(q)$.

Any function $r : \Sigma^* \to S$ is also called a *series*. We let $\mathrm{supp}(r) = \{w \in \Sigma^* \mid r(w) \neq 0\}$, the *support* of r, and $\mathrm{im}(r) = \{r(w) \mid w \in \Sigma^*\}$, the *image* of r. We note that $\mathrm{supp}(r) = \Sigma^*$ if and only if $0 \notin \mathrm{im}(r)$. The series r is called *recognizable*, if $r = ||\mathcal{A}||$ for some weighted finite automaton $\mathcal{A}$.

A semiring $(S, +, \times, 0, 1)$ is said to be *effectively given*, if S can be represented as a recursive subset of $\mathbb{N}$ and the operations $+, \times$ are effectively given. We say that S has *effective tests for equality*, if the truth value of $s = s'$ for $s, s' \in S$ given by their representations can be decided effectively. We call S a *computable* semiring, if S is effectively given with effective tests for equality.

3. Main result

Our main result will be the following.

Theorem 1. *There is a zero-sum-free commutative semiring S contained as a recursive subset in a computable commutative ring R such that for any finite alphabet Σ it is undecidable for a given weighted finite automaton $\mathcal{A}$ over Σ and S whether* $\mathrm{supp}(||\mathcal{A}||) = \Sigma^*$ *and whether* $\mathrm{supp}(||\mathcal{A}||)$ *is infinite.*

As noted before, $\mathrm{supp}(||\mathcal{A}||) = \Sigma^*$ is equivalent to $0 \notin \mathrm{im}(||\mathcal{A}||)$. Hence, equivalently, it is undecidable whether 0 occurs as value of $||\mathcal{A}||$.

By a result of Kirsten [10], since S is zero-sum-free and commutative, for any weighted finite automaton $\mathcal{A}$ over Σ and S, $\mathrm{supp}(||\mathcal{A}||)$ is recognizable. Theorem 1 shows that in general we cannot construct a finite automaton recognizing $\mathrm{supp}(||\mathcal{A}||)$, since otherwise we could decide whether $\mathrm{supp}(||\mathcal{A}||) = \Sigma^*$; moreover, we cannot construct a pushdown automaton recognizing $\mathrm{supp}(||\mathcal{A}||)$, since otherwise we could decide whether $\mathrm{supp}(||\mathcal{A}||)$ is infinite.

Proof. Let $Z = \mathbb{Z}\langle x_i | i \in \mathbb{N}\rangle$ be the commutative ring of all polynomials over the variables x_i ($i \in \mathbb{N}$) with coefficients in $\mathbb{Z}$. So, each element of Z is a finite sum of elements of the form $a_x x$ with x being a finite product of the form $x = x_1^{k_1} ... x_n^{k_n}$ where $n \in \mathbb{N}_+$ and $k_1, ..., k_n \in \mathbb{N}$, and $a_x \in \mathbb{Z}$.

Choose a recursively enumerable non-recursive subset I of $\mathbb{N}_+$. We may assume that we have an effective enumeration $I = \{j_m \mid m \in \mathbb{N}_+\}$ of I with pairwise different elements j_m ($m \in \mathbb{N}_+$).

Now consider the congruence C generated by the equivalences $x_{j_m}^m \sim 0$ ($m \in \mathbb{N}_+$) on the ring Z, and let R be the factor ring of Z modulo C. We write $[z] = [z]_C$ for the congruence class of $z \in Z$. Note that the non-zero elements of R are finite sums of elements of the form $a_x[x]$ with $a_x \in \mathbb{Z} \setminus \{0\}$ and $x = x_1^{k_1} ... x_n^{k_n}$ with $n \in \mathbb{N}_+$, $k_1, ..., k_n \in \mathbb{N}$, and $k_i < m$ in case $i = j_m$ for some $m \in \mathbb{N}_+$, for each $1 \leq i \leq n$. The operations of R are given by the operations of Z, so $[z] + [z'] = [z + z']$ and $[z] \times [z'] = [z \times z']$ for all $z, z' \in Z$.

Clearly, the operations on R are effectively given. We claim that R has an effective test for equality of two elements; that is, we can decide, given an arbitrary element $r \in R$, whether $r = 0$.

Let us first consider elements r of the particular form $r = [x_i]^k$ with $i \in \mathbb{N}_+$ and $k \in \mathbb{N}_+$. For this, we compute the elements $j_1, ..., j_k$ of I and we check whether $i = j_m$ for some $1 \leq m \leq k$. In this case, we have $r = [x_{j_m}]^k = 0$ since $m \leq k$.

In the other case, we claim that $r \neq 0$. If $i \notin I$, this is clear. Now let $i = j_m$ for some $m \in \mathbb{N}_+$. By the assumption of this case, we have $k < m$ and thus $r \neq 0$.

Next, if $r = [x]$ where $x = x_1^{k_1} ... x_n^{k_n}$ with $n \in \mathbb{N}_+$ and $k_1, ..., k_n \in \mathbb{N}$, we check whether some factor $[x_i]^{k_i} = 0$ (for some $i \in \{1, ..., n\}$).

Finally, if r is a finite sum of elements $a_x[x]$ with $a_x \in \mathbb{Z}$ and x as above, we may assume that we have simplified it so that the elements $[x]$ are pairwise different and different from 0. Then we check whether $a_x = 0$ for each $[x]$.

Now let S be the subsemiring of R comprising all elements s with non-negative coefficients, that is, s is a finite sum of elements of the form $a_x[x]$ with $a_x \in \mathbb{N}$ and $x = x_1^{k_1} ... x_n^{k_n}$ with $n \in \mathbb{N}_+$ and $k_1, ..., k_n \in \mathbb{N}$. Clearly, S is a recursive subset of R, and S is zero-sum-free.

To prove the claim of the theorem, let Σ be an arbitrary alphabet. For $i \in \mathbb{N}_+$, let $\mathcal{A}_i = (Q, \mathrm{in}, \mathrm{wt}_i, \mathrm{out})$ be the weighted automaton over Σ and S given by $Q = \{q\}$, a singleton set, $\mathrm{in}(q) = \mathrm{out}(q) = 1$ and $\mathrm{wt}_i(q, \sigma, q) = [x_i]$ for each $\sigma \in \Sigma$. Then $||\mathcal{A}_i||(w) = [x_i]^{|w|}$, where $|w|$ is the length of w. Observe that for any $n \in \mathbb{N}_+$ we have $[x_i]^n \neq 0$ if $i \notin I$, but $[x_i]^n = 0$ if $i = j_m$ for some $m \in \mathbb{N}_+$ and $m \leq n$. Hence $\mathrm{supp}(||\mathcal{A}_i||)$ is finite iff $\mathrm{supp}(||\mathcal{A}_i||) \neq \Sigma^*$ iff $0 \in \mathrm{im}(||\mathcal{A}_i||)$ iff $i \in I$. Now, since I is non-recursive, there is no procedure to decide for an arbitrary $i \in \mathbb{N}_+$ whether $\mathrm{supp}(||\mathcal{A}_i||) = \Sigma^*$ or whether $\mathrm{supp}(||\mathcal{A}_i||)$ is infinite.

Remarks. 1. For this semiring S, it also cannot be decided whether $\mathrm{im}(||\mathcal{A}||)$ is infinite. Given $\mathcal{A}_i$, we have: $\mathrm{im}(||\mathcal{A}_i||)$ is finite iff $i \in I$.

2. The above ring R is a naturally ordered ring with S as its set of non-negative elements:

For $x, y \in R$, we can put $x \leq y$ iff $y = x + s$ for some $s \in S$. Thus, $S = \{r \in R \mid r \geqslant 0\}$. Then $x \leq y$ implies $x + z \leq y + z$ for all $z \in R$, and $x \cdot z \leq y \cdot z$ for all $z \in R$ with $z \geqslant 0$.

3. There is also a computable idempotent zero-sum-free commutative semiring S' such that for any alphabet Σ, it is undecidable for a given weighted finite automaton $\mathcal{A}$ over Σ and S' whether $\mathrm{supp}(||\mathcal{A}||) = \Sigma^*$ or whether $\mathrm{supp}(||\mathcal{A}||)$ is infinite. For this, we can start with the free idempotent semiring over the free generators x_i ($i \in \mathbb{N}_+$). In the representation of the elements, the coefficients then are elements of the Boolean semiring $\{0, 1\}$. As in 2. for $x, y \in S'$ we can put $x \leq y$ iff $y = x + s$ for some $s \in S'$. Then S' becomes a naturally ordered semiring which, moreover, is *past-finite*, i.e., for each $y \in S'$, the set $\{x \in S' \mid x \leq y\}$ is finite. Here, it also cannot be decided whether $\mathrm{im}(||\mathcal{A}||)$ is infinite. This is Theorem 7 of [4].

CRediT authorship contribution statement

Manfred Droste: Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Werner Kuich:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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