TIGHT LOCAL GRAPH FOURIER FRAMES WITH FINITE SUPPORT

Philipp Reingruber and Gerald Matz

Institute of Telecommunications, TU Wien
Email: firstname.lastname@tuwien.ac.at

***Abstract*—The application of graph signal processing (GSP) methods to large-scale real-world problems has often been hampered by a high computational complexity. We therefore address the problem of designing a computationally efficient and intuitively meaningful graph signal transformation that can serve as a workhorse for a variety of GSP tasks and as an alternative to the classical graph Fourier transform. To that end, we introduce the concept of local graph Fourier frames (LGFFs), i.e., redundant bases that build on graph Fourier transforms restricted to subgraphs. We provide theoretical results regarding the existence of tight (Parseval) LGFFs, discuss simple designs that entail LGFFs with finite support, study desirable properties of LGFF, and discuss an application example in the context of graph signal denoising.**

***Index Terms*—graph signal processing; graph Fourier transform; local graph Fourier transform**

I. Introduction

During the last decade, we have witnessed a surge in interest and activity in the area of graph signal processing (GSP) [1, 2]. GSP has been found to provide avenues to deal with data embedded in irregular domains that can be captured in terms of combinatorial graphs. However, except for algorithms that exploit the sparsity of the adjacency or Laplacian matrix, many GSP methods are restricted to small to medium datasets and their practical application has been impeded by excessive computational complexity. This restriction is specifically true for graph signal expansions and filter banks that build on spectral graph theory and the graph Fourier transform (GFT) [3–8]. Approaches that aim for lower complexity by using graph partitions are described in [9, 10].

In this paper, we tackle the complexity problem by introducing local graph Fourier frames (LGFFs). LGFFs are overcomplete bases that build on graph partitions and local windowed GFTs and have finite support in the vertex domain. The latter enables substantial speedups compared to existing approaches. We provide conditions that ensure that an LGFF is tight, and we provide guidelines for the design of LGFFs. LGFFs are flexible and can be adapted to an extensive range of graph signal classes. In addition, they are norm-preserving and are shown experimentally to have excellent localization in the graph Fourier domain. We illustrate the practical applicability of LGFFs is illustrated with a graph signal denoising experiment.

Our paper is organized as follows. In Section II, we discuss the limitations of related work and review aspects of frame theory. In Section III, we introduce LGFFs along with constructive design procedures. Properties of LGFFs are discussed in Section IV. Section V showcases an exemplary application of LGFFs. Section VI concludes the paper.

II. Background

A. Related Work

Attempts to design meaningful graph signal transformations include spectral graph wavelets [3], graph filter banks [4], and spectrally localized graph frames [5–8]. All of these approaches use constructions in the graph frequency domain because the graph frequency λ provides a continuous, regularly ordered domain comparable to time and frequency in conventional signal processing. However, the resulting basis functions and transformations lack specific structure or sparsity and hence are computationally expensive (with a complexity usually scaling quadratically with the number of nodes). This can be attributed to the fact that (i) the GFT, in general, doesn't feature any symmetries similar to those of the FFT and (ii) the basis functions, in general, are supported on the full vertex set.

Methods that build on graph partitions and are related to our ideas have been discussed in [9, 10]. However, these papers take the perspective of filter banks [9] or approximation of the global GFT [10] and do not use any windows or overlapping subgraphs. As a consequence, they lack the flexibility that allows for oversampling which is crucial for spectral localization of the basis functions.

B. Frames

A frame [11] for $\mathbb{R}^N$ with frame constants $0 < A \le B < \infty$ is a set $\mathcal{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_M\}$ of basis vectors $\mathbf{f}_k \in \mathbb{R}^N$, $k = 1, \dots, M$, such that for any $\mathbf{x} \in \mathbb{R}^N$ the following double inequality is satisfied:

$$A\|\mathbf{x}\|^2 \le \sum_{k=1}^{M} \left|\mathbf{f}_k^{\mathrm{T}}\mathbf{x}\right|^2 \le B\|\mathbf{x}\|^2.$$

A and B are called frame bounds. The frame condition necessarily requires $M \ge N$. The frame is complete if $M = N$ and overcomplete/redundant if $M > N$. If $\mathcal{F}$ is a frame, there is a so-called dual frame $\{\tilde{\mathbf{f}}_1, \dots, \tilde{\mathbf{f}}_M\}$ such that any $\mathbf{x} \in \mathbb{R}^N$ has the basis representation $\mathbf{x} = \sum_{k=1}^{M} c_k \mathbf{f}_k$ with $c_k = \tilde{\mathbf{f}}_k{}^{\mathrm{T}}\mathbf{x}$. If the frame inequalities are satisfied with $A = B$, the frame is called tight. We are specifically interested in Parseval frames, i.e., tight frames with $A = B = 1$. For Parseval frames, the dual frame equals the frame itself, $\tilde{\mathbf{f}}_k = \mathbf{f}_k$, and we have the simple analysis/synthesis-relation

$$\mathbf{x} = \sum_{k=1}^{M} c_k \, \mathbf{f}_k \quad \text{with} \quad c_k = \mathbf{f}_k^{\mathrm{T}}\mathbf{x}.$$

This is equivalent to saying that $\mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{I}$, where $\mathbf{F} = (\mathbf{f}_1 \dots \mathbf{f}_M) \in \mathbb{R}^{N \times M}$.

III. Local Graph Fourier Frames

A. Basic Idea

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, \ldots, N\}$, edge set $\mathcal{E}$, and weighted adjacency matrix $\mathbf{W}$. Let $\mathbf{g}_k = (g_k[1], \ldots, g_k[N])^{\mathrm{T}}$, $k = 1, \ldots, K$, be a set of window functions on $\mathcal{G}$ and denote their vertex support by

$$\mathcal{S}_k \triangleq \{n\colon\ g_k[n] \neq 0\}.$$

Furthermore, define the indicator functions $\boldsymbol{\gamma}_k$ by

$$\gamma_k[n] = \begin{cases} 1, & n \in \mathcal{S}_k, \\ 0, & n \notin \mathcal{S}_k. \end{cases}$$

The sets $\mathcal{S}_k$ are not necessarily disjoint, i.e., the window functions $\mathbf{g}_k$ may overlap. Furthermore, let $\mathcal{G}_k$ be the subgraph induced by the vertex set $\mathcal{S}_k$. The number of vertices in $\mathcal{G}_k$ is $N_k \triangleq |\mathcal{S}_k|$. The graph Fourier basis for $\mathcal{G}_k$ is given by the eigenvectors $\mathbf{u}_{kl}$, $l = 1, \ldots, N_k$, of the associated Laplacian $\mathbf{L}_k$. We arrange these eigenvectors into the orthonormal local GFT matrix $\mathbf{U}_k = (\mathbf{u}_{k,1} \ldots \mathbf{u}_{k,N_k}) \in \mathbb{R}^{N_k \times N_k}$. With these definitions, we define the windowed local graph Fourier basis vectors ($\odot$ denotes the element-wise vector product)

$$\mathbf{f}_{kl} \triangleq \mathbf{g}_k \odot \mathbf{S}_k \mathbf{u}_{kl}, \qquad k = 1, \ldots, K,\ \ l = 1, \ldots, N_k,$$

where the matrices $\mathbf{S}_k \in \{0, 1\}^{N \times N_k}$ map the length-N_k eigenvectors $\mathbf{u}_{kl}$ on the subgraph $\mathcal{G}_k$ to length-N signals on the full graph $\mathcal{G}$ by zero-padding the vertices $\mathcal{G} \backslash \mathcal{G}_k$. Equivalently, we can express the basis vectors in terms of their elements as $f_{kl}[n] = g_k[n] u_{kl}[n]$, with $u_{kl}[n]$ being the zero-padded versions of the eigenfunctions $\mathbf{u}_{kl}$. We arrange the vectors $\mathbf{f}_{kl}$, $k = 1, \ldots, K$, $l = 1, \ldots, N_k$, into the matrix

$$\begin{aligned} \mathbf{F} &= (\mathbf{f}_{11}, \ldots, \mathbf{f}_{1N_1}, \mathbf{f}_{21}, \ldots, \mathbf{f}_{2N-2}, \ldots, \mathbf{f}_{K1} \ldots \mathbf{f}_{KN_K}) \\ &= (\mathrm{Diag}(\mathbf{g}_1)\mathbf{S}_1\mathbf{U}_1 \ldots \mathrm{Diag}(\mathbf{g}_K)\mathbf{S}_K\mathbf{U}_K). \end{aligned} \tag{1}$$

Here, $\mathrm{Diag}(\mathbf{g})$ denotes a diagonal matrix whose main diagonal is given by $\mathbf{g}$. Note that $\mathbf{F} \in \mathbb{R}^{N \times M}$ with $M \triangleq \sum_{k=1}^{K} N_k$. The oversampling factor η is defined as the ratio $\eta \triangleq M/N$. The central theoretical result of this paper is the following.

Theorem 1. $\mathbf{F}$ *is a Parseval frame (i.e.,* $\mathbf{F}\mathbf{F}^T = \mathbf{I}$*) if and only if the window functions* $\mathbf{g}_k$ *satisfy*

$$\sum_{k=1}^{K} \mathbf{g}_k \odot \mathbf{g}_k = \mathbf{1} \tag{2}$$

(equivalently, $\sum_{k=1}^{K} g_k^2[n] = 1$*).*

Proof. Using (1) we have

$$\begin{aligned} \mathbf{F}\mathbf{F}^T &= \begin{bmatrix} \mathbf{U}_1^T\mathbf{S}_1^T\mathrm{Diag}(\mathbf{g}_1) \\ \vdots \\ \mathbf{U}_K^T\mathbf{S}_K^T\mathrm{Diag}(\mathbf{g}_K) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{U}_1^T\mathbf{S}_1^T\mathrm{Diag}(\mathbf{g}_1) \\ \vdots \\ \mathbf{U}_K^T\mathbf{S}_K^T\mathrm{Diag}(\mathbf{g}_K) \end{bmatrix} \\ &= \sum_{k=1}^{K} \mathrm{Diag}(\mathbf{g}_k)\mathbf{S}_k\mathbf{U}_k\mathbf{U}_k^T\mathbf{S}_k^T\mathrm{Diag}(\mathbf{g}_k) \\ &= \sum_{k=1}^{K} \mathrm{Diag}(\mathbf{g}_k)\mathbf{S}_k\mathbf{S}_k^T\mathrm{Diag}(\mathbf{g}_k), \end{aligned}$$

where we exploited the orthonormality of the local GFT matrices ($\mathbf{U}_k{\mathbf{U}_k}^{\mathrm{T}} = \mathbf{I}$). We next note that $\mathbf{S}_k\mathbf{S}_k^T = \mathrm{Diag}(\boldsymbol{\gamma}_k)$ and $\mathrm{Diag}(\boldsymbol{\gamma}_k)\mathrm{Diag}(\mathbf{g}_k) = \mathrm{Diag}(\mathbf{g}_k)$ and hence

$$\begin{aligned} \mathbf{F}\mathbf{F}^T &= \sum_{k=1}^{K} \mathrm{Diag}(\mathbf{g}_k)\mathrm{Diag}(\boldsymbol{\gamma}_k)\mathrm{Diag}(\mathbf{g}_k) \\ &= \sum_{k=1}^{K} \mathrm{Diag}^2(\mathbf{g}_k) = \mathrm{Diag}\left(\sum_{k=1}^{K} \mathbf{g}_k \odot \mathbf{g}_k\right). \end{aligned}$$

It follows that $\mathbf{F}\mathbf{F}^T = \mathbf{I}$ if and only if $\sum_{k=1}^{K} \mathbf{g}_k \odot \mathbf{g}_k = \mathbf{1}$. □

A necessary condition for $\sum_{k=1}^{K} \mathbf{g}_k \odot \mathbf{g}_k \equiv \mathbf{1}$ is $\eta \geq 1$. For $\eta < 1$, there will be gaps because the windows cannot cover all vertices ($\bigcup_{k=1}^{K} \mathcal{S}_k \subset \mathcal{V}$). For $\eta = 1$ (critical case), constant windows with disjoint support are the only choice, in which case we recover the bases from [9, 10]; these windows have abrupt transitions and are not smooth. Smooth windows require a certain overlap and hence $\eta > 1$ (overcomplete case). The construction of the basis $\mathbf{F}$ is similar in spirit to local Fourier bases [12] and lapped orthogonal transforms [13] known from harmonic analysis and conventional signal processing. Therefore, we refer to $\mathbf{F}$ as *local graph Fourier frame* (LGFF). Furthermore, note that the global GFT is a special case of our LGFF obtained with $K = 1$ and $\mathbf{g}_1 = \mathbf{1}$.

B. Analysis and Synthesis

For a Parseval LGFF $\mathbf{F}$, the dual frame is given by $\mathbf{F}$ itself. This results in the analysis-synthesis relations

$$\xi_{kl} = \mathbf{f}_{kl}^{\mathrm{T}}\mathbf{x}, \qquad \mathbf{x} = \sum_{k=1}^{K}\sum_{l=1}^{N_k} \xi_{kl}\,\mathbf{f}_{kl}.$$

The coefficients ξ_{kl} can be computed as

$$\xi_{kl} = \mathbf{f}_{kl}^{\mathrm{T}}\mathbf{x} = \mathbf{u}_{kl}^T\mathbf{S}_k^T\mathrm{Diag}(\mathbf{g}_k)\mathbf{x} = \sum_{n \in \mathcal{S}_k} u_{kl}[n] g_k[n] x[n],$$

which amounts to a windowed local GFT, i.e., a vertex-domain windowing (pointwise multiplication with $\mathbf{g}_k$, complexity $\mathcal{O}(N_k)$) followed by a local GFT (inner products with $\mathbf{u}_{kl}$, $l = 1, \ldots, N_k$, complexity $\mathcal{O}(N_k^2)$).

Similarly, the synthesis relation amounts to an inverse local GFT followed by a windowed overlap-add stage,

$$\begin{aligned} \mathbf{x} = \sum_{k=1}^{K}\sum_{l=1}^{N_k} \xi_{kl}\,\mathbf{f}_{kl} &= \sum_{k=1}^{K}\sum_{l=1}^{N_k} \xi_{kl}\,\mathrm{Diag}(\mathbf{g}_k)\mathbf{S}_k\mathbf{u}_{kl} \\ &= \sum_{k=1}^{K} \mathrm{Diag}(\mathbf{g}_k)\mathbf{S}_k \sum_{l=1}^{N_k} \xi_{kl}\mathbf{u}_{kl}. \end{aligned}$$

The inverse local GFT (sum over l) has complexity $\mathcal{O}(N_k^2)$ and the subsequent multiplication with the window has complexity $\mathcal{O}(N_k)$. For each vertex n, the overlap-add (sum over k) involves contributions only from those subgraphs $\mathcal{G}_k$ for which $n \in \mathcal{S}_k$.

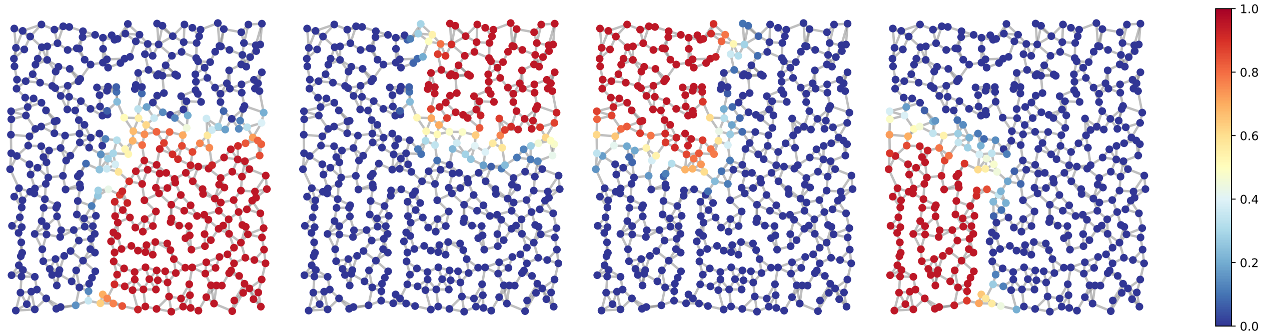

Fig. 1: *LGFF windows* $\mathbf{g}_k$*,* $k = 1, \ldots, 4$*, for a random geometric graph with* $N = 484$ *nodes designed with random walk diffusion filter of order* $I = 3$ *(color indicates signal value). The vertex support size of the windows is* $N_1 = 228$*,* $N_2 = 143$*,* $N_3 = 146$*,* $N_4 = 147$*, which entails* $M = 664$ *and an oversampling factor* $\eta = 1.37$*.*

C. Window Design

We next describe a procedure for designing windows $\mathbf{g}_k$ that satisfy the frame constraint (2). We start with a partition of the vertex set $\mathcal{V}$ into disjoint subsets, $\mathcal{V} = \bigcup_{k=1}^{K} \mathcal{V}_k$, $\mathcal{V}_k \cap \mathcal{V}_{k'} = \emptyset$ for $k \neq k'$. Such a partition can be obtained e.g. by suitable (hierarchical) graph clustering methods [10, 14, 15]. The indicator functions $\boldsymbol{\chi}_k$ of the sets $\mathcal{V}_k$, defined by

$$\chi_k[n] = \begin{cases} 1 & n \in \mathcal{V}_k, \\ 0 & n \notin \mathcal{V}_k, \end{cases}$$

satisfy $\sum_{k=1}^{K} \boldsymbol{\chi}_k = \mathbf{1}$. Next, consider a graph filter $\mathbf{G}$ that has nonnegative filter coefficients and satisfies $\mathbf{G1} = \mathbf{1}$. Define the windows $\mathbf{g}_k$ by $\mathbf{g}_k \odot \mathbf{g}_k = \mathbf{G}\boldsymbol{\chi}_k$, i.e.,

$$g_k[n] = \sqrt{\sum_{m \in \mathcal{V}_k} G[n, m]}.$$

It follows that

$$\sum_{k=1}^{K} \mathbf{g}_k \odot \mathbf{g}_k = \sum_{k=1}^{K} \mathbf{G}\boldsymbol{\chi}_k = \mathbf{G} \sum_{k=1}^{K} \boldsymbol{\chi}_k = \mathbf{G1} = \mathbf{1}$$

and hence these windows induce a Parseval LGFF. A computationally efficient choice for $\mathbf{G}$ is given by diffusion filters obtained by repeated application of the random walk weighted adjacency matrix $\widetilde{\mathbf{W}} = \mathbf{D}^{-1}\mathbf{W}$ (here, $\mathbf{D} = \mathrm{Diag}(\mathbf{W1})$ is the diagonal degree matrix), i.e.,

$$\mathbf{G} = \sum_{i=0}^{I} \eta_i \widetilde{\mathbf{W}}^i,$$

with coefficients η_i chosen such that $\sum_{i=0}^{I} \eta_i = 1$. Since $\widetilde{\mathbf{W}}\mathbf{1} = \mathbf{1}$, it follows that $\widetilde{\mathbf{W}}^i\mathbf{1} = \mathbf{1}$, and hence in turn $\mathbf{G1} = \mathbf{1}$. With this construction, the windows $\mathbf{g}_k$ have a smooth roll-off that extends into the I-hop neighborhood of $\mathcal{V}_k$, i.e.,

$$\mathcal{S}_k = \mathcal{N}^I(\mathcal{V}_k) \triangleq \{n\colon\ d(n, m) \leq I \text{ for some } m \in \mathcal{V}_k\},$$

with $d(n, m)$ the distance between nodes n and m. An example of this window design is depicted in Fig. 1.

IV. LGFF PROPERTIES

Isometry. The simple analysis/synthesis-relation of Parseval frames implies that the transformation is energy-preserving, i.e.

$$\|\boldsymbol{\xi}\|_2 = \|\mathbf{F}^T\mathbf{x}\|_2 = \|\mathbf{x}\|_2 \qquad \text{for all } \mathbf{x} \in \mathbb{R}^N.$$

This enables a direct comparison of the coefficient vectors in the two domains.

Computational Efficiency. Due to the finite support of the LGFF vectors, the analysis and the synthesis relation can both be implemented with $\mathcal{O}(\sum_k N_k^2)$ operations, which can be substantially smaller than the $\mathcal{O}(N^2)$ complexity of the global GFT (or any other unstructured/non-sparse transformation). Fig. 2 shows an example of the computational speedup of the LGFF transformation relative to the GFT. In this example, we consider random geometric graphs with average degree 3, a fixed average size of $\frac{1}{K}\sum_{k=1}^{K} |\mathcal{V}_k| = 128$ for the partitions $\mathcal{V}_k$, and an increasing number of subgraphs K (such that the graph size is $N = 128K$). After partitioning the graph using hierarchical spectral clustering, we designed the LGFF windows $\mathbf{g}_k$ using the diffusion filter $\mathbf{G} = \frac{2}{3}\mathbf{I} + \frac{1}{3}\widetilde{\mathbf{W}}$ (thus, $I = 1$). We then compared the time T_{GFT} required for obtaining the GFT coefficients to the time T_{LGFF} required for computing the the LGFF coefficients and averaged the resulting speedup $T_{\mathrm{GFT}}/T_{\mathrm{LGFF}}$ over 10 graph realizations. Fig. 2 suggests that the LGFF transform is roughly K times faster than the GFT.

Assuming homogenous partitions with $\mathrm{E}\{N_k\} = N_0$ and $\mathrm{var}\{N_k\} = v_0^2$ such that $\mathrm{E}\{N_k^2\} = N_0^2 + v_0^2$, the reciprocal LGFF speedup on average equals (note that $KN_0 = M = \eta N$)

$$\mathrm{E}\left\{\frac{\sum_k N_k^2}{N^2}\right\} = \frac{K(N_0^2 + v_0^2)}{N^2} = \frac{1}{K}\,\eta^2\left(1 + \frac{v_o^2}{N_0^2}\right).$$

Spectral Localization. By construction, our LGFF vector have finite support in the vertex. It turns out that with suitably designed smooth windows (requiring $\eta > 1$), the LGFF vectors are also well-localized in the frequency domain. We support

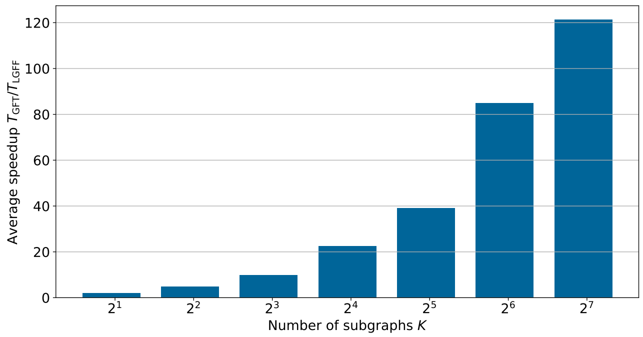

Fig. 2: *Average speedup $T_{\mathrm{GFT}}/T_{\mathrm{LGFF}}$ for different K.*

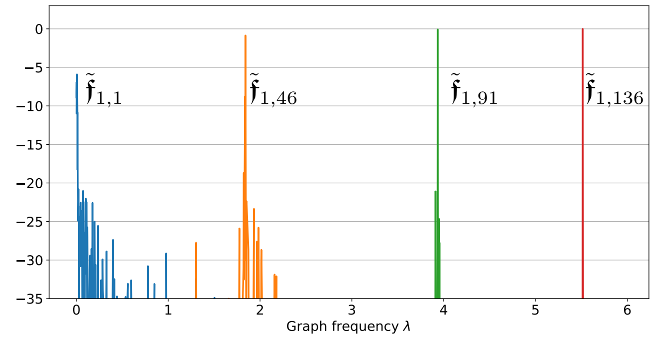

Fig. 3: *Normalized GFT $\tilde{\mathbf{f}}_{1l}$ (magnitude in dB) for LGFF vectors with $k = 1$ and $l \in \{1, 46, 91, 136\}$ (random geometric graph, $N = 1024$, $K = 8$, $N_1 = 136$, $I = 1$).*

this claim via numerical experiments. To this end, let

$$\tilde{\mathbf{f}}_{kl} = \mathbf{U}^T \mathbf{f}_{kl}/\|\mathbf{f}_{kl}\|$$

denote the GFT of (the normalized) LGFF vectors (here, $\mathbf{U}$ is the GFT matrix, i.e., the eigenvector matrix of the global Laplacian). Fig. 3 depicts four examples for these frequency-domain LGFF vectors for a random geometric graph with $N = 1024$ and $K = 8$ using the same LGFF design as described above. It is seen that these vectors are highly localized, thus suggesting that the LGFF vectors have narrow bandwidth. To corroborate this observation, we use the r.m.s. bandwidth of a graph signal $\mathbf{x}$, defined as $B = \sqrt{\sum_l (\lambda_l - \bar{\lambda})^2 \mathfrak{x}_l^2/\|\mathbf{x}\|}$, where $\mathfrak{x} = \mathbf{U}^T\mathbf{x} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_N)^{\mathrm{T}}$ is the GFT of $\mathbf{x}$ and $\bar{\lambda} = \sum_l \lambda_l \mathfrak{x}_l^2/\|\mathbf{x}\|$ is the center frequency of $\mathbf{x}$.

Fig. 4(a) plots the center frequency $\bar{\lambda}$ and the r.m.s. band edges $\bar{\lambda} \pm B$ for the $N_1 = 130$ vectors $\mathbf{f}_{1l}$ from an LGFF designed in the same way as before. With an average bandwidth of $\bar{B} = 0.21$, these basis vectors have substantially better spectral localization than the critically sampled untapered ($\mathbf{g}_k = \mathbf{1}$) local GFT with an average bandwidth of $\bar{B} = 0.33$ (see Fig. 4(b)).

V. Application

Since the LGFF captures local spectral properties of graph signals in a norm-preserving fashion, it is potentially advantageous for many GSP tasks. This holds true especially for large graphs with a meaningful partition into many subgraphs of moderate size.

We briefly discuss graph signal denoising as an exemplary use case. Here, the goal is to recover a desired signal $\mathbf{s}$ from a noisy observation $\mathbf{x} = \mathbf{s} + \mathbf{e}$. We assume that $\mathbf{e}$ is i.i.d. and perform denoising by hard-thresholding the LGFF coefficients [16, 17]. More specifically, our denoised signal is given by $\hat{\mathbf{s}} = \sum_{kl} \hat{\xi}_{kl}\mathbf{f}_{kl}$ with

$$\hat{\xi}_{kl} = \begin{cases} 0, & |\xi_{kl}| < \lambda/\|\mathbf{f}_{kl}\|, \\ \xi_{kl}, & \text{else.} \end{cases}$$

Here, $\xi_{kl} = \mathbf{f}_{kl}^{\mathrm{T}}\mathbf{x}$. We used a random geometric graph with $N = 484$ nodes and average node degree 3, $K = 4$ subgraphs, and $\mathbf{G} = \frac{5}{6}\mathbf{I} + \frac{1}{6}\widetilde{\mathbf{W}}$ ($I = 1$). The desired signal was nonstationary and generated according to $\mathbf{s} = \widetilde{\mathbf{F}}\widetilde{\boldsymbol{\xi}}$ where $\widetilde{\mathbf{F}}$ is an untapered LGFF ($\mathbf{g}_k = \mathbf{1}$), and uncorrelated uniformly distributed coefficients $\widetilde{\xi}_{kl}$ whose power decayed exponentially with the local frequency index l (local bandwidth was 10 for all subgraphs). Fig. 5 illustrates the results that correspond to an SNR improvement of $5.88\,\mathrm{dB}$ (from $-2\,\mathrm{dB}$ to $3.88\,\mathrm{dB}$).

VI. Conclusion

We introduced LGFFs that constitute Parseval frames with basis vectors of finite support in the vertex domain. LGFF can be constructed intuitively based on graph partitions and diffusion filters, thus admitting smooth windows and various oversampling factors. The finite vertex-domain support of the LGFF vectors amounts to a sparsity structure that substantially reduces computation and storage requirements. The LGFF vectors furthermore are well-localized in the frequency domain and well-suited for locally stationary graph processes. This flexibility with regard to the signal model in conjunction with the computational efficiency bears huge potential for various GSP applications.

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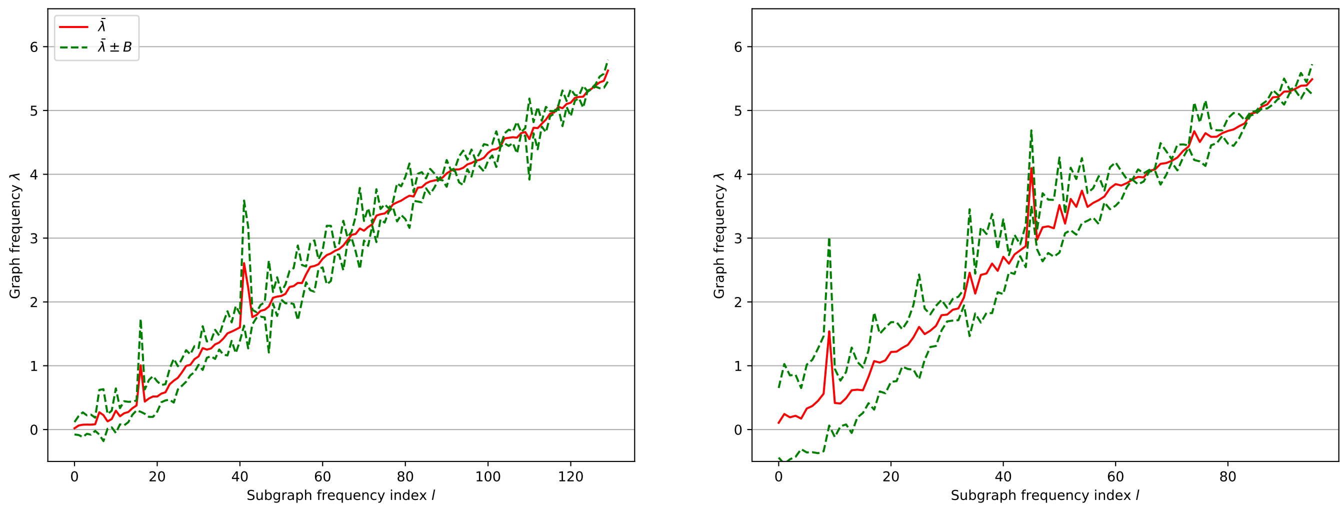

Fig. 4: *Center frequency $\bar{\lambda}$ (red solid line) and r.m.s. band edges $\bar{\lambda} \pm B$ (green dashed lines) within subgraph $\mathcal{G}_1$ for (a) an oversampled LGFF with smooth windows ($I = 1$) and (b) a critically sampled LGFF with non-smooth windows $\mathbf{g}_k = \mathbf{1}$.*

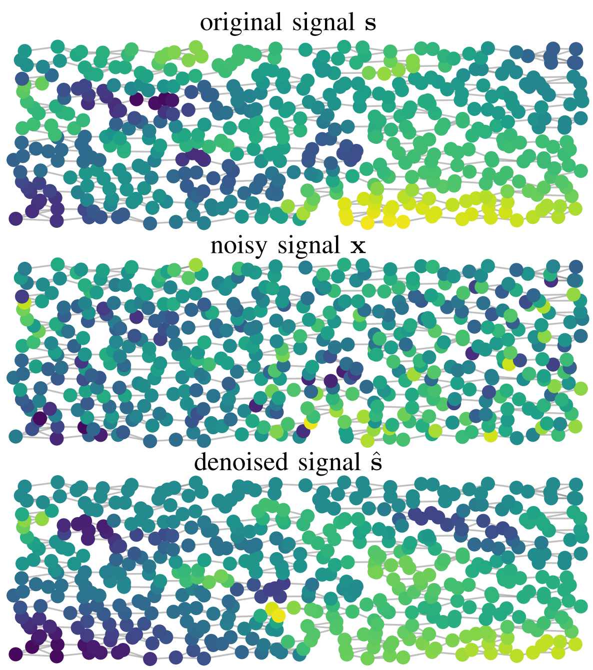

Fig. 5: *Example of the denoising of a nonstationary graph signal (random geometric graph, $N = 484$, $K = 4$, $I = 1$).*

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