DIPLOMARBEIT

The Cramer-Lundberg model under proportional and XL reinsurance

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unter der Anleitung von

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Kurzfassung

In dieser Arbeit betrachten wir ein Versicherungsunternehmen mit einem Anfangskapital x und einer Prämie, die nach dem Erwartungswertprinzip berechnet wird. Für den Überschuss wird das Cramér-Lundberg-Modell verwendet. Dieses Unternehmen hat die Möglichkeit, einen Rückversicherungsvertrag abzuschließen, der eine Kombination aus proportionaler und XL-Rückversicherung darstellt. Der proportionale Faktor ist fix gewählt, während der Selbstbehalt der XL-Rückversicherung dynamisch über die Zeit gewählt werden kann.

Die Überlebenswahrscheinlichkeit über einen unendlichen Zeitraum, definiert als die Wahrscheinlichkeit, dass der Überschussprozess über einen unendlichen Zeithorizont nicht negativ bleibt, ist ein grundlegendes Maß für die Solvenz eines Versicherers. Daraus ergibt sich die Frage: Gibt es eine optimale Rückversicherungsstrategie, die die Überlebenswahrscheinlichkeit maximiert, d.h. die Ruinwahrscheinlichkeit minimiert? Diese Arbeit beschäftigt sich mit dieser Fragestellung.

Das Problem, die optimale Rückversicherung zu finden, wird mithilfe der Hamilton-Jacobi-Bellman-Gleichung beschrieben. Anschließend beweisen wir die Existenz und Eindeutigkeit der Lösung dieses Problems. Darüber hinaus stellen wir Beispiele für Lösungen vor, die durch Computersimulationen für den Fall exponentiell verteilter Schadenhöhen erhalten wurden. Zusätzlich zeigen wir, wie Variationen anderer Parameter die optimale Rückversicherungsstrategie beeinflussen.

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Abstract

In this thesis, we consider an insurance company with initial capital x and premium calculated by expected value principle. For the risk model, the Cramér-Lundberg model is used. This company has the possibility to buy a reinsurance contract, which is a combination of proportional and excess-loss reinsurance. The proportional factor is fixed and retention level of the XL reinsurance can be chosen dynamically in time.

The infinite-time survival probability, defined as the likelihood that the surplus process remains non-negative over an infinite time horizon, is a fundamental measure of an insurer's solvency. This leads to the question: is there an optimal reinsurance strategy that maximizes the survival probability, i.e., minimizes the ruin probability? This thesis deals with this question.

The problem of finding the optimal reinsurance is defined using the Hamilton-Jacobi-Bellman equation. Then we prove the existence and uniqueness of the solution to this problem. Furthermore, we provide examples of solutions obtained through computer simulations for the case of exponentially distributed claim sizes. Additionally, we illustrate how variations in other parameters affect the optimal reinsurance strategy.

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Wien, am Datum

Name des Autors

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1 Introduction

In the insurance industry, managing risk effectively is essential to ensuring the long-term solvency and stability of companies. One critical measure of an insurer's financial stability is their survival probability, which measures the likelihood of avoiding insolvency over a given time period. Insurers face the challenge of balancing their exposure to risk while maintaining profitability, particularly when confronted with large or catastrophic claims. Reinsurance plays a crucial role in this context, allowing insurers to share part of their risks with reinsurers. However, while reinsurance provides essential risk protection, its benefits come at a cost.

Determining the optimal reinsurance strategy is crucial to maximizing the insurer's survival probability. An overly aggressive strategy results in high reinsurance premiums, which reduce the insurer's profitability. On the other hand, a strategy that is too cautious could expose the insurer to an unacceptably high risk of failure. Optimal reinsurance strategies help insurers find this balance, ensuring their solvency.

Dynamic reinsurance strategies have long been studied in the context of minimizing the ruin probability in classical risk models. The foundation in this area was established by Schmidli (2001) [9], who examined proportional reinsurance treaties. Expanding upon this, Hipp and Vogt (2003) [6] introduced dynamic excess-of-loss reinsurance. Schmidli et al. (2002) [10] further analyzed optimal investment and reinsurance strategies aimed at reducing ruin probability, particularly in scenarios involving Pareto-distributed claim sizes, and demonstrated the effectiveness of such strategies in improving the insurer's financial stability with higher initial surpluses. Schmidli's (2007) [11] extensive overview provides valuable insights into ruin probability minimization using reinsurance in both classical and diffusion risk models. Additionally, Hipp and Taksar (2010) [5] made significant contributions by focusing on non-proportional reinsurance contracts.

In this thesis, we seek to maximise the survival probability over a combination of proportional and excess-of-loss treaties. The deductible of the XL reinsurance can be changed dynamically in time. By "dynamically", we refer to a strategy that is determined and adjusted continuously over time, based on the evolving risk position of the company. Premiums for both, insurer and reinsurer, are calculated based on the expected value premium principle.

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This thesis is structured as follows:

Chapter 2 gives a brief introduction to the mathematical model used in this thesis. It starts with the classical Cramér-Lundberg model, explaining its main ideas and reviewing some common premium principles. Subsequently, ruin, the ruin probability, and the net profit condition are defined.

Chapter 3 introduces the concept of reinsurance and its possible forms. Three types of reinsurance contracts are discussed: proportional reinsurance, excess-of-loss reinsurance, and a combination of these two types, referred to here as combined reinsurance.

In Chapter 4, the main part of this thesis is presented. The risk process with a dynamic reinsurance strategy is defined, and the problem statement is formulated. Using an example of survival probability with a fixed proportional factor a and a priority level b, we demonstrate the motivation for an optimal strategy. Lastly, we prove the existence of an optimal strategy and establish the uniqueness of the solution.

Chapter 5 provides a numerical computation of the optimal strategy and the corresponding survival probability for exponentially distributed claim sizes. We provide examples with different parameters to illustrate their impact on the optimal strategy and the corresponding survival probability. All examples are simulated using Python.

Chapter 6 concludes the thesis by summarizing its findings and exploring potential directions for future research.

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2 Mathematical background

Risk management and premium calculation are critical aspects of the insurance industry, where mathematical models like the Cramér-Lundberg model play a significant role. Therefore, we start the chapter by defining the Cramér-Lundberg model, then explain some principles for premium calculation. This chapter is ended by introducing the concept of ruin, its probability, and the definition of the net profit condition. All definitions provided in this chapter are based on the lecture notes for the *Risk and Ruin Theory* [7] course by Prof. Friedrich Hubalek and Prof. Julia Eisenberg [3] from the Vienna University of Technology.

2.1 Cramér-Lundberg model

The Cramér-Lundberg model, known as the classical risk model and considered a fundamental framework in risk theory, was first proposed by Filip Lundberg and later expanded upon by Harald Cramér. The model is used to describe the risk process. The risk process represents the value of the company over time, where the company's capital increases through premium payments from policyholders and decreases through claim payments covered by insurance.

To define the Cramér-Lundberg model, we need to introduce some important concepts.

Since the claim sizes in a certain time are the most significant variable for proper management of an insurance company and as they are uncertain, we can determine only the probability of claims occurrence.

Definition 2.1.1. *Claim sizes are modeled as a sequence of a.s. positive, independent, and identically distributed random variables, denoted by $(Y_n)_{n\in\mathbb{N}}$. Each Y_n corresponds to the amount of an individual claim within an insurance portfolio. The expected value of a claim size is defined as $\mathbb{E}[Y_1] = \mu < \infty$ and the variance of a claim size is defined as $Var[Y_1] = \sigma \leq +\infty$.*

Definition 2.1.2. *Inter-occurrence times constitute a sequence of independent and identically exponentially distributed random variables $(X_n)_{n\in\mathbb{N}}$, where each X_n represents the time between the $(n-1)$-th and n-th occurrences within an insurance portfolio. The expected value (mean) of an inter-occurrence time is defined as $\mathbb{E}[X_n] = \frac{1}{\lambda}$, where λ is the rate at which events (claims) occur. It is assumed that the expected value is finite, facilitating mathematical analysis.*

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As the number of claims occurring in the portfolio in the time interval $[0, t]$ for all $t > 0$ happens, we describe it with the family of random variables $\{N_t\}$, which we call the claim arrival process.

Definition 2.1.3. *The random process $\{N_t\}_{t\geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$ if it satisfies the following conditions:*

1. *The process starts at zero, meaning $N_0 = 0$ with probability 1.*
2. *For any $0 \leq s < t$, the difference $N_t - N_s$ has the same distribution as N_{t-s}, which is termed as the process being increment stationary.*
3. *For any $0 \leq s < t$, the difference $N_t - N_s$ is independent of the past, represented by the family of random variables $\{N_u\}_{u\leq s}$, which signifies the process being increment independent.*
4. *For every $t > 0$, the random variable N_t follows a Poisson distribution with parameter λt, given by*
$$P[N_t = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad \textit{for } n \in \mathbb{N}$$

Let $T_n = \sum_{k=1}^{n} X_k$ be the occurrence time of the n-th claim. The process $(T_n)_{n\in\mathbb{N}_0}$ represents the renewal process. The claim arrival process up to a certain time t is a homogeneous Poisson Process $\{N_t\}_{t>0}$ where

$$N_t = \sup\{n \in \mathbb{N}_0 : T_n < t\}.$$

We describe the claim sizes by a sequence of identically, independent distributed random variables $\{Y_i\}$, independent of $\{N_t\}$, where Y_i is a random variable describing the size of the i-th claim. The total claim in the time interval is described by the family of random variables $\{S_t\}$, which is equal to

$$S_t = \sum_{i=1}^{N_t} Y_i.$$

The insurer of this portfolio receives a premium, and we assume that they receive it at a constant rate, denoted by c, so that the total premium received in the time interval $[0, t]$ is equal to ct. Due to the assumptions, the function that models the premium income is deterministic.

Also, we assume that at time 0, the insurer set aside an amount of money for the portfolio, which is called the initial surplus or initial capital and denoted by x. The value of R_t is known only at time $t = 0$. In other moments $t > 0$, it is a non-negative random variable, as it depends on the claims, and we can write it as follows:

$$R_t = x + c_t - S_t.$$

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Therefore, $\{R_t\}$ is a random process called the surplus process or the risk process. It should be noted that this is a simplified model in which we have neglected inflation and other dynamic changes in the portfolio. In addition, we assume that the claims are resolved immediately after they occur, and the costs of resolving the claims are included in the premium. With these assumptions, the insurer's profit is known at the end of the year. In practice, claims are usually resolved with a small delay.

Let $(\Omega, \mathcal{F}, P)$ be the probability space on which the sequences $(Y_n)_{n\in\mathbb{N}}$, $(X_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$, and $\{N_t\}_{t>0}$ are defined as in Definitions 2.1.1, 2.1.2 and 2.1.3.

Definition 2.1.4. *In the Cramér-Lundberg model, also called the classical risk model, the risk process* $\{R_t\}_{t>0}$ *is defined as*

$$R_t = x + ct - S_t, \quad t \geq 0$$

where:

- $x \geq 0$ *represents the initial capital.*
- $c > 0$ *is the constant premium rate per unit time. We adopt the net premium calculation, assuming that the expected premium over a period equals the expected total claims over the same period, i.e.,* $ct = \mathbb{E}[S_t]$.
- *The process* $\{S_t\}_{t>0}$ *denotes the aggregate claim amount at time* t*, defined as* $S_t = \sum_{k=1}^{N_t} Y_k$*, where* N_t *signifies the number of claims up to time* t*, and* Y_k *represents the size of the* k*-th claim.*

The Cramér-Lundberg model is characterized by the following assumptions:

1. The claim arrival process $\{N_t\}_{t>0}$ is a homogeneous Poisson process with intensity $\lambda > 0$.
2. The claim sizes are independent and identically distributed random variables, independent of $\{N_t\}$.
3. Premiums are received linearly over time.

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2.2 Premium calculation principles

In the world of insurance, premiums represent the payments made by policyholders to insurers in exchange for coverage against the risks they wish to insure. These payments form the cornerstone of the insurance industry, ensuring that insurers can fulfill their obligations while maintaining financial stability. There are various methods for calculating premiums, and some of these methods will be explained in this section. For more, see [1]

Denoted by P_Y, the premium charged by an insurance provider to address a risk Y is significant. By risk Y, we mean that the claims associated with it are distributed as the random variable Y. The premium P_Y is intricately linked to Y, with its determination guided by what we term a premium calculation principle.

The pure premium principle

According to the pure premium principle, premiums are set equal to the expected value of risk Y.

$$P_Y = \mathbb{E}[Y]$$

From the insurer's perspective, this principle is not particularly appealing, as it only covers the expected claims of the insurer without considering long-term viability.

The expected value principle

According to the expected value principle, premiums are set equal to the expected value of the risk Y, with an additional amount called the premium loading factor $\theta > 0$.

$$P_Y = (1 + \theta)\mathbb{E}[Y]$$

The safety loading typically refers to the additional charge applied to the pure premium to cover administrative costs, profit margins, and other expenses.

The variance principle

According to the variance principle, the variance is used as a measure of the variability of the risk.

$$P_Y = \mathbb{E}[Y] + \theta \mathrm{Var}[Y]$$

In this method, $\mathbb{E}[Y]$ represents the expected value of the claims, while $\theta \mathrm{Var}[Y]$ denotes the variance of the risk distribution associated with the risk Y with $\theta > 0$ safety loading.

The standard deviation principle

According to the standard deviation principle, the standard deviation is used as a measure of the reliability of the risk.

$$P_Y = \mathbb{E}[Y] + \theta\sqrt{\mathrm{Var}[Y]}$$

In this method, $\mathbb{E}[Y]$ represents the expected value of claims, while $\sqrt{\mathrm{Var}[Y]}$ denotes the standard deviation of risk distribution associated with the risk Y with $\theta > 0$ safety loading.

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2.3 Ruin, ruin probability, and net profit condition

In the previous section, we introduced basic concepts and defined the Cramér-Lundberg model. We are interested in determining the ruin probability.

Recall that the aggregate claim amount up to time t is denoted by $S_t = \sum_{k=1}^{N_t} Y_k$. This implies that the sequence of claim sizes (X_n) is a sequence of positive independent identically distributed random variables with a common distribution Q, and it is independent of the sequence of inter-arrival times between consecutive claim arrivals (T_n) given by $T_0 = 0, T_n = X_1 + X_2 + \ldots + X_n$ for $n \geq 1$, where the waiting times between two consecutive claims Y_n are positive independent identically distributed random variables. Additionally, the claim counter process $\{N_t\}_{t>0}$ is independent of the sequence of claim sizes (Y_n).

In addition to the above, we assume that the premium collected up to time t is equal to ct, where c is the constant premium rate.

The risk process is defined as $R_t = x + ct - S_t$, for $t \geq 0$, which represents the insurer's capital at time t, and the process $\{R_t\}_{t>0}$ describes the cash flow over time in the insurer's portfolio. If $R_t \geq 0$, the company has capital. Otherwise, the company's surplus becomes negative.

Definition 2.3.1. *The event that the value of R_t at some point becomes less than zero is called* ***ruin****, denoted as*

$$Ruin = \{R_t < 0, \text{ for some } t > 0\}.$$

Definition 2.3.2. *The time τ at which the risk process falls below zero for the first time is referred to as the* ***time of ruin****, and it is defined as*

$$\tau = \inf\{t > 0 : R_t < 0\}.$$

Note that the random variable τ may not necessarily be a real random variable, as it is possible for ruin to never occur, *i.e.*, $\tau = \infty$.

Definition 2.3.3. *The* ***ruin probability*** *given an initial capital $x \geq 0$ and a constant premium rate c is expressed as:*

$$\begin{aligned}
\psi(x) &= P\{R_t < 0, \text{ for some } t > 0 \,|\, R_0 = x\} \\
&= P\left(x + ct - \sum_{k=1}^{N_t} Y_k < 0, \text{ for some } t > 0 \,|\, R_0 = x\right) \\
&= P(\tau < \infty, \,|\, R_0 = x).
\end{aligned}$$

We denote $\delta(x)$ by the probability that ruin never occurs given initial capital x. This probability is also referred to as the ***survival probability*** *and can be expressed as*

$$\delta(x) = 1 - \psi(x).$$

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In this definition, we used the fact that

$$\text{Ruin} = \bigcup_{t\geq 0}\{R_t < 0\} = \{\inf_{t\geq 0} R_t < 0\} = \{\tau < \infty\}.$$

Based on the construction of the risk process $\{R_t\}_{t>0}$, ruin can occur only at times $t = T_n$, for some $n \geq 1$, because the value of the process $\{R_t\}_{t\geq 0}$ increases linearly in intervals $[T_n, T_{n+1})$. We call the series $R(T_n)$ the skeleton of the risk process $\{R_t\}_{t\geq 0}$. Using the skeleton of the process, we can express the probability of destruction in terms of the waiting time between two consecutive claims X_n, the size of the claims Y_n, and the premium rate c.

Based on the construction, it is clear that this probability is not determined simply, since it boils down to a very complex study of stochastic processes.

The ruin probability $\psi(x)$ is a significant indicator of the capital changes of an insurance company over time. For determining the parameters that affect the risk process the most, it is crucial to avoid ruin with probability 1, and it is likely that the random walk S_n crossing the threshold S needs to be small enough that the occurrence of ruin can be excluded from the consideration of whether the initial capital is sufficiently large.

We assume that the mathematical expectation $\mathbb{E}[Y_n]$ is finite. This assumption about the ultimate expected length of the interval of time between two consecutive claims is natural and often met in practice. Furthermore, we also know that the mathematical expectation $\mathbb{E}[Z_n] = \mathbb{E}[Y_n] - c\mathbb{E}[N_n]$ is well defined and finite. Hence the random walk S_n satisfies any **law of large numbers**, that is

$$\frac{S_n}{n} \xrightarrow{a.s.} \mathbb{E}[Z_1] \text{ as } n \to \infty,$$

which implies that

$$\frac{S_n}{n} \xrightarrow{a.s.} 0 \text{ if } \mathbb{E}[Z_1] \text{ is negative.}$$

Thus, in terms of claims with a small ruin probability, the case $\mathbb{E}[Z_1] > 0$ is unacceptable. Also, it can be shown that in the case $\mathbb{E}[Z_1] = 0$, the ruin probability is equal to 1.

We conclude the following:

If the value $\mathbb{E}[Y_n]$ is finite and the condition $\mathbb{E}[Z_n] = \mathbb{E}[Y_n] - c \cdot \frac{1}{\lambda} \geq 0$ holds, then for every fixed $x > 0$, the ruin probability is equal to 1.

From this, we can deduce that every insurance company should set the premium rate c such that $\mathbb{E}[Z_n] \leq 0$. That is the only way to avoid ruin with probability 1.

Definition 2.3.4. *We say that the Cramér-Lundberg process satisfies the* ***net profit condition*** *if:*

$$\mathbb{E}[Z_1] = \mathbb{E}[Y_1] - c \cdot \frac{1}{\lambda} < 0.$$

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Also, the interpretation of the net profit condition is intuitive because it represents the condition that, in the interval between the two claims, the expected value of the claim size $\mathbb{E}[Y_1]$ should be less than the premiums earned during that period, represented as the expected premium $c\mathbb{E}[X_1]$.

If we assume the Cramér-Lundberg model and the net premium calculation, we get that $ct = \mathbb{E}[S_t] = \mathbb{E}[N_t]\mathbb{E}[Y_1] = \lambda t\mathbb{E}[Y_1]$. However, in the case of net premium calculation, where $c = \lambda\mathbb{E}[Y_1]$, we have that $\mathbb{E}[Z_1] = 0$. Thus, the ruin probability is equal to 1. Therefore, to avoid ruin with probability 1, it is necessary to apply a different principle of premium calculation that satisfies the net profit condition $c > \lambda\mathbb{E}[Y_1]$. To achieve a satisfactory premium rate that is not net premium calculated by method, we need to apply a coefficient $\rho > 1$, where $\rho = (1 + \theta)$, which further increase it.

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3 Managing risk via reinsurance

Reinsurance plays a crucial role in risk management for insurance companies, allowing them to reduce exposure to large claims and ensure financial stability. In this chapter, we first introduce the concept of reinsurance and its forms. After defining the fundamental principles, we illustrate how proportional and non-proportional reinsurance function through practical examples. Finally, we introduce the combined reinsurance treaty, which is the main topic of this thesis. All definitions provided in this chapter are based on the lecture notes provided for *Reinsurance* [2] course by Prof. Julia Eisenberg from Vienna University of Technology.

3.1 Reinsurance

Insurance is the transfer of risk in exchange for payment. Reinsurance is the insurance of the risk assumed by the insurer. In other words, reinsurance is insurance for insurance companies. In reinsurance, the primary insurer (cedent) and the reinsurer (cessionary) share the risk. Reinsurance allows primary insurers to manage their exposure to large or unpredictable risks by transferring part of the potential claim to a reinsurer. This process not only helps in spreading risk but also provides financial stability, enabling insurers to take on more policies and protect themselves from catastrophic losses.

Reasons to buy reinsurance can be:

- protecting against model risk,
- stabilizing business results,
- reducing required capital,
- increasing underwriting capacity,
- diversification
- support in risk assessment, pricing and management,
- reducing tax payment.

Reinsurance can be classified based on various criteria, including the share of risk.

Proportional reinsurance contracts are characterized by the equal sharing of both premiums and claims between the cedent and the reinsurer. In a qauota-share (QS) reinsurance contract, the cedent and the reinsurer share a predetermined percentage of both the premiums and the claims of each policy. This type of reinsurance contract distributes the risk and reward proportionally between the insurer and reinsurer based on the agreed-upon share.

Non-proportional reinsurance contracts do not depend on fixed shares of premiums and claims between the cedent and the reinsurer. Instead, these contracts provide coverage based on the occurrence of claims that exceed a certain threshold, known as the priority level or retention limit. There are several types of non-proportional reinsurance contracts. We concentrate on excess-of-loss (XL) reinsurance contract, which provides coverage for individual claims that exceed a predetermined threshold (priority level), protecting the insurer from large, catastrophic losses.

The transfer of risk from a primary insurer to a reinsurer is characterized by the concept of an **insurance form**. Each insurance form is associated with a function $s(\cdot, Y)$, known as the **self-insurance function**, which represents the portion of a claim amount Y that remains the responsibility of the primary insurer. Consequently, the reinsurer cover the remaining amount, given by $r(\cdot, Y) = Y - s(\cdot, Y)$.

We assume that the function s is non-negative, increases with Y, and satisfies the conditions $s(\cdot, 0) = 0$ and $0 \leq s(\cdot, Y) \leq Y$. Since the self-insurance function depends on the claim size but also on a parameter specific to the chosen reinsurance form, we will further explore this dependency in the next section through concrete examples.

The relationship between the policyholder and the primary insurer is, in formal terms, similar to the relationship between the primary insurer and the reinsurer. Therefore, both areas can be described to a certain point using similar models. Reinsurance arrangements can be designed to cover either individual claims or aggregate claims.

In a reinsurance contract, the total claim amount is divided into two components:

$$S = S^E + S^R$$

Here, S represents the total claim of the primary insurer. The part retained by the primary insurer is given by $S^E = \sum_{i=1}^{N} s(\cdot, Y_i)$, while the reinsurer's share of the total claim is denoted by $S^R = \sum_{i=1}^{N} r(\cdot, Y_i)$.

In the next section, we will provide a more detailed explanation of common reinsurance forms and their combination.

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3.2 Quota-share (QS) reinsurance

As mentioned earlier, the duties of the contractual parties in proportional reinsurance agreements are determined in relation to the risks that the insurer undertake from the policyholder. The proportional factor that defines this contract specifies that the cedent's stake in the risk is valued within the range $[0, 1]$ and is denoted as a. Here $a = 0$ means full reinsurance, and $a = 1$ means no reinsurance.

Within the framework of proportional reinsurance contracts, the cedent and the reinsurer share each claim Y according to the predefined proportional factor a:

- **The cedent's share :** $s(a, Y) = aY$,
- **The reinsurer's share :** $r(a, Y) = Y - s(a, Y) = (1 - a)Y$.

To illustrate the principles discussed, consider an insurance company that enters into a quota-share reinsurance agreement to manage its risk exposure. If the proportional factor a is set at 0.25, the cedent will cover 25% of the claims, while the reinsurer will cover the remaining 75% .

Suppose a claim Y of \$10000 occurs. The cedent's share of the claim would be $0.25 \times \$10000 = \$2500$, and the reinsurer would cover the remaining \$7500 of the claim. By transferring a quarter of the risk, the insurance company can stabilize its finances and protect itself against large, aggregated losses that could arise from a disaster. In Figure 3.1, created by author using Python, we have graphically illustrated the impact of different proportional factors on paid claims.

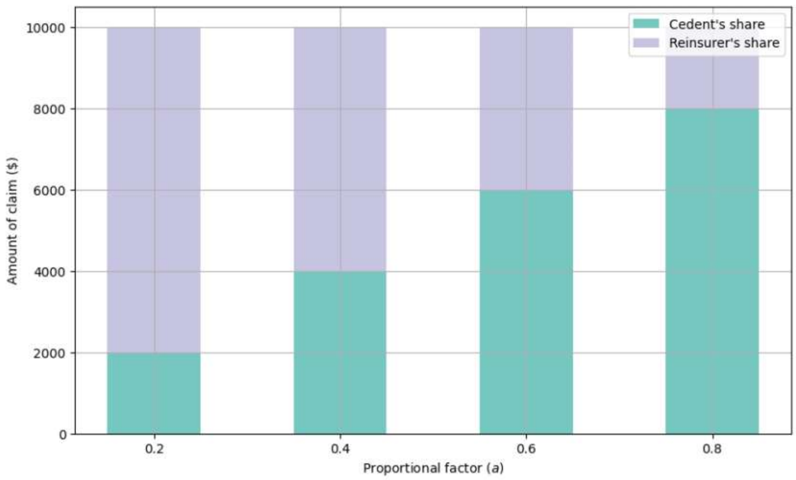

Figure 3.1: Impact of proportion factor on paid claims in proportional reinsurance.

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3.3 Excess-of-loss (XL) reinsurance

In non-proportional reinsurance contracts for excess-of-loss protection, the basis for determining the obligations of the contracting parties is the size of the claim. When concluding such contracts, the priority level of the claim by the cedent is determined, and we denote it as b for $0 \leq b \leq \infty$. Here $b = 0$ means full reinsurance, and $b = \infty$ means no reinsurance. Each claim Y is distributed between the cedent and the reinsurer based on the priority level b as follows:

- **The cedent's share:** $s(b, Y) = \min\{Y, b\}$,
- **The reinsurer's share:** $r(b, Y) = Y - s(b, Y) = \max\{Y - b, 0\} = (Y - b)^+$.

To illustrate the principles discussed, consider an insurance company that has an excess-of-loss reinsurance contract with a priority level of \$10000. The listed claim sizes in the portfolio reinsured by this contract are \$20000, \$12000, and \$7500. The cedent participates in these claims, respectively, with \$10000, \$10000, and \$7500, while the reinsurer are obligated to pay, the amounts of \$10000 , \$2000, and \$0. The provided example has been graphically illustrated in Figure 3.2, using Python.

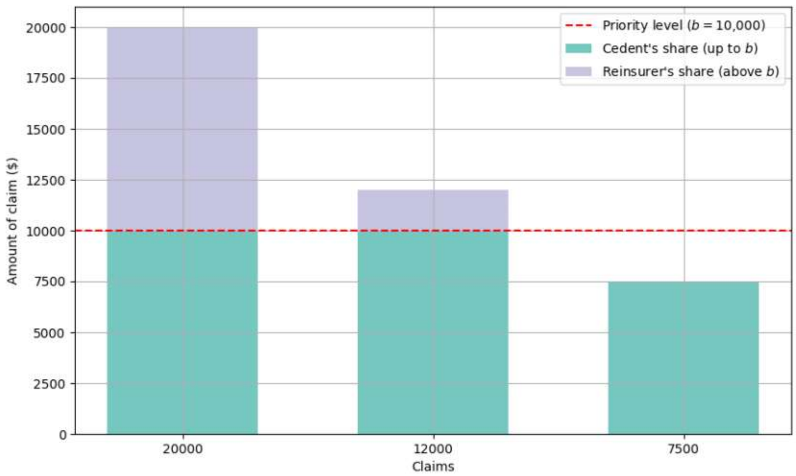

Figure 3.2: Distribution of claim payment for XL reinsurance for various claim sizes.

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3.4 Combined reinsurance

In this thesis, we combine the principles of both quota-share (QS) and excess-of-loss (XL) reinsurance under the following conditions: if the claim size Y is below or equal to a level b, the cedent pays a portion of the claim according to the proportional rate a, and the reinsurer covers the remaining portion. If the claim is above the level b, the cedent pays their portion ab, and the reinsurer covers the remaining claims above ab.

Let Y be the claim size. We define the payments of the cedent and the reinsurer as follows:

- **The cedent's share:** $s(a,b,Y) = \min\{aY, ab\} = aY \cdot \mathbf{1}_{\{Y \leq b\}} + ab \cdot \mathbf{1}_{\{Y > b\}}$,
- **The reinsurer's share:** $r(a,b,Y) = Y - s(a,b,Y) = (1-a)Y \cdot \mathbf{1}_{\{Y \leq b\}} + (Y - ab) \cdot \mathbf{1}_{\{Y > b\}}$.

To illustrate the principles discussed, consider an insurance company with a combined reinsurance contract where the priority level b is \$10000, and the proportion level is $a = 0.75$. Suppose the following three scenarios occur in the portfolio:

- *Scenario 1:* A claim of \$7500 occurs. Since $8000 \leq b$, the cedent covers: $0.75 \times 7500 =$ \$5625 and the reinsurer cover remaining $(1-0.75) \times 7,500 =$ \$1875.
- *Scenario 2:* A claim of \$12000 occurs. Since $12000 > b$, the cedent covers $0.75 \times 10000 =$ \$7500 and the reinsurer cover remaining $12000 - 7500 =$ \$4500.
- *Scenario 3:* A claim of \$20000 occurs. Since $20000 > b$, the cedent covers $0.75 \times 10000 =$ \$7500 and the reinsurer cover remaining $20000 - 7500 =$ \$12500.

The provided example has been graphically illustrated in Figure 3.3, using Python.

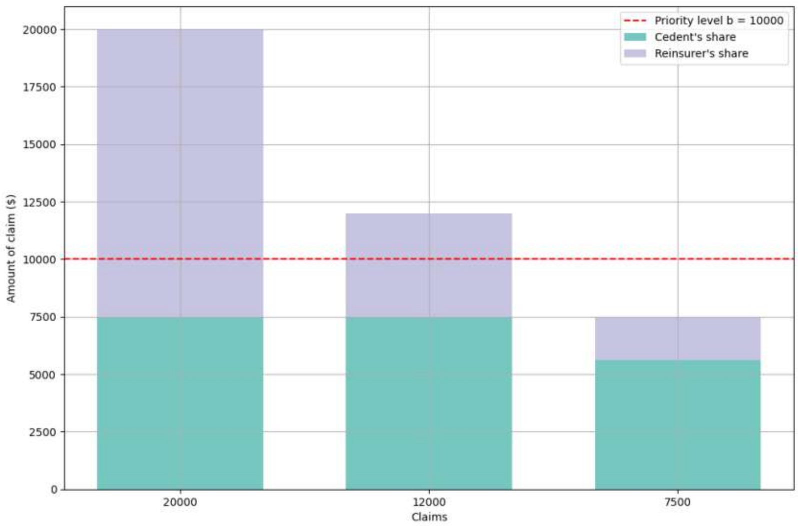

Figure 3.3: Distribution of claim payment for combined reinsurance for various claim sizes.

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Figures 3.4 and 3.5 illustrate the difference in the claims portion paid by the cedent and reinsurer for the three above-mentioned types of reinsurance. In this example, the proportional factor for quota-share and combined reinsurance is set at $a = 0.75$, while the priority level in excess-of-loss and combined reinsurance is set at $b = 100$.

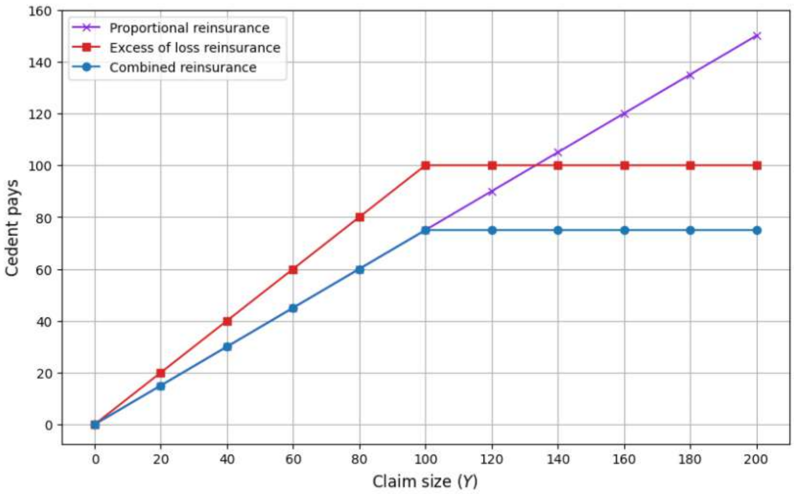

Figure 3.4: Cedent payments for all three reinsurance types.

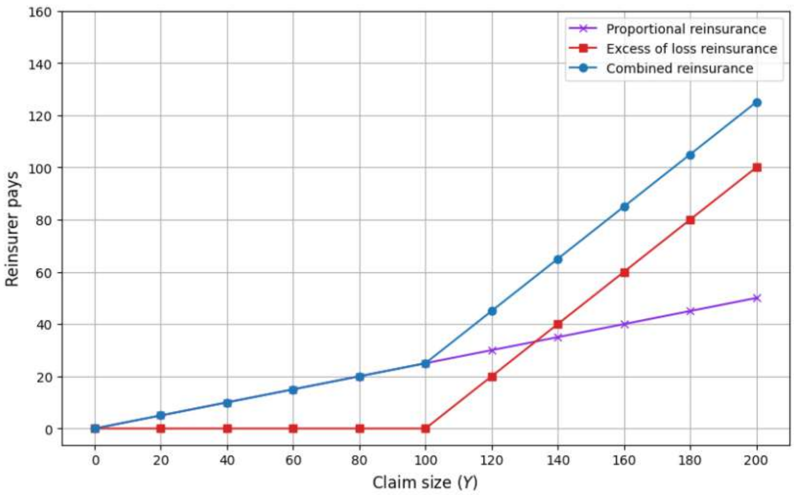

Figure 3.5: Reinsurer payments for all three reinsurance types.

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4 Optimizing combined reinsurance strategy

Assume the insurance company has the possibility and chooses to buy combined reinsurance coverage, combining excess-of-loss (XL), with a priority level b, which can be changed continuously, and quota-share (QS) reinsurance with the fixed proportional level a. We want to find the optimal strategy that minimizes the ruin probability.

4.1 Problem statement

Insurer premium rate per unit time ct is calculated on the principle of expected value, with safety loading $\eta > 0$.

$$c = (1+\eta)\lambda\mathbb{E}[Y] = (1+\eta)\lambda \int_0^\infty y\, dG(y).$$

In order for the cedent to afford this level of coverage, the cedent pays to the reinsurer the premium at the rate $h(a,b)$. The reinsurer also calculates their premium using the expected value principle with reinsurance safety loading θ, where $\theta > \eta > 0$ and $\rho = (1+\theta)$.

$$h(a,b) = \rho\lambda \left[(1-a)\mathbb{E}[Y \mid Y \leq b] \cdot P(Y \leq b) + \mathbb{E}[(Y-ab)^+ \mid Y > b] \cdot P(Y > b)\right].$$

We can write this in slightly different form

$$h(a,b) = \rho\lambda \left[(1-a)\int_0^b y\, dG(y) + (y-ab)(1-G(b))\right].$$

If $c \geq \rho\lambda\mathbb{E}[Y]$, which means reinsurance is cheaper than insurance, and the cedent wants to minimize their risk, then for all risks, they should choose the proportional factor $a = 0$ and $b = 0$, meaning to transfer all risks to reinsurance. However, this should not be the case in practice, so we further assume that it is $c < \rho\lambda\mathbb{E}[Y]$. In general, reinsurance is more expensive than insurance if $c < h(0)$, meaning that the insurance premium rate is less than the reinsurance premium rate.

We will denote premium rate left for the cedent after paying for reinsurance by $c(a,b) = c - h(a,b)$.

$$c(a,b) = \lambda \left[(1+\eta)\int_0^\infty y\, dG(y) - \rho\left[(1-a)\int_0^b y\, dG(y) + (y-ab)(1-G(b))\right]\right]$$

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In this reinsurance contract, the priority level b is assumed to be chosen dynamically, i.e., expressed as a time-dependent function b_t. The cedent adjusts the priority level b_t continuously throughout the contract's term $t \geq 0$, relying on the information accessible just before time t, and proportional factor a is fixed. This variable priority level b_t, representing the ratio at any given time t, defines the strategy.

In above-described, predictability is essential. In particular, at the time of a claim occurrence T_i, the reinsurance strategy is determined using information known up to T_{i-}. The predictability of the reinsurance strategy is a reasonable assumption in this scenario; without it, the insurer might switch the reinsurance strategy to full coverage at the moment a claim occurs. This would result in the reinsurer covering all claims while the insurer collects all the premiums.

Under these assumptions, we define R_t^b the cedent's risk process, for the sum of all claim payments up to time t, corresponding to the strategy b_t, with fixed proportional factor a and values of the priority level b_t at time t in $[0, \infty]$, modeled by the Cramér-Lundberg model as

$$R_t^b = x + \int_0^t c(a, b_v)\, dv - \sum_{i=1}^{N_t} s(a, b_{T_{i-}}, Y_i), t \geq 0.$$

The time of ruin is $\tau_b = \inf\{t \geq 0 : R_t^b < 0\}$. Let $\delta_{b_t}(x)$ be a the probability of survival with strategy b_t.

$$\delta_{b_t}(x) = P[\tau_b = \infty \mid R_0^b = x].$$

Our goal is to maximize the survival probability, by finding $\delta(x) = \sup_{b_t} \{\delta_b(x)\}$ and determining the optimal strategy, if it exists. It is necessary at every moment t to find the optimal retention level (strategy) b_t^*, such that $\delta(x) = \delta_{b_t^*}(x)$.

4.2 Motivating the Hamilton–Jacobi–Bellman equation

As demonstrated in [9], we begin by motivating the corresponding Hamilton-Jacobi-Bellman equation utilizing the following strategy:

$$b_t = \begin{cases} b, & \text{if } 0 \leq t \leq h \wedge T_1, \\ \tilde{b}_{t-h\wedge T_1}, & \text{if } t > h \wedge T_1 \text{ and } T_1 \wedge h < \tau. \end{cases}$$

Where $\epsilon > 0$ be fixed and arbitrary, $\tilde{b}_t(x)$ is a strategy such that $\delta_{\tilde{b}}(x) > \delta(x) - \epsilon$ and $h > 0$ is a small number. If ruin does not occur in the interval $(0, T_1 \wedge h)$, the new Cramér-Lundberg process begins at time $T_1 \wedge h$ with the new initial capital. Taking into account that inter-arrival times of the homogeneous Poisson process have an exponential distribution with parameter λ, it can be concluded that the density of the distribution of the random variable T_1 is of the form $\lambda e^{-\lambda t}$, for $t > 0$ and $\mathbb{P}[T_1 > h] = e^{-\lambda h}$ and we obtain

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from the law of total probability

$$\delta(x) \geq \delta_b(x) = e^{-\lambda h}\delta_{\tilde{b}}(x + c(a,b)h) + \int_0^h \int_0^\infty \delta_{\tilde{b}}(x + c(a,b)t - s(a,b,y))\, dG(y)\lambda e^{-\lambda t}\, dt$$
$$\geq e^{-\lambda h}\delta(x + c(a,b)h) + \int_0^h \int_0^\infty \delta(x + c(a,b)t - s(a,b,y))\, dG(y)\lambda e^{-\lambda t}\, dt - \epsilon.$$

Rewriting the equation using the formula for $s(a,b,y)$ along with boundary conditions for both t and y. Since we know that $\delta(x) = 0$ for all $x < 0$, this introduces additional constraints on the arguments of the δ-function in the integrals. The argument of the δ-function must be non-negative, meaning that we need to ensure the conditions inside the integrals satisfy $x + c(a,b)t - ay \geq 0$ and $x + c(a,b)t - ab \geq 0$.

$$\begin{aligned}\delta(x) \geq\ & e^{-\lambda h}\delta(x + c(a,b)h) \\ & + \int_0^h \left[\int_0^{\min\left(b, \frac{x+c(a,b)t}{a}\right)} \delta(x + c(a,b)t - ay)\, dG(y) \right. \\ & \left. + \mathbf{1}_{\{b \leq \frac{x}{a}\}} \int_b^\infty \delta(x + c(a,b)t - ab)\, dG(y) \right] \lambda e^{-\lambda t}\, dt - \epsilon.\end{aligned}$$

We can let $\epsilon = 0$ since ϵ was arbitrary. By considering the limiting value of the last expression when h tends to 0, it is concluded that the function $\delta(x)$ is right-continuous. For determining the differential equation that the function $\delta(x)$ satisfies, it is necessary to observe that

$$\begin{aligned}&\delta(x + c(a,b)h) - \delta(x) \geq \\ &\delta(x + c(a,b)h) - \left[e^{-\lambda h}\delta(x + c(a,b)h) + \int_0^h \int_0^{\min\left(b, \frac{x+c(a,b)t}{a}\right)} \delta(x + c(a,b)t - ay)\, dG(y) \right. \\ &\quad \left. + \mathbf{1}_{\{b \leq \frac{x}{a}\}} \int_b^\infty \delta(x + c(a,b)t - ab)\, dG(y) \right] \lambda e^{-\lambda t}\, dt.\end{aligned}$$

from which, by dividing by h, we obtain

$$\begin{aligned}c(a,b)\frac{\delta(x + c(a,b)h) - \delta(x)}{c(a,b)h} \geq\ & \frac{1 - e^{-\lambda h}}{h}\delta(x + c(a,b)h) \\ & + \frac{1}{h}\int_0^h \left[\int_0^{\min\left(b, \frac{x+c(a,b)t}{a}\right)} \delta(x + c(a,b)t - ay)\, dG(y) \right. \\ & \left. + \mathbf{1}_{\{b \leq \frac{x}{a}\}} \int_b^\infty \delta(x + c(a,b)t - ab)\, dG(y) \right] \lambda e^{-\lambda t}\, dt.\end{aligned}$$

By considering the limiting value when $h \to 0$, it is observed that the function $\delta(x)$ is right-differentiable and yields

$$c(a,b)\delta'(x) = \lambda \left(\delta(x) - \left(\int_0^{\min\left(b, \frac{x}{a}\right)} \delta(x - ay)\, dG(y) + \mathbf{1}_{\{b \leq \frac{x}{a}\}}\delta(x - b)(1 - G(b)) \right) \right)$$

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This must hold for every b. For h small and b optimal the corresponding survival probability is almost optimal, thus equality in the above calculations should be obtained. This yields the Hamilton–Jacobi–Bellman equation

$$\sup_{b>0}\left[c(a,b)\delta'(x)+\lambda\left(\int_0^{\min\left(b,\frac{x}{a}\right)}\delta(x-ay)\,dG(y)\right.\right.$$
$$\left.\left.+\mathbf{1}_{\{b\le\frac{x}{a}\}}\delta(x-b)(1-G(b))-\delta(x)\right)\right]=0. \tag{1}$$

The condition that the cedent retains some premium in the case of combined reinsurance is that the inequality $c\geq h(a,b)$ holds. We denote by $\underline{b}$ the value for which the equality $c=h(a,\underline{b})$ holds, so we require that $b>\underline{b}$.

As our goal is to increase the probability of survival, that is, to find an increasing solution to equation (1), it should hold that $\delta'(x)\geq 0$. In that case, equation (1) can be written in the form

$$\delta'(x)=\inf_{b>\underline{b}}\lambda\frac{\delta(x)-\left(\int_0^{\min\left(b,\frac{x}{a}\right)}\delta(x-ay)\,dG(y)+\mathbf{1}_{\{b\le\frac{x}{a}\}}\delta(x-b)(1-G(b)\right)}{c(a,b)}. \tag{2}$$

4.2.1 Survival probability for exponentially distributed claim sizes

We consider a case with exponentially distributed claims size $Exp(m)$. The complexity of the equation

$$c(a,b)\delta'(x)=\lambda\left(\delta(x)-\left(\int_0^{\min\left(b,\frac{x}{a}\right)}\delta(x-ay)\,dG(y)+\mathbf{1}_{\{b\le\frac{x}{a}\}}\delta(x-b)(1-G(b))\right)\right)$$

lies in the fact that it contains both the derivative and the integral of the function $\delta(x)$.

We will calculate the survival probability function in three different cases:

1. case with no reinsurance,
2. $x<ab$ and only proportional reinsurance is applied, and
3. $x>ab$ with excess-of-loss reinsurance.

No reinsurance

In case of no reinsurance ($a=1$ and $b=\infty$), the equation (2) can be written as follows :

$$c\delta'(x)=\lambda\left(\delta(x)-\left(\int_0^x\delta(x-y)\,dG(y)\right)\right)$$

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Assuming exponentially distributed claim sizes with parameter m, the equation simplifies to:

$$c\delta'(x) = \lambda\left(\delta(x) - \int_0^x \delta(y)\, me^{-m(x-y)}dy\right).$$

We solve this equation in a similar way as in [8], on pages 163-164. We can rewrite the integral in an equivalent form:

$$c\delta'(x) = \lambda\left(\delta(x) - e^{-mx}\int_0^x \delta(y)me^{my}dy\right).$$

Differentiating the $\delta'(x)$ gives the following:

$$c\delta''(x) = \lambda\left(\delta'(x) + me^{-mx}\int_0^x \delta(y)me^{my}dy - m\delta(x)\right)$$

$$c\delta''(x) = \lambda\left(\delta'(x) - m\left(\delta(x) - e^{-mx}\int_0^x \delta(y)me^{my}dy\right)\right)$$

$$c\delta''(x) = \lambda\left(\delta'(x) - m\frac{c}{\lambda}\delta'(x)\right)$$

$$c\delta''(x) = \lambda\delta'(x) - mc\delta'(x)$$

$$\delta''(x) = \delta'(x)\left(\frac{\lambda}{c} - m\right).$$

The general solution to this differential equation is (see Appendix 7.3)

$$\delta(x) = C_1 + C_2e^{-(m-\frac{\lambda}{c})x}.$$

Since $\lim_{x\to\infty}\delta(x) = 1$ it follows that $C_1 = 1$ and we can write $\delta(x) = 1 + C_2e^{-(m-\frac{\lambda}{c})x}$. Constant C_2 is the initial condition $\delta(0)$, i.e., the survival probability without initial capital. Firs derivative is then

$$\delta'(x) = -C_2\left(m - \frac{\lambda}{c}\right)e^{-(m-\frac{\lambda}{c})x}$$

and plugging it into

$$c\delta'(x) = \lambda\left(\delta(x) - \int_0^x \delta(y)\, me^{-m(x-y)}dy\right)$$

yields

$$-C_2(m\cdot c - \lambda)e^{-(m-\frac{\lambda}{c})x} =$$
$$= \lambda\left(1 + C_2e^{-(m-\frac{\lambda}{c})x} - (1 - e^{-mx}) + C_2e^{-mx}\frac{m\cdot c}{\lambda}\left(e^{\lambda\frac{x}{c}} - 1\right)\right).$$

For $x = 0$ yields $C_2 = -\frac{\lambda}{m\cdot c}$. Thus,

$$\delta(x) = 1 - \frac{\lambda}{m\cdot c}e^{-(m-\frac{\lambda}{c})x}.$$

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In Figure 4.1, we illustrate the survival probability function in the case with no reinsurance, where the model parameters are: claim arrival intensity $\lambda = 1$, expected claim size parameter $m = 1$, insurer safety loading $\eta = 0.5$, and reinsurer safety loading $\theta = 0.7$.

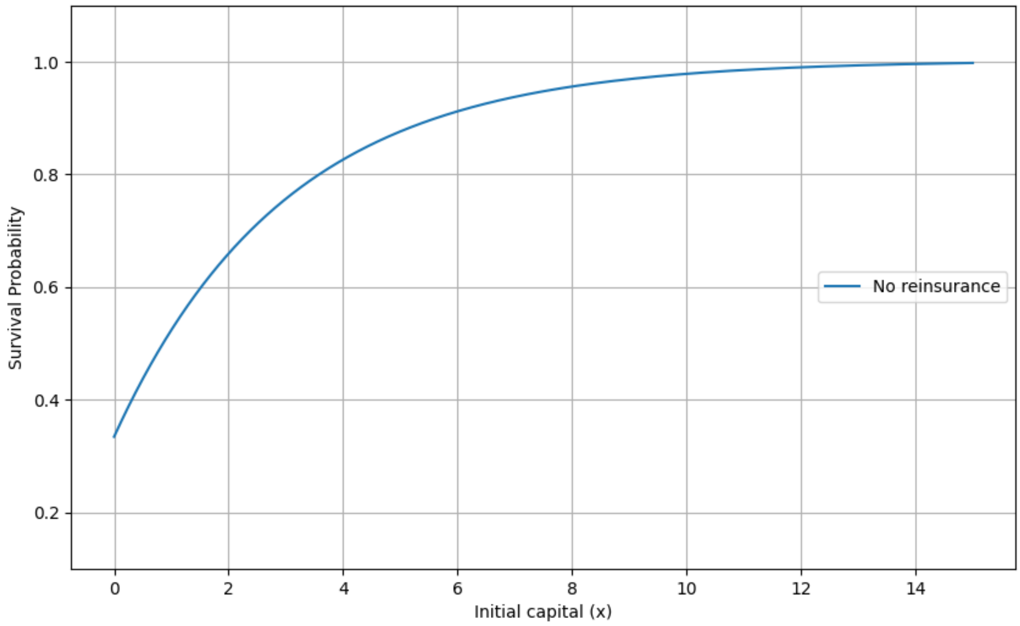

Figure 4.1: Survival probability in case of no reinsurance.

The initial capital less than ab

For $x < ab$, in equation (2), the term $\delta(x-ab)(1-G(b))$ becomes zero because the function $\delta(x-ab)$ is non-zero only in the case $x > ab$. The equation can be written as follows :

$$\frac{c(a,b)}{\lambda}\delta'(x) = \delta(x) - \int_0^{\frac{x}{a}} \delta(x-ay)\, dG(y).$$

Let $x - ay = z$, $\frac{dz}{dy} = -a$, then we obtain :

$$\frac{c(a,b)}{\lambda}\delta'(x) = \delta(x) - \frac{1}{a}\int_0^{x} \delta(z)\, dG\left(\frac{x-z}{a}\right) dz.$$

Using the exponential distribution claim size with parameter m, the equation simplifies to:

$$\frac{c(a,b)}{\lambda}\delta'(x) = \delta(x) - \frac{1}{a}\int_0^{x} \delta(z)\, me^{-m\frac{x-z}{a}} dz.$$

We can write the integral in a slightly different form.

$$\frac{c(a,b)}{\lambda}\delta'(x) = \delta(x) - e^{-m\frac{x}{a}}\frac{m}{a}\int_0^{x} \delta(z)\, e^{m\frac{z}{a}} dz.$$

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Differentiating the $\delta'(x)$ gives the following:

$$\frac{c(a,b)}{\lambda}\delta''(x) = \delta'(x) + \frac{m^2}{a^2}e^{-m\frac{x}{a}}\int_0^x \delta(z)\, e^{m\frac{z}{a}}dz - \frac{m}{a}\delta(x)$$

$$\frac{c(a,b)}{\lambda}\delta''(x) = \delta'(x) - \frac{m}{a}\left(\delta(x) - e^{-m\frac{x}{a}}\frac{m}{a}\int_0^x \delta(z)\, e^{m\frac{z}{a}}dz\right)$$

$$\frac{c(a,b)}{\lambda}\delta''(x) = \delta'(x) - \frac{m}{a}\cdot\frac{c(a,b)}{\lambda}\delta'(x)$$

$$\delta''(x) = \frac{\lambda}{c(a,b)}\delta'(x) - \frac{m}{a}\delta'(x)$$

$$\delta''(x) = \delta'(x)\left(\frac{\lambda}{c(a,b)} - \frac{m}{a}\right).$$

The general solution to this differential equation is :

$$\delta(x) = C_1 + C_2 e^{-(\frac{m}{a}-\frac{\lambda}{c(a,b)})x}.$$

The initial capital bigger than ab

For $x \geq ab$, in equation (2), the term $\delta(x-ab)(1-G(b))$ is non-zero. The equation can be written as following :

$$\frac{c(a,b)}{\lambda}\delta'(x) = \delta(x) - \left(\int_0^b \delta(x-ay)\, dG(y) + \delta(x-ab)(1-G(b))\right).$$

Integral $\int_0^b \delta(x-ay)\, dG(y)$, applying integration by parts and using the chain rule, we can write in slightly different form

$$\int_0^b \delta(x-ay)\, dG(y) = -\left[(1-G(y))\delta(x-ay)\right]_0^b + a\int_0^b \delta'(x-ay)(1-G(y))\, dy$$

$$= -(1-G(b))\delta(x-ab) + (1-G(0))\delta(x) + a\int_0^b \delta'(x-ay)(1-G(y))\, dy$$

$$= -(1-G(b))\delta(x-ab) + \delta(x) + a\int_0^b \delta'(x-ay)(1-G(y))\, dy.$$

Thus,

$$\frac{c(a,b)}{\lambda}\delta'(x) = \delta(x) - \Bigg[-(1-G(b))\delta(x-ab) + \delta(x) + a\int_0^b \delta'(x-ay)(1-G(y))\, dy + \delta(x-ab)(1-G(b))\Bigg]$$

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$$\frac{c(a,b)}{\lambda}\delta'(x) = \delta(x) - \left(\delta(x) + a\int_0^b \delta'(x-ay)(1-G(y))\,dy\right)$$

$$\frac{c(a,b)}{\lambda}\delta'(x) = -a\int_0^b \delta'(x-ay)(1-G(y))\,dy.$$

Let $x - ay = z$, $\frac{dz}{dy} = -a$, then we obtain :

$$\frac{c(a,b)}{\lambda}\delta'(x) = \int_{x-ab}^{x} \delta'(z)\left(1 - G\left(\frac{x-z}{a}\right)\right)\,dy.$$

Using exponentially distributed claim size with parameter m, the equation simplifies to:

$$\frac{c(a,b)}{\lambda}\delta'(x) = \int_{x-ab}^{x} \delta'(z)e^{-m\left(\frac{x-z}{a}\right)}\,dz.$$

We can write the integral in a slightly different form

$$\frac{c(a,b)}{\lambda}\delta'(x) = e^{-m\frac{x}{a}}\int_{x-ab}^{x} \delta'(z)e^{m\frac{z}{a}}\,dz.$$

Differentiating the $\delta'(x)$ gives the following:

$$\frac{c(a,b)}{\lambda}\delta''(x) = -\frac{m}{a}e^{-m\frac{x}{a}}\int_{x-ab}^{x} \delta'(z)e^{m\frac{z}{a}}\,dz + \delta'(x) - e^{-mb}\delta'(x-ab)$$

$$\frac{c(a,b)}{\lambda}\delta''(x) = -\frac{m}{a}\frac{c(a,b)}{\lambda}\delta'(x) + \delta'(x) - e^{-mb}\delta'(x-ab)$$

$$\delta''(x) = -\frac{m}{a}\delta'(x) + \frac{\lambda}{c(a,b)}\delta'(x) - \frac{\lambda}{c(a,b)}e^{-mb}\delta'(x-ab)$$

$$\delta''(x) = \delta'(x)\left(\frac{\lambda}{c(a,b)} - \frac{m}{a}\right) - \frac{\lambda}{c(a,b)}e^{-mb}\delta'(x-ab).$$

Unlike the case of no reinsurance, in case of combined reinsurance, it is not easy to find a general solution due to the additional term $\delta'(x-ab)$, which requires a specific solution. Therefore, we use computer simulations to illustrate the survival probability.

We consider the survival probability function under various scenarios.

1. *Scenario 1*: No reinsurance: $a = 1, b = \infty$,
2. *Scenario 2*: No XL reinsurance: $a = 0.6, b = \infty$,
3. *Scenario 3*: Pure excess-of-loss reinsurance: $a = 1, b = 2$,
4. *Scenario 4*: Combined reinsurance: $a = 0.6, b = 2$.

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Other model parameters are defined as following: claim arrival intensity $\lambda = 1$, expected claim size parameter $m = 1$, insurer safety loading $\eta = 0.5$, and reinsurer safety loading $\theta = 0.7$

Figure 4.2 illustrates all above scenarios. We can observe that for a small initial capital, the survival probability is highest in the case of no reinsurance. On the other hand, for a large initial capital, it is obvious that the combined reinsurance significantly increases the survival probability.

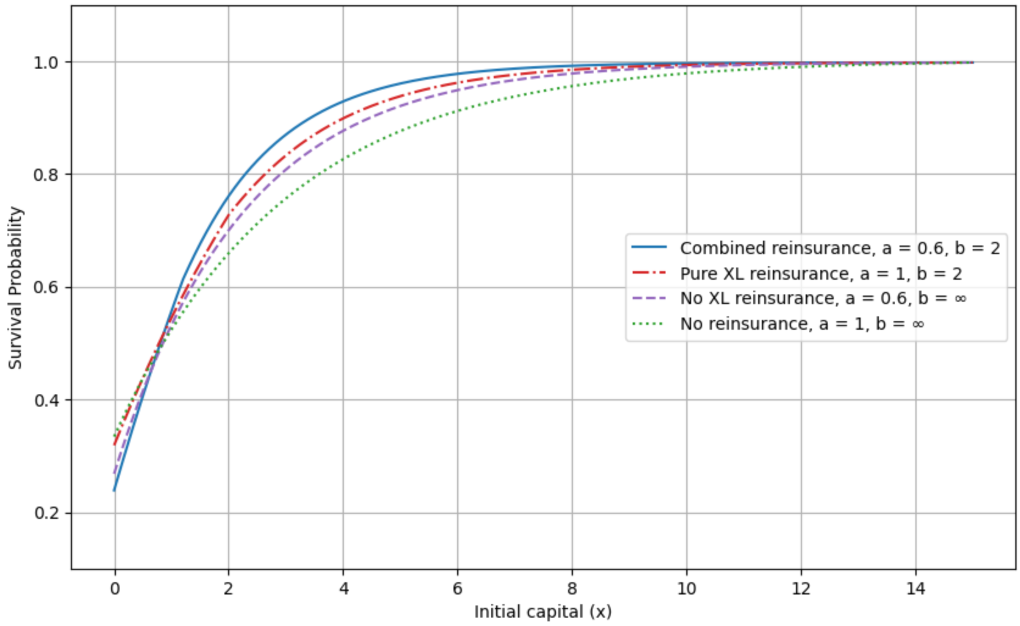

Figure 4.2: Survival probability for four different cases.

Figure 4.3 illustrates how the survival probability function depends on the proportional parameter a. We examine combined reinsurance, where in all three scenarios, the priority level is fixed at $b = 2.5$. We can observe that for small initial capital, the survival probability is higher when the proportional factor is higher. On the other hand, for large initial capital, the survival probability is higher in scenarios with a lower proportional factor.

Figures 4.4 and 4.5 illustrate how the survival probability function depends on the XL level b. In Figure 4.4, we consider the scenarios where the proportional factor is fixed at $a = 1$, meaning there is no proportional reinsurance, only pure XL reinsurance. In Figure 4.5, we examine combined reinsurance, where in all three scenarios, the proportional factor is fixed at $a = 0.75$.

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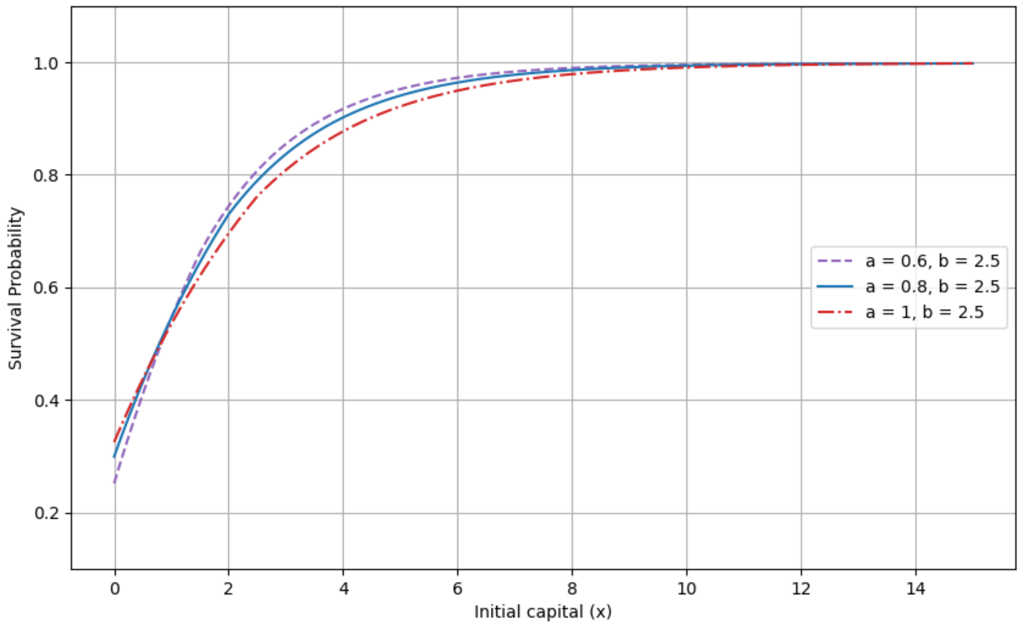

Figure 4.3: Survival probability for different proportional factor and priority level $b = 2.5$.

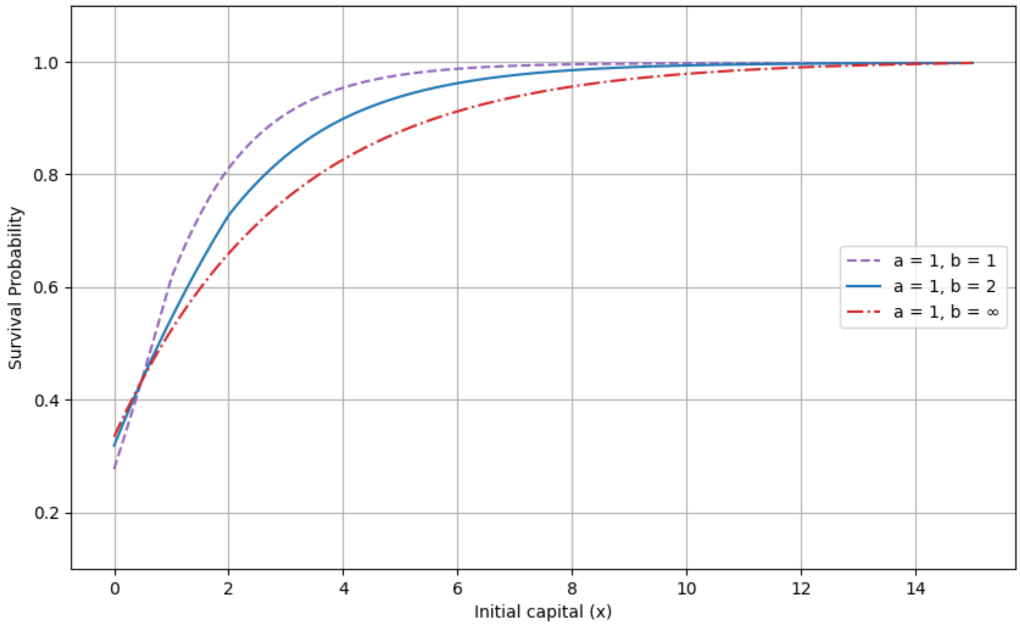

Figure 4.4: Survival probability for different priority level and no proportional reinsurance.

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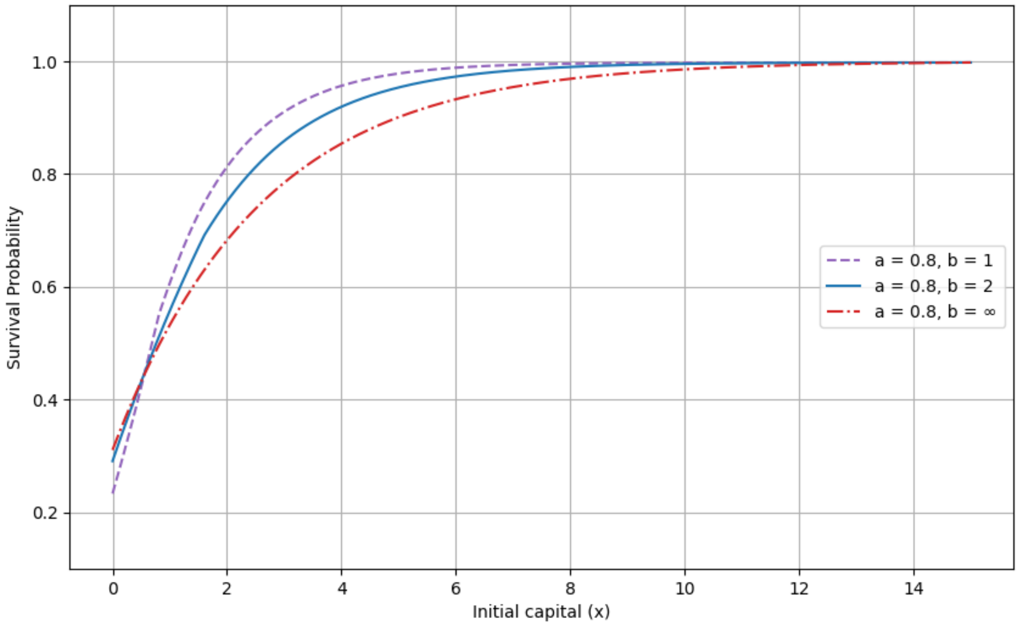

Figure 4.5: Survival probability for different priority level and proportional factor $a = 0.75$.

From these two graphs, we can observe that for small initial capital, the survival probability is higher when the priority level is large. On the other hand, for large initial capital, the survival probability is higher in scenarios with a smaller priority level. Additionally, we can observe that with a smaller proportional factor, the functions are closer to each other.

A common observation across each of these graphs is that in each of the scenarios mentioned above the functions intersect at certain points. Moreover, no single function achieves the maximum survival probability for all values of the initial capital. For this reason, we aim to find an optimal strategy that maximizes the survival probability all the time.

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4.3 Existence of an optimal strategy

In this section, we will prove the existence of a solution to equation (1), following the ideas and theorem presented in references [4] and [6].

Theorem 4.3.1. *Suppose the claim size distribution Q is absolutely continuous. Then, there exists a solution $V(x)$ that solves the Hamilton-Jacobi-Bellman (HJB) equation (1). This solution $V(x)$ is increasing function, continuous on $[0,\infty)$ and differentiable on $(0,\infty)$, and it satisfies $V(x) = 0$ for all $x < 0$ and converges to 1 as $x \to \infty$.*

Proof. Let us define a sequence of functions $V_n(x)$ as follows:

- Set $V_0(x) = \delta_0(x)$, which represents the survival probability with no reinsurance ($b = \infty$, or $b = \beta$ if $P(X \geq \beta) = 0$, and $a = 1$).
- Construct subsequent terms recursively:

$$V'_{n+1}(x) = \inf_{b>0}\left\{\lambda \frac{V_n(x) - \left(\int_0^{\min(b,\frac{x}{a})} V_n(x-ay)\,dG(y) + \mathbf{1}_{\{b\leq \frac{x}{a}\}}\int_b^\infty V_n(x-ab)\,dG(y)\right)}{c(a,b)}\right\}. \tag{3}$$

We prove by induction that the sequence $V'_n(x)$ is decreasing:

- Base case ($n = 0$): When no reinsurance is involved, the HJB equation becomes:

$$0 = \lambda\left(\int_0^x V_0(x-y)\,dG(y) - V_0(x)\right) + cV'_0(x).$$

This implies:

$$V'_0(x) = \lambda \frac{V_0(x) - \int_0^x V_0(x-y)\,dG(y)}{c}.$$

Substituting $n = 0$ into (3) yields:

$$V'_1(x) = \inf_{b>0}\left\{\lambda \frac{V_0(x) - \left(\int_0^{\min(b,\frac{x}{a})} V_0(x-ay)\,dG(y) + \mathbf{1}_{\{b\leq \frac{x}{a}\}}\int_b^\infty V_0(x-ab)\,dG(y)\right)}{c(a,b)}\right\}.$$

Then for all $x \geq 0$, we have $V'_0(x) \geq V'_1(x)$.

-Inductive Step ($n \geq 1$): Assume $V'_{n-1}(x) \geq V'_n(x)$ for all $x \geq 0$ and fixed x.

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From (3), for all b, it follows that the inequality

$$\begin{aligned}
V'_{n+1}(x)\big(c(a,b)\big) \leq{}& \lambda V_n(x) \\
&- \lambda\left(\int_0^{\min\left(b,\frac{x}{a}\right)} V_n(x-ay)\,dG(y)\right. \\
&\left. + \mathbf{1}_{\{b\leq\frac{x}{a}\}}\int_b^\infty V_n(x-ab)\,dG(y)\right) \\
={}& \lambda\mathbb{E}\left[\int_{x-s(a,b,y)}^x V'_n(u)\,dz\right] \leq \lambda\mathbb{E}\left[\int_{x-s(a,b,y)}^x V'_{n-1}(u)\,dz\right] \\
={}& \lambda V_{n-1}(x) - \lambda\left(\int_0^{\min\left(b,\frac{x}{a}\right)} V_{n-1}(x-ay)\,dG(y)\right. \\
&\left. + \mathbf{1}_{\{b\leq\frac{x}{a}\}}\int_b^\infty V_{n-1}(x-ab)\,dG(y)\right) \\
\leq{}& V'_n(x)\big(c(a,b)\big).
\end{aligned}$$

Given that b was arbitrary, we can switch to the infimum, which leads to

$$V'_n(x) \geq V'_{n+1}(x).$$

Since $V'_n(x)$ is decreasing and above 0, it converges to some function $f(x)$ with

$$V(x) = 1 - \int_x^\infty f(z)\,dz.$$

Then $V(x)$ is an increasing and continuous function satisfying:

$$f(x) = \inf_{b>0}\left\{\lambda\frac{V(x) - \left(\int_0^{\min(b,\frac{x}{a})} V(x-ay)\,dG(y) + \mathbf{1}_{\{b\leq\frac{x}{a}\}}\int_b^\infty V(x-ab)\,dG(y)\right)}{c(a,b)}\right\}.$$

Next, we want to prove continuity of $f(x)$, from which it follows that $V'(x)$ is also continuous, because $V'(x) = f(x)$, and when δ replace V, $V(x)$ satisfies equation (1).

We start by proving that $f(x) > 0$ for all $x \geq 0$. The function $f(x)$ represents the limit of the sequence $V'_n(x)$. If the infimum in (3) is not reached within the interval $[\underline{b}, \frac{x}{a}]$, it must be attained at $b = \infty$, because for all other values of priority level b, either the premium retained by the insurer is negative, or the expected losses exceed the premium in the same period.

For $0 \leq \frac{x}{a} \leq \underline{b}$, infimum in (3) is attained at a point $b = \infty$, thus

$$f(x) = V'(x) = \lambda\frac{V(x)}{c} = \lambda\frac{V_0(x)}{c} > 0.$$

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Let's suppose $x_0 = \inf\{x : f(x) = 0\} < \infty$. Therefore, $x_0 > a\underline{b}$, and there exists a value $x_0 < x < x_0 + a\underline{b}$ such that $f(x) = 0$, that is

$$\begin{aligned} 0 &= \inf_{b \geq \underline{b}} \left\{ V(x) - \left(\int_0^{\min\left(b, \frac{x}{a}\right)} V(x - ay)\, dG(y) + \mathbf{1}_{\{b \leq \frac{x}{a}\}} \int_b^\infty V(x - ab)\, dG(y) \right) \right\} \\ &= V(x) - \left(\int_0^{\min\left(\underline{b}, \frac{x}{a}\right)} V(x - ay)\, dG(y) + \mathbf{1}_{\{\underline{b} \leq \frac{x}{a}\}} \int_{\underline{b}}^\infty V(x - a\underline{b})\, dG(y) \right), \end{aligned}$$

i.e. $V(x) = V(x - a\underline{b})$. Then

$$0 = \int_{x - a\underline{b}}^{x} f(z)dz \geq \int_{x - a\underline{b}}^{x_0} f(z)dz.$$

This contradicts the choice of x_0, since $x - a\underline{b} < x_0$, and the function $f(z)$ is strictly positive on the interval $[x - a\underline{b}, x_0]$.

Now we prove that in the definition of the functions $V_n(x)$, where $\frac{x}{a} \leq \alpha$, the infimum can be restricted in the interval $[b_1, \infty]$, with $b_1 \geq \underline{b}$. To prove the contrary, assume there exists a sequence $0 \leq x_n \leq \alpha$ and $b_n \to \underline{b}$ such that

$$\begin{aligned} & V'_{n+1}(x_n) \geq \\ & \lambda \frac{V_n(x_n) - \left(\int_0^{\min\left(b_n, \frac{x}{a}\right)} V_n(x - ay)\, dG(y) + \mathbf{1}_{\{b_n \leq \frac{x}{a}\}} \int_{b_n}^\infty V_n(x - ab_n)\, dG(y) \right)}{c(a, b_n)} - \frac{1}{n} \\ & \geq V'_{n+1}(x_n) - \frac{1}{n}. \end{aligned}$$

Since

$$0 \leq V'_n(x) \leq V'_0(x) \quad \text{and} \quad c(a, b_n) \to 0,$$

we obtain

$$V_n(x_n) - \left(\int_0^{\min\left(b_n, \frac{x}{a}\right)} V_n(x - ay)\, dG(y) + \mathbf{1}_{\{b_n \leq \frac{x}{a}\}} \int_{b_n}^\infty V_n(x - ab_n)\, dG(y) \right) \to 0,$$

and thus for each accumulation point x_0 of the sequence x_n, we have

$$V(x_0) - \left(\int_0^{\min\left(\underline{b}, \frac{x}{a}\right)} V(x_0 - ay)\, dG(y) + \mathbf{1}_{\{\underline{b} \leq \frac{x}{a}\}} \int_{\underline{b}}^\infty V(x_0 - a\underline{b})\, dG(y) \right) = 0 = f(x_0),$$

which is in contradiction with the previously proven fact that the function $f(z)$ is strictly positive. This proves that the function $V(x)$ satisfies equation (1), and from its definition, it directly follows that it converges to 1 as $x \to \infty$.

$$|f(x_1) - f(x_2)| \leq$$

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$$\sup_{b \geq b_1} \left| \lambda \frac{V(x_1) - \left(\int_0^{\min\left(b, \frac{x_1}{a}\right)} V(x_1 - ay)\, dG(y) + \mathbf{1}_{\left\{b \leq \frac{x_1}{a}\right\}} \int_b^{\infty} V(x_1 - ab)\, dG(y) \right)}{c(a,b)} \right.$$
$$\left. - \lambda \frac{V(x_2) - \left(\int_0^{\min\left(b, \frac{x_2}{a}\right)} V(x_2 - ay)\, dG(y) + \mathbf{1}_{\left\{b \leq \frac{x_2}{a}\right\}} \int_b^{\infty} V(x_2 - ab)\, dG(y) \right)}{c(a,b)} \right|,$$

the continuity of the function $f(z)$ follows from the continuity of the function $V(x)$, thus proving that the function $V(x)$ is continuous on the interval $[0, \infty)$ and differentiable on the interval $(0, \infty)$.

Thus, we have shown that $V(x)$ satisfies the Hamilton-Jacobi-Bellman equation and meets all the required properties, proving the theorem. □

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4.4 Uniqueness of the optimal strategy

In this section, we will show that the strategy b_t^*, which maximizes equation (1), also maximizes the survival probability. We will follow the ideas and theorems presented in references[4] and [6].

Theorem 4.4.1. *The strategy b_t^* obtained as the solution of equation (1), when δ is replaced with V, maximizing the survival probability when applying combined reinsurance contracts. For survival probability $\delta(x)$ with any arbitrary strategy b_t and any $x \geq 0$, we have*

$$V(x) \geq \delta(x), \text{ if and only if } b_t = b_t^*.$$

Proof. Let $V(x)$ be the smooth function that is the solution of equation (2) when δ is replaced with V, constructed in the proof of the previous theorem which holds

$$0 \leq V(x) \leq 1$$

with

$$\lim_{x\to\infty} V(x) = 1.$$

Let define R_t and R_t^* as the risk processes of the insurance company with initial capital x and reinsurance strategies b_t and b_t^*. We define the stopped processes (Z_t) and (Z_t^*) with corresponding ruin times τ and τ^* and the stopped processes transformed (W_t) and (W_t^*) by $V(x)$ as:

$$W_t = V(Z_t) = V\big(R\big(\min\{t, \tau\}\big)\big),$$
$$W_t^* = V(Z_t^*) = V\big(R^*\big(\min\{t, \tau^*\}\big)\big).$$

By applying the principles as in [12], p. 80, (2.16), it follows

$$\mathbb{E}[W_t] = V(x) + \mathbb{E}\left[\int_0^t V'(Z_x)\,(c(a, b_x))\,dx + \lambda \int_0^t \mathbb{E}\left[V\left(Z_x - s(a, b_x, y)\right) - V(Z_x)\right] dx\right]$$

and

$$\mathbb{E}[W_t^*] = V(x) + \mathbb{E}\left[\int_0^t V'(Z_x^*)\,(c(a, b_x^*))\,dx + \lambda \int_0^t \mathbb{E}\left[V\left(Z_x^* - s(a, b_x^*, y)\right) - V(Z_x^*)\right] dx\right].$$

From the HJB equation (1), when we substitute δ with V, for all $t > 0$, it holds that $E[W_t^*] = V(x)$, since for b^*, as the optimal solution, it holds

$$V'(Z_x^*)\,(c(a, b_x^*)) + \lambda E[V(Z_x^* - s(a, b_x^*, y)) - V(Z_x^*)] = 0,\; x \geq 0.$$

Since b^* achieves the supremum of the previous expression, it follows that

$$V'(Z_x)\,(c(a, b_x)) + \lambda E[V(Z_x - s(a, b_x, y)) - V(Z_x)] \leq 0,\; x \geq 0,\; (a, b_x) \geq 0.$$

From the previous, it follows that

$$\mathbb{E}[W_t^*] = V(x) \geq \mathbb{E}[W_t]. \tag{4}$$

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Suppose the predictable strategy b_t satisfies the condition

$$b_t \geq \gamma > 0$$

for all $t \geq 0$, where γ is chosen such that $P(Y > \gamma) > 0$. Under this assumption, we prove that the process R_t is unbounded on the event $\{\tau = \infty\}$. Specifically, we prove for all $\beta > 0$

$$P\{R_t \leq \beta \text{ for all } t \geq 0 \text{ and } \tau = \infty\} = 0. \tag{5}$$

For $n > \frac{\beta + c}{\gamma}$, there exists a positive probability that more than n claims with sizes greater than γ occur within an interval of length 1. Given that the claims process possesses stationary and independent increments, the likelihood of observing more than n such claims in any interval $[t, t+1]$ remains strictly positive. For $R_t \leq \beta$, we have:

$$R(t+1) \leq R_t + c - n\gamma \leq \beta + c - n\gamma < 0,$$

which implies $\tau < \infty$. Hence, if $R_t \leq \beta$, it follows that $\tau < \infty$, which proves statement (5).

For any given ϵ, we define a strategy b_t^+ with an associated risk process R_t^+ and ruin time τ^+ such that $P(\tau = \infty \text{ and } \tau^+ < \infty) < \epsilon$, and $R_t^+ \to \infty$ on the set $\{\tau = \infty \text{ and } \tau^+ = \infty\}$. Choose $\beta > x$ to be enough large so that $1 - \delta_0(\beta) < \epsilon$, and define $T = \inf\{t \geq 0 : R_t = \beta\}$, which is almost surely finite on $\{\tau = \infty\}$. The strategy b_t^+ is then given by

$$b_t^+ = \begin{cases} b_t & \text{if } t \leq T, \\ \infty & \text{if } t > T. \end{cases}$$

The strategy b_t^+ is predictable, and

$$P\{\tau = \infty \text{ and } \tau^+ < \infty\} \leq P\{\tau = \infty\} P\{\tau^+ < \infty\} \leq 1 - \delta_0(\beta) < \epsilon.$$

Additionally, $T < \infty$ implies $R_t^+ \to \infty$ because ϵ is arbitrarily small, $1 - \delta_0(\beta) < \epsilon$.

Now repeat the above reasoning leading to (4) for R_t^+ instead of R_t. We obtain:

$$\mathbb{E}\Big[V\Big(R^*\big(\min\{t, \tau^*\}\big)\Big)\Big] = V(x) \geq \mathbb{E}\Big[V\Big(R^+\big(\min\{t, \tau^+\}\big)\Big)\Big].$$

Since τ^* and τ^+ are stopping times, it holds that $R^*(\tau^*) < 0$ and $R^+(\tau^+) < 0$, from which it follows that $V(R^*(\tau^*)) = 0$ and $V(R^+(\tau^+)) = 0$. When $t \to \infty$, it holds that:

$$P\Big\{\tau^* = \infty\Big\} \geq \mathbb{E}\Big[V\Big(R^*\big(\min\{t, \tau^*\}\big)\Big)\Big] = V(x) \geq \mathbb{E}\Big[V\Big(R^+\big(\min\{t, \tau^*\}\big)\Big)\Big] \geq$$

$$P\Big\{\tau = \infty \text{ and } t^+ < \infty\Big\} = P\Big\{\tau = \infty\Big\} - P\Big\{\tau = \infty \text{ and } t^+ < \infty\Big\} \geq P\Big\{\tau = \infty\Big\} - \epsilon.$$

Since ϵ was chosen arbitrarily, this proves our statement for the specific case of a strategy b_t satisfying $b_t \geq \gamma > 0$ for all $t \geq 0$. Specifically, given that every solution $V(x)$ of equation

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(2) in the context of Theorem 4.3.1 corresponds to a strategy satisfying $b_t \geq \gamma > 0$ for all $t \geq 0$, we conclude that the solution is unique, and therefore $V(x) = P(\tau^* = \infty)$.

For determining the inequality $\mathbb{E}\left[V\left(R^+\left(\min\{t, \tau^*\}\right)\right)\right] \geq P\{\tau = \infty \text{ and } \tau^+ < \infty\}$, we used the fact that $R_t^+ \to \infty$ on $\{\tau = \infty \text{ and } \tau^+ < \infty\}$ and that $\lim_{x\to\infty} V(x) = 1$.

This completes the proof. $\square$

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5 Optimal strategy for exponentially distributed claim sizes

Let us assume that the individual claim size characteristics follow the Cramér-Lundberg model for modelling the risk of an insurance company and that the cedent can buy reinsurance policies in such a way that the claim size distribution remains the same. During this chapter, we calculate the optimal strategies and optimal survival probability for the exponentially distributed claim size with mean $\mu = \frac{1}{m}$, density function $g(y) = me^{-my}, y \geq 0$ and distribution function $G(y) = 1 - e^{-my}, y \geq 0$. This distribution is a typical case of light-tails distributions. Furthermore, numerical examples are included to illustrate how different model parameters affect the optimal strategy and the optimal survival probability.

5.1 Premium under exponentially distributed claim sizes

Here, $G(y)$ is the distribution function of the claim sizes and hence:

$$G(y) = 1 - e^{-my}$$

5.1.1 Calculation of the insurer's premium

The insurer's premium rate c is given by:

$$c = (1+\eta)\lambda \int_0^\infty y\, dG(y) = (1+\eta)\lambda \int_0^\infty y \cdot me^{-my}\, dy.$$

To solve this integral, we recognize that it is the expected value of an exponential distribution with rate m:

$$\int_0^\infty y \cdot me^{-my}\, dy = \frac{1}{m}.$$

Therefore,

$$c = \frac{(1+\eta)\lambda}{m}.$$

5.1.2 Calculation of the reinsurer's premium

The reinsurer's premium rate $h(a,b)$ is given by:

$$h(a,b) = \rho\lambda \left[(1-a) \int_0^b y\, dG(y) + \int_b^\infty (y-ab)\, dG(y) \right].$$

We break this down into two parts.

First integral: $\int_0^b y\, dG(y)$

$$\int_0^b y\, dG(y) = \int_0^b y \cdot me^{-my}\, dy = m \int_0^b ye^{-my}\, dy$$

We use integration by parts. Let $u = y$ and $dv = e^{-my}\, dy$ then $du = dy$ and $v = -\frac{1}{m}e^{-my}$. Applying the integration by parts formula

$$\int u\, dv = uv - \int v\, du$$

we get

$$m \left[-\frac{y}{m}e^{-my}\Big|_0^b + \int_0^b \frac{1}{m}e^{-my}\, dy \right].$$

Evaluating this, we have

$$-\frac{b}{m}e^{-mb} + \frac{1}{m^2}(1 - e^{-mb}) = \frac{1}{m} - \left(b + \frac{1}{m}\right) e^{-mb}.$$

Thus, the final result is:

$$\int_0^b y\, dG(y) = \frac{1}{m} - \left(b + \frac{1}{m}\right) e^{-mb}.$$

Second integral: $\int_b^\infty (y - ab)\, dG(y)$

$$\int_b^\infty (y - ab)\, dG(y) = \int_b^\infty (y - ab) \cdot me^{-my}\, dy.$$

We can break it down into two parts.

$$\int_b^\infty (y - ab) \cdot me^{-my}\, dy = m \int_b^\infty ye^{-my}\, dy - ab \cdot m \int_b^\infty e^{-my}\, dy.$$

First part is :

$$m \int_b^\infty ye^{-my}\, dy = m \left[-\frac{y}{m}e^{-my}\Big|_b^\infty + \int_b^\infty \frac{1}{m}e^{-my}\, dy \right]$$
$$= m \left(\frac{b}{m}e^{-mb} + \frac{1}{m}e^{-mb} \right) = \left(b + \frac{1}{m}\right) e^{-mb}.$$

Second part :

$$ab \cdot m \int_b^\infty e^{-my}\, dy = ab \cdot m \left[\frac{-e^{-my}}{m} \right]_b^\infty = ab \cdot e^{-mb}.$$

Putting these two parts together, we obtain :

$$\int_b^\infty (y - ab) \cdot me^{-my}\, dy = e^{-mb} \left(b + \frac{1}{m} - ab\right).$$

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Substituting these results back into the formula for $h(a,b)$ we have:

$$h(a,b) = \rho\lambda\left[(1-a)\left(\frac{1}{m} - \left(b+\frac{1}{m}\right)e^{-mb}\right) + e^{-mb}\left(b+\frac{1}{m} - ab\right)\right].$$

The simplified form of $h(a,b)$ is:

$$h(a,b) = \frac{\rho\lambda}{m}\left(1-a+ae^{-mb}\right).$$

Then the premium rate left for the cedent for the exponentially distributed claim size is :

$$\begin{aligned} c(a,b) = c - h(a,b) &= \frac{(1+\eta)\lambda}{m} - \frac{\rho\lambda}{m}\left(1-a+ae^{-mb}\right) \\ &= \frac{\lambda}{m}\left[(1+\eta) - \rho\left(1-a+ae^{-mb}\right)\right]. \end{aligned}$$

5.2 Optimal strategy

We require that the condition is met for the cedent's net retained premium to be positive, namely that the condition holds $c > h(a,b)$, that is

$$\frac{\lambda(1+\eta)}{m} > \frac{\lambda\rho}{m}\left(1-a+ae^{-mb}\right)$$

which gives

$$(1+\eta) > (1+\theta)\left(1-a+ae^{-mb}\right)$$

From $c(a,b) = h(a,b)$ we determined lower boundaries.

For large b, the lower bound for a is obtained from :

$$\underline{a} = \frac{\theta-\eta}{(1+\theta)(1-e^{-mb})} = \frac{\theta-\eta}{1+\theta}.$$

For b smaller than x, the lower bound for b is obtained from :

$$\underline{b} = -\frac{1}{m}\ln\left(\frac{1+\eta-\rho(1-a)}{\rho a}\right).$$

In Appendix 7.1 and 7.2 we can find a list of tables showing the lower bounds for different parameters η, θ, and m.

In the proof of Theorem 4.3.1, we saw that the infimum in equation (2) is attained on the segment $[\underline{b}, \frac{x}{a}]$ or at the point $b = \infty$, so this will be an additional constraint for finding the optimal priority level. Therefore, we seek the b that satisfies the equation

$$\delta'(x) = \inf_{\substack{\{\frac{x}{a} > b > \underline{b}\} \\ \cup (b=\infty)}} \left\{ \lambda \frac{\delta(x) - \left(\int_0^{\min\left(b,\frac{x}{a}\right)} \delta(x-ay)\,dG(y) + \mathbf{1}_{\{b \le \frac{x}{a}\}}\delta(x-b)(1-G(b))\right)}{c(a,b)} \right\}.$$

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To further illustrate our calculations, we can rewrite this equation in a slightly different form. By definition, we have

$$\int_0^b \delta(x-ay)\,dG(y)+\delta(x-ab)(1-G(b)).$$

Let us denote by

$$f(x)=\int_0^{\frac{x}{a}} \delta(x-ay)\,dG(y)=e^{-m\frac{x}{a}}\frac{m}{a}\int_0^x \delta(z)\,e^{m\frac{z}{a}}dz.$$

Therefore,

$$f'(x)=\frac{m}{a}\left(\delta(x)-f(x)\right).$$

It is also

$$\int_0^b \delta(x-ay)\,dG(y)=f(x)-\int_b^{\frac{x}{a}} \delta(x-ay)\,dG(y)$$

$$=f(x)-e^{-mb}\int_0^{x-ab} \delta(y)\,dG(x-ab-y)=f(x)-e^{-mb}f(x-ab).$$

Thus, we find that

$$\int_0^{\min\left(b,\frac{x}{a}\right)} \delta(x-ay)\,dG(y)+\mathbf{1}_{\{b\le\frac{x}{a}\}}\delta(x-b)(1-G(b)=f(x)-e^{-mb}f(x-ab)+\delta(x-ab)e^{-mb}.$$

With the previous assumptions and notation, the challenge is to find a solution to the equation

$$\delta'(x)=\inf_{\{\frac{x}{a}>b>\underline{b}\}\cup(b=\infty)}\left\{m\frac{\delta(x)-f(x)+e^{-mb}(f(x-ab)-\delta(x-ab))}{\eta-\theta+a(1+\theta)\left(1-e^{-mb}\right)}\right\}. \qquad (6)$$

Determining a formula for the survival probability function for the reinsurance strategy analyzed in this thesis poses significant challenges and may even be practically unattainable. As a result, adopting a numerical approach was considered more feasible and effective. We will seek the solution using simulations, but with additional assumptions: As the initial condition, we will take the survival probability when there is no reinsurance, $a=1$ and $b_t=\infty$ for all t, $\delta(0)=\frac{\eta}{(1+\eta)}$.

The algorithm guiding this simulation proceeds as follows: for all initial capital values x from 0 to 15, with a step of $\frac{1}{1000}$, we determine the optimal priority level b_t^* by identifying, for all $b\in(\underline{b},\frac{x}{a})\cup\infty$, the value at which the minimum of the function (6) is achieved. This minimum value is added to the variable $\delta'(x)$. To determine $\delta(x)$, $f(x)$, and $f'(x)$, we use the following formulas:

$$\delta(x)=\delta(x-\frac{1}{1000})+\delta'(x-\frac{1}{1000})\cdot\frac{1}{1000},$$

$$f(x)=f(x-\frac{1}{1000})+f'(x-\frac{1}{1000})\cdot\frac{1}{1000}.$$

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The remaining parameters are fixed, like in [9] as follows:

Proportional factor $a = 0.8$, claim arrival intensity $\lambda = 1$, expected claim size parameter $m = 1$, insurer safety loading $\eta = 0.5$, reinsurer safety loading $\theta = 0.7$.

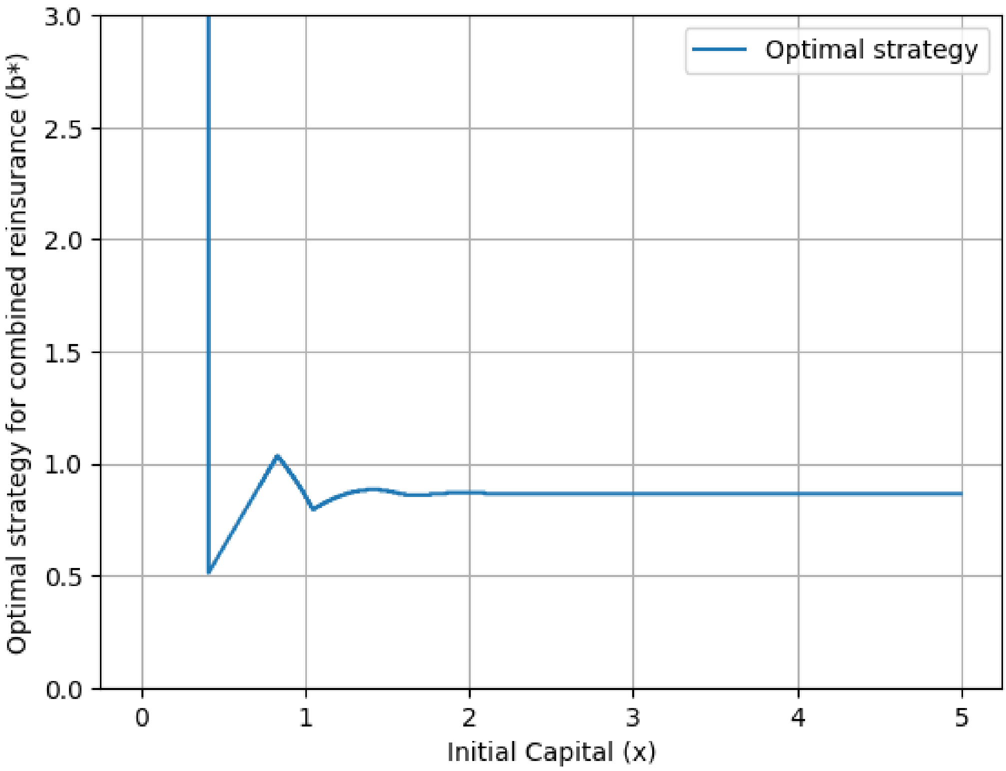

Figure 5.1: Optimal combined reinsurance strategy

The dependency of the optimal strategy on the initial capital is shown in Figure 5.1. We observe that the optimal strategy, in the case of exponentially distributed claim sizes, behaves as expected. For small capital values within the interval $x \in [0, 0.409)$, the optimal strategy satisfies $b^*(x) = \infty$, meaning “no reinsurance”. This is because the insurer, due to limited initial capital, cannot afford reinsurance. As the initial capital increases, the optimal strategy transitions to $b^*(x) = \frac{x}{a}$, which holds in the interval $x \in [0.41, 0.863)$. In this range, the following claim will not result in ruin, as the reserve remains positive even after the claim. From the point $b^*(x) < \frac{x}{a}$, starting at $x \approx 0.834$, the insurer become capable of affording more expensive reinsurance, and $b^*(x) \approx 0.869$ continues to converge.

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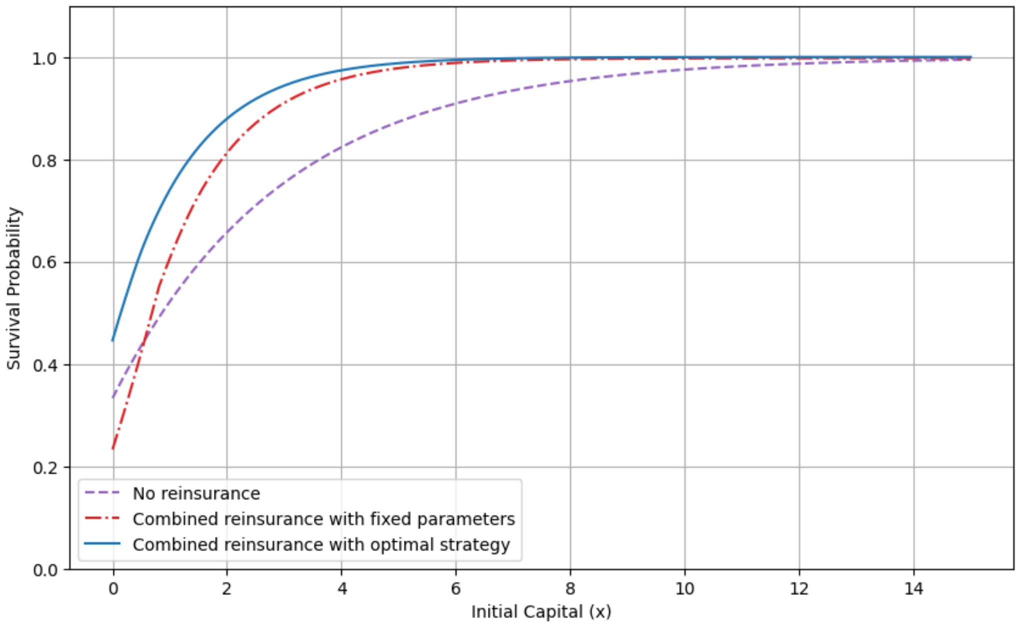

Figure 5.2: Optimal combined reinsurance strategy

In Figure 5.2, the optimal survival probability function for combined reinsurance with $a = 0.8$ and $b^*(x)$, the survival probability function for combined reinsurance with fixed parameters $a = 0.8$ and $b = 1$, and the survival probability without reinsurance with $a = 1$ are illustrated. In the case of the optimal strategy, the survival probability at initial capital zero is $\delta(0) \approx 0.4457$. As expected, the optimal survival probability is above the other two functions for every value of the initial capital. In the case without reinsurance, the survival probability at initial capital zero is $\delta(0) \approx 0.3333$, which is higher than the case with combined reinsurance with fixed parameters, where $\delta(0) \approx 0.2329$. However, at some point, this strategy will begin to provide a higher survival probability than the strategy without reinsurance, but it will still remain lower than the function with the optimal strategy.

In the following four subsections, we will demonstrate how the remaining model parameters influence both the optimal strategy and the optimal survival probability.

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5.2.1 Optimal strategy for different values of proportional factor (a)

We consider scenarios with different values of the proportional factor: $a = 0.6$, $a = 0.8$, and $a = 1$, and analyze how the proportional factor impacts the optimal strategy and the optimal survival probability. Here, $a = 0.6$ indicates that the insurer covers a smaller percentage of the claims below the retention level, meaning that reinsurance is more expensive, and the premiums retained by the insurer after paying reinsurance are lower. On the other hand, $a = 1$ means that the insurer cover the entire claim below the retention level, indicating that reinsurance is cheaper, and the premiums retained by the insurer after paying reinsurance are higher.

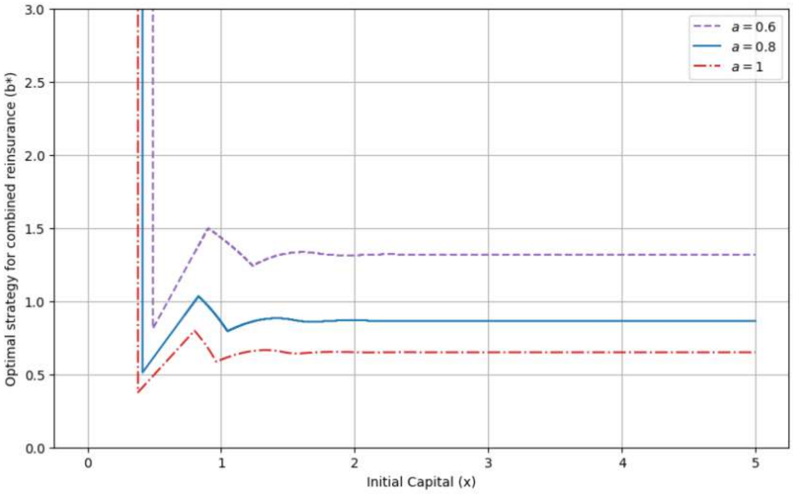

Figure 5.3: The effect of parameter a on the optimal strategy

In Figure 5.3, we can see that for $a = 0.6$, "no reinsurance" ie. $b^*(x) = \infty$, is optimal for $x \in [0, 0.487)$, and first reinsurance is bought at $x = 0.488$ with strategy $b^*(0.488) \approx 0.813$. For large initial capital, the optimal strategy is $b^*(x) \approx 1.317$. For $a = 1$, "no reinsurance" ie. $b^*(x) = \infty$, is optimal for $x \in [0, 0.375)$, and first reinsurance is bought at $x = 0.376$ with strategy $b^*(0.376) = 0.376$. For large initial capital, the optimal strategy is $b^*(x) \approx 0.65$.

Since reinsurance with $a = 1$ is cheaper than reinsurance with $a = 0.6$, this function will drop down earlier and deeper, meaning that the insurer will start with reinsurance at a lower initial capital and with reinsurance with smaller retention level.

The survival probabilities for the given scenarios are illustrated in Figure 5.4. It is evident that when the proportional factor is lower, reinsurance becomes more expensive, resulting in smaller premiums retained by the insurer after paying for reinsurance, which leads to reduced financial flexibility. Consequently, the survival probability for an initial capital at $x = 0$ is lower compared to scenarios where the proportional factor is higher. For $a = 0.6$, we have $\delta(0) \approx 0.3387$ and for $a = 1$, we have $\delta(0) \approx 0.5219$, thus the premiums retained by the insurer after paying for the cheaper reinsurance are greater.

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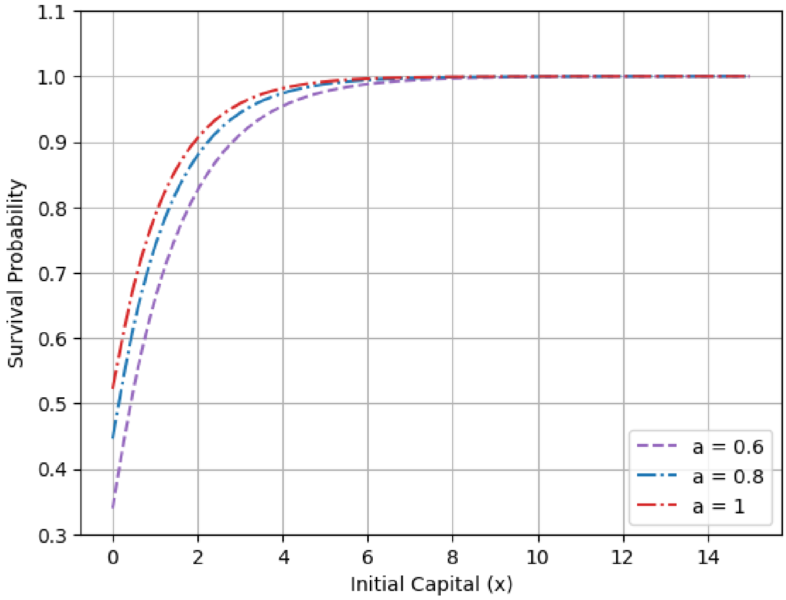

Figure 5.4: The effect of parameter a on the optimal survival probability

Additionally, it has been clearly demonstrated that the survival probability is highest when the proportional factor $a = 1$. This indicates that, in the case of combined reinsurance, the optimal strategy is a pure excess-of-loss reinsurance strategy.

5.2.2 Optimal strategy for different values of insurer's safety loading (η)

We consider scenarios with different values of the insurer safety loading: $\eta = 0.45$, $\eta = 0.5$, and $\eta = 0.55$, and analyze how the insurer's safety loading impacts the optimal strategy and the optimal survival probability. Here, $\eta = 0.45$ means that the insurance is cheaper, which means that the premiums retained by the insurer after paying reinsurance are lower, and $\eta = 0.55$ means that the insurance is more expensive, which means that the premiums retained by the insurer after paying reinsurance are higher.

In Figure 5.5, we can see that for $\eta = 0.45$, "no reinsurance" ie. $b^*(x) = \infty$, is optimal for $x \in [0, 0.581)$, and first reinsurance is bought at $x = 0.582$ with strategy $b^*(0.582) \approx 0.727$. For large initial capital, the optimal strategy is $b^*(x) \approx 1.194$. For $\eta = 0.55$, "no reinsurance" ie. $b^*(x) = \infty$, is optimal for $x \in [0, 0.275)$, and first reinsurance is bought at $x = 0.276$ with strategy $b^*(0.276) \approx 0.344$. For large initial capital, the optimal strategy is $b^*(x) \approx 0.584$.

Since insurance with $\eta = 0.55$ saves more money after paying a reinsurance than insurance with $\eta = 0.45$, this function will drop down earlier and deeper, meaning that the insurer will start with reinsurance at a lower initial capital and with reinsurance with smaller retention level.

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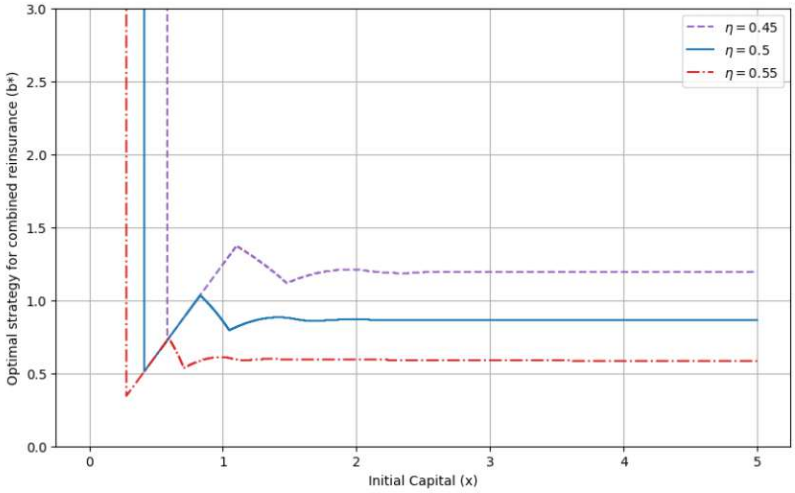

Figure 5.5: The effect of parameter η on the optimal strategy

The survival probabilities for the given scenarios are illustrated in Figure 5.6. It is clear that when insurance is cheaper, the insurer retain fewer premiums after paying for reinsurance, leaving them with less financial flexibility. As a result, the probability of survival for an initial capital at $x = 0$ is lower compared to cases where insurance is more expensive. For $\eta = 0.45$, we have $\delta(0) \approx 0.3634$, and for $\eta = 0.55$, we have $\delta(0) \approx 0.5443$, and the insurer retain a larger portion of the premiums after paying for reinsurance.

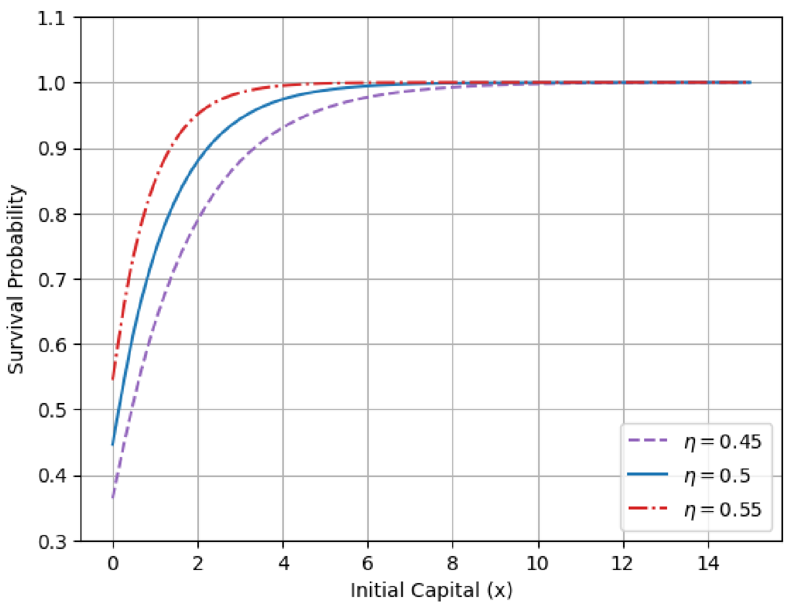

Figure 5.6: The effect of parameter η on the optimal survival probability

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5.2.3 Optimal strategy for different values of reinsurer's safety loading (θ)

We consider scenarios with different values of the reinsurer safety loading: $\theta = 0.65$, $\theta = 0.7$, and $\theta = 0.75$, and analyze how the reinsurer's safety loading impacts the optimal strategy and the optimal survival probability. Here, $\theta = 0.65$ means that the reinsurance is cheaper, which means that the premiums retained by the insurer after paying reinsurance are higher, and $\theta = 0.75$ means that the reinsurance is more expensive, which means that the premiums retained by the insurer after paying reinsurance are lower.

In Figure 5.7, we can see that for $\theta = 0.65$, "no reinsurance" ie. $b^*(x) = \infty$, is optimal for $x \in [0, 0.305)$, and first reinsurance is bought at $x = 0.306$ with strategy $b^*(0.306) \approx 0.382$. For large initial capital, the optimal strategy is $b^*(x) \approx 1.069$. For $\theta = 0.75$, "no reinsurance" ie. $b^*(x) = \infty$, is optimal for $x \in [0, 0.516)$, and first reinsurance is bought at $x = 0.517$ with strategy $b^*(0.517) \approx 0.646$. For large initial capital, the optimal strategy is $b^*(x) \approx 0.654$.

Since reinsurance with $\theta = 0.65$ is cheaper than reinsurance with $\theta = 0.75$, this function will drop down earlier and deeper, meaning that the insurer will start with reinsurance at a lower initial capital and with reinsurance with smaller retention level.

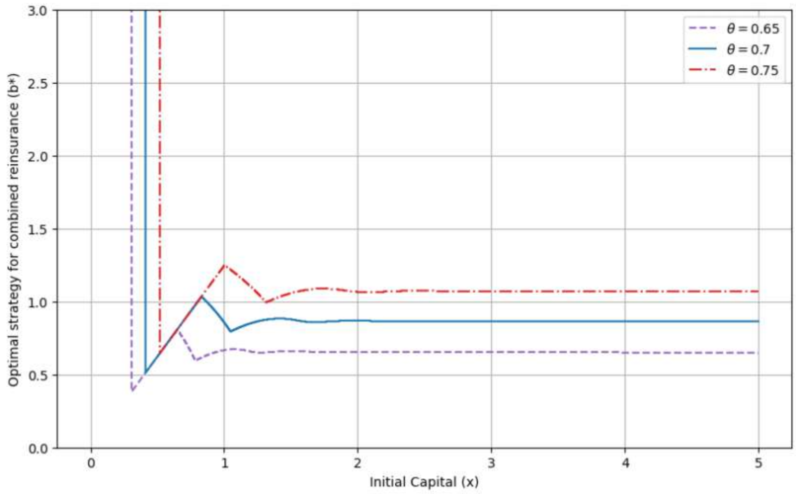

Figure 5.7: The effect of parameter θ on the optimal strategy

The survival probabilities for the given scenarios are illustrated in Figure 5.8. It is clear that when reinsurance is cheaper, the insurer retain a larger portion of the premiums after paying for reinsurance, allowing for greater financial flexibility. As a result, the probability of survival for an initial capital at $x = 0$ is higher compared to cases where reinsurance is more expensive. For $\theta = 0.65$, we have $\delta(0) \approx 0.4998$, and for $\theta = 0.75$, we have $\delta(0) \approx 0.4069$, and the insurer retain fewer premiums after paying for reinsurance.

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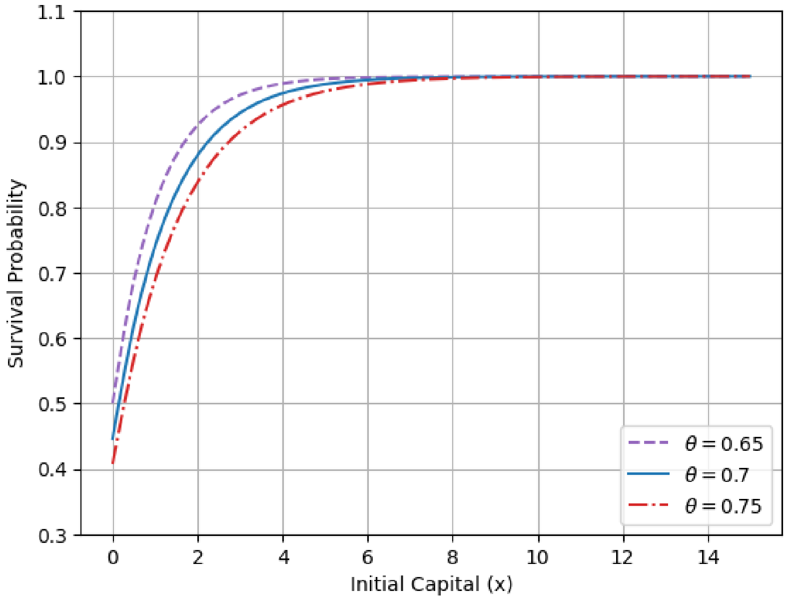

Figure 5.8: The effect of parameter θ on the optimal survival probability

5.2.4 Optimal strategy for different values of expected claim sizes

We consider scenarios with different values for expected value of exponentially distributed claim sizes with parameter: $m = 0.5$, $m = 1$, and $m = 2$, and analyze how the parameter m impacts the optimal strategy and the optimal survival probability. Here, $m = 0.5$ means that the average claim size is 2, and $m = 2$ means that the average claim size is 0.5.

In Figure 5.9, we can see that for $m = 0.5$, "no reinsurance" ie. $b^*(x) = \infty$, is optimal for $x \in [0, 0.819)$, and first reinsurance is bought at $x = 0.82$ with strategy $b^*(0.82) \approx 1.025$. For large initial capital, the optimal strategy is $b^*(x) \approx 1.734$. For $m = 2$, "no reinsurance" ie. $b^*(x) = \infty$, is optimal for $x \in [0, 0.204)$, and first reinsurance is bought at $x = 0.205$ with strategy $b^*(0.205) \approx 0.256$. For large initial capital, the optimal strategy is $b^*(x) \approx 0.429$.

Since reinsurance with $m = 2$ is cheaper than reinsurance with $m = 0.5$, this function will drop down earlier and deeper, meaning that the insurer will start with reinsurance at a lower initial capital and with reinsurance with smaller retention level.

The survival probabilities for the given scenarios are illustrated in Figure 5.10. It is evident that, across all three scenarios, the survival probability at $x = 0$ remains identical, $\delta(0) \approx 0.4457$, indicating that the parameter m does not impact $\delta(0)$ when premiums are calculated by expected value principle. However, for $x > 0$, a larger m leads to the survival probability function reaching a value close to 1 more quickly, whereas a smaller m causes it to approach 1 at a slower rate.

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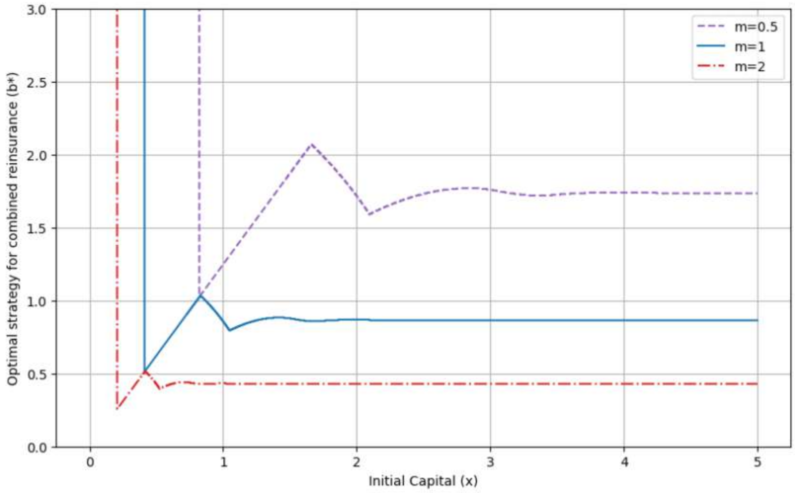

Figure 5.9: The effect of exponentially distributed claim size parameter m on the optimal strategy

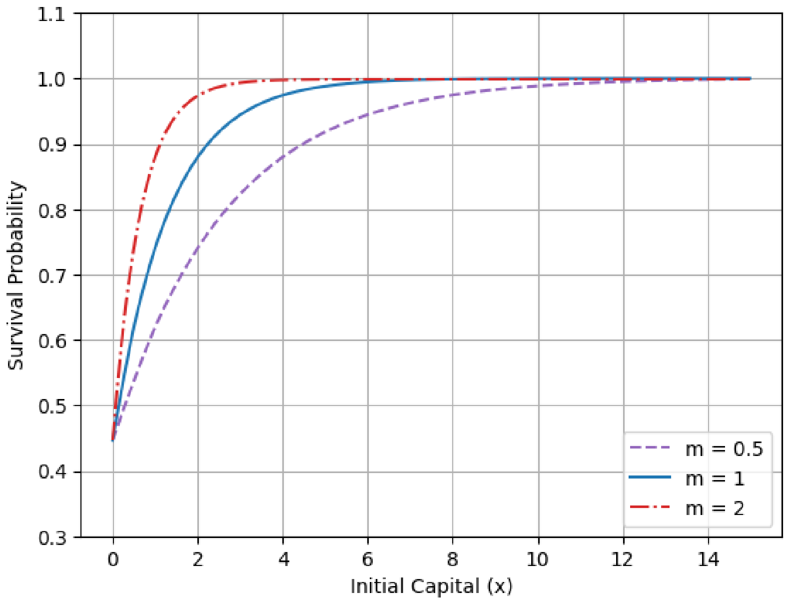

Figure 5.10: The effect of exponentially distributed claim size parameter m on the optimal survival probability

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6 Conclusion

The intended outcome of this thesis was to determine the optimal reinsurance strategy, in a continuous time, for a combination of quota-share and excess-of-loss reinsurance. The analysis was conducted under the conditions that the parameter of the XL reinsurance can be changed continuously in time, while the proportional factor remain fixed. The aim was to maximize the survival probability from the insurer's perspective. We have shown that for small initial capital, it is optimal for the insurer to retain the risk. Conversely, for larger initial capital, transferring the risk to the reinsurer becomes the optimal strategy.

Using Python software, we conducted numerical analyses to illustrate the optimal strategy and the survival probability function for exponentially distributed claims. Furthermore, we examined the impact of proportional factor, safety loading, and expected claims size on these two functions. Additionally, through examples, we have shown that the optimal strategy for maximizing the survival probability in the context of combined reinsurance is the pure excess-of-loss reinsurance strategy.

It should be noted that the techniques and methods presented in this thesis provide a theoretical perspective on the problem of determining optimal reinsurance strategies. Implementing these methods in practice requires the additional step of defining parameters λ, m, η, and θ. Even then, it is not guaranteed that the application of an optimal strategy would be practical, as the insurance and reinsurance markets operate under specific legal constraints. Also, reinsurance contracts have usually a duration of 1 year, not for $t = \infty$. Moreover, reinsurance contracts are not structured to allow for continuous adjustments in strategy. Instead, the chosen strategy remains fixed for the duration of the contract, with a predetermined premium. This premium is not based only on the methods described here but also is influenced by various indicators specific to local insurance and reinsurance markets. The approaches to determining the necessary reinsurance coverage for an insurance company rely on assessing its risk exposure. While insurers aim to develop an optimal reinsurance strategy to minimize the ruin probability, reinsurers also must manage their own risk exposure and ensure that their available capital is sufficient. This dual perspective makes it challenging to find suitable reinsurance coverage in practice.

Nevertheless, these studies can serve as a useful guideline for insurance companies in determining which reinsurance strategy to pursue, while adjusting it to the specifics of the local market. These findings also make room for future research, such as investigating the behavior of the optimal strategy function and the survival probability function when the proportional factor is dynamically determined while the retention level remains fixed, or even when both parameters are dynamic.

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7 Appendix

7.1 Tables - Lower bounds for proportional factor (a)

Table 7.1: Table of the lower bounds of the proportional factor in the case of a large priority level depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.083	0.154	0.214	0.267	0.313	0.353	0.389	0.421	0.450
0.2	-	0.077	0.143	0.200	0.250	0.294	0.333	0.368	0.400
0.3	-	-	0.071	0.133	0.188	0.235	0.278	0.316	0.350
0.4	-	-	-	0.067	0.125	0.176	0.222	0.263	0.300
0.5	-	-	-	-	0.063	0.118	0.167	0.211	0.250
0.6	-	-	-	-	-	0.059	0.111	0.158	0.200
0.7	-	-	-	-	-	-	0.056	0.105	0.150
0.8	-	-	-	-	-	-	-	0.053	0.100
0.9	-	-	-	-	-	-	-	-	0.050

7.2 Tables - Lower bounds for priority level (b)

7.2.1 Lower bounds for priority level (b) for fix $m = 1$

Table 7.2: Table of the lower bounds of the priority level in the case of a proportional factor $a = 1$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.087	0.167	0.241	0.310	0.375	0.435	0.492	0.547	0.598
0.2	-	0.080	0.154	0.223	0.288	0.348	0.405	0.460	0.511
0.3	-	-	0.074	0.143	0.208	0.268	0.325	0.379	0.431
0.4	-	-	-	0.069	0.134	0.194	0.251	0.305	0.357
0.5	-	-	-	-	0.065	0.125	0.182	0.236	0.288
0.6	-	-	-	-	-	0.061	0.118	0.172	0.223
0.7	-	-	-	-	-	-	0.057	0.111	0.163
0.8	-	-	-	-	-	-	-	0.054	0.105
0.9	-	-	-	-	-	-	-	-	0.051

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Table 7.3: Table of the lower bounds of the priority level in the case of a proportional factor $a = 0.9$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.097	0.187	0.272	0.351	0.427	0.498	0.566	0.631	0.693
0.2	-	0.089	0.173	0.251	0.325	0.396	0.463	0.527	0.588
0.3	-	-	0.083	0.160	0.234	0.303	0.369	0.432	0.492
0.4	-	-	-	0.077	0.150	0.218	0.284	0.346	0.405
0.5	-	-	-	-	0.072	0.140	0.205	0.266	0.325
0.6	-	-	-	-	-	0.068	0.132	0.193	0.251
0.7	-	-	-	-	-	-	0.064	0.124	0.182
0.8	-	-	-	-	-	-	-	0.060	0.118
0.9	-	-	-	-	-	-	-	-	0.057

Table 7.4: Table of the lower bounds of the priority level in the case of a proportional factor $a = 0.8$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.110	0.214	0.312	0.405	0.495	0.582	0.666	0.747	0.827
0.2	-	0.101	0.197	0.288	0.375	0.458	0.539	0.617	0.693
0.3	-	-	0.094	0.182	0.267	0.348	0.427	0.502	0.575
0.4	-	-	-	0.087	0.170	0.249	0.325	0.399	0.470
0.5	-	-	-	-	0.081	0.159	0.234	0.305	0.375
0.6	-	-	-	-	-	0.076	0.150	0.220	0.288
0.7	-	-	-	-	-	-	0.072	0.141	0.208
0.8	-	-	-	-	-	-	-	0.068	0.134
0.9	-	-	-	-	-	-	-	-	0.065

Table 7.5: Table of the lower bounds of the priority level in the case of a proportional factor $a = 0.7$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.127	0.248	0.365	0.480	0.591	0.702	0.811	0.920	1.030
0.2	-	0.116	0.228	0.336	0.442	0.545	0.647	0.747	0.847
0.3	-	-	0.108	0.211	0.312	0.410	0.506	0.600	0.693
0.4	-	-	-	0.100	0.197	0.290	0.382	0.472	0.560
0.5	-	-	-	-	0.094	0.184	0.272	0.358	0.442
0.6	-	-	-	-	-	0.088	0.173	0.256	0.336
0.7	-	-	-	-	-	-	0.083	0.163	0.241
0.8	-	-	-	-	-	-	-	0.078	0.154
0.9	-	-	-	-	-	-	-	-	0.074

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Table 7.6: Table of the lower bounds of the priority level in the case of a proportional factor $a = 0.6$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.150	0.296	0.442	0.588	0.736	0.887	1.045	1.210	1.386
0.2	-	0.137	0.272	0.405	0.539	0.674	0.811	0.952	1.099
0.3	-	-	0.127	0.251	0.375	0.498	0.622	0.747	0.875
0.4	-	-	-	0.118	0.234	0.348	0.463	0.577	0.693
0.5	-	-	-	-	0.110	0.218	0.325	0.432	0.539
0.6	-	-	-	-	-	0.103	0.205	0.305	0.405
0.7	-	-	-	-	-	-	0.097	0.193	0.288
0.8	-	-	-	-	-	-	-	0.092	0.182
0.9	-	-	-	-	-	-	-	-	0.087

Table 7.7: Table of the lower bounds of the priority level in the case of a proportional factor $a = 0.5$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.182	0.368	0.560	0.762	0.981	1.224	1.504	1.846	2.303
0.2	-	0.167	0.336	0.511	0.693	0.887	1.099	1.335	1.609
0.3	-	-	0.154	0.310	0.470	0.636	0.811	0.999	1.204
0.4	-	-	-	0.143	0.288	0.435	0.588	0.747	0.916
0.5	-	-	-	-	0.134	0.268	0.405	0.547	0.693
0.6	-	-	-	-	-	0.125	0.251	0.379	0.511
0.7	-	-	-	-	-	-	0.118	0.236	0.357
0.8	-	-	-	-	-	-	-	0.111	0.223
0.9	-	-	-	-	-	-	-	-	0.105

Table 7.8: Table of the lower bounds of the priority level in the case of a proportional factor $a = 0.4$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.234	0.486	0.767	1.099	1.520	2.140	3.584	-	-
0.2	-	0.214	0.442	0.693	0.981	1.329	1.792	2.539	-
0.3	-	-	0.197	0.405	0.633	0.887	1.186	1.558	2.079
0.4	-	-	-	0.182	0.375	0.582	0.811	1.073	1.386
0.5	-	-	-	-	0.170	0.348	0.539	0.747	0.981
0.6	-	-	-	-	-	0.159	0.325	0.502	0.693
0.7	-	-	-	-	-	-	0.150	0.305	0.470
0.8	-	-	-	-	-	-	-	0.141	0.288
0.9	-	-	-	-	-	-	-	-	0.134

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Table 7.9: Table of the lower bounds of the priority level in the case of a proportional factor $a = 0.3$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.325	0.719	1.253	2.197	-	-	-	-	-
0.2	-	0.296	0.647	1.099	1.792	3.932	-	-	-
0.3	-	-	0.272	0.588	0.981	1.534	2.603	-	-
0.4	-	-	-	0.251	0.539	0.887	1.350	2.097	-
0.5	-	-	-	-	0.234	0.498	0.811	1.210	1.792
0.6	-	-	-	-	-	0.218	0.463	0.747	1.099
0.7	-	-	-	-	-	-	0.205	0.432	0.693
0.8	-	-	-	-	-	-	-	0.193	0.405
0.9	-	-	-	-	-	-	-	-	0.182

Table 7.10: Table of the lower bounds of the priority level in the case of a proportional factor $a = 0.2$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.325	0.719	1.253	2.197	-	-	-	-	-
0.2	-	0.296	0.647	1.099	1.792	3.932	-	-	-
0.3	-	-	0.272	0.588	0.981	1.534	2.603	-	-
0.4	-	-	-	0.251	0.539	0.887	1.350	2.097	-
0.5	-	-	-	-	0.234	0.498	0.811	1.210	1.792
0.6	-	-	-	-	-	0.218	0.463	0.747	1.099
0.7	-	-	-	-	-	-	0.205	0.432	0.693
0.8	-	-	-	-	-	-	-	0.193	0.405
0.9	-	-	-	-	-	-	-	-	0.182

Table 7.11: Table of the lower bounds of the priority level in the case of a proportional factor $a = 0.1$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	1.792	-	-	-	-	-	-	-	-
0.2	-	1.466	-	-	-	-	-	-	-
0.3	-	-	1.253	-	-	-	-	-	-
0.4	-	-	-	1.099	-	-	-	-	-
0.5	-	-	-	-	0.981	-	-	-	-
0.6	-	-	-	-	-	0.887	-	-	-
0.7	-	-	-	-	-	-	0.811	-	-
0.8	-	-	-	-	-	-	-	0.747	-
0.9	-	-	-	-	-	-	-	-	0.693

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7.2.2 Lower bounds for priority level (b) for fix $a = 0.8$

Table 7.12: Table of the lower bounds of the priority level in the case of $m = 0.5$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.220	0.427	0.624	0.811	0.991	1.164	1.331	1.494	1.653
0.2	-	0.202	0.393	0.575	0.749	0.917	1.078	1.234	1.386
0.3	-	-	0.187	0.365	0.534	0.697	0.853	1.004	1.151
0.4	-	-	-	0.174	0.340	0.498	0.651	0.798	0.940
0.5	-	-	-	-	0.163	0.318	0.467	0.611	0.749
0.6	-	-	-	-	-	0.153	0.299	0.440	0.575
0.7	-	-	-	-	-	-	0.144	0.282	0.415
0.8	-	-	-	-	-	-	-	0.136	0.267
0.9	-	-	-	-	-	-	-	-	0.129

Table 7.13: Table of the lower bounds of the priority level in the case of $m = 2$ depending on the safety loading for insurer (η) and reinsurer (θ).

$\eta \backslash \theta$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.055	0.107	0.156	0.203	0.248	0.291	0.333	0.374	0.413
0.2	-	0.051	0.098	0.144	0.187	0.229	0.269	0.309	0.347
0.3	-	-	0.047	0.091	0.134	0.174	0.213	0.251	0.288
0.4	-	-	-	0.044	0.085	0.125	0.163	0.199	0.235
0.5	-	-	-	-	0.041	0.080	0.117	0.153	0.187
0.6	-	-	-	-	-	0.038	0.075	0.110	0.144
0.7	-	-	-	-	-	-	0.036	0.071	0.104
0.8	-	-	-	-	-	-	-	0.034	0.067
0.9	-	-	-	-	-	-	-	-	0.032

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7.3 General solution

To solve the differential equation

$$\delta''(x) = \left(\frac{\lambda}{c(a,b)} - \frac{m}{a}\right)\delta'(x).$$

we proceed with the following steps. Let $k = \frac{\lambda}{c(a,b)} - \frac{m}{a}$. The equation now simplifies to:

$$\delta''(x) = k\delta'(x).$$

This is a second-order linear homogeneous differential equation. To solve it, let's rewrite it as:

$$\frac{d^2\delta}{dx^2} - k\frac{d\delta}{dx} = 0.$$

This equation can be solved by finding the characteristic equation, which is:

$$r^2 - kr = 0.$$

Factorizing this gives:

$$r(r-k) = 0.$$

The solutions to this characteristic equation are:

$$r_1 = 0 \quad \text{and} \quad r_2 = k.$$

The general solution to the differential equation is a linear combination of the solutions corresponding to these roots:

$$\delta(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

Substituting the values of r_1 and r_2:

$$\delta(x) = C_1 e^{0\cdot x} + C_2 e^{kx}.$$

Simplifying this:

$$\delta(x) = C_1 + C_2 e^{kx},$$

where C_1 and C_2 are arbitrary constants.

The general solution to the differential equation is:

$$\delta(x) = C_1 + C_2 e^{\left(\frac{\lambda}{c(a,b)} - \frac{m}{a}\right)x},$$

where C_1 and C_2 are arbitrary constants determined by the initial conditions or boundary conditions.

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7.4 Python code for combined reinsurance model

The following Python script was used for plotting the optimal strategy and optimal survival probability functions for combined reinsurance model:

```python
import numpy as np
import matplotlib.pyplot as plt

def Comb(eta=0.5, theta=0.7, lambda_=1, m=1, a=0.8):

    # Calculate the minimum proportion factor b_ based on net retained premium conditions.
    b_ = -np.log((1 + eta - (1 + theta) * (1 - a)) / ((1 + theta) * a)) / m

    # Initialize arrays to store survival probabilities and optimal strategy.
    Delta = np.zeros(15001)
    Delta[0] = 0.44575
    f = np.zeros(15001)
    Optimal_strategy = np.zeros(15000)
    r = np.zeros(15000)

    # Loop over scaled initial capital values
    for x in range(1, 15001):
        b_optimal = 10  # Default value for b_optimal, used if no minimum is found
        if x == 1:
            f_d = m * (Delta[x-1]) / a
        else:
            f[x-1] = f[x-2] + f_d / 1000
            f_d = m * (Delta[x-1] - f[x-1]) / a

        Delta_d = 1
        b_min = int(b_)
        b_max = int(np.floor((x - 1) / a))

        # Iterate through possible 'b' values to find the optimal strategy that minimizes function Delta_d.
        for b in range(b_min, b_max + 1):
            # Ensure the index is within the valid range and is an integer
            index = int(x - (a * b) - 1)
            if 0 <= index < len(f) and 0 <= index < len(Delta):
                if lambda_ / m * ((1 + eta) - (1 + theta) + (1 + theta) * a - (1 + theta) * a * np.exp(-m * b / 1000)) < 0:
                    Delta_d = lambda_ * (Delta[x-1] - f[x-1]) / (lambda_ / m * ((1 + eta) - (1 + theta) * (1 - a)))
                    b_optimal = 10
                else:
                    helper = lambda_ * (Delta[x-1] - f[x-1] + (f[index] - Delta[index]) * np.exp(-m * b / 1000)) / (lambda_ / m * ((1 + eta) - (1 + theta) + (1 + theta) * a - (1 + theta) * a * np.exp(-m * b / 1000)))
                    if helper <= Delta_d:
                        Delta_d = helper
                        b_optimal = b / 1000
                if lambda_ * (Delta[x-1] - f[x-1]) / (lambda_ / m *((1 + eta) - (1 + theta) * (1 - a))) <= Delta_d:
```


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```
                    Delta_d = lambda_ * (Delta[x-1] - f[x-1]) / (lambda_ / m *((1 + eta) - (1 + theta) * (1 - a)))
                    b_optimal = 10
            else:
                # Handle the case where the index is out of bounds,
                continue  # Skip this iteration

        # Update survival probability and store optimal b and capital value.
        Delta[x] = Delta[x-1] + Delta_d / 1000
        Optimal_strategy[x-1] = b_optimal
        r[x-1] = (x - 1) / 1000

    # Plot the relationship between scaled initial capital (x) and optimal priority level (b*).
    plt.plot(r, Optimal_strategy, label='Optimal strategy')
    plt.xlabel('Initial Capital (x)')
    plt.ylabel('Optimal strategy for combined reinsurance (b*)')
    plt.ylim(0, 3)
    plt.legend()
    plt.grid(True)
    plt.show()

    # Plot the relationship between scaled initial capital (x) and survival probability (Delta(x)).
    plt.plot(r, Delta[1:], label='Survival Probability (Delta)')
    plt.xlabel('Initial Capital (x)')
    plt.ylabel('Survival Probability (Delta)')
    plt.ylim(0, 1.1)
    plt.legend()
    plt.grid(True)
    plt.show()

# Example usage:
Comb()
```

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