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Equilibrium Description Logics: Results on Complexity and Relations to Circumscription

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Abstract

Recently, *Equilibrium Description Logics* (EDLs) have been suggested as a promising new approach to Description Logics (DLs) with non-monotonic default negation. However, a deeper understanding of EDLs in terms of computational complexity and relations to other formalisms is still missing. Motivated by this, in this paper we investigate the computational complexity of reasoning in EDLs both in the case of expressive DLs like $\mathcal{ALCIO}$ and lightweight DLs in the $\mathcal{EL}$ and DL-Lite families. We establish a translation on EDLs into DLs with circumscription, introducing an extension of circumscribed DLs where a further set of axioms is attached to circumscribed KBs to filter out unintended minimal models. Such translation not only applies in the case of classical circumscription but can be extended to the recently introduced *pointwise circumscribed* DLs. We introduce *pointwise* EDLs where the single global minimality check on models is replaced by *local* minimality checks at the single domain elements in the style of *pointwise circumscription*. We provide preliminary results on the computational complexity of reasoning in pointwise EDLs. In particular, via the translation into pointwise circumscription, we inherit the decidability results of pointwise circumscribed DLs. Furthermore, we show that for a large class of acyclic ontologies EDLs and pointwise EDLs accept the same set of stable models. To this aim, we identify a class of ontologies where circumscription and pointwise circumscription accept the same set of minimal models, providing new decidability results for circumscribed DLs even in the presence of minimized and fixed roles.

1 Introduction

Description Logics (DLs) are a family of formalisms tailored for representing and reasoning about knowledge pertaining to a domain of interest. DLs are fragments of first-order logic and thus inherit from it many features, including its *monotonicity*. Extending DLs with non-monotonic features is challenging. In recent years, several non-monotonic extensions have been proposed in an attempt to increase the expressiveness of the formalisms while mitigating the computational costs (Baader and Hollunder 1995; Donini, Nardi, and Rosati 2002; Giordano et al. 2013; Britz et al. 2021), among which a prominent research line has been marked by *circumscribed DLs* (Bonatti, Lutz, and Wolter 2009; Bonatti et al. 2015; Bonatti 2021; Di Stefano, Ortiz, and Šimkus 2023; Lutz, Manière, and Nolte 2023; Bonatti et al. 2023).

Equilibrium Description Logics (EDLs) have been recently introduced in (Di Stefano and Šimkus 2024) as a promising approach to bring non-monotonic capabilities to DLs, among which is *default negation*. EDLs are based on *Quantified Equilibrium Logic* (QEL) (Pearce 2006; Pearce and Valverde 2008). QEL provides logical foundations to the stable model semantics of logic programs, allowing to extend it to arbitrary theories in first-order logic. QEL is based on *Here-and-There* logic with an additional minimality requirement, in the spirit of *circumscription*.

Although both EDLs and *circumscribed DLs* involve *minimal model reasoning*, the two formalisms are different. In particular, EDLs overcome some of the limitations of circumscription and capture in a more natural way the intuition of *justification* for the membership of a tuple in an extension of a predicate. Consider the following DL knowledge base stating that pizza margherita ($marg$) has two vegetarian ingredients, mozzarella ($mozz$) and tomatoes (tom), and a meal is vegetarian if all ingredients are vegetarian:

$$\mathrm{Pizza}(marg) \quad \mathrm{Veg}(mozz) \quad \mathrm{Veg}(tom)$$
$$\mathrm{hasIngredient}(marg, mozz) \quad \mathrm{hasIngredient}(marg, tom)$$
$$\forall \mathrm{hasIngredient}.\mathrm{Veg} \sqsubseteq \mathrm{Veg\_Meal}$$

If we want to apply a form of *closed-world* assumption on the ingredients of pizza margherita, we expect to infer that pizza margherita is a vegetarian meal. However, the classical semantics does not rule out the existence of further unknown ingredients for pizza margherita. We may attempt to use circumscription, requiring that all predicates are *minimized*. However, as for the classical semantics, circumscription does not derive that margherita is a vegetarian meal. Indeed, a model where margherita has further non-vegetarian ingredients is accepted as minimal. EDLs handle this scenario properly as occurrences of further ingredients in pizza margherita are not justified and can be ruled out. EDLs have been proposed in (Di Stefano and Šimkus 2024) in the context of DL *terminologies*, i.e. collections of terminological definitions. EDLs overcome some of the limitations of previous approaches, such as the syntactic monotonicity requirement in (Baader 1990; De Giacomo and Lenzerini 1997), and reasoning is not more expensive than reasoning under the classical semantics.

Reasoning with the semantics of EDLs is challenging due to the underlying minimality requirement on models, and a deep understanding of the computational complexity is still missing. In general, reasoning under forms of minimization, e.g., circumscription, is difficult. In circumscribed $\mathcal{ALC}$, if roles are fixed concept satisfiability is *undecidable* (Bonatti, Lutz, and Wolter 2009). Undecidability is typically caused by the interaction between *varying* predicates and *minimized* and *fixed* ones. However, also under the stronger assumption that all predicates are minimized, concept satisfiability becomes undecidable in $\mathcal{ALCIO}$ (Di Stefano and Šimkus 2024). Finding ways to mitigate complexity under forms of minimization is a relevant problem and many attempts to find decidable fragments have been proposed, among which we have *pointwise circumscription* (Di Stefano, Ortiz, and Šimkus 2023), circumscribed DLs in the DL-Lite family (Bonatti et al. 2023; Lutz, Manière, and Nolte 2023), and DL terminologies (Di Stefano and Šimkus 2024).

Motivated by this, in this paper we investigate the computational complexity of reasoning in EDLs, both in the case of expressive DLs like $\mathcal{ALCIO}$ and lightweight DLs in the $\mathcal{EL}$ and DL-Lite families. We extend our investigation by introducing a new formalism of *pointwise* EDLs, which inherit the computational benefits of *pointwise circumscription*. The main contributions are summarized as follows.

∘ We provide a formal translation of EDLs into circumscribed DLs. We show that an EDL knowledge base can be transformed into a circumscribed knowledge base such that there is a correspondence between the stable models of the first and a *restricted* set of models of the second one. Specifically, this restricted set of models is obtained by filtering out those models satisfying a further set of inclusions. To formalize such additional restrictions, we propose an *hybrid* form of circumscription where a circumscribed knowledge base is paired with a non-circumscribed one. This second knowledge base acts as a filter over the set of minimal models, allowing further refinement of the selection process of minimal models. We call this formalism *constrained circumscription*. We show that constrained circumscription adds expressiveness to circumscription, in some cases without an additional computational cost.

∘ We study the computational complexity of the standard reasoning tasks of concept satisfiability, subsumption, and instance checking in different EDLs. With a reduction from the *domino problem*, we show that concept satisfiability w.r.t. general KBs in $\mathcal{ALC}$ is undecidable under the stable model semantics, extending the undecidability results of (Di Stefano and Šimkus 2024). Undecidability carries over to subsumption in $\mathcal{ELIO}$, both under the stable model semantics and circumscription. Assuming roles to be fixed, we characterize the complexity in different fragments of DL-Lite$_{Bool}$. Remarkably, under the assumption that roles are fixed, concept satisfiability is Σ_p^2-complete in DL-Lite$_{Bool}$ under the stable model semantics. While concept satisfiability becomes NP-complete in DL-Lite$_{not}$, an extension of DL-Lite$_{Horn}$ with negation on the left-hand side of concept inclusions.

∘ We introduce *pointwise* EDLs where the underlining global minimization of EDLs is replaced by *local* minimizations at individual domain elements in the style of pointwise circumscription. We prove that pointwise EDLs can be translated into pointwise circumscribed DLs with constraints, by extending the analogous result for circumscription. Via such translation, we show that concept satisfiability w.r.t KBs in $\mathcal{ALCIO}$ with modal depth bounded by 1 is decidable in NExpTime. In contrast, we show that reasoning w.r.t general KBs in $\mathcal{ALCI}$ is undecidable.

∘ We identify a family of ontologies where EDLs and pointwise EDLs accept the same set of models. To this aim, we define the dependency graph of a KB induced by a set of minimized predicates. We prove that if such an induced graph is acyclic, then pointwise circumscription and circumscription agree on the set of accepted models. By recalling the translation of EDLs into constrained circumscription, we extend the result to EDLs and pointwise EDLs.

2 Preliminaries

Let N_C, N_R and N_I be countably infinite, mutually disjoint, sets of *concept names*, *role names* and *individual names*, respectively, denoted with $A, B, C, \ldots,\ r, s, t \ldots$ and $a, b, c \ldots$. Given a role name r, r^- denotes the *inverse* of r; we further set $(r^-)^- = r$. We let N_R^+ denote the set of role names and their inverses, i.e. $N_R^+ = N_R \cup \{r^- | r \in N_R\}$. Complex concepts in $\mathcal{ALCIO}$ are defined by the grammar $C := \top \mid \bot \mid A \mid \{o\} \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.C \mid \forall r.C$, where $A \in N_C$, $r \in N_R^+$ and $o \in N_I$. A concept C is in $\mathcal{ALCI}$ if it does not contain any occurrence of a *nominal* $\{o\}$. A concept C is in $\mathcal{ALC}$ if there are no occurrences of role inverses. A concept C is in $\mathcal{ELIO}$ if there are no occurrence of negation, $\bot$, $\sqcup$ and $\forall$. In $\mathcal{ELI}$ only nominal-free concepts from $\mathcal{ELIO}$ are allowed. We denote with $\mathcal{ELI}^\neg$ the extension of $\mathcal{ELI}$ allowing negation in front of concept names.

A *concept inclusion* is an expression of the form $C \sqsubseteq D$ where C and D are complex concepts. A *role inclusion* is an expression of the form $r \sqsubseteq s$, with $r, s \in {N_R}^+$. A TBox $\mathcal{T}$ in $\mathcal{ALCHIO}$ is a finite collection of concept inclusions in $\mathcal{ALCIO}$ and role inclusions. A *basic concept* in DL-Lite is an expression defined by the grammar $B := \bot \mid \top \mid A \mid \exists r$, where $A \in N_C$ and $R \in N_R^+$. A TBox in DL-Lite$_{core}^{\mathcal{H}}$ is a finite collection of role inclusions and concept inclusions of the form $C \sqsubseteq D$ or $C \sqsubseteq \neg D$, where C and D are basic concepts. A TBox in DL-Lite$_{Horn}^{\mathcal{H}}$ is a finite collection of role inclusions and concept inclusions of the form $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq B$, where $C_1, \ldots, C_n, B$ are basic concepts. A TBox is in DL-Lite$_{Bool}^{\mathcal{H}}$ if we furthermore allow for inclusions $C \sqsubseteq D$ where C and D are arbitrary boolean combinations of basic concepts.

A *concept assertion* is an expression of the form $A(a)$, with $A \in N_C$ and $a \in N_I$. A *role assertion* is an expression of the form $r(a, b)$, with $r \in N_R$ and $a, b \in N_I$. An ABox $\mathcal{A}$ is a finite collection of concept and role assertions. A *knowledge base* (KB) in a DL $\mathcal{L}$ is a pair $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ where $\mathcal{A}$ is an ABox and $\mathcal{T}$ is a TBox in $\mathcal{L}$. Given a KB $\mathcal{K}$, we denote with $N_C(\mathcal{K})$, $N_R(\mathcal{K})$, and $N_I(\mathcal{K})$ the set of concept names, role names, and individuals occurring in $\mathcal{K}$.

An interpretation is a structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$

is a non-empty domain and $\cdot^{\mathcal{I}}$ is an interpretation function associating to each $A \in N_C$ a subset $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, to each $r \in N_R$, a subset $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and to each $a \in N_I$ a domain element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. For the interpretation of complex concepts, we refer to (Baader et al. 2017). The notion of a *model* of an ABox, a TBox, and a KB are also standard. We denote with $M(\mathcal{K})$ the set of models of a KB $\mathcal{K}$. Given a concept C and a KB $\mathcal{K}$, C is *satisfiable* w.r.t. $\mathcal{K}$ if there exists $\mathcal{I} \in M(\mathcal{K})$ such that $C^{\mathcal{I}} \neq \emptyset$. Given two concepts C and D and a KB $\mathcal{K}$, C is *subsumed* by D w.r.t. $\mathcal{K}$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all $\mathcal{I} \in M(\mathcal{K})$. Given an individual a and a concept C, a is an *instance* of C w.r.t. $\mathcal{K}$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$, for all $\mathcal{I} \in M(\mathcal{K})$.

3 Circumscribed DLs

We use (Bonatti, Lutz, and Wolter 2009) and (Di Stefano, Ortiz, and Šimkus 2023) as main references respectively for *circumscribed* and *pointwise circumscribed* DLs. Given a KB, when we apply circumscription, beforehand we declare a *circumscription pattern* $\mathcal{P}$ defining which predicates are *minimized*, *varying* or *fixed*. Circumscription then simply selects the models fulfilling the requirement of the circumscription pattern, i.e. those models where the extension of the minimized predicates cannot be further reduced without violating some axioms. Thus, at the core of circumscription, there is the idea of comparing structures and discriminating which one we have to *prefer*.

Definition 1. *Given a set $F \subseteq N_C \cup N_R$ and two interpretations $\mathcal{I}$ and $\mathcal{J}$, we write $\mathcal{I} \sim_F \mathcal{J}$ if*

(i) $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$,
(ii) $a^{\mathcal{I}} = a^{\mathcal{J}}$, *for all individuals* $a \in N_I$,
(iii) $q^{\mathcal{I}} = q^{\mathcal{J}}$, *for all* $q \in F$.

The above definition simply groups interpretations sharing the same domain, interpreting individuals in the same way, and agreeing on the set of predicates in F. Note that F can be empty. In this case, we simply write $\mathcal{I} \sim \mathcal{J}$.

We now formalize the preference relation naturally induced by a circumscription pattern. Formally, a circumscription pattern is a triple $\mathcal{P} = (M, V, F)$, where M, V and F are mutually disjoint sets of predicates, called *minimized*, *varying* and *fixed* predicates. Circumscription patterns can be enriched with a *priority* relation over minimized predicates (Bonatti, Lutz, and Wolter 2009) which we do *not* assume in the following.

Definition 2. *Given a circumscription pattern $\mathcal{P} = (M, V, F)$ and two interpretations $\mathcal{I}, \mathcal{J}$ such that $\mathcal{I} \sim_F \mathcal{J}$, we write:*

- $\mathcal{I} \preceq_{\mathcal{P}} \mathcal{J}$ *if* $p^{\mathcal{I}} \subseteq p^{\mathcal{J}}$, *for all* $p \in M$;
- $\mathcal{I} \prec_{\mathcal{P}} \mathcal{J}$, *if* $\mathcal{I} \preceq_{\mathcal{P}} \mathcal{J}$ *and* $p^{\mathcal{I}} \subset p^{\mathcal{J}}$ *for some* $p \in M$.

If the circumscription pattern $\mathcal{P} = (M, V, F)$ is such that $V = \emptyset$, we denote the relations $\preceq_{\mathcal{P}}$ and $\prec_{\mathcal{P}}$ with the symbols $\subseteq_F$ and $\subset_F$, respectively.

A circumscription pattern $\mathcal{P} = (M, V, F)$ *for* a KB $\mathcal{K}$ is a partition of the predicates occurring in $\mathcal{K}$. A KB $\mathcal{K}$ equipped with a circumscription pattern $\mathcal{P}$ is called *circumscribed* and we denote it with $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$. Given a circumscribed KB, we aim to restrict its set of classical models to those where the extension of minimized predicates is the smallest possible.

Definition 3. *(Minimal model) An interpretation $\mathcal{I}$ is a* minimal model *of a KB $\mathcal{K}$ equipped with a circumscription pattern $\mathcal{P}$, in symbols $\mathcal{I} \models \mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$, if $\mathcal{I} \models \mathcal{K}$ and there is no interpretation $\mathcal{J}$ s.t. $\mathcal{J} \models \mathcal{K}$ and $\mathcal{J} \prec_{\mathcal{P}} \mathcal{I}$. We use $MM(\mathcal{K}, \mathcal{P})$ to denote the set of models of* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$.

Given a KB $\mathcal{K}$ and a model $\mathcal{I}$, applying circumscription w.r.t. a circumscription pattern $\mathcal{P}$ allows for the reconfiguration of the predicates across the entire model. The extension of a minimized predicate p can be *globally* minimized, i.e. possibly infinitely many tuples can be removed from $p^{\mathcal{I}}$. Moreover, varying predicates are not subjected to any restriction. For instance, in order to obtain a smaller model, a single tuple can be removed from a minimized predicate p, while reconfiguring across the model, possibly infinitely many times, all the varying predicates. In (Di Stefano, Ortiz, and Šimkus 2023) an approximation of circumscription based on *pointwise circumscription* (Lifschitz 1986) has been introduced. Differently from (global) circumscription, in pointwise circumscription (Lifschitz 1986) predicates can be minimized only locally at a given tuple of domain elements. In pointwise circumscribed DLs, the minimization can only affect a domain element and the roles it participates in, leaving the rest of the structure unmodified. In this setting, given a circumscribed KB $\mathcal{K}$, the minimality of a model $\mathcal{I}$ of $\mathcal{K}$ can be checked by repeatedly applying a *local* minimality check at all domain elements. First, we define a relation $\sim^\bullet$ between interpretations that may differ in terms of concept and role names involving one domain element.

Definition 4. *Given two interpretations $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \sim^\bullet \mathcal{J}$ if there exists $e \in \Delta^{\mathcal{I}}$ s.t.:*

- *for all* $A \in N_C$, $A^{\mathcal{I}} \setminus \{e\} = A^{\mathcal{J}} \setminus \{e\}$, *and*
- *for all* $r \in N_R$, $r^{\mathcal{I}} \cap (\Delta \times \Delta) = r^{\mathcal{J}} \cap (\Delta \times \Delta)$, *with* $\Delta = \Delta^{\mathcal{I}} \setminus \{e\}$.

We now use the relation in Definition 4 to give a *pointwise* flavor to the preference relation between interpretations of Definition 2.

Definition 5. *Assume a circumscription pattern $\mathcal{P} = (M, V, F)$ and two interpretations $\mathcal{I}, \mathcal{J}$ such that $\mathcal{I} \sim_F \mathcal{J}$. We write*

- $\mathcal{I} \preceq^\bullet_{\mathcal{P}} \mathcal{J}$, *if* $\mathcal{I} \sim^\bullet \mathcal{J}$ *and* $\mathcal{I} \preceq_{\mathcal{P}} \mathcal{J}$;
- $\mathcal{I} \prec^\bullet_{\mathcal{P}} \mathcal{J}$, *if* $\mathcal{I} \preceq^\bullet_{\mathcal{P}} \mathcal{J}$ *and* $p^{\mathcal{I}} \subset p^{\mathcal{J}}$ *for some* $p \in M$

Given a circumscription pattern $\mathcal{P} = (M, V, F)$ such that $V = \emptyset$, we denote $\preceq^\bullet_{\mathcal{P}}$ and $\prec^\bullet_{\mathcal{P}}$ with $\subseteq^\bullet_F$ and $\subset^\bullet_F$, respectively.

Definition 6. *An interpretation $\mathcal{I}$ is a* pointwise minimal model *of a KB $\mathcal{K}$ equipped with a circumscription pattern $\mathcal{P}$, in symbols $\mathcal{I} \models^\bullet \mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$, if $\mathcal{I} \models \mathcal{K}$ and there is no interpretation $\mathcal{J}$ s.t. $\mathcal{J} \models \mathcal{K}$ and $\mathcal{J} \prec^\bullet_{\mathcal{P}} \mathcal{I}$. We use $PMM(\mathcal{K}, \mathcal{P})$ to denote the set of pointwise minimal models of* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$.

The standard definitions of concept satisfiability, concept subsumption, and instance checking are adapted to (resp.

pointwise) minimal models in the obvious way and the corresponding decision problems are interreducible in polynomial time (Bonatti, Lutz, and Wolter 2009; Di Stefano, Ortiz, and Šimkus 2023).

Given a KB $\mathcal{K}$ and a circumscription pattern $\mathcal{P}$, we have that $MM(\mathcal{K}, \mathcal{P}) \subseteq PMM(\mathcal{K}, \mathcal{P})$, while the converse inclusion does not always hold. In Section 6.2, we shall identify a class of ontologies where pointwise minimal models are also minimal models.

4 Constrained Circumscription

In order to characterize the connection between EDL KBs and circumscribed KBs, we extend circumscription with *constraints*. The notion of constraints has been introduced in (Di Stefano, Ortiz, and Šimkus 2023) for pointwise circumscribed KBs as a tool for normalizing complex concepts without affecting minimization. A constraint there is a pair (C, D) that, roughly speaking, discards every minimal model that has some object e where C holds but D does not hold. Coupled with a circumscribed (or pointwise circumscribed) KB, constraints allow for further refinement of the set of minimal models, as the requirement expressed by constraints acts outside circumscription. As one can observe, syntactically a constraint is just a pair of concepts, and thus can be represented as a GCI. In this work, we extend the notion of constraints to KBs, i.e. allowing assertions too.

Definition 7. *Assume a circumscribed KB* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$. *We use* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}) \wedge \mathcal{C}$ *to denote an extension of* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ *with a KB* $\mathcal{C}$. *Given an interpretation* $\mathcal{I}$, *we say that* $\mathcal{I}$ *is a (pointwise) minimal model of* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}) \wedge \mathcal{C}$, *in symbols* $\mathcal{I} \models^{(\bullet)} \mathrm{Circ}_{\mathcal{P}}(\mathcal{K}) \wedge \mathcal{C}$, *if* $\mathcal{I} \models^{(\bullet)} \mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ *and* $\mathcal{I} \models \mathcal{C}$. *We call the KB* $\mathcal{C}$ *a* constraint set.

Compared to (Di Stefano, Ortiz, and Šimkus 2023) where constraints were restricted to mimicking concept inclusions, we not only allow assertions in the constraints but also role inclusions. Furthermore, we remark that the two KBs $\mathcal{K}$ and $\mathcal{C}$ can share the signature.

Expressing Closed Predicates. To further emphasize the difference between circumscription and circumscription with constraints, we show next how circumscribed DLs with constraints are able to express DLs with closed predicates (Franconi, Ibáñez-García, and İnanç Seylan 2011; Lutz, Seylan, and Wolter 2013). In DLs with closed predicates, the extension of some predicates – so-called *closed* – is restricted only to named individuals whose participation in the closed predicates is 'justified' by ABox assertions. An interpretation is a model of a KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ w.r.t. a set Σ of closed predicates, in symbols $\mathcal{I} \models (\mathcal{A}, \mathcal{T}, \Sigma)$, if $\mathcal{I} \models \mathcal{K}$ and for each $A, r \in \Sigma$:

- $d \in A^{\mathcal{I}}$ implies $d = a^{\mathcal{I}}$ for some individual a such that $A(a) \in \mathcal{A}$, and
- $(d, e) \in r^{\mathcal{I}}$ implies $d = a^{\mathcal{I}}$ and $e = b^{\mathcal{I}}$ with a, b such that $r(a, b) \in \mathcal{A}$.

Intuitively, the extension of a closed predicate p is 'circumscribed' to instances of p given by ABox assertions. Although the underlying principle is close to predicate minimization, DLs with closed predicates and (pointwise) circumscribed DLs are two different non-monotonic formalisms and we see no obvious formal translation.

On the other hand, constrained circumscription can easily capture DLs with closed predicates.

Given a KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ with a set of closed predicates Σ, let $\Sigma' = \{p' \mid p \in \Sigma\}$, i.e. we consider a copy of the signature in Σ. Let $\mathcal{A}' = \{p'(\vec{c}) \mid p(\vec{c}) \in \mathcal{A} \text{ and } p \in \Sigma\}$. We obtain this way a new KB $\mathcal{K}' = (\mathcal{A} \cup \mathcal{A}', \mathcal{T})$. We now define a circumscription pattern $\mathcal{P}$ for $\mathcal{K}'$ by stating that all predicates in Σ' are minimized. Since $\mathcal{T}$ does not contain any occurrence of symbols in Σ', in a minimal model $\mathcal{I}'$ we will have that each $p' \in \Sigma'$ contains only named individuals or pairs of named individuals, whose participation in $(p')^{\mathcal{I}}$ is justified by an assertion in $\mathcal{A}'$. To the circumscribed KB $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}')$, we add the set of constraints $\mathcal{C} = \{p \sqsubseteq p' \mid p \in \Sigma\}$. In every model $\mathcal{I}$ of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}') \wedge \mathcal{C}$, every element of $p^{\mathcal{I}}$ is an element of $(p')^{\mathcal{I}}$. In this way, we force p to be closed.

Proposition 1. *Concept satisfiability w.r.t KBs with closed predicates can be polynomially reduced to concept satisfiability w.r.t. (pointwise) circumscribed KBs with constraints.*

The result above almost directly implies that nominals can be simulated using constraints. Indeed this is already possible via closed predicates: each nominal o can be simulated by replacing every occurrence of o with a closed predicate N and adding an assertion $N(o)$.

From Proposition 1, we inherit some of the lower bounds for standard reasoning tasks in DLs with closed predicates (Ngo, Ortiz, and Šimkus 2016).

Corollary 1. *In DL-Lite*$^{\mathcal{H}}_{core}$, *deciding the consistency of a circumscribed KB* $\mathcal{K}$ *with constraints is* NP-*hard.*

In the case of circumscribed DL-Lite$_{core}$ without constraints, concept satisfiability is NLogSpace-complete (Bonatti et al. 2023).

Adding constraints to circumscription does not always increase the computational costs of circumscription. Following (Bonatti, Lutz, and Wolter 2009), we can prove that the standard *filtration technique* (see (Baader et al. 2017)) can be used to show that circumscribed $\mathcal{ALCHIO}$ with constraints has the 'small' (exponential size) model property under the assumption that roles are varying. The latter result yields the following.

Theorem 1. *Concept satisfiability in circumscribed* $\mathcal{ALCHIO}$ *with constraints is* NExpTime$^{\text{NP}}$-*complete if all roles are varying.*

Filtration has been used in (Bonatti et al. 2015) for showing that circumscribed DL-Lite$^{\mathcal{H}}_{Bool}$ has the small model property under the condition that *no* varying role is subsumed by a fixed role. A KB in DL-Lite$^{\mathcal{H}}_{Bool}$ that satisfies this condition is called *role-layered*. The model construction is the same applied for circumscribed $\mathcal{ALCHIO}$ with varying roles, however, the restricted syntax of DL-Lite$_{Bool}$ allows for handling the presence of fixed roles too. The upper bound of Theorem 1 applies also if role-layered circumscribed KBs in DL-Lite$^{\mathcal{H}}_{Bool}$ are paired with constraints in $\mathcal{ALCHIO}$.

Corollary 2. *Concept satisfiability in role-layered circumscribed DL-Lite$^{\mathcal{H}}_{Bool}$ with constraints in $\mathcal{ALCHIO}$ is in* NExpTime$^{\text{NP}}$.

5 Equilibrium Description Logics

We recall Equilibrium Description Logics (EDLs), i.e. DLs under the stable model semantics of Quantified Equilibrium Logic. *Equilibrium Logic (EL)* (Pearce 1996) is a powerful formalism that allows, e.g., extending the stable model semantics of *Answer Set Programming (ASP)* to full propositional logic. EL is built upon the logic of *Here-and-There (HT)* with an additional minimality requirement and its first-order counterpart, *Quantified Equilibrium Logic (QEL)*, has been introduced in (Pearce and Valverde 2008).

An interpretation in the HT logic is a pair of structures $(\mathcal{I}, \mathcal{J})$ sharing the same domain and interpreting individuals in the same way, where $\mathcal{I}$ is called the 'here' interpretation and $\mathcal{J}$ is called the 'there' interpretation. Roughly speaking, the two worlds are related by the inclusion relation, i.e. the 'here' is included in the 'there'. We formalize this requirement by recalling the relations $\subseteq_F$ and $\subset_F$ of Definition 2 where the set of *fixed predicates* $F \subseteq N_C \cup N_R$ specifies those predicates over which the 'here' and the 'there' interpretations must agree.

Definition 8. *A* Here-and-There (HT) interpretation *is a pair $(\mathcal{I}, \mathcal{J})$ of interpretations with $\mathcal{I} \subseteq_F \mathcal{J}$, with $F \subseteq N_C \cup N_R$. The* interpretation function $\cdot^{(\mathcal{I},\mathcal{J})}$ *is defined in Figure 1.*

In general, under the HT semantics predicates do not obey the law of the excluded middle. Already in the case of propositional HT logic, the formula $x \vee \neg x$ is not a tautology (Pearce 2006). Given an HT interpretation $(\mathcal{I}, \mathcal{J})$ as in Definition 8, we can refer to the predicates in F as *classical*: indeed since the interpretations $\mathcal{I}$ and $\mathcal{J}$ agree on the extension of predicates in F, the law of the excluded middle applies for each $p \in F$. HT interpretations satisfy the so-called *persistence* property: if a formula is true in an HT interpretation, then it is also true in the 'there' interpretation. We prove the persistence property in the setting of HT DLs.

Proposition 2 (Persistence)**.** *Given any HT interpretation $(\mathcal{I}, \mathcal{J})$, $C^{(\mathcal{I},\mathcal{J})} \subseteq C^{\mathcal{J}}$ for all concepts C.*

As already pointed out in (Di Stefano and Šimkus 2024), the universally quantified concept of the form $\forall r.C$ can be translated in FOL as $\forall y(r(x, y) \rightarrow C(y))$. Thus the interpretation must align with the interpretation of implication in quantified HT. In HT logic – whether quantified or not – the implication is intuitionistic (Pearce 2006; Pearce and Valverde 2008) and it is true in an HT interpretation when the following are satisfied: (1) if the antecedent is true in the HT interpretation, then the consequence is true in the HT interpretation, and (2) the 'there' interpretation is a classical model of it. We now give the notion of HT model for KBs in DLs.

Definition 9. *Assume a KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ and an HT interpretation $(\mathcal{I}, \mathcal{J})$. We write:*

(i) $(\mathcal{I}, \mathcal{J}) \models C \sqsubseteq D$, *if* $C^{(\mathcal{I},\mathcal{J})} \subseteq D^{(\mathcal{I},\mathcal{J})}$ *and* $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$*;*

(ii) $(\mathcal{I}, \mathcal{J}) \models \mathcal{K}$ *if* $\mathcal{I} \models \mathcal{A}$ *and* $(\mathcal{I}, \mathcal{J}) \models C \sqsubseteq D$ *for all* $C \sqsubseteq D \in \mathcal{T}$.

We can now define *stable models* of a DL KB.

Definition 10 (Stable models)**.** *Given $F \subseteq N_C \cup N_R$, an interpretation $\mathcal{J}$ is a* stable model *of a KB $\mathcal{K}$* under fixed predicates *F, if*

(i) the HT interpretation $(\mathcal{J}, \mathcal{J})$ is a model of $\mathcal{K}$, and

(ii) there is no $\mathcal{I}$ s.t. $(\mathcal{I}, \mathcal{J})$ is a model of $\mathcal{K}$ and $\mathcal{I} \subset_F \mathcal{J}$.

We denote with $SM_F(\mathcal{K})$ the set of all stable models for $\mathcal{K}$ with fixed predicates F. If $F = \emptyset$, we drop the subscript F and write $SM(\mathcal{K})$.

Similarly to the approach of this paper, in the stable model semantics of (Ferraris, Lee, and Lifschitz 2011), predicates are partitioned into *intensional* predicates, subject to minimization, and *extensional* predicates, which are kept fixed. The former are subjected to minimization, while the latter are *classical* and thus obey the law of the excluded middle. This law in classical first-order logic can be easily expressed with the disjunct $p(\vec{t}) \vee \neg p(\vec{t})$, for each extensional predicate p. However, in the setting of the DLs considered in this work, we can express fixed concept names but not fixed roles. This is not a limit, as we already parametrized the relation $\subseteq_F$ expressing the predicates that are fixed.

In the semantics given in Definition 10, the negation $\neg$ behaves as *negation as failure* or *default negation* in logic programs: the truth of a negated concept $\neg C$ at a certain domain element in a stable model intuitively means that the concept C could not be proved at such domain element. The fact that negation is not anymore classical implies that KBs that are equivalent under the classical semantics might not be equivalent under the stable model semantics.

Example 1. *Consider the following situation. The* Fair Company*, where $Steve$ works as admin, records the access to all files accessible with a password. The user Ann accesses the file f_1 with a password. Every time a user accesses with a password some file, they are required to be* password holders.

$$\text{User}(Ann) \qquad \text{acc\_psw}(Ann, f1) \qquad \text{Admin}(Steve)$$
$$\text{User} \sqcap \neg \text{Psw\_Holder} \sqsubseteq \neg \exists \text{acc\_psw}$$

Furthermore, for security reasons, assume that the company records all the people who have received a password, shared by other password holders. These people receiving a password become password holders. Admins are the primary source for passwords.

$$\exists \text{share\_psw} \sqcap \neg \text{Psw\_Holder} \sqsubseteq \bot$$
$$\exists \text{share\_psw}^{-}. \sqsubseteq \text{Psw\_Holder}$$
$$\text{Admin} \sqsubseteq \text{Psw\_Holder}$$

Assuming all roles to be fixed, under the stable model semantics, being a password holder amounts to having received the password from someone else, thus we would derive that someone shared the password with Ann. Under the stable model semantics, we derive that any password holder can be traced back to the admin, Steve, i.e., every password has been safely shared.

$$a^{(\mathcal{I},\mathcal{J})} = a^{\mathcal{I}} \qquad A^{(\mathcal{I},\mathcal{J})} = A^{\mathcal{I}} \qquad r^{(\mathcal{I},\mathcal{J})} = r^{\mathcal{I}} \qquad \top^{(\mathcal{I},\mathcal{J})} = \Delta^{\mathcal{I}} \qquad \bot^{(\mathcal{I},\mathcal{J})} = \emptyset$$

$$(r^-)^{(\mathcal{I},\mathcal{J})} = \{(e,e') \mid (e',e) \in r^{\mathcal{I}}\} \qquad (\neg C)^{(\mathcal{I},\mathcal{J})} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{J}}$$

$$(C_1 \sqcap C_2)^{(\mathcal{I},\mathcal{J})} = C_1^{(\mathcal{I},\mathcal{J})} \cap C_2^{(\mathcal{I},\mathcal{J})} \qquad (C_1 \sqcup C_2)^{(\mathcal{I},\mathcal{J})} = C_1^{(\mathcal{I},\mathcal{J})} \cup C_2^{(\mathcal{I},\mathcal{J})}$$

$$(\exists R.C)^{(\mathcal{I},\mathcal{J})} = \{e \in \Delta^{\mathcal{I}} \mid \exists e' : (e,e') \in R^{(\mathcal{I},\mathcal{J})} \wedge e' \in C^{(\mathcal{I},\mathcal{J})}\}$$

$$(\forall R.C)^{(\mathcal{I},\mathcal{J})} = \left\{ e \in \Delta^{\mathcal{I}} \mid \forall e' : \begin{array}{l} (e,e') \in R^{(\mathcal{I},\mathcal{J})} \text{ implies } e' \in C^{(\mathcal{I},\mathcal{J})} \text{ and} \\ (e,e') \in R^{\mathcal{J}} \text{ implies } e' \in C^{\mathcal{J}} \end{array} \right\}$$

Figure 1: HT semantics for DLs, with $a \in N_I$, $A \in N_C$ and $r \in N_R$.

The reasoning tasks of *concept satisfiability*, *subsumption* and *instance checking* can be adapted in the usual way to stable models. In $\mathcal{ALC}$ and its extensions and DL-Lite$_{Bool}$, these reasoning tasks are pairwise interreducible in polynomial time (Di Stefano and Šimkus 2024). In the $\mathcal{EL}$ family, this is not always possible due to the syntactic restrictions of the fragments.

5.1 Translation into Circumscribed DLs

Given a KB $\mathcal{K}$ and a set F of fixed predicates, we aim to build a KB $\bar{\mathcal{K}}$ with a circumscription pattern $\mathcal{P}$ and a set of constraints $\mathcal{C}$ such that there is a correspondence between stable models of $\mathcal{K}$ and model of $\mathrm{Circ}_{\mathcal{P}}(\bar{\mathcal{K}}) \wedge \mathcal{C}$ up to the original signature. We describe the intuition behind the translation of EDLs into Circumscription that we adapted from (Pearce, Tompits, and Woltran 2009).

The first step of our construction is to simulate HT interpretations. We consider a copy Σ' of the signature Σ of $\mathcal{K}$, which we use to split the 'here' and the 'there' interpretations. Intuitively, the interpretation of primed predicates corresponds to the interpretations of predicates in the 'there' interpretation. We construct the KB $\bar{\mathcal{K}}$ taking the union of two KBs $\mathcal{K}'$ and $\mathcal{K}^\star$. The KB $\mathcal{K}'$ is obtained from $\mathcal{K}$ by replacing each symbol with its primed counterpart. We use $\mathcal{K}'$ to require that the 'there' is a classical model of $\mathcal{K}$. Given a concept C, let C' be the concept obtained by replacing uniformly all predicates occurring in C with their primed counterpart. The KB $\mathcal{K}^\star$ is obtained by replacing each concept C occurring negatively in $\mathcal{K}$ with C'. This second KB is used to simulate the interpretation of negated concepts in the HT semantics (see Figure 1).

To capture the stable model semantics in the setting of circumscription, we require that all predicates in the original signature of $\mathcal{K}$ are minimized while all primed predicates are fixed. To synchronize the 'here' and the 'there', ensuring that the interpretations of primed and non-primed symbols coincide, we introduce constraints of the form $p \sqsubseteq p'$ and $p' \sqsubseteq p$, for each predicate symbol p. These constraints filter out minimal models that are not stable.

A natural question is: *do we really need to rely on constrained circumscription?* Circumscription and Equilibrium Logic are two orthogonal formalisms, as already argued in (Ferraris, Lee, and Lifschitz 2011).

Example 2. *Consider the TBox $\mathcal{T} = \{\neg A \sqsubseteq A\}$ with $F = \emptyset$. It is easy to show that $\mathcal{T}$ has no stable model. Indeed, given $\mathcal{I}$ such that $\Delta^{\mathcal{I}} = \{x\}$ and $\mathcal{I} \models \mathcal{T}$, then $A^{\mathcal{I}} = \{x\}$. The interpretation $\mathcal{J}$ such that $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$ and $A^{\mathcal{J}} = \emptyset$ is such that $(\mathcal{J},\mathcal{I}) \models \mathcal{T}$. If we apply (pointwise) circumscription, the most natural option to 'mimic' the stable model semantics is to minimize A. One can observe that the previous model $\mathcal{I}$ of $\mathcal{T}$ is minimal.*

Example 3. *Consider the TBox $\mathcal{T} = \{B \sqsubseteq A \sqcup \neg A\}$ and assume $F = \{B\}$. We can show that A is satisfiable in a stable model of $\mathcal{T}$. The interpretation $\mathcal{I}$ such that $\Delta^{\mathcal{I}} = \{x\}$, $B^{\mathcal{I}} = A^{\mathcal{I}} = \{x\}$ is a stable model $\mathcal{T}$ with $A^{\mathcal{I}} \neq \emptyset$. If one considers now (pointwise) circumscription, as before a natural direction is to respect the 'nature' of predicates, i.e. keep B fixed and minimize A. However, assuming that A is minimized, it is straightforward to see that there is no minimal model $\mathcal{J}$ of $\mathcal{T}$ such that $A^{\mathcal{J}} \neq \emptyset$.*

The transformation. Given a KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$, let Σ denote the set of all concept names and role names occurring in $\mathcal{K}$. We consider a copy of the signature Σ and we denote it with $\Sigma' = \{A' | A \in A\} \cup \{r' | r \in \Sigma\}$. Given a complex concept C we denote with C' the concept over Σ' obtained globally replacing each predicate symbol in C with its primed version. Following (Pearce, Tompits, and Woltran 2009), we define an operator τ recursively as follows:

- $\tau(A) = A$, $\tau(r) = r$, $\tau(\top) = \top$, $\tau(\bot) = \bot$, $\tau(\{a\}) = \{a\}$,
- $\tau(\neg C) = \neg C'$, $\tau(\exists R.C) = \exists R.\tau(C)$ and $\tau(\forall R.C) = \forall R.\tau(C) \sqcap \forall R'.C'$
- $\tau(C \circ D) = \tau(C) \circ \tau(D)$, with $\circ \in \{\sqcap, \sqcup\}$.

Given the TBox $\mathcal{T}$, we denote with $\mathcal{T}^*$ the resulting TBox, with predicates occurring in $\Sigma \cup \Sigma'$, obtained applying τ to all GCIs in $\mathcal{T}$, i.e. $\mathcal{T}^\star = \{\tau(C) \sqsubseteq \tau(D) | C \sqsubseteq D \in \mathcal{T}\}$. Observe that the assertions in the ABox can be seen as inclusions in the standard way. It is easy to observe that they are not affected by the τ transformation. We $\mathcal{K}^\star = (\mathcal{A}, \mathcal{T}^\star)$. We denote with $\mathcal{K}'$ the KB obtained from $\mathcal{K}$ by replacing each symbol $p \in \Sigma$ with its primed counterpart $p' \in \Sigma'$.

Given a set Σ of concept names and role names and interpretation $\mathcal{J}$, we denote with $\mathcal{J}'$ the interpretation such that $\Delta^{\mathcal{J}} = \Delta^{\mathcal{J}'}$, $(p')^{\mathcal{J}'} = p^{\mathcal{J}}$ for all $P \in \Sigma$. Given an HT interpretation $(\mathcal{I}, \mathcal{J})$, we denote with $\mathcal{I} \cup \mathcal{J}'$ the interpretation such that $\Delta^{\mathcal{I} \cup \mathcal{J}'} = \Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$, $P^{\mathcal{I} \cup \mathcal{J}'} = P^{\mathcal{I}}$ for all $P \in \Sigma$ and $(P')^{\mathcal{I} \cup \mathcal{J}'} = P^{\mathcal{J}}$ for all $P' \in \Sigma'$.

As mentioned at the beginning of this section, we aim to simulate the HT semantics. Roughly speaking, given an HT interpretation, we copy the 'there' interpretation using the primed symbols, while unprimed symbols are interpreted as in the 'here'.

Lemma 1. *Assume a KB $\mathcal{K}$ and two interpretations $\mathcal{I}$ and $\mathcal{J}$. Let Σ be the set of predicates occurring in $\mathcal{K}$ and $F \subseteq \Sigma$ be a set of fixed predicates. Then $(\mathcal{I}, \mathcal{J})$ is a HT model of $\mathcal{K}$ with fixed predicates in F if and only if $\mathcal{I} \cup \mathcal{J}'$ is a model of $\mathcal{K}^* \cup \mathcal{K}' \cup \{p \sqsubseteq p' \in \Sigma\} \cup \{p' \sqsubseteq p | p \in F\}$.*

We now use circumscription to capture stable models. It is worth mentioning that since EDLs do not support varying predicates, we restrict to circumscription patterns without varying predicates. With the following theorem, we formalize the intuitive explanation given at the beginning of this section.

Theorem 2. *Assume a KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ and an interpretation $\mathcal{I}$. Let Σ be the set of predicates occurring in $\mathcal{K}$ and $F \subseteq \Sigma$. We have that $\mathcal{I} \in SM_F(\mathcal{K})$ if and only if $\mathcal{I} \cup \mathcal{I}'$ is a model of $\mathrm{Circ}_{\mathcal{P}_F}(\mathcal{K}^* \cup \mathcal{K}') \wedge \mathcal{C}$, with $\mathcal{C} = \{p \sqsubseteq p', p' \sqsubseteq p | p \in \Sigma\}$ and $\mathcal{P}_F = (M, \emptyset, F \cup \Sigma')$ with $M = \Sigma \setminus F$.*

Since $\mathcal{K}'$ contains only fixed predicates, we can equivalently use $\mathrm{Circ}_{\mathcal{P}_F}(\mathcal{K}^\star) \wedge (\mathcal{C} \cup \mathcal{K}')$.

5.2 Complexity Results: Expressive DLs

In (Di Stefano and Šimkus 2024), the stable model semantics has been used in the context of DL *terminologies*. A terminology is a collection of *concept definitions* of the form $A := C$, where A is a concept name defined by C. Given a terminology $\mathcal{T}$, the signature is partitioned into the *intensional* predicates defined by a concept definition using the *extensional* predicates, always containing all the role names occurring in the terminology. The stable model semantics of Definition 10 is used in the setting of terminologies by seeing each $A := C$ as an inclusion of the form $C \sqsubseteq A$ and assuming as fixed all the extensional predicates. Remarkably, the stable model semantics overcomes the syntactic limitations of previous approaches (Baader 1990; De Giacomo and Lenzerini 1997) without increasing the complexity of standard reasoning tasks for terminologies in $\mathcal{ALCI}$. Di Stefano and Šimkus proved that for $\mathcal{ALCI}$ terminologies under the stable model semantics, the problems of satisfiability and concept satisfiability are ExpTime-complete.

In contrast to the case of terminologies where the syntax is restricted and roles are always assumed to be fixed, for general KBs in $\mathcal{ALC}$ with role minimization, we prove the following undecidability result.

Theorem 3. *Concept satisfiability w.r.t. general KBs in $\mathcal{ALC}$ under the stable model semantics is undecidable if roles are allowed to be minimized.*

Proof. The theorem above is proven using a reduction from the domino tiling problem to concept satisfiability w.r.t. a KB in $\mathcal{ALC}$ under the stable model semantics. Let $P = (T, H, V)$ be an instance of the *domino problem*, where T is a set of tiles, $H, V \subseteq T \times T$ are the horizontal and vertical matching conditions. A solution for P is a map $\tau \colon \mathbb{N} \times \mathbb{N} \to T$ such that $(\tau(i,j), \tau(i+1,j)) \in H$, and $(\tau(i,j), \tau(i,j+1)) \in V$. Consider the KB $\mathcal{K}_P$ in Figure 2 and assume the set of fixed predicates $F = \{r\}$. It is not difficult to check that P has a solution if and only if there exists $\mathcal{I} \in SM_F(\mathcal{K}_P)$ such that $G^{\mathcal{I}} \neq \emptyset$. We briefly discuss the intuition of the axiom in Figure 2. With (1), (5)-(6) we simulate the *spy-point technique*. The nominal a acts as a spy point, with (5) we force all domain elements to be connected to an S element that with (6) is forced to coincide with a. With (7)-(8) we then force that a is connected to every domain element via t. Indeed, from (7), every domain element is a B. However, using axiom (8) and the assumption that r is fixed, in order to justify all the occurrences of B we have that a must be connected to every domain element via the role r. With the axioms (9)-(10), we apply the *saturation* technique already used for $\mathcal{ALCI}$ in (Di Stefano and Šimkus 2024), without recurring to inverse roles. With (8), an X instance is placed in the model, using (9)-(10), if a proper cell of the grid is found, with X labeling the bottom-left node, then the model is saturated with X. ☐

5.3 Complexity Results: $\mathcal{EL}$ family

We now focus on DLs in the $\mathcal{EL}$ family. Logics in this family often disallow the use of $\neg$ and $\bot$, unless stated otherwise. The resulting formalisms are often unable to derive contradictions, making the reasoning tasks of concept subsumption the most interesting one to consider. With a reduction from the halting problem of a deterministic Turing machine, we prove that subsumption w.r.t. circumscribed $\mathcal{ELIO}$ KBs is undecidable if all predicates are minimized.

Theorem 4. *Subsumption in circumscribed $\mathcal{ELIO}$ with all predicates minimized is undecidable.*

Remark 1. *KBs in $\mathcal{ELIO}^\bot$ are $\neg$-free and $\forall$-free; thus, evaluating a concept in an HT interpretation is equivalent to evaluating it only in the 'here'. For this reason, given KB $\mathcal{K}$ and a set F of fixed predicates, $SM_F(\mathcal{K}) = MM(\mathcal{K}, \mathcal{P})$, where $\mathcal{P} = (M, \emptyset, F)$ with $M = (N_C(\mathcal{K}) \cup N_R(\mathcal{K})) \setminus F$ (see (Di Stefano and Šimkus 2024)).*

Under the stable model semantics nominals can be simulated using *default negation*. Let $\mathcal{ELI}^\neg$ be the extension of $\mathcal{ELI}$ allowing for negation of concept names.

From Theorem 4, the following result holds.

Corollary 3. *Subsumption in $\mathcal{ELIO}$ and $\mathcal{ELI}^\neg$ under the stable model semantics with all predicates minimized is undecidable.*

The result above does not extend to concept satisfiability in $\mathcal{ELIO}$. However, in $\mathcal{ELI}^\neg$, the result of Corollary 3 applies to concept satisfiability and instance checking, too.

5.4 Complexity Results: DL-Lite Family

The DL-Lite family is characterized by DLs where the expressiveness of the language is limited in favor of achieving good complexity bounds, resulting in many tractable fragments as DL-Lite$_{core}$ and DL-Lite$_{Horn}$. Under circumscription, the complexity of different reasoning tasks in DLs in the DL-Lite family has been studied in (Bonatti et al. 2023; Lutz, Manière, and Nolte 2023).

$$\mathrm{Spy}(a) \quad (1)$$
$$\top \sqsubseteq \exists h.\top \sqcap \exists v.\top \sqcap (\bigsqcup_{t \in T} A_t) \quad (2)$$
$$A_t \sqsubseteq \forall h. \bigsqcup_{(t,t') \in H} A_{t'} \sqcap \forall v. \bigsqcup_{(t,t') \in V} A_{t'} \quad \forall t \in T \quad (3)$$
$$A_t \sqcap A_{t'} \sqsubseteq \bot \quad \forall t, t' \in T \text{ and } t \neq t' \quad (4)$$
$$\top \sqsubseteq \exists t.S \quad (5)$$
$$(S \sqcap \neg\mathrm{Spy}) \sqsubseteq \bot \quad (6)$$
$$\neg B \sqsubseteq \bot \quad (7)$$
$$\mathrm{Spy} \sqsubseteq \forall r.B \sqcap \exists r.X \quad (8)$$
$$X \sqsubseteq \forall h.\forall v.A \quad (9)$$
$$X \sqcap \exists v.\exists h.A \sqsubseteq \forall t.(G \sqcap \forall r.X) \quad (10)$$

Figure 2: Encoding of the domino problem in $\mathcal{ALC}$.

Proposition 3. *Under the stable model semantics, concept satisfiability in DL-Lite$_{core}$ is* NLOGSPACE*-complete.*

Proof sketch. The upper bound follows from Remark 1 by recalling that concept satisfiability in circumscribed DL-Lite$_{core}$ with all predicates minimized or fixed is in NLOGSPACE (Bonatti et al. 2023). The lower bound follows from concept satisfiability in DL-Lite$_{core}$ under the classical semantics. □

Minimal model reasoning increases the complexity in DL-Lite$_{Horn}$ if fixed predicates are allowed. This is not surprising as the latter framework can express logic programs with choice rules. The NP-hardness of concept satisfiability w.r.t. KBs in circumscribed DL-Lite$_{Horn}$ with fixed and minimized predicates has been proved in (Bonatti et al. 2023). The reduction is adapted from (Cadoli and Lenzerini 1994) and fixed predicates are used to 'guess' the truth value of propositional variables. We show that the NP-hardness can be proved even without using fixed predicates.

Proposition 4. *If all predicates are minimized, concept satisfiability in circumscribed DL-Lite$_{Horn}$ is* NP*-hard.*

From Remark 1, the result above holds under the stable model semantics too. In general, role minimization is a primary source of difficulty when it comes to characterizing the complexity of reasoning, as many of the known techniques fail, e.g. *filtration*. We prove that under the stable model semantics and the assumption that all roles are fixed, DL-Lite$_{Bool}$ has the small model property.

Lemma 2. *Given a KB in DL-lite$_{Bool}$ and a concept C, if there exists $\mathcal{I} \in SM_F(\mathcal{K})$ such that $C^{\mathcal{I}} \neq \emptyset$, then there exists $\mathcal{J} \in SM_F(\mathcal{K})$ such that $|\mathcal{J}|$ is polynomial in the size of $\mathcal{K}$ and $C^{\mathcal{J}} \neq \emptyset$.*

Let us consider a fragment of DL-Lite$_{Bool}$ where we allow for inclusions of the form

$$C_1 \sqcap \cdots \sqcap C_n \sqcap \neg D_1 \sqcap \cdots \sqcap \neg D_m \sqsubseteq B$$

where $C_1, \cdots C_n, D_1, \cdots D_m$ and B are DL-Lite basic concepts. We call this fragment DL-Lite$_{not}$. Under the classical semantics, DL-Lite$_{not}$ does not differ from the full DL-Lite$_{Bool}$. Intuitively, under the stable model semantics, DL-Lite$_{not}$ is to DL-Lite$_{Bool}$ what normal logic programs are to disjunctive logic programs (Lloyd 1987). We show the following result.

Proposition 5. *Assume a DL-Lite$_{not}$ KB $\mathcal{K}$. Given a finite interpretation $\mathcal{I}$, checking that $\mathcal{I}$ is in $SM_F(\mathcal{K})$ can be done in polynomial time.*

From Lemma 2 and Proposition 5, we have the following.

Theorem 5. *Under the stable model semantics, with the assumption that all roles are fixed, we have:*

- *Concept satisfiability in DL-Lite$_{not}$ and DL-Lite$_{Bool}$ is* NP*-complete and Σ_2^p-complete, respectively;*
- *Concept subsumption in DL-Lite$_{not}$ and DL-Lite$_{Bool}$ is co*NP*-complete and Π_2^p-complete, respectively. The same applies to instance checking.*

Under the assumption that all roles are fixed, from Theorem 5 we conclude that concept satisfiability in DL-Lite$_{Horn}$ is NP-complete, where the hardness follows from (Bonatti et al. 2023).

6 Pointwise Equilibrium DLs

In Definition 4 we introduce a relation comparing interpretations that differ only at a single domain element, in terms of concept names and role names involving it. We lift this *pointwise* comparison defining *pointwise stable models* as follows. Recall the definition of $\subseteq_F^\bullet$ and $\subset_F^\bullet$.

Definition 11. *A* pointwise *HT interpretation is an HT interpretation $(\mathcal{I}, \mathcal{J})$ such that $\mathcal{I} \subseteq_F^\bullet \mathcal{J}$ and the interpretation function is defined as in Figure 1.*

Definition 12 (Pointwise Stable Models)**.** *Given $F \subseteq N_C \cup N_R$, an interpretation $\mathcal{J}$ is a* pointwise stable model *of a KB $\mathcal{K}$ under fixed predicates F, if*

(i) the HT interpretation $(\mathcal{J}, \mathcal{J})$ is a model of $\mathcal{K}$, and

(ii) there is no $\mathcal{I}$ s.t. $(\mathcal{I}, \mathcal{J})$ is a model of $\mathcal{K}$ and $\mathcal{I} \subset_F^\bullet \mathcal{J}$.

We denote with $PSM_F(\mathcal{K})$ the set of all stable models for $\mathcal{K}$ with fixed predicates F. If $F = \emptyset$, we drop the subscript F and write $PSM(\mathcal{K})$.

Pointwise stable models have been already considered for full first-order logic in (Ferraris, Lee, and Lifschitz 2011). However, there are some key differences with the above definition. In (Ferraris, Lee, and Lifschitz 2011) the stable model semantics is achieved by means of an operator associating each first-order formula Φ to a second-order formula $SM[\Phi]$ whose models are stable. In the case of pointwise circumscription, the operator associates the formula $PSM[\Phi]$ which can be expressed at the first-order level. The performed pointwise minimization in $PSM[\Phi]$ allows only for the minimization of a single predicate. In our setting, multiple predicates with different arities can be minimized, as long as they 'concern' a unique domain element.

6.1 Complexity Results

The result of Theorem 2 can be extended to the case of pointwise EDLs and pointwise circumscribed DLs with constraints. To do so, we first adapt Lemma 1 as follows.

Lemma 3. *Assume a KB $\mathcal{K}$ and two interpretations $\mathcal{I}$ and $\mathcal{J}$. Let Σ be the set of predicates occurring in $\mathcal{K}$ and $F \subseteq \Sigma$ be a set of fixed predicates. Then $(\mathcal{I}, \mathcal{J})$ is a HT model of $\mathcal{K}$ such that $\mathcal{I} \subseteq^{\bullet}_{F} \mathcal{J}$ if and only if $\mathcal{I} \cup \mathcal{J}'$ is a model of $\mathcal{K}^{*} \cup \mathcal{K}' \cup \{p \sqsubseteq p' \in \Sigma\} \cup \{p' \sqsubseteq p | p \in F\}$ and $\mathcal{I} \sim^{\bullet} \mathcal{J}$.*

Theorem 6. *Assume a KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ and an interpretation $\mathcal{I}$. Let Σ be the set of predicates occurring in $\mathcal{K}$ and $F \subseteq \Sigma$, $\mathcal{I} \in PSM_F(\mathcal{K})$ if and only if $\mathcal{I} \cup \mathcal{I}'$ is a model of $\mathrm{Circ}^{\bullet}_{\mathcal{P}_F}(\mathcal{K}^{*} \cup \mathcal{K}') \wedge \mathcal{C}$, with $\mathcal{C} = \{p \sqsubseteq p', p' \sqsubseteq p | p \in \Sigma\}$ and $\mathcal{P}_F = (M, \emptyset, F \cup \Sigma')$ with $M = \Sigma \setminus F$.*

Let $\mathcal{ALCIO}_{d\leq 1}$ be the fragment of $\mathcal{ALCIO}$ where the syntax is restricted to KBs of modal depth 1 (Di Stefano, Ortiz, and Šimkus 2023). The *mosaic technique* (Gogacz et al. 2020b; Gogacz et al. 2020a) used in (Di Stefano, Ortiz, and Šimkus 2023) can be easily extended to pointwise circumscription with constraints as defined in this work. In the *mosaic technique*, given a KB, the existence of a minimal model is reduced to checking the existence of a finite family of minimal *fragments* of models, called *abstract types*, locally satisfying the KB. These *types* can be used as building blocks for a minimal model, according to some 'plugging' instructions encoded in a system of inequalities. From Theorem 6, we obtain the following.

Theorem 7. *Concept satisfiability in $\mathcal{ALCIO}_{d\leq 1}$ under the pointwise stable semantics is in* NExpTime.

In (Di Stefano, Ortiz, and Šimkus 2023), concept satisfiability is proved to be undecidable w.r.t general KBs in $\mathcal{ALCIO}$ under pointwise circumscription. The undecidability carries over general KBs $\mathcal{ALCI}$ under the pointwise stable model semantics.

Theorem 8. *Concept satisfiability w.r.t general KBs in $\mathcal{ALCI}$ under the pointwise stable model semantics is undecidable.*

6.2 Pointwise vs Global Minimization

We can identify a large class of ontologies where pointwise circumscription coincides with global circumscription. We impose an acyclicity condition on the *dependency graph* of an ontology under which pointwise minimal models coincide with minimal models. Our approach is close to the one used in (Ferraris, Lee, and Lifschitz 2011) for arbitrary first-order theories and a weaker form of pointwise minimization.

We first collect the predicates that occur *positively* and *negatively* is a concept expression. For a concept C in NNF, we define the sets $Occ^{+}(C)$ and $Occ^{-}(C)$ based on the structure of C as follows:

- $Occ^{+}(A) = \{A\}$, $Occ^{+}(\neg A) = \emptyset$, with $A \in N_C$
- $Occ^{-}(A) = \emptyset$, $Occ^{-}(\neg A) = \{A\}$, with $A \in N_C$
- $Occ^{\pm}(C \circ D) = Occ^{\pm}(C) \cup Occ^{\pm}(D)$, with $\circ \in \{\sqcup, \sqcap\}$
- $Occ^{+}(\exists r.C) = \{r\} \cup Occ^{+}(C)$
- $Occ^{-}(\exists r.C) = Occ^{-}(C)$
- $Occ^{+}(\forall r.C) = Occ^{+}(C)$
- $Occ^{-}(\forall r.C) = \{r\} \cup Occ^{-}(C)$
- $Occ^{+}(\{o\}) = \{\{o\}\}$ and $Occ^{-}(\{o\}) = \emptyset$.

Given a concept C, we denote with $\sim C$ the concept $NNF(\neg C)$. The *dependency graph* $DG(\mathcal{T})$ of an $\mathcal{ALCIO}$ TBox $\mathcal{T}$ in NNF is a directed graph whose nodes are the nominals, concept and role names that appear in $\mathcal{T}$, and such that there is an edge from P_1 to P_2 if there is an inclusion $C \sqsubseteq D \in \mathcal{T}$ such that $P_1 \in Occ^{-}(\sim C \sqcup D)$ and $P_2 \in Occ^{+}(\sim C \sqcup D)$. Given a TBox $\mathcal{T}$ and a circumscription pattern $\mathcal{P} = (M, \emptyset, F)$, we denote with $DG(\mathcal{T})_M$ the subgraph of $DG(\mathcal{T})$ induced by the set of minimized predicates M, i.e $DG(\mathcal{T})_M = (M, E_M)$ and $(p, q) \in E_M$ if the edge (p, q) belongs to $DG(\mathcal{T})$ and $p, q \in M$.

Theorem 9. *Assume $\mathcal{T}$ is an $\mathcal{ALCIO}$ TBox and let $\mathcal{P} = (M, \emptyset, F)$. If $DG(\mathcal{T})_M$ has no directed cycle then $PMM(\mathcal{T}, \mathcal{P}) = MM(\mathcal{T}, \mathcal{P})$.*

We can extend the result to stable models and pointwise stable models too, using the following. Recall the transformation τ defined in Section 5.1. Given a TBox $\mathcal{T}$, let $\mathcal{T}^{\star} = \{\tau(C) \sqsubseteq \tau(D) \mid C \sqsubseteq D \in \mathcal{T}\}$. From the Theorems 9, 2 and 6, we have the following result.

Theorem 10. *Assume a TBox $\mathcal{T}$ in $\mathcal{ALCIO}$, let $\mathcal{P} = (M, \emptyset, F)$ be a circumscription pattern for $\mathcal{T}$. If $DG(\mathcal{T}^{\star})_M$ is acyclic, then $SM_F(\mathcal{T}) = PSM_F(\mathcal{T})$.*

With Theorem 10 and Theorem 7, we can identify a large class of ontologies in $\mathcal{ALCIO}_{d\leq 1}$ for which standard reasoning tasks are decidable under the stable model semantics, even with minimized roles. We can strengthen the result above for DLs in the DL-Lite family.

Theorem 11. *Let $\mathcal{K}$ be a KB in DL-Lite$_{Bool}$ and F be a set of fixed predicates such that $N_R(\mathcal{K}) \subseteq F$, then $PSM_F(\mathcal{K}) = SM_F(\mathcal{K})$.*

We briefly explain the intuition behind Theorem 11 in relation to Theorem 10, which uses a condition on the dependency graph induced by the minimized predicates. Under the hypothesis of Theorem 11, the dependency graphs induced by the minimized predicate may only contain cycles involving concept names. Since DLs in the DL-Lite family do not allow for qualified existentials, such cyclic dependencies between minimized predicates are detected by forms of pointwise minimization. From Theorem 11, we inherit the complexity results of Theorem 5 under the pointwise stable model semantics.

7 Conclusions

In this paper we studied some of the computational and semantic properties of EDLs—DLs equipped with a stable model semantics based on Quantified Equilibrium Logic (QEL)—both under the requirement of global minimality and pointwise minimality. EDLs and pointwise EDLs are self-contained formalisms which are in contrast with hybrid formalisms combining ontologies with rules (Eiter et al. 2008; Motik and Rosati 2010; Levy and Rousset 1998)

Fragments	Minim. roles	All roles fixed
	Stable Model Semantics	
$\mathcal{ALC}$	Undecidable (Th. 3)	?
$\mathcal{ELI}^{\neg}$	Undecidable (Cor. 3)	?
DL-Lite$_{core}$	NLogSpace-c (Prop. 3)	
DL-Lite$_{Horn}$	NP-hard[1]	NP-c[2]
DL-Lite$_{not}$	NP-hard[3]	NP-c (Th. 5)
DL-Lite$_{Bool}$	Σ_2^p-hard[3]	Σ_2^p-c (Th. 5)
	Pointwise Stable Model Semantics	
$\mathcal{ALCIO}_{d\leq 1}$	NExpTime (Th. 7)	
$\mathcal{ALCI}$	Undecidable (Th. 8)	?

Table 1: Complexity results for concept satisfiability w.r.t. generals KBs. The 'c' stands for 'complete'. [1]From Prop. 4 and Remark 1. [2]The upper bound follows from Th. 5. The lower bound follows from (Bonatti et al. 2023) and Remark 1. [3]The hardness follows by observing that, without using roles, DL-Lite$_{not}$ and DL-Lite$_{Bool}$ capture normal and disjunctive logic programs, respectively.

and do not rely on the translation into a more expressive formalism (Ferraris, Lee, and Lifschitz 2011; Donini, Nardi, and Rosati 2002).

We provided a collection of complexity results for a wide variety of DLs, ranging from lightweight to expressive DLs, summarized in Table 1. Remarkably, for DLs in the DL-Lite family, the (un)decidability of concept satisfiability under the assumption that roles are minimized is generally unknown. In particular, under the assumption that roles are minimized, we expect an increase in complexity for both DL-Lite$_{not}$ and DL-Lite$_{Bool}$ and leave the problem open for future research. Observe that under the pointwise stable model semantics the same logics trivially inherit the upper bound for $\mathcal{ALCIO}_{d\leq 1}$. For more expressive DLs, role minimization generally causes undecidability. Subsumption is proved to be undecidable in $\mathcal{ELIO}$ under the stable model semantics, allowing roles to be minimized. Without the use of the negation, the proof does not lift to concept satisfiability in $\mathcal{ELIO}$. Therefore, the problem is left open for future work. In general, the (un)decidability of concept satisfiability in DLs in the $\mathcal{EL}$ family without negation is unknown. Lastly, the (un)decidability of the standard reasoning tasks w.r.t. general KBs in $\mathcal{ALCI}$ under the (pointwise) stable model semantics with all roles fixed is still open.

One of the tools we applied was a translation from EDLs into DLs with *constrained circumscription*, which is a formalism interesting in its own right. Via such translation, we show that pointwise EDLs inherit the decidability results of pointwise circumscription. Furthermore, we identified a class of ontologies where circumscription and pointwise circumscription coincide and extended this result to EDLs and pointwise EDLs. In this way, we provided new decidability results for circumscribed DLs and EDLs.

An implementation of EDLs is left as future work. A first step towards this is to identify more fine-grained syntactic restrictions so that we can eventually use existing DL reasoners, likely coupled with standard ASP solvers. Reasoning tasks stemming from ASP, e.g. strong equivalence, could be considered for EDLs and are also left for future work.

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