Computers and Mathematics with Applications 180 (2025) 102–129

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On full linear convergence and optimal complexity of adaptive FEM with inexact solver

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ARTICLE INFO

Keywords:
Adaptive finite element method
Optimal convergence rates
Cost-optimality
Inexact solver
Full linear convergence

ABSTRACT

The ultimate goal of any numerical scheme for partial differential equations (PDEs) is to compute an approximation of user-prescribed accuracy at quasi-minimal computation time. To this end, algorithmically, the standard adaptive finite element method (AFEM) integrates an inexact solver and nested iterations with discerning stopping criteria balancing the different error components. The analysis ensuring optimal convergence order of AFEM with respect to the overall computational cost critically hinges on the concept of R-linear convergence of a suitable quasi-error quantity. This work tackles several shortcomings of previous approaches by introducing a new proof strategy. Previously, the analysis of the algorithm required several parameters to be fine-tuned. This work leaves the classical reasoning and introduces a summability criterion for R-linear convergence to remove restrictions on those parameters. Second, the usual assumption of a (quasi-)Pythagorean identity is replaced by the generalized notion of quasi-orthogonality from Feischl (2022) [22]. Importantly, this paves the way towards extending the analysis of AFEM with inexact solver to general inf-sup stable problems beyond the energy minimization setting. Numerical experiments investigate the choice of the adaptivity parameters.

1. Introduction

Over the past three decades, the mathematical understanding of adaptive finite element methods (AFEMs) has matured; see, e.g., [20,43,6,46,12,11,23] for linear elliptic PDEs, [48,19,4,29] for certain nonlinear PDEs, and [9] for an axiomatic framework summarizing the earlier references. In most of the cited works, the focus is on *(plain) convergence* in [20,43,48,19,29] and optimal convergence rates with respect to the number of degrees of freedom, i.e., *optimal rates*, in [6,12,11,4,29,23].

The adaptive feedback loop strives to approximate the unknown and possibly singular exact PDE solution $u^\star$ on the basis of *a posteriori* error estimators and adaptive mesh refinement strategies. Employing AFEM with *exact solver*, detailed in Algorithm A below, generates a sequence $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ of successively refined meshes together with the corresponding finite element solutions $u_\ell^\star \approx u^\star$ and error estimators $\eta_\ell(u_\ell^\star)$ by iterating

$$\boxed{\texttt{solve}} \longrightarrow \boxed{\texttt{estimate}} \longrightarrow \boxed{\texttt{mark}} \longrightarrow \boxed{\texttt{refine}} \tag{1}$$

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https://doi.org/10.1016/j.camwa.2024.12.013
Received 11 June 2024; Received in revised form 16 October 2024; Accepted 16 December 2024

Available online 27 December 2024

A key argument in the analysis of (1) in [12] and succeeding works for symmetric PDEs consists in showing *linear convergence* of the quasi-error

$$\Delta_\ell^\star \le q_{\rm lin}\, \Delta_{\ell-1}^\star \quad \text{with} \quad \Delta_\ell^\star := \big[|||u^\star - u_\ell^\star|||^2 + \gamma\, \eta_\ell(u_\ell^\star)^2\big]^{1/2} \quad \text{for all } \ell \in \mathbb{N}, \tag{2}$$

where $0 < q_{\rm lin}, \gamma < 1$ depend only on the problem setting and the marking parameter θ. Here, $|||\cdot|||$ is the PDE-induced energy norm providing a Pythagorean identity of the form

$$|||u^\star - u_{\ell+1}^\star|||^2 + |||u_{\ell+1}^\star - u_\ell^\star|||^2 = |||u^\star - u_\ell^\star|||^2 \quad \text{for all } \ell \in \mathbb{N}_0. \tag{3}$$

The work [9] showed that a *tail summability* of the estimator sequence

$$\sum_{\ell'=\ell+1}^{\infty} \eta_{\ell'}(u_{\ell'}^\star) \le C_{\rm lin}' \eta_\ell(u_\ell^\star) \quad \text{for all } \ell \in \mathbb{N}_0$$

or, equivalently, *R-linear convergence*

$$\eta_\ell(u_\ell^\star) \le C_{\rm lin} q_{\rm lin}^{\ell-\ell'}\, \eta_{\ell'}(u_{\ell'}^\star) \quad \text{for all } \ \ell \ge \ell' \ge 0, \tag{4}$$

with $0 < q_{\rm lin} < 1$ and $C_{\rm lin}, C_{\rm lin}' > 0$, suffices to prove convergence. An extension of the analysis to nonsymmetric linear PDEs can be done by relaxing the Pythagorean identity to a quasi-Pythagorean estimate in [11,23,5]. However, this comes at the expense that either the initial mesh has to be sufficiently fine as in [11], (2) only holds for $\ell \ge \ell_0 \in \mathbb{N}_0$ [5], or (2) holds in the general form (4) below, where the constants depend on the adaptively generated meshes in [23]. Additional to R-linear convergence (4), a sufficiently small marking parameter θ leads to optimal rates in the sense of [46,12]. This can be stated in terms of approximation classes from [6,47,12] by mathematically guaranteeing the largest possible convergence rate $s > 0$ with

$$\sup_{\ell\in\mathbb{N}} (\#\mathcal{T}_\ell)^s \eta_\ell(u_\ell^\star) < \infty. \tag{5}$$

However, due to the incremental nature of adaptivity, the mathematical question on optimal convergence rates should rather refer to the overall computational cost (resp. the cumulative computation time). This, coined as *optimal complexity* in the context of adaptive wavelet methods from [14,15], was later adopted for AFEM in [46,10]. Therein, optimal complexity is guaranteed for AFEM with *inexact* solver, provided that the computed iterates u_ℓ^k are sufficiently close to the (unavailable) exact discrete solutions $u_\ell^\star$. This theoretical result requires that the algebraic error is controlled by the discretization error multiplied by a sufficiently small solver-stopping parameter λ. However, numerical experiments in [10] indicate that also moderate choices of the stopping parameter suffice for optimal complexity. Hence, the interrelated stopping criterion led to a combined solve-estimate module in the adaptive algorithm

$$\boxed{\texttt{solve \& estimate}} \quad\longrightarrow\quad \boxed{\texttt{mark}} \quad\longrightarrow\quad \boxed{\texttt{refine}} \tag{6}$$

Driven by the interest in AFEMs for nonlinear problems in [21,16,27,34,35], recent papers [26,33,30] aimed to combine linearization and algebraic iterates into a nested adaptive algorithm. Following the latter, the algorithmic decision for either mesh refinement or linearization or algebraic solver step is steered by *a posteriori*-based stopping criteria with suitable stopping parameters. This allows to balance the error components and compute the inexact approximations $u_\ell^k \approx u_\ell^\star$ given by a contractive solver with iteration counter $k = 1, \dots, \underline{k}[\ell]$ on the mesh $\mathcal{T}_\ell$. Accordingly, the sequential loop (6) leads to a double index set $\mathcal{Q} \subset \mathbb{N}_0^2$ endowed with a lexicographic order through the step counter $|\ell, k| \in \mathbb{N}_0$ for all $(\ell, k) \in \mathcal{Q}$; see Algorithm B below.

Due to an energy identity (coinciding with (3) for symmetric linear PDEs), the works [26,33] prove full R-linear convergence for the quasi-error $\Delta_\ell^k := \big[|||u^\star - u_\ell^k|||^2 + \gamma\, \eta_\ell(u_\ell^k)^2\big]^{1/2}$ with respect to the lexicographic ordering $|\cdot,\cdot|$, i.e.,

$$\Delta_\ell^k \le C_{\rm lin} q_{\rm lin}^{|\ell,k|-|\ell',k'|}\, \Delta_{\ell'}^{k'} \quad \text{for all } \ (\ell',k'), (\ell,k) \in \mathcal{Q} \text{ with } |\ell',k'| \le |\ell,k|, \tag{7}$$

which is guaranteed for arbitrary marking parameter θ and stopping parameter λ (with constants $C_{\rm lin} > 0$ and $0 < q_{\rm lin} < 1$ depending on θ and λ). Moreover, [26] proves that full R-linear convergence is also the key argument for optimal complexity in the sense that it ensures, for all $s > 0$,

$$M(s) := \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Delta_\ell^k \le \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^k \le C_{\rm cost}(s)\, M(s), \tag{8}$$

where $C_{\rm cost}(s) > 1$ depends only on s, $C_{\rm lin}$, and $q_{\rm lin}$. Since all modules of AFEM with inexact solver as displayed in (6) can be implemented at linear cost $\mathcal{O}(\#\mathcal{T}_\ell)$, the equivalence (8) means that the quasi-error Δ_ℓ^k decays with rate s over the number of elements $\#\mathcal{T}_\ell$ if and only if it decays with rate s over the related *overall* computational work (and hence total computation time).

In essence, optimal complexity of AFEM with inexact solver thus follows from a perturbation argument (by taking the stopping parameter λ sufficiently small) as soon as full linear convergence (7) of AFEM with inexact solver and optimal rates of AFEM with exact solver (for sufficiently small θ) have been established; see, e.g., [9,26].

In this paper, we present a novel proof of full linear convergence (7) with contractive solver that, unlike [26,33], avoids the Pythagorean identity (3), but relies only on the quasi-orthogonality from [9] (even in its generalized form from [22]). The latter is known to be sufficient and necessary for linear convergence (4) in the presence of exact solvers [9]. In particular, this opens the door to proving optimal complexity for AFEM beyond symmetric energy minimization problems. Moreover, problems exhibiting additional difficulties such as nonsymmetric linear elliptic PDEs, see [7], or nonlinear PDEs, see [30], ask for more intricate (nested) solvers that treat iterative symmetrization/linearization together with inexact solving of the arising linear SPD systems. This leads to computed approximations $u_\ell^{k,j} \approx u_\ell^\star$ with symmetrization/linearization iteration counter $k = k[\ell]$ and algebraic solver index $j = j[\ell, k]$. The new proof of full linear convergence allows to improve the analysis of [7,30] by relaxing the choice of the solver-stopping parameters. Additionally, in the setting of [7], we are able to show that the full linear convergence holds from the arbitrary *initial* mesh onwards instead of the *a priori* unknown and possibly large mesh threshold level $\ell_0 > 0$. In particular, unlike the previous works [11,23,5,7] that employ a quasi-Pythagorean identity, the new analysis shows that the constants in the full R-linear convergence are independent of $\mathcal{T}_0$ and/or the sequence of adaptively generated meshes and therefore fixed *a priori*. Furthermore, the new analysis does not only improve the state-of-the-art theory of full linear convergence leading to optimal complexity, but also allows the choice of larger solver-stopping parameters which also leads to a better numerical performance in experiments.

The remainder of this work is structured as follows: As a model problem, Section 2 formulates a general second-order linear elliptic PDE together with the validity of the so-called *axioms of adaptivity* from [9] and the quasi-orthogonality from [22]. In Section 3, AFEM with exact solver (1) is presented in Algorithm A and, for completeness and eased readability of the later sections, Theorem 1 summarizes the proof of R-linear convergence (4) from [9,22]. Section 4 focuses on AFEM with inexact contractive solver (6) detailed in Algorithm B. The main contribution is the new and more general proof of full R-linear convergence of Theorem 2. Corollary 1 proves the important equivalence (8). The case of AFEM with *nested* contractive solvers, which are useful for nonlinear or nonsymmetric problems, is treated in Section 5 by presenting Algorithm C from [7] and improving its main result in Theorem 4. In Section 6, we discuss the impact of the new analysis on AFEM for nonlinear PDEs. We show that Theorem 4 applies also to the setting from [30], namely strongly monotone PDEs with scalar nonlinearity. Numerical experiments and remarks are discussed in-depth in Section 7, where the impact of the adaptivity parameters on the overall cost is investigated empirically.

Throughout the proofs, the notation $A \lesssim B$ abbreviates $A \le CB$ for some positive constant $C > 0$ whose dependencies are clearly presented in the respective theorem and $A \simeq B$ abbreviates $A \lesssim B \lesssim A$.

2. General second-order linear elliptic PDEs

On a bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \ge 1$, we consider the PDE

$$-\operatorname{div}(\boldsymbol{A}\nabla u^\star) + \boldsymbol{b} \cdot \nabla u^\star + cu^\star = f - \operatorname{div} \boldsymbol{f} \text{ in } \Omega \quad \text{subject to} \quad u^\star = 0 \text{ on } \partial\Omega, \tag{9}$$

where $\boldsymbol{A}, \boldsymbol{b}, c \in L^\infty(\Omega)$ and $\boldsymbol{f}, f \in L^2(\Omega)$ with, for almost every $x \in \Omega$, positive definite $\boldsymbol{A}(x) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\boldsymbol{b}(x), \boldsymbol{f}(x) \in \mathbb{R}^d$, and $c(x), f(x) \in \mathbb{R}$. With $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ denoting the usual $L^2(\Omega)$-scalar product, we suppose that the PDE fits into the setting of the Lax–Milgram lemma, i.e., the bilinear forms

$$a(u, v) := \langle \boldsymbol{A}\nabla u, \nabla v \rangle_{L^2(\Omega)} \quad \text{and} \quad b(u, v) := a(u, v) + \langle \boldsymbol{b} \cdot \nabla u + cu, v \rangle_{L^2(\Omega)}$$

are continuous and elliptic on $H^1_0(\Omega)$. Then, indeed, $a(\cdot, \cdot)$ is a scalar product and $|||u||| := a(u, u)^{1/2}$ defines an equivalent norm on $H^1_0(\Omega)$. Moreover, the weak formulation

$$b(u^\star, v) = F(v) := \langle f, v \rangle_{L^2(\Omega)} + \langle \boldsymbol{f}, \nabla v \rangle_{L^2(\Omega)} \quad \text{for all } v \in H^1_0(\Omega)$$

admits a unique solution $u^\star \in H^1_0(\Omega)$. Let $0 < C_{\mathrm{ell}} \le C_{\mathrm{bnd}}$ denote the continuity and ellipticity constant of $b(\cdot, \cdot)$ with respect to $|||\cdot|||$, i.e., there holds

$$C_{\mathrm{ell}}\, |||v|||^2 \le b(v, v) \quad \text{and} \quad |b(v, w)| \le C_{\mathrm{bnd}}\, |||v|||\, |||w||| \quad \text{for all } v, w \in \mathcal{X}.$$

Remark 1. We stress that the analysis below extends to problems where the associated bilinear form $b(\cdot, \cdot)$ is not coercive but satisfies only a Gårding-type inequality allowing for well-posed more general second-order linear elliptic PDEs. In this context, it is well-known that the well-posedness of the discrete FEM problems and validity of a uniform inf-sup condition require that the triangulations (i.e., the initial triangulation $\mathcal{T}_0$ below) are sufficiently fine; see, e.g., [5]. However, the question of an optimal algebraic solver for the resulting linear systems is beyond the scope of this work and is left for future research.

Let $\mathcal{T}_0$ be an initial conforming triangulation of $\Omega \subset \mathbb{R}^d$ into compact simplices. The mesh refinement employs newest-vertex bisection (NVB). We refer to [47] for NVB with admissible $\mathcal{T}_0$ and $d \ge 2$, to [39] for NVB with general $\mathcal{T}_0$ for $d = 2$, and to the recent work [18] for NVB with general $\mathcal{T}_0$ in any dimension $d \ge 2$. For $d = 1$, we refer to [1]. For each triangulation $\mathcal{T}_H$ and $\mathcal{M}_H \subseteq \mathcal{T}_H$, let $\mathcal{T}_h := \mathtt{refine}(\mathcal{T}_H, \mathcal{M}_H)$ be the coarsest conforming refinement of $\mathcal{T}_H$ such that at least all elements $T \in \mathcal{M}_H$ have been refined, i.e., $\mathcal{M}_H \subseteq \mathcal{T}_H \setminus \mathcal{T}_h$. To abbreviate notation, we write $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ if $\mathcal{T}_h$ can be obtained from $\mathcal{T}_H$ by finitely many steps of NVB and, in particular, $\mathbb{T} := \mathbb{T}(\mathcal{T}_0)$.

For each $\mathcal{T}_H \in \mathbb{T}$, we consider conforming finite element spaces

$$\mathcal{X}_H := \{v_H \in H^1_0(\Omega) \,:\, v_H|_T \text{ is a polynomial of total degree} \le p \text{ for all } T \in \mathcal{T}_H\}, \tag{10}$$

where $p \in \mathbb{N}$ is a fixed polynomial degree. We note that $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ yields nestedness $\mathcal{X}_H \subseteq \mathcal{X}_h$ of the corresponding discrete spaces.

Given $\mathcal{T}_H \in \mathbb{T}$, there exists a unique Galerkin solution $u^\star_H \in \mathcal{X}_H$ solving

$$b(u^\star_H, v_H) = F(v_H) \quad \text{for all } v_H \in \mathcal{X}_H. \tag{11}$$

Moreover, there holds the following Céa lemma

$$|||u^\star - u^\star_H||| \le C_{\text{Céa}} \min_{v_H \in \mathcal{X}_H} |||u^\star - v_H||| \quad \text{with a constant } 1 \le C_{\text{Céa}} \le C_{\text{bnd}}/C_{\text{ell}}, \tag{12}$$

where $C_{\text{Céa}} \to 1$ as adaptive mesh refinement progresses as shown in [5, Theorem 20].

We consider the residual error estimator $\eta_H(\cdot)$ defined, for $T \in \mathcal{T}_H$ and $v_H \in \mathcal{X}_H$, by

$$\begin{aligned}\eta_H(T, v_H)^2 &:= |T|^{2/d} \, \| -\text{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) + \boldsymbol{b}\cdot\nabla v_H + c\, v_H - f\|^2_{L^2(T)} \\ &\quad + |T|^{1/d} \, \|[\![(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\cdot n]\!]\|^2_{L^2(\partial T\cap\Omega)},\end{aligned} \tag{13a}$$

where $[\![\cdot]\!]$ denotes the jump over $(d-1)$-dimensional faces. Clearly, the well-posedness of (13a) requires more regularity of $\boldsymbol{A}$ and $\boldsymbol{f}$ than stated above, e.g., $\boldsymbol{A}|_T, \boldsymbol{f}|_T \in W^{1,\infty}(T)$ for all $T \in \mathcal{T}_0$. To abbreviate notation, we define, for all $\mathcal{U}_H \subseteq \mathcal{T}_H$ and all $v_H \in \mathcal{X}_H$,

$$\eta_H(v_H) := \eta_H(\mathcal{T}_H, v_H) \quad \text{with} \quad \eta_H(\mathcal{U}_H, v_H) := \Big(\sum_{T\in\mathcal{U}_H} \eta_H(T, v_H)^2\Big)^{1/2}. \tag{13b}$$

From [9], we recall that the error estimator satisfies the following properties.

Proposition 1 *(Axioms of adaptivity). There exist constants $C_{\text{stab}}, C_{\text{rel}}, C_{\text{drel}}, C_{\text{mon}} > 0$, and $0 < q_{\text{red}} < 1$ such that the following properties are satisfied for any triangulation $\mathcal{T}_H \in \mathbb{T}$ and any conforming refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ with the corresponding Galerkin solutions $u^\star_H \in \mathcal{X}_H$, $u^\star_h \in \mathcal{X}_h$ to (11) and arbitrary $v_H \in \mathcal{X}_H$, $v_h \in \mathcal{X}_h$.*

- **(A1)** ***stability.*** $|\eta_h(\mathcal{T}_h \cap \mathcal{T}_H, v_h) - \eta_H(\mathcal{T}_h \cap \mathcal{T}_H, v_H)| \le C_{\text{stab}} \, |||v_h - v_H|||$.
- **(A2)** ***reduction.*** $\eta_h(\mathcal{T}_h \backslash \mathcal{T}_H, v_H) \le q_{\text{red}}\, \eta_H(\mathcal{T}_H \backslash \mathcal{T}_h, v_H)$.
- **(A3)** ***reliability.*** $|||u^\star - u^\star_H||| \le C_{\text{rel}}\, \eta_H(u^\star_H)$.
- **(A3$^+$)** ***discrete reliability.*** $|||u^\star_h - u^\star_H||| \le C_{\text{drel}}\, \eta_H(\mathcal{T}_H \backslash \mathcal{T}_h, u^\star_H)$.
- **(QM)** ***quasi-monotonicity.*** $\eta_h(u^\star_h) \le C_{\text{mon}}\, \eta_H(u^\star_H)$.

The constant C_{rel} depends only on uniform shape regularity of all meshes $\mathcal{T}_H \in \mathbb{T}$ and the dimension d, while C_{stab} and C_{drel} additionally depend on the polynomial degree p. The constant q_{red} reads $q_{\text{red}} := 2^{-1/(2d)}$ for bisection-based refinement rules in $\mathbb{R}^d$ and the constant C_{mon} can be bounded by $C_{\text{mon}} \le \min\{1 + C_{\text{stab}}(1 + C_{Céa})C_{\text{rel}}\,,\, 1 + C_{\text{stab}}\, C_{\text{drel}}\}$. □

In addition to the estimator properties in Proposition 1, we recall the following quasi-orthogonality result from [22] as one cornerstone of the improved analysis in this paper.

Proposition 2 *(Validity of quasi-orthogonality). There exist $C_{\text{orth}} > 0$ and $0 < \delta \le 1$ such that the following holds: For any sequence $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset H^1_0(\Omega)$ of nested finite-dimensional subspaces, the corresponding Galerkin solutions $u^\star_\ell \in \mathcal{X}_\ell$ to (11) satisfy*

(A4) ***quasi-orthogonality.*** $\displaystyle\sum_{\ell'=\ell}^{\ell+N} |||u^\star_{\ell'+1} - u^\star_{\ell'}|||^2 \le C_{\text{orth}}(N+1)^{1-\delta}\, |||u^\star - u^\star_\ell|||^2$ *for all* $\ell, N \in \mathbb{N}_0$.

Here, C_{orth} and δ depend only on the dimension d, the elliptic bilinear form $b(\cdot,\cdot)$, and the chosen norm $|||\cdot|||$, but are independent of the spaces $\mathcal{X}_\ell$. □

Remark 2. Quasi-orthogonality (A4) is a generalization of the Pythagorean identity (3) for symmetric problems. Indeed, if $\boldsymbol{b} = 0$ in (9) and $a(\cdot,\cdot) := b(\cdot,\cdot)$ is a scalar product, the Galerkin method for nested subspaces $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset H^1_0(\Omega)$ guarantees (3). Thus, the telescopic series proves (A4) with $C_{\text{orth}} = 1$ and $\delta = 1$. We highlight that [22] proves (A4) even for more general linear problems and Petrov–Galerkin discretizations, where it is only needed that the continuous and discrete inf-sup constants are uniformly bounded from below. In particular, this applies to a wide range of mixed FEM formulations, but also to general second-order linear elliptic PDEs that only satisfy a Gårding-type inequality; see Remark 1 above.

A closer look at the proofs of R-linear convergence in Section 3–5 below reveals that they rely only on the properties (A1), (A2), (A3), (A4), and (QM), but not on (A3$^+$), the Céa lemma (12), or linearity of the PDE. Hence, Algorithms A, B, and C and the corresponding Theorems 1, 2, and 4 below apply beyond the linear problem (9); see Section 6 for a nonlinear PDE.

3. AFEM with exact solution

To outline the new proof strategy, we first consider the standard adaptive algorithm (see, e.g., [12]), where the arising Galerkin systems (11) are solved exactly.

Algorithm A *(AFEM with exact solver).* Given an initial mesh $\mathcal{T}_0$, polynomial degree $p > 0$, and adaptivity parameters $0 < \theta \le 1$ and $C_{\mathrm{mark}} \ge 1$, iterate the following steps for all $\ell = 0, 1, 2, 3, \dots$:

(i) **Solve:** Compute the exact solution $u_\ell^\star \in \mathcal{X}_\ell$ to (11).
(ii) **Estimate:** Compute the refinement indicators $\eta_\ell(T, u_\ell^\star)$ for all $T \in \mathcal{T}_\ell$.
(iii) **Mark:** Determine a set $\mathcal{M}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^\star]$ of quasi-minimal cardinality satisfying the Dörfler marking criterion

$$\#\mathcal{M}_\ell \le C_{\mathrm{mark}} \min_{\mathcal{U}_\ell^\star \in \mathbb{M}_\ell[\theta, u_\ell^\star]} \#\mathcal{U}_\ell^\star, \text{ where } \mathbb{M}_\ell[\theta, u_\ell^\star] := \big\{\mathcal{U}_\ell \subseteq \mathcal{T}_\ell \,:\, \theta \eta_\ell(u_\ell^\star)^2 \le \eta_\ell(\mathcal{U}_\ell, u_\ell^\star)^2\big\}. \tag{14}$$

(iv) **Refine:** Generate $\mathcal{T}_{\ell+1} := \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.

The following theorem asserts convergence of Algorithm A in the spirit of [9], and the proof given below essentially summarizes the arguments from [22]. It will, however, be the starting point for the later generalizations, i.e., for the adaptive algorithms below with inexact solvers.

Theorem 1 *(R-linear convergence of Algorithm A). Let $0 < \theta \le 1$ and $C_{\mathrm{mark}} \ge 1$ be arbitrary. Then, Algorithm A guarantees R-linear convergence of the estimators $\eta_\ell(u_\ell^\star)$, i.e., there exist constants $0 < q_{\mathrm{lin}} < 1$ and $C_{\mathrm{lin}} > 0$ such that*

$$\eta_{\ell+n}(u_{\ell+n}^\star) \le C_{\mathrm{lin}}\, q_{\mathrm{lin}}^n\, \eta_\ell(u_\ell^\star) \quad \text{for all } \ell, n \in \mathbb{N}_0. \tag{15}$$

Remark 3. For vanishing convection $\boldsymbol{b} = 0$ in (9) and $a(\cdot,\cdot) := b(\cdot,\cdot)$, [12] proves linear convergence of the quasi-error (2). Together with reliability (A3), this yields R-linear convergence of the estimator sequence

$$\eta_{\ell+n}(u_{\ell+n}^\star) \le \frac{(C_{\mathrm{rel}}^2 + \gamma)^{1/2}}{\gamma^{1/2}}\, q_{\mathrm{ctr}}^n\, \eta_\ell(u_\ell^\star) \quad \text{for all } \ell, n \in \mathbb{N}_0.$$

In this sense, Theorem 1 is weaker than linear convergence (2) from [12], but provides a direct proof of R-linear convergence even if $b(\cdot,\cdot) \ne a(\cdot,\cdot)$. Moreover, while the proof of (2) crucially relies on the Pythagorean identity (3), the works [23,5] extend the analysis to the general second-order linear elliptic PDE (9) using

$$\forall 0 < \varepsilon < 1\, \exists \ell_0 \in \mathbb{N}_0\, \forall \ell \ge \ell_0 : \quad |||u^\star - u_{\ell+1}^\star|||^2 \le |||u^\star - u_\ell^\star|||^2 - \varepsilon\, |||u_{\ell+1}^\star - u_\ell^\star|||^2. \tag{16}$$

From this, contraction (2) follows for all $\ell \ge \ell_0$ and allows to extend the AFEM analysis from [46,12] to general second-order linear elliptic PDE. However, the index ℓ_0 depends on the exact solution $u^\star$ and on the sequence of exact discrete solutions $(u_\ell^\star)_{\ell\in\mathbb{N}_0}$. Moreover, $\ell_0 = 0$ requires sufficiently fine $\mathcal{T}_0$ in [11,5] while the constants in (15) depend on $u^\star$ and the sequence $(u_\ell^\star)_{\ell\in\mathbb{N}_0}$ in [23]. In the present work however, R-linear convergence (15) from Theorem 1 holds with $\ell_0 = 0$ and any initial mesh $\mathcal{T}_0$, and the constants are independent of $u^\star$ and $(u_\ell^\star)_{\ell\in\mathbb{N}_0}$, thus clearly improving the previous state of the art.

The proof of Theorem 1 relies on the following elementary lemma that extends arguments implicitly found for the estimator sequence in [22] but that will be employed for certain quasi-errors in the present work. Its proof is found in Appendix A.

Lemma 1 *(Tail summability criterion). Let $(a_\ell)_{\ell\in\mathbb{N}_0}, (b_\ell)_{\ell\in\mathbb{N}_0}$ be scalar sequences in $\mathbb{R}_{\ge 0}$. With given constants $0 < q < 1$, $0 < \delta < 1$, and $C_1, C_2 > 0$, suppose that*

$$a_{\ell+1} \le q a_\ell + b_\ell, \quad b_{\ell+N} \le C_1\, a_\ell, \text{ and } \sum_{\ell'=\ell}^{\ell+N} b_{\ell'}^2 \le C_2\, (N+1)^{1-\delta}\, a_\ell^2 \text{ for all } \ell, N \in \mathbb{N}_0. \tag{17}$$

Then, $(a_\ell)_{\ell\in\mathbb{N}_0}$ is R-linearly convergent, i.e., there exist $C_{\mathrm{lin}} > 0$ and $0 < q_{\mathrm{lin}} < 1$ with

$$a_{\ell+n} \le C_{\mathrm{lin}}\, q_{\mathrm{lin}}^n\, a_\ell \quad \text{for all } \ell, n \in \mathbb{N}_0. \tag{18}$$

Proof of Theorem 1. We employ Lemma 1 for the sequences defined by $a_\ell = \eta_\ell(u_\ell^\star)$ and $b_\ell := C_{\mathrm{stab}}\, |||u_{\ell+1}^\star - u_\ell^\star|||$. First, we note that

$$|||u^\star_{\ell''} - u^\star_{\ell'}||| \overset{(A3)}{\lesssim} \eta_{\ell''}(u^\star_{\ell''}) + \eta_{\ell'}(u^\star_{\ell'}) \overset{(QM)}{\lesssim} \eta_\ell(u^\star_\ell) \quad \text{for all } \ell, \ell', \ell'' \in \mathbb{N}_0 \text{ with } \ell \le \ell' \le \ell''. \tag{19}$$

In particular, this proves $b_{\ell+N} \lesssim a_\ell$ for all $\ell, N \in \mathbb{N}_0$. Moreover, quasi-orthogonality (A4) and reliability (A3) show

$$\sum_{\ell'=\ell}^{\ell+N} |||u^\star_{\ell'+1} - u^\star_{\ell'}|||^2 \le C_{\mathrm{orth}}\, C^2_{\mathrm{rel}}\, (N+1)^{1-\delta}\, \eta_\ell(u^\star_\ell)^2 \quad \text{for all } \ell, N \in \mathbb{N}_0. \tag{20}$$

In order to verify (17), it thus only remains to prove the perturbed contraction of a_ℓ. To this end, let $\ell \in \mathbb{N}_0$. Then, stability (A1) and reduction (A2) show

$$\eta_{\ell+1}(u^\star_\ell)^2 \le \eta_\ell(\mathcal{T}_{\ell+1} \cap \mathcal{T}_\ell, u^\star_\ell)^2 + q^2_{\mathrm{red}} \eta_\ell(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}, u^\star_\ell)^2 = \eta_\ell(u^\star_\ell)^2 - (1 - q^2_{\mathrm{red}})\, \eta_\ell(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}, u^\star_\ell)^2.$$

Moreover, Dörfler marking (14) and refinement of (at least) all marked elements lead to

$$\theta \eta_\ell(u^\star_\ell)^2 \overset{(14)}{\le} \eta_\ell(\mathcal{M}_\ell, u^\star_\ell)^2 \le \eta_\ell(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}, u^\star_\ell)^2.$$

The combination of the two previously displayed formulas results in

$$\eta_{\ell+1}(u^\star_\ell) \le q_\theta\, \eta_\ell(u^\star_\ell) \quad \text{with} \quad 0 < q_\theta := \big[1 - (1 - q^2_{\mathrm{red}})\theta\big]^{1/2} < 1.$$

Finally, stability (A1) thus leads to the desired estimator reduction estimate

$$\eta_{\ell+1}(u^\star_{\ell+1}) \overset{(A1)}{\le} q_\theta\, \eta_\ell(u^\star_\ell) + C_{\mathrm{stab}}\, |||u^\star_{\ell+1} - u^\star_\ell||| \quad \text{for all } \ell \in \mathbb{N}_0. \tag{21}$$

Altogether, all the assumptions (17) are satisfied and Lemma 1 concludes the proof. □

4. AFEM with contractive solver

Let $\Psi_H : \mathcal{X}_H \to \mathcal{X}_H$ be the iteration mapping of a uniformly contractive solver, i.e.,

$$|||u^\star_H - \Psi_H(v_H)||| \le q_{\mathrm{alg}}\, |||u^\star_H - v_H||| \quad \text{for all } \mathcal{T}_H \in \mathbb{T} \text{ and all } v_H \in \mathcal{X}_H. \tag{22}$$

Examples of such iterative solvers include an optimally preconditioned conjugate gradient method from [13] or an optimal geometric multigrid method from [49,37]. The following algorithm is thoroughly analyzed in [26] under the assumption that the problem is symmetric (and hence the Pythagorean identity (3) holds).

Algorithm B *(AFEM with contractive solver)*. Given an initial mesh $\mathcal{T}_0$, polynomial degree $p > 0$, adaptivity parameters $0 < \theta \le 1$ and $C_{\mathrm{mark}} \ge 1$, a solver-stopping parameter $\lambda > 0$, and an initial guess $u^0_0 \in \mathcal{X}_0$, iterate the following steps (i)–(iv) for all $\ell = 0, 1, 2, 3, \dots$:

(i) **Solve & Estimate:** For all $k = 1, 2, 3, \dots$, repeat (a)–(b) until

$$|||u^k_\ell - u^{k-1}_\ell||| \le \lambda\, \eta_\ell(u^k_\ell). \tag{23}$$

(a) Compute $u^k_\ell := \Psi_\ell(u^{k-1}_\ell)$ with one step of the contractive solver.
(b) Compute the refinement indicators $\eta_\ell(T, u^k_\ell)$ for all $T \in \mathcal{T}_\ell$.

(ii) Upon termination of the iterative solver, define the index $\underline{k}[\ell] := k \in \mathbb{N}$.
(iii) **Mark:** Determine a set $\mathcal{M}_\ell \in \mathbb{M}_\ell[\theta, u^{\underline{k}}_\ell]$ satisfying (14) with $u^\star_\ell$ replaced by $u^{\underline{k}}_\ell$.
(iv) **Refine:** Generate $\mathcal{T}_{\ell+1} := \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ and employ nested iteration $u^0_{\ell+1} := u^{\underline{k}}_\ell$.

The sequential nature of Algorithm B gives rise to the countably infinite index set

$$\mathcal{Q} := \big\{(\ell, k) \in \mathbb{N}^2_0 \;:\; u^k_\ell \in \mathcal{X}_\ell \text{ is defined in Algorithm B}\big\}$$

together with the lexicographic ordering

$$(\ell', k') \le (\ell, k) \quad :\Longleftrightarrow \quad u^{k'}_{\ell'} \text{ is defined not later than } u^k_\ell \text{ in Algorithm B}$$

and the total step counter

$$|\ell, k| := \#\big\{(\ell', k') \in \mathcal{Q} \;:\; (\ell', k') \le (\ell, k)\big\} \in \mathbb{N}_0 \quad \text{for all } (\ell, k) \in \mathcal{Q}.$$

Defining the stopping indices

$$\underline{\ell} := \sup\{\ell \in \mathbb{N}_0 \;:\; (\ell, 0) \in \mathcal{Q}\} \in \mathbb{N}_0 \cup \{\infty\}, \tag{24a}$$

$$\underline{k}[\ell] := \sup\{k \in \mathbb{N}_0 \;:\; (\ell, k) \in \mathcal{Q}\} \in \mathbb{N} \cup \{\infty\}, \quad \text{whenever } (\ell, 0) \in \mathcal{Q}, \tag{24b}$$

we note that these definitions are consistent with that of Algorithm B(ii). We abbreviate $\underline{k} = \underline{k}[\ell]$, whenever the index ℓ is clear from the context, e.g., $u_\ell^{\underline{k}} := u_\ell^{\underline{k}[\ell]}$ or $(\ell, \underline{k}) = (\ell, \underline{k}[\ell])$.

As $\mathcal{Q}$ is an infinite set, the typical case is $\underline{\ell} = \infty$ and $\underline{k}[\ell] < \infty$ for all $\ell \in \mathbb{N}_0$, whereas $\underline{\ell} < \infty$ implies that $\underline{k}[\underline{\ell}] = \infty$, i.e., non-termination of the iterative solver on the mesh $\mathcal{T}_{\underline{\ell}}$. The following theorem states convergence of Algorithm B. In particular, it shows that $\underline{\ell} < \infty$ implies $\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$ and consequently $u^\star = u_{\underline{\ell}}^\star$ by reliability (A3).

Theorem 2 *(Full R-linear convergence of Algorithm B). Let $0 < \theta \le 1$, $C_{\rm mark} \ge 1$, $\lambda > 0$, and $u_0^0 \in \mathcal{X}_0$ be arbitrary. Then, Algorithm B guarantees R-linear convergence of the modified quasi-error*

$$\mathrm{H}_\ell^k := |||u_\ell^\star - u_\ell^k||| + \eta_\ell(u_\ell^k), \tag{25}$$

i.e., there exist constants $0 < q_{\rm lin} < 1$ and $C_{\rm lin} > 0$ such that

$$\mathrm{H}_\ell^k \le C_{\rm lin} q_{\rm lin}^{|\ell,k|-|\ell',k'|} \mathrm{H}_{\ell'}^{k'} \quad \text{for all } (\ell',k'),(\ell,k) \in \mathcal{Q} \text{ with } |\ell',k'| \le |\ell,k|. \tag{26}$$

Remark 4. Unlike [26] (and [12]), Theorem 2 and its proof employ the quasi-error H_ℓ^k from (25) instead of $\Delta_\ell^k := \big[|||u^\star - u_\ell^k|||^2 + \gamma\, \eta_\ell(u_\ell^k)^2\big]^{1/2}$ analogous to (2). We note that stability (A1) and reliability (A3) yield $\Delta_\ell^k \lesssim \mathrm{H}_\ell^k$, while the converse estimate follows from the Céa lemma (12).

Remark 5. The work [26] extends the ideas of [12] (that proves (2) for AFEM with exact solver) and of [25] (that extends (2) to the final iterates for AFEM with contractive solver). For the scalar product $b(\cdot,\cdot) = a(\cdot,\cdot)$ and arbitrary stopping parameters $\lambda > 0$, it shows that the quasi-error Δ_ℓ^k from Remark 4 satisfies contraction

$$\Delta_\ell^k \le q_{\rm ctr}\, \Delta_\ell^{k-1} \qquad \text{for all } (\ell,k) \in \mathcal{Q} \text{ with } 0 < k < \underline{k}[\ell], \tag{27a}$$

$$\Delta_{\ell+1}^0 \le q_{\rm ctr}\, \Delta_\ell^{\underline{k}-1} \qquad \text{for all } (\ell,\underline{k}) \in \mathcal{Q} \tag{27b}$$

with contraction constant $0 < q_{\rm ctr} < 1$, along the approximations $u_\ell^k \in \mathcal{X}_\ell$ generated by Algorithm B. The proof of (27) can be generalized similarly to Remark 3, see [7]: With the quasi-Pythagorean estimate (16), the contraction (27) transfers to general second-order linear elliptic PDEs (9) under the restriction that (27b) holds only for all $\ell \ge \ell_0$, where $\ell_0 \in \mathbb{N}_0$ exists, but is unknown in practice. While, as noted in Remark 3, contraction (27) implies full R-linear convergence (26), the proof of Theorem 2 works under much weaker assumptions than that of [26] and covers the PDE (9) with $\ell_0 = 0$.

The proof of Theorem 2 relies on Lemma 1 and the following elementary result essentially taken from [9, Lemma 4.9]. Its proof is found in Appendix A.

Lemma 2 *(Tail summability vs. R-linear convergence). Let $(a_\ell)_{\ell\in\mathbb{N}_0}$ be a scalar sequence in $\mathbb{R}_{\ge 0}$ and $m > 0$. Then, the following statements are equivalent:*

(i) ***tail summability:*** *There exists a constant $C_m > 0$ such that*

$$\sum_{\ell'=\ell+1}^{\infty} a_{\ell'}^m \le C_m\, a_\ell^m \quad \text{for all } \ell \in \mathbb{N}_0. \tag{28}$$

(ii) ***R-linear convergence:*** *There holds (18) with certain $0 < q_{\rm lin} < 1$ and $C_{\rm lin} > 0$.*

Proof of Theorem 2. We want to briefly summarize the proof strategy. First, we show that the estimator reduction together with the contraction (27) of the algebraic solver leads to tail summability of a weighted quasi-error on the mesh level ℓ. Second, we show that the quasi-error from (25) is contractive in the algebraic solver index k and is stable in the nested iteration. Finally, we combine these two steps to prove R-linear convergence of the quasi-error (26).

The proof is split into two steps.

Step 1 (tail summability with respect to ℓ). Let $\ell \in \mathbb{N}$ with $(\ell+1, \underline{k}) \in \mathcal{Q}$. Algorithm B guarantees nested iteration $u_{\ell+1}^0 = u_\ell^{\underline{k}}$ and $\underline{k}[\ell+1] \ge 1$. This and contraction of the algebraic solver (22) show

$$|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k}}||| \overset{(22)}{\le} q_{\rm alg}^{\underline{k}[\ell+1]}\, |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| \le q_{\rm alg}\, |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| \tag{29}$$

As in the proof of Theorem 1, one obtains the estimator reduction

$$\eta_{\ell+1}(u_{\ell+1}^{\underline{k}}) \overset{(21)}{\le} q_\theta\, \eta_\ell(u_\ell^{\underline{k}}) + C_{\rm stab}\, |||u_{\ell+1}^{\underline{k}} - u_\ell^{\underline{k}}||| \overset{(29)}{\le} q_\theta\, \eta_\ell(u_\ell^{\underline{k}}) + (q_{\rm alg}+1) C_{\rm stab}\, |||u_{\ell+1}^\star - u_\ell^{\underline{k}}|||. \tag{30}$$

Choosing $0 < \gamma \le 1$ with $0 < q_{\rm ctr} := \max\{q_{\rm alg} + (q_{\rm alg}+1)C_{\rm stab}\gamma\,,\, q_\theta\} < 1$, the combination of (29)–(30) reads

$$\begin{aligned} a_{\ell+1} &:= |||u^\star_{\ell+1} - u^{\underline{k}}_{\ell+1}||| + \gamma\,\eta_{\ell+1}(u^{\underline{k}}_{\ell+1}) \le q_{\rm ctr}\,\big[|||u^\star_{\ell+1} - u^{\underline{k}}_{\ell}||| + \gamma\,\eta_\ell(u^{\underline{k}}_\ell)\big] \\ &\le q_{\rm ctr}\,\big[|||u^\star_\ell - u^{\underline{k}}_\ell||| + \gamma\,\eta_\ell(u^{\underline{k}}_\ell)\big] + q_{\rm ctr}\,|||u^\star_{\ell+1} - u^\star_\ell||| =: q_{\rm ctr}\,a_\ell + b_\ell. \end{aligned} \tag{31}$$

Moreover, estimate (19) from the proof of Theorem 1 and stability (A1) prove that

$$|||u^\star_{\ell''} - u^\star_{\ell'}||| \overset{(19)}{\lesssim} \eta_\ell(u^\star_\ell) \overset{(A1)}{\lesssim} |||u^\star_\ell - u^{\underline{k}}_\ell||| + \eta_\ell(u^{\underline{k}}_\ell) \simeq a_\ell \text{ for } \ell \le \ell' \le \ell'' \le \underline{\ell} \text{ with } (\underline{\ell},\underline{k}) \in \mathcal{Q}, \tag{32}$$

which yields $b_{\ell+N} \lesssim a_\ell$ for all $0 \le \ell \le \ell + N \le \underline{\ell}$ with $(\underline{\ell},\underline{k}) \in \mathcal{Q}$. As in (20), we see

$$\begin{aligned} \sum_{\ell'=\ell}^{\ell+N} b^2_{\ell'} &\simeq \sum_{\ell'=\ell}^{\ell+N} |||u^\star_{\ell'+1} - u^\star_{\ell'}|||^2 \overset{(A4)}{\lesssim} (N+1)^{1-\delta}\,|||u^\star - u^\star_\ell|||^2 \overset{(A3)}{\lesssim} (N+1)^{1-\delta}\,\eta_\ell(u^\star_\ell)^2 \\ &\overset{(A1)}{\lesssim} (N+1)^{1-\delta}\,\big[\eta_\ell(u^{\underline{k}}_\ell) + |||u^\star_\ell - u^{\underline{k}}_\ell|||\big]^2 \simeq (N+1)^{1-\delta}\,a^2_\ell \text{ for all } 0 \le \ell \le \ell + N < \underline{\ell}. \end{aligned} \tag{33}$$

Hence, the assumptions (17) are satisfied and Lemma 1 concludes tail summability (or equivalently R-linear convergence by Lemma 2) of $\mathrm{H}^{\underline{k}}_\ell \simeq a_\ell$, i.e.,

$$\sum_{\ell'=\ell+1}^{\underline{\ell}-1} \mathrm{H}^{\underline{k}}_{\ell'} \lesssim \mathrm{H}^{\underline{k}}_\ell \quad \text{for all } 0 \le \ell < \underline{\ell}. \tag{34}$$

Step 2 (tail summability with respect to ℓ and k). First, for $0 \le k < k' < \underline{k}[\ell]$, the failure of the termination criterion (23) and contraction of the solver (22) prove that

$$\mathrm{H}^{k'}_\ell \overset{(23)}{\lesssim} |||u^\star_\ell - u^{k'}_\ell||| + |||u^{k'}_\ell - u^{k'-1}_\ell||| \overset{(22)}{\lesssim} |||u^\star_\ell - u^{k'-1}_\ell||| \overset{(22)}{\lesssim} q^{k'-k}_{\rm alg}\,|||u^\star_\ell - u^k_\ell||| \overset{(25)}{\le} q^{k'-k}_{\rm alg}\,\mathrm{H}^k_\ell.$$

Second, for $(\ell,\underline{k}) \in \mathcal{Q}$, it holds that

$$\mathrm{H}^{\underline{k}}_\ell \overset{(A1)}{\lesssim} |||u^\star_\ell - u^{\underline{k}}_\ell||| + \eta_\ell(u^{\underline{k}-1}_\ell) + |||u^{\underline{k}}_\ell - u^{\underline{k}-1}_\ell||| \le \mathrm{H}^{\underline{k}-1}_\ell + 2\,|||u^\star_\ell - u^{\underline{k}}_\ell||| \overset{(22)}{\le} (1 + 2\,q_{\rm alg})\,\mathrm{H}^{\underline{k}-1}_\ell \quad \text{for all } (\ell,\underline{k}) \in \mathcal{Q}.$$

Hence, we may conclude

$$\mathrm{H}^{k'}_\ell \lesssim q^{k'-k}_{\rm alg}\,\mathrm{H}^k_\ell \quad \text{for all } 0 \le k \le k' \le \underline{k}[\ell]. \tag{35}$$

With $|||u^\star_{\ell+1} - u^\star_\ell||| \lesssim a_\ell \simeq \mathrm{H}^{\underline{k}}_\ell$ from (19), stability (A1) and reduction (A2) show

$$\mathrm{H}^0_{\ell+1} = |||u^\star_{\ell+1} - u^{\underline{k}}_\ell||| + \eta_{\ell+1}(u^{\underline{k}}_\ell) \le \mathrm{H}^{\underline{k}}_\ell + |||u^\star_{\ell+1} - u^\star_\ell||| \lesssim \mathrm{H}^{\underline{k}}_\ell \quad \text{for all } (\ell,\underline{k}) \in \mathcal{Q}. \tag{36}$$

Overall, the geometric series proves tail summability (28) via

$$\sum_{\substack{(\ell',k') \in \mathcal{Q} \\ |\ell',k'| > |\ell,k|}} \mathrm{H}^{k'}_{\ell'} = \sum_{k'=k+1}^{\underline{k}[\ell]} \mathrm{H}^{k'}_\ell + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \mathrm{H}^{k'}_{\ell'} \overset{(35)}{\lesssim} \mathrm{H}^k_\ell + \sum_{\ell'=\ell+1}^{\underline{\ell}} \mathrm{H}^0_{\ell'} \overset{(36)}{\lesssim} \mathrm{H}^k_\ell + \sum_{\ell'=\ell}^{\underline{\ell}-1} \mathrm{H}^{\underline{k}}_{\ell'} \overset{(34)}{\lesssim} \mathrm{H}^k_\ell + \mathrm{H}^{\underline{k}}_\ell \overset{(35)}{\lesssim} \mathrm{H}^k_\ell \quad \text{for all } (\ell,k) \in \mathcal{Q}.$$

Since $\mathcal{Q}$ is countable and linearly ordered, Lemma 2 concludes the proof of (26). □

The following comments on the computational cost of implementations of standard finite element methods underline the importance of full linear convergence (26).

- ***Solve & Estimate.*** One solver step of an optimal multigrid method can be performed in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations, if smoothing is done according to the grading of the mesh as in [49,37]. Instead, one step of a multigrid method on $\mathcal{T}_\ell$, where smoothing is done on all levels and all vertex patches needs $\mathcal{O}(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'})$ operations. The same remark is valid for the preconditioned CG method with optimal additive Schwarz or BPX preconditioner from [13]. One solver step can be realized via successive updates in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations, while $\mathcal{O}\big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'}\big)$ is faced if the preconditioner does not respect the grading of the mesh hierarchy.
- ***Mark.*** The Dörfler marking strategy (14) can be realized in linear complexity $\mathcal{O}(\#\mathcal{T}_\ell)$; see [46] for $C_{\rm mark} = 2$ and [45] for $C_{\rm mark} = 1$.
- ***Refine.*** Local mesh refinement (including mesh closure) of $\mathcal{T}_\ell$ by bisection can be realized in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations; see, e.g., [6,46].

Since the adaptive algorithm depends on the full history of algorithmic decisions, the overall computational cost until step $(\ell,k) \in \mathcal{Q}$, i.e., until (and including) the computation of u^k_ℓ, is thus proportionally bounded by

$$\sum_{\substack{(\ell',k') \in \mathcal{Q} \\ |\ell',k'| \le |\ell,k|}} \#\mathcal{T}_{\ell'} \le \mathtt{cost}(\ell,k) \le \sum_{\substack{(\ell',k') \in \mathcal{Q} \\ |\ell',k'| \le |\ell,k|}} \sum_{\ell''=0}^{\ell'} \#\mathcal{T}_{\ell''}.$$

Here, the lower bound corresponds to the case that all steps of Algorithm B are done at linear cost $\mathcal{O}(\#\mathcal{T}_\ell)$. The upper bound corresponds to the case that *solve* & *estimate*, *mark*, and *refine* are performed at linear cost $\mathcal{O}(\#\mathcal{T}_\ell)$, while a suboptimal solver leads to cost $\mathcal{O}(\sum_{\ell''=0}^{\ell} \#\mathcal{T}_{\ell''})$ for each mesh $\mathcal{T}_\ell$. In any case, the following corollary shows that full R-linear convergence guarantees that convergence rates with respect to the number of degrees of freedom $\dim \mathcal{X}_\ell \simeq \#\mathcal{T}_\ell$ and with respect to the overall computational cost $\texttt{cost}(\ell, k)$ coincide even for a suboptimal solver. Moreover, the corollary shows that, under R-linear convergence, the optimal convergence rate with respect to the number of degrees of freedom as well as with respect to the overall computational cost is non-zero.

Corollary 1 *(Rates = complexity). For $s > 0$, full R-linear convergence (26) yields*

$$M(s) := \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \,\mathrm{H}_\ell^k \le \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \sum_{\ell''=0}^{\ell'} \#\mathcal{T}_{\ell''} \Big)^s \mathrm{H}_\ell^k \le C_{\mathrm{cost}}(s)\, M(s),$$

where the constant $C_{\mathrm{cost}}(s) > 0$ depends only on C_{lin}, q_{lin}, and s. Moreover, there exists $s_0 > 0$ such that $M(s) < \infty$ for all $0 < s \le s_0$ with $s_0 = \infty$ if $\underline{\ell} < \infty$.

The previous corollary is an immediate consequence of the following elementary lemma for $a_{|\ell,k|} := \mathrm{H}_\ell^k$ and $t_{|\ell,k|} := \#\mathcal{T}_\ell$.

Lemma 3 *(Rates = complexity criterion). Let $(a_\ell)_{\ell\in\mathbb{N}_0}$ and $(t_\ell)_{\ell\in\mathbb{N}_0}$ be sequences in $\mathbb{R}_{\ge 0}$ such that*

$$a_{\ell+n} \le C_1 q^n a_\ell \quad \text{and} \quad t_{\ell+1} \le C_2\, t_\ell \quad \text{for all } \ell, n \in \mathbb{N}_0. \tag{37}$$

Then, for all $s > 0$, there holds

$$M(s) := \sup_{\ell\in\mathbb{N}_0} t_\ell^s\, a_\ell \le \sup_{\ell\in\mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \sum_{\ell''=0}^{\ell'} t_{\ell''} \Big)^s a_\ell \le C_{\mathrm{cost}}(s)\, M(s), \tag{38}$$

where the constant $C_{\mathrm{cost}}(s) > 0$ depends only on C_1, q, and s. Moreover, there exists $s_0 > 0$ depending only on C_2 and q such that $M(s) < \infty$ for all $0 < s \le s_0$.

Proof. By definition, it holds that

$$t_\ell \le M(s)^{1/s}\, a_\ell^{-1/s} \quad \text{for all } \ell \in \mathbb{N}_0.$$

This, assumption (37), and the geometric series prove that

$$\sum_{\ell''=0}^{\ell'} t_{\ell''} \le M(s)^{1/s} \sum_{\ell''=0}^{\ell'} a_{\ell''}^{-1/s} \overset{(37)}{\le} M(s)^{1/s}\, C_1^{1/s} a_{\ell'}^{-1/s} \sum_{\ell''=0}^{\ell'} (q^{1/s})^{\ell'-\ell''} \le M(s)^{1/s} \frac{C_1^{1/s}}{1-q^{1/s}}\, a_{\ell'}^{-1/s} \quad \text{for all } \ell' \in \mathbb{N}_0.$$

A further application of (37) and the geometric series prove that

$$\sum_{\ell'=0}^{\ell} a_{\ell'}^{-1/s} \overset{(37)}{\le} C_1^{1/s} a_\ell^{-1/s} \sum_{\ell'=0}^{\ell} (q^{1/s})^{\ell-\ell'} \le \frac{C_1^{1/s}}{1-q^{1/s}}\, a_\ell^{-1/s} \quad \text{for all } \ell \in \mathbb{N}_0.$$

The combination of the two previously displayed formulas results in

$$\sum_{\ell'=0}^{\ell} \sum_{\ell''=0}^{\ell'} t_{\ell''} \le \Big(\frac{C_1^{1/s}}{1-q^{1/s}} \Big)^2 M(s)^{1/s}\, a_\ell^{-1/s} \quad \text{for all } \ell \in \mathbb{N}_0.$$

Rearranging this estimate, we conclude the proof of (38). It remains to verify $M(s) < \infty$ for some $s > 0$. Note that (37) guarantees that

$$0 \le t_\ell \le C_2\, t_{\ell-1} \le C_2^\ell\, t_0 \quad \text{for all } \ \ell \in \mathbb{N}.$$

Moreover, R-linear convergence (37) yields that

$$0 \le a_\ell \overset{(37)}{\le} C_1 q^\ell\, a_0 \quad \text{for all } \ \ell \in \mathbb{N}_0.$$

Multiplying the two previously displayed formulas, we see that

$$t_\ell^s\, a_\ell \le (C_2^s q)^\ell\, C_1\, t_0^s\, a_0 \quad \text{for all } \ell \in \mathbb{N}_0.$$

Note that the right-hand side is uniformly bounded, provided that $s > 0$ guarantees $C_2^s q \le 1$. This concludes the proof with $s_0 := \log(1/q)/\log(C_2)$. $\square$

With full linear convergence (26), the following theorem from [26, Theorem 8] can be applied and thus Algorithm B guarantees optimal convergence rates with respect to the overall computational cost in the case of sufficiently small adaptivity parameters θ and λ. To formalize the notion of *achievable convergence rates* $s > 0$, we introduce nonlinear approximation classes from [6,46,12,9]

$$\|u^\star\|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}_0} \Big(\big(N+1\big)^s \min_{\mathcal{T}_{\rm opt}\in\mathbb{T}_N} \eta_{\rm opt}(u^\star_{\rm opt}) \Big),$$

where $\eta_{\rm opt}(u^\star_{\rm opt})$ is the estimator for the (unavailable) exact Galerkin solution $u^\star_{\rm opt}$ on an optimal mesh $\mathcal{T}_{\rm opt} \in \mathbb{T}_N := \{\mathcal{T}_H \in \mathbb{T} : \#\mathcal{T}_H - \#\mathcal{T}_0 \le N\}$.

Theorem 3 *(Optimal complexity of Algorithm B, [26, Theorem 8]). Suppose that the estimator satisfies the axioms of adaptivity* (A1), (A2), (A3$^+$)*, and suppose that quasi-orthogonality* (A4) *holds. Suppose that the parameters θ and λ are chosen such that*

$$0 < \lambda < \lambda^\star = \min\Big\{1, \frac{1-q_{\rm alg}}{q_{\rm alg}} C_{\rm stab}^{-1}\Big\} \quad \text{and} \quad 0 < \frac{(\theta^{1/2} + \lambda/\lambda^\star)^2}{\big(1-\lambda/\lambda^\star\big)^2} < \theta^\star := \big(1 + C_{\rm stab}^2 C_{\rm drel}^2\big)^{-1}.$$

Then, Algorithm B guarantees for all $s > 0$ that

$$c_{\rm opt}\|u^\star\|_{\mathbb{A}_s} \le \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \mathrm{H}_\ell^k \le C_{\rm opt} \max\{\|u^\star\|_{\mathbb{A}_s}, \mathrm{H}_0^0\}.$$

The constant $c_{\rm opt} > 0$ depends only on $C_{\rm stab}$, the use of NVB refinement, and s, while $C_{\rm opt} > 0$ depends only on $C_{\rm stab}$, $q_{\rm red}$, $C_{\rm drel}$, $C_{\rm lin}$, $q_{\rm lin}$, $\#\mathcal{T}_0$, λ, $q_{\rm alg}$, θ, s, and the use of NVB refinement. □

Remark 6. Considering the nonsymmetric model problem (9), a natural candidate for the solver is the generalized minimal residual method (GMRES) with optimal preconditioner for the symmetric part. Another alternative would be to consider an optimal preconditioner for the symmetric part and apply a conjugate gradient method to the normal equations (CGNR). However, for both approaches, *a posteriori* error estimation and contraction in the PDE-related energy norm are still open. Instead, [7] follows the constructive proof of the Lax–Milgram lemma to derive a contractive solver. Its convergence analysis, as given in [7], is improved in the following Section 5.

5. AFEM with nested contractive solvers

While contractive solvers for SPD systems are well-understood in the literature, the recent work [7] presents contractive solvers for the nonsymmetric variational formulation (11) that essentially fit into the framework of Section 4 and allow for the numerical analysis of AFEM with optimal complexity. To this end, the proof of the Lax–Milgram lemma as proposed by [50] is exploited algorithmically (while the original proof in [41] relies on the Hahn–Banach separation theorem): For $\delta > 0$, we consider the Zarantonello mapping $\Phi_H(\delta;\cdot) : \mathcal{X}_H \to \mathcal{X}_H$ defined by

$$a(\Phi_H(\delta;u_H), v_H) = a(u_H, v_H) + \delta\big[F(v_H) - b(u_H, v_H)\big] \quad \text{for all } u_H, v_H \in \mathcal{X}_H. \tag{39}$$

Since $a(\cdot,\cdot)$ is a scalar product, $\Phi_H(\delta;u_H) \in \mathcal{X}_H$ is well-defined. Moreover, for any $0 < \delta < 2\alpha/L^2$ and $0 < q^\star_{\rm sym} := [1 - \delta(2\alpha - \delta L^2)]^{1/2} < 1$, this mapping is contractive, i.e.,

$$|||u^\star_H - \Phi_H(\delta;u_H)||| \le q^\star_{\rm sym}\, |||u^\star_H - u_H||| \quad \text{for all } u_H \in \mathcal{X}_H; \tag{40}$$

see also [34,35]. Note that (39) corresponds to a linear SPD system. For this, we employ a uniformly contractive algebraic solver with iteration function $\Psi_H(u^\sharp_H;\cdot) : \mathcal{X}_H \to \mathcal{X}_H$ to approximate the solution $u^\sharp_H := \Phi_H(\delta;u_H)$ to the SPD system (39), i.e.,

$$|||u^\sharp_H - \Psi_H(u^\sharp_H;w_H)||| \le q_{\rm alg}\, |||u^\sharp_H - w_H||| \quad \text{for all } w_H \in \mathcal{X}_H \text{ and all } \mathcal{T}_H \in \mathbb{T}, \tag{41}$$

where $0 < q_{\rm alg} < 1$ depends only on $a(\cdot,\cdot)$, but is independent of $\mathcal{X}_H$. Clearly, no knowledge of $u^\sharp_H$ is needed to compute $\Psi_H(u^\sharp_H;w_H)$ but only that of the corresponding right-hand side $a(u^\sharp_H,\cdot) : \mathcal{X}_H \to \mathbb{R}$; see, e.g., [13,49,37].

Algorithm C *(AFEM with nested contractive solvers).* Given an initial mesh $\mathcal{T}_0$, polynomial degree $p > 0$, the Zarantonello parameter $\delta > 0$, adaptivity parameters $0 < \theta \le 1$ and $C_{\rm mark} \ge 1$, solver-stopping parameters $\lambda_{\rm sym}, \lambda_{\rm alg} > 0$, and an initial guess $u_0^{0,0} := u_0^{0,\underline{j}} \in \mathcal{X}_0$, iterate the following steps (i)–(iv) for all $\ell = 0, 1, 2, 3, \ldots$:

(i) **Solve & estimate:** For all $k = 1, 2, 3, \ldots$, repeat the following steps (a)–(c) until

$$|||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| \le \lambda_{\rm sym}\, \eta_\ell(u_\ell^{k,\underline{j}}) \tag{42}$$

(a) Define $u_\ell^{k,0} := u_\ell^{k-1,\underline{j}}$ and, only as a theoretical quantity, $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$.

(b) **Inner solver loop:** For all $j = 1, 2, 3, \dots$, repeat the steps (I)–(II) until

$$|||u_\ell^{k,j} - u_\ell^{k,j-1}||| \le \lambda_{\mathrm{alg}} \big[\lambda_{\mathrm{sym}} \eta_\ell(u_\ell^{k,j}) + |||u_\ell^{k,j} - u_\ell^{k-1,\underline{j}}|||\big]. \tag{43}$$

(I) Compute one step of the contractive SPD solver $u_\ell^{k,j} := \Psi_\ell(u_\ell^{k,\star}; u_\ell^{k,j-1})$.

(II) Compute the refinement indicators $\eta_\ell(T, u_\ell^{k,j})$ for all $T \in \mathcal{T}_\ell$.

(c) Upon termination of the inner solver loop, define the index $\underline{j}[\ell, k] := j \in \mathbb{N}$.

(ii) Upon termination of the outer solver loop, define the index $\underline{k}[\ell] := k \in \mathbb{N}$.

(iii) **Mark:** Determine a set $\mathcal{M}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^{\underline{k},\underline{j}}]$ satisfying (14) with $u_\ell^\star$ replaced by $u_\ell^{\underline{k},\underline{j}}$.

(iv) **Refine:** Generate $\mathcal{T}_{\ell+1} := \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ and define $u_{\ell+1}^{0,0} := u_{\ell+1}^{0,\underline{j}} := u_\ell^{\underline{k},\underline{j}}$.

Extending the index notation from Section 4, we define the triple index set

$$\mathcal{Q} := \{(\ell, k, j) \in \mathbb{N}_0^3 \,:\, u_\ell^{k,j} \text{ is used in Algorithm C}\}$$

together with the lexicographic ordering

$$(\ell', k', j') \le (\ell, k, j) \quad :\Longleftrightarrow \quad u_{\ell'}^{k',j'} \text{ is defined not later than } u_\ell^{k,j} \text{ in Algorithm C}$$

and the total step counter

$$|\ell, k, j| := \#\{(\ell', k', j') \in \mathcal{Q} \,:\, (\ell', k', j') \le (\ell, k, j)\} \in \mathbb{N}_0 \quad \text{for } (\ell, k, j) \in \mathcal{Q}. \tag{44}$$

Moreover, we define the stopping indices

$$\underline{\ell} := \sup\{\ell \in \mathbb{N}_0 \,:\, (\ell, 0, 0) \in \mathcal{Q}\} \in \mathbb{N}_0 \cup \{\infty\}, \tag{45a}$$

$$\underline{k}[\ell] := \sup\{k \in \mathbb{N}_0 \,:\, (\ell, k, 0) \in \mathcal{Q}\} \in \mathbb{N} \cup \{\infty\}, \quad \text{whenever } (\ell, 0, 0) \in \mathcal{Q}, \tag{45b}$$

$$\underline{j}[\ell, k] := \sup\{j \in \mathbb{N}_0 \,:\, (\ell, k, j) \in \mathcal{Q}\} \in \mathbb{N} \cup \{\infty\}, \quad \text{whenever } (\ell, k, 0) \in \mathcal{Q}. \tag{45c}$$

First, these definitions are consistent with those of Algorithm C(b) and Algorithm C(ii). Second, there holds indeed $\underline{j}[\ell, k] < \infty$ for all $(\ell, k, 0) \in \mathcal{Q}$; see [7, Lemma 3.2]. Third, $\underline{\ell} < \infty$ yields $\underline{k}[\underline{\ell}] = \infty$ and $\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$ with $u_{\underline{\ell}}^\star = u^\star$; see [7, Lemma 5.2].

The following theorem improves [7, Theorem 4.1] in the following sense. First, we prove R-linear convergence for all $\ell \ge \ell_0 = 0$, while $\ell_0 \in \mathbb{N}$ is unknown in practice and depends on $u^\star$ and the non-accessible sequence $(u_\ell^\star)_{\ell \in \mathbb{N}_0}$ in [7]. Second, [7] requires severe restrictions on λ_{alg} beyond (46) below. We note that (46) is indeed satisfied, if the algebraic system is solved exactly, i.e., $\lambda_{\mathrm{alg}} = 0$, so that Theorem 4 is a consistent generalization of Theorem 2.

Theorem 4 *(Full R-linear convergence of Algorithm C). Let $0 < \theta \le 1$, $C_{\mathrm{mark}} \ge 1$, $\lambda_{\mathrm{sym}}, \lambda_{\mathrm{alg}} > 0$, and $u_0^{0,0} \in \mathcal{X}_0$. With $q_\theta := [1 - (1 - q_{\mathrm{red}}^2)\theta]^{1/2}$, suppose that*

$$0 < \frac{q_{\mathrm{sym}}^\star + \frac{2\,q_{\mathrm{alg}}}{1-q_{\mathrm{alg}}}\,\lambda_{\mathrm{alg}}}{1 - \frac{2\,q_{\mathrm{alg}}}{1-q_{\mathrm{alg}}}\,\lambda_{\mathrm{alg}}} =: q_{\mathrm{sym}} < 1 \quad \text{and} \quad \lambda_{\mathrm{alg}} \lambda_{\mathrm{sym}} < \frac{(1 - q_{\mathrm{alg}})(1 - q_{\mathrm{sym}}^\star)(1 - q_\theta)}{8\, q_{\mathrm{alg}} C_{\mathrm{stab}}}. \tag{46}$$

Then, Algorithm C guarantees R-linear convergence of the quasi-error

$$\mathrm{H}_\ell^{k,j} := |||u_\ell^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j}), \tag{47}$$

i.e., there exist constants $0 < q_{\mathrm{lin}} < 1$ and $C_{\mathrm{lin}} > 0$ such that

$$\mathrm{H}_\ell^{k,j} \le C_{\mathrm{lin}} q_{\mathrm{lin}}^{|\ell,k,j| - |\ell',k',j'|} \mathrm{H}_{\ell'}^{k',j'} \text{ for all } (\ell', k', j'), (\ell, k, j) \in \mathcal{Q} \text{ with } |\ell', k', j'| \le |\ell, k, j|. \tag{48}$$

As proven for Corollary 1 in Section 4, an immediate consequence of full linear convergence (and the geometric series) is that convergence rates with respect to the number of degrees of freedom and with respect to the overall computational cost coincide.

Corollary 2 *(Rates = complexity). For $s > 0$, full R-linear convergence (48) yields*

$$M(s) := \sup_{(\ell,k,j) \in \mathcal{Q}} (\#\mathcal{T}_\ell)^s\, \mathrm{H}_\ell^{k,j} \le \sup_{(\ell,k,j) \in \mathcal{Q}} \Bigg(\sum_{\substack{(\ell',k',j') \in \mathcal{Q} \\ |\ell',k',j'| \le |\ell,k,j|}} \; \sum_{\substack{(\ell'',k'',j'') \in \mathcal{Q} \\ |\ell'',k'',j''| \le |\ell',k',j'|}} \#\mathcal{T}_{\ell''} \Bigg)^s \mathrm{H}_\ell^{k,j} \le C_{\mathrm{cost}}(s)\, M(s),$$

where the constant $C_{\mathrm{cost}}(s) > 0$ depends only on C_{lin}, q_{lin}, and s. Moreover, there exists $s_0 > 0$ such that $M(s) < \infty$ for all $0 < s \le s_0$. □

The proof of Theorem 4 requires the following lemma (which is essentially taken from [7]). It deduces the contraction of the inexact Zarantonello iteration with computed iterates $u_\ell^{k,\underline{j}} \approx u_\ell^{k,\star}$ from the exact Zarantonello iteration. For the inexact iteration, the linear SPD system (39) is solved with the contractive algebraic solver (41), i.e., $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$ and $u_\ell^{k,j} := \Psi_\ell(u_\ell^{k,\star}; u_\ell^{k,j-1})$ guarantee

$$|||u_\ell^\star - u_\ell^{k,\star}||| \le q_{\rm sym}^\star \, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \quad \text{for all } (\ell,k,j) \in \mathcal{Q} \text{ with } k \ge 1. \tag{49}$$

We emphasize that contraction is only guaranteed for $0 < k < \underline{k}[\ell]$ in (50) below, while the final iteration $k = \underline{k}[\ell]$ leads to a perturbed contraction (51) thus requiring additional treatment in the later analysis. The proof of Lemma 4 is given in Appendix A.

Lemma 4 *(Contraction of inexact Zarantonello iteration). Under the assumptions of Theorem 4, the inexact Zarantonello iteration used in Algorithm C satisfies*

$$|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \le q_{\rm sym} \, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \quad \textit{for all } (\ell,k,\underline{j}) \in \mathcal{Q} \textit{ with } 1 \le k < \underline{k}[\ell] \tag{50}$$

as well as

$$|||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| \le q_{\rm sym}^\star \, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}||| + \frac{2\,q_{\rm alg}}{1-q_{\rm alg}} \lambda_{\rm alg}\lambda_{\rm sym}\, \eta_\ell(u_\ell^{\underline{k},\underline{j}}) \quad \textit{for all } (\ell,\underline{k},\underline{j}) \in \mathcal{Q}. \tag{51}$$

Proof of Theorem 4. The building blocks of the proof are the following: First, we show that a suitably weighted quasi-error involving the final iterates of the inexact Zarantonello iteration is tail-summable in the mesh-level index ℓ. Second, we show that the quasi-errors are tail-summable in the Zarantonello index k and, third, in the algebraic-solver index j and that they are stable in the nested iteration. Finally, combining these ideas leads to tail summability with respect to the total step counter. The proof is split into six steps. The first four steps follow the proof of Theorem 2 using

$$\mathrm{H}_\ell^k := |||u_\ell^\star - u_\ell^{k,\underline{j}}||| + \eta_\ell(u_\ell^{k,\underline{j}}) \quad \text{for all } (\ell,k,\underline{j}) \in \mathcal{Q}. \tag{52}$$

By contraction of the algebraic solver (41) as well as the stopping criteria for the algebraic solver (43) and for the symmetrization (42), it holds that

$$|||u_\ell^{\underline{k},\star} - u_\ell^{\underline{k},\underline{j}}||| \overset{(41)}{\lesssim} |||u_\ell^{\underline{k},\underline{j}} - u_\ell^{\underline{k},\underline{j}-1}||| \overset{(43)}{\lesssim} \eta_\ell(u_\ell^{\underline{k},\underline{j}}) + |||u_\ell^{\underline{k},\underline{j}} - u_\ell^{\underline{k}-1,\underline{j}}||| \overset{(42)}{\lesssim} \eta_\ell(u_\ell^{\underline{k},\underline{j}}) \le \mathrm{H}_\ell^{\underline{k}}.$$

In particular, this proves equivalence

$$\mathrm{H}_\ell^{\underline{k}} \le \mathrm{H}_\ell^{\underline{k}} + |||u_\ell^{\underline{k},\star} - u_\ell^{\underline{k},\underline{j}}||| = \mathrm{H}_\ell^{\underline{k},\underline{j}} \lesssim \mathrm{H}_\ell^{\underline{k}} \quad \text{for all } \ (\ell,\underline{k},\underline{j}) \in \mathcal{Q}. \tag{53}$$

Step 1 (auxiliary estimates & estimator reduction). For $(\ell,\underline{k},\underline{j}) \in \mathcal{Q}$, nested iteration $u_\ell^{k,0} = u_\ell^{k-1,\underline{j}}$ and $\underline{j}[\ell,\underline{k}] \ge 1$ yield

$$|||u_\ell^{k,\star} - u_\ell^{\underline{k},\underline{j}}||| \overset{(41)}{\le} q_{\rm alg}^{\underline{j}[\ell,\underline{k}]} \, |||u_\ell^{k,\star} - u_\ell^{k,0}||| \le q_{\rm alg} \, |||u_\ell^{k,\star} - u_\ell^{k-1,\underline{j}}|||. \tag{54}$$

From this, we obtain that

$$\begin{aligned} |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| &\le |||u_\ell^\star - u_\ell^{\underline{k},\star}||| + |||u_\ell^{\underline{k},\star} - u_\ell^{\underline{k},\underline{j}}||| \overset{(54)}{\le} (1+q_{\rm alg}) \, |||u_\ell^\star - u_\ell^{\underline{k},\star}||| + q_{\rm alg} \, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}||| \\ &\overset{(49)}{\le} \big[(1+q_{\rm alg})q_{\rm sym}^\star + q_{\rm alg}\big] \, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}||| \le 3\, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}|||. \end{aligned} \tag{55}$$

For $(\ell+1,\underline{k},\underline{j}) \in \mathcal{Q}$, contraction of the inexact Zarantonello iteration (50), nested iteration $u_{\ell+1}^{0,\underline{j}} = u_\ell^{\underline{k},\underline{j}}$, and $\underline{k}[\ell+1] \ge 1$, show that

$$|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k}-1,\underline{j}}||| \overset{(50)}{\le} q_{\rm sym}^{\underline{k}[\ell+1]-1} \, |||u_{\ell+1}^\star - u_{\ell+1}^{0,\underline{j}}||| \le |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}|||. \tag{56}$$

The combination of the previous two displayed formulas shows

$$|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k},\underline{j}}||| \overset{(55)}{\le} 3\, |||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k}-1,\underline{j}}||| \overset{(56)}{\le} 3\, |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}|||. \tag{57}$$

Analogous arguments to (30) in the proof of Theorem 1 establish

$$\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{j}}) \overset{(30)}{\le} q_\theta \, \eta_\ell(u_\ell^{\underline{k},\underline{j}}) + C_{\rm stab} \, |||u_{\ell+1}^{\underline{k},\underline{j}} - u_\ell^{\underline{k},\underline{j}}||| \overset{(57)}{\le} q_\theta \, \eta_\ell(u_\ell^{\underline{k},\underline{j}}) + 4C_{\rm stab} \, |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}|||. \tag{58}$$

Step 2 (tail summability with respect to ℓ). With $\lambda := \lambda_{\rm alg}\lambda_{\rm sym}$, we define

$$\gamma := \frac{q_\theta(1-q_{\rm sym}^\star)}{4\,C_{\rm stab}}, \quad C(\gamma,\lambda) := 1 + \frac{2\,q_{\rm alg}}{1-q_{\rm alg}} \frac{\lambda}{\gamma}, \quad \text{and} \quad \alpha := \frac{\lambda}{\gamma} \overset{(46)}{<} \frac{(1-q_{\rm alg})(1-q_\theta)}{2\,q_{\rm alg} q_\theta}.$$

By definition, it follows that

$$C(\gamma,\lambda) = 1 + \frac{2\,q_{\rm alg}}{1-q_{\rm alg}}\,\alpha < 1 + \frac{1-q_\theta}{q_\theta} = \frac{1}{q_\theta}.$$

This ensures that

$$q_\theta\, C(\gamma,\lambda) < 1 \quad \text{as well as} \quad q_{\rm sym}^\star + 4\,C_{\rm stab} C(\gamma,\lambda)\,\gamma < q_{\rm sym}^\star + \frac{4\,C_{\rm stab}}{q_\theta}\,\gamma = 1. \tag{59}$$

With contraction of the inexact Zarantonello iteration (51), Step 1 proves

$$\begin{aligned}
|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k},\underline{j}}||| + \gamma\,\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{j}}) &\overset{(51)}{\le} q_{\rm sym}^\star\,|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k}-1,\underline{j}}||| + C(\gamma,\lambda)\,\gamma\,\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{j}}) \\
&\overset{(56)}{\le} q_{\rm sym}^\star\,|||u_{\ell+1}^\star - u_{\ell}^{\underline{k},\underline{j}}||| + C(\gamma,\lambda)\,\gamma\,\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{j}}) \\
&\overset{(58)}{\le} \big(q_{\rm sym}^\star + 4\,C_{\rm stab}\,C(\gamma,\lambda)\,\gamma\big)\,|||u_{\ell+1}^\star - u_{\ell}^{\underline{k},\underline{j}}||| + q_\theta\,C(\gamma,\lambda)\,\gamma\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \\
&\le q_{\rm ctr}\,\big[|||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}||| + \gamma\,\eta_\ell(u_\ell^{\underline{k},\underline{j}})\big] \quad \text{for all } (\ell+1,\underline{k},\underline{j}) \in \mathcal{Q},
\end{aligned} \tag{60}$$

where (59) ensures the bound

$$0 < q_{\rm ctr} := \max\big\{q_{\rm sym}^\star + 4\,C_{\rm stab}\,C(\gamma,\lambda)\,\gamma\,,\; q_\theta\,C(\gamma,\lambda)\big\} < 1.$$

Altogether, we obtain

$$\begin{aligned}
a_{\ell+1} := |||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k},\underline{j}}||| + \gamma\,\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{j}}) &\overset{(60)}{\le} q_{\rm ctr}\,\big[|||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| + \gamma\,\eta_\ell(u_\ell^{\underline{k},\underline{j}})\big] + q_{\rm ctr}\,|||u_{\ell+1}^\star - u_\ell^\star||| \\
&=:\; q_{\rm ctr}\,a_\ell + b_\ell \quad \text{for all } (\ell,\underline{k},\underline{j}) \in \mathcal{Q},
\end{aligned}$$

which corresponds to (31) in the case of a single contractive solver (with $u_\ell^{\underline{k},\underline{j}}$ replacing $u_\ell^{\underline{k}}$ in (31)). Together with (32)–(33) (with $u_\ell^{\underline{k},\underline{j}}$ replacing $u_\ell^{\underline{k}}$), the assumptions (17) of Lemma 1 are satisfied. Therefore, Lemma 1 proves tail summability

$$\sum_{\ell'=\ell+1}^{\underline{\ell}-1} \mathrm{H}_{\ell'}^{\underline{k}} \overset{(52)}{\simeq} \sum_{\ell'=\ell+1}^{\underline{\ell}-1} \big[|||u_{\ell'}^\star - u_{\ell'}^{\underline{k},\underline{j}}||| + \gamma\,\eta_{\ell'}(u_{\ell'}^{\underline{k},\underline{j}})\big] \lesssim |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| + \gamma\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \overset{(52)}{\simeq} \mathrm{H}_\ell^{\underline{k}} \quad \text{for all } (\ell,\underline{k},\underline{j}) \in \mathcal{Q}.$$

Step 3 (auxiliary estimates). First, we employ (55) to deduce

$$\begin{aligned}
\mathrm{H}_\ell^{\underline{k}} &\overset{(A1)}{\lesssim} |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| + |||u_\ell^{\underline{k},\underline{j}} - u_\ell^{\underline{k}-1,\underline{j}}||| + \eta_\ell(u_\ell^{\underline{k}-1,\underline{j}}) \overset{(52)}{\le} \mathrm{H}_\ell^{\underline{k}-1} + 2\,|||u_\ell^{\underline{k},\underline{j}} - u_\ell^{\underline{k}-1,\underline{j}}||| \\
&\overset{(55)}{\le} \mathrm{H}_\ell^{\underline{k}-1} + 8\,|||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}||| \le 9\,\mathrm{H}_\ell^{\underline{k}-1} \quad \text{for all } (\ell,\underline{k},\underline{j}) \in \mathcal{Q}.
\end{aligned} \tag{61}$$

Second, for $0 \le k < k' < \underline{k}[\ell]$, the failure of the stopping criterion for the inexact Zarantonello symmetrization (42) and contraction (50) prove that

$$\mathrm{H}_\ell^{k'} \overset{(42)}{\lesssim} |||u_\ell^\star - u_\ell^{k',\underline{j}}||| + |||u_\ell^{k',\underline{j}} - u_\ell^{k'-1,\underline{j}}||| \overset{(50)}{\lesssim} |||u_\ell^\star - u_\ell^{k'-1,\underline{j}}||| \overset{(50)}{\lesssim} q_{\rm sym}^{k'-k}\,|||u_\ell^\star - u_\ell^{k,\underline{j}}|||. \tag{62}$$

Moreover, for $k < k' = \underline{k}[\ell]$, we combine (61) with (62) to get

$$\mathrm{H}_\ell^{\underline{k}} \overset{(61)}{\lesssim} \mathrm{H}_\ell^{\underline{k}[\ell]-1} \overset{(62)}{\lesssim} q_{\rm sym}^{(\underline{k}[\ell]-1)-k}\,|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \simeq q_{\rm sym}^{\underline{k}[\ell]-k}\,|||u_\ell^\star - u_\ell^{k,\underline{j}}|||. \tag{63}$$

The combination of (62)–(63) proves that

$$\mathrm{H}_\ell^{k'} \lesssim q_{\rm sym}^{k'-k}\,|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \lesssim q_{\rm sym}^{k'-k}\,\mathrm{H}_\ell^k \quad \text{for all } (\ell,0,0) \in \mathcal{Q} \text{ with } 0 \le k \le k' \le \underline{k}[\ell], \tag{64}$$

where the hidden constant depends only on $C_{\rm stab}$, $\lambda_{\rm sym}$, and $q_{\rm sym}$. Third, we recall

$$|||u_\ell^\star - u_{\ell-1}^\star||| \overset{(19)}{\lesssim} \eta_{\ell-1}(u_{\ell-1}^\star) \overset{(A1)}{\lesssim} \eta_{\ell-1}(u_{\ell-1}^{\underline{k},\underline{j}}) + |||u_{\ell-1}^\star - u_{\ell-1}^{\underline{k},\underline{j}}||| = \mathrm{H}_{\ell-1}^{\underline{k}}.$$

Together with nested iteration $u_{\ell-1}^{\underline{k},\underline{j}} = u_\ell^{0,\underline{j}}$, this yields that

$$\mathrm{H}_\ell^0 = |||u_\ell^\star - u_{\ell-1}^{\underline{k},\underline{j}}||| + \eta_\ell(u_{\ell-1}^{\underline{k},\underline{j}}) \le |||u_\ell^\star - u_{\ell-1}^\star||| + \mathrm{H}_{\ell-1}^{\underline{k}} \lesssim \mathrm{H}_{\ell-1}^{\underline{k}} \quad \text{for all } (\ell,0,0) \in \mathcal{Q}. \tag{65}$$

Step 4 (tail summability with respect to ℓ and k). The auxiliary estimates from Step 3 and the geometric series prove that

$$\sum_{\substack{(\ell',k',\underline{j})\in\mathcal{Q}\\ |\ell',k',\underline{j}|>|\ell,k,\underline{j}|}} \mathrm{H}_{\ell'}^{k'} = \sum_{k'=k+1}^{\underline{k}[\ell]} \mathrm{H}_{\ell}^{k'} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \mathrm{H}_{\ell'}^{k'} \overset{(64)}{\lesssim} \mathrm{H}_{\ell}^{k} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \mathrm{H}_{\ell'}^{0} \overset{(65)}{\lesssim} \mathrm{H}_{\ell}^{k} + \sum_{\ell'=\ell}^{\underline{\ell}-1} \mathrm{H}_{\ell'}^{\underline{k}} \lesssim \mathrm{H}_{\ell}^{k} + \mathrm{H}_{\ell}^{\underline{k}} \overset{(64)}{\lesssim} \mathrm{H}_{\ell}^{k} \quad \text{for all } (\ell,k,\underline{j})\in\mathcal{Q}. \tag{66}$$

Step 5 (auxiliary estimates). Recall $\mathrm{H}_{\ell}^{\underline{k}} \le \mathrm{H}_{\ell}^{\underline{k},\underline{j}}$ from (53). For $j=0$ and $k=0$, the definition $u_{\ell}^{0,0} := u_{\ell}^{0,\underline{j}} := u_{\ell}^{0,\star}$ leads to $\mathrm{H}_{\ell}^{0,0} = \mathrm{H}_{\ell}^{0}$. For $k \ge 1$, nested iteration $u_{\ell}^{k,0} = u_{\ell}^{k-1,\underline{j}}$ and contraction of the Zarantonello iteration (49) imply

$$|||u_{\ell}^{k,\star} - u_{\ell}^{k,0}||| \le |||u_{\ell}^{\star} - u_{\ell}^{k,\star}||| + |||u_{\ell}^{\star} - u_{\ell}^{k-1,\underline{j}}||| \overset{(49)}{\le} (q_{\mathrm{sym}}^{\star}+1)\, |||u_{\ell}^{\star} - u_{\ell}^{k-1,\underline{j}}||| \le 2\,\mathrm{H}_{\ell}^{k-1}.$$

Therefore, we derive that

$$\mathrm{H}_{\ell}^{k,0} \le 3\,\mathrm{H}_{\ell}^{(k-1)_+} \quad \text{for all } (\ell,k,0)\in\mathcal{Q}, \quad \text{where } (k-1)_+ := \max\{0,k-1\}. \tag{67}$$

For any $0 \le j < j' < \underline{j}[\ell,k]$, the contraction of the Zarantonello iteration (49), the contraction of the algebraic solver (41), and the failure of the stopping criterion for the algebraic solver (43) prove

$$\begin{aligned}
\mathrm{H}_{\ell}^{k,j'} &\le |||u_{\ell}^{\star} - u_{\ell}^{k,\star}||| + 2\,|||u_{\ell}^{k,\star} - u_{\ell}^{k,j'}||| + \eta_{\ell}(u_{\ell}^{k,j'}) \overset{(49)}{\lesssim} |||u_{\ell}^{k,j'} - u_{\ell}^{k-1,\underline{j}}||| + |||u_{\ell}^{k,\star} - u_{\ell}^{k,j'}||| + \eta_{\ell}(u_{\ell}^{k,j'})\\
&\overset{(41)}{\lesssim} |||u_{\ell}^{k,j'} - u_{\ell}^{k-1,\underline{j}}||| + |||u_{\ell}^{k,j'} - u_{\ell}^{k,j'-1}||| + \eta_{\ell}(u_{\ell}^{k,j'}) \overset{(43)}{\lesssim} |||u_{\ell}^{k,j'} - u_{\ell}^{k,j'-1}||| \overset{(41)}{\lesssim} |||u_{\ell}^{k,\star} - u_{\ell}^{k,j'-1}||| \overset{(41)}{\lesssim} q_{\mathrm{alg}}^{j'-j}\, |||u_{\ell}^{k,\star} - u_{\ell}^{k,j}|||\\
&\lesssim q_{\mathrm{alg}}^{j'-j}\, \mathrm{H}_{\ell}^{k,j}.
\end{aligned}$$

For $j' = \underline{j}[\ell,k]$, it follows that

$$\mathrm{H}_{\ell}^{k,\underline{j}} \overset{(A1)}{\lesssim} \mathrm{H}_{\ell}^{k,\underline{j}-1} + |||u_{\ell}^{k,\underline{j}} - u_{\ell}^{k,\underline{j}-1}||| \overset{(41)}{\lesssim} \mathrm{H}_{\ell}^{k,\underline{j}-1} + |||u_{\ell}^{k,\star} - u_{\ell}^{k,\underline{j}-1}||| \overset{(47)}{\le} 2\,\mathrm{H}_{\ell}^{k,\underline{j}-1} \lesssim q_{\mathrm{alg}}^{\underline{j}[\ell,k]-j}\, \mathrm{H}_{\ell}^{k,j}.$$

The combination of the previous two displayed formulas results in

$$\mathrm{H}_{\ell}^{k,j'} \lesssim q_{\mathrm{alg}}^{j'-j}\, \mathrm{H}_{\ell}^{k,j} \quad \text{for all } (\ell,k,0)\in\mathcal{Q} \quad \text{with} \quad 0 \le j \le j' \le \underline{j}[\ell,k], \tag{68}$$

where the hidden constant depends only on $q_{\mathrm{sym}}^{\star}$, λ_{sym}, q_{alg}, λ_{alg}, and C_{stab}.

Step 6 (tail summability with respect to ℓ, k, and j). Finally, we observe that

$$\begin{aligned}
\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|>|\ell,k,j|}} \mathrm{H}_{\ell'}^{k',j'} &= \sum_{j'=j+1}^{\underline{j}[\ell,k]} \mathrm{H}_{\ell}^{k,j'} + \sum_{k'=k+1}^{\underline{k}[\ell]} \sum_{j'=0}^{\underline{j}[\ell,k']} \mathrm{H}_{\ell}^{k',j'} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \sum_{j'=0}^{\underline{j}[\ell',k']} \mathrm{H}_{\ell'}^{k',j'} \overset{(68)}{\lesssim} \mathrm{H}_{\ell}^{k,j} + \sum_{k'=k+1}^{\underline{k}[\ell]} \mathrm{H}_{\ell}^{k',0} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell]} \mathrm{H}_{\ell'}^{k',0}\\
&\overset{(67)}{\lesssim} \mathrm{H}_{\ell}^{k,j} + \sum_{\substack{(\ell',k',\underline{j})\in\mathcal{Q}\\ |\ell',k',\underline{j}|>|\ell,k,\underline{j}|}} \mathrm{H}_{\ell'}^{k'} \overset{(66)}{\lesssim} \mathrm{H}_{\ell}^{k,j} + \mathrm{H}_{\ell}^{k} \overset{(53)}{\lesssim} \mathrm{H}_{\ell}^{k,j} + \mathrm{H}_{\ell}^{k,\underline{j}} \overset{(68)}{\lesssim} \mathrm{H}_{\ell}^{k,j} \quad \text{for all } (\ell,k,j)\in\mathcal{Q}.
\end{aligned}$$

Since $\mathcal{Q}$ is countable and linearly ordered, Lemma 2 concludes the proof of (48). □

The final theorem, following from [8, Theorem 4.3], states that for sufficiently small adaptivity parameters θ, λ_{sym}, and λ_{alg}, Algorithm C achieves optimal complexity.

Theorem 5 *(Optimal complexity of Algorithm C, [8, Theorem 4.3]). Suppose that the estimator satisfies the axioms of adaptivity* (A1)–(A3$^+$) *and suppose that quasi-orthogonality* (A4) *holds. Assume full R-linear convergence from Theorem 4. Define the constants $\theta^\star$, $\lambda_{\mathrm{sym}}^\star$ by*

$$\theta^\star := \left(1 + C_{\mathrm{stab}}^2 C_{\mathrm{drel}}^2\right)^{-1}, \quad \textit{and} \quad \lambda_{\mathrm{sym}}^\star := \min\{1, C_{\mathrm{stab}}^{-1} C_{\mathrm{alg}}^{-1}\} \quad \textit{with} \quad C_{\mathrm{alg}} := \frac{1}{1-q_{\mathrm{sym}}^\star}\Big(\frac{2\,q_{\mathrm{alg}}}{1-q_{\mathrm{alg}}}\,\lambda_{\mathrm{alg}}^\star + q_{\mathrm{sym}}^\star\Big).$$

Suppose that the constants θ, λ_{sym}, and λ_{alg} are sufficiently small in the sense that, additionally to (46)*, there holds*

$$0 < \lambda_{\mathrm{sym}} < \lambda_{\mathrm{sym}}^\star \quad \textit{and} \quad 0 < \frac{\left(\theta^{1/2} + \lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star\right)^2}{\left(1-\lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star\right)^2} < \theta^\star < 1.$$

Then, Algorithm C guarantees for all $s > 0$

$$c_{\mathrm{opt}}\|u^\star\|_{\mathbb{A}_s} \le \sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\le|\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \mathrm{H}_\ell^{k,j} \le C_{\mathrm{opt}} \max\{\|u^\star\|_{\mathbb{A}_s}, \mathrm{H}_0^{0,0}\}.$$

The constant $c_{\mathrm{opt}} > 0$ depends only on C_{stab}, the use of NVB refinement, and s, while $C_{\mathrm{opt}} > 0$ depends only on C_{stab}, q_{red}, C_{drel}, C_{lin}, q_{lin}, $\#\mathcal{T}_0$, λ_{sym}, $q^\star_{\mathrm{sym}}$, λ_{alg}, q_{alg}, θ, s, and the use of NVB refinement. □

6. Application to strongly monotone nonlinear PDEs

In the previous sections, the particular focus was on general second-order linear elliptic PDEs (9). However, the results also apply to nonlinear PDEs with strongly monotone and Lipschitz-continuous nonlinearity as considered, e.g., in [28,29,16,27,34,35,26,30,33,36,31,42] to mention only some recent works.

Given a nonlinearity $\boldsymbol{A}:\mathbb{R}^d\to\mathbb{R}^d$, we consider the nonlinear elliptic PDE

$$-\operatorname{div}\big(\boldsymbol{A}(\nabla u^\star)\big) = f - \operatorname{div}\boldsymbol{f} \text{ in } \Omega \quad \text{subject to} \quad u^\star = 0 \text{ on } \partial\Omega. \tag{69}$$

We define the nonlinear operator $\mathcal{A}: H_0^1(\Omega)\to H^{-1}(\Omega) := H_0^1(\Omega)^*$ via $\mathcal{A}u := \langle \boldsymbol{A}(\nabla u), \nabla(\cdot)\rangle_{L^2(\Omega)}$, where we suppose that the $L^2(\Omega)$ scalar product on the right-hand side is well-defined. Then, the weak formulation of (69) reads

$$\langle \mathcal{A}u^\star, v\rangle = F(v) := \langle f, v\rangle_{L^2(\Omega)} + \langle \boldsymbol{f}, \nabla v\rangle_{L^2(\Omega)} \quad \text{for all } v\in H_0^1(\Omega), \tag{70}$$

where $\langle\cdot,\cdot\rangle$ on the left-hand side denotes the duality brackets on $H^{-1}(\Omega)\times H_0^1(\Omega)$.

Let $a(\cdot,\cdot)$ be an equivalent scalar product on $H_0^1(\Omega)$ with induced norm $|||\cdot|||$. Suppose that $\mathcal{A}$ is strongly monotone and Lipschitz continuous, i.e., there exist $0<\alpha\le L$ such that, for all $u,v,w\in H_0^1(\Omega)$,

$$\alpha\,|||u-v|||^2 \le \langle \mathcal{A}u - \mathcal{A}v, u-v\rangle \quad \text{and} \quad \langle \mathcal{A}u-\mathcal{A}v, w\rangle \le L\,|||u-v|||\,|||w|||. \tag{71}$$

Under these assumptions, the Zarantonello theorem from [50] (or main theorem on strongly monotone operators [51, Section 25.4]) yields existence and uniqueness of the solution $u^\star\in H_0^1(\Omega)$ to (70). For $\mathcal{T}_H\in\mathbb{T}$ and $\mathcal{X}_H\subseteq H_0^1(\Omega)$ from (10) with polynomial degree $p=1$, it also applies to the discrete setting and yields existence and uniqueness of the discrete solution $u_H^\star\in\mathcal{X}_H$ to

$$\langle \mathcal{A}u_H^\star, v_H\rangle = F(v_H) \quad \text{for all } v_H\in\mathcal{X}_H, \tag{72}$$

which is quasi-optimal in the sense of the Céa lemma (12).

As already discussed in Section 5, the proof of the Zarantonello theorem relies on the Banach fixed-point theorem: For $0<\delta<2\alpha/L^2$, define $\Phi_H(\delta;\cdot):\mathcal{X}_H\to\mathcal{X}_H$ via

$$a(\Phi_H(\delta;u_H), v_H) = a(u_H, v_H) + \delta\big[F(v_H) - \langle \mathcal{A}(u_H), v_H\rangle\big] \quad \text{for all } u_H, v_H\in\mathcal{X}_H. \tag{73}$$

Since $a(\cdot,\cdot)$ is a scalar product, $\Phi_H(\delta;u_H)\in\mathcal{X}_H$ is well-defined. Moreover, for $0<\delta<2\alpha/L^2$ and $0<q^\star_{\mathrm{sym}} := [1-\delta(2\alpha-\delta L^2)]^{1/2} < 1$, this mapping is a contraction, i.e.,

$$|||u_H^\star - \Phi_H(\delta;u_H)||| \le q^\star_{\mathrm{sym}}\,|||u_H^\star - u_H||| \quad \text{for all } u_H\in\mathcal{X}_H;$$

see also [34,35]. Analogously to Section 5, the variational formulation (73) leads to a linear SPD system for which we employ a uniformly contractive solver (41). Overall, we note that for the nonlinear PDE (69), the natural AFEM loop consists of

- discretization via a conforming triangulation $\mathcal{T}_\ell$ (leading to the non-computable solution $u_\ell^\star$ to the discrete nonlinear system (72)),
- iterative linearization (giving rise to the solution $u_\ell^{k,\star} = \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$ of the large-scale discrete SPD system (73) obtained by linearizing (72) in $u_\ell^{k-1,\underline{j}}$),
- and an algebraic solver (leading to computable approximations $u_\ell^{k,j}\approx u_\ell^{k,\star}$).

Thus, the natural AFEM algorithm takes the form of Algorithm C in Section 5.

So far, the only work analyzing convergence of such a full adaptive loop for the numerical solution of (69) is [30], which uses the Zarantonello approach (73) for linearization and a preconditioned CG method with optimal additive Schwarz preconditioner for solving the arising SPD systems. Importantly and contrary to the present work, the adaptivity parameters θ, λ_{sym}, and λ_{alg} in [30] must be sufficiently small to guarantee full linear convergence and optimal complexity, while even plain convergence for arbitrary θ, λ_{sym}, and λ_{alg} is left open. Instead, the present work proves full R-linear convergence at least for arbitrary θ and λ_{sym} and the milder constraint (46) on λ_{alg}.

To apply the analysis from Section 5, it only remains to check the validity of Proposition 1 and Proposition 2. The following result provides the analogue of Proposition 1 for scalar nonlinearities. Note that, first, the same assumptions are made in [30] and, second, only the proof of stability (A1) (going back to [29]) is restricted to scalar nonlinearities and lowest-order discretizations, i.e., $p = 1$ in (10), while reduction (A2), reliability (A3), and discrete reliability (A3$^+$) follow as for linear PDEs and thus hold for all $p \geq 1$.

Proposition 3 *(See, e.g., [29, Section 3.2] or [9, Section 10.1]). Suppose that $\boldsymbol{A}(\nabla u) = a(|\nabla u|^2)\nabla u$, where $a \in C^1(\mathbb{R}_{\geq 0})$ satisfies*

$$\alpha(t-s) \leq a(t^2)t - a(s^2)s \leq \frac{L}{3}(t-s) \quad \textit{for all } t \geq s \geq 0.$$

Then, there holds (71) for $|||v||| := \|\nabla v\|_{L^2(\Omega)}$ and the standard residual error estimator (13) for lowest-order elements $p = 1$ (with $\boldsymbol{A}\nabla v_H$ understood as $\boldsymbol{A}(\nabla v_H)$ and $\boldsymbol{b} = 0 = c$) satisfies stability (A1), reduction (A2), reliability (A3), discrete reliability (A3$^+$), and quasi-monotonicity (QM) from Proposition 1. □

Under the same assumptions as in Proposition 3, quasi-orthogonality (A4) is satisfied. For the convenience of the reader, we include a sketch of the proof, which also shows that the quasi-orthogonality holds for any $p \geq 1$ with $C_{\mathrm{orth}} = L/\alpha$ and $\delta = 1$.

Proposition 4. *Under the assumptions of Proposition 3 and for any sequence of nested finite-dimensional subspaces $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset H^1_0(\Omega)$, the corresponding Galerkin solutions $u^\star_\ell \in \mathcal{X}_\ell$ to (72) satisfy quasi-orthogonality (A4) with $\delta = 1$ and $C_{\mathrm{orth}} = L/\alpha$, i.e.,*

$$\sum_{\ell'=\ell}^{\infty} |||u^\star_{\ell'+1} - u^\star_{\ell'}|||^2 \leq \frac{L}{\alpha} |||u^\star - u_\ell|||^2 \quad \textit{for all } \ell \in \mathbb{N}_0.$$

Sketch of proof. One can prove that the energy

$$E(v) := \frac{1}{2} \int_\Omega \int_0^{|\nabla v(x)|^2} a(t)\,\mathrm{d}t\,\mathrm{d}x - F(v) \quad \text{for all } v \in H^1_0(\Omega)$$

is Gâteaux-differentiable with $\mathrm{d}E(v) = \mathcal{A}v - F$. Then, elementary calculus (see, e.g., [27, Lemma 5.1] or [35, Lemma 2]) yields the equivalence

$$\frac{\alpha}{2} |||u^\star_H - v_H|||^2 \leq E(v_H) - E(u^\star_H) \leq \frac{L}{2} |||u^\star_H - v_H|||^2 \text{ for all } \mathcal{T}_H \in \mathbb{T} \text{ and all } v_H \in \mathcal{X}_H. \tag{74}$$

In particular, we see that $u^\star_H$ is the unique minimizer to

$$E(u^\star_H) = \min_{v_H \in \mathcal{X}_H} E(v_H), \tag{75}$$

and (74)–(75) also hold for $u^\star$ and $H^1_0(\Omega)$ replacing $u^\star_H$ and $\mathcal{X}_H$, respectively.

From this and the telescopic sum, we infer that

$$\begin{aligned}\frac{\alpha}{2} \sum_{\ell'=\ell}^{\ell+N} |||u^\star_{\ell'+1} - u^\star_{\ell'}|||^2 &\overset{(74)}{\leq} \sum_{\ell'=\ell}^{\ell+N} \big[E(u^\star_{\ell'}) - E(u^\star_{\ell'+1})\big] = E(u^\star_\ell) - E(u^\star_{\ell+N+1}) \overset{(75)}{\leq} E(u^\star_\ell) - E(u^\star) \\ &\overset{(74)}{\leq} \frac{L}{2} |||u^\star - u^\star_\ell|||^2 \quad \text{for all } \ell, N \in \mathbb{N}_0.\end{aligned}$$

Since the right-hand side is independent of N, we conclude the proof for $N \to \infty$. □

Thus, full R-linear convergence from Theorem 4 and optimal complexity from Theorem 5 apply also to the nonlinear PDE (69) under the assumptions on the nonlinearity from Proposition 3. Unlike [30], we can guarantee full R-linear convergence (48) for arbitrary θ, arbitrary λ_{sym}, and a weaker constraint (46) on λ_{alg}. As in [30, Theorem 5], optimal complexity follows if the adaptivity parameters are sufficiently small.

Remark 7. The cost-optimal numerical solution of nonlinear PDEs is widely open beyond the case of strongly monotone and Lipschitz continuous nonlinearities considered here. We stress that this problem class even excludes the p-Laplacian, for which linear convergence in [19] and optimal convergence rates in [4] are known under the constraint of the exact solution of the arising nonlinear discrete systems. Convergent linearization strategies for the p-Laplacian are the topic of recent research, e.g., [17,3,32]. However, optimal complexity appears to be still out of reach. Nevertheless, the present work could outline potential strategies also in this respect.

7. Numerical experiments

The following numerical experiments employ the MATLAB software package MooAFEM from [38].[1] The first numerical example illustrates the performance of Algorithm B for a symmetric linear elliptic PDE with a strong jump in the diffusion coefficient and compares Algorithm B to the Algorithm A with exact solution. The second numerical example demonstrates the efficiency of Algorithm C for a nonsymmetric general second-order linear elliptic PDE with a moderate convection. The numerical behavior of the adaptive algorithm proposed in Section 6 for a strongly monotone and Lipschitz continuous nonlinear PDE concludes the numerical experiments. All numerical experiments showcase the performance of the adaptive algorithms for different selections of the involved adaptivity parameters.

7.1. AFEM for a symmetric linear elliptic PDE with strong jump in the diffusion coefficient

Following [40], we consider the square domain $\Omega := (-1,1)^2$ and the jumping diffusion coefficient $\boldsymbol{A}(x) := a(x)\, I_{2\times 2} \in L^\infty(\Omega)$ for $a(x) := 161.4476387975881$ if $x_1 x_2 > 0$ and $a(x) := 1$ if $x_1 x_2 < 0$. In order to measure the performance of Algorithm B, we consider the interface problem

$$-\operatorname{div}\big(\boldsymbol{A}\,\nabla u^\star\big) = 0 \text{ in } \Omega. \tag{76}$$

The exact weak solution in polar coordinates reads $u^\star(r,\phi) := r^\alpha \mu(\phi)$ where the constants are set to be $\alpha = 0.1$, $\beta = -14.92256510455152$, $\delta = \pi/4$, and $\mu(\phi)$ is defined as

$$\mu(\phi) := \begin{cases} \cos((\pi/2-\beta)\alpha)\,\cos((\phi-\pi/2+\delta)\alpha) & \text{if } 0 \le \phi < \pi/2, \\ \cos(\delta\alpha)\,\cos((\phi-\pi+\beta)\alpha) & \text{if } \pi/2 \le \phi < \pi, \\ \cos(\beta\alpha)\,\cos((\phi-\pi-\delta)\alpha) & \text{if } \pi \le \phi < 3\pi/2, \\ \cos((\pi/2-\delta)\alpha)\,\cos((\phi-3\pi/2-\beta)\alpha) & \text{if } 3\pi/2 \le \phi < 2\pi. \end{cases}$$

The exact solution determines the inhomogeneous Dirichlet boundary conditions $u_{\mathrm{D}}(x) := u^\star(x)$ for $x \in \partial\Omega$. The parameter α gives the regularity of the solution $u \in H^{1+\alpha-\varepsilon}(\Omega)$ for all $\varepsilon > 0$, having a strong point singularity at the origin where the interfaces intersect. Fig. 1 illustrates the initial triangulation $\mathcal{T}_0$, the adaptively generated mesh $\mathcal{T}_{15}$ with 518 triangles, and the exact solution $u^\star$, and the computed solutions u_{15}^{k}. We see that the adaptive algorithm captures the singularity induced by the strong jump in the diffusion coefficient and refines around the origin. Let Π_E^{p-1} be the $L^2(E)$-orthogonal projection onto the space of polynomials of degree at most $p-1$ on the boundary face $E \subset \partial\Omega$ and $\partial u_{\mathrm{D}}/\partial s$ denote the arc-length derivative of u_{D}. We approximate the boundary data u_{D} by nodal interpolation from [44,24], leading to an additional boundary-data oscillation term (for sufficiently smooth u_{D}, see, e.g., [2]), in the error estimator

$$\eta_H(T,v_H)^2 := |T|\,\|\operatorname{div}(\boldsymbol{A}\nabla v_H)\|_{L^2(T)}^2 + |T|^{1/2}\,\|[\boldsymbol{A}\nabla v_H\cdot n]\|_{L^2(\partial T\setminus\partial\Omega)}^2 + \sum_{E\subset\partial T\cap\partial\Omega} |T|^{1/2}\|(1-\Pi_E^{p-1})\,\partial u_{\mathrm{D}}/\partial s\|_{L^2(E)}.$$

Fig. 2 shows that Algorithm B leads to optimal convergence rates $-p/2$ with respect to the number of degrees and the overall computation time for arbitrary polynomial degrees $p \in \{1,2,3,4\}$ and a moderate marking parameter $\theta = 0.5$ and fixed algebraic solver parameter $\lambda = 0.01$. Furthermore, the reduced elliptic regularity leads to convergence rates $-1/10$ for uniform mesh refinement and any polynomial degree p. In Fig. 3, we observe that even moderate values of the algebraic solver parameter λ lead to optimal convergence rates -1 for polynomial degree $p = 2$ with respect to the number of degrees and the overall computation time. Fig. 4 verifies that Algorithm B in combination with the optimal hp-robust geometric multigrid solver from [37] outperforms the MATLAB built-in `mldivide` in terms of the cumulative computation time. Table 1 summarizes the optimal selection of the adaptivity parameters for the interface problem in (76) with polynomial degree $p = 2$. The best performance is observed for the marking parameter $\theta \in \{0.5, 0.7\}$ and the algebraic solver parameter $\lambda = 0.9$.

7.2. AFEM for a general second-order linear elliptic PDE

On the L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1) \times [-1,0)$, we consider

$$-\Delta u^\star + \boldsymbol{b}\cdot\nabla u^\star + u^\star = 1 \text{ in } \Omega \quad\text{and}\quad u^\star = 0 \text{ on } \partial\Omega \quad\text{with}\quad \boldsymbol{b}(x) = x; \tag{77}$$

see Fig. 5 for the geometry and some adaptively generated meshes.

Optimality of Algorithm C with respect to large solver-stopping parameters λ_{sym} and λ_{alg}. We choose $\delta = 0.5$, $\theta = 0.3$, and the polynomial degree $p = 2$. Fig. 6 presents the convergence rates for fixed $\lambda_{\mathrm{alg}} = 0.7$ and several symmetrization parameters $\lambda_{\mathrm{sym}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. We observe that Algorithm C obtains the optimal convergence rate -1 with respect to the number of degrees of freedom and the cumulative computation time for any selection of λ_{sym}. Moreover, the same holds true for fixed $\lambda_{\mathrm{sym}} = 0.7$

[1] The experiments accompanying this paper will be provided under https://www.tuwien.at/mg/asc/praetorius/software/mooafem.

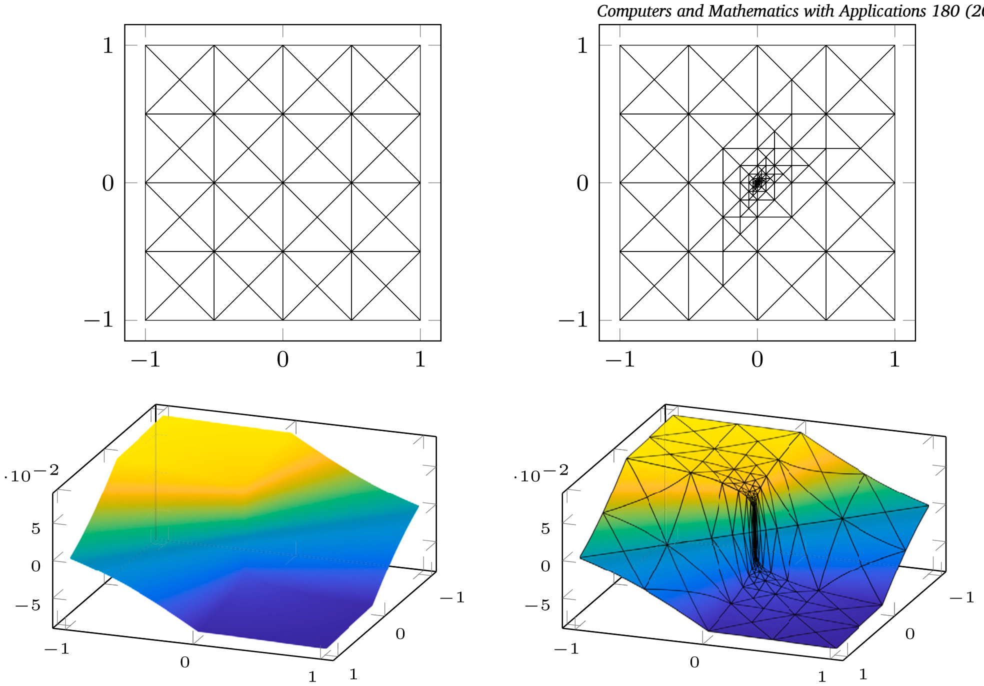

Fig. 1. Illustration of the initial triangulation $\mathcal{T}_0$, the adaptively generated mesh $\mathcal{T}_{15}$ with 518 triangles, the exact solution $u^\star$ and the computed solution $u_{15}^{\underline{k}}$ for the Kellogg benchmark problem (76) with polynomial degree $p = 2$, marking parameter $\theta = 0.5$, and algebraic solver parameter $\lambda = 0.01$.

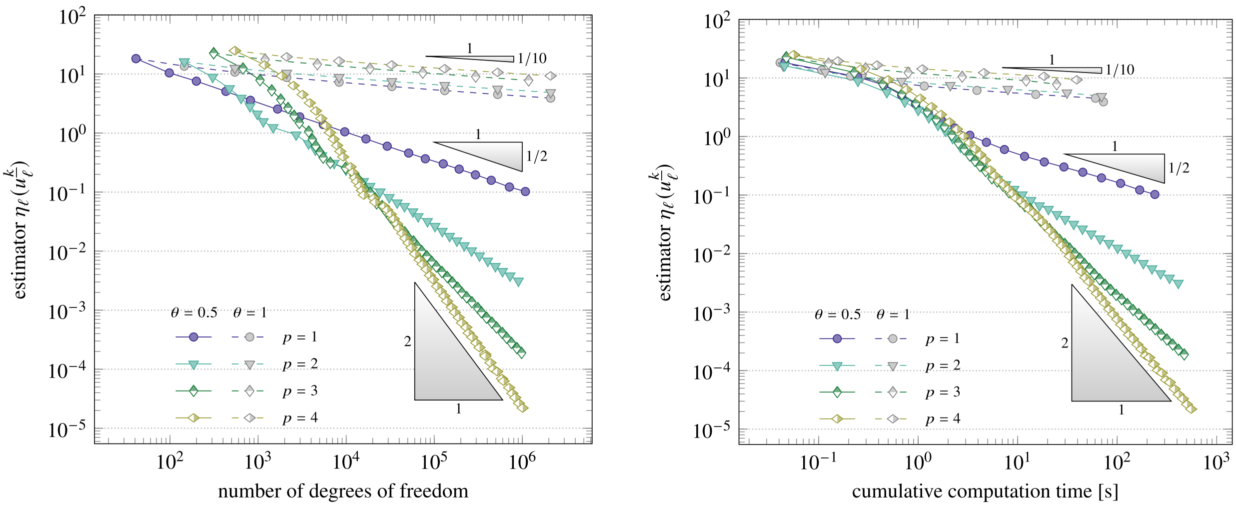

Fig. 2. Convergence history plot of the error estimator $\eta_\ell(u_\ell^{\underline{k}})$ with respect to the number of degrees of freedom (left) and the cumulative computation time (right) for the Kellogg benchmark problem (76) for different polynomial degrees $p \in \{1, 2, 3, 4\}$ with fixed marking parameter $\theta = 0.5$ and algebraic solver parameter $\lambda = 0.01$.

and any choice of the algebraic solver parameter $\lambda_{\text{alg}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ as depicted in Fig. 7. Table 2 illustrates the weighted cumulative computation time of Algorithm C and shows that a smaller marking parameter $\theta = 0.3$ in combination with larger solver-stopping parameters λ_{sym} and λ_{alg} is favorable. Furthermore, Fig. 9 shows that Algorithm C guarantees optimal convergence rates $-p/2$ for several polynomial degrees p with fixed $\delta = 0.5$, marking parameter $\theta = 0.3$, and moderate $\lambda_{\text{sym}} = \lambda_{\text{alg}} = 0.7$.

Optimality of Algorithm C with respect to large marking parameter θ. We choose the polynomial degree $p = 2$, $\delta = 0.5$, and solver-stopping parameters $\lambda_{\text{alg}} = \lambda_{\text{sym}} = 0.7$. Fig. 8 shows that also for moderate marking parameters θ, Algorithm C guarantees optimal convergence rates with respect to the number of degrees of freedom and the cumulative computation time. Moreover, we observe that a very small as well as a large choice of θ lead to a worse performance.

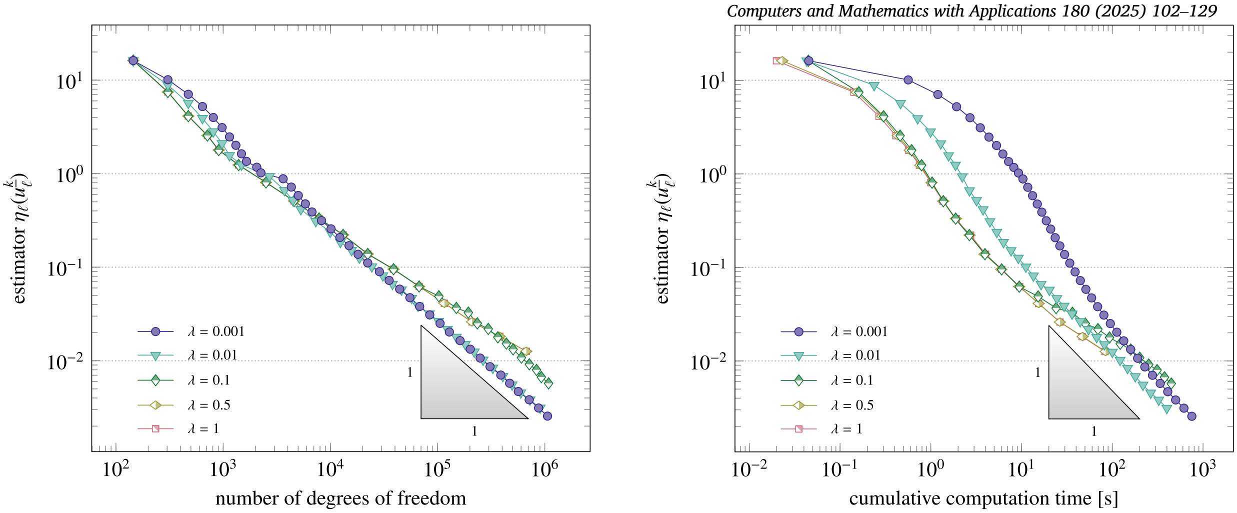

Fig. 3. Convergence history plot of the error estimator $\eta_\ell(u_\ell^{\underline{k}})$ for different algebraic solver parameters $\lambda \in \{0.001, 0.01, 0.1, 0.5, 1\}$ and fixed polynomial degree $p = 2$ and marking parameter $\theta = 0.5$ with respect to the number of degrees of freedom (left) and the cumulative computation time (right) for the Kellogg benchmark problem (76).

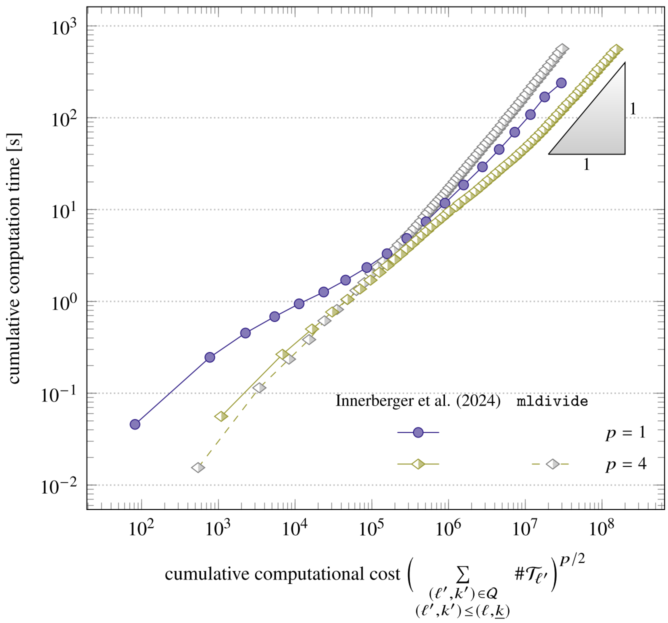

Fig. 4. Comparison of the cumulative computation time for the algebraic solver (Algorithm B) from [37] with the MATLAB built-in `mldivide` (Algorithm A) over the cumulative number of degrees of freedom to solve the Kellogg benchmark problem (76) with polynomial degree $p \in \{1, 4\}$, marking parameter $\theta = 0.5$, and algebraic solver parameter $\lambda = 0.01$.

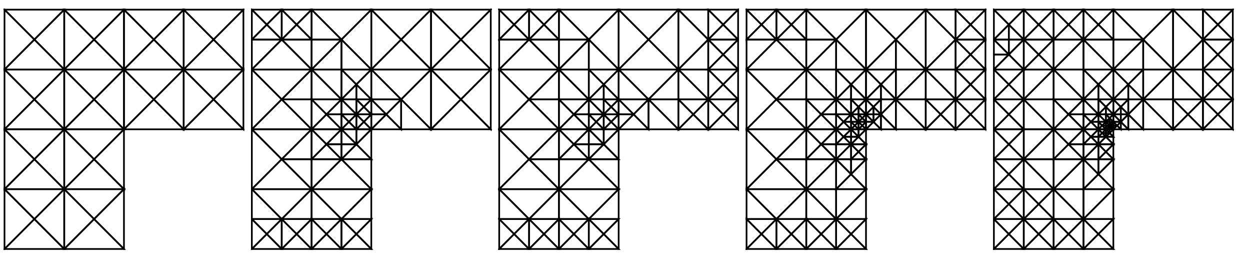

Fig. 5. Illustration of the initial triangulation $\mathcal{T}_0$ and the sequence of adaptively generated meshes $\mathcal{T}_0, \ldots, \mathcal{T}_4$ for the experiment (77).

Table 1
Optimal selection of the parameters for the Kellogg benchmark problem (76) with polynomial degree $p = 2$. For the comparison, we consider the weighted cumulative time $\big[\eta_\ell(u_\ell^{\underline{k}}) \sum_{|\ell',k'|\le|\ell,\underline{k}|} \mathrm{time}(\ell')\big]$ with stopping criterion $\eta_\ell(u_\ell^{\underline{k}}) < 10^{-2}\,\eta_0(u_0^0)$ for various choices of marking parameter θ and algebraic solver parameter λ. The best choice per column is marked in yellow, per row in blue, and for both in green. For all fixed marking parameter θ, the best performance is observed for $\lambda = 0.9$, while overall best results are achieved for $\theta \in \{0.5, 0.7\}$.

λ \ θ	0.1	0.3	0.5	0.7	0.9
0.1	1.07	1.07	1.39	1.00	1.67
0.3	1.05	1.05	1.16	0.98	1.55
0.5	1.04	1.07	1.14	0.96	1.49
0.7	1.03	1.06	1.00	1.03	1.53
0.9	1.03	1.06	0.95	0.96	1.36

Table 2
Optimal selection of parameters with respect to the computational costs for the nonsymmetric experiment (77) with $p = 2$ and $\delta = 0.5$. For the comparison, we consider the weighted cumulative time $\big[\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \sum_{|\ell',k',j'|\le|\ell,\underline{k},\underline{j}|} \mathrm{time}(\ell')\big]$ (values in 10^{-4}) with stopping criterion $\eta_\ell(u_\ell^{\underline{k},\underline{j}}) < 5\cdot 10^{-5}$ for various choices of λ_{sym}, λ_{alg}, and θ. In each θ-block, we mark in yellow the best choice per column, in blue the best choice per row, and in green when both choices coincide. The best choices for λ_{alg} and λ_{sym} are observed for $\theta = 0.3$ and $\theta = 0.5$.

$\cdot 10^{-4}$	$\theta = 0.1$					$\theta = 0.3$					$\theta = 0.5$				
λ_{alg} \ λ_{sym}	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
0.1	64.5	64.6	54.7	55.6	54.8	27.1	20.7	20.3	20.3	20.3	25.5	20.5	20.5	20.9	20.6
0.3	63.8	56.2	55.0	54.7	55.1	24.0	20.2	19.3	19.2	19.1	21.8	20.9	21.2	21.5	21.8
0.5	56.4	56.5	55.7	55.1	55.2	21.6	19.1	19.1	18.3	17.7	19.2	18.3	17.7	17.8	17.7
0.7	56.6	55.9	55.6	55.7	54.4	21.0	19.2	18.7	17.7	17.9	17.5	18.1	18.6	18.0	17.6
0.9	57.4	55.3	55.3	55.2	55.2	21.1	19.3	18.5	17.8	17.8	17.5	17.8	18.5	18.1	17.9
	$\theta = 0.7$					$\theta = 0.8$					$\theta = 0.9$				
0.1	36.2	33.4	25.8	25.7	25.8	45.8	43.1	36.1	31.3	31.3	63.5	68.6	60.8	44.6	44.2
0.3	27.4	28.0	29.5	30.2	30.9	34.3	37.1	36.7	40.4	43.2	48.4	54.7	53.5	56.1	69.7
0.5	23.8	21.5	21.0	21.5	23.1	34.2	27.4	25.9	25.8	29.6	47.1	35.9	41.9	44.6	46.4
0.7	23.0	21.0	21.7	22.1	23.3	28.9	25.9	27.0	31.0	30.0	40.0	36.3	40.7	45.6	49.8
0.9	22.9	21.0	21.8	22.1	23.0	28.8	26.3	27.0	31.0	29.8	40.7	36.4	40.6	45.5	49.8

7.3. AFEM for a strongly monotone and Lipschitz continuous nonlinearity

On the Z-shaped domain $\Omega := (-1,1)^2 \setminus \mathrm{conv}\{(-1,0),(0,0),(-1,-1)\}$ and with the nonlinearity $a(x,t) = 1 + \log(1+t)/(1+t)$ for all $x \in \Omega$ and $t \ge 0$, we consider the quasi-linear elliptic PDE with homogeneous Dirichlet boundary conditions

$$-\operatorname{div}\big(a(\cdot,|\nabla u^\star|^2)\,\nabla u^\star\big) + u^\star = 1 \text{ in } \Omega \quad \text{and} \quad u^\star = 0 \text{ on } \partial\Omega \tag{78}$$

Hence, the nonlinearity $a(\cdot)$ satisfies the growth condition in Proposition 3 with constants $\alpha \approx 0.9582898017$ and $L \approx 1.542343818$. In the experiments, we use the optimal damping parameter $\delta = 1/L$ and the fixed polynomial degree $p = 1$. Fig. 10 illustrates the initial triangulation $\mathcal{T}_0$, the adaptively generated mesh $\mathcal{T}_7$ with 1483 triangles, and the computed solution $u_7^{\underline{k},\underline{j}}$. In the Figs. 11–13, we observe that the strategy from Algorithm C applied to this nonlinear problem guarantees optimal convergence rates $-1/2$ with respect to the

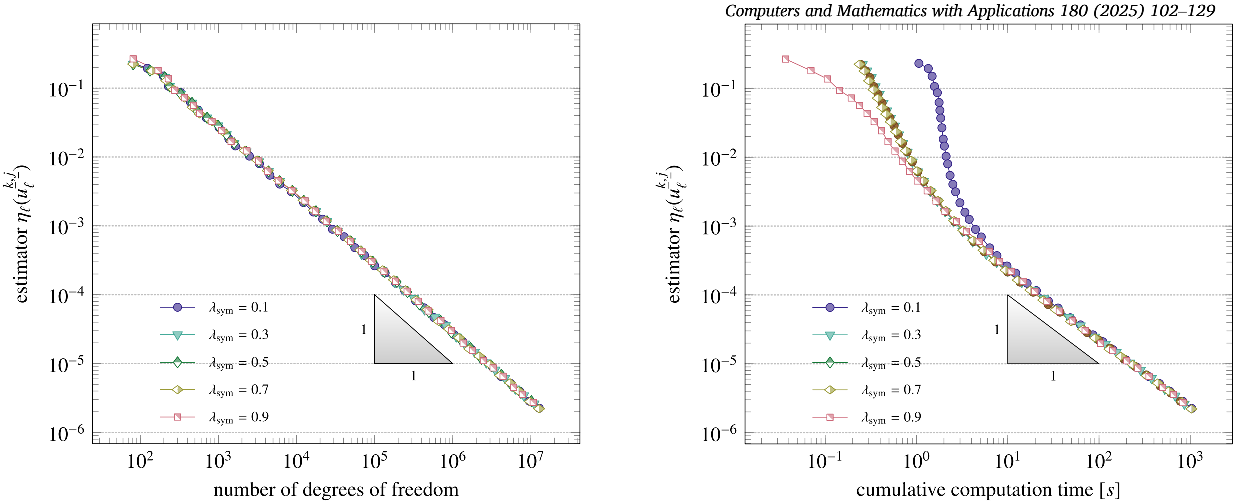

Fig. 6. Convergence history plot of the error estimator with respect to the number of degrees of freedom (left) and the computation time (right) for the nonsymmetric experiment (77) with $p = 2$ and $\delta = 0.5$ for several symmetrization parameters $\lambda_{\mathrm{sym}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and fixed algebraic solver parameter $\lambda_{\mathrm{alg}} = 0.7$ and marking parameter $\theta = 0.3$.

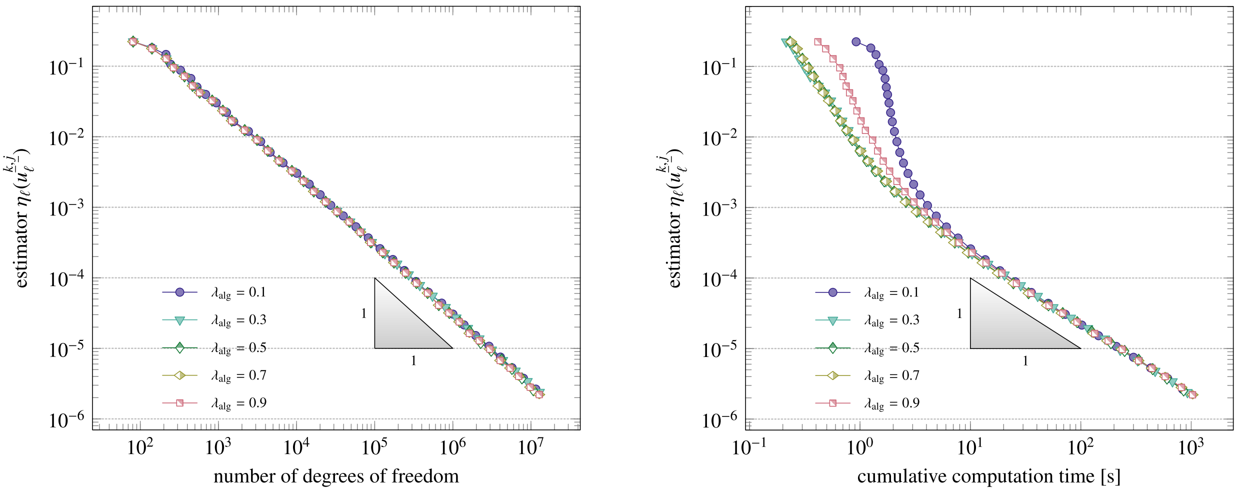

Fig. 7. Convergence history plot of the error estimator with respect to the number of degrees of freedom (left) and the computation time (right) for the nonsymmetric experiment (77) with $p = 2$ and $\delta = 0.5$ for several algebraic solver parameters $\lambda_{\mathrm{alg}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and fixed symmetrization parameter $\lambda_{\mathrm{sym}} = 0.7$ and marking parameter $\theta = 0.3$.

number of degrees of freedom and the cumulative computation time for arbitrary marking parameters θ, linearization parameters λ_{lin}, and algebraic solver parameters λ_{alg}. In particular, even large values of θ, λ_{lin}, and λ_{alg} lead to optimal convergence rates. Table 3 summarizes the optimal selection of the adaptivity parameters for the nonlinear problem (78) and indicates that moderate values of θ in combination with large values of λ_{lin} and λ_{alg} are beneficial in terms of computational cost.

Funding

This research was funded in whole or in part by the Austrian Science Fund (FWF) [10.55776/F65, 10.55776/I6667, 10.55776/I6802, and 10.55776/P33216] and the German Research Foundation (DFG) – SFB 1173 [Project-ID 258734477]. Funding was received also from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme [Grant agreement No. 101125225].

Appendix A. Proofs of Lemma 1, Lemma 2, and Lemma 4

Proof of Lemma 1. The proof is split into four steps.

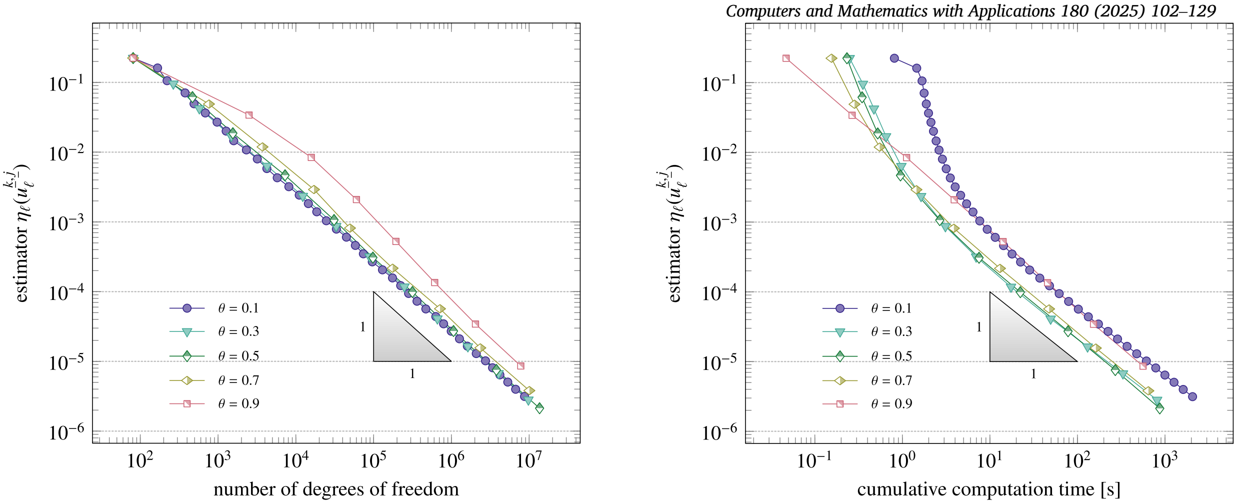

Fig. 8. Convergence history plot of the error estimator with respect to the number of degrees of freedom (left) and the computation time (right) for the nonsymmetric experiment (77) with $p = 2$ and $\delta = 0.5$ for several Dörfler marking parameters $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and fixed solver-stopping parameters $\lambda_{\mathrm{sym}} = \lambda_{\mathrm{alg}} = 0.7$.

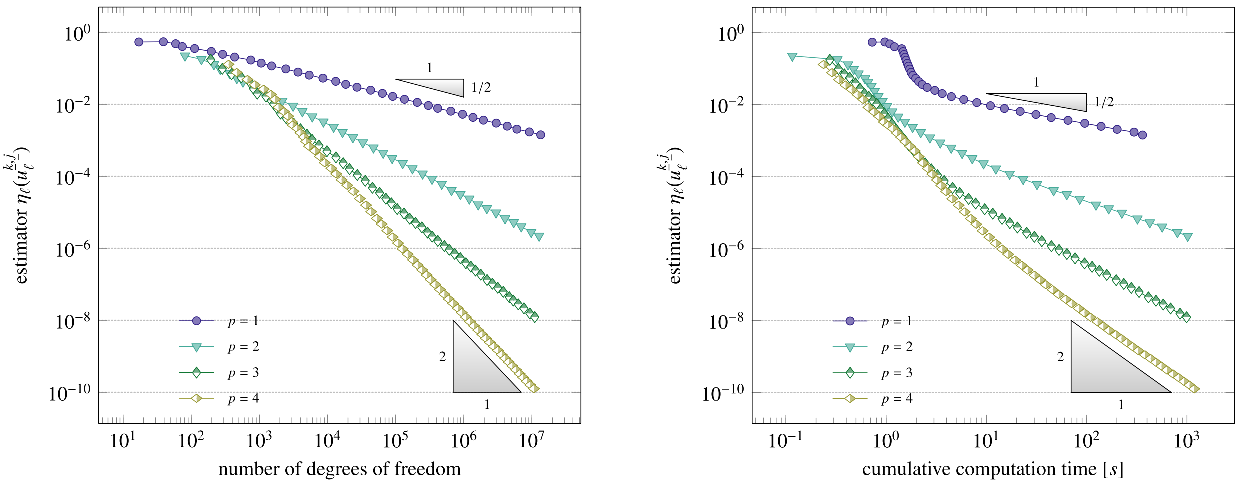

Fig. 9. Convergence history plot of the error estimator with respect to the number of degrees of freedom (left) and with respect to the overall computation time (right) for the nonsymmetric experiment (77) with $\delta = 0.5$ for several polynomial degrees $p = 1, 2, 3, 4$, and fixed marking parameter $\theta = 0.3$ and solver-stopping parameters $\lambda_{\mathrm{sym}} = \lambda_{\mathrm{alg}} = 0.7$.

Step 1. We consider the perturbed contraction of $(a_\ell)_{\ell \in \mathbb{N}_0}$ from (17). By induction on n, we see with the empty sum understood (as usual) as zero that

$$a_{\ell+n} \le q^n a_\ell + \sum_{j=1}^{n} q^{n-j} b_{\ell+j-1} \quad \text{for all } \ell, n \in \mathbb{N}_0.$$

From this and the geometric series, we infer that

$$a_{\ell+n} \le q^n a_\ell + C_1 \Big(\sum_{j=1}^{n} q^{n-j} \Big) a_\ell \le \Big(1 + \frac{C_1}{1-q} \Big) a_\ell =: C_3\, a_\ell \quad \text{for all } \ \ell, n \in \mathbb{N}_0. \tag{A.1}$$

Step 2. Next, we note that the perturbed contraction of $(a_\ell)_{\ell \in \mathbb{N}_0}$ from (17) and the Young inequality with sufficiently small $\varepsilon > 0$ ensure

$$0 < \kappa := (1+\varepsilon)\, q^2 < 1 \quad \text{and} \quad a_{\ell+1}^2 \overset{(17)}{\le} \kappa\, a_\ell^2 + (1+\varepsilon^{-1})\, b_\ell^2 \quad \text{for all } \ell \in \mathbb{N}_0.$$

This and the summability of $(b_\ell)_{\ell \in \mathbb{N}_0}$ from (17) guarantee

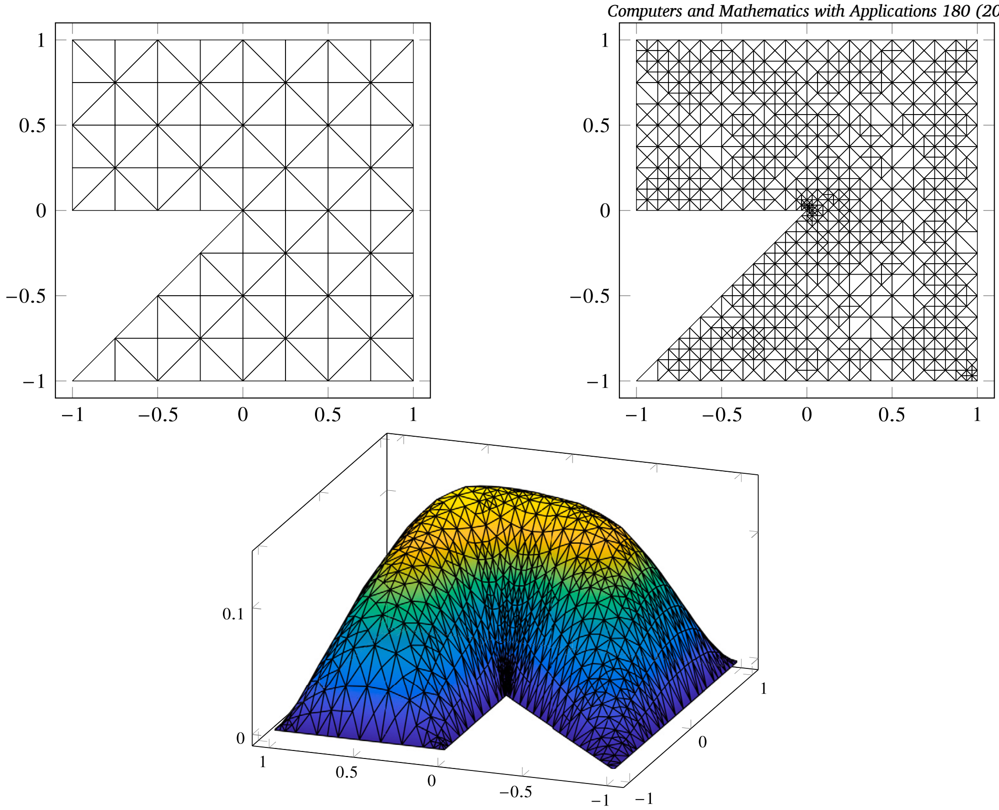

Fig. 10. Initial triangulation $\mathcal{T}_0$, adaptively generated mesh $\mathcal{T}_7$ with 1483 triangles, and the computed solution $u_7^{\underline{k},\underline{j}}$ for the nonlinear experiment (78) with polynomial degree $p = 1$, optimal damping parameter $\delta = 1/L$, marking parameter $\theta = 0.3$, linearization parameter $\lambda_{\mathrm{lin}} = 0.7$, and algebraic solver parameter $\lambda_{\mathrm{alg}} = 0.7$.

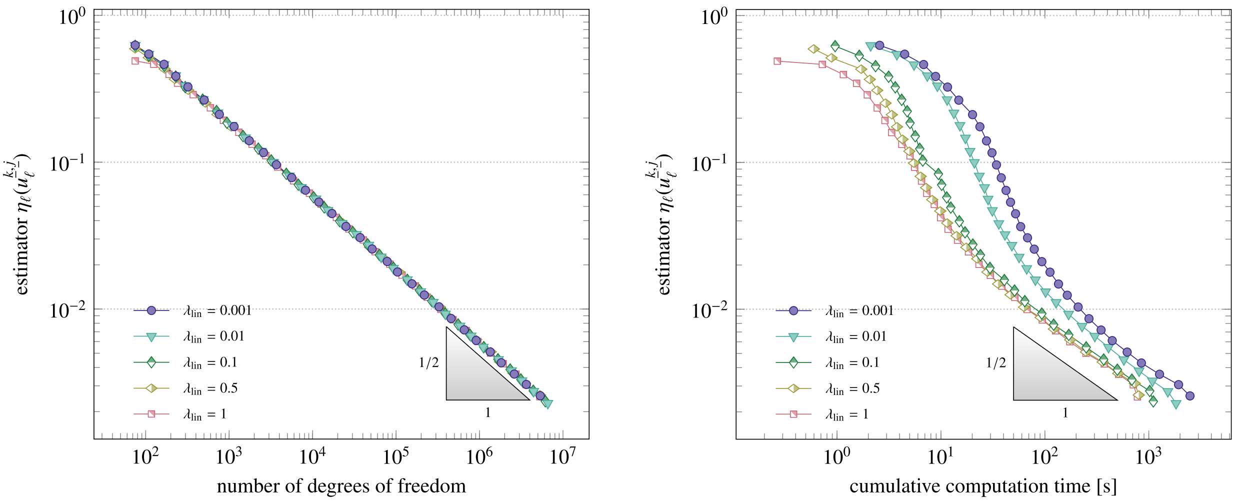

Fig. 11. Convergence history plot of the error estimator $\eta_\ell(u_\ell^{\underline{k},\underline{j}})$ with respect to the number of degrees of freedom (left) and the cumulative computation time (right) for the nonlinear experiment (78) with polynomial degree $p = 1$ and optimal damping parameter $\delta = 1/L$ for several linearization parameters $\lambda_{\mathrm{lin}} \in \{0.001, 0.01, 0.1, 0.5, 1\}$ and fixed marking parameter $\theta = 0.3$ and algebraic solver parameter $\lambda_{\mathrm{alg}} = 0.7$.

$$\sum_{\ell'=\ell+1}^{\ell+N} a_{\ell'}^2 = \sum_{\ell'=\ell}^{\ell+N-1} a_{\ell'+1}^2 \overset{(17)}{\leq} \kappa \sum_{\ell'=\ell}^{\ell+N-1} a_{\ell'}^2 + (1+\varepsilon^{-1}) C_2 \, N^{1-\delta} \, a_\ell^2.$$

Rearranging the estimate, we arrive at

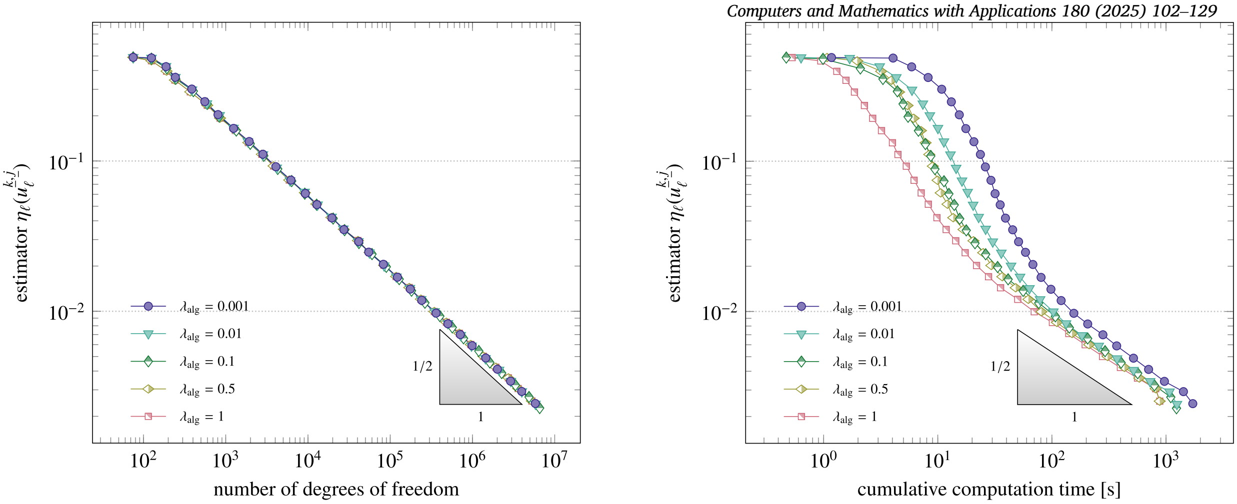

Fig. 12. Convergence history plot of the error estimator $\eta_\ell(u_\ell^{k,\underline{j}})$ with respect to the number of degrees of freedom (left) and the cumulative computation time (right) for the nonlinear experiment (78) with polynomial degree $p = 1$ and optimal damping parameter $\delta = 1/L$ for several algebraic solver parameters $\lambda_{alg} \in \{0.001, 0.01, 0.1, 0.5, 1\}$ and fixed marking parameter $\theta = 0.3$, and linearization parameter $\lambda_{lin} = 0.7$.

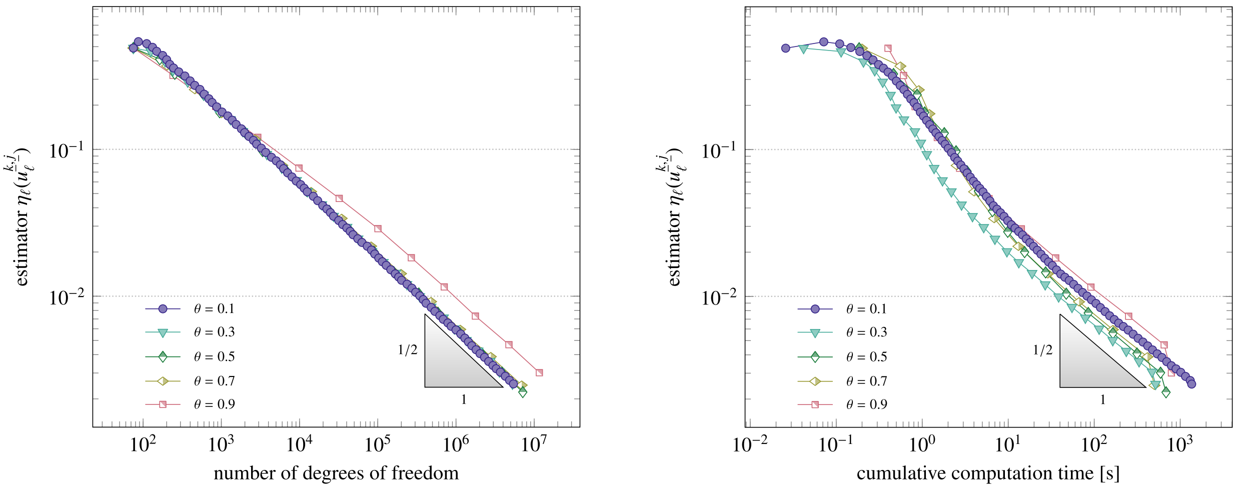

Fig. 13. Convergence history plot of the error estimator $\eta_\ell(u_\ell^{k,\underline{j}})$ with respect to the number of degrees of freedom (left) and the cumulative computation time (right) for the nonlinear experiment (78) with polynomial degree $p = 1$ and optimal damping parameter $\delta = 1/L$ for several marking parameters $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and solver parameters $\lambda_{lin} = \lambda_{alg} = 0.7$.

$$\sum_{\ell'=\ell}^{\ell+N} a_{\ell'}^2 \le \Big[1 + \frac{\kappa + (1+\varepsilon^{-1})C_2\, N^{1-\delta}}{1-\kappa}\Big]\, a_\ell^2 =: D_N\, a_\ell^2 \quad \text{for all } \ell, N \in \mathbb{N}_0, \tag{A.2}$$

where we note that $1 \le D_N \simeq N^{1-\delta}$ as $N \to \infty$. In the following, we prove that this already guarantees that (A.2) holds with an N-independent constant (instead of the constant D_N growing with N); see also Lemma 2 below.

Step 3. We show by mathematical induction on n that (A.2) implies

$$a_{\ell+n}^2 \le \Big(\prod_{j=1}^{n}(1 - D_j^{-1})\Big) \sum_{\ell'=\ell}^{\ell+n} a_{\ell'}^2 \quad \text{for all } \ell, n \in \mathbb{N}_0. \tag{A.3}$$

Note that (A.3) holds for all $\ell \in \mathbb{N}_0$ and $n = 0$ (with the empty product interpreted as 1). Hence, we may suppose that (A.3) holds for all $\ell \in \mathbb{N}_0$ and up to $n \in \mathbb{N}_0$. Then,

$$a_{\ell+(n+1)}^2 = a_{(\ell+1)+n}^2 \overset{(A.3)}{\le} \Big(\prod_{j=1}^{n}(1 - D_j^{-1})\Big) \sum_{\ell'=\ell+1}^{(\ell+1)+n} a_{\ell'}^2 = \Big(\prod_{j=1}^{n}(1 - D_j^{-1})\Big)\Big(\sum_{\ell'=\ell}^{\ell+(n+1)} a_{\ell'}^2 - a_\ell^2\Big)$$

Table 3
Optimal selection of parameters with respect to the computational costs for the nonlinear experiment (78) with $p = 1$ and $\delta = 1/L$. For the comparison, we consider the weighted cumulative time $\big[\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \sum_{|\ell',k',j'|\le|\ell,\underline{k},\underline{j}|} \mathrm{time}(\ell')\big]$ (values in 10^{-3}) with stopping criterion $\eta_\ell(u_\ell^{\underline{k},\underline{j}}) < 5\cdot 10^{-2}\,\eta_0(u_0^{0,0})$ for various choices of λ_{lin}, λ_{alg}, and θ. In each θ-block, we mark in yellow the best choice per column, in blue the best choice per row, and in green when both choices coincide. The best choices for λ_{lin} and λ_{alg} are observed for $\theta = 0.5$ and $\theta = 0.7$.

$\cdot 10^{-3}$	$\theta = 0.1$					$\theta = 0.3$					$\theta = 0.5$				
λ_{alg} \ λ_{sym}	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
0.1	16.9	16.8	15.5	15.3	15.6	11.9	11.1	10.0	9.5	9.7	9.8	8.9	9.0	8.0	8.2
0.3	17.2	14.1	14.5	14.2	14.7	9.7	8.2	8.0	8.5	8.6	7.6	7.3	6.9	6.6	6.7
0.5	15.7	14.0	14.6	14.5	14.4	9.7	8.2	8.3	8.6	8.3	8.0	6.8	7.0	7.2	6.8
0.7	14.9	14.5	14.7	13.8	15.5	9.3	8.5	8.3	8.3	8.2	7.6	7.0	6.6	6.6	7.0
0.9	14.3	14.8	14.8	14.9	14.5	9.0	8.5	8.3	8.4	8.1	7.4	6.7	6.7	6.6	6.8
	$\theta = 0.7$					$\theta = 0.8$					$\theta = 0.9$				
0.1	9.5	8.3	8.0	7.7	7.1	10.6	9.5	8.8	7.8	7.6	12.7	11.7	11.2	11.3	10.8
0.3	7.9	7.3	6.4	6.7	6.8	8.3	7.7	7.8	7.9	7.5	11.2	10.1	10.7	10.3	10.0
0.5	7.7	7.0	6.6	6.7	6.3	8.7	7.5	7.6	7.1	6.9	10.7	10.9	9.7	9.2	9.5
0.7	7.2	6.6	6.2	6.3	6.1	7.7	7.2	7.0	6.8	7.2	11.1	9.6	9.1	9.2	9.0
0.9	7.4	6.6	6.1	6.2	6.0	7.7	6.6	6.6	6.4	7.2	10.2	9.2	9.0	9.0	9.0

$$\overset{(A.2)}{\le} \Big(\prod_{j=1}^{n}(1-D_j^{-1})\Big)\Big(\sum_{\ell'=\ell}^{\ell+(n+1)} a_{\ell'}^2 - D_{n+1}^{-1}\sum_{\ell'=\ell}^{\ell+(n+1)} a_{\ell'}^2\Big) = \Big(\prod_{j=1}^{n+1}(1-D_j^{-1})\Big)\sum_{\ell'=\ell}^{\ell+(n+1)} a_{\ell'}^2.$$

This concludes the proof of (A.3).

Step 4. From (A.2)–(A.3), we infer that

$$a_{\ell+n}^2 \le \Big(\prod_{j=1}^{n}(1-D_j^{-1})\Big) D_n\, a_\ell^2 \quad \text{for all } \ell, n \in \mathbb{N}. \tag{A.4}$$

Note that

$$M_n := \log\Big[\Big(\prod_{j=1}^{n}(1-D_j^{-1})\Big) D_n\Big] = \sum_{j=1}^{n}\log(1-D_j^{-1}) + \log D_n.$$

With $1-x \le \exp(-x)$ for all $0<x<1$, it follows for $x = D_j^{-1}$ that

$$M_n \le \log D_n - \sum_{j=1}^{n} D_j^{-1} \simeq (1-\delta)\log n - \sum_{j=1}^{n}\frac{1}{j^{1-\delta}} \xrightarrow{n\to\infty} -\infty,$$

since $\log n \le \sum_{j=1}^{n}(1/j)$. Fix $n_0 \in \mathbb{N}$ such that $M_{n_0} < 0$. It follows from (A.4) that

$$a_{\ell+in_0}^2 \le q_0^i\, a_\ell^2 \quad \text{for all } \ell, i \in \mathbb{N}_0, \quad \text{where } 0 < q_0 := \exp(M_{n_0}) < 1. \tag{A.5}$$

Let $\ell \in \mathbb{N}_0$. For general $n \in \mathbb{N}_0$, choose $i, j \in \mathbb{N}$ with $j < n_0$ such that $n = in_0 + j$. With (A.5) and quasi-monotonicity (A.1) of a_ℓ, we derive

$$a_{\ell+n}^2 = a_{(\ell+j)+in_0}^2 \overset{(A.5)}{\le} q_0^i\, a_{\ell+j}^2 \overset{(A.1)}{\le} C_3^2\, q_0^i\, a_\ell^2 = C_3^2\, q_0^{-j/n_0} q_0^{n/n_0} a_\ell^2 \le (C_3^2/q_0)\,(q_0^{1/n_0})^n a_\ell^2.$$

This completes the proof of (18) with $C_{\mathrm{lin}} := C_3^2/q_0 > 0$ and $0 < q_{\mathrm{lin}} := q_0^{1/n_0} < 1$. □

Proof of Lemma 2. First, observe that $(a_\ell)_{\ell\in\mathbb{N}_0}$ is R-linearly convergent in the sense of (ii) if and only if $(a_\ell^m)_{\ell\in\mathbb{N}_0}$ is R-linearly convergent in the sense of (ii) with C_{lin} replaced by C_{lin}^m and q_{lin} replaced by q_{lin}^m. Therefore, we may restrict to $m = 1$.

The implication (ii) $\implies$ (i) follows from the geometric series, i.e.,

$$\sum_{\ell'=\ell+1}^{\infty} a_{\ell'} \overset{\text{(ii)}}{\leq} C a_\ell \sum_{\ell'=\ell+1}^{\infty} q^{\ell'-\ell} \leq \frac{Cq}{1-q} a_\ell \quad \text{for all } \ell \in \mathbb{N}_0.$$

Conversely, (i) yields that

$$(C_1^{-1}+1) \sum_{\ell'=\ell+1}^{\infty} a_{\ell'} \overset{\text{(i)}}{\leq} a_\ell + \sum_{\ell'=\ell+1}^{\infty} a_{\ell'} = \sum_{\ell'=\ell}^{\infty} a_{\ell'} \quad \text{for all } \ell \in \mathbb{N}_0.$$

Inductively, this leads to

$$a_{\ell+n} \leq \sum_{\ell'=\ell+n}^{\infty} a_{\ell'} \overset{\text{(i)}}{\leq} \frac{1}{(C_1^{-1}+1)^n} \sum_{\ell'=\ell}^{\infty} a_{\ell'} \overset{\text{(i)}}{\leq} \frac{1+C_1}{(C_1^{-1}+1)^n} a_\ell \quad \text{for all } \ell, n \in \mathbb{N}_0.$$

This proves (ii) with $C_{\text{lin}} := 1 + C_1$ and $q_{\text{lin}} := (C_1^{-1}+1)^{-1}$. □

Proof of Lemma 4. Let $(\ell, k, \underline{j}) \in \mathcal{Q}$ with $k \geq 1$. Contraction of the Zarantonello iteration (39) proves

$$|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \leq |||u_\ell^\star - u_\ell^{k,\star}||| + |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| \overset{(39)}{\leq} q_{\text{sym}}^\star |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| + |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}|||.$$

From the termination criterion of the algebraic solver (43), we see that

$$|||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| \leq \frac{q_{\text{alg}}}{1-q_{\text{alg}}} |||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}||| \overset{(43)}{\leq} \frac{q_{\text{alg}}}{1-q_{\text{alg}}} \lambda_{\text{alg}} \big[\lambda_{\text{sym}} \eta_\ell(u_\ell^{k,\underline{j}}) + |||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}|||\big].$$

With the termination criterion of the inexact Zarantonello iteration (42), it follows that

$$|||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| \overset{(42)}{\leq} \frac{2\, q_{\text{alg}}}{1-q_{\text{alg}}} \lambda_{\text{alg}} \begin{cases} \lambda_{\text{sym}} \eta_\ell(u_\ell^{k,\underline{j}}) & \text{for } k = \underline{k}[\ell], \\ |||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| & \text{for } 1 \leq k < \underline{k}[\ell]. \end{cases}$$

For $k = \underline{k}[\ell]$, the preceding estimates prove (51). For $k < \underline{k}[\ell]$, it follows that

$$|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \leq q_{\text{sym}}^\star |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| + \frac{2\, q_{\text{alg}}}{1-q_{\text{alg}}} \lambda_{\text{alg}} \big[|||u_\ell^\star - u_\ell^{k,\underline{j}}||| + |||u_\ell^\star - u_\ell^{k-1,\underline{j}}|||\big].$$

Provided that $\frac{2\, q_{\text{alg}}}{1-q_{\text{alg}}} \lambda_{\text{alg}} < 1$, this proves

$$|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \leq \frac{q_{\text{sym}}^\star + \frac{2\, q_{\text{alg}}}{1-q_{\text{alg}}} \lambda_{\text{alg}}}{1 - \frac{2\, q_{\text{alg}}}{1-q_{\text{alg}}} \lambda_{\text{alg}}} |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \overset{(46)}{=} q_{\text{sym}} |||u_\ell^\star - u_\ell^{k-1,\underline{j}}|||,$$

which is (50). This concludes the proof. □

Data availability

No data was used for the research described in the article.

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