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DIPLOMARBEIT

Reassessing Quantum Processes: A Comprehensive Analysis

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unter der Anleitung von

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durch

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Wien, am 15. Jänner 2025

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Abstract

This thesis explores the fundamental aspects of categorical quantum mechanics and its graphical calculus. We introduce the theory of classical-quantum processes to illustrate how information flows between the classical and quantum realms. With these tools in hand, we address several theorems in quantum foundations. Our goal is to express Peres-Mermin's proof of the Kochen-Specker theorem within a quantum process framework, but we encounter challenges that prevent us from fully completing the proof. This negative result suggests a potential incompatibility between categorical quantum mechanics and quantum logic.

Acknowledgements

At this point, I would like to take the opportunity to express my gratitude to everyone who supported and accompanied me during the completion of this thesis.
My thanks go to my supervisor, Karl Svozil, who has always been available for constructive feedback and my questions about the subject matter. He has not only been a knowledgeable expert in his field, but also a mentor on various different subjects extending far beyond physics. Through the several walks we took together he gave me valuable advice that I will continue to hold dear.
I want to thank my parents for their unwavering support and belief in me that made this possible. Even though they haven't discovered their love for physics yet, they always lent me an ear when I tried to recapitulate my newly learnt subject matter. They will always be my biggest source of inspiration and I aspire to be like them.
I am also very grateful for the many friends I made during my studies. Together we have struggled through exercise sheets and tried to work out difficult topics. I couldn't imagine completing my studies without the shared discussions and the exchange of ideas. You have completely enriched this experience and I would not have been able to do it without you.

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Chapter 1

Introduction

In 1932, John von Neumann's publication "Mathematische Grundlagen der Quantenmechanik" [1] provided a rigorous mathematical foundation for quantum mechanics, marking a significant milestone in the theory's development. However, just three years later, von Neumann expressed doubts about his formalism in a letter to Garrett Birkhoff, stating that "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more" - sic [2]. Despite this, the Hilbert space formalism remains a cornerstone of quantum mechanics education and practice up to this day.

Throughout the 20th century, foundational issues in quantum mechanics, such as quantum non-locality and the measurement problem, challenged the theory's validity. Quantum non-locality, exemplified by the EPR paradox [3], describes nonclassical correlations between distant quantum systems that cannot be explained by past interactions, while the measurement problem concerns the unexplained collapse of the wavefunction and the inherent non-determinism in quantum measurements [4]. These paradoxes led some to question the completeness of quantum theory, but robust experimental confirmations and discoveries, like quantum teleportation [5] and quantum key exchange [6], have reinforced the validity of the theory.

In recent decades, the field of quantum information has emerged, embracing these peculiarities of quantum mechanics not as bugs, but as features. This shift in perspective has allowed the development of fundamentally new quantum algorithms, most notably Shor's algorithm [7], which have drastically changed the course of modern cryptography.

Nevertheless, one has to wonder why it took over 60 years to discover the possibility of teleporting quantum states. A possible reason for this might be that the standard Hilbert space formalism of quantum mechanics does not paint the clearest picture of the underlying quantum processes. An attempt to alleviate this issue has been made by Abramsky and Coecke [8] with their formalism of categorical quantum mechanics. Categorical quantum mechanics describes quantum processes and their composition within a diagrammatic language. A few simple diagrammatic rules determine its graphical calculus, which enables one to argue about quantum processes in a *high-level* language.

The mathematical backbone of this formalism is *monoidal category theory*, a branch of category theory that originated from pure mathematics, that has been applied to logic in computer science [9] and semantics in natural language [10]. Monoidal categories are particularly effective in representing the algebra of linear maps in finite-dimensional Hilbert spaces, making them well-suited for quantum mechanics. Even though we will mostly discuss its applications to quantum processes, many of the theorems also have applications in some of these completely different fields of study, due to the general and abstract nature of category theory.

The following chapters will give a brief introduction to the field of categorical quantum mechanics guided by the introductory book "Picturing quantum processes" by Bob Coecke and Aleks Kissinger [11]. The definitions and proofs have been adopted from the book, although minor adjustments have been made for some of the diagrams.

Chapter 2

Process theories

Many different kinds of systems admit a description in terms of a process theory. From the computer program that takes data as input and outputs a different kind of data to proofs of propositions in formal logic. The processes we are going to be concerned with are the processes of quantum theory. Any such process theory admits a graphical description in terms of diagrams. Informally speaking:

Definition 1. *A process theory consists of:*

1. *a collection T of system-types represented by wires*
2. *a collection P of processes represented by boxes, where for each process in P the input types and output types are taken from T*
3. *a means of composition of diagrams. We can wire processes together to get different processes.*

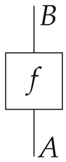
A theory of systems with sequential and parallel composition of processes is mathematically described by a *monoidal category* [12]. Physically speaking, the sequential composition of processes represents a "time-like" separation, whereas the parallel composition represents a "space-like" separation. All of the entailed algebra of a monoidal category can be fully expressed within a graphical calculus, where the equations are governed purely by diagrammatic rules. These diagrams are complete for monoidal categories [11], which means that an equational statement between formal expressions in the language of (symmetric) monoidal categories holds if and only if it is derivable in the graphical calculus.

Definition 2. *A monoidal category $\mathcal{C}$ consists of:*

1. *a collection $ob(\mathcal{C})$ of* objects.
2. *for every pair of objects A, B, a set $\mathcal{C}(A,B)$ of* morphisms.
3. *for every object A, a special identity morphism: $1_A \in \mathcal{C}(A,A)$.*
4. *a sequential composition operation for morphisms: $\circ : \mathcal{C}(B,C) \times \mathcal{C}(A,B) \to \mathcal{C}(A,C)$.*
5. *a parallel composition operation for objects: $\otimes : ob(\mathcal{C}) \times ob(\mathcal{C}) \to ob(\mathcal{C})$.*
6. *a unit object: $I \in ob(\mathcal{C})$.*
7. *a parallel composition operation for morphisms:*

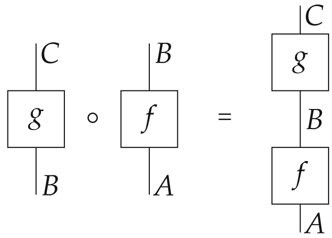
$$\otimes : \mathcal{C}(A,B) \times \mathcal{C}(C,D) \to \mathcal{C}(A \otimes B, C \otimes D).$$

$\{A, B, C, D, I\}$ (1)



(2)

(3)



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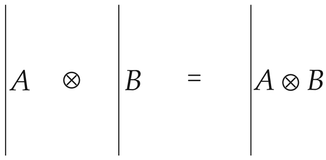
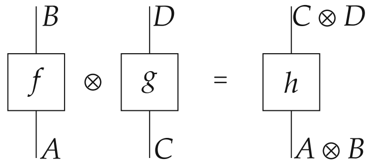


(5)

$I =$

(6)



(7)

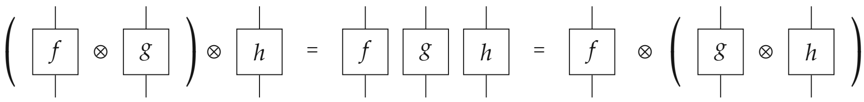
satisfying the following equations:

1. *$\otimes$ is associative and unital on objects:*

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \qquad A \otimes I = A = I \otimes A$$

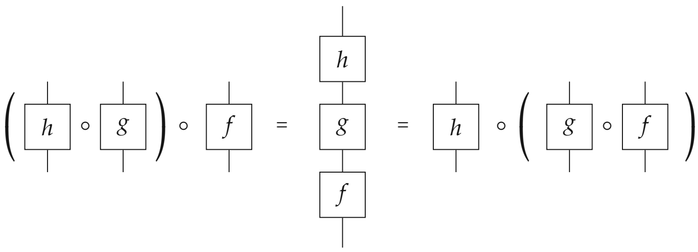
2. *$\otimes$ is associative and unital on morphisms:*

$$(f \otimes g) \otimes h = f \otimes (g \otimes h) \qquad f \otimes 1_I = f = 1_I \otimes f$$



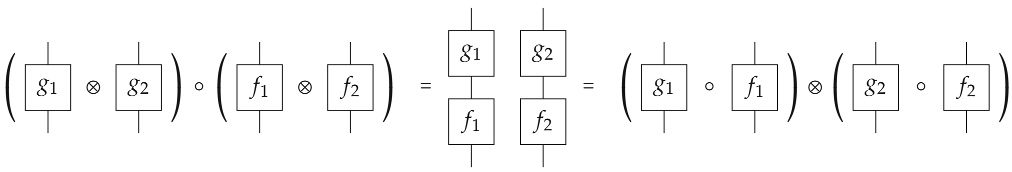
3. *$\circ$ is associative and unital on morphisms:*

$$(h \circ g) \circ f = h \circ (g \circ f) \qquad 1_B \circ f = f = f \circ 1_A$$



4. *$\otimes$ and $\circ$ satisfy the* interchange law*:*

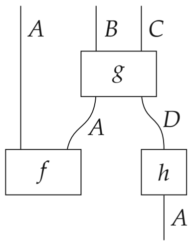
$$(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)$$



Using the above mentioned definitions, we can illustrate the following sample process:

$$(1_A \otimes g) \circ (f \otimes h) \quad : \quad A \to A \otimes B \otimes C$$

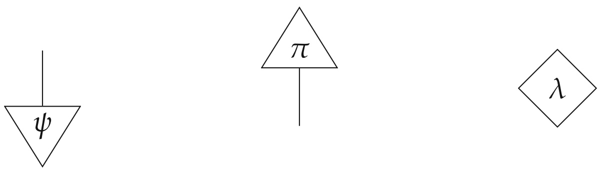
in the form of a simple diagram:



A composite process consisting of multiple processes that are combined by parallel and sequential composition.

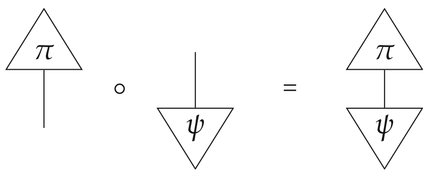
2.1 States, effects and numbers

In the process theory of quantum systems, there are some special processes we call states and effects. *States* are processes without any inputs. For a system of type A there are multiple possible states the system could be in. The particular state ψ then denotes the singular result of a certain preparation process. *Effects* are dual to states, as they are processes with no outputs. Operationally they correspond to measurements or tests, where the system is destroyed after measurement, or to processes where the system is simply just discarded. A *number* is a process without any inputs or outputs.



States have no inputs, effects have no outputs and numbers are processes without any inputs or outputs.

When one sequentially composes a state and an effect, a number results. In our theory of quantum processes, the number will be interpreted as a kind of probability measure for a positive outcome when testing the state ψ for the effect π, similar to the Born rule.

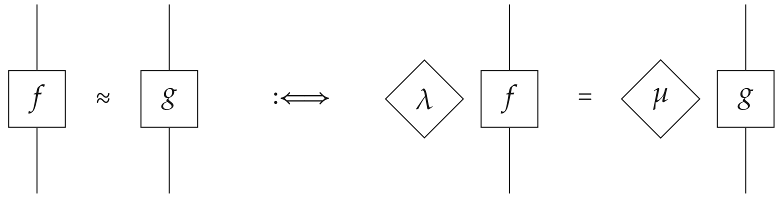


The state ψ is tested for the effect π, resulting in a number proportional to the probability of a positive test outcome.

2.2 Equality up to a number

In some instances we are only interested in some qualitative feature of a process, e.g. if a process separates into disconnected pieces. For those reasons we will define a notion of equality up to a number, which only preserves the structural features of a diagram.

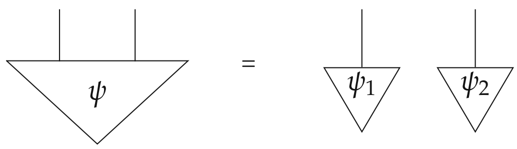
Definition 3. *Two processes are equal up to a number, if there exist two non-zero numbers* λ, μ *such that:*



2.3 Separability

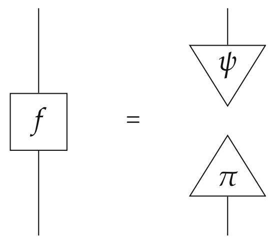
One of the most crucial and foundational aspects of quantum mechanics is entanglement. Entanglement shows that there are multipartite systems that cannot be described merely as the sum of their individual parts. To be able to translate this phenomenon into our graphical calculus, we will first define a notion of separability.

Definition 4. *A bipartite state* ψ *is a state of two systems, and we call such a state* $\otimes$*-separable if there exist states* ψ_1 *and* ψ_2 *such that:*



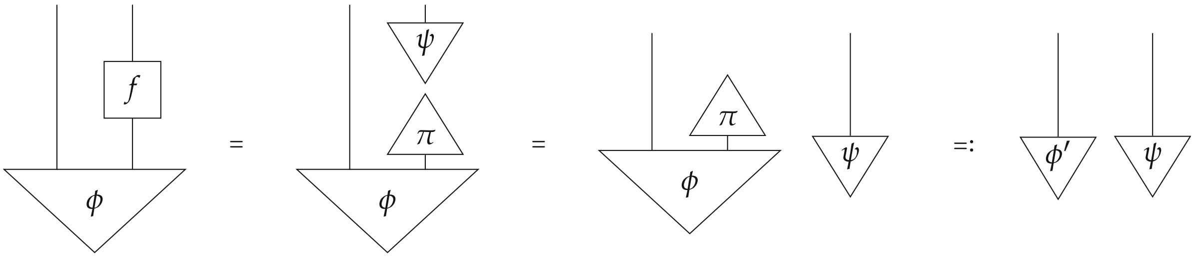
Similarly to $\otimes$-separability we will also define the following notion:

Definition 5. *A process* f *is* $\circ$*-separable if there is an effect* π *and a state* ψ *such that:*

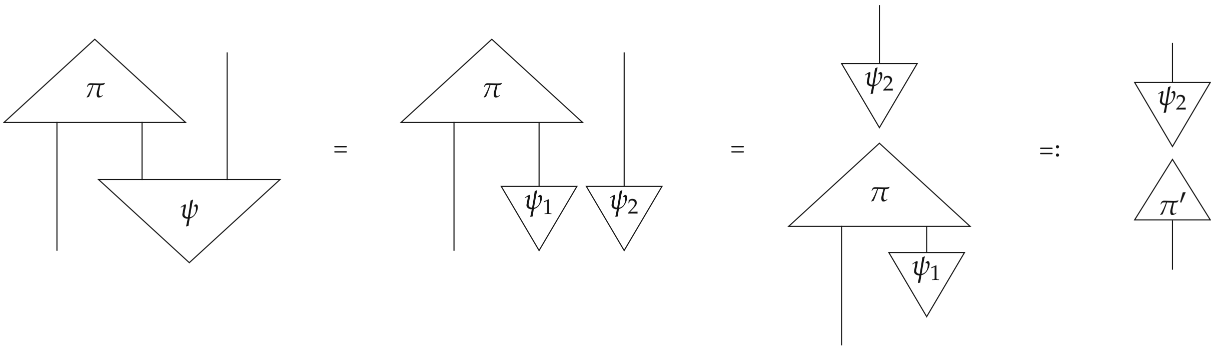


Both kinds of separability correspond to a disconnected diagram.

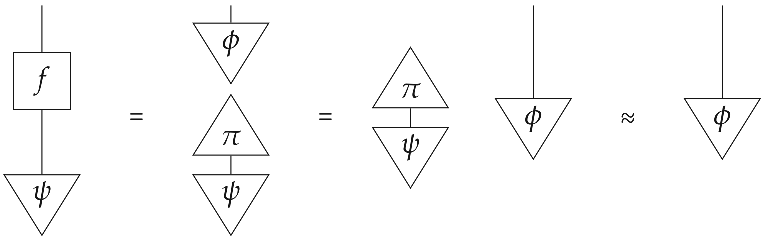
These notions of separability are also related to each other in the following way. If a process f is $\circ$-separable, then for any bipartite state ϕ also the following state is $\otimes$-separable:



And vice versa, if the bipartite state ψ is $\otimes$-separable, then for any bipartite effect π the following process is $\circ$-separable:



One might wonder how a process theory looks like, if all processes f are $\circ$-separable. The answer is that such a theory can only describe constant processes:



Since for any arbitrary input state ψ that we feed into the process f, we always receive an output state which is (up to a number) equal to ϕ. This means that all processes are essentially constant processes, or in other words nothing ever happens, as we can not describe the temporal evolution of a system. This of course contradicts our everyday experience of physical reality, meaning that *any process theory that describes physical reality has to contain processes, that are not* $\circ$*-separable.* In the later chapters of this thesis we will use this fact to prove several 'no-go' theorems. We will show that certain assumptions imply that every process is $\circ$-separable, and therefore arrive at a contradiction.

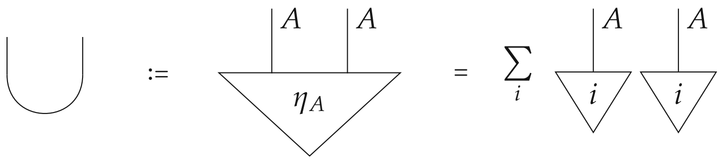
2.4 Process-state duality

Another feature of quantum mechanics is that processes and bipartite states are in one to one correspondence. This is called the Choi-Jamiolkowski isomorphism [13][14] which utilises maximally entangled states to create this bijection. In our formalism we will represent the maximally entangled state and its related effect as the cup-state and the cap-effect.

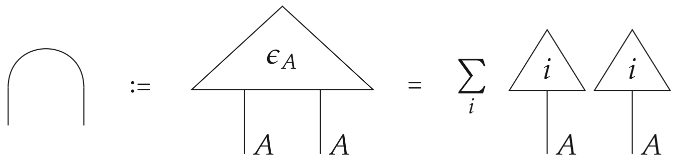
$$\sum_i |ii\rangle \qquad \longleftrightarrow \qquad I \xrightarrow{\eta_A} A \otimes A$$

$$\sum_i \langle ii| \qquad \longleftrightarrow \qquad A \otimes A \xrightarrow{\epsilon_A} I$$

Or diagrammatically the cup-state and the cap-effect will be represented as:

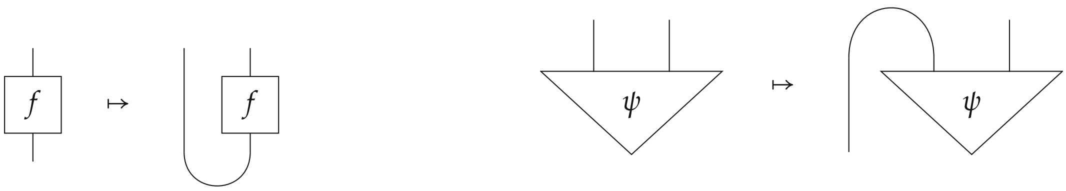


Cup-state

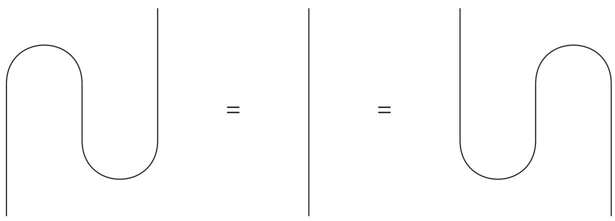


Cap-effect

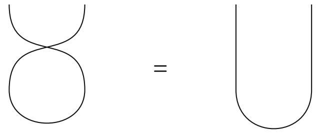
These states and effects can now be used to convert bipartite states into processes and vice versa.



To ensure that this process-state duality is really a bijection, we then further require the newly defined cup-state and cap-effect to be inverse to each other:



And that the cup-state is symmetric under exchange of its outputs:

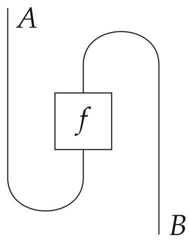


Graphically these equations can be interpreted as a bent wire being straightened by pulling on its ends, therefore we will sometimes also refer to these equations as *yanking equations*.

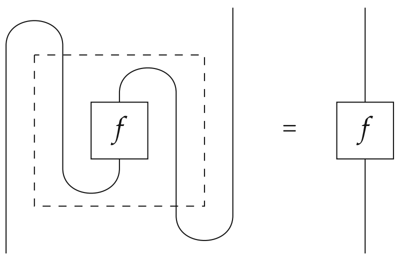
2.5 Transposition and Trace

We have already seen that we can create new processes by adjoining cups and caps to processes. The transposition and the trace of a process are two important examples of such new processes. For the *transpose*, we take all the inputs and turn them into outputs by adjoining a cup-state and turn the outputs into inputs.

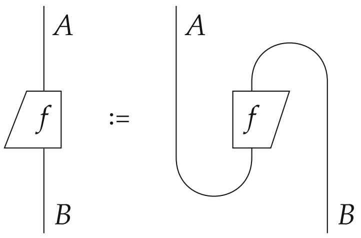
Definition 6. *The transpose f^T of a process f is another process:*



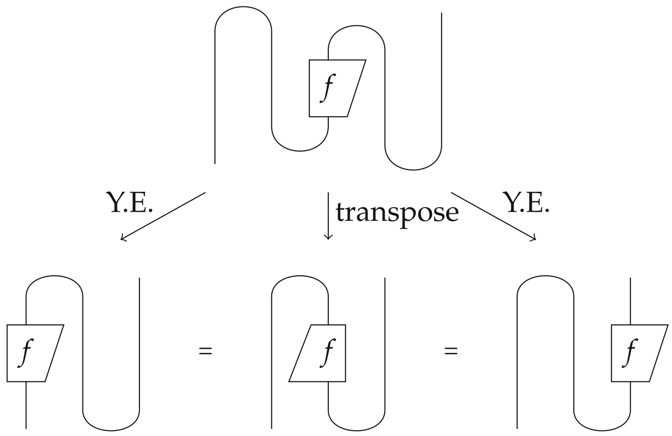
We also notice that the transposition of a process is involutive, since transposing f^T again equals f simply by applying the yanking equation.



We will now introduce a slight modification to the boxes we used to denote processes. From now on we will display them as trapezia with one side slightly skewed. By doing so, we can indicate a transposed process as a trapezium that has been rotated by 180°.

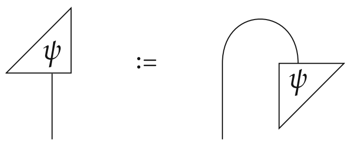


With this new notation, we can also prove the *sliding rule*, which graphically speaking tells us that we can treat processes similar to beads we can slide along a necklace:



A similar modification will be made for states to express the transpose of states:

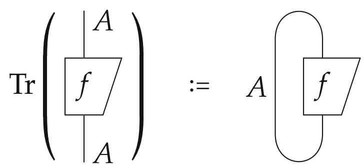
The transpose can now be expressed by a state that has been rotated by 180°:



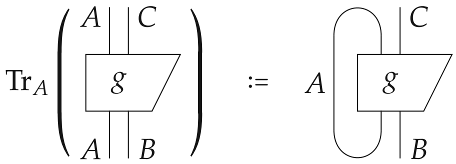
2.6 Trace and Partial Trace

Using both a cup-state and a cap-effect, we can transform a process into a number. This is also called taking the trace of a process.

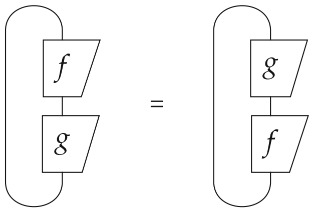
Definition 7. *For a process f where the input type matches the output type, the trace of this process is:*



For processes with multiple inputs and outputs, with a matching input and output type, we can define the *partial trace*:



One can now easily prove the cyclicity of the trace by just moving the boxes along the wire using the sliding rule we previously derived:



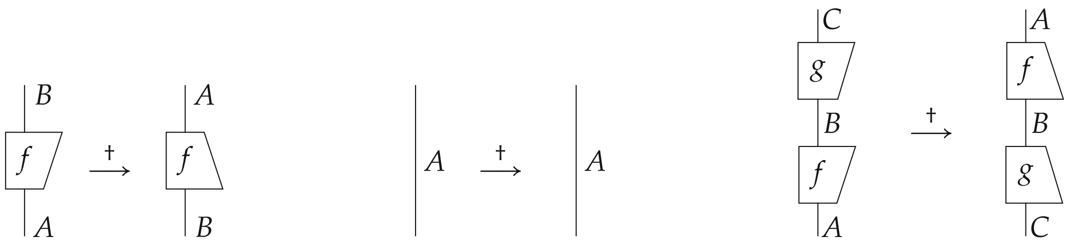
$$\Longleftrightarrow \qquad \mathrm{Tr}\{(f \circ g)\} = \mathrm{Tr}\{(g \circ f)\}$$

2.7 Adjoint Processes

Since quantum-mechanical observables are described by self-adjoint operators, we will also have to define the notion of adjoint processes. Diagrammatically, the operation of adjoining a process is represented by flipping a process upside down. We denote this transformation by a †-symbol and the resulting process theory is called a *dagger monoidal category*:

Definition 8. *A dagger monoidal category is a monoidal category C equipped with an involutive dagger structure † such that:*

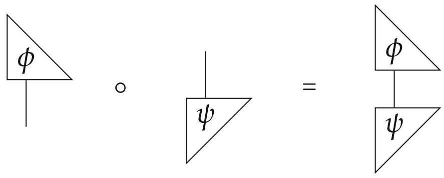
1. *for all morphisms $f : A \mapsto B$, there exists its adjoint $f^\dagger : B \mapsto A$.*
2. *for all morphisms f, $(f^\dagger)^\dagger = f$.*
3. *for all objects A, $id_A^\dagger = id_A$.*
4. *for all $f : A \mapsto B$ and $g : B \mapsto C$, $(g \circ f)^\dagger = f^\dagger \circ g^\dagger : C \mapsto A$.*



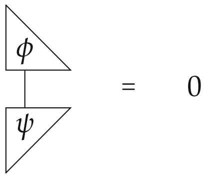
(1) (3) (4)

Using the adjoint, we can now define the inner product of two states $\psi : I \mapsto A$ and $\phi : I \mapsto A$ as the number $\phi^\dagger \circ \psi : I \mapsto I$.

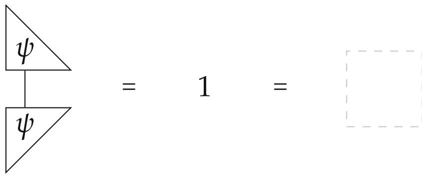
Definition 9. *The* inner product *of states ψ and ϕ of the same type is:*



two states are called orthogonal *if:*



The squared-norm *of a state ψ is the inner product of ψ with itself. A state ψ is called* normalised *if its squared-norm is equal to* 1*:*

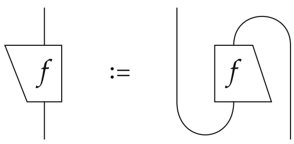


In this equation, we have also encountered the empty diagram, which is equal to 1, as it can be multiplicatively added without changing the diagram.

2.8 Conjugate Processes

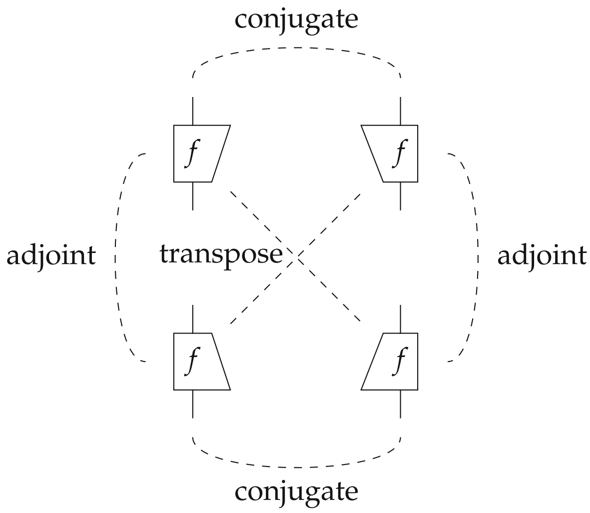
We will now go on to combine the operations of adjoining and transposing a process. This rotates the process by 180° and then reflects it vertically, resulting in a horizontal reflection. The order in which we apply these two transformations does not matter, as both operations mutually commute.

Definition 10. *The* conjugate *of a process is the transpose of its adjoint $\overline{f} := (f^T)^\dagger = (f^\dagger)^T$:*

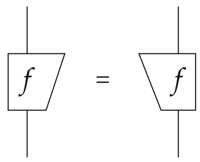


As with the adjoint or transpose, conjugation is an involutive operation, as two horizontal reflections result in the identity transformation.

The following diagram summarizes the relations between the three transformations:



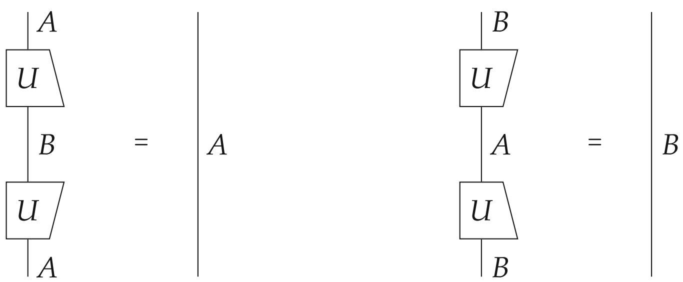
A process is termed *self-conjugate* if it is invariant under conjugation or diagrammatically:



2.9 Unitarity

We define unitary processes as those that preserve the inner product between any two states when applied to both states.

Definition 11. *A process U is* unitary *if $U^\dagger$ is a two-sided inverse of U:*

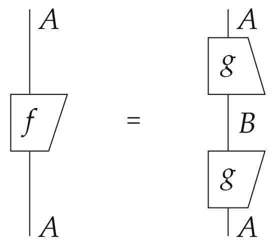


If only the weaker condition on the left side $U^\dagger U = 1_A$ is satisfied, then the process U is called isometry. The condition on the right side $UU^\dagger = 1_B$ defines a coisometry.

2.10 Positivity

In quantum theory the density operator for a system is given by a *positive* semi-definite Hermitian operator of trace one. We are now going to look into how this concept of positivity translates into our diagrammatic formalism:

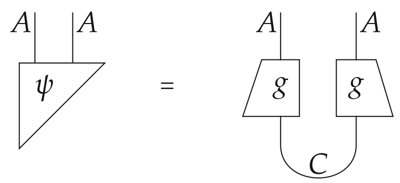
Definition 12. *A process f is* positive *if for some g we have:*



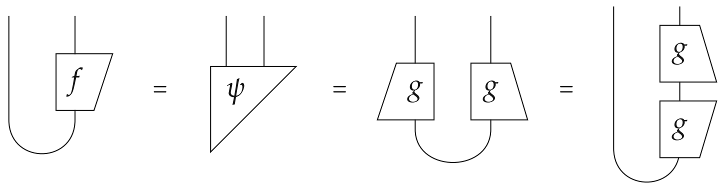
From this definition follows, that every positive process f is also *self-adjoint*, as every positive f is also invariant under vertical reflection.

We can extend the notion of positivity to include bipartite states:

Definition 13. *A bipartite state ψ is* ⊗-positive *if for some g we have:*



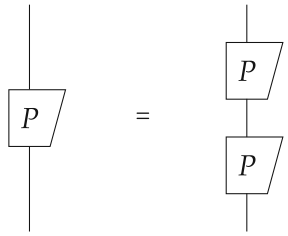
Due to process-state duality both of these traits of positivity are closely related. Since a bipartite state ψ is ⊗-positive if and only if the process corresponding to it by process-state duality is positive:



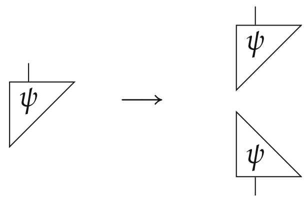
2.11 Projectors

We also have to define projectors, as they are needed to describe the quantum mechanical process of measurement.

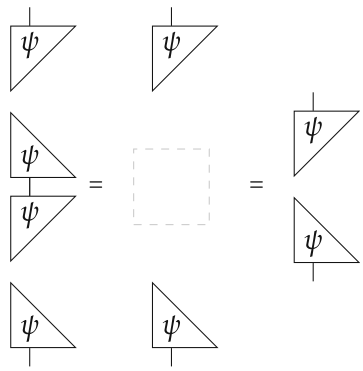
Definition 14. *A* projector *is a process P that is positive and* idempotent*:*



For every normalized state ψ we can construct its matching projections operator $\psi \circ \psi^{\dagger}$:



This projector is obviously positive and, as can be quickly checked, also idempotent:

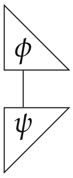


These types of projection operators are called *separable* projection operators, which correspond to projections on to the one-dimensional subspace spanned by ψ.

Chapter 3

Quantum processes

Up to this point we have mostly looked into the process theory of **linear maps**, where the system types were Hilbert spaces or vector spaces and the processes were linear maps on these spaces. To construct a theory of **quantum processes** we have to slightly modify our theory, since the process theory of **linear maps** contains redundant data in the form of a 'global phase', which has no effect on any measurable observable. Furthermore, the Born rule we previously defined:



$\in \quad \mathbb{C}$

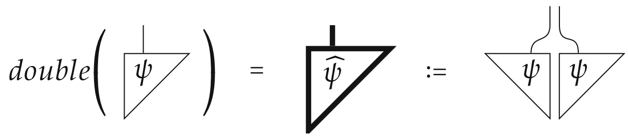
produces complex numbers and is therefore not apt to describe the probabilities of certain events, which have to be real numbers from the unit interval. To resolve both of these problems we will introduce a new procedure called doubling. When a process from **linear maps** is doubled, it turns into a **pure quantum map**.

We will then also introduce an entirely new process called *discarding*, which is a new effect, that corresponds to ignoring parts of a larger system. By combining the theory of **pure quantum maps** with the discarding effect, we obtain the theory of **quantum maps**, which also includes impure processes (= processes that cannot be written as a doubled process from **linear maps**).

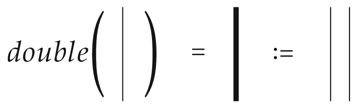
The last step in this process to describe the complete theory of **quantum processes** is to add nondeterminism to our theory. In quantum mechanics there is an irreducible uncertainty as to which quantum map (e.g. measurement outcome) actually results from a measurement. To account for this fact, we will define **quantum processes** as a set of **quantum maps** that collectively satisfy the *causality postulate* we will define later on.

3.1 Pure quantum maps

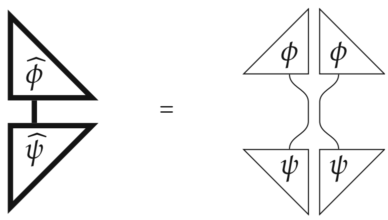
Multiplying a state ψ with a complex number of the form $e^{i\alpha}$ with $\alpha \in \mathbb{R}$ should not lead to different predictions. We can therefore get rid of these so-called 'global phases'. Since we want to produce probabilities for measurement outcomes, we also want the resulting numbers to be real numbers from the unit interval. This is done by multiplying processes of **linear maps** with their conjugate counterparts. By doing so we get a **pure quantum map** indicated by a thick border and a wedge symbol:



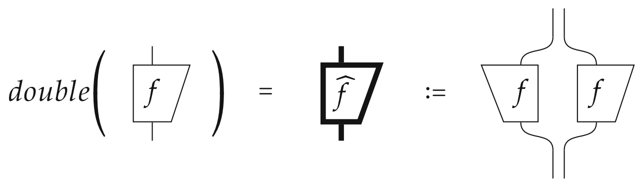
Similarly, we get our new identity for this type of quantum system by doubling the identities of **linear maps**:



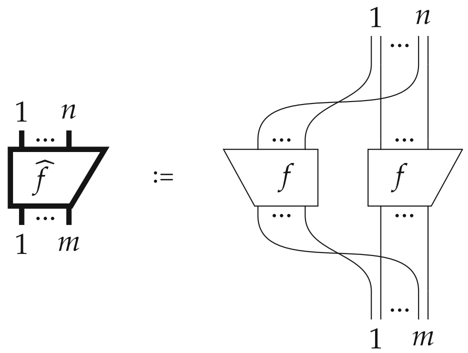
Our new Born rule for **pure quantum maps** then looks very similar to the one in **linear maps** with the difference being that we now test the doubled state ϕ for the doubled effect ϕ resulting in a real number that measures the probability of a successful test outcome, e.g. a click in a detector:



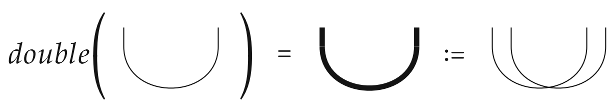
We are now going to define **pure quantum maps** by doubling processes from the theory of **linear maps**:



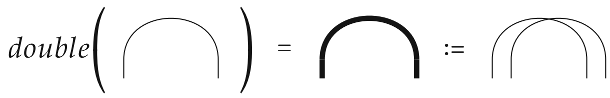
This notion then extends to processes with multiple inputs and outputs, but we have to remember to pair up the inputs and outputs that correspond to each other by conjugation:



Another very important bipartite state we have to consider is the doubled cup-state and its adjoint the doubled-cap effect:



Doubled cup-state

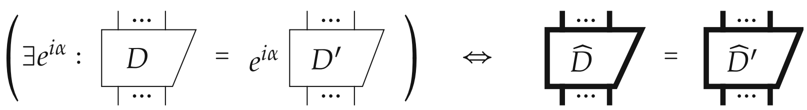


Doubled cap-effect

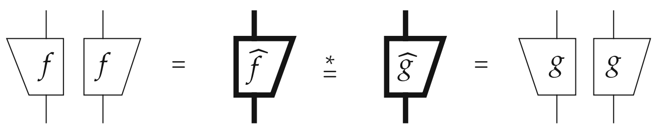
One can then show that these new cup and cap processes still satisfy the yanking equations and that doubling a transposed process f from **linear maps** is equal to transposing a doubled process $\widehat{f}$ from **pure quantum maps**.

To sum up, the same equations that hold in **pure quantum maps** also hold in **linear maps** up to a complex 'global phase':

Theorem 1. *Let D and D′ be arbitrary diagrams in* ***linear maps****, and $\widehat{D}$ and $\widehat{D}'$ be their doubled versions in* ***pure quantum maps****, then:*

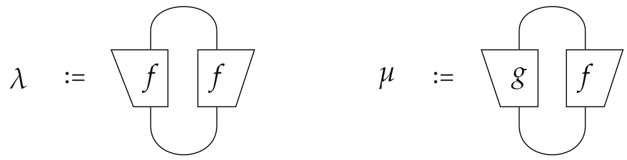


Proof. (⇒) follows directly from doubling the left hand side of the diagram. For (⇐) we need to show that:

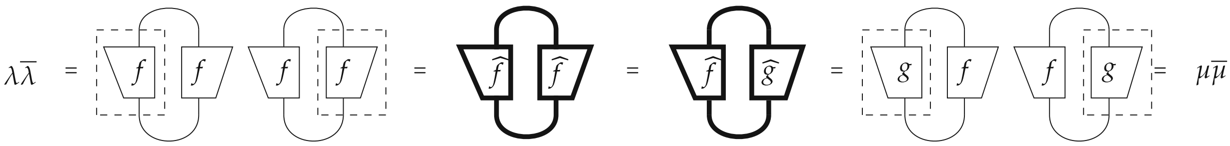


the starred equation implies that the **linear maps** f and g only differ by a global complex phase.

Let λ and μ be defined as:



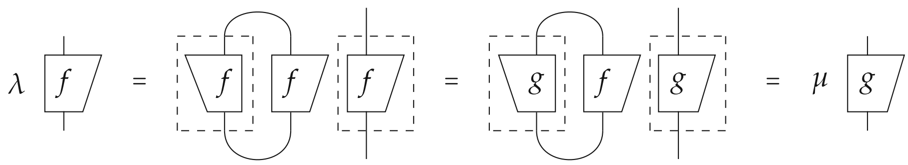
Then:



where the dashed lines indicate the use of (*). Now there are two cases, either $\lambda \neq 0$ or $\lambda = 0$. In the first case, we can divide both sides of the equation by $\lambda\overline{\lambda}$:

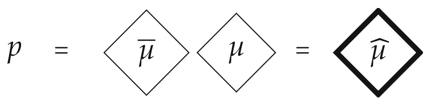
$$1 = \frac{\mu\overline{\mu}}{\lambda\overline{\lambda}} = \left(\frac{\mu}{\lambda}\right)\overline{\left(\frac{\mu}{\lambda}\right)}$$

So therefore $\frac{\mu}{\lambda} = e^{i\alpha}$ is a global phase for some $\alpha \in \mathbb{R}$. And thus:

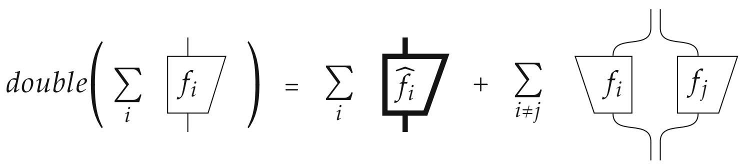


So we indeed see that $f = e^{i\alpha}g$. In the second case it follows from positive definiteness, that $f = 0$ and $\widehat{f} = 0$ and therefore also g and $\widehat{g}$ have to be 0. So $f = e^{i\alpha}g$ is also satisfied. □

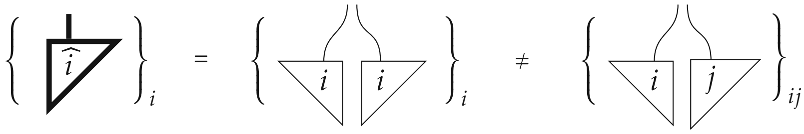
The only things that do change from the switch of **linear maps** to **pure quantum maps** are the numbers, sums and ONBs. As already mentioned the numbers of **pure quantum maps** only include positive real numbers as every doubled number is multiplied by its complex conjugate:



Another important change is that the doubled sum of processes is not equal to the sum of the doubled processes, since we get additional terms where the indices mismatch:

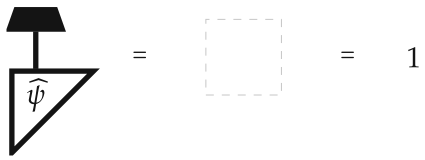


The same goes for the bases of our Hilbert spaces, since a doubled base for type A does not suffice as a base for the product space $A \times A$:



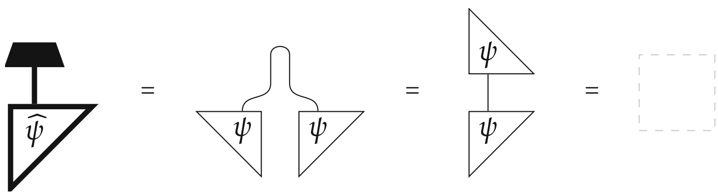
3.2 Discarding effect

For some applications in quantum theory, we are not interested in every subsystem of a larger system. This might be due to the fact that we are only interested in a specific measurement outcome of a subsystem or due to parts of our system not being accessible to us, e.g. because of spatial separation. In our theory we will model this procedure by the process of discarding a system. When any arbitrary normalized state $\widehat{\psi}$ is discarded it should just disappear from our diagram, meaning it should equal the empty diagram:



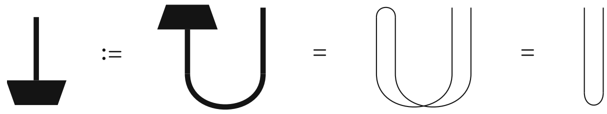
But there is no pure quantum effect that is non-orthogonal to all pure quantum states and therefore the discarding map can not be a **pure quantum process**. Thus we are going to define our first impure quantum map (:= a map that cannot be written as a doubled linear map):

When applying this discard effect to any normalised state $\widehat{\psi}$, we indeed see that we obtain the empty diagram:

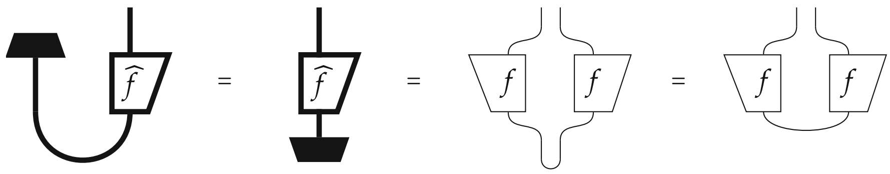


One then might ask if there is another effect with the same property of sending all normalised states to the empty diagram. It turns out that the discarding map we defined is unique, because if there was another quantum effect with the same property it would agree on all basis states of the Hilbert space and therefore has to be equal to the discarding map we defined.

We can now also consider the transpose of the discarding map by turning its input wire into an output wire with the quantum cup state:

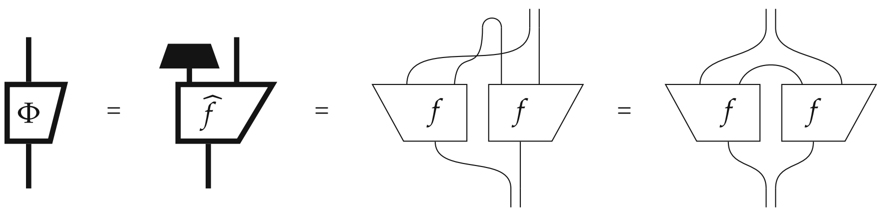


We will call this state the maximally mixed state, which results from discarding half of a Bell state. Now every impure quantum state can be seen as a result of feeding the maximally mixed state into a pure quantum map or equivalently as discarding one half of the bell state and applying a pure quantum map to the other half:



We now have all necessary tools to define the entire process theory of **quantum maps**.

Definition 15. *The process theory of **quantum maps** consists of all diagrams made of **pure quantum maps** (= linear maps that were doubled) combined with the discarding map. So every quantum map* Φ *can be written in the form of:*

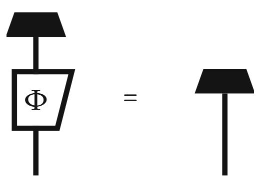


The pure quantum map $\widehat{f}$ then is called the *purification* of Φ.

3.3 Causality

As a concept causality stems from the theory of relativity, where space-like separated events are in no causal relationship, meaning that they cannot influence each other. In our theory causal processes (and especially causal states) are processes that when their outputs are discarded, they might as well have never happened. In other words, if a quantum process happens somewhere far away and we have no access to the system, we can ignore it. We can summarise this as:

Definition 16. *A quantum map* Φ *is called* causal *if it satisfies:*

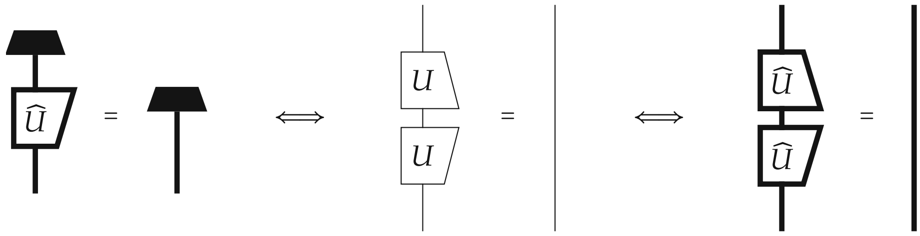


or similarly:

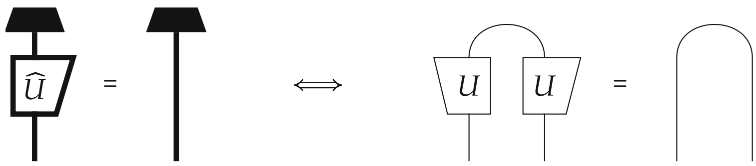
Definition 17. *A quantum state ρ is called* causal *if it satisfies:*

A very interesting consequence of this is the following theorem:

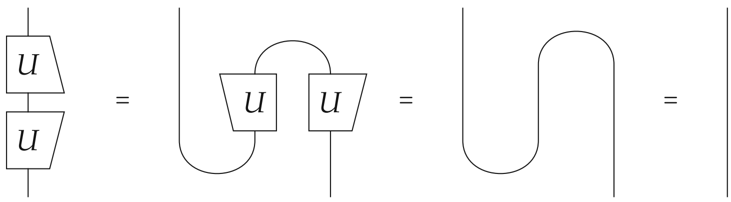
Theorem 2. *All causal* ***pure quantum maps*** *are isometries.*



Proof. For the proof of this statement, we can unfold the causality equation:



By bending up the left wire and then applying the yanking equation, we then end up with the isometry equation:



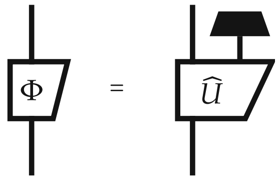
Doubling this equation then shows that this also holds for the doubled pure quantum process $\widehat{U}$.

□

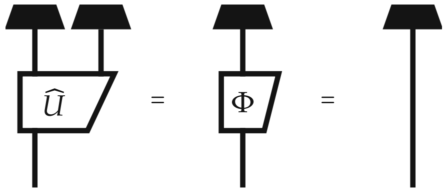
3.4 Stinespring Dilation

We already saw that any quantum map Φ can be purified, meaning that it is equal to a pure quantum map, with one of its outputs discarded. It now also follows that:

Theorem 3. *For any causal quantum map Φ there exists an isometry $\widehat{U}$ such that:*

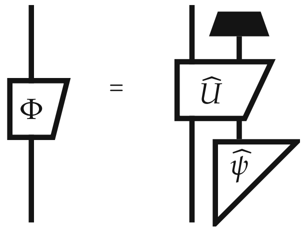


Proof. We have established that any quantum map Φ can be extended to a pure quantum process $\widehat{U}$, with one of its outputs being discarded. Furthermore, due to the causality condition of Φ, it follows that the corresponding $\widehat{U}$ must also adhere to causality:



As we already established in Theorem 2, this implies that $\widehat{U}$ is an isometry. □

We can then even extend this result by showing that every causal quantum map Φ can be realized by feeding a pure quantum state ψ into a unitary transformation $\widehat{U}$ and then discarding one of its outputs:

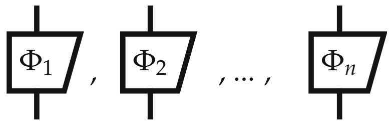


The proof of this version of the Stinespring dilation can be found in [11, p. 316].

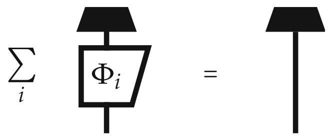
This conclusion leads some physicists to believe that there only exist unitary pure quantum processes in a larger Hilbert space, of which we can only access a smaller part. This point of view is consistent with quantum theory, and as it is neither provable nor disprovable, it is rather a question of belief rather than a scientific question. We will continue to use all kinds of quantum maps to avoid over-complicating our diagrams.

3.5 Non-deterministic quantum processes

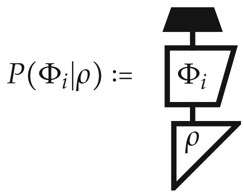
Up to this point, we have only encountered a single causal quantum effect, namely the discarding map. Furthermore, the discarding map is deterministic, as the same result always happens: the state is discarded. However, there are many more possible quantum effects that are non-deterministic, which means that the actual quantum effect that is realized is completely random and unpredictable. Such a non-deterministic quantum process consists of a collection of quantum maps:



which together satisfy the causality postulate:



The individual quantum maps Φ_i are called *branches* and when a system is passing through such a quantum process, only one of the branches actually takes place. The particular branch that occurs is called *outcome* of the process. Given a causal quantum state ρ we can assign probabilities to each of the branches by the number:



This generalized Born rule shows us the probability of each of the branches and because of the causality postulate they all have to add up to 1.

3.6 Quantum measurements

In quantum mechanics, measurements play a crucial role as they are the only way to extract information from a quantum state. But unlike the classical idea of a measurement as a mere act of observation that simply reveals a preexisting property of the system while leaving the system unchanged, this is no longer the case for a quantum measurement.

For a given state of a quantum system and a specific quantum measurement, there is a range of possible measurement outcomes. It is not possible to predict which of the potential outcomes is actually realized, but only to assign them a probability of occurring. Also, the measurement irreversibly changes the state of the system in accordance with the measurement outcome and leaves no traces behind as to what the quantum state previously was.

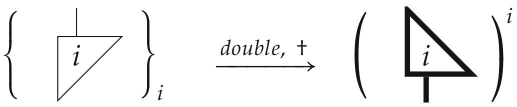
We will be describing quantum measurements as non-deterministic quantum processes, with three different kinds of possible measurements:

1. **ONB** measurements
2. **von Neumann** measurements (=**PVM**)
3. **POVM** measurements

where ONB measurements are the most specific and purest kind of measurement and the other measurements are achieved by coarse-graining them.

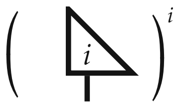
3.6.1 ONB measurements

For every ONB of a Hilbert space, we have its corresponding doubled effects:

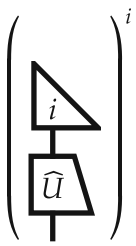


which constitute a non-deterministic quantum process. A process like this is called *demolition ONB measurement* because there is a quantum system entering the process, but only the information of the measurement outcome i exits this process and no quantum system remains. Since unitary processes are exactly those processes that send ONBs to ONBs, it can be shown that any two ONB measurements are equal up to a unitary transformation. Our put diagrammatically:

Given an ONB measurement on a system:

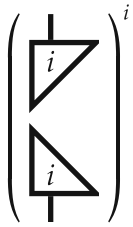


All other ONB measurements on that system are obtained as follows:



where $\widehat{U}$ is a unitary quantum map.

But we can also perform non-demolition ONB measurements consisting of effect-state pairs:

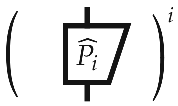


The difference being that after we have performed the measurement there is still a quantum system present and it is in the state according to the measurement outcome. When this outgoing state is discarded, we end up with a demolition ONB measurement. Experimentally, a non-demolition ONB measurement can be thought of as a demolition ONB measurement followed by a controlled preparation. The only states that are left unchanged by a non-demolition ONB measurement are the basis states of the measurement.

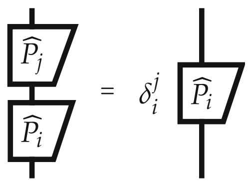
3.6.2 von Neumann measurements

Von Neumann measurements are a generalization of ONB measurements, as they do not have to be ∘-separable like the ONB measurements. But what both types of measurement have in common, is that if we perform the same type of measurement twice in a row, we are bound to get the same measurement outcome twice. This behavior can be modeled by orthogonal projection operators:

Definition 18. *A non-demolition* von Neumann measurement *is a quantum process:*



such that:



For this reason von Neumann measurements are also called *projective-valued measurements* or *PVMs* for short. PVMs can also be obtained by coarse-graining ONB measurements, where all non-demolition ONB measurements corresponding to the basis states are partitioned into a number of disjoint sets. When summing over all of the projection operators one receives the identity transformation, a procedure also called the *resolution of the identity*.

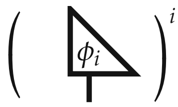
An advantage of PVMs is that they allow us to ask a larger set of questions about the system, while possibly not being as destructive as an ONB measurement. While ONB measurements are analogous to basis states, PVMs are analogous to pure quantum states. However to end up with a quantum process theory, we have to consider even more general processes than von Neumann measurements, as they are not closed under composition.

If we consider two different von Neumann measurements $\widehat{P}_i$ and $\widehat{Q}_j$, their sequential composition $\widehat{Q}_j \circ \widehat{P}_i$ only constitutes a von Neumann measurement, if their projection operators mutually commute. To resolve this issue we are going to introduce *POVM measurements*.

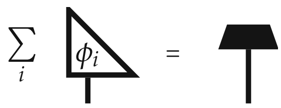
3.6.3 POVM measurements

Positive operator-valued measures or POVM represent the most general kind of measurements as they are comparable to mixed quantum states. They are described by a collection of effects that are jointly causal:

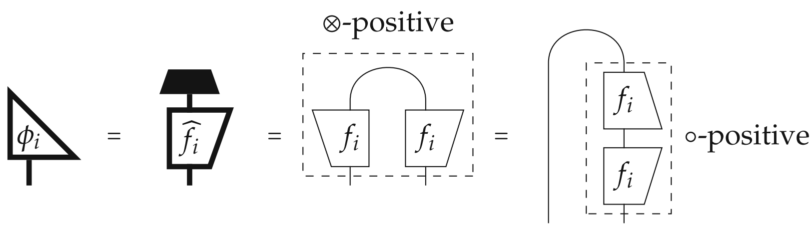
Definition 19. *A* demolition POVM measurement *is any quantum process of the form:*



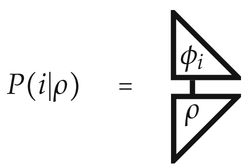
with the sum of its branches being causal:



The term *positive* refers to the fact, that the branches of a POVM are represented by positive operators. Purifying the branches ϕ_i of the POVM yields:



So every ϕ_i corresponds to a positive operator $f_i^\dagger \circ f_i$. The probability for the i-th outcome can be calculated using the Born rule for POVM measurements:

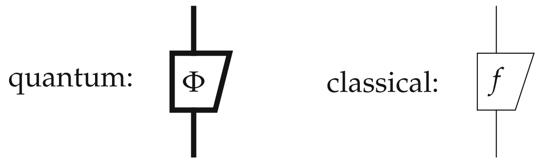


Experimentally POVM measurements are realised by performing an ONB measurement on parts of a larger system (Naimark dilation), but they can also occur due to noise or limited access to the physical system (similar to mixed quantum states). POVM measurements are also very useful for the task of unambiguously discriminating between non-orthogonal quantum states, a task that is not possible with a PVMs [15].

Chapter 4

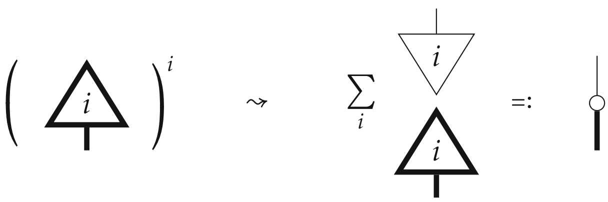
Classical-quantum processes

We have now developed a theoretical framework to describe quantum processes. Our next endeavor involves integrating an interface designed for interaction with the classical domain, enabling the transmission of data between the two realms. This integration is crucial, especially in scenarios where a quantum protocol requires utilizing the results of quantum measurements to trigger controlled operations on other quantum systems, such as in quantum teleportation. Consequently, we must incorporate the flow of classical information into our framework of *classical-quantum processes*. In this framework, quantum processes will be represented by thick or double lines, while classical systems will be denoted by thin single lines.



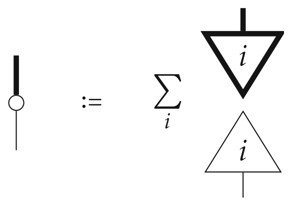
4.1 Measurement and encoding

The most important additional processes will be the measurement and the encoding process. The measurement process we introduced in the previous chapter takes in a quantum system and outputs classical information corresponding to the measurement outcome. Diagrammatically, the *measurement process* consists of a sum over all the basis states i and their doubled effects. We are therefore going to adapt the following notation:



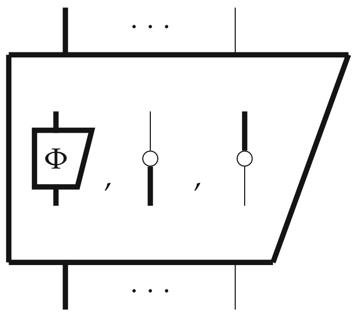
The classical (thin) basis states form a basis, whereas the doubled quantum effects do not, since they do not suffice to span the entire doubled Hilbert space $H \times H$. Therefore information is lost during the measurement process, a phenomena also known as *decoherence*.

The adjoint of the measurement process is called *encoding*. The encoding process takes in classical information in a specific basis and then prepares the corresponding quantum state:

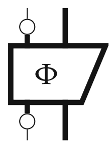


We now have all the necessary components to define the process theory of classical-quantum maps:

Definition 20. *A classical-quantum map (=cq-map) is a linear map obtained by composing quantum maps, encoding and measurement process:*

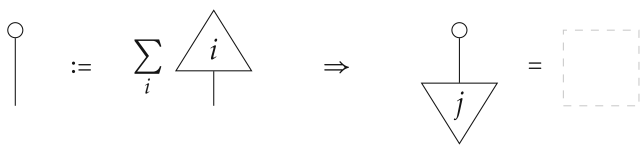


So therefore any cq-map with a single classical and quantum in- and output can be written in the form of:

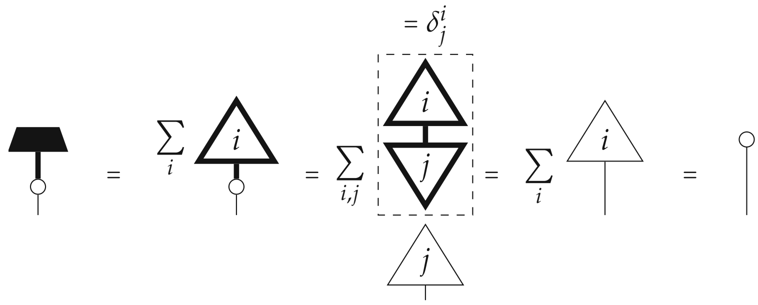


4.2 Deleting

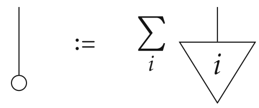
For quantum systems we already defined the discarding map, which describes the process of getting rid of or ignoring a quantum (sub-)system. We are going to extend this notion to our classical systems and call it the *deleting* process. The deleting process takes in a classical state (= classical information) and deletes it:



However, this is not a completely new map, as the deleting map can also be constructed by the encoding map followed by the discarding process:

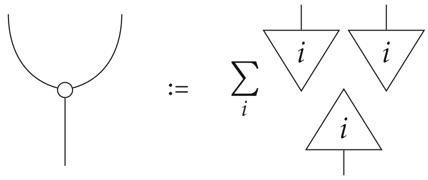


The adjoint of the deleting process is called the uniform probability distribution, which is the sum over all basis states (similar to the maximally mixed state as a quantum state):



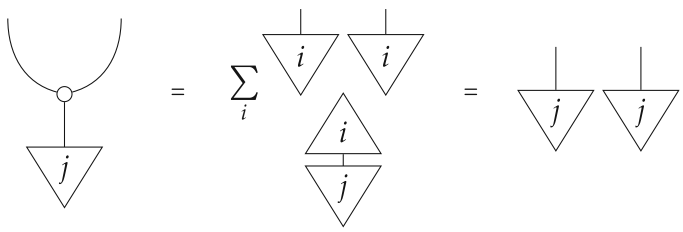
4.3 Copying and matching

For classical states, we can define the copying (= cloning) map as:

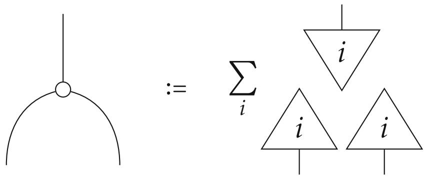


It tests for a set of basis states and if they agree, it doubles the incoming state weighted by their inner product. But, as we will show later on with the no-cloning theorem, copying or cloning arbitrary states is not possible as we are limited to copying normalized orthogonal states corresponding to a specific ONB.

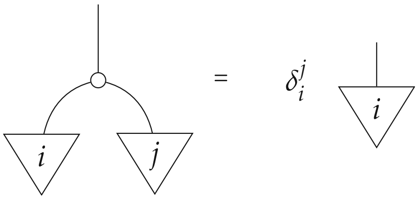
So every copying map fixes a particular ONB and it copies the incoming state if and only if it is a member of its ONB:



By adjoining the copying map, we get the *matching* process:

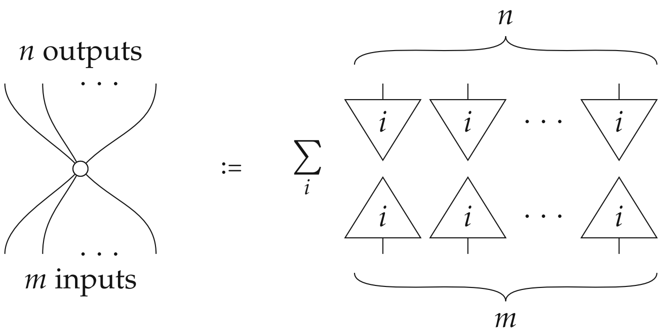


It takes in two classical states and outputs a single state. If both incoming states are from the ONB the matching process is constructed and they match, a single copy is released. If they do not match, we get the *zero process* or impossible process.



4.4 Classical spiders

We have now encountered numerous instances of so-called *spiders*. In general, a *spider* is a classical map that compares a number of m input states and outputs n identical states, if they agree. We can define a (m, n)-spider as:

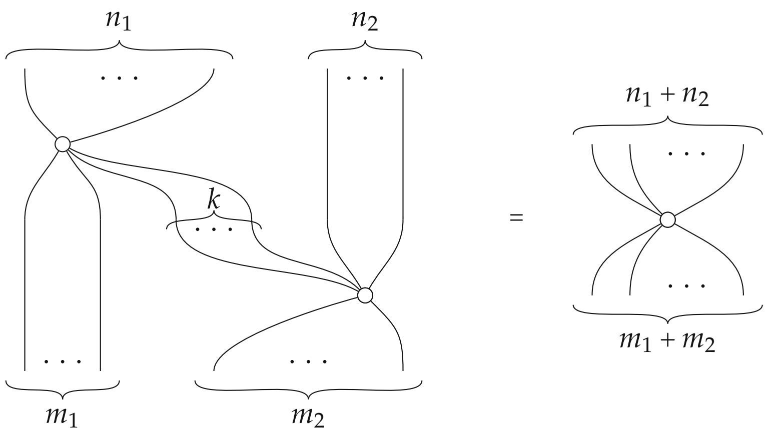


The matrix entries of a spider are the numbers that result from adjoining basis states to the in- and outputs. For a spider, they turn out to be the Kronecker delta:

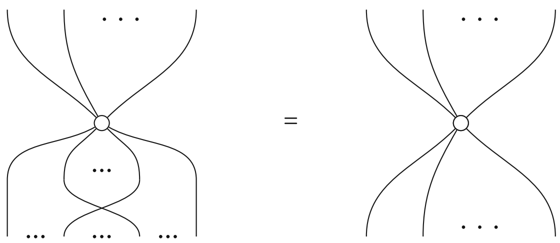
$$\delta^{j_1 \ldots j_n}_{i_1 \ldots i_m} = \begin{cases} 1 & \text{if } i_1 = \ldots = i_m = j_1 = \ldots = j_n \\ 0 & \text{otherwise} \end{cases}$$

Spiders of the same ONB compose in a very elegant way, as they simply fuse together, if they touch.

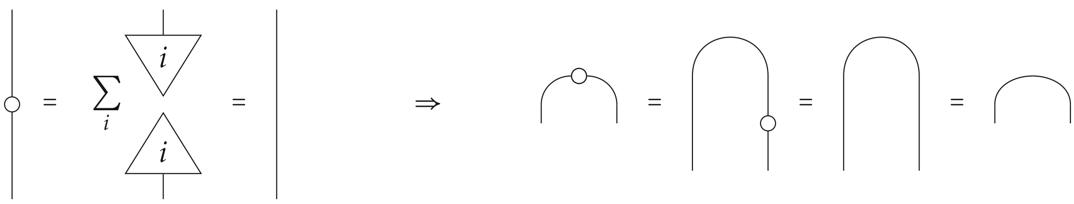
Or more formally, if a (m_1, n_1)-spider and a (m_2, n_2)-spider share at least one connection ($k \geq 1$), they form a $(m_1 + m_2, n_1 + n_2)$-spider:



Furthermore it can be shown that spiders are invariant under 'leg-swapping', meaning the exchange of the order of in- or outputs:

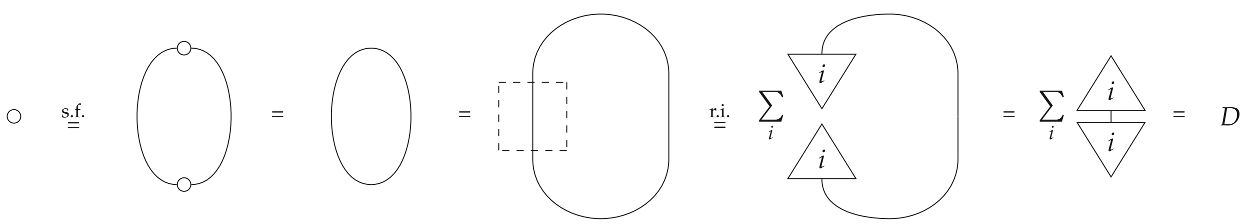


Spiders with exactly two legs are equal to a plain wire, as they form a resolution of identity (this is no longer the case for the decoherence spider, as we will see later on):



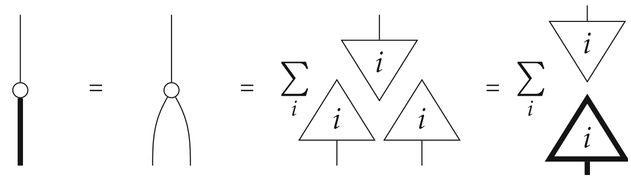
By redirecting the output wire, it follows that the maximally correlated cap- and cup-state can also be seen as $(2,0)$- and $(0,2)$-spider respectively and that both of them are base independent.

A special case of spider worth considering is the leg-less spider. Since it has no in- or outputs it is a number:



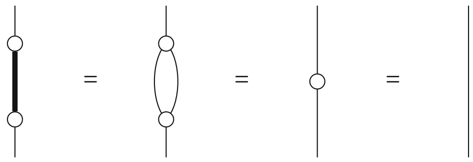
with D being the dimension of the Hilbert space.

We have now developed a set of rules we can apply to spiders. This comes in handy, as it enables us to look at the measuring and encoding process in a new light. It turns out the measuring process is just a (2, 1)-spider and the encoding process a (1, 2)-spider:

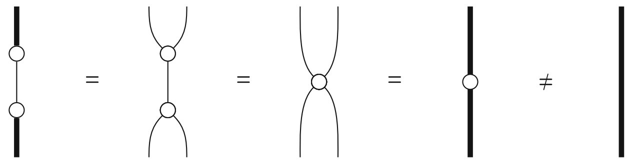


Measurement process

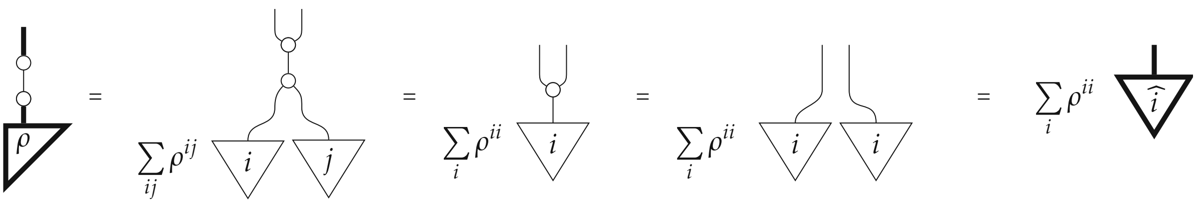
Equipped with the spider fusion rule we can now consider what happens, when we compose the measuring and the encoding process. When the encoding process is followed by the measuring process we get the identity:



So the measuring process is the inverse of the encoding process. However when these processes are in reverse order, we see that the encoding process is <u>not</u> the inverse of the measuring process:

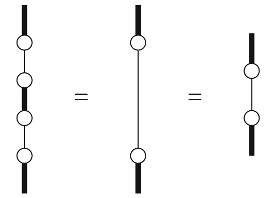


instead we get a new kind of quantum map called *decoherence*. Decoherence is a non-invertible process that happens when we force a quantum state to pass through a classical channel. During this process some information is lost as can be seen by the following diagram:



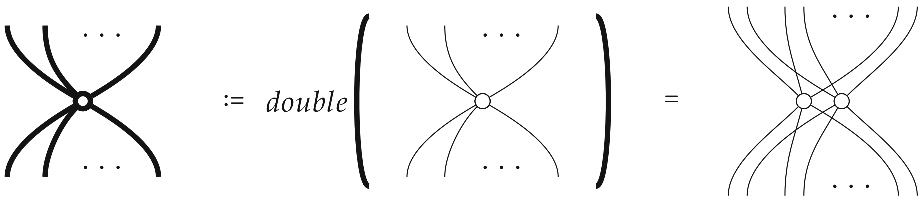
A general (possibly impure) quantum state ρ passes through a decoherence channel and as we can see the off-diagonal entries of our density matrix are lost in the process. What exits this process is a quantum state, that behaves just like a classical probability distribution.

Once a quantum state goes through the process of decoherence, it is then not affected by further decoherence, as the decoherence process is idempotent:

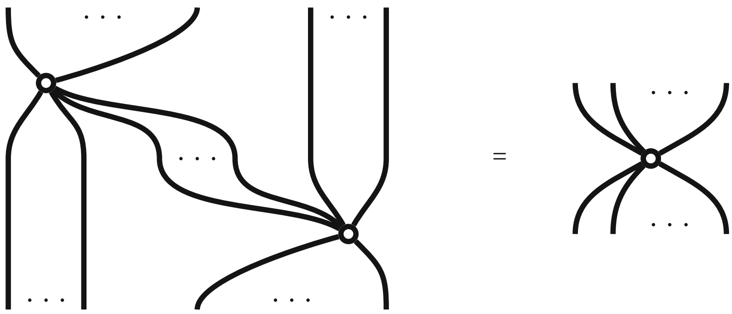


4.5 Quantum spiders

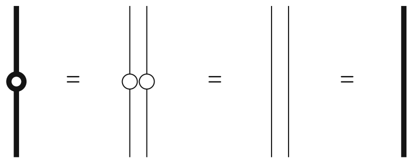
Whereas the spiders up to this point have been purely classical linear maps and the measuring and encoding processes have been transitional maps from quantum systems to classical systems and back, we will now introduce purely quantum spiders. Analogous to the construction of pure quantum maps, *quantum spiders* are created by doubling classical spiders.



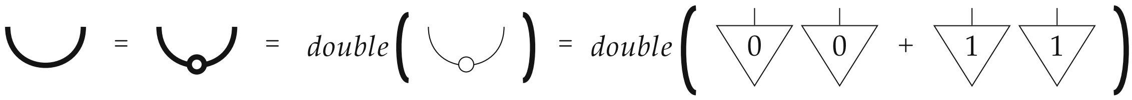
they also behave similarly to classical spiders, as the spider fusion rule also applies to quantum spiders:

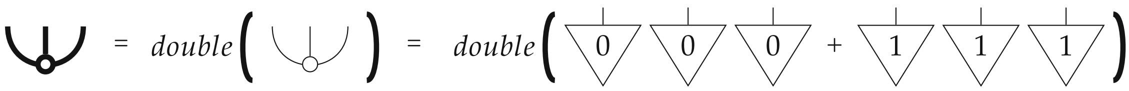


But unlike the decoherence process the $(1,1)$-quantum spider acts as an identity on quantum systems:

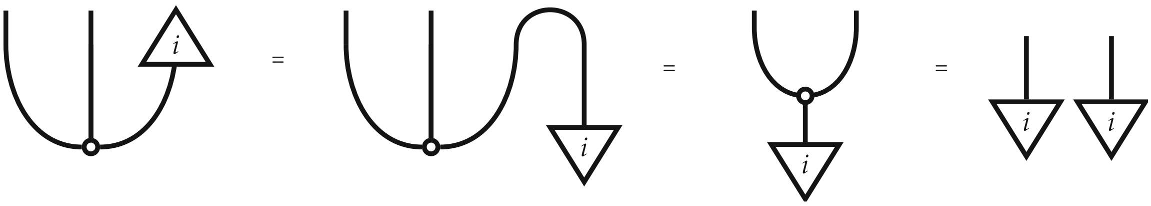


Some special cases of quantum spiders, which we will later come back to, are the two-system quantum spider state or otherwise also called *Bell-state* and the three-system quantum spider state or *Greenberger-Horner-Zeilinger state*:

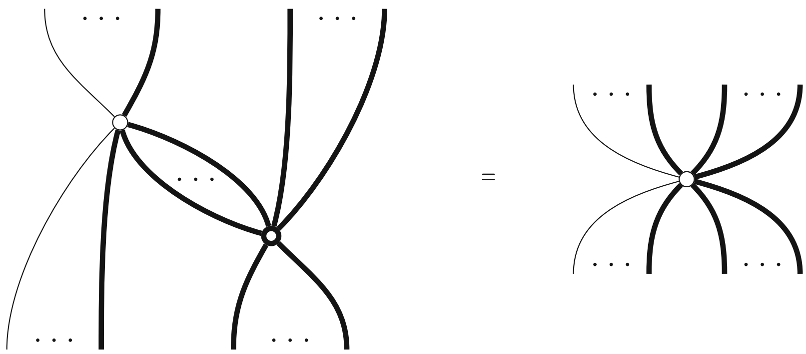




These states are examples of maximally entangled quantum states. This means that if one of the systems is measured to be in a specific basis state of the doubled ONB, the other systems will also be in the same state. This phenomenon can be understood by interpreting the GHZ state as a copying map in the following way(noting that only the orthogonal basis states of the quantum spider can be cloned):



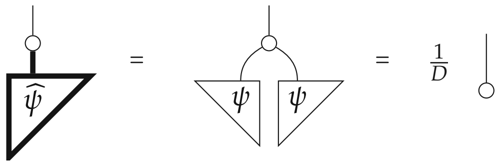
We now have two different kinds of spiders: classical spiders with a thin dot and quantum spiders with a bold dot. While classical spiders can have single legs or doubled legs (e.g. measuring process) quantum spiders can only have doubled legs. When a classical spider and a quantum spider fuse together the resulting spider is a classical spider due to decoherence:



4.6 Phase spiders

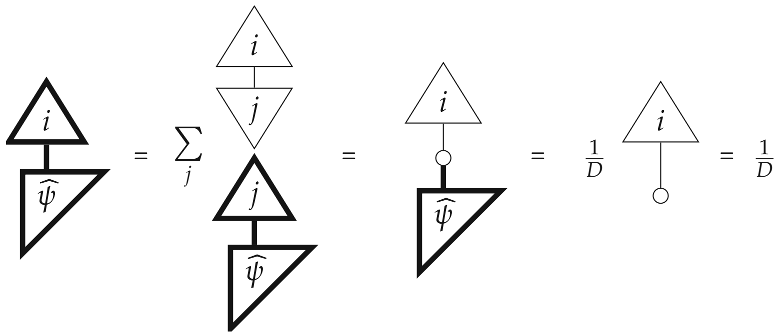
We are going to expand our repertoire of spiders to include *phase spiders*. Phase spiders are defined similarly to normal spiders except that they equip each of the basis states i with a (possibly different) complex phase $e^{i\alpha_i}$. These new spiders will enable us to construct any kind of quantum map solely from phase spiders. But before we get there we need to define the notion of *unbiased states*.

Definition 21. *A normalised pure state ψ is* unbiased *for the ○-basis if we have:*



Or in other words if we measure our unbiased state in the ○-basis, all of the possible measurement outcomes are equally likely to occur with a probability of $\frac{1}{D}$.

This result can also be directly calculated with the Born rule:



If we express our quantum state in column form through its components in the ○-basis, this fact can be stated as:

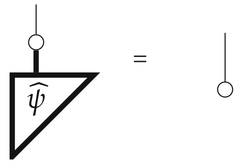
$$\begin{pmatrix} \overline{\psi^0}\psi^0 \\ \vdots \\ \overline{\psi^{D-1}}\psi^{D-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{D} \\ \vdots \\ \frac{1}{D} \end{pmatrix}$$

it then follows that the most general form our unbiased component vector ψ can have is:

$$\begin{pmatrix} \psi^0 \\ \vdots \\ \psi^{D-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{D}} e^{i\alpha_0} \\ \vdots \\ \frac{1}{\sqrt{D}} e^{i\alpha_{D-1}} \end{pmatrix}$$

for the angles $\alpha_i \in [0, 2\pi)$. To define the phase states we will drop the normalisation factor of $\frac{1}{\sqrt{D}}$ for diagrammatic simplicity, but otherwise the phase states for a given basis are exactly its unbiased states.

Definition 22. *A* phase state *for a given basis* ○ *is a pure state* $\widehat{\psi}$ *that satisfies:*

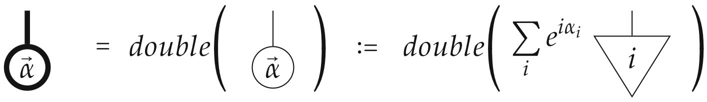


and in bracket notation a phase state can be expressed as an unnormalized column vector:

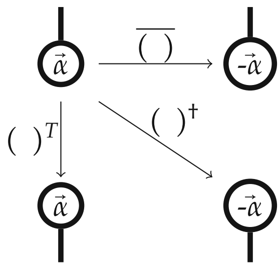
$$\begin{pmatrix} \psi^0 \\ \vdots \\ \psi^{D-1} \end{pmatrix} = \begin{pmatrix} e^{i\alpha_0} \\ \vdots \\ e^{i\alpha_{D-1}} \end{pmatrix}$$

Without loss of generality we can set $\alpha_0 = 0$, since our vector is invariant under a global phase transformation of $e^{-i\alpha_0}$. Therefore the phase state $\widehat{\psi}$ is uniquely defined by $D-1$ complex phases: $\vec{\alpha} := (\alpha_1, ..., \alpha_{D-1})$.

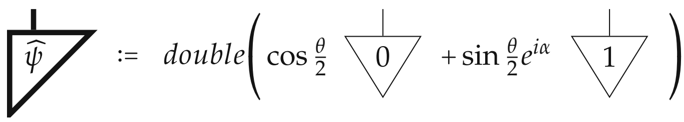
We will denote the phase state as:



Complex conjugation of a phase state is equal to putting a minus sign in front of the phases α_i, the transposed of a phase state is just the phase state rotated by 180° and the adjoint is the combination of the two:

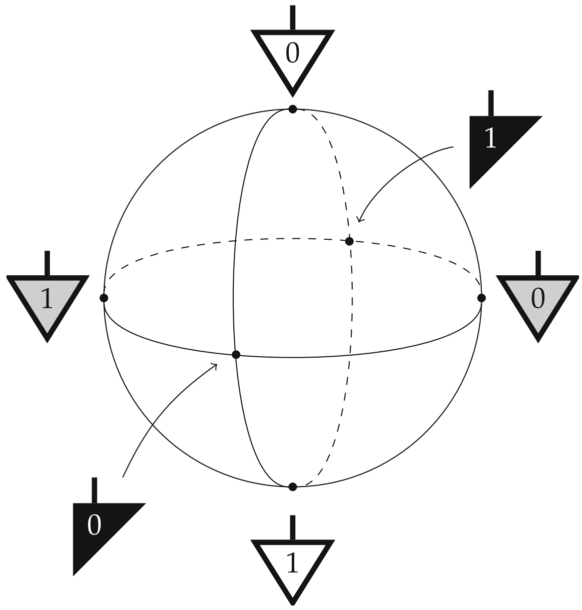


The simplest non-trivial quantum system is a two-dimensional system, also called *qubit*. Its Hilbert space is spanned by two basis vectors over the field of complex numbers. Every pure quantum state from this space can be expressed by two angle variables $0 \leq \theta \leq \pi$ and $0 \leq \alpha < 2\pi$, due to the normalization of the state vector and its invariance under a global phase transformation. Diagrammatically we can represent those states as:

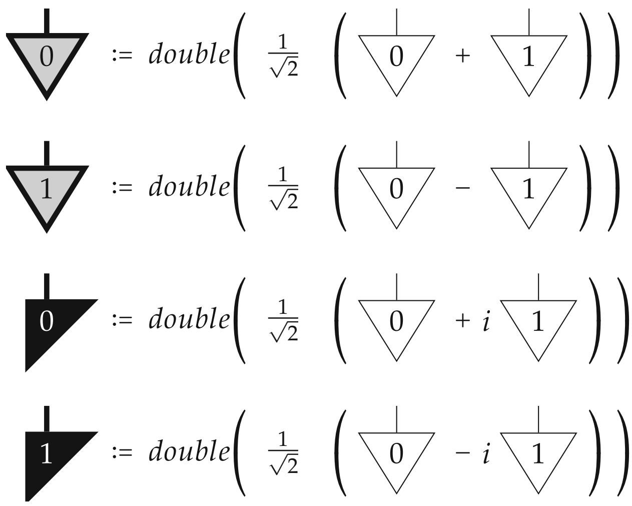


It is then very instructive to visualize these states on a three-dimensional sphere also called *Bloch sphere*. θ represents the polar angle from the north pole and α is taken as the azimuthal angle.

On the Bloch sphere pure quantum states lie on the surface of the sphere, while impure or mixed states are located on the inside of the sphere. Antipodal points on the surface of the Bloch sphere correspond to orthogonal states, which we will use to construct different bases. If embedded into a Cartesian coordinate system, the two states corresponding to the intersection points on the surface with the z-axis constitute what is called the z-basis, which is our standard basis and will diagrammatically be indicated by the white basis states. The same can be done for the x-basis and the y-basis, which we will denote as gray and black basis states respectively.



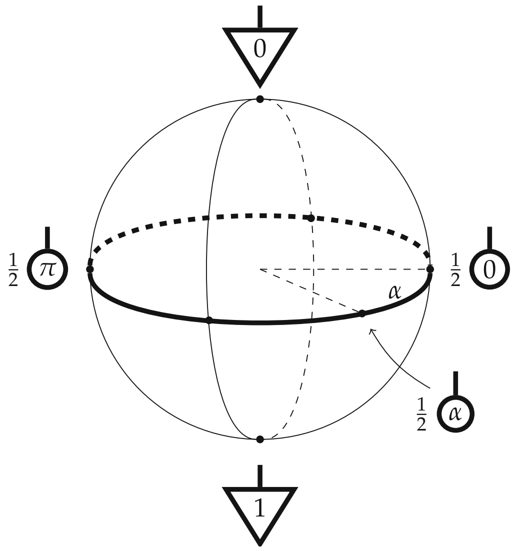
(A) Bloch sphere



(B) x- and y-basis states. Note that the black y-basis states are not self-conjugate in contrast to the other two bases.

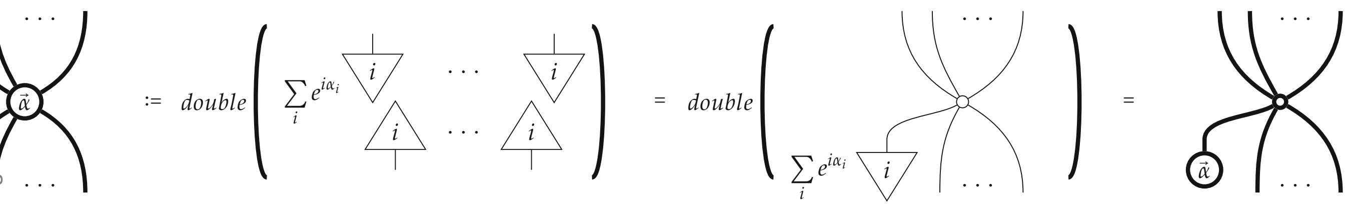
We are now returning to the topic of unbiased states or phase states and their placement on the Bloch sphere. Since we are dealing with a two-dimensional system, the phase states are parameterised by a single complex phase α:

For the z-basis the set of unbiased states is located at the equator of the Bloch sphere, equidistant from both z-basis states on the poles:

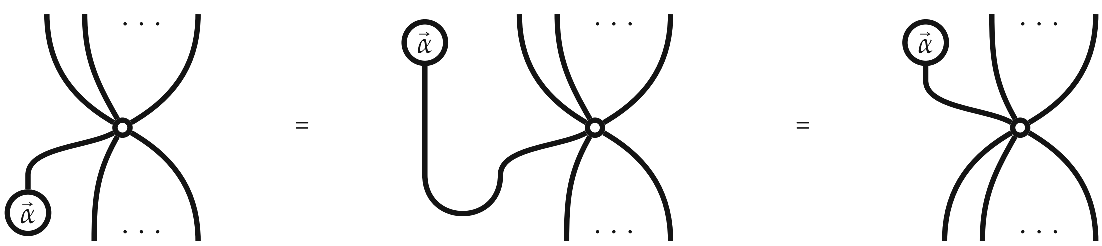


By comparing the two representations of states on the Bloch sphere, one notices that the x- and y-basis states are proportional to the z-basis phase states for phase angles that are integer multiples of $\frac{\pi}{2}$.

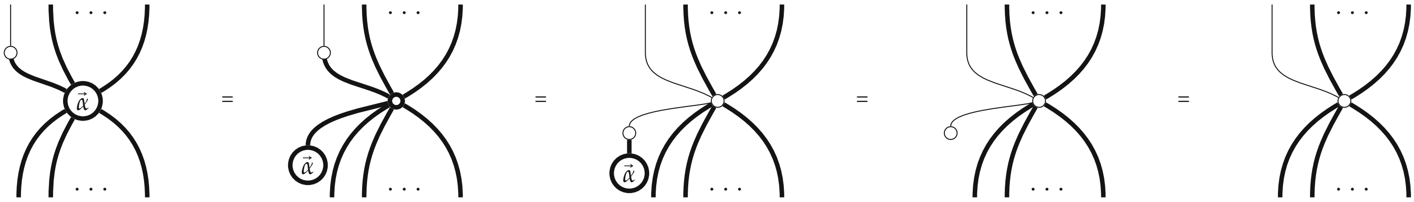
When a phase state is connected to a quantum spider, we get what is called a *phase spider*:



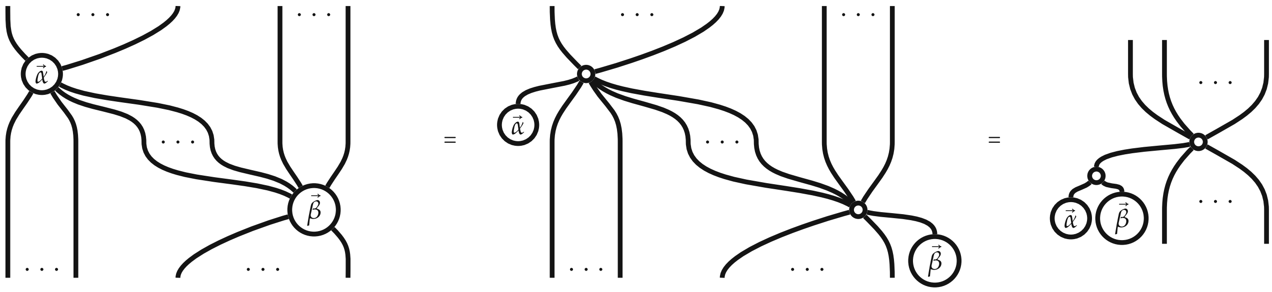
The quantum spider takes in the phase information and they fuse together. For this process it does not matter, if the phase state is connected to an input or an output or to which of the legs it is connected, as a quantum spider is also invariant under leg-swapping:



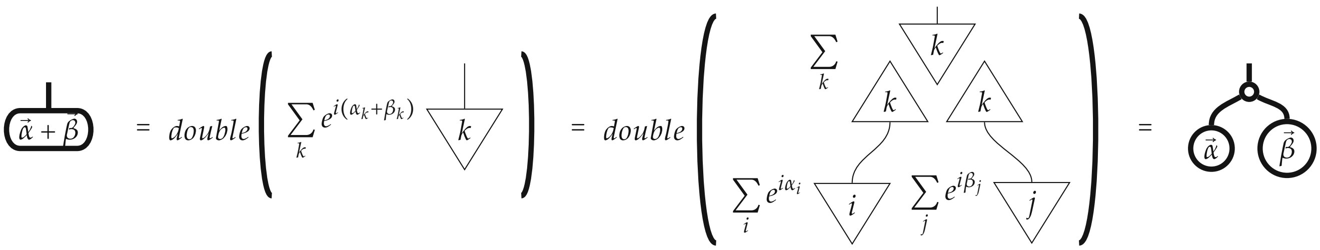
When a phase spider is measured (comes into contact with a classical spider), all of the phase information is lost, and a classical spider results:



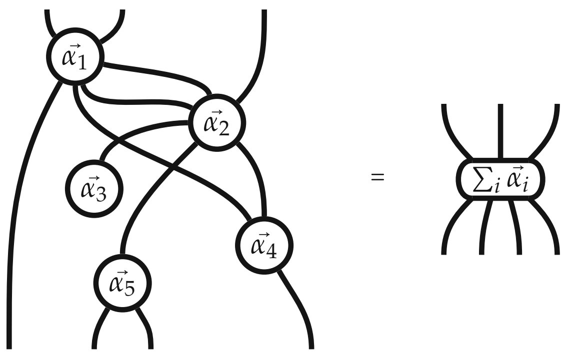
Once again, we observe that decoherence eliminates any purely quantum phase information. Next, we will examine the consequences of combining two phase spiders with each other:



the quantum spiders fuse together and we get an additional phase state of $\vec{\alpha}$ and $\vec{\beta}$ combined. When expanding this combined phase state into basis states, we notice that this fusion of phase states is equal to just adding their complex phases:

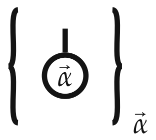


So when a couple of phase spiders touch, they all fuse together and their phases add up:

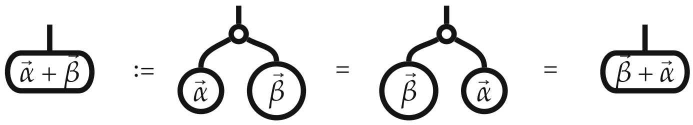


This behaviour suggests that there exists an abelian group structure within the composition of phase spiders.

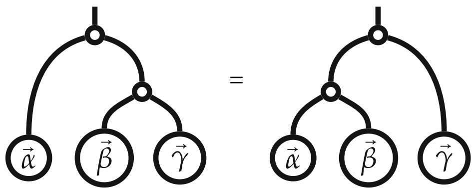
For any set of phase states, there exists a binary operation that is both commutative and associative, an identity element and an inverse element for every element:



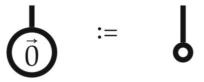
(A) Set of phase states



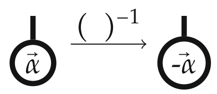
(B) Commutative group operation



(C) Associativity



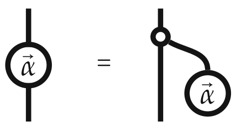
(D) Identity element



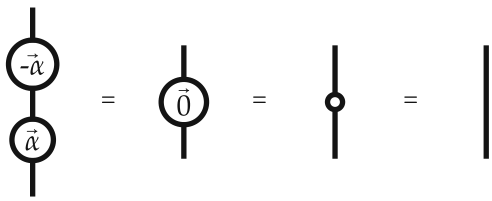
(E) Inverse element

From the periodicity of the complex exponential function it follows that the composition of phase states is isomorphic to addition modulo 2π. Thus, in total the phase group is isomorphic the $(D-1)$-fold product of the circle group: $\underbrace{U(1) \times \ldots \times U(1)}_{D-1}$

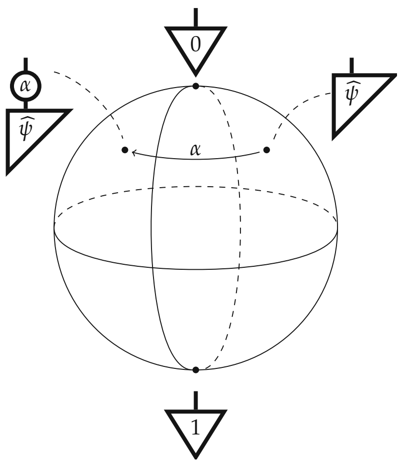
A phase spider with a single in- and output is called a *phase gate*:



A phase gate is a unitary transformation as its adjoint is also its inverse:

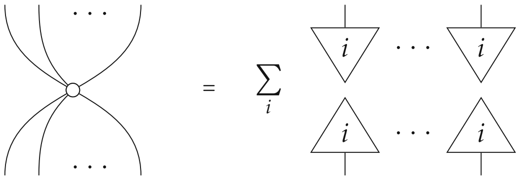


For qubit states on the Bloch sphere, the application of a phase gate corresponds to rotations along the axis of its basis states for an angle of α:

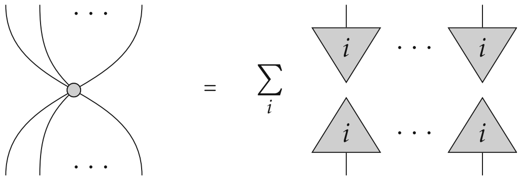


4.7 Spiders from different bases and complementarity

Until now, we have only used z-basis states to encode our spiders, but we are free to do so with any other ONB. By considering different kinds of spiders, we will be able to describe and reason about the measuring and encoding process in different ONBs and how they interact. Spiders constructed from x-basis states will be colored gray:



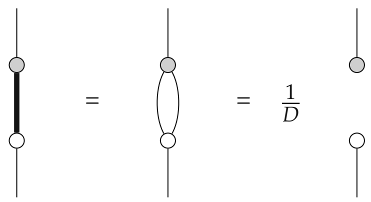
z-basis spider



x-basis spider

Whereas spiders of the same basis (or color) fuse together, this is no longer the case for spiders of different bases as they tend to disconnect, we will call these spiders *complementary*.

Definition 23. *Two different spiders are* complementary *if and only if:*

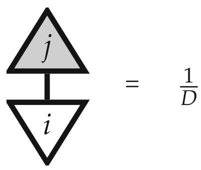


This diagram indicates that if we encode classical information in a certain basis of quantum states and then measure this quantum system in another complementary basis, the result is equivalent to deleting the classical input and producing only a uniform probability distribution, or pure noise. Consequently, no data can be transmitted through this channel, as the output is completely independent of the input.

Analogous to the uncertainty principle, one can say that if we have maximal knowledge of a state with respect to one basis, we have no information regarding another complementary basis.

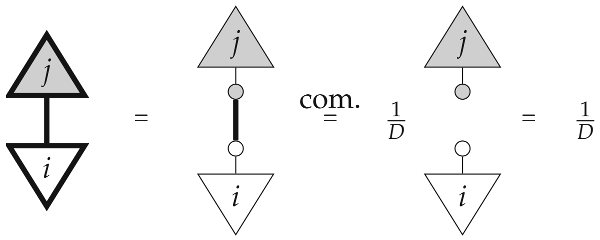
We now have an understanding of what it means for two bases to be complementary, but how do we determine if they are complementary? It turns out that the concept of complementarity of bases is closely linked to their *mutual unbiasedness*.

Definition 24. *Two ONBs are called* mutually unbiased *if for all basis states i, j we have:*

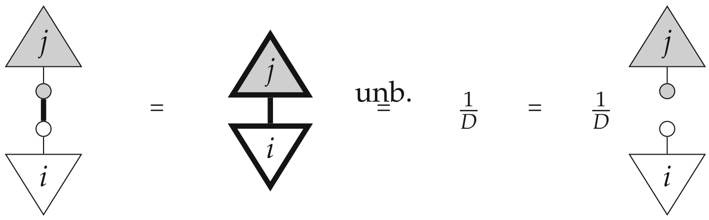


Theorem 4. *Two spiders are complementary if and only if their respective ONBs are mutually unbiased.*

Proof. If we assume that two spiders are complementary, then their mutual unbiasedness follows:



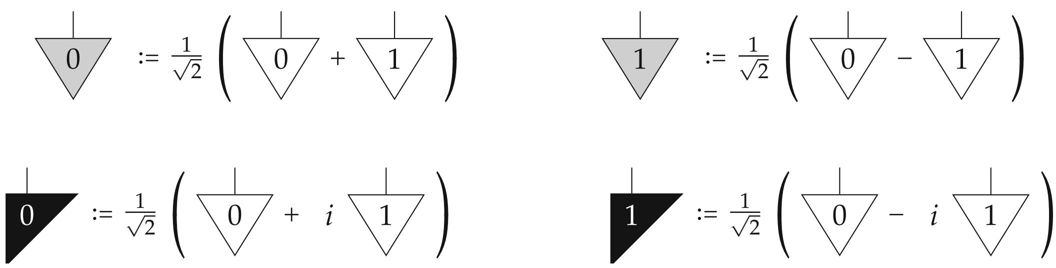
Conversely, if we assume mutual unbiasedness, then complementarity also holds:



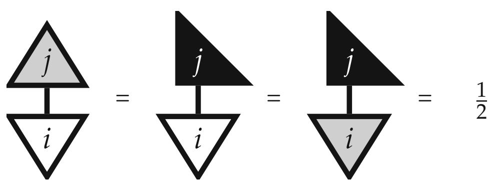
Here we have demonstrated that the matrix entries for the left-hand side and the right-hand side are identical for all basis states i, j, and therefore the process in between them must also be identical.

□

We can now proceed to verify if the x, y, and z bases are mutually unbiased by calculating their inner products. This can be done by expressing all basis states in the z-basis:



We determine their respective inner products by adjoining the states and plugging them together. It turns out that for all basis states i, j we have:



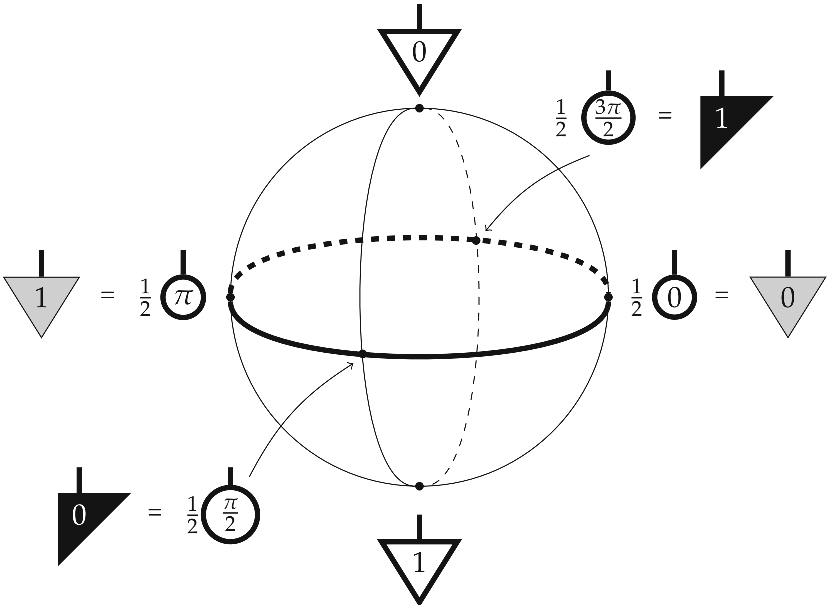
Thus, all of the three ONBs are mutually unbiased and therefore also complementary.

It can also be shown that these three pairwise mutually unbiased bases (MUBs) form a maximal set of pairwise MUBs for a qubit system, as there is no other distinct basis we can add to this set while maintaining unbiasedness.

Although determining the size of a maximal set of pairwise MUBs for a given dimension D is generally challenging, boundaries for this number have been established [16][17]. If we decompose the dimension D into its prime-power factorization $D = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ where $p_1^{n_1} < p_2^{n_2} < \cdots < p_k^{n_k}$, then the number of MUBs satisfies $p_1^{n_1} + 1 \leq \#\text{MUBs} \leq D + 1$. As a special case, if the dimension D is an integer power of a single prime number $D = p^n$, then this boundary exactly determines the number of MUBs to be $\#\text{MUBs} = p^n + 1$, or in the case of a qubit: $\#\text{MUBs} = 3$.

4.8 Strong complementarity

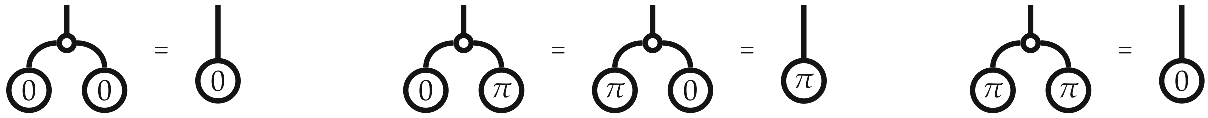
We have discussed how the set of unbiased states for a given ONB forms a set of phase states and how a commutative group structure can be developed on them. We then established that two bases are complementary if and only if their basis states are mutually unbiased. Building upon this foundation, we introduce the notion of *strong complementarity*. Two ONBs are *strongly complementary* if the basis states of one basis form a subgroup of the phase group of the other basis, and vice versa. We will illustrate this concept through an example.



The Bloch sphere is depicted with the z-basis states (white) and its respective phase states highlighted. The x-basis states (gray) and the y-basis states (black) coincide with certain z-basis phase states.

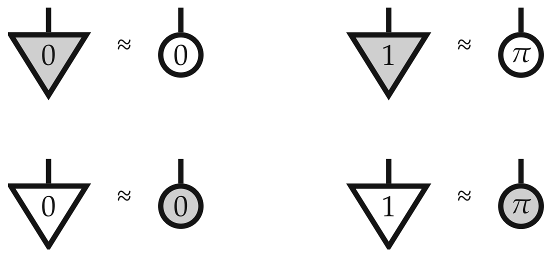
As the x- and y- basis, as well as any other basis of antipodal phase states, lie on the equator of the Bloch sphere, they form a complementary basis to the z-basis.

However only the phase states corresponding the x-basis form a subgroup of the phase group:



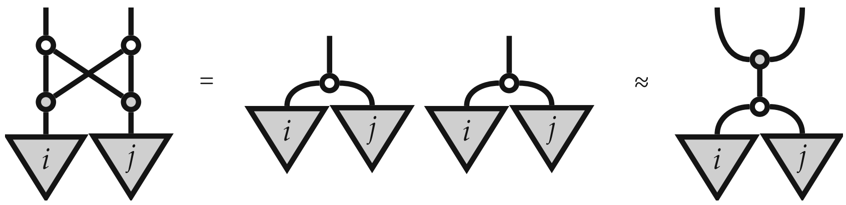
As they are the only states that are closed under the group operation. While the entire phase group is isomorphic to the circle group $U(1)$, the subgroup generated by the x-basis states is isomorphic to the two-element cyclical group $\mathbb{Z}_2$.

It can also be shown that the z-basis states form a subgroup of the phase group of the x-basis, and consequently, the x- and z-bases are strongly complementary.

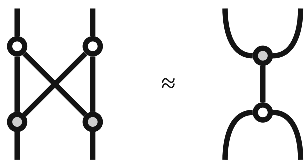


There is a bijective correspondence between basis states of the x-basis and phase states of the z-basis and vice versa.

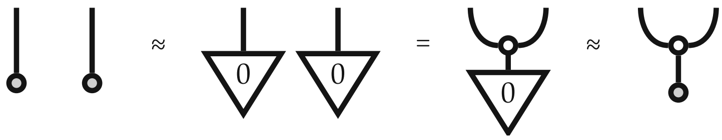
We will now establish a set of diagrammatic equations that capture the notion of strong complementarity. The first one is the following:



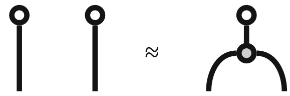
The first equation holds, as the x-basis states are cloned by the x-basis spider. The second equation holds up to a number, because the x-basis states are proportional to z-basis phase states and they are closed under the group operation due to strong complementarity, and thus the combined states can also be cloned by the x-spider. This holds for all basis states i, j, so the first characteristic equation is:



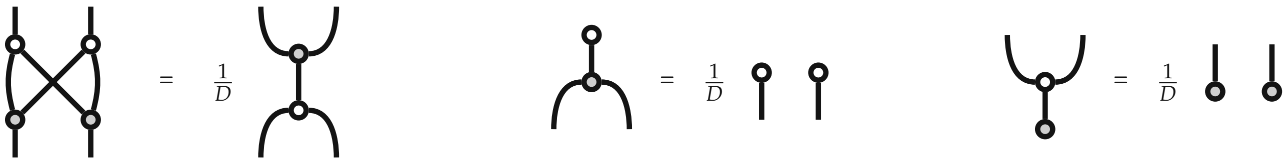
The second characteristic equation will relate to the cloning of phase states:



By adjoining the equation and swapping the colors we get the third characteristic equation:

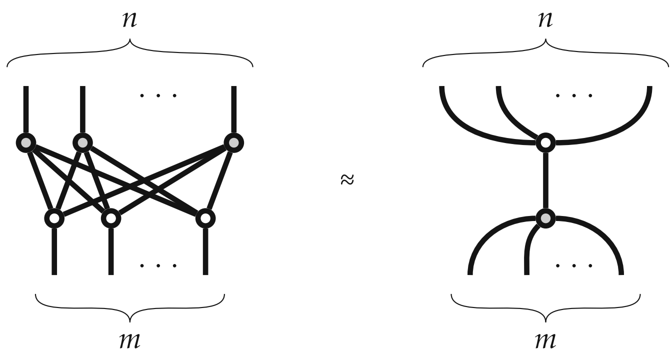


To summarize, we combine these three equations together, include the normalization constants, and state that a pair of spiders is *strongly complementary* if and only if the following set of equations is satisfied:



This set of equations can even be extended, to include all kinds of *complete bipartite diagrams*, which are diagrams where every spider of one color is connected to every spider of the other color, but not amongst each other.

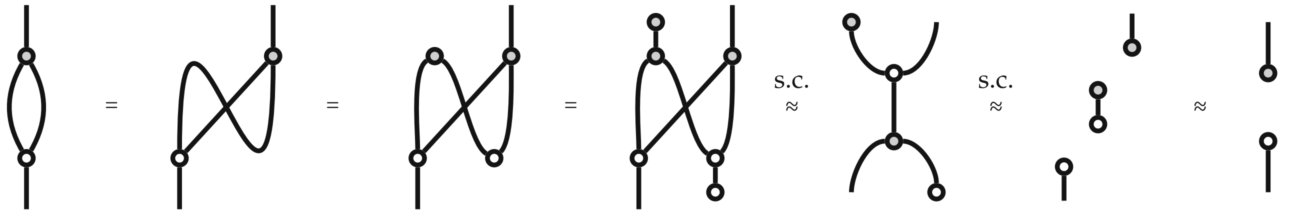
Starting from the three previous equations, it can be shown by induction, that strong complementarity is equivalent to:



This equation generalizes the previous equations and also expresses how strongly complementary spiders can commute past each other. When viewed from a perspective of category theory, this strongly complementary pair forms a bialgebra, with its defining equations describing how multiplication and comultiplication commute past each other [18].

Theorem 5. *Strong complementarity implies complementarity (the inverse is not true).*

Proof. Diagrammatically this can be shown by the following equation:



□

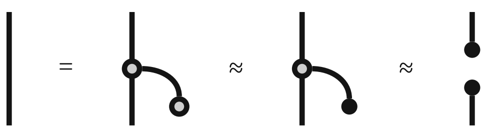
Another important difference between the two concepts is the number of bases that can be mutually complementary or mutually unbiased in other words. For a given dimension D the maximum number of mutually complementary bases is unknown, while the number of mutually strongly complementary bases is always 2, independent of D.

Theorem 6. *For a finite dimension of D, there is a maximum of 2 mutually strongly complementary bases.*

Proof. If there were three mutually strongly complementary bases  and  , then the following equations would hold:

Since both the gray phase state and the black phase state get cloned by the white spider, they both have to be equal up to a number:

This holds assuming that the number  is cancelable. However, it then follows that the identity is ∘-separable, which cannot hold for any non-trivial system:

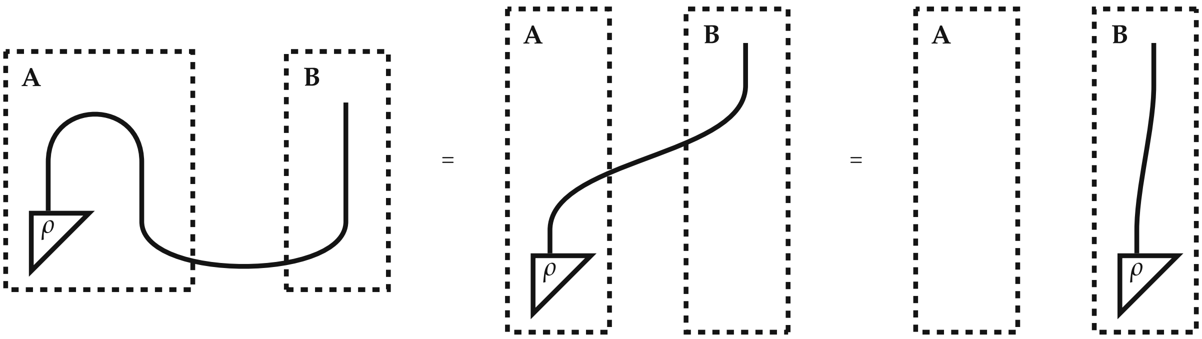


□

4.9 Quantum teleportation

We have now developed the necessary tools to fully describe quantum teleportation using only complementary quantum spiders. The general idea of quantum teleportation is the transmission of an unknown quantum state ρ across a large spatial distance. To achieve this, two parties share a maximally entangled Bell-state. Then, party A performs a Bell measurement on both the unknown quantum state and their shared qubit. This measurement yields one of four possible outcomes, which A communicates to party B over a classical information channel. With the knowledge of the measurement outcome, B can now perform a controlled operation on their entangled qubit, thus converting it into the unknown quantum state ρ.

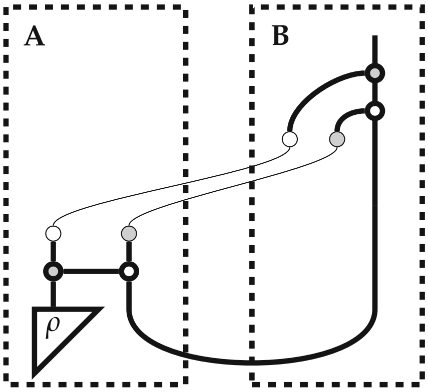
Even though this process is called quantum teleportation, it does not enable faster-than-light communication, as it relies on the transfer of classical information. Without classical communication, the only quantum state A can send is the maximally mixed state or just pure noise. Diagrammatically, the structure of the quantum teleportation protocol can be summarized by the following simplified diagram:



A possesses the arbitrary quantum state ρ and one half of the Bell-state (=cup-state) and then performs a Bell-measurement (=cap-effect) on his two states. This scenario is equivalent to B holding the ρ-state.

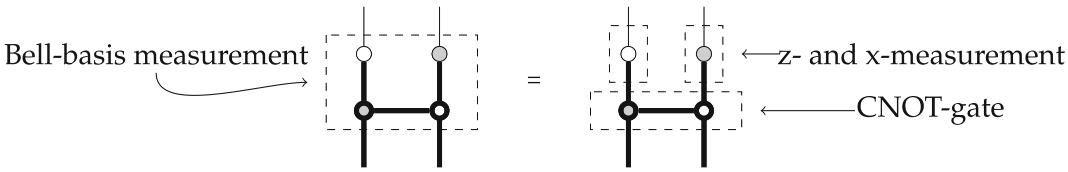
On the left-hand side of the equation, we see the Bell-state being distributed to both parties A and B, and A performing a Bell measurement. By applying the yanking equation, it follows that after the measurement, B is in possession of the unknown quantum state ρ. However, this diagram is not complete, as the Bell effect is not a deterministic quantum effect, meaning that it cannot be performed with certainty. A can only carry out a Bell measurement with the Bell effect as one of its four possible branches, with the other branches disturbing ρ. A then has to inform B which of the four possible outcomes was actually realized, allowing B to revert the change to the quantum state with a controlled unitary operation.

The four possible measurement outcomes equate to two classical bits of information, which A can send through two binary classical channels. So the complete protocol to be performed looks like this:



On the left side, A performs a non-local Bell-basis measurement on the quantum state ρ and his entangled qubit, and then transmits the measurement outcome through the two classical information channels, which B then uses to retrieve the original quantum state ρ.

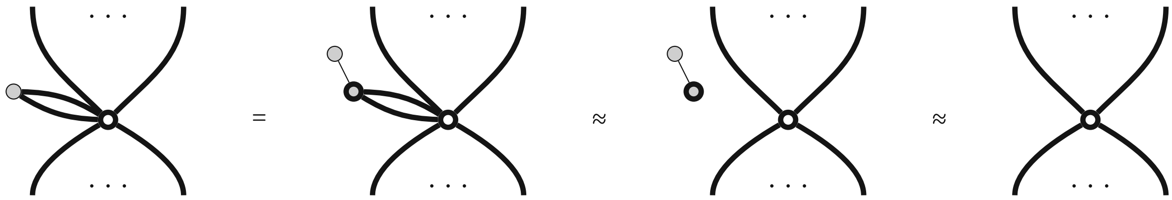
But we can also interpret this Bell-basis measurement, which takes in two quantum states and outputs two classical bits, differently:



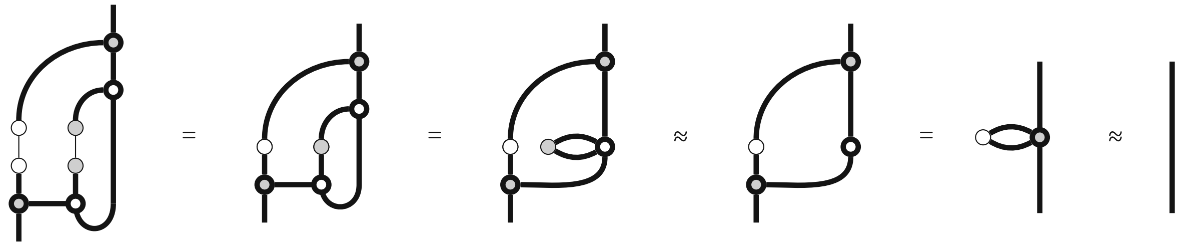
While diagrammatically there is no difference between performing a Bell-basis measurement and carrying out a quantum CNOT-gate[1] followed by two measurements in the z- and x-basis respectively, experimentally there is quite a difference. It might be harder to experimentally implement a non-local Bell measurement on two systems, than to carry out a quantum CNOT followed by two single-system measurements.

[1] a quantum CNOT-gate is a logic gate, that swaps the value of the target qubit (grey), if the control qubit (white) is 1 and does nothing if the control qubit is 0. This process can be described by connecting a quantum x- and z-spider.

We will now demonstrate that the protocol indeed describes the teleportation of a quantum state. As an intermediate step, we are going to employ the fact that a quantum spider and its complementary classical spider separate, when connected by two quantum wires:



In essence, the quantum teleportation diagram boils down to the following diagram:



After a couple of steps we notice, that this diagram reduces to the quantum identity. This implies that there is a direct and undisturbed quantum channel from A to B and thus the arbitrary quantum state ρ is transmitted.

Chapter 5

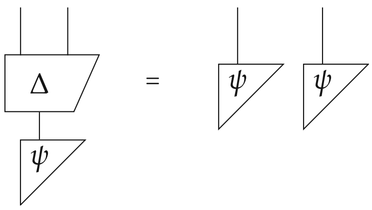
Quantum foundations

We have now developed the necessary tools to dive into some of the foundational properties of quantum mechanics. One of quantum mechanics core fundamental features is a kind of indeterminism or uncertainty, that manifests itself in various forms. Ranging from the fact, that quantum mechanics only allows one to compute probabilities for certain measurement outcomes to the inability to simultaneously predict the outcome of complementary observables. Quantum theory even raised additional philosophical questions concerning properties such as locality and realism, that were taken for granted during the era of classical physics. We are going to review some of these features in the framework of a quantum process theory.

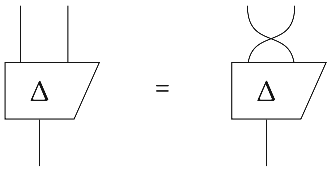
5.1 No Cloning Theorem

A fundamental principle of classical computation is that we can read out, change or copy informational bits at will, without interfering with the underlying information medium. For quantum information, this is no longer the case. We are now going to prove that it is not possible to clone any arbitrary unknown quantum state within the framework of quantum theory.

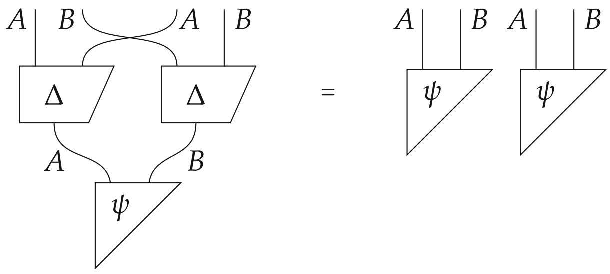
In process-theoretic terms, a *cloning process* for a system of type A is a process that takes in a single state ψ and produces two identical copies ψ:



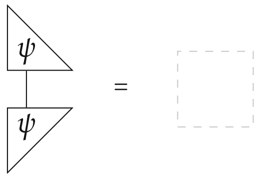
Since both copies are identical, it does not make a difference if we interchange the outputs after copying them:



If we wanted to clone a composite system of type $A \otimes B$, it should be possible to clone each subsystem of type A and B individually and then put them back together:

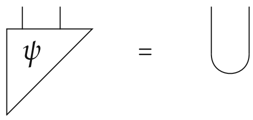


The last, but rather trivial assumption we have to make is that our process-theory contains at least one normalized state ψ, for which we have:

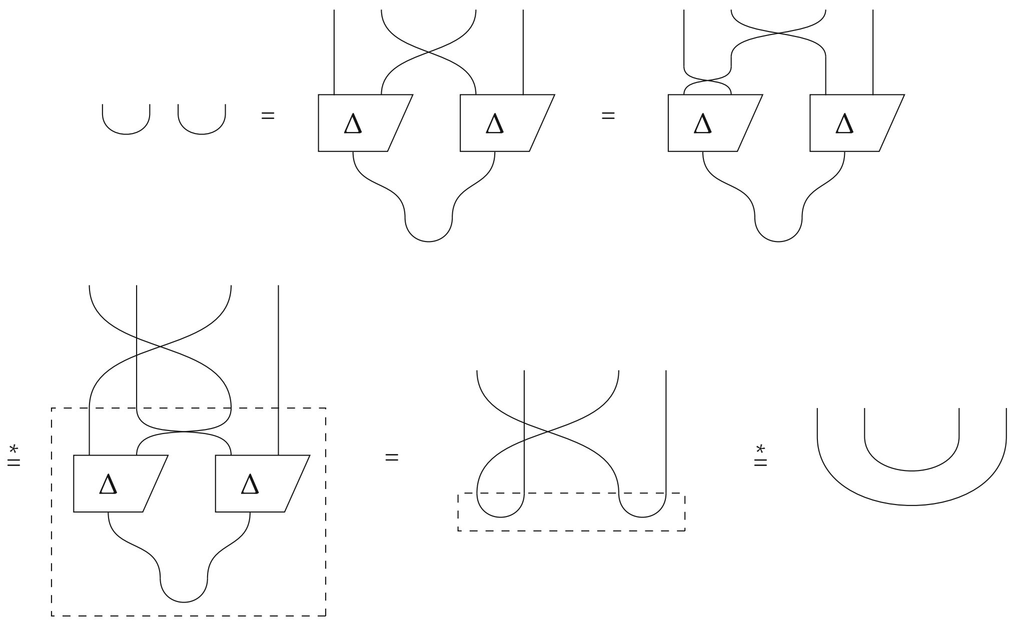


We will now show that if there existed a cloning process for a system of type A, which satisfies the three previous equations, it then follows that every process of type A is $\circ$-separable, which as we mentioned earlier is an unphysical matter of fact.

The state we are going to clone is the maximally correlated cup-state:

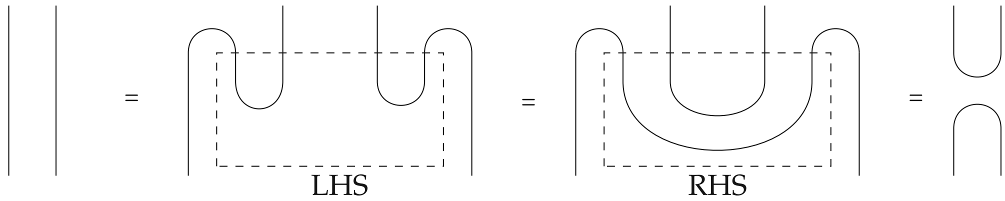


If we plug the cup-state into our cloning process we get the following equation:

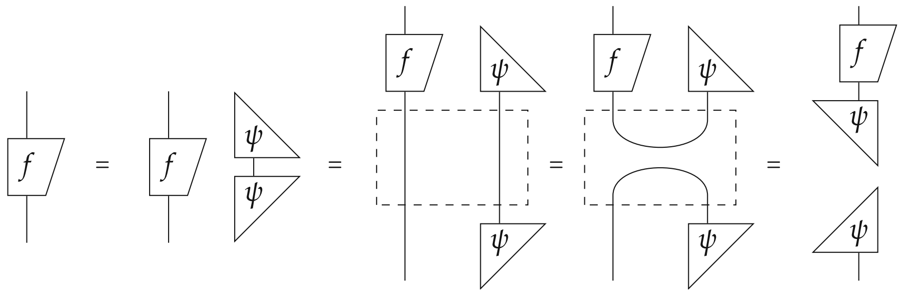


The starred equations are just deformations of the diagram.
The dashed square indicates the use of the properties of the cloning map.

For the next step, we are going to bend down the outermost outputs of the equation by adjoining cap-effects:

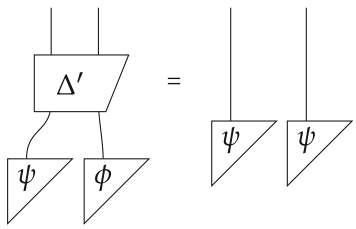


We have now shown that the parallel composition of two identities is equal to the disconnected sequential composition of a cap-effect and a cup-state. In our final step, we are going to substitute in this process in the following diagram:

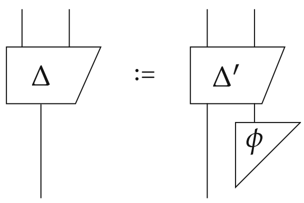


We have now shown, that any process f of type A is $\circ$-separable. As already mentioned, this would imply that there only exist constant processes and therefore we have to refute the existence of a cloning process with the above mentioned properties.

The usual no-cloning theorem in most textbooks is laid out differently, so we will also revisit the standard way of reasoning. For the standard proof, we assume the existence of a cloning process Δ' that takes in two input states, an unknown state ψ which ought to be copied onto another state ϕ, which gets overwritten in the process. It then outputs two identical copies of the input state ψ:

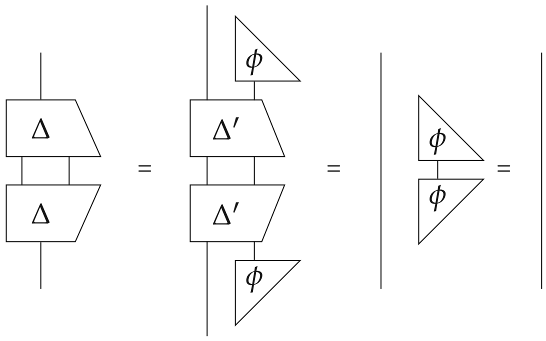


However we can define a cloning process similar to the one we previously defined, by fixing one of the inputs of the Δ' process, so there is practically no difference between those two cloning processes:

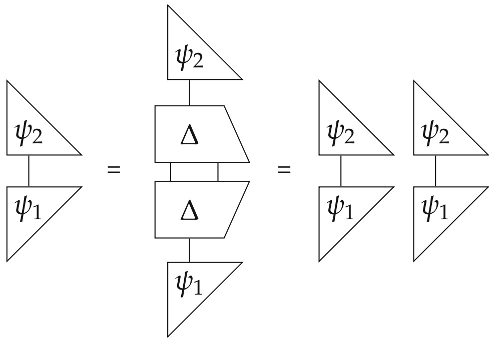


The second assumption usually made, is that Δ' is a *unitary* process. This assumption stems from the fact that in quantum mechanics the temporal evolution of a system is governed by a unitary time-evolution operator.

The unitarity of Δ' immediately implies the isometry of Δ:



We are now going to show that if two normalized states ψ_1 and ψ_2 can be cloned by the isometry we just specified, then either ψ_1 and ψ_2 have to be equal or orthogonal:



We have shown that the number $\lambda := \psi_2^\dagger \circ \psi_1$ satisfies the equation $\lambda = \lambda^2$. This leaves only two options for our states ψ_1 and ψ_2. Either their inner product is equal to 1, which means a certain positive test outcome when testing state ψ_1 for effect ψ_2 and therefore the states being equal $\psi_1 = \psi_2$, or their inner product is equal to 0, which means that both states are orthogonal. Therefore, if there is a system of type A, that contains at least two different states, only the orthogonal states can be cloned. This is closely related to the fact, that orthogonal states can be used to encode classical bits within a quantum system, due to their similar behaviour.

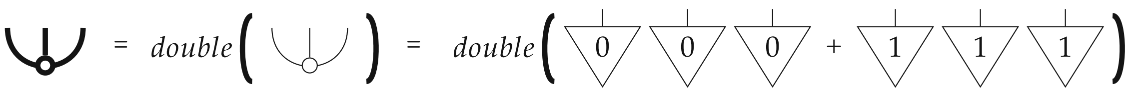
When comparing these two methods of proof, we see that for the first proof the critical assumption is the ability to clone a composite system by cloning each of its parts individually, and for the second proof we assumed that our cloning process is a unitary process. These two proofs are independent of each other but arrive at a similar result, namely, not every quantum state can be cloned.

5.2 Quantum non-locality

After the establishment of the theory of relativity, it became widely held belief that any future physical theory must adhere to the principle of *locality*. This principle posits that any interaction between two physical entities must be mediated by some form of wave or particle and is therefore limited by the speed of light. In other words, there is no instantaneous action at a distance. Another core tenet of physics was the concept of *realism*, which asserts that measurements have predetermined outcomes and that the measurement process merely reveals them to the observer.

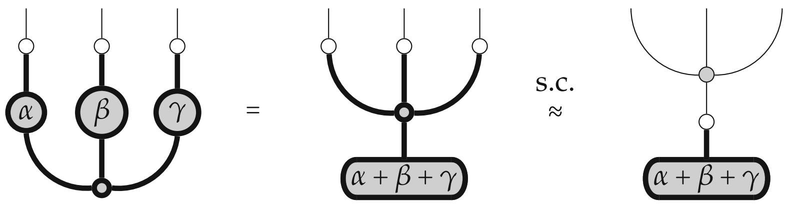
Both of these assumptions were widely accepted until 1935 when Einstein, Podolsky, and Rosen published their influential paper [3], stating that quantum theory is not a locally realistic theory. To reconcile this issue, one must abandon at least one of the assumptions, which did not sit well with Einstein, leading him to refuse to accept quantum theory as a complete theory.

We will now demonstrate that quantum theory is not compatible with local realism through the GHZ-Mermin measurement scenario. The Greenberger–Horne–Zeilinger state will be measured in various setups to investigate the resulting correlations of the measurement outcomes.



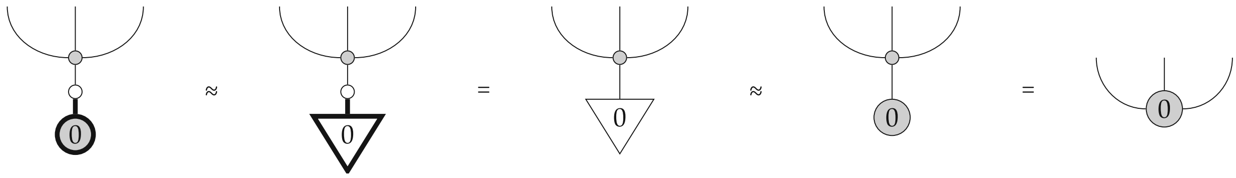
The maximally correlated GHZ-state in the z-basis.

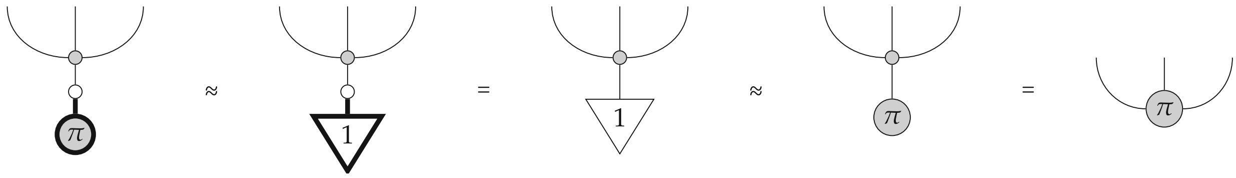
We are going to look into four different measurement scenarios of the following type:



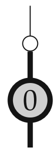
The GHZ-state is locally measured in three different bases, that are determined by the local phases α, β and γ. Due to the phase spider fusion rule and the property of strong complementarity, this measurement setup is equivalent to the one on the right-hand side of the equation. The equation also shows, that a local phase gate on a single subsystem affects all of the other subsystems as well, even if they are separated in space.

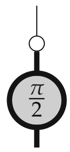
If $\alpha + \beta + \gamma$ is either 0 or π, then the phase state is in the classical subgroup for the z-basis ○ and we get the even-parity and odd-parity state respectively:





In our measurement scenarios the phases α, β and γ will either be 0 or $\frac{\pi}{2}$ to represent Z- or Y-measurements:

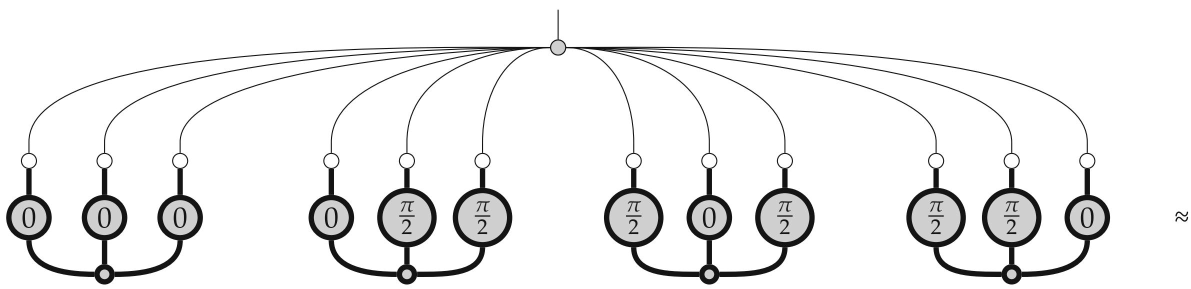
Z-measurement := 

Y-measurement := 

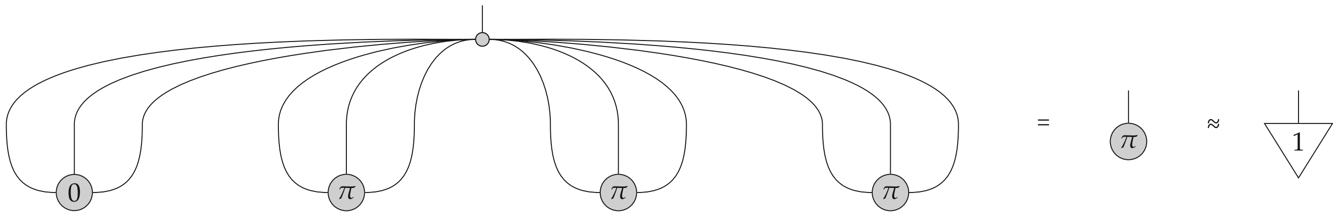
We now consider the following four measurement scenarios of the GHZ-state.

	system A	system B	system C
scenario 1	Z	Z	Z
scenario 2	Z	Y	Y
scenario 3	Y	Z	Y
scenario 4	Y	Y	Z

and combine the measurement results with an parity-gate represented by the x-basis spider ● to produce a singular valued output. The diagrammatic representation of the GHZ-Mermin scenario looks like this:

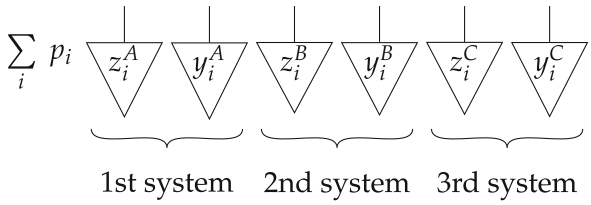


which reduces to the following diagram after employing the previously stated relations:

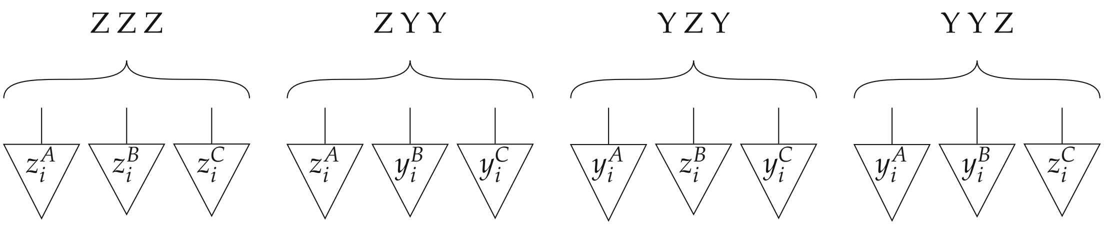


We end up with the odd-parity state, which is proportional to the second z-basis vector, in the quantum mechanical case.

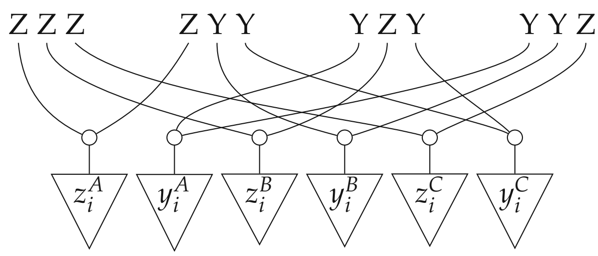
We will now calculate the result of this measurement setup for a locally realistic model. For this model we are assuming that some classical hidden-variables have already determined the measurement outcomes in advance. The most general state in this case is a tripartite classical probability distribution over all possible measurement outcomes:



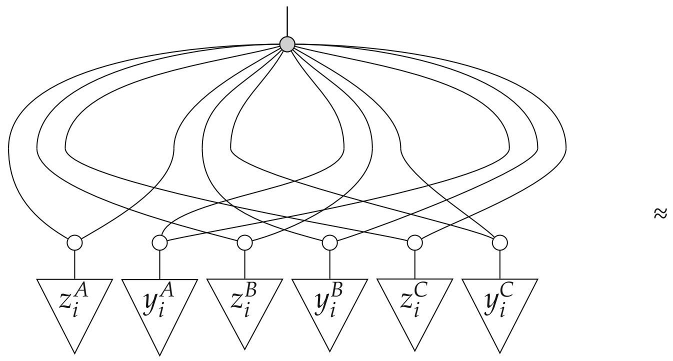
In such a case all of the possible z and y states have a pre-defined value from the set $\{0, 1\}$. Thus if we measured the same four measurement scenarios, the results would be given by the states:



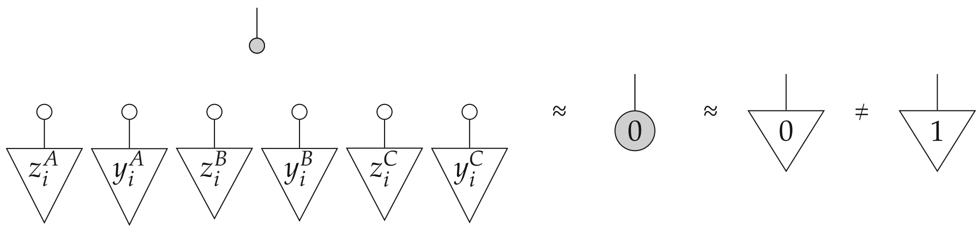
If we combine duplicate states via z-basis copy spiders, we get:



The overall parity of the measurement setup is then given by:



In this diagram every ○ spider is connected to the ● spider by exactly two legs and thus they disconnect due to complementarity.



The following state is independent of i and proportional to the first z-basis vector state. This difference in parity constitutes a contradiction and therefore quantum theory cannot be described by a locally realistic model.

5.3 Kochen-Specker Theorem

The Kochen-Specker theorem is another quantum mechanical no-go theorem, that excludes the possibility of quantum mechanics being described by a non-contextual hidden variable model. This means that there are possible measurement scenarios where the measurement of a physical property depends on its *measurement context*, namely the other compatible properties that are measured simultaneously.

In this thesis, we will try to reformulate the state-independent proof of the Kochen-Specker theorem of Peres and Mermin [19][20] diagrammatically. The proof involves the so-called Peres-Mermin square, which consists of nine measurements arranged in a square:

$$\begin{bmatrix} A & B & C \\ a & b & c \\ \alpha & \beta & \gamma \end{bmatrix} = \begin{bmatrix} \sigma_z \otimes \mathbb{1} & \mathbb{1} \otimes \sigma_z & \sigma_z \otimes \sigma_z \\ \mathbb{1} \otimes \sigma_x & \sigma_x \otimes \mathbb{1} & \sigma_x \otimes \sigma_x \\ \sigma_z \otimes \sigma_x & \sigma_x \otimes \sigma_z & \sigma_y \otimes \sigma_y \end{bmatrix}$$

Each of these two-qubit measurements is dichotomic, which means that there are only two possible outcomes +1 and -1. The three measurements in each row and column form a context, that is, a set of commuting observables that could in principle be jointly measured. We will consider the product of these measurement contexts and denote them as ABC, abc, $Aa\alpha$, etc. In a non-contextual classical model, each of the measurements takes on the same definite value, regardless of the context in which it is being measured. Therefore, the set $\{ABC, abc, \alpha\beta\gamma, Aa\alpha, Bb\beta, Cc\gamma\}$ can only contain an even number of products with the value of +1, since changing a single measurement changes the value of two products. When we define the expectation value of each context as:

$$\langle ABC \rangle \equiv \text{Prob}[ABC = +1] - \text{Prob}[ABC = -1]$$

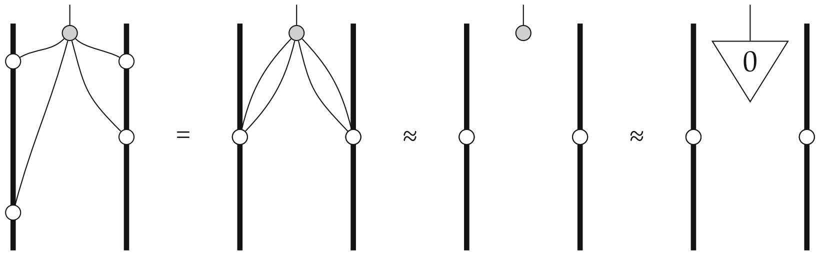
The previous argument shows that the following inequality holds:

$$\langle \text{PM} \rangle \equiv \langle ABC \rangle + \langle abc \rangle + \langle \alpha\beta\gamma \rangle + \langle Aa\alpha \rangle + \langle Bb\beta \rangle - \langle Cc\gamma \rangle \leq 4$$

An inequality that is defied by the predictions of quantum mechanics. Since all observables within each column and row mutually commute, they can be simultaneously measured, and we can therefore meaningfully speak of the expectation values of the products ABC, etc. For a given system in the normalized state $|\psi\rangle$ such an expectation value is calculated as $\langle ABC \rangle = \langle \psi | ABC | \psi \rangle$. It turns out that the product for each of the measurement contexts in each column and row $ABC = abc = ... = \mathbb{1}$, except for the third column where $Cc\gamma = -\mathbb{1}$. By summing up all of the expectation values, we get $\langle \text{PM} \rangle = 6$, which clearly violates the inequality mentioned above.

We therefore conclude that the context-independent approach is not compatible with quantum mechanics, this phenomenon is also known as *quantum contextuality*.

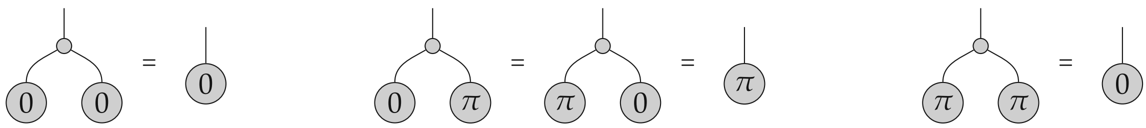
To reformulate the proof within the theory of quantum processes, we will model each column and row as three consecutive two-qubit measurements. The outcomes of the three measurements will then be combined using the parity map to calculate the product of the measurement outcomes. The first row ABC consists of $\sigma_z \otimes \mathbb{1}$, $\mathbb{1} \otimes \sigma_z$ and $\sigma_z \otimes \sigma_z$ measurements. This measurement scenario can be diagrammatically represented as:



First row: $\sigma_z \otimes \mathbb{1}$, $\mathbb{1} \otimes \sigma_z$, $\sigma_z \otimes \sigma_z$

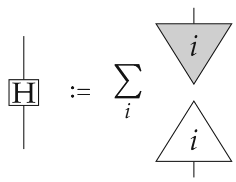
This process takes in two quantum states and performs sequential non-demolition z-measurements on each of them. The measurement outcomes are then combined using a x-basis spider, which behaves like a parity map on z-basis states. This follows directly from the fact that z-basis states are proportional to x-phase states:

together with the phase spider fusion rule, which combines the phases under addition modulo 2π:

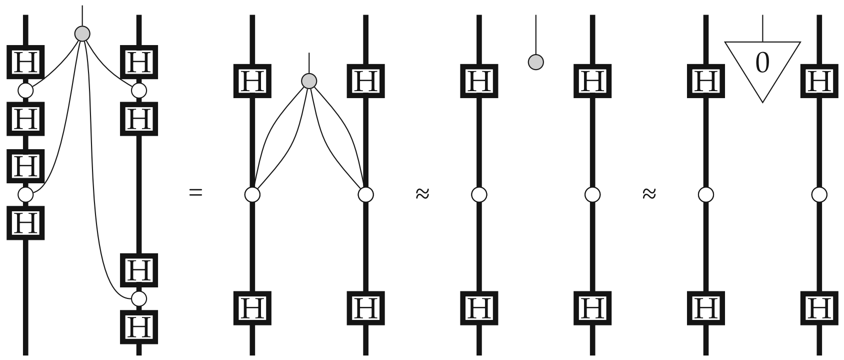


When the spider fusion rule and the complementarity rule are applied to the context of the first row, we end up with a definite outcome of the z-basis state 0 that represents a total product of +1.

For the second row $\mathbb{1} \otimes \sigma_x$, $\sigma_x \otimes \mathbb{1}$, $\sigma_x \otimes \sigma_x$, we need to exchange the z-measurements for x-measurements. This can be realized by adding additional Hadamard gates that facilitate a basis change from the z- to x-basis:

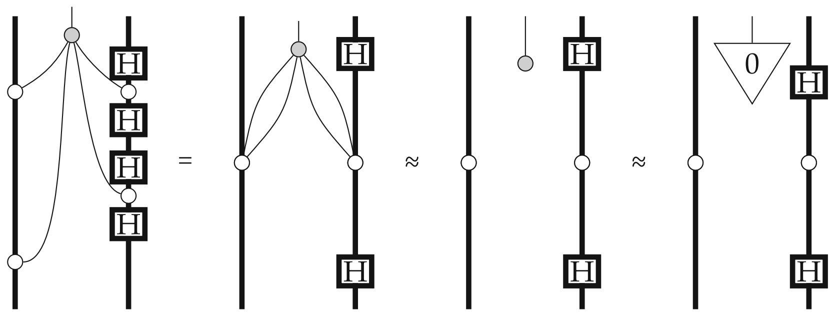


These Hadamard maps are self-adjoint and even involutory. Incorporating them into the diagram for the second row yields the following diagram:

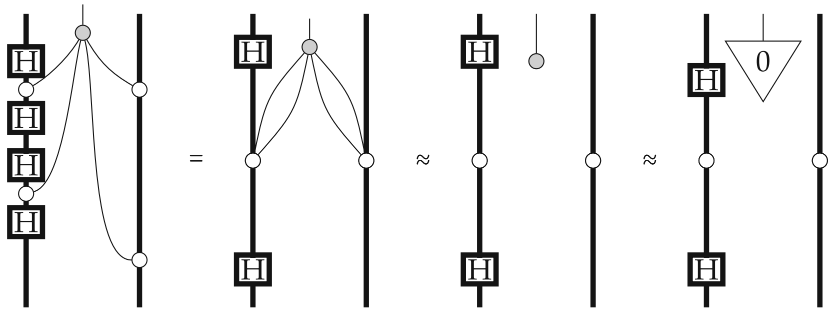


Second row: $\mathbb{1} \otimes \sigma_x$, $\sigma_x \otimes \mathbb{1}$, $\sigma_x \otimes \sigma_x$

Since the Hadamard gates are involutory they cancel out and the graph resolves in a similar manner, resulting in a total outcome of the 0 z-basis state. The calculations for the first and second columns of the Peres-Mermin square proceed analogously:



First column: $\sigma_z \otimes \mathbb{1}$, $\mathbb{1} \otimes \sigma_x$, $\sigma_z \otimes \sigma_x$

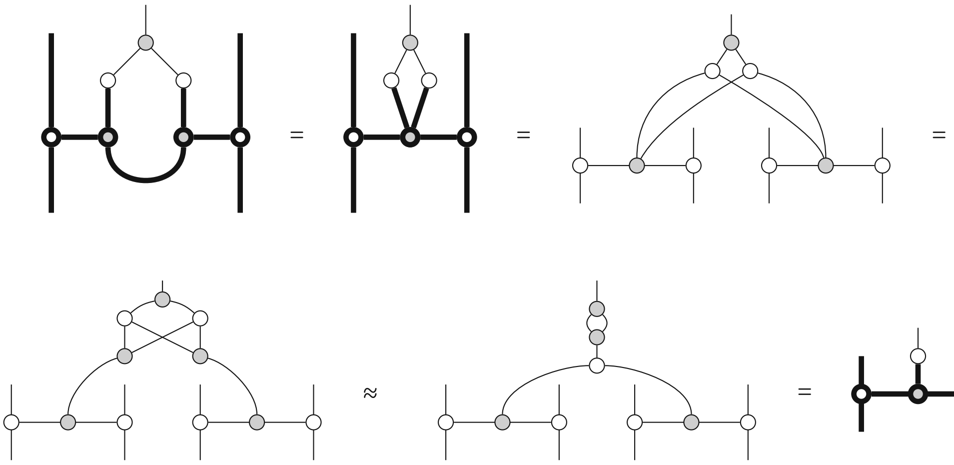


Second column: $\mathbb{1} \otimes \sigma_z$, $\sigma_x \otimes \mathbb{1}$, $\sigma_x \otimes \sigma_z$

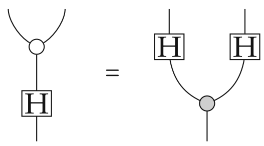
and again the 0 z-basis state results for both columns.

For the third column and row we need to adapt a different approach, as they require a global dichotomic non-local measurement, instead of two local single-qubit measurements. The last column and row involve $\sigma_x \otimes \sigma_x$ and $\sigma_y \otimes \sigma_y$ measurements, which decompose into sums of projection operators that are formed by the dyadic products of the entangled Bell states. Instead of the previous setups, where we combined the results of two local qubit measurements, we are going to implement a non-local measurement setup similar to the one described in [21].

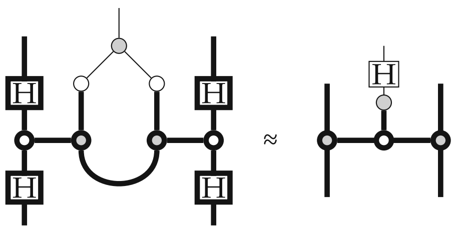
A non-local $\sigma_z \otimes \sigma_z$ measurement is realized by performing a quantum CNOT-gate on each of the incoming states and their respective half of a maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The resulting states are then measured in the z-basis and the measurement outcomes are combined by a parity map. This process is described by the following diagram, which can be further simplified by the spider fusion rule and strong complementarity.



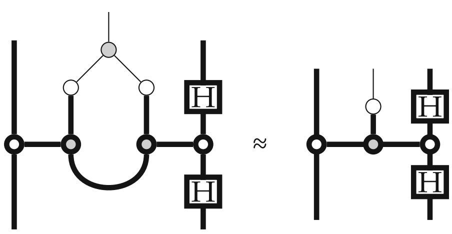
A non-local non-demolition $\sigma_x \otimes \sigma_x$ measurement is achieved by adding additional Hadamard gates before and after the quantum CNOT-gate.
Using the color change rule:

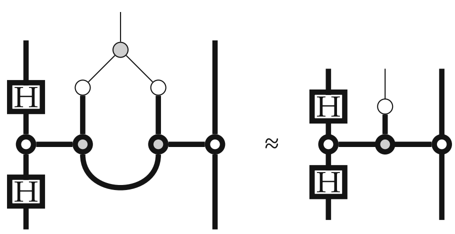


and the self-inverse property of the Hadamard gate, one can simplify the graph and push the Hadamard gates towards the middle. The simplified $\sigma_x \otimes \sigma_x$ measurement now looks like:

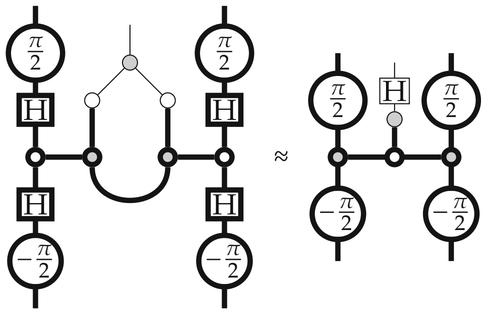


For the third row of the Peres-Mermin square we also need non-local $\sigma_z \otimes \sigma_x$ and $\sigma_x \otimes \sigma_z$ measurements, which equate to the following diagrams:

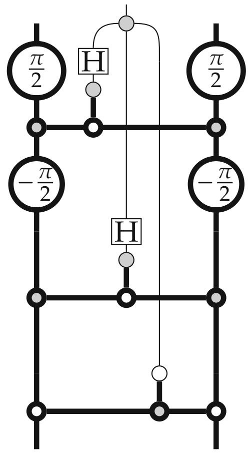




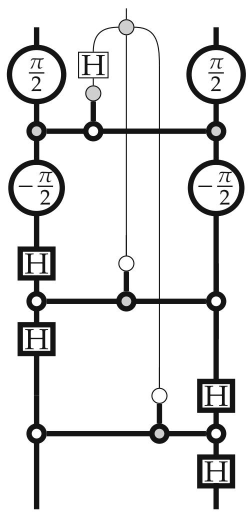
The last measurement we need to encode is the $\sigma_y \otimes \sigma_y$ measurement. To execute such a measurement we need to map the y-basis states onto the z-basis states before performing the quantum CNOT-gate. This is done by a unitary process consisting of a $-\frac{\pi}{2}$ phase gate followed by a Hadamard transformation. The simplified diagram for the $\sigma_y \otimes \sigma_y$ measurement is given by:



We now have all the necessary non-local measurements needed for the last column and row. The third column results from sequential composition of the $\sigma_z \otimes \sigma_z$, $\sigma_x \otimes \sigma_x$ and $\sigma_y \otimes \sigma_y$ measurements:



The three measurement outcomes then have to be combined using the parity map to obtain the corresponding product of outcomes. The third row involves $\sigma_z \otimes \sigma_x$, $\sigma_x \otimes \sigma_z$ and $\sigma_y \otimes \sigma_y$ measurements, which yields the following diagram:



However, in contrast to the previous diagrams, the diagrams for the last column and row do not appear to resolve in a similar fashion to produce a definite measurement outcome. Neither I myself nor the semi-automatic diagrammatic proof assistant Quantomatic[1] have been able to simplify the two graphs to arrive at a definite outcome. Therefore, the question remains whether I have missed a step in the derivation or if this is even a valid approach at all.

A possible issue might be the incompatibility of quantum logic with categorical quantum mechanics. While quantum logic describes the propositional structure of a single system, categorical quantum mechanics focuses on relationships between different systems. Therefore, categorical quantum mechanics might not be the right framework to argue about quantum contextuality.

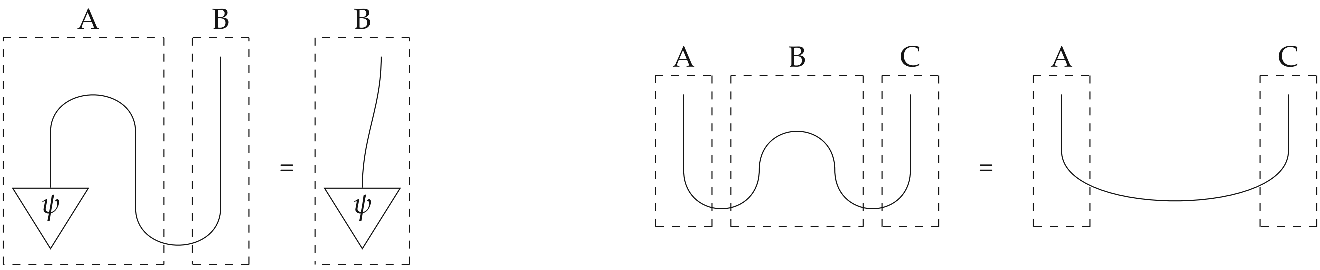
[1]https://quantomatic.github.io/

Chapter 6

Conclusion

In this thesis, we have provided an overview of categorical quantum mechanics, illustrating its capability to describe the flow of information between classical and quantum realms. This formalism is particularly useful for describing quantum computational protocols using the ZX-calculus [22]. Despite its abstract mathematical foundation, the general rules for reasoning about quantum processes are straightforward and user-friendly, making it a potential entry point into quantum theory.

We conclude that categorical quantum mechanics is a suitable formalism for high-level discussions of classical-quantum processes. In certain cases, such as quantum teleportation and entanglement swapping, the Hilbert space formalism does not clearly depict the underlying mechanisms. In contrast, the diagrammatic formalism captures the essence of these procedures, presenting them clearly and concisely.



Underlying structure of quantum teleportation and entanglement swapping

Although efforts have been made to reconcile categorical quantum mechanics with quantum logic [23][24], there may be an underlying incompatibility between the two fields. Despite numerous attempts, I have been unable to fully formalize a proof of the Kochen-Specker theorem within the framework of categorical quantum mechanics. Only time will tell if this formalization is possible and whether categorical quantum mechanics will significantly advance our understanding of quantum theory.

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