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Bowley Solution to the Retiree/Insurer game

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Abstract

In this master thesis, we consider the Bowley solution to the one-period time game between the seller and the buyer of the pension fund insurance. We examine two possible scenarios and aim to find the optimal parameters so that both the seller and the buyer are satisfied. Finally, after establishing the parameters, we analyze examples and compare the two scenarios to determine the best solution.

Kurzfassung

In dieser Masterarbeit wird die Bowley-Lösung für das Ein-Perioden-Zeitspiel zwischen dem Verkäufer und dem Käufer einer Pensionsfondsversicherung untersucht. Es werden zwei mögliche Szenarien betrachtet, und es wird versucht, die optimalen Parameter zu finden, sodass sowohl der Verkäufer als auch der Käufer zufrieden sind. Schließlich werden die Parameter festgelegt, Beispiele analysiert und die beiden Szenarien miteinander verglichen, um die beste Lösung zu finden.

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Diplomarbeit eigenständig und ohne unzulässige Hilfe verfasst habe. Ich habe ausschließlich die angegebenen Quellen und Hilfsmittel verwendet und alle wörtlich oder sinngemäß übernommenen Stellen ordnungsgemäß als solche kenntlich gemacht.

Vienna, 31.01.2025

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Contents

1 Introduction

In this thesis, we analyze a one-time-period game in which the players are the buyer and the seller of pension fund insurance, with the goal of finding the Bowley solution. The model is inspired by [1] and [2]. The Bowley solution represents the optimal pension fund insurance policy chosen by the buyer, along with the parameters that determine how the gains and losses are divided, first chosen by the buyer and then by the seller. Specifically, the buyer initially chooses between two possible scenarios regarding how they would like their pensions to be managed depending on the fund performance. We then analyze whether the chosen scenario is acceptable to the seller, based on current market conditions. Once this decision is finalized, we proceed to determine the optimal parameters that maximize the gains for both the buyer and the seller. First, we identify the conditions under which the seller would be most willing to share the gains and, conversely, the scenarios where the seller might be inclined to assist the buyer in cases of poor fund performance. Afterward, the buyer decides how they would prefer to manage the potential losses, and finally, the seller determines how the gains will be distributed, based on the buyer's decision regarding the losses.

Pension fund insurance is a type of insurance that not only provides coverage but also allows for investment in stocks or bonds. Policyholders pay regular premiums, and part of these payments goes toward insurance coverage, while the rest is invested. This investment can be in stocks, bonds, or a mix of both. This type of insurance has the potential to give higher returns than traditional insurance because it invests in financial markets. However, the returns are not guaranteed. The value of the investment can change depending on market conditions, which means there's a chance of losing the money put into the plan.
From the perspective of the insurance seller, losses in the pension fund during the payout phase (decumulation phase) can lead to reduced pensions for policyholders. This could harm the seller's reputation and cause financial losses in the future. On the other hand, pension reductions are also bad news for the buyers of the insurance. Our solution will focus on avoiding these problems and finding a balance that satisfies both the insurance company and the policyholders.

For simplicity, we will focus only on the decumulation phase of the pension scheme, which is the phase where the accumulated capital is paid out over time. We assume that the buyer's accumulated capital is invested in a fund and will be distributed gradually.

A critical factor in insurance is solvency, which refers to the financial health of the fund. Solvency is measured using the Degree of Capital Cover (DCC), which compares the value of the fund to the future pension payments that need to be made. In Germany, the law requires that the DCC must be between 100% and 125%.

- If the DCC falls below 100%, It has to be restored to the interval between

100% and 125%. If the fund still has a shortfall, pensions may need to be reduced.

- If the DCC rises above 125%, the law allows for pensions to be increased, aiming to bring the DCC back to the interval.

In our case, we will concentrate on the DCC in the interval [1, 1.25]. The challenge lies in finding the right balance that benefits both retirees and ensures the stability of the pension fund. This balance needs to consider the potential risks and rewards of investing in financial markets and the need to maintain the financial health of the fund over time.

2 Game Setup

In this section, we present a general version of the one-period game where the buyer is an individual purchasing pension fund insurance, and the seller is offering this insurance. We will examine the contract conditions and introduce some important definitions. Let us take a closer look at the contract and these key definitions.

2.1 Contract

Before any contract is made between the seller and the buyer, it is important to define the value of the fund that will be central to the pension fund insurance. The value of this fund is a critical factor, as it will influence both the returns that the buyer can expect and the potential liabilities that the seller might face. We assume that the value of the fund follows a geometric Brownian motion, which is a common model used in finance to describe the evolution of asset prices over time.

Definition 2.1
Geometric Brownian Motion (GBM) is a continuous-time stochastic process used to describe the random movement of stock prices over time. It assumes that stock prices can move up or down in small, continuous steps and that these changes are proportional to the current price. In other words, the percentage change in the stock price is random but follows a certain trend over time. A stochastic process S_t is said to follow a GBM if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

Where:

- S_t is the stock price at time t.
- μ is the drift term, representing the average rate of return.
- σ is the volatility term, representing the randomness or risk.
- dW_t is a small random change, akin to the effect of flipping the magical coin.
- W_t represents a Wiener process, also known as standard Brownian motion.

For an arbitrary initial value S_0 the above SDE has the analytic solution

$$S_t = S_0 \exp\left((\mu - \tfrac{\sigma^2}{2})t + \sigma W_t\right).$$

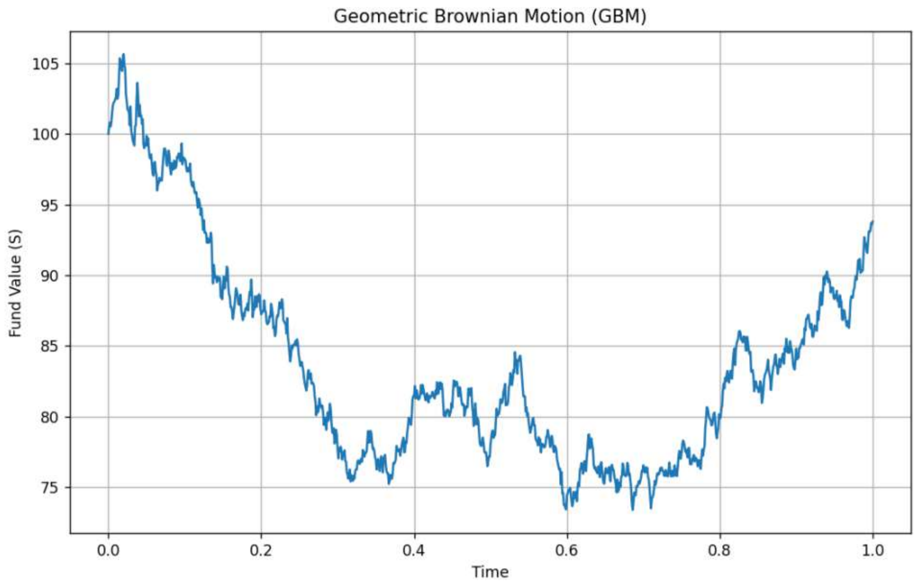


Figure 1: Example of GBM with $S_0 = 100$, $\mu = 0.1$, $\sigma = 0.2$ and $t = 1$
Source: Created by Ivana Stankic, 2024.

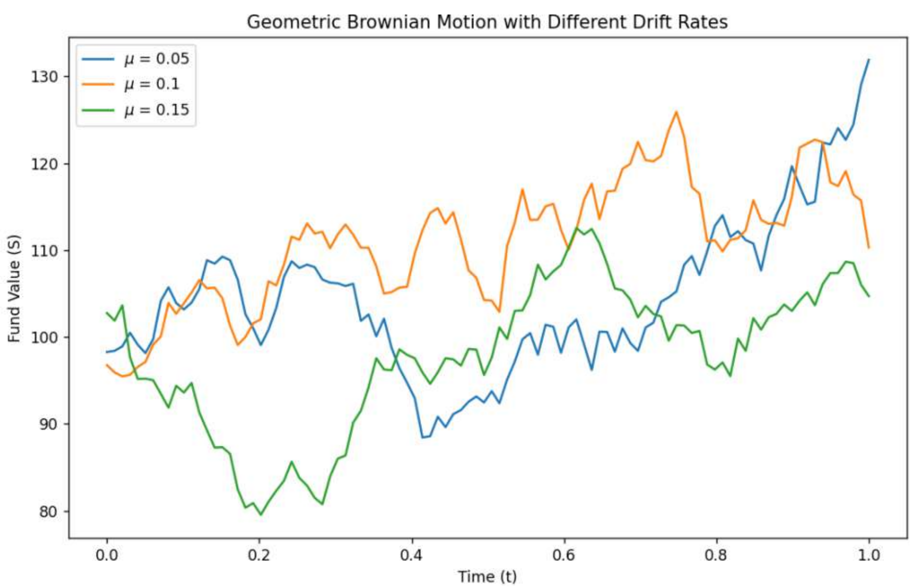


Figure 2: Example of GBM with $S_0 = 100$, $\sigma = 0.2$ and $t = 1$
Source: Created by Ivana Stankic, 2024.

So using definition 2.1 our fund value at time 1 is defined as

$$S_1 = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma W_1} \tag{1}$$

Where:

- μ tells us how fast, on average, the fund is expected to grow over time.
- σ measures how much the fund price tends to fluctate.
- t is in our case 1, since we are only interested in one time period.
- W_t is a random variable that models the uncertainty and randomness in fund price.

Definition 2.2
Since W_t is normally distributed with mean 0 and variance t, the term

$$\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

is normally distributed. Therefore, S_t is log-normal distributed

$$\ln(S_t) \sim \mathcal{N}\left(\ln(S_0) + (\mu - \frac{\sigma^2}{2}) \cdot t, \sigma^2 \cdot t\right).$$

Expected value and variance are given by:

$$\mathbb{E}[S_t] = S_0 \cdot e^{\mu t}, \tag{2}$$
$$\mathrm{Var}(S_t) = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right).$$

The probability density function of S_t is:

$$f_{S_t}(s; \mu, \sigma, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{s\sigma\sqrt{t}} \exp\left(-\frac{\left(\ln s - \ln S_0 - \left(\mu - \frac{1}{2}\sigma^2\right) t\right)^2}{2\sigma^2 t}\right). \tag{3}$$

The cumulative distribution function is:

$$F_{S_t}(x) = \Phi\left(\frac{\ln x - \ln S_0 - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right), \tag{4}$$

where Φ is the cumulative distribution function of the standard normal distribution (i.e., $\mathcal{N}(0, 1)$).

So now, let us consider the situation where the buyer is looking to purchase a pension fund. The contract is designed in such a way that the buyer will make regular premium payments to the seller until they reach the age of 65. This is the accumulation phase, where the buyer's contributions help build the fund that will later provide their pension. We assume that the buyer will retire at the age of 65, at which point the accumulation phase ends, and the decumulation phase begins. During the decumulation phase, the buyer starts receiving regular pension payments from the fund. Our focus is on this decumulation phase because it is crucial to understand how the pension fund will impact the buyer's retirement income. In the decumulation phase, the amount of pension payments the buyer receives can fluctuate based on the performance of the underlying pension fund. Since the value of the fund may vary due to market conditions, it's essential to establish a method for adjusting the pension payments accordingly. Therefore, let us introduce some important definitions, including the value of the pension for all future periods as well as the value of the pension for a single time period.

Definition 2.3
We define the future residual lifetime of a 65-year-old individual as a non-negative random variable, denoted by T_{65}. We assume that T is exponentially distributed, i.e.:

$$T \sim \exp(\lambda).$$

Here $\lambda > 0$ is the parameter of the distribution, often called the rate parameter. The cumulative distribution function of the random variable T_{65} is:

$$F_{65}(t) = \mathbb{P}[T_{65} \leq t] = 1 - e^{-\lambda t}. \tag{5}$$

$F_{65}(t)$ indicates the probability that a 65-year-old will not survive the next t years. Density function is defined as:

$$f_{65}(t) = \lambda e^{-\lambda t}, \text{for t} \geq 0. \tag{6}$$

Definition 2.4
Let us denote the pension rate by $P > 0$, and let $\delta > 0$ be the discounting rate. We can calculate the expected present value of pension payments, at time 1, for all future time periods as follows:

$$\begin{aligned} P \int_1^\infty e^{-\delta t} \mathbb{E}[\mathbb{I}_{T_{65}>t} \mid T_{65} > 1]\, dt &= P \int_1^\infty e^{-\delta t} \mathbb{P}[T_{65} > t \mid T_{65} > 1]\, dt \\ &= P \int_1^\infty e^{-\delta t} \frac{\mathbb{P}[T_{65} > t]}{\mathbb{P}[T_{65} > 1]}\, dt \\ &= P \int_1^\infty e^{-\delta t} e^{-\lambda(t-1)}\, dt \\ &= \frac{Pe^{-\delta}}{\delta + \lambda}. \end{aligned} \tag{7}$$

And expected present value of pension payments for one period would then be:

$$
\begin{aligned}
P \int_0^1 e^{-\delta t} \mathbb{P}[T_{65} > t]\, dt &= P \int_0^1 e^{-\delta t} e^{-\lambda t}\, dt \\
&= \frac{P}{(\delta + \lambda)} (1 - e^{-(\delta+\lambda)}).
\end{aligned} \tag{8}
$$

But why would the pension value need to change depending on fund movements? The answer lies in the necessity of keeping the pension plan solvent. The seller, who manages the pension fund, must ensure that the fund has enough capital to meet its future obligations to all participants. Sometimes, this requires adjusting the pension payments based on the terms agreed upon in the contract with the buyer. Solvency is crucial because it indicates the financial health of the pension fund. To measure solvency, the seller uses a metric known as the degree of capital cover. This metric compares the total value of the fund's assets to the expected future pension payments. If the fund's value is sufficient to cover these future payments, the plan is considered solvent. However, if the value of the fund decreases due to market fluctuations or other factors, the seller might need to reduce the pension payments to maintain solvency. Conversely, if the fund performs well and its value increases, there could be an opportunity to increase the pension payments, depending on the terms of the contract. In summary, the pension value may need to change in response to fund movements to ensure that the plan remains solvent and can continue to meet its obligations to the buyer.

Definition 2.5
The Degree of Capital Cover (DCC) is defined as the ratio of the current fund value to the expected present value of the pensions that are yet to be paid:

$$
DCC = \frac{\text{the current fund value}}{\text{the expected present value of pensions to be paid}}.
$$

Since we are dealing with a single buyer, we need to ensure that the person is still alive during the periods t when pensions are to be paid. If the person is no longer alive, there is no need to calculate the pensions. Division by zero is not possible, so we will use an indicator function,

$$
\mathbb{I}_{[T_{65}>t]} = \begin{cases} 1, & \text{if the person is alive at time } t, \\ 0, & \text{otherwise.} \end{cases}
$$

which specifies that if the buyer dies, the DCC is 0. So our DCC is defined as:

$$
DCC_t = \frac{\text{the current fund value at } t}{\text{the expected present value of pensions to be paid}} \cdot \mathbb{I}_{[T_{65}>t]}.
$$

We define C_1 as our DCC at time 1. In our context, the DCC at time 1, using (1) and (8) from definitions 2.3 and 2.4, is given by:

$$
\begin{aligned}
C_1 &= \frac{\text{the current fund value}}{\text{the expected present value of pensions to be paid}} \cdot \mathbb{I}_{[T_{65}>1]} \\
&= \frac{S_1}{P\int_1^\infty e^{-\delta t}\mathbb{E}[\mathbb{I}_{T_{65}>t} \mid T_{65}>1]\,dt} \cdot \mathbb{I}_{[T_{65}>1]} \\
&= \frac{S_0 e^{(-\mu-\frac{1}{2}\sigma^2)+\sigma W_1}}{\frac{Pe^{-\delta}}{\delta+\lambda}} \cdot \mathbb{I}_{[T_{65}>1]} \\
&= \frac{S_0 e^{(-\mu-\frac{1}{2}\sigma^2)+\sigma W_1}(\delta+\lambda)}{Pe^{-\delta}} \cdot \mathbb{I}_{[T_{65}>1]}.
\end{aligned} \tag{9}
$$

As already mentioned, we will consider that the DCC should remain within the interval of 1 to 1.25. It now needs to be determined what actions should be taken and what both the insurer and the insured are willing to do if at the time 1 the DCC falls outside this range. If the fund performs well and the DCC goes above 1.25, then of course we have a gain. If the buyer gets the entire gain, then in the next period, if the DCC goes below 1, the seller would have to either reduce pensions or inject money into the fund by buying additional shares, which they both would like to avoid. Therefore, we need to determine the consequences for both the buyer and the seller if the fund performs well or poorly, specifically what happens if at the time 1 the DCC leaves the interval from 1 to 1.25. We are starting our game at time zero, where the buyer can choose how they would like their pensions to be handled, depending on the movement of the fund. There are two possibilities for the buyer of insurance to consider at time 0 in their contract:

1. If the fund isn't performing well and the DCC is below 1 at time 1, the buyer could, at time zero, choose the option not to receive any help from the seller, meaning their pension would be reduced in order to bring the DCC back into the interval [1, 1.25]. If they choose this option, then in the case where the DCC is above 1.25, the seller would be willing to increase their pension by the entire gain. In this case, the buyer needs to choose some parameter $s \in [1, 1.25]$ to which the DCC will return after it has moved outside of this interval. In this scenario, there are two possible outcomes at time 1:

   - If the DCC is below 1, the pension will be reduced in order to adjust the DCC back to s,
   - If the DCC is above 1.25, the pension will be increased by the entire gain, which depends, again, on s.

   The goal for the buyer in this case is to find the optimal s^* that maximizes their gain and minimizes their loss.

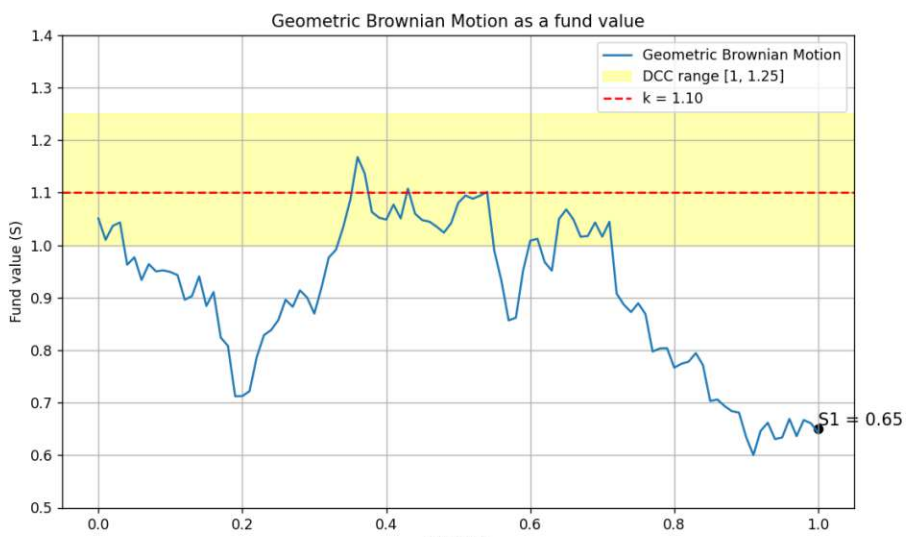
2. If, however, the buyer chooses to receive some help from the seller in case the DCC falls below 1 at time 1, meaning they do not want their pension to be reduced, there will be consequences depending on how the

fund develops. In this case, the buyer would have to share the gains with the seller in the event of good fund performance in order to cover some potential losses in case of bad fund performance. So again, it could be agreed that the buyer selects some parameter $k \in [1, 1.25]$, to which the DCC will automatically revert if, at time 1, the DCC leaves the interval $[1, 1.25]$. In this scenario, there are two possible outcomes at time 1:

- If the DCC is below 1, the pension will not be reduced, and the DCC will be adjusted to k. In this case the seller will inject the necessary amount into the pension fund in order to bring it back to k.
- Since the seller would have to inject some amount into the pension fund if the DCC falls below 1, if the DCC goes above 1.25, the buyer would have to share their gains with the seller. Depending on their choice of k, the seller would choose a parameter $\theta \in [0,1]$, which will determine the percentage the seller will get from the gain. We will determine this parameters in Section 3.

For better understanding, let's now take a look at a simple example.

Example 2.1
In the following graph, we can see a simple example of a geometric Brownian motion, with $k = 1.10$ and DCC in the range of 1 to 1.25.



Source: Created by Ivana Stankic, 2024.

Let us simplify and assume that $\frac{Pe^{-\delta}}{\delta+\lambda}$ is 1, at time 1. In the graph, we can clearly see that at time 1, S_1 is 0.65. Let us now calculate our C_1 as in the formula (9) given above.

$$C_1 = \frac{S_1}{1}$$
$$= 0.65$$

So C_1 is below 1 and there are now 2 possible scenarios, as explained above.

1. If the buyer, at time 0, chooses that if the C_1 falls below 1 at time 1, they do not want the seller's help and prefer to have their pension reduced in order to bring the C_1 back to k (for the purpose of this example and simplicity, we will use the parameter k for both scenarios, but in our further research, we will use two different parameters for these scenarios). Now, let us calculate in this example what would happen to the buyer's pension. So, in order to return to k, we will reduce the pension by some amount, which should be injected into the pension fund. The new pension would then be:

   $$\tfrac{S_1}{P^{new}} = k \Rightarrow P^{new} = \tfrac{S_1}{k}.$$

   And by plugging in the numbers from our example

   $$P^{new} = \tfrac{0.65}{1.1} = 0.59$$

   So the new pension would, instead of 1, be 0.59.

2. If the buyer, at time 0, chose the other option, meaning that their pension would not be reduced in case of the fund's bad performance, then the seller would inject the needed amount into the pension fund in order to bring the C_1 back to k. So, in this case, the seller would incur a loss of

   $$\frac{S_1 + x}{P} = k \Rightarrow x = 0.45,$$

   and the buyer's pension would remain the same.

In the previous example, we saw what happens if the C_1 goes below 1 at time 1. If, however, it goes above 1.25, the gain for the insurance buyer would be:

1. If the buyer has chosen the first case mentioned above, then in the case of the fund's good performance, the buyer will receive the entire gain, and their pension will be increased :

   $$\tfrac{S_1}{P^{new}} = k \Rightarrow P^{new} = \tfrac{S_1}{k}.$$

2. If the buyer has chosen the second case, then they would not receive the entire gain, but it would depend on the parameter θ, which was chosen by the seller. In this case, their pension will not increase, but the fund will decrease by some amount x in order to return to k. After that, the gain will be split between the buyer and the seller according to the parameter θ.

$$\frac{S_1 - x}{P} = k \Rightarrow x = S_1 - Pk.$$

   In this case, the seller's gain would be $(1-\theta) \cdot x$ and the buyer's gain would be $\theta \cdot x$.

In this simple example, we can see how all scenarios should look in practice. Our main goal here is to determine the parameters k, s and θ so that the expectations of both sides, the seller and the buyer, are fulfilled. We will take a further look at determining the optimal k, s and θ in Section 3. Now let us determine how we will theoretically treat these gains and losses.

2.2 Consequences in terms of Pensions

Let us now assume that the contract between the seller and the buyer has been made on the terms mentioned above and let us take a look at time 1 in order to see what the consequences are in terms of rents. Let w_s and w_b be random variables that denote the wealth of the buyer and the seller at time 1.

Scenario 1

$t = 0$: the pension value is fixed, and the buyer decides that if C_1 goes below 1, they prefer their pension to be reduced. However, if the fund performs well and the C_1 goes beyond 1.25, their pension will be increased. There are now 3 possible outcomes at time 1

1. $t = 1$: $0 < C_1 < 1$
2. $t = 1$: $C_1 > 1.25$
3. $t = 1$: $C_1 \in [1, 1.25]$.

Let us take a closer look at those:

1. $t = 1$: We assume that at time 1 it holds $0 < C_1 < 1$. This means that S_1 is less than the pension´s present value. In this case, the seller would need to inject some amount x into the pension fund to restore it to the desired level s. However, since the buyer chose not to require the seller's assistance in the event of $0 < C_1 < 1$, the seller had reduced their pension by the same amount x they need to inject into the pension fund. The new pension can be calculated as follows:

$$0 < C_1 < 1 \overset{(9)}{\Longrightarrow} \frac{S_1}{P^{new} \int_1^\infty e^{-\delta t} \mathbb{E}[\mathbb{I}_{T_{65}>t} \mid T_{65} > 1] \, dt} = s$$

$$\Rightarrow P^{new} \cdot \frac{e^{-\delta}}{\delta + \lambda} = \frac{S_1}{s}.$$

We are now looking only at the decumulation phase, so we assume that the buyer is no longer paying premiums for this insurance. The buyer's value of future discounted pensions at time 1 in this case could be written as
$w_b = P^{new} \cdot \frac{e^{-\delta}}{\delta+\lambda} \cdot \mathbb{I}_{[0<C_1<1]}$

2. t = 1: We now assume that at time 1 $C_1 > 1.25$. The buyer's pension will now be increased and we can use the same calculation as above:

$$C_1 > 1.25 \overset{(9)}{\Longrightarrow} \frac{S_1}{P^{new} \int_1^\infty e^{-\delta t} \mathbb{E}[\mathbb{I}_{T_{65}>t} \mid T_{65} > 1] \, dt} = s$$

$$\Rightarrow P^{new} \cdot \frac{e^{-\delta}}{\delta + \lambda} = \frac{S_1}{s}.$$

So the buyer's value of future increased pensions can be now written as
$w_b = P^{new} \cdot \frac{e^{-\delta}}{\delta+\lambda} \cdot \mathbb{I}_{[C_1>1.25]}$

3. t = 1: In case $C_1 \in [1, 1.25]$ at time 1, nothing happens; the wealth remains the same.
$w_b = \frac{Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{I}_{[C_1 \in [1,1.25]]}$

Choosing scenario 1 could be advantageous for the buyer because, in the case of good fund performance, they would have the opportunity to participate fully in the entire gain generated by the fund. This means that any positive growth or returns from the fund would directly benefit the buyer without limitations. However, this option also comes with inherent risks. If the fund performs poorly, the buyer could face significant financial losses, as they would bear the full impact of the fund's negative performance. When making this decision, it is crucial for the buyer to carefully assess the likelihood of different performance outcomes. Specifically, they should evaluate the probabilities of $0 < C_1 < 1$, which indicates a loss, or $C_1 > 1.25$, which represents a substantial gain. These probabilities can provide valuable insights into the potential risks and rewards of choosing scenario 1. Ultimately, the decision should align with the buyer's risk tolerance, financial goals, and expectations for the fund's performance. A detailed analysis of these probabilities and their implications will be presented in Section 3, where we will explore the quantitative aspects of this choice in greater depth.

Scenario 2

t = 0: Buyer decides that if the C_1 goes below 1, they wouldn't like their pension to be reduced. However, if the fund performs well and the C_1 goes beyond 1.25, they will share their gain with the seller. As already mentioned, in this case, the buyer would have to choose k to which the C_1 will return after leaving the interval [1, 1.25], and then the seller chooses θ. We now assume, that k and θ are chosen at time 0. So again, there are 3 possible outcomes at time 1

1. t = 1: $0 < C_1 < 1$
2. t = 1: $C_1 > 1.25$
3. t = 1: $C_1 \in [1, 1.25]$.

And so, let us take a closer look at those scenarios as well:

1. t = 1: If $0 < C_1 < 1$ at time 1, as agreed at the beginning, in order to come back to k, the seller would have to inject some amount x into the pension fund. The x can be determined as follows:

   $$\frac{S_1+x}{\frac{Pe^{-\delta}}{\delta+\lambda}} = k \Rightarrow x = \frac{kPe^{-\delta}}{(\delta+\lambda)} - S_1.$$

   So, in this case, there is no need to calculate a new pension because it stays the same and we are only interested in the gains and losses associated with fund performances. Since, in this case, only the seller would have to pay, the buyer incurs no losses, and their wealth remains unchanged. The seller's wealth at time 1 can be written as:
   $w_s = (S_1 - \frac{kPe^{-\delta}}{(\delta+\lambda)}) \cdot \mathbb{I}_{[0<C_1<1]}$

2. t = 1: If $C_1 > 1.25$ at time 1, now the buyer will have to share the gains with the seller. The pension fund will now be reduced by some amount x in order to return to k. Then this gain x will be split between the buyer and the seller using the already determined parameter θ from time 0.

   $$\frac{S_1-x}{\frac{Pe^{-\delta}}{\delta+\lambda}} = k \Rightarrow x = S_1 - \frac{kPe^{-\delta}}{(\delta+\lambda)}.$$

   So the wealth, at time 1, can be now written as
   $w_s = (S_1 - \frac{kPe^{-\delta}}{(\delta+\lambda)}) \cdot (1-\theta) \cdot \mathbb{I}_{[C_1>1.25]}$
   $w_b = (S_1 - \frac{kPe^{-\delta}}{(\delta+\lambda)}) \cdot \theta \cdot \mathbb{I}_{[C_1>1.25]}$

3. t = 1: In case DCC $\in$ [1, 1.25] at time 1, nothing happens; the wealth remains the same.

Now that we have listed all possible events and outcomes at the times we are considering, let us proceed to Section 3 where we will determine all those parameters.

3 Bowley solution

In the following section, we will focus on finding an optimal solution and solving the maximization problem for both the buyer's and the seller's expected wealth, as discussed earlier in Section 2. Our goal is to determine the Bowley solution using the following game setup:

1. At time 0 the buyer would have to decide how they would like their pension to be managed if the C_1 moves outside the critical interval [1,1.25]. The buyer has two main options. They can choose whether their pension should be reduced or increased when C_1 falls below or rises above this interval, respectively. Alternatively, they may prefer the seller's assistance in case of unfavorable fund movements, which would, of course, mean sharing their gains with the seller in case of favorable fund movements. In both cases the buyer select a specific value k or s to which the fund value will be adjusted once it leaves the interval. This choice influences the stability of their pension payments in the future.

2. The second step would be for the buyer to choose an optimal parameter k^* or s^*, depending on their choice of the scenarios mentioned above. In order to do that, the buyer would choose k^* or s^* which maximizes their expected wealth w_b, as defined in Section 2.2. The buyer's optimization problem is therefore:

$$\max_{k^*,s^* \in [1,1.25]} \mathbb{E}[w_b]$$

   Here, k^* and s^* is chosen from within the interval [1, 1.25]. It is very important, when making a decision on which parameters should be chosen, that the buyer takes into account the parameter θ^* , which will be chosen by the seller. Whichever parameter k^*, s^* the buyer chooses, it will directly influence the seller's choice of the parameter θ^*.

3. Once the buyer has chosen the optimal k^*, the seller then selects the optimal θ^* to maximize their own expected wealth w_s, also defined in Section 2.2. The seller's optimization problem is:

$$\max_{\theta^* \in [0,1]} \mathbb{E}[w_s]$$

   The seller's decision regarding θ^* will directly influence the adjustments made to the buyer's pension, and therefore, their own financial outcomes. By solving this maximization problem, the seller determines the best strategy to maximize their wealth while considering the buyer's choice of k^*.

3.1 Optimization problem

The buyer's goal is to find the optimal k^* or s^* such that, in case of a gain in the pension fund, they receive the highest possible return, and in case of a loss, their losses are minimized, or ideally, avoided entirely. The amount they receive from the gain also depends on the parameter θ, which will later be chosen by the seller of the insurance. Therefore, the buyer must carefully consider the seller's preferences when picking k, as choosing a value that is unfavorable to the seller could result in the seller responding with a lower θ, which would reduce the buyer's share of the gain. Let us first define the expected value mathematically, as it forms the basis for calculating the buyer's expected wealth.

Definition 3.1
Let X be a real-valued discrete random variable that takes the values $(x_i)_{i\in I}$ with corresponding probabilities $(p_i)_{i\in I}$ (where I is a countable index set). Then, assuming the expectation exists, the expected value $\mathbb{E}(X)$ is calculated as:

$$\mathbb{E}[X] = \sum_{i\in I} x_i p_i = \sum_{i\in I} x_i P[X = x_i]$$

3.1.1 Scenario 1 (Allowing for a pension reduction)

We now turn to determining the expected buyer's wealth from Scenario 1 of Section 2.2. Let T_{65} be the future residual lifetime of a 65-year-old individual as defined in Definition 2.3. The expected buyer's wealth at time 1 is defined as the difference between their increased pension and their reduced pension. Thus, the expected buyer's wealth at time 1 can be expressed as follows:

$$\begin{aligned}
\mathbb{E}[w_b] &= \mathbb{E}[w_b \cdot \mathbb{I}_{[C_1>1.25]} - w_b \cdot \mathbb{I}_{[0<C_1<1]}] \\
&= \mathbb{E}\left[\left(\frac{S_1}{s} - \frac{Pe^{-\delta}}{\delta+\lambda}\right) \cdot \mathbb{I}_{[C_1>1.25]} - \left(\frac{Pe^{-\delta}}{\delta+\lambda} - \frac{S_1}{s}\right) \cdot \mathbb{I}_{[0<C_1<1]}\right] \\
&= \frac{1}{s} \cdot \mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] - \frac{Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[C_1 > 1.25] \\
&\quad - \frac{Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[0 < C_1 < 1] + \frac{1}{s} \cdot \mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}]
\end{aligned}$$

Our goal now is to find the optimal parameter s^* to solve the following problem.

$$\max_{s^*\in[1,1.25]} \mathbb{E}[w_b].$$

To do that, we will first calculate the first derivative of the function. This will provide more information about the function itself, as well as indicate the value of s at which it reaches its maximum. The terms involving s in the function are:

$$\frac{1}{s} \cdot \left(\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1 > 1.25]}] + \mathbb{E}[S_1 \cdot \mathbb{I}_{[0 < C_1 < 1]}]. \right)$$

This is the term we need to differentiate with respect to s. The derivative of $\mathbb{E}[w_b]$ with respect to s is:

$$\frac{d}{ds}\mathbb{E}[w_b] = -\frac{1}{s^2} \cdot \left(\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1 > 1.25]}] + \mathbb{E}[S_1 \cdot \mathbb{I}_{[0 < C_1 < 1]}] \right).$$

The first derivative of the function is always negative, which implies that the function increases as s decreases. The buyer would only accept this option if their expected wealth is greater than zero, meaning that their expected gain (in this case, an increased pension) would be higher than their expected loss (a decreased pension), i.e.

$$\mathbb{E}\left[\left(\frac{S_1}{s} - \frac{Pe^{-\delta}}{\delta + \lambda} \right) \cdot \mathbb{I}_{[C_1 > 1.25]} \right] > \left[\left(\frac{Pe^{-\delta}}{\delta + \lambda} - \frac{S_1}{s} \right) \cdot \mathbb{I}_{[0 < C_1 < 1]} \right]. \tag{10}$$

Based on the derivative alone, one could conclude that the best option for s would be 1. In this case, the gain would be the highest in the event of good fund performance, and the loss would be the smallest in the case of poor fund performance. However, if s is chosen to be 1 and the fund performs poorly so that $C_1 < 1$, the buyer's pension would be decreased, and C_1 would be restored to 1. This adjustment could lead to a higher probability of the fund performing poorly again in the next period, which would be even worse for the buyer because the starting point is now even lower. We will now examine these possibilities. Since we described our fund value S_1 using geometric Brownian motion 1, we can explore more scenarios and probabilities of fund movements by considering different parameters μ and σ, as well as the known properties of geometric Brownian motion. Knowing from Definition 2.2 that S_t follows a log-normal distribution and that

$$F_{S_t}(x) = \Phi\left(\frac{\ln x - \ln S_0 - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \right),$$

we can calculate the following probabilities:

$$\mathbb{P}[0 < C_1 < 1] = \mathbb{P}[0 < S_1 < \frac{Pe^{-\delta}}{\delta + \lambda}]$$

$$= \mathbb{P}[C_1 < 1] - \mathbb{P}[C_1 = 0].$$

Since $S_1 > 0$ always holds for geometric Brownian motion, the probability simplifies to:

$$\mathbb{P}[0 < C_1 < 1] = \Phi\left(\frac{\ln(\frac{Pe^{-\delta}}{\delta+\lambda}) - \ln(S_0) - (\mu - \frac{\sigma^2}{2})}{\sigma}\right) \tag{11}$$

and

$$\begin{aligned}\mathbb{P}[C_1 > 1.25] &= 1 - \mathbb{P}[S_1 < 1.25 \cdot \frac{Pe^{-\delta}}{\delta+\lambda}] \\ &= 1 - \Phi\left(\frac{\ln(1.25 \cdot \frac{Pe^{-\delta}}{\delta+\lambda}) - \ln(S_0) - (\mu - \frac{\sigma^2}{2})}{\sigma}\right).\end{aligned} \tag{12}$$

A higher σ spreads the distribution of S_1, increasing the probabilities of extreme outcomes (both $C_1 < 1$ and $C_1 > 1.25$). With a lower σ, there is more stability in the fund's movements. A higher μ indicates a higher expected fund value, as μ contributes to an upward drift in the fund's value. Therefore, choosing s does not only depend on maximizing the buyer's wealth in a single period but also involves considering potential losses in other periods and assessing the buyer's risk tolerance. So, by choosing the right parameters and the right scenario, the buyer must consider these probabilities. Choosing s to be 1 might seem like the best option, but if σ is high, or μ very low, it could be very risky. This is because there could be a very bad outcome in the next period if the fund performs poorly, especially when starting at the low point. We will address this matter later. Let us now calculate expected fund values $\mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}]$ and $\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}]$.

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}] = \int_0^{\frac{Pe^{-\delta}}{\delta+\lambda}} l \cdot f_{S_1}(l)\, dl$$

where $f_{S_1}(l)$ is the probability density function of S_1. The PDF of S_1, a log-normal random variable, is:

$$f_{S_1}(l) = \frac{1}{l\sigma'\sqrt{2\pi}} e^{-\frac{(\ln(l)-\mu')^2}{2\sigma'^2}}$$

where:

$$\mu' = \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right), \quad \sigma'^2 = \sigma^2.$$

Substitute $f_{S_1}(l)$ into the expectation:

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}] = \int_0^{\frac{Pe^{-\delta}}{\delta+\lambda}} l \cdot \frac{1}{l\sigma'\sqrt{2\pi}} e^{-\frac{(\ln(l)-\mu')^2}{2\sigma'^2}}\, dl.$$

Simplify:

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}] = \int_0^{\frac{Pe^{-\delta}}{\delta+\lambda}} \frac{1}{\sigma'\sqrt{2\pi}} e^{-\frac{(\ln(l)-\mu')^2}{2\sigma'^2}}\, dl.$$

Let $x = \ln(l)$, so $l = e^x$ and $dl = e^x dx$. Change the integration limits:

$$l = 0 \implies x \to -\infty, \quad l = \frac{Pe^{-\delta}}{\delta + \lambda} \implies x = \ln(\frac{Pe^{-\delta}}{\delta + \lambda})$$

Substitute into the integral:

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}] = \int_{-\infty}^{\ln(\frac{Pe^{-\delta}}{\delta+\lambda})} \frac{1}{\sigma'\sqrt{2\pi}} e^{-\frac{(x-\mu')^2}{2\sigma'^2}} e^x \, dx.$$

The exponent becomes:

$$-\frac{(x-\mu')^2}{2\sigma'^2} + x = -\frac{\big(x - (\mu' + \sigma'^2)\big)^2}{2\sigma'^2} + \mu' + \frac{\sigma'^2}{2}.$$

The integral now simplifies to:

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}] = e^{\mu' + \frac{\sigma'^2}{2}} \int_{-\infty}^{\ln(\frac{Pe^{-\delta}}{\delta+\lambda})} \frac{1}{\sigma'\sqrt{2\pi}} e^{-\frac{\left(x-(\mu'+\sigma'^2)\right)^2}{2\sigma'^2}} \, dx. \qquad (*)$$

The remaining integral is the cumulative distribution function of a normal distribution:

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du.$$

The term inside the integral (*) corresponds to the probability density function of a normal distribution with mean $\mu' + \sigma'^2$ and variance σ'^2. To transform this into the standard normal form (mean 0, variance 1), we perform a change of variables. Let:

$$z = \frac{x - (\mu' + \sigma'^2)}{\sigma'}.$$

Rearranging, this implies:

$$x = z\sigma' + (\mu' + \sigma'^2).$$

When $x = -\infty$, $z \to -\infty$, and when $x = \ln\left(\frac{Pe^{-\delta}}{\delta+\lambda}\right)$, z becomes:

$$z = \frac{\ln\left(\frac{Pe^{-\delta}}{\delta+\lambda}\right) - (\mu' + \sigma'^2)}{\sigma'}.$$

Thus:

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}] = e^{\mu' + \frac{\sigma'^2}{2}} \Phi\left(\frac{\ln(\frac{Pe^{-\delta}}{\delta+\lambda}) - (\mu' + \sigma'^2)}{\sigma'}\right).$$

The expected value $\mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}]$ is defined as:

$$e^{\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma^2}{2}} \Phi\left(\frac{\ln(\frac{Pe^{-\delta}}{\delta+\lambda}) - \left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right) + \sigma^2\right)}{\sigma}\right). \qquad (13)$$

To calculate $\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}]$, we start with:

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] = \int_{\frac{1.25 \cdot Pe^{-\delta}}{\delta+\lambda}}^{\infty} l \cdot f_{S_1}(l)\, dl,$$

where $f_{S_1}(l)$ is same as above. Now using the same approach as above we can calculate:

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] = e^{\mu' + \frac{\sigma'^2}{2}} \int_{\ln(\frac{1.25 \cdot Pe^{-\delta}}{\delta+\lambda})}^{\infty} \frac{1}{\sigma' \sqrt{2\pi}} e^{-\frac{\left(x-(\mu'+\sigma'^2)\right)^2}{2\sigma'^2}} dx.$$

The remaining integral is the tail function of the normal distribution:

$$1 - \Phi\left(\frac{\ln(\frac{1.25 \cdot Pe^{-\delta}}{\delta+\lambda}) - (\mu' + \sigma'^2)}{\sigma'}\right).$$

Finally, substituting μ' and σ'^2 $\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}]$ can be written as :

$$e^{\ln(S_0)+\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\sigma^2}{2}} - \mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}]. \tag{14}$$

To make it simpler with the notations, let us denote: $\mathbb{P}[0 < C_1 < 1] = p_1$ and $\mathbb{P}[C_1 > 1.25] = p_2$. Now, there are two possible outcomes that we can examine:

- $p_2 > p_1$,
- $p_2 \leq p_1$.

We would now like to examine what happens when $p_2 > p_1$. If p_2 is significantly larger than p_1, it would be safe for the buyer to choose a smaller s, as the probability of the fund value performing poorly would be very low. However, as p_2 approaches p_1, it becomes riskier because there is an increased likelihood of the fund also performing poorly in the next period. In such a case, the starting point for subsequent performance would be very low, potentially causing the fund to decline even further. This creates a risk for the buyer of incurring greater losses, as the buyer's pension would need to be reduced to bring the C_1 back to s. However, choosing a higher s, such as $s = 1.25$, could also be even more risky. In the case of poor fund performance, the buyer's pension would need to be decreased to return to $s = 1.25$ or another high s, which could result in significant losses for the buyer because C_1 would have to return to the highest point and in the next period, if there is another loss, the decrease in the pension could be even greater. Additionally, in the case of a gain, if $s = 1.25$, the buyer would experience the lowest possible gain. So, situations where p_1 is approaching p_2 but p_2 is still higher, the optimal s^* would also be 1, but these situation would be very risky. This way, in the case of poor fund performance, the buyer's pension would decrease, but not as drastically as it would if s were closer to 1.25. Similarly, in the case of a gain, there would be a higher potential gain. If $p_2 \leq p_1$, there is a greater chance of poor fund performance in the next period as well, exposing the buyer to a much higher risk. After verifying that the condition (10) holds, we can agree that the optimal s^* for the buyer in this case would be

$$s^* = \arg \max_{1 \leq s \leq 1.25} \mathbb{E}[w_b]. \tag{15}$$

But even if (10) holds, this option could sometimes be just too risky, and the buyer should first consider all possible risks before choosing it. Let us now consider an example.

Example 3.1

Let us assume that the parameters are set as follows:

- $P = 0.05$,
- $\lambda = 0.02$,
- $\delta = 0.03$,
- $\mu = 0.5$,
- $\sigma = 0.2$,
- $S_0 = 1.1$,

First, we calculate the expected present pension value

$$\frac{Pe^{-\delta}}{\delta + \lambda} = 0.9704455335485082.$$

At time $t = 1$, S_1 follows a log-normal distribution with:

$$\ln(S_t) \sim \mathcal{N}\left(\ln(S_0) + (\mu - \frac{\sigma^2}{2}), \sigma^2\right),$$

We calculate the cumulative probabilities of C_1 using (4), (11), (12) and Phyton code 5.1:

$$\mathbb{P}[0 < C_1 < 1] = 0.001236805$$

$$\mathbb{P}[C_1 > 1.25] = 0.9719869.$$

We also calculate expected fund values using (13), (14) and Phyton code 5.1:

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}] = 0.001136143$$

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] = 1.7820489726.$$

We calculate s^* using (15):

$$s^* = \arg \max_{1 \leq s \leq 1.25} \mathbb{E}[w_b] = 1.$$

Now first check if condition (10):

$$\mathbb{E}\left[\left(\frac{S_1}{s} - \frac{Pe^{-\delta}}{\delta + \lambda}\right) \cdot \mathbb{I}_{[C_1>1.25]}\right] > \left[\left(\frac{Pe^{-\delta}}{\delta + \lambda} - \frac{S_1}{s}\right) \cdot \mathbb{I}_{[0<C_1<1]}\right]$$

holds. Left hand side:

$$\mathbb{E}\left[\left(\frac{S_1}{s}-\frac{Pe^{-\delta}}{\delta+\lambda}\right)\cdot\mathbb{I}_{[C_1>1.25]}\right]=0.8387885429759081.$$

Right hand side:

$$\mathbb{E}\left[\left(\frac{S_1}{s}-\frac{Pe^{-\delta}}{\delta+\lambda}\right)\cdot\mathbb{I}_{[C_1>1.25]}\right]=0.00006410.$$

Obviously, the condition holds, meaning that the buyer's expected gain would be higher than their expected loss in the case where s is 1. And in this case expected buyer´s wealth would be:

$$\mathbb{E}[w_b]=0.8387244335.$$

The expected gain is very high because, as previously mentioned, μ and σ are very favorable, and the fund value would almost certainly grow with these parameters. Let us now consider a different set of μ and σ parameters.

- $\mu=0.2$,
- $\sigma=0.6$,

Probabilities and expected values are calculated as follows:

$$\mathbb{P}[0<C_1<1]=0.4043189$$

$$\mathbb{P}[C_1>1.25]=0.44839308$$
$$\mathbb{E}[S_1\cdot\mathbb{I}_{[0<C_1<1]}]=0.2684971$$
$$\mathbb{E}[S_1\cdot\mathbb{I}_{[C_1>1.25]}]=0.9148476.$$

As we can see, in this case, the probabilities are very similar, and $\mathbb{P}[0<C_1<1]$ is approaching $\mathbb{P}[C_1>1.25]$. This indicates that the fund movements are highly uncertain, which is also evident from the fact that our parameter σ is high. Again, as in (15), we assume that the buyer chooses s^* :

$$s^*=1.$$

Again, we first check if condition (10):

$$\mathbb{E}\left[\left(\frac{S_1}{s}-\frac{Pe^{-\delta}}{\delta+\lambda}\right)\cdot\mathbb{I}_{[C_1>1.25]}\right]>\left[\left(\frac{Pe^{-\delta}}{\delta+\lambda}-\frac{S_1}{s}\right)\cdot\mathbb{I}_{[0<C_1<1]}\right]$$

holds. Left hand side:

$$\mathbb{E}\left[\left(\frac{S_1}{s}-\frac{Pe^{-\delta}}{\delta+\lambda}\right)\cdot\mathbb{I}_{[C_1>1.25]}\right]=0.4797065380844259$$

Right hand side:

$$\mathbb{E}\left[\left(\frac{S_1}{s} - \frac{Pe^{-\delta}}{\delta+\lambda}\right) \cdot \mathbb{I}_{[C_1>1.25]}\right] = 0.123872392353922.$$

And expected buyer's wealth in this case would be:

$$\mathbb{E}[w_b] = 0.3558341457305039.$$

But let us analyze this case further. First we will take a look at the case where we assume that at time 1, $C_1 > 1.25$. This means that the buyer's pension would increase in order for C_1 to return to $s = 1$. In this case, the new expected pension could be calculated as follows:

$$\mathbb{E}[C_1 \cdot \mathbb{I}_{[C_1>1.25]}] = \mathbb{E}\left[\frac{S_1}{\frac{Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{I}_{[C_1>1.25]}}\right] = 1$$

$$\implies \frac{Pe^{-\delta}}{\delta+\lambda} = \frac{\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}]}{\mathbb{P}[C_1 > 1.25]} = 2.0403.$$

We would now like to take a look at time 2 to see what could happen in the next period with this set of parameters and probabilities. We set the parameters as follows:

- $t = 2$,
- $s = 1$,
- $P_1 = 0.105122$,
- $\lambda = 0.02$,
- $\delta = 0.03$,
- $\mu = 0.2$,
- $\sigma = 0.6$,
- $S_1 = 2.0403$.

$$\frac{P_1 e^{-\delta}}{\delta+\lambda} = 2.0403.$$

We can easily adapt our Python code 5.1 the new values for time 2 and calculate the following:

$$\mathbb{P}[0 < C_1 < 1] = 0.4671270380$$

$$\mathbb{P}[C_1 > 1.25] = 0.4283865300.$$

We can see that the probability of poor fund performance is now higher then the probability of good fund performance. The compounding of volatility over multiple periods increases the likelihood of outcomes deviating significantly from the mean. This is often referred to as volatility drag. Volatility drag reduces actual portfolio growth below its simple average return due to the mathematical effects of volatility over time. The more volatile the portfolio, the wider this performance gap becomes. If we assume that s was higher, for example $s = 1.25$, the potential gain would be even lower, and potential losses much higher, as the pension would have to be reduced even further in order to return to a higher s, or $s = 1.25$. So, when choosing scenario 1, the buyer should probably consider choosing the option where σ is lower and μ is higher, because otherwise, they would most certainly, over time, incur a loss. In the following graphs, we can observe how different parameters μ and σ affect the fund's performance.

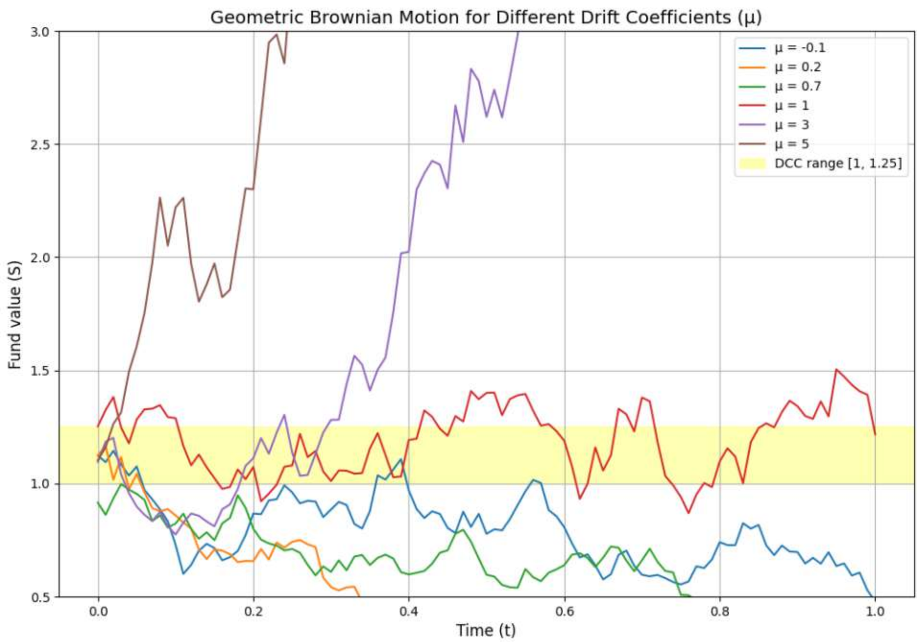


Figure 3: Example of GBM with $S_0 = 1.1$, $\sigma = 0.7$ and $t = 1$
Source: Created by Ivana Stankic, 2024.

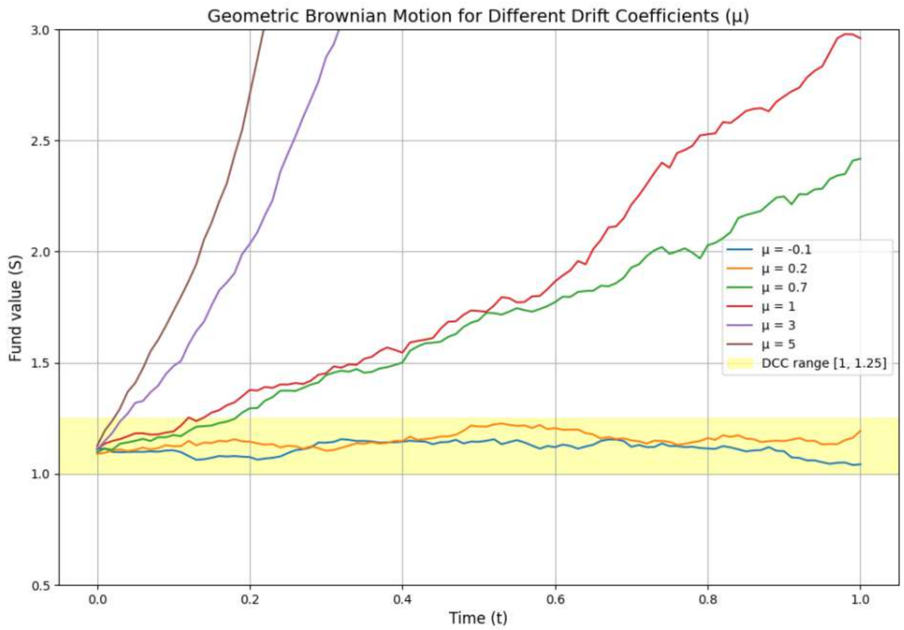


Figure 4: Example of GBM with $S_0 = 1.1$, $\sigma = 0.1$ and $t = 1$

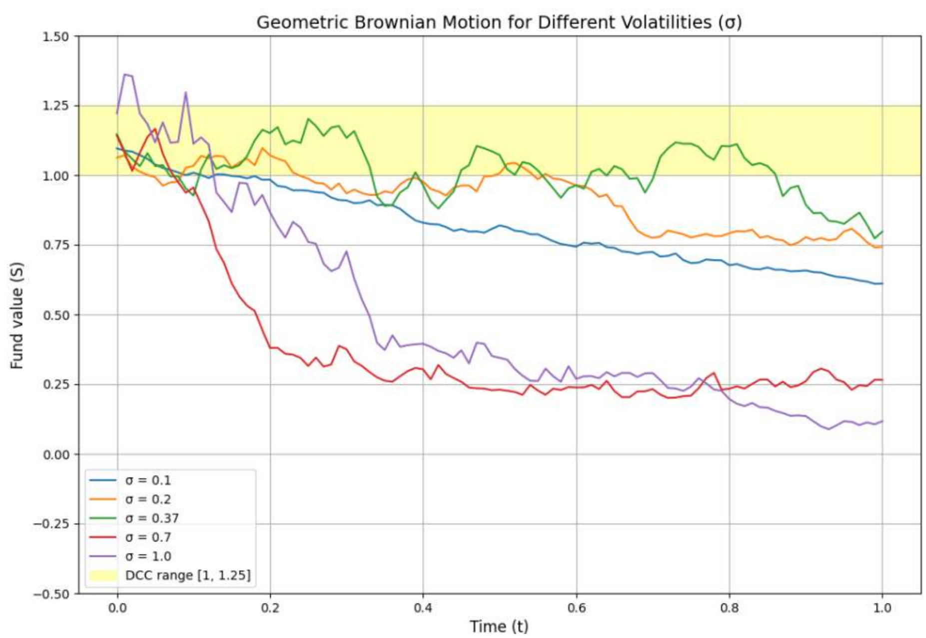


Figure 5: Example of GBM with $S_0 = 1.1$, $\mu =$ -0.5 and $t = 1$
Source: Created by Ivana Stankic, 2024.

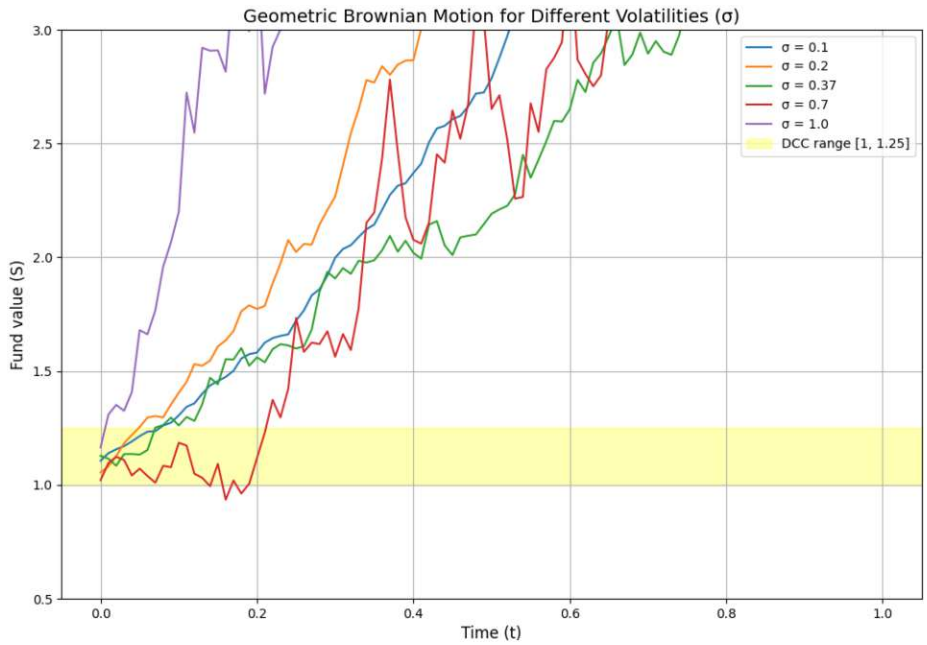


Figure 6: Example of GBM with $S_0 = 1.1$, $\mu = 2$ and $t = 1$
Source: Created by Ivana Stankic, 2024.

3.1.2 Scenario 2 (Avoiding pension reductions)

Now, let us consider the expected buyer's wealth in case of Scenario 2 from Section 2.2. As we could see in Section 2.2, in the case of Scenario 2, the buyer's wealth would only change in the event of good fund performance, as they would receive a share of the gains. In the other two cases, they will receive their pension as agreed upon at the beginning. Now, let us examine the buyer's wealth in this case:

$$\mathbb{E}[w_b] = \mathbb{E}[w_b \cdot \mathbb{I}_{[0<C_1<1]} + w_b \cdot \mathbb{I}_{[C_1>1.25]} + w_b \cdot \mathbb{I}_{[C_1 \in [1,1.25]]}]$$

$$= \mathbb{E}\bigg[\bigg(S_1 - \frac{kPe^{-\delta}}{\delta + \lambda} \bigg) \cdot \theta \cdot \mathbb{I}_{[C_1>1.25]} \bigg]$$

$$= -\frac{kPe^{-\delta}}{\delta + \lambda} \cdot \theta \cdot \mathbb{P}[C_1 > 1.25] + \theta \cdot \mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}]$$

In this scenario, the seller's wealth could also be affected by fund movements, as explained in Section 2.2. We can see that the seller's wealth would be impacted in the case of $0 < C_1 < 1$ or $C_1 > 1.25$. Let us now calculate the expected value of the seller's wealth:

$$\mathbb{E}[w_s] = \mathbb{E}[w_s \cdot \mathbb{I}_{[0<C_1<1]} + w_s \cdot \mathbb{I}_{[C_1>1.25]} + w_s \cdot \mathbb{I}_{[C_1\in[1,1.25]]}]$$

$$= \mathbb{E}\bigg[-\bigg(\frac{kPe^{-\delta}}{\delta+\lambda} - S_1\bigg) \cdot \mathbb{I}_{[0<C_1<1]} + \bigg(S_1 - \frac{kPe^{-\delta}}{\delta+\lambda}\bigg) \cdot (1-\theta) \cdot \mathbb{I}_{[C_1>1.25]}\bigg]$$

$$= -\frac{kPe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[0 < C_1 < 1] + \mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}] + (1-\theta) \cdot \mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}]$$

$$-(1-\theta) \cdot \frac{kPe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[C_1 > 1.25]$$

Our goal now is to find the optimal values of k^* and θ^* such that both the buyer and the seller are satisfied

$$\max_{k^*\in[1,1.25]} \mathbb{E}[w_b]$$

$$\max_{\theta^*\in[0,1]} \mathbb{E}[w_s].$$

Let us now calculate the first derivative of the buyer's function given above with respect to k.

$$\frac{\partial \mathbb{E}[w_b]}{\partial k} = -\frac{Pe^{-\delta}}{\delta+\lambda} \cdot \theta \cdot \mathbb{P}[C_1 > 1.25].$$

Since the derivative is negative, we can conclude that the function increases as k decreases. This means the function reaches its maximum when k is at its lowest possible value, which is $k = 1$. Let us now calculate the first derivative of the seller's function with respect to θ:

$$\frac{\partial \mathbb{E}[w_s]}{\partial \theta} = -\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] + \frac{kPe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[C_1 > 1.25].$$

The seller's function increases as θ decreases, meaning that the function reaches its maximum when θ is at its lowest possible value, which is $\theta = 0$. However, both of these solutions, $\theta = 0$ and $k = 1$, are trivial, and we would like to avoid them. If the seller chooses $\theta = 0$, the buyer would not participate in the gains at all, making this option meaningless. Similarly, choosing $k = 1$ would be very risky and unattractive for the seller. As we saw in Scenario 1, there is a higher probability of poor fund performance in the next period, if $k = 1$, and since the seller is the only one incurring losses in the case of poor fund performance, they would be exposed to significant risk. To avoid this trivial solution, we can argue that choosing $k = 1$ would imply the seller selecting $\theta = 0$, which would not benefit anyone. Choosing $k = 1.25$ or $\theta = 1$ is also trivial, as in the case where $k = 1.25$ and the probability of good fund performance is high, there will always be a gain. Additionally, if $\theta = 1$, the seller would not participate in the gain at all. We will revisit this later when we attempt to determine θ. Now, in order to calculate the parameters, we know that the seller would only prefer this option if their expected gain is greater than their expected loss, so let us compare those two:

$$\mathbb{E}[w_s \cdot \mathbb{I}_{[C_1>1.25]}] \overset{!}{>} \mathbb{E}[w_s \cdot \mathbb{I}_{[0<C_1<1]}]$$

$$\mathbb{E}\Bigg[\left(S_1 - \frac{kPe^{-\delta}}{\delta+\lambda}\right) \cdot (1-\theta) \cdot \mathbb{I}_{[C_1>1.25]}\Bigg] > \mathbb{E}\Bigg[\left(\frac{kPe^{-\delta}}{\delta+\lambda} - S_1\right) \cdot \mathbb{I}_{[0<C_1<1]}\Bigg]$$

. This implies:

$$(1-\theta)\mathbb{E}\Bigg[\left(S_1 - \frac{kPe^{-\delta}}{\delta+\lambda}\right) \cdot \mathbb{I}_{[C_1>1.25]}\Bigg] > \mathbb{E}\Bigg[\left(\frac{kPe^{-\delta}}{\delta+\lambda} - S_1\right) \cdot \mathbb{I}_{[0<C_1<1]}\Bigg] \quad (16)$$

$$\implies \theta^*(k) < 1 - \frac{\frac{kPe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[0 < C_1 < 1] - \mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}]}{\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] - \frac{kPe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[C_1 > 1.25]}. \quad (17)$$

Since $0 \leq \theta \leq 1$ term 17 has to be greater than zero. This means that it has to hold:

$$\frac{\frac{kPe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[0 < C_1 < 1] - \mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}]}{\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] - \frac{kPe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[C_1 > 1.25]} < 1. \quad (18)$$

For some arbitrarily small $\epsilon = 10^{-10}$, we can write:

$$\theta^*(k) = max\left(0, 1 - \frac{\frac{k^*Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[0 < C_1 < 1] - \mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}]}{\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] - \frac{k^*Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[C_1 > 1.25]} - \epsilon\right). \quad (19)$$

In the numerator, we can see that we have the expected loss in the case where $C_1 < 1$, and in the denominator, the expected total gain, which has not yet been shared between the buyer and the seller. This means that, in order for this option to be profitable for the seller, the expected gain should be greater than the expected loss. If it doesn't hold, the seller would incur a higher loss than gain, making it unreasonable for them to accept this option at all. Inequality 17 ensures that the θ is so chosen, depending on k, that the seller's expected gain is higher than their expected loss. However, it increases as k decreases, which means it does not account for what might happen in the next period if the probabilities of poor fund performance are very high and k is very low. The inequality 17 suggests that if the buyer chooses a lower k, the seller would choose a higher θ. From the perspective of gains, this makes sense: the lower the k, the greater the gain in the case of good fund performance. Conversely, if a loss occurs, the seller would pay less if k is lower. However, if k is low and the probability of poor fund performance is high, there is a significant chance that the fund will continue to perform poorly in the next period, resulting in greater losses for the seller. To make it simpler with the notations, let us denote: $\mathbb{P}[0 < C_1 < 1] = p_1$ and $\mathbb{P}[C_1 > 1.25] = p_2$. If $p_2 > p_1$ then we could adapt 20 so that $\theta^*(k)$ depends on how much greater p_2 is than p_1. If they are not very far apart, $\theta^*(k)$ would be smaller. If p_2 is much larger than p_1, $\theta^*(k)$ would not be significantly affected by the difference. If the probabilities are similar, this means that the fund could just as easily perform poorly, implying that a loss could occur. Additionally, it is possible that the loss might continue into the next period. So, we can write:

$$\theta_1^*(k) = \theta^*(k) \cdot (p_2 - p_1)\,,\ if\ p_2 > p_1. \tag{20}$$

In case where $p_2 \leq p_1$ we still face the problem where $\theta^*(k)$ increases as k^* decreases. p_1 is greater in this case, and as it increases, k^* should also increase to avoid being too low in the next period. To address this issue, we can write $\theta_2^*(k)$ as follows:

$$\theta_2^*(k) = \theta^*(k)_1 \cdot (k^* - 1),\ if\ p_2 \leq p_1. \tag{21}$$

This implies:

$$\theta^*(k) = \begin{cases} \theta_1^*(k), & p_2 > p_1 \\ \\ \theta_2^*(k), & p_2 \leq p_1. \end{cases} \tag{22}$$

This way, the seller ensures that their expected gain covers their expected loss. Otherwise, this option wouldn't be affordable for them, as they would have to inject additional funds into the pension fund in the case of poor fund performance to bring the value back to k. It also ensures that, in the case of a high probability of poor fund performance, the buyer would be motivated to choose a higher k if they wish to participate in the gain. Since $\theta^*(k)$ represents the portion of the gain that the buyer will receive in the case where $C_1 > 1.25$, their goal is to maximize $\theta^*(k)$. The optimal k^* for the buyer can be defined as:

$$k^* = \arg\max_{1 \leq k \leq 1.25} \theta^*(k).$$

The buyer will choose k^* depending on the probabilities and expected fund movements. Specifically, if the probability of good fund performance is higher than that of bad fund performance, the optimal k^* would be the minimal k^*, which is 1. The function $\theta^*(k)$ ensures that if k^* is 1, $\theta^*(k)$ would not equal 1 but would be less than 1. This allows the seller to also participate in the gains. Of course, as the probability of bad fund performance increases, $\theta^*(k)$ decreases so that the buyer's expected loss does not exceed their expected gain in any possible case. If the probability of bad fund performance is higher than that of good fund performance, the buyer's optimal k^* would be the maximal k^*, which is 1.25. In this scenario, $\theta^*(k)$ still would not equal 1 but would be less than 1, depending on the probabilities. However, if the buyer chooses k^* to be 1 in this case, $\theta^*(k)$ would become zero, as this represents the worst-case scenario. In this situation, where the probability of bad fund performance exceeds that of good fund performance and k^* is 1, there is a very high chance of C_1 falling below 1 again. Now, let us take a look at an example.

Example 3.2
Let us assume that the parameters are set as follows:

- $P = 0.05$,
- $\delta = 0.03$,
- $\lambda = 0.02$,
- $\mu = 0.7$,
- $\sigma = 0.2$,
- $S_0 = 1.1$

First, we calculate the expected pension value:

$$\frac{Pe^{-\delta}}{\delta + \lambda} = 0.9704455.$$

The following is calculated using the Python code from section 5.2:
Probabilities

$$\mathbb{P}[0 < C_1 < 1] = 0.0000283$$

$$\mathbb{P}[C_1 > 1.25] = 0.998197668.$$

As already mentioned, since μ is very high, the probability of good fund performance is also very high, approaching 1, while the probability of bad fund performance is very low, approaching zero. Let us now take a look at the expected fund values.

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}] = 0.00002628256$$

$$\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] = 2.21306170.$$

Let us now assume that the buyer chooses the following k at time 1:

- $k = 1$. We calculate θ using 22 as follows:
  $\theta^*(k) = max\left(0, 1 - \frac{\frac{k^* Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[0<C_1<1] - \mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}]}{\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] - \frac{k^* Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[C_1>1.25]} - \epsilon\right) \cdot$
  $(\mathbb{P}[C_1 > 1.25] - \mathbb{P}[0 < C_1 < 1])$
  $\implies \theta = 0.9981684195351516$
  $\mathbb{E}[w_b] = 1.242086079$
  $\mathbb{E}[w_s] = 0.00227797$

- $k = 1.125 \implies \theta = 0.998165266719814$
  $\mathbb{E}[w_b] = 1.12121726$
  $\mathbb{E}[w_s] = 0.00205630$

- $k = 1.25 \implies \theta = 0.9981613520435332$
  $\mathbb{E}[w_b] = 1.00034844$
  $\mathbb{E}[w_s] = 0.00183463.$

Since the probability of good fund performance is so high, the seller would choose a higher θ when k is lower. However, it wouldn't make much difference which k the buyer chooses because theta would always be high, given that the difference between the probabilities is strongly in favor of the good fund performance probability. However, it would never be 1. Even if k is chosen to be 1, they would still gain a part of the profit. The same holds for $k = 1.25$. If θ were 1, then the buyer would most likely always gain, and the seller would have nothing to benefit from. This way, they also participate in the gain.
Now, we would like to check if condition 16 holds for, for example, $k = 1$.
Left hand side:

$$(1-\theta)\mathbb{E}\Bigg[\left(S_1 - \frac{kPe^{-\delta}}{\delta+\lambda}\right)\cdot \mathbb{I}_{[C_1>1.25]}\Bigg] = 0.0018346291.$$

Right hand side:

$$\mathbb{E}\Bigg[\left(\frac{kPe^{-\delta}}{\delta+\lambda} - S_1\right)\cdot \mathbb{I}_{[0<C_1<1]}\Bigg] = 0.00000804.$$

Condition:

$$0.0018346291 > 0.00000804$$

holds. This means that this option would be acceptable to the seller for $k = 1$ and they would choose $\theta = 0.9981684195351516$ in this case. With the Python code 5.2, we can verify that the condition holds for all $k \in [1, 1.25]$ in this case. This is because μ is very high and σ is low, so the probability of a good fund movement is very high, while the probability of a bad fund movement approaches zero. Therefore, both the seller and the buyer would incur a gain in this scenario. Let us now consider an example where μ is not as high and σ is larger, making the fund movement much more uncertain and unpredictable. Let μ and σ be as follows:

- $\mu = 0.2$
- $\sigma = 0.6$.

The probabilities will now be

$$\mathbb{P}[0 < C_1 < 1] = 0.4043189382914175$$

$$\mathbb{P}[C_1 > 1.25] = 0.4483930749961499.$$

We can now see that the probabilities are similar to each other. This is because σ is high, leading to greater uncertainty in the fund value, which results in more extreme movements both upwards and downwards. This means that the fund's movement is not so easily predictable, which is why, in this case, it is more risky. Let us now assume that the buyer chooses the following k at time 1:

- $k = 1 \implies \theta = 0.032693077$
  $\mathbb{E}[w_b] = 0.015683082$
  $\mathbb{E}[w_s] = 0.340151063$

- $k = 1.125 \implies \theta = 0.026155049$
  $\mathbb{E}[w_b] = 0.011124106$
  $\mathbb{E}[w_s] = 0.241271219$

- $k = 1.25 \implies \theta = 0.017699522$
  $\mathbb{E}[w_b] = 0.0065651$
  $\mathbb{E}[w_s] = 0.1423914.$

We check again if condition 16 holds for, for example, $k = 1.25$.

$$(1-\theta)\mathbb{E}\left[\left(S_1 - \frac{kPe^{-\delta}}{\delta+\lambda}\right)\cdot \mathbb{I}_{[C_1>1.25]}\right] = 0.3643561445.$$

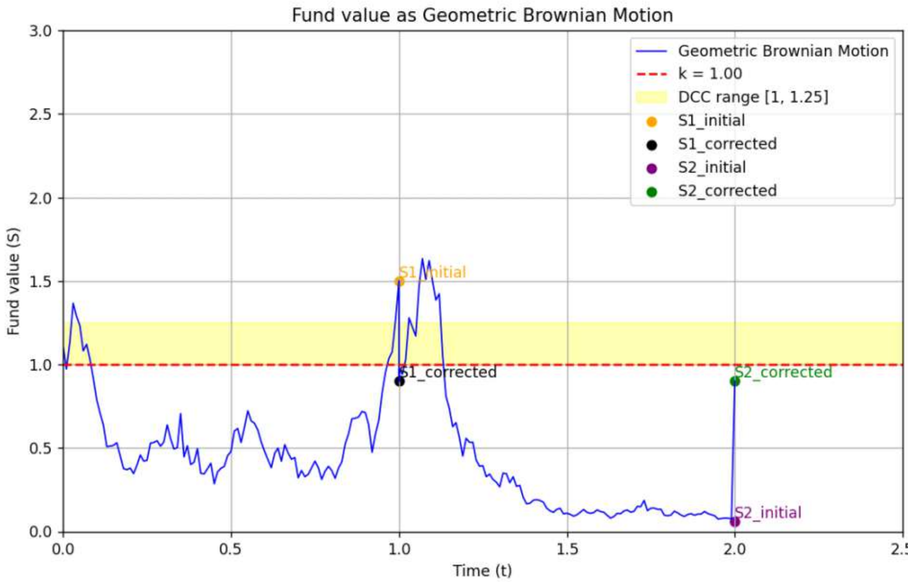
Right hand side:

$$\mathbb{E}\left[\left(\frac{kPe^{-\delta}}{\delta+\lambda} - S_1\right)\cdot \mathbb{I}_{[0<C_1<1]}\right] = 0.221964769.$$

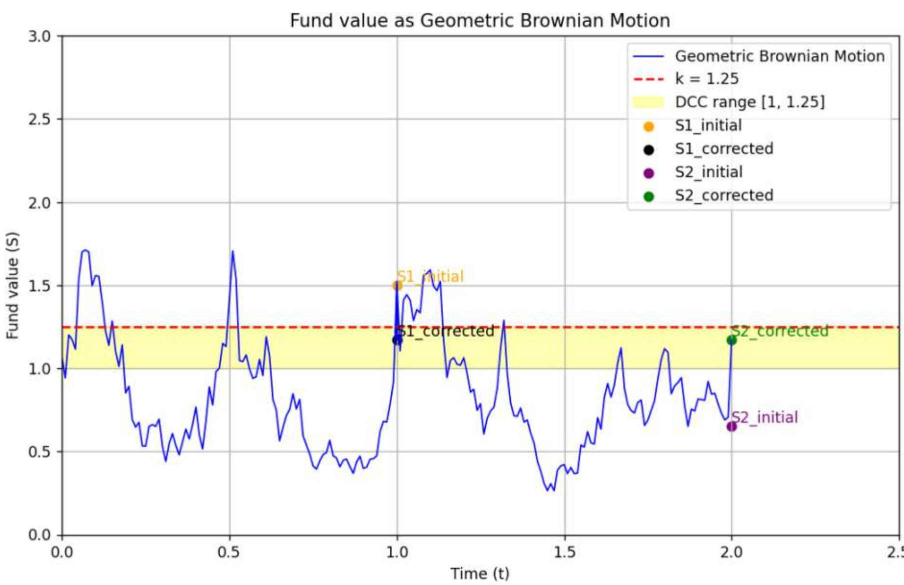
Condition:

$$0.3643561445 > 0.221964769.$$

holds. We can also check using Phyton code 5.1 for all $k \in [1, 1.25]$ here and see that the condition holds in this case as well, but only, of course, with theta calculated as described in 22. In this case, where $\mathbb{P}[0 < C_1 < 1]$ increases towards $\mathbb{P}[C_1 > 1.25]$, we can see that for both the buyer and the seller, the highest expected gain would occur if k is chosen to be 1.25. We can also see that in that case, θ is relatively low, meaning that the buyer's gain wouldn't be that high. This is because, in the case of poor fund performance, the seller would have to inject a very high amount into the pension fund in order to come back to $k = 1.25$, and in the case of good fund performance, they would need a large portion of that gain to cover their potential losses in the event of poor fund performance. Also, if $k = 1$, the seller wouldn't have to inject such a high amount into the pension fund. However, since S_1 would be at its lowest possible point, there is a high probability of the fund performing poorly again, which is why they would need to secure against that possibility as well. As already mentioned in scenario 1, this case could be very risky due to the higher σ. In the following graph, we can see the possible outcomes in this case when k is set to 1 or 1.25, for example.



Source: Created by Ivana Stankic, 2024.



Source: Created by Ivana Stankic, 2024.

We now observe another set of parameters:

- $\mu = 0.2$
- $\sigma = 0.7$.

The probabilities will now be

$$\mathbb{P}[0 < C_1 < 1] = 0.4543300316$$

$$\mathbb{P}[C_1 > 1.25] = 0.41915812219.$$

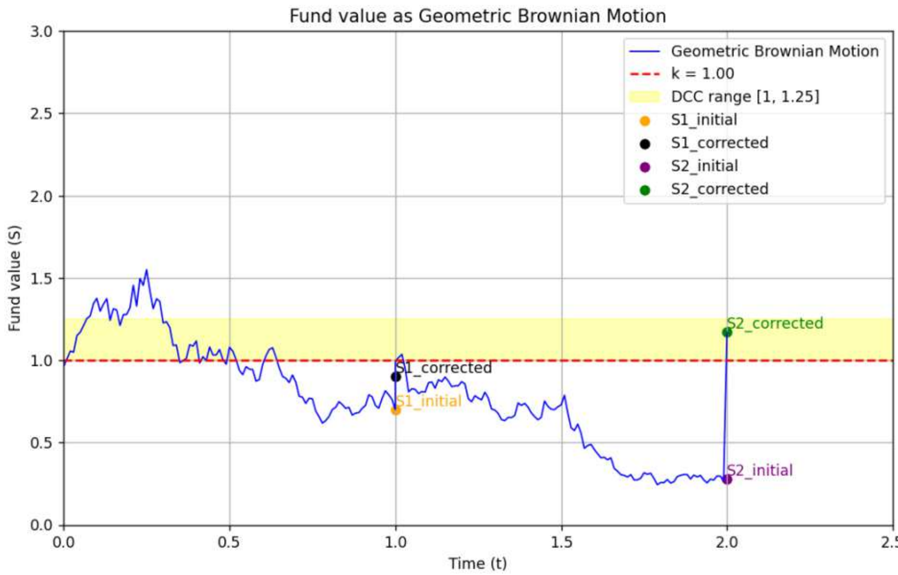
Due to the lower μ and higherσ, the probability of poor fund performance is now higher. Let us take a look at the different choices of k at time 1 again.

- $k = 1$. We calculate θ using 22 as follows:
$\theta^*(k) = max\left(0, 1 - \frac{\frac{k^* Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[0<C_1<1] - \mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}]}{\mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] - \frac{k^* Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[C_1>1.25]} - \epsilon\right)$
$\cdot (\mathbb{P}[C_1 > 1.25] - \mathbb{P}[0 < C_1 < 1]) \cdot (k^* - 1)$
$\implies \theta = 0$

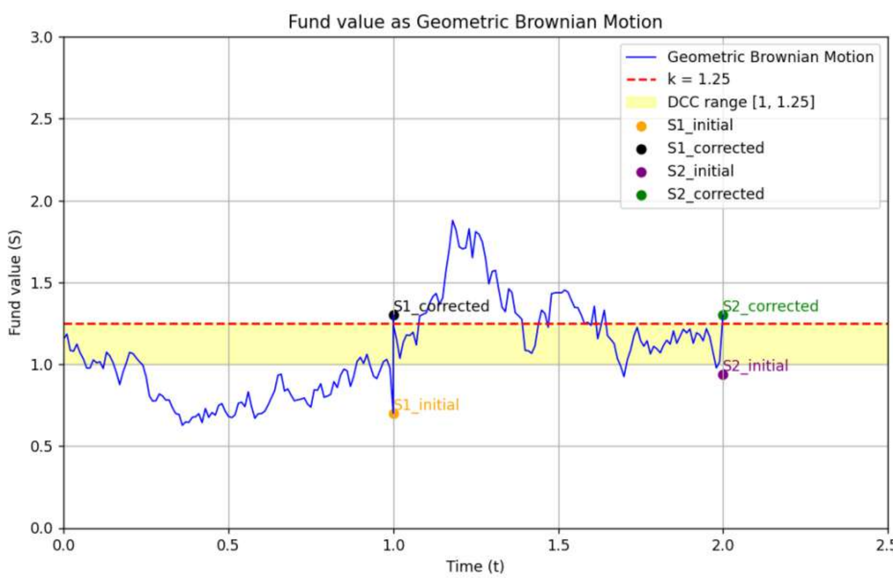
- $k = 1.125 \implies \theta = 0.00236363348$
$\mathbb{E}[w_b] = 0.001109670$
$\mathbb{E}[w_s] = 0.251289636$

- $k = 1.25 \implies \theta = 0.003075853$
$\mathbb{E}[w_b] = 0.00128764556$
$\mathbb{E}[w_s] = 0.14515257709$.

In cases where $\mathbb{P}[0 < C_1 < 1]$ is higher than $\mathbb{P}[C_1 > 1.25]$, choosing $k = 1$ is not acceptable for the seller due to the already mentioned high probability of a further decrease in the next period and in that case θ would be 0. However, the seller could still accept this option in such a case for other $k \in (1, 1.25]$, but θ would have to be very low, as we can see. It would increase as k increases, but always in such a way that the seller's expected gain is never lower than their expected loss. Also in this case, the probability of poor fund performance over time would grow, as mentioned before, due to the high volatility. That is why it is important that θ is very low and that, in case of a gain, the buyer receives a higher share of the gain to be able to cover all potential losses. In the following graphs, we can also see some potential situations where σ is so high that the probability of poor fund performance is higher than that of good fund performance.



Source: Created by Ivana Stankic, 2024.



Source: Created by Ivana Stankic, 2024.

3.2 Comparison between Scenario 1 and 2

In this section, we would like to compare Scenario 1 and Scenario 2 from the buyer's perspective, since the seller only participates in Scenario 2, and we have already seen under which conditions Scenario 2 would be possible from the seller's side. Let us take another look at the buyer's wealth in both scenarios. We will denote expected buyer´s wealth from Scenario 1 as $\mathbb{E}[w_{b_1}]$ and from Scenario 2 as $\mathbb{E}[w_{b_2}]$.

$$\mathbb{E}[w_{b_1}] = \frac{1}{s} \cdot \mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}] - \frac{Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[C_1 > 1.25]$$

$$- \frac{Pe^{-\delta}}{\delta+\lambda} \cdot \mathbb{P}[0 < C_1 < 1] + \frac{1}{s} \cdot \mathbb{E}[S_1 \cdot \mathbb{I}_{[0<C_1<1]}].$$

$$\mathbb{E}[w_{b_2}] = \mathbb{E}\left[\left(S_1 - \frac{kPe^{-\delta}}{\delta+\lambda}\right) \cdot \theta \cdot \mathbb{I}_{[C_1>1.25]}\right]$$

$$= -\frac{kPe^{-\delta}}{\delta+\lambda} \cdot \theta \cdot \mathbb{P}[C_1 > 1.25] + \theta \cdot \mathbb{E}[S_1 \cdot \mathbb{I}_{[C_1>1.25]}].$$

As previously discussed, the buyer's wealth in the first scenario consists of the expected increased pension minus the expected decreased pension. We have already determined that the buyer would only choose this option if their expected gain is higher than their expected loss. On the other hand, by choosing Scenario 2, the buyer would not have to worry about the loss because the seller would cover it. However, the buyer would need to share their gain with the seller, depending on the chosen k and the fund movements. As we can see, in cases where $\mathbb{P}[0 < C_1 < 1]$ is much higher than $\mathbb{P}[C_1 > 1.25]$, or when μ is very high and σ is low, in the first scenario, the optimal s would be 1, and the buyer would receive the entire gain. In the same case, but in the second scenario, the optimal k would also be the lowest; however, θ, which splits the gain between the buyer and the seller, would never be 1. This means the buyer would never receive the entire gain. Thus, in this case, Scenario 1 would provide a higher expected gain, as the buyer would receive the full gain. However, in the event of poor fund performance, their pension would decrease, whereas in Scenario 2, their pension would remain the same. As $\mathbb{P}[0 < C_1 < 1]$ gets higher and approaches $\mathbb{P}[C_1 > 1.25]$, both scenarios become riskier, and the optimal values change. The optimal value for the first scenario will always be $s = 1$, but for higher σ, it could be very dangerous to choose this scenario, because over time, the probability of poor fund performance will increase, and the buyer could experience significant pension decreases. Also, since the fund value would not be adjusted to a certain level, it would remain the same. If the fund performed poorly, it could perform even worse in the next period, as we have already seen above. On the other hand, choosing Scenario 2 could be less risky, as the buyer would not incur a loss in the event of poor fund performance. However, if $\mathbb{P}[0 < C_1 < 1]$ is high, the seller could choose a lower θ, and the buyer's gain would then be higher by choosing Scenario 1. So, when choosing which scenario would be best for them, the buyer has to take into account all the parameters, probabilities, and their willingness to take risks. Some buyers might be more prepared to take on higher risks and could choose Scenario 1, where they might achieve higher gains but also incur higher losses. On the other hand, other buyers might be less willing to take risks and prefer the safer Scenario 2, where they would always obtain some gain, although maybe not much, but at least they would be safe from significant losses. Let us take a look at an example where we compare both cases.

Example 3.3
Let us assume that the parameters are set as follows:

- $P = 0.05$,
- $\delta = 0.03$,
- $\lambda = 0.02$,
- $\mu = 0.7$,
- $\sigma = 0.2$,
- $S_0 = 1.1$.

We have already calculated such cases in previous examples, but we would now like to compare the expected buyer's wealth for both scenarios. To do so, we will examine a few parameter sets. The first set is one where μ is high and σ is low, meaning the gain would almost certainly occur. Let us now take a look at the expected buyer's wealth calculated using Phyton code 5.2 and 5.1 for the following parameters:

- $s = 1$, $\mathbb{E}[w_{b_1}] = 1.24436405$
- $s = 1.25$, $\mathbb{E}[w_{b_1}] = 0.80174645$
- $k = 1$, $\theta = 0.9981684$, $\mathbb{E}[w_{b_2}] = 1.24208608$
- $k = 1.25$, $\theta = 0.9981613$, $\mathbb{E}[w_{b_2}] = 1.0003484$.

As already mentioned, we can see that in this particular case, the expected buyer's wealth in Scenario 1, when $s = 1$ and $k = 1$, is higher than the expected buyer's wealth in Scenario 2. This is due to θ, which gives the seller a portion of the gain in Scenario 2. In the case where s and k are 1.25, we can see that the expected wealth in Scenario 2 is higher. This was expected because, in Scenario 1, in the case of losses, the buyer's pension would need to be increased to return to 1.25, whereas in Scenario 2, the seller would inject some amount into the pension fund to bring it back to 1.25. Let us now take a look at a different set of μ and σ parameters:

- $\mu = 0.2$,
- $\sigma = 0.6$.

We calculate:

- $s = 1$, $\mathbb{E}[w_{b_1}] = 0.355834146$
- $s = 1.25$, $\mathbb{E}[w_{b_1}] = 0.119165204$
- $k = 1$, $\theta = 0.03269308$, $\mathbb{E}[w_{b_2}] = 0.015683082$
- $k = 1.25$, $\theta = 0.0176995$, $\mathbb{E}[w_{b_2}] = 0.006565129$.

Obviously, in this case where σ is high and the fund performance is difficult to predict, the expected buyer's wealth in Scenario 1 is higher. However, because this option is very risky and there is always a higher probability over time of the fund performing poorly in the next period, the seller would choose a very low θ as a safeguard in case the fund performs poorly again. This is why the buyer's gain in Scenario 1 would be much lower. We take a look at one more set of parameters:

- $\mu = 0.2$,
- $\sigma = 0.8$.

We calculate:

- $s = 1$, $\mathbb{E}[w_{b_1}] = 0.360338359$
- $s = 1.25$, $\mathbb{E}[w_{b_1}] = 0.11552152$
- $k = 1$, $\theta = 0$, $\mathbb{E}[w_{b_2}] = 0$
- $k = 1.25$, $\theta = 0.0081133$, $\mathbb{E}[w_{b_2}] = 0.003778026$.

Now, σ is even higher, and we can see that in Scenario 2, where $k = 1$, θ is zero. This means that the probability of poor fund performance is higher than that of good fund performance, and it would not be acceptable for the seller to set $k = 1$ in this case, precisely because of the high risk of poor fund performance and further losses in the next period. We can also see that in the case where $k = 1.25$, θ is even smaller, and the buyer's expected gain is also reduced. In cases like this, and in ones where σ is even higher, or when μ is very low such that $\mathbb{P}[0 < C_1 < 1] > \mathbb{P}[C_1 > 1.25]$, the first scenario would bring a higher gain but would also be too risky because there is no protection against further losses.

4 Conclusion

In this thesis, we analyzed a one-time-period game between the buyer and the seller of pension fund insurance, aiming to determine an optimal solution for both parties based on the movements of the fund. The main objective was to balance the interests of the buyer and the seller while considering the risks and opportunities associated with the fund's performance. We modeled the value of the fund using a geometric Brownian motion, which allowed us to represent the stochastic behavior of the fund over time. Through this model, we closely examined the properties and probabilities of the fund's performance during the observed time period. This analysis provided valuable insights into how different factors influence the outcomes for both parties. Our findings show that, in order to achieve an optimal solution, it is not sufficient to focus solely on satisfying the immediate preferences of the buyer and the seller. Instead, it is essential to take into account the full range of probabilities associated with the fund's behavior. This includes not only its performance during the observed time period but also potential future scenarios. By doing so, we were able to identify conditions under which the seller might be willing to cooperate with the buyer and circumstances where the seller would reject the proposed options due to excessive risk. To support our analysis, we developed a Python code that enabled us to perform numerical calculations and run simulations on various examples. This tool proved essential in illustrating the practical applications of our theoretical findings, as it allowed us to test different scenarios and validate the proposed solutions. In conclusion, this research highlights the importance of considering both the stochastic nature of the fund's performance and the preferences of all involved parties. It provides a framework for understanding how buyers and sellers of pension fund insurance can reach mutually beneficial agreements while managing the inherent risks associated with financial markets.

5 Phyton Code

5.1 Scenario 1

```
import math
from scipy.stats import norm

def calculate_values(mu, sigma, S0, P, delta, lambd):
    # Calculate pension
    P_e_minus_delta_div_delta_plus_lambda = P * math.exp(-delta) /
        (delta + lambd)

    # Precompute constants
    ln_S0 = math.log(S0)
    sigma_sq = sigma**2
    mu_minus_sigma_sq_div_2 = mu - sigma_sq / 2

    ln_P_div_delta_plus_lambda = math.log(
        P_e_minus_delta_div_delta_plus_lambda)
    ln_1_25_P_div_delta_plus_lambda = math.log(1.25 *
        P_e_minus_delta_div_delta_plus_lambda)

    # Probabilities
    P_C1_less_1 = norm.cdf((ln_P_div_delta_plus_lambda - ln_S0 -
        mu_minus_sigma_sq_div_2) / sigma)
    P_C1_greater_1_25 = 1 - norm.cdf((
        ln_1_25_P_div_delta_plus_lambda - ln_S0 -
        mu_minus_sigma_sq_div_2) / sigma)

    # Expectations
    E_S1_C1_less_1 = math.exp(ln_S0 + mu_minus_sigma_sq_div_2 +
        sigma_sq / 2) * \
                    norm.cdf((ln_P_div_delta_plus_lambda - (ln_S0
                        + mu_minus_sigma_sq_div_2 + sigma_sq)) /
                        sigma)

    E_S1_C1_greater_1_25 = math.exp(ln_S0 + mu_minus_sigma_sq_div_2
         + sigma_sq / 2) * \
                          (1 - norm.cdf((
                              ln_1_25_P_div_delta_plus_lambda - (
                              ln_S0 + mu_minus_sigma_sq_div_2 +
                              sigma_sq)) / sigma))

    # Expected w_b
    if (1/s_star) * E_S1_C1_greater_1_25 -
        P_e_minus_delta_div_delta_plus_lambda*P_C1_greater_1_25 >
        P_e_minus_delta_div_delta_plus_lambda*P_C1_less_1 - (1/
        s_star) *  E_S1_C1_less_1:
        expected_wb = (1/s_star) * E_S1_C1_greater_1_25 -
            P_e_minus_delta_div_delta_plus_lambda*P_C1_greater_1_25
             - P_e_minus_delta_div_delta_plus_lambda*P_C1_less_1 +
            (1/s_star) * E_S1_C1_less_1
    else:
        expected_wb = 0

    # Print results
```

```
    print("\nProbabilities:")
    print("P(0 < C1 < 1):", P_C1_less_1)
    print("P(C1 > 1.25):", P_C1_greater_1_25)

    print("\nExpected values:")
    print("E[S1 * I_{0 < C1 < 1}]:", E_S1_C1_less_1)
    print("E[S1 * I_{C1 > 1.25}]:", E_S1_C1_greater_1_25)
    print("lhs:", lhs)
    print("rhs:", rhs)

    print("s_star:", s_star)

    print("\nExp. present Pension value:",
        P_e_minus_delta_div_delta_plus_lambda)
    print("\nExpected w_b:", expected_wb)
    print("P_new_C1_less_1:", P_new_C1_less_1)
    print("P_new_C1_greater_1_25:", P_new_C1_greater_1_25)

# Parameter input
mu = 0.2
sigma = 0.6
S0 = 1.1
P = 0.05
delta = 0.03
lambd = 0.02
s_star=1

calculate_values(mu, sigma, S0, P, delta, lambd)
```

Listing 1: Scenario 1

Source: Created by Ivana Stankic, 2024.

5.2 Scenario 2

```
import math
from scipy.stats import norm

def calculate_wealth_and_parameters(mu, sigma, S0, P, delta,
    lambda_, k_star):
    # Calculate pension
    P_e_minus_delta_div_delta_plus_lambda = P * math.exp(-delta) /
        (delta + lambda_)

    # Precompute constants
    ln_S0 = math.log(S0)
    sigma_sq = sigma**2
    mu_minus_sigma_sq_div_2 = mu - sigma_sq / 2

    ln_P_div_delta_plus_lambda = math.log(
        P_e_minus_delta_div_delta_plus_lambda)
    ln_1_25_P_div_delta_plus_lambda = math.log(1.25 *
        P_e_minus_delta_div_delta_plus_lambda)

    # Probabilities
    P_C1_less_1 = norm.cdf((ln_P_div_delta_plus_lambda - ln_S0 -
        mu_minus_sigma_sq_div_2) / sigma)
    P_C1_greater_1_25 = 1 - norm.cdf((
        ln_1_25_P_div_delta_plus_lambda - ln_S0 -
        mu_minus_sigma_sq_div_2) / sigma)

    # Expectations
    E_S1_C1_less_1 = math.exp(ln_S0 + mu_minus_sigma_sq_div_2 +
        sigma_sq / 2) * \
                     norm.cdf((ln_P_div_delta_plus_lambda - (ln_S0
                         + mu_minus_sigma_sq_div_2 + sigma_sq)) /
                         sigma)

    E_S1_C1_greater_1_25 = math.exp(ln_S0 + mu_minus_sigma_sq_div_2
         + sigma_sq / 2) * \
                           (1 - norm.cdf((
                               ln_1_25_P_div_delta_plus_lambda - (
                               ln_S0 + mu_minus_sigma_sq_div_2 +
                               sigma_sq)) / sigma))

    if P_C1_greater_1_25 > P_C1_less_1:
        theta_star = max(0, (1 - (k_star *
            P_e_minus_delta_div_delta_plus_lambda * P_C1_less_1 -
            E_S1_C1_less_1)/(E_S1_C1_greater_1_25 - k_star *
            P_e_minus_delta_div_delta_plus_lambda*P_C1_greater_1_25
            ) - 10**-10)*(P_C1_greater_1_25-P_C1_less_1))
    else:
        theta_star = max(0, (k_star - 1) * (1 - (k_star *
            P_e_minus_delta_div_delta_plus_lambda * P_C1_less_1 -
            E_S1_C1_less_1)/(E_S1_C1_greater_1_25 - k_star *
            P_e_minus_delta_div_delta_plus_lambda*P_C1_greater_1_25
            ) - 10**-10))
```

```
    # Check the sellers condition
    ls = (k_star * P_e_minus_delta_div_delta_plus_lambda *
        P_C1_less_1 - E_S1_C1_less_1)
    rs = (1 - theta_star) * (E_S1_C1_greater_1_25 - k_star *
        P_e_minus_delta_div_delta_plus_lambda*P_C1_greater_1_25)
    condition = ls < rs

# Calculate buyer's wealth (E[w_b])
    E_wb = - (k_star * P_e_minus_delta_div_delta_plus_lambda *
        theta_star * P_C1_greater_1_25) + \
           (theta_star * E_S1_C1_greater_1_25)

    # Calculate seller's wealth (E[w_s])
    E_ws = - (k_star * P_e_minus_delta_div_delta_plus_lambda *
        P_C1_less_1) + \
           E_S1_C1_less_1 + \
           ((1 - theta_star) * E_S1_C1_greater_1_25) - \
           ((1 - theta_star) * k_star *
               P_e_minus_delta_div_delta_plus_lambda *
               P_C1_greater_1_25)

    return {
        'P_C1_less_1': P_C1_less_1,
        'P_C1_greater_1_25': P_C1_greater_1_25,
        'E_S1_C1_less_1': E_S1_C1_less_1,
        'E_S1_C1_greater_1_25': E_S1_C1_greater_1_25,
        'theta_star': theta_star,
        'E_wb': E_wb,
        'E_ws': E_ws,
        'ls': ls,
        'rs': rs,
        'condition': condition,
        'P_e_minus_delta_div_delta_plus_lambda':
            P_e_minus_delta_div_delta_plus_lambda
    }

# Input parameters
mu = 0.2
sigma = 0.7
S0 = 1.1
P = 0.05
delta = 0.03
lambda_ = 0.02
k_star = 1.25


# Calculate wealth and parameters
output = calculate_wealth_and_parameters(mu, sigma, S0, P, delta,
    lambda_, k_star)

# Display results
print(f"P: {output['P_e_minus_delta_div_delta_plus_lambda']}")

print("Probabilities:")
print(f"P(C1 < 1): {output['P_C1_less_1']}")
print(f"P(C1 > 1.25): {output['P_C1_greater_1_25']}")
```

```
print("\nExpectations:")
print(f"E[S1 | 0 < C1 < 1]: {output['E_S1_C1_less_1']}")
print(f"E[S1 | C1 > 1.25]: {output['E_S1_C1_greater_1_25']}")

print("k: *", k_star)
print(f"Theta*: {output['theta_star']}")

print("\nWealth Calculations:")
print(f"E[w_b] (Buyer's Wealth): {output['E_wb']}")
print(f"E[w_s] (Seller's Wealth): {output['E_ws']}")

print("\nCheck if exp.sellers gain > exp.loss:")
print(f"Exp.Loss: {output['ls']}")
print(f"Exp.Gain: {output['rs']}")
print(f"Condition Holds: {output['condition']}")
```

Listing 2: Scenario 2

Source: Created by Ivana Stankic, 2024.

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