Generalizing the optimal interpolation points for IRKA ⋆

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1. INTRODUCTION

The iterative rational Krylov algorithm (IRKA) was introduced in Gugercin et al. (2008) and has become a widely adopted method in the model reduction community for computing locally optimal solutions for the $\mathcal{H}_2$ model reduction problem. IRKA relies on the first-order optimality conditions for the solution of the $\mathcal{H}_2$ optimization problem derived in Meier and Luenberger (1967). One of the main assumptions needed for IRKA to work properly is for the error transfer function between the full and reduced order models to be in $\mathcal{H}_2$. Recently, in Borghi and Breiten (2024), this assumption was relaxed to account for non asymptotically stable systems, giving the possibility of designing an extended version of IRKA by developing a new framework and deriving its optimal interpolation conditions. Throughout this paper we refer to this algorithm as extended IRKA.

The contribution of this work is twofold: (1) We build upon the findings in Borghi and Breiten (2024) and show that the optimal interpolation points are related to the Schwarz function (see Davis (1974)); (2) We show numerically that we can use the adaptive Antoulas-Anderson (AAA) algorithm developed in Nakatsukasa et al. (2018) to approximately compute the interpolation points in extended IRKA for user-defined domains in the complex plane.

2. PRELIMINARIES

We consider the large-scale minimal single-input single-output (SISO) linear time invariant (LTI) dynamical system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \\ y(t) = \mathbf{c}\mathbf{x}(t), \quad \mathbf{x}(0) = 0, \end{cases} \qquad G(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \tag{1}$$

with $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{c}^\top \in \mathbb{C}^n$, and $\mathbf{b} \in \mathbb{C}^n$. In addition, for a fixed time t, $\mathbf{x}(t) \in \mathbb{C}^n$, $u(t) \in \mathbb{C}$, and $y(t) \in \mathbb{C}$, denote the state, input, and output of the system respectively. Here, G denotes the transfer function of the system in the frequency domain. We refer to (1) as the full-order model (FOM). We assume the eigenvalues $\{\lambda_j\}_{j=1}^n$ of $\mathbf{A}$ to be in a simply-connected open set $\mathbb{A}$ in the complex plane. Most importantly, $\mathbb{A}$ does not have to be in the left half-plane $\mathbb{C}_-$, which can result in (1) not being asymptotically stable. Our work deals with the computation of a reduced-order model (ROM) of a form analogous to (1) but with system matrices $\mathbf{A}_r \in \mathbb{C}^{r\times r}$, $\mathbf{c}_r^\top \in \mathbb{C}^r$, $\mathbf{b}_r \in \mathbb{C}^r$, where $r \ll n$, and transfer function $G_r(s) = \mathbf{c}_r(s\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{b}_r$ with poles $\{\widehat{\lambda}_j\}_{j=1}^r \in \mathbb{A}$. Similar to the $\mathcal{H}_2$-optimal model reduction framework, we seek G_r such that

$$\min_{\deg(G_r)=r} \|G - G_r\|_*, \tag{2}$$

is solved, where $*$ indicates a proper norm. In Borghi and Breiten (2024) an $\mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})$ space for systems with poles in $\mathbb{A}$ and analytic in its exterior $\bar{\mathbb{A}}^\mathsf{c}$ was introduced to include error transfer functions that are not in $\mathcal{H}_2$. For the development of the $\mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})$-optimal model reduction framework, the concept of conformal map (see Theorem 6.1.2 in Wegert (2012)) is pivotal. Under proper assumptions on the conformal maps and $G_r \in \mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})$ being a local minimizer of (2), simplified $\mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})$ optimality conditions were derived in (Borghi and Breiten, 2024, Corollary 3) resulting in

$$G_r(\varphi(\widehat{\lambda}_j)) = G(\varphi(\widehat{\lambda}_j)) \text{ and } G_r'(\varphi(\widehat{\lambda}_j)) = G'(\varphi(\widehat{\lambda}_j)), \tag{3}$$

for $j = 1, \ldots, r$, with

$$\varphi(s) = \overline{\overline{\psi}(-\psi^{-1}(s))} = \psi(-\overline{\psi^{-1}(s)}), \tag{4}$$

where $\overline{\psi}(s) = \overline{\psi(\overline{s})}$. The conditions in (3) then led to the development of extended IRKA. In the next sections, we leverage the connection between (4) and Schwarz functions to approximate φ given only points on the boundary of user defined domains. However, the approximation will not necessarily satisfy the assumptions made in Borghi and Breiten (2024).

3. OUR METHOD

Before introducing the result of this work, we give a brief summary of the concepts of Schwarz reflection and Schwarz function based on Davis (1974). For a more detailed description of this topic see Davis (1974) and Shapiro (1992). We consider the analytic arc Γ given by the parametrization $z = f(\theta)$ with $\theta \in [a, b]$, $a, b \in \mathbb{R}$. Given any point $\theta_0 \in [a, b]$, f is a bijective conformal map in the disk $\mathsf{D}_{\theta_0} = \{\tau \in \mathbb{C} \big| |\tau - \theta_0| < \rho(\theta_0)\}$, with $\rho\colon [a, b] \to (0, \infty)$. Let $\tilde{z} = f(\overline{\tau})$ for $\tau \in \mathsf{D}_{\theta_0}$. Then $\tilde{z}$ is called the *Schwarz reflection* of z with respect to the analytic arc Γ. The *Schwarz function* is defined as $S(z) = \overline{\tilde{z}}$ and analytic in a neighborhood of Γ. For $z \in \Gamma$ we get $S(z) = \overline{z}$. The complex conjugate of the Schwarz function applied to a point is the Schwarz reflection of the point with respect to Γ (see Chapter 6 in Davis (1974)). For the ease of notation we use the term *anti-conformal reflection* $R(\cdot) = \overline{S(\cdot)}$ from Shapiro (1992) to indicate the

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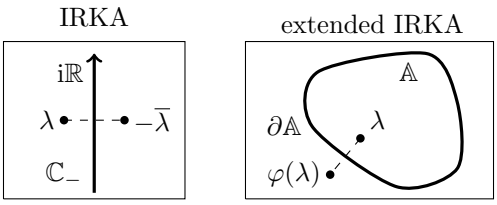

Fig. 1. Depiction of the interpolation points for IRKA (left) and extended IRKA (right).

Schwarz reflection. It is possible to use f to get an explicit formulation of the anti-conformal reflection for any point z in the neighborhood of Γ. The steps are the following

$$z \xrightarrow{f^{-1}} \tau \xrightarrow{\overline{(\cdot)}} \overline{\tau} \xrightarrow{f} \tilde{z} = R(z), \tag{5}$$

which lead to the definition

$$R(\cdot) = f(\overline{f^{-1}(\cdot)}), \tag{6}$$

and, consequently, $S(\cdot) = \overline{f}(f^{-1}(\cdot))$. Let us now set

$$f(\tau) = \psi(\mathrm{i}\tau) \text{ and so } f^{-1}(z) = -\mathrm{i}\psi^{-1}(z), \tag{7}$$

with ψ a conformal map satifying the assumptions below.

Assumption 1. Let $\mathbb{X} \subseteq \mathbb{C}$ include $\mathrm{i}\mathbb{R}$, and $\mathbb{Y} \subseteq \mathbb{C}$ include the analytic closed curve $\partial\mathbb{A}$. Furthermore, Let $\mathbb{X}_- = \mathbb{X}\backslash\mathbb{C}_+$, $\mathbb{X}_+ = \mathbb{X}\backslash\mathbb{C}_-$ such that $\left\{s \in \mathbb{C} \middle| -\overline{s} \in \mathbb{X}_-\right\} \subseteq \mathbb{X}_+$, $\partial\mathbb{A}_- = \mathbb{Y}\backslash\bar{\mathbb{A}}^{\mathsf{c}}$, and $\partial\mathbb{A}_+ = \mathbb{Y}\backslash\mathbb{A}$. We assume $\psi\colon \mathbb{X} \to \mathbb{Y}$ to be a bijective conformal map such that (i) $\psi \circ \mathrm{i}\colon \mathbb{R} \to \partial\mathbb{A}$, (ii) $\psi\colon \mathbb{X}_+ \to \partial\mathbb{A}_+$ and (iii) $\psi\colon \mathbb{X}_- \to \partial\mathbb{A}_-$.

With (7) we can connect the definition of φ in (4) and the anti-conformal reflection in (6). As a matter of fact, by substituting f in (6) with (7) we get

$$R(\cdot) = \psi(\mathrm{i}\overline{(-\mathrm{i}\psi^{-1}(\cdot))}) = \psi(-\overline{\psi^{-1}(\cdot)}) = \varphi(\cdot).$$

The composition of φ is similar to (5) but instead of applying a complex conjugation we take the mirror image with respect to the imaginary axis. In more detail, we have

$$z \xrightarrow{\psi^{-1}} \tau \xrightarrow{-\overline{(\cdot)}} -\overline{\tau} \xrightarrow{\psi} \varphi(z).$$

While $\partial\mathbb{A}$ is defined by the user, the function φ is unknown a-priori. For this reason, now that the connection between the anti-conformal reflection and (4) has been established, we use the AAA algorithm to approximate φ given $\partial\mathbb{A}$. It is important to emphasize that the function S is solely determined by the chosen $\partial\mathbb{A}$. This gives us the possibility to approximate φ with AAA using only samples of $\partial\mathbb{A}$. To do so we take points $z \in \partial\mathbb{A}$, approximate $S(z) = \overline{z}$ with AAA (see also Trefethen (2024)), and complex conjugate the resulting function. The approximated φ is then used for computing the interpolation points employed by extended IRKA. The main drawback of this approach is that, as S is defined in a neighborhood of $\partial\mathbb{A}$, the same applies for the approximation to φ.

4. NUMERICAL EXAMPLE

We test the extended IRKA with interpolation points computed through AAA on the controlled linear undamped wave equation from Borghi and Breiten (2024). After discretization by centered finite differences we get a FOM with $n = 400$ and poles on the imaginary axis. We are interested in the poles near the origin as they provide an approximately good description of the original system. For this reason, we use the 'boomerang' shape illustrated in Fig. 2 on the left for $\partial\mathbb{A}$. We parametrized $\partial\mathbb{A}$ such that it has only one segment near the FOM poles and it is close to the origin (see Fig. 2 on the right). We do so in order for extended IRKA to identify these poles as dominant and place the reduced poles accordingly. In Fig. 2 we show the resulting ROM poles $\{\widehat{\lambda}_j\}_{j=1}^r$ and interpolation points $\{\varphi(\widehat{\lambda}_j)\}_{j=1}^r$ for $r = 18$. In addition, Fig. 3 shows that the impulse response of the resulting ROM well approximates the one of the FOM.

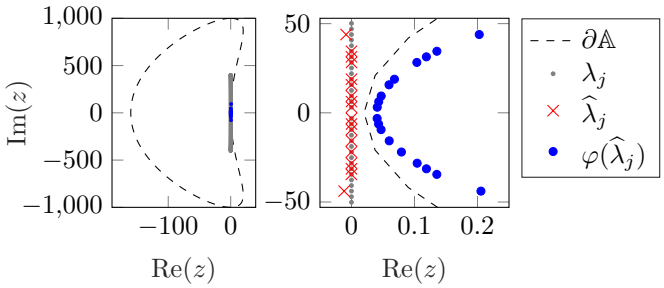

Fig. 2. Chosen $\partial\mathbb{A}$, poles of the FOM and ROM, and the computed interpolation points. The plot on the right is a magnification of the one the left near the origin.

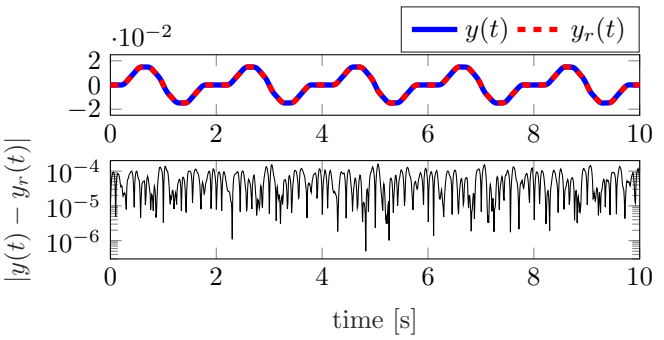

Fig. 3. (Top) output trajectories of the FOM (y) and ROM (y_r) with respect to time t. (Bottom) absolute error.

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