Calc. Var. (2025) 64:63
https://doi.org/10.1007/s00526-024-02849-8

Calculus of Variations

On time-splitting methods for gradient flows with two dissipation mechanisms

Alexander Mielke[1,2] · Riccarda Rossi[3] · Artur Stephan[4]

Received: 21 July 2023 / Accepted: 8 October 2024

Abstract

We consider generalized gradient systems in Banach spaces whose evolutions are generated by the interplay between an energy functional and a dissipation potential. We focus on the case in which the dual dissipation potential is given by a sum of two functionals and show that solutions of the associated gradient-flow evolution equation with combined dissipation can be constructed by a split-step method, i.e. by solving alternately the gradient systems featuring only one of the dissipation potentials and concatenating the corresponding trajectories. Thereby the construction of solutions is provided either by semiflows, on the time-continuous level, or by using Alternating Minimizing Movements in the time-discrete setting. In both cases the convergence analysis relies on the energy-dissipation principle for gradient systems.

Mathematics Subject Classification 34G20 · 47J20 · 49S05 · 58E30 · 35Q74

Contents

Communicated by A. Mondino.

The research of AM was partially supported by Deutsche Forschungsgemeinschaft via SPP 2256, Subproject Mi 459/9-1 (Project No. 441470105). RR has been partially supported by GNAMPA and by the MIUR - PRIN Project 2020F3NCPX "Mathematics for industry 4.0 (Math414)".

✉ Artur Stephan
artur.stephan@tuwien.ac.at

Alexander Mielke
alexander.mielke@wias-berlin.de

Riccarda Rossi
riccarda.rossi@unibs.it

1 WIAS, Berlin, Germany

2 Humboldt-Universität zu Berlin, Berlin, Germany

3 DIMI, Università degli studi di Brescia, Brescia, Italy

4 Institute for Analysis and Scientific Computing, TU Wien, Vienna, Austria

Published online: 09 January 2025

1 Introduction

This paper revolves around the application of time-splitting methods to dissipative evolutionary processes that are generated by a generalized gradient system $(\mathscr{X}, \mathscr{E}, \mathscr{R})$, which is a triple such that

1. the ambient space $(\mathscr{X}, \|\cdot\|)$ is a (separable) reflexive Banach space;
2. the energy is a lower semicontinuous, time-dependent functional $\mathscr{E} : [0,T]\times\mathscr{X} \to (-\infty,\infty]$, bounded from below and has the proper domain $[0,T]\times\mathscr{D}$;
3. and the dissipation mechanisms are encoded by a convex lower semicontinuous dissipation potential $\mathscr{R} : \mathscr{X} \to [0,\infty)$.

In what follows, we will confine the discussion to the case in which both $\mathscr{R}$ and its convex conjugate $\mathscr{R}^* : \mathscr{X}^* \to [0,\infty)$, $\xi \mapsto \sup_{v\in\mathscr{X}}(\langle\xi, v\rangle_{\mathscr{X}} - \mathscr{R}(v))$, (where $\langle\cdot,\cdot\rangle_{\mathscr{X}}$ denotes the duality pairing between $\mathscr{X}^*$ and $\mathscr{X}$), have superlinear growth at infinity. We will refer to the triple $(\mathscr{X}, \mathscr{E}, \mathscr{R})$ as a *generalized* gradient system, because for true gradient systems the dissipation potential $\mathscr{R} : \mathscr{X} \to [0,\infty)$ has to be quadratic, leading to a Hilbert-space structure, see [20]. Whenever convenient, we will alternatively write $(\mathscr{X}, \mathscr{E}, \mathscr{R}^*)$. Throughout the paper, the dissipation potential $\mathscr{R}$ will be assumed *state-independent*.

Typically, in this class there fall processes whose evolution results from a balance between the two competing mechanism of decrease of the energy $\mathscr{E}$ and dissipation of energy according to $\mathscr{R}$. Thus, they are governed by the subdifferential inclusion

$$\partial\mathscr{R}(u'(t)) + \partial\mathscr{E}(t, u(t)) \ni 0 \quad \text{in } \mathscr{X}^* \quad \text{for a.a. } t \in (0, T). \tag{1.1}$$

Indeed, (1.1) is a balance law between frictional forces in the (convex analysis) subdifferential $\partial\mathscr{R} : \mathscr{X} \rightrightarrows \mathscr{X}^*$ given by

$$\partial\mathscr{R}(v) := \{\omega \in \mathscr{X}^* \,:\, \mathscr{R}(\hat{v}) - \mathscr{R}(v) \geq \langle \omega, \hat{v}-v \rangle_{\mathscr{X}} \quad \text{for all } \hat{v} \in \mathscr{X}\},$$

and potential restoring forces in the Fréchet subdifferential $\partial\mathscr{E} : [0, T] \times \mathscr{X} \rightrightarrows \mathscr{X}^*$ of $\mathscr{E}$ with respect to its second variable: at a given $(t, u) \in [0, T]\times\mathscr{D}$ we define $\partial\mathscr{E}(t, u)$ as the set of all $\xi \in \mathscr{X}^*$ satisfying

$$\mathscr{E}(t, w) - \mathscr{E}(t, u) \geq \langle \xi, w-u \rangle_{\mathscr{X}} + \mathrm{o}(\|w-u\|) \quad \text{as } \|w-u\| \to 0. \tag{1.2}$$

Therefore, (1.1) can be recast as

$$-\xi(t) \in \partial\mathscr{R}(u'(t)) \;\; \text{in } \mathscr{X}^* \quad \text{and} \quad \xi(t) \in \partial\mathscr{E}(t, u(t)) \qquad \text{for a.a. } t \in (0, T). \tag{1.3a}$$

From the thermodynamical point of view the *primal* dissipation potential $\mathscr{R}$ has a prominent role in defining the *kinetic relation* $\eta \in \partial\mathscr{R}(v)$ between the rate $v \in \mathscr{X}$ and friction force $\eta \in \mathscr{X}^*$. By the Fenchel equivalence in convex analysis the kinetic relation can be inverted as $v \in \partial\mathscr{R}^*(\eta)$, such that it is also meaningful to reformulate (1.2) in the rate form

$$u'(t) \in \partial\mathscr{R}^*(-\xi(t)) \;\; \text{in } \mathscr{X} \quad \text{and} \quad \xi(t) \in \partial\mathscr{E}(t, u(t)) \qquad \text{for a.a. } t \in (0, T), \tag{1.3b}$$

which casts the *dual* dissipation potential under the spotlight. The first existence results for evolution equations (1.3a),(1.3b), in the Hilbert space setting and for a quadratic dissipation, date back to the late '60 s [11, 17]; in particular, we refer to the monograph [6]. Existence in general *doubly nonlinear* case has been first systematically tackled in the seminal papers [9, 10]. In the last three decades, the existence theory has been extended to encompass *nonsmooth* and *nonconvex* driving energies [26, 29], based on the *variational theory* for the analysis of gradient flows in metric spaces [2, 3, 28].

Both dissipation potentials $\mathscr{R}$ and $\mathscr{R}^*$ feature in the *energy-dissipation balance*

$$\mathscr{E}(t, u(t)) + \int_s^t \left\{\mathscr{R}(u'(r)) + \mathscr{R}^*(-\xi(r))\right\} \mathrm{d}r = \mathscr{E}(s, u(s)) + \int_s^t \partial_t\mathscr{E}(r, u(r))\mathrm{d}r \tag{1.4}$$

for all $0 \leq s \leq t \leq T$, which is equivalent to the primal and dual formulations under the validity of a suitable chain-rule property for the gradient system $(\mathscr{X}, \mathscr{E}, \mathscr{R})$. As a matter of fact, (1.4) lies at the core of our variational approach to gradient systems. Indeed, it turns out that, if the chain rule holds, a pair (u, ξ) fulfills (1.4) (and thus (1.3)) if and only if it satisfies the upper inequality $\leq$, and this has paved the way for the usage of the toolbox from Calculus of Variations in order to prove existence results for (1.3), see [2, 26, 28] and the references in the survey [20].

So far, we have used $(\mathscr{X}, \mathscr{E}, \mathscr{R})$ to denote an abstract gradient system. Now we will work with specific gradient systems $(\mathbf{X}_j, \mathcal{E}, \mathcal{R}_j)$, where we emphasize that the energy $\mathcal{E}$ stays the same, but we have two different dissipations $\mathcal{R}_j : \mathbf{X}_j \to [0, \infty)$, $j = 1, 2$. Our aim is to study the interaction between those two systems and the *effective* system $(\mathbf{X}_1, \mathcal{E}, \mathcal{R}_{\mathrm{eff}})$, where

$$\mathcal{R}^*_{\mathrm{eff}} = \mathcal{R}^*_1 + \mathcal{R}^*_2.$$

Thus, $(\mathbf{X}_1, \mathcal{E}, \mathcal{R}_{\mathrm{eff}})$ generates the subdifferential inclusion

$$u'(t) \in \partial(\mathcal{R}^*_1 + \mathcal{R}^*_2)(-\xi(t)) \;\; \text{in } \mathbf{X}_1 \quad \text{and} \quad \xi(t) \in \partial\mathcal{E}(t, u(t)) \qquad \text{for a.a. } t \in (0, T). \tag{1.5}$$

The spaces $\mathbf{X}_j$ are (separable) reflexive spaces, 'ordered' in such a way that $\mathbf{X}_2 \subset \mathbf{X}_1$ continuously. The energy functional $\mathcal{E} : [0, T] \times \mathbf{X}_2 \to (-\infty, \infty]$ is defined on the smaller space and is extended to the larger space by $+\infty$.

As we will see, under our conditions, $\mathcal{R}_{\text{eff}}$ is given by the inf-*convolution* of $\mathcal{R}_1$ and $\mathcal{R}_2$, namely

$$\mathcal{R}_{\text{eff}}(v) := \inf_{v_1,\, v_2 \in \mathbf{X}_1,\ v = v_1 + v_2} \big(\mathcal{R}_1(v_1) + \mathcal{R}_2(v_2)\big)\,. \tag{1.6}$$

In order to construct solutions to (1.5), it may then be convenient to use a time-splitting approach, capable of handling the different properties of the potentials $\mathcal{R}_1$ and $\mathcal{R}_2$. We will discuss two different approaches: (i) a time-splitting approach using exact solutions of $(\mathbf{X}_j, \mathcal{E}, \mathcal{R}_j)$ on half-intervals and (ii) an alternating minimizing approach.

The time-splitting approach

Indeed, the split-step method with time step $\tau = T/N$ and $N \gg 1$, inducing a uniform partition of the interval $[0, T]$ in sub-intervals $((k-1)\tau, k\tau)$, $k \in \{1, \ldots, N\}$, with midpoint $(k-1/2)\tau$, amounts to

(i) solving on the semi-intervals of length $\frac{1}{2}\tau$, alternately, the single-dissipation gradient systems $(\mathbf{X}_j, \mathcal{E}, 2\mathcal{R}_j^*)$, $j \in \{1, 2\}$ (i.e., with rescaled potentials $\widetilde{\mathcal{R}}_j(\cdot) = 2\mathcal{R}_j(\frac{1}{2}\,\cdot))$;
(ii) concatenating the solutions to finally fill the whole interval $[0, T]$ and obtain a trajectory $U_\tau : [0, T] \to \mathbf{X}_1$.

To be more precise (see Sect. 3 for details), we consider for $k \in \{1, \ldots, N\}$ the two Cauchy problems:

1. $\partial\widetilde{\mathcal{R}}_1(u'(t)) + \partial\mathcal{E}(t, u(t)) \ni 0$ in $\mathbf{X}_1^*$ on $((k-1)\tau, (k-1/2)\tau]$, $u((k-1)\tau) = \underline{u} \in \mathrm{dom}\,(\mathcal{E})$,
2. $\partial\widetilde{\mathcal{R}}_2(u'(t)) + \partial\mathcal{E}(t, u(t)) \ni 0$ in $\mathbf{X}_2^*$ on $((k-1/2)\tau, k\tau]$, $u((k-1/2)\tau) = \overline{u} \in \mathrm{dom}\,(\mathcal{E})$,

where the scaling factor in $\widetilde{\mathcal{R}}_j$ reflects the halved length of the intervals $((k-1)\tau, (k-1/2)\tau]$ and $((k-1/2)\tau, k\tau]$ on which the systems $(\mathbf{X}_j, \mathcal{E}, \widetilde{\mathcal{R}}_j)$ evolve. Working with the rescaled potentials $\widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_2$ will prove convenient for our analysis, see the discussion at the end of Sect. 3. Then, we define recursively the approximate solution $U_\tau : [0, T] \to \mathrm{dom}\,(\mathcal{E})$ by $U_\tau(0) := u_0$ and

U_τ solves the first Cauchy problem with $\underline{u} := U_\tau((k-1)\tau)$ on $((k-1)\tau, (k-1/2)\tau]$,
U_τ solves the second Cauchy problem with $\overline{u} := U_\tau\,((k-1/2)\tau)$ on $((k-1/2)\tau, k\tau]$.

Then, and this is the main objective of the paper, the task in the convergence analysis is to show that in the limit $N \to \infty$ the sequence of trajectories $(U_\tau)_\tau$ converge to a curve solving the *effective gradient-flow equation*

$$\partial\mathcal{R}_{\text{eff}}(u'(t)) + \partial\mathcal{E}(t, u(t)) \ni 0 \qquad \text{for a.a. } t \in (0, T), \tag{1.7}$$

which is equivalent to (1.5).

For instance, in the case of reaction-diffusion systems, with this approach the approximate solutions can be constructed by concatenating a solution obtained in the diffusion step, by a method tailored to the linear parabolic structure, and a solution arising from a pure reaction step, which exploits the distinct features of the nonlinear ODE. Split-step methods for evolution equations, where the right-hand side is given by a sum of two parts as in (1.5), have a long history starting with the works of Lie and Trotter for linear evolution equations (see e.g. [33]). A generalization for nonlinear semigroups given by subdifferentials of convex functions on a Hilbert space (i.e. quadratic dissipation and two different energies) can be found in [6] (Prop. 4.3−4.4, Ch. VI) and [15]. In [8] convergence of split-step methods for gradient flows in metric spaces as in [2] has been shown, in the case the driving energy consists of two contributions with different properties.

In contrast, the functional setup considered in this paper is significantly different from that usually addressed for time-splitting methods applied to gradient systems, because we tackle the situation in which the *dissipation* consists of two parts. Hence, we have to account for the two different geometries in the underlying space.

The alternating minimizing movement scheme

So far, we have illustrated the time-splitting approach to the generalized gradient system $(\mathbf{X}_1, \mathcal{E}, \mathcal{R}_1^* {+} \mathcal{R}_2^*)$ *on the time-continuous level.*

Nonetheless, a commonly used procedure for constructing solutions to the subdifferential inclusions for the single-dissipation gradient systems $(\mathbf{X}_j, \mathcal{E}, \mathcal{R}_j)$ is via *Minimizing Movements*. Adopting this method in the context of the time-splitting approach means that for each $j = 1, 2$ we solve the time-incremental minimization problems involving the potentials $\widetilde{\mathcal{R}}_j(\cdot) = 2\mathcal{R}_j(\frac{1}{2}\,\cdot)$, whose rescaling corresponds to the halved length $\frac{1}{2}\tau$ of the discrete intervals. Approximate solutions are then defined by piecing together these discrete solutions, i.e., the concatenation step is carried out *on the time-discrete level.*

We briefly illustrate the latter procedure in the following lines and postpone a detailed analysis to Sect. 6, where we prove our convergence result in Theorem 6.1.

For illustrative reasons, we confine ourselves to a uniform partition by intervals of equal length; the rigorous analysis in the main text allows for general partitions. Starting from an initial datum $u_0 \in \mathrm{dom}(\mathcal{E})$, we define the *piecewise constant* time-discrete solutions $\overline{U}_\tau : [0, T] \to \mathrm{dom}(\mathcal{E}) \subset \mathbf{X}_1$ via $\overline{U}_\tau(0) := u_0 =: U_0^2$ and, for $k = 1, \ldots, N$, we set

$$
\begin{aligned}
&\overline{U}_\tau(t) := U_k^1 \ \text{ for } t \in ((k{-}1)\tau, (k{-}1/2)\tau], \quad \overline{U}_\tau(t) := U_k^2 \ \text{ for } t \in ((k{-}1/2)\tau, k\tau],\\
&\text{where } U_k^1 \in \operatorname*{Argmin}_{U \in \mathbf{X}_1} \left\{ \tfrac{\tau}{2}\, \widetilde{\mathcal{R}}_1\big(\tfrac{2}{\tau}(U {-} U_{k-1}^2)\big) {+} \mathcal{E}((k{-}1/2)\tau, U) \right\},\\
&\quad \text{and } U_k^2 \in \operatorname*{Argmin}_{U \in \mathbf{X}_2} \left\{ \tfrac{\tau}{2}\, \widetilde{\mathcal{R}}_2\big(\tfrac{2}{\tau}(U {-} U_k^1)\big) {+} \mathcal{E}(k\tau, U) \right\},
\end{aligned}
\tag{1.8}
$$

cf. also (6.1) ahead; indeed, also on the time-discrete level it is useful to work with the rescaled potentials $\widetilde{\mathcal{R}}_j$. Further, we introduce the piecewise linear function $\widehat{U}_\tau : [0, T] \to \mathbf{X}_1$ so obtained by affinely interpolating the values $\overline{U}_\tau(t)$ and $\overline{U}_\tau(t{-}\tau/2)$ for t in the semi-intervals $[(k{-}1)\tau, (k{-}1/2)\tau]$ and $[(k{-}1/2)\tau, k\tau]$. We also consider the piecewise constant interpolant $\overline{\xi}_\tau$ of the discrete forces $\xi_k^1 \in \partial^{\mathbf{X}_1}\mathcal{E}((k{-}1/2)\tau, U_k^1)$ and $\xi_k^2 \in \partial^{\mathbf{X}_2}\mathcal{E}(k\tau, U_k^2)$ (with $\partial^{\mathbf{X}_j}\mathcal{E}(t, \cdot) : \mathbf{X}_j \rightrightarrows \mathbf{X}_j^*$ the Fréchet subdifferentials of $\mathcal{E}(t, \cdot)$ with respect to the $\langle \cdot, \cdot \rangle_{\mathbf{X}_j}$ pairings), that feature in the Euler-Lagrange equations for the minimum problems (1.8).

Our analysis

Let us now hint at the key points in the convergence argument for the approximate solutions arising from the alternating Minimizing Movement scheme (1.8). To simplify the exposition in this introduction we assume now that $\mathcal{E} : [0, T] \times \mathbf{X}_2 \to (-\infty, \infty]$ is λ-convex in its second variable, uniformly in $t \in [0,T]$, with respect to the coarser norm $\|\cdot\|_{\mathbf{X}_1}$, namely

$$\exists\,\lambda \in \mathbb{R}\ \forall\, t \in [0,T]\ \forall\, u_0, u_1 \in \mathrm{dom}(\mathcal{E})\ \forall\,\theta \in [0,1]\ :$$
$$\mathcal{E}(t,(1-\theta)u_0+\theta u_1) \leq (1-\theta)\mathcal{E}(t,u_0) + \theta\mathcal{E}(t,u_1) - \frac{\lambda}{2}\theta(1-\theta)\|u_0-u_1\|^2_{\mathbf{X}_1}.$$

The condition of λ-convexity will not be used in the later sections; wherever needed the piecewise linear interpolant $\widehat{U}_\tau$ will be replaced by the more advanced variational interpolants, see Sect. 6.

Staying in the simpler λ-convex case, the functions $(\overline{U}_\tau, \widehat{U}_\tau, \overline{\xi}_\tau)$ satisfy a discrete version of the energy-dissipation upper estimate, i.e. for every $0 \leq s \leq t \leq T$ we have

$$\begin{aligned}&\mathcal{E}(\bar{\mathsf{t}}_\tau(t), \overline{U}_\tau(t)) + \mathcal{D}^{\mathrm{rate}}_\tau([s,t]) + \mathcal{D}^{\mathrm{slope}}_\tau([s,t])\\ &\leq \mathcal{E}(\bar{\mathsf{t}}_\tau(s), \overline{U}_\tau(s)) + \int_s^t \partial_t\mathcal{E}(\bar{\mathsf{t}}_\tau(r), \overline{U}_\tau\left(r-\tfrac{\tau}{2}\right))\mathrm{d}r + \mathrm{Rem}_\tau([s,t]),\end{aligned} \tag{1.9}$$

where $\bar{\mathsf{t}}_\tau : [0,T] \to [0,T]$ denotes the piecewise constant interpolant of the notes $(k\tau)_{k=1}^N$ of the partition. Here, the rate contribution $\mathcal{D}^{\mathrm{rate}}_\tau$ incorporates the primal dissipation potentials depending on the rate $\widehat{U}'_\tau$ and featuring in the discrete energy-dissipation balances for the individual systems $(\mathbf{X}_j, \mathcal{E}, 2\mathcal{R}^*_j)$, namely

$$\mathcal{D}^{\mathrm{rate}}_\tau([s,t]) := 2\int_s^t \left\{\chi_\tau(r)\mathcal{R}_1\left(\tfrac{1}{2}\widehat{U}'_\tau(r)\right) + (1-\chi_\tau(r))\mathcal{R}_2\left(\tfrac{1}{2}\widehat{U}'_\tau(r)\right)\right\}\mathrm{d}r, \tag{1.10a}$$

where $\chi_\tau : [0,T] \to \{0,1\}$ is the characteristic function of the union of the left semi-intervals $((k-1)\tau, (k-1/2)\tau]$. Accordingly, the slope contribution $\mathcal{D}^{\mathrm{slope}}_\tau$ features the dual dissipation potentials evaluated at the force $\overline{\xi}_\tau(t) \in \partial\mathcal{E}(\bar{\mathsf{t}}_\tau(t), \overline{U}_\tau(t))$, i.e.

$$\mathcal{D}^{\mathrm{slope}}_\tau([s,t]) := 2\int_s^t \left\{\chi_\tau(r)\mathcal{R}^*_1(-\overline{\xi}_\tau(r)) + (1-\chi_\tau(r))\mathcal{R}^*_2(-\overline{\xi}_\tau(r))\right\}\mathrm{d}r. \tag{1.10b}$$

The previously required λ-convexity of the energy $\mathcal{E}(t,\cdot)$ plays a key role in the estimate of the remainder term via

$$\begin{aligned}\mathrm{Rem}_\tau([s,t]) &:= \frac{1}{\tau}\int_s^t \Big(\mathcal{E}(\bar{\mathsf{t}}_\tau(r), \overline{U}_\tau(r)) - \mathcal{E}(\bar{\mathsf{t}}_\tau(r), \overline{U}_\tau\left(r-\tfrac{\tau}{2}\right)) - \langle\overline{\xi}_\tau(r), \overline{U}_\tau(r) - \overline{U}_\tau\left(r-\tfrac{\tau}{2}\right)\rangle_{\mathbf{X}_1}\Big)\mathrm{d}r\\ &\leq \frac{\lambda}{2}\int_s^t \|\widehat{U}'_\tau(r)\|_{\mathbf{X}_1}\|\overline{U}_\tau(r) - \overline{U}_\tau(r-\tfrac{\tau}{2})\|_{\mathbf{X}_1}\mathrm{d}r.\end{aligned}$$

This estimate ensures that $\lim_{\tau\to 0}\mathrm{Rem}_\tau([s,t]) = 0$. However, once again we emphasize that λ-convexity of $\mathcal{E}(t,\cdot)$ is not necessary for our analysis and will not be used elsewhere in the paper. It is assumed here, only, in order to illustrate the derivation of the discrete energy-derivation upper estimate for $(\mathbf{X}_1, \mathcal{E}, \mathcal{R}^*_1+\mathcal{R}^*_2)$ in a simple and self-contained way.

Taking the limit in (1.9) leads to an upper energy-dissipation inequality that, assuming the validity of a suitable chain rule property for the energy $\mathcal{E}$, is in fact equivalent to the corresponding energy-dissipation balance and yields a solution to the generalized gradient system $(\mathbf{X}_1, \mathcal{E}, \mathcal{R}^*_1+\mathcal{R}^*_2)$. This is summarized in the following result, anticipating Theorem 6.1 ahead. In the statement below we do not detail all technical assumptions on the quintuple $(\mathbf{X}_1, \mathbf{X}_2, \mathcal{E}, \mathcal{R}_1, \mathcal{R}_2)$, but we highlight the crucial, additional requirement that the Fréchet subdifferential is a singleton, called *singleton condition* subsequently.

The counterexample constructed in Sect. 4.1 shows that convergence of the time-splitting scheme to a solution of (1.5) may in fact be *false*, if the singleton condition (1.11) is not assumed, even in the simple case $\mathbf{X}_1 = \mathbf{X}_2 = \mathbb{R}^2$.

Theorem *Under suitable conditions on* $(\boldsymbol{X}_1, \boldsymbol{X}_2, \mathcal{E}, \mathcal{R}_1, \mathcal{R}_2)$*, suppose also that*

$$\partial^{\boldsymbol{X}_1}\mathcal{E}(t,u) = \partial^{\boldsymbol{X}_2}\mathcal{E}(t,u) \text{ is a singleton for all } (t,u) \in dom(\mathcal{E}). \tag{1.11}$$

Then, for any null sequence $\tau \to 0$ *the curves* $(\overline{U}_\tau)$, $(\widehat{U}_\tau)$*, and* $(\overline{\xi}_\tau)$ *suitably converge to a pair* (U,ξ) *solving the subdifferential inclusion* (1.5) *and fulfilling the energy-dissipation balance*

$$\mathcal{E}(t,U(t)) + \int_s^t \big(\mathcal{R}_{\mathrm{eff}}(U'(r)) + (\mathcal{R}_1^* + \mathcal{R}_2^*)(-\xi(r))\big)\,\mathrm{d}r = \mathcal{E}(s,U(s)) + \int_s^t \partial_t\mathcal{E}(r,U(r))\mathrm{d}r \tag{1.12}$$

for every $0 \le s \le t \le T$*, where* $\mathcal{R}_{\mathrm{eff}} : \boldsymbol{X}_1 \to [0,\infty)$ *is the primal dissipation potential corresponding to* $(\mathcal{R}_1^*+\mathcal{R}_2^*)$*, namely the* inf*-convolution of* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *(cf.* (1.6)*).*

Without entering into the details of the proof, we now motivate how $\mathcal{R}_{\mathrm{eff}}$ naturally arises in the passage to the limit in the rate term from (1.10a). For simplicity, and with no loss in generality, we illustrate this when $[s,t] = [0,T]$. Then, for $\tau > 0$ we have

$$\begin{aligned}
\mathcal{D}_\tau^{\mathrm{rate}}([0,T]) &= \sum_{k=1}^N \left\{ \int_{(k-1)\tau}^{(k-1/2)\tau} 2\mathcal{R}_1(\tfrac{1}{2}\widehat{U}_\tau'(r))\mathrm{d}r + \int_{(k-1/2)\tau}^{k\tau} 2\mathcal{R}_2(\tfrac{1}{2}\widehat{U}_\tau'(r))\mathrm{d}r \right\} \\
&\overset{(1)}{=} \sum_{k=1}^N \tau \left\{ ⨍_{(k-1)\tau}^{(k-1/2)\tau} \mathcal{R}_1(\tfrac{1}{2}\widehat{U}_\tau'(r))\mathrm{d}r + ⨍_{(k-1/2)\tau}^{k\tau} \mathcal{R}_2(\tfrac{1}{2}\widehat{U}_\tau'(r))\mathrm{d}r \right\} \\
&\overset{(2)}{\geq} \sum_{k=1}^N \tau \left\{ \mathcal{R}_1\left(⨍_{(k-1)\tau}^{(k-1/2)\tau} \tfrac{1}{2}\widehat{U}_\tau'(r))\mathrm{d}r \right) + \mathcal{R}_2\left(⨍_{(k-1/2)\tau}^{k\tau} \tfrac{1}{2}\widehat{U}_\tau'(r)\mathrm{d}r \right) \right\} \\
&\overset{(3)}{\geq} \sum_{k=1}^N \tau\, \mathcal{R}_{\mathrm{eff}}\left(\frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} \mathbb{U}_\tau'(r)\mathrm{d}r \right) = \int_0^T \mathcal{R}_{\mathrm{eff}}\big(\mathbb{U}_\tau'(r)\big)\mathrm{d}r,
\end{aligned}$$

where $\mathbb{U} : [0,T] \to \mathbf{X}_1$ denotes the piecewise affine interpolant with $\mathbb{U}_\tau(k\tau) = U_\tau(k\tau)$ for $k \in \{0,\ldots,N\}$. In the above calculation, on the right-hand side of (1) the symbol $⨍$ denotes the integral average, for (2) we have used convexity of $\mathcal{R}_j$ and Jensen's inequality, while (3) follows from the definition of $\mathcal{R}_{\mathrm{eff}}$. Taking the limit $\tau \to 0$ we can use $\mathbb{U}_\tau' \rightharpoonup U'$ in $\mathrm{L}^1(0,T;\mathbf{X}_1)$, and obtain the first liminf estimate:

$$\liminf_{\tau\to 0} \mathcal{D}_\tau^{\mathrm{rate}}([0,T]) \geq \int_0^T \mathcal{R}_{\mathrm{eff}}\big(U'(t)\big)\mathrm{d}t.$$

Likewise, the role of the singleton condition can be understood by a perusal of the argument for taking the limit in the slope term. For this we introduce a key tool for our analysis: the *repetition operators* $\mathbb{T}_\tau^{(1)}$ and $\mathbb{T}_\tau^{(2)}$ that, applied to a given function $\zeta : [0,T] \to \mathbf{X}_j^*$, are defined via

$$(\mathbb{T}_\tau^{(1)}\zeta)(t) := \begin{cases} \zeta(t) & \text{if } t \in ((k-1)\tau, (k-1/2)\tau] \text{ for } k = 1,\ldots,N, \\ \zeta(t-\frac{\tau}{2}) & \text{if } t \in ((k-1/2)\tau, k\tau] \quad\quad \text{for } k = 1,\ldots,N, \end{cases}$$

$$(\mathbb{T}_\tau^{(2)}\zeta)(t) := \begin{cases} \zeta(t+\frac{\tau}{2}) & \text{if } t \in ((k-1)\tau, (k-1/2)\tau] \text{ for } k = 1,\ldots,N, \\ \zeta(t) & \text{if } t \in ((k-1/2)\tau, k\tau] \quad\quad \text{for } k = 1,\ldots,N. \end{cases}$$

Thus, $\mathbb{T}_\tau^{(1)}$ replicates, on the right semi-intervals $((k-1/2)\tau, k\tau]$, the restriction of ζ to the preceding left semi-intervals $((k-1/2)\tau, k\tau]$, while $\mathbb{T}_\tau^{(2)}$ does the converse. A crucial property of the repetition operators $\mathbb{T}^{(j)}$ is seen when calculating the slope part of the dissipation integral, namely (see (5.7) for more details)

$$\mathcal{D}_\tau^{\text{slope}}([0,T]) = \int_0^T \Big\{\chi_\tau(r)\, 2\mathcal{R}_1^*\big(-\overline{\xi}_\tau(r)\big) + \big(1-\chi_\tau(r)\big)\, 2\mathcal{R}_2^*\big(-\overline{\xi}_\tau(r)\big)\Big\}\mathrm{d}r$$
$$= \int_0^T \Big\{\mathcal{R}_1^*\big(-\mathbb{T}_\tau^{(1)}\overline{\xi}_\tau(r)\big)+\mathcal{R}_2^*\big(-\mathbb{T}_\tau^{(2)}\overline{\xi}_\tau(r)\big)\Big\}\mathrm{d}r\,.$$

In the first line the integrand on the right-hand side is a sum of two products, where each factor only weakly converges for $\tau \to 0$; thus convergence is not clear. However, in the second line, where the repetition operators appear, we only have one weakly converging sequence in each term. Moreover, the above expression for $\mathcal{D}_\tau^{\text{slope}}$ well motivates the crucial role of the singleton condition (1.11). In fact, the two sequences $(\mathbb{T}_\tau^{(j)}\overline{\xi}_\tau)_\tau$ can be shown to converge to limits ξ_j that fulfill $\xi_j(t) \in \partial^{\mathbf{X}_j}\mathcal{E}(t, U(t))$ for almost all $t \in (0,T)$, provided a suitable closedness property for the subdifferentials $\partial^{\mathbf{X}_j}\mathcal{E}$ is assumed. Then, condition (1.11) guarantees that $\partial^{\mathbf{X}_j}\mathcal{E}(t, U(t))$ is a singleton of $\mathbf{X}_j^*$, so that $\partial^{\mathbf{X}_1}\mathcal{E}(t, U(t)) = \{\xi_1(t)\} = \{\xi_2(t)\} = \partial^{\mathbf{X}_2}\mathcal{E}(t, U(t))$ for a.a. $t \in (0,T)$. Then, $\xi := \xi_1 = \xi_2 : (0,T) \to \mathbf{X}_1^*$ gives rise to the force term in (1.12) since

$$\liminf_{\tau\to 0} \mathcal{D}_\tau^{\text{slope}}([0,T]) \geq \int_0^T \big\{\mathcal{R}_1^*(-\xi(r))+\mathcal{R}_2^*(-\xi(r))\big\}\mathrm{d}r = \int_0^T \mathcal{R}_{\text{eff}}^*(-\xi(r))\mathrm{d}r.$$

Splitting schemes for block structures

In the second part of the paper we tackle the application of the splitting method to generalized gradient systems with a *block structure*. In such systems,

$$\text{the state variable } u \text{ is a vector } u = (y,z)^\top \in \mathbf{U} := \mathbf{Y}\times\mathbf{Z},$$

with $\mathbf{Y}$ and $\mathbf{Z}$ (separable) reflexive Banach spaces. The evolution of the system is governed by a driving energy functional $\mathcal{E} : [0,T]\times\mathbf{U} \to (-\infty,\infty]$ and by two dissipation potentials $\mathcal{R}_\text{y} : \mathbf{Y} \to [0,\infty)$ and $\mathcal{R}_\text{z} : \mathbf{Z} \to [0,\infty)$, each acting on the components of the rate vector $u' = (y', z')^\top$, namely

$$\mathcal{R}(u') = \mathcal{R}(y', z') = \big(\mathcal{R}_\text{y}\otimes\mathcal{R}_\text{z}\big)(y', z') := \mathcal{R}_\text{y}(y') + \mathcal{R}_\text{z}(z').$$

Models in solid mechanics described by the evolution of the displacement, or the deformation, of the body, coupled with that of an internal variable describing inelastic processes such as, e.g., plasticity, heat transfer, delamination, fracture, damage, typically fit into this framework. Applying the splitting method to this context boils down to letting first the variable y evolve on a semi-interval while keeping z fixed, and then letting z evolve with y fixed on the next semi-interval. On the discrete level, this can be compared with staggered minimization schemes, which have been recently used for gradient flows and rate-independent systems, cf. e.g. [1, 16, 30, 31] among others.

The analysis of this kind of systems can be framed in the context of our splitting approach by introducing the dissipation potentials $\mathcal{R}_j : \mathbf{U} \to [0,\infty]$

$$\mathcal{R}_1(u') = \mathcal{R}_1(v,w) = \mathcal{R}_\text{y}(v) + \mathcal{I}_{\{0\}}(w), \qquad \mathcal{R}_2(u') = \mathcal{R}_2(v,w) = \mathcal{I}_{\{0\}}(v) + \mathcal{R}_\text{z}(w), \tag{1.13}$$

where $\mathcal{I}_{\{0\}}$ is the indicator function of the singleton $\{0\}$, with $\mathcal{I}_{\{0\}}(0) = 0$ and ∞ otherwise. We then concatenate the (time-discrete or time-continuous) solutions to the subdifferential inclusions

$$\partial\mathcal{R}_j(u'(t)) + \partial^{\mathbf{U}}\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } \mathbf{U}^* \qquad \text{for a.a. } t \in (0,T)$$

governing the single-dissipation systems $(\mathbf{U}, \mathcal{E}, \mathcal{R}_j)$. Because of (1.13), in each step we either freeze the variable z (for $j=1$), or the variable y (for $j=2$). In particular, on the time-discrete level the corresponding Alternating Minimizing Movement scheme consists of two consecutive minimum problems in which either the variable z stays fixed and minimization only involves the variable y, or y is fixed and we minimize only with respect to z. Nonetheless, let us stress that the energy functional $\mathcal{E}(t,\cdot)$ is defined on the product space $\mathbf{U}$, and likewise the Fréchet subdifferential $\partial^{\mathbf{U}}\mathcal{E}$ involves the $\langle\cdot,\cdot\rangle_{\mathbf{U}}$ duality.

The analysis in Sects. 2 to 6 relies on the ordering assumption that $\mathbf{X}_2 \subset \mathbf{X}_1$ continuously and that $\mathcal{R}_j$ and $\mathcal{R}_j^*$ are superlinear on $\mathbf{X}_j$. This ordering property no longer holds in the case with block structure, hence Sect. 7 shows how the analysis can be adapted; in particular the singleton condition (1.11) can be replaced by the weaker *cross-product condition*

$$\partial^{\mathbf{U}}\mathcal{E}(t,y,z) = \partial_{\mathrm{y}}\mathcal{E}(t,y,z) \times \partial_{\mathrm{z}}\mathcal{E}(t,y,z) \qquad \text{for all } (t,y,z) \in [0,T] \times \mathbf{U}.$$

Under this and other conditions on the generalized gradient system $(\mathbf{U}, \mathcal{E}, \mathcal{R}_{\mathrm{y}} \otimes \mathcal{R}_{\mathrm{z}})$ we prove our two convergence results, Theorem 7.5 for the concatenation of time-continuous solutions, and Theorem 7.7 for the Alternating Minimizing Movement scheme.

Plan of the paper. Sect. 2 lays down the foundations for our analysis: after recalling some known facts about the energy-dissipation balance and its role in characterizing solutions for an abstract gradient system $(\mathscr{X}, \mathscr{E}, \mathscr{R})$ in Sects. 2.1, 2.2 we specify our working assumptions on the quintuple $(\mathbf{X}_1, \mathbf{X}_2, \mathcal{E}, \mathcal{R}_1, \mathcal{R}_2)$ and expound some of their consequences.

We devise the time-splitting scheme on the time-continuous level in Sect. 3, which also contains the statement of our first main convergence result in Theorem 3.5, proved throughout Sect. 5. Section 4 is instead devoted to examples illustrating the theory: a counterexample to convergence of the time-splitting scheme without the singleton condition, and a doubly-nonlinear PDE example. In Sect. 6 we show that the analysis carried out in Sects. 3 and 5 can be easily adapted to the Alternating Minimizing Movement scheme introduced in (1.8) and state our second convergence result in Theorem 6.1, which is the discrete counterpart to Theorem 3.5. In Sect. 7 we turn to systems with block structure and provide our last two convergence results in Theorems 7.5 and 7.7. Finally, Appendices A and B contain some auxiliary results.

2 Setup and assumptions

In the ensuing Sect. 2.1 we are going to collect some key facts about general gradient systems $(\mathscr{X}, \mathscr{E}, \mathscr{R})$ that will be used throughout the paper. Then, in Sect. 2.2 we will move to the context of two dissipation mechanisms and settle our working assumptions on the gradient systems $(\mathbf{X}_j, \mathcal{E}, \mathcal{R}_j)$, $j \in \{1,2\}$.

2.1 Preliminaries on gradient systems

By *generalized gradient system* we mean a triple $(\mathscr{X}, \mathscr{E}, \mathscr{R})$ satisfying the conditions **(1), (2), (3)** explained in the Introduction. More precisely, throughout this section, we will assume that $\mathscr{E}$ and $\mathscr{R}$ satisfy the following basic conditions:

<E> (time differentiability and power control): *The energy* $\mathscr{E} : [0, T]\times\mathscr{X} \to (-\infty, \infty]$ *is a lower semicontinuous functional, bounded from below by a positive constant, and has a proper domain* $[0, T]\times\mathscr{D}$ *with* $\mathscr{D} \subset \mathscr{X}$. *Moreover, for every* $u \in \mathscr{D}$ *the function* $t \mapsto \mathscr{E}(t, u)$ *is differentiable and*

$$\exists\, C_\# > 0 \;\; \forall\, (t, u) \in [0, T]\times\mathscr{D} : \quad |\partial_t\mathscr{E}(t, u)| \le C_\#\mathscr{E}(t, u) \,. \tag{2.1}$$

<R> (superlinear dissipation potential): *The potential* $\mathscr{R} : \mathscr{X} \to [0, \infty)$ *is lower semicontinuous and convex,* $\mathscr{R}(0) = 0$ *and, together with its convex conjugate* $\mathscr{R}^* : (\mathscr{X}^*, \|\cdot\|_*) \to [0, \infty)$, *the functional* $\mathscr{R}$ *is superlinear:*

$$\lim_{\|v\|\uparrow\infty} \frac{\mathscr{R}(v)}{\|v\|} = \lim_{\|\xi\|_*\uparrow\infty} \frac{\mathscr{R}^*(\xi)}{\|\xi\|_*} = \infty. \tag{2.2}$$

In particular, we emphasize that we will confine our study to the case of *state-independent* dissipation potentials, although we believe that all of our results could be extended to potentials $\mathscr{R} = \mathscr{R}(u, v)$ with a suitably tamed dependence on the state variable u. Instead, our analysis cannot encompass potentials taking the value ∞ (for instance, including indicator terms that force unidirectional evolution), since the treatment of the corresponding gradient systems necessitates additional estimates that are outside the scope of the present paper.

The central result of this section is Proposition 2.1, which is indeed at the core of our approach to gradient systems. It provides a characterization of the gradient-system evolution given by the subdifferential inclusion (1.1), in terms of the so-called Energy-Dissipation Principle, combined with the chain rule for the energy functional $\mathscr{E}$. The following characterization of the (1.1) via an upper energy estimate has been circulating for some time: it is in fact underlying the analysis of gradient flows and *generalized* gradient flows in Banach and metric spaces, cf. e.g. [2, 3, 18, 26, 28]. Nonetheless, in order to make the paper self-contained we detail the proof of the following result here as well.

Proposition 2.1 (Energy-dissipation principle) *Let* $u \in AC([0, T]; \mathscr{X})$ *and* $\xi \in \mathrm{L}^1([0, T]; \mathscr{X}^*)$ *with* $\xi(t) \in \partial\mathscr{E}(t, u(t))$ *for almost all* $t \in (0, T)$. *Suppose that the map* $t \mapsto \mathscr{E}(t, u(t))$ *is absolutely continuous on* $[0, T]$ *and that for* (u, ξ) *the chain rule*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t, u(t)) - \partial_t\mathscr{E}(t, u(t)) = \langle \xi(t), u'(t)\rangle_{\mathscr{X}} \quad \textit{for a.a. } t \in (0, T) \tag{2.3}$$

holds. Then, the following conditions are equivalent:

(C1) *The pair* (u, ξ) *complies with the upper energy-dissipation estimate, i.e.*

$$\mathscr{E}(T, u(T)) + \int_0^T \big\{\mathscr{R}(u'(r)) + \mathscr{R}^*(-\xi(r))\big\}\mathrm{d}r \le \mathscr{E}(0, u(0)) + \int_0^T \partial_t\mathscr{E}(r, u(r))\mathrm{d}r \,. \tag{2.4}$$

(C2) *The pair* (u, ξ) *solves*

$$-\,\xi(t) \in \partial\mathscr{R}(u'(t)) \qquad \textit{for a.a. } t \in (0, T), \tag{2.5}$$

and fulfills the energy-dissipation balance

$$\mathscr{E}(t, u(t)) + \int_s^t \big\{\mathscr{R}(u'(r)) + \mathscr{R}^*(-\xi(r))\big\}\mathrm{d}r = \mathscr{E}(s, u(s)) + \int_s^t \partial_t\mathscr{E}(r, u(r))\mathrm{d}r \tag{2.6}$$

for all $s, t \in [0, T]$ *with* $s \le t$.

Proof Obviously, (C2) implies (C1).

To show that condition (C1) implies (C2), we apply the chain rule (2.3) and deduce from the energy-dissipation upper estimate (2.4) that

$$\int_0^T \left\{\mathscr{R}(u'(r))+\mathscr{R}^*(-\xi(r))\right\} \mathrm{d}r \le \mathscr{E}(0,u(0)) - \mathscr{E}(T,u(T)) + \int_0^T \partial_t\mathscr{E}(r,u(r))\mathrm{d}r$$
$$= \int_0^T \langle -\xi(r), u'(r)\rangle_{\mathscr{X}} \, \mathrm{d}r .$$

Now, since $\mathscr{R}(v) + \mathscr{R}^*(\zeta) \ge \langle \zeta, v\rangle_{\mathscr{X}}$ for all $(v,\zeta) \in \mathscr{X}\times\mathscr{X}^*$, from the above inequality we immediately infer that

$$\mathscr{R}(u'(t)) + \mathscr{R}^*(-\xi(t)) = \langle -\xi(t), u'(t)\rangle_{\mathscr{X}} \qquad \text{for a.a. } t \in (0,T) \tag{2.7}$$

which, by a well-known convex analysis result, is equivalent to the inclusion (2.5). The energy-dissipation balance (2.6) follows by integrating (2.7) on an arbitrary time interval $[s,t] \subset [0,T]$ and again applying the chain rule (2.3). This concludes the proof. □

The chain rule and the *Quantitative Young Estimate*

The cornerstone in the proof of Proposition 2.1 is indeed the chain rule formula (2.3). Let us now gain further insight on its validity: In view of estimate (2.1), for any curve $u \in \mathrm{AC}([0,T];\mathscr{X})$ along which $t \mapsto \mathscr{E}(t,u(t))$ is absolutely continuous we even have that $t \mapsto \partial_t\mathscr{E}(t,u(t))$ is in $\mathrm{L}^\infty([0,T])$. Therefore, underlying (2.3) is the fact that the function $t \mapsto \langle \xi(t), u'(t)\rangle_{\mathscr{X}} \in \mathrm{L}^1([0,T])$.

The most general and flexible version of the chain rule for a generalized gradient system $(\mathscr{X},\mathscr{E},\mathscr{R})$ in the Banach space $\mathscr{X}$ is the following; we mention in advance that, for this paper it will be sufficient to use this version only for the effective gradient system $(\mathbf{X}_1, \mathcal{E}, \mathcal{R}_1^*+\mathcal{R}_2^*)$:

<CR> **(chain rule):** *the generalized gradient system* $(\mathscr{X},\mathscr{E},\mathscr{R})$ *with specified subdifferential* $\partial\mathscr{E}$ *satisfies the following: for all* $(u,\xi) \in \mathrm{AC}([0,T];\mathscr{X})\times\mathrm{L}^1([0,T];\mathscr{X}^*)$ *such that* $\xi(t) \in \partial\mathscr{E}(t,u(t))$ *for almost all* $t \in (0,T)$ *and*

$$\sup_{t\in[0,T]} |\mathscr{E}(t,u(t))| < \infty \text{ and } \int_0^T \big(\mathscr{R}(u'(t))+\mathscr{R}^*(\xi(t))\big)\mathrm{d}t < \infty , \tag{2.8}$$

the function $t \mapsto \mathscr{E}(t,u(t))$ *is absolutely continuous and the chain rule* (2.3) *holds.*

In practice, it is often convenient to establish **<CR>** without reference to the dissipation potential $\mathscr{R}$. This can be done by strengthening the assumption (2.8).

<CR'> **(chain rule):** *the pair* $(\mathscr{E},\partial\mathscr{E})$ *satisfies the following property: for all* $(u,\xi) \in \mathrm{AC}([0,T];\mathscr{X})\times\mathrm{L}^1([0,T];\mathscr{X}^*)$ *such that* $\xi(t) \in \partial\mathscr{E}(t,u(t))$ *for almost all* $t \in (0,T)$ *and*

$$\sup_{t\in[0,T]} |\mathscr{E}(t,u(t))| < \infty \text{ and } \int_0^T \|\xi(t)\|_* \, \|u'(t)\|\mathrm{d}t < \infty , \tag{2.9}$$

the function $t \mapsto \mathscr{E}(t,u(t))$ *is absolutely continuous and the chain rule* (2.3) *holds.*

Clearly, the stronger version **<CR'>** implies the general version **<CR>** if we provide a condition on $\mathscr{R}$ such that (2.8) implies (2.9). For this, we will introduce below the *Quantitative Young Estimate* QYE, which strengthens the classical *Young estimate*

$$\mathscr{R}(v) + \mathscr{R}^*(\xi) \ge \langle \xi, v\rangle_{\mathscr{X}} \qquad \text{for all } (v,\xi) \in \mathscr{X}\times\mathscr{X}^* .$$

<QYE> (Quantitative Young Estimate):

$$\exists\, c, C > 0\, \forall\, (v, \xi) \in \mathscr{X} \times \mathscr{X}^* : \quad \mathscr{R}(v) + \mathscr{R}^*(\xi) \geq c\|v\|\,\|\xi\|_* - C. \tag{2.10}$$

Indeed, if $(\mathscr{X}, \mathscr{E}, \mathscr{R})$ satisfies conditions **<E>**, **<R>**, **<CR'>**, and **<QYE>**, then, thanks to (2.10), for any pair $(u, \xi) \in \mathrm{AC}([0,T]; \mathscr{X}) \times \mathrm{L}^1([0,T]; \mathscr{X}^*)$ satisfying (2.8) we have that the estimates in (2.9) hold. Hence, the chain rule formula (2.3) is valid.

Let us now gain further insight into condition **<QYE>**. It is known that estimate (2.10) is not true in general, see [21, Ex. 3.4] for a counterexample. In turn, the following result provides a useful sufficient condition for (2.10).

Lemma 2.2 *Assume that there exists a continuous, convex and superlinear function* $\psi : [0, \infty[\to [0, \infty[$ *and constants* $c_1 \geq 1$ *and* $c_2, c_3 > 0$ *such that*

$$\forall\, v \in \mathscr{X} : \quad \psi\left(\frac{1}{c_3}\|v\|\right) - c_2 \leq \mathscr{R}(v) \leq c_1\, \psi(c_3\|v\|) + c_2. \tag{2.11}$$

Then, the quantitative Young estimate (2.10) *holds with* $c = 1/(c_1 c_3^2)$ *and* $C = 2c_2$.

In particular, the quantitative Young estimate (2.10) holds if $\mathscr{R}$ is given by a functional of the norm, i.e. $\mathscr{R}(v) = \psi(\|\mathbb{B}v\|)$ with a bounded, invertible operator $\mathbb{B} : \mathscr{X} \to \mathscr{X}$.

Proof Thanks to (2.11), we have $\mathscr{R}^*(\xi) \geq c_1 \psi^*\big(\|\xi\|_*/(c_1 c_3)\big) - c_2$. Combining this with the Young's inequality $\psi(r) + \psi^*(s) \geq rs$ and using that $c_1 \geq 1$ we infer

$$\begin{aligned}\mathscr{R}(v) + \mathscr{R}^*(\xi) &\geq \psi(\|v\|/c_3) - c_2 + c_1 \psi^*(\|\xi\|_*/c_1 c_3) - c_2 \\ &\geq \psi(\|v\|/c_3) + \psi^*(\|\xi\|_*/c_1 c_3) - 2c_2 \geq \frac{1}{c_1 c_3^2}\,\|v\|\,\|\xi\|_* - 2c_2.\end{aligned}$$

Thus, the result is established. □

2.2 Assumptions on the generalized gradient systems

We now depart from the general setup $(\mathscr{X}, \mathscr{E}, \mathscr{R})$ of Sect. 2.1 and tackle two *specific* (as highlighted from the change of fonts) gradient systems $(\mathbf{X}_1, \mathcal{E}, \mathcal{R}_1)$ and $(\mathbf{X}_2, \mathcal{E}, \mathcal{R}_2)$. As we will see in settling our requirements on $(\mathbf{X}_j, \mathcal{E}, \mathcal{R}_j)$, $j \in \{1, 2\}$, the conditions expounded in Sect. 2.1 will have to be adjusted to the interplay between the topologies of the spaces $(\mathbf{X}_j, \|\cdot\|_j)$.

Ordering of Banach spaces

We consider two (separable) and reflexive Banach spaces $(\mathbf{X}_1, \|\cdot\|_1)$ and $(\mathbf{X}_2, \|\cdot\|_2)$, such that

$$\mathbf{X}_2 \subset \mathbf{X}_1 \;\text{ and }\; \mathbf{X}_1^* \subset \mathbf{X}_2^* \text{ densely and continuously.} \tag{2.12a}$$

More precisely, we assume that

$$\exists\, C_{\mathrm{N}} \geq 1\, \forall\, v \in \mathbf{X}_2 : \quad \|v\|_1 \leq C_{\mathrm{N}}\|v\|_2. \tag{2.12b}$$

Driving energy functional

Clearly, the time-dependent energy functional $\mathcal{E}$ needs to have a domain contained in the smaller space $[0,T]\times\mathbf{X}_2$. It is on this domain that we require the first set of basic conditions, namely conditions **<E>** previously introduced: boundedness from below (by a constant that, up to a shift, can be assumed positive), time differentiability, and control of the power functional by means of the energy functional itself.

Hypothesis 2.3 *(Time differentiability) The energy* $\mathcal{E} : [0,T]\times\boldsymbol{X}_1 \to (-\infty,\infty]$ *has the proper domain*

$$dom(\mathcal{E}) = [0,T]\times D_0 \text{ with } D_0 \subset \boldsymbol{X}_2 \text{ and } \exists\, C_0 > 0\ \forall\,(t,u) \in [0,T]\times D_0 : \ \mathcal{E}(t,u) \geq C_0. \tag{2.13}$$

Moreover, on $[0,T]\times D_0$ *the functional* $\mathcal{E}$ *complies with* ***<E>***.

It is convenient to introduce the functional

$$\mathfrak{E} : \mathrm{D}_0 \to [0,\infty), \qquad \mathfrak{E}(u) := \sup_{t\in[0,T]} \mathcal{E}(t,u), \tag{2.14}$$

and observe that, by (2.1) and Grönwall's lemma,

$$\mathfrak{E}(u) \leq \mathrm{e}^{C_\# T}\mathcal{E}(t,u) \qquad \text{for all } (t,u) \in [0,T]\times\mathrm{D}_0. \tag{2.15}$$

We will work with the sublevel sets

$$S_E := \{u \in \mathbf{X}_2 : \mathfrak{E}(u) \leq E\}, \qquad E > 0. \tag{2.16}$$

We can now formulate our second condition.

Hypothesis 2.4 *(Lower semicontinuity & continuity of the power) We require that for every* $j \in \{1,2\}$ *there holds*

$$\begin{aligned} &\forall\, E > 0 : \Big((t_n,u_n) \rightharpoonup (t,u) \text{ in } [0,T]\times\boldsymbol{X}_1 \text{ and } u_n \in S_E \text{ for all } n \in \mathbb{N}\Big) \\ &\quad\Longrightarrow \begin{cases} \liminf_{n\to\infty} \mathcal{E}(t_n,u_n) \geq \mathcal{E}(t,u), \\ \lim_{n\to\infty} \partial_t\mathcal{E}(t_n,u_n) = \partial_t\mathcal{E}(t,u). \end{cases} \end{aligned} \tag{2.17}$$

Thanks to (2.17), the functional $\mathfrak{E}$ is weakly lower semicontinuous in $\mathbf{X}_1$ and thus the sublevel sets S_E are (sequentially) weakly closed in $\mathbf{X}_1$. We highlight that, in (2.17) lower semicontinuity of $\mathcal{E}(t,\cdot)$ is, a priori, required with respect to the coarser topology given by $\mathbf{X}_1$. Nonetheless, in concrete examples $\mathcal{E}(t,\cdot)$ may turn out to be coercive with respect to the norm $\|\cdot\|_2$, recall $\mathrm{D}_0 \subset \mathbf{X}_2$. Hence, the information that there exists $E > 0$ with $u_n \in S_E$ for all $n \in \mathbb{N}$ may yield additional compactness information on the sequence $(u_n)_n$, and in fact weaken the above lower semicontinuity/continuity requirements.

Hypothesis 2.5 below involves the Fréchet subdifferentials of $\mathcal{E}(t,\cdot)$ with respect to the duality pairings $\langle\cdot,\cdot\rangle_{\mathbf{X}_j}$, namely the operators $\partial^{\mathbf{X}_j}\mathcal{E} : [0,T]\times\mathrm{D}_j \rightrightarrows \mathbf{X}_j^*$ defined by

$$\xi \in \partial^{\mathbf{X}_j}\mathcal{E}(t,u) \iff \mathcal{E}(t,w) - \mathcal{E}(t,u) \geq \langle\xi, w-u\rangle_{\mathbf{X}_j} + \mathrm{o}(\|w-u\|_j) \quad \text{as } \|w-u\|_j \to 0,$$

where for all $t \in [0,T]$ we define $\mathrm{D}_j = \{u \in \mathrm{D}_0 : \partial^{\mathbf{X}_j}\mathcal{E}(t,u) \neq \emptyset\}$. We have $\mathrm{D}_1 \subset \mathrm{D}_2$ and

$$\partial^{\mathbf{X}_1}\mathcal{E}(t,u) \subset \partial^{\mathbf{X}_2}\mathcal{E}(t,u) \cap \mathbf{X}_1^* \qquad \text{for all } (t,u) \in [0,T]\times\mathrm{D}_1. \tag{2.18}$$

Hypothesis 2.5 *(Closedness of $\partial^{X_j}\mathcal{E}$ on energy sublevels) We require that for every $j \in \{1,2\}$ we have*

$$\forall\, E > 0: \left\{ \begin{array}{l} (t_n, u_n, \xi_n) \rightharpoonup (t, u, \xi) \text{ in } [0,T] \times X_1 \times X_j^*, \\ u_n \in S_E,\ \xi_n \in \partial^{X_j}\mathcal{E}(t_n, u_n) \text{ for all } n \in \mathbb{N} \end{array} \right\} \implies \xi \in \partial^{X_j}\mathcal{E}(t,u). \quad (2.19)$$

We emphasize that in (2.19) closedness is imposed along sequences weakly converging in $\mathbf{X}_1$; thus, for $j = 2$ condition (2.19) may also be understood as a compatibility requirement between the duality pairing of $\mathbf{X}_1$ and that of $\mathbf{X}_2$. Again, we remark that the requirement $(u_n)_n \subset S_E$ and the additional compactness information granted by it may turn (2.19) into a (more standard) closedness condition of the graph of $\partial^{\mathbf{X}_j}\mathcal{E}$ in $\mathbf{X}_j \times \mathbf{X}_j^*$.

Finally, along the footsteps of [21, 26] we will assume the validity of the following chain-rule condition for the subdifferentials $\partial^{\mathbf{X}_j}\mathcal{E}$. This condition is only used for constructing the approximate solutions $U_{\boldsymbol{\tau}}$ (cf. (3.12)) via the solution $u_j : [0,T] \to \mathbf{X}_j$ from Theorem 2.10; it will not be needed for the major convergence result in Theorem 3.5.

Hypothesis 2.6 (Chain rules) *For $j = 1, 2$, the generalized gradient systems $(X_j, \mathcal{E}, \mathcal{R}_j)$ with subdifferentials $\partial^{X_j}\mathcal{E}$ satisfy the chain rule* ***<CR>****.*

Dissipation potentials

In what follows, we will work with two dissipation potentials satisfying the following conditions.

Hypothesis 2.7 *For $j \in \{1,2\}$ the functionals $\mathcal{R}_j : X_j \to [0,\infty)$ and their conjugates $\mathcal{R}_j^* : X_j^* \to [0,\infty)$ comply with condition* ***<R>****.*

For later use, we reformulate the superlinear growth conditions (2.2) that we require for $\mathcal{R}_j$ and $\mathcal{R}_j^*$ in terms of a unique convex, superlinear and monotone function Ψ providing a lower bound for the primal and dual dissipation potentials.

Lemma 2.8 *The dissipation potentials $\mathcal{R}_j : X_j \to [0,\infty)$ fulfill (2.2) if and only if there exists a convex and increasing function $\Psi : [0,\infty) \to [0,\infty)$, with superlinear growth, such that, for $j \in \{1,2\}$,*

$$\mathcal{R}_j(v) \geq \Psi(\|v\|_j) \ \text{ and } \ \mathcal{R}_i^*(\xi) \geq \Psi(\|\xi\|_{j,*}) \ \text{ for all } v \in X_j \text{ and } \xi \in X_j^*. \quad (2.20)$$

Proof Clearly, (2.20) implies the superlinear growth (2.2) for the potentials $\mathcal{R}_j$ and $\mathcal{R}_j^*$. To check the converse implication, observe that, since $\mathcal{R}_j$ and $\mathcal{R}_j^*$ are superlinear, for each $j \in \{1,2\}$ we have

$$\forall\, K \geq 0\ \exists\, S_K^j,\ S_K^{j,*} \geq 0\ \forall\, v \in \mathbf{X}_j,\ \xi \in \mathbf{X}_j^*: \quad \begin{cases} \mathcal{R}_j(v) \geq K\|v\|_j - S_K^j, \\ \mathcal{R}_j^*(\xi) \geq K\|\xi\|_{j,*} - S_K^{j,*}, \end{cases} \quad (2.21)$$

with $S_0^j = S_0^{j,*} = 0$. For fixed $K \geq 0$, set $\mathsf{S}_K := \max_{j=1,2}\{S_K^j, S_K^{j,*}\}$ and define

$$\Psi(r) := \sup\{Kr - \mathsf{S}_K : K \geq 0\} \quad \text{for } r \in [0,\infty).$$

By construction, $\mathcal{R}_j$ and $\mathcal{R}_j^*$ satisfy (2.20). It is immediate to check that Ψ is monotone increasing, has superlinear growth at infinity, and since $\Psi(0) = 0$, satisfies $\Psi(r) \geq 0$ for all $r \geq 0$. Moreover, Ψ is convex as it is given by the supremum of linear functions. □

We now introduce some specific conditions to deal with the split-step system driven by the inf-convolution of $\mathcal{R}_1$ and $\mathcal{R}_2$. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined on two different Banach spaces, the effective dissipation potential $\mathcal{R}_{\text{eff}}$ needs to be carefully specified. It is more straightforward to first define the dual potential $\mathcal{R}^*_{\text{eff}}$ on the smaller dual space $\mathbf{X}_1^*$. Therefore, let us introduce the functional

$$\mathfrak{R}_* : \mathbf{X}_1^* \to [0,\infty), \qquad \mathfrak{R}_*(\xi) := \mathcal{R}_1^*(\xi) + \mathcal{R}_2^*(\xi) \quad \text{for } \xi \in \mathbf{X}_1^*.$$

We now identify $\mathfrak{R}_*$ as the conjugate of the potential given by the infimal convolution of $\mathcal{R}_1$ and of the functional $\overline{\mathcal{R}}_2$ that extends $\mathcal{R}_2$ to the whole of $\mathbf{X}_1$ by ∞ on $\mathbf{X}_1 \backslash \mathbf{X}_2$. We mention in advance that in (2.22) below we will directly define $\mathcal{R}_{\text{eff}}$ via a minimum: in the proof of Lemma 2.9 we will show by the *direct method* that the infimum is attained.

Lemma 2.9 *(Properties of the inf-convolution) Let* $\overline{\mathcal{R}}_2 : \boldsymbol{X}_1 \to [0,\infty)$ *be defined by* $\overline{\mathcal{R}}_2(v) := \mathcal{R}_2(v)$ *if* $v \in \boldsymbol{X}_2$*, and by* $\overline{\mathcal{R}}_2(v) := \infty$ *else. Define*

$$\mathcal{R}_{\text{eff}} : \boldsymbol{X}_1 \to [0,\infty) \qquad \mathcal{R}_{\text{eff}}(v) := \min_{v_1,\, v_2 \in \boldsymbol{X}_1,\ v = v_1 + v_2} \left(\mathcal{R}_1(v_1) + \overline{\mathcal{R}}_2(v_2)\right). \tag{2.22}$$

Then, the following statements hold:

(1) $\mathcal{R}_{\text{eff}}$ *is lower semicontinuous and convex, with* $\mathcal{R}^*_{\text{eff}} = \mathfrak{R}_*$.
(2) *With* Ψ *from Lemma 2.8 there holds*

$$2\Psi\left(\frac{1}{2C_{\text{N}}}\|v\|_1\right) \le \mathcal{R}_{\text{eff}}(v) \le \mathcal{R}_1(v) \le \Psi^*\big(\|v\|_1\big) \quad \textit{for all } v \in \boldsymbol{X}_1. \tag{2.23}$$

Estimate (2.23) highlights that, ultimately, the relevant Banach space for $\mathcal{R}_{\text{eff}}$ is $\mathbf{X}_1$. That is why, from now on we will use the notation $\mathbf{X}_{\text{eff}} := \mathbf{X}_1$.

Proof Preliminarily, observe that also the extended potential $\overline{\mathcal{R}}_2$ is lower semicontinuous on $\mathbf{X}_1$. To see this, take a sequence $v_n \to v$ in $\mathbf{X}_1$ such that $\liminf_n \overline{\mathcal{R}}_2(v_n) < \infty$. Then $v_n \in \mathbf{X}_2$ and $\overline{\mathcal{R}}_2(v_n) = \mathcal{R}_2(v_n)$. By coercivity of $\mathcal{R}_2$ and reflexivity of $\mathbf{X}_2$ we have $v_n \rightharpoonup v$ in $\mathbf{X}_2$ for a non-relabeled subsequence. Hence, $v \in \mathbf{X}_2$. Since $\mathcal{R}_2$ is convex and strongly lower semicontinuous, it is also weakly lower semicontinuous, which yields that $\liminf_n \overline{\mathcal{R}}_2(v_n) = \liminf_n \mathcal{R}_2(v_n) \ge \mathcal{R}_2(v) = \overline{\mathcal{R}}_2(v)$.

Proof of Claim (1): Applying [14, Thm. 1, p. 178] we immediately find that

$$\mathcal{R}^*_{\text{eff}} = \mathcal{R}_1^* + \overline{\mathcal{R}_2}^* = \mathcal{R}_1 + \mathcal{R}_2^*|_{\mathbf{X}_1^*} = \mathfrak{R}_*$$

(where the duality pairing with respect to all the conjugates above is that between $\mathbf{X}_1^*$ and $\mathbf{X}_1$). In turn, thanks to Hypothesis 2.7, $\text{dom}(\mathcal{R}_j^*) = \mathbf{X}_j^*$ for $j \in \{1,2\}$, hence both conjugate potentials $\mathcal{R}_j^*$ are continuous on their common domain $\mathbf{X}_1^*$. Therefore, [14, Thm. 1] again applies, yielding that $(\mathcal{R}_1^* + \mathcal{R}_2^*)^* = (\mathcal{R}_1^{**} \overset{\inf}{\circ} \mathcal{R}_2^{**})$ (where the have simply written $\mathcal{R}_2^*$ in place of $\mathcal{R}_2^*|_{\mathbf{X}_1^*}$). Thus,

$$\mathcal{R}^{**}_{\text{eff}} = \mathfrak{R}_*^* = (\mathcal{R}_1^* + \mathcal{R}_2^*)^* = (\mathcal{R}_1^{**} \overset{\inf}{\circ} \mathcal{R}_2^{**}) = (\mathcal{R}_1 \overset{\inf}{\circ} \overline{\mathcal{R}_2}) = \mathcal{R}_{\text{eff}},$$

which ensures that $\mathcal{R}_{\text{eff}}$ is convex and lower semicontinuous on $\mathbf{X}_1$.

Moreover, we see that in (2.22) the minimum is attained. Indeed, taking, for a given $v \in \mathbf{X}_1$ with $\mathcal{R}_{\text{eff}}(v) < \infty$, minimizing sequences $(v_n^1)_n$, $(v_n^2)_n \subset \mathbf{X}_1$ such that $v_n^1 + v_n^2 = v$, we again observe that, by coercivity, up to a (non-relabeled) subsequence we have $v_n^1 \rightharpoonup v^1$ in $\mathbf{X}_1$ and $v_n^2 \rightharpoonup v^2$ in $\mathbf{X}_2$ for some $v^i \in \mathbf{X}_i$. Hence, we obtain that

$$\mathcal{R}_{\text{eff}}(v) \ge \liminf_n \mathcal{R}_1(v_n^1) + \liminf_n \mathcal{R}_2(v_n^2) \ge \mathcal{R}_1(v^1) + \mathcal{R}_2(v^2),$$

and the claim follows.
Proof of Claim (2): For all $\xi \in \mathbf{X}_1^*$ we observe the following chain of inequalities:

$$\Psi\big(\|\xi\|_{1,*}\big) \overset{(1)}{\leq} \mathcal{R}_1^*(\xi) \leq \mathcal{R}_{\mathrm{eff}}^*(\xi) \overset{(2)}{\leq} \Psi^*\big(\|\xi\|_{1,*}\big) + \Psi^*\big(\|\xi\|_{2,*}\big) \tag{2.24}$$

$$\overset{(3)}{\leq} \Psi^*\big(\|\xi\|_{1,*}\big) + \Psi^*\big(C_{\mathrm{N}}\|\xi\|_{1,*}\big) \overset{(4)}{\leq} 2\Psi^*\big(C_{\mathrm{N}}\|\xi\|_{1,*}\big)\,, \tag{2.25}$$

where (1) and (2) follow from (2.20), while (3) and (4) are due to (2.12b), also taking into account the monotonicity of Ψ and $C_{\mathrm{N}} \geq 1$. Then, (2.23) follows by conjugation. □

Existence results

It follows from the Hypotheses listed above that the existence result [26, Thm. 2.2] applies to the (generalized) gradient systems $(\mathbf{X}_j, \mathcal{E}, \mathcal{R}_j)$, $j = 1, 2$. In particular, observe that, since weak convergence in $\mathbf{X}_2$ implies weak convergence in $\mathbf{X}_1$, Hypothesis 2.4 guarantees (sequential) lower semicontinuity of $\mathcal{E}$ and continuity of $\partial_t\mathcal{E}$ with respect to the weak topology of $\mathbf{X}_2$; likewise, Hypothesis 2.5 ensures (sequential) closedness of $\partial^{\mathbf{X}_2}\mathcal{E}$ with respect to the weak-weak topology of $\mathbf{X}_2\times\mathbf{X}_2^*$. Therefore, we conclude the existence of solutions to the Cauchy problems for

$$\partial\mathcal{R}_j(u'(t)) + \partial^{\mathbf{X}_j}\mathcal{E}(t, u(t)) \ni 0 \text{ in } \mathbf{X}_j^* \;\text{ a.e. in } (0, T), \quad u(0) = u_0 \in \mathrm{D}_0, \quad j \in \{1, 2\} \tag{2.26}$$

also satisfying the associated energy-dissipation balance. In fact, by Lemma 2.9 the effective dissipation potential $\mathcal{R}_{\mathrm{eff}}$ enjoys the same properties of $\mathcal{R}_1$ and $\mathcal{R}_2$, and we can likewise directly conclude an existence result for the Cauchy problem associated with

$$\partial\mathcal{R}_{\mathrm{eff}}(u'(t)) + \partial^{\mathbf{X}_1}\mathcal{E}(t, u(t)) \ni 0 \text{ in } \mathbf{X}_1^* \;\text{ a.e. in } (0, T), \quad u(0) = u_0 \in \mathrm{D}_0\,. \tag{2.27}$$

Theorem 2.10 below states the existence of solutions to the Cauchy problems for (2.26) and (2.27). In fact, the main objective of the analysis carried out in the ensuing Sects. 3 and 5 will be to demonstrate that a solution to (2.27) can be constructed via the time-splitting method.

Theorem 2.10 *Assume Hypotheses 2.3, 2.4, 2.5, 2.6, and 2.7. Then, for every $u_0 \in D_0$ and each $j \in \{1, 2, \mathrm{eff}\}$ the subdifferential inclusions (2.26) and (2.27) admit a solution $u_j \in AC([0, T]; \boldsymbol{X}_j)$ (where $\boldsymbol{X}_{\mathrm{eff}} = \boldsymbol{X}_1$) satisfying $u_j(0) = u_0$, and indeed there exist measurable selections $(0, T) \ni t \mapsto \xi_j(t) \in \partial^{\boldsymbol{X}_j}\mathcal{E}(t, u_j(t)) \in \boldsymbol{X}_j^*$ with $-\xi_j(t) \in \partial\mathcal{R}_j(u_j'(t))$ for a.a. $t \in (0, T)$, such that (u_j, ξ_j) fulfills the energy-dissipation balance*

$$\mathcal{E}(t, u_j(t)) + \int_s^t \Big(\mathcal{R}_j(u_j'(r)) + \mathcal{R}_j^*(-\xi_j(r))\Big)\,\mathrm{d}r = \mathcal{E}(s, u_j(s)) + \int_s^t \partial_t\mathcal{E}(r, u_j(r))\mathrm{d}r \tag{2.28}$$

for every $0 \leq s \leq t \leq T$.

It is important to mention that our conditions on the generalized gradient systems $(\mathbf{X}_j, \mathcal{E}, \mathcal{R}_j)$ are slightly weaker than those required in [26]. There, compactness of sublevels was required of the energy functional and, accordingly, the lower semicontinuity and closedness conditions were imposed along sequences converging with respect to the *strong* topology. Here, we work in the more general setup of Hypotheses 2.4 and 2.5: in fact, a close perusal of the proof of [26, Thm. 2.2] reveals that its arguments can be adapted to the present setup, see also [21].

3 The time-splitting approach

For the time-splitting method let us consider a (possibly non-uniform) partition of the interval $[0, T]$

$$\begin{aligned} &\mathscr{P}_{\boldsymbol{\tau}} := \{t_{\boldsymbol{\tau}}^0 = 0 < t_{\boldsymbol{\tau}}^1 < \cdots < t_{\boldsymbol{\tau}}^k < t_{\boldsymbol{\tau}}^{k+1} < \cdots < t_{\boldsymbol{\tau}}^{N_{\boldsymbol{\tau}}} = T\} \\ &\text{with } \tau_k = t_{\boldsymbol{\tau}}^k - t_{\boldsymbol{\tau}}^{k-1} \text{ and } |\tau| := \max\{\tau_k \,|\, k = 1, \ldots, N_{\boldsymbol{\tau}}\}. \end{aligned} \tag{3.1}$$

We also introduce the 'left' and 'right' semi-intervals generated by $\mathscr{P}_{\boldsymbol{\tau}}$, namely

$$\mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}} := \Big(t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^{k-1} + \tfrac{\tau_k}{2}\Big], \qquad \mathrm{I}_{\mathrm{right}}^{k,\boldsymbol{\tau}} := \Big(t_{\boldsymbol{\tau}}^{k-1} + \tfrac{\tau_k}{2}, t_{\boldsymbol{\tau}}^k\Big]. \tag{3.2}$$

In what follows, we will use the short-hand

$$t_{\boldsymbol{\tau}}^{k-1/2} := t_{\boldsymbol{\tau}}^{k-1} + \tfrac{\tau_k}{2} = \tfrac{1}{2}\big(t_{\boldsymbol{\tau}}^{k-1} + t_{\boldsymbol{\tau}}^k\big) \qquad \text{for } k = 1, \ldots, N_{\boldsymbol{\tau}}. \tag{3.3}$$

To simplify notation, we introduce the piecewise constant interpolants associated with the nodes of the partition

$$\begin{aligned} &\bar{\mathsf{t}}_{\boldsymbol{\tau}} : [0, T] \to [0, T], \qquad \bar{\mathsf{t}}_{\boldsymbol{\tau}}(0) := 0, \qquad \bar{\mathsf{t}}_{\boldsymbol{\tau}}(t) := t_{\boldsymbol{\tau}}^k \qquad \text{for } t \in (t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^k]; \\ &\underline{\mathsf{t}}_{\boldsymbol{\tau}} : [0, T] \to [0, T], \qquad \underline{\mathsf{t}}_{\boldsymbol{\tau}}(T) := T, \qquad \underline{\mathsf{t}}_{\boldsymbol{\tau}}(t) := t_{\boldsymbol{\tau}}^{k-1} \qquad \text{for } t \in [t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^k). \end{aligned} \tag{3.4}$$

We will also use the notation

$$\mathsf{k}_{\boldsymbol{\tau}}(t) := k \ \text{ for } t \in \mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}} \cup \mathrm{I}_{\mathrm{right}}^{k,\boldsymbol{\tau}} \quad \text{and} \quad \widetilde{\tau}(t) := \tau_{\mathsf{k}_{\boldsymbol{\tau}}(t)} \quad \text{for } t \in (0, T], \tag{3.5}$$

with $\widetilde{\tau}(0) := \tau_1$.

Repetition operators

A key tool for our analysis are the following operators, defined on the space $\mathrm{L}^1([0, T]; \mathscr{X})$ with a (general) separable and reflexive Banach space $\mathscr{X}$:

$$\mathbb{T}_{\boldsymbol{\tau}}^{(1)} \colon \mathrm{L}^1([0, T]; \mathscr{X}) \to \mathrm{L}^1([0, T]; \mathscr{X}); \ \ \big(\mathbb{T}_{\boldsymbol{\tau}}^{(1)} g\big)(t) := \begin{cases} g(t) & \text{for } t \in \mathrm{I}_{\mathrm{left}}^{\boldsymbol{\tau},\mathsf{k}_{\boldsymbol{\tau}}(t)}, \\ g\big(t - \frac{\widetilde{\tau}(t)}{2}\big) & \text{for } t \in \mathrm{I}_{\mathrm{right}}^{\boldsymbol{\tau},\mathsf{k}_{\boldsymbol{\tau}}(t)}, \end{cases} \tag{3.6a}$$

$$\mathbb{T}_{\boldsymbol{\tau}}^{(2)} \colon \mathrm{L}^1([0, T]; \mathscr{X}) \to \mathrm{L}^1([0, T]; \mathscr{X}); \ \ \big(\mathbb{T}_{\boldsymbol{\tau}}^{(2)} g\big)(t) := \begin{cases} g\big(t + \frac{\widetilde{\tau}(t)}{2}\big) & \text{for } t \in \mathrm{I}_{\mathrm{left}}^{\boldsymbol{\tau},\mathsf{k}_{\boldsymbol{\tau}}(t)}, \\ g(t) & \text{for } t \in \mathrm{I}_{\mathrm{right}}^{\boldsymbol{\tau},\mathsf{k}_{\boldsymbol{\tau}}(t)}. \end{cases} \tag{3.6b}$$

We shall refer to $\mathbb{T}_{\boldsymbol{\tau}}^{(1)}$ and $\mathbb{T}_{\boldsymbol{\tau}}^{(2)}$ as *repetition operators*, since $\mathbb{T}_{\boldsymbol{\tau}}^{(1)} g$ repeats, on the 'right' semi-intervals, the values of the function g from the 'left' semi-intervals, while $\mathbb{T}_{\boldsymbol{\tau}}^{(2)} g$ does the converse, see Fig. 1 for an illustration.

The following result collects some properties of the operators $\mathbb{T}_{\boldsymbol{\tau}}^{(j)}$: though straightforwardly checked, they will play a key role in our analysis. Ahead of the statement, we recall that the convergence in the space $\mathrm{C}^0([0, T]; \mathscr{X}_{\mathrm{w}})$ is, by definition, given by the convergence of $\mathrm{C}^0([0, T]; (\mathscr{X}, d_{\mathrm{weak}}))$, where the metric d_{weak} induces the weak topology on a closed bounded set of the reflexive space $\mathscr{X}$.

Lemma 3.1 *For $j \in \{1, 2\}$ we have the following properties:*

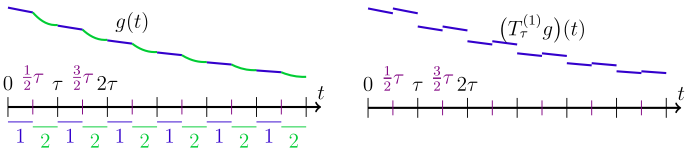

Fig. 1 Left: Schematic sketch of a function g generated by a split-step approach with equidistant partition. Right: The repetition operator $\mathbb{T}_\tau^{(1)}$ selects the left semi-intervals and repeats them in the right semi-intervals

1. *For all* $\boldsymbol{\tau} \in \Lambda$ *we have*

$$\|\mathbb{T}_{\boldsymbol{\tau}}^{(j)}\|_{Lin} \le 2\,,$$

where $\|\cdot\|_{Lin}$ *is the norm of* $Lin(\mathrm{L}^1([0,T];\mathscr{X});\mathrm{L}^1([0,T];\mathscr{X}))$.
2. $\mathbb{T}_{\boldsymbol{\tau}}^{(j)}g \to g$ *for every* $g \in \mathrm{L}^1([0,T];\mathscr{X})$ *as* $|\boldsymbol{\tau}| \to 0$.
3. *For any family* $(g_{\boldsymbol{\tau}})_{\boldsymbol{\tau}} \subset \mathrm{C}^0([0,T];\mathscr{X}_w)$*, we have*

$$g_{\boldsymbol{\tau}} \to g \text{ in } \mathrm{C}^0([0,T];\mathscr{X}_w) \text{ as } |\boldsymbol{\tau}| \to 0 \implies \mathbb{T}_{\boldsymbol{\tau}}^{(j)}g_{\boldsymbol{\tau}} \to g \text{ in } \mathrm{L}^\infty([0,T];\mathscr{X}_w)\,. \quad (3.7)$$

4. *For any family* $(g_{\boldsymbol{\tau}})_{\boldsymbol{\tau}} \subset \mathrm{L}^1([0,T];\mathscr{X})$*, we have, in the limit* $|\boldsymbol{\tau}| \to 0$*,*

$$g_{\boldsymbol{\tau}} \rightharpoonup g \text{ in } \mathrm{L}^1([0,T];\mathscr{X}) \implies \frac{1}{2}\left(\mathbb{T}_{\boldsymbol{\tau}}^{(1)} + \mathbb{T}_{\boldsymbol{\tau}}^{(2)}\right)(g_{\boldsymbol{\tau}}) \rightharpoonup g \text{ in } \mathrm{L}^1([0,T];\mathscr{X})\,. \quad (3.8)$$

The proof of Lemma 3.1 will be carried out in Appendix B.

The time-splitting algorithm

We recall the rescaled dissipation potentials $\widetilde{\mathcal{R}}_j : \mathbf{X}_j \to [0,\infty)$ given by

$$\widetilde{\mathcal{R}}_j(v) := 2\mathcal{R}_j\left(\tfrac{1}{2}v\right) \text{ with conjugates } \widetilde{\mathcal{R}}_j^*(\xi) = 2\mathcal{R}_j^*(\xi) \quad \text{for } j \in \{1,2\}. \quad (3.9)$$

For $k \in \{1,\ldots,N_{\boldsymbol{\tau}}\}$, we consider the Cauchy problems associated with $(\mathbf{X}_j,\mathcal{E},\widetilde{\mathcal{R}}_j)$:

1. the Cauchy problem for $(\mathbf{X}_1,\mathcal{E},\widetilde{\mathcal{R}}_1)$ with some initial datum $\underline{u} \in \mathrm{D}_0$:

$$\begin{cases} \partial\widetilde{\mathcal{R}}_1(u'(t)) + \partial\mathcal{E}(t,u(t)) \ni 0 & \text{in } \mathbf{X}_1^* \quad \text{for a.a. } t \in \mathrm{I}_{\text{left}}^{k,\boldsymbol{\tau}}, \\ u(t_{\boldsymbol{\tau}}^{k-1}) = \underline{u}. \end{cases} \quad (3.10)$$

2. the Cauchy problem for $(\mathbf{X}_2,\mathcal{E},\widetilde{\mathcal{R}}_2)$ with some initial datum $\underline{u} \in \mathrm{D}_0$:

$$\begin{cases} \partial\widetilde{\mathcal{R}}_2(u'(t)) + \partial\mathcal{E}(t,u(t)) \ni 0 & \text{in } \mathbf{X}_2^* \quad \text{for a.a. } t \in \mathrm{I}_{\text{right}}^{k,\boldsymbol{\tau}}, \\ u(t_{\boldsymbol{\tau}}^{k-1/2}) = \overline{u}, \end{cases} \quad (3.11)$$

(cf. (3.3) for the definition of $t_{\boldsymbol{\tau}}^{k-1/2}$).

Thanks to Theorem 2.10, both Cauchy problems admit a solution. We are now in a position to detail the time-splitting algorithm. Starting from an initial datum $u_0 \in \mathrm{D}_0$, we recursively define the approximate solution $U_{\boldsymbol{\tau}} : [0,T] \to \mathrm{D}_0$ in the following way:

$$\begin{aligned} &U_{\boldsymbol{\tau}}(0) := u_0 \qquad \text{and, for } t \in (t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^k] \text{ with } k = 1,\ldots,N_{\boldsymbol{\tau}}, \text{ we define} \\ &U_{\boldsymbol{\tau}} \text{ as a solution of Cauchy problem (3.10) with } \underline{u} = U_{\boldsymbol{\tau}}(t_{\boldsymbol{\tau}}^{k-1}) \qquad \text{on } \mathrm{I}_{\text{left}}^{k,\boldsymbol{\tau}}, \\ &U_{\boldsymbol{\tau}} \text{ as a solution of Cauchy problem (3.11) with } \overline{u} = U_{\boldsymbol{\tau}}\left(t_{\boldsymbol{\tau}}^{k-1} + \tfrac{\tau_k}{2}\right) \qquad \text{on } \mathrm{I}_{\text{right}}^{k,\boldsymbol{\tau}}. \end{aligned} \quad (3.12)$$

By construction, we have $U_{\boldsymbol{\tau}} \in \mathrm{AC}([0,T];\mathbf{X}_1)$.

Our main convergence result

We will prove the convergence of (a subsequence of) the family of curves $(U_{\boldsymbol{\tau}})_{\boldsymbol{\tau}}$ to a solution U of the Cauchy problem for the generalized gradient system (1.7), under the additional *singleton condition* on $\partial^{\mathbf{X}_2}\mathcal{E}$. We mention in advance that the fact that $\partial^{\mathbf{X}_2}\mathcal{E}(t,u)$ is a singleton does not imply Fréchet differentiability of $\mathcal{E}$ at (t,u), cf. e.g. the counterexample in [24, § 1.3, p. 90].

Hypothesis 3.2 (Singleton condition) *We assume that*

$$\partial^{X_2}\mathcal{E}(t,u) \textit{ is a singleton for all } (t,u)\in[0,T]\times D_2. \tag{3.13}$$

Obviously, due to (2.18), Hypothesis 3.2 implies that, whenever it is non-empty, $\partial^{\mathbf{X}_1}\mathcal{E}(t,u)$ is also a singleton and coincides with $\partial^{\mathbf{X}_2}\mathcal{E}(t,u)$. Indeed, in Sect. 4.1 we will present a counterexample to convergence of the split-step scheme, in the case of an energy with multivalued Fréchet subdifferentials, i.e. the singleton condition fails. As before, we have to assume a suitable chain rule for the effective generalized gradient system $(\mathbf{X}_1,\mathcal{E},\mathcal{R}_{\text{eff}})$. We emphasize that for the upcoming convergence result, the chain-rule Hypothesis 2.6 for the individual systems $(\mathbf{X}_j,\mathcal{E},\mathcal{R}_j)$ is not really needed, if we assume that the split-step approximations $U_{\boldsymbol{\tau}}$ are given.

Hypothesis 3.3 (Chain rule for $(\boldsymbol{X}_1,\mathcal{E},\mathcal{R}_{\text{eff}})$) *The effective generalized gradient system $(\boldsymbol{X}_1,\mathcal{E},\mathcal{R}_{\text{eff}})$ with subdifferential $\partial^{X_1}\mathcal{E}$ satisfies* ***<CR>***.

Remark 3.4 (QYE for $\mathcal{R}_j$ with $j\in\{1,2,\text{eff}\}$) As **<CR>** may be deduced with the help of the QYE (2.10), it is a natural question to ask whether the validity of QYE for $\mathcal{R}_1$ and $\mathcal{R}_2$ implies the validity of QYE for $\mathcal{R}_{\text{eff}}$. In general, this is false, see the example in Sect. 4.2 and the discussion in Appendix A.

Our main convergence result states the convergence of (a subsequence of) the curves $(U_{\boldsymbol{\tau}})_{\boldsymbol{\tau}}$, as $|\boldsymbol{\tau}|\downarrow 0$, to a solution of the Cauchy problem for (2.27). We highlight that we even have convergence of the 'repeated velocities' $(\mathbb{T}^{(j)}_{\boldsymbol{\tau}}U'_{\boldsymbol{\tau}})_{\boldsymbol{\tau}}$, to the *optimal* velocities contributing to $\mathcal{R}_{\text{eff}}(U')$.

Theorem 3.5 (Convergence of time-splitting method) *In addition to the assumptions of Theorem 2.10, assume the ordering condition (2.12), the singleton condition of Hypothesis 3.2, and the chain rule of Hypothesis 3.3. Starting from an initial datum $u_0\in D_0$, define the curves $(U_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\Lambda}$ as in (3.12). Then, for any sequence $(\boldsymbol{\tau}_n)_n$ with $\lim_{n\to\infty}|\boldsymbol{\tau}_n|=0$ there exist a (non-relabeled) subsequence, a curve $U\in AC([0,T];\boldsymbol{X}_1)$, some $E>0$, and $V_j\in \mathrm{L}^1([0,T];\boldsymbol{X}_j)$ for $j=1,2$ such that $U(0)=u_0$, $U(t)\in D_1\cap S_E$ for all $t\in[0,T]$,*

$$U_{\boldsymbol{\tau}_n}(t)\rightharpoonup U(t) \qquad \textit{in } \boldsymbol{X}_1 \quad \textit{for all } t\in[0,T], \tag{3.14a}$$

$$\frac{1}{2}\mathbb{T}^{(j)}_{\boldsymbol{\tau}_n}(U'_{\boldsymbol{\tau}_n})\rightharpoonup V_j \qquad \textit{in } \mathrm{L}^1([0,T];\boldsymbol{X}_j) \quad \textit{for } j=1,2, \tag{3.14b}$$

$$U'_{\boldsymbol{\tau}_n}\rightharpoonup U'=V_1+V_2 \qquad \textit{in } \mathrm{L}^1([0,T];\boldsymbol{X}_1), \tag{3.14c}$$

and there exists a function $\xi\in\mathrm{L}^1([0,T];\boldsymbol{X}_1^*)$ such that the pair (U,ξ) solves the subdifferential inclusion

$$\begin{cases}\partial\mathcal{R}_{\text{eff}}(U'(t))+\xi(t)\ni 0\\ \partial^{X_1}\mathcal{E}(t,U(t))=\{\xi(t)\}\end{cases} \qquad \textit{in } \boldsymbol{X}_1^* \quad \textit{for a.a. } t\in(0,T) \tag{3.15}$$

and fulfills the energy-dissipation balance, for every $0 \le s \le t \le T$,

$$\mathcal{E}(t, U(t)) + \int_s^t \big(\mathcal{R}_{\rm eff}(U'(r)) + \mathcal{R}^*_{\rm eff}(-\xi(r))\big)\mathrm{d}r = \mathcal{E}(s, U(s)) + \int_s^t \partial_t \mathcal{E}(r, U(r))\mathrm{d}r. \tag{3.16}$$

Moreover, (V_1, V_2) *provides an* optimal decomposition *for* U'*, namely*

$$\left.\begin{array}{l} V_1(t) + V_2(t) = U'(t) \\ \mathcal{R}_1(V_1(t)) + \mathcal{R}_2(V_2(t)) = \mathcal{R}_{\rm eff}(U'(t)) = \min\limits_{v_1+v_2=U'(t)} \mathcal{R}_1(v_1) + \mathcal{R}_2(v_2) \end{array}\right\} \text{ for a.a. } t \in (0, T). \tag{3.17}$$

The proof of Theorem 3.5 will be carried out in Sect. 5. We only mention that, as suggested by Proposition 2.1, our argument for proving that the curves $(U_{\boldsymbol{\tau}_n})_n$ converge to a solution of (3.15) will be tightly related to the proof of the energy-dissipation balance (3.16). We shall obtain (3.16) by taking the limit in its approximate version, which is in turn obtained by piecing together the energy-dissipation balances for the individual gradient systems $(\mathbf{X}_j, \mathcal{E}, \mathcal{R}_j)$. In this connection, we mention that, starting from (upper) *estimates* in place of balances for the systems $(\mathbf{X}_j, \mathcal{E}, \mathcal{R}_j)$ would be sufficient. Hence, imposing the chain-rule Hypothesis 2.6 $(\mathbf{X}_j, \mathcal{E}, \mathcal{R}_j)$ could be avoided): in fact, we will need to take the limit only in the approximate upper energy-dissipation estimate as described in the Introduction, see (1.9).

In order to obtain the approximate version of (3.16) (cf. (5.4) ahead), resorting to the rescaled dissipation potentials $\widetilde{\mathcal{R}}_j$ proves handy, for it allows us to rewrite in a concise and suggestive way the two integral terms $\mathcal{D}^{\rm rate}_{\boldsymbol{\tau}}$ and $\mathcal{D}^{\rm slope}_{\boldsymbol{\tau}}$ encoding the dissipative contributions to (5.4).

More precisely, let us bring into play the characteristic function of (the union of) the semi-intervals $\mathrm{I}^{k,\boldsymbol{\tau}}_{\rm left}$, i.e.

$$\chi_{\boldsymbol{\tau}} : (0, T) \to \{0, 1\} \qquad \chi_{\boldsymbol{\tau}}(t) := \begin{cases} 1 \text{ if } t \in \mathrm{I}^{\mathrm{k}_{\boldsymbol{\tau}}(t),\boldsymbol{\tau}}_{\rm left}, \\ 0 \text{ otherwise.} \end{cases} \tag{3.18}$$

Recalling $\widetilde{\mathcal{R}}_j = 2\mathcal{R}_j(\frac{1}{2}\,\cdot)$ and $\widetilde{\mathcal{R}}^*_j = 2\mathcal{R}^*_j$, on subintervals $[s, t] \subset [0, T]$

1. the *rate term* from (1.10a) rewrites as

$$\mathcal{D}^{\rm rate}_{\boldsymbol{\tau}}([s, t]) := \int_s^t \Big(\chi_{\boldsymbol{\tau}}(r)\, \widetilde{\mathcal{R}}_1\big(U'_{\boldsymbol{\tau}}(r)\big) + (1-\chi_{\boldsymbol{\tau}}(r))\, \widetilde{\mathcal{R}}_2\big(U'_{\boldsymbol{\tau}}(r)\big)\Big)\mathrm{d}r \tag{3.19a}$$

2. while the *slope term* from (1.10b) is given by

$$\mathcal{D}^{\rm slope}_{\boldsymbol{\tau}}([s, t]) := \int_s^t \Big(\chi_{\boldsymbol{\tau}}(r)\, \widetilde{\mathcal{R}}^*_1(-\xi_{\boldsymbol{\tau}}(r)) + (1-\chi_{\boldsymbol{\tau}}(r))\, \widetilde{\mathcal{R}}^*_2(-\xi_{\boldsymbol{\tau}}(r))\Big)\mathrm{d}r. \tag{3.19b}$$

When we take the limit $|\boldsymbol{\tau}_n| \to 0$ in the approximate energy-dissipation balance featuring the two terms above, we will obtain, a by-product, enhanced convergence information for $(U_{\boldsymbol{\tau}_n})_n$, in addition to the convergences (3.14). Indeed, we will succeed in proving that

$$\mathcal{E}(t, U_{\boldsymbol{\tau}_n}(t)) \longrightarrow \mathcal{E}(t, U(t)) \qquad \text{for all } t \in [0, T], \tag{3.20a}$$

$$\begin{cases} \mathcal{D}^{\rm rate}_{\boldsymbol{\tau}_n}([s, t]) \longrightarrow \displaystyle\int_s^t \mathcal{R}_{\rm eff}(U'(r))\mathrm{d}r, \\ \mathcal{D}^{\rm slope}_{\boldsymbol{\tau}_n}([s, t]) \longrightarrow \displaystyle\int_s^t \mathcal{R}^*_{\rm eff}(-\xi(r))\mathrm{d}r \end{cases} \qquad \text{for all } [s, t] \subset [0, T]. \tag{3.20b}$$

In fact, for the velocities we will even obtain the following, more precise, convergence statement on all subintervals $[s,t] \subset [0,T]$:

$$\int_s^t \mathcal{R}_j(\tfrac{1}{2}\mathbb{T}^{(j)}_{\boldsymbol{\tau}_n} U'_{\boldsymbol{\tau}_n}(r)) \longrightarrow \int_s^t \mathcal{R}_j(V_j(r))\mathrm{d}r \qquad \text{for } j = 1, 2. \tag{3.21}$$

4 Two examples

In the upcoming Sect. 4.1 we exhibit a counterexample to Theorem 3.5 in the case in which the Fréchet subdifferential does not comply with the singleton condition from Hypothesis 3.2. In Sect. 4.2 we provide two gradient systems $(\mathbf{X}_1, \mathcal{E}, \mathcal{R}_1)$ and $(\mathbf{X}_2, \mathcal{E}, \mathcal{R}_2)$ fulfilling all assumptions of Theorem 3.5.

4.1 Non-convergence for multi-valued $\partial\mathcal{E}$

Following [19, Section 3.1] we consider the simple case $\mathbf{X}_1 = \mathbf{X}_2 = \mathbb{R}^2$ and a one-homogeneous energy $\mathcal{E}(t,u) = \mathcal{E}(u) = \max\{|u_1|, |u_2|\}$. Clearly, $\mathcal{E}$ is convex and its convex subdifferential is multi-valued, with

$$\partial\mathcal{E}(u) = \begin{cases} \{\pm(1,0)^\top\} & \text{for } \pm u_1 > |u_2|, \\ \{\pm(0,1)^\top\} & \text{for } \pm u_2 > |u_1|, \\ \{\pm(\theta, 1{-}\theta)^\top | \theta \in [0,1]\} & \text{for } u_1 = u_2,\ \pm u_1 > 0, \\ \{\pm(\theta, \theta{-}1)^\top | \theta \in [0,1]\} & \text{for } u_1 = -u_2,\ \pm u_1 > 0, \\ [-1,1]\times[-1,1] & \text{for } u = 0. \end{cases}$$

For a general dissipation potential of the form

$$\mathcal{R}^*(\xi) = \frac{a}{2}\xi_1^2 + \frac{b}{2}\xi_2^2 \tag{4.1}$$

we can solve the gradient-flow equation for $(\mathbb{R}^2, \mathcal{E}, \mathcal{R}^*)$ and obtain a contraction semigroup on the Hilbert space $\mathbb{R}^2$ (cf. [6, Theorem 3.1]). We consider the solution with the initial condition $u^0 = u(0) = (2,1)^\top$, which leads to the piecewise affine solution

$$\begin{cases} u(t) = \binom{2-at}{1} & \text{for } t \in [0, \frac{1}{a}], \\ u(t) = (1 - (t{-}\frac{1}{a})\frac{ab}{a+b})\binom{1}{1} & \text{for } t \in \,]\frac{1}{a}, \frac{2}{a} + \frac{1}{b}[, \\ u(t) = \binom{0}{0} & \text{for } t \geq \frac{2}{a} + \frac{1}{b}. \end{cases}$$

Note that in the middle regime the solution satisfies $u_1 = u_2 > 0$ and hence the subdifferential $\partial\mathcal{E}(u(t))$ is set-valued. This multi-valuedness is necessary as the choice for $(\theta, 1{-}\theta) \in \partial\mathcal{E}(u)$ indeed depends on $\mathcal{R}^*$, namely $\theta = b/(a{+}b)$. For later use, it is nice to observe that $-u_1'(t)$ only takes three values, namely a, $ab/(a{+}b)$, and 0.

We now apply the split-step algorithm for the two dissipation potentials

$$\mathcal{R}_1^*(\xi) = \frac{a_1}{2}\xi_1^2 + \frac{b_1}{2}\xi_2^2 \quad \text{and} \quad \mathcal{R}_2^*(\xi) = \frac{a_2}{2}\xi_1^2 + \frac{b_2}{2}\xi_2^2.$$

Clearly, we have the effective potentials

$$\mathcal{R}^*_{\mathrm{eff}}(\xi) = \frac{a}{2}\xi_1^2 + \frac{b}{2}\xi_2^2 \text{ and } \mathcal{R}_{\mathrm{eff}}(\xi) = \frac{1}{2a}\xi_1^2 + \frac{1}{2b}\xi_2^2 \text{ with } a = a_1{+}a_2 \text{ and } b = b_1{+}b_2.$$

The three gradient systems $(\mathbb{R}^2, \mathcal{E}, \mathcal{R}^*_{\mathrm{eff}})$, $(\mathbb{R}^2, \mathcal{E}, 2\mathcal{R}^*_1)$, and $(\mathbb{R}^2, \mathcal{E}, 2\mathcal{R}^*_2)$ have the same form as the gradient system $(\mathbb{R}^2, \mathcal{E}, \mathcal{R}^*)$ with the general potential $\mathcal{R}$ from (4.1). Hence, the obtained solutions are again piecewise affine and $V := -u'(t)$ only takes three values.

In the first regime $u_1(t) > |u_2(t)|$ we have the three velocities:

$$\mathcal{R}^*_{\mathrm{eff}}:\ V_{\mathrm{eff}} = \begin{pmatrix} a_1{+}a_2 \\ 0 \end{pmatrix}, \quad 2\mathcal{R}^*_1:\ V_1 = \begin{pmatrix} 2a_1 \\ 0 \end{pmatrix}, \quad 2\mathcal{R}^*_2:\ V_2 = \begin{pmatrix} 2a_2 \\ 0 \end{pmatrix}.$$

Hence, we observe that the split-step solutions u_τ oscillate between the two velocities V_1 and V_2 in such a way that the weak limit of $-u'_\tau$ equals $\frac{1}{2}(V_1{+}V_2) = \binom{a_1+a_2}{0}$. Hence, in this regime (where $\partial\mathcal{E}(u)$ is single-valued), we have convergence of u_τ to u_{eff} as $V_{\mathrm{eff}} = \frac{1}{2}(V_1{+}V_2)$.

However, for $t \in \left]\frac{1}{a}, \frac{2}{a} + \frac{1}{b}\right[$ the situation is different, because for $u_1(t) = u_2(t) > 0$ the subdifferential is multi-valued. The three velocities are

$$\mathcal{R}^*_{\mathrm{eff}}:\ V_{\mathrm{eff}} = \frac{ab}{a{+}b}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad 2\mathcal{R}^*_1:\ V_1 = \frac{2a_1b_1}{a_1{+}b_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad 2\mathcal{R}^*_2:\ V_2 = \frac{2a_2b_2}{a_2{+}b_2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Clearly, for $t \in \left]\frac{1}{a}, \frac{2}{a} + \frac{1}{b}\right[$ we obtain

$$u_\tau(t) \to u_{\mathrm{lim}}(t) := \begin{pmatrix} 1 \\ 1 \end{pmatrix} - (t{-}\tfrac{1}{a})\frac{1}{2}\big(V_1{+}V_2\big), \quad \text{while } u_{\mathrm{eff}}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - (t{-}\tfrac{1}{a})\, V.$$

We always have $|V| \geq \left|\frac{1}{2}(V_1{+}V_2)\right|$ but in general with a strict inequality, e.g. for $(a_1, b_1) = (1, 3)$ and $(a_2, b_2) = (3, 1)$ we have $a = b = 4$ and obtain $V_{\mathrm{eff}} = \binom{2}{2}$ and $V_1 = V_2 = \binom{3/2}{3/2}$.

We observe that the effective solution has a higher speed and is reaching $u(t) = 0$ already at $t = 1/4 + 1/2 = 0.75$. However, the split-step solution u_τ, which is actually not oscillating with τ because of $V_1 = V_2$, is slowed down as it reaches $u_\tau(t) = 0$ only for $t = 1/4 + 2/3 \approx 0.917$.

Remark 4.1 In this example one can follow the limiting procedure in the energy-dissipation balance. One observes that the liminf estimate for the velocities always works. However, because of the limiting rate being too small, there is a true drop. With $t_1 = 1/a$ and $t_2 = 2/a + 1/b$ and $(a_1, b_1) = (1, 3)$ and $(a_2, b_2) = (3, 1)$ we have

$$\frac{1}{t_2{-}t_1}\int_{t_1}^{t_2} \widetilde{\mathcal{R}}_{J_\tau(t)}(u'_\tau)\mathrm{d}\tau = \frac{1}{2}\Big(\widetilde{\mathcal{R}}_1(V_1) + \widetilde{\mathcal{R}}_2(V_2)\Big) = \frac{1}{2}\Big(\frac{a_1b_1}{a_1{+}b_1} + \frac{a_2b_2}{a_2{+}b_2}\Big) = \frac{3}{4}.$$

However, the limit u_{lim} of u_τ satisfies $u'_{\mathrm{lim}} = V_1 = V_2 = \frac{3}{2}\binom{1}{1}$ and

$$\frac{1}{t_2{-}t_1}\int_{t_1}^{t_2} \mathcal{R}_{\mathrm{eff}}(u'_{\mathrm{lim}}(t))\mathrm{d}t = \mathcal{R}_{\mathrm{eff}}\big(V_j\big) = \frac{1}{8}|V_j|^2 = \frac{9}{16} \lneqq \frac{3}{4}.$$

Such a drop cannot be recovered if we treat the rate part of the dissipation and the slope part of the dissipation separately, as is done in [32], see also the discussion in [20, Sec. 5.4]. The approach of EDP convergence as studied in [12, 23, 25] may be capable to pass to the limit as well, but this lies outside the range of this paper.

4.2 A doubly nonlinear PDE example

In this section we consider a doubly nonlinear PDE of Allen-Cahn type where the above split-step method applies. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. On the

$$\textbf{Banach spaces} \qquad \mathbf{X}_1 = \mathrm{L}^p(\Omega) \text{ with } p \in (1, p_d], \ \text{ and } \mathbf{X}_2 = \mathrm{H}^1_0(\Omega) \qquad (4.2a)$$

with $p_d = \frac{2d}{d-2}$ for $d \geq 3$, and p_d arbitrary in $(1, \infty)$ for $d \in \{1, 2\}$, so that $\mathbf{X}_2 \subset \mathbf{X}_1$ densely and continuously, we consider the

$$\textbf{dissipation potentials} \quad \mathcal{R}_1(v) := \frac{1}{p}\|v\|^p_{\mathrm{L}^p(\Omega)} \text{ and } \mathcal{R}_2(v) := \frac{1}{2}\|\nabla v\|^2_{\mathrm{L}^2(\Omega)}, \tag{4.2b}$$

and the

$$\textbf{energy functional} \qquad \mathcal{E} : [0, T] \times \mathrm{L}^p(\Omega) \to (-\infty, \infty] \text{ defined via} \tag{4.2c}$$

$$\mathcal{E}(t, u) := \begin{cases} \int_\Omega \frac{1}{2}|\nabla u|^2 + W(u)\mathrm{d}x - \langle \ell(t), u\rangle_{\mathrm{H}^1_0(\Omega)} & \text{for } u \in \mathrm{H}^1_0(\Omega) \text{ and } W(u) \in \mathrm{L}^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

In what follows, we will suppose that

$$\ell \in \mathrm{C}^1([0, T]; \mathrm{H}^{-1}(\Omega)), \tag{4.3a}$$

and, following [28, Sec. 7], we will require that $W : \mathbb{R} \to \mathbb{R}$ satisfies $W \in \mathrm{C}^2(\mathbb{R})$ and

$$\exists\, C_{W,1}, C_{W,2}, C_{W,3} > 0 \; \exists\, s_p \in (1, \tfrac{p_d}{p_*}) \; \forall r \in \mathbb{R} : \begin{cases} W''(r) \geq -C_{W,1}, \\ W(r) \geq -C_{W,2}, \\ |W'(r)| \leq C_{W,3}(1+|r|^{s_p}), \end{cases} \tag{4.3b}$$

where with p^* is the dual exponent to p.

Theorem 4.2 below addresses the validity of the assumptions for Theorem 3.5 in the context of the two gradient systems $(\mathbf{X}_1, \mathcal{E}, \mathcal{R}_1)$ and $(\mathbf{X}_2, \mathcal{E}, \mathcal{R}_2)$. Observe that the corresponding evolutionary equations are

$$\begin{aligned} (\mathbf{X}_1, \mathcal{E}, \mathcal{R}_1) : \quad & |u'|^{p-2}u' - \Delta u + W'(u) &= \ell \quad & \text{in } (0, T)\times\Omega, \\ (\mathbf{X}_2, \mathcal{E}, \mathcal{R}_2) : \quad & -\Delta u' - \Delta u + W'(u) &= \ell \quad & \text{in } (0, T)\times\Omega. \end{aligned}$$

Theorem 4.2 *Under conditions (4.3), the gradient systems $(\boldsymbol{X}_1, \mathcal{E}, \mathcal{R}_1)$ and $(\boldsymbol{X}_2, \mathcal{E}, \mathcal{R}_2)$ from (4.2) comply with Hypotheses 2.3–2.7 and Hypothesis 3.2. The QYE (2.10) holds for $\mathcal{R}_1$ and $\mathcal{R}_2$; but it is valid for $\mathcal{R}_{\mathrm{eff}}$ if and only if $p \leq 2$. In particular, for $p \leq 2$ the solution of the gradient system $(\boldsymbol{X}_1, \mathcal{E}, \mathcal{R}_{\mathrm{eff}})$ can be constructed by the time-splitting method.*

The proof will be carried out in the two following sections, starting with a discussion of the properties of the energy functional $\mathcal{E}$.

Properties of the energy

The lower bound on W'' required in (4.3b) will be used to derive λ-convexity of $\mathcal{E}$, the bound on W shall provide the lower bound on the energy, the third bound on W will be exploited for proving the closedness of the subdifferentials of $\mathcal{E}$.

Indeed, it is immediate to check that $\mathcal{E}$, whose proper domain $\mathrm{dom}(\mathcal{E})$ is of the form $[0, T]\times \mathrm{D}_0$ with $\mathrm{D}_0 \subset \mathrm{H}^1_0(\Omega)$, is bounded from below and complies with the power-control condition (2.1). Since for every $E > 0$ the corresponding sublevel set S_E (cf. (2.16)) is contained in a bounded set in $\mathrm{H}^1_0(\Omega)$, we also immediately verify Hypothesis 2.4.

The same arguments as in the proof of [28, Lem. 7.3] (cf. also [28, Rmk. 7.4]) yield the following representations for the Fréchet subdifferentials $\partial^{\mathbf{X}_j}\mathcal{E}$ of $\mathcal{E}(t, \cdot)$:

$$\text{for } \mathbf{X}_1 = \mathrm{L}^p(\Omega): \quad \partial^{\mathbf{X}_1}\mathcal{E}(t,u) = \begin{cases} -\Delta u + W'(u) - \ell(t) & \text{if } -\Delta u + W'(u) - \ell(t) \in \mathrm{L}^{p^*}(\Omega), \\ \emptyset & \text{else;} \end{cases}$$

$$\text{for } \mathbf{X}_2 = \mathrm{H}^1_0(\Omega): \quad \partial^{\mathbf{X}_2}\mathcal{E}(t,u) = \begin{cases} -\Delta u + W'(u) - \ell(t) & \text{if } -\Delta u + W'(u) - \ell(t) \in \mathrm{H}^{-1}(\Omega), \\ \emptyset & \text{else.} \end{cases}$$

Hence, the *singleton condition* from Hypothesis 3.2 is satisfied. As for the closedness requirement from Hypothesis 2.5, let $t_n \to t$ in $[0,T]$ and let us consider a sequence $(u_n)_n \subset S_E$ for some $E > 0$, weakly converging to some u in $\mathrm{L}^p(\Omega)$. Hence, $(u_n)_n$ is bounded in $\mathrm{H}^1_0(\Omega)$ and thus $u_n \rightharpoonup u$ in $\mathrm{H}^1(\Omega)$, so that $-\Delta u_n \rightharpoonup -\Delta u$ in $\mathrm{H}^{-1}(\Omega)$, and $u_n \to u$ strongly in $\mathrm{L}^{p_d-\varepsilon}(\Omega)$ (as p_d is the critical exponent from the Rellich-Kondrachov theorem). In particular, $u_n \to u$, and thus $W'(u_n) \to W'(u)$, a.e. in Ω. Furthermore, $W'(u_n) \leq C|u_n|^{s_p}$ a.e. in Ω. Since $s_p < \frac{p_d}{p^*}$, by dominated convergence we conclude that $W'(u_n) \to W'(u)$ in $\mathrm{L}^{p^*}(\Omega)$. Also using that $\ell(t_n) \to \ell(t)$ in $\mathrm{H}^{-1}(\Omega)$, we immediately conclude the closedness of both subdifferentials $\partial^{\mathbf{X}_j}\mathcal{E}$.

Finally, it was shown in [28, Lem. 7.3] that $\mathcal{E}(t,\cdot)$ is λ-convex in $\mathrm{L}^1(\Omega)$, i.e.

$$\begin{aligned} &\exists\, \lambda < 0\ \forall t \in [0,T]\ \forall u_0, u_1 \in \mathrm{D}_0\ \forall \theta \in [0,1]: \\ &\mathcal{E}(t,u_\theta) \leq (1-\theta)\mathcal{E}(t,u_0) + \theta\mathcal{E}(t,u_1) - \frac{\lambda}{2}\theta(1-\theta)\|u_0-u_1\|^2_{\mathrm{L}^1(\Omega)} \text{ with } u_\theta = (1-\theta)u_0 + \theta u_1 \end{aligned} \tag{4.4}$$

Then, $\mathcal{E}(t,\cdot)$ are λ-uniformly convex in $\mathbf{X}_1$ and in $\mathbf{X}_2$ (namely, estimate (4.4) holds with $\|\cdot\|_{\mathrm{L}^1(\Omega)}$ replaced by $\|\cdot\|_{\mathrm{L}^p(\Omega)}$ and $\|\cdot\|_{\mathrm{H}^1_0(\Omega)}$, respectively, and λ suitably adjusted). Then, we are in a position to apply [21, Prop. A.1] and conclude the validity of the chain rule property **<CR>** for $(\mathcal{E}, \partial^{\mathbf{X}_j}\mathcal{E})$.

Properties of the dissipation potentials $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_{\mathrm{eff}}$

Finally, we discuss the validity of Hypotheses 2.7 and the QYE 2.10 for $\mathcal{R}_1$ and $\mathcal{R}_2$. Obviously, the dissipation potential $\mathcal{R}_1$ and its conjugate $\mathcal{R}_1^*$ comply with condition (2.2); also $\mathcal{R}_2$ has superlinear growth, since on $\mathbf{X}_2 = \mathrm{H}^1_0(\Omega)$ the function $\|\nabla\cdot\|_{\mathrm{L}^2(\Omega)}$ provides a norm equivalent to the $\mathrm{H}^1(\Omega)$-norm. Finally, we observe that

$$\mathcal{R}_2^*(\xi) = \sup_{v \in \mathrm{H}^1_0(\Omega)} \left\{ \langle v, \xi\rangle - \frac{1}{2}\|\nabla v\|^2_{\mathrm{L}^2(\Omega)} \right\} = \frac{1}{2}\|\nabla v_*\|^2_{\mathrm{L}^2(\Omega)},$$

where $v_* \in \mathrm{H}^1_0(\Omega)$ is the unique solution of $\xi = -\Delta v_*$ in the $\mathrm{H}^1_0 \times \mathrm{H}^{-1}$ duality. Hence,

$$\exists\, c, C > 0\ \forall \xi \in \mathrm{H}^{-1}(\Omega)\colon\ c\|\xi\|^2_{\mathrm{H}^{-1}(\Omega)} \leq \mathcal{R}_2^*(\xi) = \frac{1}{2}\|\nabla(-\Delta)^{-1}\xi\|^2_{\mathrm{L}^2(\Omega)} \leq C\|\xi\|^2_{\mathrm{H}^{-1}(\Omega)}. \tag{4.5}$$

We now discuss the validity of the Quantitative Young Estimate (2.10) for $\mathcal{R}_{\mathrm{eff}}$,

$$\mathcal{R}_{\mathrm{eff}}(u) = \min_{v \in \mathrm{L}^p(\Omega)} \left(\frac{1}{p}\|v-u\|^p_{\mathrm{L}^p(\Omega)} + \frac{1}{2}\|\nabla v\|^2_{\mathrm{L}^2(\Omega)} \right)$$

(where we omit to detail that the term $\|\nabla v\|^2_{\mathrm{L}^2(\Omega)}$ is replaced by ∞ if $v \in L^p(\Omega)\backslash \mathrm{H}^1_0(\Omega)$, cf. (2.22)). We will distinguish the cases $p \in (1,2)$, $p=2$ and $p>2$, and show that the QYE holds if and only if $p \le 2$. For this, recall $\mathbf{X}_{\text{eff}} = \mathbf{X}_1 = \mathrm{L}^p(\Omega)$.

QYE for $p \in (1,2)$. Using $\|\nabla u\|_{\mathrm{L}^2(\Omega)} \ge C\|u\|_{\mathrm{L}^{p_d}(\Omega)}$ on $\mathbf{X}_2 = \mathrm{H}^1_0(\Omega)$, we have

$$\mathcal{R}_2(v) \ge \frac{1}{2}\|\nabla v\|^2_{\mathrm{L}^2(\Omega)} \ge \frac{C}{2}\|v\|^2_{\mathrm{L}^{p_d}(\Omega)}.$$

Since $p_d > 2 > p$, we obtain

$$\exists\, \overline{c} > 0\ \forall\, v \in \mathrm{H}^1_0(\Omega): \quad \mathcal{R}_2(v) \ge \overline{C}\|v\|^2_{\mathrm{L}^{p_d}(\Omega)} \ge \overline{c}\|v\|^2_{\mathrm{L}^p(\Omega)}.$$

Therefore, $\mathcal{R}_1$ and $\mathcal{R}_2$ comply with condition (A.2) of Lemma A.2, which guarantees the validity of the QYE for $\mathcal{R}_{\text{eff}}$.

QYE for $p=2$. In this case,

$$\mathcal{R}^*_{\text{eff}}(\xi) = \mathcal{R}^*_1(\xi) + \mathcal{R}^*_2(\xi) = \frac{1}{2}\|\xi\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|\nabla(-\Delta)^{-1}\xi\|^2_{\mathrm{L}^2(\Omega)} \quad \text{for all } \xi \in \mathbf{X}^*_1 = \mathrm{L}^2(\Omega).$$

Hence, also for $\mathcal{R}_{\text{eff}}$ we have both a quadratic upper bound and a quadratic lower bound. Therefore, we may apply Lemma 2.2 and conclude that $\mathcal{R}_{\text{eff}}$ complies with the QYE.

Failure of QYE for $p>2$. Below we will establish the following two statements:

(A) $\exists\, v \neq 0\ \exists\, C_{\mathrm{A}} > 0\ \forall\, \lambda > 0: \quad \mathcal{R}_{\text{eff}}(\lambda v) \le C_{\mathrm{A}}\lambda^2\,,$

(B) $\exists\, \big(\xi_n\big)_{n\in\mathbb{N}}$ in $\mathbf{X}^*_1\ \exists\, C_{\mathrm{B}} > 0: \quad \|\xi_n\|_{\mathrm{L}^{p^*}(\Omega)} \to \infty$ and $\mathcal{R}^*_{\text{eff}}(\xi_n) \le C_{\mathrm{B}}\|\xi_n\|^{p^*}_{\mathrm{L}^{p^*}(\Omega)}\,.$

<u>*Step 1: (A) and (B) imply that QYE does not hold.*</u> In (A) we can choose $\lambda = \lambda_n = \|\xi_n\|^{p^*/2}_{\mathrm{L}^{p^*}}$. Then, (A) and (B) imply the upper bound

$$\mathcal{R}_{\text{eff}}(\lambda_n v) + \mathcal{R}^*_{\text{eff}}(\xi_n) \le \big(C_{\mathrm{A}} + C_{\mathrm{B}}\big)\|\xi_n\|^{p^*}_{\mathrm{L}^{p^*}(\Omega)} \quad \text{for all } n \in \mathbb{N}. \tag{4.6}$$

However, QYE would imply the lower bound

$$\mathcal{R}_{\text{eff}}(\lambda_n v) + \mathcal{R}^*_{\text{eff}}(\xi_n) \ge c\|\lambda_n v\|_{\mathrm{L}^p(\Omega)}\|\xi_n\|_{\mathrm{L}^{p^*}(\Omega)} - C = c\|v\|_{\mathrm{L}^p(\Omega)}\,\|\xi_n\|^{1+p^*/2}_{\mathrm{L}^{p^*}(\Omega)} - C.$$

Because of $p>2$ we have $p^* = p/(p-1) \in (1,2)$ and hence $p^* \lneq 1 + p^*/2$. Since, by (B) we can take $\|\xi_n\|_{\mathrm{L}^{p^*}}$ arbitrarily large, we see that the lower bound derived from QYE contradicts the upper bound (4.6). Hence, QYE is false.

<u>*Step 2: (A) holds.*</u> We choose any $v \in \mathrm{L}^p(\Omega) \cap \mathrm{H}^1_0(\Omega)$ with $v \neq 0$. Then, $\mathcal{R}_{\text{eff}}(\lambda v) \le \mathcal{R}_2(\lambda v) = C_{\mathrm{A}}\lambda^2$ with $C_{\mathrm{A}} = \frac{1}{2}\|\nabla v\|^2_{\mathrm{L}^2(\Omega)}$.

<u>*Step 3: (B) holds.*</u> We set $\xi_n(x) = n\sin\big(n^{1-p^*/2}x_1\big)$ and observe

$$\|\xi_n\|^{p^*}_{\mathrm{L}^{p^*}} = \int_\Omega n^{p^*}\big|\sin(n^{1-p^*/2}x_1)\big|^{p^*}\mathrm{d}x \ge n^{p^*}\int_\Omega \big|\sin(n^{1-p^*/2}x_1)\big|^2\mathrm{d}x \ge \frac{n^{p^*}}{2}\sqrt{|\Omega|},$$

where we used $|\sin\alpha|^{p^*} \ge |\sin\alpha|^2$ because of $p^* \in (1,2)$. Moreover, using $\xi_n = -\partial_{x_1}\Xi_n$ for $\Xi_n(x) = n^{p^*/2}\cos(n^{1-p^*/2}x_1)$ we find

$$\begin{aligned}\|\xi_n\|_{\mathrm{H}^1_0(\Omega)^*} &= \sup_{\|v\|_{\mathrm{H}^1_0}\le 1}\int_\Omega \xi_n\, v\mathrm{d}x = \sup_{\|v\|_{\mathrm{H}^1_0}\le 1}\int_\Omega -\partial_{x_1}\Xi_n\, v\mathrm{d}x \\ &= \sup_{\|v\|_{\mathrm{H}^1_0}\le 1}\int_\Omega \Xi_n\,\partial_{x_1}v\mathrm{d}x \le \|\Xi_n\|_{\mathrm{L}^2(\Omega)} \le n^{p^*/2}\sqrt{|\Omega|}\,.\end{aligned}$$

With the above estimates, we arrive at $\mathcal{R}^*_{\text{eff}}(\xi_n) = \mathcal{R}^*_1(\xi_n) + \mathcal{R}^*_2(\xi_n) = \frac{1}{p^*}\|\xi_n\|^{p^*}_{\mathrm{L}^{p^*}(\Omega)} + \frac{1}{2}\|\xi_n\|^2_{\mathrm{H}^1_0(\Omega)^*} \leq \widetilde{C}\, n^{p^*}$, and (B) follows with $C_{\mathrm{B}} = 2\widetilde{C}/\sqrt{|\Omega|}$.

5 Convergence proof of the time-splitting method

Our argument for the proof of Theorem 3.5 is carried out in the following steps, tackled in the upcoming Sects. 5.1–5.4. It follows the classical existence theory for solutions to gradient flow equations, see the survey [20].

(1) **A priori estimates and compactness**: From the energy-dissipation balances for the Cauchy problem (3.10) on the semi-intervals $(\mathrm{I}^{k,\boldsymbol{\tau}}_{\text{left}})_{k=1}^{N_{\boldsymbol{\tau}}}$, and for (3.11) on the semi-intervals $(\mathrm{I}^{k,\boldsymbol{\tau}}_{\text{right}})_{k=1}^{N_{\boldsymbol{\tau}}}$, we will deduce an overall energy-dissipation balance satisfied on the interval $[0, T]$ by the curves $U_{\boldsymbol{\tau}}$ from (3.12). Therefrom we will derive all the a priori estimates on the family $(U_{\boldsymbol{\tau}})_{\boldsymbol{\tau}}$, cf. Proposition 5.1 ahead. Consequently, we will deduce suitable compactness properties for a sequence $(U_{\boldsymbol{\tau}_n})_n$.

We then pass to the limit in the energy-dissipation balance, by separately addressing

(2) **the limit passage in the rate term** $\mathcal{D}^{\text{rate}}_{\boldsymbol{\tau}}([0, T])$, where we use $\mathcal{R}_{\text{eff}} = \mathcal{R}_1 \overset{\inf}{\circ} \mathcal{R}_2$, and
(3) **the limit passage in the slope term** $\mathcal{D}^{\text{slope}}_{\boldsymbol{\tau}}([0, T])$, where we exploit the singleton condition (3.13).

With Steps (1)–(3) we will thus show that (along a subsequence) the curves $(U_{\boldsymbol{\tau}_n})_n$ converge to a curve $U \in \mathrm{AC}([0, T]; \mathbf{X}_1)$ for which there exists $\xi \in \mathrm{L}^1([0, T]; \mathbf{X}^*_1)$ such that the pair (U, ξ) complies with the upper energy-dissipation estimate

$$\mathcal{E}(T, U(T)) + \int_0^T \big(\mathcal{R}_{\text{eff}}(U'(r)) + \mathcal{R}^*_{\text{eff}}(-\xi(r))\big)\mathrm{d}r \leq \mathcal{E}(0, U(0)) + \int_0^T \partial_t \mathcal{E}(r, U(r))\mathrm{d}r. \tag{5.1}$$

(4) **Conclusion of the proof**: We will apply the energy-dissipation principle from Proposition 2.1 to conclude that (U, ξ) fulfills the subdifferential inclusion (3.15) and the energy-dissipation balance (3.16). With a careful argument based on the limit passage from the approximate energy-dissipation balance to (3.16), we then derive the optimal decomposition (3.17) and the enhanced convergences (3.20) and (3.21).

5.1 Approximated energy-dissipation balance and a priori estimates

To state the approximate energy-dissipation balance we introduce a curve $\xi_{\boldsymbol{\tau}} : [0, T] \to \mathbf{X}^*_2$ encompassing the force terms that appear in the subdifferential inclusions (3.10) & (3.11). We will separately define $\xi_{\boldsymbol{\tau}}$ on the sets $\bigcup_{k=1}^{N_{\boldsymbol{\tau}}} \mathrm{I}^{k,\boldsymbol{\tau}}_{\text{left}}$ and $\bigcup_{k=1}^{N_{\boldsymbol{\tau}}} \mathrm{I}^{k,\boldsymbol{\tau}}_{\text{right}}$ whose union gives $[0, T]$. Recall that, for every $k \in \{1, \ldots, N_{\boldsymbol{\tau}}\}$, on the semi-interval $\mathrm{I}^{k,\boldsymbol{\tau}}_{\text{left}}$ the curve $U_{\boldsymbol{\tau}}$ fulfills the energy dissipation balance

$$\mathcal{E}(t, U_{\boldsymbol{\tau}}(t)) + \int_{t^{k-1}_{\boldsymbol{\tau}}}^t \Big(\widetilde{\mathcal{R}}_1(U'_{\boldsymbol{\tau}}(r)) + \widetilde{\mathcal{R}}^*_1(-\xi_{\boldsymbol{\tau}}(r))\Big)\,\mathrm{d}r = \mathcal{E}(t^{k-1}_{\boldsymbol{\tau}}, U_{\boldsymbol{\tau}}(t^{k-1}_{\boldsymbol{\tau}})) + \int_{t^{k-1}_{\boldsymbol{\tau}}}^t \partial_t \mathcal{E}(r, U_{\boldsymbol{\tau}}(r))\mathrm{d}r \quad \text{for } t^{k-1}_{\boldsymbol{\tau}} \leq t \leq t^{k-1/2}_{\boldsymbol{\tau}} \tag{5.2a}$$

(recall $t_{\boldsymbol{\tau}}^{k-1/2} = t_{\boldsymbol{\tau}}^{k-1} + \frac{\tau_k}{2}$), where $\xi_{\boldsymbol{\tau}} : \bigcup_{k=1}^{N_{\boldsymbol{\tau}}} \mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}} \to \mathbf{X}_1^*$ satisfies

$$\xi_{\boldsymbol{\tau}}(r) \in \partial^{\mathbf{X}_1}\mathcal{E}(r, U_{\boldsymbol{\tau}}(r)) \cap (-\partial\widetilde{\mathcal{R}}_1(U_{\boldsymbol{\tau}}'(r))) \quad \text{for a.a. } r \in \bigcup_{k=1}^{N_{\boldsymbol{\tau}}} \mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}}. \tag{5.2b}$$

Likewise, on each interval $\mathrm{I}_{\mathrm{right}}^{k,\boldsymbol{\tau}}$ an energy-dissipation balance involving the dissipation potential $\widetilde{\mathcal{R}}_2$ holds, namely

$$\begin{aligned} &\mathcal{E}(t, U_{\boldsymbol{\tau}}(t)) + \int_{t_{\boldsymbol{\tau}}^{k-1/2}}^{t} \left(\widetilde{\mathcal{R}}_2(U_{\boldsymbol{\tau}}'(r)) + \widetilde{\mathcal{R}}_2^*(-\xi_{\boldsymbol{\tau}}(r))\right) \mathrm{d}r \\ &= \mathcal{E}(t_{\boldsymbol{\tau}}^{k-1/2}, U_{\boldsymbol{\tau}}(t_{\boldsymbol{\tau}}^{k-1/2})) + \int_{t_{\boldsymbol{\tau}}^{k-1/2}}^{t} \partial_t\mathcal{E}(r, U_{\boldsymbol{\tau}}(r))\mathrm{d}r \quad \text{for } t_{\boldsymbol{\tau}}^{k-1/2} \le t \le t_{\boldsymbol{\tau}}^{k}, \end{aligned} \tag{5.3a}$$

where $\xi_{\boldsymbol{\tau}} : \bigcup_{k=1}^{N_{\boldsymbol{\tau}}} \mathrm{I}_{\mathrm{right}}^{k,\boldsymbol{\tau}} \to \mathbf{X}_2^*$ satisfies

$$\xi_{\boldsymbol{\tau}}(r) \in \partial^{\mathbf{X}_2}\mathcal{E}(r, U_{\boldsymbol{\tau}}(r)) \cap (-\partial\widetilde{\mathcal{R}}_2(U_{\boldsymbol{\tau}}'(r))) \qquad \text{for a.a. } r \in \bigcup_{k=1}^{N_{\boldsymbol{\tau}}} \mathrm{I}_{\mathrm{right}}^{k,\boldsymbol{\tau}}. \tag{5.3b}$$

Combining (5.2a) and (5.3a) we deduce the overall energy-dissipation balance satisfied by the curves $\overline{U}_{\boldsymbol{\tau}}$, featuring the *rate* and *slope* terms $\mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{rate}}$ and $\mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{slope}}$ from (3.19), which are defined by alternating between $\widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_2$. This energy balance is the starting point for the derivation of the first set of a priori estimates on the curves $(U_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\Lambda}$.

Proposition 5.1 *The functions* $(U_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\Lambda}$ *and* $(\xi_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\Lambda}$ *satisfy the energy-dissipation balance*

$$\mathcal{E}(t, U_{\boldsymbol{\tau}}(t)) + \mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{rate}}([s,t]) + \mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{slope}}([s,t]) = \mathcal{E}(s, U_{\boldsymbol{\tau}}(s)) + \int_s^t \partial_t\mathcal{E}(r, U_{\boldsymbol{\tau}}(r))\mathrm{d}r \tag{5.4}$$

for every $[s,t] \subset [0,T]$. *Furthermore, there exists a positive constant* $\overline{C} > 0$ *such that the following estimates are valid for all* $\boldsymbol{\tau} \in \Lambda$*:*

$$\sup_{t\in[0,T]} \mathfrak{E}(U_{\boldsymbol{\tau}}(t)) \le \overline{C}, \qquad \sup_{t\in[0,T]} |\partial_t\mathcal{E}(t, U_{\boldsymbol{\tau}}(t))| \le \overline{C}, \tag{5.5a}$$

$$\mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{rate}}([0,T]) \le \overline{C}, \tag{5.5b}$$

$$\mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{slope}}([0,T]) \le \overline{C}, \tag{5.5c}$$

and the families $\left(\chi_{\boldsymbol{\tau}} U_{\boldsymbol{\tau}}'\right)_{\boldsymbol{\tau}\in\Lambda} \subset \mathrm{L}^1([0,T];\boldsymbol{X}_1)$ *and* $\left((1-\chi_{\boldsymbol{\tau}})U_{\boldsymbol{\tau}}'\right)_{\boldsymbol{\tau}\in\Lambda} \subset \mathrm{L}^1([0,T];\boldsymbol{X}_2)$ *are uniformly integrable; in particular,* $(U_{\boldsymbol{\tau}}')_{\boldsymbol{\tau}\in\Lambda} \subset \mathrm{L}^1([0,T];\boldsymbol{X}_1)$ *is uniformly integrable.*

Proof The energy-dissipation balance (5.4) follows on $[0,t]$ simply adding (5.2a) and (5.3a) over all relevant intervals. By subtracting the result for $[0,s]$ from that for $[0,t]$ the desired result for $[s,t]$ follows.

Estimates (5.5) follow from standard arguments (cf., e.g., [26, Prop. 6.3]), which rely on the power-control estimate (2.1) giving $\int_s^t \partial_t\mathcal{E}(r, U_{\boldsymbol{\tau}}(r))\mathrm{d}r \le C_\# \int_s^t \mathcal{E}(r, U_{\boldsymbol{\tau}}(r))\mathrm{d}r$. Hence, via Grönwall's lemma, from (5.4) we derive the energy estimate in (5.5a); the power estimate immediately follows via (2.1). From (5.4) we then immediately conclude (5.5b) and (5.5c).

From (5.5b), taking into account that the terms $\mathcal{R}_j\left(\frac{1}{2}U_{\boldsymbol{\tau}}'\right)$ contribute to $\mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{rate}}$ and that each $\mathcal{R}_j$ have superlinear growth, we deduce that the families $(\chi_{\boldsymbol{\tau}} U_{\boldsymbol{\tau}}')_{\boldsymbol{\tau}\in\Lambda}$ and $((1-\chi_{\boldsymbol{\tau}})U_{\boldsymbol{\tau}}')_{\boldsymbol{\tau}\in\Lambda}$ are uniformly integrable in $\mathrm{L}^1([0,T];\mathbf{X}_1)$ and $\mathrm{L}^1([0,T];\mathbf{X}_2)$, respectively. □

It is now convenient to rewrite the 'rate' and 'slope' terms featuring in (5.4) in terms of the repetition operators $\mathbb{T}_{\boldsymbol{\tau}}^{(j)}$ from (3.6). Their role, in the context of the present time-splitting scheme, is now clear: $\mathbb{T}_{\boldsymbol{\tau}}^{(1)}$ repeats "1-steps" and omits "2-steps", while $\mathbb{T}_{\boldsymbol{\tau}}^{(2)}$ does the converse. Trivial calculations based on the definition of the repetition operators identify the contributions to $\mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{rate}}$ and $\mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{slope}}$ with quantities involving the 'repeated rates' and the 'repeated forces', namely

$$\begin{aligned} \int_0^{t_{\boldsymbol{\tau}}^k} \chi_{\boldsymbol{\tau}}^{(j)}(r)\, \widetilde{\mathcal{R}}_j\,(V(r))\,\mathrm{d}r &= \int_0^{t_{\boldsymbol{\tau}}^k} \mathcal{R}_j(\tfrac{1}{2}\mathbb{T}_{\boldsymbol{\tau}}^{(j)} V(r))\mathrm{d}r \quad \text{for } V \in \mathrm{L}^1([0,T];\mathbf{X}_j), \\ \int_0^{t_{\boldsymbol{\tau}}^k} \chi_{\boldsymbol{\tau}}^{(j)}(r)\, \widetilde{\mathcal{R}}_j^*(\Xi(r))\mathrm{d}r &= \int_0^{t_{\boldsymbol{\tau}}^k} \mathcal{R}_j^*(\mathbb{T}_{\boldsymbol{\tau}}^{(j)} \Xi(r))\mathrm{d}r \qquad \text{for } \Xi \in \mathrm{L}^1([0,T];\mathbf{X}_j^*), \end{aligned} \tag{5.6}$$

where we have used the place-holder

$$\chi_{\boldsymbol{\tau}}^{(1)} := \chi_{\boldsymbol{\tau}} \quad \text{and} \quad \chi_{\boldsymbol{\tau}}^{(2)} := 1-\chi_{\boldsymbol{\tau}}.$$

Therefore, the rate and slope parts of the dissipation take the form

$$\mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{rate}}([0,t_{\boldsymbol{\tau}}^k]) = \int_0^{t_{\boldsymbol{\tau}}^k} \left\{ \mathcal{R}_1\left(\tfrac{1}{2}\mathbb{T}_{\boldsymbol{\tau}}^{(1)} U_{\boldsymbol{\tau}}'(r)\right) + \mathcal{R}_2\left(\tfrac{1}{2}\mathbb{T}_{\boldsymbol{\tau}}^{(2)} U_{\boldsymbol{\tau}}'(r)\right)\right\} \mathrm{d}r\,, \tag{5.7a}$$

$$\mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{slope}}([0,t_{\boldsymbol{\tau}}^k]) = \int_0^{t_{\boldsymbol{\tau}}^k} \left\{ \mathcal{R}_1^*\left(-\mathbb{T}_{\boldsymbol{\tau}}^{(1)} \xi_{\boldsymbol{\tau}}(r)\right) + \mathcal{R}_2^*\left(-\mathbb{T}_{\boldsymbol{\tau}}^{(2)} \xi_{\boldsymbol{\tau}}(r)\right)\right\} \mathrm{d}r\,. \tag{5.7b}$$

We stress that, in (5.7a) the terms $\mathbb{T}_{\boldsymbol{\tau}}^{(j)} U_{\boldsymbol{\tau}}'$ are the 'repeated rates' $\mathbb{T}_{\boldsymbol{\tau}}^{(j)}(U_{\boldsymbol{\tau}}')$, not to be confused with the rates of the repeated curves $\mathbb{T}_{\boldsymbol{\tau}}^{(j)} U_{\boldsymbol{\tau}}$. As a straightforward consequence of estimates (5.5b) and (5.5c), combined with (5.7) and the superlinear growth of $\mathcal{R}_j$ and $\mathcal{R}_j^*$, we have the following

Corollary 5.2 *For $j \in \{1,2\}$ the families $(\mathbb{T}_{\boldsymbol{\tau}}^{(j)} U_{\boldsymbol{\tau}}')_{\boldsymbol{\tau}\in\Lambda} \subset \mathrm{L}^1([0,T];\boldsymbol{X}_j)$ and $(\mathbb{T}_{\boldsymbol{\tau}}^{(j)} \xi_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\Lambda} \subset \mathrm{L}^1([0,T];\boldsymbol{X}_j^*)$ are uniformly integrable.*

Relying on Proposition 5.1 and Corollary 5.2 we obtain the following compactness result. In (5.8a) below we refer to the convergence in the space $\mathrm{C}^0([0,T];\mathbf{X}_{1,\mathrm{w}})$, whose meaning has been specified prior to the statement of Lemma 3.1.

Corollary 5.3 *Let $(\boldsymbol{\tau}_n)_n \subset \Lambda$ fulfill $\lim_{n\to\infty} |\boldsymbol{\tau}_n| = 0$. Then, there exist a (non-relabeled) subsequence and a limit curve $U \in AC([0,T];\boldsymbol{X}_1)$ such that the following convergences hold as $n \to \infty$:*

$$U_{\boldsymbol{\tau}_n} \to U \qquad \textit{in } \mathrm{C}^0([0,T];\boldsymbol{X}_{1,w}), \tag{5.8a}$$

$$U_{\boldsymbol{\tau}_n}' \rightharpoonup U' \qquad \textit{in } \mathrm{L}^1([0,T];\boldsymbol{X}_1), \tag{5.8b}$$

and

$$\liminf_{n\to\infty} \mathcal{E}(t, U_{\boldsymbol{\tau}_n}(t)) \geq \mathcal{E}(t, U(t)) \qquad \textit{for all } t \in [0,T], \tag{5.9a}$$

$$\partial_t \mathcal{E}(t, U_{\boldsymbol{\tau}_n}(t)) \to \partial_t \mathcal{E}(t, U(t)) \qquad \textit{for all } t \in [0,T]. \tag{5.9b}$$

Furthermore, for $j \in \{1, 2\}$ *there exist* $V_j \in \mathrm{L}^1([0,T]; \boldsymbol{X}_j)$ *and* $\bar{\bar{\xi}}_j \in \mathrm{L}^1([0,T]; \boldsymbol{X}_j^*)$ *such that, up to a further subsequence, we have as* $n \to \infty$

$$\frac{1}{2}\mathbb{T}^{(j)}_{\boldsymbol{\tau}_n} U'_{\boldsymbol{\tau}_n} \rightharpoonup V_j \qquad \text{in } \mathrm{L}^1([0,T]; \boldsymbol{X}_j), \tag{5.10a}$$

$$\mathbb{T}^{(j)}_{\boldsymbol{\tau}_n} \xi_{\boldsymbol{\tau}_n} \rightharpoonup \bar{\bar{\xi}}_j \qquad \text{in } \mathrm{L}^1([0,T]; \boldsymbol{X}_j^*), \tag{5.10b}$$

and there holds

$$V_1(t) + V_2(t) = U'(t) \qquad \text{for a.a. } t \in (0,T). \tag{5.11}$$

Proof Convergence (5.8b) follows from the uniform integrability of the family $(U'_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\Lambda} \subset \mathrm{L}^1([0,T]; \mathbf{X}_1)$, while (5.8a) ensues, e.g., from the Arzelà-Ascoli compactness type result in [2, Prop. 3.3.1]. Then, the energy and power convergences (5.9) follow from condition (2.17).

Now, Corollary 5.2 ensures that, up to a subsequence, the sequences $\left(\frac{1}{2}\mathbb{T}^{(j)}_{\boldsymbol{\tau}_n} U'_{\boldsymbol{\tau}_n}\right)_n$ and $(\mathbb{T}^{(j)}_{\boldsymbol{\tau}_n} \xi_{\boldsymbol{\tau}_n})_n$ have a weak limit in $\mathrm{L}^1([0,T]; \mathbf{X}_j)$ and $\mathrm{L}^1([0,T]; \mathbf{X}_j^*)$, respectively. Relation (5.11) follows from combining convergence (5.8b) with item (4) in Lemma 3.1. □

In the following sections we will address the passage to the limit in the ‘rate term’ from (3.19a) and the ‘slope term’ from (3.19b).

5.2 Liminf estimate for the rate term

We are going to prove the following

$$\textbf{Claim } 1: \qquad \liminf_{n\to\infty} \mathcal{D}^{\mathrm{rate}}_{\boldsymbol{\tau}_n}([0,T]) \geq \int_0^T \mathcal{R}_{\mathrm{eff}}(U'(r))\mathrm{d}r. \tag{5.12}$$

Indeed,

$$\begin{aligned} \liminf_{n\to\infty} \mathcal{D}^{\mathrm{rate}}_{\boldsymbol{\tau}_n}([0,T]) &\overset{(1)}{=} \liminf_{n\to\infty} \int_0^T \left\{\mathcal{R}_1(\tfrac{1}{2}\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n} U'_{\boldsymbol{\tau}_n}(r)) + \mathcal{R}_2(\tfrac{1}{2}\mathbb{T}^{(2)}_{\boldsymbol{\tau}_n} U'_{\boldsymbol{\tau}_n}(r))\right\} \mathrm{d}r \\ &\overset{(2)}{\geq} \int_0^T \{\mathcal{R}_1(V_1(r)) + \mathcal{R}_2(V_2(r))\}\,\mathrm{d}r \overset{(3)}{\geq} \int_0^T \mathcal{R}_{\mathrm{eff}}(U'(r))\mathrm{d}r, \end{aligned} \tag{5.13}$$

where (1) follows from (5.7a), (2) is due to convergences (5.10a) and the convexity and lower semicontinuity of $\mathcal{R}_j$, while (3) follows by property (5.11) and the definition of $\mathcal{R}_{\mathrm{eff}}$ as an infimal convolution. Hence, Claim 1 is established.

5.3 Liminf estimate for the slope term

Claim 2: *There exists* $\xi \in \mathrm{L}^1([0,T]; \mathbf{X}_1^*)$ *such that*

$$\partial^{\mathbf{X}_1}\mathcal{E}(t, U(t)) = \partial^{\mathbf{X}_2}\mathcal{E}(t, U(t)) = \{\xi(t)\} \quad \text{for a.a. } t \in (0,T), \tag{5.14a}$$

$$\liminf_{n\to\infty} \mathcal{D}^{\mathrm{slope}}_{\boldsymbol{\tau}_n}([0,T]) \geq \int_0^T \left\{\mathcal{R}_1^*(-\xi(r)) + \mathcal{R}_2^*(-\xi(r))\right\} \mathrm{d}r\,. \tag{5.14b}$$

Clearly, recalling (5.7b) and convergences (5.10b) we immediately have, by the convexity and lower semicontinuity of $\mathcal{R}_j^*$,

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{D}_{\boldsymbol{\tau}_n}^{\text{slope}}([0,T]) &= \liminf_{n\to\infty} \int_0^T \left\{\mathcal{R}_1^*(-\mathbb{T}_{\boldsymbol{\tau}_n}^{(1)}\xi_{\boldsymbol{\tau}_n}(r)) + \mathcal{R}_2^*(-\mathbb{T}_{\boldsymbol{\tau}_n}^{(2)}\xi_{\boldsymbol{\tau}_n}(r))\right\} \mathrm{d}r \\ &\geq \int_0^T \left\{\mathcal{R}_1^*(-\bar{\bar{\xi}}_1(r)) + \mathcal{R}_2^*(-\bar{\bar{\xi}}_2(r))\right\} \mathrm{d}r,\end{aligned} \tag{5.15}$$

where $\bar{\bar{\xi}}_j \in \mathrm{L}^1([0,T];\mathbf{X}_j^*)$ is the weak limit of the sequence $(\mathbb{T}_{\boldsymbol{\tau}_n}^{(j)}\xi_{\boldsymbol{\tau}_n})_n$, cf. Corollary 5.3. In the following lines we demonstrate that $\bar{\bar{\xi}}_1$ and $\bar{\bar{\xi}}_2$ coincide by resorting to the 'singleton condition' from Hypothesis 3.2. For this we use a *Young measure* argument.

With this aim, it will be convenient to introduce the 'repeated curves'

$$\mathbb{T}_{\boldsymbol{\tau}}^{(1)}U_{\boldsymbol{\tau}} : [0,T] \to \mathrm{D}_0, \qquad \mathbb{T}_{\boldsymbol{\tau}}^{(2)}U_{\boldsymbol{\tau}} : [0,T] \to \mathrm{D}_0, \tag{5.16}$$

with the convention that $\mathbb{T}_{\boldsymbol{\tau}}^{(1)}U_{\boldsymbol{\tau}}(t=0) = \mathbb{T}_{\boldsymbol{\tau}}^{(2)}U_{\boldsymbol{\tau}}(t=0) = u_0$. Note that $\mathbb{T}_{\boldsymbol{\tau}}^{(j)}U_{\boldsymbol{\tau}}$ may no longer be continuous, but we immediately observe that, since $U_{\boldsymbol{\tau}_n} \to U$ in $\mathrm{C}^0([0,T];\mathbf{X}_{1,\mathrm{w}})$, applying the continuity of the repetition operators (cf. item (3) in Lemma 3.1) we have

$$\mathbb{T}_{\boldsymbol{\tau}_n}^{(j)}U_{\boldsymbol{\tau}_n} \to U \text{ in } \mathrm{L}^\infty([0,T];\mathbf{X}_{1,\mathrm{w}}) \quad \text{for } j \in \{1,2\}. \tag{5.17}$$

By construction and using (5.2b) and (5.3b), it now follows that

$$\partial^{\mathbf{X}_j}\mathcal{E}(t,\mathbb{T}_{\boldsymbol{\tau}_n}^{(j)}U_{\boldsymbol{\tau}_n}(t)) = \{\mathbb{T}_{\boldsymbol{\tau}_n}^{(j)}\xi_{\boldsymbol{\tau}_n}(t)\} \quad \text{for a.a. } t \in (0,T), \qquad \text{for } j \in \{1,2\}. \tag{5.18}$$

Relying on (5.18) we will infer further information on the limits $\bar{\bar{\xi}}_j$.

For this, we resort to a Young-measure compactness result, [29, Thm. 3.2], which states that, up to a (non-relabeled) subsequence, the sequences $(\mathbb{T}_{\boldsymbol{\tau}_n}^{(1)}\xi_{\boldsymbol{\tau}_n})_n$ and $(\mathbb{T}_{\boldsymbol{\tau}_n}^{(2)}\xi_{\boldsymbol{\tau}_n})_n$ admit two limiting Young measures $(\mu_t^j)_{t\in(0,T)}$, with $\mu_t^1 \in \mathrm{Prob}(\mathbf{X}_1^*)$ and $\mu_t^2 \in \mathrm{Prob}(\mathbf{X}_2^*)$ for a.a. $t \in (0,T)$, enjoying the following properties for $j \in \{1,2\}$:

1. the supports of the measures μ_t^j are contained in the set of the limit points of the sequences $(\mathbb{T}_{\boldsymbol{\tau}_n}^{(j)}\xi_{\boldsymbol{\tau}_n}(t))_n$ in the weak topology of $\mathbf{X}_j^*$, i.e. for a.a. $t \in (0,T)$ we have

$$\mathrm{supp}(\mu_t^j) \subset \mathrm{Ls}_{\mathbf{X}_j^*}^{\text{weak}}\big(\{\mathbb{T}_{\boldsymbol{\tau}_n}^{(j)}\xi_{\boldsymbol{\tau}_n}(t)\}_n\big) := \bigcap_{k\geq 1} \overline{\{\mathbb{T}_{\boldsymbol{\tau}_l}^{(j)}\xi_{\boldsymbol{\tau}_l}(t) : l \geq k\}}^{\text{weak}}, \tag{5.19a}$$

   where the notation refers to the notion of lim sup in the sense of the Kuratowski convergence of sets, cf. e.g. [4];
2. the weak limits $\bar{\bar{\xi}}_j$ of $(\mathbb{T}_{\boldsymbol{\tau}_n}^{(j)}\xi_{\boldsymbol{\tau}_n})_n$ in $\mathrm{L}^1([0,T];\mathbf{X}_j^*)$ coincide with the barycenters of the measures μ_t^j, namely

$$\bar{\bar{\xi}}_j(t) = \int_{X^*} \zeta \,\mathrm{d}\mu_t^j(\zeta) \quad \text{for a.a. } t \in (0,T). \tag{5.19b}$$

It turns out that $\mathrm{supp}(\mu_t^j)$ is a singleton for $j = 1,2$. Indeed, using the convergence of $\mathbb{T}_{\boldsymbol{\tau}_n}^{(j)}U_{\boldsymbol{\tau}_n} \to U$ in $\mathrm{L}^\infty([0,T];\mathbf{X}_{1,\mathrm{w}})$ by (5.17), the subdifferential inclusions (5.18), and the closedness property (2.19), we have that $\mathrm{Ls}_{\mathbf{X}_j^*}^{\text{weak}}\big(\{\mathbb{T}_{\boldsymbol{\tau}_n}^{(j)}\xi_{\boldsymbol{\tau}_n}(t)\}_n\big) \subset \partial^{\mathbf{X}_j}\mathcal{E}(t,U(t))$ for a.a. $t \in (0,T)$ and $j \in \{1,2\}$. From (5.19a) we then infer

$$\mathrm{supp}(\mu_t^1) = \mathrm{supp}(\mu_t^2) = \mathrm{Ls}^{\mathrm{weak}}_{\mathbf{X}_j^*}\big(\{\mathbb{T}^{(j)}_{\boldsymbol{\tau}_n}\xi_{\boldsymbol{\tau}_n}(t)\}_n\big) = \partial^{\mathbf{X}_j}\mathcal{E}(t,U(t)) = \{\xi(t)\} \quad \text{for a.a. } t\in(0,T).$$

By (5.19b) we then conclude that $\bar{\bar{\xi}}_1 = \xi = \bar{\bar{\xi}}_2$. Hence, the liminf estimate (5.14b) follows from the previously observed (5.15), and Claim 2 is established.

5.4 Conclusion of the proof of Theorem 3.5

Relying on convergences (5.8), on the lower semicontinuity and continuity properties of $\mathcal{E}$ and $\partial_t\mathcal{E}$ (cf. Hypothesis 2.4), and on the lower semicontinuity estimates (5.12) from Claim 1 and (5.14) from Claim 2, we are in a position to take the limit in the energy-dissipation balance (5.4), written on the interval $[0,T]$, and thus conclude the validity of the energy-dissipation inequality (5.1) along a curve $U\in \mathrm{AC}([0,T];\mathbf{X}_1)$ and $\xi\in \mathrm{L}^1([0,T];\mathbf{X}_1^*)$. By lower semicontinuity, we immediately have $\sup_{t\in[0,T]}\mathfrak{E}(U(t))\le \overline{C}$ with $\overline{C}>0$ from (5.5a), and thus $\sup_{t\in[0,T]}|\partial_t\mathcal{E}(t,U(t))|\le C$. Therefore, from (5.1) we infer that

$$\int_0^T \big(\mathcal{R}_{\mathrm{eff}}(U'(r)) + \mathcal{R}^*_{\mathrm{eff}}(-\xi(r))\big)\mathrm{d}r < \infty.$$

Since Hypothesis 3.3 provides the chain rule for the effective system, we deduce that $t\mapsto \mathcal{E}(t,U(t))$ is absolutely continuous on $[0,T]$, and that the chain rule formula (2.3) holds for the pair (U,ξ). Then, Proposition 2.1 allows us to conclude that (U,ξ) solves (3.15) and fulfills the energy-dissipation balance (3.16).

Now, it remains to show property (3.17), i.e. the weak limits V_j of $\big(\frac{1}{2}\mathbb{T}^{(j)}_{\boldsymbol{\tau}_n}U'_{\boldsymbol{\tau}_n}\big)_n$ provide an *optimal decomposition* of U' in the sense that $V_1+V_2=U'$ as well as $\mathcal{R}_1(V_1)+\mathcal{R}_2(V_2) = \mathcal{R}_{\mathrm{eff}}(U')$ a.e. in $(0,T)$.

To see this, we first observe that the limit passage from the approximate energy-dissipation balance (5.4) to the limit energy-dissipation balance (3.16) on the interval $[0,T]$ ultimately implies that the liminf estimates (5.12) and (5.14b) turn into convergences (see e.g. [26, Thm. 4.4] or [27, Thm. 3.11] for the standard argument). In particular, combining the convergences for the rate term with (5.13) we conclude

$$\begin{aligned}\int_0^T \mathcal{R}_{\mathrm{eff}}(U'(r))\mathrm{d}r = \lim_{n\to\infty}\mathcal{D}^{\mathrm{rate}}_{\boldsymbol{\tau}_n}([0,T]) &\ge \int_0^T\{\mathcal{R}_1(V_1(r)) + \mathcal{R}_2(V_2(r))\}\mathrm{d}r\\ &\ge \int_0^T \mathcal{R}_{\mathrm{eff}}(U'(r))\mathrm{d}r,\end{aligned}$$

which turns the above relations into a chain of equalities. Hence, from

$$\lim_{n\to\infty}\int_0^T\Big\{\mathcal{R}_1\big(\tfrac{1}{2}\mathbb{T}^{(1)}_{\boldsymbol{\tau}}U'_{\boldsymbol{\tau}}(r)\big) + \mathcal{R}_2\big(\tfrac{1}{2}\mathbb{T}^{(2)}_{\boldsymbol{\tau}}U'_{\boldsymbol{\tau}}(r)\big)\Big\}\,\mathrm{d}r = \int_0^T\{\mathcal{R}_1(V_1(r)) + \mathcal{R}_2(V_2(r))\}\,\mathrm{d}r$$

we obtain the individual convergences (3.21) on the interval $[0,T]$, as the liminf of each integral term on the left-hand side is estimated from below by the corresponding term on the right-hand side.

Finally, recall that $U' = V_1+V_2$ a.e. in $(0,T)$, so that $\mathcal{R}_{\mathrm{eff}}(U')\le \mathcal{R}_1(V_1)+\mathcal{R}_2(V_2)$ a.e. in $(0,T)$. Combining this with the fact that $\int\mathcal{R}_{\mathrm{eff}}(U')\mathrm{d}r$ equals $\int\{\mathcal{R}_1(V_1)+\mathcal{R}_2(V_2)\}\mathrm{d}r$ ultimately leads to the desired optimality property $\mathcal{R}_{\mathrm{eff}}(U') = \mathcal{R}_1(V_1)+\mathcal{R}_2(V_2)$ a.e. in $(0,T)$. Hence, we conclude the validity of (3.17), and thus, the proof of Theorem 3.5. □

Finally, we briefly comment on the enhanced convergences (3.20). It is clear that it suffices to show the convergence results on intervals of the form $[0, t]$. The technical issue arises because a general $t \in (0, T)$ is in general not aligned with the partition $\mathscr{P}_{\boldsymbol{\tau}}$ of (3.1).

To show the enhanced convergence, we recall the time-interpolants $\bar{\mathfrak{t}}_{\boldsymbol{\tau}} : [0, T] \to [0, T]$ from (3.4), which satisfy $\bar{\mathfrak{t}}_{\boldsymbol{\tau}}(t) \to t$ for $\boldsymbol{\tau} \to 0$. With this, we repeat the argument from above for proving the convergence of the rate and slope terms, while passing from the approximate energy-dissipation balance (5.4) on $[0, t^k_{\boldsymbol{\tau}}]$ to the limit energy-dissipation balance (3.16) on the interval $[0, t]$. Using the liminf estimates for energy and powers in (5.9a) and (5.9b), it again suffices to show a liminf estimate for the rate and slope terms on intervals $[0, t]$. For convenience, we consider only the rate term, because the slope can be treated analogously. In particular, it suffices to show the following liminf estimate:

$$\liminf_{\boldsymbol{\tau}\to 0} \int_0^{\bar{\mathfrak{t}}_{\boldsymbol{\tau}}(t)} \mathcal{R}_j(\tfrac{1}{2}\mathbb{T}^{(j)}_{\boldsymbol{\tau}} U'_{\boldsymbol{\tau}}(r))\mathrm{d}r \geq \int_0^t \mathcal{R}_j(V_j(r))\mathrm{d}r,$$

where $\frac{1}{2}\mathbb{T}^{(j)}_{\boldsymbol{\tau}} U'_{\boldsymbol{\tau}} =: V^{(j)}_{\boldsymbol{\tau}} \rightharpoonup V_j$ in $\mathrm{L}^1([0, T]; \mathbf{X}_j)$. Introducing $W^{(j)}_{\boldsymbol{\tau}} := \chi_{[0,\bar{\mathfrak{t}}_{\boldsymbol{\tau}}(t)]} V^{(j)}_{\boldsymbol{\tau}} \in \mathrm{L}^1([0, T]; \mathbf{X}_j)$, we get

$$\begin{aligned}\int_0^{\bar{\mathfrak{t}}_{\boldsymbol{\tau}}(t)} \mathcal{R}_j(\tfrac{1}{2}\mathbb{T}^{(j)}_{\boldsymbol{\tau}} U'_{\boldsymbol{\tau}}(r))\mathrm{d}r &= \int_0^T \chi_{[0,\bar{\mathfrak{t}}_{\boldsymbol{\tau}}(t)]}(r)\mathcal{R}_j(V^{(j)}_{\boldsymbol{\tau}}(r))\mathrm{d}r \\ &= \int_0^T \mathcal{R}_j(\chi_{[0,\bar{\mathfrak{t}}_{\boldsymbol{\tau}}(t)]}(r)V^{(j)}_{\boldsymbol{\tau}}(r))\mathrm{d}r = \int_0^T \mathcal{R}_j(W^{(j)}_{\boldsymbol{\tau}}(r))\mathrm{d}r,\end{aligned}$$

where we used $\mathcal{R}_j(0) = 0$. Moreover, $\|W^{(j)}_{\boldsymbol{\tau}}(r)\|_{\mathbf{X}_j} \leq \|V^{(j)}_{\boldsymbol{\tau}}\|_{\mathbf{X}_j}$ allows us to apply the compactness argument in Corollary 5.2 such that there is $W_j \in \mathrm{L}^1([0, T]; \mathbf{X}_j)$ with $W^{(j)}_{\boldsymbol{\tau}} \rightharpoonup W_j$ in $\mathrm{L}^1([0, T]; \mathbf{X}_j)$. Using that $\bar{\mathfrak{t}}_{\boldsymbol{\tau}}(t) \to t$ as $\boldsymbol{\tau} \to 0$, we find $W_j = \chi_{[0,t]} V_j$ and obtain

$$\begin{aligned}\liminf_{\boldsymbol{\tau}\to 0} \int_0^{\bar{\mathfrak{t}}_{\boldsymbol{\tau}}(t)} \mathcal{R}_j(\tfrac{1}{2}\mathbb{T}^{(j)}_{\boldsymbol{\tau}} U'_{\boldsymbol{\tau}}(r))\mathrm{d}r &= \liminf_{\boldsymbol{\tau}\to 0} \int_0^T \mathcal{R}_j(W^{(j)}_{\boldsymbol{\tau}}(r))\mathrm{d}r \\ \geq \int_0^T \mathcal{R}_j(W_j(r))\mathrm{d}r &= \int_0^T \mathcal{R}_j(\chi_{[0,t]} V_j(r))\mathrm{d}r = \int_0^t \mathcal{R}_j(V_j(r))\mathrm{d}r\,.\end{aligned}$$

With this, the enhanced convergences (3.20) and (3.21) are established.

6 Alternating Minimizing Movements

In this section we discuss an extension of our convergence result in Theorem 3.5. Namely, we show that the splitting scheme can be combined with the Minimizing-Movement approximations of the single-dissipation gradient systems $(\mathbf{X}_j, \mathcal{E}, \widetilde{\mathcal{R}}_j)$ with $\widetilde{\mathcal{R}}_j = 2\mathcal{R}_j(\frac{1}{2}\,\cdot)$.

6.1 Setup and convergence result

More precisely, for each $j = 1, 2$ we set up the Minimizing Movement scheme and construct discrete solutions to the subdifferential inclusions (3.10) and (3.11) by solving the time-incremental minimization problems involving the rescaled potentials $\widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_2$ in an alternating manner. Then, we define approximate solutions by suitably interpolating the discrete solutions.

Hence, let $\mathscr{P}_{\boldsymbol{\tau}}$ be a (possibly non-uniform) partition as in (3.1); recall that the sub-interval $(t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^{k}]$ is split into semi-intervals via

$$(t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^{k}] = \mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}} \cup \mathrm{I}_{\mathrm{right}}^{k,\boldsymbol{\tau}} \quad \text{with } \mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}} = \big(t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^{k-1/2}\big] \text{ and } \mathrm{I}_{\mathrm{right}}^{k,\boldsymbol{\tau}} = \big(t_{\boldsymbol{\tau}}^{k-1/2}, t_{\boldsymbol{\tau}}^{k}\big],$$

where $t_{\boldsymbol{\tau}}^{k-1/2} = t_{\boldsymbol{\tau}}^{k-1} + \frac{\tau_k}{2}$. Starting from an initial datum $u_0 \in \mathrm{D}_0$, we define the *piecewise constant* time-discrete solutions $\overline{U}_{\boldsymbol{\tau}} : [0,T] \to \mathrm{D}_0$ in the following way: We set $\overline{U}_{\boldsymbol{\tau}}(0) := u_0$ and, for $t \in (0,T)$, we define

$$\begin{aligned}
&\text{for } t \in \mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}}: \overline{U}_{\boldsymbol{\tau}}(t) := U_k^1 \text{ with} \\
&U_k^1 \in \operatorname*{Argmin}_{U \in \mathbf{X}_1} \left\{ \tfrac{\tau_k}{2} \widetilde{\mathcal{R}}_1\Big(\tfrac{2}{\tau_k} \Big(U - \overline{U}_{\boldsymbol{\tau}}\Big(t_{\boldsymbol{\tau}}^{k-1}\Big)\Big)\Big) + \mathcal{E}(t_{\boldsymbol{\tau}}^{k-1/2}, U) \right\}, \qquad (6.1a)\\
&\text{for } t \in \mathrm{I}_{\mathrm{right}}^{k,\boldsymbol{\tau}}: \overline{U}_{\boldsymbol{\tau}}(t) := U_k^2 \text{ with} \\
&U_k^2 \in \operatorname*{Argmin}_{U \in \mathbf{X}_2} \left\{ \tfrac{\tau_k}{2} \widetilde{\mathcal{R}}_2\Big(\tfrac{2}{\tau_k} \Big(U - \overline{U}_{\boldsymbol{\tau}}\Big(t_{\boldsymbol{\tau}}^{k-1/2}\Big)\Big)\Big) + \mathcal{E}(t_{\boldsymbol{\tau}}^{k}, U) \right\}. \qquad (6.1b)
\end{aligned}$$

We also define the piecewise constant interpolant $\underline{U}_{\boldsymbol{\tau}} : [0,T] \to \mathrm{D}_0$ by

$$\underline{U}_{\boldsymbol{\tau}}(t) := \begin{cases} u_0 & \text{for } t \in (0, \frac{\tau_1}{2}], \\ \overline{U}_{\boldsymbol{\tau}}\Big(t - \frac{\widetilde{\tau}(t)}{2}\Big) & \text{for } t \in (\frac{\tau_1}{2}, T] \end{cases} \tag{6.2}$$

(cf. (3.5) for the notation $\widetilde{\tau}(t)$).

Furthermore, we introduce the *piecewise linear* interpolant $\widehat{U}_{\boldsymbol{\tau}} : [0,T] \to \mathrm{D}_0$, i.e.

$$\widehat{U}_{\boldsymbol{\tau}}(t) := \begin{cases} \dfrac{t - t_{\boldsymbol{\tau}}^{k-1}}{\tau_k/2} \overline{U}_{\boldsymbol{\tau}}(t) + \dfrac{t_{\boldsymbol{\tau}}^{k-1/2} - t}{\tau_k/2} \underline{U}_{\boldsymbol{\tau}}(t) & \text{for } t \in [t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^{k-1/2}], \\ \dfrac{t - t_{\boldsymbol{\tau}}^{k-1/2}}{\tau_k/2} \underline{U}_{\boldsymbol{\tau}}(t) + \dfrac{t_{\boldsymbol{\tau}}^{k} - t}{\tau_k/2} \overline{U}_{\boldsymbol{\tau}}(t) & \text{for } t \in [t_{\boldsymbol{\tau}}^{k-1/2}, t_{\boldsymbol{\tau}}^{k}]. \end{cases} \tag{6.3}$$

Thus, the piecewise constant and linear interpolants satisfy the Euler-Lagrange equations for the minimum problems (6.1), namely

$$\begin{aligned}
&\partial\widetilde{\mathcal{R}}_1\big(\widehat{U}_{\boldsymbol{\tau}}'(t)\big) + \partial^{\mathbf{X}_1}\mathcal{E}(t_{\boldsymbol{\tau}}^{k-1/2}, \overline{U}_{\boldsymbol{\tau}}(t)) \ni 0 \qquad \text{in } \mathbf{X}_1^* \quad \text{for a.a. } t \in \mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}}, \\
&\partial\widetilde{\mathcal{R}}_2\big(\widehat{U}_{\boldsymbol{\tau}}'(t)\big) + \partial^{\mathbf{X}_2}\mathcal{E}(t_{\boldsymbol{\tau}}^{k}, \overline{U}_{\boldsymbol{\tau}}(t)) \ni 0 \qquad \text{in } \mathbf{X}_2^* \quad \text{for a.a. } t \in \mathrm{I}_{\mathrm{right}}^{k,\boldsymbol{\tau}}.
\end{aligned}$$

Nonetheless, like in the time-continuous setup we will not directly pass to the limit in the above inclusions but instead resort to a discrete energy-dissipation inequality that will act as a proxy of the energy-dissipation balance (5.4) and will be at the core of the proof of the following convergence result.

Theorem 6.1 (Alternating Minimizing Movements) *In addition to the assumptions of Theorem* 2.10*, assume the singleton condition in Hypothesis* 3.2 *and replace Hypothesis* 2.6 *(for* $(\boldsymbol{X}_j, \mathcal{E}, \mathcal{R}_j)$*) by the chain rule in Hypothesis* 3.3 *for* $(\boldsymbol{X}_1, \mathcal{E}, \mathcal{R}_{\mathrm{eff}})$*. Starting from an initial datum* $u_0 \in D_0$*, define the curves* $(\overline{U}_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\Lambda}$ *and* $(\widehat{U}_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\Lambda}$ *as in* (6.1) *and* (6.3)*.*

Then, for all sequences $(\boldsymbol{\tau}_n)_n$ *with* $\lim_{n\to\infty}|\boldsymbol{\tau}_n| = 0$ *there exist a (non-relabeled) subsequence, a curve* $U \in AC([0,T]; \boldsymbol{X}_1)$ *and functions* $V_j \in \mathrm{L}^1([0,T]; \boldsymbol{X}_j)$*,* $j = 1, 2$*, such that*

$U(0) = u_0$, *the following convergences hold (for* $n \to \infty$*)*

$$\overline{U}_{\boldsymbol{\tau}_n}(t) \rightharpoonup U(t) \ \ \textit{and} \ \ \widehat{U}_{\boldsymbol{\tau}_n}(t) \rightharpoonup U(t) \qquad \textit{in } \boldsymbol{X}_1 \qquad \textit{for all } t \in [0,T], \tag{6.4a}$$

$$\frac{1}{2}\mathbb{T}^{(j)}_{\boldsymbol{\tau}_n}(\widehat{U}'_{\boldsymbol{\tau}_n}) \rightharpoonup V_j \qquad \textit{in } \mathrm{L}^1([0,T];\boldsymbol{X}_j) \qquad \textit{for } j=1,2, \tag{6.4b}$$

$$\widehat{U}'_{\boldsymbol{\tau}_n} \rightharpoonup U' = V_1 + V_2 \qquad \textit{in } \mathrm{L}^1([0,T];\boldsymbol{X}_1), \tag{6.4c}$$

and there exists a function $\xi \in \mathrm{L}^1([0,T];\boldsymbol{X}_1^*)$ *such that the pair* (U,ξ) *solves the subdifferential inclusion* (3.15) *and fulfills the energy-dissipation balance* (3.16). *Furthermore, the functions* (V_1, V_2) *provide an* optimal decomposition *for* U' *in the sense of* (3.17).

Remark 6.2 A standard way for constructing solutions to the subdifferential inclusion (3.15), also viable under the present conditions, would be through the Minimizing Movement scheme for the gradient system $(\mathbf{X}_1, \mathcal{E}, \mathcal{R}_{\mathrm{eff}})$, featuring at the k-th step the minimum problem

$$\min_{U \in \mathbf{X}_1} \left\{ \tau_k\, \mathcal{R}_{\mathrm{eff}}\left(\tfrac{1}{\tau_k}(U - U_{\boldsymbol{\tau}}^{k-1})\right) + \mathcal{E}(t_{\boldsymbol{\tau}}^k, U) \right\}. \tag{6.5}$$

Since

$$\mathcal{R}_{\mathrm{eff}}\left(\tfrac{1}{\tau_k}(U - U_{\boldsymbol{\tau}}^{k-1})\right) = \min_{W} \left\{ \mathcal{R}_1\left(\tfrac{1}{\tau_k}(W - U_{\boldsymbol{\tau}}^{k-1})\right) + \mathcal{R}_2\left(\tfrac{1}{\tau_k}(U - W)\right) \right\}$$

the minimization scheme (6.5) reformulates as

$$\min_{U,\, W \in \mathbf{X}_1} \left\{ \tau_k\, \mathcal{R}_1\left(\tfrac{1}{\tau_k}(W - U_{\boldsymbol{\tau}}^{k-1})\right) + \tau_k\, \mathcal{R}_2\left(\tfrac{1}{\tau_k}(U - W)\right) + \mathcal{E}(t_{\boldsymbol{\tau}}^k, U) \right\}, \tag{6.6}$$

which produces two discrete solutions $U_{\boldsymbol{\tau}}^{k-1/2} := W$ and $U_{\boldsymbol{\tau}}^k := U$.

Observe that the minimization scheme (6.6) does not define a split-step method because one has to handle both dissipation potentials at the same time.

6.2 Proof of Theorem 6.1

We start by deriving the discrete analogue (in fact, an *inequality*) of the energy-dissipation balance (5.4). For this, we need to bring into the picture a further interpolant, commonly known as the *variational interpolant*, which was first introduced in the framework of the Minimizing Movement theory for metric gradient flows by E. De Giorgi, cf. [2, 3]. In the present context, the interpolant $\widetilde{U}_{\boldsymbol{\tau}} : [0,T] \to \mathrm{D}_0$ is defined in the following way: $\widetilde{U}_{\boldsymbol{\tau}}(0) := u_0$ and, for $t > 0$,

$$\begin{aligned} &\text{for } t \in \mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}},\ \ t = t_{\boldsymbol{\tau}}^{k-1} + r: \quad \widetilde{U}_{\boldsymbol{\tau}}(t) \in \operatorname*{Argmin}_{U \in X} \left\{ r \widetilde{\mathcal{R}}_1\left(\tfrac{1}{r}\left(U - \overline{U}_{\boldsymbol{\tau}}(t_{\boldsymbol{\tau}}^{k-1})\right)\right) + \mathcal{E}(t,U) \right\}; \\ &\text{for } t \in \mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}},\ \ t = t_{\boldsymbol{\tau}}^{k-1/2} + r: \quad \widetilde{U}_{\boldsymbol{\tau}}(t) \in \operatorname*{Argmin}_{U \in X} \left\{ r \widetilde{\mathcal{R}}_2\left(\tfrac{1}{r}\left(U - \overline{U}_{\boldsymbol{\tau}}(t_{\boldsymbol{\tau}}^{k-1/2})\right)\right) + \mathcal{E}(t,U) \right\}. \end{aligned} \tag{6.7}$$

The existence of a *measurable* selection in the sets of minimizers in (6.7) follows by, e.g., [7, Cor. III.3, Thm. III.6]. Since, for $t = t_{\boldsymbol{\tau}}^{k-1/2}$ and for $t = t_{\boldsymbol{\tau}}^k$ the minimum problems in (6.1) coincide with those in (6.7), we may assume that

$$\overline{U}_{\boldsymbol{\tau}}(s) = \underline{U}_{\boldsymbol{\tau}}(s) = \widehat{U}_{\boldsymbol{\tau}}(s) = \widetilde{U}_{\boldsymbol{\tau}}(s) \qquad \text{for } s = t_{\boldsymbol{\tau}}^{k-1/2},\ s = t_{\boldsymbol{\tau}}^k \ \text{ and } k = 1, \ldots, N_{\boldsymbol{\tau}}.$$

Furthermore, by [5, Thm. 8.2.9], with $\widetilde{U}_{\boldsymbol{\tau}}$ we can associate a measurable function $\widetilde{\xi}_{\boldsymbol{\tau}} : (0,T) \to \mathbf{X}_2^*$ fulfilling the Euler equation for the minimum problems (6.7), i.e.

$$\widetilde{\xi}_{\boldsymbol{\tau}}(t) \in \begin{cases} \partial^{\mathbf{X}_1}\mathcal{E}(t,\widetilde{U}_{\boldsymbol{\tau}}(t)) \cap \left(-\partial\widetilde{\mathcal{R}}_1\left(\frac{1}{t-t_{\boldsymbol{\tau}}^{k-1}}\left(\widetilde{U}_{\boldsymbol{\tau}}(t)-\underline{U}_{\boldsymbol{\tau}}(t_{\boldsymbol{\tau}}^{k-1})\right)\right)\right) & \text{for } t \in \mathrm{I}^{k,\boldsymbol{\tau}}_{\text{left}}, \\ \partial^{\mathbf{X}_2}\mathcal{E}(t,\widetilde{U}_{\boldsymbol{\tau}}(t)) \cap \left(-\partial\widetilde{\mathcal{R}}_2\left(\frac{1}{t-t_{\boldsymbol{\tau}}^{k-1/2}}\left(\widetilde{U}_{\boldsymbol{\tau}}(t)-\underline{U}_{\boldsymbol{\tau}}(t_{\boldsymbol{\tau}}^{k-1/2})\right)\right)\right) & \text{for } t \in \mathrm{I}^{k,\boldsymbol{\tau}}_{\text{right}}, \end{cases}$$

for $k=1,\ldots,N_{\boldsymbol{\tau}}$. Then, we may apply [26, Lem. 6.1] (see also [22]), and conclude that that interpolants $\overline{U}_{\boldsymbol{\tau}}$, $\widehat{U}_{\boldsymbol{\tau}}$, $\widetilde{U}_{\boldsymbol{\tau}}$, and $\widetilde{\xi}_{\boldsymbol{\tau}}$ fulfill, on the semi-intervals $\mathrm{I}^{k,\boldsymbol{\tau}}_{\text{left}} = (t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^{k-1/2}]$, the following estimate

$$\begin{aligned} &\mathcal{E}(t_{\boldsymbol{\tau}}^{k-1/2}, \overline{U}_{\boldsymbol{\tau}}(t_{\boldsymbol{\tau}}^{k-1/2})) + \int_{t_{\boldsymbol{\tau}}^{k-1}}^{t_{\boldsymbol{\tau}}^{k-1/2}} \left\{\widetilde{\mathcal{R}}_1(\widehat{U}_{\boldsymbol{\tau}}'(r)) + \widetilde{\mathcal{R}}_1^*(-\widetilde{\xi}_{\boldsymbol{\tau}}(r))\right\} \mathrm{d}r \\ &\le \mathcal{E}(t_{\boldsymbol{\tau}}^{k-1}, \overline{U}_{\boldsymbol{\tau}}(t_{\boldsymbol{\tau}}^{k-1})) + \int_{t_{\boldsymbol{\tau}}^{k-1}}^{t_{\boldsymbol{\tau}}^{k-1/2}} \partial_t\mathcal{E}(r,\widetilde{U}_{\boldsymbol{\tau}}(r))\mathrm{d}r. \end{aligned} \tag{6.8}$$

The analogue holds on $\mathrm{I}^{k,\boldsymbol{\tau}}_{\text{right}} = (t_{\boldsymbol{\tau}}^{k-1/2}, t_{\boldsymbol{\tau}}^{k}]$, involving the dissipation potential $\widetilde{\mathcal{R}}_2$.

Relying on (6.8) and its analogue on the intervals $\mathrm{I}^{k,\boldsymbol{\tau}}_{\text{right}}$, we can deduce the discrete energy-dissipation inequality (6.9) below, replacing the time-continuous energy-dissipation balance (5.4). In order to state it in a compact form, we introduce the discrete analogues of the *rate* and *slope* terms from (3.19). With slight abuse, we will denote them with the same symbols used in (3.19):

$$\begin{aligned} \mathcal{D}_{\boldsymbol{\tau}}^{\text{rate}}([0,T]) &:= \int_0^T \left\{\chi_{\boldsymbol{\tau}}(r)\,\widetilde{\mathcal{R}}_1\big(\widehat{U}_{\boldsymbol{\tau}}'(r)\big) + (1-\chi_{\boldsymbol{\tau}}(r))\,\widetilde{\mathcal{R}}_2\big(\widehat{U}_{\boldsymbol{\tau}}'(r)\big)\right\} \mathrm{d}r \\ &\overset{(5.7a)}{=} \int_0^T \left\{\mathcal{R}_1(\tfrac{1}{2}\mathbb{T}_{\boldsymbol{\tau}}^{(1)}\widehat{U}_{\boldsymbol{\tau}}'(r)) + \mathcal{R}_2(\tfrac{1}{2}\mathbb{T}_{\boldsymbol{\tau}}^{(2)}\widehat{U}_{\boldsymbol{\tau}}'(r))\right\} \mathrm{d}r, \\ \mathcal{D}_{\boldsymbol{\tau}}^{\text{slope}}([0,T]) &:= \int_0^T \left\{\chi_{\boldsymbol{\tau}}(r)\,\widetilde{\mathcal{R}}_1^*(-\widetilde{\xi}_{\boldsymbol{\tau}}(r)) + (1-\chi_{\boldsymbol{\tau}}(r))\,\widetilde{\mathcal{R}}_2^*(-\widetilde{\xi}_{\boldsymbol{\tau}}(r))\right\} \mathrm{d}r \\ &\overset{(5.7b)}{=} \int_0^T \left\{\mathcal{R}_1^*(-\mathbb{T}_{\boldsymbol{\tau}}^{(1)}\widetilde{\xi}_{\boldsymbol{\tau}}(r)) + \mathcal{R}_2^*(-\mathbb{T}_{\boldsymbol{\tau}}^{(2)}\widetilde{\xi}_{\boldsymbol{\tau}}(r))\right\} \mathrm{d}r\,. \end{aligned}$$

The following result (to be compared with Proposition 5.1 and Corollary 5.2) collects all of the a priori estimates stemming from (6.9).

Proposition 6.3 *The interpolants* $\overline{U}_{\boldsymbol{\tau}}$, $\widehat{U}_{\boldsymbol{\tau}}$, *and* $\widetilde{U}_{\boldsymbol{\tau}}$ *and* $\widetilde{\xi}_{\boldsymbol{\tau}}$ *fulfill the* discrete energy-dissipation inequality

$$\mathcal{E}(t,\overline{U}_{\boldsymbol{\tau}}(t)) + \mathcal{D}_{\boldsymbol{\tau}}^{\text{rate}}([s,t]) + \mathcal{D}_{\boldsymbol{\tau}}^{\text{slope}}([s,t]) \le \mathcal{E}(s,\overline{U}_{\boldsymbol{\tau}}(s)) + \int_s^t \partial_t\mathcal{E}(r,\widetilde{U}_{\boldsymbol{\tau}}(r))\mathrm{d}r \tag{6.9}$$

for all $0 \le s \le t \le T$. *Moreover, there exists a positive constant* $\overline{C} > 0$ *such that the following estimates hold for all* $\boldsymbol{\tau} \in \Lambda$*:*

$$\sup_{t\in[0,T]} \mathfrak{E}(\overline{U}_{\boldsymbol{\tau}}(t)) \le \overline{C}, \qquad \sup_{t\in[0,T]} \mathfrak{E}(\widetilde{U}_{\boldsymbol{\tau}}(t)) \le \overline{C}, \qquad \sup_{t\in[0,T]} |\partial_t\mathcal{E}(t,\widetilde{U}_{\boldsymbol{\tau}}(t))| \le \overline{C}, \tag{6.10a}$$

$$\mathcal{D}_{\boldsymbol{\tau}}^{\text{rate}}([0,T]) + \mathcal{D}_{\boldsymbol{\tau}}^{\text{slope}}([0,T]) \le \overline{C}. \tag{6.10b}$$

Thus, the families $(\widehat{U}_{\boldsymbol{\tau}}')_{\boldsymbol{\tau}\in\Lambda} \subset \mathrm{L}^1([0,T];\boldsymbol{X}_1)$, $(\mathbb{T}_{\boldsymbol{\tau}}^{(j)}\widehat{U}_{\boldsymbol{\tau}}')_{\boldsymbol{\tau}\in\Lambda} \subset \mathrm{L}^1([0,T];\boldsymbol{X}_j)$, *and* $(\mathbb{T}_{\boldsymbol{\tau}}^{(j)}\widetilde{\xi}_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\Lambda} \subset \mathrm{L}^1([0,T];\boldsymbol{X}_j^*)$, *for* $j \in \{1,2\}$, *are uniformly integrable. Finally, as* $|\boldsymbol{\tau}| \downarrow 0$, *we have*

$$\sup_{t\in[0,T]} \|\overline{U}_{\boldsymbol{\tau}}(t)-\underline{U}_{\boldsymbol{\tau}}(t)\|_{\mathrm{L}^1} + \sup_{t\in[0,T]} \|U_{\boldsymbol{\tau}}(t)-\overline{U}_{\boldsymbol{\tau}}(t)\|_{\mathrm{L}^1} + \sup_{t\in[0,T]} \|\widetilde{U}_{\boldsymbol{\tau}}(t)-\underline{U}_{\boldsymbol{\tau}}(t)\|_{\mathrm{L}^1} = o(1). \tag{6.11}$$

Proof Clearly, estimates (6.10) follow from (6.9) via the same arguments as in the proof of Proposition 5.1. The estimate for $\sup_{t\in[0,T]} \mathfrak{E}(\widetilde{U}_{\boldsymbol{\tau}}(t))$ can be retrieved from the discrete energy-dissipation inequality (6.8) and its analogue on the semi-intervals $\mathrm{I}^{\boldsymbol{\tau},k}_{\mathrm{right}}$. Let us just comment on the proof of (6.11): the estimates for $\|\overline{U}_{\boldsymbol{\tau}}-\underline{U}_{\boldsymbol{\tau}}\|_{\mathrm{L}^1}$ and $\|U_{\boldsymbol{\tau}}-\underline{U}_{\boldsymbol{\tau}}\|_{\mathrm{L}^1}$ derive from the uniform integrability of the family $(\widetilde{U}'_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\Lambda}$ in $\mathrm{L}^1([0,T];\mathbf{X}_1)$, while we refer to the proof of [26, Prop. 6.3] for the estimate of $\|\widetilde{U}_{\boldsymbol{\tau}}-\underline{U}_{\boldsymbol{\tau}}\|_{\mathrm{L}^1}$. □

Sketch of the passage to the limit in the energy-dissipation inequality (6.9). By repeating the very same arguments as in the proof of Corollary 5.3 and relying on (6.11), we show that for any sequence $(\boldsymbol{\tau}_n)_n$ with $|\boldsymbol{\tau}_n| \downarrow 0$ as $n\to\infty$ there exist a (non-relabeled) subsequence of $(\overline{U}_{\boldsymbol{\tau}_n})_n$, $(\underline{U}_{\boldsymbol{\tau}_n})_n$, $(U_{\boldsymbol{\tau}_n})_n$ and $(\widetilde{U}_{\boldsymbol{\tau}_n})_n$ and a curve $U \in \mathrm{AC}([0,T];\mathbf{X}_1)$ such that for all $t\in[0,T]$ there holds

$$\overline{U}_{\boldsymbol{\tau}_n}(t), \underline{U}_{\boldsymbol{\tau}_n}(t), \widehat{U}_{\boldsymbol{\tau}_n}(t), \widetilde{U}_{\boldsymbol{\tau}_n}(t) \rightharpoonup U(t) \qquad \text{in } \mathbf{X}_1.$$

Convergences (6.4c) for $(\widehat{U}'_{\boldsymbol{\tau}_n})_n$ and (6.4b) for $(\frac{1}{2}\mathbb{T}^{(j)}_{\tau_n}\widehat{U}'_{\tau_n})_n$ (to functions V_j such that $V_1 + V_2 = U'$), hold, too. Likewise, we conclude the analogues of convergences (5.10b) for the sequences $(\mathbb{T}^{(j)}_{\tau_n}\widetilde{\xi}_{\boldsymbol{\tau}_n})_n$. Therefore, we are in a position to take the limit as $n\to\infty$ in (6.9). A straightforward adaptation of the arguments from Sects. 5.2 and 5.3 leads us to conclude that there exists $\xi \in \mathrm{L}^1([0,T];\mathbf{X}_1^*)$ such that the pair (U,ξ) complies with the energy-dissipation inequality (5.1). Then, we repeat the very same arguments from Sect. 5.4 and establish that (U,ξ) in fact fulfills the energy-dissipation balance (3.16), that it solves the subdifferential inclusion (3.15), and that (V_1, V_2) provide an optimal decomposition of the rate U'.

This finishes the proof of Theorem 6.1. □

7 The time-splitting method for systems with a block structure

In this section we tackle the application of the splitting method to generalized gradient systems with a *block structure*. In such systems,

$$\text{the state variable } u \text{ is a vector } \begin{pmatrix} y \\ z \end{pmatrix} \in \mathbf{U} := \mathbf{Y}\times\mathbf{Z},$$

with $\mathbf{Y}$ and $\mathbf{Z}$ (separable) reflexive Banach spaces. The evolution of the system is governed by an energy functional

$$\mathcal{E} : [0,T]\times\mathbf{U} \to (-\infty,\infty], \qquad \mathcal{E} = \mathcal{E}(t,u) = \mathcal{E}(t,y,z)$$

(we will use both notations with slight abuse), whereas dissipation mechanisms are encoded by two dissipation potentials $\mathcal{R}_{\mathrm{y}}$ and $\mathcal{R}_{\mathrm{z}}$, each acting on one of the components of the rate vector $u' = \binom{v}{w}$, namely

$$\mathcal{R}(v,w) = \big(\mathcal{R}_{\mathrm{y}}\oplus\mathcal{R}_{\mathrm{z}}\big)(v,w) := \mathcal{R}_{\mathrm{y}}(v)+\mathcal{R}_{\mathrm{z}}(w) \ \text{ with } \mathcal{R}_{\mathrm{y}}\colon \mathbf{Y}\to[0,\infty) \text{ and } \mathcal{R}_{\mathrm{z}}\colon \mathbf{Z}\to[0,\infty).$$

The analysis of these systems can be carried out in the context of the splitting approach in the previous case of Sect. 3, by introducing the dissipation potentials $\mathcal{R}_j : \mathbf{U} \to [0, \infty]$ via

$$\mathcal{R}_1(u') = \mathcal{R}_1(v, w) = \mathcal{R}_{\mathrm{y}}(v) + \mathcal{I}_{\{0\}}(w), \qquad \mathcal{R}_2(u') = \mathcal{R}_2(v, w) = \mathcal{I}_{\{0\}}(v) + \mathcal{R}_{\mathrm{z}}(w), \tag{7.1}$$

where $\mathcal{I}_{\{0\}}$ is the indicator function of the singleton $\{0\}$ with $\mathcal{I}_{\{0\}}(0) = 0$ and ∞ else. Let

$$\partial^{\mathbf{U}}\mathcal{E} : [0, T]\times\mathbf{U} \rightrightarrows \mathbf{U}^* \text{ be the } \textit{Fréchet} \text{ subdifferential of } \mathcal{E}(t, \cdot)$$

in the duality pairing $\langle\cdot, \cdot\rangle_{\mathbf{U}}$. Our time-splitting scheme will be based on the Cauchy problems for the subdifferential inclusions

$$\partial\mathcal{R}_j(u'(t)) + \partial^{\mathbf{U}}\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } \mathbf{U}^* \qquad \text{for a.a. } t \in (0, T),$$

in which we either freeze the variable z (for $j = 1$), or the variable y (for $j = 2$). It can be easily calculated that, in this setup, the effective dissipation potential is

$$\begin{aligned} &\mathcal{R}_{\mathrm{eff}} : \mathbf{U} \to [0, \infty), \qquad \mathcal{R}_{\mathrm{eff}}(u') = \mathcal{R}_{\mathrm{y}}(v) + \mathcal{R}_{\mathrm{z}}(w) \qquad \text{with} \\ &\mathcal{R}^*_{\mathrm{eff}} : \mathbf{U}^* \to [0, \infty), \qquad \mathcal{R}^*_{\mathrm{eff}}(\xi) = \mathcal{R}^*_{\mathrm{y}}(\eta) + \mathcal{R}^*_{\mathrm{z}}(\zeta). \end{aligned} \tag{7.2}$$

7.1 Assumptions

Our conditions on $\mathcal{E}$ and on the dissipation potentials $\mathcal{R}_{\mathrm{y}}$ and $\mathcal{R}_{\mathrm{z}}$ mimic the setup of Sect. 2.2, but with some significant differences. We start by settling the properties of the energy functional.

Hypothesis 7.1 *The functional* $\mathcal{E} : [0, T]\times\boldsymbol{U} \to (-\infty, \infty]$ *has the proper domain dom*$(\mathcal{E})$ $= [0, T]\times D_0$, *on which* $\mathcal{E}$ *is bounded from below* (2.13) *and complies with the time-differentiability condition* **<E>**. *Along all sequences* $(t_n, u_n) \rightharpoonup (t, u)$ *in* $[0, T] \times \boldsymbol{U}$ *with* $(u_n)_{n\in\mathbb{N}}$ *contained in an energy sublevel* S_E

- *the lower semicontinuity and power continuity conditions from* (2.17) *hold;*
- *the closedness condition* (2.19) *for* $\partial^{\boldsymbol{U}}\mathcal{E} : [0, T]\times\boldsymbol{U} \rightrightarrows \boldsymbol{U}^*$ *holds.*

The following subsumes our requirements on the dissipation potentials $\mathcal{R}_{\mathrm{y}}$ and $\mathcal{R}_{\mathrm{z}}$.

Hypothesis 7.2 *For* $x \in \{y,z\}$ *and* $\boldsymbol{X} \in \{\boldsymbol{Y}, \boldsymbol{Z}\}$ *the functionals* $\mathcal{R}_x : \boldsymbol{X} \to [0, \infty)$ *and their conjugates* $\mathcal{R}^*_x : \boldsymbol{X}^* \to [0, \infty)$ *comply with condition* **<R>**.

Although Hypotheses 7.1 and 7.2 mimic the setup of Sect. 3, it is clear that the overall gradient system $(\mathbf{X}, \mathcal{E}, \mathcal{R}_{\mathrm{eff}})$ is different from that considered therein. The first, striking difference is that, after extending the dissipation potentials $\mathcal{R}_{\mathrm{y}}$ and $\mathcal{R}_{\mathrm{z}}$ to the dissipation potentials $\mathcal{R}_1$ and $\mathcal{R}_2$, on the *common* space $\mathbf{U} = \mathbf{Y}\times\mathbf{Z}$ (cf. (7.1)), we lose the coercivity of $\mathcal{R}^*_1 = \mathcal{R}^*_{\mathrm{y}}$ and $\mathcal{R}^*_2 = \mathcal{R}^*_{\mathrm{z}}$ required via (2.2) on $\mathbf{U}^* = \mathbf{Y}^*\times\mathbf{Z}^*$. Moreover, we emphasize that, here, the Fréchet subdifferential of $\mathcal{E}(t, \cdot)$ is considered in the duality pairing of the common space $\mathbf{U}$. Nonetheless, in what follows we are going to show that the techniques at the core of the analysis in Sect. 5 carry over to the present setting. For this, a crucial role will be played by the condition that $\partial^{\mathbf{U}}\mathcal{E}$ has a 'cross-product structure', which is weaker than the singleton condition needed in the setup of Sect. 3.

Hypothesis 7.3 (Cross-product condition) *For all* $(t, u) = (t, y, z) \in$ *dom*$(\partial\mathcal{E})$ *we have*

$$\partial^{\boldsymbol{U}}\mathcal{E}(t, y, z) = \partial_y\mathcal{E}(t, y, z)\times\partial_z\mathcal{E}(t, y, z) \tag{7.3}$$

where $\partial_x\mathcal{E} : [0,T]\times\boldsymbol{U} \rightrightarrows \boldsymbol{X}^*$*, for* $x \in \{y,z\}$ *and correspondingly* $\boldsymbol{X} \in \{\boldsymbol{Y},\boldsymbol{Z}\}$*, is the* partial subdifferential *of* $\mathcal{E}$ *with respect to the variable x, while fixing the other variable.*

We note that this condition is more general than the singleton condition because multi-valued subdifferentials are still possible. We also remark for later use that, in view of (7.3), the closedness condition for $\partial^{\mathbf{U}}\mathcal{E}$ is indeed equivalent to

$$\forall\, E>0: \begin{cases} (t_n,y_n,z_n,\eta_n,\zeta_n)\rightharpoonup(t,y,z,\eta,\zeta) \text{ in } [0,T]\times\mathbf{Y}\times\mathbf{Z}\times\mathbf{Y}^*\times\mathbf{Z}^*, \\ (y_n,z_n)\in S_E,\ \ \eta_n\in\partial_{\mathrm{y}}\mathcal{E}(t_n,y_n,z_n),\ \ \zeta_n\in\partial_{\mathrm{z}}\mathcal{E}(t_n,y_n,z_n)\ \forall\, n\in\mathbb{N} \end{cases} \\ \Longrightarrow \quad \eta\in\partial_{\mathrm{y}}\mathcal{E}(t,y,z) \text{ and } \zeta\in\partial_{\mathrm{z}}\mathcal{E}(t,y,z). \tag{7.4}$$

Obviously, in view of (7.2) and (7.3) the subdifferential inclusion

$$\partial\mathcal{R}_{\mathrm{eff}}(u'(t)) + \partial^{\mathbf{U}}\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } \mathbf{U}^* \quad \text{for a.a. } t\in(0,T) \tag{7.5}$$

is indeed equivalent to the system

$$\begin{matrix} \partial\mathcal{R}_{\mathrm{y}}(y'(t)) + \partial_{\mathrm{y}}\mathcal{E}(t,y(t),z(t)) \ni 0 \ \text{ in } \mathbf{Y}^* \\ \partial\mathcal{R}_{\mathrm{z}}(z'(t)) + \partial_{\mathrm{z}}\mathcal{E}(t,y(t),z(t)) \ni 0 \ \text{ in } \mathbf{Z}^* \end{matrix} \quad \text{for a.a. } t\in(0,T).$$

Last but not least, we need to specify our chain-rule assumption on $(\mathcal{E}, \partial^{\mathbf{U}}\mathcal{E})$.

Hypothesis 7.4 (Abstract chain rule) *The quadruple* $(\boldsymbol{U},\mathcal{E},\partial^{\boldsymbol{U}}\mathcal{E},\mathcal{R}_{\mathrm{eff}})$ *satisfies the chain rule property* ***<CR>****.*

7.2 Time-splitting for block structure systems

As in Sect. 3 we introduce the rescaled dissipation potentials $\widetilde{\mathcal{R}}_j : \mathbf{U}\to[0,\infty)$

$$\widetilde{\mathcal{R}}_1(u') = 2\mathcal{R}_1\big(\tfrac12 u'\big) = 2\mathcal{R}_{\mathrm{y}}\big(\tfrac12 v\big) + \mathcal{I}_{\{0\}}(w), \quad \widetilde{\mathcal{R}}_2(u') = 2\mathcal{R}_2\big(\tfrac12 u'\big) = 2\mathcal{R}_{\mathrm{z}}\big(\tfrac12 w\big) + \mathcal{I}_{\{0\}}(v).$$

We will construct our approximate solution to the Cauchy problem for (7.5) by solving, in suitable sub-intervals I_j of $[0,T]$, the Cauchy problems for the doubly nonlinear equations

$$\partial\widetilde{\mathcal{R}}_j(u'(t)) + \partial^{\mathbf{U}}\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } \mathbf{U}^* \quad \text{for a.a. } t\in I_j, \quad j\in\{1,2\},$$

which, thanks to the cross product condition (7.3), reformulate in the same way as the subdifferential inclusion (7.5) for $\mathcal{R}_{\mathrm{eff}}$.

More precisely, let $\mathscr{P}_{\boldsymbol{\tau}}$ be a non-uniform partition of $[0,T]$ (cf. (3.1)) and let $(\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}})_{k=1}^{N_\tau}$ and $(\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}})_{k=1}^{N_\tau}$ be the associated 'left' and 'right' semi-intervals, see (3.2). Starting from an initial datum $u_0 = (y_0,z_0)\in \mathrm{D}_0$, we define the approximate solutions $U_{\boldsymbol{\tau}} = (Y_{\boldsymbol{\tau}}, Z_{\boldsymbol{\tau}}) : [0,T]\to \mathrm{D}_0 \subset \mathbf{Y}\times\mathbf{Z}$ to (7.5) in the following way (cf. (3.12)). We start with

$$(Y_{\boldsymbol{\tau}}(0), Z_{\boldsymbol{\tau}}(0)) := (y_0,z_0) \tag{7.6a}$$

and proceed for $t\in(t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^{k}]$ and $k\in\{1,\ldots,N_{\boldsymbol{\tau}}\}$ as follows:

- On the semi-interval $\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}}$, we define $(Y_{\boldsymbol{\tau}}, Z_{\boldsymbol{\tau}})$ to be a solution of the Cauchy problem:

$$\begin{cases} \partial\mathcal{R}_{\mathrm{y}}(y'(t)) + \partial_{\mathrm{y}}\mathcal{E}(t,y(t),z(t)) \ni 0 & \text{in } \mathbf{Y}^* \ \text{ and} \\ z'(t)\equiv 0 & \text{in } \mathbf{Z} \ \text{ for a.a. } t\in \mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}}; \\ (y(t_{\boldsymbol{\tau}}^{k-1}), z(t_{\boldsymbol{\tau}}^{k-1})) = (Y_{\boldsymbol{\tau}}(t_{\boldsymbol{\tau}}^{k-1}), Z_{\boldsymbol{\tau}}(t_{\boldsymbol{\tau}}^{k-1})). \end{cases} \tag{7.6b}$$

On the semi-interval $\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}}$, we define $(Y_{\boldsymbol{\tau}}, Z_{\boldsymbol{\tau}})$ to be a solution of the Cauchy problem

$$\begin{cases} y'(t) \equiv 0 & \text{in } \mathbf{Y} \quad \text{and} \\ \partial\mathcal{R}_{\mathrm{z}}(z'(t)) + \partial_{\mathrm{z}}\mathcal{E}(t, y(t), z(t)) \ni 0 & \text{in } \mathbf{Z}^* \quad \text{for a.a. } t \in \mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}}; \\ (y(t^{k-1/2}_{\boldsymbol{\tau}}), z(t^{k-1/2}_{\boldsymbol{\tau}})) = (Y_{\boldsymbol{\tau}}(t^{k-1/2}_{\boldsymbol{\tau}}), Z_{\boldsymbol{\tau}}(t^{k-1/2}_{\boldsymbol{\tau}})). \end{cases} \tag{7.6c}$$

Existence of solutions for the Cauchy problems (7.6b) follows by noting that for fixed $z \in \mathbf{Z}$ the triple $(\mathbf{Y}, \mathcal{E}(\cdot, \cdot, z), \mathcal{R}_{\mathrm{y}})$ satisfies the required assumptions of Theorem 2.10. Analogously, the same holds for the triple $(\mathbf{Z}, \mathcal{E}(\cdot, y, \cdot), \mathcal{R}_{\mathrm{z}})$ if $y \in \mathbf{Y}$ is fixed, which provides existence of solutions of the Cauchy problems (7.6c). Hence, we conclude that the solution curve $U_{\boldsymbol{\tau}} = (Y_{\boldsymbol{\tau}}, Z_{\boldsymbol{\tau}})\colon [0, T] \to \mathbf{U}$ fulfills $U_{\boldsymbol{\tau}} \in \mathrm{AC}([0, T]; \mathbf{U})$. We also introduce the functions $\eta_{\boldsymbol{\tau}} \in \mathrm{L}^1([0, T]; \mathbf{Y}^*)$ and $\zeta_{\boldsymbol{\tau}} \in \mathrm{L}^1([0, T]; \mathbf{Z}^*)$ featuring in the force terms in the subdifferential inclusions (7.6b) and (7.6c). Namely, for $k \in \{1, \ldots, N_{\boldsymbol{\tau}}\}$ we set

$$\begin{aligned} \eta_{\boldsymbol{\tau}}(t) &\begin{cases} \in \partial_{\mathrm{y}}\mathcal{E}(t, Y_{\boldsymbol{\tau}}(t), Z_{\boldsymbol{\tau}}(t)) \cap (-\partial\mathcal{R}_{\mathrm{y}}(Y'_{\boldsymbol{\tau}}(t))) & \text{for } t \in \mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}}, \\ \equiv 0 & \text{for } t \in \mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}}, \end{cases} \\ \zeta_{\boldsymbol{\tau}}(t) &\begin{cases} \equiv 0 & \text{for } t \in \mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}}, \\ \in \partial_{\mathrm{z}}\mathcal{E}(t, Y_{\boldsymbol{\tau}}(t), Z_{\boldsymbol{\tau}}(t)) \cap (-\partial\mathcal{R}_{\mathrm{z}}(Z'_{\boldsymbol{\tau}}(t))) & \text{for } t \in \mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}}. \end{cases} \end{aligned} \tag{7.7}$$

Finally, for tailoring analysis to the block structure context, in addition to the 'overall' repetition operators $\mathbb{T}^{(j)}_{\boldsymbol{\tau}} : \mathrm{L}^1([0, T]; \mathscr{U}) \to \mathrm{L}^1([0, T]; \mathscr{U})$ for $\mathscr{U} \in \{\mathbf{U}, \mathbf{U}^*\}$, we will resort to the operators (denoted by the same symbols)

$$\mathbb{T}^{(1)}_{\boldsymbol{\tau}}\colon \mathrm{L}^1([0, T]; \mathscr{Y}) \to \mathrm{L}^1([0, T]; \mathscr{Y}); \quad \big(\mathbb{T}^{(1)}_{\boldsymbol{\tau}} g\big)(t) := \begin{cases} g(t) & \text{if } t \in \mathrm{I}^{\boldsymbol{\tau},\mathsf{k}_{\boldsymbol{\tau}}(t)}_{\mathrm{left}}, \\ g\big(t - \frac{\widetilde{\tau}(t)}{2}\big) & \text{if } t \in \mathrm{I}^{\boldsymbol{\tau},\mathsf{k}_{\boldsymbol{\tau}}(t)}_{\mathrm{right}}, \end{cases} \tag{7.8a}$$

$$\mathbb{T}^{(2)}_{\boldsymbol{\tau}}\colon \mathrm{L}^1([0, T]; \mathscr{Z}) \to \mathrm{L}^1([0, T]; \mathscr{Z}); \quad \big(\mathbb{T}^{(2)}_{\boldsymbol{\tau}} g\big)(t) := \begin{cases} g\big(t + \frac{\widetilde{\tau}(t)}{2}\big) & \text{if } t \in \mathrm{I}^{\boldsymbol{\tau},\mathsf{k}_{\boldsymbol{\tau}}(t)}_{\mathrm{left}}, \\ g(t) & \text{if } t \in \mathrm{I}^{\boldsymbol{\tau},\mathsf{k}_{\boldsymbol{\tau}}(t)}_{\mathrm{right}}, \end{cases} \tag{7.8b}$$

with $\mathscr{Y} \in \{\mathbf{Y}, \mathbf{Y}^*\}$ and $\mathscr{Z} \in \{\mathbf{Z}, \mathbf{Z}^*\}$.

Now, we are in a position to give our convergence result for the time-splitting scheme in the setup with block structure. Observe that, due to the block structure we will succeed in relating the weak limits of the repeated rates $(\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n}(Y'_{\boldsymbol{\tau}_n}))$ and $(\mathbb{T}^{(2)}_{\boldsymbol{\tau}_n}(Z'_{\boldsymbol{\tau}_n}))$ to the limiting rates Y' and Z', cf. (7.9c) below.

Theorem 7.5 (Convergence of time-splitting method for block systems) *Under Hypotheses 7.1, 7.2, 7.3, and 7.4, starting from an initial datum $u_0 = (y_0, z_0) \in D_0$, define the curves $U_{\boldsymbol{\tau}}$ as in (7.6).*

Then, for any sequence $(\boldsymbol{\tau}_n)_n$ with $\lim_{n\to\infty} |\boldsymbol{\tau}_n| = 0$ there exist a (non-relabeled) subsequence and a curve $U = (Y, Z) \in AC([0, T]; \boldsymbol{U})$ with $U(0) = u_0$ such that the following convergences hold for the sequences $(Y_{\boldsymbol{\tau}_n})_n$ and $(Z_{\boldsymbol{\tau}_n})_n$ as $n \to \infty$:

$$Y_{\boldsymbol{\tau}_n}(t) \rightharpoonup Y(t) \text{ in } \boldsymbol{Y} \quad \text{and} \quad Z_{\boldsymbol{\tau}_n}(t) \rightharpoonup Z(t) \text{ in } \boldsymbol{Z} \qquad \text{for all } t \in [0, T], \tag{7.9a}$$

$$Y'_{\boldsymbol{\tau}_n} \rightharpoonup Y' \text{ in } \mathrm{L}^1([0, T]; \boldsymbol{Y}) \text{ and } Z'_{\boldsymbol{\tau}_n} \rightharpoonup Z' \text{ in } \mathrm{L}^1([0, T]; \boldsymbol{Z}), \tag{7.9b}$$

$$\tfrac{1}{2}\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n}(Y'_{\boldsymbol{\tau}_n}) \rightharpoonup Y' \text{ in } \mathrm{L}^1([0, T]; \boldsymbol{Y}) \text{ and } \tfrac{1}{2}\mathbb{T}^{(2)}_{\boldsymbol{\tau}_n}(Z'_{\boldsymbol{\tau}_n}) \rightharpoonup Z' \text{ in } \mathrm{L}^1([0, T]; \boldsymbol{Z}), \tag{7.9c}$$

and there exists a function $\xi = (\eta, \zeta) \in \mathrm{L}^1([0,T]; \boldsymbol{U}^*)$ *such that the pair* (U, ξ) *solves the subdifferential system, for a.a.* $t \in (0,T)$,

$$\begin{aligned} &\partial\mathcal{R}_y(Y'(t)) + \eta(t) \ni 0 \textit{ and } \eta(t) \in \partial_y\mathcal{E}(t, Y(t), Z(t)) \quad \textit{in } \boldsymbol{Y}^*,\\ &\partial\mathcal{R}_z(Z'(t)) + \zeta(t) \ni 0 \textit{ and } \zeta(t) \in \partial_z\mathcal{E}(t, Y(t), Z(t)) \quad \textit{in } \boldsymbol{Z}^*, \end{aligned} \tag{7.10}$$

and fulfills the energy-dissipation balance

$$\begin{aligned} &\mathcal{E}(t, Y(t), Z(t)) + \int_s^t \big\{\mathcal{R}_y(Y'(r)) + \mathcal{R}_z(Z'(r)) + \mathcal{R}_y^*(-\eta(r)) + \mathcal{R}_z^*(-\zeta(r))\big\}\mathrm{d}r\\ &= \mathcal{E}(s, Y(s), Z(s)) + \int_s^t \partial_t\mathcal{E}(r, Y(r), Z(r))\mathrm{d}r \qquad \textit{for } 0 \le s \le t \le T. \end{aligned} \tag{7.11}$$

Like for Theorem 3.5, we obtain the enhanced convergences

$$\mathcal{E}(t, U_{\tau_n}(t)) \longrightarrow \mathcal{E}(t, U(t)) \qquad \text{for all } t \in [0,T], \tag{7.12a}$$

$$\left.\begin{aligned} \int_s^t \mathcal{R}_\mathrm{y}(\tfrac12 \mathbb{T}^{(1)}_{\tau_n} Y'_{\tau_n}(r))\mathrm{d}r &\longrightarrow \int_s^t \mathcal{R}_\mathrm{y}(Y'(r))\mathrm{d}r,\\ \int_s^t \mathcal{R}_\mathrm{z}(\tfrac12 \mathbb{T}^{(2)}_{\tau_n} Z'_{\tau_n}(r))\mathrm{d}r &\longrightarrow \int_s^t \mathcal{R}_\mathrm{z}(Z'(r))\mathrm{d}r,\\ \int_s^t \mathcal{R}^*_\mathrm{y}(-\mathbb{T}^{(1)}_{\tau_n} \eta_{\tau_n}(r))\mathrm{d}r &\longrightarrow \int_s^t \mathcal{R}^*_\mathrm{y}(-\eta(r))\mathrm{d}r,\\ \int_s^t \mathcal{R}^*_\mathrm{z}(-\mathbb{T}^{(2)}_{\tau_n} \zeta_{\tau_n}(r))\mathrm{d}r &\longrightarrow \int_s^t \mathcal{R}^*_\mathrm{y}(-\zeta(r))\mathrm{d}r \end{aligned}\right\} \text{ for all } [s,t] \subset [0,T]. \tag{7.12b}$$

Remark 7.6 (Non-convergence) It can be easily checked in our "multi-valued" counterexample of Sect. 4.1 that (i) the "cross-product condition" does not hold and (ii) that the solutions of the split-step algorithm using the block structure of $\mathbf{Y}\times\mathbf{Z} = \mathbb{R}\times\mathbb{R}$ get stuck completely when reaching the diagonal $u_1 = u_2$. Hence, we have non-convergence because solutions for the effective problem move along the diagonal until they reach $u = 0$.

7.3 Alternating minimizing movements for block structures

Last but not least, we point out that the analogue of Theorem 6.1 holds for systems with block structure, assuming Hypotheses 7.1, 7.2, 7.3, and 7.4. Let us briefly illustrate the Alternating Minimizing Movement approach to block-structured systems. As in Sect. 6, we construct approximate solutions by solving the time incremental minimization schemes for the subdifferential inclusions (7.6b) and (7.6c). This results in the *alternating minimization scheme* (7.13) below.

More precisely, starting from an initial datum $u_0 = (y_0, z_0) \in \mathrm{D}_0$, we define the piecewise constant solutions $\overline{U}_\tau : [0,T] \to \mathbf{U}$, $\overline{U}_\tau = (\overline{Y}_\tau, \overline{Z}_\tau)$, by setting $\overline{Y}_\tau(0) := y_0$, $\overline{Z}_\tau(0) := z_0$, and for $t \in (0,T)$ and $k \in \{1, \ldots, N_\tau\}$ we define

$$\begin{aligned} &\text{for } t \in \mathrm{I}^{k,\tau}_{\text{left}} : \ \overline{Y}_\tau(t) := Y_k, \quad \overline{Z}_\tau(t) := Z_{k-1},\\ &\qquad \text{with } Y_k \in \operatorname*{Argmin}_{Y\in\mathbf{Y}} \left\{ \tfrac{\tau_k}{2}\, \widetilde{\mathcal{R}}_\mathrm{y}\big(\tfrac{2}{\tau_k}(Y - Y_{k-1})\big) + \mathcal{E}(t_\tau^{k-1/2}, Y, Z_{k-1}) \right\}, \end{aligned} \tag{7.13a}$$

$$\begin{aligned} &\text{for } t \in \mathrm{I}^{k,\tau}_{\text{right}} : \ \overline{Y}_\tau(t) := Y_k, \quad \overline{Z}_\tau(t) := Z_k,\\ &\qquad \text{with } Z_k \in \operatorname*{Argmin}_{Z\in\mathbf{Z}} \left\{ \tfrac{\tau_k}{2}\, \widetilde{\mathcal{R}}_\mathrm{z}\big(\tfrac{2}{\tau_k}(Z - Z_{k-1})\big) + \mathcal{E}(t_\tau^k, Y_k, Z) \right\}. \end{aligned} \tag{7.13b}$$

We also introduce the 'delayed' piecewise constant interpolant $\underline{U}_{\boldsymbol{\tau}} = (\underline{Y}_{\boldsymbol{\tau}}, \underline{Z}_{\boldsymbol{\tau}})$ via (6.2), and the piecewise linear interpolant $\widehat{U}_{\boldsymbol{\tau}} : [0, T] \to \mathbf{U}$ of the discrete solutions by setting

$$\widehat{U}_{\boldsymbol{\tau}}(t) := (\widehat{Y}_{\boldsymbol{\tau}}, \underline{Z}_{\boldsymbol{\tau}}) \text{ for } t \in [t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^{k-1/2}] \text{ and } \widehat{U}_{\boldsymbol{\tau}}(t) := (\overline{Y}_{\boldsymbol{\tau}}, \widehat{Z}_{\boldsymbol{\tau}}) \text{ for } t \in [t_{\boldsymbol{\tau}}^{k-1/2}, t_{\boldsymbol{\tau}}^{k}]. \tag{7.14}$$

where

$$\begin{cases} \widehat{Y}_{\boldsymbol{\tau}}(t) = \frac{t-t_{\boldsymbol{\tau}}^{k-1}}{\tau_k/2}\, \overline{Y}_{\boldsymbol{\tau}}(t) + \frac{t_{\boldsymbol{\tau}}^{k-1/2}-t}{\tau_k/2}\, \underline{Y}_{\boldsymbol{\tau}}(t) \text{ for } t \in [t_{\boldsymbol{\tau}}^{k-1}, t_{\boldsymbol{\tau}}^{k-1/2}], \\ \widehat{Z}_{\boldsymbol{\tau}}(t) = \frac{t-t_{\boldsymbol{\tau}}^{k-1/2}}{\tau_k/2}\, \overline{Z}_{\boldsymbol{\tau}}(t) + \frac{t_{\boldsymbol{\tau}}^{k}-t}{\tau_k/2}\, \underline{Z}_{\boldsymbol{\tau}}(t) \text{ for } t \in [t_{\boldsymbol{\tau}}^{k-1/2}, t_{\boldsymbol{\tau}}^{k}]. \end{cases}$$

Finally, the variational interpolant $\widetilde{U}_{\boldsymbol{\tau}}$ can be defined by replacing (6.7) by an alternate minimization scheme, in analogy with (7.13).

After these preparations, we can state the result corresponding to Theorem 6.1, in the context of the block system from Sect. 7.1. We omit its proof because it follows easily by adapting the proof of Theorem 6.1 to the case with block structure, in the same way as as we will tailor the proof of Theorem 3.5 to provide a proof of Theorem 7.5 in the upcoming Sect. 7.5.

Theorem 7.7 (Alternating Minimizing Movements for block systems) *Under Hypotheses 7.1, 7.2, 7.3, and 7.4, starting from an initial datum $u_0 = (y_0, z_0) \in D_0$, define the curves $\overline{U}_{\boldsymbol{\tau}} = (\overline{Y}_{\boldsymbol{\tau}}, \overline{Z}_{\boldsymbol{\tau}})$ and $\widehat{U}_{\boldsymbol{\tau}} = (\widehat{Y}_{\boldsymbol{\tau}}, \widehat{Z}_{\boldsymbol{\tau}})$ as in (7.13) and (7.14).*

Then, for any sequence $(\boldsymbol{\tau}_n)_n$ with $\lim_{n\to\infty} |\boldsymbol{\tau}_n| = 0$ there exist a (non-relabeled) subsequence and a curve $U = (Y, Z) \in AC([0, T]; \boldsymbol{U})$ with $U(0) = u_0$ such that the following convergences hold as $n \to \infty$:

$$\left.\begin{matrix} \overline{Y}_{\boldsymbol{\tau}_n}(t),\ \widehat{Y}_{\boldsymbol{\tau}_n}(t) \rightharpoonup Y(t) \text{ in } \boldsymbol{Y} \\ \overline{Z}_{\boldsymbol{\tau}_n}(t),\ \widehat{Z}_{\boldsymbol{\tau}_n}(t) \rightharpoonup Z(t) \text{ in } \boldsymbol{Z} \end{matrix}\right\} \text{ for all } t \in [0, T], \tag{7.15a}$$

$$\widehat{Y}_{\boldsymbol{\tau}_n}' \rightharpoonup Y' \text{ in } \mathrm{L}^1([0, T]; \boldsymbol{Y}) \text{ and } \widehat{Z}_{\boldsymbol{\tau}_n}' \rightharpoonup Z' \text{ in } \mathrm{L}^1([0, T]; \boldsymbol{Z}), \tag{7.15b}$$

$$\tfrac{1}{2}\mathbb{T}_{\boldsymbol{\tau}_n}^{(1)}(\widehat{Y}_{\boldsymbol{\tau}_n}') \rightharpoonup Y' \text{ in } \mathrm{L}^1([0, T]; \boldsymbol{Y}) \text{ and } \tfrac{1}{2}\mathbb{T}_{\boldsymbol{\tau}_n}^{(2)}(\widehat{Z}_{\boldsymbol{\tau}_n}') \rightharpoonup Z' \text{ in } \mathrm{L}^1([0, T]; \boldsymbol{Z}), \tag{7.15c}$$

and there exists a function $\xi = (\eta, \zeta) \in \mathrm{L}^1([0, T]; \boldsymbol{U}^*)$ such that the pair (U, ξ) solves the subdifferential system (7.10) and fulfills the energy-dissipation balance (7.11).

7.4 An application to linearized visco-elasto-plasticity

In this section we discuss the applicability of Theorems 7.5 and 7.7 to a prototypical class of coupled systems, also considered in [21, Sec. 2]. These systems include a model combining linearized viscoelasticity and viscoplasticity, cf. Example 7.8 ahead.

Let

$$\mathbf{Y} \text{ and } \mathbf{Z} \text{ be Hilbert spaces.} \tag{7.16a}$$

The dissipation potential $\mathcal{R}_{\mathrm{y}} : \mathbf{Y} \to [0, \infty)$ is quadratic, while $\mathcal{R}_{\mathrm{z}} : \mathbf{Z} \to [0, \infty)$ consists of a 1-homogeneous part and of a contribution with p-growth for some $p > 1$, namely

$$\mathcal{R}_{\mathrm{y}}(v) = \frac{1}{2}\langle \mathbb{V}_{\mathrm{y}} v, v\rangle, \qquad \mathcal{R}_{\mathrm{z}}(w) = \Psi_1(w) + \Psi_p(w) \tag{7.16b}$$

where $\mathbb{V}_{\mathrm{y}} : \mathbf{Y} \to \mathbf{Y}^*$ is a bounded, linear, symmetric operator, $\Psi_1 : \mathbf{Z} \to [0, \infty)$ is positively 1-homogeneous and $\Psi_p(w) = \psi(\|w\|_{\mathbf{Z}})$ for some convex, increasing $\psi : [0, \infty) \to [0, \infty)$

with $cr^p \lesssim \psi(r) \lesssim Cr^p$ giving p-growth. The energy functional is of the form

$$\mathcal{E}(t, y, z) := \frac{1}{2}\langle \mathbb{A}y, y\rangle_{\mathbf{Y}} + \langle \mathbb{B}y, z\rangle_{\mathbf{Z}} + \frac{1}{2}\langle \mathbb{G}z, z\rangle_{\mathbf{Z}} - \langle f(t), y\rangle_{\mathbf{Y}} - \langle g(t), z\rangle_{\mathbf{Z}}, \tag{7.16c}$$

where $\mathbb{A} : \mathbf{Y} \to \mathbf{Y}^*$ and $\mathbb{G} : \mathbf{Z} \to \mathbf{Z}^*$ are linear, bounded, and symmetric, $\mathbb{B} : \mathbf{Y} \to \mathbf{Z}^*$ is linear and bounded such that $\begin{pmatrix} \mathbb{A} & \mathbb{B}^* \\ \mathbb{B} & \mathbb{G} \end{pmatrix}$ is positive definite. Moreover, we assume that $(f, g) \in \mathrm{C}^1([0, T]; \mathbf{Y}^* \times \mathbf{Z}^*)$ are time-dependent applied forces. In this setup, the subdifferential inclusion (7.5) translates into the system

$$\begin{aligned} \mathbb{V}_{\mathrm{y}} y' + \mathbb{A}y + \mathbb{B}^* z &= f(t) \quad \text{in } \mathbf{Y}^* \quad \text{for a.a. } t \in (0, T), && (7.17a)\\ \partial\Psi_1(z') + \partial\Psi_p(z') + \mathbb{B}y + \mathbb{G}z &= g(t) \quad \text{in } \mathbf{Z}^* \quad \text{for a.a. } t \in (0, T). && (7.17b) \end{aligned}$$

A concrete model that falls in this class of systems is provided by the following example.

Example 7.8 We consider an elastoplastic body in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. Linearized elastoplasticity is described in terms of the displacement $y : \Omega \to \mathbb{R}^d$, with $y \in \mathbf{Y} = \mathrm{H}^1_0(\Omega)$, and the symmetric, trace-free plastic strain tensor $z : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{dev}} := \left\{ z \in \mathbb{R}^{d\times d}_{\mathrm{sym}} : \mathrm{tr}(z) = 0 \right\}$. Let $\mathbf{Z} = \mathrm{L}^2(\Omega, \mathbb{R}^{d\times d}_{\mathrm{dev}})$. The energy functional $\mathcal{E} : [0, T]\times\mathbf{Y}\times\mathbf{Z} \to \mathbb{R}$ is defined by

$$\mathcal{E}(t, y, z) = \int_\Omega \left\{ \frac{1}{2}(e(y)-z) : \mathbb{C}(e(y)-z) + \frac{1}{2} z : \mathbb{H}z \right\} \mathrm{d}x - \langle f(t), y\rangle_{\mathbf{Y}}$$

where $e(y) = \frac{1}{2}(\nabla u + \nabla u^\top)$ is the linearized symmetric strain tensor, $\mathbb{C} \in \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\mathbb{H} \in \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{dev}})$ are the positive definite and symmetric elasticity and hardening tensors, respectively, and $f : [0, T] \to \mathrm{H}^{-1}(\Omega; \mathbb{R}^d)$ a time-dependent volume loading. The dissipation potentials are

$$\mathcal{R}_{\mathrm{y}}(y') = \int_\Omega \frac{1}{2} e(y') : \mathbb{D}e(y')\mathrm{d}x, \qquad \mathcal{R}_{\mathrm{z}}(z') = \int_\Omega \sigma_{\mathrm{yield}}|z'| + \tfrac{\varrho}{2}|z'|^2 \mathrm{d}x$$

with $\mathbb{D} \in \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ the positive definite viscoelasticity tensor, $\sigma_{\mathrm{yield}} > 0$ the yield stress and $\varrho > 0$ a positive coefficient. System (7.17) rephrases as

$$\begin{aligned} -\mathrm{div}\big(\mathbb{D}e(y') + \mathbb{C}(e(y)-z)\big) &= f(t) && \text{in } \Omega\times(0, T),\\ \sigma_{\mathrm{yield}}\mathrm{Sign}(z') + \varrho z' + \mathrm{dev}\big(\mathbb{C}(z-e(y))\big) + \mathbb{H}z &\ni 0 && \text{in } \Omega\times(0, T). \end{aligned}$$

where $\mathrm{dev}\, A = A - \frac{1}{d}(\mathrm{tr}\, A)\, I$ is the deviatoric part of a tensor $A \in \mathbb{R}^{d\times d}$.

It is very easy to check that the energy functional $\mathcal{E}$ from (7.16c) satisfies Hypotheses 7.1 and 7.3. Likewise, it is immediate to check that the dissipation potentials $\mathcal{R}_{\mathrm{y}}$ and $\mathcal{R}_{\mathrm{z}}$ in (7.16b) both comply with condition **<R>**. It remains to discuss the validity of the chain-rule Hypothesis 7.4. For this, let us consider a curve $u = (y, z) \in \mathrm{AC}([0, T]; \mathbf{Y}\times\mathbf{Z})$ with $\sup_{t\in[0,T]} |\mathcal{E}(t, y(t)), z(t))| < \infty$ such that

$$\sup_{t\in[0,T]} |\mathcal{E}(t, u(t))| < \infty, \text{ and } \int_0^T \big(\mathcal{R}_{\mathrm{eff}}(u'(t)) + \mathcal{R}^*_{\mathrm{eff}}(-\xi(t))\big)\, \mathrm{d}t < \infty.$$

It is immediate to check that the above estimate implies

$$\int_0^T \|\mathbb{A}y(t) + \mathbb{B}^* z(t) - f(t)\|_{\mathbf{Y}^*} \|y'(t)\|_{\mathbf{Y}} \mathrm{d}t + \int_0^T \|\mathbb{B}y(t) + \mathbb{G}z(t) - g(t)\|_{\mathbf{Z}^*} \|z'(t)\|_{\mathbf{Z}} \mathrm{d}t < \infty.$$

Now, since $(y, z) \in \mathrm{L}^\infty([0,T];\mathbf{Y}\times\mathbf{Z})$ we readily infer that $\mathbb{A}y \in \mathrm{L}^\infty([0,T];\mathbf{Y}^*)$, $\mathbb{B}^*z \in \mathrm{L}^\infty([0,T];\mathbf{Y}^*)$, $\mathbb{B}y \in \mathrm{L}^\infty([0,T];\mathbf{Z}^*)$, and $\mathbb{G}z \in \mathrm{L}^\infty([0,T];\mathbf{Z}^*)$. Therefore, the individual contributions to $\mathcal{E}$ from (7.16c), evaluated along the curve u, are absolutely continuous, and for them the following chain rules hold:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\langle \mathbb{A}y(t), y(t)\rangle_{\mathbf{Y}}\right) = \langle \mathbb{A}y(t), y'(t)\rangle_{\mathbf{Y}}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\langle \mathbb{B}y(t), z(t)\rangle_{\mathbf{Z}}\right) = \langle \mathbb{B}^*z(t), y'(t)\rangle_{\mathbf{Y}} + \langle \mathbb{B}y(t), z'(t)\rangle_{\mathbf{Z}}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\langle \mathbb{G}z(t), z(t)\rangle_{\mathbf{Z}}\right) = \langle \mathbb{G}z(t), z'(t)\rangle_{\mathbf{Z}}. \end{cases}$$

From that we immediately conclude the chain rule of Hypothesis 7.4.

All in all, we have proved that the block-structured system $(\mathbf{Y}\times\mathbf{Z}, \mathcal{E}, \mathcal{R}_{\mathrm{y}}\oplus\mathcal{R}_{\mathrm{z}})$ from (7.16) complies with Hypotheses 7.1, 7.2, 7.3, and 7.4. Thus, Theorems 7.5 and 7.7 are applicable.

7.5 Proof of Theorem 7.5

We split the argument in the following steps.

Step 1. A priori estimates: As in the proof of Theorem 3.5, the starting point for our analysis is the approximate energy-dissipation balance

$$\begin{aligned} &\mathcal{E}(t, Y_{\boldsymbol{\tau}}(t), Z_{\boldsymbol{\tau}}(t)) + \mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{rate}}([s,t]) + \mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{slope}}([s,t]) \\ &= \mathcal{E}(s, Y_{\boldsymbol{\tau}}(s), Z_{\boldsymbol{\tau}}(s)) + \int_s^t \partial_t\mathcal{E}(r, Y_{\boldsymbol{\tau}}(r), Z_{\boldsymbol{\tau}}(r))\mathrm{d}r \end{aligned} \tag{7.18}$$

along all subintervals $[s,t] \subset [0,T]$. With $\mathcal{R}_j$ given by (7.1) and $\widetilde{\mathcal{R}}_j = 2\mathcal{R}_j(\frac{1}{2}\,\cdot)$, the rate and slope terms from (3.19) now read

$$\begin{aligned} \mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{rate}}([0,T]) &:= \int_0^T \left\{\chi_{\boldsymbol{\tau}}(r)\,\widetilde{\mathcal{R}}_{\mathrm{y}}\left(\tfrac{1}{2}Y_{\boldsymbol{\tau}}'(r)\right) + (1-\chi_{\boldsymbol{\tau}}(r))\,\widetilde{\mathcal{R}}_{\mathrm{z}}\left(\tfrac{1}{2}Z_{\boldsymbol{\tau}}'(r)\right)\right\}\mathrm{d}r \\ &\overset{(5.7a)}{=} \int_0^T \left\{\mathcal{R}_{\mathrm{y}}(\tfrac{1}{2}\mathbb{T}_{\boldsymbol{\tau}}^{(1)}Y_{\boldsymbol{\tau}}'(r)) + \mathcal{R}_{\mathrm{z}}(\tfrac{1}{2}\mathbb{T}_{\boldsymbol{\tau}}^{(2)}Z_{\boldsymbol{\tau}}'(r))\right\}\mathrm{d}r, \end{aligned} \tag{7.19a}$$

$$\begin{aligned} \mathcal{D}_{\boldsymbol{\tau}}^{\mathrm{slope}}([0,T]) &:= \int_0^T \left\{\chi_{\boldsymbol{\tau}}(r)\,\widetilde{\mathcal{R}}_{\mathrm{y}}^*(-\eta_{\boldsymbol{\tau}}(r)) + (1-\chi_{\boldsymbol{\tau}}(r))\,\widetilde{\mathcal{R}}_{\mathrm{z}}^*(-\zeta_{\boldsymbol{\tau}}(r))\right\}\mathrm{d}r \\ &\overset{(5.7b)}{=} \int_0^T \left\{\mathcal{R}_{\mathrm{y}}^*(-\mathbb{T}_{\boldsymbol{\tau}}^{(1)}\eta_{\boldsymbol{\tau}}(r)) + \mathcal{R}_{\mathrm{z}}^*(-\mathbb{T}_{\boldsymbol{\tau}}^{(2)}\zeta_{\boldsymbol{\tau}}(r))\right\}\mathrm{d}r\,. \end{aligned} \tag{7.19b}$$

Then, we can mimic the arguments from from Proposition 5.1 and Corollary 5.2 and derive the analogues of the a priori estimates therein.

Step 2. Compactness: We may prove the analogue of Corollary 5.3. Namely, there exists $U \in \mathrm{AC}([0,T];\mathbf{U})$, such that, up to a subsequence,

$$U_{\boldsymbol{\tau}_n}(t) \rightharpoonup U(t) \text{ in } \mathbf{U} \text{ for all } t \in [0,T], \qquad U_{\boldsymbol{\tau}_n}' \rightharpoonup U' \text{ in } \mathrm{L}^1([0,T];\mathbf{U}), \tag{7.20}$$

whence convergences (7.9a), (7.9b). Furthermore, there exist V and Z such that

$$\begin{cases} \frac{1}{2}\mathbb{T}_{\boldsymbol{\tau}_n}^{(1)}(Y_{\boldsymbol{\tau}_n}') \rightharpoonup V & \text{in } \mathrm{L}^1([0,T];\mathbf{Y}), \\ \frac{1}{2}\mathbb{T}_{\boldsymbol{\tau}_n}^{(2)}(Z_{\boldsymbol{\tau}_n}') \rightharpoonup W & \text{in } \mathrm{L}^1([0,T];\mathbf{Z}). \end{cases} \tag{7.21}$$

In order to identify V and W we observe that, since the z (respectively, the y) variable is frozen in the Cauchy problem (7.6b) ((7.6c), resp.), we have that

$$\mathbb{T}_{\boldsymbol{\tau}}^{(1)}(U_{\boldsymbol{\tau}_n}') = \begin{pmatrix} \mathbb{T}_{\boldsymbol{\tau}}^{(1)}(Y_{\boldsymbol{\tau}_n}') \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbb{T}_{\boldsymbol{\tau}}^{(2)}(U_{\boldsymbol{\tau}_n}') = \begin{pmatrix} 0 \\ \mathbb{T}_{\boldsymbol{\tau}}^{(2)}(Z_{\boldsymbol{\tau}_n}') \end{pmatrix}.$$

In turn, (7.20) and Lemma 3.1 yield that

$$\frac{1}{2}\Big(\mathbb{T}^{(1)}_{\boldsymbol{\tau}}(U'_{\boldsymbol{\tau}_n})+\mathbb{T}^{(2)}_{\boldsymbol{\tau}}(U'_{\boldsymbol{\tau}_n})\Big)\rightharpoonup U'.$$

Therefore, by (7.21) we find that $\binom{Y'}{Z'} = U' = \binom{V}{0} + \binom{0}{W}$, whence (7.9c) holds true.

Finally, there exist $\eta \in \mathrm{L}^1([0,T];\mathbf{Y}^*)$ and $\zeta \in \mathrm{L}^1([0,T];\mathbf{Z}^*)$ such that, as $n\to\infty$,

$$\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n}\eta_{\boldsymbol{\tau}_n}\rightharpoonup\eta \ \text{ in } \mathrm{L}^1([0,T];\mathbf{Y}^*) \quad\text{and}\quad \mathbb{T}^{(2)}_{\boldsymbol{\tau}_n}\zeta_{\boldsymbol{\tau}_n}\rightharpoonup\zeta \ \text{ in } \mathrm{L}^1([0,T];\mathbf{Z}^*). \tag{7.22}$$

We now adapt the Young measure argument carried out in Sect. 5.3: up to a (non-relabeled) subsequence, the sequences $(\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n}\eta_{\boldsymbol{\tau}_n})_n$ and $(\mathbb{T}^{(2)}_{\boldsymbol{\tau}_n}\zeta_{\boldsymbol{\tau}_n})_n$ admit two limiting Young measures $(\mu_t)_{t\in(0,T)}$ and $(\nu_t)_{t\in(0,T)}$, with $\mu_t \in \mathrm{Prob}(\mathbf{Y}^*)$ and $\nu_t \in \mathrm{Prob}(\mathbf{Z}^*)$ for a.a. $t\in(0,T)$, such that

$$\begin{aligned}&\mathrm{supp}(\mu_t) \subset \mathrm{Ls}^{\mathrm{weak}}_{\mathbf{Y}^*}\big(\{\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n}\eta_{\boldsymbol{\tau}_n}(t)\}_n\big) \text{ and}\\ &\mathrm{supp}(\nu_t) \subset \mathrm{Ls}^{\mathrm{weak}}_{\mathbf{Z}^*}\big(\{\mathbb{T}^{(2)}_{\boldsymbol{\tau}_n}\zeta_{\boldsymbol{\tau}_n}(t)\}_n\big) \qquad \text{for a.a. } t\in(0,T),\end{aligned} \tag{7.23a}$$

(see (5.19a) for the definition of the limsup of sets), and the weak limits η and ζ read

$$\eta(t) = \int_{\mathbf{Y}^*}\tilde\eta\,\mathrm{d}\mu_t(\tilde\eta), \qquad \zeta(t) = \int_{\mathbf{Z}^*}\tilde\zeta\,\mathrm{d}\nu_t(\tilde\zeta) \quad \text{for a.a. } t\in(0,T). \tag{7.23b}$$

Now, arguing in the same way as in Sect. 5.3 and exploiting the closedness property (7.4) we find for a.a. $t\in(0,T)$ that

$$\mathrm{Ls}^{\mathrm{weak}}_{\mathbf{Y}^*}\big(\{\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n}\eta_{\boldsymbol{\tau}_n}(t)\}_n\big) \subset \partial_{\mathrm{y}}\mathcal{E}(t,Y(t),Z(t)) \text{ and } \mathrm{Ls}^{\mathrm{weak}}_{\mathbf{Z}^*}\big(\{\mathbb{T}^{(2)}_{\boldsymbol{\tau}_n}\zeta_{\boldsymbol{\tau}_n}(t)\}_n\big) \subset \partial_{\mathrm{z}}\mathcal{E}(t,Y(t),Z(t)).$$

Since the the subdifferentials are convex, we conclude with (7.23b) that

$$\eta(t)\in\partial_{\mathrm{y}}\mathcal{E}(t,Y(t),Z(t)) \ \text{ and } \ \zeta(t)\in\partial_{\mathrm{z}}\mathcal{E}(t,Y(t),Z(t)) \ \text{ for a.a. } t\in(0,T).$$

Step 3. Limit passage in (7.18): We take the limit as $n\to\infty$ in (7.18) on the interval $[0,T]$: thanks to (7.9a) we have $\liminf_{n\to\infty}\mathcal{E}(T,Y_{\boldsymbol{\tau}_n}(T),Z_{\boldsymbol{\tau}_n}(T)) \geq \mathcal{E}(T,Y(T),Z(T))$, while due to (7.9c) and (7.22) we have

$$\begin{aligned}\liminf_{n\to\infty}\mathcal{D}^{\mathrm{rate}}_{\boldsymbol{\tau}_n}([0,T]) &\geq \liminf_{n\to\infty}\int_0^T \big\{\mathcal{R}_{\mathrm{y}}(\tfrac12\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n}Y'_{\boldsymbol{\tau}_n}(r)) + \mathcal{R}_{\mathrm{z}}(\tfrac12\mathbb{T}^{(2)}_{\boldsymbol{\tau}_n}Z'_{\boldsymbol{\tau}_n}(r))\big\}\mathrm{d}r\\ &\geq \int_0^T \big\{\mathcal{R}_{\mathrm{y}}(Y'(r)) + \mathcal{R}_{\mathrm{z}}(Z'(r))\big\}\mathrm{d}r\,,\\ \liminf_{n\to\infty}\mathcal{D}^{\mathrm{slope}}_{\boldsymbol{\tau}_n}([0,T]) &\geq \liminf_{n\to\infty}\int_0^T \big\{\mathcal{R}^*_{\mathrm{y}}(-\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n}\eta_{\boldsymbol{\tau}_n}(r)) + \mathcal{R}^*_{\mathrm{z}}(-\mathbb{T}^{(2)}_{\boldsymbol{\tau}_n}\zeta_{\boldsymbol{\tau}_n}(r))\big\}\mathrm{d}r\\ &\geq \int_0^T \big\{\mathcal{R}^*_{\mathrm{y}}(-\eta(r)) + \mathcal{R}^*_{\mathrm{z}}(-\zeta(r))\big\}\mathrm{d}r\,.\end{aligned}$$

Relying on convergences (7.9a) we likewise take the limit as $n\to\infty$ in the power terms of the right-hand side of (7.18). All in all, we conclude that the curve $U=(Y,Z)$ and the function $\xi=(\eta,\zeta)$ fulfill the energy-dissipation inequality

$$\begin{aligned}&\mathcal{E}(T,Y(T),Z(T)) + \int_0^T\big\{\mathcal{R}_{\mathrm{y}}(Y'(r))+\mathcal{R}_{\mathrm{z}}(Z'(r)) + \mathcal{R}^*_{\mathrm{y}}(-\eta(r))+\mathcal{R}^*_{\mathrm{z}}(-\zeta(r))\big\}\mathrm{d}r\\ &\leq \mathcal{E}(0,Y(0),Z(0)) + \int_0^T \partial_t\mathcal{E}(r,Y(r),Z(r))\mathrm{d}r\,.\end{aligned}$$

Step 4. Energy-dissipation balance and conclusion of the proof: From the above inequality it immediately follows that the pair (U, ξ) complies with condition (2.8). Hence, by Hypothesis 7.4 we conclude that $t \mapsto \mathcal{E}(t, U(t))$ is absolutely continuous and the chain rule (2.3) holds for (U, ξ). Therefore, Proposition 2.1 allows us to conclude that (U, ξ) solves the subdifferential inclusion (7.5) (which rephrases as (7.10)), and that it fulfills the energy-dissipation balance (7.11) for all subintervals $[s, t] \subset [0, T]$.

As in the case of Theorem 3.14, the enhanced convergences (7.12) are a by-product of the argument for passing to the limit in (7.18); again, we refer to the proof of [26, Thm. 4.4] or [27, Thm. 3.11] for all details.

This finishes the proof of Theorem 7.5. □

A More on the quantitative young estimate

We aim to gain further insight into the connections between the QYE for the dissipation potentials $\mathcal{R}_1$ and $\mathcal{R}_2$, (for which we will always assume the validity of condition **<R>**, cf. Hypothesis 2.7), and the validity of the same property for $\mathcal{R}_{\text{eff}}$. In the particular case in which the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are indeed equivalent (i.e., $\|\cdot\|_{2,*}$ controls $\|\cdot\|_{1,*}$), we have the following result.

Lemma A.1 *Suppose that $\boldsymbol{X} := \boldsymbol{X}_1 = \boldsymbol{X}_2$ and that*

$$\exists\, \mathbb{C}_N,\ \mathbb{C}_N^* > 0\ \ \forall\, (v, \xi) \in \boldsymbol{X} \times \boldsymbol{X}^*\ : \quad \begin{cases} \|v\|_1 \le \mathbb{C}_N, \|v\|_2, \\ \|\xi\|_{1,*} \le \mathbb{C}_N^* \|\xi\|_{2,*}. \end{cases} \tag{A.1}$$

*Let $\mathcal{R}_j$ satisfy the **<QYE>** (2.10) with constants $c_j, C_j > 0$. Then, also $\mathcal{R}_{\text{eff}}$ satisfies estimate (2.10), with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_{1,*}$, with the constants $c_{\text{eff}} = \min\left\{c_1, \frac{c_2}{\mathbb{C}_N^* \mathbb{C}_N^*}\right\}$ and $C_{\text{eff}} = C_1 + C_2$.*

Proof For any $v \in \mathbf{X}$ and any $\varepsilon > 0$ there are $v_1, v_2 \in \mathbf{X}$ with $v_1 + v_2 = v$ such that $\mathcal{R}_{\text{eff}}(v) \ge \mathcal{R}_1(v_1) + \mathcal{R}_2(v_2) - \varepsilon$. Combining **<QYE>** for $\mathcal{R}_j$ with (A.1) yields

$$\begin{aligned}
\mathcal{R}_{\text{eff}}(v) + \mathcal{R}_{\text{eff}}^*(\xi) &\ge \mathcal{R}_1(v_1) + \mathcal{R}_2(v_2) - \varepsilon + \mathcal{R}_1^*(\xi) + \mathcal{R}_2^*(\xi) \\
&\ge c_1 \|v_1\|_1 \|\xi\|_{1,*} + c_2 \|v_2\|_2 \|\xi\|_{2,*} - C_1 - C_2 - \varepsilon \\
&\ge c_1 \|v_1\|_1 \|\xi\|_{1,*} + \frac{c_2}{\mathbb{C}_\mathrm{N} \mathbb{C}_\mathrm{N}^*} \|v_2\|_1 \|\xi\|_{1,*} - C_1 - C_2 - \varepsilon \\
&\ge \min\left\{c_1, \frac{c_2}{\mathbb{C}_\mathrm{N} \mathbb{C}_\mathrm{N}^*}\right\} (\|v_1\|_1 + \|v_2\|_1) \|\xi\|_{1,*} - C_1 - C_2 - \varepsilon \\
&\ge \min\left\{c_1, \frac{c_2}{\mathbb{C}_\mathrm{N} \mathbb{C}_\mathrm{N}^*}\right\} \|v\|_1 \|\xi\|_{1,*} - C_1 - C_2 - \varepsilon\,.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary the claim follows. □

In the general case in which we only have $\mathbf{X}_2 \subset \mathbf{X}_1$ (densely and) continuously (cf. (2.12b)), the next result provides some growth and coercivity conditions on $\mathcal{R}_1$ and $\mathcal{R}_2$ under which the QYE for $\mathcal{R}_1$ guarantees that for $\mathcal{R}_{\text{eff}}$.

Lemma A.2 *Assume there exist positive constants C_1, c_2, C_2 and $p, q \in (1, \infty)$ with $q < p$ such that*

$$\forall\, v \in \boldsymbol{X}_1 : \quad \mathcal{R}_1(v) \le C_1 \|v\|_1^q + C_1 \text{ and } \mathcal{R}_2(v) \ge c_2 \|v\|_1^p - C_2. \tag{A.2}$$

If additionally $\mathcal{R}_1$ satisfies **<QYE>**, *then also $\mathcal{R}_{\text{eff}}$ complies with* **<QYE>**.

Proof Consider $v_1, v_2 \in \mathbf{X}_1$ with $v_1 + v_2 = v$ and $\mathcal{R}_1(v_1) + \mathcal{R}_2(v_2) = \mathcal{R}_{\text{eff}}(v)$. Then, using (A.2) gives

$$c_2\|v_2\|_1^p - C_2 \le \mathcal{R}_2(v_2) \le \mathcal{R}_1(v_1) + \mathcal{R}_2(v_2) = \mathcal{R}_{\text{eff}}(v) \overset{(1)}{\le} \mathcal{R}_1(v-0) + \mathcal{R}_2(0) \le C_1\|v\|_1^q + C_1,$$

where $\overset{(1)}{\le}$ uses the definition of $\mathcal{R}_{\text{eff}}$ as an inf-convolution. Thus in the optimal decomposition $v = v_1 + v_2$ we have $\|v_2\|_1 \le C\|v\|_1^{q/p} + C$, which implies

$$\|v_1\|_1 \ge \|v\|_1 - \|v_2\|_1 \ge \|v\|_1 - C\|v\|_1^{q/p} - C \ge \frac{1}{2}\,\|v\|_1 - C_*, \tag{A.3}$$

where we used $q \lneqq p$.

With this we derive the lower bound

$$\mathcal{R}_{\text{eff}}(v) + \mathcal{R}^*_{\text{eff}}(\xi) = \mathcal{R}_1(v_1) + \mathcal{R}_2(v_2) + \mathcal{R}^*_1(\xi) + \mathcal{R}^*_2(\xi) \ge \mathcal{R}_1(v_1) + \mathcal{R}^*_1(\xi) \overset{\text{QYE}}{\ge} c_1^{\text{QYE}}\|v_1\|_1\,\|\xi\|_* - C^{\text{QYE}} \overset{(A.3)}{\ge} c_1^{\text{QYE}}\big(\tfrac{1}{2}\|v\|_1 - C_*\big)\,\|\xi\|_* - C^{\text{QYE}}.$$

This is almost the desired result, except for the linear term $-c_1^{\text{QYE}}C_*\|\xi\|_*$ on the right-hand side. However, since (A.2) provides an upper bound on $\mathcal{R}_1$, we have a lower bound on $\mathcal{R}^*_1$ and hence of $\mathcal{R}^*_{\text{eff}}$, viz.

$$\mathcal{R}^*_{\text{eff}}(\xi) \ge \mathcal{R}^*_1(\xi) \overset{(A.2)}{\ge} \underline{c}\|\xi\|_*^{q^*} - \underline{C} \ge \varepsilon\|\xi\|_* - C_\varepsilon \quad \text{for all } \varepsilon > 0.$$

Choosing $\varepsilon > 0$ sufficiently small, the linear term can be absorbed into the left-hand side, and the QYE for $\mathcal{R}_{\text{eff}}$ on $\mathbf{X}_1$ is established. □

B Proof of Lemma 3.1

It is sufficient to prove only items (3) and (4) of the statement; for convenience, we will show them for a sequence $(g_{\boldsymbol{\tau}_n})_n$, with $|\boldsymbol{\tau}_n| \to 0$ as $n \to \infty$.

Ad (3): To fix ideas, we will show the statement for the operators $\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n}$. Let $g_{\boldsymbol{\tau}_n} \to g$ in $\mathrm{C}^0([0,T];\mathscr{X}_{\mathrm{w}})$, namely $\lim_{n\to\infty}\sup_{t\in[0,T]} d_{\text{weak}}(g_{\boldsymbol{\tau}_n}(t), g(t)) = 0$ (where d_{weak} is the distance inducing the weak topology on a closed bounded subset of $\mathscr{X}$). In order to show that $\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n} \to g$ in $\mathrm{C}^0([0,T];\mathscr{X}_{\mathrm{w}})$, we observe that

$$\sup_{t\in[0,T]} d_{\text{weak}}(\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n}(t), g(t)) = \max\left\{S_{\text{left},\boldsymbol{\tau}_n}, S_{\text{right},\boldsymbol{\tau}_n}\right\}$$

$$\text{with} \begin{cases} S_{\text{left},\boldsymbol{\tau}_n} = \sup_{t\in\cup \mathrm{I}^{k,\tau_n}_{\text{left}}} d_{\text{weak}}(g_{\boldsymbol{\tau}_n}(t), g(t)), \\ S_{\text{right},\boldsymbol{\tau}_n} = \sup_{t\in\cup \mathrm{I}^{k,\tau_n}_{\text{right}}} d_{\text{weak}}\Big(g_{\boldsymbol{\tau}_n}\big(t - \tfrac{\tau_n(t)}{2}\big), g(t)\Big). \end{cases}$$

Now, we clearly have $S_{\mathrm{left},\boldsymbol{\tau}_n} \le \sup_{t\in[0,T]} d_{\mathrm{weak}}(g_{\boldsymbol{\tau}_n}(t), g(t)) \to 0$. In turn,

$$\begin{aligned} S_{\mathrm{right},\boldsymbol{\tau}_n} &= \sup_{t\in\cup \mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}_n}} d_{\mathrm{weak}}\Big(g_{\boldsymbol{\tau}_n}(t), g\big(t+\tfrac{\tau_n(t)}{2}\big)\Big) \\ &\le \sup_{t\in\cup \mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}_n}} d_{\mathrm{weak}}\left(g_{\boldsymbol{\tau}_n}(t), g(t)\right) + \sup_{t\in\cup \mathrm{I}_{\mathrm{left}}^{k,\boldsymbol{\tau}_n}} d_{\mathrm{weak}}\Big(g(t), g\big(t+\tfrac{\tau_n(t)}{2}\big)\Big) \longrightarrow 0 \end{aligned}$$

thanks to the fact that $g \in \mathrm{C}^0([0,T];\mathscr{X}_{\mathrm{w}})$. This concludes the proof of Claim (3).
Ad (4): Let now $g_{\boldsymbol{\tau}_n} \rightharpoonup g$ in $\mathrm{L}^1([0,T];\mathscr{X})$: we aim to show that

$$\frac{1}{2}\left(\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n} + \mathbb{T}^{(2)}_{\boldsymbol{\tau}_n}\right)(g_{\boldsymbol{\tau}_n}) \rightharpoonup g \quad \text{in } \mathrm{L}^1([0,T];\mathscr{X}). \tag{B.1}$$

In what follows, we will use the short-hand $\mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} := \frac{1}{2}\big(\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n} + \mathbb{T}^{(2)}_{\boldsymbol{\tau}_n}\big)$.

First, we observe that, by a characterization of weak compactness in $\mathrm{L}^1([0,T];\mathscr{X})$ known as the Dunford-Pettis Theorem (see [13, §IV.2, p. 101] for a version in Bochner spaces), the sequence $(g_{\boldsymbol{\tau}_n})_n$ is uniformly integrable, i.e. there exists a convex superlinear function $\Phi : [0,\infty) \to [0,\infty)$ such that

$$\sup_n \int_0^T \Phi(\|g_{\boldsymbol{\tau}_n}(r)\|)\mathrm{d}r < \infty.$$

We will now show that $(\mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n})_{n\in\mathbb{N}}$ is also uniformly integrable. Indeed, by the convexity of Φ we have

$$\begin{aligned} \sup_n \int_0^T \Phi(\|\mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n}(r)\|)\mathrm{d}r &= \sup_n \left(\int_0^T \tfrac{1}{2}\Phi(\|\mathbb{T}^{(1)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n}(r)\|)\mathrm{d}r + \int_0^T \tfrac{1}{2}\Phi(\|\mathbb{T}^{(2)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n}(r)\|)\mathrm{d}r\right) \\ &\overset{(1)}{=} \sup_n \int_0^T \chi^{(1)}_{\boldsymbol{\tau}_n}(r)\Phi(\|g_{\boldsymbol{\tau}_n}(r)\|)\mathrm{d}r + \sup_n \int_0^T \chi^{(2)}_{\boldsymbol{\tau}_n}(r)\Phi(\|g_{\boldsymbol{\tau}_n}(r)\|)\mathrm{d}r, \end{aligned}$$

where $\overset{(1)}{=}$ follows from direct calculations, cf. (5.6). Therefore, $(\mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n})_{n\in\mathbb{N}}$ is uniformly integrable as well. Hence, again by the Dunford-Pettis criterion, up to a (non-relabeled) subsequence $(\mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n})_{n\in\mathbb{N}}$ weakly converges in $\mathrm{L}^1([0,T];\mathscr{X})$ to some limit $\widetilde{g}$. In order to show that $\widetilde{g} = g$, it will be sufficient to prove that

$$\int_0^T \langle \phi, \mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n} - g\rangle_{\mathscr{X}}\,\mathrm{d}t \longrightarrow 0 \qquad \text{for all } \phi \in \mathrm{C}^0([0,T];\mathscr{X}^*).$$

In fact, since $\mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} g \to g$ in $\mathrm{L}^1([0,T];\mathscr{X})$, it will be sufficient to show that

$$\int_0^T \langle \phi, \mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n} - \mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} g\rangle_{\mathscr{X}}\,\mathrm{d}t \longrightarrow 0 \qquad \text{for all } \phi \in \mathrm{C}^0([0,T];\mathscr{X}^*).$$

With this aim, we compute the adjoint of the operator $\mathbb{T}^{(0)}_{\boldsymbol{\tau}}$. For $f \in \mathrm{L}^1([0,T],\mathbf{X})$ and $\phi \in \mathrm{C}^0([0,T],\mathbf{X}^*)$ we have

$$\begin{aligned}\int_0^T \langle \phi, \mathbb{T}^{(0)}_{\boldsymbol{\tau}} f\rangle_{\mathscr{X}}\,\mathrm{d}t &= \sum_{k=1}^{N_{\boldsymbol{\tau}}} \int_{\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}}} \langle \phi, \mathbb{T}^{(0)}_{\boldsymbol{\tau}} f\rangle_{\mathscr{X}}\,\mathrm{d}t + \sum_{k=1}^{N_{\boldsymbol{\tau}}} \int_{\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}}} \langle \phi, \mathbb{T}^{(0)}_{\boldsymbol{\tau}} f\rangle_{\mathscr{X}}\,\mathrm{d}t \\
&= \frac{1}{2}\Bigg(\sum_{k=1}^{N_{\boldsymbol{\tau}}} \int_{\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}}} \langle \phi, f\rangle_{\mathscr{X}}\,\mathrm{d}t + \int_{\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}}} \langle \phi, f(\cdot + \tau_k/2)\rangle_{\mathscr{X}}\,\mathrm{d}t \\
&\quad + \int_{\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}}} \langle \phi, f(\cdot - \tau_k/2)\rangle_{\mathscr{X}}\,\mathrm{d}t + \int_{\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}}} \langle \phi, f\rangle_{\mathscr{X}}\,\mathrm{d}t \Bigg) \\
&= \frac{1}{2}\Bigg(\sum_{k=1}^{N_{\boldsymbol{\tau}}} \int_{\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}}} \langle \phi, f\rangle_{\mathscr{X}}\,\mathrm{d}t + \int_{\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}}} \langle \phi(\cdot - \tau_k/2), f\rangle_{\mathscr{X}}\,\mathrm{d}t \\
&\quad + \int_{\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{left}}} \langle \phi(\cdot + \tau_k/2), f\rangle_{\mathscr{X}}\,\mathrm{d}t + \int_{\mathrm{I}^{k,\boldsymbol{\tau}}_{\mathrm{right}}} \langle \phi, f\rangle_{\mathscr{X}}\,\mathrm{d}t \Bigg) \\
&= \int_0^T \langle \mathbb{T}^{(0)}_{\boldsymbol{\tau}} \phi, f\rangle_{\mathscr{X}}\,\mathrm{d}t.\end{aligned}$$

Hence, for every $\phi \in \mathrm{C}^0([0,T];\mathscr{X}^*)$ we have

$$\int_0^T \langle \phi, \mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} g_{\boldsymbol{\tau}_n} - \mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} g\rangle_{\mathscr{X}}\,\mathrm{d}t = \int_0^T \langle \mathbb{T}^{(0)}_{\boldsymbol{\tau}_n} \phi, g_{\boldsymbol{\tau}_n} - g\rangle_{\mathscr{X}}\,\mathrm{d}t \longrightarrow 0,$$

because $\mathbb{T}^{(0)}_{\boldsymbol{\tau}_n}\phi \to \phi$ strongly in $\mathrm{L}^\infty([0,T];\mathscr{X}^*)$, since $\phi \in \mathrm{C}^0([0,T];\mathscr{X}^*)$ and $g_{\boldsymbol{\tau}_n} - g \rightharpoonup 0$ weakly in $\mathrm{L}^1([0,T];\mathbf{X})$. This concludes the proof of item (4) of the statement. □

Funding Open access funding provided by TU Wien (TUW).

Data availability: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

1. Artina, M., Cagnetti, F., Fornasier, M., Solombrino, F.: Linearly constrained evolutions of critical points and an application to cohesive fractures. Math. Models Methods Appl. Sci. **27**(2), 231–290 (2017)
2. Ambrosio, L., Gigli, N., Savaré, G.: Gradient flows in metric spaces and in the space of probability measures, second ed., Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel (2005)
3. Ambrosio, L.: Minimizing movements. Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **19**, 191–246 (1995)
4. Attouch, H.: Variational Convergence for Functions and Operators, Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA (1984)
5. Aubin, J.-P., Frankowska, H.: Set-Valued Analysis, Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA (reprint of the 1990 edition) (2009)
6. Brezis, H.: Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland Mathematics Studies, No. 5, North-Holland Publishing Co., Amsterdam (1973)

7. Castaing, C., Valadier, M.: Convex Analysis and Measurable Multifunctions. Lectures Notes in Mathematics, vol. 580. Springer, Berlin (1977)
8. Clément, P., Maas, J.: A Trotter product formula for gradient flows in metric spaces. J. Evol. Equ. **11**(2), 405–427 (2011)
9. Colli, P.: On some doubly nonlinear evolution equations in Banach spaces. Jpn. J. Ind. Appl. Math. **9**(2), 181–203 (1992)
10. Colli, P., Visintin, A.: On a class of doubly nonlinear evolution equations. Commun. Partial Differ. Equ. **15**(5), 737–756 (1990)
11. Crandall, M.G., Pazy, A.: Semi-groups of nonlinear contractions and dissipative sets. J. Funct. Anal. **3**, 376–418 (1969)
12. Dondl, P., Frenzel, T., Mielke, A.: A gradient system with a wiggly energy and relaxed EDP-convergence. ESAIM Control Optim. Calc. Var. **25**(68), 1–45 (2019)
13. Diestel, J., Uhl, Jr., J.J.: Vector measures, Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I. (with a foreword by B.J. Pettis) (1977)
14. Ioffe, A.D., Tihomirov, V.M.: Theory of extremal problems, Studies in Mathematics and its Applications, vol. 6. North-Holland Publishing Co., Amsterdam, Translated from the Russian by Karol Makowski (1979)
15. Kato, T., Masuda, K.: Trotter's product formula for nonlinear semigroups generated by the subdifferentials of convex functionals. J. Math. Soc. Jpn **30**(1), 155–162 (1978)
16. Knees, D., Negri, M.: Convergence of alternate minimization schemes for phase field fracture and damage. M3AS Math. Models Methods Appl. Sci. **27**(9), 1743–1794 (2017)
17. Kōmura, Y.: Nonlinear semi-groups in Hilbert space. J. Math. Soc. Jpn. **19**, 493–507 (1967)
18. Mielke, A.: On evolutionary Γ-convergence for gradient systems (Ch. 3), Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity. In: Muntean, A., Rademacher, J., Zagaris, A. (eds) Lecture Notes in Applied Mathematics Mechanics, Vol. 3. Springer, 2016, pp. 187–249. Proceedings of Summer School in Twente University (2012)
19. Mielke, A.: Relating a rate-independent system and a gradient system for the case of one-homogeneous potentials. J. Dyn. Differ. Eqns. **34**, 3143–3164 (2022)
20. Mielke A.: An introduction to the analysis of gradient systems. Script of a lecture course 100, WIAS Preprint 3022 (2023). arXiv:2306.05026
21. Mielke, A., Rossi, R.: Balanced-viscosity solutions to infinite-dimensional multi-rate systems. Arch. Ration. Mech. Anal. **247**(53), 1–100 (2023)
22. Mielke, A., Rossi, R.: On De Giorgi's lemma for variational interpolants in metric and Banach spaces. WIAS Preprint 3127 (submitted) (2024). arXiv:2409.00976
23. Mielke, A., Montefusco, A., Peletier, M.A.: Exploring families of energy-dissipation landscapes via tilting: three types of EDP convergence. Contin. Mech. Thermodyn. **33**(3), 611–637 (2021)
24. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation. I: basic theory, Grundlehren der mathematischen Wissenschaften, vol. 330. Springer, Berlin (2006)
25. Mielke, A., Peletier, M.A., Stephan, A.: EDP-convergence for nonlinear fast-slow reaction systems with detailed balance. Nonlinearity **34**(8), 5762–5798 (2021)
26. Mielke, A., Rossi, R., Savaré, G.: Nonsmooth analysis of doubly nonlinear evolution equations. Calc. Var. Partial Differ. Equ. **46**(1–2), 253–310 (2013)
27. Mielke, A., Rossi, R., Savaré, G.: Balanced viscosity (BV) solutions to infinite-dimensional rate-independent systems. J. Eur. Math. Soc. **18**(9), 2107–2165 (2016)
28. Rossi, R., Mielke, A., Savaré, G.: A metric approach to a class of doubly nonlinear evolution equations and applications. Ann. Sc. Norm. Super. Pisa Cl. Sci. **7**(1), 97–169 (2008)
29. Rossi, R., Savaré, G.: Gradient flows of non convex functionals in Hilbert spaces and applications. ESAIM Control Optim. Calc. Var. **12**(3), 564–614 (2006)
30. Roubíček, T.: Thermodynamics of rate-independent processes in viscous solids at small strains. SIAM J. Math. Anal. **42**(1), 256–297 (2010)
31. Roubíček, T., Thomas, M., Panagiotopoulos, C.G.: Stress-driven local-solution approach to quasistatic brittle delamination. Nonlinear Anal. Real World Appl. **22**, 645–663 (2015)
32. Sandier, E., Serfaty, S.: Gamma-convergence of gradient flows with applications to Ginzburg–Landau. Commun. Pure Appl. Math. **57**(12), 1627–1672 (2004)
33. Zagrebnov VA, Neidhardt H, Ichinose T (2024) Trotter–kato product formulae, Springer book series: Operator Theory: Advances and Applications