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DISSERTATION

Harmonic analysis and representations of Minkowski valuations

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Zusammenfassung

In dieser kumulativen Dissertation untersuchen wir erzeugende Funktionen rotations-äquivarianter Minkowski-Bewertungen auf ihre analytischen Eigenschaften und Entwicklung in Kugelfunktionen.

Erstens zeigen wir, dass erzeugende Funktionen, von welchen nur Integrierbarkeit bekannt war, stetig bis auf die Pole und fast überall differenzierbar sind. Für schwach monotone Minkowski-Bewertungen beschreiben wir das lokale Verhalten um die Pole und folgern, dass der Raum der konvexen Körper mit C^2 Stützfunktion in sich abgebildet wird. Für den zugehörigen Faltungsoperator weiten wir bekannte Eigenwertungleichungen auf eine größere Klasse von Minkowski-Bewertungen aus. Als Anwendung beweisen wir, dass für Schnittmittelungsoperatoren und für gerade, monotone Minkowski-Bewertungen euklidische Kugeln die einzigen Fixpunkte in einer C^2 Umgebung der Einheitskugel sind.

Zweitens betrachten wir die Wirkung von Aleskers Lefschetz-Integraloperator auf der erzeugenden Funktion. Wir zeigen, dass der Faltungskern der auftretenden Transformation eine strikt positive und bis auf den Nordpol glatte Funktion ist. Daraus schließen wir, dass alle bekannten Beispiele rotations-äquivarianter Minkowski-Bewertungen durch die Lefschetzoperatoren erhalten werden.

Allgemeiner beschreiben wir die Wirkung der Lefschetz-Operatoren auf der Klain–Schneider-Funktion skalarwertiger Bewertungen durch eine Radontransformation zwischen Fahnenmannigfaltigkeiten, womit wir ein Resultat von Schuster und Wannerer verallgemeinern. Im Zuge dessen entwickeln wir eine neue Methode, das gemischte Oberflächenmaß eines niedrigdimensionalen Körpers durch sein Oberflächenmaß bezüglich eines Unterraumes auszudrücken.

Drittens zeigen wir ein Analogon des Satzes von Klain–Schneider für Bewertungen, die invariant unter Rotationen um eine fixe Achse sind, zonal genannt, und somit auch Minkowski-Bewertungen. Daraus erhalten wir neue Beweise der Darstellungssätze für glatte und stetige Minkowski-Bewertungen, die erheblich kürzer und zugänglicher sind. Wir ermitteln auch eine neue Integraldarstellung für zonale Bewertungen, in der die Rolle des Oberflächenmaßes von einem gemischten Oberflächenmaß mit einer Scheibe übernommen wird, und präsentieren eine einfache Weise, zwischen diesen Darstellungen zu wechseln.

Als Anwendung liefern wir einen einfacheren Beweis für die Integraldarstellung der Schnittmittelungsoperatoren von Goodey und Weil mittels Bergs Funktionen. Für einen ursprungssymmetrischen Körper leiten wir eine neue Darstellung durch ein gemischtes Volumen mit einer Scheibe her. Darüber hinaus können wir diverse integralgeometrische Formeln zurückgewinnen und verallgemeinern.

Abstract

In this cumulative thesis, we examine the analytic properties and spherical harmonic decomposition of generating functions of rotationally equivariant Minkowski valuations.

First, we show that generating functions, which were merely known to be integrable, are in fact continuous up to the poles and differentiable almost everywhere. For weakly monotone Minkowski valuations, we describe the local behavior around the poles and conclude that the space of convex bodies with C^2 support functions gets mapped into itself. Regarding the corresponding spherical convolution transform, we extend the known spectral gap inequalities to a larger class of Minkowski valuations. As an application, we prove that for mean section operators and for even, monotone Minkowski valuations, Euclidean balls are the only fixed points in some C^2 neighborhood of the unit ball.

Second, we consider the action of Alesker's Lefschetz integration operator on the generating function. We show that the convolution kernel of the arising transform is a strictly positive function that is smooth up to the north pole. From this, we deduce that all known examples of rotationally equivariant Minkowski valuations are preserved by the Lefschetz operators.

More generally, we describe the action of the Lefschetz operators on the Klain–Schneider function of scalar valued valuations by a Radon type transform between flag manifolds, generalizing a result of Schuster and Wannerer. In the course of this, we introduce a new way to express the mixed area measure of a lower dimensional body in terms of its surface area measure relative to a subspace.

Third, we show an analogue of the Klain–Schneider theorem for valuations that are invariant under rotations around a fixed axis, called zonal, and thus, also Minkowski valuations. From this, we obtain new proofs of the representation theorems of smooth and continuous Minkowski valuations that are considerably shorter and more accessible. We also establish a new integral representation for zonal valuations, where the role of the area measure is taken by the mixed area measure with a disk, and we introduce an easy way to move between these two representations.

As applications, we give a simpler proof of the integral representation of mean section bodies by Goodey and Weil in terms of Berg's functions. In the case of origin symmetric bodies, we establish a new representation in terms of a mixed volume involving disks. Moreover, we recover and extend various integral geometric formulas.

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Contents

Introduction

A Minkowski valuation is an operator on the space of convex bodies in Euclidean space which is finitely additive with respect to Minkowski addition. Minkowski valuations which are continuous, homogeneous, and compatible with rigid motions occur naturally in various geometric constructions. For example, the projection body of a convex body encodes information of the size of its shadows cast in every possible direction. Since its introduction by Minkowski, it has become central in convex geometry (see, e.g., [11, 75, 76, 82, 90, 111]). A more recent instance is provided by the mean section body of Goodey and Weil [45]: it is formed as an average from sections of a convex body with all possible affine subspaces of a given dimension.

In recent years, the pursuit of a Hadwiger type theorem for Minkowski valuations (see, e.g., [60, 98, 99, 102]) has resulted in a spherical convolution representation by Schuster and Wannerer [103] and Dorrek [36]. For a homogeneous, rotationally equivariant Minkowski valuation Φ and a convex body K, the support function of the body ΦK is given as the spherical convolution of the area measure of K with a unique zonal function, called the generating function of Φ. When dealing with geometric problems involving Minkowski valuations, this representation is a very powerful tool. To illustrate this, Goodey and Weil [47] found the generating functions of the family of mean section operators; by computing their multipliers (the coefficients in the spherical harmonic decomposition), they show that every full-dimensional convex body is already determined by its mean section body.

The main objective of this cumulative thesis is to investigate the analytic properties and spherical harmonic decomposition of generating functions of Minkowski valuations. From our findings, we obtain new local uniqueness results on fixed points and lay the foundation for further progress on isoperimetric type inequalities and a complete classification of rotationally equivariant Minkowski valuations.

Article 1, which is joint work with Ortega-Moreno, is devoted to the fixed points of Minkowski valuations – a problem that is closely connected to Petty's conjectured inequality [90]. We show that generating functions, which were merely known to be integrable, are in fact locally Lipschitz outside the poles; in particular, they are continuous outside the poles and differentiable almost everywhere. If the corresponding Minkowski valuation is weakly monotone, we provide additional information on the behavior on small polar caps, from which we conclude that weakly monotone Minkowski valuations map the space of convex bodies with a C^2 support function into itself.

Regarding the spherical convolution transform of a generating function, we extend the spectral gap estimates that have been established by Ortega-Moreno and Schuster [88] to a larger class of Minkowski valuations and settle the equality cases for the second eigenvalue. We apply these findings to show that for every weakly monotone and even Minkowski valuation, there exists some C^2 neighborhood of the unit ball where its only fixed points are Euclidean balls; the same holds for the family of mean section operators. Our approach unifies and extends previous results by Ivaki [57, 58] and Ortega-Moreno and Schuster [88].

In Article 2, which is joint work with Hofstätter and Ortega-Moreno, we deal with the Lefschetz operators that were introduced by Alesker [5]. Providing a way to move between valuations of different degrees, they are a powerful tool in valuation theory that has been applied to obtain classification results and isoperimetric type inequalities (see, e.g., [7, 12, 21, 69–71, 89, 97]). The Lefschetz integration operator acts on the generating function of a Minkowski valuation as a convolution transform.

From its spherical harmonic decomposition, we show that the convolution kernel, which was merely known to be a distribution, is in fact a strictly positive function that is smooth up to the north pole. We also characterize it as the solution to some strictly elliptic Legendre type differential equation. As a direct consequence, we obtain that the Lefschetz integration operator maps the space of Minkowski valuations that are generated by some body of revolution into itself. Even more, we show that the Lefschetz operators preserve all examples of rotationally equivariant Minkowski valuations that are currently known, hinting at what a complete classification could look like.

In the same article, we also consider the Lefschetz operators on real valued valuations. These are determined by certain restrictions, which are encoded in the Klain–Schneider function [61, 62, 95]. We describe the action of the Lefschetz operators on the Klain–Schneider function by a Radon type transform between flag manifolds, generalizing a result of Schuster and Wannerer [102] from the even case. In order to compute the restrictions of valuations and more specifically, mixed volume functionals, we introduce mixed spherical liftings and projections. These allow us to express mixed area measures of lower dimensional bodies in terms of their surface area measure relative to a subspace, extending previous results of Goodey and Weil [44] for non-mixed area measures.

We pick up on this in Article 3, which is joint work with Hofstätter and Ortega-Moreno. There, we focus on restrictions of scalar valued valuations that are invariant under rotations around a fixed axis, called zonal, and thus, also Minkowski valuations. We show an analogue of the Klain–Schneider theorem for zonal valuations, characterizing those that vanish on hyperplanes containing the axis of revolution. From this, we deduce that every homogeneous, zonal valuation is determined by a restriction to one particular subspace. Conversely, we show that every zonal valuation on this subspace, if it is in addition smooth, extends to a zonal valuation on the ambient space.

Through Klain's approach, we recover the Hadwiger type theorem for smooth, zonal valuations by Schuster and Wannerer [103] as well as a recent characterization of continuous, zonal valuations by Knoerr [68]. As a consequence, the spherical convolution representation for homogeneous Minkowski valuations can also be written as a principal value integral. Let us note that our proof is considerably shorter and does not rely on any deep results from representation theory.

We also apply Klain's approach to establish a completely new classification of continuous, zonal valuations, where the role of the area measure is taken over by the mixed area measure with a disk that is orthogonal to the axis of revolution. Using the mixed spherical liftings and projections, we introduce an easy way to move between the two integral representations. The newly established integral representation involving the disk has certain benefits: the integral is always proper, the space of integral kernels is simple, and convergence is easy to understand. As a further consequence, zonal valuations are determined by their values on cones, and thus, on bodies of revolution.

We apply this to obtain various integral geometric formulas, recovering a Cauchy–Kubota type formula and extending an additive kinematic formula, both having recently been established by Hug, Mussnig, and Ulivelli [55, 56]. Moreover, we give a simpler proof of the integral representation of mean section bodies by Goodey and Weil [47] in terms of Berg's functions. In the case of origin symmetric bodies, we establish a new representation in terms of a mixed volume involving disks.

Articles included in this thesis

Article 1 **Fixed points of mean section operators** [27]
with Oscar Ortega-Moreno
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Article 2 **Lefschetz operators on convex valuations** [25]
with Georg C. Hofstätter and Oscar Ortega-Moreno
submitted
`arxiv.org/abs/2402.14731`

Article 3 **The Klain approach to zonal valuations** [26]
with Georg C. Hofstätter and Oscar Ortega-Moreno
submitted
`arxiv.org/abs/2410.18651`

1 Fixed points of mean section operators

1.1 Introduction

Sections and projections of convex bodies play an essential role in the field of geometric tomography. By taking measurements in lower dimensions, one seeks to recover information about the geometry of the original object. A common procedure to work with these measurements is to assemble them into a new convex body. For instance, the projection body of a convex body K is built from the volumes of shadows of K cast from every possible direction. To give the exact definition, recall that a convex body K (that is, a convex, compact subset) in $\mathbb{R}^n$, where throughout $n \geq 3$, can be defined by its support function $h(K,u) = \max\{\langle x,u\rangle : x \in K\}$, $u \in \mathbb{S}^{n-1}$. The *projection body* ΠK of a convex body K is defined by

$$h(\Pi K, u) = V_{n-1}(K|u^\perp), \qquad u \in \mathbb{S}^{n-1},$$

where $K|u^\perp$ denotes the orthogonal projection of K onto the hyperplane $u^\perp$ and V_{n-1} denotes the $(n-1)$-dimensional volume. The geometric operator Π was already introduced by Minkowski and has since become central to convex geometry (see, e.g., [11, 72, 75, 80–83, 90, 111, 112]).

A more recent instance of the procedure described above is the family of *mean section operators* introduced by Goodey and Weil [45, 46]. For $0 \leq j \leq n$, the j-th mean section body $\mathrm{M}_j K$ of a convex body K is essentially the average of all j-dimensional sections of K with respect to Minkowski addition. More precisely,

$$h(\mathrm{M}_j K, u) = \int_{\mathrm{AG}(n,j)} h(K \cap E, u)\,dE, \qquad u \in \mathbb{S}^{n-1},$$

where $\mathrm{AG}(n,j)$ denotes the affine Grassmannian (that is, the space of j-dimensional affine subspaces of $\mathbb{R}^n$) and integration is with respect to a suitably normalized, positive, rigid motion invariant measure.

The projection body and mean section operators belong to a rich class of geometric operators acting on the space $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$. A *Minkowski valuation* is a map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ satisfying

$$\Phi K + \Phi L = \Phi(K \cup L) + \Phi(K \cap L)$$

with respect to Minkowski addition whenever $K \cup L \in \mathcal{K}^n$. Scalar valued valuations have a long history in convex geometry (see, e.g., [3, 4, 6, 18, 23, 51, 63, 79]). Their systematic study goes back to Hadwiger's [52] famous characterization of the intrinsic volumes V_i,

$0 \leq i \leq n$, (see Section 1.2) as a basis for the space of continuous, rigid motion invariant scalar valuations.

The investigation of Minkowski valuations has originated from Schneider's [92] research on Minkowski endomorphisms. However, it was the seminal work by Ludwig [75, 76] which prompted further development. In [75], Ludwig identifies Minkowski's projection body map as the unique (up to a positive constant) continuous, translation invariant, affine contravariant Minkowski valuation, solving a problem posed by Lutwak. Following Ludwig's steps, contributions of several authors (e.g., [1, 24, 30, 50, 78, 101, 109]) show that the convex cone of Minkowski valuations compatible with affine transformations is in many instances finitely generated. In contrast, a less restrictive condition such as rotation equivariance produces a significantly larger class of valuations, making their classification challenging.

Denote by **MVal** the space of all continuous, translation invariant Minkowski valuations intertwining rotations and by $\mathbf{MVal}_i$ the subspace of Minkowksi valuations homogeneous of degree i. A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is said to have *degree* i if $\Phi(\lambda K) = \lambda^i \Phi K$ for every $K \in \mathcal{K}^n$ and $\lambda \geq 0$. By a classical result of McMullen [85], continuous, translation invariant, homogeneous valuations can only have integer degree $i \in \{0, \dots, n\}$. In recent years, substantial progress (e.g., [60, 98, 99, 102, 103]) to obtain a Hadwiger-type theorem for the space **MVal** has led to the following representation by Dorrek [36] involving the *spherical convolution* of an integrable function and the *area measures* $S_i(K, \cdot)$ of a convex body K (see Section 1.2): for every $\Phi_i \in \mathbf{MVal}_i$ of degree $1 \leq i \leq n-1$, there exists a unique centered, $\mathrm{SO}(n-1)$ invariant function $f \in L^1(\mathbb{S}^{n-1})$ such that for every $K \in \mathcal{K}^n$,

$$h(\Phi_i K, \cdot) = S_i(K, \cdot) * f. \tag{1.1.1}$$

A function on $\mathbb{S}^{n-1}$ is said to be *centered* if it is orthogonal to all linear functions. We call the function f in (1.1.1) the *generating function* of Φ_i. If $\Phi_i K = \{o\}$ for all $K \in \mathcal{K}^n$, we call Φ_i *trivial*.

For $1 \leq i \leq n-1$, the i-th projection body map Π_i is defined by $h(\Pi_i K, u) = V_i(K|u^\perp)$, $u \in \mathbb{S}^{n-1}$. Note that $\Pi_{n-1} = \Pi$ is Minkowski's projection body map. Each operator Π_i belongs to $\mathbf{MVal}_i$ and is generated by the support function of a line segment, as can be easily deduced from Cauchy's projection formulas (see, e.g., [43, p. 408]). The j-th mean section operator M_j, up to a suitable translation, also belongs to $\mathbf{MVal}_i$, where $i = n - j + 1$. However, unlike for the projection body map, the determination of their generating functions is non-trivial and involves the functions employed by Berg in his solution of the Christoffel problem [29]. For each dimension $n \geq 2$, Berg [13] constructed a function $g_n \in C^\infty(-1, 1)$ with the property that $h(K, \cdot) = S_1(K, \cdot) * \breve{g}_n + \langle s(K), \cdot \rangle$ for every convex body $K \in \mathcal{K}^n$, where $\breve{g}_n$ is the $\mathrm{SO}(n-1)$ invariant function on $\mathbb{S}^{n-1}$ associated to g_n and $s(K)$ denotes the *Steiner point* of K (see Section 1.2). Goodey and Weil [45, 47] showed that for $2 \leq j \leq n$ and every $K \in \mathcal{K}^n$,

$$h(\mathrm{M}_j K, \cdot) = S_{n-j+1}(K, \cdot) * \breve{g}_j + c_{n,j} V_{n-j}(K) \langle s(K), \cdot \rangle, \tag{1.1.2}$$

where $c_{n,j} > 0$ is some constant.

Fixed points of geometric operators are closely related to a range of open problems in convex geometry (see, e.g., [28, 43, 90]). For instance, *Petty's conjecture* [90] can be expressed in terms of fixed points of Π^2 up to affine transformations. This was first observed by Schneider [94] and later extended by Lutwak [81] to projection body maps of all degrees. The global classification of fixed points of Π_i^2 has only been settled in the polytopal case by Weil [111] and in the 1-homogeneous case by Schneider [93]. It is conjectured that the only smooth fixed points of Π^2 are ellipsoids. Locally around the unit ball, this was recently confirmed independently by Saroglou and Zvavitch [91] and Ivaki [58], motivated by the work of Fish, Nazarov, Ryabogin, and Zvavitch [41].

For degree $1 < i < n-1$, Ivaki [57] showed that in some C^2 neighborhood of the unit ball, the only fixed points of Π_i^2 are Euclidean balls. The second author and Schuster [87, 88] have shown that this phenomenon also holds for the class of even C^2_+ *regular* Minkowski valuations, that is, Minkowski valuations generated by the support function of an origin-symmetric convex body of revolution that has a C^2 boundary with positive Gauss curvature. A line segment is clearly not of this kind, so the results by Ivaki and by the second author and Schuster appear to be disconnected. In this paper, we bridge this gap with our first main result.

Theorem 1.A. *Let $1 < i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ be generated by an origin-symmetric convex body of revolution. Then there exists a C^2 neighborhood of the unit ball where the only fixed points of Φ_i^2 are Euclidean balls, unless Φ_i is a multiple of the projection body map, in which case ellipsoids are also fixed points.*

The case when $i = 1$ (that is, $\Phi_1 \in \mathbf{MVal}_1$ is a Minkowski endomorphism) has been settled globally by Kiderlen [60]. With Theorem 1.A, we unify the previous results on C^2_+ regular Minkowski valuations and projection body maps obtained in [88] and [57, 58], respectively. However, none of them (including Theorem 1.A) cover any local uniqueness of fixed points of mean section operators. This is because Berg's functions are neither even nor support functions. By further extending the techniques employed in the proof of Theorem 1.A, we obtain the following.

Theorem 1.B. *For $2 \leq j < n$, there exists a C^2 neighborhood of the unit ball where the only fixed points of M_j^2 are Euclidean balls.*

Throughout, this is to be understood as follows: there exists some $\varepsilon > 0$ such that if $K \in \mathcal{K}^n$ has a C^2 support function satisfying $\|h(\alpha K + x, \cdot) - 1\|_{C^2(\mathbb{S}^{n-1})} < \varepsilon$ for some $\alpha > 0$ and $x \in \mathbb{R}^n$, and if $\mathrm{M}_j^2 K$ is a dilated and translated copy of K, then K is a Euclidean ball. We want to emphasize that we will obtain both Theorem 1.A and 1.B from a more general result (Theorem 1.5.2) that applies to all weakly monotone, homogeneous Minkowski valuations in the space $\mathbf{MVal}$. A Minkowski valuation $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called *weakly monotone* if $\Phi K \subseteq \Phi L$ whenever $K \subseteq L$ and the Steiner points of K and L are at the origin. The proof of Theorem 1.5.2 requires the convolution transform defined by its generating function to be a bounded operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$. The following theorem provides necessary and sufficient conditions for the boundedness of convolution transforms.

Theorem 1.C. *Let $f \in L^1(\mathbb{S}^{n-1})$ be $\mathrm{SO}(n-1)$ invariant. Then the convolution transform $\phi \mapsto \phi * f$ is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$ if and only if $\Box_n f$ is a signed measure and*

$$\int_{(0,\frac{\pi}{2})} \frac{1}{r}\, |(\Box_n f)(\{u \in \mathbb{S}^{n-1} : \langle \pm\bar{e}, u\rangle > \cos r\})|\, dr < \infty. \tag{1.1.3}$$

Here, $\bar{e} \in \mathbb{S}^{n-1}$ denotes the north pole of the unit sphere fixing the axis of revolution of f and $\Box_n f = \frac{1}{n-1}\Delta_{\mathbb{S}} f + f$, where $\Delta_{\mathbb{S}}$ is the spherical Laplacian on $\mathbb{S}^{n-1}$. Theorem 1.C tells us that the regularity of the convolution transform defined by f is determined by the mass distribution of $\Box_n f$ on small polar caps. It turns out that generating functions of weakly monotone Minkowski valuations exhibit precisely this behavior.

Theorem 1.D. *Let $1 \le i \le n-1$ and $\Phi_i \in \mathbf{MVal}_i$ with generating function f. Then f is locally Lipschitz outside the poles and $\Box_n f$ is a signed measure on $\mathbb{S}^{n-1}$. Moreover, if Φ_i is in addition weakly monotone, then there exists $C > 0$ such that for all $r \ge 0$,*

$$|\Box_n f|\big(\{u \in \mathbb{S}^{n-1} : |\langle \bar{e}, u\rangle| > \cos r\}\big) \le C r^{i-1}. \tag{1.1.4}$$

As an immediate consequence of Theorems 1.C and 1.D, we obtain the following.

Corollary. *Let $1 < i \le n-1$ and $\Phi_i \in \mathbf{MVal}_i$ be weakly monotone with generating function f. Then the convolution transform $\phi \mapsto \phi * f$ is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$. In particular, Φ_i maps the space of convex bodies with a C^2 support function into itself.*

To the best of our knowledge, apart from smooth Minkowski valuations, this was previously only known for the projection body operators: it was shown by Martinez-Maure [84] that the cosine transform is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$, which is an essential tool in the proof of Ivaki's [57, 58] fixed point results.

We want to remark that the continuity of f proven in Theorem 1.D confirms a conjecture by Dorrek. Moreover, note that (1.1.4) relates the regularity of f to the degree of homogeneity. It has been shown by Parapatits and Schuster [89] that if a function generates a Minkowski valuation of a certain degree, then it also generates Minkowski valuations of all lower degrees. Using (1.1.4), it can be shown that for each $1 \le i \le n-1$, the Berg function g_{n-i+1} generates a weakly monotone Minkowski valuation of degree i but not higher.

Organization of the article. In Section 1.2, we collect the required background on convex geometry and analysis on the unit sphere. In Section 1.3, we investigate regularity of zonal measures and convolution transforms, proving Theorem 1.C. In Section 1.4, we show that weak monotonicity of Minkowski valuations implies additional regularity of the generating function, proving Theorem 1.D. Finally, in Section 1.5 we apply our results from the previous sections to the study of fixed points. There we prove Theorems 1.A and 1.B as well as a general result on even Minkowski valuations.

1.2 Background material

In the following, we collect basic facts about convex bodies, mixed volumes and area measures. We also discuss differential geometry and the theory of distributions on the unit sphere. In the final part of this section, we gather the required material from harmonic analysis, including spherical harmonics and the convolution of measures. As general references for this section, we cite the monographs by Gardner [43], Schneider [96], Hörmander [54], Lee [73, 74], and Groemer [49].

Convex geometry. The space $\mathcal{K}^n$ of convex bodies naturally carries an algebraic and topological structure. The so-called Minkowski operations, dilation and the Minkowski addition, are given by $\lambda K = \{\lambda x : x \in K\}$, $\lambda \geq 0$, and $K + L = \{x + y : x \in K, y \in L\}$. The Hausdorff metric can be defined as

$$d(K, L) = \max\{t \geq 0 : K \subseteq L + tB^n \text{ and } L \subseteq K + tB^n\},$$

where B^n denotes the unit ball of $\mathbb{R}^n$.

As was pointed out before, every convex body $K \in \mathcal{K}^n$ is uniquely determined by its support function $h_K(x) = h(K, x) = \max\{\langle x, y\rangle : y \in K\}$, $x \in \mathbb{R}^n$, which is homogeneous of degree one and subadditive. Conversely, every function with these two properties is the support function of a unique body $K \in \mathcal{K}^n$. Associating a convex body with its support function is compatible with the structure of $\mathcal{K}^n$, that is, $h_{\lambda K+L} = \lambda h_K + h_L$ and $d(K, L) = \|h_K - h_L\|_\infty$, where $\|\cdot\|_\infty$ denotes the maximum norm on the unit sphere. Moreover, $K \subseteq L$ if and only if $h_K \leq h_L$. In addition, $h_{\vartheta K+x}(u) = h_K(\vartheta^{-1}u) + \langle x, u\rangle$ for every $\vartheta \in \mathrm{SO}(n)$ and $x \in \mathbb{R}^n$.

The Steiner formula expresses the volume of the parallel set of a convex body K at distance $t \geq 0$ as a polynomial in t. To be precise,

$$V_n(K + tB^n) = \sum_{i=0}^{n} t^{n-i}\kappa_{n-i}V_i(K), \tag{1.2.1}$$

where κ_i denotes the i-dimensional volume of B^i and the coefficient $V_i(K)$ is called the i-th *intrinsic volume* of K for $0 \leq i \leq n$. The intrinsic volumes are important quantities carrying geometric information on convex bodies. For instance, V_n is the volume, V_{n-1} the surface area, and V_1 the mean width.

The *surface area measure* $S_{n-1}(K, \cdot)$ of a convex body is the positive measure on $\mathbb{S}^{n-1}$ defined as follows: the measure $S_{n-1}(K, A)$ of a measurable subset $A \subseteq \mathbb{S}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure of all boundary points of K with outer unit normal in A. Analogously to (1.2.1), there is a Steiner-type formula for surface area measures:

$$S_{n-1}(K + tB^n, \cdot) = \sum_{i=0}^{n-1} t^{n-1-i}\tbinom{n-1}{i}S_i(K, \cdot),$$

where the measure $S_i(K, \cdot)$ is called the i-th *area measure* of K for $0 \leq i \leq n-1$. Each of the area measures is *centered*, meaning that they integrate all linear functions to zero. By a theorem of Alexandrov-Fenchel-Jessen (see, e.g., [96, Section 8.1]), if K has non-empty interior, then each area measure $S_i(K,\cdot)$ determines K up to translations.

The *Steiner point* of a convex body K is defined as $s(K) = \int_{\mathbb{S}^{n-1}} h(K,u)u\,du$. The Steiner point map $s : \mathcal{K}^n \to \mathbb{R}^n$ is the unique continuous, vector valued valuation intertwining rigid motions (see, e.g., [96, p. 181]).

Differential geometry. As an embedded submanifold of $\mathbb{R}^n$, the unit sphere $\mathbb{S}^{n-1}$ naturally inherits the structure of an $(n-1)$-dimensional Riemannian manifold. We identify the tangent space at each point $u \in \mathbb{S}^{n-1}$ with $u^\perp \subseteq \mathbb{R}^n$, which allows us to interpret tensor fields as maps from $\mathbb{S}^{n-1}$ into some Euclidean space.

Throughout, we will only work with tensor fields up to order two. That is, we define a vector field on $\mathbb{S}^{n-1}$ as a map $X : \mathbb{S}^{n-1} \to \mathbb{R}^n$ such that $X(u) \in u^\perp$ for every $u \in \mathbb{S}^{n-1}$, and a 2-tensor field on $\mathbb{S}^{n-1}$ as a map $Y : \mathbb{S}^{n-1} \to \mathbb{R}^{n\times n}$ such that $Y(u)(u^\perp) \subseteq u^\perp$ and $Y(u)u = 0$ for every $u \in \mathbb{S}^{n-1}$. For instance, let $Y(u) = P_{u^\perp}$ be the orthogonal projection onto $u^\perp$ for each $u \in \mathbb{S}^{n-1}$. Then the 2-tensor field Y acts as the identity on each tangent space. The inner product of two 2-tensors Y_1 and Y_2 on $u^\perp$ is given by $\langle Y_1, Y_2\rangle = \operatorname{tr}(Y_1Y_2)$.

We denote by $\nabla_{\mathbb{S}}$ the standard covariant derivative and by $\operatorname{div}_{\mathbb{S}}$ the divergence operator on $\mathbb{S}^{n-1}$. The operators $\nabla_{\mathbb{S}}$ and $\operatorname{div}_{\mathbb{S}}$ are related via the *spherical divergence theorem*, which states that

$$\int_{\mathbb{S}^{n-1}} \langle X(u), \nabla_{\mathbb{S}}\phi(u)\rangle du = -\int_{\mathbb{S}^{n-1}} \operatorname{div}_{\mathbb{S}} X(u)\phi(u)du$$

and

$$\int_{\mathbb{S}^{n-1}} \langle Y(u), \nabla_{\mathbb{S}}X(u)\rangle du = -\int_{\mathbb{S}^{n-1}} \langle \operatorname{div}_{\mathbb{S}} Y(u), X(u)\rangle du$$

for every smooth function ϕ, smooth vector field X, and smooth 2-tensor field Y on $\mathbb{S}^{n-1}$.

The *spherical gradient* $\nabla_{\mathbb{S}}\phi$ and *spherical Hessian* $\nabla^2_{\mathbb{S}}\phi$ of a smooth function ϕ can be expressed in terms of derivatives along smooth curves. If $\gamma : I \to \mathbb{S}^{n-1}$ is a geodesic in $\mathbb{S}^{n-1}$, then

$$\left.\frac{d}{ds}\right|_0 \phi(\gamma(s)) = \langle \nabla_{\mathbb{S}}\phi(\gamma(0)), \gamma'(0)\rangle \qquad \text{and} \qquad \left.\frac{d^2}{ds^2}\right|_0 \phi(\gamma(s)) = \langle \nabla^2_{\mathbb{S}}\phi(\gamma(0)), \gamma'(0)\otimes\gamma'(0)\rangle.$$

For the first identity, γ does not actually need to be a geodesic; for the second identity, the fact that γ is a geodesic eliminates an additional first order term compared to a general smooth curve. All geodesics γ in the unit sphere are of the form

$$\gamma(s) = \cos(cs)u + \sin(cs)v \tag{1.2.2}$$

for some orthogonal vectors $u, v \in \mathbb{S}^{n-1}$, where $c \geq 0$ is the constant speed of γ.

Distributions. For an open interval $(a,b) \subseteq \mathbb{R}$, we denote by $\mathcal{D}(a,b)$ the space of *test functions* (that is, compactly supported smooth functions) on (a,b), endowed with the standard Fréchet topology. The elements of the continuous dual space $\mathcal{D}'(a,b)$ are called *distributions* on (a, b). Moreover, we denote the pairing of a test function $\psi \in \mathcal{D}(a, b)$ and a distribution $g \in \mathcal{D}'(a, b)$ by $\langle \psi, g \rangle_{\mathcal{D}'}$. The derivative of g and the product of a smooth function $\eta \in C^\infty(a, b)$ with g are defined by

$$\langle \psi, g' \rangle_{\mathcal{D}'} = -\langle \psi', g \rangle_{\mathcal{D}'} \qquad \text{and} \qquad \langle \psi, \eta \cdot g \rangle_{\mathcal{D}'} = \langle \eta \cdot \psi, g \rangle_{\mathcal{D}'},$$

The space $C^{-\infty}(\mathbb{S}^{n-1})$ of distributions on the unit sphere is defined as the continuous dual space of the space $C^\infty(\mathbb{S}^{n-1})$ of smooth functions, endowed with the standard Fréchet topology. We denote the pairing of a test function $\phi \in C^\infty(\mathbb{S}^{n-1})$ and a distribution $\mu \in C^{-\infty}(\mathbb{S}^{n-1})$ by $\langle \phi, \mu \rangle_{C^{-\infty}}$. By virtue of the spherical divergence theorem, we define the spherical gradient and spherical Hessian of a distribution $\mu \in C^{-\infty}(\mathbb{S}^{n-1})$ by

$$\langle X, \nabla_{\mathbb{S}} \mu \rangle_{C^{-\infty}} = -\langle \operatorname{div}_{\mathbb{S}} X, \mu \rangle_{C^{-\infty}} \qquad \text{and} \qquad \langle Y, \nabla_{\mathbb{S}}^2 \mu \rangle_{C^{-\infty}} = \langle \operatorname{div}_{\mathbb{S}}^2 Y, \mu \rangle_{C^{-\infty}},$$

respectively, where X is an arbitrary smooth vector field and Y is an arbitrary smooth 2-tensor field on $\mathbb{S}^{n-1}$.

The group $\mathrm{SO}(n)$ acts on the space $C^\infty(\mathbb{S}^{n-1})$ in a natural way: for $\vartheta \in \mathrm{SO}(n)$ and $\phi \in C^\infty(\mathbb{S}^{n-1})$, we define $\vartheta\phi$ by $(\vartheta\phi)(u) = \phi(\vartheta^{-1}u)$. By duality, the action of $\mathrm{SO}(n)$ extends to distributions: for $\mu \in C^{-\infty}(\mathbb{S}^{n-1})$, we define $\vartheta\mu$ by $\langle \phi, \vartheta\mu \rangle_{C^{-\infty}} = \langle \vartheta^{-1}\phi, \mu \rangle_{C^{-\infty}}$. A map T is said to be $\mathrm{SO}(n)$ *equivariant* if it intertwines rotations, that is, $\mathrm{T}(\vartheta\mu) = \vartheta\mathrm{T}\mu$ for every μ in the domain of T.

We may identify the space $\mathcal{M}(a, b)$ of finite signed measures on (a, b) with a subspace of $\mathcal{D}'(a, b)$. By virtue of to the *Riesz-Markov-Kakutani representation theorem*, a distribution is defined by a finite signed measure on (a, b) if and only if it is continuous on $\mathcal{D}(a, b)$ with respect to uniform convergence. Similarly, the space $\mathcal{M}(\mathbb{S}^{n-1})$ of signed measures on $\mathbb{S}^{n-1}$ corresponds to the subspace of distributions which are continuous on $C^\infty(\mathbb{S}^{n-1})$ with respect to uniform convergence.

Harmonic analysis. Denote by $\mathcal{H}_k^n$ the space of *spherical harmonics* of dimension n and degree $k \geq 0$, that is, the space of harmonic, k-homogeneous polynomials on $\mathbb{R}^n$, restricted to the unit sphere $\mathbb{S}^{n-1}$. The spherical Laplacian $\Delta_{\mathbb{S}} = \operatorname{tr} \nabla_{\mathbb{S}}^2 = \operatorname{div}_{\mathbb{S}} \nabla_{\mathbb{S}}$ is a second-order uniformly elliptic self-adjoint operator on $\mathbb{S}^{n-1}$ that intertwines rotations. It turns out that the spaces $\mathcal{H}_k^n$ are precisely the eigenspaces of $\Delta_{\mathbb{S}}$. Consequently, $L^2(\mathbb{S}^{n-1})$ decomposes into a direct orthogonal sum of them. Each space $\mathcal{H}_k^n$ is a finite dimensional and irreducible $\mathrm{SO}(n)$ invariant subspace of $L^2(\mathbb{S}^{n-1})$ and for every $Y_k \in \mathcal{H}_k^n$, we have that $\Delta_{\mathbb{S}} Y_k = -k(k+n-2)Y_k$. For the box operator $\square_n = \frac{1}{n-1}\Delta_{\mathbb{S}} + \mathrm{Id}$, this implies

$$\square_n Y_k = -\frac{(k-1)(k+n-1)}{n-1} Y_k, \qquad Y_k \in \mathcal{H}_k^n. \tag{1.2.3}$$

Throughout, we use $\bar{e}$ to denote a fixed but arbitrarily chosen pole of $\mathbb{S}^{n-1}$ and write $\mathrm{SO}(n-1)$ for the subgroup of rotations in $\mathrm{SO}(n)$ fixing $\bar{e}$. Functions, measures, and distributions on $\mathbb{S}^{n-1}$ that are invariant under the action of $\mathrm{SO}(n-1)$ are called *zonal*. Clearly the value of a zonal function at $u \in \mathbb{S}^{n-1}$ depends only on the value of $\langle \bar{e}, u \rangle$, so there is a natural correspondence between zonal functions on $\mathbb{S}^{n-1}$ and functions on

$[-1,1]$. For a zonal function $f \in C(\mathbb{S}^{n-1})$, we define $\bar{f} \in C[-1,1]$ by $f(u) = \bar{f}(\langle \bar{e}, u\rangle)$ and for $g \in C[-1,1]$, we define $\breve{g} \in C(\mathbb{S}^{n-1})$ by $\breve{g}(u) = g(\langle \bar{e}, u\rangle)$.

By identifying the unit sphere $\mathbb{S}^{n-1}$ with the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, the natural convolution structure on $C^\infty(\mathrm{SO}(n))$ can be used to define a convolution structure on $C^\infty(\mathbb{S}^{n-1})$. For an extensive exposition of this construction, we refer the reader to the excellent article by Grinberg and Zhang [48]. The spherical convolution $\phi * \nu$ of a smooth function $\phi \in C^\infty(\mathbb{S}^{n-1})$ and a zonal distribution $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$ is defined by

$$(\phi * \nu)(\vartheta \bar{e}) = \langle \phi, \vartheta\nu\rangle_{C^{-\infty}} = \langle \vartheta^{-1}\phi, \nu\rangle_{C^{-\infty}}, \qquad \vartheta \in \mathrm{SO}(n).$$

Note that this definition does not depend on the special choice of ϑ and that $\phi * \nu \in C^\infty(\mathbb{S}^{n-1})$.

The convolution transform $\mathrm{T}_\nu : f \mapsto f * \nu$ is a self-adjoint endomorphism of $C^\infty(\mathbb{S}^{n-1})$ intertwining rotations and thus extends by duality to an endomorphism of $C^{-\infty}(\mathbb{S}^{n-1})$ which also intertwines rotations. That is, for a distribution $\mu \in C^{-\infty}(\mathbb{S}^{n-1})$,

$$\langle \phi, \mu * \nu\rangle_{C^{-\infty}} = \langle \phi, \mathrm{T}_\nu \mu\rangle_{C^{-\infty}} = \langle \mathrm{T}_\nu \phi, \mu\rangle_{C^{-\infty}} = \langle \phi * \nu, \mu\rangle_{C^{-\infty}}.$$

This definition includes the convolution of signed measures. Moreover, the convolution product is Abelian on zonal distributions. In the special case when $\phi \in C(\mathbb{S}^{n-1})$ and $f \in L^1(\mathbb{S}^{n-1})$ is zonal, the convolution product can be expressed as

$$(\phi * f)(u) = \int_{\mathbb{S}^{n-1}} \phi(v)\bar{f}(\langle u, v\rangle)dv, \qquad u \in \mathbb{S}^{n-1}.$$

For each $k \geq 0$, the space of zonal spherical harmonics in $\mathcal{H}_k^n$ is one-dimensional and spanned by $\breve{P}_k^n$, where $P_k^n \in C[-1,1]$ denotes the Legendre polynomial of dimension $n \geq 3$ and degree $k \geq 0$. The orthogonal projection $\pi_k : L^2(\mathbb{S}^{n-1}) \to \mathcal{H}_k^n$ onto the space $\mathcal{H}_k^n$ turns out to be the convolution transform associated with $\breve{P}_k^n$, that is,

$$\pi_k \phi = \frac{\dim \mathcal{H}_k^n}{\omega_{n-1}}\, \phi * \breve{P}_k^n, \qquad \phi \in L^2(\mathbb{S}^{n-1}),$$

where ω_n is the surface area of $\mathbb{S}^{n-1}$. By duality, π_k extends to a map from $C^{-\infty}(\mathbb{S}^{n-1})$ onto $\mathcal{H}_k^n$. Moreover, the formal Fourier series $\sum_{k=0}^\infty \pi_k \mu$ of a distribution $\mu \in C^{-\infty}(\mathbb{S}^{n-1})$ converges to μ in the weak sense. If $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$ is zonal, then

$$\nu = \sum_{k=0}^{\infty} \frac{\dim \mathcal{H}_k^n}{\omega_{n-1}} a_k^n[\nu]\breve{P}_k^n,$$

where $a_k^n[\nu] = \langle \breve{P}_k^n, \nu\rangle_{C^{-\infty}}$.

Throughout this work, we repeatedly use spherical cylinder coordinates $u = t\bar{e} + \sqrt{1-t^2}v$ on $\mathbb{S}^{n-1}$. For $\phi \in C(\mathbb{S}^{n-1})$ and $g \in C[-1,1]$,

$$\int_{\mathbb{S}^{n-1}} \phi(u)g(\langle \bar{e}, u\rangle)du = \int_{(-1,1)} \int_{\mathbb{S}^{n-1}\cap \bar{e}} \phi(t\bar{e} + \sqrt{1-t^2}v)dvg(t)(1-t^2)^{\frac{n-3}{2}}dt. \qquad (1.2.4)$$

For a signed measure $\nu \in \mathcal{M}(\mathbb{S}^{n-1})$ that carries no mass at the poles, we denote by $\bar{\nu} \in \mathcal{M}(-1,1)$ the unique finite signed measure on $(-1,1)$ such that

$$\int_{\mathbb{S}^{n-1}} \breve{g}(u)\nu(du) = \omega_{n-1} \int_{(-1,1)} g(t)(1-t^2)^{\frac{n-3}{2}} \bar{\nu}(dt), \qquad g \in C[-1,1].$$

By (1.2.4), this naturally extends the notation $\bar{f}$ for $f \in C(\mathbb{S}^{n-1})$. From the above, the Fourier coefficient $a_k^n[\nu]$ can be computed as

$$a_k^n[\nu] = \omega_{n-1} \int_{(-1,1)} P_k^n(t)(1-t^2)^{\frac{n-3}{2}} \bar{\nu}(dt).$$

The *Funk-Hecke Theorem* states that the spherical harmonic expansion of the convolution product of a signed measure μ and a zonal signed measure ν is given by

$$\mathrm{T}_\nu \mu = \mu * \nu = \sum_{k=0}^{\infty} a_k^n[\nu] \pi_k \mu.$$

Hence the convolution transform T_ν acts as a multiple of the identity on each space $\mathcal{H}_k^n$ of spherical harmonics. The Fourier coefficients $a_k^n[\nu]$ are called the *multipliers* of T_ν.

For the explicit computations of multipliers, the following identity relating Legendre polynomials of different dimensions and degrees is useful:

$$\frac{d}{dt} P_k^n(t) = \frac{k(k+n-2)}{n-1} P_{k-1}^{n+2}(t). \tag{1.2.5}$$

The Legendre polynomials also satisfy the following second-order differential equation, which also determines them up to a constant factor:

$$(1-t^2)\frac{d^2}{dt^2} P_k^n(t) - (n-1)t\frac{d}{dt} P_k^n(t) + k(k+n-2)P_k^n(t) = 0. \tag{1.2.6}$$

1.3 Regularity of the spherical convolution

1.3.1 Zonal measures

In this section, we investigate the regularity of zonal signed measures. We provide necessary and sufficient conditions to decide whether $\nabla_{\mathbb{S}}\mu$ and $\nabla_{\mathbb{S}}^2\mu$ are signed measures and provide explicit formulas for them. As one might expect, this can be expressed in terms of the corresponding measure $\bar{\mu}$ on $(-1,1)$. In the smooth case, we have the following.

Lemma 1.3.1 ([88]). *Let $f \in C^\infty(\mathbb{S}^{n-1})$ be zonal. Then for all $u, v \in \mathbb{S}^{n-1}$,*

$$\nabla_{\mathbb{S}} f^v(u) = \bar{f}'(\langle u, v\rangle) P_{u^\perp} v, \tag{1.3.1}$$

$$\nabla_{\mathbb{S}}^2 f^v(u) = \bar{f}''(\langle u, v\rangle)(P_{u^\perp} v \otimes P_{u^\perp} v) - \langle u, v\rangle \bar{f}'(\langle u, v\rangle) P_{u^\perp}. \tag{1.3.2}$$

Throughout, $P_{u^\perp}$ denotes the orthogonal projection onto $u^\perp$, and f^v denotes the rotated copy of f with axis of revolution $v \in \mathbb{S}^{n-1}$, that is, $f^v = \vartheta f$ where $\vartheta \in \mathrm{SO}(n)$ is such that $\vartheta \bar{e} = v$. Moreover, for every $v \in \mathbb{S}^{n-1}$, we define two operators $\mathsf{J}^v : C[-1,1] \to C(\mathbb{S}^{n-1})$ and $\mathsf{J}_v : C(\mathbb{S}^{n-1}) \to C[-1,1]$ by

$$\mathsf{J}^v[\psi](u) = \psi(\langle u, v\rangle) \qquad \text{and} \qquad \mathsf{J}_v[\phi](t) = (1-t^2)^{\frac{n-3}{2}} \int_{\mathbb{S}^{n-1}\cap v^\perp} \phi(tv + \sqrt{1-t^2}w)dw.$$

By a change to spherical cylinder coordinates (see (1.2.4)), we obtain

$$\int_{\mathbb{S}^{n-1}} \phi(u)\mathsf{J}^v[\psi](u)du = \int_{[-1,1]} \mathsf{J}_v[\phi](t)\psi(t)dt,$$

which shows that J_v and J^v are adjoint to each other. Hence, by continuity and duality, both operators naturally extend to signed measures. With these notations in place, we prove the following dual version of Lemma 1.3.1.

Lemma 1.3.2. *Let $v \in \mathbb{S}^{n-1}$, let X be a smooth vector field on $\mathbb{S}^{n-1}$, and Y be a smooth 2-tensor field on $\mathbb{S}^{n-1}$. Then for all $t \in (-1,1)$,*

$$\mathsf{J}_v[\operatorname{div}_{\mathbb{S}} X](t) = \frac{d}{dt}\mathsf{J}_v[\langle X, v\rangle](t), \tag{1.3.3}$$

$$\mathsf{J}_v[\operatorname{div}^2_{\mathbb{S}} Y](t) = \frac{d^2}{dt^2}\mathsf{J}_v[\langle Y, v\otimes v\rangle](t) + \frac{d}{dt}\left(t\mathsf{J}_v[\operatorname{tr} Y](t)\right). \tag{1.3.4}$$

Proof. Let $\psi \in \mathcal{D}(-1,1)$ be an arbitrary test function. Due to the spherical divergence theorem and (1.3.1),

$$\int_{\mathbb{S}^{n-1}} \operatorname{div}_{\mathbb{S}} X(u)\psi(\langle u,v\rangle)du = -\int_{\mathbb{S}^{n-1}} \langle X(u), v\rangle\psi'(\langle u,v\rangle)du.$$

We transform both integrals to spherical cylinder coordinates. For the left hand side, we have

$$\int_{\mathbb{S}^{n-1}} \operatorname{div}_{\mathbb{S}} X(u)\psi(\langle u,v\rangle)du = \int_{(-1,1)} \mathsf{J}_v[\operatorname{div}_{\mathbb{S}} X](t)\psi(t)dt$$

and for the right hand side,

$$-\int_{\mathbb{S}^{n-1}} \langle X(u), v\rangle\psi'(\langle u,v\rangle)du = -\int_{(-1,1)} \mathsf{J}_v[\langle X,v\rangle]\psi'(t)dt = \int_{(-1,1)} \frac{d}{dt}\mathsf{J}_v[\langle X,v\rangle](t)\psi(t)dt,$$

where the final equality is obtained from integration by parts. This yields (1.3.3).

For the second part of the lemma, let $\psi \in \mathcal{D}(-1,1)$ be an arbitrary test function. Due to the spherical divergence theorem for 2-tensor fields and (1.3.2),

$$\int_{\mathbb{S}^{n-1}} \operatorname{div}^2_{\mathbb{S}} Y(u)\psi(\langle u,v\rangle)du = \int_{\mathbb{S}^{n-1}} \langle Y(u), v\otimes v\rangle\psi''(\langle u,v\rangle) - \operatorname{tr} Y(u)\langle u,v\rangle\psi'(\langle u,v\rangle)du.$$

We transform both integrals to spherical cylinder coordinates. For the left hand side, we have

$$\int_{\mathbb{S}^{n-1}} \operatorname{div}^2_{\mathbb{S}} Y(u)\psi(\langle u,v\rangle)du = \int_{(-1,1)} \mathsf{J}_v[\operatorname{div}^2_{\mathbb{S}} Y](t)\psi(t)dt,$$

and for the right hand side,

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \langle Y(u), v\otimes v\rangle \psi''(\langle u,v\rangle) - \operatorname{tr} Y(u)\langle u,v\rangle\psi'(\langle u,v\rangle)du \\
&\quad= \int_{(-1,1)} \mathsf{J}_v[\langle Y, v\otimes v\rangle](t)\psi''(t) - \mathsf{J}_v[\operatorname{tr} Y](t)t\psi'(t)dt \\
&\quad= \int_{(-1,1)} \left(\frac{d^2}{dt^2}\mathsf{J}_v[\langle Y, v\otimes v\rangle](t) + \frac{d}{dt}\left(t\mathsf{J}_v[\operatorname{tr} Y](t)\right)\right)\psi(t)dt,
\end{aligned}$$

where the final equality is obtained from integration by parts. This yields (1.3.4). □

Throughout Sections 1.3 and 1.4, we repeatedly apply the following two technical lemmas. Their proofs are given in Appendix 1.A.

Lemma 1.3.3. *Let $\beta > 0$ and $g \in \mathcal{D}'(-1,1)$ such that $(1-t^2)^{\frac{\beta}{2}}g'(t) \in \mathcal{M}(-1,1)$. Then g is a locally integrable function and $(1-t^2)^{\frac{\beta-2}{2}}g(t) \in L^1(-1,1)$. Moreover, whenever $\psi \in C^1(-1,1)$ is such that both $(1-t^2)^{-\frac{\beta-2}{2}}\psi'(t)$ and $(1-t^2)^{-\frac{\beta}{2}}\psi(t)$ are bounded on $(-1,1)$, then*

$$\int_{(-1,1)} \psi(t)g'(dt) = -\int_{(-1,1)} \psi'(t)g(t)dt. \tag{1.3.5}$$

Lemma 1.3.4. *Let $v \in \mathbb{S}^{n-1}$, $w \in v^\perp$, and $\phi \in C^\infty(\mathbb{S}^{n-1}\backslash\{\pm v\})$. Then for all $k \geq 0$ and $\alpha, \beta \geq 0$, there exists a constant $C_{n,k,\alpha,\beta} > 0$ such that for all $t \in (-1,1)$,*

$$\left|\frac{d^k}{dt^k}\mathsf{J}_v[\langle\cdot, w\rangle^\alpha(1-\langle\cdot,v\rangle^2)^{\frac{\beta}{2}}\phi](t)\right| \leq C_{n,k,\alpha,\beta}|w|^\alpha\|\phi\|_{C^k(\mathbb{S}^{n-1}\backslash\{\pm v\})}(1-t^2)^{\frac{n-3+\alpha+\beta}{2}-k}. \tag{1.3.6}$$

For now, we only the need the following instances of Lemma 1.3.4.

Lemma 1.3.5. *Let $v \in \mathbb{S}^{n-1}$, X be a smooth vector field, and Y be smooth 2-tensor field on $\mathbb{S}^{n-1}$. Then for all $k \geq 0$, there exists a constant $C_{n,k} > 0$ such that for all $t \in (-1,1)$,*

$$\left|\frac{d^k}{dt^k}\mathsf{J}_v[\langle X, v\rangle](t)\right| \leq C_{n,k}\|X\|_{C^k}(1-t^2)^{\frac{n-2}{2}-k}, \tag{1.3.7}$$

$$\left|\frac{d^k}{dt^k}\mathsf{J}_v[\langle Y, v\otimes v\rangle](t)\right| \leq C_{n,k}\|Y\|_{C^k}(1-t^2)^{\frac{n-1}{2}-k}. \tag{1.3.8}$$

Proof. For the proof of (1.3.7), note that $\langle X(u), v\rangle = (1-\langle u,v\rangle^2)^{\frac{1}{2}}\phi(u)$, where

$$\phi(u) = \left\langle X(u), \frac{P_{u^\perp}v}{|P_{u^\perp}v|}\right\rangle, \qquad u \in \mathbb{S}^{n-1}\backslash\{\pm v\}.$$

Clearly, $\phi \in C^\infty(\mathbb{S}^{n-1}\backslash\{\pm v\})$, so we may apply (1.3.6) for $\alpha = 0$ and $\beta = 1$. The proof of (1.3.8) is analogous. □

In the following proposition, we characterize the zonal signed measures for which their spherical gradient is a (vector-valued) signed measure and show that identity (1.3.1) extends to this case in the weak sense.

Proposition 1.3.6. *Let $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ be zonal. Then $\nabla_{\mathbb{S}}\mu \in \mathcal{M}(\mathbb{S}^{n-1}, \mathbb{R}^n)$ if and only if μ does not carry any mass at the poles and $(1-t^2)^{\frac{n-2}{2}}\bar{\mu}'(t) \in \mathcal{M}(-1,1)$. In this case, $\mu(du) = f(u)du$ for some zonal $f \in L^1(\mathbb{S}^{n-1})$ such that for all $v \in \mathbb{S}^{n-1}$,*

$$\nabla_{\mathbb{S}}\mu^v(du) = P_{u^\perp}v(\mathsf{J}^v\bar{f}')(du). \tag{1.3.9}$$

Proof. First, let X be a smooth vector field on $\mathbb{S}^{n-1}$ and note that if μ does not carry any mass on the poles or if $\operatorname{supp} X \subseteq \mathbb{S}^{n-1}\backslash\{\pm v\}$, then

$$\begin{aligned}\langle X, \nabla_{\mathbb{S}}\mu^v\rangle_{C^{-\infty}} &= -\int_{\mathbb{S}^{n-1}} \operatorname{div}_{\mathbb{S}} X(u)\mu^v(du) = -\int_{(-1,1)} \mathsf{J}_v[\operatorname{div}_{\mathbb{S}} X](t)\bar{\mu}(dt) \\ &= -\int_{(-1,1)} \frac{d}{dt}\mathsf{J}_v[\langle X, v\rangle](t)\bar{\mu}(dt),\end{aligned} \tag{1.3.10}$$

where the second equality is obtained from a change to spherical cylinder coordinates and the final equality from (1.3.3).

Suppose now that μ does not carry any mass on the poles and that $(1-t^2)^{\frac{n-2}{2}}\bar{\mu}'(t) \in \mathcal{M}(-1,1)$. Then $\bar{\mu}$ and thus μ is absolutely continuous, that is, $\mu(du) = f(u)du$ for some zonal $f \in L^1(\mathbb{S}^{n-1})$. By (1.3.7), we have that $(1-t^2)^{-\frac{n-2}{2}}\mathsf{J}_v[\langle X, v\rangle](t)$ and $(1-t^2)^{-\frac{n-4}{2}}\frac{d}{dt}\mathsf{J}_v[\langle X, v\rangle](t)$ are bounded, so for every smooth vector field X on $\mathbb{S}^{n-1}$, (1.3.10) and (1.3.5) yield

$$\langle X, \nabla_{\mathbb{S}}\mu^v\rangle_{C^{-\infty}} = \int_{(-1,1)} \mathsf{J}_v[\langle X, v\rangle](t)\bar{f}'(dt) = \int_{\mathbb{S}^{n-1}} \langle X(u), P_{u^\perp}v\rangle(\mathsf{J}^v\bar{f}')(du),$$

where we applied a change to cylinder coordinates in the second equality. This proves identity (1.3.9) and in particular that $\nabla_{\mathbb{S}}\mu \in \mathcal{M}(\mathbb{S}^{n-1}, \mathbb{R}^n)$.

Conversely, suppose that $\nabla_{\mathbb{S}}\mu \in \mathcal{M}(\mathbb{S}^{n-1}, \mathbb{R}^n)$. Take an arbitrary test function $\psi \in \mathcal{D}(-1,1)$ and define a smooth vector field X by

$$X(u) = \psi(u\cdot v)\frac{P_{u^\perp}v}{|P_{u^\perp}v|}, \qquad u \in \mathbb{S}^{n-1}.$$

Then $\operatorname{supp} X \subseteq \mathbb{S}^{n-1}\backslash\{\pm v\}$ and $\mathsf{J}_v[\langle X, v\rangle](t) = \omega_{n-1}(1-t^2)^{\frac{n-2}{2}}\psi(t)$, thus (1.3.10) yields

$$\omega_{n-1}\left\langle \psi(t), (1-t^2)^{\frac{n-2}{2}}\bar{\mu}'(t)\right\rangle_{\mathcal{D}'} = -\int_{(-1,1)} \frac{d}{dt}\mathsf{J}_v[\langle X, v\rangle](t)\bar{\mu}(dt) = \int_{\mathbb{S}^{n-1}} \langle X(u), \nabla_{\mathbb{S}}\mu^v(du)\rangle.$$

Therefore, we obtain the estimate

$$\left|\left\langle \psi(t), (1-t^2)^{\frac{n-2}{2}}\bar{\mu}'(t)\right\rangle_{\mathcal{D}'}\right| \leq \omega_{n-1}^{-1}\|\nabla_{\mathbb{S}}\mu\|_{\mathrm{TV}}\|X\|_\infty = \omega_{n-1}^{-1}\|\nabla_{\mathbb{S}}\mu\|_{\mathrm{TV}}\|\psi\|_\infty,$$

where $\|\nabla_{\mathbb{S}}\mu\|_{\mathrm{TV}}$ denotes the total variation of $\nabla_{\mathbb{S}}\mu$. Hence, $(1-t^2)^{\frac{n-2}{2}}\bar{\mu}'(t) \in \mathcal{M}(-1,1)$. Denoting $\mu_0 = \mathbb{1}_{\mathbb{S}^{n-1}\setminus\{\pm\bar{e}\}}\mu$, the first part of the proof shows that $\nabla_{\mathbb{S}}\mu_0 \in \mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^n)$, and thus,

$$\mu(\{\bar{e}\})\nabla_{\mathbb{S}}\delta_{\bar{e}} + \mu(\{-\bar{e}\})\nabla_{\mathbb{S}}\delta_{-\bar{e}} = \nabla_{\mathbb{S}}\mu - \nabla_{\mathbb{S}}\mu_0 \in \mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^n).$$

Since $\nabla_{\mathbb{S}}\delta_{\bar{e}}$ and $\nabla_{\mathbb{S}}\delta_{-\bar{e}}$ are distributions of order one (see, e.g., [54, Section 2.1]), this is clearly possible only if μ carries no mass at the poles. $\square$

Employing the same technique as in Proposition 1.3.6, we can characterize signed measures for which their spherical Hessian is a (matrix-valued) signed measure. Identities (1.3.1) and (1.3.2) extend to this case in the weak sense.

Proposition 1.3.7. *Let $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ be zonal. Then $\nabla^2_{\mathbb{S}}\mu \in \mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^n)$ if and only if μ carries no mass at the poles and $(1-t^2)^{\frac{n-1}{2}}\bar{\mu}''(t) \in \mathcal{M}(-1,1)$. In this case, $\mu(du) = f(u)du$ for some zonal $f \in L^1(\mathbb{S}^{n-1})$ such that $\nabla_{\mathbb{S}}\mu \in L^1(\mathbb{S}^{n-1},\mathbb{R}^n)$ and for all $v \in \mathbb{S}^{n-1}$,*

$$\nabla_{\mathbb{S}}\mu^v(du) = P_{u^\perp}v\bar{f}'(\langle u,v\rangle), \tag{1.3.11}$$

$$\nabla^2_{\mathbb{S}}\mu^v(du) = (P_{u^\perp}v \otimes P_{u^\perp}v)(\mathsf{J}^v\bar{f}'')(du) - \langle u,v\rangle\bar{f}'(\langle u,v\rangle)P_{u^\perp}du. \tag{1.3.12}$$

Proof. First, take a smooth 2-tensor field Y on $\mathbb{S}^{n-1}$ and note that if μ does not carry any mass on the poles or if $\operatorname{supp} Y \subseteq \mathbb{S}^{n-1}\setminus\{\pm v\}$, then

$$\begin{aligned}\langle Y, \nabla^2_{\mathbb{S}}\mu^v\rangle_{C^{-\infty}} &= \int_{\mathbb{S}^{n-1}} \operatorname{div}^2_{\mathbb{S}} Y(u)\mu^v(du) = \int_{(-1,1)} \mathsf{J}_v[\operatorname{div}^2_{\mathbb{S}} Y](t)\bar{\mu}(dt)\\ &= -\left\langle \frac{d}{dt}\mathsf{J}_v[\langle Y, v\otimes v\rangle](t) + t\mathsf{J}_v[\operatorname{tr} Y](t), \bar{\mu}'(t)\right\rangle_{\mathcal{D}'},\end{aligned} \tag{1.3.13}$$

where the second equality is obtained from a change to spherical cylinder coordinates and the final equality from (1.3.4).

Suppose now that μ carries no mass at the poles and that $(1-t^2)^{\frac{n-1}{2}}\bar{\mu}''(t) \in \mathcal{M}(-1,1)$. Then $\bar{\mu}$, and thus, μ are absolutely continuous, that is, $\mu(du) = f(u)du$ for some zonal $f \in L^1(\mathbb{S}^{n-1})$. Moreover, Lemma 1.3.3 implies that $(1-t^2)^{\frac{n-3}{2}}\bar{f}'(t) \in L^1(-1,1)$, and thus, Proposition 1.3.6 implies $\nabla_{\mathbb{S}}\mu \in L^1(\mathbb{S}^{n-1},\mathbb{R}^n)$ and identity (1.3.11). By (1.3.8), we have that $(1-t^2)^{-\frac{n-1}{2}}\mathsf{J}_v[\langle Y, v\otimes v\rangle](t)$ and $(1-t^2)^{-\frac{n-3}{2}}\frac{d}{dt}\mathsf{J}_v[\langle Y, v\otimes v\rangle](t)$ are bounded, so for every smooth 2-tensor field Y on $\mathbb{S}^{n-1}$, (1.3.13) and (1.3.5) yield

$$\begin{aligned}\langle Y, \nabla^2_{\mathbb{S}}\mu^v\rangle_{C^{-\infty}} &= \int_{(-1,1)} \mathsf{J}_v[\langle Y, v\otimes v\rangle](t)\bar{f}''(dt) - \int_{(-1,1)} t\mathsf{J}_v[\operatorname{tr} Y](t)\bar{f}'(t)dt\\ &= \int_{\mathbb{S}^{n-1}} \langle Y(u), P_{u^\perp}v \otimes P_{u^\perp}v\rangle(\mathsf{J}^v\bar{f}'')(du) - \int_{\mathbb{S}^{n-1}} \langle Y(u), P_{u^\perp}\rangle\langle u,v\rangle\bar{f}'(\langle u,v\rangle)du,\end{aligned}$$

where we applied a change to cylinder coordinates in the second equality. This proves (1.3.12) and in particular that $\nabla^2_{\mathbb{S}}\mu \in \mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^n)$.

Conversely, suppose now that $\nabla^2_{\mathbb{S}}\mu \in \mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^n)$. Take an arbitrary test function $\psi \in \mathcal{D}(-1,1)$ and define a smooth 2-tensor field Y on $\mathbb{S}^{n-1}$ by

$$Y(u) = \psi(\langle u,v\rangle)\left(\frac{P_{u^\perp}v}{|P_{u^\perp}v|} \otimes \frac{P_{u^\perp}v}{|P_{u^\perp}v|} - \frac{1}{n-2}\left(P_{u^\perp} - \frac{P_{u^\perp}v}{|P_{u^\perp}v|} \otimes \frac{P_{u^\perp}v}{|P_{u^\perp}v|}\right)\right), \qquad u \in \mathbb{S}^{n-1}.$$

Then $\operatorname{supp} Y \subseteq \mathbb{S}^{n-1}\backslash\{\pm v\}$, and Y satisfies $\mathsf{J}_v[\langle Y, v\otimes v\rangle](t) = \omega_{n-1}(1-t^2)^{\frac{n-1}{2}}\psi(t)$ and $\operatorname{tr} Y = 0$, thus

$$\begin{aligned}\omega_{n-1}\left\langle \psi(t), (1-t^2)^{\frac{n-1}{2}}\bar{\mu}''(t)\right\rangle_{\mathcal{D}'} &= \omega_{n-1}\left\langle \mathsf{J}_v[Y, v\otimes v], \bar{\mu}''(t)\right\rangle_{\mathcal{D}'} \\ = -\left\langle \frac{d}{dt}\mathsf{J}_v[\langle Y, v\otimes v\rangle](t) + t\mathsf{J}_v[\operatorname{tr} Y](t), \bar{\mu}'(t)\right\rangle_{\mathcal{D}'} &= \int_{\mathbb{S}^{n-1}}\langle Y(u), \nabla^2_{\mathbb{S}}\mu^v(du)\rangle.\end{aligned}$$

Therefore, we obtain the estimate:

$$\left|\left\langle \psi(t), (1-t^2)^{\frac{n-1}{2}}\bar{\mu}''(t)\right\rangle_{\mathcal{D}'}\right| \leq \omega_{n-1}^{-1}\|\nabla^2_{\mathbb{S}}\mu\|_{\mathrm{TV}}\|Y\|_\infty = \omega_{n-1}^{-1}\|\nabla^2_{\mathbb{S}}\mu\|_{\mathrm{TV}}\|\psi\|_\infty,$$

where $\|\nabla^2_{\mathbb{S}}\mu\|_{\mathrm{TV}}$ denotes the total variation of $\nabla^2_{\mathbb{S}}\mu$. Hence, $(1-t^2)^{\frac{n-1}{2}}\bar{\mu}''(t) \in \mathcal{M}(-1,1)$. Denoting $\mu_0 = \mathbb{1}_{\mathbb{S}^{n-1}\backslash\{\pm\bar{e}\}}\mu$, the first part of the proof shows that $\nabla^2_{\mathbb{S}}\mu_0 \in \mathcal{M}(\mathbb{S}^{n-1}, \mathbb{R}^{n\times n})$, and thus,

$$\mu(\{\bar{e}\})\nabla^2_{\mathbb{S}}\delta_{\bar{e}} + \mu(\{-\bar{e}\})\nabla^2_{\mathbb{S}}\delta_{-\bar{e}} = \nabla^2_{\mathbb{S}}\mu - \nabla^2_{\mathbb{S}}\mu_0 \in \mathcal{M}(\mathbb{S}^{n-1}, \mathbb{R}^{n\times n}).$$

Since $\nabla^2_{\mathbb{S}}\delta_{\bar{e}}$ and $\nabla^2_{\mathbb{S}}\delta_{-\bar{e}}$ are distributions of order two (see, e.g., [54, Section 2.1]), this is clearly possible only if μ carries no mass at the poles. □

For later purposes, it will be useful to describe the regularity of zonal functions $f \in L^1(\mathbb{S}^{n-1})$ in terms of their Laplacian.

Lemma 1.3.8. *If $f \in L^1(\mathbb{S}^{n-1})$ is zonal and $\Delta_{\mathbb{S}} f \in \mathcal{M}(\mathbb{S}^{n-1})$, then for almost all $t \in (-1,1)$,*

$$(\Delta_{\mathbb{S}} f)(\{u \in \mathbb{S}^{n-1} : \langle \bar{e}, u\rangle > t\}) = -\omega_{n-1}(1-t^2)^{\frac{n-1}{2}}\bar{f}'(t). \tag{1.3.14}$$

Proof. Let $\psi \in \mathcal{D}(-1,1)$ be an arbitrary test function. Define $\eta(t) = \int_{(-1,t)}\psi(s)ds$ and note that $\eta(\langle \bar{e}, \cdot\rangle) \in C^\infty(\mathbb{S}^{n-1})$. Then Lebesgue-Stieltjes integration by parts yields

$$\begin{aligned}\int_{(-1,1)}(\Delta_{\mathbb{S}} f)(\{u \in \mathbb{S}^{n-1} : \langle \bar{e}, u\rangle > t\})\psi(t)dt &= \int_{[-1,1]}\mathsf{J}_{\bar{e}}[\Delta_{\mathbb{S}} f]((t,1])\psi(t)dt \\ = \int_{[-1,1]}\eta(t)\mathsf{J}_{\bar{e}}[\Delta_{\mathbb{S}} f](dt) = \int_{\mathbb{S}^{n-1}}\eta(\langle \bar{e}, u\rangle)(\Delta_{\mathbb{S}} f)(du) &= \int_{\mathbb{S}^{n-1}}\Delta_{\mathbb{S}}\eta(\langle \bar{e}, \cdot\rangle)(u)f(u)du,\end{aligned}$$

where the third equality follows from the characteristic property of the pushforward measure and the final equality, from the definition of the distributional spherical Laplacian. Taking the trace in (1.3.2),

$$\Delta_{\mathbb{S}}\eta(\langle \bar{e}, \cdot\rangle)(u) = (1-\langle \bar{e}, u\rangle^2)\eta''(\langle \bar{e}, u\rangle) - (n-1)\langle \bar{e}, u\rangle\eta'(\langle \bar{e}, u\rangle).$$

By a change to spherical cylinder coordinates, we obtain

$$\begin{aligned}&\int_{(-1,1)}(\Delta_{\mathbb{S}} f)(\{u \in \mathbb{S}^{n-1} : \langle \bar{e}, u\rangle > t\})\psi(t)dt \\ &= \omega_{n-1}\int_{(-1,1)}\left((1-t^2)\eta''(t) - (n-1)t\eta'(t)\right)(1-t^2)^{\frac{n-3}{2}}\bar{f}(t)dt \\ &= \omega_{n-1}\int_{(-1,1)}\frac{d}{dt}\left((1-t^2)^{\frac{n-1}{2}}\psi(t)\right)\bar{f}(t)dt = -\omega_{n-1}\left\langle \psi(t), (1-t^2)^{\frac{n-1}{2}}\bar{f}'(t)\right\rangle_{\mathcal{D}'}.\end{aligned}$$

Since ψ was arbitrary, identity (1.3.14) holds for almost all $t \in (-1,1)$. □

From now on, we denote by

$$\mathrm{C}_r^{\mathbb{S}}(u) = \{v \in \mathbb{S}^{n-1} : \langle u, v\rangle > \cos r\} \tag{1.3.15}$$

the *spherical cap* around $u \in \mathbb{S}^{n-1}$ with radius $r \geq 0$. The following proposition classifies zonal functions f for which their spherical Hessian is a signed measure in terms of the behavior of $\Delta_{\mathbb{S}} f$ on small polar caps.

Proposition 1.3.9. *For a zonal function $f \in L^1(\mathbb{S}^{n-1})$, the following are equivalent:*

(a) $\nabla^2_{\mathbb{S}} f \in \mathcal{M}(\mathbb{S}^{n-1}, \mathbb{R}^{n\times n})$,

(b) $\Delta_{\mathbb{S}} f \in \mathcal{M}(\mathbb{S}^{n-1})$ *and* $\int_{(0,\frac{\pi}{2})} \frac{|(\Delta_{\mathbb{S}} f)(\mathrm{C}_r^{\mathbb{S}}(\pm\bar{e}))|}{r} dr < \infty$,

(c) $\square_n f \in \mathcal{M}(\mathbb{S}^{n-1})$ *and* $\int_{(0,\frac{\pi}{2})} \frac{|(\square_n f)(\mathrm{C}_r^{\mathbb{S}}(\pm\bar{e}))|}{r} dr < \infty$.

Proof. Each of the three statements above implies that $\Delta_{\mathbb{S}} f \in \mathcal{M}(\mathbb{S}^{n-1})$. Due to Lemma 1.3.8, we have that $\nabla_{\mathbb{S}} f \in L^1(\mathbb{S}^{n-1}, \mathbb{R}^n)$ and for almost all $t \in (-1, 1)$,

$$\mathsf{J}_{\bar{e}}[\Delta_{\mathbb{S}} f]((t, 1]) = -\omega_{n-1}(1 - t^2)^{\frac{n-1}{2}} \bar{f}'(t). \tag{1.3.16}$$

Taking the distributional derivative on both sides yields

$$\mathbb{1}_{(-1,1)}(t)\mathsf{J}_{\bar{e}}[\Delta_{\mathbb{S}} f](dt) = \omega_{n-1}\left((1 - t^2)^{\frac{n-1}{2}} \bar{f}''(t) - (n - 1)(1 - t^2)^{\frac{n-3}{2}} \bar{f}'(t)\right) \tag{1.3.17}$$

in $\mathcal{D}'(-1, 1)$. According to Proposition 1.3.7, condition (a) is fulfilled if and only if $(1 - t^2)^{\frac{n-1}{2}} \bar{f}''(t)$ is a finite signed measure. Due to (1.3.16) and (1.3.17), this is the case if and only if

$$\int_{(0,1)} \frac{|(\Delta_{\mathbb{S}} f)(\{u \in \mathbb{S}^{n-1} : \langle\bar{e}, u\rangle > t\})|}{1 - t^2} dt + \int_{(0,1)} \frac{|(\Delta_{\mathbb{S}} f)(\{u \in \mathbb{S}^{n-1} : \langle-\bar{e}, u\rangle > t\})|}{1 - t^2} dt < \infty.$$

The substitution $t = \cos r$ then shows that conditions (a) and (b) are equivalent.

For the equivalence of (b) and (c), it suffices to show that $(1-t^2)^{-1}|\int_{\{u:|\langle\bar{e},u\rangle|>t\}} f(u)du|$ is integrable on $(0, 1)$. To that end, using spherical cylinder coordinates, we estimate

$$\int_{\{u:|\langle\bar{e},u\rangle|>t\}} |f(u)|du = \int_{(t,1)} (1 - s^2)^{\frac{n-3}{2}} |\bar{f}(s)|ds \leq (1 - t^2)^{\frac{1}{2}} \int_{(-1,1)} (1 - s^2)^{\frac{n-4}{2}} |\bar{f}(s)|ds,$$

for $t \in (0, 1)$. Since $\nabla_{\mathbb{S}} f \in L^1(\mathbb{S}^{n-1}, \mathbb{R}^n)$, Proposition 1.3.6 and Lemma 1.3.3 imply that $(1 - t^2)^{\frac{n-4}{2}} \bar{f}(t)$ is integrable. This completes the proof. □

We want to note that Lemma 1.3.8 and Proposition 1.3.9 still hold if the zonal function f is replaced by a zonal signed measure that carries no mass at the poles.

Example 1.3.10. For Berg's function g_n we have that $\breve{g}_n = g_n(\langle\bar{e}, \cdot\rangle)$ is an integrable function on the sphere and $\square_n \breve{g}_n = (\mathrm{Id} - \pi_1)\delta_{\bar{e}}$ is a finite signed measure. Hence Lemma 1.3.8 implies that $\nabla_{\mathbb{S}} \breve{g}_n$ is integrable on $\mathbb{S}^{n-1}$. At the same time, the integrability condition in Proposition 1.3.9 (c) is clearly violated, so the distributional spherical Hessian $\nabla^2_{\mathbb{S}} \breve{g}_n$ is not a finite signed measure.

1.3.2 Convolution transforms

Linear operators on functions on the unit sphere intertwining rotations can be identified with convolution transforms, as the following theorem shows.

Theorem 1.3.11 ([97])**.** *If $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ is zonal, then the convolution transform T_μ is a bounded linear operator on $C(\mathbb{S}^{n-1})$. Conversely, if T is an $\mathrm{SO}(n)$ equivariant bounded linear operator on $C(\mathbb{S}^{n-1})$, then there exists a unique zonal $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ such that $\mathrm{T} = \mathrm{T}_\mu$.*

The regularizing properties of a convolution transform correspond to the regularity of its integral kernel. In this section, we classify rotation equivariant bounded linear operators from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$ in terms of their integral kernel, proving Theorem 1.C. We require the following two lemmas.

Lemma 1.3.12. *Let $\phi \in C^\infty(\mathbb{S}^{n-1})$ and let $\gamma : I \subseteq \mathbb{R} \to \mathbb{S}^{n-1}$ be a smooth curve in $\mathbb{S}^{n-1}$. Then for all $s \in I$ and $t \in (-1,1)$,*

$$\frac{d}{ds}\mathsf{J}_{\gamma(s)}[\phi](t) = -\frac{d}{dt}\mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle\phi](t), \tag{1.3.18}$$

$$\frac{d}{ds}\mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle\phi](t) = -\frac{d}{dt}\mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle^2\phi](t) + \mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma''(s)\rangle\phi](t). \tag{1.3.19}$$

Proof. Let $\psi \in \mathcal{D}(-1,1)$ be an arbitrary test function. On the one hand, spherical cylinder coordinates yield

$$\frac{d}{ds}(\phi * \breve{\psi})(\gamma(s)) = \frac{d}{ds}\int_{(-1,1)} \mathsf{J}_{\gamma(s)}[\phi](t)\psi(t)dt = \int_{(-1,1)} \frac{d}{ds}\mathsf{J}_{\gamma(s)}[\phi](t)\psi(t)dt.$$

On the other hand,

$$\frac{d}{ds}(\phi * \breve{\psi})(\gamma(s)) = \frac{d}{ds}\int_{\mathbb{S}^{n-1}} \phi(v)\mathsf{J}^v[\psi](\gamma(s))dv = \int_{\mathbb{S}^{n-1}} \phi(v)\langle\nabla_{\mathbb{S}}\mathsf{J}^v[\psi](\gamma(s)),\gamma'(s)\rangle dv.$$

By (1.3.1) and a change to spherical cylinder coordinates, we obtain

$$\begin{aligned}\frac{d}{ds}(\phi * \breve{\psi})(\gamma(s)) &= \int_{\mathbb{S}^{n-1}} \phi(v)\langle v,\gamma'(s)\rangle\psi'(\langle\gamma(s),v\rangle)dv = \int_{(-1,1)} \mathsf{J}_{\gamma(s)}[\phi\langle\cdot,\gamma'(s)\rangle](t)\psi'(t)dt \\ &= -\int_{(-1,1)} \frac{d}{dt}\mathsf{J}_{\gamma(s)}[\phi\langle\cdot,\gamma'(s)\rangle](t)\psi(t)dt,\end{aligned}$$

where the final equality follows from integration by parts. This implies (1.3.18).

For the second part of the lemma, let $\psi \in \mathcal{D}(-1,1)$ be an arbitrary test function. On the one hand, spherical cylinder coordinates yield

$$\begin{aligned}\frac{d}{ds}((\langle\cdot,\gamma'(s)\rangle\psi) * \breve{\phi})(\gamma(s)) &= \frac{d}{ds}\int_{(-1,1)} \mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle\phi](t)\psi(t)dt \\ &= \int_{(-1,1)} \frac{d}{ds}\mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle\phi](t)\psi(t)dt.\end{aligned}$$

On the other hand,

$$\frac{d}{ds}((\langle\cdot,\gamma'(s)\rangle\psi)*\breve{\phi})(\gamma(s)) = \frac{d}{ds}\int_{\mathbb{S}^{n-1}}\phi(v)\langle v,\gamma'(s)\rangle \mathsf{J}^v[\psi](\gamma(s))dv$$
$$= \int_{\mathbb{S}^{n-1}}\phi(v)\langle v,\gamma'(s)\rangle\langle\nabla_{\mathbb{S}}\mathsf{J}^v[\psi](\gamma(s)),\gamma'(s)\rangle dv + \int_{\mathbb{S}^{n-1}}\phi(v)\langle v,\gamma''(s)\rangle\psi(\langle v,\gamma(s)\rangle)dv.$$

By (1.3.1) and a change to cylinder coordinates, we obtain

$$\begin{aligned}
&\frac{d}{ds}((\langle\cdot,\gamma'(s)\rangle\psi)*\breve{\phi})(\gamma(s))\\
&\quad= \int_{\mathbb{S}^{n-1}}\phi(v)\langle v,\gamma'(s)\rangle^2\psi'(\langle v,\gamma(s)\rangle)dv + \int_{\mathbb{S}^{n-1}}\phi(v)\langle v,\gamma''(s)\rangle\psi(\langle v,\gamma(s)\rangle)dv\\
&\quad= \int_{(-1,1)}\mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle^2\phi](t)\psi'(t)dt + \int_{(-1,1)}\mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma''(s)\rangle\phi](t)\psi(t)dt\\
&\quad= \int_{(-1,1)}\left(-\frac{d}{dt}\mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle^2\phi](t) + \mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma''(s)\rangle\phi](t)\right)\psi(t)dt,
\end{aligned}$$

where the final equality follows from integration by parts. This implies (1.3.19). $\square$

Lemma 1.3.13. *Let $\phi\in C^\infty(\mathbb{S}^{n-1})$ and let $\gamma: I\subseteq\mathbb{R}\to\mathbb{S}^{n-1}$ be a smooth curve in $\mathbb{S}^{n-1}$. Then for all $s\in I$ and $t\in(-1,1)$,*

$$\left|\frac{d}{ds}\mathsf{J}_{\gamma(s)}[\phi](t)\right| \le C_n|\gamma''(s)|\|\phi\|_{C^1(\mathbb{S}^{n-1})}(1-t^2)^{\frac{n-4}{2}}, \tag{1.3.20}$$

$$\left|\frac{d}{ds}\mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle\phi](t)\right| \le C_n\left(|\gamma'(s)|^2+|\gamma''(s)|\right)\|\phi\|_{C^1(\mathbb{S}^{n-1})}(1-t^2)^{\frac{n-3}{2}}. \tag{1.3.21}$$

Proof. For the proof of (1.3.20), apply estimate (1.3.6) to the right hand side of (1.3.18) in the instance where $(k,\alpha,\beta)=(1,1,0)$. To obtain (1.3.21), apply estimate (1.3.6) to the right hand side of (1.3.19) in the instances where $(k,\alpha,\beta)=(1,2,0)$ and $(k,\alpha,\beta)=(0,0,0)$. $\square$

Theorem 1.3.14. *If $f\in L^1(\mathbb{S}^{n-1})$ is zonal and $\nabla_{\mathbb{S}}f\in\mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^n)$, then the convolution transform T_f is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^1(\mathbb{S}^{n-1})$. Conversely, if T is an $\mathrm{SO}(n)$ equivariant bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^1(\mathbb{S}^{n-1})$, then there exists a unique zonal $f\in L^1(\mathbb{S}^{n-1})$ satisfying $\nabla_{\mathbb{S}}f\in\mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^n)$ such that $\mathrm{T}=\mathrm{T}_f$. In this case, for every $\phi\in C(\mathbb{S}^{n-1})$ and $u\in\mathbb{S}^{n-1}$,*

$$\nabla_{\mathbb{S}}(\phi*f)(u) = \int_{\mathbb{S}^{n-1}}\phi(v)P_{u^\perp}v(\mathsf{J}^u\bar{f}')(dv). \tag{1.3.22}$$

Proof. Suppose that $f\in L^1(\mathbb{S}^{n-1})$ is zonal and that $\nabla_{\mathbb{S}}f\in\mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^n)$. Theorem 1.3.11 implies that T_f is a bounded linear operator on $C(\mathbb{S}^{n-1})$. First, we will verify identity (1.3.22) for an arbitrary smooth function $\phi\in C^\infty(\mathbb{S}^{n-1})$. Take a point $u\in\mathbb{S}^{n-1}$ and a tangent vector $w \in u^\perp$. Choosing a smooth curve γ in $\mathbb{S}^{n-1}$ such that $\gamma(0) = u$ and $\gamma'(0) = w$ yields

$$\langle \nabla_{\mathbb{S}}(\phi * f)(u), w\rangle = \frac{d}{ds}\Big|_0 (\phi * f)(\gamma(s)) = \frac{d}{ds}\Big|_0 \int_{(-1,1)} \mathsf{J}_{\gamma(s)}[\phi](t)\bar{f}(t)dt.$$

Note that Proposition 1.3.6 and Lemma 1.3.3 imply that $(1-t^2)^{\frac{n-2}{2}}\bar{f}'(t)$ is a finite signed measure and $(1-t^2)^{\frac{n-4}{2}}\bar{f}(t)$ is an integrable function on $(-1,1)$. Due to the estimate (1.3.20), we have that $(1-t^2)^{-\frac{n-4}{2}}\frac{d}{ds}\mathsf{J}_{\gamma(s)}[\phi](t)$ is bounded uniformly in s for all sufficiently small s, so we may interchange differentiation and integration and obtain

$$\begin{aligned}\langle \nabla_{\mathbb{S}}(\phi * f)(u), w\rangle &= \int_{(-1,1)} \frac{d}{ds}\Big|_0 \mathsf{J}_{\gamma(s)}[\phi](t)\bar{f}(t)dt = -\int_{(-1,1)} \frac{d}{dt}\mathsf{J}_u[\langle\cdot, w\rangle\phi](t)\bar{f}(t)dt\\ &= \int_{(-1,1)} \mathsf{J}_u[\langle\cdot, w\rangle\phi](t)\bar{f}'(dt) = \int_{\mathbb{S}^{n-1}} \phi(v)\langle P_{u^\perp}v, w\rangle(\mathsf{J}^u\bar{f}')(dv),\end{aligned}$$

where the second equality follows from (1.3.18), the third from (1.3.5), and the final equality from a change to spherical cylinder coordinates. This proves identity (1.3.22) for all $\phi \in C^\infty(\mathbb{S}^{n-1})$.

As an immediate consequence, $\|\mathrm{T}_f\phi\|_{C^1(\mathbb{S}^{n-1})} \leq C\|\phi\|_{C(\mathbb{S}^{n-1})}$ for some constant $C \geq 0$ and all $\phi \in C^\infty(\mathbb{S}^{n-1})$. Thus, T_f extends to a bounded linear operator $\overline{\mathrm{T}} : C(\mathbb{S}^{n-1}) \to C^1(\mathbb{S}^{n-1})$. Since $\mathrm{T}_f : C(\mathbb{S}^{n-1}) \to C(\mathbb{S}^{n-1})$ is a bounded operator and the inclusion $C^1(\mathbb{S}^{n-1}) \subseteq C(\mathbb{S}^{n-1})$ is continuous, $\overline{\mathrm{T}}$ agrees with T_f. By density and continuity, (1.3.22) is valid for all $\phi \in C(\mathbb{S}^{n-1})$.

For the second part of the theorem, suppose that T is an $\mathrm{SO}(n)$ equivariant bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^1(\mathbb{S}^{n-1})$. Then Theorem 1.3.11 implies that $\mathrm{T} = \mathrm{T}_\mu$ for a unique zonal $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$. According to Proposition 1.3.6, it suffices to show that μ carries no mass at the poles and that $(1-t^2)^{\frac{n-2}{2}}\bar{\mu}'(t) \in \mathcal{M}(-1,1)$. To that end, take an arbitrary test function $\psi \in \mathcal{D}(-1,1)$, a point $u \in \mathbb{S}^{n-1}$, a unit tangent vector $w \in u^\perp$, and define

$$\phi(v) = \frac{\langle v, w\rangle}{\sqrt{1-\langle v, u\rangle^2}}\psi(\langle v, u\rangle), \qquad v \in \mathbb{S}^{n-1}.$$

Then ϕ is a smooth function satisfying $\mathsf{J}_u[\langle\cdot, w\rangle\phi](t) = C_n(1-t^2)^{\frac{n-2}{2}}\psi(t)$, where $C_n > 0$ is given by $C_n = \int_{\mathbb{S}^{n-1}\cap u^\perp}\langle w, v\rangle^2 dv$. Choosing a smooth curve γ in $\mathbb{S}^{n-1}$ such that $\gamma(0) = u$ and $\gamma'(0) = w$ yields

$$\begin{aligned}C_n\left\langle \psi(t), (1-t^2)^{\frac{n-2}{2}}\bar{\mu}'(t)\right\rangle_{\mathcal{D}'} &= \langle \mathsf{J}_u[\langle\cdot, w\rangle\phi](t), \bar{\mu}'(t)\rangle_{\mathcal{D}'}\\ &= -\int_{(-1,1)} \frac{d}{dt}\mathsf{J}_u[\langle\cdot, w\rangle\phi](t)\bar{\mu}(dt) = \int_{(-1,1)} \frac{d}{ds}\Big|_0 \mathsf{J}_{\gamma(s)}[\phi](t)\bar{\mu}(dt),\end{aligned}$$

where the final equality is due to (1.3.18). Observe that $\operatorname{supp}\mathsf{J}_{\gamma(s)}[\phi] \subseteq [-1+\varepsilon, 1-\varepsilon]$ and that $|\frac{d}{ds}\mathsf{J}_{\gamma(s)}[\phi]| \leq C$ uniformly in s for all sufficiently small s, so we may interchange differentiation and integration and obtain

$$C_n\left\langle \psi(t), (1-t^2)^{\frac{n-2}{2}}\bar{\mu}'(t)\right\rangle_{\mathcal{D}'} = \frac{d}{ds}\Big|_0 \int_{(-1,1)} \mathsf{J}_{\gamma(s)}[\phi_j](t)\bar{\mu}(dt) = \langle \nabla_{\mathbb{S}}(\phi * \mu)(u), w\rangle.$$

Therefore, we arrive at the following estimate:

$$\left|\left\langle \psi(t),(1-t^2)^{\frac{n-2}{2}}\bar{\mu}'(t)\right\rangle_{\mathcal{D}'}\right| \leq C_n^{-1}\|\mathrm{T}\|\|\phi\|_{C(\mathbb{S}^{n-1})} = C_n^{-1}\|\mathrm{T}\|\|\psi\|_\infty.$$

This shows that $(1-t^2)^{\frac{n-2}{2}}\bar{\mu}'(t) \in \mathcal{M}(-1,1)$. Denoting $\mu_0 = \mathbb{1}_{\mathbb{S}^{n-1}\setminus\{\pm\bar{e}\}}\mu$, Proposition 1.3.6 and the first part of the proof show that T_{μ_0} is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^1(\mathbb{S}^{n-1})$. Hence also

$$\mu(\{\bar{e}\})\mathrm{Id} + \mu(\{-\bar{e}\})\mathrm{Refl} = \mathrm{T} - \mathrm{T}_{\mu_0}$$

is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^1(\mathbb{S}^{n-1})$, where $\mathrm{Refl} = \mathrm{T}_{\delta_{-\bar{e}}}$ is the reflection at the origin. Clearly, this is possible only if μ carries no mass at the poles. □

Theorem 1.3.15. *If $f \in L^1(\mathbb{S}^{n-1})$ is zonal and $\nabla^2_{\mathbb{S}} f \in \mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^{n\times n})$, then the convolution transform T_f is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$. Conversely, if T is an $\mathrm{SO}(n)$ equivariant bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$, then there exists a unique zonal $f \in L^1(\mathbb{S}^{n-1})$ satisfying $\nabla^2_{\mathbb{S}} f \in \mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^{n\times n})$ such that $\mathrm{T} = \mathrm{T}_f$. In this case, for every $\phi \in C(\mathbb{S}^{n-1})$ and $u \in \mathbb{S}^{n-1}$,*

$$\nabla_{\mathbb{S}}(\phi * f)(u) = \int_{\mathbb{S}^{n-1}} \phi(v) P_{u^\perp} v \bar{f}'(\langle u,v\rangle)dv, \tag{1.3.23}$$

$$\nabla^2_{\mathbb{S}}(\phi * f)(u) = \int_{\mathbb{S}^{n-1}} \phi(v)(P_{u^\perp}v \otimes P_{u^\perp}v)(\mathrm{J}^u\bar{f}'')(dv) - \int_{\mathbb{S}^{n-1}} \phi(v)\langle u,v\rangle \bar{f}'(\langle u,v\rangle)dv\, P_{u^\perp}. \tag{1.3.24}$$

Proof. Suppose that $f \in L^1(\mathbb{S}^{n-1})$ is zonal and that $\nabla^2_{\mathbb{S}} f \in \mathcal{M}(\mathbb{S}^{n-1},\mathbb{R}^{n\times n})$. Then $\nabla_{\mathbb{S}} f \in L^1(\mathbb{S}^{n-1},\mathbb{R}^n)$ due to Proposition 1.3.7. According to Theorem 1.3.14, the convolution transform T_f is a bounded linear operator from $C(\mathbb{S}^{n-1})$ from $C^1(\mathbb{S}^{n-1})$ and identity (1.3.23) holds for every $\phi \in C(\mathbb{S}^{n-1})$. Next, we will verify identity (1.3.24) for an arbitrary smooth function $\phi \in C^\infty(\mathbb{S}^{n-1})$. Let $u \in \mathbb{S}^{n-1}$ be a point and $w \in u^\perp$ a tangent vector. Choosing a geodesic γ in $\mathbb{S}^{n-1}$ such that $\gamma(0) = u$ and $\gamma'(0) = w$ yields

$$\langle \nabla^2_{\mathbb{S}}(\phi * f)(u), w \otimes w\rangle = \frac{d^2}{ds^2}\bigg|_0 (\phi * f)(\gamma(s)) = \frac{d^2}{ds^2}\bigg|_0 \int_{(-1,1)} \mathrm{J}_{\gamma(s)}[\phi](t)\bar{f}(t)dt.$$

Note that Proposition 1.3.7 and Lemma 1.3.3 imply that $(1-t^2)^{\frac{n-1}{2}}\bar{f}''(t)$ is a finite signed measure and both $(1-t^2)^{\frac{n-3}{2}}\bar{f}'(t)$ and $(1-t^2)^{\frac{n-4}{2}}\bar{f}(t)$ are integrable functions on $(-1,1)$. Due to (1.3.20), we have that $(1-t^2)^{-\frac{n-4}{2}}\frac{d}{ds}\mathrm{J}_{\gamma(s)}[\phi](t)$ is bounded uniformly in s for all sufficiently small s, so we may interchange differentiation and integration and obtain

$$\begin{aligned}\langle \nabla^2_{\mathbb{S}}(\phi * f)(u), w \otimes w\rangle &= \frac{d}{ds}\bigg|_0 \int_{(-1,1)} \frac{d}{ds}\mathrm{J}_{\gamma(s)}[\phi](t)\bar{f}(t)dt \\ &= -\frac{d}{ds}\bigg|_0 \int_{(-1,1)} \frac{d}{dt}\mathrm{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle](t)\bar{f}(t)dt = \frac{d}{ds}\bigg|_0 \int_{(-1,1)} \mathrm{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle](t)\bar{f}'(t)dt,\end{aligned}$$

where the second equality follows from (1.3.18) and the final equality from (1.3.5). Due to (1.3.21), we have that $(1-t^2)^{-\frac{n-3}{2}}\frac{d}{ds}\mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle\phi](t)$ is uniformly bounded in s for all sufficiently small s, so we may again interchange differentiation and integration and obtain

$$\begin{aligned}\langle\nabla^2_{\mathbb{S}}(\phi*f)(u),w\otimes w\rangle &= \int_{(-1,1)}\frac{d}{ds}\Big|_0 \mathsf{J}_{\gamma(s)}[\langle\cdot,\gamma'(s)\rangle](t)\bar f'(t)dt\\
&=\int_{(-1,1)}\left(-\frac{d}{dt}\mathsf{J}_u[\langle\cdot,w\rangle\phi](t)-|w|^2t\mathsf{J}_u[\phi](t)\right)\bar f'(t)dt\\
&=\int_{(-1,1)}\mathsf{J}_u[(.\cdot w)^2\phi](t)\bar f''(dt)-|w|^2\int_{(-1,1)}\mathsf{J}_u[\phi](t)\bar f'(t)dt\\
&=\int_{\mathbb{S}^{n-1}}\phi(v)\langle P_{u^\perp}v\otimes P_{u^\perp}v,w\otimes w\rangle(\mathsf{J}^u\bar f'')(dv)-\int_{\mathbb{S}^{n-1}}\langle u,v\rangle\bar f'(\langle u,v\rangle)dv\,\langle P_{u^\perp},w\otimes w\rangle,\end{aligned}$$

where the second equality follows from (1.3.19) and (1.2.2), the third from (1.3.5), and the final equality from a change to spherical cylinder coordinates. Since the space of 2-tensors on the tangent space $u^\perp$ is spanned by pure tensors $w\otimes w$, this proves identity (1.3.24) for all $\phi\in C^\infty(\mathbb{S}^{n-1})$.

As an immediate consequence, $\|\mathrm{T}_f\phi\|_{C^2(\mathbb{S}^{n-1})}\le C\|\phi\|_{C(\mathbb{S}^{n-1})}$ for some constant $C\ge 0$ and all $\phi\in C^\infty(\mathbb{S}^{n-1})$. Thus, T_f extends to a bounded linear operator $\overline{\mathrm{T}}:C(\mathbb{S}^{n-1})\to C^2(\mathbb{S}^{n-1})$. Since $\mathrm{T}_f:C(\mathbb{S}^{n-1})\to C^1(\mathbb{S}^{n-1})$ is a bounded operator and the inclusion $C^2(\mathbb{S}^{n-1})\subseteq C^1(\mathbb{S}^{n-1})$ is continuous, $\overline{\mathrm{T}}$ agrees with T_f. By density and continuity, (1.3.24) is valid for all $\phi\in C(\mathbb{S}^{n-1})$.

For the second part of the theorem, suppose that T is an $\mathrm{SO}(n)$ equivariant bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$. Then Theorem 1.3.14 implies that $\mathrm{T}=\mathrm{T}_f$ for a unique zonal $f\in L^1(\mathbb{S}^{n-1})$. According to Proposition 1.3.7, it suffices to show that $(1-t^2)^{\frac{n-1}{2}}\bar f''(t)$ is a finite signed measure. To that end, take an arbitrary test function $\psi\in\mathcal{D}(-1,1)$, a point $u\in\mathbb{S}^{n-1}$, a unit tangent vector $w\in u^\perp$ and define

$$\phi(v)=P_2^{n-1}\left(\frac{\langle v,w\rangle}{\sqrt{1-\langle v,u\rangle^2}}\right)\psi(\langle v,u\rangle),\qquad v\in\mathbb{S}^{n-1}.$$

Then ϕ is a smooth function satisfying $\mathsf{J}_u[\phi](t)=0$ and $\mathsf{J}_u[\langle\cdot,w\rangle^2\phi](t)=C_n(1-t^2)^{\frac{n-1}{2}}\psi(t)$, where $C_n>0$ is given by $C_n=\int_{\mathbb{S}^{n-1}\cap u^\perp}P_2^{n-1}(\langle w,v\rangle)\langle w,v\rangle^2dv$. Choosing a geodesic γ in $\mathbb{S}^{n-1}$ such that $\gamma(0)=u$ and $\gamma'(0)=w$ yields

$$\begin{aligned}C_n\left\langle\psi(t),(1-t^2)^{\frac{n-1}{2}}\bar f''(t)\right\rangle_{\mathcal{D}'} &= \left\langle\mathsf{J}_u[\langle\cdot,w\rangle^2\phi](t),\bar f''(t)\right\rangle_{\mathcal{D}'}-\left\langle\mathsf{J}_u[\phi](t),t\bar f'(t)\right\rangle_{\mathcal{D}'}\\
&=\int_{(-1,1)}\left(\frac{d^2}{dt^2}\mathsf{J}_u[\langle\cdot,w\rangle^2\phi](t)+\frac{d}{dt}\left(t\mathsf{J}_u[\phi](t)\right)\right)\bar f(dt)=\int_{(-1,1)}\frac{d^2}{ds^2}\Big|_0\mathsf{J}_{\gamma(s)}[\phi](t)\bar f(t)dt,\end{aligned}$$

where the final equality is due to (1.3.19). Observe that $\operatorname{supp}\mathsf{J}_{\gamma(s)}[\phi]\subseteq[-1+\varepsilon,1-\varepsilon]$ and that $|\frac{d^2}{ds^2}\mathsf{J}_{\gamma(s)}[\phi](t)|\le C$ uniformly in s for all sufficiently small s, so we may interchange differentiation and integration and obtain

$$C_n\left\langle\psi(t),(1-t^2)^{\frac{n-1}{2}}\bar f''(t)\right\rangle_{\mathcal{D}'}=\frac{d^2}{ds^2}\Big|_0\int_{(-1,1)}\mathsf{J}_{\gamma(s)}[\phi](t)\bar f(t)dt=\langle\nabla^2_{\mathbb{S}}(\phi*f)(u),w\otimes w\rangle.$$

Therefore, we arrive at the following estimate:

$$\left|\left\langle \psi(t), (1-t^2)^{\frac{n-1}{2}}\bar{f}''(t)\right\rangle_{\mathcal{D}'}\right| \leq C_n^{-1}\|\mathrm{T}\|\|\phi\|_{C(\mathbb{S}^{n-1})} \leq C_n^{-1}\|\mathrm{T}\|\|\psi\|_\infty.$$

This shows that $(1-t^2)^{\frac{n-1}{2}}\bar{f}''(t) \in \mathcal{M}(-1,1)$, which completes the proof. □

In Theorems 1.3.14 and 1.3.15, we identify convolution transforms with zonal functions for which the spherical gradient and Hessian are signed measures, respectively. In general, checking these conditions directly can be difficult. However, Propositions 1.3.6, 1.3.7, and 1.3.9 provide more practical equivalent conditions. In this way, we obtain Theorem 1.C.

Proof of Theorem 1.C. According to Theorem 1.3.15, the convolution transform T_f is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$ if and only if $\nabla^2_{\mathbb{S}}f \in \mathcal{M}(\mathbb{S}^{n-1}, \mathbb{R}^{n\times n})$. Due to Lemma 1.3.8, this is the case precisely when $\square_n f$ is a finite signed measure and satisfies (1.1.3). □

For a zonal $f \in C^2(\mathbb{S}^{n-1})$, integration and differentiation can be interchanged, and thus,

$$\nabla^2_{\mathbb{S}}(\phi * f)(u) = \int_{\mathbb{S}^{n-1}} \phi(v)\nabla^2_{\mathbb{S}}\bar{f}(\langle\cdot, v\rangle)(u)dv$$

for every $\phi \in C(\mathbb{S}^{n-1})$. In light of (1.3.12), we see that (1.3.24) naturally extends this identity to general $f \in L^1(\mathbb{S}^{n-1})$. Denote by $D^2\phi$ the Hessian of the 1-homogeneous extension of a function ϕ on $\mathbb{S}^{n-1}$. Since $D^2\phi(u) = \nabla_{\mathbb{S}}\phi(u) + \phi(u)P_{u^\perp}$, as a direct consequence of (1.3.24), we obtain the following formula for:

$$\begin{aligned} D^2(\phi * f)(u) = &\int_{\mathbb{S}^{n-1}} \phi(v)(P_{u^\perp}v \otimes P_{u^\perp}v)(\mathsf{J}^u\bar{f}'')(dv) \\ &+ \int_{\mathbb{S}^{n-1}} \phi(v)\left(\bar{f}(\langle u, v\rangle) - \langle u, v\rangle \bar{f}'(\langle u, v\rangle)\right) dv\, P_{u^\perp}. \end{aligned} \tag{1.3.25}$$

As an instance of Theorem 1.3.15, we obtain Martinez-Maure's [84] result on the cosine transform, which is discussed in the following example.

Example 1.3.16. The *cosine transform* is the convolution transform T_f generated by the $L^1(\mathbb{S}^{n-1})$ function $f(u) = |\langle \bar{e}, u\rangle|$, that is, $\bar{f}(t) = |t|$. Thus $\bar{f}'' = 2\delta_0$ in the sense of distributions, so Proposition 1.3.7 and Theorem 1.3.15 imply that the cosine transform is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$.

Moreover, $\bar{f}(t) - t\bar{f}'(t) = 0$ and $\mathsf{J}^u\bar{f}'' = 2\lambda_{\mathbb{S}^{n-1}\cap u^\perp}$, where $\lambda_{\mathbb{S}^{n-1}\cap u^\perp}$ denotes the Lebesgue measure on the $(n-2)$-dimensional subsphere $\mathbb{S}^{n-1} \cap u^\perp$. Hence (1.3.25) shows that for every $\phi \in C(\mathbb{S}^{n-1})$,

$$D^2(\phi * f)(u) = 2\int_{\mathbb{S}^{n-1}\cap u^\perp} \phi(v)(P_{u^\perp}v \otimes P_{u^\perp}v)dv = 2\int_{\mathbb{S}^{n-1}\cap u^\perp} \phi(v)(v \otimes v)dv.$$

Example 1.3.17. For Berg's function g_n, we have seen in Example 1.3.10 that $\nabla_{\mathbb{S}}\breve{g}_n$ is an integrable function on $\mathbb{S}^{n-1}$ while the distributional spherical Hessian $\nabla^2_{\mathbb{S}}\breve{g}_n$ is not a finite signed measure. Thus, Theorems 1.3.14 and 1.3.15 imply that the convolution transform $\mathrm{T}_{\breve{g}_n}$ is a bounded operator from $C(\mathbb{S}^{n-1})$ to $C^1(\mathbb{S}^{n-1})$ but not a bounded operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$.

1.4 Regularity of Minkowski valuations

In this section, we study the regularity of Minkowski valuations $\Phi_i \in \mathbf{MVal}_i$ of degrees $1 \leq i \leq n-1$, proving Theorem 1.D. In the $(n-1)$-homogeneous case, Schuster [98] showed that Φ_{n-1} is generated by a continuous function. For other degrees of homogeneity, all that is known about the regularity of a generating function f is that $\square_n f$ is a signed measure and f is integrable (due to Dorrek [36]). Using our study of regularity of zonal functions in Section 1.3, we are able to refine Dorrek's results.

Theorem 1.4.1. *Let $1 \leq i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ with generating function f. Then*

(i) *$\square_n f$ is a signed measure on $\mathbb{S}^{n-1}$,*

(ii) *f is a locally Lipschitz function on $\mathbb{S}^{n-1} \backslash \{\pm \bar{e}\}$,*

(iii) *f is differentiable almost everywhere on $\mathbb{S}^{n-1}$, and $\nabla_{\mathbb{S}} f \in L^1(\mathbb{S}^{n-1}, \mathbb{R}^n)$.*

Proof. By [102, Theorem 6.1(i)], the function f also generates a Minkowski valuation of degree one. It follows from [36, Theorem 1.2] that $\square_n f$, and thus $\Delta_{\mathbb{S}} f$, is a signed measure on $\mathbb{S}^{n-1}$. Therefore, (1.3.14) shows that $(1-t^2)^{\frac{n-1}{2}} \bar{f}'(t)$ is an $L^\infty(-1,1)$ function. This implies that $\bar{f}$ is locally Lipschitz on $(-1,1)$, and thus, f is locally Lipschitz on $\mathbb{S}^{n-1} \backslash \{\pm \bar{e}\}$. Moreover, $(1-t^2)^{\frac{n-2}{2}} \bar{f}'(t)$ is in $L^1(-1,1)$, so due to Proposition 1.3.6, the distributional gradient $\nabla_{\mathbb{S}} f$ is in $L^1(\mathbb{S}^{n-1}, \mathbb{R}^n)$. Since f is locally Lipschitz on $\mathbb{S}^{n-1} \backslash \{\pm \bar{e}\}$, according to Rademacher's theorem, the classical gradient of f exists almost everywhere on $\mathbb{S}^{n-1}$ and agrees with the distributional gradient. $\square$

As a consequence of Theorems 1.3.14 and 1.4.1, we obtain the following.

Corollary 1.4.2. *For $1 \leq i \leq n-1$, every Minkowski valuation $\Phi_i \in \mathbf{MVal}_i$ maps convex bodies with a C^2 support function to strictly convex bodies.*

Proof. Denote by f the generating function of Φ_i. Theorem 1.4.1 (iii) implies that $\nabla_{\mathbb{S}} f \in L^1(\mathbb{S}^{n-1}, \mathbb{R}^n)$. According to Theorem 1.3.14, the convolution transform T_f is a bounded operator from $C(\mathbb{S}^{n-1})$ to $C^1(\mathbb{S}^{n-1})$. Suppose now that $K \in \mathcal{K}^n$ has a C^2 support function. Then $S_i(K,\cdot)$ has a continuous density, so $h(\Phi_i K, \cdot) = S_i(K,\cdot) * f$ is a $C^1(\mathbb{S}^{n-1})$ function, and thus, $\Phi_i K$ is strictly convex (see, e.g., [96, Section 2.5]). $\square$

We now turn to weakly monotone Minkowski valuations, for which we will obtain additional regularity of their generating functions. Recall that $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ is called *weakly monotone* if $\Phi_i K \subseteq \Phi_i L$ whenever $K \subseteq L$ and the Steiner points of K and L are at the origin.

Theorem 1.4.3. *Let $1 \leq i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ be weakly monotone with generating function f. Then $\square_n f$ is a weakly positive measure on $\mathbb{S}^{n-1}$ and there exists $C > 0$ such that for all $r \geq 0$,*

$$|\square_n f|(\{u \in \mathbb{S}^{n-1} : |\langle \bar{e}, u \rangle| > \cos r\}) \leq C r^{i-1}. \tag{1.4.1}$$

Note that Theorems 1.4.1 and 1.4.3 together yield Theorem 1.D. Here and in the following, a distribution on $\mathbb{S}^{n-1}$ is called *weakly positive* if it can be written as the sum of a positive measure and a linear function. In particular, every weakly positive distribution is a signed measure. The following characterization of weak positivity is a simple consequence of the Hahn-Banach separation theorem. For completeness, we provide a proof in Appendix 1.A.

Lemma 1.4.4. *A distribution $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$ is weakly positive if and only if $\langle \phi, \nu \rangle_{C^{-\infty}} \geq 0$ for every positive centered smooth function $\phi \in C^{\infty}(\mathbb{S}^{n-1})$.*

The proof of Theorem 1.4.3 relies on the behavior of area measures of convex bodies on spherical caps. We need the following classical result by Firey (recall the notation introduced in (1.3.15)).

Theorem 1.4.5 ([40]). *Let $1 \leq i \leq n-1$ and $K \in \mathcal{K}^n$ be a convex body. Then for every $u \in \mathbb{S}^{n-1}$,*

$$S_i(K, \mathrm{C}_r^{\mathbb{S}}(u)) \leq C_{n,i} (\operatorname{diam} K)^i r^{n-1-i} \tag{1.4.2}$$

where $C_{n,i} > 0$ depends only on n and i and $\operatorname{diam} K$ denotes the diameter of K.

The area measures of the $(n-1)$-dimensional disk in $\bar{e}^{\perp}$, which we denote by D^{n-1}, exhibit the worst possible asymptotic behavior in (1.4.2). This is shown in the example below.

Example 1.4.6. We seek to compute the area measures of D^{n-1}. For a convex body $K \in \mathcal{K}^n$ it is well known that $S_{n-1}(K, A)$ is the area of the reverse spherical image of a measurable subset $A \subseteq \mathbb{S}^{n-1}$. Thus,

$$S_{n-1}(D^{n-1}, \cdot) = \kappa_{n-1} \left(\delta_{-\bar{e}} + \delta_{\bar{e}} \right). \tag{1.4.3}$$

In order to compute the area measures of lower order, note that if a convex body $K \in \mathcal{K}^n$ with absolutely continuous area measure of order $1 \leq i < n-1$ lies in a hyperplane $u^{\perp}$ (where $u \in \mathbb{S}^{n-1}$), then

$$s_i(K, v) = \frac{n-1-i}{n-1} \left(1 - \langle u, v \rangle^2\right)^{-\frac{i}{2}} s_i^{u^{\perp}} \left(K, \frac{P_{u^{\perp}} v}{|P_{u^{\perp}} v|}\right), \qquad v \in \mathbb{S}^{n-1} \backslash \{\pm u\},$$

where $s_i(K, \cdot)$ and $s_i^{u^{\perp}}(K, \cdot)$ denote the densities of the i-th area measure of K with respect to the ambient space and the hyperplane $u^{\perp}$, respectively (see [59, Lemma 3.15]). Hence, for $1 \leq i < n-1$,

$$S_i(D^{n-1}, dv) = \frac{n-1-i}{n-1} \left(1 - \langle \bar{e}, v \rangle^2\right)^{-\frac{i}{2}} dv.$$

By a change to spherical cylinder coordinates, the measure of a polar cap can be estimated by

$$S_i(D^{n-1}, \mathrm{C}_r^{\mathbb{S}}(\bar{e})) \geq \frac{n-1-i}{n-1} \omega_{n-1} \int_{[\cos r, 1]} t (1 - t^2)^{\frac{n-3-i}{2}} dt = \kappa_{n-1} (\sin r)^{n-1-i}. \tag{1.4.4}$$

We require the following simple lemma, which relates the behavior of two positive measures μ and ν on small polar caps to the behavior of their convolution product.

Lemma 1.4.7. *Let $\mu \in \mathcal{M}_+(\mathbb{S}^{n-1})$ and let $\nu \in \mathcal{M}_+(\mathbb{S}^{n-1})$ be zonal. Then for all $u \in \mathbb{S}^{n-1}$ and $r \geq 0$,*

$$\mu(\mathrm{C}_r^{\mathbb{S}}(u))\nu(\mathrm{C}_r^{\mathbb{S}}(\bar{e})) \leq (\mu * \nu)(\mathrm{C}_{2r}^{\mathbb{S}}(u)), \tag{1.4.5}$$

$$\mu(\mathrm{C}_r^{\mathbb{S}}(u))\nu(\mathrm{C}_r^{\mathbb{S}}(-\bar{e})) \leq (\mu * \nu)(\mathrm{C}_{2r}^{\mathbb{S}}(-u)). \tag{1.4.6}$$

Proof. First observe that (1.4.5) implies (1.4.6) by reflecting ν at the origin. Moreover, we may assume that $u = \bar{e}$. The general case can be obtained from this by applying a suitable rotation to the measure μ and exploiting the $\mathrm{SO}(n)$ equivariance of the convolution transform T_ν. Next, note that

$$(\mu * \nu)(\mathrm{C}_{2r}^{\mathbb{S}}(\bar{e})) = \int_{\mathbb{S}^{n-1}} \mu(\mathrm{C}_{2r}^{\mathbb{S}}(u))\nu(du),$$

as can be easily shown by approximating $\mathbb{1}_{\mathrm{C}_{2r}^{\mathbb{S}}(\bar{e})}$ with smooth functions from below and applying the principle of monotone convergence. Thus

$$(\mu * \nu)(\mathrm{C}_{2r}^{\mathbb{S}}(\bar{e})) \geq \int_{\mathrm{C}_r^{\mathbb{S}}(\bar{e})} \mu(\mathrm{C}_{2r}^{\mathbb{S}}(u))\nu(du) \geq \int_{\mathrm{C}_r^{\mathbb{S}}(\bar{e})} \mu(\mathrm{C}_r^{\mathbb{S}}(\bar{e}))\nu(du) = \mu(\mathrm{C}_r^{\mathbb{S}}(\bar{e}))\nu(\mathrm{C}_r^{\mathbb{S}}(\bar{e})),$$

where the first inequality follows from shrinking the domain of integration and the second from the fact that $\mathrm{C}_{2r}^{\mathbb{S}}(u) \supseteq \mathrm{C}_r^{\mathbb{S}}(\bar{e})$ for all $u \in \mathrm{C}_r^{\mathbb{S}}(\bar{e})$ combined with the monotonicity of μ. $\square$

Now we are in a position to prove Theorem 1.4.3.

Proof of Theorem 1.4.3. Define a functional ξ_i on the space of smooth support functions by $\xi_i(h_K) = h(\Phi_i K, \bar{e})$. For every $\phi \in C^\infty(\mathbb{S}^{n-1})$, the function $1+t\phi$ is a support function whenever $t \in \mathbb{R}$ is sufficiently small. Therefore, we may compute the first variation of ξ_i at $\phi_0 = 1$. To that end, note that as a consequence of (1.1.1) and the polynomiality of area measures (see, e.g., [96, Section 5.1]),

$$\xi_i(1 + t\phi) = a_0^n[f] + i \langle \square_n \phi, f \rangle_{C^{-\infty}} t + O(t^2) \qquad \text{as } t \to 0.$$

Thus, for the first variation we obtain

$$\delta\xi_i(1, \phi) = \left.\frac{d}{dt}\right|_0 \xi_i(1 + t\phi) = i \langle \square_n \phi, f \rangle_{C^{-\infty}} = i \langle \phi, \square_n f \rangle_{C^{-\infty}}.$$

Since Φ_i is weakly monotone, the functional ξ_i is monotone on the subspace of centered functions, that is, $\xi_i(\phi_1) \leq \xi_i(\phi_2)$ whenever ϕ_1 and ϕ_2 are smooth centered support functions and $\phi_1 \leq \phi_2$. Consequently, the first variation $\delta\xi_i(1, \phi)$ must be non-negative for every positive and centered $\phi \in C^\infty(\mathbb{S}^{n-1})$. Lemma 1.4.4 implies that $\square_n f$ is a weakly positive measure.

Since $\square_n f$ is weakly positive, there exists some positive measure $\nu \in \mathcal{M}_+(\mathbb{S}^{n-1})$ and $x \in \mathbb{R}^n$ such that $\square_n f = \nu + \langle x, \cdot\rangle$. Observe that for every $K \in \mathcal{K}^n$,

$$S_1(\Phi_i K, \cdot) = \square_n h(\Phi_i K, \cdot) = \square_n (S_i(K, \cdot) * f) = S_i(K, \cdot) * \square_n f = S_i(K, \cdot) * \nu.$$

Hence, (1.4.5) and (1.4.6) imply that for all $r \geq 0$,

$$S_i(K, \mathrm{C}_r^{\mathbb{S}}(\bar{e}))\nu(\mathrm{C}_r^{\mathbb{S}}(\pm\bar{e})) \leq S_1(\Phi_i K, \mathrm{C}_{2r}^{\mathbb{S}}(\pm\bar{e})).$$

Due to (1.4.2), the right hand side is bounded from above by a constant multiple of r^{n-2}. If we choose K to be the $(n-1)$-dimensional disk D^{n-1}, then $S_i(K, \mathrm{C}_r^{\mathbb{S}}(\bar{e}))$ is bounded from below by a multiple of r^{n-i-1}, as is shown in (1.4.3) and (1.4.4). Thus,

$$\nu(\mathrm{C}_r^{\mathbb{S}}(\pm\bar{e})) \leq \frac{S_1(\Phi_i D^{n-1}, \mathrm{C}_{2r}^{\mathbb{S}}(\pm\bar{e}))}{S_i(D^{n-1}, \mathrm{C}_r^{\mathbb{S}}(\bar{e}))} \leq C' \frac{r^{n-2}}{r^{n-i-1}} = C' r^{i-1}.$$

Since $|\square_n f| \leq \nu + |\langle x, \cdot\rangle|$, we have that

$$|\square_n f|(\mathrm{C}_r^{\mathbb{S}}(\pm\bar{e})) \leq \nu(\mathrm{C}_r^{\mathbb{S}}(\pm\bar{e})) + \int_{\mathrm{C}_r^{\mathbb{S}}(\pm\bar{e})} |\langle x, u\rangle| du \leq C' r^{i-1} + |x|\kappa_{n-1} r^{n-1} \leq C r^{i-1}$$

for a suitable constant $C \geq 0$, which proves (1.4.1). $\square$

By combining Theorem 1.4.3 with Theorem 1.C, we immediately obtain the following.

Corollary 1.4.8. *Let $1 < i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ be weakly monotone with generating function f. Then the convolution transform T_f is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$.*

As was pointed out in Proposition 1.3.9, the behavior of zonal measures on small polar caps determines their regularity. In the following, we show that this behavior also determines the rate of convergence of their multipliers, which is another way of expressing regularity. We use the following classical asymptotic estimate for Legendre polynomials.

Theorem 1.4.9 ([106, 7.33])**.** *For all $n \geq 3$ and $\delta > 0$, there exists $M > 0$ such that for all $k \geq 0$,*

$$|P_k^n(t)| \leq M k^{-\frac{n-2}{2}}(1-t^2)^{-\frac{n-2}{4}} \qquad \text{for } t \in \left[-\cos\tfrac{\delta}{k}, \cos\tfrac{\delta}{k}\right]. \tag{1.4.7}$$

Theorem 1.4.10. *Let $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ be zonal and suppose that there exist $C > 0$ and $\alpha \geq 0$ such that*

$$|\mu|(\{u \in \mathbb{S}^{n-1} : |\langle \bar{e}, u\rangle| > \cos r\}) \leq C r^{\alpha}$$

for all $r \geq 0$. Then

$$a_k^n[\mu] \in \begin{cases} O(k^{-\alpha}), & \alpha < \frac{n-2}{2}, \\ O(k^{-\frac{n-2}{2}} \ln k), & \alpha = \frac{n-2}{2}, \\ O(k^{-\frac{n-2}{2}}), & \alpha > \frac{n-2}{2}. \end{cases}$$

Proof. Since μ can be decomposed into two signed measures that are each supported on one hemisphere, we may assume that $\operatorname{supp}\mu \subseteq \{u \in \mathbb{S}^{n-1} : \langle \bar{e}, u\rangle \geq 0\}$. Denoting by $\rho = \mathsf{J}_{\bar{e}}[|\mu|]$ the pushforward measure of $|\mu|$ with respect to the map $u \mapsto \langle \bar{e}, u\rangle$, we have that

$$|a_k^n[\mu]| = \left|\int_{\mathbb{S}^{n-1}} P_k^n(\langle \bar{e}, u\rangle)\mu(du)\right| \leq \int_{\mathbb{S}^{n-1}} |P_k^n(\langle \bar{e}, u\rangle)|\, |\mu|(du) = \int_{[0,1]} |P_k^n(t)|\rho(dt).$$

Our aim now is to find suitable bounds for the integral on the right hand side. To that end, we fix some arbitrary $\delta > 0$ and split it into the two integrals

$$I_1(k) = \int_{[0,\cos\frac{\delta}{k}]} |P_k^n(t)|\rho(dt) \qquad \text{and} \qquad I_2(k) = \int_{(\cos\frac{\delta}{k},1]} |P_k^n(t)|\rho(dt).$$

Observe that our assumption on μ implies that $\rho((t,1]) \leq C(1-t^2)^{\frac{\alpha}{2}}$ for all $t \in [0,1]$. Since $|P_k^n(t)| \leq 1$ for all $t \in [0,1]$, we obtain

$$I_2(k) \leq \rho\left(\left(\cos\tfrac{\delta}{k}, 1\right]\right) \leq C\left(\sin\tfrac{\delta}{k}\right)^{\alpha} \in O(k^{-\alpha}).$$

For the integral $I_1(k)$, estimate (1.4.7) and Lebesgue-Stieltjes integration by parts yield

$$\begin{aligned} I_1(k) &\leq Mk^{-\frac{n-2}{2}} \int_{[0,\cos\frac{\delta}{k}]} (1-t^2)^{-\frac{n-2}{4}} \rho(dt) \\ &= Mk^{-\frac{n-2}{2}} \left(\rho\left(\left[0, \cos\tfrac{\delta}{k}\right]\right) - \left(\sin\tfrac{\delta}{k}\right)^{-\frac{n-2}{2}} \rho\left(\left(\cos\tfrac{\delta}{k}, 1\right]\right) + \tilde{I}_1(k)\right) \\ &\leq Mk^{-\frac{n-2}{2}} \left(\rho([0,1]) + \tilde{I}_1(k)\right), \end{aligned}$$

where we defined

$$\tilde{I}_1(k) = \tfrac{n-2}{2} \int_{[0,\cos\frac{\delta}{k}]} \rho\left((t,1]\right) t(1-t^2)^{-\frac{n+2}{4}} dt.$$

Employing again our estimate on $\rho((t,1])$ and performing a simple computation shows that

$$\tilde{I}_1(k) \leq C\tfrac{n-2}{2} \int_{[0,\cos\frac{\delta}{k}]} t(1-t^2)^{\frac{1}{2}(\alpha - \frac{n-2}{2} - 1)} dt \in \begin{cases} O(k^{\frac{n-2}{2}-\alpha}), & \alpha < \frac{n-2}{2}, \\ O(\ln k), & \alpha = \frac{n-2}{2}, \\ O(1), & \alpha > \frac{n-2}{2}. \end{cases}$$

Combining the estimates for $I_1(k)$ and $I_2(k)$ completes the proof. $\square$

As an immediate consequence of Theorems 1.4.3 and 1.4.10, we obtain the following.

Corollary 1.4.11. *Let $1 < i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ be weakly monotone with generating function f. Then $a_k^n[\square_n f] \in O(k^{-1/2})$ as $k \to \infty$.*

1.5 Fixed points

In this section, we prove a range of results regarding local uniqueness of fixed points of Minkowski valuations $\Phi_i \in \mathbf{MVal}_i$ of degree $1 < i \leq n-1$ (the 1-homogeneous case has been settled globally by Kiderlen [60]). This section is divided into three subsections. In Section 1.5.1, we prove Theorem 1.B concerning the mean section operators. Section 1.5.2 is dedicated to Minkowski valuations Φ_i generated by origin-symmetric convex bodies of revolution. There we prove Theorem 1.A, unifying previous results by Ivaki [57, 58] and the second author and Schuster [88]. Finally, in Section 1.5.3 we consider general even Minkowski valuations for which we obtain information about the fixed points of Φ_i (as opposed to Φ_i^2).

The proofs given in this section utilize the following result for general Minkowski valuations $\Phi_i \in \mathbf{MVal}_i$. It provides three sufficient conditions on the generating function of Φ_i to obtain the desired local uniqueness of fixed points of Φ_i^2. It contains however no information on when these conditions are fulfilled. For instance, in the particular case when Φ_i is generated by an origin symmetric C^2_+ convex body of revolution, checking condition (C3) turns out to be rather involved.

Theorem 1.5.1 ([88]). *Let $1 < i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ with generating function f satisfying the following conditions:*

(C1) *the convolution transform T_f is a bounded linear operator from $C(\mathbb{S}^{n-1})$ to $C^2(\mathbb{S}^{n-1})$,*

(C2) *there exists $\alpha > 0$ such that $a_k^n[\Box_n f] \in O(k^{-\alpha})$ as $k \to \infty$,*

(C3) *for all $k \geq 2$,*

$$\frac{|a_k^n[\Box_n f]|}{a_0^n[\Box_n f]} < \frac{1}{i}.$$

Then there exists a C^2 neighborhood of B^n where the only fixed points of Φ_i^2 are Euclidean balls.

Now we can apply our results on regularity of weakly monotone Minkowski valuations Φ_i to these fixed point problems. Corollaries 1.4.8 and 1.4.11 show that conditions (C1) and (C2) are fulfilled in the weakly monotone case, which yields the following.

Theorem 1.5.2. *Let $1 < i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ be weakly monotone with generating function f satisfying condition* (C3)*. Then there exists a C^2 neighborhood of B^n where the only fixed points of Φ_i^2 are Euclidean balls.*

Remark 1.5.3. The way Theorem 1.5.1 was stated in [88] additionally required Φ_i to be *even* as the proof employs the following classical result by Strichartz [105]. Denote by $H^s(\mathbb{S}^{n-1})$, $s \in \mathbb{N}$, the Sobolev space of functions on $\mathbb{S}^{n-1}$ with weak covariant derivatives up to order s in $L^2(\mathbb{S}^{n-1})$. Strichartz showed that

$$\|\phi\|^2_{H^s} \approx \sum_{k=0}^{\infty} (k^2+1)^s \|\pi_k \phi\|^2_{L^2} \tag{1.5.1}$$

for every *even* $\phi \in H^s(\mathbb{S}^{n-1})$, where $\|\cdot\|_{H^s}$ is the standard norm of $H^s(\mathbb{S}^{n-1})$. However, the classical theory on the Dirichlet problem on compact Riemannian manifolds (see, e.g., [108, Section 5.1]) implies that (1.5.1) holds for *every* $\phi \in H^s(\mathbb{S}^{n-1})$. Therefore, by a minor modification of the proof of [88, Theorem 6.1], the assumption on Φ_i to be even can be omitted.

1.5.1 Mean section operators

As a first application, we show local uniqueness of fixed points of the mean section operators M_j, which were defined at the beginning of this article. As was pointed out in the introduction, the mean section operators are not generated by a convex body of revolution. This is the main reason why they have not been included in previous results. Due to our extensive study of regularity, we obtain Theorem 1.B as a simple consequence of Theorem 1.5.2.

Theorem B. *For $2 \leq j < n$, there exists a C^2 neighborhood of B^n where the only fixed points of M_j^2 are Euclidean balls.*

Proof. Define the j-th *centered* mean section operator by $\tilde{\mathrm{M}}_j K = \mathrm{M}_j(K - s(K))$. Then $\tilde{\mathrm{M}}_j \in \mathbf{MVal}_i$ for $i = n+1-j$ and due to (1.1.2) its generating function is given by $(\mathrm{Id} - \pi_1)\breve{g}_j$. Clearly M_j is monotone, and thus, $\tilde{\mathrm{M}}_j$ is weakly monotone. By Theorem 1.5.2, it suffices to check condition (C3) for $\breve{g}_j$.

It was shown in [12] and [20] independently that the multipliers of $\breve{g}_j$ are given by

$$a_k^n[\breve{g}_j] = -\frac{\pi^{\frac{n-j}{2}}(j-1)}{4}\frac{\Gamma\big(\frac{n-j+2}{2}\big)\Gamma\big(\frac{k-1}{2}\big)\Gamma\big(\frac{k+j-1}{2}\big)}{\Gamma\big(\frac{k+n-j+1}{2}\big)\Gamma\big(\frac{k+n+1}{2}\big)}$$

for $k \neq 1$. A simple computation using (1.2.3) and the functional equation $\Gamma(x+1) = x\Gamma(x)$ yields

$$\frac{a_k^n[\square_n \breve{g}_j]}{a_0^n[\square_n \breve{g}_j]} = \frac{1}{i}\frac{\Gamma\big(\frac{n-1}{2}\big)\Gamma\big(\frac{i+2}{2}\big)\Gamma\big(\frac{k+1}{2}\big)\Gamma\big(\frac{k+n-i}{2}\big)}{\Gamma\big(\frac{n-i}{2}\big)\Gamma\big(\frac{3}{2}\big)\Gamma\big(\frac{k+i}{2}\big)\Gamma\big(\frac{k+n-1}{2}\big)}.$$

Since the Gamma function is strictly positive and strictly increasing on $[\frac{3}{2}, \infty)$, it follows that $\breve{g}_j$ satisfies condition (C3). $\square$

1.5.2 Convex bodies of revolution

We now turn to Minkowski valuations that are generated by a convex body of revolution, that is, their generating function is a support function. This class includes all even Minkowski valuations in $\mathbf{MVal}_{n-1}$, as was shown in [98]. Our aim for this section is to prove Theorem 1.A which is restated below.

Theorem A. *Let $1 < i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ be generated by an origin-symmetric convex body of revolution. Then there exists a C^2 neighborhood of B^n where the only fixed points of Φ_i^2 are Euclidean balls, unless Φ_i is a multiple of the projection body operator, in which case ellipsoids are also fixed points.*

Note that if $\Phi_i \in \mathbf{MVal}_i$ is generated by a convex body of revolution L, then for every $K \in \mathcal{K}^n$,

$$h(\Phi_i K, u) = V(K[i], L(u), B^n[n-i-1]), \qquad u \in \mathbb{S}^{n-1},$$

where $L(u)$ denotes a suitably rotated copy of L, and $V(K_1, \ldots, K_n)$ is the mixed volume of the convex bodies $K_1, \ldots, K_n \in \mathcal{K}^n$ (see, e.g., [96, Section 5.1]). Due to the monotonicity of the mixed volume we see that every Minkowski valuation generated by a convex body of revolution is monotone. In light of Theorem 1.5.2, it is natural to ask when condition (C3) is fulfilled. The following result shows that L being origin symmetric is already sufficient up to the second multiplier.

Theorem 1.5.4 ([88]). *Let L be a convex body of revolution. Then for all even $k \geq 4$,*

$$\frac{|a_k^n[\Box_n h_L]|}{a_0^n[\Box_n h_L]} < \frac{1}{n-1}, \tag{1.5.2}$$

and

$$-\frac{1}{n-1} \leq \frac{a_2^n[\Box_n h_L]}{a_0^n[\Box_n h_L]} < \frac{1}{n-1}, \tag{1.5.3}$$

where the left hand side inequality in (1.5.3) *is strict if L is of class C^2_+.*

If $i < n-1$, then condition (C3) is fulfilled. If $i = n-1$, then Theorem 1.5.4 shows that condition (C3) is fulfilled under the additional assumption that L is of class C^2_+. We will show that imposing this regularity is not necessary: line segments are the only bodies for which equality is attained in the left hand side of inequality (1.5.3).

Definition 1.5.5. On $(-1,1)$, we define the two differential operators

$$\mathcal{A}_1 = \mathrm{Id} - t\frac{d}{dt} \qquad \text{and} \qquad \mathcal{A}_2 = (1-t^2)\frac{d^2}{dt^2} + \mathrm{Id} - t\frac{d}{dt}. \tag{1.5.4}$$

These operators come up naturally in the study of zonal functions. For a zonal function $f \in C^2(\mathbb{S}^{n-1})$, the Hessian of its 1-homogeneous extension $D^2 f$ at each point only has two eigenvalues: $\mathcal{A}_1 \bar{f}$ is the eigenvalue of multiplicity $n-2$, and $\mathcal{A}_2 \bar{f}$ the eigenvalue of multiplicity one (see (1.3.25)). The following lemma is a simple consequence of this fact.

Lemma 1.5.6 ([88]). *Let $g \in C[-1,1]$. Then $g(\langle \bar{e}, \cdot \rangle)$ is the support function of a convex body of revolution if and only if $\mathcal{A}_1 g \geq 0$ and $\mathcal{A}_2 g \geq 0$ in the weak sense.*

In [88], this lemma was proven only for $C^2[-1,1]$ functions, however it extends to $C[-1,1]$ by a simple approximation argument. Next, we determine the kernels of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Lemma 1.5.7. *Let g be a locally integrable function on $(-1,1)$.*

(i) *$\mathcal{A}_1 g = 0$ in the weak sense if and only if $g(t) = c_1|t| + c_2 t$ for some $c_1, c_2 \in \mathbb{R}$.*

(ii) *$\mathcal{A}_2 g = 0$ in the weak sense if and only if $g(t) = c_1\sqrt{1-t^2} + c_2 t$ for some $c_1, c_2 \in \mathbb{R}$.*

Proof. Clearly, $g(t) = c_1|t| + c_2 t$ is a weak solution of the differential equation $\mathcal{A}_1 g = 0$. Conversely, suppose that $\mathcal{A}_1 g = 0$ in the weak sense. Observe that on the interval $(0,1)$, the change of variables $t = e^s$ transforms the differential operator $\mathcal{A}_1$ as follows:

$$\tilde{\mathcal{A}}_1 = \mathrm{Id} - \frac{d}{ds}.$$

Therefore, there exists $c_+ \in \mathbb{R}$ such that $g(e^s) = c_+ e^s$ in $\mathcal{D}'(-\infty, 0)$. Reversing the change of variables yields $g(t) = c_+ t$ in $\mathcal{D}'(0,1)$. Similarly, there exists $c_- \in \mathbb{R}$ such that $g(t) = c_- t$ in $\mathcal{D}'(-1,0)$. Since g is locally integrable, choosing $c_1 = \frac{1}{2}(c_+ + c_-)$ and $c_2 = \frac{1}{2}(c_+ - c_-)$, we obtain that $g(t) = c_1|t| + c_2 t$ in $\mathcal{D}'(-1,1)$.

For the second part of the lemma, note that $g(t) = c_1\sqrt{1-t^2} + c_2 t$ solves the differential equation $\mathcal{A}_2 g = 0$. Conversely, suppose that $\mathcal{A}_2 g = 0$ in the weak sense. Observe that the change of variables $t = \sin\theta$ transforms the differential operator $\mathcal{A}_2$ as follows:

$$\tilde{\mathcal{A}}_2 = \mathrm{Id} + \frac{d^2}{d\theta^2}.$$

Therefore, there exist $c_1, c_2 \in \mathbb{R}$ such that $g(\sin\theta) = c_1\cos\theta + c_2\sin\theta$ in $\mathcal{D}'(-\frac{\pi}{2}, \frac{\pi}{2})$. Reversing the change of variables yields $g(t) = c_1\sqrt{1-t^2} + c_2 t$ in $\mathcal{D}'(-1,1)$. □

The following lemma describes the action of $\mathcal{A}_1$ and $\mathcal{A}_2$ on Legendre polynomials.

Lemma 1.5.8. *For every $k \geq 2$,*

$$\frac{n-1}{(k-1)(k+n-1)}\mathcal{A}_1 P_k^n = -\frac{k}{2k+n-2}P_{k-2}^{n+2} - \frac{k+n-2}{2k+n-2}P_k^{n+2}, \tag{1.5.5}$$

$$\frac{n-1}{(k-1)(k+n-1)}\mathcal{A}_2 P_k^n = \frac{k(k+n-3)}{2k+n-2}P_{k-2}^{n+2} - \frac{(k+1)(k+n-2)}{2k+n-2}P_k^{n+2}. \tag{1.5.6}$$

Proof. We need the following two identities:

$$(2k+n-2)(n-1)P_k^n(t) = (k+n-2)(k+n-1)P_k^{n+2}(t) - (k-1)kP_{k-2}^{n+2}(t), \tag{1.5.7}$$

$$(k-1)P_{k-2}^n(t) - (2k+n-4)tP_{k-1}^n(t) + (k+n-3)P_k^n(t) = 0. \tag{1.5.8}$$

Both follow from (1.2.5) by a simple inductive argument (see [49, Section 3.3]). By (1.2.5) and (1.5.8),

$$(2k+n-2)(n-1)t\frac{d}{dt}P_k^n(t) = k(k+n-2)((k-1)P_{k-2}^{n+2}(t) + (k+n-1)P_k^{n+2}(t)). \tag{1.5.9}$$

Combining (1.2.6) with (1.5.7) and (1.5.9) yields

$$(2k+n-2)(n-1)(1-t^2)\frac{d^2}{dt^2}P_k^n(t) = (k-1)k(k+n-2)(k+n-1)(P_{k-2}^{n+2}(t) - P_k^{n+2}(t)). \tag{1.5.10}$$

By (1.5.4) and a combination of identities (1.5.7), (1.5.9), and (1.5.10), we obtain (1.5.5) and (1.5.6). □

We use (1.5.5) and (1.5.6) to derive the following recurrence relation for multipliers.

Lemma 1.5.9. *Let g be a locally integrable function on $(-1,1)$.*

(i) *If $(1-t^2)^{\frac{n-1}{2}}g'(t) \in \mathcal{M}(-1,1)$, then for all $k \geq 0$,*

$$\frac{k-1}{2k+n}a_k^n[g] + \frac{k+n+1}{2k+n}a_{k+2}^n[g] = -\frac{1}{2\pi}a_k^{n+2}[\mathcal{A}_1 g]. \tag{1.5.11}$$

(ii) *If $(1-t^2)^{\frac{n+1}{2}}g''(t) \in \mathcal{M}(-1,1)$, then for all $k \geq 0$,*

$$\frac{(k-1)(k+1)}{2k+n}a_k^n[g] - \frac{(k+n-1)(k+n+1)}{2k+n}a_{k+2}^n[g] = -\frac{1}{2\pi}a_k^{n+2}[\mathcal{A}_2 g]. \tag{1.5.12}$$

Proof. In the following, we use that the family $(P_k^n)_{k=0}^\infty$ of Legendre polynomials is an orthogonal system with respect to the inner product $[\psi, g]_n = \int_{[-1,1]} \psi(t)g(t)(1-t^2)^{\frac{n-3}{2}}dt$ on $[-1,1]$. Moreover, $[P_k^n, P_k^n]_n = \frac{\omega_n(k+n-2)}{\omega_{n-1}(2k+n-2)}\binom{k+n-2}{n-2}^{-1}$ (see, e.g., [49, Section 3.3]).

For the first part of the lemma, note that due to (1.3.5), for every $\psi \in C^1[-1,1]$,

$$[\psi, \mathcal{A}_1 g]_{n+2} = [\mathcal{A}_1^* \psi, g]_n, \tag{1.5.13}$$

where $\mathcal{A}_1^*$ denotes the differential operator

$$\mathcal{A}_1^* = (1-(n+1)t^2)\mathrm{Id} + (1-t^2)t\frac{d}{dt}.$$

Clearly $\mathcal{A}_1^*$ increases the degree of a polynomial at most by two, so there exist $x_{k,j} \in \mathbb{R}$ such that for every $k \geq 0$,

$$\mathcal{A}_1^* P_k^{n+2} = \sum_{j=0}^{k+2} x_{k,j} P_j^n.$$

Choosing $\psi = P_k^{n+2}$ in (1.5.13) yields

$$\frac{1}{\omega_{n+1}}a_k^{n+2}[\mathcal{A}_1 g] = [P_k^{n+2}, \mathcal{A}_1 g]_{n+2} = [\mathcal{A}_1^* P_k^{n+2}, g]_n = \sum_{j=0}^{k+2} x_{k,j}[P_j^n, g]_n = \frac{1}{\omega_{n-1}}\sum_{j=0}^{k+2} x_{k,j} a_j^n[g].$$

Hence, it only remains to determine the numbers $x_{k,j}$. By applying the identity above to $g = P_j^n$ for $0 \leq j \leq k+2$, and employing (1.5.5), we obtain that

$$x_{k,k} = -\frac{(k-1)(n-1)}{2k+n}, \qquad x_{k,k+2} = -\frac{(k+n+1)(n-1)}{2k+n},$$

and $x_{k,j} = 0$ for $j \notin \{k, k+2\}$, which proves (1.5.11).

For the second part of the proof, note that due to (1.3.5), for every $\psi \in C^2[-1,1]$,

$$[\psi, \mathcal{A}_2 g]_{n+2} = [\mathcal{A}_2^* \psi, g]_n, \tag{1.5.14}$$

where $\mathcal{A}_2^*$ denotes the differential operator

$$\mathcal{A}_2^* = (1-t^2)^2\frac{d^2}{dt^2} + (n-1)((n+1)t^2-1)\mathrm{Id} - (2n+1)(1-t^2)t\frac{d}{dt}.$$

Clearly $\mathcal{A}_2^*$ increases the degree of a polynomial at most by two, so there exist $y_{k,j} \in \mathbb{R}$ such that for every $k \geq 0$,

$$\mathcal{A}_2^* P_k^{n+2} = \sum_{j=0}^{k+2} y_{k,j} P_j^n.$$

Choosing $\psi = P_k^{n+2}$ in (1.5.14) yields

$$\frac{1}{\omega_{n+1}} a_k^{n+2}[\mathcal{A}_2 g] = [P_k^{n+2}, \mathcal{A}_2 g]_{n+2} = [\mathcal{A}_2^* P_k^{n+2}, g]_n = \sum_{j=0}^{k+2} y_{k,j}[P_j^n, g]_n = \frac{1}{\omega_{n-1}} \sum_{j=0}^{k+2} y_{k,j} a_j^n[g].$$

Hence, it only remains to determine the numbers $y_{k,j}$. By applying the identity above to $g = P_j^n$ for $0 \leq j \leq k+2$, and employing (1.5.6), we obtain that

$$y_{k,k} = -\frac{(k-1)(k+1)(n-1)}{2k+n}, \qquad y_{k,k+2} = -\frac{(k+n-1)(k+n+1)(n-1)}{2k+n},$$

and $y_{k,j} = 0$ for $j \notin \{k, k+2\}$, which proves (1.5.12). □

We arrive at the following geometric inequality for convex bodies of revolution. This shows that equality is attained in (1.5.3) only by line segments, which completes the proof of Theorem A.

Theorem 1.5.10. *Let $L \in \mathcal{K}^n$ be a convex body of revolution. Then*

$$-\frac{1}{n-1} \leq \frac{a_2^n[\Box_n h_L]}{a_0^n[\Box_n h_L]} \leq \frac{1}{(n-1)^2} \tag{1.5.15}$$

with equality in the left hand inequality if and only if L is a line segment and equality in the right hand inequality if and only if L is an $(n-1)$-dimensional disk.

Proof. As an instance of (1.5.11),

$$a_2^n[\Box_n h_L] + \frac{1}{n-1} a_0^n[\Box_n h_L] = \frac{n}{2\pi(n-1)} a_0^{n+2}[\mathcal{A}_1 \overline{h_L}].$$

Due to Lemma 1.5.6, we have that $\mathcal{A}_1\overline{h_L} \geq 0$, which proves the first inequality in (1.5.15). Moreover, equality holds precisely when $\mathcal{A}_1\overline{h_L} = 0$. According to Lemma 1.5.7 (i), this is the case if and only if $\overline{h_L}(t) = c_1|t| + c_2 t$ for some $c_1, c_2 \in \mathbb{R}$, which means that L is a line segment.

As an instance of (1.5.12),

$$a_2^n[\Box_n h_L] - \frac{1}{(n-1)^2} a_0^n[\Box_n h_L] = \frac{n}{2\pi(n-1)^2} a_0^{n+2}[\mathcal{A}_2 \overline{h_L}].$$

Due to Lemma 1.5.6, we have that $\mathcal{A}_2\overline{h_L} \geq 0$, which proves the second inequality in (1.5.15). Moreover, equality holds precisely when $\mathcal{A}_2\overline{h_L} = 0$. According to Lemma 1.5.7 (ii), this is the case if and only if $\overline{h_L}(t) = c_1\sqrt{1-t^2} + c_2 t$ for some $c_1, c_2 \in \mathbb{R}$, which means that L is an $(n-1)$-dimensional disk. □

1.5.3 Even Minkowski valuations

This section is dedicated to even Minkowski valuations $\Phi_i \in \mathbf{MVal}_i$ of degree $1 < i \leq n-1$. In the previous subsection, we have shown that if Φ_i is generated by an origin-symmetric convex body of revolution, then condition (C3) is fulfilled, unless Φ_i is a multiple of the projection body map (see Theorems 1.5.4 and 1.5.10).

In general, the generating function of an even Minkowski valuation does not need to be a support function. In this broader setting, we prove a weaker condition than (C3), which we use to obtain information about the fixed points of the map Φ_i itself as opposed to Φ_i^2. To that end, we require the following version of Theorem 1.5.1, which can be obtained from a minor modification of its proof, as was observed in [88].

Theorem 1.5.11 ([88])**.** *Let $1 < i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ with generating function f satisfying conditions* (C1)*,* (C2)*, and*

(C3') *for all $k \geq 2$,*

$$\frac{a_k^n[\square_n f]}{a_0^n[\square_n f]} < \frac{1}{i}.$$

Then there exists a C^2 neighborhood of B^n where the only fixed points of Φ_i are Euclidean balls.

Again, Corollaries 1.4.8 and 1.4.11 show that conditions (C1) and (C2) are fulfilled in the weakly monotone case. Hence we obtain the following.

Theorem 1.5.12. *Let $1 < i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ be weakly monotone with generating function f satisfying condition* (C3')*. Then there exists a C^2 neighborhood of B^n where the only fixed points of Φ_i are Euclidean balls.*

The main result of this section will be that if $\Phi_i \in \mathbf{MVal}_i$ is even, then its generating function satisfies condition (C3'). We require the following lemma, which is a consequence of a classical result by Firey [39]. We call a convex body of revolution *smooth* if it has a $C^2(\mathbb{S}^{n-1})$ support function and $\mathcal{A}_2\overline{h_K} > 0$ on $[-1, 1]$.

Lemma 1.5.13. *Let $1 \leq i < n-1$ and let $\phi \in C(\mathbb{S}^{n-1})$ be zonal and centered. Then ϕ is the density of the i-th area measure of a smooth convex body of revolution if an only if for all $t \in (-1, 1)$,*

$$\bar{\phi}(t) > \frac{n-1-i}{n-1}\mathcal{A}_1(\overline{\phi * \breve{g}_n})(t) > 0.$$

Proof. It was proved in [39] that ϕ is the density of the i-th area measure of a smooth convex body if and only if for all $t \in (-1, 1)$,

$$\bar{\phi}(t) > (n-1-i)(1-t^2)^{-\frac{n-1}{2}} \int_{(t,1)} \bar{\phi}(s) s (1-s^2)^{\frac{n-3}{2}} ds > 0.$$

Therefore it only remains to show that for all $t \in (-1, 1)$,

$$\int_{(t,1)} \bar{\phi}(s) s (1-s^2)^{\frac{n-3}{2}} ds = \frac{1}{n-1}(1-t^2)^{\frac{n-1}{2}} \mathcal{A}_1(\overline{\phi * \breve{g}_n})(t). \tag{1.5.16}$$

We have seen in Example 1.3.17 that the convolution transform $\mathrm{T}_{\breve{g}_n}$ is a bounded operator from $C(\mathbb{S}^{n-1})$ to $C^1(\mathbb{S}^{n-1})$, so both sides of (1.5.16) depend continuously on $\phi \in C(\mathbb{S}^{n-1})$ with respect to uniform convergence. Therefore it suffices to show (1.5.16) only for smooth ϕ.

To that end, let $\zeta = \phi * \breve{g}_n \in C^\infty(\mathbb{S}^{n-1})$ and observe that according to (1.3.2),

$$\bar{\phi}(s) = \overline{\Box_n \zeta}(s) = \frac{1}{n-1}\overline{\Delta_{\mathbb{S}}\zeta}(s) + \bar{\zeta}(s) = \frac{1}{n-1}(1-s^2)\bar{\zeta}''(s) + \bar{\zeta}(s) - s\bar{\zeta}'(s).$$

A direct computation yields

$$\bar{\phi}(s)s(1-s^2)^{\frac{n-3}{2}} = -\frac{1}{n-1}\frac{d}{ds}\left((1-s^2)^{\frac{n-1}{2}}\mathcal{A}_1\bar{\zeta}(s)\right).$$

Hence, we obtain (1.5.16), which completes the proof. $\square$

Next, we prove the following two technical lemmas. For smooth functions $\psi \in C^\infty[-1,1]$, we define $\overline{\Box}_n\psi = \overline{\Box_n\psi(\langle \bar{e}, \cdot\rangle)}$. Note that $\overline{\Box}_n\psi(t) = \frac{1}{n-1}(1-t^2)\psi''(t) + \psi(t) - t\psi'(t)$ due to (1.3.2).

Lemma 1.5.14. *For every $\psi \in C^\infty[-1,1]$,*

$$\max_{[-1,1]} \mathcal{A}_1\psi \le \max_{[-1,1]} \overline{\Box}_n\psi. \tag{1.5.17}$$

Proof. Let $t_0 \in [-1,1]$ be a maximum point of $\mathcal{A}_1\psi$. We will show that

$$(1-t_0^2)\psi''(t_0) \ge 0. \tag{1.5.18}$$

If $t_0 = \pm 1$, then clearly we have (1.5.18). If $t_0 \in (-1,1)$, then

$$-t_0\psi''(t_0) = (\mathcal{A}_1\psi)'(t_0) = 0 \qquad \text{and} \qquad -\psi''(t_0) - t_0\psi'''(t_0) = (\mathcal{A}_1\psi)''(t_0) \le 0,$$

which implies that $t_0 = 0$ or that $\psi''(t_0) = 0$. In the latter case, we obtain (1.5.18) again. In the case where $t_0 = 0$, we obtain that $-\psi''(t_0) \le 0$, which also yields (1.5.18). Therefore

$$\mathcal{A}_1\psi(t_0) = \overline{\Box}_n\psi(t_0) - \tfrac{1}{n-1}(1-t_0^2)\psi''(t_0) \le \overline{\Box}_n\psi(t_0),$$

which proves (1.5.17). $\square$

Lemma 1.5.15. *For every $k \ge 2$,*

$$\min_{[-1,1]} \mathcal{A}_1 P_k^n = \mathcal{A}_1 P_k^n(1) = -\frac{(k-1)(k+n-1)}{n-1}. \tag{1.5.19}$$

Proof. According to (1.5.5), the function $a_k^n[\Box_n]^{-1}\mathcal{A}_1 P_k^n$ is a convex combination of the two Legendre polynomials P_{k-2}^{n+2} and P_k^{n+2}. They both have 1 as their maximum value on $[-1,1]$ and they both attain it at $t_0 = 1$. Therefore, this must also be the case for $a_k^n[\Box_n]^{-1}\mathcal{A}_1 P_k^n$, which proves (1.5.19). $\square$

We now define a family of polynomials that turns out to be instrumental in the following.

Definition 1.5.16. For $1 \leq i \leq n-1$, $k \geq 0$ and $k \neq 1$, we define

$$Q^n_{k,i} = P^n_k + \frac{n-1-i}{(k-1)(k+n-1)} \mathcal{A}_1 P^n_k. \tag{1.5.20}$$

Observe that for $i = n-1$, the polynomial $Q^n_{k,n-1}$ is the classical Legendre polynomial P^n_k. Denote the extrema of $Q^n_{k,i}$ on the interval $[-1,1]$ by

$$m^n_{k,i} = \min_{[-1,1]} Q^n_{k,i} \qquad \text{and} \qquad M^n_{k,i} = \max_{[-1,1]} Q^n_{k,i}.$$

The following lemma about the minima $m^n_{k,i}$ is why we require k to be even.

Lemma 1.5.17. *Let $k \geq 2$ be even. Then the sequence $(m^n_{k,i})_{i=1}^{n-1}$ is strictly increasing, that is,*

$$m^n_{k,1} < m^n_{k,2} < \cdots < m^n_{k,n-1}. \tag{1.5.21}$$

Proof. For fixed even $k \geq 2$, define a family $(\eta_t)_{t\in[-1,1]}$ of affine functions by

$$\eta_t(s) = P^n_k(t) + s\mathcal{A}_1 P^n_k(t)$$

and observe that it suffices to show that the function η defined by

$$\eta(s) = \min_{t\in[-1,1]} \eta_t(s) = \min_{[-1,1]} \{P^n_k + s\mathcal{A}_1 P^n_k\}$$

is strictly decreasing on $[0,\infty)$.

To that end, note that as the point-wise minimum of a family of affine functions, η is a concave function. Next, note that since P^n_k is an even Legendre polynomial, it is minimized in the interior of $[-1,1]$, that is, there exists $t_0 \in (-1,1)$ such that

$$\eta(0) = \min_{[-1,1]} P^n_k = P^n_k(t_0) < 0.$$

Moreover, $\frac{d}{dt} P^n_k(t_0) = 0$, so for every $s > 0$,

$$\eta(s) \leq \eta_{t_0}(s) = P^n_k(t_0) + s\mathcal{A}_1 P^n_k(t_0) = (1+s)P^n_k(t_0) < P^n_k(t_0) = \eta(0).$$

Since η is concave, this implies that η is strictly decreasing on $[0,\infty)$, which completes the proof. $\square$

The following two propositions are an extension of [88, Proposition 5.4].

Proposition 1.5.18. *For $1 \leq i \leq n-1$ and $k \geq 2$, denote by $J^n_{k,i}$ the set of all $\lambda \in \mathbb{R}$ for which $1 + \lambda P^n_k(\langle \bar{e}, \cdot\rangle)$ is the density of the i-th area measure of a smooth convex body of revolution. If $1 \leq i < n-1$ and $k \geq 2$ is even, then*

$$\left(-\frac{i}{(n-1)M^n_{k,i}}, -\frac{i}{(n-1)m^n_{k,i}}\right) \subseteq J^n_{k,i} \subseteq \left[-\frac{i}{(n-1)M^n_{k,i}}, -\frac{i}{(n-1)m^n_{k,i}}\right]. \tag{1.5.22}$$

Moreover, if $i = n-1$, then the interval on the right hand side of (1.5.22) is precisely the set of all $\lambda \in \mathbb{R}$ for which $1 + \lambda P^n_k(\langle \bar{e}, \cdot\rangle)$ is the density of the surface area measure of a convex body of revolution.

Proof. To simplify notation, all minima and maxima in this proof refer to the interval $[-1,1]$. Lemma 1.5.13 shows that $\lambda \in J_{k,i}^n$ if and only if

$$1+\lambda P_k^n(t) > \frac{n-1-i}{n-1} - \lambda\frac{n-1-i}{(k-1)(k+n-1)}\mathcal{A}_1 P_k^n(t) > 0 \tag{1.5.23}$$

for all $t \in (-1,1)$. An easy rearrangement of (1.5.23) implies the right hand set inclusion in (1.5.22). For the other set inclusion, let $\lambda \in \mathbb{R}$ and suppose that

$$-\frac{i}{(n-1)M_{k,i}^n} < \lambda < -\frac{i}{(n-1)m_{k,i}^n}. \tag{1.5.24}$$

Due to (1.5.19), we have that

$$\frac{(k-1)(k+n-1)}{(n-1)\min \mathcal{A}_1 P_k^n} = -1 = -\frac{i}{(n-1)Q_{k,i}^n(1)} \leq -\frac{i}{(n-1)M_{k,i}^n} < \lambda. \tag{1.5.25}$$

Moreover, (1.5.21) combined with (1.5.17) applied to P_k^n yields

$$\lambda < -\frac{i}{(n-1)m_{k,i}^n} \leq -\frac{1}{m_{k,i}^n} < -\frac{1}{m_{k,n-1}^n} = \frac{(k-1)(k+n-1)}{(n-1)\max \overline{\square}_n P_k^n} \leq \frac{(k-1)(k+n-1)}{(n-1)\max \mathcal{A}_1 P_k^n}. \tag{1.5.26}$$

Finally, observe that (1.5.24), (1.5.25), and (1.5.26) jointly imply (1.5.23), thus $\lambda \in J_{k,i}^n$. This shows the left hand set inclusion in (1.5.22).

For the second part of the proposition, observe that λ lies in the interval on the right hand side of (1.5.22) precisely when $1+\lambda P_k^n(\langle \bar{e},\cdot\rangle) \geq 0$. According to Minkowski's existence theorem (see, e.g., [96, p. 455]), this is the case if and only if $1+\lambda P_k^n(\langle \bar{e},\cdot\rangle)$ is the density of the surface area measure of a convex body. $\square$

Proposition 1.5.19. *Let $k \geq 2$ be even and I_k^n denote the set of all $\lambda \in \mathbb{R}$ for which $1+\lambda P_k^n(\langle \bar{e},\cdot\rangle)$ is the support function of a convex body of revolution K_λ. Then*

$$I_k^n = \left[-\frac{1}{(k-1)(k+n-1)m_{k,1}^n}, -\frac{1}{(k-1)(k+n-1)M_{k,1}^n}\right]. \tag{1.5.27}$$

Proof. Denote the interval on the right hand side of (1.5.27) by $\tilde{I}_k^n$. Since the space of support functions is a closed convex cone of $C(\mathbb{S}^{n-1})$, the set I_k^n must be a closed interval. Recall that $S_1(K,\cdot) = \square_n h(K,\cdot)$ for every convex body $K \in \mathcal{K}^n$. Hence, (1.5.22) shows that

$$I_k^n \supseteq a_k^n[\square_n]^{-1}\,\mathrm{cl}(J_{k,1}^n) = \tilde{I}_k^n,$$

where cl denotes the closure. For the converse set inclusion, let $\lambda \in I_k^n$. Then $D^2 h_{K_\lambda}(u) = P_{u^\perp} + \lambda D^2 \breve{P}_k^n(u)$ is positive semidefinite for all $u \in \mathbb{S}^{n-1}$. This implies that for every $\varepsilon > 0$, the matrix $P_{u^\perp} + (1-\varepsilon)\lambda D^2 \breve{P}_k^n(u)$ is positive definite for all $u \in \mathbb{S}^{n-1}$. Therefore, $1+(1-\varepsilon)\lambda \breve{P}_k^n$ is the support function of a convex body of revolution which is of class C_+^∞, and thus, strictly convex (see, e.g., [96, Section 2.5]). Hence, (1.5.22) implies that $(1-\varepsilon)\lambda \in \tilde{I}_k^n$ for all $\varepsilon > 0$, and thus, $\lambda \in \tilde{I}_k^n$. $\square$

We are now in a position to prove the main result of this section.

Theorem 1.5.20. *Let $1 < i \le n-1$ and $\Phi_i \in \mathbf{MVal}_i$ be non-trivial. Then its generating function f satisfies for all even $k \ge 2$,*

$$\frac{a_k^n[\Box_n f]}{a_0^n[\Box_n f]} < \frac{1}{i}.$$

Proof. First, observe that for every convex body $K \in \mathcal{K}^n$,

$$S_i(K,\cdot) * \Box_n f = \Box_n(S_i(K,\cdot) * f) = \Box_n h(\Phi_i K,\cdot) = S_1(\Phi_i K,\cdot),$$

thus the convolution transform $\mathrm{T}_{\Box_n f}$ maps i-th order area measures to first order area measures. Moreover, for every $\lambda \in \mathbb{R}$,

$$(1 + \lambda P_k^n(\langle \bar{e},\cdot\rangle)) * \Box_n f = a_0^n[\Box_n f] + \lambda a_k^n[\Box_n f] P_k^n(\langle \bar{e},\cdot\rangle).$$

Hence, we obtain that

$$\frac{a_k^n[\Box_n f]}{a_0^n[\Box_n f]} J_{k,i}^n \subseteq I_k^n.$$

The descriptions of the intervals $J_{k,i}^n$ and I_k^n given in (1.5.22) and (1.5.27) imply that

$$\frac{a_k^n[\Box_n f]}{a_0^n[\Box_n f]} \le \frac{1}{i}\frac{m_{k,i}^n}{m_{k,1}^n} < \frac{1}{i},$$

where the strict inequality is due to (1.5.21). $\Box$

Combining Theorem 1.5.20 with Theorem 1.5.12, we obtain the following.

Corollary 1.5.21. *Let $1 < i \le n-1$ and $\Phi_i \in \mathbf{MVal}_i$ be weakly monotone and even. Then there exists a C^2 neighborhood of B^n where the only fixed points of Φ_i are Euclidean balls.*

Remark 1.5.22. Computational simulations suggest that for every $1 \le i \le n-1$ and $k \ge 2$, the maximum of $Q_{k,i}^n$ on $[-1,1]$ is attained in $t = 1$, that is,

$$M_{k,i}^n = Q_{k,i}^n(1) = \frac{i}{n-1}. \tag{1.5.28}$$

As an immediate consequence, the intervals in (1.5.22) could be simplified.

Computational simulations also suggest that for $1 \le i \le n-1$ and for every even $k \ge 4$,

$$-\frac{1}{n-1} < m_{k,i}^n. \tag{1.5.29}$$

If both (1.5.28) and (1.5.29) were shown to be true, then the argument in the proof of Theorem 1.5.20 would immediately imply that whenever $1 < i \le n-1$ and $\Phi_i \in \mathbf{MVal}_i$ is non-trivial with generating function f, then for all even $k \ge 4$,

$$-\frac{1}{i} < \frac{a_k^n[\Box_n f]}{a_0^n[\Box_n f]}.$$

1.A Appendix

Proof of Lemma 1.3.3. We may assume that $(1-t^2)^{\frac{\beta}{2}} g'(t)$ is a positive measure: all statements of the lemma follow from this case by the Jordan decomposition theorem and linearity. Thus g' itself is a locally finite positive measure on $(-1,1)$, so there exists some constant $c \in \mathbb{R}$ such that for almost all $t \in (-1,1)$,

$$g(t) = \begin{cases} c - g'((t,0]), & t < 0, \\ c + g'((0,t]), & t \geq 0. \end{cases}$$

We may assume that $c = g(0) = 0$. Since g is an increasing function, we have that $g \leq 0$ on $(-1,0]$ and $g \geq 0$ on $[0,1)$.

For $0 < a < 1$, Lebesgue-Stieltjes integration by parts yields

$$\begin{aligned} \beta \int_{(-a,a]} t(1-t^2)^{\frac{\beta-2}{2}} g(t)dt &= \int_{(-a,a]} (1-t^2)^{\frac{\beta}{2}} g'(dt) - g'((-a,a])(1-a^2)^{\frac{\beta}{2}} \\ &\leq \int_{(-a,a]} (1-t^2)^{\frac{\beta}{2}} g'(dt). \end{aligned}$$

By passing to the limit $a \to 1^-$ and applying the monotone convergence theorem, we obtain that $(1-t^2)^{\frac{\beta-2}{2}} g(t)$ is integrable on $(-1,1)$.

For the second part of the lemma, note that since g is increasing,

$$g(a)(1-a^2)^{\frac{\beta}{2}} = \beta g(a) \int_{(a,1)} t(1-t^2)^{\frac{\beta-2}{2}} dt \leq \beta \int_{(a,1)} g(t) t (1-t^2)^{\frac{\beta-2}{2}} dt$$

and the right hand side tends to zero as a tends to 1. An analogous argument applies to $-a$, thus

$$\lim_{a\to 1^-} g(a)(1-a^2)^{\frac{\beta}{2}} = \lim_{a \to 1^-} g(-a)(1-a^2)^{\frac{\beta}{2}} = 0.$$

Suppose now that ψ is as stated above. For $0 < a < 1$, Lebesgue-Stieltjes integration by parts yields

$$\int_{(-a,a]} \psi(t) g'(dt) + \int_{(-a,a]} \psi'(t) g(t) dt = (g(a) - g(-a))(\psi(a) - \psi(-a)).$$

Due to our assumptions on ψ, the right hand hand side tends to zero as a tends to 1. Thus, by passing to the limit $a \to 1^-$ and applying the dominated convergence theorem, we obtain (1.3.5). $\square$

Proof of Lemma 1.3.4. First, fix v, α and β and observe that it suffices to find a family of bounded linear operators $D_k : C^k(\mathbb{S}^{n-1} \backslash \{\pm v\}) \to C(\mathbb{S}^{n-1} \backslash \{\pm v\})$ such that for every $\phi \in C^\infty(\mathbb{S}^{n-1} \backslash \{\pm v\})$ and $w \in v^\perp$,

$$\frac{d^k}{dt^k} \mathsf{J}_v[\langle \cdot, w\rangle^\alpha (1 - \langle \cdot, v\rangle^2)^{\frac{\beta}{2}} \phi](t) = (1-t^2)^{-k} \mathsf{J}_v[\langle \cdot, w\rangle^\alpha (1-\langle \cdot, v\rangle^2)^{\frac{\beta}{2}} D_k \phi](t).$$

We will construct this family D_k inductively, starting with $D_0 = \mathrm{Id}$.

For the induction step, define a first order differential operator $\tilde{D}_k$ by

$$\tilde{D}_k\phi(u) = \langle \nabla_{\mathbb{S}}\phi(u), P_{u^\perp}v\rangle - 2\left(\tfrac{n-3+\alpha+\beta}{2} - (k-1)\right)\langle u, v\rangle\phi(u).$$

A straightforward computation using spherical cylinder coordinates shows that

$$\begin{aligned}
&\frac{d}{dt}\left((1-t^2)^{-(k-1)}\mathsf{J}_v[\langle\cdot,w\rangle^\alpha(1-\langle\cdot,v\rangle^2)^{\frac{\beta}{2}}\phi](t)\right)\\
&\quad= \frac{d}{dt}\left((1-t^2)^{\frac{n-3+\alpha+\beta}{2}-(k-1)}\int_{\mathbb{S}^{n-1}\cap v}\phi(tv+\sqrt{1-t^2}u)du\right)\\
&\quad= (1-t^2)^{-k}\mathsf{J}_v[\langle\cdot,w\rangle^\alpha(1-\langle\cdot,v\rangle^2)^{\frac{\beta}{2}}\tilde{D}_k\phi](t),
\end{aligned}$$

thus we see that the operators $D_k = \tilde{D}_k\tilde{D}_{k-1}\cdots\tilde{D}_1$ have the desired property. Since every $\tilde{D}_j$ is a bounded linear operator from $C^j(\mathbb{S}^{n-1}\backslash\{\pm v\})$ to $C^{j-1}(\mathbb{S}^{n-1}\backslash\{\pm v\})$, it follows by induction that every D_k is a bounded linear operator from $C^k(\mathbb{S}^{n-1}\backslash\{\pm v\})$ to $C(\mathbb{S}^{n-1}\backslash\{\pm v\})$. ◻

Proof of Lemma 1.4.4. Suppose that $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$ is weakly positive, that is, $\nu = \mu + y$ for some positive measure μ and some linear function y. Then for every positive centered $\phi \in C^\infty(\mathbb{S}^{n-1})$, we have that $\langle\phi,\nu\rangle_{C^{-\infty}} = \langle\phi,\mu+y\rangle_{C^{-\infty}} \geq 0$.

Conversely, suppose that $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$ is not weakly positive. Observe that the set of weakly positive distributions is a closed convex cone of $C^{-\infty}(\mathbb{S}^{n-1})$. Due to the Hahn-Banach separation theorem there exists some $\phi \in C^\infty(\mathbb{S}^{n-1})$ such that

$$\langle\phi,\nu\rangle_{C^{-\infty}} < \langle\phi,\mu+y\rangle_{C^{-\infty}}$$

for every positive measure μ and linear function y. By fixing $\mu = 0$ and varying $y \in \mathcal{H}^n_1$, we see that ϕ is centered. By fixing $y = 0$ and varying $\mu \in \mathcal{M}_+(\mathbb{S}^{n-1})$, we see that $\phi \geq 0$. Finally, by choosing $\mu = y = 0$, we see that $\langle\phi,\nu\rangle_{C^{-\infty}} < 0$. ◻

2 Lefschetz operators on convex valuations

2.1 Introduction

Scalar valued valuations. A *valuation* on the space $\mathcal{K}(\mathbb{R}^n)$ of convex bodies (convex, compact subsets) in $\mathbb{R}^n$ is a functional $\varphi : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ such that

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}(\mathbb{R}^n)$. Valuations have a long history in convex and integral geometry (see, e.g., [3–6, 18, 38, 51, 52, 63, 79]). We denote by **Val** the space of continuous, translation invariant valuations. By a classical result of McMullen [86], the space **Val** is the direct sum of the subspaces $\mathbf{Val}_i$ of valuations that are homogeneous of degree $i \in \{0, \dots, n\}$ (that is, $\varphi(\lambda K) = \lambda^i \varphi(K)$ for all $K \in \mathcal{K}(\mathbb{R}^n)$ and $\lambda \geq 0$).

Motivated by the Hard Lefschetz theorem from Kähler geometry, Alesker [5] introduced *Lefschetz operators* on valuations. For $\varphi \in \mathbf{Val}$ and $K \in \mathcal{K}(\mathbb{R}^n)$,

$$(\Lambda\varphi)(K) = \left.\frac{d}{dt}\right|_{t=0^+} \varphi(K + tB^n) \qquad \text{and} \qquad (\mathfrak{L}\varphi)(K) = \int_{\overline{\mathrm{Gr}}_{n,n-1}} \varphi(K \cap H)\, dH,$$

where B^n is the unit ball in $\mathbb{R}^n$ and $\overline{\mathrm{Gr}}_{n,j}$ is the Grassmann manifold of affine j-dimensional subspaces of $\mathbb{R}^n$, endowed with a (suitably normalized) rigid motion invariant measure. The Lefschetz operators are a powerful tool in valuation theory, since they allow to transfer results between valuations of different degrees (see, e.g., [5, 7, 12, 14, 21, 69–71, 89, 97]). The derivation operator Λ decreases the degree of a valuation by one; the integral operator $\mathfrak{L}$ increases it by one.

In this article, we investigate the action of the Lefschetz operators on two well-known representations of valuations: Klain–Schneider functions for scalar valued valuations and generating functions for Minkowski valuations. Denoting by $\mathrm{Gr}_{n,i}$ the Grassmann manifold of i-dimensional subspaces of $\mathbb{R}^n$, Klain [61] showed that if $\varphi \in \mathbf{Val}_i$ is *even* and $E \in \mathrm{Gr}_{n,i}$, then $\varphi|_E = c_E \mathrm{vol}_E$ for some $c_E \in \mathbb{R}$, where $\varphi|_E$ is the restriction of φ to convex bodies in E. Its *Klain function* $\mathrm{Kl}_\varphi \in C(\mathrm{Gr}_{n,i})$, $\mathrm{Kl}_\varphi : E \mapsto c_E$, uniquely determines $\varphi \in \mathbf{Val}_i$, as was proved by Klain [62].

Schuster and Wannerer [102] showed that for *smooth valuations* (see Section 2.5.1), the Lefschetz operators act on the Klain function by a Radon transform between the different Grassmannians. In the following, κ_k denotes the volume of B^k and $\mathrm{Gr}^E_{n,i} \subseteq \mathrm{Gr}_{n,i}$ denotes the submanifold of spaces that are contained in E or that contain E, depending on the dimension of E. Integration is with respect to the unique probability measure invariant under rotations fixing E.

Theorem ([102])**.** *Let $1 \le i \le n-1$ and $\varphi \in \mathbf{Val}_i$ be smooth and even.*

(i) $\mathrm{Kl}_{\Lambda\varphi}(E) = \dfrac{(n-i+1)\kappa_{n-i+1}}{\kappa_{n-i}} \displaystyle\int_{\mathrm{Gr}^E_{n,i}} \mathrm{Kl}_\varphi(F)dF$ *for* $E \in \mathrm{Gr}_{n,i-1}$*, if* $i > 1$.

(ii) $\mathrm{Kl}_{\mathfrak{L}\varphi}(E) = \dfrac{(i+1)\kappa_{n-1}\kappa_{i+1}}{n\kappa_n\kappa_i} \displaystyle\int_{\mathrm{Gr}^E_{n,i}} \mathrm{Kl}_\varphi(F)dF$ *for* $E \in \mathrm{Gr}_{n,i+1}$*, if* $i < n-1$.

In the odd case, Schneider [95] introduced the natural counterpart to the Klain function, called the *Schneider function*, and showed that it determines the respective valuation uniquely. We combine these representations for the even and odd part of a valuation $\varphi \in \mathbf{Val}_i$ into one common continuous function on the *flag manifold* $\mathrm{Fl}_{n,i+1} = \{(E,u) : E \in \mathrm{Gr}_{n,i+1}, u \in \mathbb{S}^i(E)\}$, where $\mathbb{S}^i(E)$ denotes the unit sphere in E. This function, called its *Klain–Schneider function* and denoted by KS_φ, has the property that for every $E \in \mathrm{Gr}_{n,i+1}$,

$$\varphi(K) = \int_{\mathbb{S}^i(E)} \mathrm{KS}_\varphi(E,u)\, S_i^E(K,du), \qquad K \in \mathcal{K}(E),$$

where $S_i^E(K,\,\cdot\,)$ is the surface area measure of K relative to E (cf. [96, Section 4.2]). This determines $\mathrm{KS}_\varphi(E,\,\cdot\,)$ only up to the addition of linear functions, so for now, we define it as the corresponding equivalence class. By the results of Klain and Schneider, φ is uniquely determined by KS_φ.

With our first main result, we describe the action of the Lefschetz operators on the Klain–Schneider function, generalizing the theorem above. Here, $\cdot\,|E$ denotes the orthogonal projection onto a linear subspace $E \subseteq \mathbb{R}^n$ and $\mathrm{pr}_E u = \|u|E\|^{-1}u|E$ for $u \in \mathbb{S}^{n-1}\setminus E^\perp$.

Theorem 2.A. *Let $1 \le i \le n-1$ and $\varphi \in \mathbf{Val}_i$.*

(i) *If $i > 1$, then for all $(E,u) \in \mathrm{Fl}_{n,i}$,*

$$\mathrm{KS}_{\Lambda\varphi}(E,u) = \frac{(n-i+1)\kappa_{n-i+1}}{\kappa_{n-i}} \int_{\mathrm{Gr}^{E\cap u^\perp}_{n,i}} \mathrm{KS}_\varphi(\mathrm{span}(F\cup u), \mathrm{pr}_{F^\perp}u)dF.$$

(ii) *If $i < n-1$, then for all $(E,u) \in \mathrm{Fl}_{n,i+2}$,*

$$\mathrm{KS}_{\mathfrak{L}\varphi}(E,u) = \frac{(i+2)\kappa_{n-1}\kappa_{i+2}}{n\kappa_n\kappa_{i+1}} \int_{\mathrm{Gr}^E_{n,i+1}} \mathrm{KS}_\varphi(F, \mathrm{pr}_F u)\|u|F\|dF.$$

In fact, our results extend to more general Lefschetz operators: we can describe the Alesker product and the Bernig–Fu convolution with an even valuation of degree and codegree one, respectively (see Remarks 2.3.5 and 2.3.7).

We also want to emphasize that our approach is very different from that in [102]. Our proof of (ii) only uses simple geometric properties of polytopes. For (i), we reduce the general case to valuations of the form $\varphi(K) = V(K^{[i]},\mathcal{C})$ defined in terms of a mixed volume with a family $\mathcal{C} = (C_1,\dots,C_{n-i})$ of fixed reference bodies. Indeed, as a consequence of his irreducibility theorem [4], Alesker proved that such valuations are dense in $\mathbf{Val}_i$, confirming a conjecture by McMullen [86] (this was recently refined by Knoerr [66] for smooth valuations).

The computation of the Klain–Schneider function of a mixed volume boils down to a relation between mixed area measures and surface are measures relative to a subspace. In order to establish the required relation, we generalize the *spherical projections and liftings* that were introduced by Goodey, Kiderlen, and Weil [44].

Theorem 2.B. *Let $0 \leq i \leq n-1$, $E \in \mathrm{Gr}_{n,i+1}$, and $\mathcal{C} = (C_1, \dots, C_{n-i-1})$ be a family of convex bodies with C^2 support functions. Then for all $K \in \mathcal{K}(E)$,*

$$S(K^{[i]}, \mathcal{C}, \,\cdot\,) = \frac{1}{\binom{n-1}{i}} \pi^*_{E,\mathcal{C}} S^E_i(K, \,\cdot\,). \tag{2.1.1}$$

Here, $S(K^{[i]}, \mathcal{C}, \,\cdot\,)$ denotes the mixed area measure and $\pi^*_{E,\mathcal{C}}$ denotes the $\mathcal{C}$-*mixed spherical lifting*, which we define in Section 2.2 as a linear operator mapping measures on $\mathbb{S}^i(E)$ to measures on $\mathbb{S}^{n-1}$. The special case where the reference bodies are balls was treated in [44].

Minkowski valuations. We now turn to valuations that are convex body valued: A *Minkowski valuation* is a map $\Phi : \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ such that

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi K + \Phi L$$

whenever $K, L, K \cup L \in \mathcal{K}(\mathbb{R}^n)$, where addition on $\mathcal{K}(\mathbb{R}^n)$ is the usual Minkowski addition. In the last two decades, starting with the seminal work of Ludwig [75, 76], considerable advances in the theory of Minkowski valuations have been made, including classification results and isoperimetric type inequalities (see, e.g., [12, 27, 50, 53, 77, 87–89, 97]). We denote by **MVal** the space of continuous, translation invariant Minkowski valuations that intertwine all rotations and by $\mathbf{MVal}_i$ the subspace of i-homogeneous Minkowski valuations.

Through the correspondence between a convex body $K \in \mathcal{K}(\mathbb{R}^n)$ and its support function $h_K \in C(\mathbb{S}^{n-1})$, $h_K(u) = \max_{x \in K} \langle u, x \rangle$ with the Euclidean inner product $\langle \,\cdot\,, \,\cdot\, \rangle$, it is natural to define the Lefschetz operators on the space **MVal** by

$$h_{(\Lambda\Phi)(K)} = \frac{d}{dt}\Big|_{t=0^+} h_{\Phi(K+tB^n)} \qquad \text{and} \qquad h_{(\mathfrak{L}\Phi)(K)} = \int_{\mathrm{Gr}_{n,n-1}} h_{\Phi(K \cap H)} \, dH,$$

where these identities are to be understood pointwise. The integration operator $\mathfrak{L}$ clearly preserves convexity, whereas the non-trivial fact that the derivation operator Λ is well-defined on **MVal** is due to Parapatits and Schuster [89].

Every $\Phi \in \mathbf{MVal}$ is determined by its *associated real valued valuation*, which is the $\mathrm{SO}(n-1)$-invariant valuation $\varphi \in \mathbf{Val}$ defined by $\varphi(K) = h_{\Phi K}(e_n)$, where e_n is the north pole of $\mathbb{S}^{n-1}$. In this sense, Minkowski valuations present a special case of real valued valuations and in addition to the Klain–Schneider function, we have a more specific representation available involving the *spherical convolution* (see Appendix 2.A) and the i-th order area measure $S_i(K, \,\cdot\,) = S(K^{[i]}, B^{n[n-i-1]}, \,\cdot\,)$.

Theorem ([36, 103]). *Let $1 \leq i \leq n-1$ and $\Phi \in \mathbf{MVal}_i$. Then there exists a unique centered,* $\mathrm{SO}(n-1)$*-invariant function $f_\Phi \in L^1(\mathbb{S}^{n-1})$ such that*

$$h_{\Phi K} = S_i(K, \cdot\,) * f_\Phi, \qquad K \in \mathcal{K}(\mathbb{R}^n). \tag{2.1.2}$$

The function f_Φ is called the *generating function* of Φ, and by *centered*, we mean that f_Φ is orthogonal to all linear functions. The representation above was first established by Schuster and Wannerer [103] for a measure f_Φ and then improved by Dorrek [36] to $f_\Phi \in L^1(\mathbb{S}^{n-1})$. More recently, the first- and third-named author [27] refined this further by showing that f_Φ is in fact locally Lipschitz continuous outside the poles $\pm e_n$.

The action of the Lefschetz derivation operator Λ on the generating function is a simple multiplication by a constant, which is a direct consequence of the Steiner formula for area measures. For the integration operator $\mathfrak{L}$ however, things are far more intricate: For $\Phi \in \mathbf{MVal}_i$,

$$f_{\Lambda\Phi} = i f_\Phi \qquad \text{and} \qquad f_{\mathfrak{L}\Phi} = f_\Phi * \rho_i \tag{2.1.3}$$

with some unique $\mathrm{SO}(n-1)$-invariant, centered distribution ρ_i on $\mathbb{S}^{n-1}$, as was shown in [103]. Since this is a purely structural statement, describing the action of the Lefschetz integration operator $\mathfrak{L}$ on Minkowski valuations boils down to describing the distribution ρ_i. This is the content of our second main result.

Theorem 2.C. *Let $1 \leq i < n-1$. Then ρ_i is an $L^1(\mathbb{S}^{n-1})$ function that is smooth on $\mathbb{S}^{n-1} \setminus \{e_n\}$ and strictly positive up to the addition of a linear function. Moreover, the function $\bar{\rho}_i$, defined by $\rho_i(u) = \bar{\rho}_i(\langle e_n, u\rangle)$, satisfies for all $t \in (-1,1)$,*

$$(1-t^2)\frac{d^2}{dt^2}\bar{\rho}_i(t) - nt\frac{d}{dt}\bar{\rho}_i(t) - i(n-i-1)\bar{\rho}_i(t) = 0. \tag{2.1.4}$$

In particular, we obtain a representation of ρ_i in terms of a hypergeometric function. The positivity of ρ_i has several notable consequences. Whenever Φ is generated by the support function of a body of revolution, then so is $\mathfrak{L}\Phi$. This answers a question of Schuster [100]. Moreover, the combined Lefschetz operators $\Lambda \circ \mathfrak{L}$ and $\mathfrak{L} \circ \Lambda$ act on Minkowski valuations as a composition with some *Minkowski endomorphism*, that is, a Minkowski valuation in $\mathbf{MVal}_1$.

Corollary. *Let $1 \leq i < n-1$. There exists $\Psi^{(i)} \in \mathbf{MVal}_1$ such that*

$$\Lambda(\mathfrak{L}\Phi_i) = \Psi^{(i)} \circ \Phi_i \qquad \text{and} \qquad \mathfrak{L}(\Lambda\Phi_{i+1}) = \Psi^{(i)} \circ \Phi_{i+1}$$

for every $\Phi_i \in \mathbf{MVal}_i$ and $\Phi_{i+1} \in \mathbf{MVal}_{i+1}$.

Subsequently, the examples of Minkowski valuations that are currently known are preserved under the Lefschetz operators (see Sections 2.4.2 and 2.5.3). Moreover, we want to point out that the notion of generating functions and the validity of (2.1.3) extends to the much larger class of *spherical valuations* (see Section 2.5.1), and thus, so does the scope of Theorem 2.C.

Organization of the article In Section 2.2, we introduce mixed spherical projections and liftings and prove Theorem 2.B. We apply this in Section 2.3 to the Klain–Schneider function of scalar valued valuations; in there, we show Theorem 2.A. Then, in Section 2.4 we turn to Minkowski valuations, discussing the structure of the space **MVal** and making some key preparations. Section 2.5 is devoted to the action of the Lefschetz operators on generating functions; in there, we prove Theorem 2.C and its consequences. Finally, in Section 2.6, we discuss the connection between generating functions and Klain–Schneider functions.

2.2 Mixed spherical projections and liftings

In this section, we recall weighted spherical projections and lifting and introduce a mixed version of them with the main goal of proving Theorem 2.B. To that end, we need to fix some notation: For $E \in \mathrm{Gr}_{n,k}$,

$$\mathbb{H}^{n-k}(E,u) = \{v \in \mathbb{S}^{n-1} \setminus E^\perp : \mathrm{pr}_E v = u\}$$

denote the relatively open $(n-k)$-dimensional half-sphere generated by $E^\perp$ and $u \in \mathbb{S}^{k-1}(E)$. Note that $\mathbb{H}^{n-k}(E,u) = \mathbb{S}^{n-k}(E^\perp \vee u) \cap u^+$, where we denote by $E' \vee u = \mathrm{span}(E' \cup u)$ the subspace generated by E' and $u \in \mathbb{S}^{n-1}$, and by $u^+ = \{x \in \mathbb{R}^n : \langle x, u\rangle > 0\}$ the open half-space in the direction of $u \in \mathbb{S}^{n-1}$. Throughout, integration on j-dimensional spheres or half-spheres, unless indicated otherwise, is with respect to the j-dimensional Hausdorff measure $\mathcal{H}^j$.

Now we want to recall the weighted spherical liftings and projections that were introduced by Goodey, Kiderlen, and Weil [44]. These prove to be helpful when it comes to relating the geometry of convex bodies in a subspace to their geometry relative to the ambient space. In the following, $\mathcal{M}(\mathbb{S}^{n-1})$ denotes the space of finite signed measures on $\mathbb{S}^{n-1}$.

Definition 2.2.1 ([44])**.** Let $1 \le k \le n$, $E \in \mathrm{Gr}_{n,k}$, and $m > -k$. The *m-weighted spherical projection* is the bounded linear operator

$$\pi_{E,m} : C(\mathbb{S}^{n-1}) \to C(\mathbb{S}^{k-1}(E)),$$
$$(\pi_{E,m} f)(u) = \int_{\mathbb{H}^{n-k}(E,u)} f(v)\langle u, v\rangle^{k+m-1} dv, \qquad u \in \mathbb{S}^{k-1}(E).$$

The *m-weighted spherical lifting* is its adjoint operator

$$\pi^*_{E,m} : \mathcal{M}(\mathbb{S}^{k-1}(E)) \to \mathcal{M}(\mathbb{S}^{n-1}).$$

If $m > 0$, then $(\pi^*_{E,m} f)(u) = \|u|E\|^m f(\mathrm{pr}_E u)$ for $f \in C(\mathbb{S}^{k-1}(E))$ and $u \in \mathbb{S}^{n-1}$. Hence, $\pi^*_{E,m}$ restricts to a bounded linear operator $C(\mathbb{S}^{k-1}(E)) \to C(\mathbb{S}^{n-1})$, and by duality, $\pi_{E,m}$ naturally extends to an operator $\mathcal{M}(\mathbb{S}^{n-1}) \to \mathcal{M}(\mathbb{S}^{k-1}(E))$.

Goodey, Kiderlen, and Weil [44] showed the following formula for convex bodies of a subspace, expressing its i-th order area measure relative to the ambient space in terms of the one relative to the subspace. As a general reference for mixed area measures and volumes, we cite [96, Chapters 4 and 5].

Theorem 2.2.2 ([44, Theorem 6.2])**.** *Let $1 \le i < k < n$ and $E \in \mathrm{Gr}_{n,k}$. Then for all $K \in \mathcal{K}(E)$,*

$$S_i(K, \cdot\,) = \frac{\binom{k-1}{i}}{\binom{n-1}{i}} \pi^*_{E,-i} S_i^E(K, \cdot\,). \tag{2.2.1}$$

For our purposes, the relevant instance of this is when $i = k-1$ and the measure $S_i^E(K, \cdot\,)$ is the surface area measure of K relative E. In this instance, we propose a generalization of Definition 2.2.1. Here and in the following, the notational convention $(L_1, \dots, L_i)|E = (L_1|E, \dots, L_i|E)$ where $L_1, \dots, L_i \in \mathcal{K}(\mathbb{R}^n)$ and $E \subseteq \mathbb{R}^n$ is a linear subspace, will be used frequently.

Definition 2.2.3. Let $1 \le k \le n$ and $E \in \mathrm{Gr}_{n,k}$. Also, let $C_1, \dots, C_{n-k} \in \mathcal{K}(\mathbb{R}^n)$ and set $\mathcal{C} = (C_1, \dots, C_{n-k})$. The $\mathcal{C}$*-mixed spherical projection* is the bounded linear operator

$$\pi_{E,\mathcal{C}} : C(\mathbb{S}^{n-1}) \to C(\mathbb{S}^{k-1}(E)),$$
$$(\pi_{E,\mathcal{C}} f)(u) = \int_{\mathbb{H}^{n-k}(E,u)} f(v)\; S^{E^\perp \vee u}(\mathcal{C}|(E^\perp \vee u), dv), \qquad u \in \mathbb{S}^{k-1}(E).$$

The $\mathcal{C}$*-mixed spherical lifting* is its adjoint operator

$$\pi^*_{E,\mathcal{C}} : \mathcal{M}(\mathbb{S}^{k-1}(E)) \to \mathcal{M}(\mathbb{S}^{n-1}).$$

In the instance where $k = n$ and the family $\mathcal{C}$ is empty, the above is to be understood as $\pi_{E,\mathcal{C}} = \mathrm{Id}$ and $\pi^*_{E,\mathcal{C}} = \mathrm{Id}$. Next, as an intermediate step, we prove a polytopal version of Theorem 2.B. The following reduction theorem for mixed volumes will be a key ingredient.

Theorem 2.2.4 ([96, Theorem 5.3.1])**.** *Let $1 \le k < n$ and $E \in \mathrm{Gr}_{n,k}$. For all convex bodies $K \in \mathcal{K}(E)$ and $C_1, \dots, C_{n-k} \in \mathcal{K}(\mathbb{R}^n)$,*

$$V(K^{[k]}, C_1, \dots, C_{n-k}) = \frac{1}{\binom{n}{k}} V_k(K) V^{E^\perp}(C_1|E^\perp, \dots, C_{n-k}|E^\perp), \tag{2.2.2}$$

where $V^{E^\perp}$ denotes the mixed volume relative to the subspace $E^\perp$.

Theorem 2.2.5. *Let $1 \le i < n-1$, $E \in \mathrm{Gr}_{n,i+1}$, and $\mathcal{Q} = (Q_1, \dots, Q_{n-i-1})$ be a family of polytopes in $\mathbb{R}^n$. Then for every polytope $P \in \mathcal{K}(E)$,*

$$\mathbb{1}_{\mathbb{S}^{n-1} \setminus E^\perp} S(P^{[i]}, \mathcal{Q}, \cdot\,) = \frac{1}{\binom{n-1}{i}} \pi^*_{E,\mathcal{Q}} S_i^E(P, \cdot\,). \tag{2.2.3}$$

Proof. First, observe that both sides of (2.2.3) are multilinear in the reference polytopes $Q_1, \dots, Q_{n-i-1}$. Thus, by polarization, it suffices to consider the case where $Q_1 = \dots = Q_{n-i-1} = Q$ for some polytope $Q \in \mathcal{K}(\mathbb{R}^n)$.

The mixed area measure of polytopes $P_1, \dots, P_{n-1} \in \mathcal{K}(\mathbb{R}^n)$ can be written as

$$S(P_1, \dots, P_{n-1}, \,\cdot\,) = \sum_{u \in \mathbb{S}^{n-1}} V^{u^\perp}(F(P_1, u), \dots, F(P_{n-1}, u))\delta_u, \tag{2.2.4}$$

where $V^{u^\perp}$ denotes the mixed volume relative to $u^\perp$ and $F(P_j, u)$ denotes the *support set* of P_j with outer unit normal $u \in \mathbb{S}^{n-1}$ (cf. [96, (5.22)]). Note that the sum in fact extends only over the outer unit normals of the facets of $P_1 + \dots + P_{n-1}$, and thus, is a finite sum. By (2.2.4), we have that for every $f \in C(\mathbb{S}^{n-1})$,

$$\begin{aligned}
\int_{\mathbb{S}^{n-1} \setminus E^\perp} f(u) S(P^{[i]}, Q^{[n-i-1]}, du) &= \sum_{u \in \mathbb{S}^{n-1} \setminus E^\perp} f(u) V^{u^\perp}(F(P, u)^{[i]}, F(Q, u)^{[n-i-1]}) \\
&= \frac{1}{\binom{n-1}{i}} \sum_{u \in \mathbb{S}^{n-1} \setminus E^\perp} f(u) V_i(F(P, u)) V_{n-i-1}(F(Q, u) | (E \cap u^\perp)^{\perp_{(u^\perp)}}),
\end{aligned}$$

where in the second equality, we applied reduction formula (2.2.2) relative to $u^\perp$, using the fact that $E \cap u^\perp \in \mathrm{Gr}_{n,i}^{u^\perp}$ for $u \in \mathbb{S}^{n-1} \setminus E^\perp$, and denoting by $(E \cap u^\perp)^{\perp_{(u^\perp)}}$ the orthogonal complement of $E \cap u^\perp$ relative to $u^\perp$.

Every $x \in \mathbb{R}^n$ can be written as the orthogonal sum $x = x|(E \cap u^\perp) + x|(E^\perp \vee u)$; by choosing $x \in u^\perp$, this yields $x|(E \cap u^\perp)^{\perp_{(u^\perp)}} = x|(E^\perp \vee u)$. Thus,

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1} \setminus E^\perp} f(u) S(P^{[i]}, Q^{[n-i-1]}, du) \\
&\qquad = \frac{1}{\binom{n-1}{i}} \sum_{u \in \mathbb{S}^{n-1} \setminus E^\perp} f(u) V_i(F(P, u)) V_{n-i-1}(F(Q, u) | E^\perp \vee u) \\
&\qquad = \frac{1}{\binom{n-1}{i}} \sum_{u \in \mathbb{S}^i(E)} V_i(F(P, u)) \sum_{v \in \mathbb{H}^{n-i-1}(E, u)} f(v) V_{n-i-1}(F(Q, v) | (E^\perp \vee u)),
\end{aligned} \tag{2.2.5}$$

where in the second equality, we used that for every $v \in \mathbb{S}^{n-1} \setminus E^\perp$, there exists a unique $u \in \mathbb{S}^i(E)$ such that $v \in \mathbb{H}^{n-i-1}(E, u)$; moreover, $F(P, v) = F(P, u)$ and $E^\perp \vee v = E^\perp \vee u$. Next, note that an instance of (2.2.4),

$$(\pi_{E,Q} f)(u) = \sum_{v \in \mathbb{H}^{n-i-1}(E, u)} f(v) V_{n-i-1}(F(Q | (E^\perp \vee u), v)). \tag{2.2.6}$$

Note that $F(C, v)|E' = F(C|E', v)$ for every convex body $C \in \mathcal{K}(\mathbb{R}^n)$, subspace $E' \subseteq \mathbb{R}^n$, and direction $v \in E'$. Hence, plugging (2.2.6) into (2.2.5) yields

$$\begin{aligned}
\int_{\mathbb{S}^{n-1} \setminus E^\perp} f(u) S(P^{[i]}, Q^{[n-i-1]}, du) &= \frac{1}{\binom{n-1}{i}} \sum_{u \in \mathbb{S}^i(E)} (\pi_{E,Q} f)(u) V_i(F(P, u)) \\
&= \frac{1}{\binom{n-1}{i}} \int_{\mathbb{S}^i(E)} (\pi_{E,Q} f)(u) S_i^E(P, du) = \frac{1}{\binom{n-1}{i}} \int_{\mathbb{S}^{n-1}} f(u) \, [\pi_{E,Q}^* S_i^E(P, \,\cdot\,)](du),
\end{aligned}$$

where the second equality is another instance of (2.2.4). Since $f \in C(\mathbb{S}^{n-1})$ was arbitrary, this yields $\mathbb{1}_{\mathbb{S}^{n-1} \setminus E^\perp} S(P^{[i]}, Q^{[n-i-1]}, \,\cdot\,) = 1/\binom{n-1}{i} \pi_{E,Q}^* S_i^E(P, \,\cdot\,)$. As was noted at the beginning of the proof, this shows the theorem. $\square$

In the theorem above, we have deliberately avoided the set $\mathbb{S}^{n-1} \cap E^\perp$. This is because in the polytopal case, $S(P^{[i]}, \mathcal{Q}, \cdot\,)$ assigns positive mass to this set which can not be captured by the mixed spherical lifting. However, if we replace the polytopes Q_j by smooth bodies C_j, then this is no longer the case, as the following lemma shows.

Lemma 2.2.6. *Let $0 \le i < n-1$ and $\mathcal{C} = (C_1, \dots, C_{n-i-1})$ be a family of convex bodies with C^2 support functions. Then for all $K \in \mathcal{K}(\mathbb{R}^n)$, the mixed area measure $S(K^{[i]}, \mathcal{C}, \cdot\,)$ is absolutely continuous with respect to $\mathcal{H}^{n-i-1}$.*

Proof. First, note that the i-th area measure of K is absolutely continuous with respect to $\mathcal{H}^{n-i-1}$ (cf. [96, Theorem 4.5.5]). In order to pass to the mixed area measure, recall that if $L_1, \dots, L_{n-1} \in \mathcal{K}(\mathbb{R}^n)$ have C^2 support functions, then

$$S(L_1, \dots, L_{n-1}, du) = \mathsf{D}(D^2 h_{L_1}(u), \dots, D^2 h_{L_{n-1}}(u))du,$$

where $D^2 h_L(u)$ denotes the restriction of the Hessian of the support function h_L (as a one-homogeneous function on $\mathbb{R}^n$) to the hyperplane $u^\perp$ and D denotes the mixed discriminant (cf. [96, (2.68) and (5.48)]). Thus, whenever K also has a C^2 support function,

$$\begin{aligned} S(K^{[i]}, C_1, \dots, C_{n-i-1}, du) &= \mathsf{D}(D^2 h_K(u)^{[i]}, D^2 h_{C_1}(u), \dots, D^2 h_{C_{n-i-1}}(u))du \\ &\le \|h_{C_1}\|_{C^2} \cdots \|h_{C_{n-i-1}}\|_{C^2} \mathsf{D}(D^2 h_K(u)^{[i]}, \mathrm{Id}^{[n-i-1]})du = M S_i(K, du), \end{aligned}$$

where we used the monotonicity of mixed discriminants and $M > 0$ is a constant depending on $C_1, \dots, C_{n-i-1}$ but not on K. The obtained inequality

$$S(K^{[i]}, C_1, \dots, C_{n-i-1}, \cdot\,) \le M S_i(K, \cdot\,),$$

by continuity of the mixed area measure, extends to all convex bodies $K \in \mathcal{K}(\mathbb{R}^n)$, which concludes the proof. ☐

Now we want to pass from the polytopal to the smooth case. To that end, we need the following formulation of the classical Portmanteau theorem that characterizes weak convergence of measures.

Theorem 2.2.7 ([64, Theorem 13.16])**.** *Let μ_m, μ be finite positive measures on a compact metric space X. Then the following are equivalent:*

(a) *$\mu_m \to \mu$ weakly.*
(b) *For every $f \in C(X)$, we have $\lim_{m\to\infty} \int_X f d\mu_m = \int_X f d\mu$.*
(c) *For every bounded, measurable function f on X such that its discontinuity points are a set of μ-measure zero, $\lim_{m\to\infty} \int_X f d\mu_m = \int_X f d\mu$.*

We are now ready to prove Theorem 2.B, which we state here again.

Theorem 2.2.8. *Let $0 \le i < n-1$, $E \in \mathrm{Gr}_{n,i+1}$, and $\mathcal{C} = (C_1, \dots, C_{n-i-1})$ be a family of convex bodies with C^2 support functions. Then for all $K \in \mathcal{K}(E)$,*

$$S(K^{[i]}, \mathcal{C}, \cdot\,) = \frac{1}{\binom{n-1}{i}} \pi^*_{E,\mathcal{C}} S_i^E(K, \cdot\,). \tag{2.2.7}$$

Proof. As in the proof of Theorem 2.2.5, it suffices to consider the case where $C_1 = \cdots = C_{n-i-1} = C$ for some body $C \in \mathcal{K}(\mathbb{R}^n)$ with a C^2 support function. Next, let $K = P \in \mathcal{K}(E)$ be a polytope and let $Q_m \in \mathcal{K}(\mathbb{R}^n)$ be a sequence of polytopes such that $Q_m \to C$ in the Hausdorff metric. By Theorem 2.2.5, for every $f \in C(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} \mathbb{1}_{\mathbb{S}^{n-1}\setminus E^\perp}(u) f(u) S(P^{[i]}, Q_m^{[n-i-1]}, du) = \frac{1}{\binom{n-1}{i}} \int_{\mathbb{S}^i(E)} (\pi_{E,Q_m} f)(u) S_i^E(P, du).$$

Now we want to pass to the limit $m \to \infty$ on both sides. On the left hand side, we have weak convergence of the mixed area measures and according to Lemma 2.2.6, the mixed area measure $S(P^{[i]}, C^{[n-i-1]}, \cdot\,)$ vanishes on $\mathbb{S}^{n-1} \cap E^\perp$, as it is of Hausdorff dimension $n-i-2$. Thus, by Theorem 2.2.7,

$$\lim_{m\to\infty} \int_{\mathbb{S}^{n-1}} \mathbb{1}_{\mathbb{S}^{n-1}\setminus E^\perp}(u) f(u) S(P^{[i]}, Q_m^{[n-i-1]}, du) = \int_{\mathbb{S}^{n-1}} f(u) S(P^{[i]}, C^{[n-i-1]}, du).$$

For the limit on the right hand side, note that for every fixed $u \in \mathbb{S}^i(E)$, the projections $Q_m|(E^\perp \vee u)$ converge to $C|(E^\perp \vee u)$. This implies weak convergence of the respective surface area measures relative to $E^\perp \vee u$. The support function of $C|(E^\perp \vee u)$ is just the restriction to $\mathbb{S}^{n-i-1}(E^\perp \vee u)$ of the support function of C, and thus, is again of class C^2. Consequently, the respective surface area measure relative to $E^\perp \vee u$ is absolutely continuous; in particular, it vanishes on $\mathbb{S}^{n-i-1}(E^\perp \vee u) \cap E^\perp$, so Theorem 2.2.7 implies that

$$\begin{aligned}\lim_{m\to\infty} &\int_{\mathbb{S}^{n-i-1}(E^\perp \vee u)} \mathbb{1}_{\mathbb{H}^{n-i-1}(E,u)}(v) f(v) S_{n-i-1}^{E^\perp \vee u}(Q_m|(E^\perp \vee u), dv) \\ &= \int_{\mathbb{S}^{n-i-1}(E^\perp \vee u)} \mathbb{1}_{\mathbb{H}^{n-i-1}(E,u)}(v) f(v) S_{n-i-1}^{E^\perp \vee u}(C|(E^\perp \vee u), dv).\end{aligned}$$

That is, $\lim_{m\to\infty} \pi_{E,Q_m} f = \pi_{E,C} f$ pointwise on $\mathbb{S}^i(E)$. Since the measure $S_i^E(P, \cdot\,)$ has finite support, passing to the limit $m \to \infty$ yields

$$\int_{\mathbb{S}^{n-1}} f(u) S(P^{[i]}, C^{[n-i-1]}, du) = \frac{1}{\binom{n-1}{i}} \int_{\mathbb{S}^i(E)} (\pi_{E,C} f)(u) S_i^E(P, du).$$

Observe that both sides of this identity depend continuously on P, and thus, by approximation, we may replace P by any convex body $K \in \mathcal{K}(\mathbb{R}^n)$. Since $f \in C(\mathbb{S}^{n-1})$ was arbitrary, we obtain for every $K \in \mathcal{K}(\mathbb{R}^n)$,

$$S(K^{[i]}, C^{[n-i-1]}, \cdot\,) = \frac{1}{\binom{n-1}{i}} \pi_{E,C}^* S_i^E(K, \cdot\,).$$

As was noted at the beginning of the proof, this shows the theorem. $\square$

Remark 2.2.9. Comparing Theorems 2.2.2 and 2.2.8, it might be natural to ask for a unification that deals with mixed area measures of the form $S(K^{[i]}, B^{n[k-i-1]}, \mathcal{C}, \cdot\,)$, where $E \in \mathrm{Gr}_{n,k}$, $K \in \mathcal{K}(E)$, and $\mathcal{C} = (C_1, \ldots, C_{n-k})$. However, the methods in the proofs of Theorems 2.2.2 and 2.2.8 seem to be disconnected, and thus, it is not immediately clear to the authors how such a unification would look like and how it could be obtained.

2.3 Action on the Klain–Schneider function

This section is devoted to describing the action of the Lefschetz operators on the Klain–Schneider function, which from now on we require to be centered. First, we introduce the aforementioned Radon type transforms. Then we consider the operator Λ, where we apply our findings from the previous section; afterwards we turn to the operator $\mathfrak{L}$.

2.3.1 Two Radon type transforms

Let us now formally introduce the integral transforms that occur in Theorem 2.A. Throughout, integration on compact Grassmann manifolds, unless indicated otherwise, is with respect to the unique rotationally invariant probability measure.

Definition 2.3.1. Let $1 \le k \le n$.

(i) If $k > 1$, we define $\tilde{\mathcal{R}}_{k,k-1} : C(\mathrm{Fl}_{n,k}) \to C(\mathrm{Fl}_{n,k-1})$ by

$$[\tilde{\mathcal{R}}_{k,k-1}\zeta](E,u) = \int_{\mathrm{Gr}_{n,k-1}^{E\cap u^\perp}} \zeta(F \vee u, \mathrm{pr}_{F^\perp} u)\, dF, \qquad (E,u) \in \mathrm{Fl}_{n,k-1}.$$

(ii) If $k < n$, we define $\tilde{\mathcal{R}}_{k,k+1} : C(\mathrm{Fl}_{n,k}) \to C(\mathrm{Fl}_{n,k+1})$ by

$$[\tilde{\mathcal{R}}_{k,k+1}\zeta](E,u) = \int_{\mathrm{Gr}_{n,k}^{E}} \zeta(F, \mathrm{pr}_F u)\|u|F\|\, dF, \qquad (E,u) \in \mathrm{Fl}_{n,k+1}.$$

Subsequently, we will need the fact that these transforms map linear functions to linear functions. To that end, recall that the orthogonal projection π_1 from $L^2(\mathbb{S}^{n-1})$ onto the space of linear functions is given by

$$(\pi_1 f)(u) = \frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}} \langle u, v\rangle f(v)\, dv, \qquad u \in \mathbb{S}^{n-1},$$

where $\omega_k = k\kappa_k$ denotes the surface area of $\mathbb{S}^{k-1}$. For $E \in \mathrm{Gr}_{n,k}$, we denote by π_1^E the respective orthogonal projection relative to E. Moreover, we define an operator $\pi_1^{\langle k\rangle}$ on the space $C(\mathrm{Fl}_{n,k})$ by $[\pi_1^{\langle k\rangle}\zeta](E,u) = [\pi_1^E\zeta(E,\,\cdot\,)](u)$ for $\zeta \in C(\mathrm{Fl}_{n,k})$ and $(E,u) \in \mathrm{Fl}_{n,k}$. The interplay between the Radon type transforms defined above and linear functions is summarized in the following proposition, which we prove in Appendix 2.C.

Proposition 2.3.2. *Let* $1 \le k \le n$.

(i) *If* $k > 1$, *then* $\tilde{\mathcal{R}}_{k,k-1}\pi_1^{\langle k\rangle} = \pi_1^{\langle k-1\rangle}\tilde{\mathcal{R}}'_{k,k-1}$.

(ii) *If* $k < n$, *then* $\tilde{\mathcal{R}}_{k,k+1}\pi_1^{\langle k\rangle} = \frac{k+1}{k}\pi_1^{\langle k+1\rangle}\tilde{\mathcal{R}}_{k,k+1}$.

Here, we also defined an auxiliary transform $\tilde{\mathcal{R}}'_{k,k-1} : C(\mathrm{Fl}_{n,k}) \to C(\mathrm{Fl}_{n,k-1})$ by

$$[\tilde{\mathcal{R}}'_{k,k-1}\zeta](E,u) = \frac{\omega_{n-k+1}\omega_{n-k+3}\omega_{k-1}}{\omega_{n-k+2}^2\omega_k}\int_{\mathrm{Gr}_{n,k}^E} [\pi_{E,1}^F\zeta(F,\,\cdot\,)](u)dF, \quad (E,u) \in \mathrm{Fl}_{n,k-1},$$

where $\pi_{E,1}^F : C(\mathbb{S}^{k-1}(F)) \to C(\mathbb{S}^{k-2}(E))$ denotes the 1-weighted spherical projection relative to F.

2.3.2 The Lefschetz derivation operator

Our strategy to prove Theorem 2.A (i) is to reduce the general case to certain mixed volume valuations: For $1 \leq i \leq n-1$, consider valuations of the form $V(\,\cdot^{[i]}, \mathcal{C}, f) \in \mathbf{Val}_i$, defined by

$$V(K^{[i]}, \mathcal{C}, f) = \int_{\mathbb{S}^{n-1}} f(u)\, S(K^{[i]}, \mathcal{C}, du), \qquad K \in \mathcal{K}(\mathbb{R}^n), \tag{2.3.1}$$

where $\mathcal{C} = (C_1, \ldots, C_{n-i-1})$ is a family of convex bodies with C^2 support functions and $f \in C(\mathbb{S}^{n-1})$.

The reduction will require an approximation argument. To that end, recall first that by a classical result of McMullen [86],

$$\mathbf{Val} = \bigoplus_{i=0}^{n} \mathbf{Val}_i.$$

The space $\mathbf{Val}_0$ is spanned by the Euler characteristic and, due to a famous characterization by Hadwiger [52], the space $\mathbf{Val}_n$ is spanned by the volume. We endow the space $\mathbf{Val}$ with the norm $\|\varphi\| = \max\{|\varphi(K)| : K \subseteq B^n\}$. It is easy to see that this is a complete norm on each space $\mathbf{Val}_i$. By virtue of the homogeneous decomposition, the space $\mathbf{Val}$ is a Banach space.

Moreover, there is a natural representation of the general linear group $\mathrm{GL}(n)$ on the space $\mathbf{Val}$. For $g \in \mathrm{GL}(n)$ and $\varphi \in \mathbf{Val}$, we set

$$(g \cdot \varphi)(K) = \varphi(g^{-1}(K)), \qquad K \in \mathcal{K}(\mathbb{R}^n).$$

Clearly, each space of valuations with a given degree and parity is $\mathrm{GL}(n)$-invariant. By Alesker's *irreducibility theorem* [4], these are the building blocks of all closed $\mathrm{GL}(n)$-invariant subspaces of $\mathbf{Val}$:

Theorem 2.3.3 ([4, Theorem 1.3])**.** *For $0 \leq i \leq n$, the natural representation of $\mathrm{GL}(n)$ on the spaces $\mathbf{Val}_i^{\mathrm{even}}$ and $\mathbf{Val}_i^{\mathrm{odd}}$ is irreducible.*

This result has far-reaching consequences in valuation theory. Especially relevant for our purposes, it is not hard to see that for fixed $1 \leq i \leq n-1$, the valuations of the form (2.3.1) span a $\mathrm{GL}(n)$-invariant subspace of $\mathbf{Val}_i$ containing non-trivial even and odd elements. Therefore, by Theorem 2.3.3, the linear span of such mixed volumes is dense in $\mathbf{Val}_i$.

As a direct consequence of Theorem 2.B, the Klain–Schneider function of a valuation of the form (2.3.1) is given by

$$\mathrm{KS}_{V(\cdot^{[i]}, \mathcal{C}, f)}(E, u) = \frac{1}{\binom{n-1}{i}} \left[(\mathrm{Id} - \pi_1^E)(\pi_{E,\mathcal{C}} f)\right](u), \qquad (E, u) \in \mathrm{Fl}_{n,i+1}. \tag{2.3.2}$$

In order to describe the action of the Lefschetz derivation operator Λ on these Klain–Schneider functions, we need the following Cauchy–Kubota type formula for mixed area measures (cf. [96, (4.78)]): For every family $\mathcal{C} = (C_1, \dots, C_{n-2})$ of convex bodies in $\mathbb{R}^n$ and all $f \in C(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} f(u) S(\mathcal{C}, B, du) = \frac{\omega_n}{\omega_{n-1}} \int_{\mathrm{Gr}_{n,n-1}} \int_{\mathbb{S}^{n-2}(H)} f(u) S^H(\mathcal{C}|H, du)\, dH. \tag{2.3.3}$$

We are now ready to prove Theorem 2.A (i).

Theorem 2.3.4. *Let $1 < i \leq n-1$ and $\varphi \in \mathbf{Val}_i$. Then*

$$\mathrm{KS}_{\Lambda\varphi} = \frac{(n-i)\omega_{n-i+1}}{\omega_{n-i}} (\mathrm{Id} - \pi_1^{\langle i \rangle}) \tilde{\mathcal{R}}_{i+1,i} \mathrm{KS}_\varphi. \tag{2.3.4}$$

Proof. As a consequence of Theorem 2.3.3, valuations of the form $\varphi = V(\,\cdot\,^{[i]}, \mathcal{C}, f)$, where $\mathcal{C} = (C_1, \dots, C_{n-i-1})$ is a family of convex bodies with C^2 support functions and $f \in C(\mathbb{S}^{n-1})$, span a dense subspace of $\mathbf{Val}_i$. Hence, by linearity and by continuity of the Klain–Schneider map and the operator Λ, it suffices to consider these valuations.

Due to the Steiner formula for mixed area measures, $\Lambda\varphi = iV(\,\cdot\,^{[i-1]}, \mathcal{C}, B^n, f)$. Hence, by (2.3.2), we have that $\mathrm{KS}_\varphi = (\mathrm{Id} - \pi_1^{\langle i+1 \rangle})\zeta_\varphi$ and $\mathrm{KS}_{\Lambda\varphi} = (\mathrm{Id} - \pi_1^{\langle i \rangle})\zeta_{\Lambda\varphi}$, where we defined

$$\zeta_\varphi(F, v) = \frac{1}{\binom{n-1}{i}} (\pi_{F,\mathcal{C}} f)(v) \qquad \text{and} \qquad \zeta_{\Lambda\varphi}(E, u) = \frac{i}{\binom{n-1}{i-1}} (\pi_{E,(\mathcal{C},B^n)} f)(u)$$

for $(F, v) \in \mathrm{Fl}_{n,i+1}$ and $(E, u) \in \mathrm{Fl}_{n,i}$. By applying the definition of $\pi_{E,(\mathcal{C},B^n)}$ and an instance of (2.3.3) relative to the space $E^\perp \vee u$, we obtain

$$\begin{aligned}
\zeta_{\Lambda\varphi}(E, u) &= \frac{i}{\binom{n-1}{i-1}} \int_{\mathbb{H}^{n-i}(E,u)} f(v) S^{E^\perp \vee u}(\mathcal{C}|(E^\perp \vee u), B^n|(E^\perp \vee u), dv) \\
&= \frac{i}{\binom{n-1}{i-1}} \frac{\omega_{n-i+1}}{\omega_{n-i}} \int_{\mathrm{Gr}_{n,i}^{E \cap u^\perp}} \int_{\mathbb{H}^{n-i-1}(F \vee u, \mathrm{pr}_{F^\perp} u)} f(v) S^{F^\perp}(\mathcal{C}|F^\perp, dv)\, dF,
\end{aligned}$$

where we used that whenever $F \in \mathrm{Gr}_{n,i}^{E \cap u^\perp}$, then $(\mathcal{C}|(E^\perp \vee u))|F^\perp = \mathcal{C}|F^\perp$ and

$$\mathbb{H}^{n-i}(E, u) \cap F^\perp = \mathbb{H}^{n-i-1}(F \vee u, \mathrm{pr}_{F^\perp} u).$$

Moreover, since $F^\perp = (F \vee u)^\perp \vee \mathrm{pr}_{F^\perp} u$ for almost all $F \in \mathrm{Gr}_{n,i}^{E \cap u^\perp}$, applying again the definition of the mixed spherical projection yields

$$\begin{aligned}
\zeta_{\Lambda\varphi}(E, u) &= \frac{i}{\binom{n-1}{i-1}} \frac{\omega_{n-i+1}}{\omega_{n-i}} \int_{\mathrm{Gr}_{n,i}^{E \cap u^\perp}} (\pi_{F \vee u, \mathcal{C}} f)(\mathrm{pr}_{F^\perp} u) dF \\
&= \frac{(n-i)\omega_{n-i+1}}{\omega_{n-i}} [\tilde{\mathcal{R}}_{i+1,i} \zeta_\varphi](E, u).
\end{aligned}$$

Finally, the linear components remain to be eliminated. Due to Proposition 2.3.2 (i),

$$\tilde{\mathcal{R}}_{i+1,i} \mathrm{KS}_\varphi = \tilde{\mathcal{R}}_{i+1,i} \zeta_\varphi - \tilde{\mathcal{R}}_{i+1,i} \pi_1^{\langle i+1 \rangle} \zeta_\varphi = \frac{\omega_{n-i}}{(n-i)\omega_{n-i+1}} \zeta_{\Lambda\varphi} - \pi_1^{\langle i \rangle} \tilde{\mathcal{R}}'_{i+1,i} \zeta_\varphi.$$

By applying $\mathrm{Id} - \pi_1^{\langle i \rangle}$ to both sides of this equation, we obtain (2.3.4). $\square$

Remark 2.3.5. For an even measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$, we define a generalized Lefschetz operator on $\mathbf{Val}$ by

$$(\Lambda_\mu \varphi)(K) = \frac{d}{dt}\Big|_{t=0^+} \varphi(K + tZ_\mu)$$

where Z_μ is the zonoid generated by the measure μ (cf. [96, p. 193]). It is not too difficult to generalize the Cauchy–Kubota type formula (2.3.3) to zonoids:

$$\int_{\mathbb{S}^{n-1}} f(v) S(\mathcal{C}, Z_\mu, dv) = \frac{2}{n-1} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-2}(u^\perp)} f(v) S^{u^\perp}(\mathcal{C}|u^\perp, dv)\ \mu(du).$$

Note also that $Z_\mu|E' = Z_{\pi_{E',1}\mu}$ for every subspace $E' \subseteq \mathbb{R}^n$. We introduce a generalized transform $\tilde{\mathcal{R}}^\mu_{i+1,i} : C(\mathrm{Fl}_{n,i+1}) \to C(\mathrm{Fl}_{n,i})$ by

$$[\tilde{\mathcal{R}}^\mu_{i+1,i}\zeta](E,u) = \int_{\mathrm{Gr}^{E\cap u^\perp}_{n,i}} \zeta(F \vee u, \mathrm{pr}_{F^\perp} u)\ [J^{(E,u)}_*(\pi_{E^\perp \vee u, 1}\mu)](dF), \quad (E,u) \in \mathrm{Fl}_{n,i},$$

where we defined the map $J^{(E,u)} : \mathbb{S}^{n-i}(E^\perp \vee u) \to \mathrm{Gr}^{E\cap u^\perp}_{n,i} : w \mapsto (E \cap u^\perp) \vee w$. Then, by carrying out the same argument as in the proof of Theorem 2.3.4, we obtain that $\mathrm{KS}_{\Lambda_\mu\varphi}(E,u) = 2(\mathrm{Id} - \pi_1^{\langle i\rangle})\tilde{\mathcal{R}}^\mu_{i+1,i}\mathrm{KS}_\varphi$.

This has the following application: Every even, smooth valuation $\psi \in \mathbf{Val}_{n-1}$ (see Section 2.5.1) admits a representation

$$\psi(K) = \int_{\mathbb{S}^{n-1}} V_{n-1}(K|u^\perp)\mu(du), \qquad K \in \mathcal{K}(\mathbb{R}^n),$$

where μ is a signed measure on $\mathbb{S}^{n-1}$ with a smooth density, called *Crofton measure* of ψ. Then for every smooth valuation $\varphi \in \mathbf{Val}$, its *Bernig–Fu convolution* (see [9]) with ψ is given by $\varphi * \psi = (\Lambda_{\mu^+} - \Lambda_{\mu^-})\varphi$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ. In this sense, we obtain a description of the Bernig–Fu convolution of a smooth valuation with an even, smooth valuation of codegree one.

2.3.3 The Lefschetz integration operator

We now turn to the action of the operator $\mathfrak{L}$ on the Klain–Schneider function, proving Theorem 2.A (ii). We want to stress that the proof utilizes only basic tools from convex and integral geometry.

On affine Grassmann manifolds, there exists a positive rigid motion invariant Radon measure which is unique up to normalization. For $0 \le k \le n$, we fix this normalization by setting

$$\int_{\overline{\mathrm{Gr}}_{n,k}} f(E)dE = \int_{\mathrm{Gr}_{n,k}} \int_{E'^\perp} f(E'+x)dx\ dE'$$

for $f \in C_c(\overline{\mathrm{Gr}}_{n,k})$. Below, we will need the following change of variables rule for affine Grassmannians: If $0 \le j \le k \le n$ and $E \in \overline{\mathrm{Gr}}_{n,k}$, then

$$\int_{\overline{\mathrm{Gr}}_{n,n-j}} f(E\cap F)dF = \frac{\left[{n-j \atop k-j}\right]}{\left[{n \atop k}\right]} \int_{\overline{\mathrm{Gr}}^E_{n,k-j}} f(F)dF \tag{2.3.5}$$

for all $f \in C_c(\overline{\mathrm{Gr}}^E_{n,k-j})$, where $\overline{\mathrm{Gr}}^E_{n,k-j}$ denotes the Grassmann manifold of affine $(k-j)$-dimensional affine subspaces of E and $\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k} \frac{\kappa_n}{\kappa_k \kappa_{n-k}}$, the *flag coefficient*.

Now we prove Theorem 2.A (ii).

Theorem 2.3.6. *Let* $1 \le i < n-1$ *and* $\varphi \in \mathbf{Val}_i$. *Then*

$$\mathrm{KS}_{\mathfrak{L}\varphi} = \frac{\begin{bmatrix} n-1 \\ i+1 \end{bmatrix}}{\begin{bmatrix} n \\ i+2 \end{bmatrix}} \tilde{\mathcal{R}}_{i+1,i+2} \mathrm{KS}_{\varphi}. \tag{2.3.6}$$

Proof. First, we fix a subspace $E \in \mathrm{Gr}_{n,i+2}$. We will show (2.3.6) by evaluating $\mathfrak{L}\varphi$ on arbitrary polytopes $P \in \mathcal{K}(E)$. By the definition of the Lefschetz integration operator $\mathfrak{L}$ and (2.3.5),

$$\begin{aligned}(\mathfrak{L}\varphi)(P) &= \int_{\overline{\mathrm{Gr}}_{n,n-1}} \varphi(P \cap (E \cap H)) dH = \frac{\begin{bmatrix} n-1 \\ i+1 \end{bmatrix}}{\begin{bmatrix} n \\ i+2 \end{bmatrix}} \int_{\overline{\mathrm{Gr}}^E_{n,i+1}} \varphi(P \cap F) dF \\ &= \frac{\begin{bmatrix} n-1 \\ i+1 \end{bmatrix}}{\begin{bmatrix} n \\ i+2 \end{bmatrix}} \int_{\mathrm{Gr}^E_{n,i+1}} \int_{E \cap H'^{\perp}} \varphi(P \cap (H'+x)) dx \, dH'.\end{aligned} \tag{2.3.7}$$

Next, denote by F_j the facets of P relative to E with corresponding outer unit normals $u_j \in \mathbb{S}^{i+1}(E)$, where $j \in \{1, \dots, N\}$. Observe that for $H' \in \mathrm{Gr}^E_{n,i+1}$ and $x \in E \cap H'^{\perp}$, the outer unit normals and facets of $P \cap (H'+x)$ relative to $H'+x$ are given by the vectors $\mathrm{pr}_{H'} u_j$ and sets $F_j \cap (H'+x)$ whenever they are i-dimensional. Thus, by the translation invariance of φ,

$$\begin{aligned}\varphi(P \cap (H'+x)) &= \int_{\mathbb{S}^i(H')} \mathrm{KS}_{\varphi}(H', u) S_i^{H'}((P-x) \cap H', du) \\ &= \sum_{j=1}^{N} \mathrm{KS}_{\varphi}(H', \mathrm{pr}_{H'} u_j) V_i(F_j \cap (H'+x)).\end{aligned}$$

Note that for every $u \in \mathbb{S}^{i+1}(E)$ and convex body $L \in \mathcal{K}(E \cap u^{\perp})$,

$$\int_{E \cap H'^{\perp}} V_i(L \cap (H'+x)) dx = \|u|H'\| V_{i+1}(L),$$

as follows from Fubini's theorem. By combining these identities, it follows that

$$\begin{aligned}\int_{E \cap H'^{\perp}} \varphi(P \cap (H'+x)) dx &= \sum_{j=1}^{N} \mathrm{KS}_{\varphi}(H', \mathrm{pr}_{H'} u_j) \|u_j|H'\| V_{i+1}(F_j) \\ &= \int_{\mathbb{S}^{i+1}(E)} \mathrm{KS}_{\varphi}(H', \mathrm{pr}_{H'} u) \|u|H'\| S^E_{i+1}(P, du).\end{aligned}$$

Plugging this into (2.3.7) and changing the order of integration yields

$$(\mathfrak{L}\varphi)(P) = \frac{\begin{bmatrix} n-1 \\ i+1 \end{bmatrix}}{\begin{bmatrix} n \\ i+2 \end{bmatrix}} \int_{\mathbb{S}^{i+1}(E)} \int_{\mathrm{Gr}^E_{n,i+1}} \mathrm{KS}_{\varphi}(H', \mathrm{pr}_{H'} u) \|u|H'\| dH' \; S^E_{i+1}(P, du).$$

By the definition of the Klain–Schneider function, this proves (2.3.6) up to the addition of a linear function. By Proposition 2.3.2 (ii) however, $\tilde{\mathcal{R}}_{i+1,i+2}\mathrm{KS}_\varphi$ is centered, and thus, we obtain (2.3.6). □

Remark 2.3.7. For a translation invariant signed Radon measure μ on $\overline{\mathrm{Gr}}_{n,n-1}$, define a generalized Lefschetz operator on **Val** by

$$(\mathfrak{L}_\mu\varphi)(K) = \int_{\overline{\mathrm{Gr}}_{n,n-1}} \varphi(K \cap H)\, \mu(dH), \qquad K \in \mathcal{K}(\mathbb{R}^n).$$

We define a generalized transform $\tilde{\mathcal{R}}^\mu_{i+1,i+2} : C(\mathrm{Fl}_{n,i+1}) \to C(\mathrm{Fl}_{n,i+2})$ by

$$[\tilde{\mathcal{R}}^\mu_{i+1,i+2}\zeta](E,u) = \int_{\mathrm{Gr}^E_{n,i+1}} \zeta(F, \mathrm{pr}_F u)\|u|F\|\; \mu^E(dF), \qquad (E,u) \in \mathrm{Fl}_{n,i+2},$$

where μ^E is the unique signed measure on $\mathrm{Gr}^E_{n,i+1}$ such that for all $f \in C_c(\overline{\mathrm{Gr}}^E_{n,i+1})$,

$$\int_{\overline{\mathrm{Gr}}_{n,n-1}} f(E \cap H)\mu(dH) = \int_{\mathrm{Gr}^E_{n,i+1}} \int_{E\cap H'^\perp} f(H' + x)dx\; \mu^E(dH').$$

Then the argument in the proof of (2.3.6) shows that $\mathrm{KS}_{\mathfrak{L}_\mu\varphi} = \tilde{\mathcal{R}}^\mu_{i+1,i+2}\mathrm{KS}_\varphi$.

This has the following application: Every even, smooth valuation $\psi \in \mathbf{Val}_1$ (see Section 2.5.1) admits a representation

$$\psi(K) = \int_{\overline{\mathrm{Gr}}_{n,n-1}} \chi(K \cap H)\mu(dH),$$

where χ denotes the Euler characteristic and μ is a translation invariant signed Radon measure with a smooth density on $\overline{\mathrm{Gr}}_{n,n-1}$, called a *Crofton measure* of ψ. Then for every valuation $\varphi \in \mathbf{Val}$, its *Alesker product* (see [8]) with ψ is given by $\varphi \cdot \psi = \mathfrak{L}_\mu\varphi$. In this sense, we obtain a description of the Alesker product of a continuous valuation with an even, smooth valuation of degree one.

2.4 Minkowski valuations

In this section, we turn to Minkowski valuations and more specifically, the *mean section operators*. This family of geometric operators, which was introduced by Goodey and Weil [45], are of particular interest to us as they will play a crucial role in the proof of Theorem 2.C.

Definition 2.4.1 ([45])**.** For $0 \le j \le n$, the *j-th mean section body* M_jK of a convex body $K \in \mathcal{K}(\mathbb{R}^n)$ is defined by

$$h_{\mathrm{M}_jK}(u) = \int_{\overline{\mathrm{Gr}}_{n,j}} h_{K\cap E}(u)\; dE, \qquad u \in \mathbb{S}^{n-1}.$$

The j-th mean section operator M_j is a continuous, rotationally equivariant Minkowski valuation, but it is not translation invariant. By defining the j-th *centered* mean section operator $\tilde{\mathrm{M}}_j$ as

$$\tilde{\mathrm{M}}_j K = \mathrm{M}_j(K - s(K)),$$

where $s(K)$ is the Steiner point of K (cf. [96, p. 50]), it follows that $\tilde{\mathrm{M}}_j \in \mathbf{MVal}_i$ for $2 \le j \le n$, where $i + j = n + 1$. Determining their generating functions is highly non-trivial and requires the family of functions introduced by Berg [13] in his solution to the Christoffel problem [29]. The *Berg functions* $g_j \in C^\infty(-1,1)$, for $j \ge 2$, are defined recursively as follows:

$$\begin{aligned} g_2(t) &= \frac{1}{2\pi}(\pi - \arccos t)(1-t^2)^{\frac{1}{2}} - \frac{1}{4\pi} t, \\ g_3(t) &= \frac{1}{2\pi}\left(1 + t\log(1-t) + \left(\frac{4}{3} - \log 2\right) t\right), \\ g_{j+2}(t) &= \frac{j+1}{2\pi} g_j(t) + \frac{j+1}{2\pi(j-1)} t g_j'(t) + \frac{j+1}{2\pi\omega_j} t, \end{aligned} \tag{2.4.1}$$

where $\omega_j = j\kappa_j$ is the $(j-1)$-dimensional Hausdorff measure of $\mathbb{S}^{j-1}$. Goodey and Weil [47] showed that essentially, the j-th Berg function generates the j-th mean section operator – independently of the dimension $n \ge 3$ of the ambient space.

Theorem 2.4.2 ([47])**.** *For $n \ge 3$ and $2 \le j \le n$,*

$$f_{\tilde{\mathrm{M}}_j} = m_{n,j}(\mathrm{Id} - \pi_1) g_j(\langle e_n, \cdot \rangle). \tag{2.4.2}$$

Here, $m_{n,j}$ is a constant which depends only on n and j, and was explicitly determined in [47]. Moreover, $\pi_1 f$ denotes the linear component of a function $f \in L^1(\mathbb{S}^{n-1})$ (see Section 2.3.1).

2.4.1 The Berg functions

Next, we discuss analytic properties of the Berg functions. Berg [13] showed that they satisfy the following second order differential equation: For every $j \ge 2$ and $t \in (-1,1)$,

$$\frac{1}{j-1}(1-t^2) g_j''(t) - t g_j'(t) + g_j(t) = -\frac{j}{\omega_j} t. \tag{2.4.3}$$

Moreover, g_2 extends to a continuous function on $[-1,1]$ and for all $j \ge 3$, the function g_j extends to a continuous function on $[-1,1)$ and $\lim_{t\to 1} g_j(t) = -\infty$. Berg showed that for every $\varepsilon > 0$,

$$\lim_{t \to \pm 1} (1-t^2)^{\frac{j-3+\varepsilon}{2}} g_j(t) = 0. \tag{2.4.4}$$

In the following, we investigate the behavior of the Berg functions at the end points in more detail. In particular, we determine the precise order of the singularity at $t = 1$ as well as the scaling limit.

Lemma 2.4.3. *For every $j \geq 2$ and $m \geq 0$ where $(j,m) \notin \{(2,0),(3,0)\}$,*

$$\begin{aligned} &\lim_{t\to -1}(1-t^2)^{\frac{j-3}{2}+m} g_j^{(m)}(t) = 0, \\ &\lim_{t\to +1}(1-t^2)^{\frac{j-3}{2}+m} g_j^{(m)}(t) = -(j-1)2^{m-2}\frac{\Gamma(\frac{j-3}{2}+m)}{\pi^{\frac{j-1}{2}}}. \end{aligned} \tag{2.4.5}$$

Proof. We show the lemma by induction on $j \geq 2$. First, note that for $j \in \{2,3\}$ and $m = 1$, one can show (2.4.5) by a simple direct computation. In order to apply L'Hôpital's rule, we need to justify the existence of the limits for the higher order derivatives. To that end, observe that we can express them as

$$\begin{aligned} g_2^{(m)}(t) &= p_{2,m}(t)(1-t^2)^{\frac{1}{2}-m} + q_{2,m}(t)(1-t^2)^{\frac{1}{2}-m}\arccos t + r_{2,m}(t), \\ g_3^{(m)}(t) &= p_{3,m}(t)\log(1-t) + q_{3,m}(t)(1-t)^{-m} + r_{3,m}(t), \end{aligned}$$

with some polynomials $p_{j,m}$, $q_{j,m}$ and $r_{j,m}$, where $j \in \{2,3\}$ and $m \geq 0$. This can be shown by a simple induction on $m \geq 0$. Consequently, the limits in (2.4.5) exist for $j \in \{2,3\}$ and $m \geq 1$, and by applying L'Hôpital's rule repeatedly, we obtain the desired identities, which proves the lemma in the instance where $j \in \{2,3\}$.

For the inductive step, suppose that (2.4.5) holds for some fixed $j \geq 2$ and all $m \geq 0$. Next, observe that m-fold differentiation of the recurrence relation in (2.4.1) yields the following recurrence relation for the m-th order derivatives:

$$\frac{2\pi}{j+1}g_{j+2}^{(m)}(t) = \frac{j+m-1}{j-1}g_j^{(m)}(t) + \frac{1}{j-1}tg_j^{(m+1)}(t) + \frac{1}{\omega_j}\frac{d^m t}{dt^m},$$

as can be shown by a simple induction on $m \geq 0$. Now we multiply both sides with $(1-t^2)^{\frac{j-1}{2}+m}$, pass to the limit $t \to \pm 1$, and employ both the induction hypothesis and (2.4.4). In this way, we obtain the desired identities for $j+2$ and all $m \geq 0$, which concludes the argument. □

As was mentioned before, the functions g_j extend continuously to $t = -1$. In order to investigate the behavior at $t = -1$ further, the change of variables $t = -\cos\theta$ turns out to be quite useful. That is, we define a family of functions $\hat{g}_j$, where $j \geq 2$, by

$$\hat{g}_j(\theta) = g_j(-\cos\theta), \qquad \theta \in (-\pi,\pi)\backslash\{0\}.$$

Transforming the recursion (2.4.1) yields the following recursion for the functions $\hat{g}_j$:

$$\begin{aligned} \hat{g}_2(\theta) &= \frac{1}{2\pi}\theta\sin\theta + \frac{1}{4\pi}\cos\theta, \\ \hat{g}_3(\theta) &= \frac{1}{2\pi}\left(1 - \cos\theta\log(1+\cos\theta) - \left(\frac{4}{3} - \log 2\right)\cos\theta\right), \\ \hat{g}_{j+2}(\theta) &= \frac{j+1}{2\pi}\hat{g}_j(\theta) - \frac{j+1}{2\pi(j-1)}\frac{\hat{g}_j'(\theta)}{\tan\theta} - \frac{j+1}{2\pi\omega_j}\cos\theta. \end{aligned} \tag{2.4.6}$$

This recursion shows that the functions $\hat{g}_j$ extend to even analytic functions on $(-\pi, \pi)$. For $\hat{g}_2$ and $\hat{g}_3$, this is obvious. If $\hat{g}_j$ extends to an even analytic function on $(-\pi, \pi)$, then its derivative $\hat{g}_j'$ has a simple zero in $\theta = 0$ which cancels with the zero of the tangent function, and thus, $\hat{g}_{j+2}$ also extends to an even analytic function on $(-\pi, \pi)$.

This observation allows us to obtain new information about the regularity of the generating functions of the mean section operators. To that end, denote by $d_{\mathbb{S}^{n-1}}(u, v) := \arccos\langle u, v\rangle$ the geodesic distance on $\mathbb{S}^{n-1}$. The following lemma is a classical fact from Riemannian geometry; for the convenience of the reader, we provide an elementary proof.

Lemma 2.4.4. *For $v \in \mathbb{S}^{n-1}$, the function $d_{\mathbb{S}^{n-1}}(v, \cdot\,)^2$ is smooth on $\mathbb{S}^{n-1} \setminus \{-v\}$.*

Proof. Clearly, the function $q : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{R}$, defined as

$$q(u, v) = d_{\mathbb{S}^{n-1}}(u, v)^2 = (\arccos\langle u, v\rangle)^2,$$

is smooth on the set of all pairs (u, v) for which $|\langle u, v\rangle| < 1$. Next, observe that $\sin^2\theta$, as an even analytic function, can be written as a power series of θ^2. Thus, there is a unique analytic function f such that $\sin^2\theta = f(\theta^2)$ for $\theta \in \mathbb{R}$. Since

$$f'(0) = \lim_{\theta\to 0} \frac{f(\theta^2)}{\theta^2} = \lim_{\theta\to 0} \left(\frac{\sin\theta}{\theta}\right)^2 = 1 \neq 0,$$

there exists $\delta > 0$ such that f is invertible on $(-\delta, \delta)$ and its inverse f^{-1} is also smooth. Consequently, whenever $d_{\mathbb{S}^{n-1}}(u, v)^2 < \delta$, we can express $q(u, v)$ as

$$q(u, v) = d_{\mathbb{S}^{n-1}}(u, v)^2 = f^{-1}(\sin^2 d_{\mathbb{S}^{n-1}}(u, v)) = f^{-1}(1 - \langle u, v\rangle^2).$$

This shows that q is smooth on the set of all pairs (u, v) for which $d_{\mathbb{S}^{n-1}}(u, v)^2 < \delta$. Hence, we have found two open sets covering the space $\{(u, v) \in \mathbb{S}^{n-1} : u \neq -v\}$ such that q is smooth on each set, which proves the lemma. □

Remark 2.4.5. Let $n \geq 2$ and $j \geq 2$ and consider the function $g_j(\langle e_n, \cdot\,\rangle)$ on $\mathbb{S}^{n-1}$. Clearly, this function is smooth on $\mathbb{S}^{n-1} \setminus \{\pm e_n\}$. Since $\hat{g}_j$ is even and analytic, there exists a smooth function q_j such that $\hat{g}_j(\theta) = q_j(\theta^2)$, and thus,

$$g_j(\langle e_n, u\rangle) = g_j(-\cos d_{\mathbb{S}^{n-1}}(-e_n, u)) = q_j(d_{\mathbb{S}^{n-1}}(-e_n, u)^2).$$

Combined with Lemma 2.4.4, this shows that $g_j(\langle e_n, \cdot\rangle)$ is a smooth function on $\mathbb{S}^{n-1}\setminus\{e_n\}$. Consequently, the generating functions of the mean section operators are smooth outside the north pole.

Next, we turn to the value that is attained at the south pole. To that end, we apply the change of variables $t = -\cos\theta$ to the differential equation (2.4.3), which yields the following: For all $\theta \in (-\pi, \pi) \setminus \{0\}$,

$$\frac{1}{j-1}\hat{g}_j''(\theta) + \frac{j-2}{j-1}\frac{\hat{g}_j'(\theta)}{\tan\theta} + \hat{g}_j(\theta) = \frac{j}{\omega_j}\cos\theta. \tag{2.4.7}$$

This allows us to deduce a recurrence relation for the value $\hat{g}_j(0)$, which is instrumental in the proof of the following lemma.

Lemma 2.4.6. *Let $j \geq 3$. Then* $\lim_{t\to -1} g_j(t) < 0$.

Proof. We show that $\hat{g}_j(0) < 0$ by induction on $j \geq 3$. Direct computation shows that $\hat{g}_3(0) = -\frac{1}{6\pi}$ and $\hat{g}_4(0) = -\frac{3}{2\pi^2}$. Letting $\theta \to 0$ in (2.4.7) yields

$$\hat{g}_j''(0) + \hat{g}_j(0) = \frac{j}{\omega_j}.$$

By taking the limit $\theta \to 0$ in the recurrence relation in (2.4.6) and combining this with the identity above, we obtain that

$$\hat{g}_{j+2}(0) = \frac{j+1}{2\pi}\hat{g}_j(0) - \frac{j+1}{2\pi(j-1)}\hat{g}_j''(0) - \frac{j+1}{2\pi\omega_j} = \frac{j+1}{2\pi(j-1)}\left(j\hat{g}_j(0) - \frac{2j-1}{\omega_j}\right),$$

which shows that if $\hat{g}_j(0) < 0$, then also $\hat{g}_{j+2}(0) < 0$. □

2.4.2 The space $\mathbf{MVal}_i$

We want to make some remarks about the structure of the space $\mathbf{MVal}_i$, which we will come back to in the subsequent section. Before that, since it is convenient and will also be needed later on, we want to recall the natural action of $\mathrm{SO}(n)$ on (generalized) functions on the unit sphere.

For a rotation $\vartheta \in \mathrm{SO}(n)$ and a smooth function $\phi \in C^\infty(\mathbb{S}^{n-1})$, we set $(\vartheta\phi)(u) = \phi(\vartheta^{-1}u)$ for $u \in \mathbb{S}^{n-1}$. For a distribution $\gamma \in C^{-\infty}(\mathbb{S}^{n-1})$, we set $\langle \phi, \vartheta\gamma\rangle_{\mathbb{S}^{n-1}} = \langle \vartheta^{-1}\phi, \gamma\rangle_{\mathbb{S}^{n-1}}$, where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{S}^{n-1}}$ denotes the natural pairing of a distribution and a smooth function. An $\mathrm{SO}(n-1)$-invariant (generalized) function on $\mathbb{S}^{n-1}$ is also called *zonal*. This also encompasses continuous functions, Lebesgue integrable functions, and signed measures, by virtue of the chain of identifications

$$C^\infty(\mathbb{S}^{n-1}) \subseteq C(\mathbb{S}^{n-1}) \subseteq L^2(\mathbb{S}^{n-1}) \subseteq \mathcal{M}(\mathbb{S}^{n-1}) \subseteq C^{-\infty}(\mathbb{S}^{n-1}). \tag{2.4.8}$$

To date, a complete classification of continuous, translation invariant and rotationally equivariant Minkowski valuations is not known. In view of the integral representation (2.1.2), this open problem reduces to a characterization of generating functions. It is not hard to see that the support function h_L of a convex body of revolution $L \in \mathcal{K}(\mathbb{R}^n)$ generates a Minkowski valuation $\Phi \in \mathbf{MVal}_i$ in every degree $i \in \{1, \dots, n-1\}$. However, not all Minkowski valuations in $\mathbf{MVal}_i$ are of this form, as is exemplified by the mean section operators.

Since the space $\mathbf{MVal}_i$, endowed with the pointwise Minkowski operations, has the structure of a convex cone, we may consider generating functions of the form

$$f = h_L + cf_{\tilde{\mathrm{M}}_j}, \tag{2.4.9}$$

where L is a convex body of revolution, $i + j = n + 1$, and $c \geq 0$. Another family of operations on $\mathbf{MVal}_i$ is the composition with *Minkowski endomorphisms*. These are a continuous, translation invariant, rotationally equivariant Minkowski additive operators on $\mathcal{K}(\mathbb{R}^n)$. Note that the space of Minkowski endomorphisms is precisely $\mathbf{MVal}_1$. Their action on the support function can be described in terms of the *spherical convolution* (see Appendix 2.A). Kiderlen [60] and Dorrek [36] showed that for every $\Psi \in \mathbf{MVal}_1$, there is a unique centered, zonal signed measure $\mu_\Psi \in \mathcal{M}(\mathbb{S}^{n-1})$ such that for all $K \in \mathcal{K}(\mathbb{R}^n)$,

$$h_{\Psi K} = h_K * \mu_\Psi. \tag{2.4.10}$$

We call μ_Ψ the *generating measure* of Ψ. Kiderlen [60] showed that whenever a zonal measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ is *weakly positive*, that is, positive up to the addition of a linear function, then it generates some Minkowski endomorphism.

Observe that whenever $\Phi_i \in \mathbf{MVal}_i$ and Ψ is a Minkowski endomorphism, then $\Psi \circ \Phi_i \in \mathbf{MVal}_i$ and its generating function is given by

$$f_{\Psi\circ\Phi_i} = f_{\Phi_i} * \mu_\Psi. \tag{2.4.11}$$

That is, the space of generating functions of Minkowski valuations is closed with respect to convolution with generating measures of Minkowski endomorphisms, including all weakly positive zonal measures. Consequently, we may consider generating functions of the form

$$f = h_L + f_{\tilde{\mathrm{M}}_j} * \mu_\Psi, \tag{2.4.12}$$

where L is a convex body of revolution, $i+j = n+1$, and Ψ is a Minkowski endomorphism. In fact, this encompasses all instances of generating functions that are known to date.

The following example demonstrates that generating functions of the form (2.4.12) create a much greater class than those of the form (2.4.9). This indicates the importance of taking compositions with Minkowski endomorphisms into account.

Example 2.4.7. Let $n \geq 3$, $1 \leq i < n-1$, and $i + j = n + 1$. Due to Lemma 2.4.6 and the fact that $\lim_{t\to 1} g_j(t) = -\infty$, there exists some $\delta > 0$ such that $g_j(t) < 0$ whenever $|t| > 1 - \delta$. Choose some non-trivial, zonal, positive, even measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ with a smooth density that is supported on $\{u \in \mathbb{S}^{n-1} : |\langle e_n, u\rangle| > 1 - \delta\}$.

Then μ is the generating measure of a Minkowski endomorphism, and thus, $f = g_j(\langle e_n, \cdot\rangle) * \mu$ is a generating function of the form (2.4.12). Moreover, f is smooth and $f(e_n) = f(-e_n) < 0$, as follows from the definition of the convolution. Suppose now that f is of the form (2.4.9). Since f is smooth, we must have $c = 0$, but since f is not weakly positive, we must also have $L = \{o\}$; consequently $f = 0$, which is a contradiction.

2.5 Action on the generating function

In this section, we investigate the action of the Lefschetz integration operator $\mathfrak{L}$ on generating functions of Minkowski valuations, proving Theorem 2.C and its consequences. As was already indicated in the introduction, we consider a broader framework, which we introduce below.

2.5.1 Spherical valuations

A valuation $\varphi \in \mathbf{Val}$ is called *smooth* if the map $\mathrm{GL}(n) \to \mathbf{Val} : g \mapsto g \cdot \varphi$ is infinitely differentiable, where $g \cdot \varphi$ denotes the natural action of $\mathrm{GL}(n)$ on $\mathbf{Val}$ (see Section 2.3.2). Then, for $0 \leq i \leq n$, the subspace $\mathbf{Val}_i^\infty \subseteq \mathbf{Val}_i$ of smooth valuations is dense in $\mathbf{Val}_i$ and the map

$$\mathbf{Val}_i^\infty \to C^\infty(\mathrm{GL}(n), \mathbf{Val}_i) : \varphi \mapsto (g \mapsto g \cdot \varphi)$$

leads to an identification of $\mathbf{Val}_i^\infty$ with a closed subspace of $C^\infty(\mathrm{GL}(n), \mathbf{Val}_i)$, endowed with the standard Fréchet topology (cf. [110, Section 4.4]). Hence, $\mathbf{Val}_i^\infty$ also becomes a Fréchet space by endowing it with the induced topology, called the Gårding topology. The space $\mathbf{Val}_i^{\infty,\mathrm{sph}}$ of smooth *spherical* valuations is the closure of the direct sum of $\mathrm{SO}(n)$-irreducible subspaces of $\mathbf{Val}_i^\infty$ that contain a non-trivial $\mathrm{SO}(n-1)$-invariant element.

This representation theoretic and somewhat implicit definition does not provide a clear idea of what a spherical valuation looks like. However, Schuster and Wannerer [103] showed that these are precisely the valuations in $\mathbf{Val}_i^\infty$ that admit an integral representation on $\mathbb{S}^{n-1}$ with respect to the i-th area measure. In the following, we denote by $C_o^\infty(\mathbb{S}^{n-1})$ and $C_o^{-\infty}(\mathbb{S}^{n-1})$ the spaces of centered smooth functions and distributions, respectively.

Theorem 2.5.1 ([103]). *Let $1 \leq i \leq n-1$. For every valuation $\varphi \in \mathbf{Val}_i^{\infty,\mathrm{sph}}$, there exists a unique function $f_\varphi \in C_o^\infty(\mathbb{S}^{n-1})$ such that for all $K \in \mathcal{K}(\mathbb{R}^n)$,*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f_\varphi(u)\; S_i(K, du). \tag{2.5.1}$$

The map $\mathbf{Val}_i^{\infty,\mathrm{sph}} \to C_o^\infty(\mathbb{S}^{n-1}) : \varphi \mapsto f_\varphi$ is an $\mathrm{SO}(n)$-equivariant isomorphism of topological vector spaces.

We call the function f_φ the *generating function* of φ. As was already pointed out in the introduction, the Lefschetz derivation operator acts on the generating function as a multiple of the identity. That is, $f_{\Lambda\varphi} = i f_\varphi$ for $\varphi \in \mathbf{Val}_i^{\infty,\mathrm{sph}}$, which is a simple consequence of the Steiner formula for area measures. For this reason, we set the operator Λ aside and turn to the Lefschetz integration operator $\mathfrak{L}$.

As a direct consequence of Theorem 2.5.1, there exists a unique $\mathrm{SO}(n)$-equivariant endomorphism T_i on the topological vector space $C_o^\infty(\mathbb{S}^{n-1})$ such that $f_{\mathfrak{L}\varphi} = \mathrm{T}_i f_\varphi$ for $\varphi \in \mathbf{Val}_i^{\infty,\mathrm{sph}}$. By setting $\rho_i = \mathrm{T}_i'(\mathrm{Id} - \pi_1)\delta_{e_n}$, where T_i' is the continuous transpose map of T_i on $C_o^{-\infty}(\mathbb{S}^{n-1})$, one obtains the following.

Corollary 2.5.2. *For $1 \leq i < n-1$, there exists a unique centered, zonal distribution $\rho_i \in C^{-\infty}(\mathbb{S}^{n-1})$ such that for all $\varphi \in \mathbf{Val}_i^{\infty,\mathrm{sph}}$,*

$$f_{\mathfrak{L}\varphi} = f_\varphi * \rho_i.$$

Interestingly enough, all the information that is needed to determine and subsequently describe the distribution ρ_i can be obtained from the action of $\mathfrak{L}$ on the mean section operators. For $1 \leq i < n-1$.

$$\mathfrak{L}\tilde{\mathrm{M}}_j = \frac{\left[{j \atop j-1}\right]}{\left[{n \atop n-1}\right]}\tilde{\mathrm{M}}_{j-1},$$

where $i+j=n+1$, as was shown in [102]. By approximating the associated real valued valuations of the centered mean section operators with smooth valuations, it follows that

$$f_{\tilde{\mathrm{M}}_j} * \rho_i = \frac{\left[{j \atop j-1}\right]}{\left[{n \atop n-1}\right]} f_{\tilde{\mathrm{M}}_{j-1}}. \tag{2.5.2}$$

In the following, we use methods from harmonic analysis on $\mathbb{S}^{n-1}$ (see Appendix 2.B) to determine the spherical harmonic expansion of ρ_i. To that end, recall that by (2.4.2), the centered mean section operators are generated by a constant multiple of the centered Berg functions. Their multipliers were explicitly computed in [20] and [12] independently: for $2 \leq j \leq n$ and $k \neq 1$,

$$a_k^n[g_j] = -\frac{\pi^{\frac{n-j}{2}}(j-1)}{4} \frac{\Gamma\left(\frac{n-j+2}{2}\right)\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{k+j-1}{2}\right)}{\Gamma\left(\frac{k+n-j+1}{2}\right)\Gamma\left(\frac{k+n+1}{2}\right)}.$$

These numbers are all non-zero, and thus, by a simple division, for all $k \neq 1$,

$$a_k^n[\rho_i] = \frac{\left[{j \atop j-1}\right]}{\left[{n \atop n-1}\right]} \frac{a_k^n[f_{\tilde{\mathrm{M}}_{j-1}}]}{a_k^n[f_{\tilde{\mathrm{M}}_j}]} = c_{n,i} \frac{\Gamma\left(\frac{k+i}{2}\right)\Gamma\left(\frac{k+n-i-1}{2}\right)}{\Gamma\left(\frac{k+i+1}{2}\right)\Gamma\left(\frac{k+n-i}{2}\right)},$$

where $i+j=n+1$ and the constant $c_{n,i}>0$ only depends on n and i. The above was already done by Berg, Parapatits, Schuster, and Weberndorfer [12]. Our contribution will be to extract the information of Theorem 2.C from the spherical harmonic expansion of ρ_i.

2.5.2 A Legendre type differential equation

Our aim is now to establish the differential equation (2.1.4). To that end, it is convenient to renormalize ρ_i and manipulate its linear part: We define an auxiliary distribution $\tilde{\rho}_i \in C^{-\infty}(\mathbb{S}^{n-1})$ in terms of its multipliers by

$$a_k^n[\tilde{\rho}_i] = \frac{\Gamma\left(\frac{k+i}{2}\right)\Gamma\left(\frac{k+n-i-1}{2}\right)}{\Gamma\left(\frac{k+i+1}{2}\right)\Gamma\left(\frac{k+n-i}{2}\right)}, \qquad k \geq 0. \tag{2.5.3}$$

Next, note that we can lift every test function $\psi \in \mathcal{D}(-1,1)$ to a smooth function $\psi(\langle e_n, \cdot\rangle) \in C^\infty(\mathbb{S}^{n-1})$. In the following, we let

$$I_* = \{(n,i) \in \mathbb{N} \times \mathbb{N} : n \geq 3,\ 1 \leq i < n-1\}$$

denote the set of admissible index pairs. Then we define a family of distributions $\rho_{n,i} \in \mathcal{D}'(-1,1)$, where $(n,i) \in I_*$, by

$$\langle \psi, \rho_{n,i}\rangle = \langle \psi(\langle e_n, \cdot\rangle), \tilde{\rho}_i\rangle, \qquad \psi \in \mathcal{D}(-1,1).$$

This construction on the interval $(-1,1)$ will allow us to relate distributions $\tilde{\rho}_i$ that live on spheres $\mathbb{S}^{n-1}$ of different dimensions.

We continue with two simple, yet crucial observations about the family $\rho_{n,i}$ that are immediate from their definition and the fact that every distribution is uniquely determined by its sequence of multipliers. Firstly, for $(n,i) \in I_*$,

$$\rho_{n,i} = \rho_{n,n-i-1}. \tag{2.5.4}$$

Secondly, we compare the multipliers of $\rho_{n,n-2}$ with those of g_{n-1} and the *box operator* $\square_n = \frac{1}{n-1}\Delta_{\mathbb{S}^{n-1}} + \mathrm{Id}$ (see Appendix 2.B). Its action on zonal functions (see, e.g., [88, Lemma 5.3]) can be described in terms of the differential operator

$$\overline{\square}_n = \frac{1}{n-1}(1-t^2)\frac{d^2}{dt^2} - t\frac{d}{dt} + \mathrm{Id}.$$

Namely, for all test functions $\psi \in \mathcal{D}(-1,1)$, we have that

$$[\square_n \psi(\langle e_n, \cdot\,\rangle)](u) = (\overline{\square}_n \psi)(\langle e_n, u\rangle), \qquad u \in \mathbb{S}^{n-1}.$$

In this way, for all $n \geq 3$, we obtain that the distribution $\rho_{n,n-2}$ is a $C^\infty(-1,1)$ function and that for all $t \in (-1,1)$,

$$\rho_{n,n-2}(t) = \frac{n-1}{2(n-2)}\overline{\square}_n g_{n-1}(t) + \frac{n}{2\omega_{n-1}}t. \tag{2.5.5}$$

Next, we establish the smoothness of the distributions $\rho_{n,i}$ on $(-1,1)$, their behavior at the end points, and a remarkably simple recurrence relation.

Lemma 2.5.3. *Let $(n,i) \in I_*$.*

(i) $\rho_{n,i} \in C^\infty(-1,1)$ *and for all $m \geq 0$,*

$$\begin{aligned} &\lim_{t\to-1}(1-t^2)^{\frac{n-2}{2}+m}\rho_{n,i}^{(m)}(t) = 0, \\ &\lim_{t\to+1}(1-t^2)^{\frac{n-2}{2}+m}\rho_{n,i}^{(m)}(t) = 2^{m-2}\frac{\Gamma\left(\frac{n-2}{2}+m\right)}{\pi^{\frac{n-2}{2}}}. \end{aligned} \tag{2.5.6}$$

(ii) *For all $t \in (-1,1)$,*

$$\rho_{n+2,i+1}(t) = \frac{1}{2\pi}\rho_{n,i}'(t). \tag{2.5.7}$$

In order to show the recurrence relation (2.5.7), we need the following multiplier relation, which we prove in Appendix 2.B.

Lemma 2.5.4. *If $n \geq 3$ and $g \in C^1(-1,1)$ is such that $(1-t^2)^{\frac{n-1}{2}}g'(t)$ is integrable on $(-1,1)$, then $(1-t^2)^{\frac{n-3}{2}}g(t)$ is integrable on $(-1,1)$ and for all $k \geq 0$,*

$$a_k^{n+2}[g'] = 2\pi a_{k+1}^n[g]. \tag{2.5.8}$$

We want to point out that (2.5.4), (2.5.5), and (2.5.7) together result in a complete recursion such that every $\rho_{n,i}$ can be traced back to the Berg functions. In this way, we will obtain many of the desired properties inductively. The following lemma provides the necessary induction scheme on I_*; the statement is obvious, so we omit the proof.

Lemma 2.5.5. *Let $I \subseteq I_*$ and suppose that for all $(n,i) \in I_*$, the following holds:*

- $(n, n-2) \in I$,
- $(n,i) \in I$ *if and only if* $(n, n-i-1) \in I$,
- *whenever* $(n,i) \in I$, *then also* $(n+2, i+1) \in I$.

Then $I = I_*$.

Proof of Lemma 2.5.3. First, we show that for some fixed $(n,i) \in I_*$, the assertion of Item (i) implies the assertion of Item (ii). For this purpose, suppose that $\rho_{n,i} \in C^\infty(-1,1)$ and satisfies (2.5.6). Then the requirements of Lemma 2.5.4 are met, and thus, (2.5.8) shows that for all $k \geq 0$,

$$\frac{1}{2\pi} a_k^{n+2}[\rho_{n,i}'] = a_{k+1}^n[\rho_{n,i}] = \frac{\Gamma\left(\frac{k+i+1}{2}\right)\Gamma\left(\frac{k+n-i}{2}\right)}{\Gamma\left(\frac{k+i+2}{2}\right)\Gamma\left(\frac{k+n-i+1}{2}\right)} = a_k^{n+2}[\rho_{n+2,i+1}].$$

This proves (2.5.7) in the sense of distributions. Since we assumed $\rho_{n,i}$ to be smooth, it follows that $\rho_{n+2,i+1}$ is also smooth, and thus, identity (2.5.7) holds pointwise.

We will now prove Lemma 2.5.3 through the induction scheme from above. To that end, denote by $I \subseteq I_*$ the set of admissible index pairs for which Lemma 2.5.3 (or equivalently, Item (i)) is true. Due to (2.5.4), we have that $(n,i) \in I$ if and only if $(n, n-i-1) \in I$. From (2.5.5) and by applying the general Leibniz rule, it follows that $\rho_{n,n-2} \in C^\infty(-1,1)$ and for all $m \geq 0$,

$$\begin{aligned}\rho_{n,n-2}^{(m)}(t) = \frac{1}{2(n-2)}(1-t^2)g_{n-1}^{(m+2)}(t) - \frac{2m+n-1}{2(n-2)} t g_{n-1}^{(m+1)}(t)& \\ - \frac{(m+n-1)(m-1)}{2(n-2)} g_{n-1}^{(m)}(t) + \frac{n}{2\omega_{n-1}} \frac{d^m t}{dt^m}&.\end{aligned}$$

Now we multiply both sides with $(1-t^2)^{\frac{n-2}{2}+m}$, pass to the limit $t \to \pm 1$, and employ (2.4.5). This verifies (2.5.6) in the instance where $i = n-2$ for all $m \geq 0$, and thus, $(n, n-2) \in I$.

Suppose now that $(n,i) \in I$. Then, as we have established at the beginning of the proof, $\rho_{n+2,i+1} \in C^\infty(-1,1)$ and for all $m \geq 0$,

$$\begin{aligned}\lim_{t\to+1}(1-t^2)^{\frac{n}{2}+m}\rho_{n+2,i+1}^{(m)}(t) &= \frac{1}{2\pi}\lim_{t\to+1}(1-t^2)^{\frac{n-2}{2}+(m+1)}\rho_{n,i}^{(m+1)}(t) \\ &= \frac{1}{2\pi}2^{(m+1)-2}\frac{\Gamma\left(\frac{n-2}{2}+(m+1)\right)}{\pi^{\frac{n-2}{2}}} = 2^{m-2}\frac{\Gamma\left(\frac{n}{2}+m\right)}{\pi^{\frac{n}{2}}},\end{aligned}$$

where we applied the appropriate instance of recurrence relation (2.5.7). Similarly,

$$\lim_{t\to-1}(1-t^2)^{\frac{n}{2}+m}\rho_{n+2,i+1}^{(m)}(t) = \frac{1}{2\pi}\lim_{t\to-1}(1-t^2)^{\frac{n-2}{2}+(m+1)}\rho_{n,i}^{(m+1)}(t) = 0,$$

and thus, we obtain that also $(n+2, i+1) \in I$. Lemma 2.5.5 now yields $I = I_*$, which concludes the argument. □

We now establish the Legendre type differential equation (2.1.4).

Theorem 2.5.6. *Let $(n,i) \in I_*$. Then for all $t \in (-1,1)$,*

$$(1-t^2)\rho''_{n,i}(t) - nt\rho'_{n,i}(t) - i(n-i-1)\rho_{n,i}(t) = 0. \tag{2.5.9}$$

Proof. First, define a second order differential operator

$$D_{n,i} = (1-t^2)\frac{d^2}{dt^2} - nt\frac{d}{dt} - i(n-i-1)\mathrm{Id}$$

and denote by I the set of all index pairs $(n,i) \in I_*$ for which $D_{n,i}\rho_{n,i} = 0$. Due to (2.5.4) and the fact that $D_{n,i} = D_{n,n-i-1}$, we have that $(n,i) \in I$ if and only if $(n, n-i-1) \in I$. Note that

$$(n-1)D_{n,n-2}\overline{\square}_n = (n-2)D_{n+1,n-1}\overline{\square}_{n-1},$$

as can be shown by direct computation. Therefore, by applying (2.5.5), we obtain

$$\begin{aligned} D_{n,n-2}\rho_{n,n-2}(t) &= \frac{n-1}{2(n-2)}D_{n,n-2}\overline{\square}_n g_{n-1}(t) + \frac{n}{2\omega_{n-1}}D_{n,n-2}t \\ &= \frac{1}{2}D_{n+1,n-1}\overline{\square}_{n-1}g_{n-1}(t) + \frac{n}{2\omega_{n-1}}D_{n,n-2}t \\ &= -\frac{n-1}{2\omega_{n-1}}D_{n+1,n-1}t + \frac{n}{2\omega_{n-1}}D_{n,n-2}t = 0, \end{aligned}$$

where the third equality is due to (2.4.3). It follows that $(n, n-2) \in I$.

Suppose now that $(n,i) \in I$. Note that $D_{n+2,i+1} \circ \frac{d}{dt} = \frac{d}{dt} \circ D_{n,i}$, as can again be shown by direct computation. By employing recurrence relation (2.5.7), we get that for all $t \in (-1,1)$,

$$D_{n+2,i+1}\rho_{n+2,i+1}(t) = \frac{1}{2\pi}D_{n+2,i+1}\rho'_{n,i}(t) = \frac{1}{2\pi}(D_{n,i}\rho_{n,i})'(t) = 0,$$

and thus, $(n+2, i+1) \in I$. By Lemma 2.5.5, we have that $I = I_*$. □

The differential equation (2.5.9), combined with the behavior at the end points has the following notable interpretation: If we lift $\rho_{n,i}$ to a zonal function on the unit sphere in $\mathbb{R}^{n+1}$, then it is the Green's function of some strictly elliptic Helmholtz type operator.

Corollary 2.5.7. *Let $(n,i) \in I_*$. Then $\rho_{n,i}(\langle e_{n+1}, \cdot\rangle) \in L^1(\mathbb{S}^n)$ and*

$$(-\Delta_{\mathbb{S}^n} + i(n-i-1)\mathrm{Id})\rho_{n,i}(\langle e_{n+1}, \cdot\rangle) = \pi\delta_{e_{n+1}} \tag{2.5.10}$$

in the sense of distributions on $\mathbb{S}^n$.

Proof. Due to Lemma 2.5.3 (i), the function $(1-t^2)^{\frac{n-2}{2}}\rho_{n,i}(t)$ is smooth and bounded, and thus, integrable on $(-1,1)$. By a change to spherical cylinder coordinates, this implies that $\rho_{n,i}(\langle e_{n+1}, \cdot\rangle)$ is integrable on $\mathbb{S}^n$.

Next, observe that both sides of (2.5.10) are zonal distributions on $\mathbb{S}^n$. Hence, it suffices to test the identity on zonal smooth functions $\phi \in C^\infty(\mathbb{S}^n)$. Then ϕ is of the form $\phi(u) = \psi(\langle e_{n+1}, u\rangle)$ for some $\psi \in C^\infty(-1,1)$. By a change to spherical cylinder coordinates,

$$\begin{aligned}
&\langle \phi, (\Delta_{\mathbb{S}^n} - i(n-i-1)\mathrm{Id})\rho_{n,i}(\langle e_{n+1}, \cdot\rangle)\rangle_{\mathbb{S}^n} \\
&\quad = \langle (\Delta_{\mathbb{S}^n} - i(n-i-1)\mathrm{Id})\phi, \rho_{n,i}(\langle e_{n+1}, \cdot\rangle)\rangle_{\mathbb{S}^n} \\
&\quad = \omega_n \int_{(-1,1)} \left(((1-t^2)^{\frac{n}{2}}\psi'(t))' - i(n-i-1)\psi(t)(1-t^2)^{\frac{n-2}{2}} \right) \rho_{n,i}(t)dt.
\end{aligned}$$

We consider the first part of the integral and perform integration by parts. Then

$$\begin{aligned}
&\int_{(-1,1)} ((1-t^2)^{\frac{n}{2}}\psi'(t))'\rho_{n,i}(t)dt \\
&\qquad = \int_{(-1,1)} \psi'(t)\rho_{n,i}'(t)(1-t^2)^{\frac{n}{2}}dt + \left[(1-t^2)^{\frac{n}{2}}\psi'(t)\rho_{n,i}(t)\right]_{t=-1}^{t=1},
\end{aligned}$$

where the marginal terms are to be understood as limits. If γ is a geodesic in $\mathbb{S}^n$ such that $\gamma(0) = -e_{n+1}$, then

$$\frac{d}{d\theta}(\phi \circ \gamma)(\theta) = \frac{d}{d\theta}\psi(-\cos\theta) = (\sin\theta)\psi'(-\cos\theta) = (1-t^2)^{\frac{1}{2}}\psi'(t),$$

where we applied the change of variables $t = -\cos\theta$ in the final equality. Since $\phi \in C^\infty(\mathbb{S}^n)$, the final expression is bounded, so by (2.5.6), the marginal terms vanish. Integrating by parts a second time yields

$$\begin{aligned}
&\int_{(-1,1)} ((1-t^2)^{\frac{n}{2}}\psi'(t))'\rho_{n,i}(t)dt \\
&\qquad = \int_{(-1,1)} \psi(t)((1-t^2)^{\frac{n}{2}}\rho_{n,i}'(t))'dt - \left[(1-t^2)^{\frac{n}{2}}\psi(t)\rho_{n,i}'(t)\right]_{t=-1}^{t=1} \\
&\qquad = \int_{(-1,1)} \psi(t)((1-t^2)^{\frac{n}{2}}\rho_{n,i}'(t))'dt - \frac{\pi}{\omega_n}\psi(1),
\end{aligned}$$

where the marginal terms are again to be understood as limits. The final equality is due to (2.5.6) and the fact that ψ is bounded. By the differential equation (2.5.9), the remaining integral terms add up to zero, and thus, we obtain that

$$\langle \phi, (\Delta_{\mathbb{S}^n} - i(n-i-1)\mathrm{Id})\rho_{n,i}(\langle e_{n+1}, \cdot\rangle)\rangle_{\mathbb{S}^n} = \pi\phi(e_{n+1}),$$

which completes the proof. □

2.5.3 Description of ρ_i

In the following, we use the Legendre type differential equation (2.5.9) to obtain the desired description of the distributions ρ_i; we prove what is left to prove of Theorem 2.C and discuss its consequences.

First, we want to point out that the differential equation (2.5.9), combined with the behavior at the end points $t = 1$ and $t = -1$, determines each function $\rho_{n,i}$ uniquely. In particular, we obtain an interesting representation for the family $\rho_{n,i}$. For parameters $\lambda, \mu \in \mathbb{R}$, the *associated Legendre function* $\tilde{P}^\mu_\lambda$ is defined as

$$\tilde{P}^\mu_\lambda(t) = \frac{e^{\mathrm{i}\pi\frac{\mu}{2}}}{\Gamma(1-\mu)} \left(\frac{1+t}{1-t}\right)^{\frac{\mu}{2}} {}_2F_1\Big(-\lambda, \lambda+1, 1-\mu, \frac{1-t}{2}\Big),$$

where ${}_2F_1$ is the hypergeometric function (cf. [2, 8.1.2]). We show that $\rho_{n,i}$ is a constant multiple of a reflection of an associated Legendre function.

Proposition 2.5.8. *Let $(n,i) \in I_*$. Then for all $t \in (-1,1)$,*

$$\rho_{n,i}(t) = \frac{\Gamma(i)\Gamma(n-i-1)}{4(2\pi)^{\frac{n-2}{2}}} e^{\mathrm{i}\pi\frac{n-2}{4}} (1-t^2)^{-\frac{n-2}{4}} \tilde{P}^{1-\frac{n}{2}}_{i-\frac{n}{2}}(-t). \tag{2.5.11}$$

Proof. For fixed $(n,i) \in I_*$, we consider the function $y(t) = (1-t^2)^{\frac{n-2}{4}} \rho_{n,i}(t)$. A simple transformation of (2.5.9) shows that y is a solution to Legendre's differential equation: For all $t \in (-1,1)$,

$$(1-t^2)y''(t) - 2ty'(t) + \Big(\lambda(\lambda+1) - \frac{\mu^2}{1-t^2}\Big)y(t) = 0 \tag{2.5.12}$$

with parameters $\lambda = i - \frac{n}{2}$ and $\mu = 1 - \frac{n}{2}$.

Moreover, the associated Legendre function $\tilde{P}^\mu_\lambda(t)$, and thus, also its reflection $\tilde{P}^\mu_\lambda(-t)$ solve this differential equation (cf. [2, 8.1.1]). In order to describe their behavior at the end points, we recall Gauss' summation theorem (cf. [2, 15.1.20]), which states that

$$\lim_{t\to+1} {}_2F_1(a,b,c,t) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

whenever $c - a - b > 0$. As an instance of this identity and due to the simple fact that ${}_2F_1(a,b,c,0) = 1$, we obtain that

$$\begin{aligned} &\lim_{t\to-1} (1-t^2)^{\frac{n-2}{4}} \tilde{P}^\mu_\lambda(t) = e^{-\mathrm{i}\pi\frac{n-2}{4}} \frac{2^{\frac{n-2}{2}}\Gamma(\frac{n-2}{2})}{\Gamma(i)\Gamma(n-i-1)}, \\ &\lim_{t\to+1} (1-t^2)^{\frac{n-2}{4}} \tilde{P}^\mu_\lambda(t) = 0. \end{aligned} \tag{2.5.13}$$

This entails that $\tilde{P}^\mu_\lambda(t)$ and $\tilde{P}^\mu_\lambda(-t)$ are linearly independent, and thus, they form a basis of the two-dimensional space of solutions to (2.5.12). Consequently, the function $y(t)$ is a linear combination of those two functions. By comparing (2.5.6) and (2.5.13), we observe that $y(t)$ must in fact be a constant multiple of $\tilde{P}^\mu_\lambda(-t)$, and we obtain the respective multiplicative constant from a simple division. □

In (2.5.6), we have shown that the functions $\rho_{n,i}$ have a singularity at $t = 1$ and determined the precise asymptotic behavior. In order to investigate their behavior at $t = -1$, we perform the substitution $t = -\cos\theta$ (just like we did for the Berg functions in Section 2.4.1). That is, we define a family of functions $\hat{\rho}_{n,i}$, where $(n,i) \in I_*$, by

$$\hat{\rho}_{n,i}(\theta) = \rho_{n,i}(-\cos\theta), \qquad \theta \in (-\pi,\pi)\backslash\{0\}.$$

Applying the transformation $t = -\cos\theta$ to the recursion for the family $\rho_{n,i}$ yields the following recursion for the family $\hat{\rho}_{n,i}$:

$$\hat{\rho}_{n,i} = \hat{\rho}_{n,n-i-1}, \tag{2.5.14}$$

$$\hat{\rho}_{n,n-2}(\theta) = \frac{1}{2(n-2)}\hat{g}''_{n-1}(\theta) + \frac{1}{2}\frac{\hat{g}'_{n-1}(\theta)}{\tan\theta} + \frac{n-1}{2(n-2)}\hat{g}_{n-1}(\theta) - \frac{n}{2\omega_{n-1}}\cos\theta, \tag{2.5.15}$$

$$\hat{\rho}_{n+2,i+1}(\theta) = \frac{1}{2\pi}\frac{\hat{\rho}'_{n,i}(\theta)}{\sin\theta}. \tag{2.5.16}$$

This recursion shows that the functions $\hat{\rho}_{n,i}$ extend to even analytic functions on $(-\pi,\pi)$. To that end, denote by I the set of all index pairs $(n,i) \in I_*$ for which this is the case. Due to (2.5.14), we have that $(n,i) \in I$ if and only if $(n, n-i-1) \in I$. We have shown in Section 2.4.1 that $\hat{g}_{n-1}$ is an even analytic function on $(-\pi,\pi)$. Consequently, its derivative $\hat{g}'_{n-1}$ has a simple zero in $\theta = 0$ which cancels with the zero of the tangent function, thus $(n, n-2) \in I$. Applying a similar argument to the recurrence relation (2.5.16) shows that if $(n,i) \in I$, then also $(n+2, i+1) \in I$. Lemma 2.5.5 now yields $I = I_*$.

We can now complete the proof of Theorem 2.C.

Theorem 2.5.9. *Let $1 \le i < n-1$. Then ρ_i is an $L^1(\mathbb{S}^{n-1})$ function which is smooth on $\mathbb{S}^{n-1} \setminus \{e_n\}$ and strictly positive up to the addition of a linear function.*

Proof. Due to Lemma 2.5.3 (i), the function $\rho_{n,i}(\langle e_n, \cdot\rangle)$ is integrable on $\mathbb{S}^{n-1}$ and smooth on $\mathbb{S}^{n-1}\backslash\{\pm e_n\}$. In particular, it defines a zonal distribution on $\mathbb{S}^{n-1}$ which must coincide with $\tilde{\rho}_i$, since zonal distributions are uniquely determined by their multipliers.

Observe that $\hat{\rho}_{n,i}(\theta)$, as an even analytic function on $(-\pi,\pi)$, can be written as a power series of θ^2. Thus, there exists a unique analytic smooth function $q_{n,i}$ such that $\rho_{n,i}(-\cos\theta) = q_{n,i}(\theta^2)$. Consequently,

$$\tilde{\rho}_i(u) = \rho_{n,i}(\langle e_n, u\rangle) = \rho_{n,i}(-\cos d_{\mathbb{S}^{n-1}}(-e_n, u)) = q_{n,i}(d_{\mathbb{S}^{n-1}}(-e_n,u)^2)$$

for all $u \in \mathbb{S}^{n-1} \setminus \{e_n\}$. Together with Lemma 2.4.4, this shows that $\tilde{\rho}_i$ is smooth around the south pole.

We have shown that $\rho_{n,i}$ satisfies the second-order elliptic differential equation (2.5.9) on $(-1,1)$. Moreover, due to the representation formula (2.5.11),

$$\lim_{t\to-1} \rho_{n,i}(t) = \frac{\Gamma(i)\Gamma(n-i-1)}{2^n\pi^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}.$$

Combined with (2.5.6), we see that

$$\lim_{t\to-1}\rho_{n,i}(t)>0 \qquad \text{and} \qquad \lim_{t\to+1}\rho_{n,i}(t)>0,$$

so the strong maximum principle (cf. [37, Section 6.4.2]) implies that the function $\rho_{n,i}$ attains a strictly positive minimum on $(-1,1)$. Since $\tilde{\rho}_i$ is just a renormalization of ρ_i up to the addition of a linear function, the claim follows. □

As was pointed out in the introduction, this result has some interesting consequences for Minkowski valuations that we will now touch upon, revisiting our discussion in Section 2.4.2. Since ρ_i is weakly positive, it is the density of the generating measure of some Minkowski endomorphism $\Psi^{(i)}$. Hence, if a Minkowski valuation $\Phi_i \in \mathbf{MVal}_i$ is generated by some body of revolution $L \in \mathcal{K}(\mathbb{R}^n)$, then, by (2.1.3) and (2.4.11),

$$f_{\mathfrak{L}\Phi_i} = f_{\Phi_i} * \rho_i = h_L * \mu_{\Psi^{(i)}} = h_{\Psi^{(i)}L},$$

and thus, $\mathfrak{L}\Phi$ is again generated by some body of revolution. More generally, we obtain that all known examples of generating functions are preserved under the action of the Lefschetz operators.

Corollary 2.5.10. *Generating functions of the form* (2.4.12) *are preserved under the action of the Lefschetz operators.*

Proof. Consider a Minkowski valuation $\Phi_i \in \mathbf{MVal}_i$ with a generating function of the form (2.4.12), that is, $f_{\Phi_i} = h_L + f_{\tilde{\mathrm{M}}_j} * \mu_\Psi$, where L is a convex body of revolution, $\Psi \in \mathbf{MVal}_1$, and $i+j=n+1$. Due to (2.1.3), (2.4.10), and (2.5.2),

$$\begin{aligned} f_{\mathfrak{L}\Phi_i} &= h_L * \rho_i + (f_{\tilde{\mathrm{M}}_j} * \rho_i) * \mu_\Psi = h_{\Psi^{(i)}L} + a_{n,i} f_{\tilde{\mathrm{M}}_{j-1}} * \mu_\Psi, \\ f_{\Lambda\Phi_i} &= i f_{\Phi_i} = h_{iL} + f_{\tilde{\mathrm{M}}_{j+1}} * (\rho_{i-1} * \mu_\Psi) = h_{iL} + b_{n,i} f_{\tilde{\mathrm{M}}_{j+1}} * \mu_{\Psi \circ \Psi^{(i-1)}}, \end{aligned}$$

with some constants $a_{n,i}$ and $b_{n,i}$. Consequently, both $f_{\mathfrak{L}\Phi_i}$ are $f_{\Lambda\Phi_i}$ are again of the form (2.4.12). □

Lastly, the combined Lefschetz operators $\Lambda \circ \mathfrak{L}$ and $\mathfrak{L} \circ \Lambda$ act on Minkowski valuations as a composition with Minkowski endomorphisms.

Corollary 2.5.11. *Let* $1 \le i < n-1$. *There exists* $\Psi^{(i)} \in \mathbf{MVal}_1$ *such that*

$$\Lambda(\mathfrak{L}\Phi_i) = i\Psi^{(i)} \circ \Phi_i \qquad \textit{and} \qquad \mathfrak{L}(\Lambda\Phi_{i+1}) = i\Psi^{(i)} \circ \Phi_{i+1}$$

for every $\Phi_i \in \mathbf{MVal}_i$ *and* $\Phi_{i+1} \in \mathbf{MVal}_{i+1}$.

Proof. Due to (2.1.3) and (2.4.11),

$$\begin{aligned} f_{\Lambda(\mathfrak{L}\Phi_i)} &= i f_{\mathfrak{L}\Phi_i} = i f_{\Phi_i} * \rho_i = f_{i\Psi^{(i)}\circ\Phi_i}, \\ f_{\mathfrak{L}(\Lambda\Phi_{i+1})} &= f_{\Lambda\Phi_i} * \rho_i = i f_{\Phi_i} * \rho_i = f_{i\Psi^{(i)}\circ\Phi_i}. \end{aligned}$$

Since a Minkowski valuation is uniquely determined by its generating function, this proves the corollary. □

2.6 Klain–Schneider functions of Minkowski valuations

In this final section, we discuss the connection between Klain–Schneider functions and generating functions. For a smooth spherical valuation $\varphi \in \mathbf{Val}_i^{\infty,\mathrm{sph}}$, where $1 \le i \le n-1$, we can express its Klain–Schneider function in terms of its generating function by

$$\mathrm{KS}_\varphi(E,u) = \frac{1}{\binom{n-1}{i}}(\pi_{E,-i} f_\varphi)(u), \qquad (E,u) \in \mathrm{Fl}_{n,i+1}, \tag{2.6.1}$$

as can easily be deduced from (2.2.1), and is a special case of (2.3.2).

Now we turn to Minkowski valuations. For $\Phi \in \mathbf{MVal}$ and $v \in \mathbb{S}^{n-1}$, we define a valuation $\varphi^v \in \mathbf{Val}$ by $\varphi^v(K) = h_{\Phi K}(v)$, generalizing its associated real valued valuation. The Klain function of an *even* Minkowski valuation $\Phi \in \mathbf{MVal}_i$ can then be defined as a continuous function on $\mathrm{Gr}_{n,i} \times \mathbb{S}^{n-1}$ by

$$\mathrm{Kl}_\Phi(F,v) = \mathrm{Kl}_{\varphi^v}(F), \qquad F \in \mathrm{Gr}_{n,i}, \qquad v \in \mathbb{S}^{n-1}.$$

For fixed $F \in \mathrm{Gr}_{n,i}$, the function $\mathrm{Kl}_\Phi(F,\,\cdot\,)$ is the support function of a convex body that is invariant under all rotations stabilizing F, called the *Klain body* of Φ. It was introduced by Schuster and Wannerer [102] and is given by $1/\kappa_i \Phi(B^n \cap F)$. Every even $\Phi \in \mathbf{MVal}_i$ is uniquely determined by its Klain body.

In order to also encompass non-even Minkowski valuations, like the mean section operators, we define the Klain–Schneider function of a Minkowski valuation $\Phi \in \mathbf{MVal}_i$ as

$$\mathrm{KS}_\Phi((E,u),v) = \mathrm{KS}_{\varphi^v}(E,u), \qquad (E,u) \in \mathrm{Fl}_{n,i+1}, \qquad v \in \mathbb{S}^{n-1}.$$

Then KS_Φ is a continuous function on $\mathrm{Fl}_{n,i+1} \times \mathbb{S}^{n-1}$ that determines Φ uniquely. Moreover, due to the rotationally equivariance of Φ, we have that

$$\mathrm{KS}_\Phi(\vartheta(E,u),\vartheta v) = \mathrm{KS}_\Phi((E,u),v)$$

for every $\vartheta \in \mathrm{SO}(n)$. Since the group of rotations $\mathrm{SO}(n)$ acts transitively on $\mathrm{Fl}_{n,i+1}$, this implies that whenever we fix some $(E,u) \in \mathrm{Fl}_{n,i+1}$, the function $\mathrm{KS}_\Phi((E,u),\,\cdot\,) \in C(\mathbb{S}^{n-1})$ already contains all the information of KS_Φ. In this way, it also makes sense to consider the Klain–Schneider function of Φ as this continuous function on the unit sphere that is invariant under all rotations stabilizing (E,u).

We consider some concrete examples. In [102], the Klain bodies of the projection body maps and the even parts of the mean section operators were computed. Below, we compute the respective Klain–Schneider functions (taking into account the odd part of the mean section operators) in a very direct way.

Example 2.6.1. For $1 \le i \le n-1$, the i-th projection body map Π_i is even, and thus, for every $(E,u) \in \mathrm{Fl}_{n,i+1}$ and $v \in \mathbb{S}^{n-1}$,

$$\begin{aligned}\mathrm{KS}_{\Pi_i}((E,u),v) &= \frac{1}{2}\mathrm{Kl}_{\Pi_i}(E \cap u^\perp, v)\\ &= \frac{1}{2\kappa_i} V_i((B^n \cap (E \cap u^\perp)^\perp)|v^\perp) = \frac{1}{2}\|v|(E^\perp \vee u)\|.\end{aligned}$$

Example 2.6.2. Let $1 \leq i \leq n-1$ and $i+j=n+1$. For a subspace $E \in \mathrm{Gr}_{n,i+1}$ and a convex body $K \in \mathcal{K}(E)$, an application of (2.3.5) shows that for $v \in \mathbb{S}^{n-1}$,

$$h(\tilde{\mathrm{M}}_j K, v) = \frac{\left[\begin{smallmatrix} j \\ 2 \end{smallmatrix}\right]}{\left[\begin{smallmatrix} n \\ i+1 \end{smallmatrix}\right]} h(\tilde{\mathrm{M}}_2^E K, v) = \frac{\left[\begin{smallmatrix} j \\ 2 \end{smallmatrix}\right]}{\left[\begin{smallmatrix} n \\ i+1 \end{smallmatrix}\right]} \|v|E\| h(\tilde{\mathrm{M}}_2^E K, \mathrm{pr}_E v),$$

where $\tilde{\mathrm{M}}_2^E$ denotes the centered mean section operator $\tilde{\mathrm{M}}_2$ relative to E. By (2.4.2),

$$h(\tilde{\mathrm{M}}_2^E K, \mathrm{pr}_E v) = m_{i+1,2} \int_{\mathbb{S}^i(E)} g_2(\langle \mathrm{pr}_E v, w\rangle) S_i^E(K, dw).$$

In conclusion, for every $(E,u) \in \mathrm{Fl}_{n,i+1}$ and $v \in \mathbb{S}^{n-1}$,

$$\mathrm{KS}_{\tilde{\mathrm{M}}_j}((E,u),v) = \frac{\left[\begin{smallmatrix} j \\ 2 \end{smallmatrix}\right]}{\left[\begin{smallmatrix} n \\ i+1 \end{smallmatrix}\right]} m_{i+1,2} \|v|E\| g_2(\langle \mathrm{pr}_E v, u\rangle).$$

Next, we show that the Klain–Schneider function of a Minkowski valuation can be expressed in terms of its generating function via some hemispherical transform. As an application, we can describe how the Klain–Schneider function is transformed under composition with Minkowski endomorphisms.

Theorem 2.6.3. *Let $1 \leq i \leq n-1$ and $\Phi \in \mathbf{MVal}_i$. Then for $(E,u) \in \mathrm{Fl}_{n,i+1}$,*

$$\mathrm{KS}_\Phi((E,u), \cdot\,) = \frac{1}{\binom{n-1}{i}} \lambda_{\mathbb{H}^{n-i-1}(E,u)} * f_\Phi, \tag{2.6.2}$$

where $\lambda_{\mathbb{H}^{n-i-1}(E,u)}$ denotes the restriction of $\mathcal{H}^{n-i-1}$ to $\mathbb{H}^{n-i-1}(E,u)$.

Proof. Suppose first that (the associated real valued valuation of) Φ is smooth, and thus, f_Φ is smooth. Then (2.6.1) yields for all $\vartheta \in \mathrm{SO}(n)$,

$$\begin{aligned} \mathrm{KS}_\Phi((E,u),\vartheta e_n) &= \frac{1}{\binom{n-1}{i}} (\pi_{E,-i}(\vartheta f_\Phi))(u) = \frac{1}{\binom{n-1}{i}} \int_{\mathbb{H}^{n-i-1}(E,u)} (\vartheta f_\Phi)(w) dw \\ &= \frac{1}{\binom{n-1}{i}} \int_{\mathbb{S}^{n-1}} (\vartheta f_\Phi)(w)\ \lambda_{\mathbb{H}^{n-i-1}(E,u)}(dw) = \frac{1}{\binom{n-1}{i}} (\lambda_{\mathbb{H}^{n-i-1}(E,u)} * f_\Phi)(\vartheta e_n). \end{aligned}$$

To pass from the smooth to the general case, we follow the argument of the proof of [99, Theorem 6.5]. Let $\eta_\delta \in C^\infty(\mathbb{S}^{n-1})$ be a spherical approximate identity of zonal functions. Then $f_\Phi * \eta_\delta$ converges to f_Φ in $L^1(\mathbb{S}^{n-1})$ as $\delta \to 0$. Moreover, the functions $f_\Phi * \eta_\delta$ generate Minkowski valuations $\Phi^{(\delta)} \in \mathbf{MVal}_i$ and $\Phi^{(\delta)}$ converges to Φ uniformly on compact subsets of $\mathcal{K}(\mathbb{R}^n)$. Thus, by continuity of the Klain–Schneider map, $\mathrm{KS}_{\Phi^{(\delta)}}$ converges uniformly to KS_Φ. □

Corollary 2.6.4. *Let $1 \leq i \leq n-1$ and $\Phi \in \mathbf{MVal}_i$. Then for every Minkowski endomorphism Ψ and $(E,u) \in \mathrm{Fl}_{n,i+1}$,*

$$\mathrm{KS}_{\Psi\circ\Phi}((E,u), \cdot\,) = \mathrm{KS}_\Phi(E,u), \cdot\,) * \mu_\Psi. \tag{2.6.3}$$

Proof. By combining (2.4.11) and (2.6.2),

$$\begin{aligned}\mathrm{KS}_{\Psi\circ\Phi}((E,u),\,\cdot\,) &= \frac{1}{\binom{n-1}{i}}\lambda_{\mathbb{H}^{n-i-1}(E,u)} * f_{\Psi\circ\Phi} = \frac{1}{\binom{n-1}{i}}\lambda_{\mathbb{H}^{n-i-1}(E,u)} * (f_\Phi * \mu_\Psi) \\ &= \frac{1}{\binom{n-1}{i}}(\lambda_{\mathbb{H}^{n-i-1}(E,u)} * f_\Phi) * \mu_\Psi = \mathrm{KS}_\Phi((E,u),\,\cdot\,) * \mu_\Psi,\end{aligned}$$

and thus, we obtain (2.6.3). □

This corollary shows that composition with Minkowski endomorphisms acts on the Klain–Schneider function of a Minkowski valuation in the same way as on its generating function: by convolution with the corresponding generating measure from the right. Consequently, the discussion of the structure of the space of generating functions in Section 2.4.2 also translates to Klain–Schneider functions of Minkowski valuations.

2.A The spherical convolution

We can identify $\mathbb{S}^{n-1}$ with the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, where we denote by $\mathrm{SO}(n-1) \subseteq \mathrm{SO}(n)$ the the subgroup of rotations that stabilize the north pole e_n. Due to this identification, the convolution structure on the compact Lie group $\mathrm{SO}(n)$ naturally induces a convolution structure on $\mathbb{S}^{n-1}$. For an in-depth exposition, we recommend the article by Grinberg and Zhang [48] and the book by Takeuchi [107].

The convolution of some $\phi \in C^\infty(\mathbb{S}^{n-1})$ and $\gamma \in C^{-\infty}(\mathbb{S}^{n-1})^{\mathrm{zonal}}$ is defined by

$$(\phi * \gamma)(\vartheta e_n) = \langle \phi, \vartheta\gamma\rangle_{\mathbb{S}^{n-1}} = \langle \vartheta^{-1}\phi, \gamma\rangle_{\mathbb{S}^{n-1}}, \qquad \vartheta \in \mathrm{SO}(n).$$

Since $\mathrm{SO}(n)$ operates transitively on the unit sphere and γ is zonal, this is well-defined and it turns out that $\phi * \gamma \in C^\infty(\mathbb{S}^{n-1})$. Thus, we may define the convolution of spherical distributions by

$$\begin{gathered} C^{-\infty}(\mathbb{S}^{n-1}) \times C^{-\infty}(\mathbb{S}^{n-1})^{\mathrm{zonal}} \to C^{-\infty}(\mathbb{S}^{n-1}) : (\nu, \gamma) \mapsto \nu * \gamma \\ \langle \phi, \nu * \gamma\rangle_{\mathbb{S}^{n-1}} = \langle \phi * \gamma, \nu\rangle_{\mathbb{S}^{n-1}}, \qquad \phi \in C^\infty(\mathbb{S}^{n-1}).\end{gathered}$$

Clearly, the convolution product is linear in each of its arguments. Although its definition is fundamentally asymmetrical (in that the right factor must always be zonal), it enjoys the nice properties one might expect. For all distributions $\nu \in C^\infty(\mathbb{S}^{n-1})$ and $\gamma, \gamma_1, \gamma_2 \in C^{-\infty}(\mathbb{S}^{n-1})^{\mathrm{zonal}}$, we have the following:

- $(\nu * \gamma_1) * \gamma_2 = \nu * (\gamma_1 * \gamma_2)$. (associativity)
- $\gamma_1 * \gamma_2 = \gamma_2 * \gamma_1$. (commutativity)
- $\nu * \delta_{e_n} = \nu$ and $\delta_{e_n} * \gamma = \gamma$. (neutral element)
- $(\vartheta\nu) * \gamma = \vartheta(\nu * \gamma)$ for all $\vartheta \in \mathrm{SO}(n)$. ($\mathrm{SO}(n)$-equivariance)

Next, note that by virtue of the chain of identifications (2.4.8), the definition above also encompasses the convolution of continuous functions, Lebesgue integrable functions, and signed measures. We want to note that the spherical convolution of two signed measures is again a signed measure. Also, the convolution of a signed measure and an $L^1(\mathbb{S}^{n-1})$ function is an $L^1(\mathbb{S}^{n-1})$ function. In particular, identities such as (2.5.1) are to be understood in a weak sense, as an equality in the space $L^1(\mathbb{S}^{n-1})$.

Moreover, the convolution of a signed measure and a continuous function is again a continuous function that can be expressed by an integral representation. If $f \in C(\mathbb{S}^{n-1})$ and $\mu \in \mathcal{M}(\mathbb{S}^{n-1})^{\mathrm{zonal}}$, then

$$(f * \mu)(\vartheta e_n) = \int_{\mathbb{S}^{n-1}} f(\vartheta v)\ \mu(dv), \qquad \vartheta \in \mathrm{SO}(n).$$

In the case where $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ and $f \in C(\mathbb{S}^{n-1})^{\mathrm{zonal}}$, there is a unique function $\bar{f} \in C[-1,1]$ such that $f(w) = \bar{f}(\langle e_n, w\rangle)$ for all $w \in \mathbb{S}^{n-1}$, and then,

$$(\mu * f)(u) = \int_{\mathbb{S}^{n-1}} \bar{f}(\langle u, v\rangle)\ \mu(dv), \qquad u \in \mathbb{S}^{n-1}.$$

2.B Spherical Harmonics

As a general reference for this section, we cite the monograph by Groemer [49]. Denote by $\mathcal{H}_k^n$ the space of *spherical harmonics* of dimension $n \geq 3$ and degree $k \geq 0$, that is, harmonic, k-homogeneous polynomials, restricted to the unit sphere. These turn out to be precisely the eigenspaces of the spherical Laplacian, that is, $\Delta_{\mathbb{S}^{n-1}} Y_k = -k(k+n-2)Y_k$ for every $Y_k \in \mathcal{H}_k^n$. Recall that $\Delta_{\mathbb{S}^{n-1}}$ is a second-order uniformly elliptic self-adjoint differential operator that has compact resolvent and intertwines rotations. Consequently, $L^2(\mathbb{S}^{n-1})$ decomposes into an orthogonal direct sum of the spaces $\mathcal{H}_k^n$, each of which is finite dimensional and $\mathrm{SO}(n)$-irreducible.

For each $k \geq 0$, the subspace of zonal spherical harmonics in $\mathcal{H}_k^n$ is of dimension one and spanned by $P_k^n(\langle e_n, \cdot\,\rangle)$, where P_k^n denotes the Legendre polynomial of dimension $n \geq 3$ and degree $k \geq 0$. It can be defined by *Rodrigues' formula*, which states that

$$P_k^n(t) = (-1)^k \frac{\Gamma\left(\frac{n-1}{2}\right)}{2^k \Gamma\left(\frac{n-1}{2}+k\right)} (1-t^2)^{-\frac{n-3}{2}} \left(\frac{d}{dt}\right)^k (1-t^2)^{\frac{n-3}{2}+k}. \tag{2.B.1}$$

The orthogonal projection π_k from $L^2(\mathbb{S}^{n-1})$ onto the subspace $\mathcal{H}_k^n$ is given by

$$\pi_k \phi = \frac{\dim \mathcal{H}_k^n}{\omega_{n-1}}\ \phi * P_k^n(\langle e_n, \cdot\,\rangle), \qquad \phi \in L^2(\mathbb{S}^{n-1}).$$

As a convolution transform, π_k extends to a map from $C^{-\infty}(\mathbb{S}^{n-1})$ onto $\mathcal{H}_k^n$, and for every distribution $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$, its formal Fourier series $\sum_{k=0}^{\infty} \pi_k \nu$, also called its *spherical harmonic expansion*, converges to ν in the weak sense. Moreover, if $\gamma \in C^{-\infty}(\mathbb{S}^{n-1})^{\mathrm{zonal}}$, then

$$\gamma = \sum_{k=0}^{\infty} \frac{\dim \mathcal{H}_k^n}{\omega_{n-1}} a_k^n[\gamma] P_k^n(\langle e_n, \cdot\,\rangle),$$

where $a_k^n[\gamma] = \langle P_k^n(\langle e_n, \cdot \rangle), \gamma\rangle_{\mathbb{S}^{n-1}}$. The *Funk–Hecke Theorem* states that the convolution product of two distributions $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$ and $\gamma \in C^{-\infty}(\mathbb{S}^{n-1})^{\mathrm{zonal}}$ has the spherical harmonic expansion

$$\nu * \gamma = \sum_{k=0}^{\infty} a_k^n[\gamma]\pi_k \nu.$$

That is, the convolution transform $\nu \mapsto \nu*\gamma$ acts as a multiple of the identity on each space $\mathcal{H}_k^n$ of spherical harmonics. The Fourier coefficients $a_k^n[\gamma]$ are thus also called *multipliers*.

If γ is defined by some $L^1(\mathbb{S}^{n-1})$ function, then the multipliers can be computed using cylindrical coordinates on the sphere: If $g : (-1,1) \to \mathbb{R}$ is measurable and $(1-t^2)^{\frac{n-3}{2}} g(t)$ is integrable on $(-1,1)$, then $g(\langle e_n, \cdot \rangle) \in L^1(\mathbb{S}^{n-1})^{\mathrm{zonal}}$ and

$$a_k^n[g(\langle e_n, \cdot \rangle)] = \omega_{n-1} \int_{(-1,1)} P_k^n(t)(1-t^2)^{\frac{n-3}{2}} g(t)dt. \tag{2.B.2}$$

For convenience, we use the convention $a_k^n[g] = a_k^n[g(\langle e_n, \cdot \rangle)]$.

Finally, we prove Lemma 2.5.4, which is a refinement of [87, Lemma 3.6]. To that end, we need the following technical integration by parts lemma.

Lemma 2.B.1 ([27, Lemma 3.3])**.** *Let $\alpha > 0$ and let $g \in \mathcal{D}'(-1,1)$ be such that $(1-t^2)^{\frac{\alpha}{2}} g'(t)$ is a finite signed measure on $(-1,1)$. Then g is a locally integrable function and $(1-t^2)^{\frac{\alpha-2}{2}} g(t) \in L^1(-1,1)$. Moreover, whenever $\psi \in C^1(-1,1)$ such that both $(1-t^2)^{-\frac{\alpha-2}{2}} \psi'(t)$ and $(1-t^2)^{\frac{\alpha}{2}} \psi(t)$ are bounded, then*

$$\int_{(-1,1)} \psi(t) g'(dt) = -\int_{(-1,1)} \psi'(t) g(t)dt. \tag{2.B.3}$$

Proof of Lemma 2.5.4. By Lemma 2.B.1, we have that $(1-t^2)^{\frac{n-3}{2}} g(t) \in L^1(-1,1)$. Combining (2.B.2) with identity (2.B.3) in the instance where $\psi = P_k^{n+2}$ yields

$$\begin{aligned} \frac{1}{\omega_{n+1}} a_k^{n+2}[g'] &= \int_{(-1,1)} P_k^{n+2}(t)(1-t^2)^{\frac{n-1}{2}} g'(t)dt \\ &= (n-1)\int_{(-1,1)} P_{k+1}^n(t)(1-t^2)^{\frac{n-3}{2}} g(t)dt = \frac{n-1}{\omega_{n-1}} a_{k+1}^n[g], \end{aligned}$$

where in the second equality, we also employed Rodrigues' formula (2.B.1). Since $(n-1)\omega_{n+1} = 2\pi\omega_{n-1}$, this yields (2.5.8). $\square$

2.C Proof of Proposition 2.3.2

For the proof of Item (i), we express the Radon type transform $\tilde{\mathcal{R}}_{k,k-1}$ in terms of an actual Radon transform of some weighted spherical projections. To that end, if $E \subseteq F \subseteq \mathbb{R}^n$ are nested subspaces, we denote by $\pi_{E,m}^F$ and $\pi_{E,m}^{F*}$ the m-weighted spherical projection and lifting relative to F. Moreover, if $F \in \mathrm{Gr}_{n,k}$ and $E \in \mathrm{Gr}_{n,k-1}^F$, then for all $f \in C(\mathbb{S}^{k-1}(F))$

$$\int_{\mathbb{S}^{k-1}(F)} f(u)du = \int_{\mathbb{S}^{k-2}(E)} \int_{\mathbb{H}^1(F;E,u)} f(v)\langle u, v\rangle^{k-2} dv\, du, \tag{2.C.1}$$

where $\mathbb{H}^1(F;E,u) = \{v \in \mathbb{S}^{k-1}(F) \setminus E^\perp : \mathrm{pr}_E v = u\}$ (cf. [44, (3.3)]).

Lemma 2.C.1. *Let $1 < k \leq n$. Then for all $\zeta \in C(\mathrm{Fl}_{n,k})$ and $(E,u) \in \mathrm{Fl}_{n,k-1}$,*

$$[\tilde{\mathcal{R}}_{k,k-1}\zeta](E,u) = \frac{\omega_{n-k+1}}{\omega_{n-k+2}} \int_{\mathrm{Gr}_{n,k}^E} [\pi_{E,n-2k+2}^F \zeta(F,\,\cdot\,)](u)\; dF \tag{2.C.2}$$

Proof. First, we parametrize the Grassmann manifold $\mathrm{Gr}_{n,k-1}^{E\cap u^\perp}$ through setting $F = (E \cap u^\perp) \vee w$ for $F \in \mathrm{Gr}_{n,k-1}^{E\cap u^\perp}$ and $w \in \mathbb{S}^{n-k+1}(E^\perp \vee u)$. This yields

$$\begin{aligned}
[\tilde{\mathcal{R}}_{k,k-1}\zeta](E,u) &= \frac{1}{\omega_{n-k+2}} \int_{\mathbb{S}^{n-k+1}(E^\perp \vee u)} \zeta(E \vee w, \mathrm{pr}_{w^\perp} u) dw \\
&= \frac{1}{\omega_{n-k+2}} \int_{\mathbb{S}^{n-k}(E^\perp)} \int_{\mathbb{H}^1(E^\perp \vee u; E^\perp, w)} \zeta(E \vee v, \mathrm{pr}_{v^\perp} u) \langle w, v\rangle^{n-k} dv\; dw \\
&= \frac{1}{\omega_{n-k+2}} \int_{\mathbb{S}^{n-k}(E^\perp)} \int_{\mathbb{S}^1(u \vee w)\cap w^+} \zeta(E \vee w, \mathrm{pr}_{v^\perp} u) \langle u, \mathrm{pr}_{v^\perp} u\rangle^{n-k} dv\; dw,
\end{aligned}$$

where the second equality is an application of (2.C.1) relative to the space $E^\perp \vee u$ and in the final equality, we simply rewrote the inner integral. Next, observe that for every vector $w \in \mathbb{S}^{n-k}(E^\perp)$ and function $f \in C(\mathbb{S}^1(u \vee w))$,

$$\int_{\mathbb{S}^1(u\vee w)\cap w^+} f(\mathrm{pr}_{v^\perp} u) dv = \int_{\mathbb{S}^1(u\vee w)\cap u^+} f(v) dv = \int_{\mathbb{H}^1(E\vee w; E, u)} f(v) dv,$$

as follows from applying a change of variables with respect to a rotation by $\pi/2$ in the plane $u \vee w$. Applying this to $f(v) = \zeta(E \vee w, v)\langle u, v\rangle^{n-k}$ yields

$$\begin{aligned}
[\tilde{\mathcal{R}}_{k,k-1}\zeta](E,u) &= \frac{1}{\omega_{n-k+2}} \int_{\mathbb{S}^{n-k}(E^\perp)} \int_{\mathbb{H}^1(E\vee w; E, u)} \zeta(E \vee w, v) \langle u, v\rangle^{n-k} dv\; dw \\
&= \frac{\omega_{n-k+1}}{\omega_{n-k+2}} \int_{\mathrm{Gr}_{n,k}^E} \int_{\mathbb{H}^1(F;E,u)} \zeta(F, v) \langle u, v\rangle^{n-k} dy\; dF,
\end{aligned}$$

where in the second equality, we applied the parametrization $F = E \vee w$ with $F \in \mathrm{Gr}_{n,k}^E$ and $w \in \mathbb{S}^{n-k}(E^\perp)$. □

We will need the fact that weighted spherical projections map linear functions to linear functions, which was shown in [47].

Lemma 2.C.2 ([47, Lemma 2.3])**.** *Let $1 \leq k \leq n$, $E \in \mathrm{Gr}_{n,k}$, and $m > -k$. Then for all $u \in \mathbb{S}^{n-1}$ and $v \in \mathbb{S}^{k-1}(E)$,*

$$[\pi_{E,m}\langle u,\,\cdot\,\rangle](v) = \frac{\omega_{n+m+1}}{\omega_{k+m+1}} \langle u, v\rangle. \tag{2.C.3}$$

Proof of Proposition 2.3.2 (i). Let $\zeta \in C(\mathrm{Fl}_{n,k})$ and denote by $z : \mathrm{Gr}_{n,k} \to \mathbb{R}^n$ the continuous function with the property that $\pi_1^F \zeta(F, \cdot\,) = \langle z(F), \cdot\,\rangle$ for $F \in \mathrm{Gr}_{n,k}$. By combining (2.C.2) and (2.C.3), we have that for all $(E,u) \in \mathrm{Fl}_{n,k-1}$,

$$\begin{aligned}\frac{\omega_{n-k+2}}{\omega_{n-k+1}}[\tilde{\mathcal{R}}_{k,k-1}\pi_1^{\langle k\rangle}\zeta](E,u) &= \int_{\mathrm{Gr}_{n,k}^E}[\pi^F_{E,n-2k+2}\langle z(F), \cdot\,\rangle](u)dF \\ &= \frac{\omega_{n-k+3}}{\omega_{n-k+2}}\int_{\mathrm{Gr}_{n,k}^E}\langle z(F), u\rangle dF = \frac{\omega_{n-k+3}}{\omega_{n-k+2}}\frac{1}{\omega_k}\int_{\mathrm{Gr}_{n,k}^E}\int_{\mathbb{S}^{k-1}(F)}\zeta(F,v)\langle u,v\rangle dv\, dF,\end{aligned}$$

Note that whenever $F \in \mathrm{Gr}_{n,k}^E$, then $[\pi^{F*}_{E,1}\langle u, \cdot\,\rangle](v) = \langle u, v\rangle$ for all $u \in \mathbb{S}^{k-2}(E)$ and $v \in \mathbb{S}^{k-1}(F)$, and thus,

$$\begin{aligned}\int_{\mathbb{S}^{k-1}(F)}\zeta(F,v)\langle u,v\rangle dv &= \int_{\mathbb{S}^{k-1}(F)}\zeta(F,v)[\pi^{F*}_{E,1}\langle u, \cdot\,\rangle](v)dv \\ &= \int_{\mathbb{S}^{k-2}(E)}[\pi^F_{E,1}\zeta(F, \cdot\,)](v)\langle u,v\rangle dv.\end{aligned}$$

By changing the order of integration, we obtain that

$$\begin{aligned}[\tilde{\mathcal{R}}_{k,k-1}\pi_1^{\langle k\rangle}\zeta](E,u) &= \frac{\omega_{n-k+1}\omega_{n-k+3}}{\omega_{n-k+2}^2}\frac{1}{\omega_k}\int_{\mathbb{S}^{k-2}(E)}\int_{\mathrm{Gr}_{n,k}^E}[\pi^F_{E,1}\zeta(F, \cdot\,)](v)dF\ \langle u,v\rangle dv \\ &= \frac{1}{\omega_{k-1}}\int_{\mathbb{S}^{k-2}(E)}[\tilde{\mathcal{R}}'_{k,k-1}\zeta](E,v)\langle u,v\rangle dv = [\pi_1^{\langle k-1\rangle}\tilde{\mathcal{R}}'_{k,k-1}\zeta](E,u),\end{aligned}$$

which concludes the argument. $\square$

Similar as for $\tilde{\mathcal{R}}_{k,k-1}$, we can express the Radon type transform $\tilde{\mathcal{R}}_{k,k+1}$ in terms of an actual Radon transform and the 1-weighted spherical lifting. More precisely, if $1 \leq k < n$, then for all $\zeta \in C(\mathrm{Fl}_{n,k})$ and $(E,u) \in \mathrm{Fl}_{n,k+1}$,

$$[\tilde{\mathcal{R}}_{k,k+1}\zeta](E,u) = \int_{\mathrm{Gr}_{n,k}^E}[\pi^{E*}_{F,1}\zeta(F, \cdot\,)](u)\ dF, \tag{2.C.4}$$

which is immediate from the definition. Now we prove Item (ii) of Proposition 2.3.2.

Proof of Proposition 2.3.2 (ii). Let $\zeta \in C(\mathrm{Fl}_{n,k})$ and $(E,u) \in \mathrm{Fl}_{n,k+1}$. Then by (2.C.4) and a change of order of integration,

$$\begin{aligned}[\pi_1^{\langle k+1\rangle}\tilde{\mathcal{R}}_{k,k+1}\zeta](E,u) &= \frac{1}{\omega_{k+1}}\int_{\mathrm{Gr}_{n,k}^E}\int_{\mathbb{S}^k(E)}[\pi^{E*}_{F,1}\zeta(F, \cdot\,)](v)\langle u,v\rangle dv\ dF \\ &= \frac{1}{\omega_{k+1}}\int_{\mathrm{Gr}_{n,k}^E}\int_{\mathbb{S}^{k-1}(F)}\zeta(F,v)[\pi^E_{F,1}\langle u, \cdot\,\rangle](v)dv\ dF \\ &= \frac{\omega_{k+3}}{\omega_{k+2}}\frac{1}{\omega_{k+1}}\int_{\mathrm{Gr}_{n,k}^E}\int_{\mathbb{S}^{k-1}(F)}\zeta(F,v)\langle u,v\rangle dv\ dF,\end{aligned}$$

where the final equality is due to (2.C.3). Note that for every $F \in \mathrm{Gr}_{n,k}^E$,

$$\frac{1}{\omega_k} \int_{\mathbb{S}^{k-1}(F)} \zeta(F,v)\langle u,v\rangle dv = [\pi_1^{\langle k\rangle}\zeta](F,u) = [\pi_{F,1}^{E*}(\pi_1^{\langle k\rangle}\zeta)(F,\,\cdot\,)](u).$$

Plugging this into the expression above and applying (2.C.4) again, we obtain

$$\begin{aligned}
[\pi_1^{\langle k+1\rangle}\tilde{\mathcal{R}}_{k,k+1}\zeta](E,u) &= \frac{\omega_{k+3}}{\omega_{k+2}} \frac{\omega_k}{\omega_{k+1}} \int_{\mathrm{Gr}_{n,k}^E} [\pi_{F,1}^{E*}(\pi_1^{\langle k\rangle}\zeta)(F,\,\cdot\,)](u)\, dF \\
&= \frac{k}{k+1}[\tilde{\mathcal{R}}_{k,k+1}\pi_1^{\langle k\rangle}\zeta](E,u),
\end{aligned}$$

which concludes the argument. □

3 The Klain approach to zonal valuations

3.1 Introduction

A *valuation* on the space $\mathcal{K}(\mathbb{R}^n)$ of convex bodies (that is, convex, compact subsets) of $\mathbb{R}^n$, where $n \geq 3$, is a map $\varphi : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ satisfying

$$\varphi(K) + \varphi(L) = \varphi(K \cup L) + \varphi(K \cap L)$$

whenever $K, L \in \mathcal{K}(\mathbb{R}^n)$ and $K \cup L$ is convex. We denote by $\mathbf{Val}(\mathbb{R}^n)$ the space of continuous, translation-invariant valuations on $\mathbb{R}^n$. Valuations in $\mathbf{Val}(\mathbb{R}^n)$ play a central role in convex and integral geometry, appearing naturally in a wide range of applications (see the monographs [43, 96]). Notable examples include the intrinsic volumes – fundamental geometric quantities that encode information about the size and shape of convex bodies, such as volume, surface area, and mean width – and, more generally, mixed volumes.

Among the most celebrated results in valuation theory is Hadwiger's characterization of rigid motion invariant valuations in terms of intrinsic volumes. This foundational theorem has sparked a long and ongoing line of research (see, e.g., [3, 18, 32, 61, 79]). Below, we state it in a homogeneous form: for the subspaces $\mathbf{Val}_i(\mathbb{R}^n) \subseteq \mathbf{Val}(\mathbb{R}^n)$ of valuations that are homogeneous of degree $i \in \{0, \ldots, n\}$ (that is, $\varphi(\lambda K) = \lambda^i \varphi(K)$ for all $\lambda > 0$).

Theorem 3.1.1 ([52]). *For $0 \leq i \leq n$, a valuation $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$ is rotation invariant if and only if it is a constant multiple of the i-th intrinsic volume.*

Theorem 3.1.1 was originally proved without assuming homogeneity. However, by McMullen's homogeneous decomposition theorem [86], the space $\mathbf{Val}(\mathbb{R}^n)$ is the direct sum of the spaces $\mathbf{Val}_i(\mathbb{R}^n)$ for $i \in \{0, \ldots, n\}$. Since $\mathbf{Val}_0(\mathbb{R}^n)$ consists only of constant valuations and $\mathbf{Val}_n(\mathbb{R}^n)$ is spanned by the volume functional V_n [52], the problem reduces to understanding the intermediate degrees, $1 \leq i \leq n-1$.

Classification theorems, such as Theorem 3.1.1, reveal the underlying geometric structure of valuations, providing conceptual proofs of central integral geometric formulas, such as the classical Crofton, Cauchy–Kubota, and kinematic formulas (see, e.g., [63]). Motivated by this, recent efforts have focused on classification theorems for valuations invariant under different linear groups (see, e.g., [3, 10, 15–19, 22, 23, 42, 79, 104]). One of these is the subgroup $\mathrm{SO}(n-1)$ of $\mathrm{SO}(n)$: the stabilizer of the n-th canonical basis vector e_n. Valuations invariant under $\mathrm{SO}(n-1)$, called *zonal*, appear naturally in the study of convex-body valued (Minkowski) valuations (see, e.g., [1, 27, 36, 50, 60, 75, 76, 78, 87, 88, 99, 102, 103]) and possess similarities to rigid motion invariant valuations on convex functions (see, e.g., [31–35, 65, 67]).

Recently in [68], a full characterization of zonal valuations in $\mathbf{Val}(\mathbb{R}^n)$ was obtained using the following function spaces: We set $\mathcal{D}^0 = C[-1,1]$ and for $\alpha > 0$, we define $\mathcal{D}^\alpha$ as the class of functions $\bar{f} \in C(-1,1)$ such that

$$\lim_{s\to\pm 1} \bar{f}(s)(1-s^2)^{\frac{\alpha}{2}} = 0 \quad \text{and} \quad \lim_{s\to\pm 1} \int_0^s \bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}} dt \text{ exists and is finite.}$$

Knoerr's main result in [68] is the following Hadwiger-type theorem for zonal valuations, which gives an improper integral representation in terms of the i-th order *area measure* $S_i(K, \cdot\,)$ of a convex body $K \in \mathcal{K}(\mathbb{R}^n)$ (cf. [96, Ch. 4]). In the following, a function on $\mathbb{S}^{n-1}$ is called zonal if it is invariant under $\mathrm{SO}(n-1)$.

Theorem 3.1.2 ([68])**.** *For $1 \leq i \leq n-1$, a valuation $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$ is zonal if and only if there exists a function $f = \bar{f}(\langle e_n, \cdot\,\rangle) \in C(\mathbb{S}^{n-1} \setminus \{\pm e_n\})$ with $\bar{f} \in \mathcal{D}^{n-i-1}$ such that*

$$\varphi(K) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{S}^{n-1}\setminus U_\varepsilon} f(u)\, dS_i(K,u), \qquad K \in \mathcal{K}(\mathbb{R}^n), \tag{3.1.1}$$

where $U_\varepsilon = \{u \in \mathbb{S}^{n-1} : |\langle e_n, u\rangle| > 1-\varepsilon\}$. Moreover, f is unique up to the addition of a zonal linear function.

Here, we denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the standard Euclidean inner product on $\mathbb{R}^n$. For degrees $1 \leq i < n-1$, Theorem 3.1.2 is obtained by approximation from an earlier result of Schuster and Wannerer [103] for *smooth valuations* (see Section 3.4.1). For $i = n-1$, it is an immediate consequence of a classical result by McMullen [86] and the principal value is in fact a proper integral.

Inspired by the Hadwiger type theorem for convex functions with Monge–Ampère measures [33], our first main result is an analogue of Theorem 3.1.2, where the role of the i-th area measure is replaced by the mixed area measure

$$S_i(K,\mathbb{D},\,\cdot\,) = S(K^{[i]},\mathbb{D}^{[n-i-1]},\,\cdot\,),$$

see [96, Sec. 5.1], where $\mathbb{D}$ denotes the $(n-1)$-dimensional unit disk in $e_n^\perp$ and $L^{[j]}$ denotes the tuple consisting of j copies of the body L.

Theorem 3.A. *For $1 \leq i \leq n-1$, a valuation $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$ is zonal if and only if there exists a zonal function $g \in C(\mathbb{S}^{n-1})$ such that*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} g(u)\, dS_i(K,\mathbb{D},u), \qquad K \in \mathcal{K}(\mathbb{R}^n). \tag{3.1.2}$$

Moreover, g is unique up to the addition of a zonal linear function.

This integral representation has certain benefits. In contrast to (3.1.1), the integral is always proper and the corresponding class of integral kernels is simpler, depending neither on the dimension nor the degree. Moreover, the values of a zonal valuation φ on cones with axis e_n determine the integral kernel g (up to linear maps), and therefore, the valuation itself. As a further consequence, we can easily express convergence of zonal valuations as uniform convergence of the integral kernels (see Section 3.2.1).

The Klain approach. To establish Theorem 3.A, we follow the approach of Klain [61], who provided a significantly simpler proof of Hadwiger's classical Theorem 3.1.1. At the core of Klain's approach lies the following characterization of valuations vanishing on all hyperplanes, also known as *simple* valuations. Here and throughout, we say that a valuation vanishes on a subspace E if it vanishes on all convex bodies $K \subseteq E$.

Theorem 3.1.3 ([61, 95])**.** *A valuation $\varphi \in \mathbf{Val}(\mathbb{R}^n)$ vanishes on all hyperplanes if and only if there exist a constant $c \in \mathbb{R}$ and an odd function $f \in C(\mathbb{S}^{n-1})$ such that*

$$\varphi(K) = cV_n(K) + \int_{\mathbb{S}^{n-1}} f(u)\, dS_{n-1}(K, u), \qquad K \in \mathcal{K}(\mathbb{R}^n).$$

Let us point out that Theorem 3.1.3 is due to Klain [61] for even valuations and due to Schneider [95] for odd valuations. Denoting by $\mathrm{Gr}_k(\mathbb{R}^n)$ the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$, a simple corollary of Theorem 3.1.3 can be formulated as follows.

Corollary 3.1.4. *Let $0 \le i \le n-1$ and $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$. If φ vanishes on all subspaces $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$, then $\varphi = 0$.*

Recently, Klain's approach was employed by Colesanti, Ludwig, and Mussnig in [35] to give a new and simpler proof of their previously established Hadwiger-type theorem for valuations on convex functions [32]. To adapt Klain's method to our context, we require an analogue of Theorem 3.1.3 for zonal valuations.

Theorem 3.B. *A zonal valuation $\varphi \in \mathbf{Val}(\mathbb{R}^n)$ vanishes on some hyperplane containing e_n if and only if there exist a constant $c \in \mathbb{R}$ and a zonal function $f \in C(\mathbb{S}^{n-1})$ vanishing on $\mathbb{S}^{n-1} \cap e_n^\perp$ such that*

$$\varphi(K) = cV_n(K) + \int_{\mathbb{S}^{n-1}} f(u)\, dS_{n-1}(K, u), \qquad K \in \mathcal{K}(\mathbb{R}^n). \tag{3.1.3}$$

Let us note that a zonal valuation that vanishes on one hyperplane containing e_n must already vanish on all such hyperplanes. In a similar way to Theorem 3.1.3, Theorem 3.B implies that every i-homogeneous valuation is already determined by its restriction to a *single* $(i+1)$-dimensional subspace.

Corollary 3.C. *Let $0 \le i \le n-1$ and $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$ be zonal. If φ vanishes on some subspace $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$ containing e_n, then $\varphi = 0$.*

As will be demonstrated later, Corollary 3.C proves extremely helpful in showing integral geometric formulas for $\mathrm{SO}(n-1)$-invariant quantities.

Another important step in Klain's approach is the extension of valuations from proper subspaces. In the setting of rigid motion invariant valuations, where only intrinsic volumes appear, this is trivial: the restriction of the i-th intrinsic volume to an i-dimensional subspace is exactly the volume on that subspace. In general, the problem of extending valuations is more delicate (see, e.g., [38]). In fact, for zonal valuations this is not always possible; a difficulty that also arises in the functional setting [35]. For smooth valuations, however, we can always extend the integral representations that naturally emerge from Klain's approach.

Theorem 3.D. *Let $1 \leq i \leq n-1$ and $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$ be such that $e_n \in E$. Then for every zonal function $f_E \in C^\infty(\mathbb{S}^i(E))$, there exists a zonal function $f \in C^\infty(\mathbb{S}^{n-1})$ such that*

$$\int_{\mathbb{S}^{n-1}} f(u)\, dS_i(K,u) = \int_{\mathbb{S}^i(E)} f_E(u)\, dS_i^E(K,u), \qquad K \in \mathcal{K}(E).$$

Here, $\mathbb{S}^i(E) = \mathbb{S}^{n-1} \cap E$ and by $S_i^E(K,\cdot)$ we denote the i-th order area measure of $K \subseteq E$ relative to $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$. The Hadwiger-type theorem for smooth, zonal valuations of [103] is now a direct consequence of Corollary 3.C and Theorem 3.D (see Section 3.4.1). Let us point out that our proof does not rely on any deep results from representation theory such as the irreducibility theorem [4].

Moving between representations. In order to deduce the general Hadwiger-type theorems (for continuous valuations) from the statement for smooth valuations, it is crucial to understand how to move between the integral representations (3.1.1) and (3.1.2). Indeed, in Section 3.2.1, we define a map T_{n-i-1} that transforms the integral kernel from one representation to the other. In order to show that this is in fact the right transform, by Corollary 3.C, it suffices to check that the corresponding integral representations coincide on some subspace containing e_n.

These restrictions can be made explicit by certain maps π_{n-i-1} and $\pi_{n-i-1,\mathbb{D}}$, derived from the mixed spherical projections, which were recently introduced by the authors in [25] to describe the relations between (mixed) area measures of lower dimensional bodies in different ambient spaces (see Section 3.2.2).

$$\begin{tikzcd}
\mathcal{D}^{n-i-1} \arrow[rr, "T_{n-i-1}", "\cong"'] \arrow[dr, "\pi_{n-i-1}"'] & & C[-1,1] \arrow[dl, "\pi_{n-i-1,\mathbb{D}}"] \\
& C[-1,1] &
\end{tikzcd}$$

Figure 3.1

Once all the elements are in place, it is easy to check that the diagram in Fig. 3.1 commutes. By Corollary 3.C, we can thus move between the different representations for zonal valuations via the transform T_{n-i-1}.

Furthermore, using the simpler description of convergence of zonal valuations in terms of the integral representation (3.1.2), and the fact that every continuous zonal function trivially defines a valuation in this way, we obtain Theorem 3.A by a simple approximation argument. Having established Theorem 3.A, we can use the diagram of Fig. 3.1 to recover Theorem 3.1.2 without further difficulty.

Applications Just as their counterparts for rigid motion invariant valuations, Theorems 3.A and 3.B have various applications to integral geometry that we will now discuss. First, we show the following additive kinematic formula for $\mathrm{SO}(n-1)$, extending a recent result by Hug, Mussnig, and Ulivelli [55, Thm. 1.5] from the even to the general case. Throughout, we denote by κ_m the volume of the m-dimensional unit ball in $\mathbb{R}^m$.

Theorem 3.E. *Let $1 \leq j \leq n-1$ and $g \in C(\mathbb{S}^{n-1})$ be zonal. For all $K, L \in \mathcal{K}(\mathbb{R}^n)$,*

$$\begin{aligned}\int_{\mathrm{SO}(n-1)} \int_{\mathbb{S}^{n-1}} & g(u)\, dS_j(K + \vartheta L, \mathbb{D}, u)\, d\vartheta \\ &= \frac{1}{\kappa_{n-1}} \sum_{i=0}^{j} \binom{j}{i} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} q(u,v)\, dS_i(K, \mathbb{D}, u)\, dS_{j-i}(L, \mathbb{D}, v),\end{aligned} \tag{3.1.4}$$

where $g = \bar{g}(\langle e_n, \cdot \rangle)$ and $q(u,v) = \max\{\langle e_n, u\rangle, \langle e_n, v\rangle\} \bar{g}(\min\{\langle e_n, u\rangle, \langle e_n, v\rangle\})$.

In [55], the statement is derived for even g from an additive kinematic formula for convex functions. As the functional setting is related to the even geometrical setting (see Remark 3.3.10), this leads to an additional symmetry assumption. Here, as the left-hand side of (3.1.4) is a zonal valuation, Theorem 3.A yields (3.1.4) for an unknown integral kernel q, depending a priori on i and j. The map q can then be easily determined by plugging in cones with axis e_n.

In a similar way, we can recover the following Kubota-type formula from [56, Thm. 3.2]: For $1 \leq i \leq n-1$, $K \in \mathcal{K}(\mathbb{R}^n)$, and $f \in C(\mathbb{S}^{n-1})$,

$$\int_{\mathrm{Gr}_i(\mathbb{R}^n, e_n)} \int_{\mathbb{S}^{i-1}(E)} f(u)\, dS_{i-1}^E(K|E, u)\, dE = \frac{\kappa_{i-1}}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} f(u)\, dS_{i-1}(K, \mathbb{D}, u). \tag{3.1.5}$$

where integration on $\mathrm{Gr}_i(\mathbb{R}^n, e_n) = \{E \in \mathrm{Gr}_i(\mathbb{R}^n) : e_n \in E\}$ is with respect to the unique rotation invariant probability measure and $K|E$ denotes the orthogonal projection of K onto E. From (3.1.5), in turn, we can retrieve the following Crofton-type formula. Here, we denote by $\mathrm{AGr}_j(\mathbb{R}^n)$ the affine j-Grassmannian, endowed with the rigid motion invariant measure, normalized so that the set of j-flats intersecting the unit ball has measure κ_{n-j}, and by $h_L(u) = \max\{\langle u, x\rangle : x \in L\}$ the support function of $L \in \mathcal{K}(\mathbb{R}^n)$.

Corollary 3.F. *Let $1 \leq j \leq n-1$. If $K \in \mathcal{K}(\mathbb{R}^n)$ is origin-symmetric, then*

$$\int_{\mathrm{AGr}_j(\mathbb{R}^n)} h_{K \cap E}(e_n)\, dE = a_{n,j} V(K^{[n-j+1]}, \mathbb{D}^{[j-1]}), \tag{3.1.6}$$

where $a_{n,j} = \frac{\pi j \kappa_j \kappa_{n-j}}{(j+1)(n-j+1)\kappa_{j+1}\kappa_n}$.

Our primary interest in (3.1.6) stems from the fact that the expression on the left-hand side coincides with the support function of the mean section body of K. Indeed, the j-th mean section operator $\mathrm{M}_j : \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ is defined by

$$h_{\mathrm{M}_j K}(u) = \int_{\mathrm{AGr}_j(\mathbb{R}^n)} h_{K \cap E}(u)\, dE, \qquad u \in \mathbb{S}^{n-1}. \tag{3.1.7}$$

These operators were introduced by Goodey and Weil [45], motivated by the question whether every convex body can be reconstructed from the mean of random sections. They gave a positive answer to this question by finding an integral representation of M_j using the Berg functions $g_j \in C^\infty(-1,1)$, $j \geq 2$. The functions g_j were constructed by Berg [13] in his solution to the Christoffel problem [29] so that for every dimension $n \geq 2$ and $K \in \mathcal{K}(\mathbb{R}^n)$,

$$h_{K-s(K)}(u) = \int_{\mathbb{S}^{n-1}} g_n(\langle u, v\rangle)\, dS_1(K, v), \qquad u \in \mathbb{S}^{n-1}, \tag{3.1.8}$$

where $s(K)$ denotes the *Steiner point* of K (cf. [96, p. 50]). Interestingly enough, the integral representation of M_j of type (3.1.1) arises by lifting Berg's functions to the unit sphere in different dimensions.

Theorem 3.1.5 ([45, 47]). *Let $2 \leq j < n$. Then for every $K \in \mathcal{K}(\mathbb{R}^n)$,*

$$h_{\mathrm{M}_j(K-s(K))}(u) = c_{n,j} \int_{\mathbb{S}^{n-1}} g_j(\langle u, v\rangle)\, dS_{n-j+1}(K, v), \qquad u \in \mathbb{S}^{n-1}, \tag{3.1.9}$$

where $c_{n,j} = \frac{j\kappa_j\kappa_{n-j}}{(n-j+1)n\kappa_n}$.

Let us note that the case $j = 2$ of Theorem 3.1.5 was settled in [45], while the cases $2 < j < n$ were deduced from this more recently in [47]. The proofs in [45, 47] rely on heavy tools from harmonic analysis. Applying our Corollary 3.C, we can give a new, shorter proof of the results in [47] using the case $j = 2$ from [45].

Corollary 3.G. *Theorem 3.1.5 holds for all $2 < j < n$.*

Organization of the article. In Section 3.2, we introduce the transform T_{n-i-1} and examine restrictions of integral representations; in there, we establish the commuting diagram above and Theorem 3.D. In Section 3.3, we prove Theorem 3.B and the subsequent corollary. Section 3.4 is devoted to the Hadwiger type theorems for zonal valuations; using our findings from the previous sections, we show Theorem 3.1.2 and Theorem 3.A. Finally, in Section 3.5, we discuss the applications to integral geometry and the mean section operators.

3.2 Moving between integral representations

In this section, we investigate how we can move between different integral representations of zonal valuations in $\mathbf{Val}_i(\mathbb{R}^n)$ – in terms of the i-th area measure and the mixed area measure with the disk – and how to restrict to and extend from an $(i+1)$-dimensional subspace containing e_n. For $1 \leq i \leq n-1$ and zonal functions $f, g \in C(\mathbb{S}^{n-1})$, we define zonal valuations $\varphi_{i,f}, \psi_{i,g} \in \mathbf{Val}_i(\mathbb{R}^n)$ by

$$\varphi_{i,f}(K) := \int_{\mathbb{S}^{n-1}} f(u)\, dS_i(K, u) \qquad \text{and} \qquad \psi_{i,g}(K) := \int_{\mathbb{S}^{n-1}} g(u)\, dS_i(K, \mathbb{D}, u)$$

for $K \in \mathcal{K}(\mathbb{R}^n)$. We want to find a transform T_{n-i-1} with the property that $\varphi_{i,f} = \psi_{i,g}$, whenever $T_{n-i-1}\bar{f} = \bar{g}$, where $f = \bar{f}(\langle e_n, \cdot\rangle)$ and $g = \bar{g}(\langle e_n, \cdot\rangle)$. We will evaluate $\varphi_{i,f}$ and $\psi_{i,g}$ on a certain family of cones to see how T_{n-i-1} needs to be defined. Then, we show that this transform ensures that $\varphi_{i,f}$ and $\psi_{i,g}$ also coincide on subspaces $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$ containing e_n.

3.2.1 Evaluation on cones

For $s \in [-1,1] \setminus \{0\}$, we denote by C_s the cone with basis $\mathbb{D}$ and apex $\frac{\sqrt{1-s^2}}{s} e_n$, that is,

$$C_s := \operatorname{conv}\left(\mathbb{D} \cup \left\{\tfrac{\sqrt{1-s^2}}{s} e_n\right\}\right).$$

Observe that $C_{-s} = -C_s$; for $s > 0$, the cone C_s is pointing "up", for $s < 0$, it is pointing "down". As $s \to 0$, the height of C_s tends to infinity, and $C_{-1} = C_1 = \mathbb{D}$. Moreover, the support function of C_s is given by

$$h_{C_s}(u) = \begin{cases} \sqrt{1-\langle e_n, u\rangle^2}, & \operatorname{sgn}(s)\langle e_n, u\rangle \le |s|, \\ \frac{\sqrt{1-s^2}}{s}\langle e_n, u\rangle, & \operatorname{sgn}(s)\langle e_n, u\rangle \ge |s|, \end{cases} \qquad u \in \mathbb{S}^{n-1}.$$

Evaluating $\varphi_{i,f}$ on the cones C_s boils down to computing their area measures. This has been done recently in [68, Lemma 2.2].

Lemma 3.2.1 ([68])**.** *Let $1 \le i < n-1$ and $f = \bar{f}(\langle e_n, \cdot\rangle) \in C(\mathbb{S}^{n-1})$. For $s \in [-1,1] \setminus \{0\}$,*

$$\varphi_{i,f}(C_s) = \kappa_{n-1}\left(\frac{1}{|s|}(1-s^2)^{\frac{n-i-1}{2}} \bar{f}(s) + (n-i-1)\operatorname{sgn} s \int_{-\operatorname{sgn} s}^{s} \bar{f}(t)(1-t^2)^{\frac{n-i-3}{2}} dt\right).$$

Next, we evaluate the valuation $\psi_{i,g}$ on the family C_s.

Lemma 3.2.2. *Let $1 \le i \le n-1$ and $g = \bar{g}(\langle e_n, \cdot\rangle) \in C(\mathbb{S}^{n-1})$. For $s \in [-1,1] \setminus \{0\}$,*

$$\psi_{i,g}(C_s) = \kappa_{n-1}\left(\bar{g}(-\operatorname{sgn} s) + \frac{1}{|s|}\bar{g}(s)\right). \tag{3.2.1}$$

Proof. First, note that $C_{-s} = -C_s$, and thus, the case when $s < 0$ can easily be deduced from the case when $s > 0$. Therefore, we will now restrict ourselves to the case when $s > 0$. For degree $i = n-1$, we deduce (3.2.1) from the classical facts that the surface area measure $S_{n-1}(C_s, \cdot)$ is the area of the reverse spherical image of C_s and that it integrates all linear functions to zero (cf. [96, Section 4.2].

For degrees $1 \le i < n-1$, note that for $\lambda \ge 0$, the body $\lambda C_s + \mathbb{D}$ is a truncated cone with basis $(\lambda+1)\mathbb{D}$ which is cut off at a unit disk of radius one. That is,

$$\lambda C_s + \mathbb{D} = ((\lambda+1)C_s \setminus (C_s + \tfrac{\sqrt{1-s^2}}{s} e_n)) \cup (\mathbb{D} + \tfrac{\sqrt{1-s^2}}{s} e_n).$$

Hence, by the valuation property and the translation invariance of the surface area measure, we have that

$$S_{n-1}(\lambda C_s + \mathbb{D}, \cdot\,) = S_{n-1}((\lambda+1)C_s, \cdot\,) - S_{n-1}(C_s, \cdot\,) + S_{n-1}(\mathbb{D}, \cdot\,).$$

Subsequently, applying (3.2.1) for degree $n-1$, we obtain that

$$\psi_{n-1,g}(\lambda C_s + \mathbb{D}) = \kappa_{n-1}\Big(((\lambda+1)^{n-1} - 1)\Big(\bar{g}(-1) + \frac{1}{s}\bar{g}(s)\Big) + (\bar{g}(-1) + \bar{g}(1))\Big).$$

Moreover, by the multilinearity of the surface area measure (cf. [96, p. 280]),

$$\psi_{n-1,g}(\lambda C_s + \mathbb{D}) = \sum_{i=0}^{n-1} \binom{n-1}{i} \lambda^i \psi_{i,g}(C_s).$$

By a comparison of coefficients, we obtain (3.2.1) for degrees $1 \leq i < n-1$. $\square$

Note that (3.2.1) allows us to extract the function g easily from the valuation $\psi_{i,g}$. To pursue this further, for $1 \leq i \leq n-1$ and $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$, we define a function $\bar{g}_\varphi : [-1,1] \to \mathbb{R}$ by

$$\bar{g}_\varphi(s) := \frac{1}{\kappa_{n-1}} \cdot \begin{cases} s(\varphi(\mathbb{D}) - \varphi(C_s)), & s \in [-1,0), \\ \frac{1}{i}(\varphi(\mathbb{D} + [0,e_n]) - \varphi(\mathbb{D})), & s = 0, \\ s\varphi(C_s), & s \in (0,1]. \end{cases} \tag{3.2.2}$$

It is initially unclear whether $\bar{g}_\varphi$ is continuous at $s = 0$. This will be ensured by the following elementary lemma; it was shown in [68, Proposition 4.4], but we also provide a proof for the convenience of the reader. Here and in the following, we denote by $\mathbb{B}$ the unit ball in $\mathbb{R}^n$ and by $\|\varphi\| := \sup\{|\varphi(K)| : K \subseteq \mathbb{B}\}$ the Banach norm on $\mathbf{Val}(\mathbb{R}^n)$.

Lemma 3.2.3 ([68])**.** *Let $1 \leq i \leq n-1$ and $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$. Then $\bar{g}_\varphi \in C[-1,1]$ and $\|\bar{g}_\varphi\|_\infty \leq M_i\|\varphi\|$, where $M_i > 0$ is a constant depending only on i.*

Proof. Observe that for $s \in (0, \frac{1}{\sqrt{2}})$, the cone C_s has height strictly larger than one, so making a cut at height one splits it into two parts. The upper part is the cone $a_sC_s + e_n$ and the lower part is the truncated cone $Z_s = \operatorname{conv}(\mathbb{D} \cup (a_s\mathbb{D} + e_n))$, where we defined $a_s = 1 - s/\sqrt{1-s^2}$. That is,

$$Z_s \cup (a_sC_s + e_n) = C_s \qquad \text{and} \qquad Z_s \cap (a_sC_s + e_n) = a_s\mathbb{D} + e_n.$$

Hence, by the valuation property, translation invariance, and homogeneity of φ,

$$s\varphi(C_s) = \frac{s}{1-a_s^i}(\varphi(C_s) - \varphi(a_sC_s)) = \frac{s}{1-a_s^i}(\varphi(Z_s) - \varphi(a_s\mathbb{D})).$$

First, note that if we pass to the limit $s \to 0^+$, then $a_s \to 1$, and by L'Hôpital's rule, $s/(1-a_s^i) \to 1/i$. Moreover Z_s converges to the cylinder $Z_0 = \mathbb{D} + [0,e_n]$, so by the continuity of φ, the right hand side converges to $\frac{1}{i}(\varphi(Z_0) - \varphi(\mathbb{D}))$. Repeating this argument for $s \in (-\frac{1}{\sqrt{2}}, 0)$ yields the first claim.

For the second claim, note that for $s \in (0, \frac{1}{\sqrt{2}})$, the term $s/(1-a_s^i)$ is bounded by some number $M_i' > 0$. Thus,

$$|s\varphi(C_s)| \leq M_i'(|\varphi(Z_s)| + |\varphi(a_s\mathbb{D})|) \leq M_i'(\sqrt{2}^i + 1)\|\varphi\|,$$

due to the fact that $a_s\mathbb{D} \subseteq \mathbb{B}$ and $Z_s \subseteq Z_0 \subseteq \sqrt{2}\mathbb{B}$. If $s \in [\frac{1}{\sqrt{2}}, 1]$, then $C_s \subseteq \mathbb{B}$, and thus, $|s\varphi(C_s)| \leq \|\varphi\|$. Repeating this argument for $s \in [-1,0)$ yields the second claim. $\square$

Note that the lemma above does not require the valuation φ to be zonal. For the converse estimate on zonal valuations, let $1 \leq i \leq n-1$ and $g \in C(\mathbb{S}^{n-1})$ be zonal. For every $K \in \mathcal{K}(\mathbb{R}^n)$ such that $K \subseteq \mathbb{B}$,

$$\begin{aligned}\psi_{i,g}(K) &= \int_{\mathbb{S}^{n-1}} g(u)\, dS_i(K, \mathbb{D}, u) \leq \int_{\mathbb{S}^{n-1}} dS_i(K, \mathbb{D}, u)\, \|g\|_\infty \\ &= nV(K^{[i]}, \mathbb{D}^{[n-i-1]}, \mathbb{B})\|g\|_\infty \leq nV_n(\mathbb{B})\|g\|_\infty,\end{aligned} \tag{3.2.3}$$

which shows that $\|\psi_{i,g}\| \leq n\kappa_n\|g\|_\infty$. From Theorem 3.A, we will obtain that every zonal valuation $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$ is of the form $\varphi = \psi_{i,g}$, where we can choose g to be $g = \bar{g}_\varphi(\langle e_n, \cdot\rangle)$. From this, we will deduce that zonal valuations are determined on cones, and that convergence in the Banach norm is equivalent to uniform convergence of the corresponding integral kernels (see Section 3.4.2).

3.2.2 Restricting to subspaces

Now we investigate how the integral representations of $\varphi_{i,f}$ and $\psi_{i,g}$ behave when restricted to subspaces containing e_n. In the following, for $E \in \mathrm{Gr}_k(\mathbb{R}^n)$ and $u \in \mathbb{S}^{k-1}(E)$, we define the relatively open $(n-k)$-dimensional half-sphere generated by $E^\perp$ and u as

$$\begin{aligned}\mathbb{H}^{n-k}(E, u) &= \{v \in \mathbb{S}^{n-1} \setminus E^\perp : (v|E)/\|v|E\| = u\} \\ &= \{v \in \mathbb{S}^{n-k}(E^\perp \vee u) : \langle u, v\rangle > 0\}.\end{aligned}$$

Here, $E^\perp \vee u = \mathrm{span}(E^\perp \cup u)$ denotes the subspace generated by $E^\perp$ and u. When dealing with mixed area measures of lower dimensional bodies, a key tool will be provided by the mixed spherical projections and liftings that were introduced recently by the authors [25].

Definition 3.2.4 ([25, Definition 2.3])**.** Let $1 \leq k < n$ and $E \in \mathrm{Gr}_k(\mathbb{R}^n)$. Also, let $C_1, \ldots, C_{n-k} \in \mathcal{K}(\mathbb{R}^n)$ and set $\mathcal{C} = (C_1, \ldots, C_{n-k})$. The *$\mathcal{C}$-mixed spherical projection* is the bounded linear operator $\pi_{E,\mathcal{C}} : C(\mathbb{S}^{n-1}) \to C(\mathbb{S}^{k-1}(E))$,

$$(\pi_{E,\mathcal{C}} f)(u) = \int_{\mathbb{H}^{n-k}(E,u)} f(v)\, dS^{E^\perp \vee u}(\mathcal{C}|(E^\perp \vee u), v), \qquad u \in \mathbb{S}^{k-1}(E).$$

We call its adjoint operator $\pi^*_{E,\mathcal{C}} : \mathcal{M}(\mathbb{S}^{k-1}(E)) \to \mathcal{M}(\mathbb{S}^{n-1})$ the *$\mathcal{C}$-mixed spherical lifting.* That is, for $\mu \in \mathcal{M}(\mathbb{S}^{k-1}(E))$ and $f \in C(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} f\, d(\pi^*_{E,\mathcal{C}}\mu) = \int_{\mathbb{S}^{k-1}(E)} \pi_{E,\mathcal{C}} f\, d\mu.$$

Here, we used the abbreviation $\mathcal{C}|E' = (C_1|E', \ldots, C_{n-k}|E')$. Moreover, in the case when $C_1 = \cdots = C_{n-k} = C$, we will write $\pi_{E,\mathcal{C}} = \pi_{E,C}$. In [25], the authors established the following theorem, expressing the mixed area measure of a lower dimensional body in terms of its surface area measure relative to a subspace.

Theorem 3.2.5 ([25, Theorem B])**.** *Let $1 \leq i < n-1$ and $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$. Also, let $\mathcal{C} = (C_1, \dots, C_{n-i-1})$ be a family of convex bodies with C^2 support functions. Then for all $K \in \mathcal{K}(E)$,*

$$S(K^{[i]}, \mathcal{C}, \,\cdot\,) = \frac{1}{\binom{n-1}{i}} \pi^*_{E,\mathcal{C}} S_i^E(K, \,\cdot\,). \tag{3.2.4}$$

In the instance where the reference bodies $C_1, \dots, C_{n-i-1}$ are Euclidean balls, this coincides with a particular case of a result by Goodey, Kiderlen, and Weil (see [44, Theorem 6.2]). In order to compute the $\mathbb{B}$-mixed and $\mathbb{D}$-mixed spherical projection of zonal functions, we will need spherical cylinder coordinates: For every $\bar{f} \in C[-1,1]$,

$$\int_{\mathbb{S}^{n-1}} \bar{f}(\langle e_n, u\rangle)\, du = \omega_{n-1} \int_{-1}^{1} \bar{f}(t)(1-t^2)^{\frac{n-3}{2}}\, dt,$$

where ω_m denotes the surface area of the unit sphere in $\mathbb{R}^m$ (cf. [49, p. 9]). In the following, we also define $\omega_\alpha := 2\pi^{\frac{\alpha}{2}}/\Gamma(\frac{\alpha}{2})$ for $\alpha > 0$.

Lemma 3.2.6. *Let $1 \leq i < n-1$ and $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$ be such that $e_n \in E$. Then for every function $f = \bar{f}(\langle e_n, \cdot\,\rangle) \in C(\mathbb{S}^{n-1})$, we have $\pi_{E,\mathbb{B}} f = (\pi_{n-i-1,\mathbb{B}} \bar{f})(\langle e_n, \cdot\,\rangle)$, where we define for $\alpha > 0$,*

$$(\pi_{\alpha,\mathbb{B}} \bar{f})(s) = \omega_\alpha \int_0^1 \bar{f}(st)(1-t^2)^{\frac{\alpha-2}{2}} dt, \qquad s \in (-1,1).$$

Proof. By definition of the mixed spherical projection,

$$(\pi_{E,\mathbb{B}} f)(u) = \int_{\mathbb{H}^{n-i-1}(E,u)} \bar{f}(\langle e_n, v\rangle)\, dv = \int_{\mathbb{H}^{n-i-1}(E,u)} \bar{f}(s\langle u, v\rangle)\, dv,$$

where dv denotes the spherical Lebesgue measure on $\mathbb{S}^{n-i-1}(E \vee u^\perp)$ and the second equality is due to the fact that $\langle e_n, v\rangle = \langle e_n | (E^\perp \vee u), v\rangle = \langle e_n, u\rangle\langle u, v\rangle$. Applying spherical cylinder coordinates in $\mathbb{S}^{n-i-1}(E^\perp \vee u)$ then yields the desired identity. $\square$

Next, we want to do the same for $\pi_{E,\mathbb{D}}$. However, the disk $\mathbb{D}$ does not have a C^2 support function, so we can not apply Theorem 3.2.5 directly. By an approximation argument, we can obtain the required formula as a corollary of Theorem 3.2.5. For this, we recall the classical Portmanteau theorem.

Theorem 3.2.7 ([64, Theorem 13.16])**.** *Let μ_k, μ be finite positive measures on a compact metric space X. Then the following are equivalent:*

(a) *$\mu_k \to \mu$ weakly.*

(b) *For every $f \in C(X)$, we have $\lim_{k\to\infty} \int_X f\, d\mu_k = \int_X f\, d\mu$.*

(c) *For every bounded, measurable function f on X such that its discontinuity points are a set of μ-measure zero, $\lim_{k\to\infty} \int_X f\, d\mu_k = \int_X f\, d\mu$.*

Corollary 3.2.8. *Let $1 \leq i < n-1$ and $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$ be such that $E \not\subseteq e_n^\perp$. Then for all $K \in \mathcal{K}(E)$,*

$$S(K^{[i]}, \mathbb{D}^{[n-i-1]}, \,\cdot\,) = \frac{1}{\binom{n-1}{i}} \pi^*_{E,\mathbb{D}} S_i^E(K, \,\cdot\,). \tag{3.2.5}$$

Proof. Take a sequence of convex bodies $D_k \in \mathcal{K}(\mathbb{R}^n)$ with C^2 support functions, converging to $\mathbb{D}$ in the Hausdorff metric. By Theorem 3.2.5, for every $f \in C(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} f(u)\, dS(K^{[i]}, D_k^{[n-i-1]}, u) = \frac{1}{\binom{n-1}{i}} \int_{\mathbb{S}^i(E)} (\pi_{E,D_k} f)(u)\, dS_i^E(K,u).$$

We want to pass to the limit $k \to \infty$ on both sides. On the left hand side, as mixed area measures are weakly continuous,

$$\int_{\mathbb{S}^{n-1}} f(u)\, dS(K^{[i]}, D_k^{[n-i-1]}, u) \to \int_{\mathbb{S}^{n-1}} f(u)\, dS(K^{[i]}, \mathbb{D}^{[n-i-1]}, u).$$

Next we want to establish pointwise convergence of $\pi_{E,D_k} f$ to $\pi_{E,\mathbb{D}} f$. To this end, we first show that for all $u \in \mathbb{S}^i(E)$,

$$S_{n-i-1}^{E^\perp \vee u}(\mathbb{D}|(E^\perp \vee u), \mathbb{S}^{n-i-1}(E^\perp \vee u) \cap E^\perp) = 0.$$

We consider the surface area measure $S_{n-i-1}^F(\mathbb{D}|F, \cdot\,)$ on the unit sphere $\mathbb{S}^{n-i-1}(F)$ of $F = E^\perp \vee u$, and distinguish two cases. If $e_n \notin F$, then $\mathbb{D}|F$ is a smooth convex body in F, so $S_{n-i-1}^F(\mathbb{D}|F, \cdot\,)$ is absolutely continuous with respect to the spherical Lebesgue measure. If $e_n \in F$, then $\mathbb{D}|F$ is a disk in F, and thus, $S_{n-i-1}^F(\mathbb{D}|F, \cdot\,)$ is concentrated on the two points e_n and $-e_n$. In either case, the set $\mathbb{S}^i(F) \cap E^\perp$, which is a great sphere of $\mathbb{S}^i(F)$ containing neither e_n nor $-e_n$, is a null set of $S_{n-i-1}^F(\mathbb{D}|F, \cdot\,)$.

Consequently, by the definition of the mixed spherical projection, Theorem 3.2.7, and the fact that the set $\mathbb{S}^{n-i-1}(E^\perp \vee u) \cap E^\perp$ contains all possible discontinuity points of $\mathbb{1}_{\mathbb{H}^{n-i-1}(E,u)} f$, we obtain that $(\pi_{E,D_k} f)(u) \to (\pi_{E,\mathbb{D}} f)(u)$ for all $u \in \mathbb{S}(E)$. By dominated convergence,

$$\int_{\mathbb{S}^i(E)} (\pi_{E,D_k} f)(u)\, dS_i^E(K,u) \to \int_{\mathbb{S}^i(E)} (\pi_{E,\mathbb{D}} f)(u)\, dS_i^E(K,u),$$

which yields the desired identity. □

Remark 3.2.9. Note that the proof of Corollary 3.2.8 works verbatim if the disk $\mathbb{D}$ is replaced by any smooth convex body in $e_n^\perp$.

We also want to comment on the condition that $E \not\subseteq e_n^\perp$. Since we mainly consider restrictions so subspaces containing e_n, this is not an obstacle to our purposes. However, we want to point out that the condition is necessary. Indeed, if $E \subseteq e_n^\perp$, then for all $K \in \mathcal{K}(E)$,

$$S(K^{[i]}, \mathbb{D}^{[n-i-1]}, \cdot\,) = \frac{\kappa_{n-1}\kappa_{n-i-1}}{\binom{n-1}{i}} V_i(K)\,(\delta_{e_n} + \delta_{-e_n})\,.$$

This follows by polarization from the fact that $S_{n-1}(C, \cdot\,) = V_{n-1}(C)(\delta_{e_n} + \delta_{-e_n})$ for every convex body $C \in \mathcal{K}(e_n^\perp)$. This also exemplifies that the regularity condition in Theorem 3.2.5 can not be dropped completely.

Next, we prove the analogue of Lemma 3.2.6 for the disk. For this, we need the following formula of the surface area measure of smooth convex bodies of revolution. It is an easy consequence of the proof of [88, Lemma 5.3].

Lemma 3.2.10 ([88])**.** *Let $L \in \mathcal{K}(\mathbb{R}^n)$ be a convex body of revolution with support function $h_L = \eta(\langle e_n, \cdot \rangle) \in C^2(\mathbb{S}^{n-1})$. Then*

$$dS_{n-1}(L, u) = (\mathcal{A}_1\eta)(\langle e_n, u\rangle)^{n-2}(\mathcal{A}_2\eta)(\langle e_n, u\rangle)du, \tag{3.2.6}$$

where $(\mathcal{A}_1\eta)(t) = \eta(t) - t\eta'(t)$ and $(\mathcal{A}_2\eta)(t) = (1-t^2)\eta''(t) + \eta(t) - t\eta'(t)$.

Lemma 3.2.11. *Let $0 \le i < n-1$ and $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$ be such that $e_n \in E$. Then for every $f = \bar{f}(\langle e_n, \cdot \rangle) \in C(\mathbb{S}^{n-1})$, we have that $\pi_{E,\mathbb{D}} f = (\pi_{n-i-1,\mathbb{D}}\bar{f})(\langle e_n, \cdot \rangle)$, where we define for $\alpha > 0$,*

$$(\pi_{\alpha,\mathbb{D}}\bar{f})(s) = \omega_\alpha(1-s^2)\int_0^1 \bar{f}(st)(1-s^2t^2)^{-\frac{\alpha+2}{2}}(1-t^2)^{\frac{\alpha-2}{2}}dt, \qquad s \in (-1,1).$$

Proof. By definition of the mixed spherical projection, for all $u \in \mathbb{S}^i(E)$,

$$(\pi_{E,\mathbb{D}} f)(u) = \int_{\mathbb{H}^{n-i-1}(E,u)} \bar{f}(\langle e_n, v\rangle)\, dS_{n-i-1}^{E^\perp \vee u}(\mathbb{D}|(E^\perp \vee u), v).$$

Note that for all $v \in \mathbb{S}^{n-i-1}(E^\perp \vee u)$, we have that for $s = \langle e_n, u\rangle$,

$$h_{\mathbb{D}|(E^\perp \vee u)}(v) = h_{\mathbb{D}}(v|(E^\perp \vee u)) = h_{\mathbb{D}}(v) = \sqrt{1-\langle e_n, v\rangle^2} = \sqrt{1-s^2\langle u, v\rangle^2},$$

since $\langle e_n, v\rangle = \langle e_n|(E^\perp \vee u), v\rangle = \langle e_n, u\rangle\langle u, v\rangle$. Consequently, the projected disk $\mathbb{D}|(E^\perp \vee u)$ is a smooth body of revolution in $E^\perp \vee u$ with axis of revolution u. Its support function is given by $h_{\mathbb{D}|(E^\perp \vee u)} = \eta_s(\langle u, \cdot \rangle)$, where $\eta_s(t) = \sqrt{1-s^2t^2}$. Direct computation shows that

$$\mathcal{A}_1\eta_s(t) = (1-s^2t^2)^{-\frac{1}{2}} \qquad \text{and} \qquad \mathcal{A}_2\eta_s(t) = (1-s^2)(1-s^2t^2)^{-\frac{3}{2}}.$$

Therefore, by (3.2.6), we obtain that

$$(\pi_{E,\mathbb{D}} f)(u) = (1-s^2)\int_{\mathbb{H}^{n-i-1}(E,u)} \bar{f}(s\langle u, v\rangle)(1-s^2\langle u, v\rangle^2)^{-\frac{n-i+1}{2}}dv.$$

Applying spherical cylinder coordinates in $\mathbb{S}^{n-i-1}(E^\perp \vee u)$ then yields the desired identity. □

3.2.3 The commuting diagram

Comparing the expressions for $\varphi_{i,f}(C_s)$ found in Lemma 3.2.1 and for $\psi_{i,g}(C_s)$ found in Lemma 3.2.2 motivates the following definition of a family of integral transforms.

Definition 3.2.12. Let $\alpha \ge 0$ and $\bar{f} \in C(-1,1)$. We define $T_0\bar{f} := \bar{f}$ and for $\alpha > 0$,

$$T_\alpha\bar{f}(s) := (1-s^2)^{\frac{\alpha}{2}}\bar{f}(s) + \alpha s\int_0^s \bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}dt, \qquad s \in (-1,1).$$

Note that integrating on $(0,s)$ instead of $(-1,s)$ alters the outcome only by a linear function, however this domain of integration turns out to be convenient in later computations. We will prove that if $\varphi_{i,f}=\psi_{i,g}$, then $\bar{g}$ must be related to $\bar{f}$ (up to the addition of linear functions) via the transform T_{n-i-1}.

Next, as we have defined the transform T_{n-i-1} and computed the $\mathbb{B}$-mixed and $\mathbb{D}$-mixed spherical projections of zonal functions, we can show that the diagram in Fig. 3.1 commutes. This will ensure that whenever $\bar{g}=T_{n-i-1}\bar{f}$, then the valuations $\varphi_{i,f}$ and $\psi_{i,g}$ agree on subspaces $E\in\mathrm{Gr}_{i+1}(\mathbb{R}^n)$ containing e_n. We require the following technical lemma.

Lemma 3.2.13. *For all $\alpha>0$ and $x,t\in(-1,1)$,*

$$\int_x^t s(1-s^2)^{-\frac{\alpha+2}{2}}|s^2-t^2|^{\frac{\alpha-2}{2}}\,ds=\frac{(1-x^2)^{-\frac{\alpha}{2}}|t^2-x^2|^{\frac{\alpha}{2}}}{\alpha(1-t^2)}. \tag{3.2.7}$$

Proof. Fix the parameters $\alpha>0$ and $t\in(-1,1)$ and observe that the right hand side of (3.2.7) defines a continuous function of $x\in(-1,1)$ that vanishes at $x=t$ and is differentiable on $(-1,1)\setminus\{t\}$. Differentiating the right hand side at $x\in(-1,1)\setminus\{t\}$ yields

$$\frac{d}{dx}\frac{(1-x^2)^{-\frac{\alpha}{2}}|t^2-x^2|^{\frac{\alpha}{2}}}{\alpha(1-t^2)}=-x(1-x^2)^{-\frac{\alpha+2}{2}}|x^2-t^2|^{\frac{\alpha-2}{2}}.$$

Hence, by the fundamental theorem of calculus, we obtain (3.2.7). □

Lemma 3.2.14. *Let $\alpha>0$ and $\bar{f}\in C(-1,1)$. Then $\pi_{\alpha,\mathbb{D}}T_\alpha\bar{f}=\pi_{\alpha,\mathbb{B}}\bar{f}$.*

Proof. Define a function $\bar{g}\in C(-1,1)$ by $\bar{g}:=T_\alpha\bar{f}$, that is,

$$\bar{g}(s)=(1-s^2)^{\frac{\alpha}{2}}\bar{f}(s)+\alpha s\int_0^s\bar{f}(x)(1-x^2)^{\frac{\alpha-2}{2}}\,dx,\qquad s\in(-1,1).$$

By a change of variables, we have that

$$\begin{aligned}\pi_{\alpha,\mathbb{D}}\bar{g}(t)&=\omega_\alpha(1-t^2)\int_0^1\bar{g}(st)(1-s^2t^2)^{-\frac{\alpha+2}{2}}(1-s^2)^{\frac{\alpha-2}{2}}\,ds\\&=\frac{\omega_\alpha}{t^{\alpha-1}}(1-t^2)\int_0^t\bar{g}(s)(1-s^2)^{-\frac{\alpha+2}{2}}(t^2-s^2)^{\frac{\alpha-2}{2}}\,ds.\end{aligned}$$

Next, inserting one integral expression into the other and changing the order of integration yields

$$\begin{aligned}&\int_0^t s\int_0^s\bar{f}(x)(1-x^2)^{\frac{\alpha-2}{2}}dx\;(1-s^2)^{-\frac{\alpha+2}{2}}(t^2-s^2)^{\frac{\alpha-2}{2}}ds\\&\quad=\int_0^t\bar{f}(x)(1-x^2)^{\frac{\alpha-2}{2}}\int_x^t s(1-s^2)^{-\frac{\alpha+2}{2}}(t^2-s^2)^{\frac{\alpha-2}{2}}\,ds\,dx\\&\quad=\frac{1}{\alpha(1-t^2)}\int_0^t\bar{f}(x)\frac{1}{1-x^2}(t^2-x^2)^{\frac{\alpha}{2}}\,dx,\end{aligned}$$

where the final equality is due to (3.2.7). Consequently, we obtain that

$$\begin{aligned}\pi_{\alpha,\mathbb{D}}\bar{g}(t) &= \frac{\omega_\alpha}{t^{\alpha-1}}\int_0^t \bar{f}(x)\frac{1}{1-x^2}\left((t^2-x^2)^{\frac{\alpha}{2}} + (1-t^2)(t^2-x^2)^{\frac{\alpha-2}{2}}\right)dx \\ &= \frac{\omega_\alpha}{t^{\alpha-1}}\int_0^t \bar{f}(x)(t^2-x^2)^{\frac{\alpha-2}{2}}dx = \pi_{\alpha,\mathbb{B}}\bar{f}(t),\end{aligned}$$

where the final equality is again due to a change of variables. □

The uniqueness of the respective integral kernels in Theorem 3.1.2 and Theorem 3.A will be deduced from the following.

Proposition 3.2.15. *For $\alpha > 0$, the maps $\pi_{\alpha,\mathbb{B}}$ and $\pi_{\alpha,\mathbb{D}}$ are injective and map linear functions to linear functions.*

Proof. First, observe that the map $\pi_{\alpha,\mathbb{B}}$ is an instance of the $R_{a,b}$ transform defined in Section 3.A, which are all injective by Proposition 3.A.4. Similarly, the map $\pi_{\alpha,\mathbb{D}}$, as a composition of an $R_{a,b}$ transform and two maps of the form $\bar{f} \mapsto (1-t^2)^\beta \bar{f}(t)$, is injective.

Clearly, $\pi_{\alpha,\mathbb{B}}$ maps linear functions to linear functions. A direct computation shows that T_α maps the function $\bar{f}(t) := t$ to itself, so Lemma 3.2.14 yields that $\pi_{\alpha,\mathbb{D}}$ also maps linear functions to linear functions. □

Lemma 3.2.16. *Let $1 \le i \le n-1$ and $f, g \in C(\mathbb{S}^{n-1})$ be zonal.*

(i) *If $\varphi_{i,f} = 0$, then f is a zonal linear function.*

(ii) *If $\psi_{i,g} = 0$, then g is a zonal linear function.*

Proof. Statement (i) follows immediately from Theorem 3.2.5, Lemma 3.2.6, the uniqueness result in Theorem 3.3.5, and Proposition 3.2.15. Similarly, statement (ii) follows immediately from Corollary 3.2.8, Lemma 3.2.11, the uniqueness result in Theorem 3.3.5, and Proposition 3.2.15. □

3.2.4 Extending from subspaces

Next, for $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$ containing e_n, we show that a zonal valuation φ^E_{i,f_E} on E always extends to a zonal valuation $\varphi_{i,f}$ on $\mathbb{R}^n$, provided that f_E is smooth; this is the content of Theorem 3.D. For the proof, we need the following basic lemma. We denote by $C^\infty[-1,1]$ the space of $C[-1,1]$ functions that are infinitely differentiable on $(-1,1)$ and also posses all (one-sided) higher order derivatives at ± 1.

Lemma 3.2.17. *Let $f = \bar{f}(\langle e_n, \cdot\rangle) : \mathbb{S}^{n-1} \to \mathbb{R}$ be zonal. Then $f \in C^\infty(\mathbb{S}^{n-1})$ if and only if $\bar{f} \in C^\infty[-1,1]$.*

Proof. We can parametrize the unit sphere by $u = (\cos\theta)e_n + (\sin\theta)v$ with $\theta \in \mathbb{R}$ and $v \in \mathbb{S}^{n-2}(e_n^\perp)$. Then $f(u) = \bar{f}(\cos\theta)$, which shows that $f \in C^\infty(\mathbb{S}^{n-1})$ if and only if $\bar{f}(\cos\theta)$ is a smooth function of θ. If $\bar{f} \in C^\infty[-1,1]$, then $\bar{f}(\cos\theta)$ is a smooth function of θ by the chain rule.

Conversely, suppose that $\bar{f}(\cos\theta)$ is a smooth function of θ. Then clearly $\bar{f} \in C^\infty(-1,1)$, and it remains to show the existence of all higher order derivatives at ± 1. Since $\bar{f}(\cos\theta)$ is an even, smooth function, there is a smooth function $\tilde{f}$ such that $\bar{f}(\cos\theta) = \tilde{f}(\theta^2)$. Similarly, there is a smooth function $\tilde{q}$ such that $\cos\theta = \tilde{q}(\theta^2)$. By L'Hôpital's rule,

$$\tilde{q}'(0) = \lim_{\theta\to 0} \frac{\tilde{q}(\theta^2) - \tilde{q}(0)}{\theta^2} = \lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta^2} = -\frac{1}{2} \neq 0,$$

so there exists some neighborhood of zero where $\tilde{q}$ is invertible and its inverse is also smooth. Hence, if t is close to 1, then $\bar{f}(t) = \tilde{f}((\arccos t)^2) = \tilde{f}(\tilde{q}^{-1}(t))$, so by the chain rule $\bar{f} \in C^\infty(-1,1]$. The argument for $\bar{f} \in C^\infty[-1,1)$ is analogous. □

Theorem 3.D is now an easy consequence of what we have shown so far and our study of integral transforms in Section 3.A.

Proof of Theorem 3.D. Due to Theorem 3.2.5, proving the theorem corresponds to finding a zonal function $f \in C^\infty(\mathbb{S}^{n-1})$ such that

$$f_E = \frac{1}{\binom{n-1}{i}} \pi_{E,\mathbb{B}} f.$$

Writing $f = \bar{f}(\langle e_n, \cdot\rangle)$ and $f_E = \bar{f}_E(\langle e_n, \cdot\rangle)$, by Lemmas 3.2.6 and 3.2.17, this is equivalent to finding a function $\bar{f} \in C^\infty[-1,1]$ such that

$$\bar{f}_E = \frac{1}{\binom{n-1}{i}} \overline{\pi}_{n-i-1,\mathbb{B}} \bar{f} = \frac{\omega_{n-i-1}}{\binom{n-1}{i}} R_{1,\frac{n-i-1}{2}} \bar{f},$$

where $R_{a,b}$ is the transform defined in Section 3.A. According to Proposition 3.A.7, such a function $\bar{f}_E \in C^\infty[-1,1]$ exists, concluding the argument. □

3.3 The Klain–Schneider theorem for zonal valuations

In this section we establish Theorem 3.B, the zonal analogue of the classical Klain–Schneider theorem and centerpiece of the Klain approach. The main step in the proof is to eliminate the $(n-2)$-homogeneous component, which is subsumed in the following theorem.

Theorem 3.3.1. *Let $\varphi \in \mathbf{Val}_{n-2}(\mathbb{R}^n)$ be zonal. If φ vanishes on some hyperplane $H \in \mathrm{Gr}_{n-1}(\mathbb{R}^n)$ such that $e_n \in H$, then $\varphi = 0$.*

We will prove this theorem by induction on the dimension $n \geq 3$; the three-dimensional case will be the induction base.

3.3.1 The three-dimensional case

First, we consider Theorem 3.3.1 in three dimensions. To this end, we prove the one-homogeneous instance of Theorem 3.A, using our computation of the area measures of cones.

Proposition 3.3.2. *For every zonal valuation $\varphi \in \mathbf{Val}_1(\mathbb{R}^n)$, there exists a zonal function $g \in C(\mathbb{S}^{n-1})$ such that*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} g(u)\, dS_1(K, \mathbb{D}, u), \qquad K \in \mathcal{K}(\mathbb{R}^n). \tag{3.3.1}$$

Proof. Take g to be $g = \bar{g}_\varphi(\langle e_n, \cdot \rangle)$, where $\bar{g}_\varphi$ is defined as in (3.2.2). Due to Lemma 3.2.3, the function g is continuous on $\mathbb{S}^{n-1}$, and by (3.2.1), the valuations φ and $\psi_{i,g}$ coincide on the family of cones C_s for $s \in [-1, 1] \setminus \{0\}$.

Next, observe that the valuation property implies that φ and $\psi_{i,g}$ coincide on truncated cones and subsequently, on all bodies of revolution with axis e_n that have a polytopal cross-section by two-dimensional planes containing e_n. By continuity, φ and $\psi_{i,g}$ agree on all bodies of revolution with axis e_n.

For a general body $K \in \mathcal{K}(\mathbb{R}^n)$, we define a body of revolution $\overline{K} \in \mathcal{K}(\mathbb{R}^n)$ by

$$h_{\overline{K}}(x) = \int_{\mathrm{SO}(n-1)} h_K(\vartheta^{-1}x)\; d\vartheta = \int_{\mathrm{SO}(n-1)} h_{\vartheta K}(x)\; d\vartheta, \qquad x \in \mathbb{R}^n,$$

where integration is with respect to the unique invariant probability measure on $\mathrm{SO}(n-1)$. Hence, by the invariance, Minkowski additivity, and continuity of the valuations φ and $\psi_{i,g}$,

$$\varphi(K) = \int_{\mathrm{SO}(n-1)} \varphi(\vartheta K)\; d\vartheta = \varphi(\overline{K}) = \psi_{i,g}(\overline{K}) = \int_{\mathrm{SO}(n-1)} \psi_{i,g}(\vartheta K)\; d\vartheta = \psi_{i,g}(K).$$

This shows that $\varphi = \psi_{i,g}$, which concludes the argument. □

Lemma 3.3.3. *Let $\varphi \in \mathbf{Val}_1(\mathbb{R}^3)$ be zonal. If φ vanishes on some plane $E \in \mathrm{Gr}_2(\mathbb{R}^3)$ such that $e_3 \in E$, then $\varphi = 0$.*

Proof. By Proposition 3.3.2, φ admits an integral representation (3.3.1) with some zonal function $g = \bar{g}(\langle e_3, \cdot \rangle) \in C(\mathbb{S}^2)$. Suppose now that φ vanishes on a plane $E \in \mathrm{Gr}_2(\mathbb{R}^3)$ containing e_3. Then Corollary 3.2.8 and Lemma 3.2.11 imply that for all $K \in \mathcal{K}(E)$,

$$\int_{\mathbb{S}^1(E)} (\pi_{1,\mathbb{D}}\bar{g})(\langle e_3, u\rangle)\, dS_1^E(K, u) = 2\varphi(K) = 0,$$

and thus, $\pi_{1,\mathbb{D}}\bar{g}$ is a linear function. Hence, by Proposition 3.2.15, g is a linear function, and subsequently $\varphi = 0$. □

3.3.2 The induction step

Now we pass from three dimensions to general dimensions. One of the main ideas is to show that the valuation in question vanishes on certain orthogonal sums of convex bodies. First, we need the following easy lemma. Recall that we globally assumed the dimension to be $n \geq 3$.

Lemma 3.3.4. *Let $\ell : \mathbb{R}^n \to \mathbb{R}$ be such that its restriction to $e_n^\perp$ and to each hyperplane containing e_n is a linear function. Then ℓ is a linear function.*

Proof. By assumption, we find $x_0 \in e_n^\perp$ such that $\ell|_{e_n^\perp} = \langle x_0, \cdot \rangle$ and for every hyperplane H containing e_n, we find $x_H \in H$ such that $\ell|_H = \langle x_H, \cdot \rangle$. As $n \geq 3$, we have $e_n^\perp \cap H \neq \{o\}$ and we can consider $\ell|_{e_n^\perp \cap H}$ to deduce that

$$P_H x_0 = P_{H\cap e_n^\perp} x_0 = P_{H\cap e_n^\perp} x_H = P_{e_n^\perp} x_H,$$

where P_E denotes the orthogonal projection onto a subspace $E \subseteq \mathbb{R}^n$ and we used the fact that $P_{H\cap e_n^\perp} = P_H P_{e_n^\perp} = P_{e_n^\perp} P_H$. Next, by plugging e_n into ℓ, we see that $\ell(e_n) = \langle x_H, e_n \rangle$ for each hyperplane H containing e_n. Consequently,

$$x_H = \langle x_H, e_n \rangle e_n + P_{e_n^\perp} x_H = \ell(e_n) e_n + P_H x_0 = P_H(\ell(e_n) e_n + x_0),$$

and we conclude that the linear function $\langle \ell(e_n) e_n + x_0, \cdot \rangle$ coincides with ℓ on every hyperplane H containing e_n, and thus, everywhere. $\square$

For the next lemma, we require the following classical result of McMullen.

Theorem 3.3.5 ([86])**.** *For every valuation $\varphi \in \mathbf{Val}_{n-1}(\mathbb{R}^n)$, there exists a function $f \in C(\mathbb{S}^{n-1})$ such that*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u)\, dS_{n-1}(K, u), \qquad K \in \mathcal{K}(\mathbb{R}^n).$$

Moreover, f is unique up to the addition of a linear function.

Lemma 3.3.6. *Let $\varphi \in \mathbf{Val}_{n-1}(\mathbb{R}^n)$ and suppose that*

$$\varphi(K + I) = 0 \qquad \text{for all } K \in \mathcal{K}(H) \text{ and } I \in \mathcal{K}(H^\perp)$$

whenever $H = e_n^\perp$ or H is a hyperplane containing e_n. Then $\varphi = 0$.

Proof. By Theorem 3.3.5, there exists a function $f \in C(\mathbb{S}^{n-1})$ such that

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u)\, dS_{n-1}(K, u), \qquad K \in \mathcal{K}(\mathbb{R}^n).$$

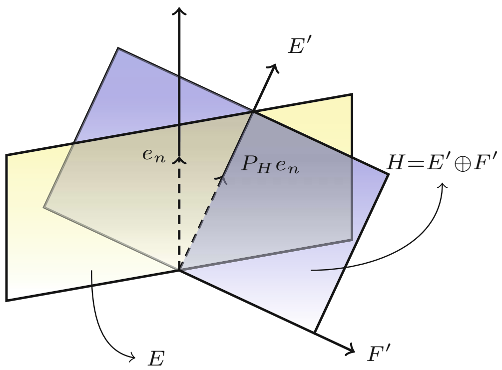


Figure 3.2: We extend the orthogonal sum $H = E' \oplus F'$, where $P_H e_n \in E'$, to an orthogonal sum $\mathbb{R}^n = E \oplus F'$.

Let now H be some hyperplane such that $\varphi(K+I) = 0$ for all $K \in \mathcal{K}(H)$ and $I \in \mathcal{K}(H^\perp)$. Since $K+I$ is a cylinder over K, its boundary splits naturally, and thus, so does its surface area measure, yielding

$$\begin{aligned} 0 = \varphi(K+I) &= \int_{\mathbb{S}^{n-1}} f(u)\, dS_{n-1}(K+I, u) \\ &= V_{n-1}(K)(f(w) + f(-w)) + V_1(I) \int_{\mathbb{S}^{n-2}(H)} f(u)\, dS^H_{n-2}(K, u), \end{aligned}$$

where $w \in \mathbb{S}^{n-1}$ is such that $H = w^\perp$. By choosing $I = \{o\}$, we see that $f(w)+f(-w) = 0$. By choosing $I \neq \{o\}$, we see that the final integral expression vanishes for all $K \in \mathcal{K}(H)$, and thus, the restriction $f|_{\mathbb{S}^{n-2}(H)}$ is linear.

Finally, let $\ell : \mathbb{R}^n \to \mathbb{R}$ denote the one-homogeneous extension of f to $\mathbb{R}^n$, that is, $\ell(o) = 0$ and $\ell(x) = \|x\| f(x/\|x\|)$ for $x \neq o$. Lemma 3.3.4 shows that ℓ is a linear function on $\mathbb{R}^n$, so f is a linear function on $\mathbb{S}^{n-1}$, and thus, $\varphi = 0$. □

We are now ready to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. We prove the theorem by induction on the dimension $n \geq 3$. The three-dimensional case is precisely the content of Lemma 3.3.3.

For the induction step, let $n > 3$ and take some zonal $\varphi \in \mathbf{Val}_{n-2}(\mathbb{R}^n)$ that vanishes on some, and thus, on every hyperplane $H \in \mathrm{Gr}_{n-1}(\mathbb{R}^n)$ containing e_n. Consider a proper orthogonal sum $\mathbb{R}^n = E \oplus F$, where $e_n \in E$. We claim that $\varphi(K+L) = 0$ for all $K \in \mathcal{K}(E)$ and $L \in \mathcal{K}(F)$. To show this, observe that for fixed K, the map $\varphi(K + \,\cdot\,)$ defines a continuous and rigid motion invariant valuation on F. According to Theorem 3.1.1, it must be a linear combination of intrinsic volumes. Since $\varphi(K + \,\cdot\,)$ is also a simple valuation on F, it is a multiple of vol_F, the only simple intrinsic volume on F. That is, there exists some map $\psi_E : \mathcal{K}(E) \to \mathbb{R}$ such that $\varphi(K+L) = \psi_E(K)\mathrm{vol}_F(L)$ for all $K \in \mathcal{K}(E)$ and $L \in \mathcal{K}(F)$. Fixing the body L reveals that ψ_E is a continuous, translation invariant, and zonal valuation on E. Moreover, ψ_E is homogeneous of degree $\dim E - 2$

and vanishes on all hyperplanes of E containing e_n. By induction hypothesis, $\psi_E = 0$, and thus, $\varphi(K + L) = 0$ for all $K \in \mathcal{K}(E)$ and $L \in \mathcal{K}(F)$.

Next, we want to show that φ vanishes on every hyperplane $H \in \mathrm{Gr}_{n-1}(\mathbb{R}^n)$. If $e_n \in H$, then this is due to our assumption on φ; for $H = e_n^\perp$, this is due to the previous step. Otherwise, consider a proper orthogonal sum $H = E' \oplus F'$, where $P_H e_n \in E'$ and P_H denotes the orthogonal projection onto H. Note that for every $x \in F'$,

$$\langle x, e_n \rangle = \langle P_H x, e_n \rangle = \langle x, P_H e_n \rangle = 0,$$

hence $F' \subseteq e_n^\perp$. Consequently, if we define $E = \mathrm{span}(E' \cup \{e_n\})$ and $F = F'$, we obtain the proper orthogonal sum $\mathbb{R}^n = E \oplus F$ (see Fig. 3.2). Since $E' \subseteq E$, the previous step implies that $\varphi(K + L) = 0$ for all $K \in \mathcal{K}(E')$ and $L \in \mathcal{K}(F')$. Therefore, the restriction $\varphi|_H$ meets the requirements of Lemma 3.3.6, so $\varphi|_H = 0$. This shows that the valuation φ is simple, so Theorem 3.1.3 implies that $\varphi = 0$, concluding the proof. □

As we have announced at the beginning of this section, the main part in the proof of Theorem 3.B is to eliminate the $(n-2)$-homogeneous component. Now that we have dealt with this case, it remains to handle the other homogeneous cases and reduce the general case to these.

Lemma 3.3.7. *If a valuation $\varphi \in \mathbf{Val}(\mathbb{R}^n)$ vanishes on a subspace $E \subseteq \mathbb{R}^n$, then so do all of its homogeneous components.*

Proof. Suppose that $\varphi \in \mathbf{Val}(\mathbb{R}^n)$ vanishes on the subspace $E \subseteq \mathbb{R}^n$ and let $\varphi = \varphi_0 + \cdots + \varphi_n$ denote its homogeneous decomposition. Then for $K \in \mathcal{K}(E)$ and $\lambda \geq 0$,

$$0 = \varphi(\lambda K) = \sum_{i=0}^{n} \varphi_i(\lambda K) = \sum_{i=0}^{n} \lambda^i \varphi_i(K).$$

By comparison of coefficients, we see that the homogeneous components of φ vanish on E. □

We can now finally prove Theorem 3.B.

Proof of Theorem 3.B. It is easy to see that every zonal valuation $\varphi \in \mathbf{Val}(\mathbb{R}^n)$ of the form (3.1.3) vanishes on all hyperplanes containing e_n. Conversely, take a zonal valuation $\varphi \in \mathbf{Val}(\mathbb{R}^n)$ and suppose that it vanishes on some hyperplane H containing e_n. Letting $\varphi = \varphi_0 + \cdots + \varphi_n$ denote its homogeneous decomposition, each component $\varphi_i \in \mathbf{Val}_i(\mathbb{R}^n)$ is again zonal and vanishes on H. In particular, φ_i vanishes on all subspaces $E \in \mathrm{Gr}_{n-2}(\mathbb{R}^n)$, so according to Corollary 3.1.4, all homogeneous components but φ_{n-2}, φ_{n-1}, and φ_n must vanish.

By Theorem 3.3.1, $\varphi_{n-2} = 0$. Due to Theorem 3.3.5 and Hadwiger's classification of n-homogeneous valuations [52], the valuation φ is thus of the form (3.1.3) for some constant $c \in \mathbb{R}$ and function $g \in C(\mathbb{S}^{n-1})$ which must clearly be zonal. In order to see that $g(u) = 0$

for $u \in \mathbb{S}^{n-1} \cap e_n^\perp$, we evaluate φ on the body $K = \mathbb{B} \cap u^\perp$, where $\mathbb{B}$ denotes the Euclidean unit ball, which yields

$$0 = \varphi(K) = \int_{\mathbb{S}^{n-1}} g(v)\, dS_{n-1}(K, v) = \kappa_{n-1}(g(u) + g(-u)).$$

As g is zonal, this shows that g vanishes on $\mathbb{S}^{n-1} \cap e_n^\perp$. $\square$

Note that the assumption of φ being zonal cannot be dropped in Theorem 3.B. For instance, consider the valuation $\varphi \in \mathbf{Val}_{n-2}(\mathbb{R}^n)$ defined by

$$\varphi(K) = \int_{\mathbb{S}^{n-2}(e_n^\perp)} f(u)\, dS_{n-2}^{e_n^\perp}(K|e_n^\perp, u), \qquad K \in \mathcal{K}(\mathbb{R}^n),$$

for some odd function $f \in C(\mathbb{S}^{n-2}(e_n^\perp))$. Then φ vanishes on all hyperplanes containing e_n, but it is not of the form (3.1.3). This raises the question of how to characterize valuations in $\mathbf{Val}(\mathbb{R}^n)$ that vanish on all hyperplanes containing e_n.

In the case of even valuations, however, it turns out that the assumption of $\mathrm{SO}(n-1)$-invariance can be dropped and the proof is a simple application of the following corollary of Theorem 3.1.3.

Corollary 3.3.8. *Let $0 \le i \le n-1$ and $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$ be even. If φ vanishes on all subspaces $E \in \mathrm{Gr}_i(\mathbb{R}^n)$, then $\varphi = 0$.*

Proposition 3.3.9. *An even valuation $\varphi \in \mathbf{Val}(\mathbb{R}^n)$ vanishes on all hyperplanes containing e_n if and only if there exist a constant $c \in \mathbb{R}$ and an even function $f \in C(\mathbb{S}^{n-1})$ vanishing on $\mathbb{S}^{n-1} \cap e_n^\perp$ such that*

$$\varphi(K) = cV_n(K) + \int_{\mathbb{S}^{n-1}} f(u)\, dS_{n-1}(K, u), \qquad K \in \mathcal{K}(\mathbb{R}^n). \tag{3.3.2}$$

Proof. As every subspace of dimension less or equal $n-2$ is contained in a hyperplane containing e_n, by Corollary 3.3.8, the proof reduces to considering valuations of degree n and $n-1$. The claim then follows directly from Hadwiger's characterization of n-homogeneous valuations [52] and Theorem 3.3.5. $\square$

Remark 3.3.10. In the Klain–Schneider type theorem in the functional setting, [35, Theorem 1.2], the valuations are merely assumed to be epi-translation invariant. It might seem that this is somehow more general than Theorem 3.B since there is no additional rotational invariance imposed. However, it turns out that from Proposition 3.3.9 and the correspondence between the functional and the even geometrical setting (see [65, Section 3]), one can deduce [35, Theorem 1.2]. This will be discussed in more detail in future work.

Like the classical Klain–Schneider theorem, Theorem 3.B entails that zonal valuations are determined by certain restrictions; this is the content of Corollary 3.C.

Proof of Corollary 3.C. For degree $i = n-1$, the statement is trivial. For degrees $1 \leq i < n-1$, we prove the claim by induction on the dimension $n \geq 3$. For dimension $n = 3$ and degree $i = 1$, this is precisely the content of Lemma 3.3.3.

For the induction step, let $n > 3$ and $1 \leq i < n-1$. Take a zonal valuation $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$ that vanishes on some subspace $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$ containing e_n. Choose a hyperplane $H \in \mathrm{Gr}_{n-1}(\mathbb{R}^n)$ such that $H \supseteq E$. Then $\varphi|_H$ is a zonal valuation on H and by the induction hypothesis, $\varphi|_H = 0$. Consequently, φ meets the requirements of Theorem 3.B, so due to its homogeneity, $\varphi = 0$. □

As a consequence of Corollary 3.C and the commuting diagram in Fig. 3.1, we obtain that the transform T_{n-i-1} allows us to move between integral representations as expected.

Corollary 3.3.11. *Let $1 \leq i \leq n-1$ and $f = \bar{f}(\langle e_n, \cdot \rangle), g = \bar{g}(\langle e_n, \cdot \rangle) \in C(\mathbb{S}^{n-1})$. If $\bar{g} = T_{n-i-1}\bar{f}$, then $\varphi_{i,f} = \psi_{i,g}$.*

Proof. Consider a subspace $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$ containing e_n. By Theorem 3.2.5 and Lemma 3.2.6,

$$\varphi_{i,f}(K) = \frac{1}{\binom{n-1}{i}} \int_{\mathbb{S}^i(E)} (\pi_{n-i-1,\mathbb{B}}\bar{f})(\langle e_n, u\rangle)\, dS_i^E(K,u), \qquad K \in \mathcal{K}(E).$$

Similarly, by Corollary 3.2.8 and Lemma 3.2.11,

$$\psi_{i,g}(K) = \frac{1}{\binom{n-1}{i}} \int_{\mathbb{S}^i(E)} (\pi_{n-i-1,\mathbb{D}}\bar{g})(\langle e_n, u\rangle)\, dS_i^E(K,u), \qquad K \in \mathcal{K}(E).$$

Since $\bar{g} = T_{n-i-1}\bar{f}$, Lemma 3.2.14 yields $\pi_{n-i-1,\mathbb{B}}\bar{f} = \pi_{n-i-1,\mathbb{D}}\bar{g}$, so the valuations $\varphi_{i,f}$ and $\psi_{i,g}$ coincide on E. Hence, Corollary 3.C implies that $\varphi_{i,f} = \psi_{i,g}$. □

3.4 Hadwiger type theorems for zonal valuations

In this section, we establish several integral representations for zonal valuations using the Klain approach. First, we recover a Hadwiger type theorem for smooth, zonal valuations by Schuster and Wannerer [103]. From this we deduce Theorem 3.A, and, finally, we also obtain Theorem 3.1.2.

3.4.1 Smooth Valuations

Recall that the space $\mathbf{Val}(\mathbb{R}^n)$ is a Banach space, when endowed with the norm $\|\varphi\| = \sup\{|\varphi(K)| : K \subseteq \mathbb{B}\}$. Moreover, there is a natural representation of the group $\mathrm{GL}(n)$ on this space: For $\varphi \in \mathbf{Val}(\mathbb{R}^n)$ and $\vartheta \in \mathrm{GL}(n)$, we set

$$(\vartheta \cdot \varphi)(K) = \varphi(\vartheta^{-1}(K)), \qquad K \in \mathcal{K}(\mathbb{R}^n).$$

A valuation $\varphi \in \mathbf{Val}(\mathbb{R}^n)$ is called *smooth* if the map $\mathrm{GL}(n) \to \mathbf{Val}(\mathbb{R}^n) : \vartheta \mapsto \vartheta \cdot \varphi$ is infinitely differentiable.

For $(n-1)$-homogeneous smooth valuations, we have the following integral representation. It is a corollary of the classical Theorem 3.3.5 by McMullen [86].

Corollary 3.4.1 ([86]). *For every smooth valuation $\varphi \in \mathbf{Val}_{n-1}(\mathbb{R}^n)$, there exists a function $f \in C^\infty(\mathbb{S}^{n-1})$ such that*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u)\, dS_{n-1}(K,u), \qquad K \in \mathcal{K}(\mathbb{R}^n).$$

Moreover, f is unique up to the addition of a linear function.

The fact that $f \in C^\infty(\mathbb{S}^{n-1})$ whenever φ is smooth can be easily obtained in a similar fashion as in the proof of McMullen for the continuity of f. By combining this with Corollary 3.C and Theorem 3.D, we recover the following Hadwiger type theorem about smooth, zonal valuations by Schuster and Wannerer [103].

Theorem 3.4.2 ([103]). *Let $1 \le i \le n-1$. Then for every smooth, zonal valuation $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$, there exists a zonal function $f \in C^\infty(\mathbb{S}^{n-1})$ such that*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u)\, dS_i(K,u), \qquad K \in \mathcal{K}(\mathbb{R}^n). \tag{3.4.1}$$

Moreover, f is unique up to the addition of a zonal linear function.

Proof. The uniqueness of f follows from Lemma 3.2.16 (i).

For $i = n-1$, we apply Corollary 3.4.1 to obtain integral representation (3.4.1) with a function $f \in C^\infty(\mathbb{S}^{n-1})$. Since φ is zonal, so is f.

For $i < n-1$, choose some subspace $E \in \mathrm{Gr}_{i+1}(\mathbb{R}^n)$ such that $e_n \in E$ and consider the restriction $\varphi|_E \in \mathbf{Val}_i(E)$. Then $\varphi|_E$, as a valuation on E, is smooth (with respect to the natural representation of $\mathrm{GL}(E)$) and zonal. Therefore, by the first part of the proof, there exists a zonal function $f_E \in C^\infty(\mathbb{S}^i(E))$ such that

$$\varphi(K) = \int_{\mathbb{S}^i(E)} f_E(v)\, dS_i^E(K,v), \qquad K \in \mathcal{K}(E).$$

By Theorem 3.D, there exists a zonal function $f \in C^\infty(\mathbb{S}^{n-1})$ such that

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u)\, dS_i(K,du), \qquad K \in \mathcal{K}(E).$$

Observe now that the right hand side defines a valuation $\tilde{\varphi} \in \mathbf{Val}_i(\mathbb{R}^n)$ which agrees with φ on E. According to Corollary 3.C, this already implies that $\varphi = \tilde{\varphi}$, yielding the desired integral representation. □

3.4.2 Continuous valuations

We now turn to the Hadwiger type theorems for continuous zonal valuations that we have presented in the introduction: Theorem 3.A involving mixed area measures with the disk and Theorem 3.1.2 involving the classical area measures. First, we obtain Theorem 3.A by an approximation argument from Theorem 3.4.2. This requires the following lemma which can be proved using a standard convolution argument.

Lemma 3.4.3. *Let $1 \leq i \leq n-1$. Then for every zonal valuation $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$, there exists a sequence of smooth, zonal valuations in $\mathbf{Val}_i(\mathbb{R}^n)$ converging to φ in the Banach norm.*

Proof of Theorem 3.A. The uniqueness of g follows from Lemma 3.2.16 (ii).

For the existence, by Lemma 3.4.3, we may choose a family of smooth, zonal valuations $\varphi^k \in \mathbf{Val}_i(\mathbb{R}^n)$ converging to φ in the Banach norm as $k \to \infty$. By Theorem 3.4.2, there exist zonal functions $f_k = \bar{f}_k(\langle e_n, \cdot \rangle) \in C^\infty(\mathbb{S}^{n-1})$ such that $\varphi^k = \varphi_{i,f_k}$. If we let $\bar{g}_k := T_{n-i-1}\bar{f}_k$, then $g_k = \bar{g}_k(\langle e_n, \cdot \rangle) \in C(\mathbb{S}^{n-1})$ and Corollary 3.3.11 yields $\varphi^k = \psi_{i,g_k}$.

Modifying each g_k by a linear function, if necessary, by the uniqueness part above and Lemma 3.2.3, the functions $\bar{g}_k$ form a Cauchy sequence in $C[-1,1]$, so by completeness, they converge uniformly to some function $\bar{g} \in C[-1,1]$ as $k \to \infty$. If we set $g := \bar{g}(\langle e_n, \cdot \rangle)$, then

$$\varphi(K) = \lim_{k\to\infty} \varphi^k(K) = \lim_{k\to\infty} \psi_{i,g_k}(K) = \psi_{i,g}(K), \qquad K \in \mathcal{K}(\mathbb{R}^n).$$

Thus $\varphi = \psi_{i,g}$, which concludes the argument. $\square$

As was already indicated in the introduction, we obtain the following two corollaries as a direct consequence of Theorem 3.A, Lemma 3.2.3, and (3.2.3).

Corollary 3.4.4. *Let $1 \leq i \leq n-1$ and $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$ be zonal. If $\varphi(C_s) = 0$ for all $s \in [-1,1] \setminus \{0\}$, then $\varphi = 0$.*

Corollary 3.4.5. *Let $1 \leq i \leq n-1$ and $\varphi^k, \varphi \in \mathbf{Val}_i(\mathbb{R}^n)$ be zonal for $k \in \mathbb{N}$, and let $g_k, g \in C(\mathbb{S}^{n-1})$ be as in (3.1.2). Then $\varphi^k \to \varphi$ in the Banch norm if and only if there exist constants $a_k \in \mathbb{R}$ such that $g_k + a_k\langle e_n, \cdot \rangle \to g$ uniformly on $\mathbb{S}^{n-1}$.*

Next, we want to recover Theorem 3.1.2. One key step of the proof is to extend the definition of $\varphi_{i,f}$ from continuous $\bar{f} \in C[-1,1]$ to $\bar{f} \in \mathcal{D}^{n-i-1}$ and to extend Corollary 3.3.11 about moving between the different integral representations. For this, we need the following classical result by Firey [40].

Theorem 3.4.6 ([40])**.** *Let $1 \leq i \leq n-1$. Then there exists a constant $A_{n,i} > 0$ such that for all $K \in \mathcal{K}(\mathbb{R}^n)$, $v \in \mathbb{S}^{n-1}$, and $\varepsilon \geq 0$,*

$$S_i(K, \{u \in \mathbb{S}^{n-1} : \langle u, v \rangle > 1 - \varepsilon\}) \leq A_{n,i}(\operatorname{diam} K)^i \varepsilon^{\frac{n-i-1}{2}}.$$

Proposition 3.4.7. *Let $1 \leq i \leq n-1$, $\bar{f} \in \mathcal{D}^{n-i-1}$, and $f = \bar{f}(\langle e_n, \cdot \rangle)$. Then $f \in C(\mathbb{S}^{n-1} \setminus \{\pm e_n\})$ and there exists a zonal valuation $\varphi_{i,f} \in \mathbf{Val}_i(\mathbb{R}^n)$ such that*

$$\varphi_{i,f}(K) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{S}^{n-1}\setminus U_\varepsilon} f(u)\, dS_i(K,u), \qquad K \in \mathcal{K}(\mathbb{R}^n),$$

where $U_\varepsilon = \{u \in \mathbb{S}^{n-1} : |\langle e_n, u \rangle| > 1 - \varepsilon\}$.

Moreover, if $\bar{g} = T_{n-i-1}\bar{f}$, then $g = \bar{g}(\langle e_n, \cdot \rangle) \in C(\mathbb{S}^{n-1})$ and $\varphi_{i,f} = \psi_{i,g}$.

Proof. The idea is to obtain the statement by approximation and Corollary 3.3.11. To this end, take a family of bump functions $\bar{\eta}_\varepsilon \in C[-1,1]$, $\varepsilon > 0$, such that

$$\bar{\eta}_\varepsilon(s) = 1 \text{ for } |s| \leq 1-\varepsilon, \qquad \bar{\eta}_\varepsilon(s) = 0 \text{ for } |s| \geq 1-\tfrac{\varepsilon}{2}, \qquad \text{and} \qquad 0 \leq \bar{\eta}_\varepsilon \leq 1.$$

For $\varepsilon > 0$, we define $\bar{f}_\varepsilon = \bar{\eta}_\varepsilon \bar{f} \in C[-1,1]$. We also define $\bar{g}_\varepsilon = T_{n-i-1}\bar{f}_\varepsilon$ and $\bar{g} = T_{n-i-1}\bar{f}$, and observe that $\bar{g}_\varepsilon, \bar{g} \in C[-1,1]$. We denote the respective zonal extensions of these functions to the unit sphere by $\eta_\varepsilon, f_\varepsilon, g_\varepsilon, g \in C(\mathbb{S}^{n-1})$.

According to Corollary 3.3.11, we have that $\varphi_{i,f_\varepsilon} = \psi_{i,g_\varepsilon}$ for all $\varepsilon > 0$, and due to Lemma 3.B.1, the functions g_ε converge to g uniformly on $\mathbb{S}^{n-1}$ as $\varepsilon \to 0^+$. Hence, by Corollary 3.4.5,

$$\lim_{\varepsilon \to 0^+} \varphi_{i,f_\varepsilon}(K) = \lim_{\varepsilon \to 0^+} \psi_{i,g_\varepsilon}(K) = \psi_{i,g}(K), \qquad K \in \mathcal{K}(\mathbb{R}^n).$$

Consequently, it remains to show that for every given $K \in \mathcal{K}(\mathbb{R}^n)$, the principal value integral in the statement of the proposition exists and agrees with the limit of $\varphi_{i,f_\varepsilon}(K)$ as $\varepsilon \to 0^+$. To this end, observe that

$$\varphi_{i,f_\varepsilon}(K) - \int_{\mathbb{S}^{n-1}\setminus U_\varepsilon} f \, dS_i(K,\cdot\,) = \int_{U_\varepsilon^+} \eta_\varepsilon f \, dS_i(K,\cdot\,) + \int_{U_\varepsilon^-} \eta_\varepsilon f \, dS_i(K,\cdot\,),$$

where $U_\varepsilon^\pm = \{u \in \mathbb{S}^{n-1} : \pm\langle e_n, u\rangle > 1-\varepsilon\}$ are spherical caps around the individual poles. Since U_ε^+ is connected, by the mean value theorem for integrals, there exists some $t \in (1-\varepsilon, 1-\frac{\varepsilon}{2})$ such that

$$\int_{U_\varepsilon^+} \eta_\varepsilon f \, dS_i(K,\cdot\,) = \bar{f}(t) \int_{U_\varepsilon^+} \eta_\varepsilon \, dS_i(K,\cdot\,).$$

Consequently, using Theorem 3.4.6, we can estimate that

$$\begin{aligned}\left| \int_{U_\varepsilon^+} \eta_\varepsilon f \, dS_i(K,\cdot\,) \right| &= |\bar{f}(t)| \int_{U_\varepsilon^+} \eta_\varepsilon \, dS_i(K,\cdot\,) \leq |\bar{f}(t)| S_i(K, U_\varepsilon^+) \\ &\leq A_{n,i} (\operatorname{diam} K)^i |\bar{f}(t)| \varepsilon^{\frac{n-i-1}{2}} \leq 2^{\frac{n-i-1}{2}} A_{n,i} (\operatorname{diam} K)^i |\bar{f}(t)| (1-t^2)^{\frac{n-i-1}{2}},\end{aligned}$$

where the final inequality uses that $\frac{\varepsilon}{2} < 1-t < 1-t^2$. Since $\bar{f} \in \mathcal{D}^{n-i-1}$, the final term tends to zero as $\varepsilon \to 0^+$. The argument on U_ε^- is completely analogous. As we have pointed out before, this concludes the argument. □

For our later applications, we also note the following immediate consequence of Proposition 3.4.7 and the commuting diagram in Fig. 3.1.

Corollary 3.4.8. *Let $1 \leq i \leq n-1$, $\bar{f} \in \mathcal{D}^{n-i-1}$, and $f = \bar{f}(\langle e_n, \cdot\,\rangle)$. Then for every subspace $E \in \operatorname{Gr}_{i+1}(\mathbb{R}^n)$ with $e_n \in E$,*

$$\varphi_{i,f}(K) = \int_{\mathbb{S}^i(E)} (\pi_{n-i-1,\mathbb{B}} \bar{f})(\langle e_n, u\rangle) \, dS_i^E(K,u), \qquad K \in \mathcal{K}(E).$$

We are now ready to prove Theorem 3.1.2.

Proof of Theorem 3.1.2. Let $\varphi \in \mathbf{Val}_i(\mathbb{R}^n)$ be zonal. By Theorem 3.A, there exists a zonal function $g = \bar{g}(\langle e_n, \cdot \rangle) \in C(\mathbb{S}^{n-1})$ such that $\varphi = \psi_{i,g}$. According to Proposition 3.B.2, there exists some $\bar{f} \in \mathcal{D}^{n-i-1}$ such that $\bar{g} = T_{n-i-1}\bar{f}$. Due to Proposition 3.4.7, the function $f = \bar{f}(\langle e_n, \cdot \rangle)$ then provides the desired improper integral representation.

For the uniqueness, suppose that $\varphi = 0$. Then Theorem 3.A implies that $\bar{g}$ is a linear function. According to Proposition 3.B.2, the transform T_{n-i-1} is injective, and direct computation shows that it maps linear functions to linear functions. Hence, $\bar{f}$ is a linear function. $\square$

3.5 Applications

3.5.1 Integral geometric formulas

In the following, we apply the Hadwiger type Theorem 3.A for zonal valuations, and more specifically, the determination by their values on cones (see Section 3.2.1), to prove some integral geometric formulas. First, we establish the additive kinematic formula (3.1.4) in Theorem 3.E.

Proof of Theorem 3.E. For convenience, we define functionals φ and φ_i as follows.

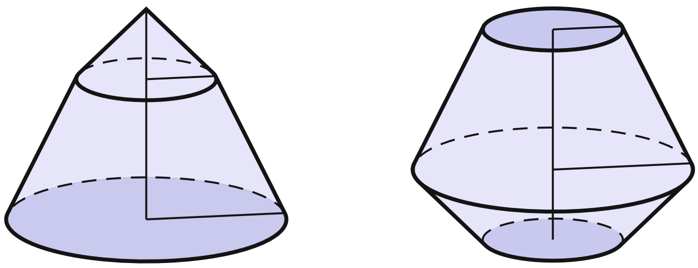
$$\varphi(K, L) := \frac{1}{\kappa_{n-1}} \int_{\mathrm{SO}(n-1)} \int_{\mathbb{S}^{n-1}} g(u)\, dS_j(K + \vartheta L, \mathbb{D}, u)\, d\vartheta,$$
$$\varphi_i(K, L) := \frac{1}{\kappa_{n-1}^2} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} q(u, v)\, dS_i(K, \mathbb{D}, u)\, dS_{j-i}(L, \mathbb{D}, v),$$

where $q(u, v) = \bar{q}(\langle e_n, u\rangle, \langle e_n, v\rangle)$ and $\bar{q}(s, t) = \max\{s, t\}\bar{g}(\min\{s, t\})$. Observe that φ is a translation-invariant, continuous, and zonal valuation in both of its arguments. The same is true for φ_i, which is also homogeneous in each argument; that is, $\varphi_i(\,\cdot\,, L) \in \mathbf{Val}_i(\mathbb{R}^n)$ and $\varphi_i(K, \cdot) \in \mathbf{Val}_{j-i}(\mathbb{R}^n)$. Thus, by Corollary 3.4.4 and the homogeneous decomposition theorem by McMullen [86], it suffices to show that for all $s, t \in [-1, 1] \setminus \{0\}$ and $\lambda, \mu \geq 0$,

$$\varphi(\lambda C_s, \mu C_t) = \sum_{i=0}^{j} \binom{j}{i} \lambda^i \mu^{j-i} \varphi_i(C_s, C_t). \tag{3.5.1}$$

First, we consider the case where both cones are pointing in the same direction. To this end, let $0 < s \leq t$. Then the Minkowski sum $\lambda C_s + \mu C_t$ is the cone $(\lambda + \mu)C_s$, truncated and glued together with a translate of μC_t (see Fig. 3.3). More precisely,

$$\lambda C_s + \mu C_t = ((\lambda + \mu)C_s \setminus (\mu C_s + \lambda \tfrac{\sqrt{1-s^2}}{s} e_n)) \cup (\mu C_t + \lambda \tfrac{\sqrt{1-s^2}}{s} e_n).$$



$0 < s \leq t$ $\qquad$ $t < 0 < s$

Figure 3.3: The Minkowski sum $\lambda C_s + \mu C_t$.

From (3.2.1), the valuation property, homogeneity, and translation invariance of $\psi_{j,g}$ (see the definition at the beginning of Section 3.2), the left hand side of (3.5.1) becomes

$$\begin{aligned}\varphi(\lambda C_s, \mu C_t) &= \frac{1}{\kappa_{n-1}}(\psi_{j,g}((\lambda+\mu)C_s) - \psi_{j,g}(\mu C_s) + \psi_{j,g}(\mu C_t)) \\ &= ((\lambda+\mu)^j - \mu^j)\Big(\bar{g}(-1) + \frac{\bar{g}(s)}{s}\Big) + \mu^j\Big(\bar{g}(-1) + \frac{\bar{g}(t)}{t}\Big).\end{aligned}$$

For the right hand side, applying again (3.2.1),

$$\varphi_i(C_s, C_t) = \begin{cases} \bar{q}(-1,-1) + \bar{q}(1,-1) + \dfrac{\bar{q}(-1,t)}{t} + \dfrac{\bar{q}(1,t)}{t}, & i = 0, \\ \bar{q}(-1,-1) + \dfrac{\bar{q}(s,-1)}{s} + \dfrac{\bar{q}(-1,t)}{t} + \dfrac{\bar{q}(s,t)}{st}, & 0 < i < j, \\ \bar{q}(-1,-1) + \dfrac{\bar{q}(s,-1)}{s} + \bar{q}(-1,1) + \dfrac{\bar{q}(s,1)}{s}, & i = j. \end{cases}$$

Plugging in the definition of $\bar{q}$ in terms of $\bar{g}$, one readily verifies (3.5.1). If $0 < t \leq s$, the argument is analogous.

Next, we consider the case where the two cones have opposite orientations. To this end, let $t < 0 < s$. Then the Minkowski sum $\lambda C_s + \mu C_t$ consists of two truncated cones glued together (see Fig. 3.3). More precisely,

$$\begin{aligned}\lambda C_s + \mu C_t = &((\lambda+\mu)C_s \setminus (\mu C_s + \lambda\tfrac{\sqrt{1-s^2}}{s}e_n)) \cup (\mu\mathbb{D} + \lambda\tfrac{\sqrt{1-s^2}}{s}e_n) \\ &\cup ((\lambda+\mu)C_t \setminus (\lambda C_t + \mu\tfrac{\sqrt{1-t^2}}{t}e_n)) \cup (\lambda\mathbb{D} + \mu\tfrac{\sqrt{1-t^2}}{t}e_n).\end{aligned}$$

Hence, similarly as before, the left hand side of (3.5.1) amounts to

$$\varphi(\lambda C_s, \mu C_t) = ((\lambda+\mu)^j - \mu^j)\frac{\bar{g}(s)}{s} + \mu^j\bar{g}(1) + ((\lambda+\mu)^j - \lambda^j)\frac{\bar{g}(t)}{|t|} + \lambda^j\bar{g}(-1).$$

For the right hand side, applying (3.2.1),

$$\varphi_i(C_s,C_t)=\begin{cases}\bar{q}(-1,1)+\bar{q}(1,1)+\dfrac{\bar{q}(-1,t)}{|t|}+\dfrac{\bar{q}(1,t)}{|t|}, & i=0,\\[2ex] \bar{q}(-1,1)+\dfrac{\bar{q}(s,1)}{s}+\dfrac{\bar{q}(-1,t)}{|t|}+\dfrac{\bar{q}(s,t)}{s|t|}, & 0<i<j,\\[2ex] \bar{q}(-1,1)+\dfrac{\bar{q}(s,1)}{s}+\bar{q}(-1,-1)+\dfrac{\bar{q}(s,-1)}{s}, & i=j.\end{cases}$$

Plugging in the definition of $\bar{q}$ in terms of $\bar{g}$, one readily verifies (3.5.1). If $s<0<t$, the argument is analogous. Thus, we have shown (3.5.1) for all $s,t\in[-1,1]\setminus\{0\}$ and $\lambda,\mu\geq 0$, which concludes the proof. $\square$

Note that the choice of the function $\bar{q}$ in the proof of Theorem 3.E is far from unique. In fact, one can add any function of the form $\bar{q}_1(s)t+s\bar{q}_2(t)+c\cdot st$, where $\bar{q}_1,\bar{q}_2\in C[-1,1]$ and $c\in\mathbb{R}$.

We now turn to the Kubota-type formula (3.1.5). First, we prove a version of this formula involving intrinsic volumes of projections.

Theorem 3.5.1. *Let $1\leq i\leq n-1$. Then for all $K\in\mathcal{K}(\mathbb{R}^n)$,*

$$\int_{\mathrm{Gr}_i(\mathbb{R}^n,e_n)} V_i(K|E)\,dE=\frac{n\kappa_{i-1}}{i\kappa_{n-1}}V(K^{[i]},\mathbb{D}^{[n-i]}). \tag{3.5.2}$$

Proof. Observe that both sides define zonal valuations in $\mathbf{Val}_i(\mathbb{R}^n)$. Therefore, according to Corollary 3.4.4, it suffices to show the identity on the family C_s of cones for $s\in[-1,1]\setminus\{0\}$. According to (3.2.1), for $1\leq i\leq n$,

$$V(C_s^{[i]},\mathbb{D}^{[n-i]})=\frac{1}{n}\int_{\mathbb{S}^{n-1}}h_{C_s}(u)\,dS(C_s^{[i-1]},\mathbb{D}^{[n-i]},u)=\frac{\kappa_{n-1}}{n}\frac{\sqrt{1-s^2}}{|s|}.$$

Clearly, the orthogonal projection of C_s onto $E\in\mathrm{Gr}_i(\mathbb{R}^n,e_n)$ is precisely the cone with base $\mathbb{D}\cap E$ and apex $\frac{\sqrt{1-s^2}}{s}e_n$, so

$$\int_{\mathrm{Gr}_i(\mathbb{R}^n,e_n)} V_i(C_s|E)\,dE=\frac{\kappa_{i-1}}{i}\frac{\sqrt{1-s^2}}{|s|}.$$

Hence, (3.5.2) holds for all C_s, which, by Corollary 3.4.4, concludes the proof. $\square$

By a classical polarization argument, we obtain from (3.5.2) that for all functions $f\in C(\mathbb{S}^{n-1})$ and convex bodies $K_1,\dots,K_{i-1}\in\mathcal{K}(\mathbb{R}^n)$,

$$\int_{\mathrm{Gr}_i(\mathbb{R}^n,e_n)}\int_{\mathbb{S}^{i-1}(E)} f(u)\,dS^E(K_1|E,\dots,K_{i-1}|E,u)\,dE \\ =\frac{\kappa_{i-1}}{\kappa_{n-1}}\int_{\mathbb{S}^{n-1}} f(u)\,dS(K_1,\dots,K_{i-1},\mathbb{D}^{[n-i]},u).$$

This is the formulation of the Kubota-type formula in [56, Theorem 3.2]. In particular, by setting $K_1 = \cdots = K_{i-1} = K$, we obtain (3.1.5).

Next, we want to deduce the Crofton-type formula (3.1.6). To this end, we recall the following formula about integration over affine Grassmannians. If $1 \leq j \leq n$ and $F \in \mathrm{AGr}_k(\mathbb{R}^n)$ is some fixed affine subspace, where $n - j \leq k \leq n$, then for every measurable function $\xi : \mathrm{AGr}_{j+k-n}(F) \to [0, \infty)$,

$$\int_{\mathrm{AGr}_j(\mathbb{R}^n)} \xi(E \cap F)\, dE = \frac{\omega_{j+k-n+1}\omega_{n+1}}{\omega_{j+1}\omega_{k+1}} \int_{\mathrm{AGr}_{j+k-n}(F)} \xi(E)\, dE. \tag{3.5.3}$$

This follows from the uniqueness of the invariant measure on $\mathrm{AGr}_{j+k-n}(F)$, where the multiplicative constant can be computed from the classical Crofton formula (see, e.g., [47, p. 481]). We require the following integral identity.

Lemma 3.5.2. *Let $1 \leq j \leq n$ and $\xi : \mathrm{AGr}_{j-1}(\mathbb{R}^n) \to [0, \infty)$ be measurable. Then*

$$\int_{\mathrm{AGr}_j(\mathbb{R}^n)} \int_{\mathbb{R}} \xi(E \cap (e_n^\perp + te_n))\, dt\, dE = \frac{\omega_j \omega_{n+1}}{\omega_{j+1}\omega_n} \int_{\mathrm{Gr}_{j-1}(e_n^\perp)} \int_{E^\perp} \xi(E + x)\, dx\, dE.$$

Proof. As an instance of (3.5.3), with $k = n - 1$ and $F = e_n^\perp + te_n$,

$$\begin{aligned}\frac{\omega_{j+1}\omega_n}{\omega_j\omega_{n+1}} \int_{\mathrm{AGr}_j(\mathbb{R}^n)} \xi(E \cap (e_n^\perp + te_n))\, dE &= \int_{\mathrm{AGr}_{j-1}(e_n^\perp + te_n)} \xi(E)\, dE \\ = \int_{\mathrm{AGr}_{j-1}(e_n^\perp)} \xi(E + te_n)\, dE &= \int_{\mathrm{Gr}_{j-1}(e_n^\perp)} \int_{E^{\perp(e_n^\perp)}} \xi(E + y + te_n)\, dy\, dE,\end{aligned}$$

where the final equality is by the uniqueness of the invariant measure on $\mathrm{AGr}_{j-1}(e_n^\perp)$ and $E^{\perp(e_n^\perp)}$ denotes the orthogonal complement of E relative to $e_n^\perp$. Hence,

$$\begin{aligned}&\int_{\mathrm{AGr}_j(\mathbb{R}^n)} \int_{\mathbb{R}} \xi(E \cap (e_n^\perp + te_n))\, dt\, dE \\ &\qquad = \frac{\omega_j\omega_{n+1}}{\omega_{j+1}\omega_n} \int_{\mathrm{Gr}_{j-1}(e_n^\perp)} \int_{E^{\perp(e_n^\perp)}} \int_{\mathbb{R}} \xi(E + y + te_n)\, dt\, dy\, dE \\ &\qquad = \frac{\omega_j\omega_{n+1}}{\omega_{j+1}\omega_n} \int_{\mathrm{Gr}_{j-1}(e_n^\perp)} \int_{E^\perp} \xi(E + x)\, dx\, dE,\end{aligned}$$

where we applied Fubini's theorem. □

Proof of Corollary 3.F. First, note that for every convex body $C \in \mathcal{K}(\mathbb{R}^n)$,

$$h_C(e_n) + h_C(-e_n) = V_1(C|\mathrm{span}\, e_n) = \int_{\mathbb{R}} V_0(C \cap (e_n^\perp + te_n))\, dt.$$

Consequently, since K is origin symmetric, we have that

$$\begin{aligned}\int_{\mathrm{AGr}_j(\mathbb{R}^n)} h_{K\cap E}(e_n)\, dE &= \frac{1}{2} \int_{\mathrm{AGr}_j(\mathbb{R}^n)} (h_{K\cap E}(e_n) + h_{K\cap E}(-e_n))\, dE \\ &= \frac{1}{2} \int_{\mathrm{AGr}_j(\mathbb{R}^n)} \int_{\mathbb{R}} V_0(K \cap E \cap (e_n^\perp + te_n))\, dt\, dE.\end{aligned}$$

Applying Lemma 3.5.2 to $\xi(E) := V_0(K \cap E)$ yields

$$\begin{aligned}\frac{\omega_{j+1}\omega_n}{\omega_j\omega_{n+1}} \int_{\mathrm{AGr}_j(\mathbb{R}^n)} h_{K\cap E}(e_n)\, dE &= \frac{1}{2}\int_{\mathrm{Gr}_{j-1}(e_n^\perp)} \int_{E^\perp} V_0(K\cap(E+x))\, dx\, dE \\ &= \frac{1}{2}\int_{\mathrm{Gr}_{j-1}(e_n^\perp)} V_{n-j+1}(K|E^\perp)\, dE = \frac{1}{2}\int_{\mathrm{Gr}_{n-j+1}(\mathbb{R}^n, e_n)} V_{n-j+1}(K|E)\, dE.\end{aligned}$$

By combining this with the Kubota-type formula (3.5.2), we obtain (3.1.6). □

3.5.2 Mean section operators

We now turn to the application of Corollary 3.C to the mean section operators. Our aim is to deduce Theorem 3.1.5 for $j > 2$ from the instance where $j = 2$. In our argument, we use the fact that for a convex body of some linear subspace E, its Steiner point relative to E agrees with its Steiner point relative to the ambient space (cf. [96, p. 315]).

We require the following relation between Berg's functions.

Lemma 3.5.3. *For every $j > 2$, there exists $a_j \in \mathbb{R}$ such that*

$$\pi_{j-2,\mathbb{B}} g_j(s) = (j-1)g_2(s) + a_j s, \qquad s \in (-1,1). \tag{3.5.4}$$

Proof. Take $E \in \mathrm{Gr}_2(\mathbb{R}^n)$ with $e_n \in E$ and $K \in \mathcal{K}(E)$. Then, by combining Berg's identity (3.1.8) with Lemma 3.2.6 and (3.2.4), we have that

$$\begin{aligned}h_{K-s(K)}(e_n) &= \int_{\mathbb{S}^{n-1}} g_n(\langle e_n, u\rangle)\, dS_1(K,u) \\ &= \frac{1}{n-1}\int_{\mathbb{S}^1(E)} [\pi_{n-2,\mathbb{B}} g_n](\langle e_n, u\rangle)\, dS_1^E(K,u).\end{aligned}$$

On the other hand, applying (3.1.8) in the subspace E yields

$$h_{K-s(K)}(e_n) = \int_{\mathbb{S}^1(E)} g_2(\langle e_n, u\rangle)\, dS_1^E(K,u).$$

Since $K \in \mathcal{K}(E)$ was arbitrary, $\pi_{n-2,\mathbb{B}} g_n - (n-1)g_2$ is a linear function. □

Note that the lemma yields the following integral identity, which might be of independent interest. For every $j > 2$, there exists $a_j \in \mathbb{R}$ such that

$$\omega_j \int_0^1 g_j(st)(1-t^2)^{\frac{j-4}{2}}\, dt = (j-1)g_2(s) + a_j s, \qquad s \in (-1,1).$$

We are now ready to recover [47, Theorem 4.4]. Let us remark that the existence of the integrals on the right hand side of (3.1.8) and (3.1.9) was shown in [13] and [47] using a certain regularization procedure. More recently, Knoerr [68] gave a simpler argument that these integrals exist and also define continuous valuations.

Proof of Corollary 3.G. Let $2 < j < n$ and write $i = n - j + 1$. Due to the rotational equivariance, it suffices to show the claim only for $u = e_n$. Let $E \in \mathrm{Gr}_{i+1}(E)$ with $e_n \in E$ and take $K \in \mathcal{K}(E)$. As a consequence of (3.5.3), $\mathrm{M}_j K = \frac{\omega_3 \omega_{n+1}}{\omega_{j+1}\omega_{n-j+3}} \mathrm{M}_2^E K$, where M_2^E denotes the mean section operator relative to E (see [47, Lemma 3.3]). Consequently, by an application of Theorem 3.1.5 for $j = 2$,

$$\begin{aligned} h_{\mathrm{M}_j(K-s(K))}(e_n) &= \frac{\omega_3\omega_{n+1}}{\omega_{j+1}\omega_{n-j+3}} h_{\mathrm{M}_2^E(K-s(K))}(e_n) \\ &= \frac{\omega_3\omega_{n+1}}{\omega_{j+1}\omega_{n-j+3}} c_{n-j+2,2} \int_{\mathbb{S}^i(E)} g_2(\langle e_n, u\rangle)\, dS_i^E(K,u). \end{aligned}$$

Moreover, by relation (3.5.4), we have that

$$\begin{aligned} \int_{\mathbb{S}^i(E)} g_2(\langle e_n, u\rangle)\, dS_i^E(K,u) &= \frac{1}{j-1} \int_{\mathbb{S}^i(E)} [\pi_{j-2,\mathbb{B}} g_j](\langle e_n, u\rangle)\, dS_i^E(K,u) \\ &= \frac{\binom{n-1}{n-j+1}}{j-1} \int_{\mathbb{S}^{n-1}} g_j(\langle e_n, u\rangle)\, dS_i(K,u) \end{aligned}$$

where the final equality is due to Lemma 3.2.6 and (3.2.4). Finally, note that $h_{\mathrm{M}_j(K-s(K))}(e_n)$ defines a zonal valuation in $\mathbf{Val}_i(\mathbb{R}^n)$ and the final integral expression does too, as was shown in [68, Section 3]. Our argument shows that they coincide on E, so by Corollary 3.C, they coincide on all convex bodies in $\mathbb{R}^n$. $\square$

3.A The transform $R_{a,b}$

In this section, we consider the following integral transform that comes up naturally when dealing with restrictions of zonal valuations to proper subspaces.

Definition 3.A.1. For $a, b > 0$, we define $R_{a,b} : C(-1,1) \to C(-1,1)$ by

$$(R_{a,b}\bar{f})(t) := \int_0^1 \bar{f}(st) s^{a-1}(1-s^2)^{b-1} ds, \qquad t \in (-1,1).$$

In order to see that the transform $R_{a,b}$ is well-defined, note that for every compact set $I \subseteq (-1,1)$, there exists $C > 0$ such that $\max_{s\in[0,1]} |f(st)| \le C$ for all $t \in I$. Thus, the integral exists and by dominated convergence, $R_{a,b} f \in C(-1,1)$. We want to show that the map is injective, for which we require two lemmas.

Lemma 3.A.2. *Let $a > 0$, $b \ge 1$, and $f \in C(-1,1)$. Then for all $t \in (-1,1) \setminus \{0\}$, the function $|t|^a R_{a,b} f(t)$ is differentiable at t and*

$$\frac{d}{dt}\left[|t|^a (R_{a,b} f)(t)\right] = \begin{cases} |t|^{a-1} f(t), & b = 1, \\ 2(b-1)t^{-3}(R_{a+2,b-1} f)(t), & b > 1. \end{cases} \tag{3.A.1}$$

Proof. By a change of variables,

$$|t|^a(R_{a,b}\bar{f})(t) = \int_0^t \bar{f}(s)|s|^{a-1}\left(1-\frac{s^2}{t^2}\right)^{b-1} ds.$$

For $b = 1$, the claim follows directly from the fundamental theorem of calculus. For $b > 1$, note that at every $t \in (-1,1)\setminus\{0\}$ and $s \in [0,t]$ (or $[t,0]$, respectively), the partial derivative of $(1-s^2/t^2)^{b-1}$ with respect to t exists. Hence, the Leibniz integral rule implies that $R_{a,b}\bar{f}$ is differentiable at t and

$$\begin{aligned}\frac{d}{dt}\left[|t|^a(R_{a,b}\bar{f})(t)\right] &= \bar{f}(t)|t|^{a-1}\left(1-\frac{t^2}{t^2}\right)^{b-1} + \int_0^t \bar{f}(s)|s|^{a-1}\frac{\partial}{\partial t}\left(1-\frac{s^2}{t^2}\right)^{b-1} ds\\ &= 2(b-1)t^{-3}\int_0^t \bar{f}(s)|s|^{a+1}\left(1-\frac{s^2}{t^2}\right)^{b-2} ds = 2(b-1)t^{-3}(R_{a+2,b-1}\bar{f})(t),\end{aligned}$$

which yields the claim for $b > 1$. $\square$

Lemma 3.A.3. *Let $a_1, b_1, a_2, b_2 > 0$ and suppose that $a_1 = a_2 + 2b_2$. Then*

$$R_{a_1,b_1}R_{a_2,b_2} = \frac{1}{2}\mathrm{B}(b_1,b_2)R_{a_2,b_1+b_2}.$$

Proof. For every $\bar{f} \in C(-1,1)$ and $t \in (-1,1)$, we have that

$$(R_{a_1,b_1}R_{a_2,b_2}\bar{f})(t) = \int_0^1\int_0^1 \bar{f}(xst)x^{a_2-1}(1-x^2)^{b_2-1}dx\ s^{a_1-1}(1-s^2)^{b_1-1}ds.$$

By applying the change of variables $r = sx$ to the inner integral, we obtain

$$\begin{aligned}(R_{a_1,b_1}R_{a_2,b_2}\bar{f})(t) &= \int_0^1\int_0^s \bar{f}(rt)\frac{r^{a_2-1}}{s^{a_2-1}}\left(1-\frac{r^2}{s^2}\right)^{b_2-1} dr\ s^{a_1-1}(1-s^2)^{b_1-1}\frac{1}{s}ds\\ &= \int_0^1 \bar{f}(rt)r^{a_2-1}\int_r^1 s^{a_1-a_2-1}\left(1-\frac{r^2}{s^2}\right)^{b_2-1}(1-s^2)^{b_1-1}ds\ dr,\end{aligned}$$

where the second equality is due to a change of the order of integration. It remains to compute the inner integral. To that end, we rearrange the integrand using the identity $a_1 = a_2 + 2b_2$ and then apply the change of variables $y = (1-s^2)/(1-r^2)$, which yields

$$\begin{aligned}\int_r^1 s^{a_1-a_2-1}\left(1-\frac{r^2}{s^2}\right)^{b_2-1}(1-s^2)^{b_1-1}ds &= \int_r^1 (s^2-r^2)^{b_2-1}(1-s^2)^{b_1-1}sds\\ &= \frac{1}{2}(1-r^2)^{b_1+b_2-1}\int_0^1 y^{b_1-1}(1-y)^{b_2-1}dy = \frac{1}{2}\mathrm{B}(b_1,b_2)(1-r^2)^{b_1+b_2-1},\end{aligned}$$

where $\mathrm{B}(\,\cdot\,,\,\cdot\,)$ denotes the classical beta function. Plugging this into the expression above yields the desired result. $\square$

Proposition 3.A.4. *For each $a, b > 0$, the map $R_{a,b}$ is injective.*

Proof. First note, that by Lemma 3.A.2, $R_{a,1}$ is injective for every $a > 0$. Next, if $0 < b < 1$, Lemma 3.A.3 applied with $(a_2, b_2) = (a, b)$ and $(a_1, b_1) = (a + 2b, 1 - b)$ implies that

$$R_{a+2b,1-b}R_{a,b} = \frac{1}{2}\mathrm{B}(1-b,b)R_{a,1}.$$

Consequently, using that the beta function takes always positive values on real arguments, we deduce that $R_{a,b}$ is injective for every $a > 0$ and $0 < b < 1$, since $R_{a,1}$ is. Lemma 3.A.2 shows that $R_{a,b}$ is injective whenever $R_{a+2,b-1}$ is. Finally, we complete the proof by induction. $\square$

Next, we want to show that the transform $R_{a,b}$ maps the space $C^\infty[-1,1]$ into itself bijectively.

Lemma 3.A.5. *For $a, b > 0$, the map $R_{a,b}$ maps the space $C^\infty[-1,1]$ into itself.*

Proof. Take $\bar{f} \in C^\infty[-1,1]$ and note that for $k \geq 0$ and $t \in (-1,1)$,

$$\frac{d^k}{dt^k}R_{a,b}\bar{f}(t) = \int_0^1 \bar{f}^{(k)}(st)s^{a+k-1}(1-s^2)^{b-1}ds.$$

This follows by induction on $k \geq 0$ by interchanging integral and derivative, using the fact that $\bar{f}$ is infinitely differentiable and all of its derivatives are bounded. Thus, $R_{a,b}\bar{f}$ is a $C^\infty(-1,1)$ function. By interchanging the above integral with the limit $\lim_{t\to\pm 1}$, we see that all higher order derivatives of $R_{a,b}\bar{f}$ extend continuously to $[-1,1]$, that is, $R_{a,b}\bar{f} \in C^\infty[-1,1]$. $\square$

We want to show that the transform $R_{a,b}$ also maps the space $C^\infty[-1,1]$ onto itself. We define the right candidate for the inverse operator recursively.

Definition 3.A.6. For $a, b > 0$, we define $Q_{a,b} : C^\infty[-1,1] \to C^\infty[-1,1]$ recursively as follows. Let $\bar{g} \in C^\infty[-1,1]$.

(i) If $b = 1$, then $Q_{a,1}\bar{g}(s) := a\bar{g}(s) + s\bar{g}'(s)$, $s \in [-1,1]$.

(ii) If $0 < b < 1$, then $Q_{a,b}\bar{g} := \dfrac{2}{B(b,1-b)}Q_{a,1}R_{a+2b,1-b}\bar{g}$.

(iii) If $b > 1$, then $Q_{a,b}\bar{g} := \dfrac{1}{2(b-1)}Q_{a,b-1}Q_{a+2b-2,1}\bar{g}$.

From Lemma 3.A.5 and an inductive argument, it is immediate that $Q_{a,b}$ is well-defined as a transform mapping the space $C^\infty[-1,1]$ into itself.

Proposition 3.A.7. *For each $a, b > 0$, the map $R_{a,b} : C^\infty[-1,1] \to C^\infty[-1,1]$ is a bijection and $R_{a,b}^{-1} = Q_{a,b}$.*

Proof. We want to show inductively that $R_{a,b}Q_{a,b}\bar{g} = \bar{g}$ for all $a, b > 0$ and $\bar{g} \in C^\infty[-1,1]$. If $b = 1$, then

$$[R_{a,1}Q_{a,1}\bar{g}](t) = \int_0^1 [Q_{a,1}\bar{g}](st)s^{a-1}ds = \int_0^1 \left(a\bar{g}(st) + st\bar{g}'(st)\right) s^{a-1}ds$$
$$= \int_0^1 \frac{d}{ds}\left[\bar{g}(st)s^a\right] ds = \bar{g}(t)$$

for all $t \in [-1,1]$, and thus, $R_{a,1}Q_{a,1}\bar{g} = \bar{g}$. For $0 < b < 1$, by definition

$$Q_{a,b}\bar{g} = \frac{2}{B(b,1-b)} Q_{a,1}R_{a+2b,1-b}\bar{g}.$$

Applying the operator $R_{a+2b,1-b}R_{a,b}$ on both sides yields

$$R_{a+2b,1-b}R_{a,b}Q_{a,b}\bar{g} = \frac{2}{B(b,1-b)} R_{a+2b,1-b}R_{a,b}Q_{a,1}R_{a+2b,1-b}\bar{g}$$
$$= R_{a,1}Q_{a,1}R_{a+2b,1-b}\bar{g} = R_{a+2b,1-b}\bar{g},$$

where the second equality is an instance of Lemma 3.A.3 and the final equality is due to the previous step. According to Proposition 3.A.4, the map $R_{a+2b,1-b}$ is injective, which implies that $R_{a,b}Q_{a,b}\bar{g} = \bar{g}$.

For the induction step, let $b > 1$ and note that $R_{a,b} = 2(b-1)R_{a+2b-2,1}R_{a,b-1}$ due to Lemma 3.A.3. Therefore, by the recursive definition of $Q_{a,b}$,

$$R_{a,b}Q_{a,b}\bar{g} = R_{a+2b-2,1}R_{a,b-1}Q_{a,b-1}Q_{a+2b-2,1}\bar{g} = R_{a+2b-2,1}Q_{a+2b-2,1}\bar{g} = \bar{g},$$

where the second equality is due to the induction hypothesis, and the final equality is due to the case where $b = 1$. In conclusion, $R_{a,b}Q_{a,b}\bar{g} = \bar{g}$ for all $a, b > 0$.

Finally, for all $a, b > 0$ and $\bar{f} \in C^\infty[-1,1]$,

$$R_{a,b}(Q_{a,b}R_{a,b}\bar{f}) = R_{a,b}Q_{a,b}(R_{a,b}\bar{f}) = R_{a,b}\bar{f},$$

and since $R_{a,b}$ is injective by Proposition 3.A.4, we also have that $Q_{a,b}R_{a,b}\bar{f} = \bar{f}$. □

3.B The transform T_α

In this section, we investigate the transform T_α from Definition 3.2.12.

Lemma 3.B.1. *Let $\alpha \geq 0$, $\bar{f} \in \mathcal{D}^\alpha$, and let $\bar{\eta}_\varepsilon \in C[-1,1]$, $\varepsilon > 0$, be a family of bump functions such that*

$$\bar{\eta}_\varepsilon(s) = 1 \text{ for } |s| \leq 1-\varepsilon, \qquad \bar{\eta}_\varepsilon(s) = 0 \text{ for } |s| \geq 1-\tfrac{\varepsilon}{2}, \qquad \text{and} \qquad 0 \leq \bar{\eta}_\varepsilon \leq 1.$$

Then $T_\alpha(\bar{\eta}_\varepsilon\bar{f}) \to T_\alpha\bar{f}$ uniformly on $[-1,1]$ as $\varepsilon \to 0^+$.

Proof. We define $\bar{\zeta}_\varepsilon := 1 - \bar{\eta}_\varepsilon$ and will show that $T_\alpha(\bar{\zeta}_\varepsilon \bar{f}) \to 0$ uniformly on $[-1,1]$. For all $s \in [-1,1]$,

$$T_\alpha(\bar{\zeta}_\varepsilon \bar{f})(s) = (1-s^2)^{\frac{\alpha}{2}} \bar{\zeta}_\varepsilon(s)\bar{f}(s) + \alpha s \int_0^s \bar{\zeta}_\varepsilon(t)\bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}\,dt.$$

For the first term, observe that for all $s \geq 0$,

$$\left|(1-s^2)^{\frac{\alpha}{2}}\bar{\zeta}_\varepsilon(s)\bar{f}(s)\right| \leq \sup_{x \geq 1-\varepsilon} |\bar{f}(x)|(1-x^2)^{\frac{\alpha}{2}}.$$

The right hand side is independent of s, and since $\lim_{s\to 1}(1-s^2)^{\frac{\alpha}{2}}\bar{f}(s) = 0$, it tends to zero as $\varepsilon \to 0^+$. We now turn to the integral expression. For $|s| \leq 1-\varepsilon$, it vanishes. For $1-\varepsilon \leq s \leq 1-\frac{\varepsilon}{2}$, by the mean value theorem, there exists $s_0 \in (1-\varepsilon, 1-\frac{\varepsilon}{2})$ such that

$$\alpha s \int_{1-\varepsilon}^s \bar{\zeta}_\varepsilon(t)\bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}\,dt = \bar{f}(s_0)\alpha s \int_{1-\varepsilon}^s \bar{\zeta}_\varepsilon(t)(1-t^2)^{\frac{\alpha-2}{2}}\,dt.$$

For the integral on the right hand side, we find the following estimate.

$$\begin{aligned}\alpha \int_{1-\varepsilon}^s \bar{\zeta}_\varepsilon(t)(1-t^2)^{\frac{\alpha-2}{2}}\,dt &\leq \alpha \int_{1-\varepsilon}^1 (1-t^2)^{\frac{\alpha-2}{2}}\,dt \leq \frac{\alpha}{1-\varepsilon}\int_{1-\varepsilon}^1 t(1-t^2)^{\frac{\alpha-2}{2}}\,dt \\ &= \frac{(1-(1-\varepsilon)^2)^{\frac{\alpha}{2}}}{1-\varepsilon} \leq \frac{2^{\frac{\alpha}{2}}(1-(1-\frac{\varepsilon}{2})^2)^{\frac{\alpha}{2}}}{1-\varepsilon} \leq \frac{2^{\frac{\alpha}{2}}(1-s_0^2)^{\frac{\alpha}{2}}}{1-\varepsilon}.\end{aligned}$$

By combining these computations, we obtain

$$\begin{aligned}\left|\alpha s \int_0^s \bar{\zeta}_\varepsilon(t)\bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}\,dt\right| &= \left|\alpha s \int_{1-\varepsilon}^s \bar{\zeta}_\varepsilon(t)\bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}\,dt\right| \\ &\leq \frac{2^{\frac{\alpha}{2}}}{1-\varepsilon}|f(s_0)|(1-s_0^2)^{\frac{\alpha}{2}} \leq \frac{2^{\frac{\alpha}{2}}}{1-\varepsilon}\sup_{x\geq 1-\varepsilon}|\bar{f}(x)|(1-x^2)^{\frac{\alpha}{2}}.\end{aligned}$$

The final expression is again independent of s an converges to zero as $\varepsilon \to 0^+$.

For $s \geq 1-\frac{\varepsilon}{2}$, we split the integral at $1-\frac{\varepsilon}{2}$, which yields

$$\begin{aligned}&\left|\int_0^s \bar{\zeta}_\varepsilon(t)\bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}\,dt\right| \\ &\qquad \leq \left|\int_0^{1-\frac{\varepsilon}{2}} \bar{\zeta}_\varepsilon(t)\bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}\,dt\right| + \sup_{x\geq 1-\frac{\varepsilon}{2}}\left|\int_{1-\frac{\varepsilon}{2}}^x \bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}\,dt\right|.\end{aligned}$$

For the integral on $[0, 1-\frac{\varepsilon}{2}]$, we can use our estimate from above to show that it tends to zero as $\varepsilon \to 0^+$. The final term is independent of s and also tends to zero because the limit $\lim_{s\to 1}\int_0^s \bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}\,dt$ exists. In conclusion, we have shown that $T_\alpha(\bar{\zeta}_\varepsilon\bar{f})(s) \to 0$ uniformly for $s \in [0,1]$. For $s \in [-1,0]$, the argument is completely analogous. $\square$

Proposition 3.B.2. *For each $\alpha > 0$, the map $T_\alpha : \mathcal{D}^\alpha \to C[-1,1]$ is a bijection and for $\bar{g} \in C[-1,1]$,*

$$T_\alpha^{-1}\bar{g}(t) = (1-t^2)^{-\frac{\alpha}{2}}\bar{g}(t) - \alpha t \int_0^t \bar{g}(s)(1-s^2)^{-\frac{\alpha+2}{2}}\,ds, \qquad t \in (-1,1).$$

Proof. Clearly, T_α maps $C(-1,1)$ functions to $C(-1,1)$ functions and a direct computation verifies the inverse transform on the space $C(-1,1)$. We observe that T_α maps the subspace $\mathcal{D}^\alpha$ into the subspace $C[-1,1]$. Thus, it remains to show that T_α^{-1} maps $C[-1,1]$ into $\mathcal{D}^\alpha$.

To this end, take some $\bar{g} \in C[-1,1]$ and let $\bar{f} := T_\alpha^{-1}\bar{g}$. Then for all $t \in (-1,1)$,

$$(1-t^2)^{\frac{\alpha}{2}}\bar{f}(t) = \bar{g}(t) - \alpha t(1-t^2)^{\frac{\alpha}{2}} \int_0^t \bar{g}(s)(1-s^2)^{-\frac{\alpha+2}{2}}ds.$$

Note that whenever $0 < t_0 < t < 1$, by the mean value theorem for integrals, there exists $t_1 \in (t_0, t) \subseteq (t_0, 1)$ such that

$$\begin{aligned}\alpha t(1-t^2)^{\frac{\alpha}{2}} \int_{t_0}^t \bar{g}(s)(1-s^2)^{-\frac{\alpha+2}{2}}ds &= \alpha t(1-t^2)^{\frac{\alpha}{2}} \int_{t_0}^t \frac{\bar{g}(s)}{s} s(1-s^2)^{-\frac{\alpha+2}{2}}ds \\ &= \frac{\bar{g}(t_1)}{t_1}\alpha t(1-t^2)^{\frac{\alpha}{2}} \int_{t_0}^t s(1-s^2)^{-\frac{\alpha+2}{2}}ds = \frac{\bar{g}(t_1)}{t_1} t\Big(1-\Big(\frac{1-t^2}{1-t_0^2}\Big)^{\frac{\alpha}{2}}\Big).\end{aligned}$$

Let now $\varepsilon > 0$ be arbitrary. Then, by the continuity of $\bar{g}$ at $t = 1$, we may choose $t_0 \in (0,1)$ such that for all $t, t_1 \in (t_0, 1)$,

$$\left|\bar{g}(t) - \frac{\bar{g}(t_1)}{t_1} t\Big(1-\Big(\frac{1-t^2}{1-t_0^2}\Big)^{\frac{\alpha}{2}}\Big)\right| < \varepsilon.$$

Consequently, with this choice of t_0, we obtain that

$$\begin{aligned}\lim_{t\to 1}\left|(1-t^2)^{\frac{\alpha}{2}}\bar{f}(t)\right| &= \lim_{t\to 1}\left|\bar{g}(t) - \alpha t(1-t^2)^{\frac{\alpha}{2}} \int_0^t \bar{g}(s)(1-s^2)^{-\frac{\alpha+2}{2}}dt\right| \\ &= \lim_{t\to 1}\left|\bar{g}(t) - \alpha t(1-t^2)^{\frac{\alpha}{2}} \int_{t_0}^t \bar{g}(s)(1-s^2)^{-\frac{\alpha+2}{2}}dt\right| \le \varepsilon,\end{aligned}$$

where in the second equality, we used that changing the lower integral bound does not affect the limit. Since $\varepsilon > 0$ was arbitrary, $\lim_{t\to 1}(1-t^2)^{\frac{\alpha}{2}}\bar{f}(t) = 0$. The argument for the limit as $t \to -1$ is completely analogous. For the second condition, note that

$$\alpha s \int_0^s \bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}dt = \bar{g}(s) - (1-s^2)^{\frac{\alpha}{2}}\bar{f}(s),$$

so from what we have shown and the continuity of $\bar{g}$ at ± 1, we obtain that the limits $\lim_{s\to\pm 1}\int_0^s \bar{f}(t)(1-t^2)^{\frac{\alpha-2}{2}}dt$ exist and are finite, and thus, $\bar{f} \in \mathcal{D}^\alpha$. □

Bibliography

[1] J. Abardia and A. Bernig, Projection bodies in complex vector spaces, Adv. Math. **227** (2011), no. 2, 830–846.

[2] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, U. S. Government Printing Office, Washington, DC, 1964.

[3] S. Alesker, Continuous rotation invariant valuations on convex sets, Ann. of Math. (2) **149** (1999), no. 3, 977–1005.

[4] S. Alesker, Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture, Geom. Funct. Anal. **11** (2001), no. 2, 244–272.

[5] S. Alesker, Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations, J. Differential Geom. **63** (2003), no. 1, 63–95.

[6] S. Alesker, Introduction to the theory of valuations, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2018.

[7] S. Alesker, Kotrbatý's theorem on valuations and geometric inequalities for convex bodies, Israel J. Math. **247** (2022), no. 1, 361–378.

[8] S. Alesker and A. Bernig, The product on smooth and generalized valuations, Amer. J. Math. **134** (2012), no. 2, 507–560.

[9] S. Alesker and A. Bernig, Convolution of valuations on manifolds, J. Differential Geom. **107** (2017), no. 2, 203–240.

[10] S. Alesker and D. Faifman, Convex valuations invariant under the Lorentz group, J. Differential Geom. **98** (2014), no. 2, 183–236.

[11] K. Ball, Shadows of convex bodies, Trans. Amer. Math. Soc. **327** (1991), no. 2, 891–901.

[12] A. Berg, L. Parapatits, F. E. Schuster, and M. Weberndorfer, Log-concavity properties of Minkowski valuations, Trans. Amer. Math. Soc. **370** (2018), no. 7, 5245–5277, With an appendix by Semyon Alesker.

[13] C. Berg, Corps convexes et potentiels sphériques, Mat.-Fys. Medd. Danske Vid. Selsk. **37** (1969), no. 6, 64 pp. (1969).

[14] A. Bernig and L. Bröcker, Valuations on manifolds and Rumin cohomology, Journal of Differential Geometry **75** (2007), no. 3, 433 –457.

[15] A. Bernig, Integral geometry under G_2 and Spin(7), Israel J. Math. **184** (2011), 301–316.

[16] A. Bernig, Invariant valuations on quaternionic vector spaces, J. Inst. Math. Jussieu **11** (2012), no. 3, 467–499.

[17] A. Bernig and D. Faifman, Valuation theory of indefinite orthogonal groups, J. Funct. Anal. **273** (2017), no. 6, 2167–2247.

[18] A. Bernig and J. H. G. Fu, Hermitian integral geometry, Ann. of Math. (2) **173** (2011), no. 2, 907–945.

[19] A. Bernig, J. H. G. Fu, and G. Solanes, Integral geometry of complex space forms, Geom. Funct. Anal. **24** (2014), no. 2, 403–492.

[20] A. Bernig and D. Hug, Kinematic formulas for tensor valuations, J. Reine Angew. Math. **736** (2018), 141–191.

[21] A. Bernig, J. Kotrbatý, and T. Wannerer, Hard Lefschetz theorem and Hodge-Riemann relations for convex valuations (2024), arXiv: 2312.12294, `2312.12294`.

[22] A. Bernig and G. Solanes, Classification of invariant valuations on the quaternionic plane, J. Funct. Anal. **267** (2014), no. 8, 2933–2961.

[23] A. Bernig and G. Solanes, Kinematic formulas on the quaternionic plane, Proc. Lond. Math. Soc. (3) **115** (2017), no. 4, 725–762.

[24] K. J. Böröczky and M. Ludwig, Minkowski valuations on lattice polytopes, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 1, 163–197.

[25] L. Brauner, G. C. Hofstätter, and O. Ortega-Moreno, Lefschetz operators on convex valuations (2024), `arXiv:2402.14731`.

[26] L. Brauner, G. C. Hofstätter, and O. Ortega-Moreno, The Klain approach to zonal valuations (2024), `arXiv:2410.18651`.

[27] L. Brauner and O. Ortega-Moreno, Fixed points of mean section operators, Trans. Amer. Math. Soc. (electronically published on October 31, 2024), `https://doi.org/10.1090/tran/9270`.

[28] H. Busemann and C. M. Petty, Problems on convex bodies, Math. Scand. **4** (1956), 88–94.

[29] E. B. Christoffel, Ueber die Bestimmung der Gestalt einer krummen Oberfläche durch lokale Messungen auf derselben, J. Reine Angew. Math. **64** (1865), 193–209.

[30] A. Colesanti, M. Ludwig, and F. Mussnig, Minkowski valuations on convex functions, Calc. Var. Partial Differential Equations **56** (2017), no. 6, Paper No. 162, 29.

[31] A. Colesanti, M. Ludwig, and F. Mussnig, A homogeneous decomposition theorem for valuations on convex functions, J. Funct. Anal. **279** (2020), no. 5, 108573, 25.

[32] A. Colesanti, M. Ludwig, and F. Mussnig, The Hadwiger theorem on convex functions, I, Geom. Funct. Anal. (in press) (2022), `arXiv:2009.03702`.

[33] A. Colesanti, M. Ludwig, and F. Mussnig, The Hadwiger theorem on convex functions, III: Steiner formulas and mixed Monge-Ampère measures, Calc. Var. Partial Differential Equations **61** (2022), no. 5, Paper No. 181, 37.

[34] A. Colesanti, M. Ludwig, and F. Mussnig, The Hadwiger theorem on convex functions, II: Cauchy-Kubota formulas, Amer. J. Math. (in press) (2023), `arXiv:2109.09434`.

[35] A. Colesanti, M. Ludwig, and F. Mussnig, The Hadwiger theorem on convex functions, IV: The Klain approach, Adv. Math. **413** (2023), Paper No. 108832, 35.

[36] F. Dorrek, Minkowski endomorphisms, Geom. Funct. Anal. **27** (2017), no. 3, 466–488.

[37] L. C. Evans, Partial differential equations, American Mathematical Society, Providence, RI, 2010.

[38] D. Faifman and G. C. Hofstätter, Convex valuations, from Whitney to Nash (2023), `arXiv:2306.07390`.

[39] W. J. Firey, Intermediate Christoffel-Minkowski problems for figures of revolution, Israel J. Math. **8** (1970), 384–390.

[40] W. J. Firey, Local behaviour of area functions of convex bodies, Pacific J. Math. **35** (1970), 345–357.

[41] A. Fish, F. Nazarov, D. Ryabogin, and A. Zvavitch, The unit ball is an attractor of the intersection body operator, Adv. Math. **226** (2011), no. 3, 2629–2642.

[42] J. H. G. Fu, Structure of the unitary valuation algebra, J. Differential Geom. **72** (2006), no. 3, 509–533.

[43] R. J. Gardner, Geometric tomography, Cambridge University Press, New York, 2006.

[44] P. Goodey, M. Kiderlen, and W. Weil, Spherical projections and liftings in geometric tomography, Adv. Geom. **11** (2011), no. 1, 1–47.

[45] P. Goodey and W. Weil, The determination of convex bodies from the mean of random sections, Math. Proc. Cambridge Philos. Soc. **112** (1992), no. 2, 419–430.

[46] P. Goodey and W. Weil, A uniqueness result for mean section bodies, Adv. Math. **229** (2012), no. 1, 596–601.

[47] P. Goodey and W. Weil, Sums of sections, surface area measures, and the general Minkowski problem, J. Differential Geom. **97** (2014), no. 3, 477–514.

[48] E. Grinberg and G. Zhang, Convolutions, transforms, and convex bodies, Proc. London Math. Soc. (3) **78** (1999), no. 1, 77–115.

[49] H. Groemer, Geometric applications of Fourier series and spherical harmonics, Cambridge University Press, Cambridge, 1996.

[50] C. Haberl, Minkowski valuations intertwining with the special linear group, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 5, 1565–1597.

[51] C. Haberl and L. Parapatits, The centro-affine Hadwiger theorem, J. Amer. Math. Soc. **27** (2014), no. 3, 685–705.

[52] H. Hadwiger, Vorlesungen über Inhalt, Oberfläche und Isoperimetrie, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

[53] G. C. Hofstätter and F. E. Schuster, Blaschke-Santaló inequalities for Minkowski and Asplund endomorphisms, Int. Math. Res. Not. IMRN **2** (2023), 1378–1419.

[54] L. Hörmander, The analysis of linear partial differential operators. I, Springer-Verlag, Berlin, 1990.

[55] D. Hug, F. Mussnig, and J. Ulivelli, Additive kinematic formulas for convex functions (2024), `arXiv:2403.06697`.

[56] D. Hug, F. Mussnig, and J. Ulivelli, Kubota-type formulas and supports of mixed measures (2024), `arXiv:2401.16371`.

[57] M. N. Ivaki, The second mixed projection problem and the projection centroid conjectures, J. Funct. Anal. **272** (2017), no. 12, 5144–5161.

[58] M. N. Ivaki, A local uniqueness theorem for minimizers of Petty's conjectured projection inequality, Mathematika **64** (2018), no. 1, 1–19.

[59] M. Kiderlen, Schnittmittelungen und äquivariante endomorphismen konvexer körper, Ph.D. Thesis, 1999.

[60] M. Kiderlen, Blaschke- and Minkowski-endomorphisms of convex bodies, Trans. Amer. Math. Soc. **358** (2006), no. 12, 5539–5564.

[61] D. A. Klain, A short proof of Hadwiger's characterization theorem, Mathematika **42** (1995), no. 2, 329–339.

[62] D. A. Klain, Even valuations on convex bodies, Trans. Amer. Math. Soc. **352** (2000), no. 1, 71–93.

[63] D. A. Klain and G.-C. Rota, Introduction to geometric probability, Cambridge University Press, Cambridge, 1997.

[64] A. Klenke, Probability theory—a comprehensive course, Springer, Cham, 2020.

[65] J. Knoerr, The support of dually epi-translation invariant valuations on convex functions, J. Funct. Anal. **281** (2021), no. 5, Paper No. 109059, 52.

[66] J. Knoerr, Smooth valuations on convex bodies and finite linear combinations of mixed volumes (2023), arXiv: 2312.08183, `2312.08183`.

[67] J. Knoerr, Smooth valuations on convex functions, J. Differential Geom. **126** (2024), no. 2, 801–835.

[68] J. Knoerr, Zonal valuations on convex bodies (2024), `arXiv:2409.01897`.

[69] J. Kotrbatý, On Hodge-Riemann relations for translation-invariant valuations, Adv. Math. **390** (2021), Paper No. 107914, 28.

[70] J. Kotrbatý and T. Wannerer, On mixed Hodge-Riemann relations for translation-invariant valuations and Aleksandrov-Fenchel inequalities, Commun. Contemp. Math. **24** (2022), no. 7, Paper No. 2150049, 24.

[71] J. Kotrbatý and T. Wannerer, From harmonic analysis of translation-invariant valuations to geometric inequalities for convex bodies, Geom. Funct. Anal. **33** (2023), no. 2, 541–592.

[72] D. Langharst, M. Roysdon, and A. Zvavitch, General measure extensions of projection bodies, Proc. Lond. Math. Soc. (3) **125** (2022), no. 5, 1083–1129.

[73] J. M. Lee, Introduction to smooth manifolds, Springer, New York, 2013.

[74] J. M. Lee, Introduction to Riemannian manifolds, Springer, Cham, 2018.

[75] M. Ludwig, Projection bodies and valuations, Adv. Math. **172** (2002), no. 2, 158–168.

[76] M. Ludwig, Minkowski valuations, Trans. Amer. Math. Soc. **357** (2005), no. 10, 4191–4213.

[77] M. Ludwig, Intersection bodies and valuations, Amer. J. Math. **128** (2006), no. 6, 1409–1428.

[78] M. Ludwig, Minkowski areas and valuations, J. Differential Geom. **86** (2010), no. 1, 133–161.

[79] M. Ludwig and M. Reitzner, A classification of SL(n) invariant valuations, Ann. of Math. (2) **172** (2010), no. 2, 1219–1267.

[80] E. Lutwak, Centroid bodies and dual mixed volumes, Proc. London Math. Soc. (3) **60** (1990), no. 2, 365–391.

[81] E. Lutwak, On quermassintegrals of mixed projection bodies, Geom. Dedicata **33** (1990), no. 1, 51–58.

[82] E. Lutwak, D. Yang, and G. Zhang, L_p affine isoperimetric inequalities, J. Differential Geom. **56** (2000), no. 1, 111–132.

[83] E. Lutwak, D. Yang, and G. Zhang, Orlicz projection bodies, Adv. Math. **223** (2010), no. 1, 220–242.

[84] Y. Martinez-Maure, Hedgehogs and zonoids, Adv. Math. **158** (2001), no. 1, 1–17.

[85] P. McMullen, Valuations and Euler-type relations on certain classes of convex polytopes, Proc. London Math. Soc. (3) **35** (1977), no. 1, 113–135.

[86] P. McMullen, Continuous translation-invariant valuations on the space of compact convex sets, Arch. Math. (Basel) **34** (1980), no. 4, 377–384.

[87] O. Ortega-Moreno, Iterations of Minkowski valuations, J. Funct. Anal. **284** (2023), no. 10, Paper No. 109887, 30.

[88] O. Ortega-Moreno and F. E. Schuster, Fixed points of Minkowski valuations, Adv. Math. **392** (2021), Paper No. 108017, 33.

[89] L. Parapatits and F. E. Schuster, The Steiner formula for Minkowski valuations, Adv. Math. **230** (2012), no. 3, 978–994.

[90] C. M. Petty, Isoperimetric problems, Proceedings of the Conference on Convexity and Combinatorial Geometry (Univ. Oklahoma, Norman, Okla., 1971), 1971, pp. 26–41. MR0362057

[91] C. Saroglou and A. Zvavitch, Iterations of the projection body operator and a remark on Petty's conjectured projection inequality, J. Funct. Anal. **272** (2017), no. 2, 613–630.

[92] R. Schneider, Equivariant endomorphisms of the space of convex bodies, Trans. Amer. Math. Soc. **194** (1974), 53–78.

[93] R. Schneider, Rekonstruktion eines konvexen Körpers aus seinen Projektionen, Math. Nachr. **79** (1977), 325–329.

[94] R. Schneider, Geometric inequalities for Poisson processes of convex bodies and cylinders, Results Math. **11** (1987), no. 1-2, 165–185.

[95] R. Schneider, Simple valuations on convex bodies, Mathematika **43** (1996), no. 1, 32–39.

[96] R. Schneider, Convex bodies: the Brunn-Minkowski theory, Cambridge University Press, Cambridge, 2014.

[97] F. E. Schuster, Volume inequalities and additive maps of convex bodies, Mathematika **53** (2006), no. 2, 211–234 (2007).

[98] F. E. Schuster, Convolutions and multiplier transformations of convex bodies, Trans. Amer. Math. Soc. **359** (2007), no. 11, 5567–5591.

[99] F. E. Schuster, Crofton measures and Minkowski valuations, Duke Math. J. **154** (2010), no. 1, 1–30.

[100] F. E. Schuster, Private communication, 2023.

[101] F. E. Schuster and T. Wannerer, GL(n) contravariant Minkowski valuations, Trans. Amer. Math. Soc. **364** (2012), no. 2, 815–826.

[102] F. E. Schuster and T. Wannerer, Even Minkowski valuations, Amer. J. Math. **137** (2015), no. 6, 1651–1683.

[103] F. E. Schuster and T. Wannerer, Minkowski valuations and generalized valuations, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 8, 1851–1884.

[104] G. Solanes and T. Wannerer, Integral geometry of exceptional spheres, J. Differential Geom. **117** (2021), no. 1, 137–191.

[105] R. S. Strichartz, L^p estimates for Radon transforms in Euclidean and non-Euclidean spaces, Duke Math. J. **48** (1981), no. 4, 699–727.

[106] G. Szegő, Orthogonal polynomials, American Mathematical Society, Providence, R.I., 1975.

[107] M. Takeuchi, Modern spherical functions, American Mathematical Society, Providence, RI, 1994.

[108] M. E. Taylor, Partial differential equations I. Basic theory, Springer, New York, 2011.

[109] T. Wannerer, GL(n) equivariant Minkowski valuations, Indiana Univ. Math. J. **60** (2011), no. 5, 1655–1672.

[110] G. Warner, Harmonic analysis on semi-simple Lie groups. I, Springer-Verlag, New York-Heidelberg, 1972.

[111] W. Weil, Über die Projektionenkörper konvexer Polytope, Arch. Math. (Basel) **22** (1971), 664–672.

[112] G. Y. Zhang, Restricted chord projection and affine inequalities, Geom. Dedicata **39** (1991), no. 2, 213–222.