Data based modeling and reduced order modeling for port-Hamiltonian descriptor systems

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1. INTRODUCTION

We present a method to construct low order linear time-invariant (LTI) port-Hamiltonian (pH) descriptor systems from observed response data in the frequency domain.

The simplest form of an LTI pH descriptor system is

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\mathbf{x}+\mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{B}^{\mathbf{T}}\mathbf{x} \end{aligned} \quad \text{and} \quad \mathbf{x}(0)=\mathbf{x}_0, \tag{1}$$

with $\mathbf{J}=-\mathbf{J}^T$, $\mathbf{E}=\mathbf{E}^T\geq 0$, $\mathbf{R}=\mathbf{R}^T\geq 0\in\mathbb{R}^{n,n}$, $\mathbf{B}\in\mathbb{R}^{n,m}$, and the quadratic Hamiltonian (modeling the internal energy stored in the system) is $\mathcal{H}=\frac{1}{2}\mathbf{x}^T\mathbf{E}\mathbf{x}$, see e.g. Mehrmann and Unger (2023). PH descriptor systems are closely related to passive and positive-real systems, see Cherifi et al. (2023). We assume that the underlying physical system is passive and that it can be represented as a pH system of the form (1) with positive definite $\mathbf{E}$.

A classical approach to derive a low-order pH descriptor system from observed data is to first derive a realization

$$\begin{aligned} \dot{\mathbf{x}} &= \widetilde{\mathbf{A}}\mathbf{x}+\widetilde{\mathbf{B}}\mathbf{u},\ \mathbf{x}(0)=\mathbf{x}_0, \\ \mathbf{y} &= \widetilde{\mathbf{C}}\mathbf{x}, \end{aligned} \tag{2}$$

with $\widetilde{\mathbf{A}}\in\mathbb{R}^{n,n}$, $\widetilde{\mathbf{B}},\widetilde{\mathbf{C}}^T\in\mathbb{R}^{n,m}$, e.g. via the Loewner approach Mayo and Antoulas (2007). The resulting system is typically close to a passive system. If the system is passive, see e.g. Cherifi et al. (2023), then there exists a matrix $\mathbf{E}=\mathbf{E}^T>\mathbf{0}$ such that

$$\begin{bmatrix} -\mathbf{E}\widetilde{\mathbf{A}}-\widetilde{\mathbf{A}}^T\mathbf{E} & \widetilde{\mathbf{C}}^T-\mathbf{E}\widetilde{\mathbf{B}} \\ \widetilde{\mathbf{C}}-\mathbf{E}\widetilde{\mathbf{B}}^T & \mathbf{0} \end{bmatrix}\geq\mathbf{0}. \tag{3}$$

Setting $\mathbf{B}:=\mathbf{E}\widetilde{\mathbf{B}}$ and $\mathbf{E}\widetilde{\mathbf{A}}=\mathbf{J}-\mathbf{R}$, with $\mathbf{J}=-\mathbf{J}^T$ and $\mathbf{R}=\mathbf{R}^T\geq 0$, see e.g. Beattie et al. (2019), gives the pH descriptor system

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\mathbf{x}+\mathbf{B}\mathbf{u}, \\ \mathbf{y} &= \mathbf{B}^T\mathbf{x}. \end{aligned} \tag{4}$$

For a non-passive realization one typically applies small perturbations Alam et al. (2011); Brüll and Schröder (2013); Grivet-Talocia (2004); Freund et al. (2007) to obtain a nearby passive system for which the approach can be applied.

Unfortunately, proceeding in this way requires the solution of an optimization problem to enforce passivity and a solution of (3), both of which is prohibitive for large scale systems. So typically it is necessary to first perform a model reduction, see e.g. Antoulas et al. (2020), which may, however, destroy the pH structure. For pH descriptor systems, such model reduction methods are well established, see e.g. Beattie et al. (2022b); Hauschild et al. (2019).

All these approaches, however, require an explicit realization of the system. An ideal procedure would directly generate a pH descriptor system from data. But so far only indirect methods are available Benner et al. (2020); Cherifi et al. (2019). We propose a direct approach.

2. A DATA BASED DIRECT APPROACH

Suppose that we are able to sample the system response of a passive system at (complex) frequencies $\{\sigma_1,\sigma_2,\ldots,\sigma_r\}\subset\mathbb{C}_+$ with corresponding input profiles $\{b_1,b_2,\ldots,b_r\}$. The input-output map of (2) in frequency domain is associated to the transfer function $\boldsymbol{\mathcal{G}}(s)=\widetilde{\mathbf{C}}(s\,\mathbf{I}-\widetilde{\mathbf{A}})^{-1}\widetilde{\mathbf{B}}$, so we have access to the data, $g_k=\boldsymbol{\mathcal{G}}(\sigma_k)b_k$, $k=1,\ldots,r$ but not to a realization (2).

We may construct an interpolating subspace, $\mathfrak{V}_r=\mathsf{Ran}(\mathbf{V}_r)$, where $\mathbf{V}_r=[\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_r]$ and $\mathbf{v}_k=(\sigma_k\mathbf{I}-\widetilde{\mathbf{A}})^{-1}\widetilde{\mathbf{B}}b_k$, $k=1,\ldots,r$; or equivalently,

$$\mathbf{V}_r\boldsymbol{\Sigma}_r-\widetilde{\mathbf{A}}\mathbf{V}_r=\widetilde{\mathbf{B}}\mathbb{B}_r \tag{5}$$

with $\boldsymbol{\Sigma}_r=\mathsf{diag}\{\sigma_1,\sigma_2,\ldots,\sigma_r\}$ and $\mathbb{B}_r=[b_1,b_2,\ldots,b_r]$. Then define $\mathbf{E}_r=\mathbf{V}_r^*\mathbf{E}\mathbf{V}_r$ and the auxiliary modeling space, $\mathfrak{W}_r=\mathsf{Ran}(\mathbf{W}_r)$, with $\mathbf{W}_r=\mathbf{E}\mathbf{V}_r$. Using the ansatz, $\mathbf{x}\approx\mathbf{V}_r\mathbf{x}_r$, in (4) and applying a Petrov-Galerkin condition that forces residual orthogonality (in $\mathbb{C}^n$) to $\mathfrak{W}_r$, we obtain a reduced order model

$$\begin{aligned} \mathbf{E}_r\dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\mathbf{x}_r+\mathbf{C}_r^T\mathbf{u}, \\ \mathbf{y} &= \mathbf{C}_r\mathbf{x}_r, \end{aligned} \tag{6}$$

where $\mathbf{J}_r=\mathbf{V}_r^*\mathbf{J}\mathbf{V}_r=-\mathbf{J}_r^*$, $\mathbf{R}_r=\mathbf{V}_r^*\mathbf{R}\mathbf{V}_r=\mathbf{R}_r^*\geq\mathbf{0}$, and $\mathbf{C}_r=\widetilde{\mathbf{C}}\mathbf{V}_r=\widetilde{\mathbf{B}}^T\mathbf{E}\mathbf{V}_r=\widetilde{\mathbf{B}}^T\mathbf{W}_r$. Note that $g_k=\boldsymbol{\mathcal{G}}(\sigma_k)b_k=\widetilde{\mathbf{C}}(\sigma_k\mathbf{I}-\widetilde{\mathbf{A}})^{-1}\widetilde{\mathbf{B}}b_k=\widetilde{\mathbf{C}}\mathbf{v}_k$, so with $\mathbb{G}_r=[g_1,g_2,\ldots,g_r]$, we have $\mathbf{C}_r=\mathbb{G}_r$, and likewise, applying the same Petrov-Galerkin condition to (5) implies

$$\mathbf{V}_r^*\mathbf{E}\mathbf{V}_r\boldsymbol{\Sigma}_r-\mathbf{V}_r^*\mathbf{E}\widetilde{\mathbf{A}}\mathbf{V}_r=\mathbf{V}_r^*\mathbf{E}\widetilde{\mathbf{B}}\mathbb{B}_r=\mathbf{C}_r^*\mathbb{B}_r$$

which gives

$$\mathbf{E}_r\boldsymbol{\Sigma}_r-(\mathbf{J}_r-\mathbf{R}_r)=\mathbb{G}_r^*\mathbb{B}_r. \tag{7}$$

Summing with the conjugate transpose of (7) we have

$$\mathbf{E}_r\,\boldsymbol{\Sigma}_r+\overline{\boldsymbol{\Sigma}_r}\,\mathbf{E}_r+2\,\mathbf{R}_r=\mathbb{G}_r^*\mathbb{B}_r+\mathbb{B}_r^*\mathbb{G}_r. \tag{8}$$

DOI: 10.34726/9006

Denote by $\mathbb{S}^{r\times r}$ the set of Hermitian $r \times r$ matrices, by $\mathcal{P}_r$ the closed, convex cone of positive semidefinite (Hermitian) $r \times r$ matrices, and by $\mathcal{P}_r^\circ$ its interior, i.e., the open cone of $r \times r$ strictly positive-definite Hermitian matrices. Then define the linear operator $\mathcal{L} : \mathbb{S}^{r\times r} \to \mathbb{S}^{r\times r}$ as

$$\mathcal{L}(\mathbf{M}) = \mathbf{M}\,\boldsymbol{\Sigma}_r + \overline{\boldsymbol{\Sigma}}_r\,\mathbf{M}$$

and observe that since $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ is contained in the open right half-plane, $\mathcal{L}^{-1}$ is well-defined and cone-preserving: $\mathcal{L}^{-1} : \mathcal{P}_r \to \mathcal{P}_r$ and $\mathcal{L}^{-1} : \mathcal{P}_r^\circ \to \mathcal{P}_r^\circ$. Then (8) leads to

$$\mathbf{E}_r + 2\,\mathcal{L}^{-1}(\mathbf{R}_r) = \mathcal{L}^{-1}(\mathbb{G}_r^*\mathbb{B}_r + \mathbb{B}_r^*\mathbb{G}_r). \quad (9)$$

Note that in all nontrivial circumstances, $\mathbb{G}_r^*\mathbb{B}_r + \mathbb{B}_r^*\mathbb{G}_r$ is indefinite, however, since $\mathbf{E}_r > 0$ and $\mathcal{L}^{-1}(\mathbf{R}_r) \geq \mathbf{0}$, whenever the original system (2) is passive, then $\mathcal{L}^{-1}(\mathbb{G}_r^*\mathbb{B}_r + \mathbb{B}_r^*\mathbb{G}_r)$ itself must be positive definite if the original system (2) is passive. Note also that the condition $\mathcal{L}^{-1}(\mathbb{G}_r^*\mathbb{B}_r + \mathbb{B}_r^*\mathbb{G}_r) > 0$ involves only observed quantities, and it can checked computationally using methods for the numerical solution of Sylvester equations Golub and Van Loan (1996).

Theorem 1. Given complex frequencies $\{\sigma_1, \sigma_2, \ldots, \sigma_r\} \subset \mathbb{C}_+$ and corresponding input profiles $\{b_1, b_2, \ldots, b_r\}$ together with the induced system responses $\{g_1, g_2, \ldots, g_r\}$, where $g_k = \boldsymbol{\mathcal{G}}(\sigma_k)b_k$, $k = 1, 2, \ldots, r$. Let the matrices, $\mathbb{B}_r$ and $\mathbb{G}_r$, as well as the linear operator, $\mathcal{L}$, be defined as above. If the original system, (2), is passive then $\mathcal{L}^{-1}(\mathbb{G}_r^*\mathbb{B}_r + \mathbb{B}_r^*\mathbb{G}_r)$ is positive-definite.

Equivalently, if the data-based quantity, $\mathcal{L}^{-1}(\mathbb{G}_r^*\mathbb{B}_r + \mathbb{B}_r^*\mathbb{G}_r)$, *fails* to be positive-definite then the observed responses of the original system (2) are incompatible with passivity of the system; it cannot be expressed as a port-Hamiltonian system.

3. ANALYSIS OF THE PROCEDURE AND OPEN QUESTIONS

The described approach allows to produce a reduced order port-Hamiltonian from input-output data in frequency domain. However, there is freedom in the approach. since the representation of a standard system as port-Hamiltonian system is not unique. Any solution $\mathbf{E} > 0$ of (3) will lead to a port-Hamiltonian formulation. This freedom can be used to make the representation robust against perturbations, see Bankmann et al. (2020); Mehrmann and Dooren (2020). So one could first produce any reduced order model of the form (7) and then make it robust. We can also use a perturbation approach if the conditions of Theorem 1 are not met, see Beattie et al. (2022a).

Another open question is whether we can take a greedy approach by first taking a small number of sampling data and then increase the model order to achieve better approximations.

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