

D I P L O M A R B E I T

Isoperimetric Inequalities for Affine and Dual Affine Quermassintegrals

zur Erlangung des akademischen Grades

Diplom-Ingenieur

ausgeführt am

Institut für

Diskrete Mathematik und Geometrie

TU Wien

unter der Anleitung von

Univ.Prof. Dipl.-Ing. Dr.techn. Franz Schuster

durch

Florian Mielke-Sulz, BSc

Wien, am 28. Oktober 2024

Betreuer

Student

Contents

1	Introduction	1
2	Notation	2
3	Geometric Preliminaries	3
3.1	The Löwner Ellipsoid	5
3.2	Steiner Symmetrization	8
3.3	Characterization of Centered Ellipsoids	10
4	Two Essential Stochastic Results	14
5	Affine and Dual Affine Quermassintegrals - Definitions and Basic Properties	21
5.1	Log-Concavity	21
5.1.1	Log-Concavity of the Dual Affine Quermassintegral	22
5.1.5	Log-Concavity of the Affine Quermassintegral	24
5.2	Proof of the Affine Invariance of $\tilde{\Phi}_k$ and Φ_k	27
6	Isoperimetric Inequality for Dual Affine Quermassintegrals	30
6.0.8	Relation to the Isoperimetric Inequality for Dual Quermassintegrals	35
7	Isoperimetric Inequalities for Affine Quermassintegrals	37
7.1	Preliminaries	37
7.2	Convexity of Projections of Shadow System	40
7.3	A Blaschke-Petkantschin-Type Formula	42
7.4	$\Phi_k(K)$ Decreases under Steiner Symmetrization	43
7.5	The Isoperimetric Inequality	47
7.5.4	Relation to the Isoperimetric Inequality for Quermassintegrals . .	48
7.6	Some Additional Convexity Properties	49
7.6.1	A Generalization of Proposition 7.2.5	49
7.6.4	The s -Moment Function is Convex	51

Contents

7.6.9	A Dichotomy for $t \mapsto \Phi_k(K_u(t))$	52
8	Cases of Equality of $\Phi_k(K) \geq \Phi_k(B_K)$	53
8.1	The Case $k < n - 1$	53
8.2	The Case $k = n - 1$	60
	Bibliography	67

1 Introduction

In convex geometry, affine quermassintegrals are important quantities in the for study of geometric inequalities. Lutwak introduced two types of these integrals: the dual affine quermassintegrals, denoted by $\tilde{\Phi}_k$ and the affine quermassintegrals, denoted by Φ_k . Of special interest are the isoperimetric inequalities associated with these integrals. For the dual affine quermassintegrals $\tilde{\Phi}_k$, this involves finding sharp upper bounds and identifying the convex bodies of a given volume that achieve equality. For the affine quermassintegrals Φ_k , the goal is to establish sharp lower bounds and determine the convex bodies that minimize them.

The isoperimetric inequality for the dual affine quermassintegrals $\tilde{\Phi}_k$ was proven earlier, with the inequality shown in [BS60] and the cases of equality discussed by Grinberg in [Gri91]. In contrast, the isoperimetric inequality for the affine quermassintegrals Φ_k remained an unsolved problem for many years. It was only in 2022 that this inequality was finally proven by E. Milman and Yehudayoff in [MY23].

In this thesis, we give a self-contained presentation of the proofs of the isoperimetric inequalities for $\tilde{\Phi}_k$ and Φ_k , establish the equality cases, and examine some of their properties. Additionally, we will discuss some consequences of the isoperimetric inequality and highlight important special cases that serve as fundamental tools in affine convex geometry.

Acknowledgments I would like to thank my supervisor, Professor Franz Schuster, for his support and guidance during the writing of this thesis. My sincere thanks also go to my parents for their continuous support throughout my studies.

2 Notation

As usual, we denote the n -dimensional Euclidean space by $\mathbb{R}^n$ with origin 0 and standard basis $e_1, \dots, e_n$, equipped with inner product $(x, y) = x \cdot y = xy$ and induced norm $|\cdot|$. Let $|x|$ also denote the absolute value of $x \in \mathbb{R}$, and let $\mathbb{R}^+$ denote the positive reals, whereas $\mathbb{R}^-$ denotes the negative reals. For $x \in \mathbb{R}$ we denote $x^+ = x_+ = \max(x, 0)$ and $x^- = x_- = \max(-x, 0)$, where we use the $+$ index either as a superscript or a subscript to avoid collisions with other indices. Additionally, x^+ should be understood component-wise if x is a vector. Furthermore, for $u \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$, we set $u^\perp = \{x \in \mathbb{R}^n : x \cdot u = 0\}$ and $A^\perp = \{x \in \mathbb{R}^n : x \cdot a = 0 \ \forall a \in A\}$. With $\text{int}(A)$, $\text{cl}(A)$ and $\text{bd}(A)$, we denote the topological interior, closure and boundary of A .

We denote by GL_n the set of linear and bijective maps $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, represented by matrices with non-zero determinant, $\det(\phi) \neq 0$. The subgroup SL_n of GL_n consists of matrices with $\det(\phi) = 1$. We write SO_n for the set of all orthogonal matrices ϕ with $\det(\phi) = 1$. If ϕ is a linear map we write ϕ^* for its adjoint, ϕ^{-1} for its inverse and ϕ^t for its transposed. We denote by $\text{Ker}\phi$ the kernel of ϕ and by Im the image of ϕ . Furthermore, the composition of two maps ϕ, ψ in general will be denoted $\phi \circ \psi$, for the restriction of ϕ on a set E we write $\phi|_E$.

The set of k -dimensional linear subspaces of $\mathbb{R}^n$ will be denoted as $\mathcal{G}(n, k)$ and, the set of k -dimensional linear subspaces contained in a linear subspace $L \subseteq \mathbb{R}^n$ will be denoted $\mathcal{G}(L, k)$. With $A \oplus B$ we will denote the direct sum of two linear subspaces A and B . Furthermore, for $A \subseteq \mathbb{R}^n$ we write $\mathbb{1}_A$ for the indicator function, i.e. $\mathbb{1}_A(x) = 1$ for $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise.

3 Geometric Preliminaries

We start with some basic definitions. For $A, B \subseteq \mathbb{R}^n$, we define the *Minkowski sum* of A and B

$$A + B = \{a + b : a \in A, b \in B\}.$$

A set $A \subseteq \mathbb{R}^n$ is called *convex* if, for every $x, y \in A$, the line segment from x to y

$$[x, y] = \{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$$

is contained in A . We denote by $\mathcal{K}(\mathbb{R}^n)$ the set of non-empty, compact, and convex sets. Moreover, a *convex body* $K \subseteq \mathbb{R}^n$ is a compact, convex set with a non-empty interior. The set of convex bodies will be denoted $\mathcal{K}^n$. Given a convex body $K \subseteq \mathbb{R}^n$, we denote the support function of K by $h_K(x) = \max\{x \cdot y : y \in K\}$, $x \in \mathbb{R}^n$. Furthermore, we can equip $\mathcal{K}(\mathbb{R}^n)$ with the *Hausdorff metric* δ ,

$$\delta(K, D) = \max\{\max\{d(k, D) : k \in K\}, \max\{d(K, d) : d \in D\}\}, \quad K, D \in \mathcal{K}^n,$$

where $d(k, D) = d(D, k) = \min\{|k - d| : d \in D\}$. The *polar body* K° of a convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior is defined as follows,

$$K^\circ = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

With the n -dimensional Lebesgue measure λ_n we have a notion of volume on $\mathcal{K}^n$ and denote by κ_n the volume of the n -dimensional unit ball $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, i.e. $\kappa_n = \lambda_n(B^n)$. If K is a measurable set in $\mathbb{R}^n$, we define B_K as the ball centered at the origin and with the same Lebesgue measure as K . Furthermore, the $n - 1$ -dimensional sphere is denoted by $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. If H is a subspace of $\mathbb{R}^n$ then B_H is the unit ball in H , $P_H : \mathbb{R}^n \rightarrow H$ is the projection onto H , and we often write $A|H = P_H(A)$ for $A \subseteq \mathbb{R}^n$. By $\text{span}(H)$, we denote the linear hull of H , i.e. the set of all linear combinations of elements of H . The dimension $\dim(A)$ of a set $A \subseteq \mathbb{R}^n$ is defined as the dimension of the affine hull of A . For $u \in S^{n-1}$, the set $\{y \in \mathbb{R}^n : y \cdot u = h_K(u)\}$

is called the *supporting hyperplane to K with outer normal u* .

We call a set $L \subseteq \mathbb{R}^n$ *star-shaped at the origin* if $[0, x] \subseteq L$ for every $x \in L$. For such L , we define its *radial function* $\rho_L(x) = \max\{\lambda \geq 0 : \lambda x \in L\}$. Furthermore, a *star body* is a compact, star-shaped set with a positive, continuous radial function. The set of star bodies will be denoted by $\mathcal{S}^n$. For $K, L \in \mathcal{S}^n$, we define the *radial sum* $K \tilde{+} L$ as the star body with radial function $\rho_K + \rho_L$. The *projection body* ΠK of K is defined via $h_{\Pi K}(x) = \lambda_{n-1}(K|x^\perp)$ and the *intersection body* of K is the convex body IK with $\rho_{IK}(x) = \lambda_{n-1}(K \cap x^\perp)$. Furthermore, the *Minkowski functional* of $K \in \mathcal{K}^n$ will be denoted

$$\|x\|_K = \inf\{t > 0 : x \in tK\}.$$

For $L \in \mathcal{S}^n$, the polar coordinate formula for volume holds

$$\lambda_n(L) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^n du. \quad (3.1)$$

We call a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ *convex* if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y),$$

for all $x, y \in \mathbb{R}^n$ and $0 \leq t \leq 1$, and *concave*, if $-f$ is convex.

We will need the following important classical theorems. Theorem 3.0.1, Blaschke's Selection Theorem, can be found in [Had57], Theorem 3.0.2, the Brunn-Minkowski Inequality, Theorem 3.0.3 and Theorem 3.0.4, the Dual Brunn-Minkowski Inequality, can be found in [Sch13] and for Theorem 3.0.5, Brunn's concavity principle, we refer to [Gru07].

Theorem 3.0.1. *Every uniformly bounded sequence in $\mathcal{K}(\mathbb{R}^n)$ admits a convergent subsequence.*

Theorem 3.0.2. *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies with non-empty interior, then*

$$\lambda_n(K + L)^{1/n} \geq \lambda_n(K)^{1/n} + \lambda_n(L)^{1/n}$$

with equality if and only if K and L are homothetic (i.e. $L = \alpha K + x$ for some $\alpha > 0$ and $x \in \mathbb{R}^n$).

Theorem 3.0.3. *For convex bodies $K_i \subseteq \mathbb{R}^n, i \in \{1, \dots, m\}$, the volume of their Minkowski*

sum is a polynomial with non-negative coefficients in the scaling parameters, i.e.

$$\lambda_n(t_1 K_1 + \dots + t_m K_m) = \sum_{1 \leq i_1, \dots, i_n \leq m} t_{i_1} \cdot \dots \cdot t_{i_n} V(K_{i_1}, \dots, K_{i_n}) \quad (3.2)$$

for $t_i \geq 0$. The non-negative coefficients $V(K_{i_1}, \dots, K_{i_n})$ in (3.2) are called mixed volumes of $(K_{i_1}, \dots, K_{i_n})$.

If in particular the first k entries of $V(\dots)$ are K_1 and the following $n - k$ entries are K_2 , then we simply write $V(K_1, k; K_2, n - k)$.

Theorem 3.0.4. Let $K, L \subseteq \mathbb{R}^n$ be star bodies. Then,

$$\lambda_n(K \tilde{+} L)^{1/n} \leq \lambda_n(K)^{1/n} + \lambda_n(L)^{1/n},$$

where equality holds if and only if K arises from L through scaling.

Theorem 3.0.5. If $K \subseteq \mathbb{R}^{n+1}$ is a convex body and $u \in S^n$, then the function

$$t \mapsto \lambda_n \left(K \cap (tu + u^\perp) \right)^{1/n}, \quad t \in \mathbb{R}$$

is concave on its support.

Finally, the following formula of Fedotov can be found in [Sch13], Theorem 5.3.1.

Theorem 3.0.6. Let $F \in \mathcal{G}(n, k)$ for $1 \leq k \leq n - 1$, and let $L_1, \dots, L_{n-k} \subseteq F^\perp$ be non-empty compact convex sets. Then, for $K_1, \dots, K_k \in \mathcal{K}(\mathbb{R}^n)$ it holds that

$$\binom{n}{k} V(K_1, \dots, K_k, L_1, \dots, L_{n-k}) = V_F(P_F K_1, \dots, P_F K_k) V_{F^\perp}(L_1, \dots, L_{n-k}),$$

with $V_E, E \in \{F, F^\perp\}$, denoting the mixed volume in the subspace E .

3.1 The Löwner Ellipsoid

We will prove that for a compact set C with positive measure, there exists a unique centered ellipsoid of minimal volume that contains C . This result will be used to prove some important characterization theorems for ellipsoids. The results of this section can be found in [Gar06].

Definition 3.1.1. An *affine transformation* is a map $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $\phi(x) = Ax + t$, where A is $n \times n$ -matrix with $\det(A) = \pm 1$ and $t \in \mathbb{R}^n$. An *ellipsoid* in $\mathbb{R}^n$ is the image of a ball under an affine transformation. We call an ellipsoid E *centered* if $-x \in E$ whenever $x \in E$.

Remark 3.1.2. Note that in the above definition, $\det(A) = \pm 1$ ensures that affine transformations are volume-preserving.

Theorem 3.1.3. (*Weighted arithmetic-geometric mean inequality*) For numbers $x_1, \dots, x_n \geq 0$ and weights $w_1, \dots, w_n \geq 0$ with $w = w_1 + \dots + w_n > 0$ we have

$$\frac{w_1 x_1 + \dots + w_n x_n}{w} \geq \sqrt[w]{x_1^{w_1} \dots x_n^{w_n}}$$

with equality if and only if all the x_k with $w_k > 0$ are equal.

Proof. We can assume that all w_k are positive, since the terms with zero weights have no influence on the inequality. By the fact that the natural logarithm is concave, Jensen's inequality yields

$$\begin{aligned} \ln \left(\frac{w_1 x_1 + \dots + w_n x_n}{w} \right) &\geq \frac{w_1}{w} \ln x_1 + \dots + \frac{w_n}{w} \ln x_n \\ &= \ln \sqrt[w]{x_1^{w_1} \dots x_n^{w_n}}. \end{aligned}$$

Since the natural logarithm is increasing, we obtain the desired inequality.

Clearly, equality holds if all x_k with $w_k > 0$ are equal. If at least two of the x_k are not equal, the above inequality obtained by Jensen is strict, and since the natural logarithm is strictly increasing we also have strict inequality in $\frac{w_1 x_1 + \dots + w_n x_n}{w} > \sqrt[w]{x_1^{w_1} \dots x_n^{w_n}}$. $\square$

Theorem 3.1.4. Let $C \subseteq \mathbb{R}^n$ be compact with $\lambda_n(C) > 0$. There exists a unique centered n -dimensional ellipsoid, known as the *Löwner ellipsoid*, of minimal volume containing C .

Proof. To prove the existence let $\mathcal{E}$ be the class of centered n -dimensional ellipsoids containing C and $a = \inf\{\lambda_n(E) : E \in \mathcal{E}\} > 0$. There is a sequence E_m in $\mathcal{E}$ with $\lambda_n(E_m) \rightarrow a$ for $m \rightarrow \infty$. Because each of the E_m is an n -dimensional Ellipsoid we obtain a sequence $A_m \in GL_n$ of regular $n \times n$ matrices such that $E_m = A_m B^n$. Since C is bounded, the entries of A_m are also bounded and therefore there exists a subsequence $m(j)$ such that $A_{m(j)}$ converges to some $n \times n$ matrix A . Furthermore, we get from

$\lambda_n(E_m) = |\det A_m| \kappa_n$ that

$$|\det A_{m(j)}| = \frac{\lambda_n(E_{m(j)})}{\kappa_n} \longrightarrow \frac{a}{\kappa_n}, \quad \text{for } j \rightarrow \infty.$$

Thus $|\det A| \neq 0$ and $E = AB^n$ is the n -dimensional ellipsoid we were looking for.

To prove uniqueness, suppose that E_1, E_2 are two such centered n -dimensional ellipsoids containing C with least volume a . We can find $\phi \in SL_n$ such that $E'_1 = \phi E_1$ is a ball with radius b and $E'_2 = \phi E_2$ a centered ellipsoid, i.e.

$$E'_1 = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq b^2\},$$

$$E'_2 = \{x \in \mathbb{R}^n : \sum_{i=1}^n \frac{x_i^2}{a_i^2} \leq 1\}$$

for some $a_1, \dots, a_n > 0$. Since ϕ is volume preserving, we get

$$a = \lambda_n(E'_1) = b^n \kappa_n = \lambda_n(E'_2) = a_1 a_2 \cdots a_n \kappa_n.$$

From $\phi C \subseteq E'_1$ and $\phi C \subseteq E'_2$ we obtain $\sum_{i=1}^n x_i^2 \leq b^2$ and $\sum_{i=1}^n \frac{x_i^2}{a_i^2} \leq 1$ for all $x \in \phi C$. Since such $x \in \phi C$ also satisfy

$$\frac{1}{2} \sum_{i=1}^n x_i^2 (b^{-2} + a_i^{-2}) \leq 1$$

we conclude that ϕC is also contained in the centered ellipsoid $E'_3 = \{x \in \mathbb{R}^n : \frac{1}{2} \sum_{i=1}^n x_i^2 (b^{-2} + a_i^{-2}) \leq 1\}$. This gives us $\lambda_n(E'_3) \geq a$. On the other hand the weighted arithmetic-geometric mean inequality 3.1.3 yields

$$\begin{aligned} \lambda_n(E'_3) &= \kappa_n \prod_{i=1}^n \sqrt{2ba_i} (b^2 + a_i^2)^{-1/2} \\ &\leq \kappa_n \prod_{i=1}^n (ba_i)^{1/2} = \lambda_n(E'_1)^{1/2} \lambda_n(E'_2)^{1/2} = a, \end{aligned}$$

Hence $\lambda_n(E'_3) = a$, and from the equality cases in the arithmetic geometric mean inequality, we obtain $b = a_i$ for $1 \leq i \leq n$. This gives $E'_1 = E'_2$, and therefore $E_1 = E_2$. $\square$

3.2 Steiner Symmetrization

To establish some characterizations of ellipsoids, we introduce the important concept of Steiner symmetrization and specifically prove that a convex body converges to a ball under successive Steiner symmetrizations. As a reference, we cite [Gar06], [Gru07], and [Sch05].

Definition 3.2.1. Let $K \subseteq \mathbb{R}^n$ be a convex body, $u \in S^{n-1}$ and $l_u \subseteq \mathbb{R}^n$ the line through the origin spanned by u . The *Steiner symmetral* $S_u K$ of K in the direction of u is defined as follows: For every $x \in u^\perp$ with the property that $(l_u + x) \cap K \neq \emptyset$, let $c(x)$ be the line segment parallel to u with center x and length $\lambda_1((l_u + x) \cap K)$. If $(l_u + x) \cap K = \emptyset$, then set $c(x) = \emptyset$. The union of all $c(x)$ is $S_u K$.

More formally, since $(l_u + x) \cap K$ is a line segment, we may write it in the form

$$\{x + zu : z \in \mathbb{R}, \underline{z}(x) \leq z \leq \bar{z}(x)\}$$

with two functions $\underline{z} : K|u^\perp \rightarrow \mathbb{R}$ and $\bar{z} : K|u^\perp \rightarrow \mathbb{R}$. In particular, the difference $\bar{z}(x) - \underline{z}(x)$ is the length of the line segment $(l_u + x) \cap K$. Therefore

$$c(x) = \{x + zu : z \in \mathbb{R}, -\frac{1}{2}(\bar{z}(x) - \underline{z}(x)) \leq z \leq \frac{1}{2}(\bar{z}(x) - \underline{z}(x))\},$$

and we obtain the Steiner symmetral as

$$S_u(K) = \bigcup_{x \in K|u^\perp} c(x).$$

Proposition 3.2.2. Suppose $K, D \subseteq \mathbb{R}^n$ are convex bodies and $u \in S^{n-1}$ is a given direction. The Steiner symmetral has the following properties:

- (i) $\underline{z}$ is convex and lower semi-continuous, $\bar{z}$ is concave and upper semi-continuous,
- (ii) $S_u K$ is symmetric with respect to reflection in $u^\perp$,
- (iii) $S_u K \subseteq \mathbb{R}^n$ is a convex body,
- (iv) $S_u(\lambda K) = \lambda S_u K$ (up to translations) for $\lambda \geq 0$,
- (v) $S_u(K) + S_u(D) \subseteq S_u(K + D)$ (up to translations),
- (vi) if $K \subseteq D$, then $S_u(K) \subseteq S_u(D)$,

(vii) $S_u : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is a continuous map.

(viii) $\lambda_n(S_u K) = \lambda_n(K)$.

Proof. (i): Since K is convex, it is easy to check that also $\underline{z}$ is convex. To prove the lower semi-continuity, suppose that $x \in K|u^\perp$ such that there is an $\epsilon > 0$ and a sequence $y_i, i \in \mathbb{N}$ in $K|u^\perp$ with $y_i \rightarrow x$ but $\underline{z}(y_i) < \underline{z}(x) - \epsilon$. We can assume that $\underline{z}(y_i)$ converges to some real number a , since if not, we replace y_i with a suitable subsequence. Since $y_i + \underline{z}(y_i)u \in K$ and $y_i + \underline{z}(y_i)u \rightarrow x + au$ we obtain $x + au \in K$ and therefore, the contradiction $\underline{z}(x) \leq a \leq \underline{z}(x) - \epsilon$. The statements for $\bar{z}$ follow analogously.

(ii): Trivial.

(iii): The boundedness of $S_u K$ and $S_u K \neq \emptyset$ is obvious. To show that $S_u K$ is closed let $y_i, i \in \mathbb{N}$ be a sequence in $S_u K$ converging to some $y \in \mathbb{R}^n$. Then $y_i = x_i + z_i u$ for some $x_i \in K|u^\perp, z_i \in \mathbb{R}$ and $x_i \rightarrow x, z_i \rightarrow z, y = x + zu$. Since $K|u^\perp$ is closed, we obtain $x \in K|u^\perp$. The inequality $2|z_i| \leq \bar{z}(x_i) - \underline{z}(x_i)$ and the semi-continuity yield

$$2|z| = \lim_{i \rightarrow \infty} 2|z_i| \leq \limsup_{i \rightarrow \infty} \bar{z}_i(x_i) - \liminf_{i \rightarrow \infty} \underline{z}(x_i) \leq \bar{z}(x) - \underline{z}(x),$$

and thus $y \in S_H(K)$. The convexity of $S_u K$ follows from the fact that the function $\bar{z} - \underline{z}$ is concave.

(iv): Trivial.

(v): Let $x + y \in S_u(K) + S_u(D)$, that means we can write $x = h + l$ and $y = k + m$ for some $h, k \in u^\perp$ and $l, m \in l_u$ with

$$|l| \leq \frac{1}{2} \lambda_1(K \cap (l_u + x)) \quad \text{and} \quad |m| \leq \frac{1}{2} \lambda_1(D \cap (l_u + y)).$$

Since l_u and $u^\perp$ are subspaces, we have $h + k \in H$ and $l + m \in L$ and therefore

$$x + y = (h + k) + (l + m) \in u^\perp + l_u.$$

Because of

$$\begin{aligned}
 |l + m| &\leq |l| + |m| \\
 &\leq \frac{1}{2} (\lambda_1(K \cap (l_u + x)) + \lambda_1(D \cap (l_u + y))) \\
 &= \frac{1}{2} \lambda_1(K \cap (l_u + x) + D \cap (l_u + y)) \\
 &\leq \frac{1}{2} \lambda_1((K + D) \cap (l_u + x + y))
 \end{aligned}$$

we obtain $x + y \in S_u(K + D)$.

(vi): Trivial.

(vii): Let $C_n, n \in \mathbb{N}$, be a sequence in $\mathcal{K}^n$ converging to some $C \in \mathcal{K}^n$ with respect to the Hausdorff metric. We may assume $0 \in \text{int}(C)$. For $\epsilon > 0$ and sufficiently large n we have

$$(1 - \epsilon)C \subseteq C_n \subseteq (1 + \epsilon)C.$$

Applying (iv) and (v) yields

$$(1 - \epsilon)S_u C \subseteq S_u(C_n) \subseteq (1 + \epsilon)S_u C.$$

Because of $0 \in \text{int}(S_u(C))$ we obtain the convergence of $S_u(C_n)$ to $S_u(C)$ in $\mathcal{K}^n$.

(viii): Applying Fubini's Theorem, we obtain

$$\lambda_n(S_u K) = \int_{\mathbb{R}} \lambda_1(c(x)) dx = \int_{\mathbb{R}} \lambda_1((l_u + x) \cap K) dx = \lambda_n(K),$$

where we set $c(x) = 0$ for $x \notin K|u^\perp$. □

3.3 Characterization of Centered Ellipsoids

With the notion of the Löwner ellipsoid and the results regarding Steiner symmetrization in hand, we can now prove some important characterization theorems for ellipsoids. As a reference for this section, we cite [Gar06].

Theorem 3.3.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body and assume the sequence of directions $u_m \in S^{n-1}, m \in \mathbb{N}$, with the property outlined in Lemma 6.0.4. If for every $m \in \mathbb{N}$, the set of midpoints M_m of all chords of K parallel to u_m lie in a hyperplane passing through the origin, then K is a centered ellipsoid.*

Proof. We will show, that for every $m \in \mathbb{N}$ there is a $\phi \in SL_n$ with $S_{u_m}K = \phi K$. Therefore fix m and let $x \in \mathbb{R}^n$. There are $y \in u^\perp$ and $s \in \mathbb{R}$ such that $x = y + su_m$. Denote by S the hyperplane which contains M_m , and let $z \in S$ be the unique point with $z = y + tu$ for some $t \in \mathbb{R}$. We define $\phi x = x - (z - y)$. Since y is the orthogonal projection of x onto $u^\perp$ and z depends linearly on y , we get that also ϕ is linear. By the Cavalieri principle and the property of ϕ being a translation in each line parallel to u , it follows that $\phi \in SL_n$. It is easy to check that $S_u K = \phi K$ holds.

Like in the previous lemma let $K_m = S_{u_m} S_{u_{m-1}} \dots S_{u_1} K$. We just showed that K_m is the image of finitely many volume preserving linear transformations of K and therefore of just one such transformation which we denote by ψ_m . Let E be the Löwner ellipsoid of K . Then $\psi_m E$ is the Löwner ellipsoid of K_m with $\lambda_n(\psi_m E) = \lambda_n(E)$. Since K_m converge to a ball rB^n with the same volume as K and the ellipsoids $\psi_m E$ converge to the Löwner ellipsoid of rB^n , which is rB^n itself, we obtain $\lambda_n(E) = \lambda_n(rB^n) = \lambda_n(K)$. Since $K \subseteq E$ we get $K = E$. $\square$

Lemma 3.3.2. *If E is an ellipsoid containing the origin, then the polar E° is also an ellipsoid that contains the origin.*

Proof. First, if $a \in \text{int}B$, then, by the relation between the support and the radial function, we obtain for $u \in S^{n-1}$

$$\rho_{(B^n+a)^\circ}(u) = \frac{1}{h_{(B+a)}(u)} = \frac{1}{1 + a \cdot u}.$$

In particular, it holds that

$$\rho_{(B^n+a)^\circ}(u) = 1 - \rho_{(B^n+a)^\circ}(u) u \cdot a,$$

which if $x = (x_1, \dots, x_n) = \rho_{(B^n+a)^\circ}(u)u$ can be rewritten as

$$\sum_{i=1}^n x_i^2 = \left(1 - \sum_{i=1}^n a_i x_i\right)^2.$$

The above equation is quadratic, $(B+a)^\circ$ is convex, thus, by 15.4.7 in [BCL09], $(B+a)^\circ$ is an ellipsoid. Let $\phi \in GL_n$ be such that $E = \phi(B+a)$, $a \in \text{int}B$. Because of

$$h_{\phi(B+a)}(u) = \frac{1}{\rho_{\phi(B+a)}(u)} = \frac{1}{\rho_{B+a}(\phi^{-1}u)} = h_{(B+a)^\circ}(\phi^{-1}u) = h_{\phi^{-t}((B+a)^\circ)}(u),$$

we obtain

$$E^\circ = (\phi(B + a))^\circ = \phi^{-t}((B + a)^\circ),$$

which shows that E° is an ellipsoid containing the origin. $\square$

Remark 3.3.3. Suppose that (P) is a property of convex bodies and we proved that K has property (P) whenever all its projections $K|S, S \in \mathcal{G}(n, n-1)$ have property (P). Then if $1 < k \leq n-1$ and it holds that K has property (P) whenever all $K|S, S \in \mathcal{G}(n, k)$ have property (P). To see this, we argue by induction as follows: Assume that $H \in \mathcal{G}(n, 3)$ and all projections $K|S, S \in \mathcal{G}(n, 2)$ fulfill property (P). If we identify H with $\mathbb{R}^3$ we can apply our assumption (with $n = 3$ and $k = 2$) to get that $K|H$ has property (P). By induction on the dimension of the subspaces, we conclude that K possesses property (P).

Lemma 3.3.4. *Let $1 < k \leq n-1$ and $K \subseteq \mathbb{R}^n$ be a compact convex set such that $K|S$ is an ellipsoid for every $S \in \mathcal{G}(n, k)$. Then K is an ellipsoid.*

Proof. By Remark 3.3.3 it suffices to show it for $k = n-1$. If $\dim K < n$ there is $S \in \mathcal{G}(n, n-1)$ containing a translate of K . Thus K is just a translate of $K|S$, and therefore an ellipsoid.

Now let $\dim K = n$, that means K is a convex body. We can find p_1, p_2 in $\text{bd}K$ such that the line segment $[p, p']$ is a diameter of K . Without loss of generality let $p = (-1, 0, \dots, 0)$ and $p' = (1, 0, \dots, 0)$. Suppose H is a supporting plane to K which is parallel to $[p, p']$ and let $S \in \mathcal{G}(n, n-1)$ be such that $[p, p'] \subseteq S$ and S is orthogonal to H . Thus, $K|S$ is an ellipsoid with $[p, p']$ as one axis. The body $K|S$ is supported by $H \cap S$ at one point, which is contained in $\{x_1 = 0\} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 = 0\}$, hence $H \cap K \subseteq \{x_1 = 0\}$. Therefore, $E = K|\{x_1 = 0\} = K \cap \{x_1 = 0\}$, so E is an ellipsoid and as readily observed also centered.

Now let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an affine map, which lets $[p, p']$ invariant and maps E onto a ball D in $\{x_1 = 0\}$ with the origin as its center. Since ϕ maps ellipsoids onto ellipsoids, showing that $\phi(K)$ is an ellipsoid of revolution finishes the proof. So, for every $P \in \mathcal{G}(n, 2)$ the projection $\phi(K)|P$ is an ellipse. Let q, q' be so that $[q, q']$ is a diameter of the ball D . If we apply the above argument with K replaced by $\phi(K)$ and $[p, p']$ by $[q, q']$, we get that the intersection of the hyperplane, which is orthogonal to $[q, q']$, and $\phi(K)$ is an ellipsoid E' . Furthermore, the line $[p, p']$ is an axis of E' , and for every subspace $P \in \mathcal{G}(n, 2)$ containing $[p, p']$, the intersection $P \cap \phi(K)$ is an ellipse

with the two axes $[p, p']$ and a diameter of D . That shows that $\phi(K)$ is an ellipsoid of revolution. $\square$

Lemma 3.3.5. *Suppose that $K \subseteq \mathbb{R}^n$ is compact, convex and containing the origin in its relative interior and let $1 < k \leq n - 1$. If for all $S \in \mathcal{G}(n, k)$ the body $K \cap S$ is an ellipsoid, then K is an ellipsoid.*

Proof. Due to Remark 3.3.3 we can assume $k = n - 1$. For $\dim K < n$, that means $K \subseteq S$ for some $S \in \mathcal{G}(n, n - 1)$, it follows directly that K is an ellipsoid. So let K be full dimensional, i.e. $K \in \mathcal{K}^n$. For every $S \in \mathcal{G}(n, n - 1)$ and $u \in S \cap S^{n-1}$ we have

$$\rho_{(K^\circ|S)^\circ}(u) = \frac{1}{h_{K^\circ|S}(u)} = \frac{1}{h_{K^\circ}(u)} = \rho_{K^{\circ\circ}}(u) = \rho_{K \cap S}(u),$$

where we used $K^{\circ\circ} = K$ and the relation $\rho_K = 1/h_K$ between radial and support function. Thus $(K^*|S)^\circ = K \cap S$ is an ellipsoid and therefore, by Lemma 3.3.2, $(K^\circ|S)^{\circ\circ} = K^\circ|S$ is also an ellipsoid for every $S \in \mathcal{G}(n, n - 1)$. Lemma 3.3.4 yields that K° must be an ellipsoid and thus this also holds for K , again by Lemma 3.3.2. $\square$

4 Two Essential Stochastic Results

In this chapter, we will prove two crucial stochastic results. The first result will be essential for proving the isoperimetric inequality for dual affine quermassintegrals, Theorem 6.0.6, while the second will be used for the corresponding inequality for affine quermassintegrals, Theorem 7.5.1. As a reference for the first stochastic theorem and the preliminary work we cite [SW92], Section 7.3, and for the second we refer to [SW08], Theorem 7.2.6.

But first, recall that $\mathcal{G}(n, k)$ is the set of k -dimensional linear subspaces of $\mathbb{R}^n$. On $\mathcal{G}(n, k)$ we assume the usual topology. Furthermore, we equip $\mathcal{G}(n, k)$ with a suitable normalized SO_n invariant Borel measure and denote the integral of a function $f : \mathcal{G}(n, k) \rightarrow \mathbb{R}$ with respect to the measure by

$$\int_{\mathcal{G}(n, k)} f(F) dF.$$

We begin with a few definitions and statements that are not yet of a stochastic nature.

Definition 4.0.1. Let G be a topological group, that is, a group G with a topology $\mathcal{T}$ such that the group operation and the inversion map are continuous. A *homogeneous G -space* is a pair (X, ϕ) , where X is a topological space and $\phi : G \times X \rightarrow X$ a transitive and continuous operation of G on X with $\phi(., p)$ being an open map for every $p \in X$.

Remark 4.0.2. We will argue that every homogeneous G -space X can be viewed as a quotient space G/H with a subgroup H of G . Indeed, let H be a subgroup of G , equipped with the subspace topology, and $G/H = \{aH : a \in G\}$, the quotient space with the quotient topology. We define a *natural operation* ζ of G on G/H as follows

$$\zeta(g, aH) = gaH \quad \text{for } g \in G, aH \in G/H.$$

With that, $(G/H, \zeta)$ becomes a homogeneous G -space. On the other hand if (X, ϕ) is a homogeneous G -space, fix an arbitrary $p \in X$ and set $S_p = \{g \in G : \phi(g, p) = p\}$. Then

the map

$$\beta : G/S_p \rightarrow X, \quad gS_p \mapsto \phi(g, p)$$

is a homeomorphism from G/S_p to X with $\beta(gaS_p) = \phi(g, \beta(aS_p))$ for all $g \in G$ and $aS_p \in G/S_p$. In this sense the homogeneous G -spaces (X, ϕ) and $(G/S_p, \zeta)$ are isomorphic.

Definition 4.0.3. Let G be a locally compact topological group and $C_c(G)$ the set of continuous functions $f : G \rightarrow \mathbb{R}$. Then, for $a \in G$ we define the function $a.f : G \rightarrow \mathbb{R}$ as follows

$$(a.f)(x) = f(a^{-1}x), \quad x \in G.$$

Furthermore we call a functional I on $C_c(G)$ an *integral on G* if I is linear, positive and non-zero everywhere. If H is a subgroup of G and $\chi : G \rightarrow \mathbb{R}$ a given function, we call an integral I on G/H *relative invariant with multiplier χ* if $I(a.f) = \chi(a)I(f)$ for all $f \in C_c(G/H)$ and $a \in G$. By the Riesz–Markov–Kakutani representation theorem every Integral on G/H corresponds to a unique regular Borel measure ρ on G/H which satisfies $I(f) = \int_{G/H} f(x) d\rho(x)$ for $f \in C_c(G/H)$. Analogous, ρ is said to be *relative invariant with multiplier $\chi : G \rightarrow \mathbb{R}$* if

$$\rho(gA) = \chi(g)\rho(A), \quad \text{for } g \in G \text{ and a Borel set } A \text{ on } G/H.$$

Note that the measure ρ is relative invariant if and only if the corresponding integral I is relative invariant.

Definition 4.0.4. A Borel measure ρ on G is called *left invariant* if $\rho(gA) = \rho(A)$ for all $g \in G$ and Borel sets A on G .

For the following Theorem, we refer to [Coh80].

Theorem 4.0.5. *Let G be a locally compact topological group. Then there is (up to a positive factor) a unique left invariant regular Borel measure on G , the so-called left Haar measure.*

Our aim is to show that with given multiplier χ there is up to a factor only one relative invariant measure on G/H . Therefore we establish a connection between $C_c(G)$ and $C_c(G/H)$ in the following way: For $f \in C_c(G)$ we define

$$f'(x) = \int_H f(xy) d\eta(y), \quad x \in G,$$

where η denotes the Haar measure as in Theorem 4.0.5. The function f' is constant on every left coset. Indeed, for $x = zh \in zH$ we have

$$f'(x) = \int_H f(zhy) d\eta(y) = \int_H f(zy) d\eta(y) = f'(z).$$

Therefore, we obtain a unique function $f^+ : G/H \rightarrow \mathbb{R}$ with $f'(x) = f^+(xH)$ for all $x \in G$.

The following can be found in [SW92], 7.3.2.

Lemma 4.0.6. *The map $f \mapsto f^+$ is a linear bijective function from $C_c(G)$ onto $C_c(G/H)$.*

With this in hand, we can prove the following important theorem.

Theorem 4.0.7. *Let G be a locally compact topological group, H a closed subgroup of G and χ a given multiplier. Up to a constant factor there is only one relative invariant measure on G/H , or equivalent, up to a constant factor there is only one relative invariant integral on G/H .*

Proof. Let ρ be a relative invariant measure on G/H and $a \in G$. By the invariance of ρ we obtain

$$\int_{G/H} (a.h)(x) d\rho(x) = \chi(a) \int_{G/H} h(x) d\rho(x)$$

for $h \in C_c(G/H)$. Moreover, for $f \in C_c(G)$ it is readily seen that $(a.f)^+ = a.(f^+)$. Since χ is a homomorphism $\chi = \chi(a)a.\chi$. Therefore,

$$\left(\frac{a.f}{\chi}\right)^+ = \frac{1}{\chi(a)} \left(\frac{a.f}{a.\chi}\right)^+ = \chi(a^{-1})a. \left(\frac{f}{\chi}\right)^+.$$

For readability, we will not explicitly note the integration variable in the following. Let us define a positive linear functional

$$I(f) = \int_{G/H} \left(\frac{f}{\chi}\right)^+ d\rho \quad \text{for } f \in C_c(G).$$

Since

$$\begin{aligned} I(a.f) &= \int_{G/H} \left(\frac{a.f}{\chi}\right)^+ d\rho = \chi(a^{-1}) \int_{G/H} a. \left(\frac{f}{\chi}\right)^+ d\rho \\ &= \chi(a^{-1})\chi(a) \int_{G/H} \left(\frac{f}{\chi}\right)^+ d\rho = I(f), \end{aligned}$$

I is a left invariant integral on $C_c(G)$ and therefore, up to a constant factor, uniquely determined. Suppose that $\hat{\rho}$ is another relative invariant measure on G/H with multiplier χ . Then

$$\int_{G/H} \left(\frac{f}{\chi} \right)^+ d\rho = c \int_{G/H} \left(\frac{f}{\chi} \right)^+ d\hat{\rho}$$

for all $f \in C_c(G)$ and some constant c . By Lemma 4.0.6, every function of $C_c(G/H)$ corresponds to suitable $(f/\chi)^+$ and therefore $\rho = c\hat{\rho}$. $\square$

With these tools established, we can now prove the important stochastic result mentioned earlier. The corollary that follows from this result will be used to prove the isoperimetric inequality for dual affine quermassintegrals.

Theorem 4.0.8. *Let f be a Borel function on the product space $(\mathbb{R}^n)^i$, where $1 \leq i \leq n-1$. Then there is a constant $c \in \mathbb{R}$, depending only on n and i , such that*

$$\begin{aligned} & \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(p_1, \dots, p_i) dp_1 \cdots dp_i \\ &= c \int_{\mathcal{G}(n,i)} \int_S \cdots \int_S f(p_1, \dots, p_i) \lambda_i([o, p_1, \dots, p_i])^{n-i} dp_1 \cdots dp_i dS. \end{aligned}$$

Proof. The idea of the proof is to view both sides as functionals of f and then apply Theorem 4.0.7 to obtain the constant. For that we have to show that

$$I_1(f) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(p_1, \dots, p_i) dp_1 \cdots dp_i$$

and

$$I_2(f) = \int_{\mathcal{G}(n,i)} \int_S \cdots \int_S f(p_1, \dots, p_i) \lambda_i([o, p_1, \dots, p_i])^{n-i} dp_1 \cdots dp_i dS$$

are positive, linear and relative invariant integrals on $C_c(X)$ with the same multiplier. Here X denotes the set of all tuples $(p_1, \dots, p_i)$, where the p_j are linear independent, i.e. X is the set of (n, i) matrices with rank i . Let us define a topological group $G = SO_n \times GL_i$, where the topology is the product of the standard topology. We can now equip X with a transitive operation

$$\phi : ((D, M), (p_1, \dots, p_i)) \mapsto D(p_1, \dots, p_i)M^t,$$

for $(D, M) \in G$ and $(p_1, \dots, p_i) \in X$. Therefore (X, ϕ) is a homogeneous G -space. Note that by Remark 4.0.2, (X, ϕ) can be seen as a quotient space G/H for a suitable subgroup

H and therefore the assumptions in Theorem 4.0.7 are fulfilled for X .

Positivity and linearity of I_1 and I_2 are clear. By the rotation invariance of the Lebesgue measure and the fact that the linear map $(p_1, \dots, p_i) \mapsto (p_1, \dots, p_i)M$ has determinant $(\det M)^n$, we obtain for $f \in C_c(X)$ and $(D, M) \in G$,

$$\begin{aligned} I_1((D, M).f) &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(D^{-1}(p_1, \dots, p_i)M^{-t}) dp_1 \cdots dp_i \\ &= |\det M|^n I_1(f). \end{aligned}$$

Furthermore, if we define $\xi(p_1, \dots, p_i) = \lambda_i([o, p_1, \dots, p_i])$ and fix $S \in \mathcal{G}(n, i)$ we have

$$\begin{aligned} &\int_S \cdots \int_S f(D^{-1}(p_1, \dots, p_i)M^{-t}) \lambda_i([o, p_1, \dots, p_i])^{n-i} dp_1 \cdots dp_i \\ &= |\det M|^i \int_S \cdots \int_S f(D^{-1}(p_1, \dots, p_i)) \xi((p_1, \dots, p_i)M^t)^{n-i} dp_1 \cdots dp_i \\ &= |\det M|^n \int_{D^{-1}S} \cdots \int_{D^{-1}S} f(p_1, \dots, p_i) \xi(p_1, \dots, p_i)^{n-i} dp_1 \cdots dp_i \end{aligned}$$

By the rotation invariance of the measure on $\mathcal{G}(n, i)$ we obtain

$$I_2((D, M).f) = |\det M|^n I_2(f),$$

and thus, I_1, I_2 are relative invariant with the same multiplier. Theorem 4.0.7 yields $I_1 = cI_2$ for some constant. □

For the following corollary, which will be crucial for the proof of the isoperimetric inequality for dual affine quermassintegrals, we will use functions $g_{m,k}$, which will be defined later in Definition 6.0.1.

Corollary 4.0.9. *If $C \subseteq \mathbb{R}^n$ is a compact set and $1 \leq i \leq n-1$, it holds that*

$$\lambda_n(C)^i = c \int_{\mathcal{G}(n,i)} g_{n-i,i}(C \cap S) dS$$

where $c \in \mathbb{R}$ depends only on n and i .

Proof. Applying Theorem 7.3.1 to the function $f(p_1, \dots, p_i) = \mathbb{1}_C(p_1) \cdots \mathbb{1}_C(p_i)$ and

the fact that

$$\begin{aligned} & \int_S \cdots \int_S \mathbf{1}_C(p_1) \cdots \mathbf{1}_C(p_i) \lambda_i([o, p_1, \dots, p_i])^{n-i} dp_1 \cdots dp_i \\ &= \int_{S \cap C} \cdots \int_{S \cap C} \lambda_i([o, p_1, \dots, p_i])^{n-i} dp_1 \cdots dp_i dS \\ &= g_{n-i,i}(C \cap S) \end{aligned}$$

for $S \in \mathcal{G}(n, i)$, yields the claimed equality. $\square$

We now turn to the second stochastic result.

Definition 4.0.10. For a linear subspace $L \subseteq \mathbb{R}^n$ let us denote by $G(L, q)$ the set of all q -dimensional linear subspaces contained in L if $q \leq \dim(L)$, and if $q > \dim(L)$ containing L . Furthermore, let $SO_{n,L} := \{U \in SO_n : UL = L, Ux = x \text{ for } x \in L^\perp\}$. Then, a $SO_{n,L}$ -invariant measure can be defined on $G(L, q)$, as done in reference [SW08], Section 13.2, allowing us to integrate functions defined on $G(L, q)$.

Let $L_1, \dots, L_k$ be linear subspaces of $\mathbb{R}^n$ with either

$$\sum_{i=1}^k \dim L_i =: m \leq n \quad (4.1)$$

or

$$\sum_{i=1}^k \dim L_i \geq (k-1)n. \quad (4.2)$$

We call $L_1, \dots, L_k$ in general position if, in case (4.1)

$$\dim \operatorname{lin}(L_1 \cup \dots \cup L_k) = \dim L_1 + \dots + \dim L_k$$

holds, or in case (4.2) if

$$\dim(L_1 \cap \dots \cap L_k) = \dim L_1 + \dots + \dim L_k - (k-1)n.$$

Note that, $L_1, \dots, L_k$ are in general position if and only if $L_1^\perp, \dots, L_k^\perp$ are in general position. Now, we define the *subspace determinant*, denoted by $[L_1, \dots, L_k]$, as follows. In case of (4.1) we choose an orthonormal basis in each L_i and define $[L_1, \dots, L_k]$ as the m -dimensional volume of the parallelepiped spanned by the union of these bases. When (4.2) holds, we define

$$[L_1, \dots, L_k] := [L_1^\perp, \dots, L_k^\perp].$$

Furthermore, if $q \in \mathbb{N}$, $r_1, \dots, r_q \in \{1, \dots, n-1\}$ and $(L_1, \dots, L_q) \in \mathcal{G}(n, r_1) \times \dots \times \mathcal{G}(n, r_q)$, we denote by

$$[L_1, \dots, L_q]_r := [L_1, \dots, L_q],$$

where $r := (r_1, \dots, r_q)$ serves as a multi-index. Moreover, if we fix a linear subspace L_0 , we may also set

$$[L_1, \dots, L_q, L_0]_r := [L_1, \dots, L_q, L_0]$$

Therefore, for $r := (r_1, \dots, r_q)$, the determinants $[\cdot, \dots, \cdot]_r$ and $[\cdot, \dots, \cdot, L_0]_r$ are both functions defined on $\mathcal{G}(n, r_1) \times \dots \times \mathcal{G}(n, r_q)$.

The proof of the following can be found in [SW08], Theorem 7.2.6.

Theorem 4.0.11. *For given integers $s_1, \dots, s_q \in \{1, \dots, n-1\}$ and $s_0 \in \{1, \dots, n\}$ with*

$$s_1 + \dots + s_q - (q-1)n =: m \geq n - s_0,$$

a linear subspace $L_0 \in \mathcal{G}(n, s_0)$, and a non-negative measurable function $f : \mathcal{G}(n, s_1) \times \dots \times \mathcal{G}(n, s_q) \rightarrow \mathbb{R}$, we have

$$\begin{aligned} & \int_{\mathcal{G}(n, s_1) \times \dots \times \mathcal{G}(n, s_q)} f(F_1, \dots, F_q) d(F_1, \dots, F_q) \\ &= \bar{c} \int_{G(L_0, m+s_0-n)} \int_{G(L, s_1) \times \dots \times G(L, s_q)} f(F_1, \dots, F_q) \cdot [F_1, \dots, F_q, L_0]_s^{m+s_0-n} d(F_1, \dots, F_q) dL, \end{aligned} \quad (4.3)$$

where $s := (s_1, \dots, s_q)$, $b_{dq} := \frac{\kappa_{d-q+1} \dots \kappa_d}{\kappa_1 \dots \kappa_q}$ and

$$\bar{c} := b_{s_0(d-m)} \prod_{j=1}^q \frac{b_{(2d-m-s_0)(d-s_j)}}{b_{d(d-s_j)}}.$$

In [SW08], Theorem 7.2.1, one can also find an alternative proof of Theorem 7.3.1, as well as a demonstration of how Theorem 4.0.11 follows from Theorem 7.3.1.

5 Affine and Dual Affine Quermassintegrals - Definitions and Basic Properties

In this section, we recall the definition of the affine quermassintegral $\Phi_k(K)$ and the dual affine quermassintegral $\tilde{\Phi}_k(K)$ for a convex body K , and establish some basic properties. These include the log-concavity of Φ_k with respect to the Minkowski sum, and the corresponding properties of $\tilde{\Phi}_k$ with respect to the radial sum. Additionally, we will prove the affine invariance of both Φ_k and $\tilde{\Phi}_k$ and conclude by exploring their connections to other classical isoperimetric inequalities.

The k -th affine quermassintegral Φ_k was originally defined by Lutwak in [Lut84].

Definition 5.0.1. Let $K \subseteq \mathbb{R}^n$ be a convex body, and $1 \leq k \leq n$. The k -th *affine quermassintegral* of K is defined by

$$\Phi_k(K) = \frac{\kappa_n}{\kappa_k} \left(\int_{\mathcal{G}(n,k)} \lambda_k(K|F)^{-n} dF \right)^{-\frac{1}{n}}.$$

Lutwak also defined the dual affine quermassintegrals.

Definition 5.0.2. Let $C \subseteq \mathbb{R}^n$ be a compact set and $1 \leq i \leq n$. The quantities

$$\tilde{\Phi}_{n-i}(C) = \frac{\kappa_n}{\kappa_i} \left(\int_{\mathcal{G}(n,i)} \lambda_i(C \cap S)^n dS \right)^{1/n}$$

are called the *dual affine quermassintegrals* of C .

5.1 Log-Concavity

In this section, we will prove the log-concavity of $\tilde{\Phi}_k$ and Φ_k with respect to the Minkowski sum and the radial sum, respectively.

5.1.1 Log-Concavity of the Dual Affine Quermassintegral

As a reference, we cite [Gar07].

Definition 5.1.2. Let us denote by $\mathcal{B}^n$ the class of bounded Borel sets in $\mathbb{R}^n$ and $\mathcal{B}_{so}^n$ the class of sets of $\mathcal{B}^n$, which are star-shaped with respect to the origin. If a subset $\mathcal{C}^n \subseteq \mathcal{B}^n$ is closed under intersections with linear subspaces, radial sums, and dilatations we call it *admissible*. A function f defined on such $\mathcal{C}^n$ is said to be radially convex if, for all $C, D \in \mathcal{C}^n$ and $0 \leq t \leq 1$,

$$f((1-t)C \tilde{+} tD) \leq (1-t)f(C) + tf(D).$$

Furthermore, we call f *positively homogeneous of degree 1* if $f(rC) = rf(C)$ for all $r \geq 0$.

To prove log-concavity of the dual affine quermassintegrals we need the following lemma.

Lemma 5.1.3. Fix $p \geq 1$ and $1 \leq i \leq n-1$. If f is a non-negative function on an admissible class $\mathcal{C}^n$ that is homogeneous and radially convex on any i -dimensional subspace, and such that $f(C \cap \cdot)^p$ is integrable on $\mathcal{G}(n, i)$, then the function $g : \mathcal{C}^n \rightarrow \mathbb{R}$,

$$g(C) = \left(\int_{\mathcal{G}(n, i)} f(C \cap S)^p dS \right)^{1/p},$$

is homogeneous and radially convex.

Proof. It is easy to check that g is homogeneous. To show that g is radially convex, let $0 \leq t \leq 1$ and $C, D \in \mathcal{C}^n$. Since the radial sum $x \tilde{+} y$ of $x, y \in \mathbb{R}^n$ can be defined by

$$x \tilde{+} y = \begin{cases} x + y & \text{if } x, y, \text{ and } o \text{ are collinear,} \\ o & \text{otherwise,} \end{cases}$$

we have, for each $S \in \mathcal{G}(n, i)$,

$$((1-t)C \tilde{+} tD) \cap S = (1-t)(C \cap S) \tilde{+} t(D \cap S).$$

Minkowski's inequality for integrals yields

$$\begin{aligned}
 g((1-t)C \tilde{+} tD) &= \left(\int_{\mathcal{G}(n,i)} f(((1-t)C \tilde{+} tD) \cap S)^p dS \right)^{1/p} \\
 &\leq \left(\int_{\mathcal{G}(n,i)} ((1-t)f(C \cap S) + tf(D \cap S))^p dS \right)^{1/p} \\
 &\leq (1-t) \left(\int_{\mathcal{G}(n,i)} f(C \cap S)^p dS \right)^{1/p} + t \left(\int_{\mathcal{G}(n,i)} f(D \cap S)^p dS \right)^{1/p} \\
 &= (1-t)g(C) + tg(D),
 \end{aligned} \tag{5.1}$$

which proves the statement. $\square$

With this in hand, we are able to prove log-concavity of the dual affine quermassintegrals.

Theorem 5.1.4. *For $K, L \in \mathcal{B}_{so}^n$ and $0 \leq i \leq n-1$ it holds that*

$$\tilde{\Phi}_i(K \tilde{+} L)^{1/(n-i)} \leq \tilde{\Phi}_i(K)^{1/(n-i)} + \tilde{\Phi}_i(L)^{1/(n-i)}$$

where equality holds if and only if K is a dilatate of L , modulo a set of measure zero.

Proof. The case $i = 0$ corresponds to the usual dual Brunn-Minkowski inequality, Theorem 3.0.4. So, let $1 \leq i \leq n-1$ and $0 \leq t \leq 1$. Note that the class $\mathcal{B}_{so}^n$ is admissible. Furthermore, let $S \in \mathcal{G}(n, i)$ and $C, D \in \mathcal{B}_{so}^n$ with $C, D \subseteq S$. The function $\lambda_i^{1/i}$ defined on S is homogeneous. If we identify S with $\mathbb{R}^i$, the dual Brunn-Minkowski inequality yields

$$\lambda_i((1-t)C \tilde{+} tD)^{1/i} \leq (1-t)\lambda_i(C)^{1/i} + t\lambda_i(D)^{1/i}$$

where equality holds if and only if C is a dilatate of D , modulo a set of measure zero. The function $f = \lambda_i^{1/i}$ is homogeneous and radially convex on S . Therefore, Lemma 5.1.3 with $p = ni$ implies that the function $(\kappa_i/\kappa_n)\tilde{\Phi}_{n-i}^i$ is radially convex. Replacing i by $n-i$ yields the claimed inequality.

We are left to address the cases of equality. If that is the case, equality also holds in (5.1) for $f = \lambda_i^{1/i}$ and $p = ni$. This implies that we have equality in the dual Brunn-Minkowski inequality, Theorem 3.0.4, where $n = i$ and K and L are replaced by $K \cap S$ and $L \cap S$, respectively, for almost every $S \in \mathcal{G}(n, i)$. We conclude that $K \cap S$ is a dilatate of $L \cap S$, modulo a set of λ_i -measure zero, for almost every $S \in \mathcal{G}(n, i)$. But λ_{n-1} is the

unique Borel-regular, rotation-invariant measure on S^{n-1} so that the measure of S^{n-1} is $n\kappa_n$. Thus,

$$\int_{S^{n-1}} f(u) du = \frac{n\kappa_n}{i\kappa_i} \int_{\mathcal{G}(n,i)} \int_{S^{n-1} \cap S} f(u) dudS$$

If we substitute $f = \rho_K$ on the left-hand side and $f = c_S \rho_L$ on the right-hand side, where c_S is a constant possibly depending on S , the above equality remains true. However, the same equation must also be true if $\rho_K(u) = c_S \rho_L(u)$ for all $u \in S^{n-1} \cap S$. We deduce that the constant c_S is independent of S . But this means that K is a dilatate of L , modulo a set of measure zero. $\square$

5.1.5 Log-Concavity of the Affine Quermassintegral

As a reference, we cite [Had57].

Definition 5.1.6. A *functional* on a subset $\mathcal{M} \subseteq \mathcal{K}(\mathbb{R}^n) \cup \{\emptyset\}$ with $\emptyset \in \mathcal{M}$ is a map $\phi : \mathcal{M} \rightarrow \mathbb{R}$ with $\phi(\emptyset) = 0$. We will call ϕ *homogeneous of degree 1* if $\phi(\alpha A) = \alpha \phi(A)$ for every $\alpha \in \mathbb{R}$ and $A \in \mathcal{M}$. Furthermore, ϕ is said to be *concave* if

$$\phi(\alpha A + \beta B) \geq \alpha \phi(A) + \beta \phi(B)$$

for every $A, B \in \mathcal{M}$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Finally, we call ϕ *strictly defined* if $\phi(A) > 0$ for all $A \in \mathcal{M}, A \neq \emptyset$.

For proving the log-concavity of the affine quermassintegrals, we need two lemmas.

Lemma 5.1.7. Let $\mathcal{M} \subseteq \mathcal{K}(\mathbb{R}^n) \cup \{\emptyset\}$ be a set of convex bodies, such that $\emptyset \in \mathcal{M}$ and $A \subseteq C$ for every $A \in \mathcal{M}$ and a suitable cube C . If ϕ is continuous on $\mathcal{M}$, then it is also uniformly continuous on $\mathcal{M}$, i.e. for every $\epsilon > 0$ there is an $\alpha > 0$ with $|\phi(A) - \phi(B)| < \epsilon$ whenever $\delta(A, B) < \alpha$.

Proof. Assume that the statement is false. There would be sequences $A_k, B_k, k \in \mathbb{N}$ in $\mathcal{M}$ with $\delta(A_n, B_n) \rightarrow 0$ but $|\phi(A_n) - \phi(B_n)| \geq \epsilon$ for a suitable choice of ϵ . Concerning the selection theorem of Blaschke, Theorem 3.0.1, we can assume $A_n \rightarrow A$ and $B_n \rightarrow B$ in $\mathcal{M}$, which yields $\delta(A, B) = 0$, so $A = B$. Since ϕ is continuous, we have $\phi(A_n) \rightarrow \phi(A)$ and $\phi(B_n) \rightarrow \phi(B)$, thus $|\phi(A_n) - \phi(B_n)| \rightarrow 0$, a contradiction. $\square$

Lemma 5.1.8. Let ϕ be a functional homogeneous of degree 1 on $\mathcal{K}(\mathbb{R}^n) \cup \{\emptyset\}$ with $\phi(K) > 0$ for every $K \neq \emptyset$. Then ϕ is concave if and only if there exists $p > 0$, such

that the function $\Psi(K) = \phi(A)^p$, $K \in \mathcal{K}(\mathbb{R}^n)$ fulfills

$$\Psi(\alpha K + \beta D) \geq 1$$

for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $\Psi(K) = \Psi(D) = 1$.

Proof. First, let ϕ be concave, and K, D, α , and β as above. Then $\phi(K) = \phi(D) = 1$ and therefore

$$\phi(\alpha K + \beta D) \geq \alpha \phi(K) + \beta \phi(D) = 1,$$

so $\Psi(\alpha K + \beta D) \geq 1$. Conversely, let $K, D \in \mathcal{K}(\mathbb{R}^n)$ be non-empty, and let $\phi(K) = 1/k$ and $\phi(D) = 1/d$ for $k, d > 0$. Since ϕ is homogeneous of degree 1, we have $\phi(kK) = \phi(dD) = 1$ and therefore $\Psi(kK) = \Psi(dD) = 1$. For given $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, we define

$$\xi = \frac{\alpha k}{\beta k + \alpha d} \quad \text{and} \quad \eta = \frac{\beta \alpha}{\beta k + \alpha d}.$$

Since $\xi + \eta = 1$, we can apply our assumption and get $\Psi(\xi kK + \eta dD) \geq 1$, thus $\phi(\xi kK + \eta dD) \geq 1$. Finally,

$$\phi(\alpha K + \beta D) \geq \frac{\beta k + \alpha d}{kd} = \alpha \phi(K) + \beta \phi(D).$$

□

Since we aim to deduce the concavity of functionals on all convex bodies from their concavity on lower-dimensional convex bodies, we introduce the following definition.

Definition 5.1.9. For a k -dimensional subspace $H \in \mathcal{G}(n, k)$ of $\mathbb{R}^n$, $1 \leq k \leq n-1$, we define $\mathcal{K}_{H,o}^n \subseteq \mathcal{K}(\mathbb{R}^n)$ as the set of all convex bodies in H , in particular, with non-empty relative interior. Furthermore, set

$$\mathcal{K}_{k,o}^n := \bigcup_{H \in \mathcal{G}(n,k)} \mathcal{K}_{H,o}^n.$$

Theorem 5.1.10. Let $p \geq -1$, $p \neq 0$, $1 \leq k \leq n-1$ and let ϕ be a strictly defined, continuous functional on $\mathcal{K}_{k,o}^n \cup \{\emptyset\}$, which is homogeneous of degree 1 and concave on $\mathcal{K}_{H,o}^n$ for every $H \in \mathcal{G}(n, k)$. Then, the map

$$\psi(A) = \left(\int_{\mathcal{G}(n,k)} (\phi(A|H))^{-p} dH \right)^{-1/p}, \quad A \in \mathcal{K}^n,$$

is a strictly defined, continuous, homogeneous of degree 1 and concave functional on $\mathcal{K}^n \cup \{\emptyset\}$.

Proof. Since ϕ and the map $H \mapsto A|H$ are continuous, the integrand is continuous. It is also bounded due to the fact that ϕ is strictly defined, and therefore, $\phi(A|H)$ remains uniformly positive for $H \in \mathcal{G}(n, k)$. Thus, the integral exists for $A \in \mathcal{K}^n$ and ψ is well-defined.

That ψ is homogeneous of degree 1 and strictly defined is readily seen. Since ϕ is continuous, we can apply Lemma 5.1.7, yielding the uniform continuity of $A \mapsto \phi(A|H)$ on every uniformly bounded $\mathcal{M} \subseteq \mathcal{K}^n$. Therefore, we conclude the continuity of ψ .

We are left to show that ψ is concave. Let $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Since ϕ is concave and for $A, B \in \mathcal{K}^n$ and $H \in \mathcal{G}(n, k)$

$$(\alpha A + \beta B)|H = \alpha(A|H) + \beta(B|H),$$

we obtain

$$\phi((\alpha A + \beta B)|H) \geq \alpha\phi(A|H) + \beta\phi(B|H).$$

In case of $-1 \leq p < 0$, the concavity of the power function $x \mapsto x^{-p}$ yields

$$\phi((\alpha A + \beta B)|H)^{-p} \geq \alpha\phi(A|H)^{-p} + \beta\phi(B|H)^{-p}.$$

If we set $\Psi(A) = \psi(A)^{-p}$, then

$$\Psi(\alpha A + \beta B) \geq \alpha\Psi(A) + \beta\Psi(B).$$

In particular, whenever A, B are such that $\Psi(A) = \Psi(B) = 1$, we get $\Psi(\alpha A + \beta B) \geq 1$. Applying Lemma 7.3 yields the concavity of ψ . If $p > 0$, the power function $x \mapsto x^{-p}$ is convex. Thus,

$$\phi((\alpha A + \beta B)|H)^{-p} \leq \alpha\phi(A|H)^{-p} + \beta\phi(B|H)^{-p}.$$

We now set $\Psi(A) = \psi(A)^p$ and get

$$\Psi(\alpha A + \beta B)^{-1} \geq \alpha\Psi(A)^{-1} + \beta\Psi(B)^{-1}.$$

Again, $\Psi(A) = \Psi(B) = 1$ implies $\Psi(\alpha A + \beta B) \geq 1$ and by Lemma 7.3 the concavity of ψ follows. $\square$

With this theorem in hand, the log-concavity of the affine quermassintegrals becomes an easy consequence.

Corollary 5.1.11. *Let $K, L \in \mathcal{K}^n$ and $0 \leq i \leq n-1$. Then*

$$\Phi_i(K+L)^{1/(n-i)} \leq \Phi_i(K)^{1/(n-i)} + \Phi_i(L)^{1/(n-i)}.$$

Proof. We will prove the equivalent statement that the function $K \mapsto \Phi_i(K)^{1/(n-i)}$ is concave. But that is just the case $p = n$ and $\phi(K) = \lambda_i(K|F)$, $F \in \mathcal{G}(n, i)$ of Theorem 5.1.10. □

5.2 Proof of the Affine Invariance of $\tilde{\Phi}_k$ and Φ_k

To do this, we introduce the concept of a multiplier function. The results of this section can be found in [Gri91].

Definition 5.2.1. Let G denote a topological group and M a topological G -space, meaning that we have a continuous map $G \times M \rightarrow M$ denoted by $(g, x) \rightarrow gx$, which satisfies $ex = x$ and $(g'g'')x = g'(g''x)$. Furthermore, let $Z(G, M)$ be the group of functions $\sigma : G \times M \rightarrow \mathbb{R}^+$ satisfying

$$\sigma(g'g'', x) = \sigma(g', g''x) \sigma(g'', x)$$

for every $g', g'' \in G, x \in M$ and let $B(G, M) \subseteq Z(G, M)$ be the subgroup of *multiplier functions* that are of the form

$$\sigma(g, x) = f(gx)/f(x),$$

where $f : M \rightarrow \mathbb{R}^+$ is a continuous function (not to be confused with the notion of multipliers as in Definition 4.0.3).

Theorem 5.2.2. *If $K \subseteq \mathbb{R}^n$ is a convex body and g an affine transformation, then $\tilde{\Phi}_k(K) = \tilde{\Phi}_k(gK)$ for $1 \leq k \leq n-1$.*

Proof. As shown in [FT71], the function

$$\sigma_k(g, H) \equiv \frac{\lambda_k(g(K \cap H))}{\lambda_k(K \cap H)},$$

where g is an affine transformation and $H \in \mathcal{G}(n, k)$, is a multiplier function as well as the Radon-Nikodym derivative

$$\sigma_{\mathcal{G}(n, k)}(g, H) = \frac{dg^{-1}H}{dH}.$$

Furthermore, it is shown in [FT71] that the following equation holds

$$\sigma_{\mathcal{G}(n, k)}(g, H) = \sigma_k(g, H)^{-n}.$$

We conclude the proof by computing

$$\begin{aligned} \tilde{\Phi}_{n-k}(gK) &= \kappa_n \left(\int_{H \in \mathcal{G}(n, k)} (\lambda_k(gK \cap H) / \kappa_k)^n dH \right)^{1/n} \\ &= \kappa_n \left(\int_{gH \in \mathcal{G}(n, k)} (\lambda_k(gK \cap gH) / \kappa_k)^n d(gH) \right)^{1/n} \\ &= \kappa_n \left(\int_{H \in \mathcal{G}(n, k)} \sigma_k(g, H)^n (\lambda_k(K \cap H) / \kappa_k)^n d(gH) \right)^{1/n} \\ &= \kappa_n \left(\int_{H \in \mathcal{G}(n, k)} (\lambda_k(K \cap H) / \kappa_k)^n dH \right)^{1/n} \\ &= \tilde{\Phi}_{n-k}(K). \end{aligned}$$

□

To prove affine invariance for Φ_k we again use the notion of multipliers. But first, we need to establish a projection volume identity.

Lemma 5.2.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body, $H \in \mathcal{G}(n, k)$, and $\phi \in SL_n$. Then*

$$\lambda_k(\phi K | H) = \lambda_k(K | \phi^t H) \sigma_k(\phi^t, H).$$

Proof. In case that ϕ is an orthogonal matrix, the above is trivial. If ϕ is a lower triangular matrix with respect to the coordinate system $\{H, H^\perp\}$, then $\phi^t H = H$ and the matrix of ϕ has the form

$$\phi = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}.$$

We deduce

$$\lambda_k(\phi K|H) = \lambda_k(A(K|H)) = \lambda_k(K|H)\sigma_k(A, H),$$

where the last equation follows from the definition of σ_k . This completes the proof for the lower triangular case. For the general case $\phi \in SL_n$ we may write ϕ as the product of a lower triangular matrix and an orthogonal matrix and apply the above arguments. $\square$

Theorem 5.2.4. *If $K \subseteq \mathbb{R}^n$ is a convex body and g an affine transformation, then $\Phi_k(K) = \Phi_k(gK)$ for $1 \leq k \leq n-1$.*

Proof. Applying Lemma 5.2.3 yields

$$\begin{aligned} \Phi_{n-k}(gK) &= \kappa_n \left(\int_{H \in \mathcal{G}(n,k)} \left(\frac{\lambda_k(gK|H)}{\kappa_k} \right)^{-n} dH \right)^{-1/n} \\ &= \kappa_n \left(\int_{H \in \mathcal{G}(n,k)} \left(\frac{\lambda_k(K|g^t H)}{\kappa_k} \right)^{-n} (\sigma_k(g^t, H))^{-n} dH \right)^{-1/n} \\ &= \kappa_n \left(\int_{H \in \mathcal{G}(n,k)} \left(\frac{\lambda_k(K|H)}{\kappa_k} \right)^{-n} (\sigma_k(g^t, g^{-t} H))^{-n} d(g^{-t} H) \right)^{-1/n} \\ &= \kappa_n \left(\int_{H \in \mathcal{G}(n,k)} \left(\frac{\lambda_k(K|H)}{\kappa_k} \right)^{-n} (\sigma_k(g^t, g^{-t} H))^{-n} (\sigma_k(g^{-t}, H))^{-n} dH \right)^{-1/n}. \end{aligned}$$

And finally, due to the multiplier property of σ_k ,

$$\begin{aligned} \Phi_{n-k}(gK) &= \kappa_n \left(\int_{H \in \mathcal{G}(n,k)} \left(\frac{\lambda_k(K|H)}{\kappa_k} \right)^{-n} dH \right)^{-1/n} \\ &= \Phi_{n-k}(K). \end{aligned}$$

$\square$

6 Isoperimetric Inequality for Dual Affine Quermassintegrals

With the geometric and stochastic preliminaries in place, we are now ready to prove the isoperimetric inequality for dual affine quermassintegrals. As a reference, we cite [Gar06].

We begin with the definition of the useful functions $g_{m,k}$ and then proceed by showing that $g_{m,k}$ decreases under Steiner symmetrization.

Definition 6.0.1. For $1 \leq k \leq n$ the k -dimensional simplex in $\mathbb{R}^n$ with vertices $o, p_1, \dots, p_k$ is denoted by $[o, p_1, \dots, p_k]$. Let C be a compact set with $C \subseteq S$ for some $S \in \mathcal{G}(n, k)$. For each $m \in \mathbb{N}$ we set

$$g_{m,k}(C) = \int_C \cdots \int_C \lambda_k([o, p_1, \dots, p_k])^m dp_1 \cdots dp_k.$$

For $k = n$ we simply write

$$g_m(C) = g_{m,n}(C) = \int_C \cdots \int_C \lambda_n([o, p_1, \dots, p_n])^m dp_1 \cdots dp_n.$$

Lemma 6.0.2. For a convex body $K \subseteq \mathbb{R}^n$ and $u \in S^{n-1}$, we have

$$g_m(K) \geq g_m(S_u K), \quad m \in \mathbb{N},$$

with equality if and only if the midpoints of all chords of K parallel to u lie in a hyperplane containing the origin.

Proof. Let $1 \leq j \leq n$ and $p_j \in S_u K$. For every j , we find unique $y_j \in u^\perp$ and $s_j \in \mathbb{R}$ with $p_j = y_j + s_j u$, and unique $t_j \in \mathbb{R}^n$ such that $q_j = y_j + t_j u \in K$ transforms to p_j by the process of Steiner symmetrization in $u^\perp$. With l_u being the line spanned by u , let z_j be the center of the line segment $K \cap (l_u + y_j)$. Thus $q_j \in K \cap (l_u + y_j)$. Furthermore, let us denote by $q'_j = y_j + t'_j u$ the reflection of q_j in z_j on the line l_u . We obtain $s_j = (t_j - t'_j)/2$, $1 \leq j \leq n$.

Without loss of generality, we can assume that $u^\perp$ is the hyperplane $\{x \in \mathbb{R}^n : x_n = 0\}$. We write $p_{k,j}$ for the k th component of p_j and similarly for the other vectors. The fact that u is parallel to the x_n axis yields $p_{k,j} = y_{k,j} = q_{k,j} = q'_{k,j}$ for $1 \leq k \leq n-1$ and $p_{n,j} = s_j$, $q_{n,j} = t_j$, $q'_{n,j} = t'_j$. With that and the formula for the volume of a simplex, we obtain

$$\begin{aligned} \lambda_n([o, p_1, \dots, p_n]) &= \frac{1}{n!} \left| \det \begin{pmatrix} p_{1,1} & \dots & p_{1,n} \\ \vdots & \ddots & \vdots \\ p_{n-1,1} & \dots & p_{n-1,n} \\ p_{n,1} & \dots & p_{n,n} \end{pmatrix} \right| \\ &= \frac{1}{n!} \left| \det \begin{pmatrix} y_{1,1} & \dots & y_{1,n} \\ \vdots & \ddots & \vdots \\ y_{n-1,1} & \dots & y_{n-1,n} \\ s_{n,1} & \dots & s_{n,n} \end{pmatrix} \right| \end{aligned}$$

Because of $s_j = (t_j - t'_j)/2$ and the linearity of the determinant, the last expression is equal to

$$\begin{aligned} &\frac{1}{2n!} \left| \det \begin{pmatrix} q_{1,1} & \dots & q_{1,n} \\ \vdots & \ddots & \vdots \\ q_{n-1,1} & \dots & q_{n-1,n} \\ t_{n,1} & \dots & t_{n,n} \end{pmatrix} - \det \begin{pmatrix} q'_{1,1} & \dots & q'_{1,n} \\ \vdots & \ddots & \vdots \\ q'_{n-1,1} & \dots & q'_{n-1,n} \\ t'_{n,1} & \dots & t'_{n,n} \end{pmatrix} \right| \\ &\leq \frac{1}{2} (\lambda_n([o, q_1, \dots, q_n]) + \lambda_n([o, q'_1, \dots, q'_n])). \end{aligned}$$

Applying Jensen's inequality leads to

$$\lambda_n([o, p_1, \dots, p_n])^m \leq \frac{1}{2} \lambda_n([o, q_1, \dots, q_n])^m + \frac{1}{2} \lambda_n([o, q'_1, \dots, q'_n])^m.$$

Finally we get

$$\begin{aligned}
 g_m(S_u K) &= \int_{S_u K} \cdots \int_{S_u K} \lambda_n([o, p_1, \dots, p_n])^m dp_1 \cdots dp_n \\
 &\leq \frac{1}{2} \int_K \cdots \int_K \lambda_n([o, q_1, \dots, q_n])^m dq_1 \cdots dq_n \\
 &\quad + \frac{1}{2} \int_K \cdots \int_K \lambda_n([o, q'_1, \dots, q'_n])^m dq'_1 \cdots dq'_n \\
 &= \int_K \cdots \int_K \lambda_n([o, q_1, \dots, q_n])^m dq_1 \cdots dq_n \\
 &= g_m(K).
 \end{aligned}$$

It remains to prove the equality cases. Let us assume equality. Since the volume of a simplex is a function of its vertices, the previous inequalities must hold as equalities for every choice of $p_j, 1 \leq j \leq n$, which yields $\det(q_{jk}) = -\det(q'_{jk})$. Since

$$\lambda_n([o, z_1, \dots, z_n]) = |\det(z_{jk})|/n!$$

where $z_{jk} = y_{jk}$ for $1 \leq k \leq n-1$ and $z_{jn} = (t_j + t'_j)/2$ leads to

$$\lambda_n([o, z_1, \dots, z_n]) = 0.$$

In particular the points $z_j, 1 \leq j \leq n$, lie in a hyperplane containing the origin. $\square$

Lemma 6.0.3. *Let $K_m, m \in \mathbb{N}$ be a sequence in $\mathcal{K}^n$ and $K \in \mathcal{K}^n$ its limit. If there is a convex body $\tilde{K}$ with $S_u K_m \rightarrow \tilde{K} \in \mathcal{K}^n$ for a fixed direction $u \in S^{n-1}$, then $\tilde{K} \subseteq S_u K$.*

Proof. Fix $x \in \tilde{K}$. There is a sequence x_m in $S_u K_m$ with $x_m \rightarrow x$. Indeed, if we define x_m as the unique point in $S_u K_m$ with least distance to x , we obtain $|x - x_m| \leq \delta(K, K_m) \rightarrow 0$, where here δ denotes the Hausdorff metric. Let σx_m be the reflection of x_m with respect to $u^\perp$, then we can choose $y_m, z_m \in (l_u + x_m) \cap K_m$ with $|y_m - z_m| \geq |x_m - \sigma x_m|$. If we replace y_m and z_m with suitable subsequences, we find that $y_m \rightarrow y$ and $z_m \rightarrow z$ for some $y, z \in G_x \cap K$. Thus $|y - z| \geq |x - \sigma x|$ and therefore $x \in S_u K$. $\square$

The following theorem demonstrates that any convex body converges to a ball through successive suitable Steiner symmetrizations.

Lemma 6.0.4. *Suppose that $K \subseteq \mathbb{R}^n$ is a convex body. Then there exist directions $u_m \in S^{n-1}, m \in \mathbb{N}$, such that the sequence of successive Steiner symmetrals $K_m = S_{u_m} S_{u_{m-1}} \cdots S_{u_1} K$ converge with respect to the Hausdorff metric to a centered ball with the same volume as K .*

Proof. For $L \in \mathcal{K}^n$ let us denote by $R(L)$ the radius of the smallest ball centered at the origin that contains L . Furthermore let $\mathcal{S}(K)$ be the set of convex bodies obtained by finitely many Steiner symmetrizations of K in various directions and

$$R_0 = \inf\{R(K') : K' \in \mathcal{S}(K)\}.$$

Since $\mathcal{S}(K)$ is bounded, we obtain a sequence K_m in $\mathcal{S}(K)$ with

$$\lim_{m \rightarrow \infty} R(K_m) = R_0 \quad \text{and} \quad \lim_{m \rightarrow \infty} K_m = K_0 \in \mathcal{K}^n.$$

The continuity of R implies $R(K_0) = R_0$, and since Steiner symmetrization preserves volume we have $\lambda_n(K) = \lambda_n(K_0)$.

We will prove that K_0 is the (closed) ball B_0 centered at the origin with radius R_0 . Suppose, for the sake of contradiction, that there is $z \in B_0$ with $z \notin K_0$. Let C be a ball centered at z with $C \cap K_0 = \emptyset$. For an arbitrary direction $u \in S^{n-1}$ we have

$$B_0 \cap C \cap S_u K_0 = \emptyset \quad \text{and} \quad B_0 \cap \sigma C \cap S_u K_0 = \emptyset,$$

where as in the proof before σ denotes the reflection with respect to $u^\perp$. We can cover B_0 by finitely many balls $C_1, \dots, C_j$ which are congruent to C . Let us write $H_i = u_i^\perp$ for the hyperplane, which cuts C and C_i in half. The body $S^*(K_0) = S_{u_j} \dots S_{u_1} K_0$ satisfies $S^*(K_0) \cap B_0 = \emptyset$. Moreover, since $S^*(K_0)$ is compact, we obtain $R(S^*(K_0)) < R_0$. Without loss of generality we can assume $S^*(K_m) \rightarrow \tilde{K} \in \mathcal{K}^n$, and by Lemma 6.0.3 we get $\tilde{K} \subseteq S^*(K_0)$. Thus $R(\tilde{K}) < R_0$ and $R(S^*(K_m)) < R_0$ for m sufficiently large. Since $S^*(K_m) \in \mathcal{S}(K)$ we get a contradiction. $\square$

Corollary 6.0.5. *Suppose that $K \subseteq \mathbb{R}^n$ is a convex body and $r > 0$ such that $\lambda_n(K) = \lambda_n(rB^n)$. Then for $m \in \mathbb{N}$*

$$g_m(K) \geq g_m(rB^n),$$

where equality holds if and only if K is a centered ellipsoid.

Proof. Lemma 6.0.4 shows that the sequence K_j of successive Steiner symmetrals of K converge to a ball rB^n with $\lambda_n(rB^n) = \lambda_n(K)$. By Lemma 6.0.2 we get

$$g_m(K) \geq g_m(K_1) \geq \dots \geq g_m(K_j).$$

Taking the limit $j \rightarrow \infty$ leads to the claimed inequality. The equality cases follow directly from those of Lemma 6.0.2 and Theorem 3.3.1. $\square$

We are now able to prove the isoperimetric inequality for dual affine quermassintegrals.

Theorem 6.0.6. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $1 \leq i \leq n-1$. Then*

$$\kappa^{(n-i)/n} \lambda_n(K)^{i/n} \geq \tilde{\Phi}_{n-i}(K).$$

For $i > 1$, equality holds if and only if K is a centered ellipsoid, and for $i = 1$ if and only if K is centered.

Proof. We will prove the following equivalent version of the inequality

$$\lambda_n(K)^i \geq \frac{\kappa_n^i}{\kappa_i^n} \int_{\mathcal{G}(n,i)} \lambda_i(K \cap S)^n dS.$$

Therefor we fix $S \in \mathcal{G}(n, i)$. By Corollary 6.0.5 applied to $K \cap S$ with dimension i and $m = n - i$ we obtain

$$g_{n-i,i}(K \cap S) \geq g_{n-i,i}(r_S B^n \cap S),$$

for $r_S > 0$ with $\lambda_i(K \cap S) = \lambda_i(r_S B^n \cap S)$. Furthermore, according to Corollary 4.0.9, we have

$$\lambda_n(K)^i \geq c \int_{\mathcal{G}(n,i)} g_{n-i,i}(r_S B^n \cap S) dS$$

and

$$\lambda_n(r_S B^n)^i = c \int_{\mathcal{G}(n,i)} g_{n-i,i}(r_S B^n \cap S) dS = c g_{n-i,i}(r_S B^n \cap S),$$

with $c \in \mathbb{R}$ only depending on n and i . The last equality holds because the integral is taken over a constant function. Combining these yields

$$\begin{aligned} \lambda_n(K)^i &\geq \int_{\mathcal{G}(n,i)} \lambda_n(r_S B^n)^i dS \\ &= \frac{\kappa_n^i}{\kappa_i^n} \int_{\mathcal{G}(n,i)} \lambda_i(r_S B^n \cap S)^n dS \\ &= \frac{\kappa_n^i}{\kappa_i^n} \int_{\mathcal{G}(n,i)} \lambda_i(K \cap S)^n dS. \end{aligned}$$

It remains to prove the equality cases. Since all the integrands from above are continuous in S , equality holds if and only if $g_{n-i,i}(K \cap S) = g_{n-i,i}(r_S B^n \cap S)$ for all $S \in \mathcal{G}(n, i)$.

This is, according to Corollary 6.0.5, equivalent to $K \cap S$ being a centered ellipsoid for all $S \in \mathcal{G}(n, i)$. By Lemma 3.3.5 if $1 < i \leq n - 1$ this holds precisely when K is a centered ellipsoid. In case of $i = 1$, it is easy to check that equality holds if and only if K is centered. $\square$

As a corollary of the preceding theorem, we obtain the Busemann intersection inequality involving the intersection body IK of K .

Corollary 6.0.7. *Suppose that $K \subseteq \mathbb{R}^n$ is a convex body with the origin in its interior. Then*

$$\lambda_n(IK) \leq \frac{\kappa_{n-1}^n}{\kappa_n^{n-2}} \lambda_n(K)^{n-1},$$

where equality holds if and only if K is a centered ellipsoid.

Proof. By the polar coordinate formula for volume (3.1), we obtain

$$\begin{aligned} \lambda_n(IK) &= \frac{1}{n} \int_{S^{n-1}} \rho_{IK}(u)^n du = \frac{1}{n} \int_{S^{n-1}} \lambda_{n-1}(K \cap u^\perp)^n du \\ &= \kappa_n \int_{\mathcal{G}(n, n-1)} \lambda_{n-1}(K \cap S)^n dS. \end{aligned}$$

Applying Theorem 6.0.6 with $i = n - 1$ yields the claimed inequality. The equality cases follow directly from those of Theorem 6.0.6. $\square$

6.0.8 Relation to the Isoperimetric Inequality for Dual Quermassintegrals

Dual quermassintegrals were introduced by Lutwak in [Lut75] in a slightly different way, and the formula in the following definition was later demonstrated by Lutwak in [Lut79].

Definition 6.0.9. Let $K \subseteq \mathbb{R}^n$ be a convex body, and $1 \leq k \leq n$. The k -th dual quermassintegral of K is defined by

$$\tilde{W}_{n-k}(K) = \frac{\kappa_n}{\kappa_k} \int_{\mathcal{G}(n, k)} \lambda_k(K \cap S) dS.$$

Remark 6.0.10. Note that the $\tilde{W}_k$ are invariant under rotations, see [Lut75], whereas the $\tilde{\Phi}_k$ are invariant under affine transformations, see Theorem 5.2.2.

By Jensen Inequality and the fact $\tilde{W}_k(B_K) = \tilde{\Phi}_k(B_K)$ it is easy to see that Theorem 6.0.6 implies the isoperimetric inequality for dual affine quermassintegrals.

Theorem 6.0.11. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $1 \leq k \leq n$. Then*

$$\tilde{W}_k(K) \leq \tilde{W}_k(B_K)$$

with equality if and only if K is a ball.

7 Isoperimetric Inequalities for Affine Quermassintegrals

In this chapter, we delve into the proof of the isoperimetric inequality for Φ_k , a result that was recently established in 2023, see [MY23]. To summarize, we will prove that the function Φ_k decreases under Steiner symmetrization and then, using a standard argument, we will derive the desired inequality. Since the cases of equality involve significantly more work, we will cover them in Chapter 8.

7.1 Preliminaries

In this section, we will introduce the basic definitions needed for what follows and prove some lemmas.

Definition 7.1.1. Let $u \in S^{n-1}$ and $T_t^u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ denote the projection onto $\mathbb{R}^n$ parallel to $e_{n+1} + tu$, i.e., T_t^u is a projection which is the identity on $\mathbb{R}^n$ and sends e_{n+1} to $-tu$. A *shadow system in the direction of u* is a family $K(t) \in \mathcal{K}(\mathbb{R}^n), t \in \mathbb{R}$ such that there exists $\tilde{K} \in \mathcal{K}^{n+1}$ with $K(t) = T_t^u(\tilde{K})$.

Definition 7.1.2. Let $K \subseteq \mathbb{R}^n$ be a convex body and $u \in S^{n-1}$ a direction. For a set $A \subseteq \mathbb{R}^n$ and $y \in u^\perp$, we denote by $A^{(y)}$ the one-dimensional section of A in the direction of u over y . Furthermore, we define the following *linear reflection shadow system associated to K in the direction of u*

$$(K_u(t))^{(y)} = \frac{1+t}{2}K^{(y)} + \frac{1-t}{2}(R_u K)^{(y)}, \quad \text{for } t \in \mathbb{R} \text{ and } y \in u^\perp, \quad (7.1)$$

where R_u denotes the reflection about $u^\perp$.

We obtain the following lemma in a straightforward manner.

Lemma 7.1.3. Suppose that $K \subseteq \mathbb{R}^n$ is a convex body, $u \in S^{n-1}$ is a direction, and $K_u(t), t \in \mathbb{R}$ are defined by (7.1). Then the following holds

- (i) $R_u(K_u(t)) = K_u(-t)$ for all $t \in \mathbb{R}$,
- (ii) in particular $K_u(1) = K$ and $K_u(-1) = R_u K$,
- (iii) $K_u(0) = S_u K$,
- (iv) $\lambda_n(K_u(t)) = \lambda_n(K)$ for $t \in [-1, 1]$ and
- (v) $t \mapsto K_u(t), t \in \mathbb{R}$ is a continuous map.

Lemma 7.1.4. *If $\{K(t)\}_{t \in \mathbb{R}}$ is a shadow system in the direction of u , and $y \in u^\perp$, then*

$$(K((1 - \alpha)t_0 + \alpha t_1))^{(y)} \subseteq (1 - \alpha)(K(t_0))^{(y)} + \alpha(K(t_1))^{(y)} \quad (7.2)$$

for all $\alpha \in \mathbb{R}$ and $t_0, t_1 \in \mathbb{R}$.

Proof. Since for $t_0 = t_1$ there is nothing to prove, let $t_0 \neq t_1$ and set $K_i = K(t_i), i = 0, 1$. Shepard showed that this implies $P_{u^\perp} K_0 = P_{u^\perp} K_1$, see reference [She64], (4). Furthermore, also as proved by Shepard, there is a maximal shadow system $\{K_{\max}(t)\}_{t \in \mathbb{R}}$ implying $\bar{K}(t) \subseteq K_{\max}(t), t \in \mathbb{R}$ for every shadow system $\{\bar{K}(t)\}_{t \in \mathbb{R}}$, and having the form $K_{\max}(t) = T_t^u(\tilde{K}_{\max})$, where

$$\tilde{K}_{\max} := (T_{t_0}^u)^{-1}(K_0) \cap (T_{t_1}^u)^{-1}(K_1),$$

or equivalently,

$$(K_{\max}((1 - \alpha)t_0 + \alpha t_1))^{(y)} := (1 - \alpha)(K_0)^{(y)} + \alpha(K_1)^{(y)}, \quad \text{for } \alpha \in \mathbb{R}, y \in u^\perp$$

In particular, $K(t) \subseteq K_{\max}(t)$ for all $t \in \mathbb{R}$, which proves the assertion. $\square$

Definition 7.1.5. Let $1 \leq k \leq n - 1$, $E \in \mathcal{G}(n, k - 1)$ and $x \in \mathbb{R}^n$. For $K \in \mathcal{K}^n$ we define

$$|P_{E \wedge x} K| := |P_{E^\perp} x| \lambda_k(P_{\text{span}(E, x)} K),$$

and in case of $E = \{0\}$, we will simply write $|P_x K| := |x| \lambda_1(P_{\text{span}(x)} K)$.

Lemma 7.1.6. *With the notation of the previous definition, the map defined by*

$$\mathcal{G}(n, k - 1) \times \mathbb{R}^n \rightarrow \mathbb{R}^+, \quad (E, x) \mapsto |P_{E \wedge x} K|$$

is continuous.

Proof. If we apply Theorem 3.0.6 to $F = \text{span}(E, x)$, we obtain for $x \notin E$,

$$\binom{n}{k} V\left(K, k; |P_{E^\perp} x|^{\frac{1}{n-k}} B_{E^\perp \cap x^\perp}, n-k\right) = \lambda_k(P_{\text{span}(E, x)} K) |P_{E^\perp} x| \lambda_k(B_{F^\perp}),$$

where $V(\dots)$ denotes the mixed volume as defined in Theorem 3.0.3. Therefore, if $x \notin E$, then

$$|P_{E \wedge x} K| = \frac{\binom{n}{k}}{\kappa_{n-k}} V\left(K, k; |P_{E^\perp} x|^{\frac{1}{n-k}} B_{E^\perp \cap x^\perp}, n-k\right).$$

Since for $x \in E$ both sides are zero, the above holds for all $x \in \mathbb{R}^n$. Obviously, the map $(E, x) \mapsto |P_{E^\perp} x|^{\frac{1}{n-k}} B_{E^\perp \cap x^\perp}$ is continuous in (E_0, x_0) whenever $x_0 \notin E_0$. In case $x_0 \in E_0$, it is also continuous, since $|P_{E^\perp} x|^{\frac{1}{n-k}}$ converges to zero as $(E, x) \rightarrow (E_0, x_0)$. By the continuity of mixed volume, we also get that $(E, x) \mapsto |P_{E \wedge x} K|$ is continuous. $\square$

Definition 7.1.7. Let $K \subseteq \mathbb{R}^n$ be a convex body and $E \in \mathcal{G}(n, k-1)$ with $1 \leq k \leq n-1$. We call the set

$$L_E(K) = \left\{x \in E^\perp : |P_{E \wedge x} K| \leq 1\right\} \subseteq E^\perp$$

the E -projected polar body of K .

As can easily be shown, $L_E(K)$ is origin-symmetric, closed, and contains the origin in its interior. Since K has non-empty interior $L_E(K)$, is bounded in $E^\perp$ and hence compact. Thus, the following Lemma shows that $L_E(K)$ is convex, i.e. a convex body in $E^\perp$.

Lemma 7.1.8. Let $K \subseteq \mathbb{R}^n$ be a convex body and $E \in \mathcal{G}(n, k-1)$ with $1 \leq k \leq n-1$. Then, the map $x \mapsto |P_{E \wedge x} K|$, $x \in \mathbb{R}^n$ is convex and its level set $L_E(K)$ is also convex.

Proof. If $E = \{0\}$, clearly $|P_x K| = h_K(x) + h_K(-x)$ is convex. So let $E \neq \{0\}$. For $w \in P_E K$, we define

$$K^w := (K - w) \cap E^\perp.$$

We obtain for $x \in E^\perp$,

$$P_{\text{span}(E, x)} K = \bigcup_{w \in P_E K} (w + P_{\text{span}(x)} K^w).$$

By Fubini's Theorem and homogeneity, we get for $x \in E^\perp$,

$$|P_{E \wedge x} K| = \int_{P_E K} |P_x K^w| dw = \int_{P_E K} (h_{K^w}(x) + h_{K^w}(-x)) dw. \quad (7.3)$$

Observe that the above gives a useful expression for $\|x\|_{L_E(K)}$. Since $|P_{E \wedge x} K|$ is only depending on $P_{E^\perp} x$, it suffices to show convexity for $x \in E^\perp$. But this is, by the above expression, now clear since the support functions h_{K^w} are convex. $\square$

Definition 7.1.9. Suppose that $E \in \mathcal{G}(u^\perp, k-1)$, with $1 \leq k \leq n-1$. We define, for a convex body $K \subseteq \mathbb{R}^n$,

$$V_{k,u} = \left\{ (E, x) : E \in \mathcal{G}(u^\perp, k-1), x \in E^\perp \right\},$$

$$L_{k,u}(K) = \left\{ (E, x) : E \in \mathcal{G}(u^\perp, k-1), x \in L_E(K) \right\} \subseteq V_{k,u},$$

and for $s \in \mathbb{R}$

$$L_{E,u,s}(K) = \left\{ y \in E^\perp \cap u^\perp : |P_{E \wedge (y+su)} K| \leq 1 \right\}.$$

The set $L_{k,u}(K)$ is called the *k-dimensional projection rolodex of K relative to $u^\perp$* .

Remark 7.1.10. $L_{E,u,s}(K)$ is the section of $L_E(K)$ perpendicular to u at height $s \in \mathbb{R}$, and hence convex. Due to $\dim(E^\perp \cap u^\perp) = n-k$, the dimension of $L_{E,u,s}(K)$ is $n-k$. By Brunn's concavity principle, Theorem 3.0.5, the map $s \mapsto \lambda_{n-k}(L_{E,u,s}(K))^{\frac{1}{n-k}}$ is concave on its support, and so we obtain that $s \mapsto \lambda_{n-k}(L_{E,u,s}(K))$ is measurable on $\mathbb{R}$.

Furthermore, the set $V_{k,u}$ is closed, and for a convex body $K \subseteq \mathbb{R}^n$, Lemma 7.1.6 implies that the map

$$V_{k,u} \rightarrow \mathbb{R}_+, \quad (E, x) \mapsto |P_{E \wedge x} K|$$

is continuous, and therefore the sub-level set

$$\{(E, x) \in V_{k,u} : |P_{E \wedge x} K| \leq 1\} = \left\{ (E, x) : E \in \mathcal{G}(u^\perp, k-1), x \in L_E(K) \right\} = L_{k,u}(K)$$

is closed in $V_{k,u}$. In case of $\text{int } K \neq \emptyset$, the *k-dimensional projection rolodex* $L_{k,u}(K)$ is bounded and therefore compact.

7.2 Convexity of Projections of Shadow System

In this section we will prove that the function $(y, t) \mapsto |P_{E \wedge (y+su)} K(t)|$ is jointly convex.

Definition 7.2.1. For $x_1, \dots, x_k \in \mathbb{R}^n$, let $\Delta(x_1, \dots, x_k)$ denote the *k-dimensional Lebesgue measure* of the parallelepiped $[0, x_1] + \dots + [0, x_k]$. Furthermore, we write

$$|P_{x_1 \wedge \dots \wedge x_k} K| := \lambda_k(P_{\text{span}\{x_1, \dots, x_k\}} K) \Delta(x_1, \dots, x_k) \quad K \in \mathcal{K}^n.$$

Considering Definition 7.1.5, if $E \in \mathcal{G}(n, k-1)$ is spanned by an orthonormal basis $\{x_1, \dots, x_{k-1}\}$ we have

$$|P_{E \wedge x_k} K| = |P_{x_1 \wedge \dots \wedge x_{k-1} \wedge x_k} K|.$$

We need the following linear algebra lemma.

Lemma 7.2.2. *Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map, and $E \subseteq \mathbb{R}^n$ a subspace such that $T^*|_E : E \rightarrow T^*E$ is injective. Then, there is a linear map $S : T^*E \rightarrow E$ with*

$$P_E \circ T = S \circ P_{T^*E}.$$

Proof. Since the operator defined as $M = T^* \circ P_E \circ T$ is self-adjoint, we obtain that $\text{Im } M \subseteq T^*E$ is an invariant subspace with respect to M . Because of $\text{Ker } M = (\text{Im } M)^\perp \supseteq (T^*E)^\perp$, we find a self-adjoint linear map $N : T^*E \rightarrow T^*E$ such that

$$T^* \circ P_E \circ T = M = N \circ P_{T^*E},$$

The claimed equality now holds for $S = (T^*|_E)^{-1} \circ N$. $\square$

Lemma 7.2.3. *Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear map and $x_1, \dots, x_k \in \mathbb{R}^n$. Then, for a compact set $A \subseteq \mathbb{R}^n$, we have*

$$|P_{x_1 \wedge \dots \wedge x_k} T(A)| = |P_{T^*(x_1) \wedge \dots \wedge T^*(x_k)} A|.$$

Proof. If the x_i are linearly dependent, both sides are zero, so we may assume linear independence. Let $E = \text{span}\{x_1, \dots, x_k\}$. Since $P_E \circ T$ maps onto E if and only if $(\text{Im } T)^\perp \cap E = \{0\}$ and, therefore, precisely if $\text{Ker } T^* \cap E = \{0\}$, we may also assume $T^*|_E$ to be injective, because otherwise both sides would be zero again. Applying Lemma 7.2.2, yields

$$\lambda_k(P_E \circ T(A)) = \lambda_k(S \circ P_{T^*E}(A)) = |\det_{T^*E \rightarrow E} S| \cdot \lambda_k(P_{T^*E}(A)),$$

where $|\det_{P \rightarrow Q} L|$ denotes the Jacobian of a linear map $L : P \rightarrow Q$ between two isomorphic vector spaces P and Q . Note that $|\det_{P \rightarrow Q} L| = \sqrt{\det_{P \rightarrow P} L^* L} = \sqrt{\det_{Q \rightarrow Q} L L^*}$. Combining this with again Lemma 7.2.2 and $SS^* = SP_{T^*E}S^*$, we obtain

$$|\det_{T^*E \rightarrow E} S| = \sqrt{\det_{E \rightarrow E} SS^*} = \sqrt{\det_{E \rightarrow E} P_E T T^* P_E} = |\det_{E \rightarrow T^*E}(T^*|_E)|,$$

and therefore

$$\begin{aligned}
 & |P_{x_1 \wedge \dots \wedge x_k} T(A)| \\
 &= \Delta(x_1, \dots, x_k) \lambda_k(P_E T(A)) \\
 &= \Delta(x_1, \dots, x_k) \cdot |\det_{E \rightarrow T^*E} T^*|_E \cdot \lambda_k(P_{T^*E}(A)) \\
 &= \Delta(T^*x_1, \dots, T^*x_k) \cdot \lambda_k(P_{T^*E}(A)) \\
 &= |P_{T^*(x_1) \wedge \dots \wedge T^*(x_k)} A|.
 \end{aligned}$$

□

Definition 7.2.4. Let V, W be two vector spaces over $\mathbb{R}$. A function $f : V \times W \rightarrow \mathbb{R}$ is said to be *jointly convex* if

$$f(\lambda v + (1 - \lambda)v', \lambda w + (1 - \lambda)w') \leq \lambda f(v, w) + (1 - \lambda)f(v', w')$$

for every $\lambda \in [0, 1]$ and $v, v' \in V, w, w' \in W$.

Proposition 7.2.5. Suppose that $K(t), t \in \mathbb{R}$ is a shadow system in the direction of $u \in S^{n-1}$, and let $E \in \mathcal{G}(u^\perp, k - 1)$. For fixed $s \in \mathbb{R}$, the function

$$u^\perp \times \mathbb{R} \rightarrow \mathbb{R}_+, \quad (y, t) \mapsto |P_{E \wedge (y + su)} K(t)|$$

is jointly convex.

Proof. Let $\tilde{K} \in \mathcal{K}(\mathbb{R}^{n+1})$ be such that $K(t) = T_t^u(\tilde{K})$. Recall that T_t^u is a projection which is the identity on $\mathbb{R}^n$ and sends e_{n+1} to $-tu$, see Definition 7.1.1. It is easy to see that $(T_t^u)^*$ is the identity on $u^\perp$ and therefore on $E \subseteq u^\perp$. Furthermore $(T_t^u)^*(u) = u - te_{n+1}$ and by Lemma 7.2.3

$$|P_{E \wedge (su + y)} K(t)| = |P_{E \wedge (su - ste_{n+1} + y)} \tilde{K}|,$$

for $y \in u^\perp$. (for the case $E = \{0\}$, see also reference [CG06], (5)). Since the map $(t, y) \mapsto su - ste_{n+1} + y$ is affine for every $s \in \mathbb{R}$, Lemma 7.1.8 yields that $|P_{E \wedge (y + su)} K(t)|$ is jointly convex. □

7.3 A Blaschke-Petkantschin-Type Formula

Now, we will invoke the stochastic result from Chapter 4 to prove the following theorem, which will be essential for the proof of the isoperimetric inequality.

Theorem 7.3.1. *Let $u \in S^{n-1}$. Then there is a constant $c_{n,k} > 0$, only depending on n and k , with*

$$c_{n,k} \int_{\mathcal{G}(n,k)} f(F) dF = \int_{\mathcal{G}(u^\perp, k-1)} \int_{S^{n-1} \cap E^\perp} f(\text{span}(E, \theta)) \cdot |\theta \cdot u|^{k-1} d\theta dE$$

for every measurable function $f : \mathcal{G}(n, k) \rightarrow \mathbb{R}^+$.

Proof. The proof is an application of Theorem 4.0.11, with $q = 1, m = s_1 = k, s_0 = n - 1$ and $L_0 = u^\perp$. Indeed, the inner integral of (4.3) has the form

$$\int_{\mathcal{G}(E, k)} f(F) [F, u^\perp]^{k-1} \sigma_{E, k}(dF)$$

whereas in Definition 7.6.7 $G(E, k)$ denotes the set $\{F \in \mathcal{G}(n, k) : F \supset E\}$ equipped with its uniform Haar probability measure, denoted by $\sigma_{E, k}$, which is invariant under the action of $SO_{n, E} := \{U \in SO_n : UE = E\}$, and $[F, u^\perp]$ is the subspace determinant also defined in Definition 7.6.7. Every $F \in G(E, k)$ can be written as $F = \text{span}(E, \theta)$ for some $\theta \in S^{n-1} \cap E^\perp$. That, and because of the invariance of the Haar measure $d\theta$ on $S^{n-1} \cap E^\perp$ under the action of $SO_{n, E}$, the uniqueness of the Haar measure up to a multiplicative constant implies that we can rewrite the above integral as

$$\frac{1}{\lambda_{n-k}(S^{n-1} \cap E^\perp)} \int_{S^{n-1} \cap E^\perp} f(\text{span}(E, \theta)) [\text{span}(E, \theta), u^\perp]^{k-1} d\theta.$$

By definition we have $[\text{span}(E, \theta), u^\perp] = [\text{span}(E, \theta)^\perp, \text{span}(u)]$, where, by Definition 7.6.7, $[L_1, L_2]$ is (when $b = \dim L_1 + \dim L_2 \leq n$) the b -dimensional volume of the parallelepiped spanned by the union of any orthonormal bases of L_1 and L_2 . Finally, due to $E \subseteq u^\perp$ and $\theta \in E^\perp$,

$$[\text{span}(E, \theta)^\perp, \text{span}(u)] = [\text{span}(E, \theta)^\perp \oplus E, \text{span}(u)] = [\text{span}(\theta)^\perp, \text{span}(u)] = |\theta \cdot u|.$$

The specific value of $c_{n,k}$ is not important for us, but can be found in [SW08], Theorem 7.2.6. $\square$

7.4 $\Phi_k(K)$ Decreases under Steiner Symmetrization

We start by defining a Borel measure on $V_{k, u}$.

Definition 7.4.1. For $u \in S^{n-1}$ and $1 \leq k \leq n-1$, we define the Borel measure $\mu_{k,u}$ on $V_{k,u}$ as follows. Let $\sigma_{u^\perp, k-1}$ denote the usual normalized SO_n invariant Borel measure on $\mathcal{G}(u^\perp, k-1)$, and let $\mathcal{H}^{n-k+1}$ denote the $n-k+1$ -dimensional Hausdorff measure on $\mathbb{R}^n$. Then $\mu_{k,u}$ is defined as the restriction of the product measure

$$\sigma_{u^\perp, k-1}(dE) \otimes \left(|x \cdot u|^{k-1} \mathcal{H}^{n-k+1}(dx) \right) \quad \text{on } \mathcal{G}(u^\perp, k-1) \times \mathbb{R}^n$$

to the closed subset $V_{k,u}$. Note that, since $L_{k,u}(K)$ is a closed subset of $V_{k,u}$, it is Borel measurable.

Lemma 7.4.2. Suppose that $K \subseteq \mathbb{R}^n$ is a convex body, and $u \in S^{n-1}$. Then

$$\mu_{k,u}(L_{k,u}(K)) = \frac{c_{n,k}}{n} \int_{\mathcal{G}(n,k)} \frac{1}{\lambda_k(P_F K)^n} dF. \quad (7.4)$$

Proof. If we set $p(x) = |x \cdot u|^{k-1}$, then integrating in polar coordinates on $E^\perp$ and Theorem 7.3.1 yield

$$\begin{aligned} \mu_{k,u}(L_{k,u}(K)) &= \int_{\mathcal{G}(u^\perp, k-1)} \int_{E^\perp} \mathbb{1}_{L_{k,u}(K)}(E, x) \cdot p(x) dx dE \\ &= \int_{\mathcal{G}(u^\perp, k-1)} \int_{S^{n-k} \cap E^\perp} \int_0^\infty \mathbb{1}_{L_{k,u}(K)}(E, r\theta) \cdot p(r\theta) r^{n-k} dr d\theta dE \\ &= \int_{\mathcal{G}(u^\perp, k-1)} \int_{S^{n-k} \cap E^\perp} p(\theta) \int_0^{1/\lambda_k(P_{\text{span}(E, \theta)}(K))} r^{n-1} dr d\theta dE \\ &= \frac{1}{n} \int_{\mathcal{G}(u^\perp, k-1)} \int_{S^{n-k} \cap E^\perp} \frac{1}{\lambda_k(P_{\text{span}(E, \theta)} K)^n} \cdot |\theta \cdot u|^{k-1} d\theta dE \\ &= \frac{c_{n,k}}{n} \int_{\mathcal{G}(n,k)} \frac{1}{\lambda_k(P_F K)^n} dF. \end{aligned}$$

□

Remark 7.4.3. According to the Fubini-Tonelli theorem, the equation (7.4) applies to any $K \in \mathcal{K}(\mathbb{R}^n)$. However, if K has empty interior, notice that both sides of (7.4) could be infinite. But these expressions will definitely be finite when K has non-empty interior, i.e. when K is a convex body.

With this Lemma in hand we are able to show that Φ_k decreases under Steiner symmetrization, which will be essential in the proof of the isoperimetric inequality.

Theorem 7.4.4. *Suppose $K \subseteq \mathbb{R}^n$ is a convex body, and $u \in S^{n-1}$ a fixed direction. Then*

$$\Phi_k(K) \geq \Phi_k(S_u K)$$

for every $1 \leq k \leq n-1$.

Proof. By Lemma 7.4.2, it suffices to show that

$$\mu_{k,u}(L_{k,u}(K)) \leq \mu_{k,u}(L_{k,u}(S_u K)). \quad (7.5)$$

Decomposing $E^\perp$ into $\text{span}(u) \oplus (E^\perp \cap u^\perp)$, and applying Fubini's theorem yields

$$\begin{aligned} \mu_{k,u}(L_{k,u}(K)) &= \int_{\mathcal{G}(u^\perp, k-1)} \int_{E^\perp} \mathbb{1}_{L_{k,u}(K)}(E, x) \cdot |x \cdot u|^{k-1} dx dE \\ &= \int_{\mathcal{G}(u^\perp, k-1)} \int_{\mathbb{R}} \int_{E^\perp \cap u^\perp} \mathbb{1}_{|P_{E \wedge (y+su)} K| \leq 1}(E, y, s) \cdot |(y+su) \cdot u|^{k-1} dy ds dE \\ &= \int_{\mathcal{G}(u^\perp, k-1)} \int_{\mathbb{R}} |s|^{k-1} \int_{E^\perp \cap u^\perp} \mathbb{1}_{|P_{E \wedge (y+su)} K| \leq 1}(E, y, s) dy ds dE \\ &= \int_{\mathcal{G}(u^\perp, k-1)} \int_{\mathbb{R}} |s|^{k-1} |L_{E,u,s}(K)| ds dE, \end{aligned} \quad (7.6)$$

where $L_{E,u,s}(K)$ was shown to be convex, see Remark 7.1.10, and hence measurable, and $\mathbb{R} \ni s \mapsto \lambda_{n-k}(L_{E,u,s}(K))$ was shown to be measurable as well.

The Borel measurability of the inner integral in $E \in G_{u^\perp, k-1}$ is a consequence of the Fubini-Tonelli theorem, applied to the iterated integral of the Borel function $\mathbb{1}_{L_{k,u}(K)}(E, x)$ with respect to $\sigma_{u^\perp, k-1}(dE) \otimes (|x \cdot u|^{k-1} \mathcal{H}^{n-k+1}(dx))$.

If we apply Proposition 7.2.5 to the linear reflection shadow system associated to K in the direction of u , see 7.1.2, we obtain for every fixed $s \in \mathbb{R}$ that the function

$$(E^\perp \cap u^\perp) \times \mathbb{R} \rightarrow \mathbb{R}^+,$$

$$(y, t) \mapsto f^{(s)}(y, t) := |P_{E \wedge (y+su)} K_u(t)|$$

is jointly convex. Moreover, since $K_u(-t) = R_u(K_u(t))$, the function $f^{(s)}(y, t)$ is even.

We conclude,

$$\begin{aligned} f^{(s)}(-y, -t) &= |P_{E \wedge (-y+su)} K_u(-t)| = |P_{R_u E \wedge R_u(-y+su)} K_u(t)| \\ &= |P_{E \wedge (-y-su)} K_u(t)| = |P_{E \wedge (y+su)} K_u(t)| = f^{(s)}(y, t), \end{aligned} \quad (7.7)$$

and therefore the level sets

$$\tilde{L}_{E,u,s} := \left\{ (y, t) \in (E^\perp \cap u^\perp) \times \mathbb{R} : |P_{E \wedge (y+su)} K_u(t)| \leq 1 \right\},$$

are origin-symmetric convex bodies. Note that the t -section of $\tilde{L}_{E,u,s}$ is precisely $L_{E,u,s}(K_u(t))$. By Lemma 7.1.3 we have $K_u(1) = K$ and $K_u(0) = S_u K$. Since $\tilde{L}_{E,u,s}$ is convex and origin-symmetric, we obtain

$$L_{E,u,s}(S_u K) \supseteq \frac{1}{2} (L_{E,u,s}(K) - L_{E,u,s}(K)). \quad (7.8)$$

and therefore, by the Brunn-Minkowski inequality,

$$\lambda_{n-k}(L_{E,u,s}(S_u K)) \geq \lambda_{n-k}(L_{E,u,s}(K)). \quad (7.9)$$

Note that the last equation is also true in case $L_{E,u,s}(K) = \emptyset$. If we now plug this back into (7.6) and roll everything back, we finally obtain (7.5). $\square$

Furthermore, we can prove a theorem that we will need in the proof of the equality cases.

Theorem 7.4.5. *Let $K \subseteq \mathcal{K}^n$ be a convex body and let $u \in S^{n-1}$. Then,*

$$t \mapsto \Phi_k(K_u(t)) = \Phi_k(K_u(-t)), \quad t \in \mathbb{R}^+,$$

is a monotone non-decreasing function.

Proof. As noted in Remark 7.1.10, the function $t \mapsto \lambda_{n-k}(L_{E,u,s}(K_u(t)))^{\frac{1}{n-k}}, t \in \mathbb{R}$ is concave on its support. Furthermore, since $\tilde{L}_{E,u,s}$ is origin-symmetric, the above function is even. So,

$$t \mapsto \lambda_{n-k}(L_{E,u,s}(K_u(t))) = \lambda_{n-k}(L_{E,u,s}(K_u(-t))), \quad t \in \mathbb{R},$$

is non-increasing. Now, if we integrate this according to (7.6) and apply Lemma 7.4, we get the desired assertion. $\square$

7.5 The Isoperimetric Inequality

With Theorem 7.4.4 in hand, we are now able to prove the isoperimetric inequality for affine quermassintegrals.

Theorem 7.5.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body, and $1 \leq k \leq n-1$. Then*

$$\Phi_k(K) \geq \Phi_k(B_K),$$

where equality holds if and only if K is an ellipsoid.

Proof. By Theorem 6.0.4, we obtain a sequence K_i of Steiner symmetrals of K such that $K_i \rightarrow B_K$, where B_K has the same volume as K . Applying Theorem 7.4.4, and the continuity of Φ_k on $\mathcal{K}^n$, yields

$$\Phi_k(K) \geq \Phi_k(K_1) \geq \dots \geq \Phi_k(K_i) \searrow \Phi_k(B_K).$$

Since the cases of equality require considerably more work, we will address them in the separate Chapter 8. $\square$

Since for origin-symmetric convex bodies K , the $(-n)$ -power of the first affine quermassintegral Φ_1^{-n} is proportional to the volume of the polar body K° , we get that the case $k=1$ in Theorem 7.5.1 corresponds to the Blaschke-Santaló inequality $\lambda_n(K)\lambda_n(K^\circ) \leq \kappa_n^2$.

Corollary 7.5.2. *Let $K \subseteq \mathbb{R}^n$ be a convex body. Then*

$$\lambda_n(K)\lambda_n(K^\circ) \leq \kappa_n^2,$$

where equality holds if and only if K is an ellipsoid.

Note that for general convex bodies K , Theorem 7.5.1 is weaker than the Blaschke-Santaló inequality, since the latter holds for K , which are first centered at their Santaló point, see [San49] for a reference.

Conversely, since the $(-n)$ -power of the $n-1$ -th affine quermassintegral Φ_{n-1}^{-n} is proportional to the volume of the polar of the projection body $(\Pi K)^\circ$, the case $k=n-1$ in Theorem 7.5.1 amounts to the Petty projection inequality stating $\lambda_n((\Pi K)^\circ) \leq \lambda_n((\Pi B_K)^\circ)$ with equality if and only if K is an ellipsoid, see [Gar06] for a proof.

Corollary 7.5.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body. Then*

$$\lambda_n((\Pi K)^\circ) \leq \lambda_n((\Pi B_K)^\circ),$$

where equality holds if and only if K is an ellipsoid.

7.5.4 Relation to the Isoperimetric Inequality for Quermassintegrals

Theorem 7.5.5. *If $K \subseteq \mathbb{R}^n$ is a convex body, then the volume of the outer parallel body $K + tB^n$ is a polynomial in $t > 0$ of degree n ,*

$$\lambda_n(K + tB^n) = \sum_{k=0}^n \binom{n}{k} W_k(K) t^{n-k}. \quad (7.10)$$

Moreover, Kubota's formula holds, i.e., the coefficients $W_k(K)$, called quermassintegrals, can be computed as

$$W_k(K) = \frac{\kappa_n}{\kappa_k} \int_{\mathcal{G}(n,k)} \lambda_k(K|F) dF.$$

A proof of (7.10) can be found in [Sch13], page 223 and (4.8). For Kubota's formula, see [Sch13], page 301, and [SW08], page 222.

Remark 7.5.6. Note that the W_k are invariant under motions (translations and rotations), see [Had57], page 210, whereas the Φ_k are invariant under affine transformations, see Theorem 5.2.4.

One can easily derive the isoperimetric inequality for quermassintegrals, Theorem 7.5.7, from the isoperimetric inequality for affine quermassintegrals, Theorem 7.5.1 by applying Jensen's inequality and the fact $W_k(B_K) = \Phi_k(B_K)$. Alternatively, one can deduce the above inequality by applying the Alexandrov-Fenchel inequality for the mixed volumes, see Theorem 7.3.1 in [Sch13].

Theorem 7.5.7. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $1 \leq k \leq n-1$. Then*

$$W_k(K) \geq W_k(B_K)$$

with equality if and only if K is a ball.

Remark 7.5.8. The case $k = n-1$ in Theorem 7.5.7 corresponds to the sharp isoperimetric inequality for surface area, while the case $k = 1$ recovers the sharp isoperimetric inequality for the mean width.

7.6 Some Additional Convexity Properties

We will now establish further convexity properties to prove Theorem 7.6.10, which will be a crucial ingredient for analyzing the equality cases of Theorem 7.5.1.

7.6.1 A Generalization of Proposition 7.2.5

Definition 7.6.2. If $\{K(t)\}_{t \in \mathbb{R}}$ is a shadow system in the direction of $u \in S^{n-1}$, we may introduce the following notation. For $\alpha \in (0, 1)$ and $s_0, s_1 \in \mathbb{R}^+$, we define

$$\frac{1}{s_\alpha} := \frac{1-\alpha}{s_0} + \frac{\alpha}{s_1}, \quad \lambda = \lambda_\alpha(s_0, s_1) := \frac{\alpha s_0}{\alpha s_0 + (1-\alpha)s_1},$$

and for $y_0, y_1 \in u^\perp$ and $t_0, t_1 \in \mathbb{R}$,

$$y_\lambda := (1-\lambda)y_0 + \lambda y_1, \quad t_\alpha := (1-\alpha)t_0 + \alpha t_1.$$

Furthermore, we will use the conventions $s_\alpha = 0$ if $s_0 s_1 = 0$, and $\lambda = \alpha$ if $s_0 = s_1 = 0$.

Proposition 7.6.3. *With the notation from above, for any $E \in \mathcal{G}(u^\perp, k-1)$,*

$$|P_{E \wedge (y_\lambda + s_\alpha u)} K(t_\alpha)| \leq (1-\lambda) |P_{E \wedge (y_0 + s_0 u)} K(t_0)| + \lambda |P_{E \wedge (y_1 + s_1 u)} K(t_1)|. \quad (7.11)$$

and therefore

$$L_{E, u, s_\alpha}(K(t_\alpha)) \geq (1-\lambda) L_{E, u, s_0}(K(t_0)) + \lambda L_{E, u, s_1}(K(t_1)) \quad (7.12)$$

as well as

$$\lambda_{n-k}(L_{E, u, s_\alpha}(K(t_\alpha))) \geq \lambda_{n-k}(L_{E, u, s_0}(K(t_0)))^{1-\lambda} \cdot \lambda_{n-k}(L_{E, u, s_1}(K(t_1)))^\lambda. \quad (7.13)$$

Proof. We first prove (7.11) for $y_0, y_1 \in E^\perp \cap u^\perp$, which is sufficient, because $|P_{E \wedge x} K|$ only depends on $P_{E^\perp} x$. Recall, $K^w = (K - w) \cap E^\perp$ for $w \in P_E K$. In case the shadow system has the form $K(t) = T_t^u(\tilde{K})$ for some $\tilde{K} \in \mathcal{K}(\mathbb{R}^{n+1})$, we may define $\tilde{K}^w := (\tilde{K} - w) \cap \tilde{E}^\perp$, with $\tilde{E}^\perp$ denoting the orthogonal complement of E in $\mathbb{R}^{n+1}$. Furthermore, if we set $K^w(t) := T_t^u(\tilde{K}^w)$ for $w \in P_E K$, due to $u \in E^\perp$, we have $K(t)^w = K^w(t)$. Therefore to prove (7.11), by (7.3), we may show that

$$|P_{y_\lambda + s_\alpha u} K^w(t_\alpha)| \leq (1-\lambda) |P_{y_0 + s_0 u} K^w(t_0)| + \lambda |P_{y_1 + s_1 u} K^w(t_1)|$$

for every $w \in P_E K$. Since all projections are one-dimensional intervals, we can parameterize $E^\perp$ by

$$\left\{ (a, b) := a + bu : a \in E^\perp \cap u^\perp, b \in \mathbb{R} \right\}.$$

If we write a plus (or minus) next to a real number x , it means $\max(x, 0)$ (or $\max(-x, 0)$). If x is a vector, this is to be understood component-wise. Now, let $* \in \{+, -\}$ and $i \in \{0, 1\}$, we compute

$$\begin{aligned} & |P_{y_\lambda + s_\alpha u} K^w(t_\alpha)| \\ &= \max_{(a_*, b_*) \in K^w(t_\alpha)} \{y_\lambda \cdot (a_+ - a_-) + s_\alpha (b_+ - b_-)\} \\ &\leq \max_{(a_*, r_i^*) \in K^w(t_i)} \left\{ y_\lambda \cdot (a_+ - a_-) + s_\alpha \begin{pmatrix} ((1-\alpha)r_0^+ + \alpha r_1^+) \\ -((1-\alpha)r_0^- + \alpha r_1^-) \end{pmatrix} \right\} \\ &= \max_{(a_*, r_i^*) \in K^w(t_i)} \left\{ ((1-\lambda)y_0 + \lambda y_1) \cdot (a_+ - a_-) + \begin{pmatrix} (1-\lambda)s_0 (r_0^+ - r_0^-) \\ +\lambda s_1 (r_1^+ - r_1^-) \end{pmatrix} \right\} \\ &\leq (1-\lambda) \max_{(a_*, r_0^*) \in K^w(t_0)} \{y_0 \cdot (a_+ - a_-) + s_0 (r_0^+ - r_0^-)\} \\ &\quad + \lambda \max_{(a_*, r_1^*) \in K^w(t_1)} \{y_1 \cdot (a_+ - a_-) + s_1 (r_1^+ - r_1^-)\} \\ &= (1-\lambda) |P_{y_0 + s_0 u} K^w(t_0)| + \lambda |P_{y_1 + s_1 u} K^w(t_1)|. \end{aligned}$$

The first of the above inequalities is a consequence of Lemma 7.1.4, and the following equation follows from

$$(1-\alpha)s_\alpha = (1-\lambda)s_0, \quad \alpha s_\alpha = \lambda s_1.$$

By Definition 7.1.9, and (7.11), we immediately get (7.12). Furthermore, (7.13) is obtained as an application of (7.12) and the Brunn-Minkowski inequality, Theorem 3.0.2. $\square$

The above Proposition is indeed a generalization of Proposition 7.2.5 since setting $s_0 = s_1 = s \in \mathbb{R}^+$, which implies $\lambda = \alpha$, leads to the joint convexity of $(y, t) \mapsto |P_{E \wedge (y+su)} K(t)|$.

7.6.4 The s -Moment Function is Convex

Definition 7.6.5. Let $u \in S^{n-1}$ and $E \in \mathcal{G}(u^\perp, k-1)$, then we define

$$M_k(L_E(K_u(t))) = \left(\int_{L_E(K_u(t))} |x \cdot u|^{k-1} dx \right)^{-1/k} = \left(\int_{\mathbb{R}} |s|^{k-1} \lambda_{n-k}(L_{E,u,s}(K_u(t))) ds \right)^{-1/k}.$$

Note that this expression appeared in (7.6).

For the following Theorem by Ball, we refer to [Bal88], page 74.

Theorem 7.6.6. Let $f, g, h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be measurable functions with the property that

$$h(s_\alpha) \geq f(s_0)^{1-\lambda_\alpha(s_0, s_1)} g(s_1)^{\lambda_\alpha(s_0, s_1)}$$

for some $\alpha \in (0, 1)$, and all $s_0, s_1 \in \mathbb{R}^+$, where s_α and $\lambda_\alpha(s_0, s_1)$ are as in Definition 7.6.2. Then, if we set $I_p(w) = \left(\int_0^\infty s^{p-1} w(s) ds \right)^{-1/p}$, it holds that for all $p > 0$

$$I_p(h) \leq (1-\alpha)I_p(f) + \alpha I_p(g),$$

A simple consequence is the following corollary.

Corollary 7.6.7. Let $\{K(t)\}_{t \in \mathbb{R}}$ be a shadow system in the direction of $u \in S^{n-1}$, and let $E \in \mathcal{G}(u^\perp, k-1)$, then the function

$$t \mapsto \left(\int_{L_E(K(t))} (x \cdot u)_+^{k-1} dx \right)^{-1/k} = \left(\int_0^\infty s^{k-1} \lambda_{n-k}(L_{E,u,s}(K(t))) ds \right)^{-1/k}, \quad t \in \mathbb{R},$$

is convex.

Proof. This follows by setting $h(s) = w_{t_\alpha}(s)$, $f(s) = w_{t_0}(s)$ and $g(s) = w_{t_1}(s)$, where $w_t(s) = \lambda_{n-k}(L_{E,u,s}(K(t)))$ and $t_\alpha = (1-\alpha)t_0 + \alpha t_1$, and applying Theorem 7.6.6 and (7.13). Observe that, due to Remark 7.1.10, the map $s \mapsto \lambda_{n-k}(L_{E,u,s}(K(t)))$, $s \in \mathbb{R}$ is measurable. $\square$

Theorem 7.6.8. The function $t \mapsto M_k(L_E(K_u(t)))$, $t \in \mathbb{R}$, is convex and even.

Proof. Due to Corollary 7.6.7, applied to $\{K_u(t)\}$, and due to Lemma 7.1.3, $K_u(-t) = R_u K_u(t)$, and the definition of $L_{E,u,s}(K)$, Definition 7.1.9, and (7.7) yields

$$L_{E,u,s}(K_u(-t)) = L_{E,u,-s}(K_u(t)) = -L_{E,u,s}(K_u(t)) = -L_{E,u,-s}(K_u(-t)),$$

which concludes the proof. $\square$

7.6.9 A Dichotomy for $t \mapsto \Phi_k(K_u(t))$

We are now able to prove the following theorem, which we will need for proving the equality cases.

Theorem 7.6.10. *Let $K \subseteq \mathbb{R}^n$ be a convex body, $u \in S^{n-1}$, and $t_0 \in \mathbb{R}$. Then the equality $\Phi_k(K_u(t_1)) = \Phi_k(K_u(t_0))$ holds for some t_1 with $|t_1| < |t_0|$ if and only if it holds for every t_1 with $|t_1| < |t_0|$.*

Proof. Suppose that $\Phi_k(K_u(t_1)) = \Phi_k(K_u(t_0))$ holds true for some t_1 with $|t_1| < |t_0|$. Due to Lemma 7.4.2, this means $\mu_{k,u}(L_{k,u}(K_u(t_1))) = \mu_{k,u}(L_{k,u}(K_u(t_0)))$. From (7.6) and the subsequent discussion, we obtain

$$\mu_{k,u}(L_{k,u}(K_u(t))) = \int_{\mathcal{G}(u^\perp, k-1)} M_k(L_E(K_u(t)))^{-k} dE. \quad (7.14)$$

Since by Theorem 7.6.8, the map $t \mapsto M_k(L_E(K_u(t))) = M_k(L_E(K_u(-t)))$, $t \in \mathbb{R}_+$ is monotone non-decreasing for all $E \in G_{u^\perp, k-1}$, we deduce that

$$M_k(L_E(K_u(\pm t_1))) = M_k(L_E(K_u(\pm t_0)))$$

for almost every $E \in \mathcal{G}(u^\perp, k-1)$. Applying again Theorem 7.6.8 yields that the map $t \mapsto M_k(L_E(K_u(t)))$, $t \in [-|t_0|, |t_0|]$ has to be constant for almost every $E \in \mathcal{G}(u^\perp, k-1)$. Finally, due to the expression (7.14) and Lemma 7.4.2, we get $\Phi_k(K_u(t)) = \Phi_k(K_u(t_0))$ for all $t \in [-|t_0|, |t_0|]$. $\square$

8 Cases of Equality of $\Phi_k(K) \geq \Phi_k(B_K)$

To analyze the cases of equality, we will establish the following Theorem 8.0.1, which is proven in two cases $k < n - 1$ and $k = n - 1$. Once we have this, the actual proof of the equality cases of $\Phi_k(K) \geq \Phi_k(B_K)$ can be easily conducted as described below.

Theorem 8.0.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $1 \leq k \leq n - 1$. Then $\Phi_k(K) = \Phi_k(S_u K)$ for all $u \in S^{n-1}$ if and only if K is an ellipsoid.*

Proof. The straightforward part of Theorem 8.0.1 follows from the well-known fact (see Lemma 2 in [BLM06]), that Steiner symmetrization transforms an ellipsoid into another ellipsoid of the same volume, along with Theorem 5.2.4, the affine invariance of Φ_k . The proof of the nontrivial part involves multiple steps, which are elaborated on in the following sections. $\square$

If Theorem 8.0.1 is proven, we can derive the equality cases of Theorem 7.5.1 as follows.

Theorem 8.0.2. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $1 \leq k \leq n - 1$. Then $\Phi_k(K) = \Phi_k(B_K)$ holds true if and only if K is an ellipsoid.*

Proof. As stated above, if K is an ellipsoid, then $\Phi_k(K) = \Phi_k(B_K)$. So, let us assume that equality holds for a convex body $K \subseteq \mathbb{R}^n$, and denote by $\{K_u(t)\}$ the linear reflection shadow system from (7.1). Since $\lambda_n(K_u(t)) = \lambda_n(K)$ for every $t \in [-1, 1]$, the continuity of the map $t \mapsto K_u(t)$ with respect to the Hausdorff metric implies that for every $u \in S^{n-1}$ there is $\epsilon \in (0, 1)$ with $\Phi_k(K_u(1 - \epsilon)) \geq \Phi_k(K)$ for all $t \in [-1, 1]$. Conversely, applying Theorem 7.4.5, yields $\Phi_k(K) \geq \Phi_k(K_u(t))$ for $t \in [-1, 1]$, and therefore equality holds at $t = 1 - \epsilon$. In fact, due to Theorem 7.6.10, we get equality for every $t \in [-1, 1]$, especially for $t = 0$, that means $\Phi_k(K) = \Phi_k(S_u K)$. This holds for every $u \in S^{n-1}$ and therefore, by Theorem 8.0.1, K is an ellipsoid. $\square$

8.1 The Case $k < n - 1$

Step 1 - Point of Symmetry

Lemma 8.1.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $1 \leq k \leq n-1$. Then the map*

$$\mathcal{G}(n, k-1) \rightarrow \mathcal{K}^n, \quad E \mapsto L_E(K),$$

is continuous.

Proof. Since by Remark 7.1.10, $g_{L_E(K)}(\theta) = |P_{\text{span}(E, \theta)} K|$ for all $\theta \in S^{n-1} \cap E^\perp$, we obtain

$$\frac{1}{R} B_{E^\perp} \subseteq L_E(K) \subseteq \frac{1}{r} B_{E^\perp}$$

for all $E \in \mathcal{G}(n, k-1)$, where $r = \min_{F \in \mathcal{G}(n, k)} |P_F K| > 0$ and $R = \max_{F \in \mathcal{G}(n, k)} |P_F K| > 0$. The map $E \mapsto \frac{1}{r} B_{E^\perp}$ is continuous for $E \in \mathcal{G}(n, k-1)$ and therefore, to conclude the proof, it suffices to show that $F \mapsto |P_F K|$ is continuous for $F \in \mathcal{G}(n, k)$. But this is immediate by the continuity of mixed volumes and, as in the proof of Lemma 7.1.6,

$$|P_F K| = \frac{\binom{n}{k}}{\kappa_{n-k}} V(K, k; B_{F^\perp}, n-k) \quad \forall F \in \mathcal{G}(n, k).$$

□

Note that $L_{E, u, s}(K)$ is the section of $L_E(K)$ perpendicular to u at height $s \in \mathbb{R}$. In particular

$$L_{E, u, s}(K) = (L_E(K) - su) \cap u^\perp \subseteq E^\perp \cap u^\perp, \quad (8.1)$$

and hence $L_{E, u, s}(K)$ is convex and compact. So, if $L_{E, u, s}(K) \neq \emptyset$, then $L_{E, u, s}(K)$ is a convex body. The set of these (E, s) will be denoted by

$$\mathcal{G}_u^k(K) = \left\{ (E, s) \in \mathcal{G}(u^\perp, k-1) \times \mathbb{R} : L_{E, u, s}(K) \neq \emptyset \right\}.$$

Since all $L_E(K)$ are compact, Lemma 8.1.1 yields that $\mathcal{G}_u^k(K)$ is a closed (in fact compact) set (since, if $L_{E, u, s}(K) = \emptyset$, then $su + u^\perp$ has to be at a positive distance from $L_E(K)$).

We get the following

Corollary 8.1.2. *If $(E, s) \in \mathcal{G}(u^\perp, k-1) \times \mathbb{R}$ is such that $\text{int } L_E(K) \cap (su + u^\perp) \neq \emptyset$, we have $(E, s) \in \text{int } \mathcal{G}_u^k(K)$.*

Lemma 8.1.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body, $u \in S^{n-1}$ and $1 \leq k \leq n-1$. Then the map*

$$\mathcal{G}_u^k(K) \rightarrow \mathbb{R}_+, \quad (E, s) \mapsto \lambda_{n-k}(L_{E, u, s}(K)),$$

is continuous.

Proof. By (8.1) and Lemma 8.1.1 the map $(E, s) \mapsto L_{E,u,s}(K) \in \mathcal{K}^n$ is continuous on $\mathcal{G}_u(K)$. Also, the map $E \mapsto B_{\text{span}(E,u)} \in \mathcal{K}^n$ is continuous on $\mathcal{G}(u^\perp, k-1)$, where B_F denotes the unit ball in the subspace F . Since $L_{E,u,s}(K)$ is a convex body in $E^\perp \cap u^\perp$ and $\dim(E^\perp \cap u^\perp) = n - k$, we have, as in the proof of Lemma 7.1.6,

$$\lambda_{n-k}(L_{E,u,s}(K)) = \frac{\binom{n}{k}}{\kappa_k} V(B_{\text{span}(E,u)}, k; L_{E,u,s}(K), n-k).$$

The asserted continuity now follows from the continuity of mixed volume. $\square$

Corollary 8.1.4. *The map*

$$\mathcal{G}(u^\perp, k-1) \times \mathbb{R} \rightarrow \mathbb{R}^+, \quad (E, s) \mapsto \lambda_{n-k}(L_{E,u,s}(K)),$$

is Borel measurable.

Proof. This follows since the map is continuous on the compact set $\mathcal{G}_u^k(K)$ and zero outside of it. $\square$

In (7.8) and (7.9) we showed for all $s \in \mathbb{R}$,

$$L_{E,u,s}(S_u K) \supseteq \frac{1}{2}(L_{E,u,s}(K) - L_{E,u,s}(K)) \quad (8.2)$$

and

$$\lambda_{n-k}(L_{E,u,s}(S_u K)) \geq \lambda_{n-k}(L_{E,u,s}(K)). \quad (8.3)$$

So we obtain $\mathcal{G}_u^k(K) \subseteq \mathcal{G}_u^k(S_u K)$.

Proposition 8.1.5. *Suppose $K \subseteq \mathbb{R}^n$ is a convex body, $u \in S^{n-1}$, and $1 \leq k \leq n-1$ with $\Phi_k(K) = \Phi_k(S_u K)$. Then*

$$\lambda_{n-k}(L_{E,u,s}(S_u K)) = \lambda_{n-k}(L_{E,u,s}(K)) \quad (8.4)$$

for all $s \in \mathbb{R}$ and $E \in \mathcal{G}(u^\perp, k-1)$. Furthermore, we have for some $\alpha_{E,u,s} \in E^\perp \cap u^\perp$,

$$L_{E,u,s}(K) = L_{E,u,s}(S_u K) + \alpha_{E,u,s}. \quad (8.5)$$

Proof. Due to Lemma 7.4.2 and Remark 7.4.3 the assumption implies $\mu_{k,u}(L_{k,u}(K)) = \mu_{k,u}(L_{k,u}(S_u K)) < \infty$.

Considering (8.3) and (7.6), equation (8.4) must hold for $\sigma_{u^\perp, k-1} \otimes \lambda_1$ -almost every $(E, s) \in \mathcal{G}(u^\perp, k-1) \times \mathbb{R}$, where $\sigma_{u^\perp, k-1} \otimes \mathcal{L}$ denotes the product measure of the usual measure $\sigma_{u^\perp, k-1}$ on $\mathcal{G}(u^\perp, k-1)$ and λ_1 the one-dimensional Lebesgue measure on $\mathbb{R}$. Due to Corollary 8.1.2 and Lemma 8.1.1, applied to K and $S_u K$, the inclusion $\mathcal{G}_u^k(K) \subseteq \mathcal{G}_u^k(S_u K)$ yields that (8.4) is true for all $(E, s) \in \mathcal{G}(u^\perp, k-1) \times \mathbb{R}$. Thus, the set $\text{int } L_E(K) \cap (su + u^\perp)$ is non-empty.

Let $E \in \mathcal{G}(u^\perp, k-1)$ and let $\Sigma_E \neq \emptyset$ be the interior of the compact interval $P_{\text{span}(u)} L_E(K)$, which we can view as an open interval in $\mathbb{R}$, via $su \leftrightarrow s$. The equation (8.4) holds for every $s \in \Sigma_E$, and furthermore, both sides are continuous maps in s on their support, by (8.1). We conclude that (8.4) holds true for all $s \in \text{cl}(\Sigma_E)$, where $\text{cl}(\Sigma_E)$ denotes the closure of Σ_E .

In case of $s \notin \text{cl}(\Sigma_E)$, $L_E(K) \cap (su + u^\perp) = \emptyset$ and $\lambda_{n-k}(L_{E,u,s}(K)) = 0$, for almost every $E \in \mathcal{G}(u^\perp, k-1)$ it holds that

$$\lambda_{n-k}(L_{E,u,s}(S_u K)) = 0$$

for all $s \notin \text{cl}(\Sigma_E)$, that means

$$L_E(S_u K) \cap (su + u^\perp) = \emptyset.$$

Thus, for almost every $E \in \mathcal{G}(u^\perp, k-1)$, we obtain

$$L_E(S_u K) \subseteq \text{cl}(\Sigma_E)u + u^\perp. \quad (8.6)$$

But concerning Lemma 8.1.1, the maps

$$E \mapsto \text{cl}(\Sigma_E) \quad \text{and} \quad E \mapsto L_E(S_u K), \quad E \in \mathcal{G}(u^\perp, k-1),$$

are continuous with respect to the Hausdorff topology, and therefore (8.6) holds true for $E \in \mathcal{G}(u^\perp, k-1)$. To put it in another way,

$$\lambda_{n-k}(L_{E,u,s}(S_u K)) = 0 = \lambda_{n-k}(L_{E,u,s}(K))$$

for all $E \in \mathcal{G}(u^\perp, k-1)$ and $s \notin \text{cl}(\Sigma_E)$, that means we proved (8.4) for all $E \in \mathcal{G}(u^\perp, k-1)$ and $s \in \mathbb{R}$.

By the equality case of the Brunn-Minkowski inequality, (8.2) and (8.4) imply that for

every $E \in \mathcal{G}(u^\perp, k-1)$ and $s \in \Sigma_E$, equality (8.5) must hold with some $\alpha_{E,u,s} \in E^\perp \cap u^\perp$. Conversely, we already know $L_{E,u,s}(K) = L_{E,u,s}(S_u K) = \emptyset$ for $s \notin \text{cl}(\Sigma_E)$, therefore (8.5) also holds true for $s \notin \text{cl}(\Sigma_E)$ as well. By (8.1) and continuity of the section of a convex body with respect to the Hausdorff topology, (8.5) is true for $s_0 \in \text{bd}(\Sigma_E)$ with $\alpha_{E,u,s_0} = \lim_{\Sigma_E \ni s \rightarrow s_0} \alpha_{E,u,s}$. $\square$

Definition 8.1.6. We say that a set $K \subseteq \mathbb{R}^n$ has a *point of symmetry* if there is $v \in \mathbb{R}^n$ with $K - v = -(K - v)$.

Corollary 8.1.7. For $E \in \mathcal{G}(n, k-1)$, $u \in S^{n-1} \cap E^\perp$ and $s \in \mathbb{R}$, the set $L_{E,u,s}(K)$ has a point of symmetry.

Proof. Suppose that $\Phi_k(K) = \Phi_k(S_u K)$ for all $u \in S^{n-1}$. In the proof of Theorem 7.4.4, we showed that $L_{E,u,s}(S_u K)$ is the level-set of the even function $f^{(s)}(\cdot, 0)$, see (7.7), and therefore origin-symmetric. Now, (8.5) implies that $L_{E,u,s}(K)$ has a point of symmetry for all $E \in \mathcal{G}(n, k-1)$, $u \in S^{n-1} \cap E^\perp$, and $s \in \mathbb{R}$. $\square$

Step 2 - Brunn's Characterization Theorem We now make use of the following characterization theorem of ellipsoids by Brunn. As a reference, we cite [Bru89].

Theorem 8.1.8. If $K \subseteq \mathbb{R}^n$ is a convex body, $n \geq 3$, and let $2 \leq k \leq n-1$, then K is an ellipsoid if and only if every k -dimensional section of K that passes through its interior has a point of symmetry.

If $E \in \mathcal{G}(n, k-1)$, then (8.1) and the fact that for every $u \in S^{n-1} \cap E^\perp$ and $s \in \mathbb{R}$ the set $L_{E,u,s}(K)$ has a point of symmetry, implies that in case $\dim E^\perp = n - k + 1 \geq 3$, i.e. $k \leq n - 2$, that $L_E(K)$ must be an ellipsoid in $E^\perp$.

Step 3 - Distinguished Orthonormal Basis

Lemma 8.1.9. Let $K \subseteq \mathbb{R}^n$ be a convex body and $E \in \mathcal{G}(n, k-1)$ with $1 \leq k \leq n-2$. Then there is a positive-definite linear map $T_E : \mathbb{R}^n \rightarrow \mathbb{R}^n$, which is the identity on E , and on $E^\perp$ it maps the ball $B_{E^\perp}$ onto $L_E(K)$. For an orthonormal basis $u_i, i = 1, \dots, n - k + 1$ of $E^\perp$ consisting of eigenvectors of T_E we have

$$L_E(S_{u_i} T_E K) = L_E(T_E K) = B_{E^\perp},$$

for all $1 \leq i \leq n - k + 1$.

Proof. Lemma 7.2.3 yields

$$\begin{aligned} L_E(T(A)) &= \left\{ x \in E^\perp; |P_{E \wedge x} T(A)| \leq 1 \right\} \\ &= \left\{ x \in E^\perp; |P_{E \wedge T^*(x)} A| \leq 1 \right\} = T^{-*}(L_E(A)) \end{aligned}$$

for any non-empty compact and convex $A \in \mathcal{K}(\mathbb{R}^n)$ and an invertible linear map T , which is invariant on E . Because of the fact that $L_E(K) \subseteq E^\perp$ is an origin-symmetric ellipsoid, we can find a positive-definite linear map T_E on $\mathbb{R}^n$, which is the identity on E , and on $E^\perp$ it maps the ball $B_{E^\perp}$ onto $L_E(K)$. If we set $K_E := T_E K$, we obtain

$$L_E(K_E) = T_E^{-*}(L_E(K)) = B_{E^\perp}. \quad (8.7)$$

Let $u_i, i \in \{1, \dots, n-k+1\}$ represent an orthonormal basis of $E^\perp$ made up of eigenvectors of T_E . Therefore, since T_E acts diagonally in this basis and consequently the actions of S_{u_i} and T_E commute, we derive

$$L_E(S_{u_i} K_E) = L_E(S_{u_i} T_E(K)) = L_E(T_E(S_{u_i} K)) = T_E^{-*}(L_E(S_{u_i} K)). \quad (8.8)$$

By (8.5) and (8.1) for every $s \in \mathbb{R}$ there is an $\alpha_{E, u_i, s} \in u_i^\perp$ so that

$$(L_E(K) - su_i) \cap u_i^\perp = (L_E(S_{u_i} K) - su_i) \cap u_i^\perp + \alpha_{E, u_i, s}. \quad (8.9)$$

The map T_E^{-*} acts invariantly on $\text{span}(u_i)$ and $u_i^\perp$, and therefore, along with (8.7) and (8.8), applying T_E^{-*} to (8.9) yields

$$(B_{E^\perp} - su_i) \cap u_i^\perp = (L_E(S_{u_i} K_E) - su_i) \cap u_i^\perp + T_E^{-*}(\alpha_{E, u_i, s})$$

for all $s \in \mathbb{R}$ and $i \in \{1, \dots, n-k+1\}$. The fact that $(L_E(S_u K) - su) \cap u^\perp$ is origin-symmetric in $E^\perp \cap u^\perp$, and this property does not change if applying a linear transformation, we deduce that $(L_E(S_{u_i} K_E) - su_i) \cap u_i^\perp$ is also origin-symmetric in $E^\perp \cap u_i^\perp$ for every $s \in \mathbb{R}$. Furthermore, $(B_{E^\perp} - su_i) \cap u_i^\perp$ is origin-symmetric too, which implies $T_E^{-*}(\alpha_{E, u_i, s}) = 0$. Finally,

$$L_E(S_{u_i} K_E) = L_E(K_E) = B_{E^\perp}$$

for all $i \in \{1, \dots, n-k+1\}$. □

Step 4 - Invariance under Reflections of K^w

Recall, $K^w := (K - w) \cap E^\perp$ for $w \in P_E K$.

Lemma 8.1.10. *Suppose that $K \subseteq \mathbb{R}^n$ is a convex body, and let $E \in \mathcal{G}(n, k-1)$ with $1 \leq k \leq n-1$. If $L_E(S_u K) = L_E(K)$ for some $u \in E^\perp$, then, up to translation in the direction of u , we have $S_u K^w = K^w$ for every $w \in \text{int } P_E K$. In particular, K^w is invariant under reflections about $u^\perp$.*

Proof. Fix $x \in E^\perp$. In (7.3), we showed that

$$\|x\|_{L_E(K)} = |P_{E \wedge x} K| = \int_{P_E K} |P_x K^w| dw = \int_{P_E K} (h_{K^w}(x) + h_{K^w}(-x)) dw.$$

The assumption $L_E(S_u K) = L_E(K)$ implies $(S_u K)^w = S_u K^w$ for all $w \in P_E K$ and therefore

$$\int_{P_E K} (h_{K^w}(x) + h_{K^w}(-x)) dw = \int_{P_E K} (h_{S_u K^w}(x) + h_{S_u K^w}(-x)) dw$$

Since $S_u K^w \subseteq \frac{1}{2}(K^w + R_u K^w)$, we get

$$h_{S_u K^w} \leq \frac{1}{2}(h_{K^w} + h_{R_u K^w}). \quad (8.10)$$

Furthermore, $h_{R_u K^w}(\xi) = h_{K^w}(R_u \xi)$ implies

$$\begin{aligned} & \int_{P_E K} (h_{K^w}(\xi) + h_{K^w}(-\xi)) dw \\ & \leq \frac{1}{2} \int_{P_E K} (h_{K^w}(\xi) + h_{K^w}(R_u \xi) + h_{K^w}(-\xi) + h_{K^w}(-R_u \xi)) dw. \end{aligned}$$

If we apply this to $\xi = \theta$ and $\xi = R_u \theta$ for $\theta \in E^\perp$ and sum both equalities, we get

$$\begin{aligned} & \int_{P_E K} (h_{K^w}(\theta) + h_{K^w}(-\theta) + h_{K^w}(R_u \theta) + h_{K^w}(-R_u \theta)) dw \\ & \leq \int_{P_E K} (h_{K^w}(\theta) + h_{K^w}(R_u \theta) + h_{K^w}(-\theta) + h_{K^w}(-R_u \theta)) dw. \end{aligned} \quad (8.11)$$

Both sides are equal, so we must have equality for a.e. $w \in P_E K$ in the 4 instances of the inequality (8.10) we used in the directions $\xi \in \{\theta, -\theta, R_u \theta, -R_u \theta\}$ to derive (8.11). Since the corresponding functions are continuous on their support we get equality for

all $w \in P_E K$. Hence

$$h_{S_u K^w}(\xi) = \frac{1}{2} (h_{K^w}(\xi) + h_{R_u K^w}(\xi))$$

for all $w \in P_E K$ and $\xi \in \{\theta, -\theta, R_u \theta, -R_u \theta\}$. Given that θ was arbitrary, we get for all $w \in P_E K$,

$$S_u K^w = \frac{1}{2} (K^w + R_u K^w)$$

Applying the Brunn-Minkowski inequality yields

$$\lambda_{n-(k-1)}(K^w) = \lambda_{n-(k-1)}(S_u K^w) \geq \lambda_{n-(k-1)}(K^w)^{\frac{1}{2}} \lambda_{n-(k-1)}(R_u K^w)^{\frac{1}{2}} = \lambda_{n-(k-1)}(K^w),$$

and from the equality cases we deduce that $R_u K^w$ and K^w are translates whenever $\text{int } K^w \neq \emptyset$. And in particular, whenever $w \in \text{int } P_E K$. As no translation perpendicular to u is possible, the proof is complete. $\square$

Corollary 8.1.11. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $w \in \text{int } P_E K$. Then, K^w has a point of symmetry.*

Proof. For $w \in \text{int } P_E K$, Lemma 8.1.10 and Lemma 8.1.9 yield that $R_{u_i} K_E^w = K_E^w$ up to translations in the direction of u_i . But the u_i 's are all orthogonal, so there is a single translation of K_E^w such that $R_{u_i} K_E^w = K_E^w$ holds for all $i = 1, \dots, n - k + 1$. Since the composition of all R_{u_i} 's is $-\text{Id}$ on $E^\perp$, we obtain that K_E^w has a point of symmetry. Furthermore, since $K_E = T_E(K)$ and T_E is the identity Id on E , we find that K^w must also have a point of symmetry. $\square$

Step 5 - Concluding the Proof when $1 \leq k < n - 1$

We proved that for every $E \in G_{n,k-1}$ the section $K \cap (w + E^\perp) = w + K^w$ of K through its interior has a point of symmetry. Thus, by Brunn's Theorem 8.1.8, in case of $n \geq 3$ and $\dim E^\perp = n - k + 1 \geq 2$, i.e. $k \leq n - 1$, K has to be an ellipsoid.

This concludes the proof for the case $k \leq n - 1$ and we will now address the case $k = n - 1$.

8.2 The Case $k = n - 1$

Since $\dim E^\perp = 2$ for $E \in \mathcal{G}(n, k - 1)$, we cannot use Theorem 8.1.8 for the case $k = n - 1$. In the following, we establish a way to circumvent Step 2 from the previous section

Linear Boundary Segments

Recall that

$$L(K) = L_{\{0\}}(K) = (K - K)^\circ.$$

Proposition 8.2.1. *Suppose that $K(t), t \in \mathbb{R}$, is a shadow system in $\mathbb{R}^2$ in the direction of e_2 and let $S, T \subseteq \mathbb{R}$ be non-empty open intervals. If there are functions $a, \Psi : S \rightarrow \mathbb{R}$ with $(a(s) + \Psi(s)t, s) \in \text{bd}(L(K(t)))$ for all $s \in S$ and $t \in T$, then there exist $c_+, c_- \in \mathbb{R}$ such that $\Psi(s) = c_+s_+ - c_-s_-$ for all $s \in S$.*

Proof. By definition, we have $K(t) = T_t^{e_2}(\tilde{K})$ $\tilde{K} \in \mathcal{K}(\mathbb{R}^3)$, where $T_t^{e_2} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ denotes a projection onto $\mathbb{R}^2$ parallel to $e_3 + te_2$. As in the proof of Proposition 7.2.5, we obtain

$$\begin{aligned} \|(y, s)\|_{L(K(t))} &= h_{K(t)}(y, s) + h_{K(t)}(-y, -s) \\ &= h_{\tilde{K}}(y, s, -st) + h_{\tilde{K}}(-y, -s, st) = \|(y, s, -st)\|_{L(\tilde{K})}. \end{aligned}$$

Now, due to our assumption, we have a local parametrization of the surface $\text{bd}(L(\tilde{K}))$ by

$$F(s, t) := (a(s) + \Psi(s)t, s, -st) \in \text{bd}(L(\tilde{K})), \quad s \in S \quad t \in T.$$

Since $L(\tilde{K})$ is convex, we may represent its boundary locally by a convex function f , which is therefore Lipschitz. Rademacher's theorem, see e.g. [Mag12] for a reference, implies that f is differentiable almost-everywhere. Furthermore, due to Alexandrov's theorem (for a reference, we cite [Gru07], Chapter 2), f is twice differentiable in Alexandrov's sense almost-everywhere. At points of first differentiability, we have two tangent vectors to the boundary, which are linearly independent and given by

$$\partial_s F(s, t) = (a'(s) + \Psi'(s)t, 1, -t), \quad \partial_t F(s, t) = (\Psi(s), 0, -s),$$

therefore, the normal to the boundary in the direction is

$$N := (s, -sa'(s) - s\Psi'(s)t + t\Psi(s), \Psi(s)).$$

Furthermore, at points of second differentiability, the surface has a second-order Taylor expansion

$$\text{II} := \langle \partial_s^2 F, N/|N| \rangle ds^2 + 2 \langle \partial_s \partial_t F, N/|N| \rangle dt ds + \langle \partial_t^2 F, N/|N| \rangle dt^2,$$

which is governed by the second fundamental form. Concerning $\partial_t^2 F \equiv 0$ and $\langle \partial_s \partial_t F, N \rangle =$

$s\Psi'(s) - \Psi(s)$, we obtain that unless that last term above vanishes, the form II has strictly negative determinant. This implies that the surface must have a saddle at this point, which is a contradiction to convexity.

We state that the only locally Lipschitz functions Ψ , solving $s\Psi'(s) - \Psi(s) = 0$ for almost every $s \in S$ are of the form $\Psi(s) = c_+s_+ - c_-s_-$. Therefore, let us write S_+ and S_- for the open subsets of S where the continuous Ψ is positive and negative, respectively. Since on S_+ the equation $(\log \Psi)'(s) = (\log s)'$ holds true, the local absolute continuity of $\log \Psi$ implies that $\Psi(s) = c_+s$, $c_+ \neq 0$ on each connected component S_+^i of S_+ . Furthermore, Ψ vanishes at the end-points of each connected component which lie in S . We deduce that there can be at most one connected component in each of $S \cap \mathbb{R}^+$ and $S \cap \mathbb{R}^-$, and that the end-points of them in S have to be at $s = 0$. Similarly, one can conclude for S_- , and thus Ψ must be of the stated form. $\square$

Step 1 - Segments of Constant Projections of K

Recall that the definition of $f^{(s)}$ for $s \in \mathbb{R}$, $E \in \mathcal{G}(n, k-1)$ and $u \in S^{n-1} \cap E^\perp$ is given by

$$\begin{aligned} & (E^\perp \cap u^\perp) \times \mathbb{R} \rightarrow \mathbb{R}^+, \\ & (y, t) \mapsto f^{(s)}(y, t) := |P_{E \wedge (y+su)} K_u(t)|. \end{aligned}$$

Lemma 8.2.2. *For every $y \in E^\perp \cap u^\perp$ such that $f^{(s)}(y, 0) = R$, it holds that $f^{(s)} = R$ on the two segments*

$$\{(\pm y + R\alpha_{E,u,s/R}t, t) : t \in [-1, 1]\}$$

for some $\alpha_{E,u,s/R} \in \mathbb{R}$.

Proof. Let $E \in \mathcal{G}(n, k-1)$ and $u \in S^{n-1} \cap E^\perp$. The argument from Step 1 in the previous section reveals more information than was previously mentioned. For $s \in \mathbb{R}$ the function $f^{(s)}$ is convex and even in (y, t) , and therefore, its level set

$$\tilde{L}_{E,u,s} := \left\{ (y, t) \in (E^\perp \cap u^\perp) \times \mathbb{R} : f^{(s)}(y, t) \leq 1 \right\}$$

is convex and origin-symmetric. Obviously, $\tilde{L}_{E,u,s}(t) = L_{E,u,s}(K_u(t))$, where $\tilde{L}_{E,u,s}(t)$ the t -section of $\tilde{L}_{E,u,s}$. Analogous to the proof of Theorem 7.6.10, Brunn's concavity principle, Theorem 3.0.5, implies that the function $t \mapsto \lambda_{n-k}(\tilde{L}_{E,u,s}(t))^{\frac{1}{n-k}}$, $t \in \mathbb{R}$, is even and concave on its support.

When $\Phi_k(K) = \Phi_k(S_u K)$ holds true, we obtain

$$\lambda_{n-k}(\tilde{L}_{E,u,s}(1)) = \lambda_{n-k}(\tilde{L}_{E,u,s}(-1)) = \lambda_{n-k}(\tilde{L}_{E,u,s}(0)),$$

and therefore the function $t \mapsto \lambda_{n-k}(\tilde{L}_{E,u,s}(t))$, $t \in [-1, 1]$ has to be constant. Let us write Σ_E for the non-empty interior of the compact interval $P_{\text{span}(u)} L_E(K)$, viewed as an open interval in $\mathbb{R}$. The equality case of the Brunn-Minkowski inequality, Theorem 3.0.2, implies that for all $s \in \Sigma_E$, the set $\tilde{L}_{E,u,s} \cap \{t \in [-1, 1]\}$ has to be a tilted cylinder over the origin-symmetric base $\tilde{L}_{E,u,s}(0) = L_{E,u,s}(S_u K) \subseteq E^\perp \cap u^\perp$, i.e.,

$$L_{E,u,s}(K_u(t)) = L_{E,u,s}(S_u K) + \alpha_{E,u,s} t,$$

for all $s \in \Sigma_E$ and $t \in [-1, 1]$. Proceeding as in Proposition 8.1.5 yields that the above extends to all $s \in \mathbb{R}$, a generalization of (8.5).

Let us define for $R > 0$

$$\tilde{L}_{E,u,s,R} := \left\{ (y, t) \in (E^\perp \cap u^\perp) \times \mathbb{R} : f^{(s)}(y, t) \leq R \right\}.$$

Since $f^{(s)}(y, t)$ is homogeneous in (y, s) , a rescaling yields

$$\tilde{L}_{E,u,s,R}(t) = R \tilde{L}_{E,u,s/R,1}(t),$$

for all $t \in [-1, 1]$ and therefore

$$\tilde{L}_{E,u,s,R}(t) = R L_{E,u,s/R}(S_u K) + R \alpha_{E,u,s/R} t,$$

again for all $t \in [-1, 1]$. Concerning evenness, we finally conclude

$$f^{(s)} \equiv R \text{ on both segments } \{(\pm y + R \alpha_{E,u,s/R} t, t) : t \in [-1, 1]\}$$

for every $y \in E^\perp \cap u^\perp$ with $f^{(s)}(y, 0) = R$. □

Step 2 - Segments of Constant Projections of K^w

For given $w \in \text{int } P_E K$, the set $K^w = (K - w) \cap E^\perp$ has a non-empty relative interior in $E^\perp$ and thus is a convex body. Since all $(K^w)_u(t)$, $t \in \mathbb{R}$, are convex bodies as well, $\{L((K^w)_u(t))\}_{t \in \mathbb{R}}$ are origin-symmetric convex bodies in $E^\perp$. Since $(K^w)_u(t) = (K_u(t))^w$ holds true, we may simply write $K_u^w(t)$ for the latter. If for $s \in \mathbb{R}$, we define

the function

$$f_w^{(s)}(y, t) := |P_{y+su} K_u^w(t)|, \quad (y, t) \in (E^\perp \cap u^\perp) \times \mathbb{R},$$

we have

$$L(K_u^w(t)) = \left\{ y + su : f_w^{(s)}(y, t) \leq 1 \right\},$$

and, due to (7.3),

$$f^{(s)}(y, t) = \int_{P_E K} f_w^{(s)}(y, t) dw.$$

Furthermore, Proposition 7.2.5 implies that $f_w^{(s)}$ is convex and even in (y, t) . Let us write $\Sigma_w(t)$ for the non-empty interior of the compact interval $P_{\text{span}(u)} L(K_u^w(t))$, viewed as an open interval in $\mathbb{R}$. We claim that

$$\Sigma_w := \Sigma_w(0) = \Sigma_w(t)$$

holds true for all $t \in [-1, 1]$. Since the projection of the polar body equals the polar body of the section, we obtain

$$P_{\text{span}(u)} L(K_u^w(t)) = P_{\text{span}(u)} (K_u^w(t) - K_u^w(t))^\circ = ((K_u^w(t) - K_u^w(t)) \cap \text{span}(u))^\circ.$$

Because of

$$K_u^w(t) = \bigcup_{y \in P_{u^\perp} K^w} (y + (c_w(y)t + [-\ell_w(y), \ell_w(y)])u)$$

for all $t \in [-1, 1]$, we deduce that

$$(K_u^w(t) - K_u^w(t)) \cap \text{span}(u) = \bigcup_{y \in P_{u^\perp} K^w} [-2\ell_w(y), 2\ell_w(y)]u$$

is independent of t .

Now, let $w_0 \in \text{int } P_E K$ and let $\pm y_s + su \in \text{bd}(L(K_{w_0}^{w_0}(0)))$ for $s \in \Sigma_{w_0}$, amounting to $f_{w_0}^{(s)}(\pm y_s, 0) = 1$. Furthermore, let us define $R_{y_s, s} := f^{(s)}(\pm y_s, 0)$. Because of the fact that the function $w \mapsto f_w^{(s)}(x)$, $w \in P_E K$ is continuous for every $x = (y, t)$ as well as $f_w^{(s)}$ are all convex and $f^{(s)} \equiv R_{y_s, s}$ on both segments $\{(\pm y_s + R_{y_s, s} \alpha_{E, u, s} / R_{y_s, s} t, t) : t \in [-1, 1]\}$, we can deduce from Step 2, that each $f_w^{(s)}$ must be constant on these segments as well. Especially

$$f_{w_0}^{(s)} \equiv 1 \text{ on both segments } \{(\pm y_s + R_{y_s, s} \alpha_{E, u, s} / R_{y_s, s} t, t) : t \in [-1, 1]\}.$$

Since $f_{w_0}^{(s)}$ is convex, we can conclude that

$$\pm y_s + R_{y_s, s} \alpha_{E, u, s / R_{y_s, s}} t + su \in \text{bd}(L(K_u^{w_0}(t))),$$

for all $s \in \Sigma_{w_0}, t \in [-1, 1]$ and $\pm y_s + su \in \text{bd}(L(K_u^{w_0}(0)))$.

Step 3 - Using $k = n - 1$

If $k = n - 1$, it holds that $\dim E^\perp = 2$, and therefore, for all $w \in \text{int } P_E K$, the set K^w is a two-dimensional convex body. For a set $A \subseteq E^\perp$, we write $A(s)$ for the one-dimensional chord $(A - su) \cap (E^\perp \cap u^\perp)$, which can be identified as a subset of $\mathbb{R}$. Step 2's analysis leads to the conclusion that

$$L(K_u^w(t))(s) = [-a_w(s), a_w(s)] + \Psi_w(s)t, \quad s \in \Sigma_w, \quad t \in [-1, 1] \quad (8.12)$$

for all $w \in \text{int } P_E K$.

Proposition 8.2.1 implies that $\Psi_w(s) = c_+^w s_+ - c_-^w s_-$ for some $c_\pm^w \in \mathbb{R}$ and all $s \in \text{cl}(\Sigma_w) = P_{\text{span}(u)} L(K_u^w(t))$, where the assertion on Σ_w extends by continuity of the mid-point to $\text{cl}(\Sigma_w)$. Since $L(K^w)$ is origin-symmetric as well as setting $t = 1$ in (8.12), we obtain that Ψ_w must be odd, and therefore we may define $c^w := c_+^w = c_-^w$. Finally, the mid-point of the chord of $L(K^w)$, which is perpendicular to u at height s , is $c^w s$ for all those s for which the chord is non-empty. That means, all mid-points lie on one single line. Notice that this holds true for every $u \in S^{n-1} \cap E^\perp$.

The following Theorem by Bertrand-Brunn can be found in [MMO19], Theorem 2.12.1.

Theorem 8.2.3. *A convex body $K \subseteq \mathbb{R}^n$ is an ellipsoid if and only if for any $u \in S^{n-1}$ the mid-points of all (one-dimensional) chords of K parallel to u lie in a hyperplane.*

The above Theorem of Bertrand-Brunn implies that for all $w \in \text{int } P_E K$, the set $L(K^w)$ must be an (origin-symmetric) ellipsoid. We are now able to conclude the proof for the case $k = n - 1$.

Step 4 - Finalizing the Proof

We are now able to establish the equality cases for $k = n - 1$. Since for every $w \in \text{int } P_E K$ we have $L(K^w) = T_w(B_{E^\perp})$ for some linear $T_w : E^\perp \rightarrow E^\perp$ and furthermore

$$L(K^w)(s) = L(S_u K^w)(s) + c_w s \quad \forall s,$$

applying Lemma 8.1.9, yields the existence of two orthogonal directions $u_1, u_2 \in S^{n-1} \cap E^\perp$ with

$$L(S_{u_i}T_w(K^w)) = L(T_w(K^w)) = B_{E^\perp} \quad i = 1, 2.$$

Moreover, due to Lemma 8.1.10, the map $T_w(K^w)$ is invariant (up to translation in the direction of u_i) under reflection about $u_i^\perp$. Thus, $T_w(K^w)$ has a point of symmetry for all $w \in \text{int } P_E K$, and so this holds for K^w as well. Since this is true for all $E \in \mathcal{G}(n, n-2)$, every two-dimensional section of K through its interior has a point of symmetry. In the case $n \geq 3$, applying Theorem 8.1.8 yields that K must be an ellipsoid. If $n = 2$, we have $E = \{0\}$ and therefore $\text{int } P_E K = \{0\}$ and $K = K^w$ for $w = 0$. Furthermore, since $T_0(K) = x_0 + C$ for an origin-symmetric convex body C , and

$$(2C)^\circ = (T_0(K) - T_0(K))^\circ = L(T_0(K)) = B_2^2,$$

this finally yields $C = \frac{1}{2}B_2^2$, which implies that K is an ellipsoid.

Bibliography

- [Bal88] Keith Ball. “Logarithmically concave functions and sections of convex sets in $R^{\{n\}}$ ”. eng. In: *Studia Mathematica* 88.1 (1988), pp. 69–84.
- [BCL09] M. Berger, M. Cole, and S. Levy. *Geometry II*. Universitext. Springer Berlin Heidelberg, 2009. ISBN: 9783540170150.
- [BLM06] J. Bourgain, J. Lindenstrauss, and V.D. Milman. “Estimates related to Steiner symmetrizations”. In: vol. 1376. Nov. 2006, pp. 264–273. ISBN: 978-3-540-51303-2.
- [Bru89] H. Brunn. *Über Curven ohne Wendepunkte*. Ackermann, 1889.
- [BS60] H. Busemann and E. G. Straus. “Area and Normality”. In: *Pacific Journal of Mathematics* (1960).
- [CG06] Stefano Campi and Paolo Gronchi. “On volume product inequalities for convex sets”. In: *Proceedings of the American Mathematical Society* 134 (Aug. 2006).
- [Coh80] D.L. Cohn. *Measure Theory*. Birkhäuser, 1980.
- [FT71] H. Furstenberg and I. Tzkon. “Spherical functions and integral geometry”. In: *Israel J. Math.* (1971).
- [Gar06] Richard J. Gardner. *Geometric Tomography*. 2nd ed. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2006.
- [Gar07] R.J. Gardner. “The dual Brunn–Minkowski theory for bounded Borel sets: Dual affine quermassintegrals and inequalities”. In: *Advances in Mathematics* 216.1 (2007), pp. 358–386. ISSN: 0001-8708.
- [Gri91] Eric L. Grinberg. “Isoperimetric inequalities and identities fork-dimensional cross-sections of convex bodies”. In: *Mathematische Annalen* 291 (1991), pp. 75–86.
- [Gru07] P.M. Gruber. *Convex and Discrete Geometry*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2007. ISBN: 9783540711339.

- [Had57] H. Hadwiger. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Comprehensive studies in mathematics. Springer, 1957. ISBN: 9780387021515.
- [Lut75] Erwin Lutwak. “Dual Mixed Volumes”. In: *Pacific Journal of Mathematics* 58 (1975), pp. 531–538.
- [Lut79] Erwin Lutwak. “Mean dual and harmonic cross-sectional measures.” In: *Ann. Mat. Pura Appl* (1979).
- [Lut84] Erwin Lutwak. “A General Isephipanic Inequality”. In: *Proceedings of the American Mathematical Society* 90.3 (1984), pp. 415–421. ISSN: 00029939, 10886826.
- [Mag12] Francesco Maggi. *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2012.
- [MMO19] Horst Martini, L. Montejano, and Deborah Oliveros. *Bodies of Constant Width: An Introduction to Convex Geometry with Applications*. Jan. 2019. ISBN: 978-3-030-03866-3.
- [MY23] Emanuel Milman and Amir Yehudayoff. “Sharp Isoperimetric Inequalities for Affine Quermassintegrals”. In: *Journal of the American Mathematical Society* 36, 1061-1101, (2023).
- [San49] L. A. Santaló. “An affine invariant for convex bodies of n-dimensional space (Spanish)”. In: *Portugal. Math.* 8 (1949).
- [Sch05] R. Schneider. *Konvexgeometrie, Vorlesung im WS 2004/05*. 2004/05.
- [Sch13] Rolf Schneider. *Convex Bodies: The Brunn–Minkowski Theory*. 2nd ed. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2013.
- [She64] Geoffrey C. Shephard. “Shadow systems of convex sets”. In: *Israel Journal of Mathematics* 2 (1964), pp. 229–236.
- [SW08] Rolf Schneider and Wolfgang Weil. *Stochastic and integral geometry*. Vol. 1. Springer, 2008.
- [SW92] R. Schneider and W. Weil. *Integralgeometrie*. Teubner Skripten zur Mathematischen Stochastik. Vieweg+Teubner Verlag, 1992. ISBN: 9783519027348.