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Multi-Attribute Search with General Value Function and Jointly Distributed Attributes

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supervised by

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Affidavit

I, Amir Kazempouresmati

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22 pages, bound, and that I have not used any source or tool other than those referenced or any other illicit aid or tool, and that I have not prior to this date submitted this Master's Thesis as an examination paper in any form in Austria or abroad.

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Abstract

A usual assumption in the multi-attribute search literature is independent distribution of attributes. This assumption is necessary for the tractability of the model. However, in reality many economic search problems show some degree of correlation among the attributes, i.e., by searching an attribute the decision maker not only resolves the uncertainty with regards to that attribute but also updates her beliefs about the distribution of other values. Furthermore, unlike it is usually assumed the decision maker's utility function does not need to be linear in all attributes, in fact, in most cases it is sensible to consider other functional forms. The search order is thus decided based on the marginal utility and informativeness of each attribute. In this thesis, I introduce a simplified search model where the decision maker seeks to choose between two objects which are described by their two attributes. These attributes are assumed to be jointly distributed according to a distribution function known by the decision maker. Moreover, I will try to analyze the optimal search and stopping rule.

1 Introduction

The economic search models have been around for a long time. However, most of studies have focused on the search problems where the Decision Maker (DM) is choosing a single attribute alternative. In these problems the value of an alternative to the DM is characterized by its single attribute, thus, the alternatives can be ranked from the best to worst. Although the optimal stopping time in these problems under some mild assumptions can be derived analytically, many interesting economic problems are dealing with multi-attribute objects. For instance a DM who is searching in the housing market to buy a property does not only consider the advertised price as the only factor in her decision. Other attributes such as the age, amenities, neighbors, and the growth plans for the neighborhood are also decisive. Thus, other than the observed price, the DM needs to acquire further information about the other attributes which can affect the desirability of the property.

As the dimensionality of the problem increases in a multi-attribute search problems, the simplifying assumptions are an important part of the models in the literature. However, the unrealistic implications of these assumptions are usually overlooked.

- **Independent distribution of attributes**: It is assumed that attribute m of every alternative is independently drawn from distribution F_m. Thus, the DM cannot acquire information about other attributes $m' \neq m$ of an alternative by inspecting the attribute m. However, the level of attributes are usually correlated. Higher rental rate is often an indicator of desirability of a neighborhood and the age of the building. In the model presented in this paper, it is only assumed that the joint distribution of the attributes is known by the DM.

- **Hard-wired order of search** An analogue of the optimal stopping rule in the multi-attribute search problems is the trade-off between searching deeply in different attributes of an alternative (depth) and searching attributes of different alternatives (breadth). It is usual to assume the order of search is hard-wired. For instance, the DM can only search the attributes of an alternative in a predefined order and she can only start searching the next alternative after rejecting the current one. The second assumption is known as "*No Recall*". Allowing for recall, the DM selects the alternative with the maximum expected payoff. However, If the optimal search rule is of a threshold form (e.g. select an alternative if the observed values of attributes are higher than a predefined threshold), the DM will never pick a previously searched and discarded alternative. As she immediately selects the alternative which its searched attributes exceed the thresholds, she will never use the recall option. If the search order is not hard-wired, then the DM can revisit a previously searched alternative and further search the remaining attributes in the light of the new information acquired while searching the other available alternatives. In this paper, the search order is decided by the DM and recall is allowed.

In this paper, I introduce a simple search problem with only two alternatives, each described by its two attributes. Since the attribute levels are not independent draws, the search order is not solely decided by the marginal utility of each attribute, but the correlation with other attributes is also important. Thus, the decision maker also considers the informativeness of an attribute about the other unsearched attributes. Further, two special joint distribution function are considered, first, perfect correlation with-in alternatives and no correlation across alternatives, and second, perfect correlation across alternatives and no correlation with-in alternatives. In line with intuition, it is shown that when attributes of an alternative are correlated, the DM tends to search across alternatives, and when the attributes are correlated across alternatives, the DM tends to search with-in alternatives.

2 Literature Review

Different studies on multi-attribute search problem can be categorized based on the model assumptions. The simple environment of single attribute search problems such as the one in (Weitzman 1979), makes it possible to have an analytic form for the optimal stopping time strategy. Weitzman considers a model where the DM seeks to choose a single attribute alternative among $n < \infty$ choices with a known and independent distribution. To reveal the reward of each alternative a fixed cost $c > 0$ should be paid. He shows that the optimal strategy is given by a reservation price rule. However, he acknowledges an unrealistic underlying assumption in his model which is of great importance in my paper: correlated probability distributions. It is worth noting that even though recall is allowed in this model, the reservation price rule implies that a previously rejected alternative will never be recalled.

A variant of the multi-attribute search problem was used in (MacQueen 1964) for experimental analysis of a boundedly rational model, namely, the directed cognition model. They introduce an *N-good game* where the subjects are facing an m by n matrix where each row represents a consumption good which is characterized by its n attributes. Even though, the payoff of an alternative is simply the sum of the values of the selected row, the attributes are ordered by their variability.

In a more recent study, Lim, Bearden, and Smith (Lim, Bearden, and Smith 2006) focus on a problem where the DM faces a sequential search problem without recall. For the sake of tractability they restrict their attention to the case with independently distributed attributes. They further assume the payoff function exhibits *expectation-monotonic* property, i.e., if the expected alternative value under the observation vector x is lower than the expected value under x', then the realization of a common outcome x'' does not make the former preferable to the latter. Thus, the model assume value functions which are separable in attribute values. Moreover, the order in which the DM encounters the alternatives is fixed. The fixed order and no recall assumptions together rule out strategies such as attributes of two alternatives parallel to each other. They finally propose a threshold policy which is shown to be optimal for the certain class of value functions which exhibit the expectation-monotonic property.

Shortly after, (Bearden and Connolly 2007) propose two new multi-attribute search problems. In one variant of the problem the DM pays to learn the precise value of an attribute (*continuous* version) and in the second variant she pays to learn whether the value of an attribute is higher than a threshold chosen by her (*threshold* version). Again, the values of attributes are independent draws from a known distribution. In line with the necessary condition for the value function in (Lim, Bearden, and Smith 2006), the payoff of choosing an alternative is given by the sum its attributes. Further, they assume all attributes have the same distribution, thus, the order in which the attributes in each alternative are searched is irrelevant in the optimal strategy. As the experimental portion of their paper focuses on the over-searching with-in and across alternatives the above assumptions seem harmless to the results of their experiments.

Sanjurjo in (Sanjurjo 2014a) provides the first partial characterization of optimal sequential search in a variant of the problem where full recall is allowed and no restrictions on search order is assumed. The value of an attribute is assumed to be an independent random variable with a distribution centered at zero. Moreover, values of an attribute is drawn from the same distribution across the alternatives. Similar to (Klabjan, Olszewski, and Wolinsky 2014), it is assumed that for every two attributes, the distribution of one can be obtained by a symmetric mean-preserving spread of the other one.

There is a fixed cost to be paid for searching an additional attribute, however, the first attribute of all alternative can be viewed at no cost. The author introduces 4 necessary conditions for optimal search rule. Two of these conditions are trivial and follow from the assumption of rationality and risk neutrality, namely, the DM does not pay to view an attribute which has already been searched and upon termination, the DM chooses an alternative which maximizes the *expected* payoff.

The two other conditions focus on the depth and breadth trade-off and distribution of attributes. The first condition holds for the case in which there are at least three alternatives which is different from the model considered in this paper. The first condition states that attributes in an alternative cannot be searched if there is another alternative that has both a weakly higher number of unsearched attributes and a weakly higher expected value, with at least one of these relations strict.

The second condition exploits the ordering of attributes' distributions with respect to the second order stochastic dominance relation. This condition states that the value of searching an attribute with-in an alternative which is second order stochastically dominated by all the other attributes is the highest. This result is similar to the one from (Weitzman 1979). Sanjurjo analyzes the results of the experiment in and finds out that agents behaviors show a systematic deviations from the partial optimality conditions explained above. In particular, he finds evidence that people search too deeply with-in too few alternatives and switch too adjacently between alternative (Sanjurjo 2014a).

In his other paper, (Sanjurjo 2014b) tries to explain these deviations from the optimal search rule by introducing the concept of Working Memory Load. Consider an agent who is searching through different elements of the alternative-attribute matrix. Clearly, after exhaustively searching all the alternatives, her problem is simply to choose the alternative i with the highest realized value V_i. Possibility of recall and exhaustive searching reduce the problem of optimal search to a trivial one, where the order of search does not play a role in the outcome. However, this will be different if one considers partial search of a larger matrix.

An agent who seeks to choose an alternative with the highest value, at each stage of the search process needs to remember the search history. This includes location and values of the searched alternatives and attributes. Moreover, if she is following a search order which specifically determines the next alternative and attribute to be searched, she has to remember these instructions. As the search proceeds the DM will experience an overload of information needed to be remembered in the next stages, which in turn gives rise to choice errors. The author manages to explain the choice errors observed in (Sanjurjo 2014a) by the model of working memory load.

3 The model

A decision maker (DM) seeks to maximize her expected utility of choosing a multi-attribute object. An object x_m is characterized by its n attributes, $x_{m1}, x_{m2}, \ldots, x_{mn}$. Unlike the usual practice in the literature the attributes are not assumed to be independent random variables. Thus, there exists a joint density function f over the attributes of all available alternatives. In this paper, I will focus on a simplified version of the search problem, where, the DM decides between two alternatives which are characterized by their two attributes. Moreover, it is assumed that the each attribute can take on two values, *high* and *low*. Ex-ante the DM does not have any information about the realized values of the attributes. However, at each search step, she can reveal the value of one attribute by paying a constant cost $c > 0$. After observing the value she can decide whether to stop the search and choose the best alternative based on her current information, or to continue the search by revealing the value of another attribute.

It is clear that the maximum number of search steps is bounded above by total number of attributes, in which case, all the attributes are searched exhaustively and the DM simply picks the alternative that maximizes her utility. The lower bound on search steps is zero, in which case, the DM pick an alternative solely based on her belief about the joint distribution. The utility function of the DM who chooses to select alternative m is given by:

$$u(x_{m1}, x_{m2}, s) \quad m \in \{1, 2\}, s \in \{0, 1, 2, 3, 4\},$$

where, s denotes the number of search steps undertaken by the DM prior to picking the alternative. x_{mn} denotes the realized level of the attribute n of alternative m, where, $x_{mn} \in \{L, H\}$. The utility function is increasing in its first two arguments and decreasing in s. Further, I assume that one attribute, say the second attribute, has a higher marginal utility,i.e., in particular

$$u(\omega, H, s) - u(\omega, L, s) > u(H, \omega, s) - u(L, \omega, s),$$

for $\omega \in \{H, L\}$. Thus, $u(H, H, s) > u(L, H, s) > u(H, L, s) > u(L, L, s)$.

3.1 Batch Search

In this section, I consider a variant of the search problem. Consider a DM who is given s search coupons. So, the DM need not consider the optimal stopping rule. For instance, for $s = 1$, the DM is allowed to search one attribute and pick the best alternative after learning about the value of that attribute. Let's define $u(x_{m1}, x_{m2}) \equiv u(x_{m1}, x_{m2}, 0)$.

3.1.1 $s = 0$

The DM picks an alternative which maximizes her expected utility with respect to the joint probability density function $P(x_{11}, x_{12}, x_{21}, x_{22})$. The expected utility of choosing alternative m is given by:

$$\mathbb{E}[u(x_{m1}, x_{m2})] = \sum_{\omega,\omega' \in \Omega} \Pr(x_{m1} = \omega, x_{m2} = \omega')u(\omega, \omega'), \tag{1}$$

where, $\Omega = \{L, H\}$. The DM will then choose $x_{m'}$, where,

$$m' \in \arg\max_{m \in \Omega} \mathbb{E}[u(x_{m1}, x_{m2})].$$

3.1.2 $s = 1$

The DM first chooses an attribute of an alternative to be inspected. After observing the level of the attribute she chooses one of the alternatives. Let m be the alternative and n be the attribute chosen to be inspected. The expected utility of the agent from inspecting the attribute x_{mn} is given by:

$$\sum_{\omega \in \{H,L\}} \Pr(x_{mn} = \omega) \max\Bigg(\mathbb{E}[u(x_{11}, x_{12})|x_{mn} = \omega]; \mathbb{E}[u(x_{21}, x_{22})|x_{mn} = \omega]\Bigg).$$

Denote the above value by V_{mn}. The DM will choose $m, n \in \{1, 2\}$ so as to maximize V_{mn}.

3.1.3 $s = 2$

The DM first gets to observe the value of two attributes sequentially,i.e., she observes the level of the first chosen attribute (L or H), then she selects another attribute and then chooses the (expected) utility maximizing alternative. Suppose the DM chooses to observe the level of x_{11} which turns out to be H. At this stage, she needs to choose among the three remaining attributes. Let $V_{mn}(x_{11} = H)$ denote the value of searching the attribute x_{mn} after observing the high value for x_{11}.

$$V_{mn}(x_{11} = H) := \mathbb{E}\Bigg[\max(\mathbb{E}[u(x_{11}, x_{12})|x_{11} = H]; \mathbb{E}[u(x_{21}, x_{22})|x_{11} = H])\Bigg| x_{mn}\Bigg] =$$

$$= \sum_{\omega \in \{H,L\}} \Pr(x_{mn} = \omega | x_{11} = H) \max_{m \in \{1,2\}} \{\mathbb{E}[u(x_{m1}, x_{m2})|x_{11} = H, x_{mn} = \omega]\}.$$

In particular, if she decides to reveal the level of x_{12}, the exact utility of choosing alternative 1 is known to her. Thus, the expected payoff of choosing x_{12} is given by:

$$\begin{aligned}
&V_{12}(x_{11} = H) = \\
&\quad \Pr(x_{12} = H|x_{11} = H) \max(u(H, H); \mathbb{E}[u(x_{21}, x_{22})|x_{11} = H, x_{12} = H])+ \\
&\quad + \Pr(x_{12} = L|x_{11} = H) \max(u(H, L); \mathbb{E}[u(x_{21}, x_{22})|x_{11} = H, x_{12} = L]).
\end{aligned}$$

The DM chooses m and n to maximize $V_{mn}(x_{11} = H)$. Let $V^*(x_{11} = H)$ to be the maximum value, then the value of choosing x_{11} in the first search step is given by:

$$V_{11} = \Pr(x_{11} = H)V^*(x_{11} = H) + \Pr(x_{11} = L)V^*(x_{11} = L).$$

Therefore, in the first step DM chooses mn to maximize V_{mn}, where,

$$V_{x_{mn}} = \sum_{\omega} \Pr(x_{mn} = \omega)V^*(x_{mn} = \omega).$$

3.2 Sequential search and optimal stopping time

In this section, the number of search steps undertaken by the DM is chosen endogenously. To this end, at each decision node, the DM also has the option to terminate the search and pick the optimal alternative. The terminal nodes in the tree correspond to different search steps. If the DM decides to pick an alternative without search, the payoff is similar to the batch-search with $s = 0$.

3.3 Recursive formulation of the value function

Let A to be the set of all attributes, i.e., $A = \{x_{11}, x_{12}, x_{21}, x_{22}\}$, and $S \subseteq A$ to be the set of attributes not searched by the decision maker so far. Note that s, the number of search steps undertaken by the DM so far is simply $|A \setminus S|$. Define $V_{x_{mn}}(S, I)$ as the value of searching attribute x_{mn} given the set of not searched attributes S and the history of searched attributes I. And let $V_e(S, I)$ to denote the value of terminating the search given S and

I.

$$V_{x_{mn}}(S,I) = \max\Bigg(\Pr(x_{mn} = H|I)V_e(S \setminus \{x_{mn}\}, I \cup \{x_{mn} = H\}) + \\ + \Pr(x_{mn} = L|I)V_e(S \setminus \{x_{mn}\}, I \cup \{x_{mn} = L\}); \\ \Pr(x_{mn} = H|I) \max_{x' \in S \setminus \{x_{mn}\}} (V_{x'}(S \setminus \{x_{mn}\}, I \cup \{x_{mn} = H\}) + \\ + \Pr(x_{mn} = L|I) \max_{x' \in S \setminus \{x_{mn}\}} (V_{x'}(S \setminus \{x_{mn}\}, I \cup \{x_{mn} = L\}) \Bigg),$$

and,

$$V_e(S,I) = \max(\mathbb{E}[u(x_{11}, x_{12}, |A \setminus S|)|I], \mathbb{E}[u(x_{21}, x_{22}, |A \setminus S)|I]).$$

The terminal nodes are reached either after a termination decision or after exhaustively searching all attributes, i.e., $S = \emptyset$. At such nodes all uncertainty is resolved and the agent simply picks the alternative which gives her higher utility.

Consider a node where the DM decides whether to search the last attribute or terminate the search, Therefore, for some $m, n \in \{1,2\}$, $S = \{x_{mn}\}$, and let $\tilde{x}_{m'n'}$ to be the realized and revealed level of the other attributes for $m', n' \in \{1,2\}$, $m' \neq m$ or $n' \neq n$. Without loss of generality let $m = 2, n = 2$, then:

$$V_{x_{mn}}(\{x_{mn}\}, I) = \mathbb{E}\Big[\max(u(x_{11}, x_{12}, |A|), u(x_{21}, x_{22}, |A|)|I \cup \{x_{mn}\} \Big] \\ = \Pr(x_{22} = H|I) \max(u(\tilde{x}_{11}, \tilde{x}_{12}, |A|); u(\tilde{x}_{21}, H, |A|)) + \\ \Pr(x_{22} = L|I) \max(u(\tilde{x}_{11}, \tilde{x}_{12}, |A|); u(\tilde{x}_{21}, L, |A|)).$$

Given the value of all terminal decision nodes in the tree, value of any other node can be computed. Note that there are two classes of decision nodes in this tree. First, those nodes where the DM decides between continuation and termination of search. Second, nodes where the DM picks an attribute to be searched. However, in the recursive formulation of the value function the two decisions happen simultaneously.

3.4 From independent attributes to correlated attributes

The model in this paper allows for correlated attributes. Thus, the decision maker updates her beliefs about the joint distribution after each search step. In this section we compare a model where the attributes are independent draws from a known distribution with a model where the attributes are correlated. Suppose that the first attribute of both alternatives is drawn from distribution F_1 and the second attribute is drawn from distribution F_2. Thus, $\Pr(x_{m1} = H) = p_1$ and $\Pr(x_{m2} = H) = p_2$ for $m \in \{1, 2\}$. The value of terminating the search at the initial node is given by:

$$\mathbb{E}[u(x_{m1}, x_{m2})] = \sum_{\omega,\omega' \in \Omega} \Pr(x_{m1} = \omega, x_{m2} = \omega')u(\omega, \omega')$$

Therefore, the value of no search in this model with no search is the same as the one with correlation. However, if the DM decides to search the attribute x_{mn} for some m and n and terminate the search afterward, the value of searching x_{mn} for $m = 1, n = 1$ is given by:

$$\begin{aligned}
V_{11} &= \mathbb{E}[\max(\mathbb{E}[u(x_{11}, x_{12}))]; \mathbb{E}[u(x_{21}, x_{22})]|x_{11}] \\
&= \Pr(x_{11} = H) \max(\mathbb{E}[u(H, x_{12})]; \mathbb{E}[u(x_{21}, x_{22})])+ \\
&+ \Pr(x_{11} = L) \max(\mathbb{E}[u(L, x_{12})]; \mathbb{E}[u(x_{21}, x_{22})]).
\end{aligned}$$

However, the value of searching x_{11} with correlated attributes as discussed in the batch search with $s = 1$ is given by:

$$\begin{aligned}
&\Pr(x_{11} = H) \max(\mathbb{E}[u(H, x_{12})|x_{11} = H]; \mathbb{E}[u(x_{21}, x_{22})|x_{11} = H])+ \\
&\Pr(x_{11} = L) \max(\mathbb{E}[u(L, x_{12})|x_{11} = L]; \mathbb{E}[u(x_{21}, x_{22})|x_{11} = L]).
\end{aligned}$$

Clearly, the uncorrelated case is derived as a special case of the general model in this paper, where, conditioning on the level of the searched attribute does not alter the distribution with respect to which the expectation is taken.

3.5 Two extreme cases

To better illustrate the trade off between the depth and breadth of search as a result of the joint distribution, in this section I consider two extreme cases:

3.5.1 Perfect correlation of attributes with-in alternatives and no correlation across alternatives

Consider a model with two alternatives and two attributes. Assume that the two attributes of each alternative are perfectly correlated with each other. Therefore, given a positive search cost the DM will never pay to observe more than one attribute of each alternative, i.e., the maximum number of search steps is bounded above by 2. First, I will show the optimal search rule when the attributes are positively and perfectly correlated ,i.e., $\Pr(x_{m1} = \omega | x_{m2} = \omega) = \Pr(x_{m2} = \omega | x_{m1} = \omega) = 1$ for $m \in \{1, 2\}$ and $\omega \in \{H, L\}$. Let $p_1 := \Pr(x_{1n} = H)$ and $p_2 := \Pr(x_{2n} = H)$ and assume $p_1 > p_2$.Then, if the agent decides to reveal the level of one attribute of the alternative m, she will immediately choose the alternative m if the realized level is H and select the other alternative if the level is L. Therefore, the DM will never pay to view more than one attribute.

Lemma 1. *When the attributes are perfectly and positively correlated with-in alternatives and uncorrelated across alternatives, then the order of search does not matter.*

Proof. First, note that since attributes are perfectly correlated, the order of search with-in an alternative is irrelevant. The expected payoff of searching an attribute of the first alternative is given by:

$$p_1 u(H, H, 1) + (1 - p_1)[p_2 u(H, H, 1) + (1 - p_2)u(L, L, 1)], \tag{2}$$

and the expected payoff of searching an attribute of the second alternative is given by:

$$p_2 u(H, H, 1) + (1 - p_2)[p_1 u(H, H, 1) + (1 - p_1)u(L, L, 1)].$$

It immediately follows that the expected payoff of searching an attribute of the first and the second alternatives are equal. As the order of search with-in attributes does not matter, if it is optimal for the DM to search, she can randomly choose any attribute from any alternative. □

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If the DM decides to pick an alternative without any search effort, she will choose the one with higher expected payoff (the first alternative in here) which is given by:

$$p_1 u(H, H, 0) + (1 - p_1)u(L, L, 0). \tag{3}$$

The following proposition fully characterizes the optimal search rule of the DM in this extreme case.

Proposition 1. *In the above model, the DM picks the alternative with a higher expected payoff without any search if*

$$u(H,H,1) - u(L,L,1) < \frac{p_1}{(1-p_1)p_2}\big[u(H,H,0) - u(H,H,1)\big] + \frac{1}{p_2}\big[u(L,L,0) - u(L,L,1)\big],$$

otherwise, she will randomly choose an attribute of any alternative to be searched, and selects that alternative if the level is realized to be H and selects the other alternative if the level is realized to be L.

Proof. By lemma (1) if the DM decides to search the order does not matter, let's say she will search an attribute of the first alternative. The expected value of search is then given by eq.2 and if she picks the alternative with higher expected payoff the value of no search is given by eq.3. Subtracting (3) from (2) and setting it equal zero makes the DM indifferent between search and no search. Thus, when the above inequality holds the DM terminates the search in the initial node. □

Thus, for a given distribution, if the marginal utility of attributes increases, search becomes more attractive, and if the marginal disutility of search increases, termination of search becomes more desirable. Moreover, for a given utility function, if p_1 or p_2 are close enough to zero or one, then the DM terminates the search in the initial step. Since $u(L,H,s) > u(H,L,s)$ under negatively and perfectly correlated attributes, the same argument as above still holds.

3.5.2 Perfect correlation of attributes across alternatives and no correlation with-in alternatives

In contrast with perfect correlation with-in an alternative, as the DM can perfectly observe the level of an attribute in other attributes she will never search attributes of more than one alternative. Therefore, by decreasing the correlation with-in alternatives and at the same time increasing the correlation across alternatives the DM maker tends to increase the breadth of the search. In this version of the model, the DM will not undertake any search as the two alternatives are equivalent.

3.6 Comparative statics

Suppose for a joint distribution function P: $\{L,H\}^4 \mapsto [0,1]$, the optimal search rule for the DM is to search attribute x_{mn} for some $x_{mn} \in A$

and terminate the search and pick the best alternative regardless of the observed level of x_{mn}. This implies $V_{x_{mn}}(A, I) \geq V_e(A, I)$, and $x_{mn} \in \arg\max_{x \in A} V_x(A, I)$. This condition guarantees that the DM decides to (i) continue searching (ii) pick x_{mn} to be searched. After the x_{mn}'s level is observed, the DM needs to decide whether to terminate the search or pick another attribute to be searched, i.e. continue the search optimally. The four possible policies are to:

(i) terminate the search for $\tilde{x}_{mn} = H$ or $\tilde{x}_{mn} = L$

(ii) continue searching for $\tilde{x}_{mn} = H$, terminate the search for $\tilde{x}_{mn} = L$

(iii) continue searching for $\tilde{x}_{mn} = L$, terminate the search for $\tilde{x}_{mn} = H$

(iv) continue searching for $\tilde{x}_{mn} = H$ or $\tilde{x}_{mn} = L$

3.6.1 Increasing the search cost c

Let's increase the fixed search cost from c to $\tilde{c}$, $\tilde{c} > c$ under the two following assumptions.

i) $\tilde{u}(x_{m1}, x_{m2}, s) < u(x_{m1}, x_{m2}, s)$,

ii) $\tilde{u}(x_{m1}, x_{m2}, s) - \tilde{u}(x_{m1}, x_{m2}, s+1) > u(x_{m1}, x_{m2}, s) - u(x_{m1}, x_{m2}, s+1)$,

where, $\tilde{u}$ is the utility function of the DM under the search cost $\tilde{c}$. Thus, for a fixed number of search steps the utility of choosing alternative m is lower under higher search cost. Moreover, the loss of utility from an additional search step is higher under a higher search cost.

Proposition 2. *Consider a decision node where the DM decides between termination and continuation of search and it can be reached with a positive probability. If the optimal action of the DM at this node is continuation of search, then there exists $\bar{c} \geq c$ for which the DM is indifferent between continuation and termination. Further, for any search cost above $\bar{c}$ the optimal action is termination of search, and for any search cost below $\bar{c}$ the optimal action is to continue the search.*

Proof. If the DM chooses to continue searching at this node under cost c then $V_{x_{mn}}(S, I) \geq V_e(S, I)$ for some $m, n \in \{1, 2\}$. Without loss of generality, suppose conditional on termination at this node (not necessarily an optimal decision) the maximum expected payoff is achieved by selecting the first alternative, i.e., $V_e(S, I) = \mathbb{E}[u(x_{11}, x_{12}, |A \setminus S|)|I]$. Increasing the search cost from c to $\tilde{c}$ the decrease in the value of termination at this node is bounded above by $\psi \equiv \mathbb{E}\left[u(x_{11}, x_{12}, |A \setminus S|) - \tilde{u}(x_{11}, x_{12}, |A \setminus S|)\middle| I\right] < 0$.

The value of continuing the search (given that the x_{mn} is the optimal decision at this node) is a linear combination of value of each terminal node that can be reached with a positive probability after searching the attribute x_{mn}. Consider the terminal nodes that are reached after a termination decision, the value of these nodes is of the form $V_e(S \setminus W, I \cup I_c)$ where $W \neq \emptyset$ is the set of searched attributes before the termination decision, and I_c is the set that contains the history of realizations of attributes $x \in W$. The decrease in the value of such terminal nodes is given by $\psi' \equiv \mathbb{E}\left[u(x_{11}, x_{12}, |A \setminus (S \setminus W)|) - \tilde{u}(x_{11}, x_{12}, |A \setminus (S \setminus W)|)\middle| I \cup I_c\right]$. Since $x_{mn} \in W$ then $|A \setminus (S \setminus W)| > |A \setminus S|$ and by assumption (ii) it is easy to show $\psi > \psi'$, i.e., the value of each of such terminal nodes is decreased more than the termination before inspecting x_{mn}. Similarly, the value of terminal nodes which are achieved after exhaustively searching all the attributes is also decreased more than the value of termination at the starting node. Therefore, for some $c' > c$, $V_{x_{mn}}(S, I) < V_e(S, I)$. □

Proposition 3. *Consider a Termination-Continuation(TC) decision node n_1 and another TC node n_2 that is a successor of n_1. Let $\bar{c}(n)$ denote the $\bar{c}$ at node n as defined in proposition(2). If the optimal decision at both nodes is to continue the search, then $c(n_1) > c(n_2)$.*

Proof. Let $x \in S$ to be the chosen attribute at n_1 and $x' \in S \setminus \{x\}$ to be the chosen attribute at n_2. Since, the optimal decision is to continue the search at both nodes, then,

$$V_x(S, I) > V_e(S, I),$$

and,

$$V_{x'}(S \setminus D, I') > V_e(S \setminus D, I'),$$

where, $D \subset S$ is the set of searched attributes after the node n_1 such that $x \in D$ and $I' \subset I$ is the updated history such that either $x = H \in I'$ or $x = L \in I'$. Also, given the search decision at n_1 the optimal choice of the DM was to search x while x' was also available, the value of searching

x' at n_1 under S and I should be less than or equal to value of searching alternative x, i.e.,

$$V_{x'}(S, I) \leq V_x(S, I),$$

As the DM continues the search the difference between the value of continuing the search and termination of search decreases. The convexity of disutility of search implies that as the search cost increases the value of search at n_2 decreases at a faster rate than the one at n_1 therefore, the threshold search cost at n_2 is smaller than n_1 $\square$

3.6.2 Decreasing the marginal utility of the attribute n

For a given joint distribution, suppose that the marginal utility of the attribute n is decreased, e.g.,for $n = 1$:

$$\upsilon_1(\tilde{u}) \equiv \tilde{u}(H, \omega, s) - \tilde{u}(L, \omega, s) < u(H, \omega, s) - u(L, \omega, s).$$

Let's further assume $V_x(A) < V_e(A)$ for all $x \in A \setminus \{x_{mn}\}$, i.e., value of searching any other attribute rather than x_{mn} is less than the value of terminating the search.

Conjecture 1. *Suppose at a decision node, searching x_{m1} is the optimal decision, then there exists a utility function $\tilde{u}$ such that $\upsilon_2(\tilde{u}) = \upsilon_2(u)$ and $\upsilon_1(\tilde{u}) < \upsilon_1(u)$ for which x_{m1} is not the optimal decision, i.e., the optimal decision is to search another attribute or terminate the search.*

3.7 Informativeness and Marginal Utility of an Attribute

In the 2 attribute model let's assume the first attribute of each alternative is the most informative. This means that upon knowing the level of the first attribute, the DM knows the true level of the second attribute with a high probability, say $1 - \epsilon$. So, $\Pr(x_{m2} = \omega | x_{m1} = \omega') = 1 - \epsilon$ for some $\epsilon > 0$ and $\omega, \omega' \in \{H, L\}$. To further focus on the trade off between information and marginal utility contribution, let's assume $\Pr(x_{m1} = \omega | x_{m2} = \omega') < 1 - \epsilon$ for all $\omega, \omega' \in \Omega$. This simply says the DM cannot ascertain the level of the first attribute via searching the second attribute equally well.

If there exists no correlation between the attributes of the first and the second alternative, then:

$$\Pr(x_{21}, x_{22} | x_{11}, x_{12}) = \Pr(x_{21}, x_{22}).$$

At the initial decision node, the DM needs to decide whether to start searching an attribute or terminate the search and take the status-quo expected utility maximizing alternative. The condition for a positive search effort by the DM is given by:

$$\max_{m,n \in \{1,2\}} V_{mn}(A, \emptyset) > V_e(A, \emptyset),$$

where, V_e is given by:

$$V_e \equiv \max(\mathbb{E}[u(x_{11}, x_{12}, 0)], \mathbb{E}[u(x_{21}, x_{22}, 0)])$$

Conjecture 2. *Consider a model where the attributes are uncorrelated across alternative and suppose at a TC decision node the DM's optimal decision is to search the attribute* x_{m2} *while* $x_{m1} \in S$ *and* $\Pr(x_{m2} = \omega | x_{m1} = \omega') = 1 - \epsilon$ *for some* $\epsilon > 0$ *and* $\omega, \omega' \in \{H, L\}$*. There exists an* $\epsilon > 0$ *for which the DM optimally searches* x_{m1} *instead of* x_{m2}*.*

4 Conclusion

This paper departs from the recent works on multi-attribute search problems by relaxing the assumption that the attributes are independently distributes. Further, no specific functional form is assumed for the utility function, however, few assumptions on the marginal utility of attributes and dis-utility of search are needed for main results of the paper.

First, I looked at the batch search case where the DM is endowed with a number of search coupons which gives her the possibility of searching an attribute for free. In this version of the problem, the DM does not need to decide on the optimal stopping time as the number of search steps are given exogenously. Hence, the optimal order of search is the main focus of this problem.

After, a multi-attribute search problem with endogenous stopping rule is introduced. There are two classes of decision nodes in this tree, the one where she decides whether to continue or terminate the search and the one where upon a continuation she decides which attribute to be inspected. Moreover, a recursive formulation of the value function at each node as a function of the realized history is given. Equally important, two extreme cases of the joint distribution, namely, perfect correlation of attributes across and with-in alternatives are discussed. The condition for optimality of a search decision is given as a function of primitives of the model.

Finally, the last section provides the reader with a comparative statics analysis of a continuation-termination decision. It is shown that at each TC decision node there exists a threshold search cost for which the DM chooses to terminate the search. Moreover, these reservation costs are decreasing for successor nodes.

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