On Stewartson's collision problem on a sphere

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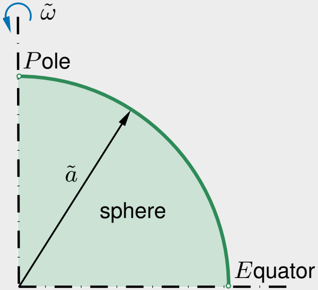
Overview

- Motivation & problem formulation
- Stationary equatorial collision: an old controversy
- Spin-up from rest
- Equatorial finite-time break-up of boundary layer
- Achievements & outlook

Rotation-induced flow around sphere: a long-standing challenge

Problem formulation

- rigid impervious sphere in unbounded quiescent Newtonian fluid with uniform properties
- angular speed: $0 \ldots \tilde{\omega} = const$
- $\nu = 1/Re = \tilde{\nu}/(\tilde{a}^2\tilde{\omega}) \to 0$
- canonical representation of oblate/prolate/asymmetric bodies of revolution, $O(1)$-ratios of principal curvatures



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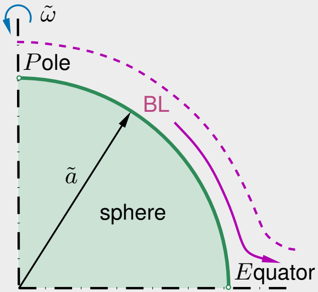
- canonical representation of oblate/prolate/asymmetric bodies of revolution, $O(1)$-ratios of principal curvatures
- pioneers in equatorial collision/BL breakaway

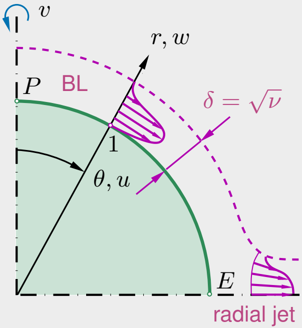
  Howarth (Phil. Mag. Ser., 1951)
  Stewartson (IUTAM Symp., 1958)
  Smith & Duck (QJMAM, 1977)
  } steady state

  Banks & Zaturska (JEM, 1979)
  Simpson & Stewartson (ZAMP, 1982)
  } impulsive spin-up

- theoretical progress slow & controversial!



Problem formulation



stationary reference frame

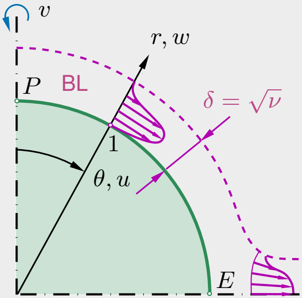
$\nu \to 0$: BL forms

$\theta \to 0$: von Kármán swirl flow

$\theta \to \frac{\pi}{2}$: collison $\to$ jet eruption

- $r = \tilde{r}/\tilde{a}, \quad t = \tilde{t}\,\tilde{\omega}, \quad [u, v, w] = [\tilde{u}, \tilde{v}, \tilde{w}]/(\tilde{a}\tilde{\omega}), \quad [\psi, u, v, w, p](\theta, r, t; \nu)$

Problem formulation



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- continuity, full NS eqs., BCs

$$u = \frac{\psi_r}{r \sin\theta}, \quad w = -\frac{\psi_\theta}{r^2 \sin\theta}, \quad \nabla^2(*) = \frac{[r^2(*)_r]_r}{r^2} + \frac{[(*)_\theta \sin\theta]_\theta}{r^2 \sin\theta}$$

$$u_t + \frac{u u_\theta}{r} + \frac{w(ru)_r}{r} - \frac{v^2 \cot\theta}{r} = -\frac{p_\theta}{r} + \nu\left[\nabla^2 u + \frac{2w_\theta}{r^2} - \frac{u}{(r\sin\theta)^2}\right]$$

$$v_t + \frac{u(v\sin\theta)_\theta}{r\sin\theta} + \frac{w(rv)_r}{r} = \nu\left[\nabla^2 v - \frac{v}{(r\sin\theta)^2}\right]$$

$$w_t + \frac{u w_\theta}{r} + w w_r - \frac{u^2+v^2}{r} = -p_r + \nu\left[\nabla^2 w - \frac{2w}{r^2} - \frac{2(u\sin\theta)_\theta}{r^2\sin\theta}\right]$$

$$r = 1: \quad \psi = \psi_r = v - \omega(t)\sin\theta = 0, \qquad r = \infty: \quad \psi = v = 0$$

Rotation-induced flow around sphere: a long-standing challenge

Central questions as $Re \to \infty$ tackled by numerical & rigorous perturbation analysis

- steady state? if existent:
  - structure of equatorial flow collision & jet formation: toroidal vortex?
  - approached by transient unsteadiness, by spin-up from rest?
- if not:
  - turning-point loss for $Re = Re_{\mathrm{crit}} \approx$?
  - manifest in equatorial finite-time break-up of BL
  - theoretical challenges: regularisation, radial jet

Axial & equatorial symmetries revalued

solution regular for $\nu > 0$ but NS eqs.

- singular at $\theta = 0 \quad \Rightarrow$

  $\partial_\theta^{2n}[u,v] = \partial_\theta^{2n+1}[w,p] = 0, \quad n = 0, 1, \dots$

  $[u,v,w,p](\theta,r,t) \equiv [-u,-v,w,p](-\theta,r,t)$

  natural BCs $\; u = v = 0, \; w_\theta = p_\theta = 0$

- regular at $\theta = \pi/2$

  essential BCs $\; u = 0, \; v_\theta = w_\theta = p_\theta = 0 \quad \Rightarrow$

  $\partial_\theta^{2n} u = \partial_\theta^{2n+1}[v,w,p] = 0$

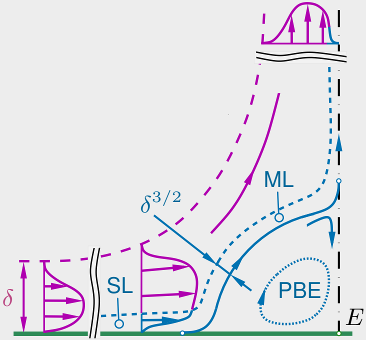
  $[u,v,w,p](\pi/2+\theta,r,t) \equiv [-u,v,w,p](\pi/2-\theta,r,t)$

Stationary collision & jet formation: completing proposed flow structures

Canonical: colliding symmetric wall jets

K. Stewartson (IUTAM Symp., 1958)

- Euler bending on BL scale δ
- slip layer (SL) undergoes separation
- triple-deck structure $\rightarrow$ mixing layer (ML)
- Prandtl–Batchelor eddy (PBE): $[u, w] = O(\delta)$



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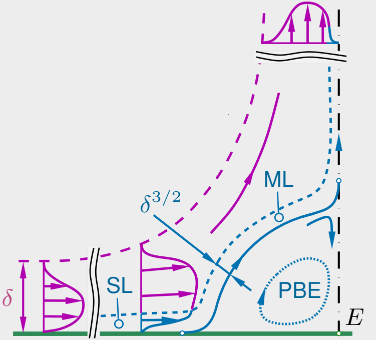
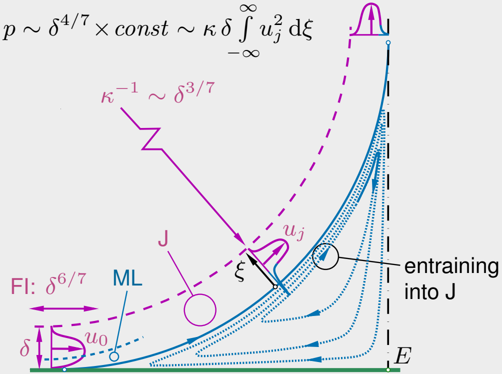


Smith & Duck (QJMAM **30**, 1977)

- compressive branching: free interaction (FI)
- predicts separation $\rightarrow$ ML + jet (J) merge
- $\sim$ inviscid larger eddy: $[u, w] = O(\delta^{6/7})$,



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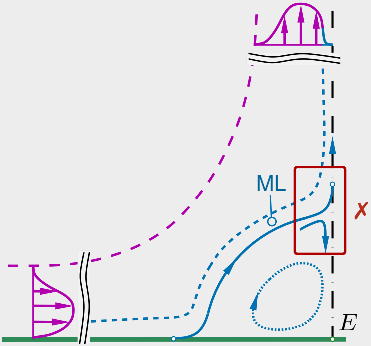

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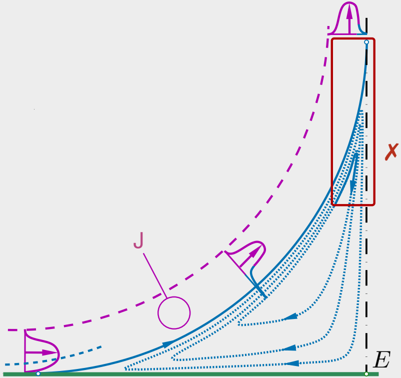


- predominantly inviscid ML merging
- ✗ inconsistent with weak backflow, reverse BL would separate

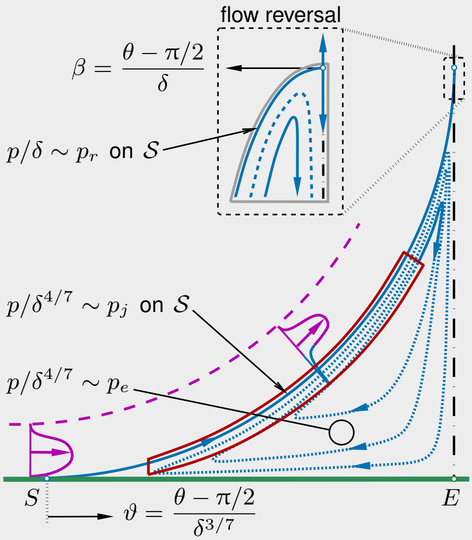
No steady state

Utilising global conservation of momentum as $\delta \to 0$

- ✗ A varying $p = O(\delta)$ would trigger backflow separation inside PBE.
- ✓ Smith & Duck (1977) recognised correctly $p/\delta^{4/7} \sim p_e = const$ inside eddy.
- ✗ However, inconsistent though . . .

No steady state

Utilising global conservation of momentum as $\delta \to 0$



dominant inviscid vertical balances

$$-p_e\, x_S - \int_{\mathcal{S}} p_j \,\mathrm{d}\vartheta = M_r = \int_{-\infty}^{0} u_j|_{\vartheta=0}\,\mathrm{d}\xi \qquad \text{(inner J)}$$

$$-p_e\, x_S - \int_{\mathcal{S}} p_j \,\mathrm{d}\vartheta = P_r = \int_{-\infty}^{0} p_r\,\mathrm{d}\beta \qquad \text{(eddy)}$$

$P_r = M_r$ but

$P_r = 2\,M_r$ (flow reversal) □

cf. Sychev (1982): no slender wake eddy behind bluff body

No steady state locally

- recombination of detached jets breaks stationarity locally first
- hence: radial jet unsteady, spreads into undisturbed fluid ahead for all $t > 0$
- but: incident wall jet potentially steady in BL limit
- if so: which of 2 possible collision structures provoked – and how?

Exponential branch-off upstream

- BL limit $\eta = (r-1)/\delta = O(1), \quad \delta \to 0$

$$\left[\frac{\psi}{\delta}, v\right] = \underbrace{[\Psi, V](\theta,\eta)}_{\text{base flow}} + \underbrace{\left\{\left[1, \frac{V_\eta}{\Psi_\eta}\right] a_0(\theta) R^0(\theta,\eta) + \delta\,[a_1, b_1](\theta) R^1(\theta,\eta)\right\} \exp \int_{\frac{\pi}{2}}^{\theta} \frac{k(s)}{\delta}\,\mathrm{d}s + O(\delta)}_{\text{non-}\|\text{ upstream influence: short-scale, WKBJ-type, } \operatorname{Re} k > 0}$$

- base flow problem

$$\Psi_\eta (\Psi_\eta/\sin\theta)_\theta - \Psi_\theta\, \Psi_{\eta\eta}/\sin\theta - V^2 \cos\theta = \Psi_{\eta\eta\eta}, \quad \Psi_\eta (V \sin\theta)_\theta - \Psi_\theta V_\eta \sin\theta = (\sin\theta)^2 V_{\eta\eta}$$

$$\eta = 0: \quad \Psi = \Psi_\eta = V - \sin\theta = 0, \quad \eta = \infty: \quad \Psi_\eta = V = 0$$

$\theta \to 0: \quad [\Psi/\theta^2, V] = [F, G](\eta) + O(\theta) \quad \ldots$ von Kármán swirl

$\theta \to \pi/2: \quad \Psi,\ V$ regular!

Exponential branch-off upstream

- BL limit

$$\left[\frac{\psi}{\delta}, v\right] = \underbrace{[\Psi, V](\theta,\eta)}_{\text{base flow}} + \underbrace{\left\{\left[1, \frac{V_\eta}{\Psi_\eta}\right] a_0(\theta) R^0(\theta,\eta) + \delta\,[a_1, b_1](\theta) R^1(\theta,\eta)\right\} \exp \int_{\frac{\pi}{2}}^{\theta} \frac{k(s)}{\delta}\,\mathrm{d}s + O(\delta)}_{\text{non-}\|\text{ upstream influence: short-scale, WKBJ-type, } \operatorname{Re} k > 0}$$

- Rayleigh problems: self-adjoint, regular, normed

$$R^{0,1}_{\eta\eta} + \left(k^2 - \Psi_{\eta\eta\eta}/\Psi_\eta\right) R^{0,1} = I^{0,1}, \quad \eta = 0: \quad R^{0,1} = 0, \quad R^{0,1}_\eta = 1, \quad \eta = \infty: \quad R^{0,1}_\eta = 0$$

- $I^0 = 0 \quad \Rightarrow$ eigenvalues $k = k_0 \underset{\delta\to 0}{\to} 0$, $k_1, \ldots, k_n < \sqrt{(\Psi_{\eta\eta\eta}/\Psi_\eta)(\theta,\infty)}$

- $I^1 = a_0'\, f(\theta,\eta) - a_0\, g(\theta,\eta) \quad \Rightarrow$ solvability condition

$$\int_0^\infty I^1 R^0\,\mathrm{d}\eta = 0 \quad \Rightarrow \quad \frac{a_0(\pi/2)}{a_0(\theta)} = \exp \int_\theta^{\pi/2} \frac{\int_0^\infty g(s,\eta)\,\mathrm{d}\eta}{\int_0^\infty f(s,\eta)\,\mathrm{d}\eta}\,\mathrm{d}s,$$ eddy structure fixes $a_0(\pi/2)$

Exponential branch-off upstream

- viscous correction: singular BL limit $\epsilon = [\delta \sin\theta/(k\lambda)]^{1/3} \to 0$

$$\frac{\psi}{\delta} = \left\{ \begin{matrix} \Psi(\theta,\eta) \\ \lambda(\theta)\,\eta^2/2 \end{matrix} \right\} + \left\{ \begin{matrix} R^0(\theta,\eta) \\ \epsilon S(\zeta) \end{matrix} \right\} a_0(\theta) \exp \int_{\pi/2}^{\theta} \frac{k(s)}{\delta}\,\mathrm{d}s + \cdots \quad \left\{ \begin{matrix} \eta = O(1) \\ \zeta = \eta/\epsilon = O(1) \end{matrix} \right.$$

- sublayer problem $\zeta S'' = S'''',\ \ S(0) = S'(0) = 0,\ \ S'(\infty) = 1$

$$S = 3\int_0^{\zeta} (\zeta - s)\mathrm{Ai}(s)\,\mathrm{d}s \underset{\zeta\to\infty}{\sim} \zeta + \Pi, \quad \text{induced pressure } \Pi = -\frac{3^{2/3}}{\Gamma(\frac{1}{3})}$$

Exponential branch-off upstream

- sublayer problem $\quad \zeta S'' = S'''', \quad S(0) = S'(0) = 0, \quad S'(\infty) = 1$

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(1) degenerate case $\quad k = k_1 = O(1)$, sublayer passive

$$-\beta = \frac{\pi/2 - \theta}{\delta} = O(1):$$ Stewartson's structure, via non-interactive 3-tiered BL

Exponential branch-off upstream

- sublayer problem $\zeta S'' = S'''', \quad S(0) = S'(0) = 0, \quad S'(\infty) = 1$

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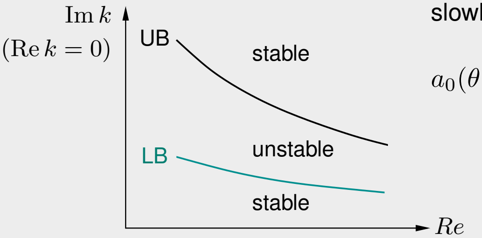
(2) least-degenerate long-wave limit $k = k_0$

$$R^0 = \frac{\Psi_\eta}{\lambda} - k_0^2 F(\theta,\eta) + O(k_0^4), \quad F_{\eta\eta} - \frac{\Psi_{\eta\eta\eta}}{\Psi_\eta} F = \frac{\Psi_\eta}{\lambda}, \quad F'(\theta,0) = F(\theta,\infty) = 0$$

$$\lambda^2 F(\theta,0) = J = \int_0^\infty \Psi_\eta^2\,\mathrm{d}s \overset{!}{\sim} -\frac{\epsilon\lambda^2}{k_0^2}\Pi \quad \Rightarrow \quad k_0(\theta) \sim \left[\frac{9\lambda^5 \sin\theta}{\Gamma(\frac{1}{3})^3 J^3}\,\delta\right]^{1/7}$$

$\pi/2 - \theta = O(\delta^{6/7})$: Smith & Duck's interactive branching

Time-dependent extension: BL instability



slowly modulated mode

$$a_0(\theta, t)\, \exp\left[\int_{\pi/2}^{\theta} \frac{k(s)}{\delta}\,\mathrm{d}s + \frac{\sigma t}{\delta}\right]$$

	lower branch LB (TS)		upper branch UB (TS)		inflectional (Rayleigh)	azimuthal modes ⋆
wavenumber $\mathrm{Im}\,k$	$O(\delta^{1/7})$	$\ll$	$O(\delta^{1/11})$	$\ll$	$O(1)$	$\frac{n\delta}{2\pi},\quad n = 1, 2, \ldots$
linear growth rate $\mathrm{Re}\,\sigma$	$O(\delta^{6/7})$	$\ll$	$O(\delta^{3/11})$	$\ll$	$O(1)$	$O(1)$

⋆ Garrett & Peake (JFM, 2002), Segalini & Garrett (JFM, 2017)

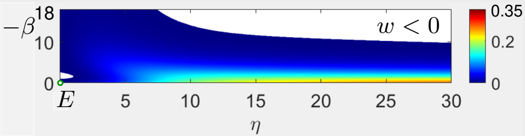
What DNS tells us

	max. Re	steady state?	collision vortex?	method
Dennis, Ingham & Singh (QJMAM, 1981)	5000	✓	no	direct (FD)
Dennis & Duck (Comput. Fluids, 1988)	5000	✓	no	impulsive spin-up (FD, inverting r)
Calabretto et al. (Proc. R. Soc. A, 2015)	$32\,000$	symmetry lost: conv. unstable		impulsive spin-up (FEM)
Calabretto & Denier (PRF, 2019)	$30\,000$	✓	no	”
Calabretto, Denier & Levy (JFM, 2019)	10^5	✓	tiny	”
B. J. Smith et al. (JFM **984**, 2024)	10^5	✓	very flat	”
Ch. K., Open∇FOAM® (2025)	3×10^5	✗		” (FV)

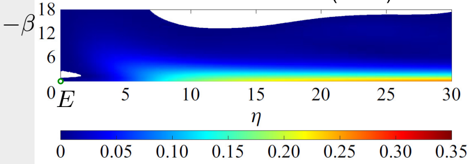
- previously in favour of Stewartson's structure, but
- no steady state for sufficiently large Re and $t > 30$

Spin-up from rest: what very recent DNS tells us

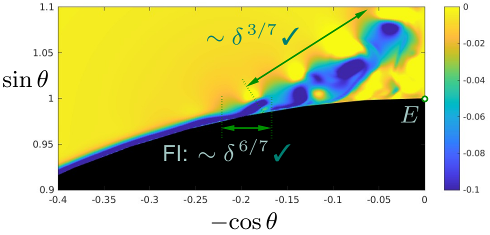
w, Euler scales β, η: $t = 11$, $Re = 10^5$



cf. B. J. Smith et al. (2024)

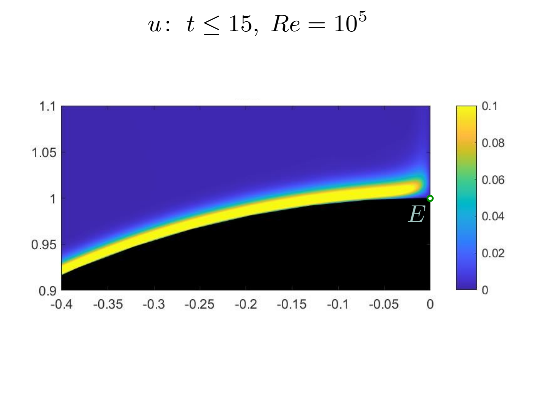


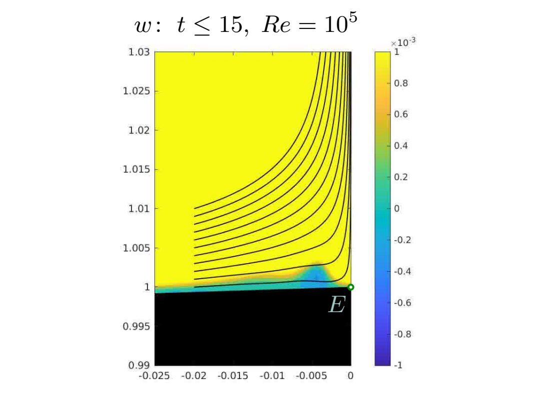
u: $0 \le t \le 34$, $Re = 10^5 \times [1 + 2H(t)]$



- vortex shedding engendered by separated shear layers trying to recombine
- turbulence not fully resolved due to symmetry restraints
- Smith & Duck's structure lurking transiently

Subcritical impulsive spin-up



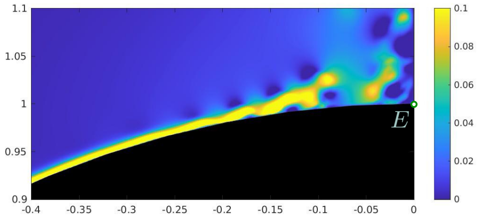


- complex eddy structure settles to steady state – but eddies die out
- surpringly large $Re_{\text{crit}} \approx 1.25 \times 10^5$ – but too small for asymptotic regime

Supercritical impulsive spin-up

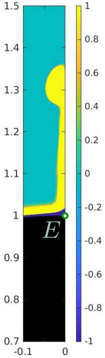
Equatorial break-up & jet generation

$u\colon\ t \le 15,\ Re = 3 \times 10^5$



burst at $t = t_b \approx 4.5$

azimuthal vorticity: $t \le 9,\ Re = 1.4 \times 10^5$



Expected finite-time break-up of BL

- theoretical progress slow, regularisation?

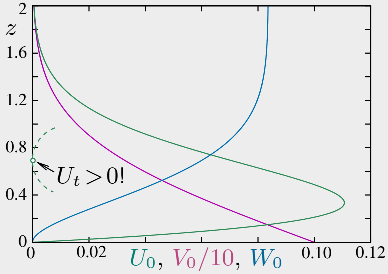
  Banks & Zaturska (1979), Simpson & Stewartson (1982), Dennis & Duck (1988), Van Dommelen (1987, 1990)

- well-posed BL/wall jet problem $[u, v, w/\delta] = [U, V, W](\theta, \eta, t) + O(\delta)$

$$U_\theta + W_\eta + U \cot\theta = 0$$

$$U_t + UU_\theta + WU_\eta - V^2 \cot\theta = U_{\eta\eta}, \quad V_t + UV_\theta + WV_\eta + UV \cot\theta = V_{\eta\eta}$$

$$\eta = 0\colon \quad U = W = V - H(t) \sin\theta = 0, \quad \eta = \infty\colon \quad U = V = 0$$



$t \to 0, \; z = \eta/(2t^{1/2}) = O(1)\colon$

$U \sim t \sin(2\theta)\, U_0(z)/2, \; V \sim \sin\theta\, V_0(z),$

$W \sim t^{3/2}[3(\cos\theta)^2 - 1] W_0(z)$

$0 < \theta < \pi/2, \; t > 0\colon \quad U \geq 0$ & regular!

Expected finite-time break-up of BL

- BL stops & limit intact & symmetries met & isolated problems for

  $\theta \to 0$: $[U/\theta, V/\theta, W] \to [-\hat{W}_\eta/2, \hat{V}, \hat{W}](\eta, t) \underset{t\to\infty}{\longrightarrow}$ forward von Kármán swirl

  $\theta \to \pi/2$: $[U/(\pi/2-\theta), V, W] \to [\bar{W}_\eta, \bar{V}, \bar{W}](\eta, t)$

- $\bar{W}_{\eta t} + \bar{W}\bar{W}_{\eta\eta} - \bar{W}_\eta^2 - \bar{V}^2 = \bar{W}_{\eta\eta\eta}, \quad \bar{V}_t + \bar{W}\bar{V}_\eta = \bar{V}_{\eta\eta}$

  $\eta = 0$: $\bar{W} = \bar{W}_\eta = \bar{V} - H(t) = 0, \quad \eta = \infty$: $\bar{V} = \bar{W}_\eta = 0$

  $\bar{W} \geq 0$

Expected finite-time break-up of BL

$t = t_b \approx 4.57446$

- BL stops & limit intact & symmetries met & isolated problems for

  $\theta \to 0$: $[U/\theta, V/\theta, W] \to [-\hat{W}_\eta/2, \hat{V}, \hat{W}](\eta, t) \underset{t\to\infty}{\longrightarrow}$ forward von Kármán swirl

  $\theta \to \pi/2$: $[U/(\pi/2-\theta), V, W] \to [\bar{W}_\eta, \bar{V}, \bar{W}](\eta, t)$

- $\bar{W}\bar{W}_{\eta\eta} - \bar{W}_\eta^2 - \bar{V}^2 \sim \bar{W}_{\eta\eta\eta}, \quad \bar{W}\bar{V}_\eta \sim \bar{V}_{\eta\eta}$

  $\eta = 0$: $\bar{W} = \bar{W}_\eta = \bar{V} - 1 = 0, \quad \eta = \infty$: $\bar{V} = \bar{W}_\eta = 0$

  $\bar{W} \geq 0$ but $\bar{W}(\infty, \infty) < 0$ ✗

- inviscid radial burst creates jet $\quad \tau = t_b - t \to 0, \ Z = \tau^{1/2}\eta = O(1)$

  $\bar{W} = \tau^{-3/2}[Z/2 - \sin(\alpha Z)/(2\alpha)] + O(\tau^{-1}\ln\tau), \quad \alpha \approx 0.712$

Finite-time break-up: oblate/prolate spheroids

canonical local analysis recovered by rescaling $(\eta, t) \mapsto (\gamma\eta, \gamma^2 t)$ for

- oblate, $\gamma < 1$: faster & closer to equator (extreme: finite disc)
- prolate, $\gamma > 1$: slower & further away (extreme: ∞ cylinder)

Results & to-dos

Asymptotic & numerical large-Re analysis of steady state

- 2 structures triggerable via exponential branching:

  (1) Stewartson (1958): on Euler scale, $Re^{-1/2}$

  (2) Smith & Duck (1977): interactive branching yields larger scale, $Re^{-3/14}$
- (1) pathological exception
- (1) & (2) suffer from inconsistent jet formation $\Rightarrow$
- no steady state for $Re > Re_{\mathrm{crit}} \approx 1.25 \times 10^5$

  contrasts previous findings!

Upcoming

- saving (2) via unsteady jet formation
- regularising equatorial finite-time break-up in BL computations
- route to turbulence

There is much more... Thank you!