DIPLOMARBEIT

Optimal adaptive FEM with iterative solver driven by non-residual estimators

ausgeführt am

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unter der Anleitung von

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[redacted]

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Kurzfassung

Das Ziel jedes numerischen Verfahrens für partielle Differentialgleichungen ist die Berechnung einer Näherungslösung mit einer vorgeschriebenen Genauigkeit bei minimaler Rechenzeit. Zu diesem Zweck umfasst die adaptive Finite-Elemente-Methode (AFEM) neben einer schätzerbasierten lokalen Netzverfeinerung einen inexakten Löser mit einem ausgeklügelten Abbruchkriterium, um die verschiedenen Fehlerkomponenten auszugleichen.

Zur *a posteriori* Fehlerschätzung des Diskretisierungsfehlers setzt die *state-of-the-art* Analysis für AFEM mit inexaktem Löser auf den Residualschätzer. Dieser erfüllt die sogenannten *axioms of adaptivity* aus [Carstensen, Feischl, Page, Praetorius: Comput. Math. Appl. 67, 2014]. Das Ziel dieser Arbeit ist es, die aktuelle Analysis von [Bringmann, Feischl, Miraçi, Praetorius, Streitberger: Comput. Math. Appl. 180, 2025] auf adaptive Algorithmen mit inexaktem Löser zu erweitern, die durch nicht-residualbasierte Fehlerschätzer gesteuert werden. Dies wird durch die Tatsache motiviert, dass es viele andere Fehlerschätzer mit wünschenswerten praktischen und numerischen Eigenschaften gibt. Basierend auf einer Idee von [Kreuzer, Siebert: Numer. Math. 117, 2011], die AFEM für nicht-residualbasierte Schätzer, aber mit exaktem Löser analysieren, werden in dieser Arbeit Fehlerschätzer betrachtet, die zwar nicht die *axioms of adaptivity* erfüllen, aber in einem gewissen Sinne lokal äquivalent zum Residualschätzer sind.

Im abstrakten Rahmen der *axioms of adaptivity* betrachten wir allgemeine lineare elliptische PDEs zweiter Ordnung. Das Hauptresultat ist der Beweis der parameter-unabhängigen vollen R-linearen Konvergenz von AFEM mit inexaktem Löser, die durch einen lokal äquivalenten Schätzer gesteuert wird, d.h. Kontraktion eines geeigneten Quasi-Fehlers in jedem Schritt des Algorithmus unabhängig von den vom Benutzer gewählten Parametern. Dies verifiziert die unbedingte Konvergenz des adaptiven Algorithmus und erlaubt es, in einem weiteren Schritt die optimale Komplexität des adaptiven Algorithmus zu zeigen, d.h. optimale Konvergenzraten bezüglich der kumulierten Rechenzeit. Zudem zeigt die Arbeit, dass der ZZ-Schätzer von [Zienkiewicz, Zhu: Int. J. Numer. Methods Eng. 24, 1987] und Schätzer basierend auf Fluss-Equilibrierung (siehe z.B. [Ern, Vohralík: SIAM J. Numer. Anal. 53, 2015] und die dort zitierten Arbeiten) lokal äquivalent zum Residualschätzer sind, wodurch die optimale Komplexität für AFEM, die durch die diese Schätzer gesteuert wird, bewiesen wird. Die Arbeit schließt mit numerischen Experimenten, die die theoretischen Resultate bestätigen und eine praktische Anwendung anderer Fehlerschätzer in AFEM aufzeigen.

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Abstract

The ultimate goal of any numerical scheme for partial differential equations (PDEs) is to compute an approximation of user-prescribed accuracy at minimal computational cost. To this end, the adaptive finite element method (AFEM) employs an estimator-steered local mesh-refinement strategy and an inexact solver with a cleverly designed stopping criterion to balance the different error components.

The state-of-the-art analysis for AFEM with inexact solver hinges on the standard residual-based estimator for *a posteriori* error estimation of the discretization error. This estimator satisfies the so-called *axioms of adaptivity* from [Carstensen, Feischl, Page, Praetorius: Comput. Math. Appl. 67, 2014]. The goal of this thesis is to extend the current analysis from [Bringmann, Feischl, Miraçi, Praetorius, Streitberger: Comput. Math. Appl. 180, 2025] to adaptive algorithms with inexact solver steered by non-residual error estimators. This is motivated by the fact that there are many other error estimators with desirable practical and numerical properties. Based on an idea of [Kreuzer, Siebert: Numer. Math. 117, 2011], that considers AFEM for non-residual-based estimators yet exact solver, we consider estimators that do not satisfy the axioms of adaptivity directly, but are locally equivalent to the residual-based estimator in a certain sense.

In the abstract framework of the axioms of adaptivity, we consider general second-order linear elliptic PDEs in this thesis. Our main contribution is proving parameter-robust full R-linear convergence of AFEM with inexact solver steered by a locally equivalent estimator, i.e., contraction of a suitable quasi-error in every step of the algorithm independently of the user-chosen parameters. This proves unconditional convergence of the adaptive algorithm and allows to show optimal complexity, i.e., optimal convergence rates with respect to the total computational time. Moreover, the thesis shows that the ZZ-estimator from [Zienkiewicz, Zhu: Int. J. Numer. Methods Eng. 24, 1987] and the equilibrated flux estimator (see, e.g., [Ern, Vohralík: SIAM J. Numer. Anal. 53, 2015] and the references therein) are locally equivalent to the residual-based estimator, thus proving optimal complexity for AFEM steered by these estimators. The thesis closes with numerical experiments that support the theoretical results and demonstrate the practical application of other error estimators in AFEM.

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe. Hilfsmittel der künstlichen Intelligenz wurden nur verwendet, um sprachliche Formulierungen und Zeichensetzung Korrektur zu lesen und gegebenenfalls zu verbessern.

Wien, am 15. Mai 2025

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Contents

1 Introduction

Partial differential equations (PDEs) are used to model complex real-world phenomena across various scientific disciplines such as physics, engineering, finance, and biology. Since most PDEs do not have closed-form solutions, a variety of numerical methods has been developed yielding computable approximations instead. One such numerical method is the *finite element method (FEM)*, which approximates the solution of a PDE by discretizing the function space to which the solution belongs. First, the domain is meshed into a finite set $\mathcal{T}_H$ of elements. For second-order elliptic PDEs with solution in $H^1(\Omega)$, the approximate solution is usually computed as a $\mathcal{T}_H$-piecewise polynomial function defined on each element while ensuring inter-element continuity. The accuracy of this approximation depends on the quality of the mesh and the order of the polynomial degree chosen for the discretization, with finer meshes and higher-order polynomials generally producing more accurate solutions.

However, many PDEs exhibit singularities or sharp gradients that locally require very fine meshes to be resolved accurately. In such cases, using uniformly fine meshes is computationally expensive and often unnecessary. The *adaptive finite element method (AFEM)* is a technique that aims to reduce the computational cost of numerically solving PDEs by adaptively refining the mesh only in regions where the solution has low accuracy. Given an initial mesh and polynomial degree, the standard AFEM algorithm can be described by modules `SOLVE`, `ESTIMATE`, `MARK`, and `REFINE`, illustrated in Figure 1.

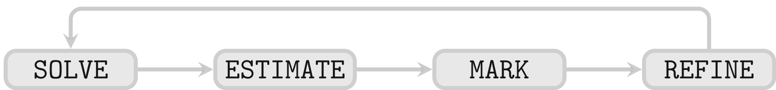


Figure 1: Schematic for standard AFEM.

The `SOLVE` module computes the FEM solution for the given mesh and polynomial degree. This is then used by the `ESTIMATE` module to compute *a posteriori* error estimates used as refinement indicators for each element of the mesh. The `MARK` module then marks elements for refinement, where the estimated error is large. Finally, the `REFINE` module produces a finer mesh by refining (at least) the marked elements. The loop is repeated until a stopping criterion is met, which in real-world applications is usually determined by the available computational resources or a user-prescribed mandatory accuracy.

Over the past three decades, the convergence theory of AFEM has matured significantly. The groundwork was laid by [Dör96], who proved *plain convergence* of AFEM, i.e., convergence of the *a posteriori* error estimator, for the 2D Poisson problem under certain assumptions, most notably on the `MARK` module. The marking strategy introduced by [Dör96] is known as *Dörfler marking* and used by most AFEM algorithms. In the subsequent years, plain convergence of AFEM was shown for more general problems under weaker assumptions; see, e.g., [MNS00; MSV08]. The notion of *optimal convergence rates* for AFEM was first put into a rigorous mathematical framework by [BDD04], but the proof therein required an additional coarsening step that was later proven unnecessary by [Ste07]. In both works, optimal convergence rates for AFEM with respect to the number of *degrees of freedom*, i.e., the dimension of the discrete finite element space, were shown for the 2D Poisson problem under considerably strong constraints on the refinement strategy. This result was generalized by [CKNS08] to general second-order symmetric linear elliptic PDEs in arbitrary dimensions and standard refinement by *newest vertex bisection*. The extension to general (nonsymmetric) linear elliptic PDEs was achieved in [CN12] for a sufficiently fine initial mesh and in [FFP14] without any further assumptions. Later, [CFPP14] introduced an axiomatic approach to the convergence theory of AFEM, which provides a unified framework for proving optimal convergence rates with respect to the number of degrees of freedom for a wide range of PDEs and adaptive algorithms.

However, due to the incremental nature of AFEM, optimal converge rates should rather be considered with respect to the overall computational cost (and, in practice, overall computational time) than the number of degrees of freedom. Beyond the 1D case, optimality in this sense, usually referred to as *optimal complexity*, can only be achieved for *AFEM with inexact solvers*: Instead of solving the FEM problem exactly in the `SOLVE` module, such algorithms employ an *iterative solver* and a cleverly designed stopping criterion in order to balance the *solver error*, resulting from the iterative solver, against the *discretization error*, resulting from a locally too coarse mesh and under-resolved singularity. In simple terms, this enables the algorithm to save computational time in the `SOLVE` module to not unnecessarily iterate the solver if the discretization error dominates. Since this requires an alternating computation of solver-steps and error estimates, the `SOLVE` and `ESTIMATE` modules should be considered as a single module `SOLVE & ESTIMATE`, as illustrated in Figure 2.

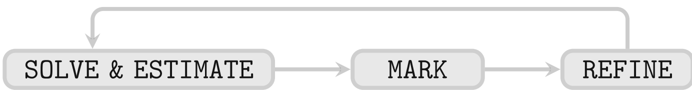


Figure 2: Schematic for AFEM with inexact solver.

Optimal complexity of AFEM with inexact solver was already shown for certain model problems in [Ste07; CG12] under the assumption that the iterative solution of the solver is sufficiently close to the (unavailable) exact solution. This algorithmic restriction was removed in [GHPS21], who also showed that *full R-linear convergence*, i.e., contraction of a suitable quasi-error in every step of the algorithm, is the key to proving optimal complexity. Recently, [BFM$^+$25] proposed a novel proof of full R-linear convergence that, unlike [GHPS21], does not rely on the Pythagorean identity and thus extends to nonsymmetric problems by employing the *generalized quasi-orthogonality* from [Fei22].

The state-of-the-art analysis of [GHPS21; BFM$^+$25] requires that the error estimator steering the adaptive algorithm satisfies the so-called *axioms of adaptivity* from [CFPP14]. While the standard *residual-based error estimator* fits into this framework, there are many other error estimators with desirable practical and numerical properties that do not satisfy the axioms of adaptivity. Examples include *recovery-based estimators*, often referred to as *ZZ-estimators* (see, e.g., [ZZ87]), or *estimators based on flux equilibration* (see, e.g., [EV15] and the references therein).

The paper [KS11] proposed the idea to consider estimators that satisfy certain *local equivalence* properties with respect to the residual-based estimator. Under the restriction that the Galerkin solution is computed exactly (and hence excluding inexact solvers), they showed that AFEM steered by such estimators can achieve optimal convergence rates (with respect to the number of degrees of freedom), but their analysis is only concerned with the Poisson problem. Later, [CFPP14] extended this result to the axiomatic framework, which allows for a wider range of problems and a broader notion of local equivalence. However, optimal complexity of AFEM with inexact solver steered by non-residual-based estimators remained an open question.

The goal of this thesis is to address this question by extending the state-of-the-art analysis of [BFM$^+$25] to AFEM with inexact solvers steered by non-residual-based error estimators. As in [KS11; CFPP14], the considered error estimators do not satisfy the axioms of adaptivity (in particular, the *reduction axiom* (A2)), but are in a some sense locally equivalent to the residual-based estimator. The main contribution of this thesis is the proof of full R-linear convergence independently of the user-chosen adaptivity parameters, thus showing unconditional convergence of the adaptive algorithm. This then allows to show optimal complexity of AFEM steered by such locally equivalent estimators with respect to the total computational time. As potential application, the thesis shows that, for general second-order linear elliptic PDEs in the setting of the Lax-Milgram lemma, the ZZ-estimator and the equilibrated flux estimator fit into the framework, thus proving optimal complexity for AFEM steered by the these estimators.

The thesis is structured as follows: In Chapter 2, we present the AFEM algorithm with iterative contractive solver and the underlying mathematical assumptions. In particular, we show how to derive a contractive solver for *nonsymmetric* problems from contractive solvers for *symmetric* problems by means of the Zarantonello iteration [Zar60]; see Section 2.4.2. Moreover, we introduce two notions of local equivalence: one where the considered estimator is equivalent to the residual-based estimator for all discrete functions, and a weaker one where the estimator is only equivalent for the exact Galerkin solution; see Section 2.3. In Chapter 3, we show that AFEM steered by an equivalent estimator guarantees a perturbed version of the so-called *estimator reduction* from [CKNS08, Corollary 3.4] for a modified residual-based estimator that uses the *generalized mesh-size function* from [CFPP14, Proposition 8.6]. This is the key to proving unconditional full R-linear convergence (Theorem 3.11 and Theorem 3.14). In Chapter 4, we employ the full R-linear convergence to show optimal complexity of the adaptive algorithm steered by an equivalent estimator (Theorem 4.3) provided that, as usual in this context, the adaptivity parameters are sufficiently small. Afterwards, we discuss the ZZ-estimator and the equilibrated flux estimator in Chapter 5, where we show that these estimators are locally equivalent to the residual-based estimator. This allows us to conclude optimal complexity for AFEM steered by these estimators (Corollary 5.7 and Corollary 5.19). Finally, in Chapter 6, we present numerical experiments that support the analysis of the preceding chapters.

2 Adaptive FEM with contractive solver

The algorithm presented in this chapter follows the general structure of the adaptive loop shown in Figure 2. We begin by introducing the abstract model problem in Section 2.1. Following this, we discuss our assumptions on the submodules `REFINE`, `ESTIMATE`, `SOLVE` and `MARK` in that specific order, which results from their interdependencies. Finally, we formulate the algorithm itself in Section 2.6.

2.1 Abstract model problem

Throughout this chapter, $\mathcal{X}$ is a real Hilbert space with norm $\|\cdot\|_{\mathcal{X}}$ and scalar product $\langle\cdot,\cdot\rangle_{\mathcal{X}}$. Let $a : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a bounded and elliptic bilinear form, i.e., there exist constants $C_{\mathrm{ell}}, C_{\mathrm{bnd}} > 0$ such that

$$a(u,u) \geq C_{\mathrm{ell}} \|u\|_{\mathcal{X}} \quad \text{and} \quad a(u,v) \leq C_{\mathrm{bnd}} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} \quad \text{for all } u,v \in \mathcal{X}. \tag{2.1}$$

The symmetric part of $a(\cdot,\cdot)$ defined by

$$b(u,v) := \frac{a(u,v)+a(v,u)}{2} \quad \text{for all } u,v \in \mathcal{X} \tag{2.2}$$

is a symmetric, bounded, and elliptic bilinear form on $\mathcal{X}$. In particular, $b(\cdot,\cdot)$ is a scalar product on $\mathcal{X}$ and the induced norm $b(\cdot,\cdot)^{1/2}$ is an equivalent norm on $\mathcal{X}$. Since $a(u,u) = b(u,u)$ for all $u \in \mathcal{X}$, this norm is indeed the *energy norm* $|||\cdot||| := a(\cdot,\cdot)^{1/2}$. For a bounded linear functional $F : \mathcal{X} \to \mathbb{R}$, we seek the solution $u^\star \in \mathcal{X}$ to the variational problem

$$a(u^\star, v) = F(v) \quad \text{for all } v \in \mathcal{X}. \tag{2.3}$$

The existence and uniqueness of the solution $u^\star$ to (2.3) is guaranteed by the Lax-Milgram theorem [Eva98, Section 6.2.1].

Assume that Ω is a bounded polyhedral Lipschitz domain in $\mathbb{R}^d$ with $d \geq 1$, i.e., Ω is open and connected and, for $d \geq 2$, the boundary $\partial\Omega$ of Ω is locally the graph of a piecewise affine and Lipschitz continuous function. Given a symmetric diffusion tensor $\boldsymbol{A} \in [L^\infty(\Omega)]^{d\times d}_{\mathrm{sym}}$, a convection coefficient $\boldsymbol{b} \in [L^\infty(\Omega)]^d$, a reaction coefficient $c \in L^\infty(\Omega)$, and data $\boldsymbol{f} \in [L^2(\Omega)]^d$ and $f \in L^2(\Omega)$, a possible application would be the nonsymmetric second-order linear elliptic PDE

$$-\operatorname{div}(\boldsymbol{A}\nabla u^\star) + \boldsymbol{b}\cdot\nabla u^\star + cu^\star = f - \operatorname{div}(\boldsymbol{f}) \text{ in } \Omega \subseteq \mathbb{R}^d \quad \text{subject to} \quad u^\star = 0 \text{ on } \partial\Omega. \tag{2.4}$$

We use the notation for Sobolev spaces from [Eva98, Chapter 5] and write $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ for the usual $L^2(\Omega)$-scalar product. Multiplying (2.4) with a test-function $v \in H^1_0(\Omega)$, integrating over Ω, and performing integration by parts, we obtain (2.3) with space $\mathcal{X} := H^1_0(\Omega)$, bilinear form $a(u,v) := \langle \boldsymbol{A}\nabla u, \nabla v\rangle_{L^2(\Omega)} + \langle \boldsymbol{b}\cdot\nabla u + cu, v\rangle_{L^2(\Omega)}$, and right-hand side functional $F(v) := \langle f, v\rangle_{L^2(\Omega)} + \langle \boldsymbol{f}, \nabla v\rangle_{L^2(\Omega)}$. We suppose that $\boldsymbol{A}$, $\boldsymbol{b}$ and c guarantee that $a(\cdot,\cdot)$ is bounded and elliptic on $H^1_0(\Omega)$ in order to fit into the setting of (2.3). This is for instance satisfied if $\operatorname{div}\boldsymbol{b} \in L^\infty(\Omega)$ and $-\frac{1}{2}\operatorname{div}\boldsymbol{b} + c \geq 0$ almost everywhere in Ω.

The Lax-Milgram theorem also applies to any closed subspace $\mathcal{X}_H \subset \mathcal{X}$ and guarantees the existence and uniqueness of $u^\star_H \in \mathcal{X}_H$ solving

$$a(u^\star_H, v_H) = F(v_H) \quad \text{for all } v_H \in \mathcal{X}_H. \tag{2.5}$$

The so-called *Galerkin method* considers the problem in (2.5) restricted to finite-dimensional subspaces $\mathcal{X}_H \subset \mathcal{X}$. In this case, the unique solution $u^\star_H \in \mathcal{X}_H$ to (2.5) is called *Galerkin solution*. The subtraction of (2.5) from (2.3) results in the *Galerkin orthogonality*

$$a(u^\star - u^\star_H, v_H) = 0 \quad \text{for all } v_H \in \mathcal{X}_H. \tag{2.6}$$

If $a(\cdot,\cdot)$ is additionally symmetric and thus a scalar product on $H^1_0(\Omega)$, the Galerkin orthogonality (2.6) implies the Pythagorean identity

$$|||u^\star - v_H|||^2 = |||u^\star - u^\star_H|||^2 + |||u^\star_H - v_H|||^2 \quad \text{for all } v_H \in \mathcal{X}_H. \tag{2.7}$$

The following proposition from [Fei22] provides a generalization of the Pythagorean identity (2.7) that also applies to nonsymmetric problems.

Proposition 2.1 (quasi-orthogonality). *Under the assumptions (2.1), there exist constants $C_{\rm orth} > 0$ and $0 < \delta < 1$ such that the following property holds for any sequence $(\mathcal{X}_\ell)_{\ell\in\mathbb{N}_0}$ of nested finite-dimensional subspaces $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset \mathcal{X} = H^1_0(\Omega)$:*

- **(QO)** ***quasi-orthogonality:*** *The corresponding Galerkin solutions $u^\star_\ell \in \mathcal{X}_\ell$ to (2.5) satisfy*

$$\sum_{\ell'=\ell}^{\ell+N} |||u^\star_{\ell'+1} - u^\star_{\ell'}|||^2 \leq C_{\rm orth}(N+1)^{1-\delta}|||u^\star - u^\star_\ell|||^2 \quad \textit{for all } \ell, N \in \mathbb{N}_0.$$

Here, $C_{\rm orth}$ and δ depend only on the dimension d, the elliptic bilinear form $a(\cdot,\cdot)$, and the chosen norm $|||\cdot|||$, but are independent of the spaces $\mathcal{X}_\ell$ and $F \in \mathcal{X}'$. □

Finally, we want to mention the following well-known result, which states that the *Galerkin error* $|||u^\star - u^\star_H|||$ is quasi-optimal, i.e., it behaves like the best approximation error up to a multiplicative constant.

Lemma 2.2 (Céa). *With the constants $C_{\rm ell}, C_{\rm bnd} > 0$ from (2.1) and $C_{\text{Céa}} := C_{\rm bnd}/C_{\rm ell}$, it holds*

$$|||u^\star - u^\star_H||| \leq C_{\text{Céa}} \min_{v_H\in\mathcal{X}_H} |||u^\star - v_H|||, \tag{2.8}$$

i.e., the Galerkin error is quasi-optimal. □

The proof of the Céa lemma can be found in any introductory finite-element textbook, e.g., [EG21b, Lemma 26.13].

2.2 Refinement

Before we can discuss the refinement step, we need to introduce the notion of simplices and triangulations.

Definition 2.3 (simplex). Let $d \geq 2$. For $1 \leq k \leq d$, a subset $T \subset \mathbb{R}^d$ is called a (compact) *k-dimensional simplex* if there exist vertices $z_0, \dots z_k \in \mathbb{R}^d$ such that $T = \operatorname{conv}\{z_0, \dots, z_k\}$, i.e., T is the convex hull of $z_0, \dots, z_k$. Its *set of vertices* is denoted by $\mathcal{V}(T) := \{z_0, \dots, z_k\}$. The k-dimensional simplex is *non-degenerate* if $\{z_1 - z_0, \dots, z_k - z_0\}$ is linearly independent and hence the k-dimensional measure is positive. We say that $T \subseteq \mathbb{R}^d$ is a (compact) *simplex*, if T is a d-dimensional simplex. A k'-dimensional simplex T' is a subsimplex of a k-dimensional simplex T, if $\mathcal{V}(T') \subseteq \mathcal{V}(T)$. The 1-dimensional subsimplices of a simplex T are called *edges*, whereas the $(d-1)$-dimensional subsimplices of T are called *faces*. We denote the set of faces of T with $\mathcal{E}(T)$.

Definition 2.4 (conforming triangulation). Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral Lipschitz domain. A finite set $\mathcal{T}_H$ is a conforming (simplicial) triangulation of Ω if and only if

- every $T \in \mathcal{T}_H$ is a non-degenerate simplex,
- the closure of Ω is covered by $\mathcal{T}_H$, i.e., $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_H} T$,
- and the intersection of all pairwise different simplices $T, T' \in \mathcal{T}_H$ is either empty or a joint k-dimensional subsimplex of T and T' with $1 \leq k \leq d-1$.

We denote the set of vertices of $\mathcal{T}_H$ by $\mathcal{V}_H := \{\mathcal{V}(T) : T \in \mathcal{T}_H\}$ and the set of faces by $\mathcal{E}_H := \{\mathcal{E}(T) : T \in \mathcal{T}_H\}$. Moreover, we write $\mathcal{E}_H^\Omega$ for the faces which lie inside Ω, i.e., for $E \in \mathcal{E}_H^\Omega$, there exist $T, T' \in \mathcal{T}_H$ with $E = T \cap T' \in \mathcal{E}_H$.

Definition 2.5 (uniform shape regularity). Let $T \subset \mathbb{R}^d$ be a simplex. The shape regularity constant $\sigma(T)$ involves the diameter

$$\operatorname{diam}(T) := \max\{|x - y| : x, y \in T\}$$

and reads

$$\sigma(T) := \frac{\operatorname{diam}(T)}{|T|^{1/d}}.$$

Since the volume of T is clearly smaller than the volume of the d-dimensional cube with side length $\operatorname{diam}(T)$, we have $|T| \leq \operatorname{diam}(T)^d$, which implies $\sigma(T) \geq 1$. We say that a triangulation $\mathcal{T}_H$ is *σ-shape regular* if

$$\sigma(\mathcal{T}_H) := \max_{T \in \mathcal{T}_H} \sigma(T) \leq \sigma < \infty. \tag{2.9}$$

Given a triangulation $\mathcal{T}_H$ and a set $\mathcal{M}_H \subseteq \mathcal{T}_H$ of marked elements, the fixed refinement strategy $\texttt{refine}(\cdot, \cdot)$ generates a new triangulation $\mathcal{T}_h := \texttt{refine}(\mathcal{T}_H, \mathcal{M}_H)$ such that

- at least all marked elements $\mathcal{M}_H \subseteq \mathcal{T}_H$ are refined, i.e., there holds $\mathcal{M}_H \subseteq \mathcal{T}_h \setminus \mathcal{T}_H$,

- and each refined element $T \in \mathcal{T}_H \setminus \mathcal{T}_h$ is the union of its children, i.e., there holds $T = \bigcup\{T' \in \mathcal{T}_h : T' \subseteq T\}$.

We write $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ if $\mathcal{T}_h$ can be obtained from $\mathcal{T}_H$ by finitely many applications of `refine`. Let $\mathcal{T}_0$ be a conforming initial triangulation of Ω and define $\mathbb{T} := \mathbb{T}(\mathcal{T}_0)$ as the set of all meshes that can be generated from the initial mesh $\mathcal{T}_0$ by use of `refine`. We suppose that `refine` preserves conformity in the sense of Definition 2.4 and ensures uniform shape regularity, i.e., there exists a constant $\sigma \geq 1$ depending only on the initial triangulation $\mathcal{T}_0$ such that (2.9) holds for all $\mathcal{T}_H \in \mathbb{T}$. Finally, we assume that `refine` satisfies the following properties, which are sufficient to prove optimality (cf. [CFPP14]).

(R1) child estimate: There exists a constant $C_{\text{child}} \geq 1$, such that for all $\mathcal{T}_H \in \mathbb{T}$ and all $\emptyset \neq \mathcal{M}_H \subseteq \mathcal{T}_H$ it holds

$$\mathcal{T}_h = \texttt{refine}(\mathcal{T}_H, \mathcal{M}_H) \quad \Longrightarrow \quad \#\mathcal{T}_H < \#\mathcal{T}_h \leq C_{\text{child}}\, \#\mathcal{T}_H.$$

(R2) overlay estimate: For all meshes $\mathcal{T}_H, \mathcal{T}_h \in \mathbb{T}$, there exists a coarsest common refinement $\mathcal{T}_H \oplus \mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H) \cap \mathbb{T}(\mathcal{T}_h)$ such that

$$\#(\mathcal{T}_H \oplus \mathcal{T}_h) \leq \#\mathcal{T}_H + \#\mathcal{T}_h - \#\mathcal{T}_0.$$

(R3) closure estimate: There exists a constant $C_{\text{closure}} \geq 1$ depending only on $\mathcal{T}_0$ and `refine` such that for any sequence $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$ of successive refinements of $\mathcal{T}_0$, i.e., $\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ for all $\ell \in \mathbb{N}_0$ with appropriate $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$, it holds

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \leq C_{\text{closure}} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j.$$

We suppose that, for any $\mathcal{T}_H \in \mathbb{T}$ and any set of marked elements $\mathcal{M}_H \subseteq \mathcal{T}_H$, the computation of $\mathcal{T}_h = \texttt{refine}(\mathcal{T}_H, \mathcal{M}_H)$ can be accomplished at linear cost $\mathcal{O}(\#\mathcal{T}_H)$ (cf. [BDD04; Ste07]). In particular, the child estimate (R1) guarantees that this is possible. For instance, the refinement strategy *newest vertex bisection* (NVB) satisfies (R1)–(R3) (cf. [AFF⁺15] for $d = 1$, [KPP13] for $d = 2$, and [DGS25] for $d \geq 2$). In the following, we illustrate the procedure in the two-dimensional case.

Example 2.6 (newest vertex bisection). In the 2D NVB algorithm, the refinement of a triangulation $\mathcal{T}_\ell \in \mathbb{T}$ is generated by repeated bisection of its triangles $T \in \mathcal{T}_\ell$. A triangle is bisected by introducing a new edge between the midpoint of the so-called *refinement edge* and its opposite vertex. For the initial conforming triangulation $\mathcal{T}_0$, each triangle $T \in \mathcal{T}_0$ is assigned a refinement edge. After each bisection, the edges opposite of the newly created vertex become the refinement edges of the resulting children triangles, explaining the name of the algorithm. For a triangulation $\mathcal{T}_\ell$ with marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$, the refinement process proceeds as follows.

Algorithm A (2D newest vertex bisection).
Input: *Triangulation $\mathcal{T}_\ell$ and set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$.*
(i) *For all $T \in \mathcal{M}_\ell$, mark its refinement edge.*

(ii) ***Repeat recursively***
If a triangle $T \in \mathcal{T}_\ell$ has a marked edge that is not the refinement edge, also mark the refinement edge of T.
until
No triangle $T \in \mathcal{T}_\ell$ with marked edges has an unmarked refinement edge.
(iii) *For all $T \in \mathcal{T}_\ell$, refine according to the pattern from Figure 3.*

Output: *Refined triangulation $\mathcal{T}_{\ell+1}$.*

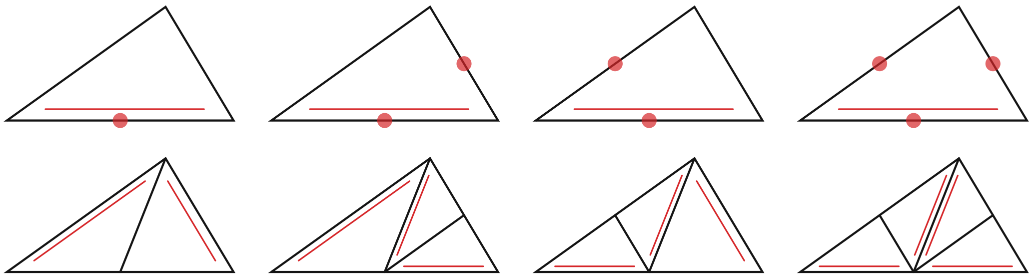


Figure 3: Visualization of the refinement pattern of 2D newest vertex bisection in 2D. Marked edges are indicated by red dots and refinement edges by red lines. The upper triangles display the configuration prior to refinement, while the lower triangles show the corresponding refined elements.

The application of NVB as refinement routine ensures that all resulting triangulations are conforming and uniformly shape-regular. The algorithm can also be formulated for the general case in $\mathbb{R}^d$ with $d \geq 2$, but it becomes more complicated. For more details, we refer to [Ste08; KPP13; DGS25; Mau95; Tra97; Mit91; Sew72].

With every triangulation $\mathcal{T}_H$, we associate a finite-dimensional subspace $\mathcal{X}_H \subset \mathcal{X}$. We require nestedness of the discrete spaces $\mathcal{X}_H$:

(N) nestedness of discrete spaces: Nestedness of meshes $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ implies nestedness $\mathcal{X}_H \subseteq \mathcal{X}_h$ of the corresponding discrete spaces.

For the PDE problem (2.4), a common choice for $\mathcal{X}_H$ would be the space of globally continuous and piecewise polynomials of total degree $p \in \mathbb{N}$, i.e.,

$$\mathcal{S}_0^p(\mathcal{T}_H) := \{v_H \in H_0^1(\Omega) \,:\, v_H|_T \text{ is a polynomial of degree} \leq p \text{ for all } T \in \mathcal{T}_H\}. \tag{2.10}$$

In particular, the choice $\mathcal{X}_H := \mathcal{S}_0^p(\mathcal{T}_H)$ guarantees the nestedness (N) of $\mathcal{X}_H$. Moreover, this ensures that the discrete spaces $\mathcal{X}_\ell$ associated to any sequence $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$ of successive refinements of $\mathcal{T}_0$ satisfy the assumptions of quasi-orthogonality (QO).

Remark 2.7 (discrete objects). The mesh-level index H historically refers to meshes with a uniform mesh-size $H \in \mathbb{R}$. In this thesis, however, it denotes all discrete objects, including those related to highly adapted meshes with non-uniform mesh-sizes. In that case, the mesh-size is a function of the triangulation $\mathcal{T}_H$ denoted by H, i.e., the size of an element $T \in \mathcal{T}_H$ is given by $H(T)$. Related discrete objects share the same index, e.g., v_H is a discrete function in the space $\mathcal{X}_H$ corresponding to the triangulation $\mathcal{T}_H$. Many estimates, especially those used in the later convergence analysis, should not depend on the mesh-level H, which is why we introduce the following notation: For two expressions A and B, we write $A \lesssim B$ if there holds $A \le C\,B$ for a constant $C > 0$ that is independent of H. In case that both $A \lesssim B$ and $B \lesssim A$ hold, we write $A \simeq B$.

2.3 Error estimators

For every mesh $\mathcal{T}_H \in \mathbb{T}$, suppose that we can compute refinement indicators

$$\mu_H(T, v_H) \quad \text{for all } T \in \mathcal{T}_H \text{ and all } v_H \in \mathcal{X}_H. \tag{2.11}$$

For each $T \in \mathcal{T}_H$, the indicator $\mu_H(T, u_H^\star)$ should, at least heuristically, measure the *discretization error* $|||u^\star - u_H^\star|||$ locally on the element T. To abbreviate the notation, we define

$$\mu_H(\mathcal{U}_H, v_H)^2 \coloneqq \sum_{T \in \mathcal{U}_H} \mu_H(T, v_H)^2 \quad \text{for any } \mathcal{U}_H \subseteq \mathcal{T}_H \tag{2.12}$$

and set $\mu_H(v_H) \coloneqq \mu_H(\mathcal{T}_H, v_H)$ as well as $\mu_H \coloneqq \mu_H(u_H^\star)$. We suppose that, for all $\mathcal{T}_H \in \mathbb{T}$, all $T \in \mathcal{T}_H$, and all $v_H \in \mathcal{X}_H$, the refinement indicators $\mu_H(T, v_H)$ can be computed in constant time $\mathcal{O}(1)$, i.e., the computation of $\mu_H(v_H)$ has linear complexity $\mathcal{O}(\#\mathcal{T}_H)$. A possible choice is the so-called *residual-based error estimator*.

Example 2.8 (residual-based error estimator). In addition to the assumptions in Section 2.1, suppose that $\boldsymbol{A}|_T \in \left[W^{1,\infty}(T)\right]^{d\times d}$ and $\boldsymbol{f}|_T \in \left[H^1(T)\right]^d$ for all $T \in \mathcal{T}_0$. For all $\mathcal{T}_H \in \mathbb{T}$ and all $T \in \mathcal{T}_H$, we define the corresponding *mesh-size function* $H : \mathcal{T}_H \to \mathbb{R}_{>0}$ by $H(T) \coloneqq |T|^{1/d}$. Furthermore, for neighboring simplices $T, T' \in \mathcal{T}_H$ with joint face $E \coloneqq T \cap T' \in \mathcal{E}_H^\Omega$ and corresponding normal vectors $\boldsymbol{n}_T|_E$ and $\boldsymbol{n}_{T'}|_E$, we define the *normal jump* of $v_H \in \mathcal{X}_H$ across the face E by

$$[\![\nabla v_H \cdot \boldsymbol{n}]\!]|_E \coloneqq (\nabla v_H|_T)|_E \cdot \boldsymbol{n}_T|_E + (\nabla v_H|_{T'})|_E \cdot \boldsymbol{n}_{T'}|_E. \tag{2.13}$$

Then, for all triangulations $\mathcal{T}_H \in \mathbb{T}$, all $T \in \mathcal{T}_H$, and all $v_H \in \mathcal{X}_H$, the refinement indicators

$$\begin{aligned}
\eta_H(T, v_H)^2 &= H(T)^2\, \| -\operatorname{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) + \boldsymbol{b}\cdot\nabla v_H + c\,v_H - f\|_{L^2(T)}^2 \\
&\qquad + H(T)\, \|[\![(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\cdot\boldsymbol{n}]\!]\|_{L^2(\partial T\cap\Omega)}^2 \\
&= |T|^{2/d}\, \| -\operatorname{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) + \boldsymbol{b}\cdot\nabla v_H + c\,v_H - f\|_{L^2(T)}^2 \\
&\qquad + |T|^{1/d}\, \|[\![(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\cdot\boldsymbol{n}]\!]\|_{L^2(\partial T\cap\Omega)}^2
\end{aligned} \tag{2.14}$$

are well-defined, since $v_H \in C^\infty(T)$ for all $T \in \mathcal{T}_H$. The error estimator η_H is known as *residual-based error estimator*. The term

$$\| -\operatorname{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) + \boldsymbol{b}\cdot\nabla v_H + c\,v_H - f\|_{L^2(T)}^2 \eqqcolon \|R_H(v_H)\|_{L^2(T)}^2 \tag{2.15}$$

is the so-called *local volume residual*, since it measures the residual associated with problem (2.4) on an element $T \in \mathcal{T}_H$. The term $\|[\![(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\cdot\boldsymbol{n}]\!]\|_{L^2(\partial T\cap\Omega)}^2 \eqqcolon \|J_H(v_H)\|_{L^2(T)}^2$

is usually referred to as *jump term* or *consistency error* and arises from piecewise integration by parts of the residual in the weak formulation (2.3). The residual-based error estimator can also be formulated with other mesh-size functions. For example, a potentially more natural choice than the mesh-size function $H(T)$ from above would be $\widehat{H}(T) := \mathrm{diam}(T)$. Correspondingly, we define $\widehat{\eta}_H$ as the error estimator that is obtained by replacing $|T|^{1/d}$ by $\mathrm{diam}(T)$ in (2.14), i.e., for all triangulations $\mathcal{T}_H \in \mathbb{T}$, all $T \in \mathcal{T}_H$, and all $v_H \in \mathcal{X}_H$, it holds

$$\begin{aligned}\widehat{\eta}_H(T, v_H)^2 = \mathrm{diam}(T)^2 \, \| -\mathrm{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) + \boldsymbol{b}\cdot\nabla v_H + c\, v_H - f\|^2_{L^2(T)} \\ + \mathrm{diam}(T)\, \|[\![(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\cdot \boldsymbol{n}]\!]\|^2_{L^2(\partial T\cap\Omega)}.\end{aligned} \tag{2.16}$$

For more details on residual-based estimators, we refer to [AO00; Ver94].

The residual-based error estimator η_H has many useful properties. Most importantly, it satisfies the so-called *axioms of adaptivity* from [CFPP14, Section 3], allowing for optimal rates of an adaptive algorithm steered by η_H. There exist constants $C_{\mathrm{stab}}, C_{\mathrm{rel}} \geq 1$ and $0 < q_{\mathrm{red}} < 1$ such that the following properties hold for any triangulation $\mathcal{T}_H \in \mathbb{T}$ and any refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ with corresponding Galerkin solutions $u_H^\star \in \mathcal{X}_H$ and $u_h^\star \in \mathcal{X}_h$ to (2.5), any subset $\mathcal{U}_H \subseteq \mathcal{T}_H \cap \mathcal{T}_h$ of non-refined elements, and for arbitrary $v_H \in \mathcal{X}_H$ and $v_h \in \mathcal{X}_h$:

(A1) stability: $|\eta_H(\mathcal{U}_H, v_H) - \eta_h(\mathcal{U}_H, v_h)| \leq C_{\mathrm{stab}} \, |\!|\!| v_h - v_H |\!|\!|$,
(A2) reduction: $\eta_h(\mathcal{T}_h \setminus \mathcal{T}_H, v_H) \leq q_{\mathrm{red}}\, \eta_H(\mathcal{T}_H \setminus \mathcal{T}_h, v_H)$,
(A3) reliability: $|\!|\!| u^\star - u_H^\star |\!|\!| \leq C_{\mathrm{rel}}\, \eta_H(u_H^\star)$.

Furthermore, there exist constants $C_{\mathrm{drel}}, C_{\mathrm{ref}} \geq 1$ such that the following property holds for any triangulation $\mathcal{T}_H \in \mathbb{T}$ and any refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$:

(A3$^+$) discrete reliability: There exists a subset $\mathcal{R}_{Hh} \subseteq \mathcal{T}_H$ with $\mathcal{T}_H \setminus \mathcal{T}_h \subseteq \mathcal{R}_{Hh}$ and $\#\mathcal{R}_{Hh} \leq C_{\mathrm{ref}}\, \#(\mathcal{T}_H \setminus \mathcal{T}_h)$ such that $|\!|\!| u_h^\star - u_H^\star |\!|\!| \leq C_{\mathrm{drel}}\, \eta_H(\mathcal{R}_{Hh}, u_H^\star)$.

Finally, there holds quasi-monotonicity of the error estimator η_H in the sense that there exists a constant $C_{\mathrm{mon}} \geq 0$ such that the following property is satisfied for any triangulation $\mathcal{T}_H \in \mathbb{T}$ and any refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$:

(QM) quasi-monotonicity: $\eta_h(u_h^\star) \leq C_{\mathrm{mon}}\, \eta_H(u_H^\star)$.

Remark 2.9 (quasi-monotonicity). In the present setting, quasi-monotonicity (QM) is already implied in two ways: either by stability (A1), reduction (A2), and discrete reliability (A3$^+$) (cf. [CFPP14, Lemma 3.5]) or by stability (A1), reduction (A2), reliability (A3), and the Ceá lemma (2.8) (cf. [CFPP14, Lemma 3.6]). In particular, the two approaches lead to the bound

$$C_{\mathrm{mon}} \leq \min\Big\{ \big(2 + 2C_{\mathrm{stab}}^2 C_{\mathrm{drel}}^2\big)^{1/2},\ \big(2 + 4C_{\mathrm{stab}}^2 C_{\mathrm{rel}}^2 (1 + C_{\mathrm{Ceá}}^2)\big)^{1/2} \Big\}.$$

The following remark discusses some implications of the stability axiom (A1) which are frequently used in this thesis.

Remark 2.10 (implications of stability (A1)). For $v_h := v_H \in \mathcal{X}_H$, stability (A1) yields

$$\eta_h(\mathcal{U}_H, v_H) = \eta_H(\mathcal{U}_H, v_H) \quad \text{for all } \mathcal{U}_H \subseteq \mathcal{T}_H \cap \mathcal{T}_h, \tag{2.17}$$

i.e., on any subset $\mathcal{U}_H$ of non-refined elements, the error estimators η_H and η_h are identical for any coarse-mesh function $v_H \in \mathcal{X}_H$. Conversely, setting $\mathcal{T}_h := \mathcal{T}_H =: \mathcal{U}_H$ implies

$$|\eta_H(v_h) - \eta_H(v_H)| \le C_{\text{stab}} \, |||v_h - v_H||| \quad \text{for all } v_h, v_H \in \mathcal{X}_H. \tag{2.18}$$

Furthermore, we assume that the error estimator μ_H is in some sense locally equivalent to the residual-based error estimator η_H. We consider two types of estimator equivalence, detailed in Definition 2.13 below. For that, we need to introduce the notion of m-patches.

Definition 2.11 (m-patches). For any subset $\mathcal{U}_H \subseteq \mathcal{T}_H$ of a triangulation $\mathcal{T}_H$, we define the *patch* of $\mathcal{U}_H$ by

$$\mathcal{T}_H[\mathcal{U}_H] := \mathcal{T}_H^{(1)}[\mathcal{U}_H] := \{T \in \mathcal{T}_H : \exists T' \in \mathcal{U}_H, T \cap T' \neq \emptyset\}.$$

Let $m \in \mathbb{N}$ with $m \ge 2$. The *m-patch* of $\mathcal{U}_H$ is defined inductively by

$$\mathcal{T}_H^{(m)}[\mathcal{U}_H] := \mathcal{T}_H[\mathcal{T}_H^{(m-1)}[\mathcal{U}_H]].$$

To simplify notation, we define $\mathcal{T}_H^{(0)}[\mathcal{U}_H] := \mathcal{U}_H$ and $\Omega_H^{(m)}[\mathcal{U}_H] := \bigcup \mathcal{T}_H^{(m)}[\mathcal{U}_H] \subseteq \overline{\Omega}$. Furthermore, we write $\mathcal{T}_H^{(m)}[T]$ and $\Omega_H^{(m)}[T]$ in case of $\mathcal{U}_H := \{T\}$. Figure 4 shows the m-patch of an element for $m = 1$ and $m = 2$, while Figure 5 illustrates the m-patch of a subset of $\mathcal{T}_H$ for $m = 1$ and $m = 2$.

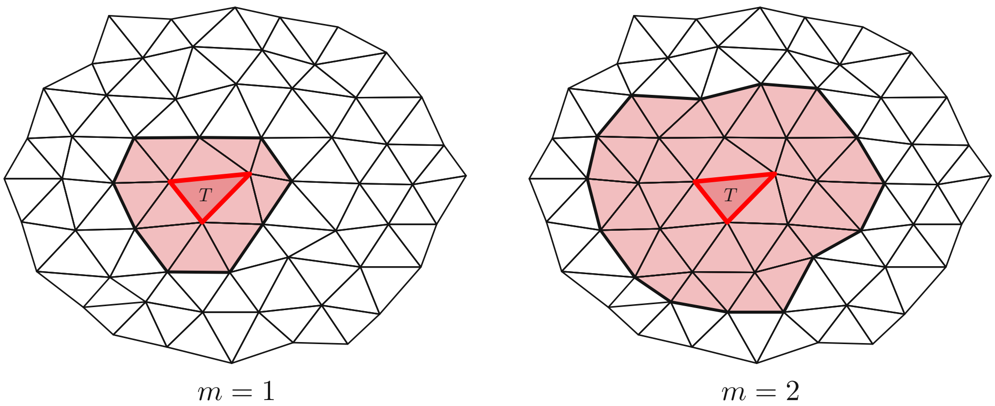


Figure 4: Illustration of m-patches $\mathcal{T}_H^{(m)}[T]$ of $T \in \mathcal{T}_H$ for $m \in \{1, 2\}$.

Remark 2.12 (cardinality of m-patches). Uniform shape regularity implies that for each $\mathcal{T}_H \in \mathbb{T}$, the solid angle formed by d faces sharing a common vertex z is uniformly bounded away from 0 by a constant depending only on the shape-regularity constant $\sigma > 0$ and the dimension $d \in \mathbb{N}$ (cf. [EG21a, Remark 11.5]). As a consequence, the number of elements sharing a vertex z is uniformly bounded. This in turn implies that also the number

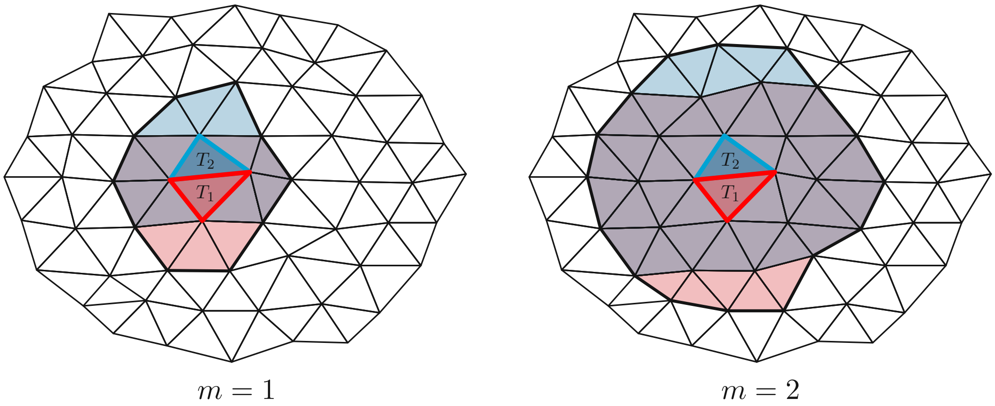


Figure 5: Illustration of m-patches $\mathcal{T}_H^{(m)}[\mathcal{U}_H]$ of $\mathcal{U}_H := \{T_1, T_2\} \subseteq \mathcal{T}_H$ for $m \in \{1, 2\}$.

of elements per patch $\mathcal{T}_H[T]$ is bounded by a constant $C_{\text{patch}} \geq 1$ depending only on the shape-regularity constant σ and the dimension d, i.e., there holds that

$$\#\mathcal{T}_H[T] \leq C_{\text{patch}} \quad \text{for all } \mathcal{T}_H \in \mathbb{T} \text{ and all } T \in \mathcal{T}_H. \tag{2.19}$$

The inductive definition of m-patches $\mathcal{T}_H^{(m)}[T]$ leads to

$$\#\mathcal{T}_H^{(m)}[T] \leq C_{\text{patch}}^m \quad \text{for all } m \in \mathbb{N}, \text{ all } \mathcal{T}_H \in \mathbb{T}, \text{ and all } T \in \mathcal{T}_H, \tag{2.20}$$

i.e., the number of elements in m-patches is also uniformly bounded. Since for all $T, T' \in \mathcal{T}_H$ it holds $T \in \mathcal{T}_H^{(m)}[T']$ if and only if $T' \in \mathcal{T}_H^{(m)}[T]$, the number of m-patches each element $T \in \mathcal{T}_H$ belongs to is also bounded by C_{patch}^m, i.e.,

$$\#\{T' \in \mathcal{T}_H : T \in \mathcal{T}_H^{(m)}[T']\} \leq C_{\text{patch}}^m \quad \text{for all } m \in \mathbb{N}, \text{ all } \mathcal{T}_H \in \mathbb{T}, \text{ and all } T \in \mathcal{T}_H. \tag{2.21}$$

Definition 2.13 (locally equivalent estimators). A pair of error estimators μ_H and η_H is *locally equivalent (in the strong sense)* if there exists a constant $C_{\text{loc}} \geq 1$ and a patch level $m \in \mathbb{N}_0$ such that for all triangulations $\mathcal{T}_H \in \mathbb{T}$, all simplices $T \in \mathcal{T}_H$, and all $v_H \in \mathcal{X}_H$, it holds

$$\eta_H(T, v_H) \leq C_{\text{loc}}\, \mu_H\big(\mathcal{T}_H^{(m)}[T], v_H\big), \tag{2.22a}$$
$$\mu_H(T, v_H) \leq C_{\text{loc}}\, \eta_H\big(\mathcal{T}_H^{(m)}[T], v_H\big). \tag{2.22b}$$

We also consider a weaker local equivalence, where (2.22) holds only for the Galerkin solution $u_H^\star$ to (2.5): A pair of error estimators μ_H and η_H is *locally equivalent in the weak sense* if there exists a constant $C_{\text{loc}} \geq 1$ and a patch level $m \in \mathbb{N}_0$ such that for all triangulations $\mathcal{T}_H \in \mathbb{T}$ and all simplices $T \in \mathcal{T}_H$, it holds

$$\eta_H(T, u_H^\star) \leq C_{\text{loc}}\, \mu_H\big(\mathcal{T}_H^{(m)}[T], u_H^\star\big), \tag{2.23a}$$
$$\mu_H(T, u_H^\star) \leq C_{\text{loc}}\, \eta_H\big(\mathcal{T}_H^{(m)}[T], u_H^\star\big). \tag{2.23b}$$

Example 2.14. Recall the mesh-size functions from Example 2.8. By uniform shape regularity of $\mathcal{T}_H \in \mathbb{T}$, it holds $H(T) \le \widehat{H}(T) \le \sigma\, H(T)$ for all $T \in \mathcal{T}_H$. This shows that the estimators η_H from (2.14) and $\widehat{\eta}_H$ from (2.16) are locally equivalent in the strong sense (2.22) with $m = 0$ and $C_{\mathrm{loc}} = \sigma$.

Remark 2.15 (non element-based estimators). Many popular error estimators are not defined element-wise such as in (2.11), but rather using an index set $\mathcal{I}_H$ depending on the triangulation $\mathcal{T}_H$. Let ϱ_H be such an error estimator, i.e., for all indices $I \in \mathcal{I}_H$ and all $v_H \in \mathcal{X}_H$, suppose that we can compute refinement indicators $\varrho_H(I, v_H)$. In particular, the choice $\mathcal{I}_H := \mathcal{T}_H$ results in an element-based estimator as in (2.11). Other common index sets include the set of all vertices $\mathcal{I}_H := \mathcal{V}_H$, the set of all edges $\mathcal{I}_H := \mathcal{E}_H$, or any union of the aforementioned sets. In order to define equivalence between ϱ_H and η_H, there must exist some “translation” between the index set $\mathcal{I}_H$ and the triangulation $\mathcal{T}_H$. With $\mathrm{pow}(\cdot)$ denoting the power set, let $\iota : \mathcal{T}_H \to \mathrm{pow}(\mathcal{I}_H)$ and $\tau : \mathcal{I}_H \to \mathrm{pow}(\mathcal{T}_H)$ be such translation functions, i.e., it holds

$$\iota(T) \subseteq \mathcal{I}_H \quad \text{and} \quad \tau(I) \subseteq \mathcal{T}_H \quad \text{for all } T \in \mathcal{T}_H \text{ and } I \in \mathcal{I}_H.$$

To abbreviate notation, we write

$$\iota(\mathcal{U}_H) := \bigcup_{T \in \mathcal{U}_H} \iota(T) \quad \text{and} \quad \tau(\mathcal{J}_H) := \bigcup_{I \in \mathcal{J}_H} \tau(I) \quad \text{for all } \mathcal{U}_H \subseteq \mathcal{T}_H \text{ and } \mathcal{J}_H \subseteq \mathcal{I}_H.$$

Suppose that ι and τ satisfy the following properties:

- **locality:** There exists a constant $s \in \mathbb{N}_0$ such that for all $\mathcal{T}_H \in \mathbb{T}$ with corresponding index set $\mathcal{I}_H$, it holds

$$\tau(\iota(T)) \subseteq \mathcal{T}_H^{(s)}[T] \quad \text{for all } T \in \mathcal{T}_H. \tag{2.24}$$

- **boundedness:** There exists a constant C_{index} such that

$$\#\iota(T) \le C_{\mathrm{index}} \quad \text{for all } T \in \mathcal{T}_H. \tag{2.25}$$

We say that ϱ_H is locally equivalent to η_H (in the strong sense) if there exists a constant $C_{\mathrm{loc}} \ge 1$ and a patch-level $m \in \mathbb{N}_0$ such that for all triangulations $\mathcal{T}_H \in \mathbb{T}$ with corresponding index set $\mathcal{I}_H$, all simplices $T \in \mathcal{T}_H$, all indices $I \in \mathcal{I}_H$, and all $v_H \in \mathcal{X}_H$, it holds

$$\eta_H(T, v_H) \le C_{\mathrm{loc}}\, \varrho_H\big(\iota(\mathcal{T}_H^{(m)}[T]), v_H\big), \tag{2.26a}$$

$$\varrho_H(I, v_H) \le C_{\mathrm{loc}}\, \eta_H\big(\mathcal{T}_H^{(m)}[\tau(I)], v_H\big). \tag{2.26b}$$

In that case, we can prove that the error estimator μ_H defined by

$$\mu_H(T, v_H)^2 := \varrho_T(\iota(T), v_H)^2 = \sum_{I \in \iota(T)} \varrho_H(I, v_H)^2 \quad \text{for all } T \in \mathcal{T}_H \tag{2.27}$$

is strongly equivalent to the error estimator η_H in the sense of (2.22), as the following calculations show: For all $T \in \mathcal{T}_H$ and all $v_H \in \mathcal{X}_H$, it holds

$$\begin{aligned}
\eta_H(T, v_H)^2 &\overset{(2.26a)}{\leq} C_{\mathrm{loc}}^2\, \varrho_H\big(\iota(\mathcal{T}_H^{(m)}[T]), v_H\big)^2 \\
&= C_{\mathrm{loc}}^2 \sum_{I \in \iota(\mathcal{T}_H^{(m)}[T])} \varrho_H(I, v_H)^2 \\
&\overset{(2.25)}{\leq} C_{\mathrm{loc}}^2\, C_{\mathrm{index}} \sum_{T \in \mathcal{T}_H^{(m)}[T]} \sum_{I \in \iota(T)} \varrho_H(I, v_H)^2 \\
&= C_{\mathrm{loc}}^2\, C_{\mathrm{index}}\, \mu_H(\mathcal{T}_H^{(m)}[T], v_H)^2 \\
&\leq C_{\mathrm{loc}}^2\, C_{\mathrm{index}}\, \mu_H(\mathcal{T}_H^{(m+s)}[T], v_H)^2.
\end{aligned}$$

Conversely, for all $T \in \mathcal{T}_H$ and all $v_H \in \mathcal{X}_H$ we have

$$\begin{aligned}
\mu_H(T, v_H)^2 &\overset{(2.26b)}{\leq} C_{\mathrm{loc}}^2 \sum_{I \in \iota(T)} \eta_H(\mathcal{T}_H^{(m)}[\tau(I)], v_H)^2 \\
&\leq C_{\mathrm{loc}}^2 \sum_{I \in \iota(T)} \eta_H(\mathcal{T}_H^{(m)}[\tau(\iota(T))], v_H)^2 \\
&\overset{(2.25)}{\leq} C_{\mathrm{loc}}^2\, C_{\mathrm{index}}\, \eta_H(\mathcal{T}_H^{(m)}[\tau(\iota(T))], v_H)^2 \\
&\overset{(2.24)}{\leq} C_{\mathrm{loc}}^2\, C_{\mathrm{index}}\, \eta_H(\mathcal{T}_H^{(m+s)}[T], v_H)^2.
\end{aligned}$$

Hence, μ_H is locally equivalent to η_H in the strong sense of (2.22) with constant $C_{\mathrm{loc}}\, C_{\mathrm{index}}^{1/2}$ and patch-level $m + s$. Analogously to (2.23), we define local equivalence of ϱ_H and η_H in the weak sense by restricting v_H to $u_H^\star$ in (2.26), i.e., there exists a constant $C_{\mathrm{loc}} \geq 1$ and a patch-level $m \in \mathbb{N}_0$ such that for all triangulations $\mathcal{T}_H \in \mathbb{T}$ with corresponding index set $\mathcal{I}_H$, all simplices $T \in \mathcal{T}_H$, and all indices $I \in \mathcal{I}_H$, it holds

$$\eta_H(T, u_H^\star) \leq C_{\mathrm{loc}}\, \varrho_H\big(\iota(\mathcal{T}_H^{(m)}[T]), u_H^\star\big), \tag{2.28a}$$

$$\varrho_H(I, u_H^\star) \leq C_{\mathrm{loc}}\, \eta_H\big(\mathcal{T}_H^{(m)}[\tau(I)], u_H^\star\big). \tag{2.28b}$$

Replacing v_H with $u_H^\star$ in the above calculations, we can show that weak local equivalence of ϱ_H and η_H in the sense of (2.28) implies weak local equivalence of μ_H and η_H in the sense of (2.23). Moreover, given that the refinement indicators $\varrho_H(I, v_H)$ can be computed in constant time $\mathcal{O}(1)$, boundedness (2.25) guarantees that the computation of $\mu_H(T, v_H)$ also requires only constant time $\mathcal{O}(1)$, i.e., the computation of $\mu_H(v_H)$ has linear complexity $\mathcal{O}(\#\mathcal{T}_H)$. Therefore, all subsequent results can be extended to index-based error estimators ϱ_H via the error estimator μ_H defined in (2.27).

An implication of the following lemma is that the local equivalences (2.22)–(2.23) imply global equivalence of the estimators μ_H and η_H.

Lemma 2.16. *Suppose local equivalence (2.22) of the estimators μ_H and η_H. Then, there exists a constant $C_{\rm eq} \geq 1$ such that for all $\mathcal{T}_H \in \mathbb{T}$, all $\mathcal{U}_H \subseteq \mathcal{T}_H$ and all $v_H \in \mathcal{X}_H$, it holds*

$$\eta_H(\mathcal{U}_H, v_H) \leq C_{\rm eq}\, \mu_H(\mathcal{T}_H^{(m)}[\mathcal{U}_H], v_H) \tag{2.29a}$$

$$\mu_H(\mathcal{U}_H, v_H) \leq C_{\rm eq}\, \eta_H(\mathcal{T}_H^{(m)}[\mathcal{U}_H], v_H). \tag{2.29b}$$

Analogously, if μ_H and η_H are weakly locally equivalent in the sense of (2.23), there exists a constant $C_{\rm eq} \geq 1$ such that for all $\mathcal{T}_H \in \mathbb{T}$ and all $\mathcal{U}_H \subseteq \mathcal{T}_H$ it holds

$$\eta_H(\mathcal{U}_H, u_H^\star) \leq C_{\rm eq}\, \mu_H(\mathcal{T}_H^{(m)}[\mathcal{U}_H], u_H^\star) \tag{2.30a}$$

$$\mu_H(\mathcal{U}_H, u_H^\star) \leq C_{\rm eq}\, \eta_H(\mathcal{T}_H^{(m)}[\mathcal{U}_H], u_H^\star). \tag{2.30b}$$

In particular, setting $\mathcal{U}_H := \mathcal{T}_H$ yields the global equivalence of the estimators μ_H and η_H.

Proof. In (2.21), we have shown that each $T \in \mathcal{T}_H$ is contained in at most $C_{\rm patch}^m$ many m-patches of $\mathcal{T}_H$. Hence, local equivalence (2.22) of μ_H and η_H with $C_{\rm eq} := C_{\rm loc}\, C_{\rm patch}^{m/2}$ reveals, for all $\mathcal{T}_H \in \mathbb{T}$, all $\mathcal{U}_H \in \mathcal{T}_H$, and all $v_H \in \mathcal{X}_H$,

$$\begin{aligned}\eta_H(\mathcal{U}_H, v_H)^2 = \sum_{T \in \mathcal{U}_H} \eta_H(T, v_H)^2 &\overset{(2.22a)}{\leq} C_{\rm loc}^2 \sum_{T \in \mathcal{U}_H} \sum_{T' \in \mathcal{T}_H^{(m)}[T]} \mu_H(T', v_H)^2 \\ &\leq C_{\rm loc}^2\, C_{\rm patch}^m \sum_{T \in \mathcal{T}_H^{(m)}[\mathcal{U}_H]} \mu_H(T, v_H)^2 = C_{\rm eq}^2\, \mu_H(\mathcal{T}_H^{(m)}[\mathcal{U}_H], v_H)^2,\end{aligned}$$

which verifies (2.29a). Switching the roles of μ_H and η_H provides (2.29b). Analogously, setting $v_H := u_H^\star$ establishes (2.30), which concludes the proof. □

Remark 2.17 (inheritance of axioms). From (2.29)–(2.30), it immediately follows that the locally equivalent estimator μ_H satisfies reliability (A3) with constant $\widetilde{C}_{\rm rel} := C_{\rm rel}\, C_{\rm eq}$. Moreover, for any triangulation $\mathcal{T}_H \in \mathbb{T}$ and any refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, it holds

$$|||u_h^\star - u_H^\star||| \overset{(A3^+)}{\leq} C_{\rm drel}\, \eta_H(\mathcal{R}_{Hh}, u_H^\star) \overset{(2.29)}{\underset{(2.30)}{\leq}} C_{\rm drel}\, C_{\rm eq}\, \mu_H(\mathcal{T}_H^{(m)}[\mathcal{R}_{Hh}], u_H^\star).$$

Due to (2.20), the set $\widetilde{\mathcal{R}}_{Hh} := \mathcal{T}_H^{(m)}[\mathcal{R}_{Hh}] \supseteq \mathcal{T}_H \setminus \mathcal{T}_h$ fulfills $\#\widetilde{\mathcal{R}}_{Hh} \leq C_{\rm patch}^m C_{\rm ref}\, \#(\mathcal{T}_H \setminus \mathcal{T}_h)$. Hence, the estimator μ_H also satisfies discrete reliability (A3$^+$) with $\widetilde{\mathcal{R}}_{Hh} \subseteq \mathcal{T}_H$ and constants $\widetilde{C}_{\rm drel} := C_{\rm drel}\, C_{\rm eq} \geq 1$ and $\widetilde{C}_{\rm ref} := C_{\rm patch}^m C_{\rm ref} \geq 1$. In general, however, μ_H does not inherit stability (A1) and reduction (A2) from the residual-based estimator η_H.

If μ_H is locally equivalent to η_H in the strong sense (2.22), a combination of global estimator equivalence (2.29) and stability (A1) validates for $\widetilde{C}_{\rm stab} := \max\{C_{\rm eq}^2, C_{\rm eq}\, C_{\rm stab}\}$, all subsets $\mathcal{U}_H \subseteq \mathcal{T}_H$, and all $v_H, w_H \in \mathcal{X}_H$ that

$$\begin{aligned}\mu_H(\mathcal{U}_H, v_H) &\overset{(2.29b)}{\leq} C_{\rm eq}\, \eta_H(\mathcal{T}_H^{(m)}[\mathcal{U}_H], v_H) \\ &\overset{(A1)}{\leq} C_{\rm eq}\, \big(\eta_H(\mathcal{T}_H^{(m)}[\mathcal{U}_H], w_H) + C_{\rm stab}\, |||v_H - w_H|||\big) \\ &\overset{(2.29a)}{\leq} C_{\rm eq}^2\, \mu_H(\mathcal{T}_H^{(2m)}[\mathcal{U}_H], w_H) + C_{\rm eq}\, C_{\rm stab}\, |||v_H - w_H||| \\ &\leq \widetilde{C}_{\rm stab} \Big[\mu_H(\mathcal{T}_H^{(2m)}[\mathcal{U}_H], w_H) + |||v_H - w_H|||\Big].\end{aligned} \tag{2.31}$$

For $\mu_H := \eta_H$, inequality (2.31) is implied by stability (A1) with $\widetilde{C}_{\rm stab} := C_{\rm stab}$. This gives rise to the following weak stability axiom (W1): There exist constants $\widetilde{C}_{\rm stab} \geq 1$ and $r \in \mathbb{N}_0$ such that for any mesh $\mathcal{T}_H \in \mathbb{T}$, any subset $\mathcal{U}_H \subseteq \mathcal{T}_H$, and arbitrary functions $v_H, w_H \in \mathcal{X}_H$, there holds

$$\textbf{(W1) weak stability:} \quad \mu_H(\mathcal{U}_H, v_H) \leq \widetilde{C}_{\rm stab} \Big[\mu_H(\mathcal{T}_H^{(r)}[\mathcal{U}_H], w_H) + |||v_H - w_H||| \Big]$$

In particular, the calculation in (2.31) shows that weak stability (W1) is already implied by strong estimator equivalence (2.22) and stability (A1) with $\widetilde{C}_{\rm stab} := \max\{C_{\rm eq}^2, C_{\rm eq}\, C_{\rm stab}\}$ and $r := 2m$. However, we suppose that weak stability (W1) is also satisfied in case of weak equivalence (2.23).

Concrete examples of estimators μ_H that are locally equivalent to the residual-based estimator η_H are discussed in Chapter 5 below.

2.4 Contractive solver

Since the direct computation of the exact solution $u_H^\star$ of (2.5) is expensive in practice, one resorts to iterative solvers. For all $\mathcal{T}_H \in \mathbb{T}$, we write $\Psi_H : \mathcal{X}_H \to \mathcal{X}_H$ for the iteration operator of such a solver. Hence, starting from an initial guess $u_H^0 \in \mathcal{X}_H$, the discrete function $u_H^k := \Psi_H(u_H^{k-1})$ denotes the new approximation of $u_H^\star$ constructed from the previous approximation u_H^{k-1} by one solver step of the algebraic solver. We suppose that each solver step has linear complexity, i.e., the computation of $\Psi_H(u_H^{k-1})$ requires $\mathcal{O}(\#\mathcal{T}_H)$ operations. An essential assumption for the later convergence analysis is uniform contraction of the solver, i.e., there exists a constant $0 < q_{\rm ctr} < 1$ independent of the mesh level H such that

$$|||u_H^\star - u_H^k||| \leq q_{\rm ctr}\, |||u_H^\star - u_H^{k-1}||| \quad \text{for all } k \in \mathbb{N}. \tag{2.32}$$

Since the exact solution $u_H^\star$ is never computed explicitly but only approximated by u_H^k, controlling the *algebraic solver error* $|||u_H^\star - u_H^k|||$ is crucial. Together with the (reverse) triangle inequality, contraction (2.32) implies, for all $k \in \mathbb{N}$,

$$\begin{aligned} \frac{1 - q_{\rm ctr}}{q_{\rm ctr}}\, |||u_H^\star - u_H^k||| &\overset{(2.32)}{\leq} (1 - q_{\rm ctr})\, |||u_H^\star - u_H^{k-1}||| \overset{(2.32)}{\leq} |||u_H^\star - u_H^{k-1}||| - |||u_H^\star - u_H^k||| \\ &\leq |||u_H^k - u_H^{k-1}||| \leq |||u_H^\star - u_H^k||| + |||u_H^\star - u_H^{k-1}||| \overset{(2.32)}{\leq} (1 + q_{\rm ctr})\, |||u_H^\star - u_H^{k-1}|||. \end{aligned} \tag{2.33}$$

This means that $|||u_H^k - u_H^{k-1}|||$ acts as a computable quantity to measure the algebraic solver error $|||u_H^\star - u_H^k|||$ from above and $|||u_H^\star - u_H^{k-1}|||$ from above and below. Using the triangle inequality, the *total error* $|||u^\star - u_H^k|||$ can thus be bounded by the sum of the *discretization error* $|||u^\star - u_H^\star|||$ and the algebraic solver error $|||u_H^\star - u_H^k|||$ via

$$|||u^\star - u_H^k||| \leq |||u^\star - u_H^\star||| + |||u_H^\star - u_H^k|||. \tag{2.34}$$

Therefore, the algorithm should control both error contributions simultaneously. However, both terms are not available in practice. In Remark 2.17, we have observed that the locally equivalent error estimator μ_H satisfies reliability (A3). As a result, the discretization error

$|||u^\star - u_H^\star|||$ is bounded by $\widetilde{C}_{\rm rel}\, \mu_H(u_H^\star)$. Moreover, the combination with weak stability (W1) yields, for $k \geq 1$,

$$\begin{aligned}
|||u^\star - u_H^k||| &\overset{(2.34)}{\leq} |||u^\star - u_H^\star||| + |||u_H^\star - u_H^k||| \\
&\overset{(A3)}{\leq} \widetilde{C}_{\rm rel}\, \mu_H(u_H^\star) + |||u_H^\star - u_H^k||| \\
&\overset{(W1)}{\leq} \widetilde{C}_{\rm rel}\, \widetilde{C}_{\rm stab}\, \mu_H(u_H^k) + (1 + \widetilde{C}_{\rm rel}\, \widetilde{C}_{\rm stab})\, |||u_H^\star - u_H^k||| \\
&\overset{(2.33)}{\leq} \widetilde{C}_{\rm rel}\, \widetilde{C}_{\rm stab}\, \mu_H(u_H^k) + (1 + \widetilde{C}_{\rm rel}\, \widetilde{C}_{\rm stab}) \frac{q_{\rm ctr}}{1 - q_{\rm ctr}}\, |||u_H^k - u_H^{k-1}||| \\
&\lesssim \mu_H(u_H^k) + |||u_H^k - u_H^{k-1}|||.
\end{aligned}$$

This means that the sum $\mu_H(u_H^k) + |||u_H^k - u_H^{k-1}|||$ is a computable upper bound for the total error $|||u^\star - u_H^k|||$. Since solver iterations only reduce the (estimated) algebraic solver error $|||u_H^k - u_H^{k-1}|||$, this motivates to stop the solver once the second summand $|||u_H^k - u_H^{k-1}|||$ is smaller than a fixed multiple $\lambda > 0$ of the first summand $\mu_H(u_H^k)$, i.e.,

$$|||u_H^k - u_H^{k-1}||| \leq \lambda\, \mu_H(u_H^k). \tag{2.35}$$

This functions as the stopping criterion for the algebraic solver used in the adaptive algorithm in Section 2.6. With $\underline{k}[H]$ we denote the minimal index $k \in \mathbb{N}$ such that u_H^k satisfies (2.35). Whenever it is clear from the context, we will omit the dependency on the discretization parameter H in this notation and write only $\underline{k} = \underline{k}[H]$ for the final solver index, e.g., $u_H^{\underline{k}} = u_H^{\underline{k}[H]}$.

2.4.1 Symmetric elliptic PDEs

For the symmetric case, iterative linear solvers are well-understood. Examples include optimally preconditioned CG methods [CNX12] or optimal geometric multigrid methods [WZ17; IMPS24].

2.4.2 Nonsymmetric elliptic PDEs

In case that $a(\cdot,\cdot)$ is nonsymmetric, we consider its symmetric part $b(\cdot,\cdot)$ from (2.2) and the so-called *Zarantonello mapping* introduced in the state-of-the-art proof of the Lax-Milgram theorem by Zarantonello [Zar60]. For a triangulation $\mathcal{T}_H \in \mathbb{T}$ and $\delta > 0$, the Zarantonello mapping $\Phi_H(\delta;\cdot) : \mathcal{X}_H \to \mathcal{X}_H$ is implicitly defined by

$$b(\Phi_H(\delta; u_H), v_H) = b(u_H, v_H) + \delta\big[F(v_H) - a(u_H, v_H)\big] \quad \text{for all } u_H, v_H \in \mathcal{X}_H. \tag{2.36}$$

Since $b(\cdot,\cdot)$ is a scalar product, the Riesz representation theorem guarantees the existence and uniqueness of $\Phi_H(\delta; u_H) \in \mathcal{X}_H$ satisfying (2.36), i.e., the mapping is well-defined. In particular, the Galerkin solution $u_H^\star$ of (2.5) is the only fixpoint of $\Phi_H(\delta;\cdot)$ for any $\delta > 0$. For a sufficiently small $\delta > 0$, the Zarantonello mapping $\Phi_H(\delta;\cdot)$ is contractive, i.e., there exists a constant $0 < q_{\rm sym} < 1$ such that

$$|||u_H^\star - \Phi_H(\delta; u_H)||| \leq q_{\rm sym}\, |||u_H^\star - u_H||| \quad \text{for all } u_H \in \mathcal{X}_H. \tag{2.37}$$

Solving for $\Phi_H(\delta; u_H)$ in (2.36) leads to a symmetric problem, which means that $\Phi_H(\delta; u_H)$ can be approximated with the aforementioned linear solvers from Section 2.4.1. For any $k \in \mathbb{N}$, let j denote the index of such a solver with iteration operator $\Psi_H^{\mathrm{sym}}(u_H^{k,\star}; \cdot)$ and contraction constant $0 < q_{\mathrm{alg}} < 1$ approximating $u_H^{k,\star} := \Phi_H(\delta; u_H^{k-1})$. Starting from $u_H^{k,0} := u_H^{k-1} \in \mathcal{X}_H$, the discrete function $u_H^{k,j} = \Psi_H^{\mathrm{sym}}(u_H^{k,\star}; u_H^{k,j-1})$ denotes the new approximation of $u_H^{k,\star}$ constructed from the previous approximation $u_H^{k,j-1}$ by one step of the solver for the symmetric problem (2.36), i.e., it holds

$$|||u_H^{k,\star} - u_H^{k,j}||| \leq q_{\mathrm{alg}}\, |||u_H^{k,\star} - u_H^{k,j-1}||| \quad \text{for all } j \in \mathbb{N}. \tag{2.38}$$

We stress that $u_H^{k,\star}$ is never computed explicitly and that $\Psi_H^{\mathrm{sym}}(u_H^{k,\star}; \cdot)$ depends only on the right-hand side of (2.36) and the scalar product $b(\cdot,\cdot)$, while $0 < q_{\mathrm{alg}} < 1$ depends only on $b(\cdot,\cdot)$. With $\underline{j}$ denoting the final solver index, we define the iteration operator Ψ_H of the algebraic solver for the nonsymmetric problem (2.5) via $u_H^k := \Psi_H(u_H^{k-1}) := u_H^{k,\underline{j}}$, i.e., one step of the algebraic solver for the nonsymmetric problem (2.5) corresponds to $\underline{j}$ steps of the solver for the symmetric problem (2.36). We aim to determine a lower bound j_0 for the total number of solver iterations $\underline{j}$ needed to ensure that Ψ_H is a uniformly contractive linear solver approximating $u_H^\star$. This is the content of the following proposition.

Proposition 2.18 (solver contraction for nonsymmetric elliptic PDEs). *Suppose that the iteration operator $\Psi_H^{\mathrm{sym}}(u_H^{k,\star}; \cdot)$ of the solver for the symmetric problem (2.36) is contractive, i.e., there exists a constant $0 < q_{\mathrm{alg}} < 1$ independent of the mesh level H and the index k such that (2.38) holds. Suppose that total number of solver iterations $\underline{j} \in \mathbb{N}$ for the symmetric problem (2.36) is sufficiently large in the sense that*

$$\underline{j} > j_0 := \left| \frac{\log\left(\frac{1-q_{\mathrm{sym}}}{1+q_{\mathrm{sym}}}\right)}{\log(q_{\mathrm{alg}})} \right|. \tag{2.39}$$

Then, there exists a constant $0 < q_{\mathrm{ctr}} < 1$ such that

$$|||u_H^\star - u_H^{k,\underline{j}}||| \leq q_{\mathrm{ctr}}\, |||u_H^\star - u_H^{k-1,\underline{j}}||| \quad \textit{for all } 1 \leq k \leq \underline{k}[H], \tag{2.40}$$

i.e., the proposed solver for the nonsymmetric problem (2.5) is contractive.

Proof. We follow the reasoning from [BIM+24, Section 2]. The triangle inequality and the contraction of Zarantonello mapping (2.37) show for all $k \in \mathbb{N}$ that

$$|||u_H^\star - u_H^{k,\underline{j}}||| \leq |||u_H^\star - u_H^{k,\star}||| + |||u_H^{k,\star} - u_H^{k,\underline{j}}||| \leq q_{\mathrm{sym}}\, |||u_H^\star - u_H^{k-1,\underline{j}}||| + |||u_H^{k,\star} - u_H^{k,\underline{j}}|||. \tag{2.41}$$

Contraction (2.38) together with nested iteration $u_H^{k,0} = u_H^{k-1,\underline{j}}$ and $\underline{j} \geq 1$ proves

$$|||u_H^{k,\star} - u_H^{k,\underline{j}}||| \leq q_{\mathrm{alg}}^{\underline{j}}\, |||u_H^{k,\star} - u_H^{k-1,\underline{j}}|||. \tag{2.42}$$

Furthermore, the triangle inequality and contraction (2.37) yield

$$|||u_H^{k,\star} - u_H^{k-1,\underline{j}}||| \leq |||u_H^\star - u_H^{k,\star}||| + |||u_H^\star - u_H^{k-1,\underline{j}}||| \leq (q_{\mathrm{sym}} + 1)\, |||u_H^\star - u_H^{k-1,\underline{j}}|||. \tag{2.43}$$

Combining the last three estimates (2.41)–(2.43), we obtain

$$|||u_H^\star - u_H^{k,\underline{j}}||| \le \Big[q_{\rm sym} + q_{\rm alg}^{\underline{j}}\,(q_{\rm sym}+1)\Big]\, |||u_H^\star - u_H^{k-1,\underline{j}}|||.$$

We observe that $q_{\rm ctr} := q_{\rm sym} + q_{\rm alg}^{\underline{j}}\,(q_{\rm sym}+1) < 1$ is equivalent to

$$q_{\rm alg}^{\underline{j}} < \frac{1-q_{\rm sym}}{1+q_{\rm sym}}.$$

Hence, the choice of $\underline{j} \in \mathbb{N}$ in (2.39) guarantees contraction (2.40) of each solver iteration $u_H^{k,\underline{j}} = \Psi_H\big(u_H^{k-1,\underline{j}}\big)$ for all $\mathcal{T}_H \in \mathbb{T}$ and $1 \le k \le \underline{k}[H]$. This concludes the proof. □

Hence, we set the number of solver iterations $\underline{j}$ to the minimal index satisfying (2.39), i.e., $\underline{j} := \min\{j \in \mathbb{N} : j > j_0\}$. Since $\underline{j}$ is independent of the triangulation $\mathcal{T}_H \in \mathbb{T}$ and the index k, and each solver step $\Psi_H^{\rm sym}$ has linear cost $\mathcal{O}(\#\mathcal{T}_H)$, the resulting solver for the nonsymmetric problem has also linear complexity $\mathcal{O}(\#\mathcal{T}_H)$ and thus fits into the framework from above.

2.5 Marking

Given the local error indicators $\mu_H(T, u_H^{\underline{k}})$ of the final iterate $u_H^{\underline{k}}$ on each element $T \in \mathcal{T}_H$, the goal in the marking step is to determine a set $\mathcal{M}_H \subseteq \mathcal{T}_H$ of marked elements that should be refined in the consecutive refinement step. For this, we will employ the so-called *Dörfler marking criterion* introduced in [Dör96]: Given a marking parameter $0 < \theta \le 1$, find a set $\mathcal{M}_H \subseteq \mathcal{T}_H$ such that

$$\theta\,\mu_H(u_H^{\underline{k}})^2 \le \mu_H(\mathcal{M}_H, u_H^{\underline{k}})^2. \tag{2.44}$$

This can be interpreted as choosing a set of elements $\mathcal{M}_H \subseteq \mathcal{T}_H$ whose corresponding estimated discretization error accounts for a θ-fraction of the total estimated discretization error. Smaller choices of θ lead to fewer marked elements and thus highly adapted meshes. On the contrary, the selection $\theta = 1$ essentially corresponds to uniform mesh refinement since (generically) all elements are marked for refinement. In order to obtain optimal convergence rates, it seems natural to select $\mathcal{M}_H$ with minimal cardinality. A naive approach involves sorting the computed refinement indicators, which results in a suboptimal log-linear complexity in terms of the number of elements $\#\mathcal{T}_H$. However, [Ste07] showed that determining a set of elements

$$\mathcal{M}_H \in \mathbb{M}_H[\theta, u_H^{\underline{k}}] := \big\{\mathcal{U}_H \subseteq \mathcal{T}_H : \theta\mu_H(u_H^{\underline{k}})^2 \le \mu_H(\mathcal{U}_H, u_H^{\underline{k}})^2\big\} \tag{2.45a}$$

with quasi-minimal cardinality

$$\#\mathcal{M}_H \le C_{\rm mark} \min_{\mathcal{U}_H \in \mathbb{M}_H[\theta, u_H^{\underline{k}}]} \#\mathcal{U}_H \quad \text{where} \quad C_{\rm mark} \ge 1, \tag{2.45b}$$

suffices to prove optimality. [Ste07] proposed an algorithm based on binning that achieves (2.45) with $C_{\rm mark} = 2$ in linear complexity $\mathcal{O}(\#\mathcal{T}_H)$. [PP20] later improved on that by showing that a modified QuickSelect algorithm on average even allows for $C_{\mathrm{mark}} = 1$ with linear complexity.

Dörfler marking is sufficient for linear convergence (cf. [CKNS08]). However, the following proposition even shows that it is essentially necessary to achieve linear convergence, which is the central reason why the adaptive algorithm in Section 2.6 employs Dörfler marking.

Proposition 2.19 (optimality of Dörfler marking [CFPP14, Proposition 4.12]). *Suppose that the estimator η_H satisfies stability (A1) and discrete reliability (A3$^+$). Recall the set $\mathcal{R}_{Hh} \subseteq \mathcal{T}_H$ from discrete reliability (A3$^+$) satisfying $\mathcal{T}_H \setminus \mathcal{T}_h \subseteq \mathcal{R}_{Hh}$ and $\#\mathcal{R}_{Hh} \le C_{\mathrm{ref}}\, \#(\mathcal{T}_H \setminus \mathcal{T}_h)$. Let $\theta^\star := (1 + C_{\mathrm{drel}}^2\, C_{\mathrm{stab}}^2)^{-1}$. Then, for all $0 < \theta_0 < \theta^\star$, there exists a constant $0 < q_{\mathrm{opt}} < 1$ such that for all $0 < \theta < \theta_0$, all triangulations $\mathcal{T}_H \in \mathbb{T}$, and all refinements $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ it holds*

$$\eta_h(u_h^\star)^2 \le q_{\mathrm{opt}}\, \eta_H(u_H^\star)^2 \quad \Longrightarrow \quad \theta\, \eta_H(u_H^\star)^2 \le \eta_H(\mathcal{R}_{Hh}, u_H^\star)^2. \qquad \square$$

The proposition above asserts that whenever the error estimator is contracted, the set $\mathcal{R}_{Hh} \subseteq \mathcal{T}_H$ satisfies the Dörfler marking criterion (2.44). Since the set $\mathcal{R}_{Hh}$ is essentially the set of refined elements $\mathcal{T}_H \setminus \mathcal{T}_h$, this implies that, regardless of the marking strategy used, the set of refined elements will always satisfy the Dörfler marking criterion (2.44) if the marking strategy guarantees contraction of the estimator. Therefore, using any marking strategy other than Dörfler would be pointless.

2.6 Formulation of the algorithm

With the preliminary discussion from Section 2.1–2.5 above, we can formulate the following adaptive algorithm (cf. [GHPS21; BFM$^+$25]), which is steered by the estimator μ_H.

Algorithm B (AFEM with contractive solver).
Input: *Initial mesh $\mathcal{T}_0$, marking parameters $0 < \theta \le 1$ and $C_{\mathrm{mark}} \ge 1$, solver-stopping parameter $\lambda > 0$, and initial guess $u_0^0 \in \mathcal{X}_0$.*
For all $\ell = 0, 1, 2, \ldots$, repeat (i)–(iii):

(i) ***Solve & Estimate: For all $k = 1, 2, 3, \ldots$, repeat (a)–(b):***
  (a) *Compute $u_\ell^k := \Psi_\ell(u_\ell^{k-1})$ using one step of the contractive solver Ψ_ℓ.*
  (b) *Compute refinement indicators $\mu_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$.*
  until
$$|||u_\ell^k - u_\ell^{k-1}||| \le \lambda\, \mu_\ell(u_\ell^k). \tag{2.46}$$
  Upon termination, define the index $\underline{k}[\ell] := k$ and abbreviate $u_\ell^{\underline{k}} := u_\ell^{\underline{k}[\ell]}$.

(ii) ***Mark:*** *Determine a set $\mathcal{M}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^{\underline{k}}]$ that satisfies (2.45).*

(iii) ***Refine:*** *Generate $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ and use nested iteration $u_{\ell+1}^0 := u_\ell^{\underline{k}}$.*

We define
$$\mathcal{Q} := \{(\ell, k) \in \mathbb{N}_0^2 : u_\ell^k \text{ is defined in Algorithm B}\}$$
as the countably infinite index set of all iterates generated by Algorithm B. We order the indices sequentially by their appearance in the algorithm, i.e.,

$$(\ell, k) \le (\ell', k') \quad :\Longleftrightarrow \quad u_\ell^k \text{ appears not later than } u_{\ell'}^{k'} \text{ in Algorithm B},$$

or in explicit terms,

$$(\ell, k) \leq (\ell', k') \quad :\Longleftrightarrow \quad \ell < \ell' \quad \text{or} \quad (\ell = \ell' \text{ and } k \leq k').$$

The latter definition is usually referred to as lexicographical ordering. Correspondingly, we define the total step counter

$$|\ell, k| := \#\{(\ell', k') \in \mathcal{Q} : (\ell', k') \leq (\ell, k)\} = k + \sum_{\ell'=0}^{\ell-1} \underline{k}[\ell']$$

and the stopping indices

$$\underline{\ell} := \sup\{\ell \in \mathbb{N}_0 : (\ell, 0) \in \mathcal{Q}\}, \tag{2.47}$$

$$\underline{k}[\ell] := \sup\{k \in \mathbb{N} : (\ell, k) \in \mathcal{Q}\}. \tag{2.48}$$

We note that this definition of $\underline{k}[\ell]$ is consistent with the definition in Algorithm B.

3 Parameter-robust full R-linear convergence

Our objective in the following chapter is to show full R-linear convergence of Algorithm B independently of the adaptivity parameters θ and λ. This property is the key to cost-optimality as described in [GHPS21] and Chapter 4 below. While [BFM+25] establishes parameter-robust full R-linear convergence for the standard residual-based estimator that fulfills the axioms of adaptivity (A1)–(A3), we aim to prove this for locally equivalent estimators μ_ℓ that lack reduction (A2) and thus seemingly do not fit into this framework.

First, we prove a weaker version of the estimator reduction in the spirit of [BFM+25, Equation (36)] for a modified residual-based estimator $\overline{\eta}_\ell$ that still applies if Algorithm B is steered by the locally equivalent estimator μ_ℓ. For that, we suppose that $\overline{\eta}_\ell$ satisfies a modified stability axiom (M1) and a modified reduction axiom (M2). In Section 3.2, we define $\overline{\eta}_\ell$ through the appropriate mesh-size function from Proposition 3.8 and prove that the estimator satisfies the modified axioms. Finally, we apply the estimator reduction from Section 3.1 to prove parameter-robust full R-linear convergence for the locally equivalent estimator μ_ℓ in Section 3.3.

3.1 Estimator reduction

In general, Dörfler marking

$$\theta\, \mu_\ell(u_\ell^k)^2 \le \mu_\ell(\mathcal{M}_\ell, u_\ell^k)^2 \tag{3.1}$$

for an estimator μ_ℓ does not imply Dörfler marking

$$\theta\, \eta_\ell(u_\ell^k)^2 \le \eta_\ell(\mathcal{M}_\ell, u_\ell^k)^2 \tag{3.2}$$

for a locally equivalent estimator η_ℓ. However, for strong estimator equivalence (2.22), we can prove (3.2) for a different marking parameter $\widehat{\theta}$ and a suitable larger set of marked elements $\mathcal{T}_\ell^{(m)}[\mathcal{M}_\ell]$.

Lemma 3.1 (equivalence of Dörfler-marking for strongly equivalent estimators). *Let $0 < \theta \le 1$ and $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ be the marked set associated with the triangulation $\mathcal{T}_\ell \in \mathbb{T}$ from Step (ii) in Algorithm B. Suppose that the error estimator μ_ℓ is strongly equivalent to an estimator η_ℓ in the sense of (2.22). Then, it holds $0 < \widehat{\theta} := C_{\mathrm{eq}}^{-4}\theta \le 1$ and the Dörfler marking criterion (3.1) for μ_ℓ implies the Dörfler marking criterion for η_ℓ in the sense that*

$$\widehat{\theta}\, \eta_\ell(u_\ell^k)^2 \le \eta_\ell(\mathcal{T}_\ell^{(m)}[\mathcal{M}_\ell], u_\ell^k)^2. \tag{3.3}$$

Proof. Strong estimator equivalence (2.29) and Dörfler marking (3.1) provide

$$\theta\eta_\ell(u_\ell^{\underline{k}})^2 \overset{(2.29a)}{\leq} C_{\mathrm{eq}}^2\,\theta\mu_\ell(u_\ell^{\underline{k}})^2 \overset{(3.1)}{\leq} C_{\mathrm{eq}}^2\,\mu_\ell(\mathcal{M}_\ell, u_\ell^{\underline{k}})^2 \overset{(2.29b)}{\leq} C_{\mathrm{eq}}^4\,\eta_\ell(\mathcal{T}_\ell^{(m)}[\,\mathcal{M}_\ell\,], u_\ell^{\underline{k}})^2.$$

Since $C_{\mathrm{eq}} \geq 1$, a rearrangement of this concludes the proof. □

Dörfler marking is crucial to prove the *estimator reduction* for the inexact iterate $u_\ell^{\underline{k}}$ from [BFM+25, Equation (36)], which states that there exists a constant $0 < q_\theta < 1$ such that

$$\eta_{\ell+1}(u_{\ell+1}^{\underline{k}}) \leq q_\theta\,\eta_\ell(u_\ell^{\underline{k}}) + C_{\mathrm{stab}}|||u_{\ell+1}^{\underline{k}} - u_\ell^{\underline{k}}||| \quad \text{for all } \ell \in \mathbb{N}_0 \text{ with } (\ell+1, \underline{k}) \in \mathcal{Q}. \tag{3.4}$$

In the proof, the reduction axiom (A2) is combined with the Dörfler marking criterion (3.2), which is possible since all marked elements are refined. However, this is not feasible if only (3.3) holds, because not all elements of $\mathcal{T}_\ell^{(m)}[\,\mathcal{M}_\ell\,]$ are refined in general, i.e., we cannot guarantee that $\mathcal{T}_\ell^{(m)}[\,\mathcal{M}_\ell\,] \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$. As a remedy, we introduce a modified residual-based estimator $\overline{\eta}_H$, which satisfies a modified stability axiom (M1) and a modified reduction axiom (M2). This requires some additional notation.

Definition 3.2. For a refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ and $M \in \mathbb{N}_0$, we define $\Omega_{H,h}^{(M)}$ as the union of all refined elements plus M additional layers

$$\Omega_{H,h}^{(M)} \coloneqq \bigcup \mathcal{T}_H^{(M)}[\,\mathcal{T}_H \setminus \mathcal{T}_h\,] \subseteq \overline{\Omega}.$$

We write $\mathcal{T}_h|_{\Omega_{H,h}^{(M)}}$ for the set of fine-mesh simplices $T \in \mathcal{T}_h$ that are contained in $\Omega_{H,h}^{(M)}$, i.e.,

$$\mathcal{T}_h|_{\Omega_{H,h}^{(M)}} \coloneqq \{T \in \mathcal{T}_h : T \subseteq \Omega_{H,h}^{(M)}\}.$$

Note that for the triangulation $\mathcal{T}_H$, the set $\mathcal{T}_H|_{\Omega_{H,h}^{(M)}}$ coincides with $\mathcal{T}_H^{(M)}[\,\mathcal{T}_H \setminus \mathcal{T}_h\,]$. Moreover, since $\mathcal{T}_H \setminus \mathcal{T}_H^{(M)}[\,\mathcal{T}_H \setminus \mathcal{T}_h\,] \subseteq \mathcal{T}_H \cap \mathcal{T}_h$, it holds

$$\mathcal{T}_h \setminus \mathcal{T}_h|_{\Omega_{H,h}^{(M)}} = \mathcal{T}_H \setminus \mathcal{T}_H|_{\Omega_{H,h}^{(M)}} \subseteq \mathcal{T}_H \cap \mathcal{T}_h.$$

Let $\overline{C}_{\mathrm{stab}} \geq 0$ and $0 < \overline{q}_{\mathrm{red}} < 1$ be constants such that the following properties hold for any triangulation $\mathcal{T}_H \in \mathbb{T}$, any refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, any subset $\mathcal{U}_H \subseteq \mathcal{T}_H \setminus \mathcal{T}_H^{(M)}[\,\mathcal{T}_H \setminus \mathcal{T}_h\,]$, and arbitrary $v_H \in \mathcal{X}_H$ and $v_h \in \mathcal{X}_h$:

- **(M1) patch stability:** $|\overline{\eta}_h(\mathcal{U}_H, v_h) - \overline{\eta}_H(\mathcal{U}_H, v_H)| \leq \overline{C}_{\mathrm{stab}}\,|||v_h - v_H|||$,
- **(M2) patch reduction:** $\overline{\eta}_h(\mathcal{T}_h|_{\Omega_{H,h}^{(M)}}, v_H) \leq \overline{q}_{\mathrm{red}}\,\overline{\eta}_H(\mathcal{T}_H|_{\Omega_{H,h}^{(M)}}, v_H)$.

Remark 3.3 (implications of patch stability (M1)). Patch stability (M1) yields implications comparable to those of stability (A1) discussed in Remark 2.10. Similar to (2.17), the choice $v_h \coloneqq v_H$ provides

$$\overline{\eta}_h(\mathcal{U}_H, v_H) = \overline{\eta}_H(\mathcal{U}_H, v_H) \quad \text{for all } \mathcal{U}_H \subseteq \mathcal{T}_H \setminus \mathcal{T}_H^{(M)}[\,\mathcal{T}_H \setminus \mathcal{T}_h\,].$$

Analogously to (2.18), setting $\mathcal{T}_h \coloneqq \mathcal{T}_H$ implies

$$|\overline{\eta}_H(v_h) - \overline{\eta}_H(v_H)| \leq \overline{C}_{\mathrm{stab}}\,|||v_h - v_H|||.$$

In Section 3.2 below, we show that the residual-based estimator $\overline{\eta}_H$ defined in (3.22) satisfies the modified axioms (M1)–(M2) in addition to reliability (A3), discrete reliability (A3$^+$), and quasi-monotonicity (QM). Contrary to η_H, the estimator $\overline{\eta}_H$ satisfies estimator reduction (3.4) in case of strong local equivalence (2.22).

Lemma 3.4 (estimator reduction via strong local equivalence (2.22)). *Let $0 < \theta \leq 1$, $C_{\mathrm{mark}} \geq 1$, $\lambda > 0$, and $u_0^0 \in \mathcal{X}_0$ in Algorithm B be arbitrary. Suppose that the error estimator μ_ℓ is equivalent to an estimator $\overline{\eta}_\ell$ in the sense of (2.22). With $m \in \mathbb{N}_0$ from the equivalence (2.22), suppose that $\overline{\eta}_\ell$ satisfies the axioms (M1)–(M2) with $M := m$. Recall $\widehat{\theta} := C_{\mathrm{eq}}^{-4}\theta$ from Lemma 3.1. Then, with the constant*

$$0 < q_{\widehat{\theta}} := \Big[1 - \big(1 - (\overline{q}_{\mathrm{red}})^2\big)\widehat{\theta}\Big]^{1/2} < 1, \tag{3.5}$$

the estimator $\overline{\eta}_\ell$ satisfies

$$\overline{\eta}_{\ell+1}(v_{\ell+1}) \leq q_{\widehat{\theta}}\, \overline{\eta}_\ell(u_\ell^{\underline{k}}) + \overline{C}_{\mathrm{stab}} |\!|\!| v_{\ell+1} - u_\ell^{\underline{k}} |\!|\!| \quad \text{for all } \ell < \underline{\ell} \text{ and all } v_{\ell+1} \in \mathcal{X}_{\ell+1}. \tag{3.6}$$

In particular, if $(\ell+1, \underline{k}) \in \mathcal{Q}$, setting $v_{\ell+1} := u_{\ell+1}^{\underline{k}}$ yields the estimator reduction (3.4).

Proof. Let $\ell < \underline{\ell}$ be arbitrary. Using patch stability (M1) and patch reduction (M2), we get

$$\begin{aligned}
\overline{\eta}_{\ell+1}(u_\ell^{\underline{k}})^2 &= \overline{\eta}_{\ell+1}(\mathcal{T}_{\ell+1} \setminus \mathcal{T}_{\ell+1}|_{\Omega_{\ell,\ell+1}^{(m)}}, u_\ell^{\underline{k}})^2 + \overline{\eta}_{\ell+1}(\mathcal{T}_{\ell+1}|_{\Omega_{\ell,\ell+1}^{(m)}}, u_\ell^{\underline{k}})^2 \\
&\overset{(M1)}{=} \overline{\eta}_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\ell|_{\Omega_{\ell,\ell+1}^{(m)}}, u_\ell^{\underline{k}})^2 + \overline{\eta}_{\ell+1}(\mathcal{T}_{\ell+1}|_{\Omega_{\ell,\ell+1}^{(m)}}, u_\ell^{\underline{k}})^2 \\
&\overset{(M2)}{\leq} \overline{\eta}_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\ell|_{\Omega_{\ell,\ell+1}^{(m)}}, u_\ell^{\underline{k}})^2 + \overline{q}_{\mathrm{red}}^2\, \overline{\eta}_\ell(\mathcal{T}_\ell|_{\Omega_{\ell,\ell+1}^{(m)}}, u_\ell^{\underline{k}})^2 \\
&= \overline{\eta}_\ell(u_\ell^{\underline{k}})^2 - \big(1 - \overline{q}_{\mathrm{red}}^2\big)\, \overline{\eta}_\ell(\mathcal{T}_\ell|_{\Omega_{\ell,\ell+1}^{(m)}}, u_\ell^{\underline{k}})^2.
\end{aligned} \tag{3.7}$$

Since $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$, Lemma 3.1 implies for the marking parameter $0 < \widehat{\theta} \leq 1$

$$\widehat{\theta}\, \overline{\eta}_\ell(u_\ell^{\underline{k}})^2 \overset{(3.3)}{\leq} \overline{\eta}_\ell(\mathcal{T}_\ell^{(m)}[\mathcal{M}_\ell], u_\ell^{\underline{k}})^2 \leq \overline{\eta}_\ell(\mathcal{T}_\ell^{(m)}[\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}], u_\ell^{\underline{k}})^2 = \overline{\eta}_\ell(\mathcal{T}_\ell|_{\Omega_{\ell,\ell+1}^{(m)}}, u_\ell^{\underline{k}})^2.$$

By definition (3.5) of $q_{\widehat{\theta}}$, the combination of this estimate with (3.7) yields

$$\overline{\eta}_{\ell+1}(u_\ell^{\underline{k}}) \leq q_{\widehat{\theta}}\, \overline{\eta}_\ell(u_\ell^{\underline{k}}). \tag{3.8}$$

Together with patch stability (M1), we obtain for any $v_{\ell+1} \in \mathcal{X}_{\ell+1}$ that

$$\overline{\eta}_{\ell+1}(v_{\ell+1}) \overset{(M1)}{\leq} \overline{\eta}_{\ell+1}(u_\ell^{\underline{k}}) + \overline{C}_{\mathrm{stab}} |\!|\!| v_{\ell+1} - u_\ell^{\underline{k}} |\!|\!| \overset{(3.8)}{\leq} q_{\widehat{\theta}}\, \overline{\eta}_\ell(u_\ell^{\underline{k}}) + \overline{C}_{\mathrm{stab}} |\!|\!| v_{\ell+1} - u_\ell^{\underline{k}} |\!|\!|.$$

This concludes the proof. □

Contrary to [BFM+25], we use a weaker estimator reduction than (3.4) in the full R-linear convergence proof, which has two reasons: First, we employ a different (but equivalent) quasi-error compared to [BFM+25], detailed in Section 3.3 below. Second, it is not possible to prove estimator reduction (3.6) in case of weak local equivalence (2.23). Instead, we will use the following estimator reduction, which is an immediate consequence of Lemma 3.4.

Corollary 3.5 (perturbed estimator reduction via strong local equivalence (2.22)). *Let $0 < \theta \le 1$, $C_{\mathrm{mark}} \ge 1$, $\lambda > 0$ and $u_0^0 \in \mathcal{X}_0$ in Algorithm B be arbitrary. With $m \in \mathbb{N}_0$ from the strong equivalence (2.22), suppose that $\overline{\eta}_\ell$ satisfies the axioms (M1)–(M2) with $M := m$. Then, with the constant $0 < q_{\widehat{\theta}} < 1$ defined in (3.5) and $C_{\widehat{\theta}} := (1 + q_{\widehat{\theta}})\,\overline{C}_{\mathrm{stab}} \ge 1$, the estimator $\overline{\eta}_\ell$ satisfies*

$$\overline{\eta}_{\ell+1}(u_{\ell+1}^\star) \le q_{\widehat{\theta}}\,\overline{\eta}_\ell(u_\ell^\star) + C_{\widehat{\theta}}\,|||u_\ell^\star - u_\ell^{\underline{k}}||| + \overline{C}_{\mathrm{stab}}|||u_{\ell+1}^\star - u_\ell^\star||| \quad \textit{for all } \ell < \underline{\ell}. \tag{3.9}$$

Proof. Let $\ell < \underline{\ell}$ be arbitrary. Patch stability (M1), the triangle inequality and Lemma 3.4 applied to the function $v_{\ell+1} := u_{\ell+1}^\star$ yield

$$\begin{aligned}
\overline{\eta}_{\ell+1}(u_{\ell+1}^\star) &\overset{(3.6)}{\le} q_{\widehat{\theta}}\,\overline{\eta}_\ell(u_\ell^{\underline{k}}) + \overline{C}_{\mathrm{stab}}|||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| \\
&\overset{(M1)}{\le} q_{\widehat{\theta}}\,\overline{\eta}_\ell(u_\ell^\star) + q_{\widehat{\theta}}\overline{C}_{\mathrm{stab}}|||u_\ell^\star - u_\ell^{\underline{k}}||| + \overline{C}_{\mathrm{stab}}|||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| \\
&\le q_{\widehat{\theta}}\,\overline{\eta}_\ell(u_\ell^\star) + C_{\widehat{\theta}}\,|||u_\ell^\star - u_\ell^{\underline{k}}||| + \overline{C}_{\mathrm{stab}}|||u_{\ell+1}^\star - u_\ell^\star|||.
\end{aligned}$$

This concludes the proof. □

In case that μ_ℓ is locally equivalent to $\overline{\eta}_\ell$ in the weak sense (2.23), utilizing Dörfler marking becomes problematic: Since the Dörfler marking criterion (2.45) involves only the final iterate $u_\ell^{\underline{k}}$ and weak local equivalence considers only the Galerkin solution $u_\ell^\star$, they cannot be combined directly. However, using the weak stability (W1) that we suppose for μ_ℓ in Section 2.3, we can prove Dörfler marking for $\overline{\eta}_\ell$ up to a perturbation term.

Lemma 3.6 (Dörfler-like inequality for weakly equivalent estimators). *Let $0 < \theta \le 1$. For a triangulation $\mathcal{T}_\ell \in \mathbb{T}$ from Algorithm B, let $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ be the corresponding set of marked elements. Suppose that the error estimator μ_ℓ is weakly locally equivalent to an estimator $\overline{\eta}_\ell$ in the sense of (2.23) with $m \in \mathbb{N}_0$ and satisfies weak stability (W1) with $r \in \mathbb{N}_0$. Furthermore, suppose that $\overline{\eta}_\ell$ satisfies the axioms (M1)–(M2) with $M := m + r$. Then, with the constants*

$$0 < \overline{\theta} := \frac{1}{2}\,\theta\, C_{\mathrm{eq}}^{-4}\widetilde{C}_{\mathrm{stab}}^{-4} \le \frac{1}{2} \quad \textit{and} \quad 0 < C_{\mathrm{per}} := C_{\mathrm{eq}}^{-1} + \theta^{1/2}C_{\mathrm{eq}}^{-1}\widetilde{C}_{\mathrm{stab}}^{-1} \le 2 \tag{3.10}$$

the Dörfler marking criterion (3.1) implies

$$\overline{\theta}\,\overline{\eta}_\ell(u_\ell^\star)^2 \le \overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{M}_\ell], u_\ell^\star)^2 + C_{\mathrm{per}}^2|||u_\ell^\star - u_\ell^{\underline{k}}|||^2. \tag{3.11}$$

Moreover, if the estimator $\overline{\eta}_\ell$ additionally satisfies patch stability (M1) for any $M \in \mathbb{N}_0$, the Dörfler marking criterion (3.1) also implies for $\overline{C}_{\mathrm{per}} := \overline{C}_{\mathrm{stab}}(1 + \theta^{1/2}C_{\mathrm{eq}}^{-2}\widetilde{C}_{\mathrm{stab}}^{-2}) + C_{\mathrm{per}} > 0$

$$\overline{\theta}\,\overline{\eta}_\ell(u_\ell^{\underline{k}})^2 \le \overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{M}_\ell], u_\ell^{\underline{k}})^2 + \overline{C}_{\mathrm{per}}^2|||u_\ell^\star - u_\ell^{\underline{k}}|||^2. \tag{3.12}$$

Proof. Weak estimator equivalence (2.30), weak stability (W1) of μ_ℓ, and Dörfler marking (3.1) lead to

$$
\begin{aligned}
\theta^{1/2} C_{\rm eq}^{-2} \widetilde{C}_{\rm stab}^{-2} \overline{\eta}_\ell(u_\ell^\star) &\overset{(2.30a)}{\leq} \theta^{1/2} C_{\rm eq}^{-1} \widetilde{C}_{\rm stab}^{-2}\, \mu_\ell(u_\ell^\star) \\
&\overset{(W1)}{\leq} \theta^{1/2} C_{\rm eq}^{-1} \widetilde{C}_{\rm stab}^{-1}\, \mu_\ell(u_\ell^k) + \theta^{1/2} C_{\rm eq}^{-1} \widetilde{C}_{\rm stab}^{-1}\, |||u_\ell^\star - u_\ell^k||| \\
&\overset{(3.1)}{\leq} C_{\rm eq}^{-1} \widetilde{C}_{\rm stab}^{-1}\, \mu_\ell(\mathcal{M}_\ell, u_\ell^k) + \theta^{1/2} C_{\rm eq}^{-1} \widetilde{C}_{\rm stab}^{-1}\, |||u_\ell^\star - u_\ell^k||| \\
&\overset{(W1)}{\leq} C_{\rm eq}^{-1} \mu_\ell(\mathcal{T}_\ell^{(r)}[\mathcal{M}_\ell], u_\ell^\star) + \Big[C_{\rm eq}^{-1} + \theta^{1/2} C_{\rm eq}^{-1} \widetilde{C}_{\rm stab}^{-1} \Big]\, |||u_\ell^\star - u_\ell^k||| \\
&\overset{(2.30b)}{\leq} \overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{M}_\ell], u_\ell^\star) + C_{\rm per}\, |||u_\ell^\star - u_\ell^k|||.
\end{aligned}
\tag{3.13}
$$

Since

$$
\frac{1}{2}(a+b)^2 \leq a^2 + b^2 \quad \text{for any } a, b \geq 0, \tag{3.14}
$$

inequality (3.13) results in

$$
\begin{aligned}
\overline{\theta}\, \overline{\eta}_\ell(u_\ell^\star)^2 &\overset{(3.13)}{\leq} \frac{1}{2} \big(\overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{M}_\ell], u_\ell^\star) + C_{\rm per} |||u_\ell^\star - u_\ell^k||| \big)^2 \\
&\leq \overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{M}_\ell], u_\ell^\star)^2 + C_{\rm per}^2 |||u_\ell^\star - u_\ell^k|||^2.
\end{aligned}
\tag{3.15}
$$

The fact $C_{\rm eq}, \widetilde{C}_{\rm stab} \geq 1$ guarantees $0 < \overline{\theta} \leq \frac{1}{2}$ and $0 < C_{\rm per} \leq 2$. This concludes the proof of (3.11). For the proof of (3.12), we combine patch stability (M1) with (3.13) to obtain

$$
\begin{aligned}
\theta^{1/2} C_{\rm eq}^{-2} \widetilde{C}_{\rm stab}^{-2}\, \overline{\eta}_\ell(u_\ell^k) &\overset{(M1)}{\leq} \theta^{1/2} C_{\rm eq}^{-2} \widetilde{C}_{\rm stab}^{-2}\, \overline{\eta}_\ell(u_\ell^\star) + \theta^{1/2} C_{\rm eq}^{-2} \widetilde{C}_{\rm stab}^{-2} \overline{C}_{\rm stab}\, |||u_\ell^\star - u_\ell^k||| \\
&\overset{(3.13)}{\leq} \overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{M}_\ell], u_\ell^\star) + (\theta^{1/2} C_{\rm eq}^{-2} \widetilde{C}_{\rm stab}^{-2} \overline{C}_{\rm stab} + C_{\rm per})\, |||u_\ell^\star - u_\ell^k||| \\
&\overset{(M1)}{\leq} \overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{M}_\ell], u_\ell^k) + \overline{C}_{\rm per}\, |||u_\ell^\star - u_\ell^k|||.
\end{aligned}
$$

With an analogous calculation as in (3.15), we conclude the proof of (3.12). □

The following lemma shows that the Dörfler-like inequality (3.11) is sufficient to prove the perturbed estimator reduction (3.9).

Lemma 3.7 (perturbed estimator reduction via weak local equivalence (2.23)). *Let $0 < \theta \leq 1$, $C_{\rm mark} \geq 1$, $\lambda > 0$ and $u_0^0 \in \mathcal{X}_0$ in Algorithm B be arbitrary. Suppose that the error estimator μ_ℓ is equivalent to an estimator $\overline{\eta}_\ell$ in the sense of (2.23) and satisfies weak stability (W1). With $m \in \mathbb{N}_0$ from the equivalence (2.23) and $r \in \mathbb{N}_0$ from weak stability (W1), suppose that $\overline{\eta}_\ell$ satisfies (M1)–(M2) with $M := m + r$. Recall $\overline{\theta} := \frac{1}{2}\, \theta\, C_{\rm eq}^{-4} \widetilde{C}_{\rm stab}^{-4}$ and $0 < C_{\rm per} \leq 2$ from Lemma 3.6. Then, with the constants*

$$
\frac{1}{\sqrt{2}} < q_{\overline{\theta}} := \Big[1 - (1 - \overline{q}_{\rm red}^2) \overline{\theta} \Big]^{1/2} < 1 \quad \text{and} \quad 0 \leq C_{\overline{\theta}} := C_{\rm per} \big(1 - \overline{q}_{\rm red}^2 \big)^{1/2} < 2,
$$

the estimator $\overline{\eta}_\ell$ satisfies

$$
\overline{\eta}_{\ell+1}(u_{\ell+1}^\star) \leq q_{\overline{\theta}}\, \overline{\eta}_\ell(u_\ell^\star) + C_{\overline{\theta}}\, |||u_\ell^\star - u_\ell^k||| + \overline{C}_{\rm stab} |||u_{\ell+1}^\star - u_\ell^\star||| \quad \text{for all } \ell < \underline{\ell}. \tag{3.16}
$$

Proof. Let $\ell < \underline{\ell}$ be arbitrary. Analogously to (3.7), patch stability (M1) and patch reduction (M2) verify

$$\begin{aligned}\overline{\eta}_{\ell+1}(u_\ell^\star)^2 &= \overline{\eta}_{\ell+1}(\mathcal{T}_{\ell+1} \setminus \mathcal{T}_{\ell+1}|_{\Omega^{(m+r)}_{\ell,\ell+1}}, u_\ell^\star)^2 + \overline{\eta}_{\ell+1}(\mathcal{T}_{\ell+1}|_{\Omega^{(m+r)}_{\ell,\ell+1}}, u_\ell^\star)^2 \\ &\overset{\text{(M1)}}{=} \overline{\eta}_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\ell|_{\Omega^{(m+r)}_{\ell,\ell+1}}, u_\ell^\star)^2 + \overline{\eta}_{\ell+1}(\mathcal{T}_{\ell+1}|_{\Omega^{(m+r)}_{\ell,\ell+1}}, u_\ell^\star)^2 \\ &\overset{\text{(M2)}}{\leq} \overline{\eta}_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\ell|_{\Omega^{(m+r)}_{\ell,\ell+1}}, u_\ell^\star)^2 + \overline{q}_{\mathrm{red}}^2\, \overline{\eta}_\ell(\mathcal{T}_\ell|_{\Omega^{(m+r)}_{\ell,\ell+1}}, u_\ell^\star)^2 \\ &= \overline{\eta}_\ell(u_\ell^\star)^2 - \big(1 - \overline{q}_{\mathrm{red}}^2\big)\, \overline{\eta}_\ell(\mathcal{T}_\ell|_{\Omega^{(m+r)}_{\ell,\ell+1}}, u_\ell^\star)^2.\end{aligned} \tag{3.17}$$

The Dörfler-like inequality (3.11) and $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ imply

$$\begin{aligned}\overline{\theta}\, \overline{\eta}_\ell(u_\ell^\star)^2 &\overset{(3.11)}{\leq} \overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{M}_\ell], u_\ell^\star)^2 + C_{\mathrm{per}}^2 |||u_\ell^\star - u_\ell^{\underline{k}}|||^2 \\ &\leq \overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}], u_\ell^\star)^2 + C_{\mathrm{per}}^2 |||u_\ell^\star - u_\ell^{\underline{k}}|||^2 \\ &= \overline{\eta}_\ell(\mathcal{T}_\ell|_{\Omega^{(m+r)}_{\ell,\ell+1}}, u_\ell^\star)^2 + C_{\mathrm{per}}^2 |||u_\ell^\star - u_\ell^{\underline{k}}|||^2.\end{aligned} \tag{3.18}$$

The combination of the last two estimates results in

$$\overline{\eta}_{\ell+1}(u_\ell^\star)^2 \leq q_{\overline{\theta}}^2\, \overline{\eta}_\ell(u_\ell^\star)^2 + C_{\overline{\theta}}^2\, |||u_\ell^\star - u_\ell^{\underline{k}}|||^2. \tag{3.19}$$

Since $\big(a^2 + b^2\big)^{1/2} \leq a + b$ for all $a, b \geq 0$, inequality (3.19) yields

$$\overline{\eta}_{\ell+1}(u_\ell^\star) \leq q_{\overline{\theta}}\, \overline{\eta}_\ell(u_\ell^\star) + C_{\overline{\theta}}\, |||u_\ell^\star - u_\ell^{\underline{k}}|||. \tag{3.20}$$

Together with patch stability (M1), we obtain

$$\begin{aligned}\overline{\eta}_{\ell+1}(u_{\ell+1}^\star) &\overset{\text{(M1)}}{\leq} \overline{\eta}_{\ell+1}(u_\ell^\star) + \overline{C}_{\mathrm{stab}} |||u_{\ell+1}^\star - u_\ell^\star||| \\ &\overset{(3.20)}{\leq} q_{\overline{\theta}}\, \overline{\eta}_\ell(u_\ell^\star) + C_{\overline{\theta}}\, |||u_\ell^\star - u_\ell^{\underline{k}}||| + \overline{C}_{\mathrm{stab}} |||u_{\ell+1}^\star - u_\ell^\star|||.\end{aligned}$$

This concludes the proof. □

3.2 Generalized mesh-size function

A central property in the proof of the reduction axiom (A2) for the residual-based estimator η_H from Example 2.8 is that the mesh-size function $H(T)$ contracts whenever an element $T \in \mathbb{T}_H$ is refined, i.e., there exists a constant $0 < \rho < 1$ such that for all triangulations $\mathcal{T}_H \in \mathbb{T}$, all refinements $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, all refined elements $T \in \mathcal{T}_H \setminus \mathcal{T}_h$, and all $T' \in \mathcal{T}_h$ with $T' \subset T$ it holds $H(T) \leq \rho\, h(T')$. This is the reason why [CKNS08] introduced the mesh-size function $H(T) := |T|^{1/d}$ over the (arguably) more natural mesh-size function $\widehat{H}(T) := \mathrm{diam}(T)$: While the diameter of refined elements $T \in \mathcal{T}_H \setminus \mathcal{T}_h$ does not necessarily decrease, their volume $|T|$ is always halved with every bisection. Consequently, it holds $|T'| \leq |T|/2$ for all refined elements $T \in \mathcal{T}_H \setminus \mathcal{T}_h$ and all $T' \in \mathcal{T}_h$ with $T' \subset T$, which implies $H(T) \leq 2^{-1/d}\, h(T')$.

In the previous section, however, we introduced the patch reduction axiom (M2) because we did not only require the reduction property for refined elements $\mathcal{T}_H \setminus \mathcal{T}_h$, but also for elements in the M-patch of refined elements $\mathcal{T}_H^{(M)}[\mathcal{T}_H \setminus \mathcal{T}_h]$. Since the mesh-size function H does not contract for all elements in $\mathcal{T}_H^{(M)}[\mathcal{T}_H \setminus \mathcal{T}_h]$, the original proof cannot be applied to the residual-based estimator η_H directly. Instead, we switch to a mesh-size function $\overline{H} : \mathcal{T}_H \to \mathbb{R}_{>0}$ which contracts for an element $T \in \mathcal{T}_H$ if at least one element in its M-patch $\mathcal{T}_H^{(m)}[T]$ is refined. A mesh-size function with this property is constructed in [CFPP14].

Proposition 3.8 (generalized mesh-size function [CFPP14, Proposition 8.6]). *Let $M \in \mathbb{N}_0$. For a given initial triangulation $\mathcal{T}_0$, suppose that the triangulations $\mathcal{T}_H \in \mathbb{T}$ are conforming and uniformly shape regular with constant $\sigma \geq 1$. Then, there exist constants $C_{\mathrm{mesh}} \geq 1$ and $0 < \rho_{\mathrm{mesh}} < 1$ and, for all $\mathcal{T}_H \in \mathbb{T}$, a mesh-size function $\overline{H} : \mathcal{T}_H \to \mathbb{R}_{>0}$, such that the following properties are satisfied for all $T \in \mathcal{T}_H$, all $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, and all $T' \in \mathcal{T}_h$ with $T' \subseteq T$:*

- **(H1)** ***local equivalence:*** $\overline{H}(T) \leq |T|^{1/d} \leq C_{\mathrm{mesh}}\, \overline{H}(T)$,
- **(H2)** ***monotonicity:*** $\overline{h}(T') \leq \overline{H}(T)$,
- **(H3)** ***contraction:*** $\overline{h}(T') \leq \rho_{\mathrm{mesh}}\, \overline{H}(T)$, *if* $T \in \mathcal{T}_H^{(M)}[\mathcal{T}_H \setminus \mathcal{T}_h]$.

The constants C_{mesh} and ρ_{mesh} depend only on the shape-regularity constant σ and M. □

Remark 3.9 (construction of the generalized mesh-size function). In the proof of Proposition 3.8 given in [CFPP14], the inductive construction of the generalized mesh-size function ensures that for all triangulations $\mathcal{T}_H \in \mathbb{T}$ and all refinements $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, the corresponding mesh-size functions $\overline{H}$ and $\overline{h}$ satisfy

$$\overline{H}(T) = \overline{h}(T) \quad \text{for all } T \in \mathcal{T}_H \setminus \mathcal{T}_H^{(M)}[\mathcal{T}_H \setminus \mathcal{T}_h]. \tag{3.21}$$

Let $\overline{\eta}_H$ be the residual-based error estimator that is obtained by replacing the mesh-size function H in (2.14) with the generalized mesh-size function $\overline{H}$ from Proposition 3.8, i.e., for all triangulations $\mathcal{T}_H \in \mathbb{T}$, all $T \in \mathcal{T}_H$, and all $v_H \in \mathcal{X}_H$, it holds

$$\begin{aligned} \overline{\eta}_H(T, v_H)^2 = \overline{H}(T)^2 \,\| - \operatorname{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) + \boldsymbol{b} \cdot \nabla v_H + c\, v_H - f\|_{L^2(T)}^2 \\ + \overline{H}(T)\, \|[\![(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) \cdot \boldsymbol{n}]\!]\|_{L^2(\partial T \cap \Omega)}^2. \end{aligned} \tag{3.22}$$

Given that the residual-based error estimator η_H is locally equivalent to the error estimator μ_H in the sense of (2.22) or (2.23), our goal in this section is therefore to show that

- the estimator $\overline{\eta}_H$ is also locally equivalent to μ_H and
- the estimator $\overline{\eta}_H$ satisfies patch stability (M1), patch reduction (M2), reliability (A3), and discrete reliability (A3$^+$).

For all $\mathcal{T}_H \in \mathbb{T}$, all $T \in \mathcal{T}_H$, and all $v_H \in \mathcal{X}_H$, local equivalence (H1) and $C_{\mathrm{mesh}} \geq 1$ lead to

$$\overline{\eta}_H(T, v_H) \overset{\text{(H1)}}{\leq} \eta_H(T, v_H) \overset{\text{(H1)}}{\leq} C_{\mathrm{mesh}}\, \overline{\eta}_H(T, v_H). \tag{3.23}$$

This implies that both strong local equivalence (2.22) and weak local equivalence (2.23) of η_H and μ_H extends to $\overline{\eta}_H$ and μ_H with $\overline{C}_{\mathrm{loc}} := C_{\mathrm{mesh}}\, C_{\mathrm{loc}}$. Similarly, quasi-monotonicity (QM) extends to $\overline{\eta}_H$ for all triangulations $\mathcal{T}_H \in \mathbb{T}$ and all refinements $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ via

$$\overline{\eta}_h(u_H^\star) \overset{\text{(H1)}}{\leq} \eta_h(u_h^\star) \overset{\text{(QM)}}{\leq} C_{\mathrm{mon}}\, \eta_H(u_H^\star) \overset{\text{(H1)}}{\leq} C_{\mathrm{mesh}}\, C_{\mathrm{mon}}\, \overline{\eta}_H(u_H^\star) =: \overline{C}_{\mathrm{mon}}\, \overline{\eta}_H(u_H^\star). \tag{3.24}$$

Using (3.23), we can also conclude both reliability (A3) and discrete reliability (A3$^+$) for the estimator $\overline{\eta}_H$ via

$$|||u^\star - u_H^\star||| \overset{\text{(A3)}}{\leq} C_{\mathrm{rel}}\, \eta_H(u_H^\star) \overset{\text{(H1)}}{\leq} C_{\mathrm{mesh}}\, C_{\mathrm{rel}}\, \overline{\eta}_H(u_H^\star) =: \overline{C}_{\mathrm{rel}}\, \overline{\eta}_H(u_H^\star), \tag{3.25a}$$

$$\begin{aligned} |||u_h^\star - u_H^\star||| &\overset{\text{(A3}^+\text{)}}{\leq} C_{\mathrm{drel}}\, \eta_H(R_{Hh}, u_H^\star) \\ &\overset{\text{(H1)}}{\leq} C_{\mathrm{mesh}}\, C_{\mathrm{drel}}\, \overline{\eta}_H(R_{Hh}, u_H^\star) =: \overline{C}_{\mathrm{drel}}\, \overline{\eta}_H(R_{Hh}, u_H^\star). \end{aligned} \tag{3.25b}$$

Patch stability (M1) and patch reduction (M2), however, require modifications in their original proofs.

Proposition 3.10 (axioms of adaptivity). *Let η_H be the standard residual-based error estimator (2.14) which satisfies the axioms of adaptivity (A1)–(A3$^+$) and (QM) with constants $C_{\mathrm{stab}}, C_{\mathrm{rel}}, C_{\mathrm{drel}}, C_{\mathrm{ref}} \geq 1$, $C_{\mathrm{mon}} \geq 0$ and $0 < q_{\mathrm{red}} < 1$. For $M \in \mathbb{N}_0$ and $\mathcal{T}_H \in \mathbb{T}$, let $\overline{H}$ be the corresponding generalized mesh-size function from Proposition 3.8. Let $\overline{\eta}_H$ from (3.22) be the residual-based error estimator with the mesh-size function $\overline{H}$. Then, the estimator $\overline{\eta}_H$ satisfies patch stability (M1), patch reduction (M2), reliability (A3), discrete reliability (A3$^+$), and quasi-monotonicity (QM) with the constants $\overline{C}_{\mathrm{rel}}, \overline{C}_{\mathrm{drel}} \geq 1$ from (3.25), $\overline{C}_{\mathrm{mon}} \geq 0$ from (3.24), $\overline{C}_{\mathrm{stab}} := C_{\mathrm{stab}}$, $\overline{C}_{\mathrm{ref}} := C_{\mathrm{ref}}$ and $0 < \overline{q}_{\mathrm{red}} := \rho_{\mathrm{mesh}}^{1/2} < 1$.*

Proof. Reliability (A3), discrete reliability (A3$^+$), and quasi-monotonicity of $\overline{\eta}_H$ are proven in (3.24)–(3.25). Thus, it remains to establish patch stability (M1) and patch reduction (M2) of $\overline{\eta}_H$.

Step 1 (proof of patch stability (M1)). Let $\mathcal{T}_H \in \mathbb{T}$ be a triangulation and $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ be any refinement of $\mathcal{T}_H$. For any $\mathcal{U}_H \subseteq \mathcal{T}_H \setminus \mathcal{T}_H^{(M)}[\mathcal{T}_H \setminus \mathcal{T}_h]$, any $v_H \in \mathcal{X}_H$ and any $v_h \in \mathcal{X}_h$, the reverse triangle inequalities on the sequence space ℓ^2 and the Lebesgue space L^2 together with equality (3.21) yield

$$\begin{aligned} \big|\overline{\eta}_H(\mathcal{U}_H, v_H) - \overline{\eta}_h(\mathcal{U}_H, v_h)\big|^2 &\leq \sum_{T \in \mathcal{U}_H} \big|\overline{\eta}_H(T, v_H) - \overline{\eta}_h(T, v_h)\big|^2 \\ &\overset{(3.21)}{\leq} \sum_{T \in \mathcal{U}_H} \Big(\overline{H}(T)^2\, \| - \operatorname{div}\big(\boldsymbol{A}\nabla(v_H - v_h)\big) + \boldsymbol{b} \cdot \nabla(v_H - v_h) + c\,(v_H - v_h)\|_{L^2(T)}^2 \\ &\qquad + \overline{H}(T)\, \|[\![(\boldsymbol{A}\nabla(v_H - v_h)) \cdot \boldsymbol{n}]\!]\|_{L^2(\partial T \cap \Omega)}^2 \Big). \end{aligned}$$

This and local equivalence (H1) validate

$$\begin{aligned}\big|\overline{\eta}_H(\mathcal{U}_H, v_H) - \overline{\eta}_h(\mathcal{U}_H, v_h)\big|^2 \\ \overset{(H1)}{\le} \sum_{T\in\mathcal{U}_H} \Big(|T|^{2/d}\, \| - \operatorname{div}\big(\boldsymbol{A}\nabla(v_H - v_h)\big) + \boldsymbol{b}\cdot\nabla(v_H - v_h) + c\,(v_H - v_h)\|^2_{L^2(T)} \\ + |T|^{1/d}\, \|[\![(\boldsymbol{A}\nabla(v_H - v_h))\cdot \boldsymbol{n}]\!]\|^2_{L^2(\partial T\cap\Omega)}\Big).\end{aligned} \tag{3.26}$$

The latter sum also appears in the original stability proof, where the estimate

$$\begin{aligned}\sum_{T\in\mathcal{U}_H} \Big(|T|^{2/d}\, \| - \operatorname{div}\big(\boldsymbol{A}\nabla(v_H - v_h)\big) + \boldsymbol{b}\cdot\nabla(v_H - v_h) + c\,(v_H - v_h)\|^2_{L^2(T)} \\ + |T|^{1/d}\, \|[\![(\boldsymbol{A}\nabla(v_H - v_h))\cdot \boldsymbol{n}]\!]\|^2_{L^2(\partial T\cap\Omega)}\Big) \le C_{\mathrm{stab}} \|\!|v_H - v_h|\!\|^2\end{aligned}$$

is shown using an inverse estimate and the trace inequality. For details, we refer to [CKNS08, Corollary 3.4]. The combination of that result with (3.26) thus proves patch stability (M1) of $\overline{\eta}_H$ with the same stability constant C_{stab} as for η_H.

Step 2 (proof of patch reduction (M2)). Let $\mathcal{T}_H \in \mathbb{T}$ be an arbitrary triangulation and $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ any refinement of $\mathcal{T}_H$. By Definition 3.2, it holds

$$\mathcal{T}_h|_{\Omega^{(M)}_{H,h}} = \bigcup_{T\in\mathcal{T}^{(M)}_H[\mathcal{T}_H\setminus\mathcal{T}_h]} \{T' \in \mathcal{T}_h : T' \subseteq T\} \quad \text{and} \quad \mathcal{T}_H|_{\Omega^{(M)}_{H,h}} = \mathcal{T}^{(M)}_H[\mathcal{T}_H \setminus \mathcal{T}_h].$$

With the notation from Example 2.8, contraction (H3) of $\overline{H}$ with $0 < \rho_{\mathrm{mesh}} < 1$ results for any $v_H \in \mathcal{X}_H$ in

$$\begin{aligned}\overline{\eta}_h(\mathcal{T}_h|_{\Omega^{(M)}_{H,h}}, v_H)^2 &= \sum_{T\in\mathcal{T}_H|_{\Omega^{(M)}_{H,h}}} \sum_{\substack{T'\in\mathcal{T}_h\\ T'\subseteq T}} \overline{\eta}_h(T, v_H)^2 \\ &= \sum_{T\in\mathcal{T}_H|_{\Omega^{(M)}_{H,h}}} \sum_{\substack{T'\in\mathcal{T}_h\\ T'\subseteq T}} \Big(\overline{h}(T')^2\, \|R_H(v_H)\|^2_{L^2(T')} + \overline{h}(T')\, \|J_H(v_H)\|^2_{L^2(T')}\Big) \\ &\overset{(H3)}{\le} \sum_{T\in\mathcal{T}_H|_{\Omega^{(M)}_{H,h}}} \sum_{\substack{T'\in\mathcal{T}_h\\ T'\subseteq T}} \Big(\rho^2_{\mathrm{mesh}}\, \overline{H}(T)^2\, \|R_H(v_H)\|^2_{L^2(T')} + \rho_{\mathrm{mesh}}\, \overline{H}(T)\, \|J_H(v_H)\|^2_{L^2(T')}\Big) \\ &\le \rho_{\mathrm{mesh}} \sum_{T\in\mathcal{T}_H|_{\Omega^{(M)}_{H,h}}} \sum_{\substack{T'\in\mathcal{T}_h\\ T'\subseteq T}} \Big(\overline{H}(T)^2\, \|R_H(v_H)\|^2_{L^2(T')} + \overline{H}(T)\, \|J_H(v_H)\|^2_{L^2(T')}\Big).\end{aligned}$$

Since $v_H \in \mathcal{X}_H$ is smooth on the new edges of the refinement $\mathcal{T}_h$ and $T = \bigcup\{T' \in \mathcal{T}_h : T' \subseteq T\}$, linearity of the integral shows

$$\begin{aligned}\sum_{\substack{T'\in\mathcal{T}_h\\ T'\subseteq T}} \Big(\overline{H}(T)^2\, \|R_H(v_H)\|^2_{L^2(T')} + \overline{H}(T)\, \|J_H(v_H)\|^2_{L^2(T')}\Big) \\ = \overline{H}(T)^2\, \|R_H(v_H)\|^2_{L^2(T)} + \overline{H}(T)\, \|J_H(v_H)\|^2_{L^2(T)} = \overline{\eta}_H(T, v_H)^2.\end{aligned}$$

Hence, the combination of the previous equations leads to

$$\overline{\eta}_h(\mathcal{T}_h|_{\Omega^{(M)}_{H,h}}, v_H)^2 \leq \rho_{\mathrm{mesh}} \sum_{T \in \mathcal{T}_H|_{\Omega^{(M)}_{H,h}}} \overline{\eta}_H(T, v_H)^2 = \rho_{\mathrm{mesh}}\, \overline{\eta}_H(\mathcal{T}_H|_{\Omega^{(M)}_{H,h}}, v_H)^2.$$

This concludes the proof. □

3.3 Proof of full R-linear convergence

The two notions of local estimator equivalence require different assumptions and are therefore covered in separate theorems. In both cases, we consider the *quasi-error*

$$\mathrm{M}_\ell^k := \mu_\ell(u_\ell^\star) + |||u_\ell^\star - u_\ell^k|||, \tag{3.27}$$

which measures the algebraic error plus discretization error. This differs from the quasi-error

$$\widetilde{\mathrm{M}}_\ell^k := \mu_\ell(u_\ell^k) + |||u_\ell^\star - u_\ell^k|||,$$

proposed in [BFM+25], which uses $\mu_\ell(u_\ell^k)$ instead of $\mu_\ell(u_\ell^\star)$ in (3.27). However, weak stability (W1) provides for all $(\ell, k) \in \mathcal{Q}$ the estimates

$$\mu_\ell(u_\ell^\star) \leq \widetilde{C}_{\mathrm{stab}} \Big[\mu_\ell(u_\ell^k) + |||u_\ell^\star - u_\ell^k|||\Big] \quad \text{and} \quad \mu_\ell(u_\ell^k) \leq \widetilde{C}_{\mathrm{stab}} \Big[\mu_\ell(u_\ell^\star) + |||u_\ell^\star - u_\ell^k|||\Big]. \tag{3.28}$$

This implies equivalence of both quasi-errors, i.e.,

$$\mathrm{M}_\ell^k = \mu_\ell(u_\ell^\star) + |||u_\ell^\star - u_\ell^k||| \simeq \mu_\ell(u_\ell^k) + |||u_\ell^\star - u_\ell^k||| = \widetilde{\mathrm{M}}_\ell^k \quad \text{for all } (\ell, k) \in \mathcal{Q}.$$

The following theorem states that in case of strong local equivalence (2.22), there holds parameter-robust full R-linear convergence of M_ℓ^k, i.e., quasi-contraction of the quasi-error M_ℓ^k in each step of Algorithm B for any parameter θ and λ.

Theorem 3.11 (full R-linear convergence for strong local equivalence (2.22)). *Let $0 < \theta \leq 1$, $C_{\mathrm{mark}} \geq 1$, $\lambda > 0$ and $u_0^0 \in \mathcal{X}_0$ be arbitrary. Suppose that the error estimator μ_ℓ is strongly equivalent to an estimator $\overline{\eta}_\ell$ in the sense of (2.22). With $m \in \mathbb{N}_0$ from the equivalence (2.22), suppose that $\overline{\eta}_\ell$ satisfies (M1), (M2), (A3), and (QM) with $M := m$. Then, Algorithm B guarantees full R-linear convergence of the quasi-error M_ℓ^k, i.e., there exist constants $0 < q_{\mathrm{lin}} < 1$ and $C_{\mathrm{lin}} > 0$ such that*

$$\mathrm{M}_\ell^k \leq C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\ell,k| - |\ell',k'|} \mathrm{M}_{\ell'}^{k'} \quad \textit{for all } (\ell', k'), (\ell, k) \in \mathcal{Q} \textit{ with } |\ell', k'| \leq |\ell, k|. \tag{3.29}$$

The constants C_{lin} and q_{lin} depend only on $\overline{C}_{\mathrm{stab}}$, $\overline{q}_{\mathrm{red}}$, C_{rel}, C_{eq}, q_{ctr}, C_{mon}, C_{orth}, θ, and λ.

The proof follows the reasoning of [BFM+25, Theorem 7] and employs the following characterizations of full R-linear convergence.

Lemma 3.12 (tail summability vs. R-linear convergence [BFM+25, Lemma 10]). *Let $(a_\ell)_{\ell \in \mathbb{N}_0}$ be a nonnegative sequence of real numbers and $s > 0$. Then, the following statements are equivalent:*

(i) ***tail summability:*** *There exists a constant* $C_s > 0$ *such that*

$$\sum_{\ell'=\ell+1}^{\infty} a_{\ell'}^s \le C_s \, a_\ell^s \quad \textit{for all } \ell \in \mathbb{N}_0.$$

(ii) ***R-linear convergence:*** *There exist* $0 < q_{\rm lin} < 1$ *and* $C_{\rm lin} > 0$ *with*

$$a_{\ell+n} \le C_{\rm lin}\, q_{\rm lin}^n \, a_\ell \quad \textit{for all } \ell, n \in \mathbb{N}_0. \qquad \square$$

Lemma 3.13 (tail summability criterion [BFM⁺25, Lemma 6]). *Let* $(a_\ell)_{\ell\in\mathbb{N}_0}$, $(b_\ell)_{\ell\in\mathbb{N}_0}$ *be nonnegative sequences of real numbers. With given constants* $0 < q < 1$, $0 < \delta < 1$, *and* $C_1, C_2 > 0$, *suppose that* $(a_\ell)_{\ell\in\mathbb{N}_0}$ *and* $(b_\ell)_{\ell\in\mathbb{N}_0}$ *satisfy the following conditions:*

(i) $a_{\ell+1} \le q\, a_\ell + b_\ell$ *for all* $\ell \in \mathbb{N}_0$

(ii) $b_{\ell+N} \le C_1\, a_\ell$ *for all* $\ell, N \in \mathbb{N}_0$

(iii) $\sum_{\ell'=\ell}^{\ell+N} b_{\ell'}^2 \le C_2\,(N+1)^{1-\delta}\, a_\ell^2$ *for all* $\ell, N \in \mathbb{N}_0$

Then, $(a_\ell)_{\ell\in\mathbb{N}_0}$ *is R-linearly convergent, i.e., there exist* $0 < q_{\rm lin} < 1$ *and* $C_{\rm lin} > 0$ *with*

$$a_{\ell+n} \le C_{\rm lin}\, q_{\rm lin}^n \, a_\ell \quad \textit{for all } \ell, n \in \mathbb{N}_0. \qquad \square$$

Proof of Theorem 3.11. First, we observe that global estimator equivalence (2.29) with $v_H := u_\ell^\star$ implies

$$\mathrm{M}_\ell^k = |||u_\ell^\star - u_\ell^k||| + \mu_\ell(u_\ell^\star) \simeq |||u_\ell^\star - u_\ell^k||| + \overline{\eta}_\ell(u_\ell^\star) =: \mathrm{H}_\ell^k. \tag{3.30}$$

Hence, to prove full R-linear convergence of M_ℓ^k, it suffices to show full R-linear convergence of H_ℓ^k. The proof is split into two steps.

Step 1 (tail summability of H_ℓ^k in ℓ). Let $\ell \in \mathbb{N}_0$ with $(\ell+1, \underline{k}) \in \mathcal{Q}$ be arbitrary. We note that $u_{\ell+1}^0 = u_\ell^{\underline{k}}$ by nested iteration and Algorithm B ensures $\underline{k}[\ell+1] \ge 1$. Thus, contraction of the algebraic solver (2.32) yields

$$|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k}}||| \overset{(2.32)}{\le} q_{\rm ctr}^{\underline{k}[\ell+1]}\, |||u_{\ell+1}^\star - u_{\ell+1}^0||| = q_{\rm ctr}^{\underline{k}[\ell+1]}\, |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| \le q_{\rm ctr}\, |||u_{\ell+1}^\star - u_\ell^{\underline{k}}|||. \tag{3.31}$$

We first prove contraction of the weighted quasi-error

$$\mathrm{H}_\ell := |||u_\ell^\star - u_\ell^{\underline{k}}||| + \gamma\, \overline{\eta}_\ell(u_\ell^\star)$$

with a suitable $0 < \gamma < 1$ chosen below. By definition, we have $\mathrm{H}_\ell \le \mathrm{H}_\ell^{\underline{k}} \le \gamma^{-1}\,\mathrm{H}_\ell$, i.e., it holds $\mathrm{H}_\ell \simeq \mathrm{H}_\ell^{\underline{k}}$. The error estimator $\overline{\eta}_\ell$ satisfies the assumptions of Lemma 3.4. Hence, the combination of inequality (3.31), estimator reduction (3.9) and the triangle inequality results in

$$\begin{aligned}
\mathrm{H}_{\ell+1} &\overset{(3.31)}{\le} q_{\rm ctr} |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| + \gamma\, \overline{\eta}_{\ell+1}(u_{\ell+1}^\star) \\
&\overset{(3.9)}{\le} q_{\rm ctr} |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| + \gamma \Big[q_{\widehat{\theta}}\, \overline{\eta}_\ell(u_\ell^\star) + C_{\widehat{\theta}}\, |||u_\ell^\star - u_\ell^{\underline{k}}||| + \overline{C}_{\rm stab} |||u_{\ell+1}^\star - u_\ell^\star||| \Big] \\
&\le (q_{\rm ctr} + \gamma\, C_{\widehat{\theta}})\, |||u_\ell^\star - u_\ell^{\underline{k}}||| + q_{\widehat{\theta}}\, \gamma\, \overline{\eta}_\ell(u_\ell^\star) + (q_{\rm ctr} + \gamma\, \overline{C}_{\rm stab})\, |||u_{\ell+1}^\star - u_\ell^\star||| \\
&\le \max\{q_{\rm ctr} + \gamma\, C_{\widehat{\theta}},\ q_{\widehat{\theta}}\} \Big[|||u_\ell^\star - u_\ell^{\underline{k}}||| + \gamma\, \overline{\eta}_\ell(u_\ell^\star) \Big] + (q_{\rm ctr} + \gamma\, \overline{C}_{\rm stab})\, |||u_{\ell+1}^\star - u_\ell^\star|||.
\end{aligned} \tag{3.32}$$

We set $q := \max\{q_{\rm ctr} + \gamma\, C_{\widehat{\theta}},\ q_{\widehat{\theta}}\}$. The choice $0 < \gamma < C_{\widehat{\theta}}^{-1}(1-q_{\rm ctr}) < 1$ ensures $0 < q < 1$. Thus, this verifies condition (i) from Lemma 3.13 for the sequences defined by $a_\ell := \mathrm{H}_\ell$ and $b_\ell := (q_{\rm ctr} + \gamma\,\overline{C}_{\rm stab})\,|||u_{\ell+1}^\star - u_\ell^\star|||$. The triangle inequality, reliability (A3), and quasi-monotonicity (QM) show for all $\ell, \ell', \ell'' \in \mathbb{N}_0$ with $\ell \le \ell' \le \ell''$ the estimate

$$
\begin{aligned}
|||u_{\ell''}^\star - u_{\ell'}^\star||| &\le |||u_{\ell''}^\star - u^\star||| + |||u^\star - u_{\ell'}^\star||| \\
&\overset{\text{(A3)}}{\le} C_{\rm rel}\big(\overline{\eta}_{\ell''}(u_{\ell''}^\star) + \overline{\eta}_{\ell'}(u_{\ell'}^\star)\big) \overset{\text{(QM)}}{\le} 2\,C_{\rm mon}C_{\rm rel}\,\overline{\eta}_\ell(u_\ell^\star) \le 2\,C_{\rm mon}C_{\rm rel}\,\gamma^{-1}\,\mathrm{H}_\ell.
\end{aligned}
\tag{3.33}
$$

In particular, this proves the second condition (ii) from Lemma 3.13. Quasi-orthogonality (QO) and reliability (A3) prove condition (iii) from Lemma 3.13 via

$$
\begin{aligned}
\sum_{\ell'=\ell}^{\ell+N} b_{\ell'}^2 \simeq \sum_{\ell'=\ell}^{\ell+N} |||u_{\ell+1}^\star - u_\ell^\star|||^2 &\overset{\text{(QO)}}{\lesssim} (N+1)^{1-\delta}\,|||u^\star - u_\ell^\star|||^2 \\
&\overset{\text{(A3)}}{\lesssim} (N+1)^{1-\delta}\,\overline{\eta}_\ell(u_\ell^\star)^2 \lesssim (N+1)^{1-\delta}(\mathrm{H}_\ell)^2.
\end{aligned}
$$

Since $\mathrm{H}_\ell^{\underline{k}} \simeq \mathrm{H}_\ell$, Lemma 3.13 thus concludes tail summability of $\mathrm{H}_\ell^{\underline{k}}$, i.e.,

$$
\sum_{\ell'=\ell+1}^{\underline{\ell}-1} \mathrm{H}_{\ell'}^{\underline{k}} \simeq \sum_{\ell'=\ell+1}^{\underline{\ell}-1} \mathrm{H}_{\ell'} \lesssim \mathrm{H}_\ell \simeq \mathrm{H}_\ell^{\underline{k}} \quad \text{for all } 0 \le \ell < \underline{\ell} - 1. \tag{3.34}
$$

Step 2 (tail summability of H_ℓ^k in ℓ and k). First, we consider $0 \le k < k' \le \underline{k}[\ell] - 1$ for $0 \le \ell \le \underline{\ell} - 1$. Recall that weak stability (W1) implies

$$
\mu_\ell(u_\ell^\star) \overset{\text{(W1)}}{\le} \widetilde{C}_{\rm stab}\Big[\mu_\ell(u_\ell^{k'}) + |||u_\ell^\star - u_\ell^{k'}|||\Big]. \tag{3.35}
$$

Hence, failure of the stopping criterion (2.46), the triangle inequality, and contraction of the solver (2.32) show

$$
\begin{aligned}
\mathrm{M}_\ell^{k'} &\overset{(3.35)}{\le} (1 + \widetilde{C}_{\rm stab})\,|||u_\ell^\star - u_\ell^{k'}||| + \widetilde{C}_{\rm stab}\,\mu_\ell(u_\ell^{k'}) \\
&\overset{(2.46)}{\le} (1 + \widetilde{C}_{\rm stab})\,|||u_\ell^\star - u_\ell^{k'}||| + \widetilde{C}_{\rm stab}\,\lambda^{-1}|||u_\ell^{k'} - u_\ell^{k'-1}||| \\
&\le \big(1 + \widetilde{C}_{\rm stab} + \widetilde{C}_{\rm stab}\,\lambda^{-1}\big)\,|||u_\ell^\star - u_\ell^{k'}||| + \widetilde{C}_{\rm stab}\,\lambda^{-1}|||u_\ell^\star - u_\ell^{k'-1}||| \\
&\overset{(2.32)}{\le} \Big[q_{\rm ctr}\,\big(1 + \widetilde{C}_{\rm stab} + \widetilde{C}_{\rm stab}\,\lambda^{-1}\big) + \widetilde{C}_{\rm stab}\,\lambda^{-1}\Big]\,|||u_\ell^\star - u_\ell^{k'-1}||| \\
&\overset{(2.32)}{\le} q_{\rm ctr}^{k'-k}\,\Big[1 + \widetilde{C}_{\rm stab} + \widetilde{C}_{\rm stab}\,\lambda^{-1} + q_{\rm ctr}^{-1}\,\widetilde{C}_{\rm stab}\,\lambda^{-1}\Big]\,|||u_\ell^\star - u_\ell^k||| \\
&\overset{(3.27)}{\le} q_{\rm ctr}^{k'-k}\,\Big[1 + \widetilde{C}_{\rm stab} + \widetilde{C}_{\rm stab}\,\lambda^{-1}\,\big(1 + q_{\rm ctr}^{-1}\big)\Big]\,\mathrm{M}_\ell^k \\
&\lesssim q_{\rm ctr}^{k'-k}\,\mathrm{M}_\ell^k
\end{aligned}
$$

Clearly, the inequality above applies for $k = k'$ as well. Due to (3.30), this contraction property also holds for H_ℓ^k, i.e.,

$$
\mathrm{H}_\ell^{k'} \lesssim q_{\rm ctr}^{k'-k}\,\mathrm{H}_\ell^k \quad \text{for all } 0 \le k \le k' < \underline{k}[\ell].
$$

Furthermore, contraction (2.32) yields

$$\mathrm{H}_\ell^k \overset{(2.32)}{\leq} q_{\mathrm{ctr}}\, |||u_\ell^\star - u_\ell^{k-1}||| + \overline{\eta}_\ell(u_\ell^\star) \lesssim \mathrm{H}_\ell^{k-1} \simeq q_{\mathrm{ctr}}\, \mathrm{H}_\ell^{k-1}.$$

Therefore, it follows that

$$\mathrm{H}_\ell^{k'} \lesssim q_{\mathrm{ctr}}^{k'-k}\, \mathrm{H}_\ell^k \quad \text{for all } 0 \leq k \leq k' \leq \underline{k}[\ell]. \tag{3.36}$$

Analogous to (3.7) and (3.17), patch stability (M1) and patch reduction (M2) yield monotonicity of the estimator

$$\overline{\eta}_{\ell+1}(u_\ell^\star) \leq \overline{\eta}_\ell(u_\ell^\star) \quad \text{for all } \ell < \underline{\ell}. \tag{3.37}$$

Together with patch stability (M1), the triangle inequality, and estimate (3.33), this proves

$$\begin{aligned}
\mathrm{H}_{\ell+1}^0 &= |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| + \overline{\eta}_{\ell+1}(u_{\ell+1}^\star) \\
&\overset{(M1)}{\leq} |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| + \overline{C}_{\mathrm{stab}} |||u_{\ell+1}^\star - u_\ell^\star||| + \overline{\eta}_{\ell+1}(u_\ell^\star) \\
&\overset{(3.37)}{\leq} |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| + \overline{C}_{\mathrm{stab}} |||u_{\ell+1}^\star - u_\ell^\star||| + \overline{\eta}_\ell(u_\ell^\star) \\
&\leq (1 + \overline{C}_{\mathrm{stab}})\, |||u_{\ell+1}^\star - u_\ell^\star||| + \mathrm{H}_\ell^{\underline{k}} \\
&\overset{(3.33)}{\leq} \big[1 + (1 + \overline{C}_{\mathrm{stab}})\, 2\, C_{\mathrm{mon}} C_{\mathrm{rel}}\, \gamma^{-1}\big]\, \mathrm{H}_\ell^{\underline{k}}.
\end{aligned} \tag{3.38}$$

Therefore, tail summability follows from the geometric series via

$$\begin{aligned}
\sum_{\substack{(\ell',k') \in \mathcal{Q} \\ |\ell',k'| > |\ell,k|}} \mathrm{H}_{\ell'}^{k'} &= \sum_{k'=k+1}^{\underline{k}[\ell]} \mathrm{H}_\ell^{k'} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \mathrm{H}_{\ell'}^{k'} \\
&\overset{(3.36)}{\lesssim} \mathrm{H}_\ell^k + \sum_{\ell'=\ell+1}^{\underline{\ell}} \mathrm{H}_{\ell'}^0 \overset{(3.38)}{\lesssim} \mathrm{H}_\ell^k + \sum_{\ell'=\ell}^{\underline{\ell}-1} \mathrm{H}_{\ell'}^{\underline{k}} \overset{(3.34)}{\lesssim} \mathrm{H}_\ell^k + \mathrm{H}_\ell^{\underline{k}} \overset{(3.36)}{\lesssim} \mathrm{H}_\ell^k
\end{aligned}$$

for all $(\ell, k) \in \mathcal{Q}$. Since $\mathcal{Q}$ is countable and linearly ordered with respect to $|\cdot,\cdot|$, we can employ Lemma 3.12 to conclude full R-linear convergence of H_ℓ^k, which by (3.30) implies full R-linear convergence of M_ℓ^k. This concludes the proof. □

In case of weak local equivalence (2.23), we must additionally require weak stability (W1) in order to obtain the estimator reduction (3.9). With that assumption, we can still show parameter-robust full R-linear convergence for M_ℓ^k using the same arguments as for strong local equivalence (2.22) in Theorem 3.11.

Theorem 3.14 (full R-linear convergence for weak local equivalence (2.23)). *Let $0 \leq \theta \leq 1$, $C_{\mathrm{mark}} \geq 1$, $\lambda > 0$ and $u_0^0 \in \mathcal{X}_0$ be arbitrary. Suppose that the error estimator μ_ℓ is weakly equivalent to an estimator $\overline{\eta}_\ell$ in the sense of (2.23) and satisfies weak stability (W1). With $m \in \mathbb{N}_0$ from the equivalence (2.23) and $r \in \mathbb{N}_0$ from weak stability (W1), suppose that $\overline{\eta}_\ell$ satisfies (M1), (M2), (A3), and (QM) with $M := m + r$. Then, Algorithm B* *guarantees full R-linear convergence of the quasi-error* M_ℓ^k *from* (3.27), *i.e., there exist constants* $0 < q_{\mathrm{lin}} < 1$ *and* $C_{\mathrm{lin}} > 0$ *such that*

$$\mathrm{M}_\ell^k \le C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\ell,k|-|\ell',k'|} \mathrm{M}_{\ell'}^{k'} \quad \text{for all } (\ell',k'),(\ell,k) \in \mathcal{Q} \text{ with } |\ell',k'| \le |\ell,k|.$$

The constants C_{lin} *and* q_{lin} *depend only on* $\overline{C}_{\mathrm{stab}}$, $\widetilde{C}_{\mathrm{stab}}$, $\overline{q}_{\mathrm{red}}$, C_{rel}, C_{eq}, q_{ctr}, C_{mon}, C_{orth}, θ *and* λ.

Proof. We note that the global estimator equivalence (2.30) implies $\mathrm{M}_\ell^k \simeq \mathrm{H}_\ell^k$. Hence, we can proceed as in the proof of Theorem 3.11. In the following, we will only highlight the changes that are necessary to account for the different assumptions on the estimators μ_ℓ and $\overline{\eta}_\ell$.

In Step 1, we used the estimator equivalence only in terms of estimator reduction (3.9). This perturbed estimator reduction is also provided by Lemma 3.7, whose assumptions are satisfied by μ_ℓ and $\overline{\eta}_\ell$. Consequently, we only need to change the constants $q_{\widehat{\theta}}$ and $C_{\widehat{\theta}}$ in inequality (3.32) to $q_{\overline{\theta}}$ and $C_{\overline{\theta}}$, which then reads

$$\mathrm{H}_{\ell+1} \;\le\; \max\{q_{\mathrm{ctr}} + \gamma\, C_{\overline{\theta}},\; q_{\overline{\theta}}\} \Big[|||u_\ell^\star - u_\ell^{\underline{k}}||| + \gamma\, \overline{\eta}_\ell(u_\ell^\star) \Big] + (q_{\mathrm{ctr}} + \gamma\, \overline{C}_{\mathrm{stab}})\, |||u_{\ell+1}^\star - u_\ell^\star|||.$$

Defining $q := \max\{q_{\mathrm{ctr}} + \gamma\, C_{\overline{\theta}},\; q_{\overline{\theta}}\}$ analogously with $0 < \gamma < \min\big\{1,\, C_{\overline{\theta}}^{-1}(1 - q_{\mathrm{ctr}})\big\} \le 1$, we can therefore conclude tail summability of $\mathrm{H}_\ell^{\underline{k}}$ in ℓ identically as in the first step of the original proof.

Step 2 does not make use of the equivalence of μ_ℓ and $\overline{\eta}_\ell$. Therefore, all arguments of the second step hold verbatim. Overall, we can thus conclude full R-linear convergence of M_ℓ^k in the same way as in the proof of Theorem 3.11. □

In case that $a(\cdot,\cdot)$ is a scalar product, and thus the Pythagorean identity (2.7) holds, we can prove an even stronger result. In the following, we prove that the quasi-error Λ_ℓ^k introduced below contracts with every step of Algorithm B. As for the proof of full R-linear convergence, the two notions of estimator equivalence require different assumptions and are therefore covered in separate theorems.

Theorem 3.15 (contraction of quasi-error for strong local equivalence (2.22)). *Suppose that* $a(\cdot,\cdot)$ *is a scalar product so that the Pythagorean identity* (2.7) *holds. Let* $0 < \theta \le 1$, $C_{\mathrm{mark}} \ge 1$, $\lambda > 0$ *and* $u_0^0 \in \mathcal{X}_0$ *be arbitrary. Suppose that the error estimator* μ_ℓ *is strongly equivalent to an estimator* $\overline{\eta}_\ell$ *in the sense of* (2.22). *With* $m \in \mathbb{N}_0$ *from the equivalence* (2.22), *suppose that* $\overline{\eta}_\ell$ *satisfies* (M1), (M2), (A3) *with* $M := m$. *Let* $\mathcal{Q}^\# := \{(\ell,k) \in \mathcal{Q} : 0 \le k < \underline{k}[\ell]\}$ *be the set of all iterates generated by Algorithm B without the nested iterates* $u_\ell^{\underline{k}}$. *Then, Algorithm B guarantees the existence of constants* $0 < q_{\mathrm{lin}} < 1$ *and* $0 < \gamma < 1$ *such that the quasi-error*

$$\Lambda_\ell^k := |||u^\star - u_\ell^k|||^2 + \gamma\, \overline{\eta}_\ell(u_\ell^k)^2 \quad \text{for all } (\ell,k) \in \mathcal{Q}^\#$$

satisfies the following statements:

(i) $\Lambda_\ell^{k+1} \le q_{\mathrm{lin}}\, \Lambda_\ell^k$ *for all* $(\ell, k+1) \in \mathcal{Q}^\#$.

(ii) $\Lambda_{\ell+1}^{0} \leq q_{\mathrm{lin}}\, \Lambda_{\ell}^{\underline{k}-1}$ *for all* $(\ell+1, 0) \in \mathcal{Q}^{\#}$.

The constants q_{lin} *and* γ *depend only on* $\overline{C}_{\mathrm{stab}}$, $\overline{q}_{\mathrm{red}}$, C_{rel}, q_{ctr}, C_{eq}, θ *and* λ.

Proof. The proof follows the reasoning of [GHPS21, Lemma 10] and is split into three steps. In the first two steps, we prove the statements (i) and (ii) under certain assumptions on parameters ε, γ and δ. In the last step, we fix those parameters such that the assumptions from the previous two steps are satisfied.

Step 1 (proof of Theorem 3.15 (i)). Let ε and γ be free parameters, which will be fixed below. For arbitrary $(\ell, k+1) \in \mathcal{Q}^{\#}$, reliability (A3), patch stability (M1), and the Young inequality (3.14) provide

$$|||u^\star - u_\ell^\star|||^2 \overset{(A3)}{\leq} C_{\mathrm{rel}}^2\, \overline{\eta}_\ell(u_\ell^\star)^2 \overset{(M1)}{\underset{(3.14)}{\leq}} 2\, C_{\mathrm{rel}}^2\, \overline{\eta}_\ell(u_\ell^{k+1})^2 + 2\, C_{\mathrm{rel}}^2\, \overline{C}_{\mathrm{stab}}^2\, |||u_\ell^\star - u_\ell^{k+1}|||^2. \tag{3.39}$$

We define $C_1 \coloneqq 2\, C_{\mathrm{rel}}^2$ and $C_2 \coloneqq 2\, C_{\mathrm{rel}}^2\, \overline{C}_{\mathrm{stab}}^2$. Together with the Pythagorean identity (2.7) and contraction of the algebraic solver (2.32), this leads to

$$\begin{aligned}
\Lambda_\ell^{k+1} &\overset{(2.7)}{=} (1-\varepsilon)\, |||u^\star - u_\ell^\star|||^2 + \varepsilon\, |||u^\star - u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^{k+1}|||^2 + \gamma\, \overline{\eta}_\ell(u_\ell^{k+1})^2 \\
&\overset{(3.39)}{\leq} (1-\varepsilon)\, |||u^\star - u_\ell^\star|||^2 + (\gamma + \varepsilon\, C_1)\, \overline{\eta}_\ell(u_\ell^{k+1})^2 + (1 + \varepsilon\, C_2)\, |||u_\ell^\star - u_\ell^{k+1}|||^2. \\
&\overset{(2.32)}{\leq} (1-\varepsilon)\, |||u^\star - u_\ell^\star|||^2 + (\gamma + \varepsilon\, C_1)\, \overline{\eta}_\ell(u_\ell^{k+1})^2 + q_{\mathrm{ctr}}^2 (1 + \varepsilon\, C_2)\, |||u_\ell^\star - u_\ell^{k}|||^2.
\end{aligned}$$

Since $k+1 < \underline{k}[\ell]$ by the definition of $\mathcal{Q}^{\#}$, failure of the stopping criterion (2.46) and estimate (2.33) thus yield

$$\begin{aligned}
\Lambda_\ell^{k+1} &\overset{(2.46)}{\leq} (1-\varepsilon)\, |||u^\star - u_\ell^\star|||^2 + (\gamma + \varepsilon\, C_1)\, \lambda^{-2}\, |||u_\ell^{k+1} - u_\ell^{k}|||^2 + q_{\mathrm{ctr}}^2 (1 + \varepsilon\, C_2)\, |||u_\ell^\star - u_\ell^{k}|||^2 \\
&\overset{(2.33)}{\leq} (1-\varepsilon)\, |||u^\star - u_\ell^\star|||^2 + \Big[(\gamma + \varepsilon\, C_1)\, \lambda^{-2}\, (1 + q_{\mathrm{ctr}})^2 + q_{\mathrm{ctr}}^2 (1 + \varepsilon\, C_2)\Big]\, |||u_\ell^\star - u_\ell^{k}|||^2.
\end{aligned}$$

Provided that ε, γ are chosen such that

$$(\gamma + \varepsilon\, C_1)\, \lambda^{-2}\, (1 + q_{\mathrm{ctr}})^2 + q_{\mathrm{ctr}}^2 (1 + \varepsilon\, C_2) \;\leq\; (1-\varepsilon), \tag{3.40}$$

the Pythagorean identity (2.7) verifies

$$\Lambda_\ell^{k+1} \overset{(3.40)}{\leq} (1-\varepsilon)\, \big(|||u^\star - u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^{k}|||^2\big) \overset{(2.7)}{=} (1-\varepsilon)\, |||u^\star - u_\ell^{k}|||^2 \leq (1-\varepsilon)\, \Lambda_\ell^{k}.$$

Up to the final choice of ε and γ, this concludes the proof of Theorem 3.15 (i).

Step 2 (proof of Theorem 3.15 (ii)). Let ε, γ, and δ be free parameters, which will be fixed below. Analogous to estimate (3.39), reliability (A3), patch stability (M1), and the Young inequality (3.14) provide

$$|||u^\star - u_\ell^\star|||^2 \overset{(A3)}{\leq} C_{\mathrm{rel}}^2\, \overline{\eta}_\ell(u_\ell^\star)^2 \overset{(M1)}{\underset{(3.14)}{\leq}} C_1\, \overline{\eta}_\ell(u_\ell^{\underline{k}-1})^2 + C_2\, |||u_\ell^\star - u_\ell^{\underline{k}-1}|||^2. \tag{3.41}$$

Together with the Pythagorean identity (2.7) and contraction of the algebraic solver (2.32), this leads to

$$\begin{aligned} |||u^\star - u_\ell^k|||^2 &\overset{(2.7)}{=} (1-\varepsilon)\,|||u^\star - u_\ell^\star|||^2 + \varepsilon\,|||u^\star - u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^k|||^2 \\ &\overset{(3.41)}{\le} (1-\varepsilon)\,|||u^\star - u_\ell^\star|||^2 + \varepsilon\,C_1\,\overline{\eta}_\ell(u_\ell^{k-1})^2 + \varepsilon\,C_2\,|||u_\ell^\star - u_\ell^{k-1}|||^2 + |||u_\ell^\star - u_\ell^k|||^2 \\ &\overset{(2.32)}{\le} (1-\varepsilon)\,|||u^\star - u_\ell^\star|||^2 + \varepsilon\,C_1\,\overline{\eta}_\ell(u_\ell^{k-1})^2 + (\varepsilon\,C_2 + q_{\mathrm{ctr}}^2)\,|||u_\ell^\star - u_\ell^{k-1}|||^2. \end{aligned} \tag{3.42}$$

The triangle inequality and contraction of the algebraic solver (2.32) prove

$$|||u_\ell^k - u_\ell^{k-1}||| \le |||u_\ell^k - u_\ell^\star||| + |||u_\ell^\star - u_\ell^{k-1}||| \overset{(2.32)}{\le} (1+q_{\mathrm{ctr}})\,|||u_\ell^\star - u_\ell^{k-1}|||. \tag{3.43}$$

Together with patch stability (M1) and the Young inequality for $\delta > 0$, this yields for $C_3 := \overline{C}_{\mathrm{stab}}^2\,(1+q_{\mathrm{ctr}})^2$

$$\begin{aligned} \overline{\eta}_\ell(u_\ell^k)^2 &\overset{(M1)}{\le} (1+\delta)\,\overline{\eta}_\ell(u_\ell^{k-1})^2 + (1+\delta^{-1})\,\overline{C}_{\mathrm{stab}}^2\,|||u_\ell^k - u_\ell^{k-1}|||^2 \\ &\overset{(3.43)}{\le} (1+\delta)\,\overline{\eta}_\ell(u_\ell^{k-1})^2 + (1+\delta^{-1})\,C_3\,|||u_\ell^\star - u_\ell^{k-1}|||^2. \end{aligned} \tag{3.44}$$

Recall estimate (3.8) and $0 < q_{\widehat{\theta}} < 1$ from Lemma 3.4, which was derived using patch stability (M1) and patch reduction (M2). Furthermore, note that Algorithm B guarantees nested iteration $u_{\ell+1}^0 = u_\ell^k$. Combining this with estimates (3.42) and (3.44), we obtain

$$\begin{aligned} \Lambda_{\ell+1}^0 &\overset{(3.8)}{\le} |||u^\star - u_\ell^k|||^2 + \gamma\,q_{\widehat{\theta}}^2\,\overline{\eta}_\ell(u_\ell^k)^2 \\ &\overset{(3.42)}{\le} (1-\varepsilon)\,|||u^\star - u_\ell^\star|||^2 + \varepsilon\,C_1\,\overline{\eta}_\ell(u_\ell^{k-1})^2 + (\varepsilon\,C_2 + q_{\mathrm{ctr}}^2)\,|||u_\ell^\star - u_\ell^{k-1}|||^2 + \gamma\,q_{\widehat{\theta}}^2\,\overline{\eta}_\ell(u_\ell^k)^2 \\ &\overset{(3.44)}{\le} (1-\varepsilon)\,|||u^\star - u_\ell^\star|||^2 + \big[\varepsilon\,C_1\,\gamma^{-1} + (1+\delta)\,q_{\widehat{\theta}}^2\big]\,\gamma\,\overline{\eta}_\ell(u_\ell^{k-1})^2 \\ &\qquad + \big[\varepsilon\,C_2 + q_{\mathrm{ctr}}^2 + \gamma\,q_{\widehat{\theta}}^2\,(1+\delta^{-1})\,C_3\big]\,|||u_\ell^\star - u_\ell^{k-1}|||^2. \end{aligned}$$

Provided that

$$\varepsilon\,C_1\,\gamma^{-1} + (1+\delta)\,q_{\widehat{\theta}}^2 \le 1-\varepsilon \quad \text{and} \quad \varepsilon\,C_2 + q_{\mathrm{ctr}}^2 + \gamma\,q_{\widehat{\theta}}^2\,(1+\delta^{-1})\,C_3 \le 1-\varepsilon, \tag{3.45}$$

the Pythagorean identity (2.7) verifies

$$\Lambda_{\ell+1}^0 \overset{(3.45)}{\le} (1-\varepsilon)\,\Big[|||u^\star - u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^{k-1}|||^2 + \gamma\,\overline{\eta}_\ell(u_\ell^{k-1})^2\Big] \overset{(2.7)}{=} (1-\varepsilon)\,\Lambda_\ell^{k-1}.$$

Up to the final choice of ε, γ, and δ, this concludes the proof of Theorem 3.15 (ii).

Step 3 (fixing the free parameters). Note that the constants C_1, C_2, C_3, and $q_{\widehat{\theta}}$ depend only on the problem setting. We proceed as follows:

- Choose $\delta > 0$ such that $(1+\delta)\,q_{\widehat{\theta}}^2 < 1$.

- Choose $\gamma > 0$ such that $q_{\mathrm{ctr}}^2 + \gamma\, q_{\overline{\theta}}^2\,(1+\delta^{-1})\, C_3 < 1$ and $\gamma\,\lambda^{-2}\,(1+q_{\mathrm{ctr}})^2 + q_{\mathrm{ctr}}^2 < 1$.
- Finally, choose $\varepsilon > 0$ sufficiently small such that (3.40) and (3.45) are satisfied.

This concludes the proof of Theorem 3.15 with $q_{\mathrm{lin}} := 1 - \varepsilon$. □

As with the proof of full R-linear convergence, we need to additionally suppose weak stability (W1) in case of weak local equivalence (2.23) in order to prove contraction of Λ_ℓ^k.

Theorem 3.16 (contraction of quasi-error for weak local equivalence (2.23)). *Suppose that $a(\cdot,\cdot)$ is a scalar product so that the Pythagorean identity (2.7) holds. Let $0 < \theta \le 1$, $C_{\mathrm{mark}} \ge 1$, $\lambda > 0$, and $u_0^0 \in \mathcal{X}_0$ be arbitrary. Suppose that the error estimator μ_ℓ is weakly equivalent to an estimator $\overline{\eta}_\ell$ in the sense of (2.23) and satisfies weak stability (W1). With $m \in \mathbb{N}_0$ from the equivalence (2.22) and $r \in \mathbb{N}_0$ from weak stability (W1), suppose that $\overline{\eta}_\ell$ satisfies (M1), (M2), (A3) with $M := m + r$. Let $\mathcal{Q}^\# := \{(\ell,k) \in \mathcal{Q} : 0 \le k < \underline{k}[\ell]\}$ be the set of all iterates generated by Algorithm B without the nested iterates $u_\ell^{\underline{k}}$. Then, Algorithm B guarantees the existence of constants $0 < q_{\mathrm{lin}} < 1$ and $0 < \gamma < 1$ such that the quasi-error*

$$\Lambda_\ell^k := |||u^\star - u_\ell^k|||^2 + \gamma\,\overline{\eta}_\ell(u_\ell^k)^2 \quad \text{for all } (\ell,k) \in \mathcal{Q}^\#$$

satisfies the following statements:

(i) $\Lambda_\ell^{k+1} \le q_{\mathrm{lin}}\,\Lambda_\ell^k$ *for all* $(\ell, k+1) \in \mathcal{Q}^\#$.

(ii) $\Lambda_{\ell+1}^0 \le q_{\mathrm{lin}}\,\Lambda_\ell^{\underline{k}-1}$ *for all* $(\ell+1, 0) \in \mathcal{Q}^\#$.

The constants q_{lin} and γ depend only on $\overline{C}_{\mathrm{stab}}$, $\overline{q}_{\mathrm{red}}$, C_{rel}, $\widetilde{C}_{\mathrm{stab}}$, q_{ctr}, C_{eq}, θ and λ.

Proof. Similar to Theorem 3.15, the proof follows the reasoning of [GHPS21, Lemma 10] and is split into three steps. In the first two steps, we prove the statements (i) and (ii) under certain assumptions on the parameters ε, γ and δ. In the last step, we fix those parameters such that the assumptions from the previous two steps are satisfied. We highlight only the changes that are necessary to account for the different assumptions on the estimators μ_ℓ and $\overline{\eta}_\ell$.

Step 1 (proof of Theorem 3.16 (i)). All arguments in Step 1 of the proof of Theorem 3.15 do not make use of the equivalence of μ_ℓ and $\overline{\eta}_\ell$ and therefore hold verbatim. Provided that (3.40) is satisfied, this concludes the proof of Theorem 3.16 (i) up to the final choice of ε and γ.

Step 2 (proof of Theorem 3.16 (ii)). Let ε, γ, and δ be free parameters, which will be fixed below. Analogously to estimate (3.17), patch stability (M1) and patch reduction (M2) verify, for all $\ell < \underline{\ell}$, that

$$\begin{aligned}
\overline{\eta}_{\ell+1}(u_\ell^{\underline{k}})^2 &= \overline{\eta}_{\ell+1}(\mathcal{T}_{\ell+1} \setminus \mathcal{T}_{\ell+1}|_{\Omega_{\ell,\ell+1}^{(m+r)}}, u_\ell^{\underline{k}})^2 + \overline{\eta}_{\ell+1}(\mathcal{T}_{\ell+1}|_{\Omega_{\ell,\ell+1}^{(m+r)}}, u_\ell^{\underline{k}})^2 \\
&\overset{\text{(M1)}}{=} \overline{\eta}_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\ell|_{\Omega_{\ell,\ell+1}^{(m+r)}}, u_\ell^{\underline{k}})^2 + \overline{\eta}_{\ell+1}(\mathcal{T}_{\ell+1}|_{\Omega_{\ell,\ell+1}^{(m+r)}}, u_\ell^{\underline{k}})^2 \\
&\overset{\text{(M2)}}{\le} \overline{\eta}_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\ell|_{\Omega_{\ell,\ell+1}^{(m+r)}}, u_\ell^{\underline{k}})^2 + \overline{q}_{\mathrm{red}}^2\,\overline{\eta}_\ell(\mathcal{T}_\ell|_{\Omega_{\ell,\ell+1}^{(m+r)}}, u_\ell^{\underline{k}})^2 \\
&= \overline{\eta}_\ell(u_\ell^{\underline{k}})^2 - (1 - \overline{q}_{\mathrm{red}}^2)\,\overline{\eta}_\ell(\mathcal{T}_\ell|_{\Omega_{\ell,\ell+1}^{(m+r)}}, u_\ell^{\underline{k}})^2.
\end{aligned} \tag{3.46}$$

Recall $\overline{C}_{\mathrm{per}} > 0$ from Lemma 3.6. The Dörfler-like inequality (3.12) and $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ imply

$$\begin{aligned} \bar\theta\, \overline{\eta}_\ell(u_\ell^k)^2 &\overset{(3.12)}{\le} \overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{M}_\ell], u_\ell^k)^2 + \overline{C}_{\mathrm{per}}^2 |||u_\ell^\star - u_\ell^k|||^2 \\ &\le \overline{\eta}_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}], u_\ell^k)^2 + \overline{C}_{\mathrm{per}}^2 |||u_\ell^\star - u_\ell^k|||^2 \\ &= \overline{\eta}_\ell(\mathcal{T}_\ell|_{\Omega_{\ell,\ell+1}^{(m+r)}}, u_\ell^k)^2 + \overline{C}_{\mathrm{per}}^2 |||u_\ell^\star - u_\ell^k|||^2. \end{aligned} \tag{3.47}$$

With $1/2 < q_{\overline{\theta}}^2 := 1 - (1 - \overline{q}_{\mathrm{red}}^2)\overline{\theta} < 1$ and $\overline{C}_{\overline{\theta}}^2 := \overline{C}_{\mathrm{per}}^2(1 - \overline{q}_{\mathrm{red}}^2) > 0$, the combination of the last two estimates results in

$$\overline{\eta}_{\ell+1}(u_\ell^k)^2 \le q_{\overline{\theta}}^2\, \overline{\eta}_\ell(u_\ell^k)^2 + \overline{C}_{\overline{\theta}}^2\, |||u_\ell^\star - u_\ell^k|||^2. \tag{3.48}$$

Moreover, the estimates (3.42) and (3.44) from Step 2 of the proof of Theorem 3.15 hold verbatim. Together with nested iteration and contraction (2.32), the combination of above estimates yields

$$\begin{aligned} \Lambda_{\ell+1}^0 &\overset{(3.48)}{\le} |||u^\star - u_\ell^k|||^2 + \gamma\, q_{\overline{\theta}}^2\, \overline{\eta}_\ell(u_\ell^k)^2 + \gamma\, \overline{C}_{\overline{\theta}}^2\, |||u_\ell^\star - u_\ell^k|||^2 \\ &\overset{(2.32)}{\le} |||u^\star - u_\ell^k|||^2 + \gamma\, q_{\overline{\theta}}^2\, \overline{\eta}_\ell(u_\ell^k)^2 + \gamma\, \overline{C}_{\overline{\theta}}^2\, q_{\mathrm{ctr}}^2\, |||u_\ell^\star - u_\ell^{k-1}|||^2 \\ &\overset{(3.42)}{\le} (1-\varepsilon)\, |||u^\star - u_\ell^\star|||^2 + \varepsilon\, C_1\, \overline{\eta}_\ell(u_\ell^{k-1})^2 \\ &\qquad + \big[\varepsilon\, C_2 + q_{\mathrm{ctr}}^2 + \gamma\, \overline{C}_{\overline{\theta}}^2\, q_{\mathrm{ctr}}^2\big]\, |||u_\ell^\star - u_\ell^{k-1}|||^2 + \gamma\, q_{\overline{\theta}}^2\, \overline{\eta}_\ell(u_\ell^k)^2 \\ &\overset{(3.44)}{\le} (1-\varepsilon)\, |||u^\star - u_\ell^\star|||^2 + \big[\varepsilon\, C_1\, \gamma^{-1} + (1+\delta)\, q_{\overline{\theta}}^2\big]\, \gamma\, \overline{\eta}_\ell(u_\ell^{k-1})^2 \\ &\qquad + \Big[\varepsilon\, C_2 + q_{\mathrm{ctr}}^2 + \gamma\, \big(\overline{C}_{\overline{\theta}}^2\, q_{\mathrm{ctr}}^2 + q_{\overline{\theta}}^2\, (1+\delta^{-1})\, C_3\big)\Big]\, |||u_\ell^\star - u_\ell^{k-1}|||^2. \end{aligned}$$

Provided that

$$\varepsilon\, C_1\, \gamma^{-1} + (1+\delta)\, q_{\overline{\theta}}^2 \le 1-\varepsilon \quad \text{and} \quad \varepsilon\, C_2 + q_{\mathrm{ctr}}^2 + \gamma\, \big(\overline{C}_{\overline{\theta}}^2\, q_{\mathrm{ctr}}^2 + q_{\overline{\theta}}^2\, (1+\delta^{-1})\, C_3\big) \le 1-\varepsilon, \tag{3.49}$$

the Pythagorean identity (2.7) verifies

$$\Lambda_{\ell+1}^0 \overset{(3.45)}{\le} (1-\varepsilon)\, \Big[|||u^\star - u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^{k-1}|||^2 + \gamma\, \overline{\eta}_\ell(u_\ell^{k-1})^2\Big] \overset{(2.7)}{=} (1-\varepsilon)\, \Lambda_\ell^{k-1}.$$

Up to the final choice of ε, γ, and δ, this concludes the proof of Theorem 3.16 (ii).

Step 3 (fixing the free parameters). Note that the constants C_1, C_2, C_3, and $q_{\overline{\theta}}$ depend only on the problem setting. We proceed as follows:

- Choose $\delta > 0$ such that $(1+\delta)\, q_{\overline{\theta}}^2 < 1$.
- Choose $\gamma > 0$ such that

$$\gamma\, \big(\overline{C}_{\overline{\theta}}^2\, q_{\mathrm{ctr}}^2 + q_{\overline{\theta}}^2\, (1+\delta^{-1})\, C_3\big) + q_{\mathrm{ctr}}^2 < 1 \quad \text{and} \quad \gamma\, \lambda^{-2}\, (1+q_{\mathrm{ctr}})^2 + q_{\mathrm{ctr}}^2 < 1.$$

- Finally, choose $\varepsilon > 0$ sufficiently small such that (3.40) and (3.49) are satisfied.

This concludes the proof of Theorem 3.16 with $q_{\mathrm{lin}} := 1 - \varepsilon$. □

4 Optimal complexity

Recall the quasi-error M_ℓ^k from (3.27) that satisfies full R-linear convergence (3.29) from Theorem 3.11 (for strong local equivalence (2.22)) and Theorem 3.14 (for weak local equivalence (2.23)). This section reveals that this is the crucial component to link optimal convergence rates with respect to the number of degrees of freedom and optimal convergence rates with respect to the total computational cost (and hence time). To this end, we say that M_ℓ^k decays with rate $s > 0$ over the number of elements $\#\mathcal{T}_\ell$ if and only if $\mathrm{M}_\ell^k \in \mathcal{O}((\#\mathcal{T}_\ell)^s)$, i.e., it holds that

$$A(s) \coloneqq \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \,\mathrm{M}_\ell^k < \infty. \tag{4.1}$$

As discussed in Sections 2.2–2.5, the modules *Solve & Estimate*, *Mark* and *Refine* from Algorithm B can be realized at linear cost $\mathcal{O}(\#\mathcal{T}_\ell)$. Since the adaptive algorithm relies on the full history of prior algorithmic decisions, the overall computational cost $\texttt{cost}(\ell,k)$ until step $(\ell,k)\in\mathcal{Q}$ (i.e., the cost to compute u_ℓ^k) is thus proportional to

$$\texttt{cost}(\ell,k) \;\simeq \sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'}. \tag{4.2}$$

The goal of this chapter is to prove that Algorithm B ensures optimal convergence of M_ℓ^k with respect to the total computational cost, i.e., the quasi-error decays with the best possible convergence rate $s > 0$. To this end, we first prove two crucial corollaries of full R-linear convergence in Section 4.1. Then, we formalize the notion of optimality and prove optimal complexity as the main result of this chapter in Theorem 4.3. The analysis in this chapter proceeds along the lines of similar results in [CFPP14] and [MPS24].

4.1 Corollaries of full R-linear convergence

A first crucial consequence of full R-linear convergence is that if the rate of convergence $s > 0$ is attainable with respect to the degrees of freedom $\dim\mathcal{X}_\ell \simeq \#\mathcal{T}_\ell$, it is also achievable with respect to the total computational cost.

Corollary 4.1 (rates = complexity). *Suppose full R-linear convergence (3.29) of the quasi-error M_ℓ^k. Then, for all $s > 0$, it holds that*

$$A(s) \le \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'}\Big)^s \mathrm{M}_\ell^k \le \frac{C_{\mathrm{lin}}}{(1-q_{\mathrm{lin}}^{1/s})^s}\, A(s). \tag{4.3}$$

Moreover, there exists an $s_0 > 0$ such that $A(s) < \infty$ for all $0 < s \le s_0$.

Proof. The first inequality of (4.3) is clear by definition of $A(s)$ in (4.1). Moreover, the definition shows that

$$\#\mathcal{T}_{\ell'} \leq A(s)^{1/s} \big(\mathrm{M}_{\ell'}^{k'}\big)^{-1/s} \quad \text{for all } (\ell', k') \in \mathcal{Q}. \tag{4.4}$$

Hence, full R-linear convergence (3.29) and the geometric series show

$$\begin{aligned} \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} \#\mathcal{T}_{\ell'} \overset{(4.4)}{\leq} A(s)^{1/s} \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} \big(\mathrm{M}_{\ell'}^{k'}\big)^{-1/s} &\overset{(3.29)}{\leq} A(s)^{1/s}\, C_{\mathrm{lin}}^{1/s} \big(\mathrm{M}_{\ell}^{k}\big)^{-1/s} \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} \big(q_{\mathrm{lin}}^{1/s}\big)^{|\ell,k|-|\ell',k'|} \\ &\leq A(s)^{1/s} \frac{C_{\mathrm{lin}}^{1/s}}{\big(1-q_{\mathrm{lin}}^{1/s}\big)} \big(\mathrm{M}_{\ell}^{k}\big)^{-1/s}. \end{aligned}$$

The upper bound from (4.3) thus follows from a rearrangement of this estimate. Therefore, it only remains to verify that there exists an $s_0 > 0$ such that $A(s) < \infty$ for $0 < s \leq s_0$. Note that successive application of the child estimate (R1) implies

$$0 \leq \#\mathcal{T}_{\ell} \overset{(R1)}{\leq} C_{\mathrm{child}}\, \#\mathcal{T}_{\ell-1} \overset{(R1)}{\leq} C_{\mathrm{child}}^{\ell}\, \#\mathcal{T}_0 \leq C_{\mathrm{child}}^{|\ell,k|}\, \#\mathcal{T}_0 \quad \text{for all } (\ell,k) \in \mathcal{Q}.$$

Since full R-linear convergence (3.29) guarantees

$$0 \leq \mathrm{M}_{\ell}^{k} \overset{(3.29)}{\leq} C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\ell,k|}\, \mathrm{M}_0^0 \quad \text{for all } (\ell,k) \in \mathcal{Q},$$

the multiplication of the previous two estimates thus yields

$$(\#\mathcal{T}_{\ell})^s\, \mathrm{M}_{\ell}^{k} \leq (C_{\mathrm{child}}^s\, q_{\mathrm{lin}})^{|\ell,k|}\, C_{\mathrm{lin}}\, (\#\mathcal{T}_0)^s\, \mathrm{M}_0^0 \quad \text{for all } (\ell,k) \in \mathcal{Q}.$$

The right-hand side is uniformly bounded, provided that $s > 0$ is sufficiently small such that $C_{\mathrm{child}}^s\, q_{\mathrm{lin}} \leq 1$. This concludes the proof with $s_0 \coloneqq \log(1/q_{\mathrm{lin}})/\log(C_{\mathrm{child}})$. □

Another important implication of full R-linear convergence (3.29) is the following result, which characterizes the limit of Algorithm B in case of a finite number of mesh levels $\underline{\ell} < \infty$.

Corollary 4.2 (lucky breakdown). *Suppose full R-linear convergence (3.29) of the quasi-error M_{ℓ}^{k}. Furthermore, suppose reliability (A3) of the estimator μ_{ℓ}. Then, $\underline{\ell} < \infty$ guarantees that $u^\star = u_{\underline{\ell}}^\star$ and $\mu_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$.*

Proof. From the definition (2.47) of the stopping index $\underline{\ell}$, it follows that $k \to \infty$ on the mesh level $\underline{\ell}$. By full R-linear convergence (3.29), it thus holds

$$0 \leq \mu_{\underline{\ell}}(u_{\underline{\ell}}^\star) \leq \mathrm{M}_{\underline{\ell}}^{k} \leq C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\underline{\ell},k|}\, \mathrm{M}_0^0 \longrightarrow 0 \quad \text{as } k \to \infty.$$

Hence, the estimator satisfies $\mu_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$ and reliability (A3) verifies

$$|||u^\star - u_{\underline{\ell}}^\star||| \leq C_{\mathrm{rel}}\, \mu_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0.$$

This concludes the proof. □

4.2 Proof of optimal complexity

To formalize the idea of attainable convergence rates, we introduce the notion of approximation classes from [BDD04; Ste07; CKNS08; CFPP14]. For $N \in \mathbb{N}$, let $\mathbb{T}_N$ denote the set of all triangulations with at most N more elements than the initial triangulation $\mathcal{T}_0$, i.e.,

$$\mathbb{T}_N := \{\mathcal{T}_H \in \mathbb{T} : \#\mathcal{T}_H - \#\mathcal{T}_0 \leq N\}.$$

For any rate $s > 0$ and the exact solution $u^\star$, we define

$$\|u^\star\|_{\mathbb{A}_s} := \sup_{N \in \mathbb{N}_0} \Big((N+1)^s \min_{\mathcal{T}_H \in \mathbb{T}_N} \mu_H(u_H^\star) \Big).$$

It can be shown that $\|u^\star\|_{\mathbb{A}_s} < \infty$ implies the existence of a sequence of meshes $(\widetilde{\mathcal{T}}_\ell)_{\ell \in \mathbb{N}_0}$ along which the corresponding error estimators $\mu_\ell(u_\ell^\star)$ decay with rate s over the number of elements $\#\widetilde{\mathcal{T}}_\ell$. Consequently, we say that Algorithm B is optimal with respect to the number of degrees of freedom $\dim \mathcal{X}_\ell \simeq \#\mathcal{T}_\ell$ if the generated sequence of meshes $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$ satisfies

$$\forall s > 0 : \quad \Big(\|u^\star\|_{\mathbb{A}_s} < \infty \quad \Longrightarrow \quad \sup_{(\ell,k) \in \mathcal{Q}} (\#\mathcal{T}_\ell)^s \, \mathrm{M}_\ell^k < \infty \Big),$$

i.e., the adaptive algorithm realizes any possible convergence rate. Likewise, we say that Algorithm B is optimal with respect to the total computational cost if the generated sequence of meshes $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$ satisfies

$$\forall s > 0 : \quad \Big(\|u^\star\|_{\mathbb{A}_s} < \infty \quad \Longrightarrow \quad \sup_{(\ell,k) \in \mathcal{Q}} \Big(\sum_{\substack{(\ell',k') \in \mathcal{Q} \\ |\ell',k'| \leq |\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \mathrm{M}_\ell^k < \infty \Big).$$

The goal of this section is to prove the following theorem, which states that Algorithm B is optimal with respect to the total computational cost, provided that the adaptivity parameters θ and λ are chosen sufficiently small. Since strong estimator equivalence (2.22) implies weak estimator equivalence (2.23) by definition and, according to (2.31), also implies weak stability (W1), we formulate the theorem only for the latter two assumptions.

Theorem 4.3 (optimal complexity). *Let $C_{\mathrm{mark}} \geq 1$ and $u_0^0 \in \mathcal{X}_0$ be arbitrary. Suppose that the error estimator μ_ℓ is weakly equivalent to an estimator η_ℓ in the sense of* (2.23). *Furthermore, suppose that μ_ℓ satisfies weak stability* (W1) *and that η_ℓ satisfies the axioms* (A1)–(A3$^+$). *With*

$$\lambda^\star := \frac{1 - q_{\mathrm{ctr}}}{q_{\mathrm{ctr}} \, \widetilde{C}_{\mathrm{stab}}} \quad \text{and} \quad \theta^\star := (1 + C_{\mathrm{stab}}^2 \, C_{\mathrm{drel}}^2)^{-1}, \tag{4.5}$$

let $0 \leq \theta \leq 1$ and $\lambda > 0$ be sufficiently small in the sense that

$$0 < \lambda < \lambda^\star \quad \text{and} \quad 0 < \theta_{\mathrm{mark}} := \frac{\widetilde{C}_{\mathrm{stab}}^4 \, C_{\mathrm{eq}}^4 \, \big(\widetilde{C}_{\mathrm{stab}}^{-1} \, \lambda/\lambda^\star + \theta^{1/2}\big)^2}{\big(1 - \lambda/\lambda^\star\big)^2} < \theta^\star. \tag{4.6}$$

Then, for all $s > 0$, there exist constants $0 < c_{\mathrm{opt}}, C_{\mathrm{opt}}$ such that

$$c_{\mathrm{opt}} \, \|u^\star\|_{\mathbb{A}_s} \leq \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \, \mathrm{M}_\ell^k \leq C_{\mathrm{opt}} \, \max\big\{\|u^\star\|_{\mathbb{A}_s}, \, \mathrm{M}_0^0\big\}. \tag{4.7}$$

In particular, Algorithm B is optimal with respect to the total computational cost. The constant c_{opt} depends only on C_{child} and s, while the constant C_{opt} depends only on C_{stab}, $\widetilde{C}_{\mathrm{stab}}$, C_{drel}, C_{mark}, C_{patch}, C_{mesh}, q_{alg}, λ, C_{lin}, q_{lin}, $\#\mathcal{T}_0$, and s.

The proof follows the lines of [CFPP14, Proposition 4.15] and [MPS24, Theorem 10]. To this end, we need the following three auxiliary results and start with an upper bound on the algebraic error by means of the discretization error $\eta_\ell(u_\ell^\star)$ due to the stopping criterion (2.35).

Lemma 4.4 (upper bound on the algebraic error by discretization error). *Suppose that the error estimator μ_ℓ is weakly equivalent to an estimator η_ℓ in the sense of (2.23) and satisfies weak stability (W1). With $\lambda^\star > 0$ from (4.5), it holds*

$$|||u_\ell^\star - u_\ell^k||| \leq \frac{C_{\mathrm{eq}} \, \lambda}{\lambda^\star - \lambda} \, \eta_\ell(u_\ell^\star) \quad \textit{for all } 0 < \lambda < \lambda^\star. \tag{4.8}$$

Proof. The a-posteriori error estimate of the algebraic error in (2.33) and the stopping criterion (2.35) result in

$$|||u_\ell^\star - u_\ell^k||| \overset{(2.33)}{\leq} \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \, |||u_\ell^k - u_\ell^{k-1}||| \overset{(2.35)}{\leq} \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \, \lambda \, \mu_\ell(u_\ell^k). \tag{4.9}$$

Hence, weak stability (W1) yields

$$|||u_\ell^\star - u_\ell^k||| \overset{(4.9)}{\underset{(W1)}{\leq}} \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \, \widetilde{C}_{\mathrm{stab}} \, \lambda \Big[\mu_\ell(u_\ell^\star) + |||u_\ell^\star - u_\ell^k||| \Big] = \frac{\lambda}{\lambda^\star} \Big[\mu_\ell(u_\ell^\star) + |||u_\ell^\star - u_\ell^k||| \Big]. \tag{4.10}$$

Since $\lambda < \lambda^\star$, estimator equivalence (2.30) establishes

$$|||u_\ell^\star - u_\ell^k||| \overset{(4.10)}{\leq} \frac{\lambda/\lambda^\star}{1-\lambda/\lambda^\star} \, \mu_\ell(u_\ell^\star) = \frac{\lambda}{\lambda^\star - \lambda} \, \mu_\ell(u_\ell^\star) \overset{(2.30)}{\leq} \frac{C_{\mathrm{eq}} \, \lambda}{\lambda^\star - \lambda} \, \eta_\ell(u_\ell^\star).$$

This concludes the proof. □

The following lemma shows that Dörfler marking for $\eta_\ell(u_\ell^\star)$ with θ_{mark} from (4.6) implies Dörfler marking for $\mu_\ell(u_\ell^k)$ with θ for a slightly larger set of marked elements.

Lemma 4.5 (Dörfler marking for η_ℓ implies Dörfler marking for μ_ℓ). *Suppose that the error estimator μ_ℓ is weakly equivalent to an estimator η_ℓ in the sense of (2.23) and satisfies weak stability (W1). With $m \in \mathbb{N}_0$ from the equivalence (2.23), $r \in \mathbb{N}_0$ from weak stability (W1), and $0 < \theta_{\mathrm{mark}} < 1$ from (4.6), the following implication holds for any triangulation $\mathcal{T}_\ell \in \mathbb{T}$ and any subset $\mathcal{R}_\ell \subseteq \mathcal{T}_\ell$:*

$$\theta_{\mathrm{mark}} \, \eta_\ell(u_\ell^\star)^2 \leq \eta_\ell(\mathcal{R}_\ell, u_\ell^\star)^2 \quad \Longrightarrow \quad \theta \, \mu_\ell(u_\ell^k)^2 \leq \mu_\ell\big(\mathcal{T}_\ell^{(m+r)}[\mathcal{R}_\ell], u_\ell^k\big)^2. \tag{4.11}$$

Proof. Weak stability (W1), weak estimator equivalence (2.30) and estimate (4.9) provide

$$\begin{aligned}
\widetilde{C}_{\rm stab}^{-2}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2}\, \mu_\ell(u_\ell^k) &\overset{\rm (W1)}{\leq} \widetilde{C}_{\rm stab}^{-1}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2} \Big[\mu_\ell(u_\ell^\star) + |||u_\ell^\star - u_\ell^k|||\Big] \\
&\overset{\rm (2.30b)}{\leq} \widetilde{C}_{\rm stab}^{-1}\, C_{\rm eq}^{-1}\, \theta_{\rm mark}^{1/2}\, \eta_\ell(u_\ell^\star) + \widetilde{C}_{\rm stab}^{-1}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2}\, |||u_\ell^\star - u_\ell^k||| \\
&\overset{\rm (4.11)}{\leq} \widetilde{C}_{\rm stab}^{-1}\, C_{\rm eq}^{-1}\, \eta_\ell(\mathcal{R}_\ell, u_\ell^\star) + \widetilde{C}_{\rm stab}^{-1}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2}\, |||u_\ell^\star - u_\ell^k||| \\
&\overset{\rm (2.30a)}{\leq} \widetilde{C}_{\rm stab}^{-1}\, \mu_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{R}_\ell], u_\ell^\star) + \widetilde{C}_{\rm stab}^{-1}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2}\, |||u_\ell^\star - u_\ell^k||| \\
&\overset{\rm (W1)}{\leq} \mu_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{R}_\ell], u_\ell^k) + \big(1 + \widetilde{C}_{\rm stab}^{-1}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2}\big)\, |||u_\ell^\star - u_\ell^k||| \\
&\overset{\rm (4.9)}{\leq} \mu_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{R}_\ell], u_\ell^k) + \big(1 + \widetilde{C}_{\rm stab}^{-1}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2}\big)\, \frac{q_{\rm ctr}}{1-q_{\rm ctr}}\, \lambda\, \mu_\ell(u_\ell^k).
\end{aligned}$$

Rearrangement of the second term on the right-hand side verifies

$$\Big[\widetilde{C}_{\rm stab}^{-2}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2} - \big(1 + \widetilde{C}_{\rm stab}^{-1}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2}\big)\, \frac{q_{\rm ctr}}{1-q_{\rm ctr}}\, \lambda\Big]\, \mu_\ell(u_\ell^k) \leq \mu_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{R}_\ell], u_\ell^k). \tag{4.12}$$

The definition of $\theta_{\rm mark}$ and $\lambda^\star$ in (4.5)–(4.6) reveals

$$\begin{aligned}
&\widetilde{C}_{\rm stab}^{-2}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2} - \big(1 + \widetilde{C}_{\rm stab}^{-1}\, C_{\rm eq}^{-2}\, \theta_{\rm mark}^{1/2}\big)\, \frac{q_{\rm ctr}}{1-q_{\rm ctr}}\, \lambda \\
&\quad \overset{(4.5)}{\underset{(4.6)}{\leq}} \frac{\widetilde{C}_{\rm stab}^{-1}\, \lambda/\lambda^\star + \theta^{1/2}}{1-\lambda/\lambda^\star} - \Big(1 + \frac{\lambda/\lambda^\star + \widetilde{C}_{\rm stab}\, \theta^{1/2}}{1-\lambda/\lambda^\star}\Big)\, \widetilde{C}_{\rm stab}^{-1}\, \lambda/\lambda^\star \\
&\quad = \frac{\widetilde{C}_{\rm stab}^{-1}\, \lambda/\lambda^\star + \theta^{1/2}}{1-\lambda/\lambda^\star} - \widetilde{C}_{\rm stab}^{-1}\, \lambda/\lambda^\star - \lambda/\lambda^\star\, \frac{\widetilde{C}_{\rm stab}^{-1}\, \lambda/\lambda^\star + \theta^{1/2}}{1-\lambda/\lambda^\star} \\
&\quad = (1-\lambda/\lambda^\star)\, \frac{\widetilde{C}_{\rm stab}^{-1}\, \lambda/\lambda^\star + \theta^{1/2}}{1-\lambda/\lambda^\star} - \widetilde{C}_{\rm stab}^{-1}\, \lambda/\lambda^\star = \theta^{1/2}.
\end{aligned}$$

Thus, estimate (4.12) reduces to

$$\theta^{1/2}\, \mu_\ell(u_\ell^k) \leq \mu_\ell(\mathcal{T}_\ell^{(m+r)}[\mathcal{R}_\ell], u_\ell^k),$$

which concludes the proof. □

Finally, we need the following comparison lemma for the error estimator of the Galerkin solution $\eta_\ell(u_\ell^\star)$, which is found in [CFPP14, Lemma 4.14] and relies only on the optimality of Dörfler marking (Proposition 2.19) and the overlay estimate (R2).

Lemma 4.6 (comparison lemma [CFPP14, Lemma 4.14.]). *Suppose that the error estimator η_ℓ satisfies (A1)–(A3⁺). Let $0 < \theta' < \theta^\star := (1 + C_{\rm stab}^2\, C_{\rm drel}^2)^{-1}$. Then, there exist constants $C_1, C_2 > 0$ depending only on the constants of (A1)–(A3⁺) such that for all $s > 0$ with $\|u^\star\|_{\mathbb{A}_s} < \infty$ and all $\mathcal{T}_H \in \mathbb{T}$, there exists a subset $\mathcal{R}_H \subseteq \mathcal{T}_H$ satisfying*

$$\#\mathcal{R}_H \leq C_1\, C_2^{-1/s}\, \|u^\star\|_{\mathbb{A}_s}^{1/s}\, \eta_H(u_H^\star)^{-1/s} \quad \text{and} \quad \eta_H(\mathcal{R}_H, u_H^\star) \leq \theta'\, \eta_H(u_H^\star). \qquad \square$$

With these auxiliary results at hand, we are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. By Corollary 4.1, it suffices to show

$$\|u^\star\|_{\mathbb{A}_s} \lesssim \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\, \mathrm{M}_\ell^k \lesssim \max\{\|u^\star\|_{\mathbb{A}_s}, \mathrm{M}_0^0\}. \tag{4.13}$$

The proof is divided into four steps.

Step 1 (lower bound in (4.7) for $\underline{\ell} = \infty$). Let $\underline{\ell} = \infty$. Since $\mathcal{M}_\ell \neq \emptyset$ and hence $\#\mathcal{T}_{\ell+1} > \#\mathcal{T}_\ell$ by the child estimate (R1), Algorithm B guarantees $\#\mathcal{T}_\ell \to \infty$ as $\ell \to \infty$. Following the proof of [CFPP14, Proposition 4.15], we choose for any $N \in \mathbb{N}$ the maximal index $\ell' \in \mathbb{N}_0$ with $\mathcal{T}_{\ell'} \in \mathbb{T}_N$. Since the child estimate (R1) provides $\#\mathcal{T}_{\ell'+1} \le C_{\mathrm{child}}\, \#\mathcal{T}_{\ell'}$, it holds

$$N + 1 < \#\mathcal{T}_{\ell'+1} - \#\mathcal{T}_0 + 1 \le \#\mathcal{T}_{\ell'+1} \le C_{\mathrm{child}}\, \#\mathcal{T}_{\ell'}.$$

The fact that $\mathcal{T}_{\ell'} \in \mathbb{T}_N$ verifies

$$\min_{\mathcal{T}_H\in\mathbb{T}_N} \mu_H(u_H^\star) \le \mu_{\ell'}(u_{\ell'}^\star) \le \mathrm{M}_{\ell'}^{k'} \quad \text{for all } k' \in \mathbb{N}_0 \text{ with } (\ell', k') \in \mathcal{Q}.$$

A combination of the two previous estimates thus shows

$$(N+1)^s \min_{\mathcal{T}_H\in\mathbb{T}_N} \mu_H(u_H^\star) \le C_{\mathrm{child}}^s\, (\#\mathcal{T}_{\ell'})^s\, \mathrm{M}_{\ell'}^{k'} \le C_{\mathrm{child}}^s \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\, \mathrm{M}_\ell^k.$$

Hence, taking the supremum over all $N \in \mathbb{N}$ yields the lower bound in (4.13).

Step 2 (lower bound in (4.7) for $\underline{\ell} < \infty$). In the case that $\underline{\ell} < \infty$, Corollary 4.2 ensures $u^\star = u_{\underline{\ell}}^\star$ and $\mu_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$. As discussed in Remark 2.9, the estimator η_ℓ satisfies quasi-monotonicity (QM), which by Remark 2.17 implies that μ_ℓ also satisfies quasi-monotonicity. Hence, it holds $\min_{\mathcal{T}_H\in\mathbb{T}_N} \mu_H(u_H^\star) = 0$ for all $N \ge \#\mathcal{T}_{\underline{\ell}} - \#\mathcal{T}_0$ and thus the definition of the approximation class $\|\cdot\|_{\mathbb{A}_s}$ reduces to

$$\|u^\star\|_{\mathbb{A}_s} = \sup_{0\le N<\#\mathcal{T}_{\underline{\ell}}-\#\mathcal{T}_0} \Big((N+1)^s \min_{\mathcal{T}_H\in\mathbb{T}_N} \mu_H(u_H^\star) \Big).$$

Treating all $N \in \mathbb{N}_0$ with $0 \le N < \#\mathcal{T}_{\underline{\ell}} - \#\mathcal{T}_0$ analogously to Step 1 establishes the lower bound in (4.13).

Step 3 (estimate of marked elements). Since the upper bound in (4.13) is clear if $\|u^\star\|_{\mathbb{A}_s} = \infty$, we may suppose $\|u^\star\|_{\mathbb{A}_s} < \infty$. Let $(\ell'+1, 0) \in \mathcal{Q}$ be arbitrary. By Lemma 4.6, there exists a subset $\mathcal{R}_{\ell'} \subseteq \mathcal{T}_{\ell'}$ such that

$$\#\mathcal{R}_{\ell'} \lesssim \|u^\star\|_{\mathbb{A}_s}^{1/s}\, \eta_{\ell'}(u_{\ell'}^\star)^{-1/s} \quad \text{and} \quad \theta_{\mathrm{mark}}\, \eta_{\ell'}(u_{\ell'}^\star)^2 \le \eta_{\ell'}(\mathcal{R}_{\ell'}, u_{\ell'}^\star)^2. \tag{4.14}$$

As a consequence of Lemma 4.5, the Dörfler marking criterion in (4.14) implies

$$\theta\, \mu_{\ell'}(u_{\ell'}^{\underline{k}})^2 \le \mu_{\ell'}\big(\mathcal{T}_{\ell'}^{(m+r)}[\mathcal{R}_{\ell'}], u_{\ell'}^{\underline{k}}\big)^2.$$

Hence, the Dörfler marking criterion (2.45) holds on the enlarged set $\mathcal{T}_{\ell'}^{(m+r)}[\mathcal{R}_{\ell'}]$. Estimate (2.20) thus ensures that the set of marked elements $\mathcal{M}_{\ell'}$ satisfies

$$\#\mathcal{M}_{\ell'} \overset{(2.45)}{\leq} C_{\mathrm{mark}}\, \#\mathcal{T}_{\ell'}^{(m+r)}[\mathcal{R}_{\ell'}] \overset{(2.20)}{\leq} C_{\mathrm{mark}}\, C_{\mathrm{patch}}^{m+r}\, \#\mathcal{R}_{\ell'}. \tag{4.15}$$

Full R-linear convergence (3.29), estimator equivalence (2.30b), and the upper bound on the algebraic error from (4.8) result in

$$\mathrm{M}_{\ell'+1}^{0} \overset{(3.29)}{\lesssim} \mathrm{M}_{\ell'}^{\underline{k}} = |||u_{\ell'}^\star - u_{\ell'}^{\underline{k}}||| + \mu_{\ell'}(u_{\ell'}^\star) \overset{(2.30b)}{\lesssim} |||u_{\ell'}^\star - u_{\ell'}^{\underline{k}}||| + \eta_{\ell'}(u_{\ell'}^\star) \overset{(4.8)}{\lesssim} \eta_{\ell'}(u_{\ell'}^\star). \tag{4.16}$$

A combination of the previous estimates (4.14)–(4.16) therefore validates for all $(\ell'+1, 0) \in \mathcal{Q}$

$$\#\mathcal{M}_{\ell'} \overset{(4.15)}{\lesssim} \#\mathcal{R}_{\ell'} \overset{(4.14)}{\lesssim} \|u^\star\|_{\mathbb{A}_s}^{1/s}\, \eta_{\ell'}(u_{\ell'}^\star)^{-1/s} \overset{(4.16)}{\lesssim} \|u^\star\|_{\mathbb{A}_s}^{1/s}\, (\mathrm{M}_{\ell'+1}^{0})^{-1/s}. \tag{4.17}$$

Step 4 (upper bound in (4.7)). Let $(\ell,k) \in \mathcal{Q}$ be arbitrary. Full R-linear convergence (3.29) and the geometric series verify

$$\sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\leq|\ell,k|}} (\mathrm{M}_{\ell'}^{k'})^{-1/s} \overset{(3.29)}{\lesssim} (\mathrm{M}_{\ell}^{k})^{-1/s} \sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\leq|\ell,k|}} (q_{\mathrm{lin}}^{1/s})^{|\ell,k|-|\ell',k'|} \lesssim (\mathrm{M}_{\ell}^{k})^{-1/s}. \tag{4.18}$$

A combination with the mesh-closure estimate (R3) and estimate (4.17) proves, for all $\ell \geq 1$ with $(\ell, 0) \in \mathcal{Q}$, that

$$\begin{aligned}\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1 \leq 2\,(\#\mathcal{T}_\ell - \#\mathcal{T}_0) &\overset{(R3)}{\lesssim} \sum_{\ell'=0}^{\ell-1} \#\mathcal{M}_{\ell'} \overset{(4.17)}{\lesssim} \|u^\star\|_{\mathbb{A}_s}^{1/s} \sum_{\ell'=0}^{\ell-1} (\mathrm{M}_{\ell'+1}^{0})^{-1/s}\\ &\leq \|u^\star\|_{\mathbb{A}_s}^{1/s} \sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\leq|\ell,k|}} (\mathrm{M}_{\ell'}^{k'})^{-1/s} \overset{(4.18)}{\lesssim} \|u^\star\|_{\mathbb{A}_s}^{1/s}\, (\mathrm{M}_{\ell}^{k})^{-1/s}.\end{aligned}$$

A rearrangement of this estimate thus results in

$$(\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^s\, \mathrm{M}_\ell^k \lesssim \|u^\star\|_{\mathbb{A}_s} \quad \text{for all } (\ell,k) \in \mathcal{Q} \text{ with } \ell \geq 1. \tag{4.19a}$$

In the case $\ell = 0$, full R-linear convergence (3.29) trivially provides

$$(\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^s\, \mathrm{M}_\ell^k = \mathrm{M}_0^k \overset{(3.29)}{\lesssim} \mathrm{M}_0^0 \quad \text{for all } (\ell,k) \in \mathcal{Q} \text{ with } \ell = 0. \tag{4.19b}$$

As in [BHP17, Lemma 22], a rearrangement of

$$(\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1) - \frac{\#\mathcal{T}_\ell}{\#\mathcal{T}_0} = (\#\mathcal{T}_\ell - \#\mathcal{T}_0)(1 - \frac{1}{\#\mathcal{T}_0}) \geq 0$$

leads to the estimate

$$\#\mathcal{T}_\ell \leq \#\mathcal{T}_0\, (\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1). \tag{4.20}$$

Overall, the previous estimates (4.19)–(4.20) prove

$$(\#\mathcal{T}_\ell)^s\, \mathrm{M}_\ell^k \overset{(4.20)}{\leq} (\#\mathcal{T}_0)^s\, (\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^s\, \mathrm{M}_\ell^k \overset{(4.19)}{\lesssim} \max\{\|u^\star\|_{\mathbb{A}_s}, \mathrm{M}_0^0\}.$$

Taking the supremum over all $(\ell,k) \in \mathcal{Q}$ thus concludes the proof of the upper bound in (4.13). □

5 Applications

In Example 2.14, we have already seen that the residual-based estimator η_H (2.14) is strongly equivalent in the sense of (2.22) to the residual-based estimator $\widehat{\eta}_H$ (2.16), which uses the diameter as mesh-size function. As a consequence of Theorem 3.11 and Theorem 4.3, we can conclude full R-linear convergence and optimal complexity for Algorithm B steered by $\widehat{\eta}_H$.

In this chapter, we want to introduce further error estimators, that satisfy the requirements for full R-linear convergence and optimal complexity laid out in the previous chapters. Specifically, we will consider recovery-based estimators and estimators based on flux equilibration. The analysis of recovery-based estimators is based on [ZZ87; KS11; CFPP14], while the section on flux equilibration uses results and ideas from [BPS09; EV15; EV20].

Throughout this chapter, we employ newest vertex bisection (cf. Example 2.6) as the mesh-refinement strategy and, for a polynomial degree $p \in \mathbb{N}$, define the discrete spaces $\mathcal{X}_H := \mathcal{S}_0^p(\mathcal{T}_H)$ as in (2.10).

5.1 Recovery-based estimators

In this section, we consider recovery-based error estimators, which are also referred to as ZZ-estimators after Zienkiewicz and Zhu [ZZ87]. These estimators are widely used in computational science and engineering due to their ease of implementation and impressive performance in various applications.

Throughout this section, we suppose $\boldsymbol{A} = \alpha \boldsymbol{I}$ for $\alpha \in C(\overline{\Omega})$ and $\boldsymbol{f} = 0$ in problem (2.4), i.e., we consider the PDE

$$-\operatorname{div}(\alpha \nabla u^\star) + \boldsymbol{b} \cdot \nabla u^\star + c u^\star = f \text{ in } \Omega \quad \text{subject to} \quad u^\star = 0 \text{ on } \partial\Omega. \tag{5.1}$$

In particular, the refinement indicators of the residual-based estimator η_H (2.14) read

$$\eta_H(T, v_H^\star)^2 = H(T)^2 \, \| R_H(v_H) \|_{L^2(T)}^2 + H(T) \, \| J_H(v_H) \|_{L^2(\partial T \cap \Omega)}^2$$

with local volume residuals $R_H(v_H) = -\operatorname{div}(\alpha \nabla v_H) + \boldsymbol{b} \cdot \nabla v_H + c v_H - f$ and jump terms $J_H(v_H) = [\![\alpha \nabla v_H \cdot \boldsymbol{n}]\!]$. In order to define the ZZ-estimator, we need to introduce further notation.

Definition 5.1 (patches and stars). Let $\mathcal{T}_H \in \mathbb{T}$ be a triangulation and $z \in \mathcal{V}_H \cap \Omega$ an interior vertex of $\mathcal{T}_H$. We define the corresponding vertex-patch $\mathcal{T}_H[\,z\,]$ as the set of elements $T \in \mathcal{T}_H$ such that $z \in T$, i.e.,

$$\mathcal{T}_H[\,z\,] := \{T \in \mathcal{T}_H : z \in T\}.$$

Similarly, the star $\Sigma_H[\,z\,]$ is defined as the set of faces $E \in \mathcal{E}_H^\Omega$ such that $z \in E$, i.e.,

$$\Sigma_H[\,z\,] := \{E \in \mathcal{E}_H^\Omega : z \in E\}.$$

To abbreviate notation, we write $\Omega_H[\,z\,] := \bigcup \mathcal{T}_H[\,z\,]$ and $\omega_H[\,z\,] := \bigcup \Sigma_H[\,z\,]$.

For a triangulation $\mathcal{T}_H \in \mathbb{T}$, the definition of the ZZ-estimator employs a local averaging operator $G_H : L^2(\Omega) \to \mathcal{S}^p(\mathcal{T}_H)$ which is defined as follows:

- For lowest-order polynomials $p = 1$, the operator G_H maps any function $v \in L^2(\Omega)$ to the unique discrete function $G_H(v) \in \mathcal{S}^1(\mathcal{T}_H)$ satisfying

$$G_H(v)(z) := \frac{1}{|\Omega_H[\,z\,]|} \int_{\Omega_H[\,z\,]} v \,\mathrm{d}x \quad \text{for all vertices } z \in \mathcal{V}_H. \tag{5.2}$$

- For the general case $p \in \mathbb{N}$, we define the operator G_H as L^2-stable variant of the Scott–Zhang projection from [SZ90]. With $J := \dim(\mathcal{S}^p(\mathcal{T}_H))$, let $\{\phi_j\}_{j=1}^J$ denote the nodal basis of $\mathcal{S}^p(\mathcal{T}_H)$ with associated nodes $\{a_j\}_{j=1}^J$, i.e.,

$$\phi_j(a_k) = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k. \end{cases} \quad \text{for all } j, k = 1, \dots, J.$$

  With each node a_j, we associate an element $S_j \in \mathcal{T}_H$ with $a_j \in S_j$. For each $j \in \{1, \dots, J\}$, linear algebra yields the existence of a unique dual basis function $\psi_j \in \operatorname{span}\{\phi_i|_{S_j} : a_i \in S_j\}$ satisfying

$$(\psi_k, \phi_j)_{L^2(S_j)} = \delta_{jk} \quad \text{for all } j, k = 1, \dots, J. \tag{5.3}$$

  The Scott–Zhang projection $G_H : L^2(\Omega) \to \mathcal{S}^p(\mathcal{T}_H)$ is then defined as the unique function $G_H(v) \in \mathcal{S}^p(\mathcal{T}_H)$ satisfying $G_H(v)(a_j) = (\psi_j, v)_{L^2(S_j)}$ for all $j = 1, \dots, J$, i.e.,

$$G_H(v) = \sum_{j=1}^{J} (\psi_j, v)_{L^2(S_j)}\, \phi_j \quad \text{for all } v \in L^2(\Omega). \tag{5.4}$$

  By (5.3), it holds for all $v_H \in \mathcal{S}^p(\mathcal{T}_H)$ with $v_H = \sum_{k=1}^J v_k \phi_k$ that

$$G_H(v_H) = \sum_{j=1}^{J} (\psi_j, v_H)_{L^2(S_j)}\, \phi_j = \sum_{j,k=1}^{J} v_k (\psi_j, \phi_k)_{L^2(S_j)}\, \phi_j \overset{(5.3)}{=} \sum_{k=1}^{J} v_k \phi_k = v_H,$$

  i.e., the Scott–Zhang projection is indeed a projection from $L^2(\Omega)$ onto $\mathcal{S}^p(\mathcal{T}_H)$.

Let $q \in \mathbb{N}_0$ with $0 \leq q \leq p - 1$. For each subset $U \subseteq \overline{\Omega}$, let $\Pi_q(U) : L^2(U) \to \mathcal{P}^q(U)$ denote the L^2-orthogonal projection onto the space of polynomials of degree q, i.e.,

$$\Pi_q(U)(v) := \min_{w \in \mathcal{P}^q(U)} \|v - w\|_{L^2(U)}^2 \quad \text{for all } v \in L^2(U).$$

For each interior vertex $z \in \mathcal{V}_H \cap \Omega$ and each discrete function $v_H \in \mathcal{X}_H$, we define

$$r_{H,z}(v_H) := \Pi_q(\Omega_H[\,z\,])\, R_H(v_H) = \operatorname*{argmin}_{w \in \mathcal{P}^q(\Omega_H[\,z\,])} \|R_H(v_H) - w\|_{L^2(\Omega_H[\,z\,])}^2. \tag{5.5}$$

With the index set $\mathcal{I}_H := \mathcal{T}_H \cup \mathcal{V}_H$ and $H(z) := \operatorname{diam}(\Omega_H[\,z\,])$, the refinement indicators of the ZZ-estimator read

$$\varrho_H(I, v_H)^2 := \begin{cases} \|\alpha^{1/2}(1 - G_H)\nabla v_H\|_{L^2(T)}^2 & \text{for } I = T \in \mathcal{T}_H, \\ H(z)^2\, \|R_H(v_H) - r_{H,z}(v_H)\|_{L^2(\Omega_H[\,z\,])}^2 & \text{for } I = z \in \mathcal{V}_H \cap \Omega. \end{cases}$$

As detailed in Remark 2.15, we consider the corresponding element-based estimator

$$\mu_H(T, v_H)^2 := \|\alpha^{1/2}(1 - G_H)\nabla v_H\|_{L^2(T)}^2 + \sum_{\substack{z \in \mathcal{V}_H \cap \Omega \\ z \in T}} H(z)^2\, \|R_H(v_H) - r_{H,z}(v_H)\|_{L^2(\Omega_H[\,z\,])}^2. \tag{5.6}$$

Our goal is to show that the ZZ-estimator satisfies the requirements for full R-linear convergence (Theorem 3.14) and optimal complexity (Theorem 4.3). To this end, we need to prove that the ZZ-estimator is weakly equivalent (in the sense of (2.23)) to the residual-based estimator η_H from 2.14, and that it satisfies weak stability (W1). This is the content of the following two subsections.

5.1.1 Weak equivalence of ZZ-estimator and residual-based estimator

In order to show weak equivalence of the ZZ-estimator (5.6) and the residual-based estimator (2.14), we first need to establish a few auxiliary results. The following lemma provides a bound for the residual in terms of normal jumps and volume oscillations.

Lemma 5.2. *There exists a constant $C_{\mathrm{res}} > 0$ such that for any interior vertex $z \in \mathcal{V}_H \cap \Omega$ and any element $T \in \mathcal{T}_H$ with $z \in T$, it holds that*

$$\begin{aligned} &H(z)^2\, \|R_H(u_H^\star)\|_{L^2(T)}^2 \\ &\quad \le C_{\mathrm{res}}\Big[H(z)\, \|[\![\alpha \nabla u_H^\star \cdot \boldsymbol{n}]\!]\|_{L^2(\omega_H[\,z\,])}^2 + H(z)^2\, \|R_H(u_H^\star) - r_{H,z}(u_H^\star)\|_{L^2(\Omega_H[\,z\,])}^2\Big]. \end{aligned} \tag{5.7}$$

The constant C_{res} depends only on the polynomial degree q and the initial triangulation $\mathcal{T}_0$.

Proof. For every $z \in \mathcal{V}_H$, let $\phi_z \in \mathcal{S}^1(\mathcal{T}_H)$ denote the nodal basis function characterized by $\phi_z(z) = 1$ and $\phi_z(z') = 0$ for all $z' \in \mathcal{V}_H$ with $z' \neq z$. To abbreviate notation in the proof, we write $r_{H,z}^\star := r_{H,z}(u_H^\star)$, $R_H^\star := R_H(u_H^\star)$ and $\Pi_q := \Pi_q(\Omega_H[\,z\,])$. Recall that $r_{H,z}^\star = \Pi_q\, R_H^\star$. A scaling argument shows

$$\begin{aligned} \|r_{H,z}^\star\|_{L^2(\Omega_H[\,z\,])} &\lesssim \|\phi_z^{1/2} r_{H,z}^\star\|_{L^2(\Omega_H[\,z\,])} \\ &= \int_{\Omega_H[\,z\,]} R_H^\star \phi_z r_{H,z}^\star \,\mathrm{d}x - \int_{\Omega_H[\,z\,]} \big((1 - \Pi_q) R_H^\star\big) \phi_z r_{H,z}^\star \,\mathrm{d}x. \end{aligned}$$

The hidden constant depends only on σ-shape regularity and the polynomial degree q. Together with $\|\phi_z\|_{L^\infty(\Omega)} = 1$, we obtain

$$\|r_{H,z}^\star\|_{L^2(\Omega_H[\,z\,])} \lesssim \int_{\Omega_H[\,z\,]} R_H^\star \phi_z r_{H,z}^\star \,\mathrm{d}x + \|(1 - \Pi_q) R_H^\star\|_{L^2(\Omega_H[\,z\,])} \|r_{H,z}^\star\|_{L^2(\Omega_H[\,z\,])}. \tag{5.8}$$

We first consider the left term of the right-hand side in (5.8). Since $\operatorname{supp}(\phi_z) = \Omega_H[z]$, it holds $v := \phi_z r^\star_{H,z} \in S^p_0(\mathcal{T}_H[z])$. Combining this with the weak formulation (2.5) and element-wise integration by parts, we calculate

$$\begin{aligned}
\int_{\Omega_H[z]} R^\star_H \phi_z r^\star_{H,z} \,\mathrm{d}x &= \int_{\Omega_H[z]} \big(-\operatorname{div}(\alpha\nabla u^\star_H) + \boldsymbol{b}\cdot\nabla u^\star_H + c u^\star_H\big) v \,\mathrm{d}x - \int_{\Omega_H[z]} f v \,\mathrm{d}x \\
&\overset{(2.5)}{=} \int_{\Omega_H[z]} -\operatorname{div}(\alpha\nabla u^\star_H) v \,\mathrm{d}x - \int_{\Omega_H[z]} \alpha\nabla u^\star_H \cdot \nabla v \,\mathrm{d}x \\
&= \int_{\omega_H[z]} [\![\alpha\nabla u^\star_H \cdot \boldsymbol{n}]\!] v \,\mathrm{d}x \\
&\le \|[\![\alpha\nabla u^\star_H \cdot \boldsymbol{n}]\!]\|_{L^2(\omega_H[z])} \, \|r^\star_{H,z}\|_{L^2(\omega_H[z])}.
\end{aligned}$$

Since $r^\star_{H,z} \in \mathcal{P}^q(\Omega_H[z])$, an inverse-type inequality provides

$$\|r^\star_{H,z}\|_{L^2(\omega_H[z])} \lesssim H(z)^{-1/2} \, \|r^\star_{H,z}\|_{L^2(\Omega_H[z])}.$$

Again, the hidden constant depends only on σ-shape regularity and the polynomial degree q. Combining the previous estimates with (5.8), we obtain

$$\begin{aligned}
\|r^\star_{H,z}\|^2_{L^2(\Omega_H[z])} \lesssim \big[H(z)^{-1/2} \, &\|[\![\alpha\nabla u^\star_H \cdot \boldsymbol{n}]\!]\|_{L^2(\omega_H[z])} \\
&+ \|R^\star_H - r^\star_{H,z}\|_{L^2(\Omega_H[z])}\big] \, \|r^\star_{H,z}\|_{L^2(\Omega_H[z])}.
\end{aligned} \tag{5.9}$$

Thus, the triangle inequality shows

$$\begin{aligned}
H(z)^2 \, \|R^\star_H\|_{L^2(T)} &\lesssim H(z)^2 \, \|r^\star_{H,z}\|^2_{L^2(\Omega_H[z])} + H(z)^2 \, \|R^\star_H - r^\star_{H,z}\|_{L^2(\Omega_H[z])} \\
&\overset{(5.9)}{\lesssim} H(z) \, \|[\![\alpha\nabla u^\star_H \cdot \boldsymbol{n}]\!]\|_{L^2(\omega_H[z])} + H(z)^2 \, \|R^\star_H - r^\star_{H,z}\|_{L^2(\Omega_H[z])}.
\end{aligned}$$

This concludes the proof. □

The next lemma shows that the normal jumps are locally equivalent to averaging.

Lemma 5.3 (averaging is equivalent to jumps). *There exists a constant $C_{\mathrm{avg}} > 0$ such that*

$$\begin{aligned}
C^{-1}_{\mathrm{avg}} \, H(z) \|[\![\nabla v_H \cdot \boldsymbol{n}]\!]\|^2_{L^2(\omega_H[z])} &\le \|(1 - G_H)\nabla v_H\|^2_{L^2(\Omega_H[z])} \\
&\le C_{\mathrm{avg}} \sum_{z' \in \Sigma[z] \cap \mathcal{V}_H \cap \Omega} H(z') \, \|[\![\nabla v_H \cdot \boldsymbol{n}]\!]\|^2_{L^2(\omega_H[z'])}.
\end{aligned} \tag{5.10}$$

The constant C_{avg} depends only the polynomial degree p and the use of newest vertex bisection.

The proof of Lemma 5.3 uses a seminorm argument, which is based on the following elementary result.

Lemma 5.4 (equivalence of seminorms). *For any two seminorms $|\cdot|_1, |\cdot|_2$ on a finite dimensional space V, there exists a constant $C > 0$ such that $|v|_1 \le C|v|_2$ for all $v \in V$ if and only if it holds*

$$\{v \in V : |v|_2 = 0\} \subseteq \{v \in V : |v|_1 = 0\}.$$ □

Proof of Lemma 5.3. All terms in (5.10) define seminorms on $\mathcal{S}^p(\mathcal{T}_H[\,z\,])$. By Lemma 5.4, it thus suffices to show that the chain of inequalities holds true if one term is zero.

First, we assume that $(1 - G_H)\nabla v_H = 0$ on $\Omega_H[\,z\,]$, which implies $\nabla v_H \in \mathcal{S}^p(\mathcal{T}_H[\,z\,])$. Thus, ∇v_H is continuous on $\Omega_H[\,z\,]$ and hence $[\![\nabla v_H \cdot \boldsymbol{n}]\!] = 0$ on $\omega_H[\,z\,]$.

Next, assume that $[\![\nabla v_H \cdot \boldsymbol{n}]\!] = 0$ on $\omega_H[\,z'\,]$ for all inner vertices z' of $\Sigma_H[\,z\,]$. Since $v_H \in H^1(\Omega)$, also all tangential jumps of ∇v_H vanish over $\Sigma_H[\,z'\,]$. Overall, this implies that $\nabla v_H \in \mathcal{S}^{p-1}(\mathcal{T}_H[\,z'\,]) = \mathcal{P}^{p-1}(\mathcal{T}_H[\,z'\,]) \cap C(\overline{\Omega_H[\,z'\,]})$ for all inner vertices z' of $\Sigma_H[\,z\,]$. If $p \geq 1$ and the averaging operator G_H is defined via the Scott–Zhang projection, this results in $G_H \nabla v_H = \nabla v_H$. In case that $p = 1$ and patch averaging (5.2) is used, $\nabla v_H \in \mathcal{S}^0(\mathcal{T}_H[\,z'\,])$ yields that ∇v_H is constant on $\Omega_H[\,z'\,]$ for all inner vertices z' of $\Sigma_H[\,z\,]$, and thus $G_H \nabla v_H = \nabla v_H$. In any case, it therefore holds $(1 - G_H)\nabla v_H = 0$ on $\Omega_H[\,z\,]$.

The constant C_{avg} initially depends on the shape of the patches $\Omega_H[\,z'\,]$. However, since NVB leads to finitely many patch shapes, a scaling argument proves that C_{avg} depends only on the use of newest vertex bisection and the polynomial degree p. □

With Lemma 5.2 and Lemma 5.3 at hand, we can finally prove that the ZZ-estimator is weakly equivalent to the residual-based estimator η_H in the sense of (2.23).

Proposition 5.5 (ZZ-estimator is weakly equivalent to residual-based estimator). *Suppose that the triangulation $\mathcal{T}_H \in \mathbb{T}$ is sufficiently fine in the sense that each element $T \in \mathcal{T}_H$ contains at least one interior vertex $z \in \mathcal{V}_H \cap \Omega \cap T$. For any polynomial degree $p \in \mathbb{N}$, the ZZ-estimator μ_H defined in (5.6) is weakly equivalent to the residual-based estimator η_H (2.14) in the sense of (2.23) with $m = 2$. The equivalence constant C_{eq} depends only on the polynomial degrees p and q, the bounds $\|\alpha\|_{L^\infty(\Omega)}$ and $\alpha_{\min} := \min_{x\in\Omega} \alpha(x) > 0$, the initial triangulation $\mathcal{T}_0$, and the use of newest vertex bisection.*

Proof. For an arbitrary triangulation $\mathcal{T}_H \in \mathbb{T}$, let $T \in \mathcal{T}_H$ and $z \in \mathcal{V}_H \cap \Omega \cap T$. Since $\boldsymbol{A} = \alpha \boldsymbol{I}$ with $\alpha \in C(\overline{\Omega})$, it holds that

$$
\begin{aligned}
\|[\![\alpha \nabla u_H^\star \cdot \boldsymbol{n}]\!]\|_{L^2(\omega_H[\,z\,])} &= \|\alpha\, [\![\nabla u_H^\star \cdot \boldsymbol{n}]\!]\|_{L^2(\omega_H[\,z\,])} \\
&\leq \|\alpha\|_{L^\infty(\Omega)}\, \|[\![\nabla u_H^\star \cdot \boldsymbol{n}]\!]\|_{L^2(\omega_H[\,z\,])},
\end{aligned} \tag{5.11}
$$

Furthermore, with $\alpha_{\min} := \min_{x\in\Omega} \alpha(x) > 0$ we obtain

$$
\|[\![\nabla u_H^\star \cdot \boldsymbol{n}]\!]\|_{L^2(\omega_H[\,z\,])} \leq \frac{1}{\alpha_{\min}}\, \|[\![\alpha \nabla u_H^\star \cdot \boldsymbol{n}]\!]\|_{L^2(\omega_H[\,z\,])} \tag{5.12}
$$

and

$$
\|(1 - G_H)\nabla u_H^\star\|_{L^2(T)} \leq \alpha_{\min}^{-1/2}\, \|\alpha^{1/2}(1 - G_H)\nabla u_H^\star\|_{L^2(T)}. \tag{5.13}
$$

Uniform σ-shape regularity implies $H(z) \simeq H(T)$. Thus, finite overlap of patches (2.21) and estimates (5.7), (5.11), and (5.10) yield

$$
\begin{aligned}
\eta(T, u_H^\star)^2 &= H(T)^2\, \|R_H(u_H^\star)\|_{L^2(T)}^2 + H(T)\, \|[\![\alpha \nabla u_H^\star \cdot \boldsymbol{n}]\!]\|_{L^2(\partial T \cap \Omega)}^2 \\
&\overset{(5.7)}{\lesssim} \sum_{\substack{z \in \mathcal{V}_H \cap \Omega \\ z \in T}} \Big[H(z)\, \|[\![\alpha \nabla u_H^\star \cdot \boldsymbol{n}]\!]\|_{L^2(\omega_H[\,z\,])}^2 + H(z)^2\, \|R_H(u_H^\star) - r_{H,z}(u_H^\star)\|_{L^2(\Omega_H[\,z\,])}^2 \Big]
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(5.11)}{\lesssim} \sum_{\substack{z \in \mathcal{V}_H \cap \Omega \\ z \in T}} \Big[H(z) \, \| [\![\nabla u_H^\star \cdot \boldsymbol{n}]\!] \|_{L^2(\omega_H[z])}^2 + H(z)^2 \, \| R_H(u_H^\star) - r_{H,z}(u_H^\star) \|_{L^2(\Omega_H[z])}^2 \Big] \\
&\overset{(5.10)}{\lesssim} \sum_{\substack{z \in \mathcal{V}_H \cap \Omega \\ z \in T}} \Big[\| (1 - G_H) \nabla u_H^\star \|_{L^2(\Omega_H[z])}^2 + H(z)^2 \, \| R_H(u_H^\star) - r_{H,z}(u_H^\star) \|_{L^2(\Omega_H[z])}^2 \Big] \\
&\overset{(5.13)}{\lesssim} \sum_{\substack{z \in \mathcal{V}_H \cap \Omega \\ z \in T}} \Big[\| \alpha^{1/2} (1 - G_H) \nabla u_H^\star \|_{L^2(\Omega_H[z])}^2 + H(z)^2 \, \| R_H(u_H^\star) - r_{H,z}(u_H^\star) \|_{L^2(\Omega_H[z])}^2 \Big] \\
&\overset{(2.21)}{\lesssim} \sum_{T' \in \mathcal{T}_H[T]} \Big[\| \alpha^{1/2} (1 - G_H) \nabla u_H^\star \|_{L^2(T')}^2 + \sum_{\substack{z \in \mathcal{V}_H \cap \Omega \\ z \in T'}} H(z)^2 \, \| R_H(u_H^\star) - r_{H,z}(u_H^\star) \|_{L^2(\Omega_H[z])}^2 \Big] \\
&= \mu_H(\mathcal{T}_H[T], u_H^\star)^2.
\end{aligned}
$$

The hidden constant depends only on the polynomial degrees p and q, the bounds $\|\alpha\|_{L^\infty(\Omega)}$ and $\alpha_{\min}$, the initial triangulation $\mathcal{T}_0$, and the use of newest vertex bisection. This concludes the proof of (2.23a). To prove (2.23b), we treat the two terms of (5.6) individually. For the first term, estimates (5.10) and (5.12), $\alpha \in L^\infty(\Omega)$, and uniform σ-shape regularity verify

$$
\begin{aligned}
\| \alpha^{1/2} (1 - G_H) \nabla u_H^\star \|_{L^2(T)}^2 &\le \| \alpha^{1/2} (1 - G_H) \nabla u_H^\star \|_{L^2(\Omega_H[z])}^2 \\
&\lesssim \| (1 - G_H) \nabla u_H^\star \|_{L^2(\Omega_H[z])}^2 \\
&\overset{(5.10)}{\lesssim} \sum_{z' \in \Sigma[z] \cap \mathcal{V}_H \cap \Omega} H(z') \, \| [\![\nabla u_H^\star \cdot \boldsymbol{n}]\!] \|_{L^2(\omega_H[z'])}^2 \\
&\overset{(5.12)}{\lesssim} \sum_{z' \in \Sigma[z] \cap \mathcal{V}_H \cap \Omega} H(z') \, \| [\![\alpha \nabla u_H^\star \cdot \boldsymbol{n}]\!] \|_{L^2(\omega_H[z'])}^2 \\
&\lesssim \eta_H(\mathcal{T}_H^{(2)}[T], u_H^\star)^2.
\end{aligned}
$$

The hidden constant depends only on the polynomial degree p, the constant $\|\alpha\|_{L^\infty(\Omega)} > 0$, the use of newest vertex bisection, and the initial triangulation $\mathcal{T}_0$. Since $1 - \Pi_q(\Omega_H[z])$ is an L^2-orthogonal projection and $R_H(u_H^\star) - r_{H,z}(u_H^\star) = [1 - \Pi_q(\Omega_H[z])] \, R_H(u_H^\star)$, finite overlap of patches (2.21) yields

$$
\begin{aligned}
\sum_{\substack{z \in \mathcal{V}_H \cap \Omega \\ z \in T}} H(z)^2 \, \| R_H(u_H^\star) - r_{H,z}(u_H^\star) \|_{L^2(\Omega_H[z])}^2 &\le \sum_{\substack{z \in \mathcal{V}_H \cap \Omega \\ z \in T}} H(T)^2 \, \| R_H(u_H^\star) \|_{L^2(\Omega_H[z])}^2 \\
&\overset{(2.21)}{\lesssim} \eta_H(\mathcal{T}_H[T], u_H^\star)^2.
\end{aligned}
$$

The hidden constant depends only on the initial triangulation $\mathcal{T}_0$ and the use of newest vertex bisection. Finally, combining these two estimates, we get

$$
\begin{aligned}
\mu(T, u_H^\star)^2 &= \| (1 - G_H) \nabla u_H^\star \|_{L^2(T)}^2 + \sum_{\substack{z \in \mathcal{V}_H \cap \Omega \\ z \in T}} H(z)^2 \, \| R_H(u_H^\star) - r_{H,z}(u_H^\star) \|_{L^2(\Omega_H[z])}^2 \\
&\lesssim \eta_H(\mathcal{T}_H^{(2)}[T], u_H^\star)^2.
\end{aligned}
$$

Overall, the hidden constant depends only on the polynomial degrees p and q, the bounds $\|\alpha\|_{L^\infty(\Omega)}$ and $\alpha_{\min}$, the initial triangulation $\mathcal{T}_0$, and the use of newest vertex bisection. This concludes the proof of (2.23b) and thus of (2.23). $\square$

5.1.2 Weak stability of the ZZ-estimator

It only remains to prove weak stability (W1) for the ZZ-estimator (5.6) in order to fulfill the assumptions of Theorem 3.14 and Theorem 4.3. This is the content of the following Proposition.

Proposition 5.6 (weak stability of ZZ-estimator). *The ZZ-estimator defined in (5.6) satisfies weak stability (W1) with $r = 0$, where the constant $\widetilde{C}_{\mathrm{stab}}$ depends only on the polynomial degree p, the bounds $\|\alpha\|_{L^\infty(\Omega)}$ and $\alpha_{\min} := \min_{x\in\Omega}\alpha(x) > 0$, the ellipticity constant $C_{\mathrm{ell}} > 0$, and uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$.*

Proof. The proof is divided into five steps.

Step 1 (stability of averaging term). Let $\mathcal{T}_H \in \mathbb{T}$ and $v_H, w_H \in \mathcal{X}_H$ be arbitrary. For the first term of (5.6), the triangle inequality and the Young inequality (3.14) yield

$$\begin{aligned}&\|\alpha^{1/2}(1-G_H)\nabla v_H\|_{L^2(T)}^2\\&\qquad\lesssim \|\alpha^{1/2}(1-G_H)\nabla w_H\|_{L^2(T)}^2 + \|\alpha^{1/2}(1-G_H)\nabla(v_H-w_H)\|_{L^2(T)}^2.\end{aligned} \tag{5.14}$$

Both patch averaging (5.2) and the Scott–Zhang projection (5.4) are locally L^2-stable, i.e., it holds

$$\|\alpha^{1/2}(1-G_H)v\|_{L^2(T)}^2 \lesssim \|\alpha^{1/2}v\|_{L^2(\Omega_H[T])}^2 \quad \text{for all } v \in L^2(\Omega), \tag{5.15}$$

where the hidden constant depends only on the bounds $\|\alpha\|_{L^\infty(\Omega)}$ and $\alpha_{\min}$, and, in the case of the Scott–Zhang projection, on the polynomial degree p and uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$. We will prove (5.15) in the next two steps for both patch averaging and the Scott–Zhang projection.

Step 2 (L^2-stability (5.15) of patch averaging). Let $v \in L^2(\Omega)$. For every $z \in \mathcal{V}_H$, let $\phi_z \in \mathcal{S}^1(\mathcal{T}_H)$ denote the nodal basis function characterized by $\phi_z(z) = 1$ and $\phi_z(z') = 0$ for all $z' \in \mathcal{V}_H$ with $z' \neq z$. Note that $\|\phi_z\|_{L^\infty(\Omega)} = 1$ and that $|\operatorname{supp}(\phi_z) \cap T| > 0$ implies $z \in T$. Hence, the triangle inequality, the Cauchy–Schwarz inequality, and $\alpha \in L^\infty(\Omega)$ provide

$$\begin{aligned}\|\alpha^{1/2}G_H v\|_{L^2(T)} &= \Big\|\alpha^{1/2}\sum_{z\in\mathcal{V}_H}\Big(\frac{1}{|\Omega_H[z]|}\int_{\Omega_H[z]} v\,\mathrm{d}x\Big)\phi_z\Big\|_{L^2(T)}\\&\le \sum_{z\in\mathcal{V}_H\cap T}|\Omega_H[z]|^{-1}\|v\|_{L^1(\Omega_H[z])}\|\alpha^{1/2}\phi_z\|_{L^2(T)}\\&\lesssim \sum_{z\in\mathcal{V}_H\cap T}|\Omega_H[z]|^{-1/2}\|\alpha^{1/2}v\|_{L^2(\Omega_H[z])}\|\phi_z\|_{L^2(T)}\\&\le \sum_{z\in\mathcal{V}_H\cap T}\|\alpha^{1/2}v\|_{L^2(\Omega_H[z])} \le (d+1)\,\|\alpha^{1/2}v\|_{L^2(\Omega_H[T])}.\end{aligned}$$

Combining this once more with the triangle inequality, we obtain

$$\|\alpha^{1/2}(1-G_H)v\|_{L^2(T)}^2 \le \|\alpha^{1/2}v\|_{L^2(T)}^2 + \|\alpha^{1/2}G_Hv\|_{L^2(T)}^2 \le (d+2)\,\|\alpha^{1/2}v\|_{L^2(\Omega_H[T])}.$$

This proves L^2-stability (5.15) for patch averaging.

Step 3 (L^2-stability (5.15) of Scott–Zhang projection). With $J := \dim(\mathcal{S}^p(\mathcal{T}_H))$, recall that $\{\phi_j\}_{j=1}^J$ denotes the nodal basis of $\mathcal{S}^p(\mathcal{T}_H)$ with associated nodes $\{a_j\}_{j=1}^J$ and $a_j \in S_j \in \mathcal{T}_H$. Let $j \in \{1,\dots,J\}$ be arbitrary. Note that $|\operatorname{supp}(\phi_j)\cap T| > 0$ implies $a_j \in T$. Hence, with $\mathcal{J}_T := \{j : a_j \in T\}$, the triangle inequality, the Cauchy–Schwarz inequality, and $\alpha \in L^\infty(\Omega)$ yield

$$\begin{aligned}\|\alpha^{1/2}G_Hv\|_{L^2(T)} &\le \sum_{j\in\mathcal{J}_T} \big|(\psi_j, v)_{L^2(S_j)}\big|\,\|\alpha^{1/2}\phi_j\|_{L^2(T)}\\ &\le \sum_{j\in\mathcal{J}_T} \|v\|_{L^2(S_j)}\,\|\psi_j\|_{L^2(S_j)}\,\|\alpha^{1/2}\phi_j\|_{L^2(T)}\\ &\le \sum_{j\in\mathcal{J}_T} |S_j|^{1/2}\,|T|^{1/2}\,\|v\|_{L^2(T)}\,\|\psi_j\|_{L^\infty(S_j)}\,\|\phi_j\|_{L^\infty(T)}\,\|\alpha\|_{L^\infty(T)}\end{aligned} \tag{5.16}$$

A scaling argument shows

$$\|\phi_j\|_{L^\infty(\Omega)} \lesssim 1 \quad \text{and} \quad \|\psi_j\|_{L^\infty(S_j)} \lesssim |S_j|^{-1}, \tag{5.17}$$

where the hidden constants depend only on the polynomial degree p. Since $S_j \in \mathcal{T}_H[T]$ by the definition of the Scott–Zhang projection (5.4), uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$ implies $|S_j|^{-1/2}|T|^{1/2} \lesssim 1$. Together with $\#\mathcal{J}_T \le \frac{1}{2}(p+1)(p+2)$, we can further estimate (5.16) by

$$\begin{aligned}\|\alpha^{1/2}G_Hv\|_{L^2(T)} &\overset{(5.17)}{\lesssim} \sum_{j\in\mathcal{J}_T} |S_j|^{-1/2}\,|T|^{1/2}\,\|v\|_{L^2(T)}\\ &\lesssim \sum_{j\in\mathcal{J}_T} \|v\|_{L^2(T)} \lesssim \sum_{j\in\mathcal{J}_T} \|\alpha^{1/2}v\|_{L^2(T)} \lesssim \|\alpha^{1/2}v\|_{L^2(\Omega_H[T])}.\end{aligned}$$

Here, the hidden constant depends only on the polynomial degree p, the bounds $\|\alpha\|_{L^\infty(\Omega)}$ and $\alpha_{\min}$, and uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$. This proves L^2-stability (5.15) for the Scott–Zhang projection.

Step 4 (stability of oscillation term). Next, we consider the second term of (5.6). Since $1 - \Pi_q(\Omega_H[z])$ is an L^2-orthogonal projection, the triangle inequality and the Young inequality (3.14) show

$$\begin{aligned}&\sum_{\substack{z\in\mathcal{V}_H\cap\Omega\\ z\in T}} H(z)^2\,\|R_H(v_H) - r_{H,z}(v_H)\|_{L^2(\Omega_H[z])}^2\\ &\quad\lesssim \sum_{\substack{z\in\mathcal{V}_H\cap\Omega\\ z\in T}} H(z)^2\Big[\|(1-\Pi_q(\Omega_H[z]))R_H(w_H)\|_{L^2(\Omega_H[z])}^2\\ &\qquad\qquad + \|R_H(v_H) - R_H(w_H)\|_{L^2(\Omega_H[z])}^2\Big].\end{aligned} \tag{5.18}$$

As in the proof of stability (A1) of the residual-based estimator (see [CKNS08, Corollary 3.4]), an inverse estimate proves

$$H(z)^2 \, \|R_H(v_H) - R_H(w_H)\|^2_{L^2(\Omega_H[z])} \lesssim \|\alpha^{1/2}\nabla(v_H - w_H)\|^2_{L^2(\Omega_H[z])}. \tag{5.19}$$

Step 5 (combination of estimates). A combination of the previous estimates together with uniform σ-shape regularity and uniform ellipticity of $a(\cdot,\cdot)$ yields

$$\begin{aligned}
\mu(\mathcal{U}_H, v_H)^2 &= \sum_{T\in\mathcal{U}_H} \Big[\|\alpha^{1/2}(1-G_H)\nabla v_H\|^2_{L^2(T)} + \sum_{\substack{z\in\mathcal{V}_H\cap\Omega\\ z\in T}} H(z)^2 \, \|R_H(v_H) - r_{H,z}(v_H)\|^2_{L^2(\Omega_H[z])}\Big] \\
&\overset{(5.14)}{\underset{(5.18)}{\lesssim}} \mu(\mathcal{U}_H, w_H)^2 + \sum_{T\in\mathcal{U}_H} \Big[\|\alpha^{1/2}(1-G_H)\nabla(v_H - w_H)\|^2_{L^2(T)} \\
&\qquad + \sum_{\substack{z\in\mathcal{V}_H\cap\Omega\\ z\in T}} H(z)^2 \|R_H(v_H) - R_H(w_H)\|^2_{L^2(\Omega_H[z])}\Big] \\
&\overset{(5.15)}{\underset{(5.19)}{\lesssim}} \mu(\mathcal{U}_H, w_H)^2 + |||v_H - w_H|||^2.
\end{aligned}$$

By finally applying the Young inequality (3.14) to the above inequality, we conclude weak stability (W1) for the ZZ-estimator. Overall, the constant $\widetilde{C}_{\mathrm{stab}}$ depends only on the polynomial degree p, the bounds $\|\alpha\|_{L^\infty(\Omega)}$ and $\alpha_{\min} := \min_{x\in\Omega} \alpha(x) > 0$, and uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$. $\square$

Finally, we fulfill all conditions of Theorem 3.14 and Theorem 4.3 and can therefore conclude full R-linear convergence of the quasi-error (3.27) and optimal complexity of Algorithm B steered by the ZZ-estimator.

Corollary 5.7. *Let $0 \le \theta \le 1$, $C_{\mathrm{mark}} \ge 1$, $\lambda > 0$, and $u_0^0 \in \mathcal{X}_0$ be arbitrary. Suppose that $\boldsymbol{A} = \alpha\boldsymbol{I}$ for $\alpha \in C(\overline{\Omega})$ and $\boldsymbol{f} = 0$ so that the model problem (2.4) reads as in (5.1). Let Algorithm B be steered by the ZZ-estimator μ_ℓ defined in (5.6). Then, Theorem 3.14 guarantees full R-linear convergence of the quasi-error (3.27) and Theorem 4.3 ensures optimal complexity of Algorithm B.*

5.2 Estimator based on local flux equilibration

The equilibrated flux estimator is a popular choice for error estimation due to its remarkable numerical properties, which include a known reliability constant. The motivation for the equilibrated flux estimator is based on the following observation.

We consider the variational problem (2.3) for the PDE (2.4). It is well-known (see, e.g., [Bra07, Theorem 3.6]) that the well-posedness of this problem is equivalent to the inf-sup condition

$$\alpha := \inf_{v\in H^1_0(\Omega)\setminus\{0\}} \sup_{w\in H^1_0(\Omega)\setminus\{0\}} \frac{|a(v,w)|}{\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}} > 0 \tag{5.20a}$$

and the non-degeneracy condition

$$\forall w \in H_0^1(\Omega) \setminus \{0\} \; \exists v \in H_0^1(\Omega) : |a(v,w)| > 0. \tag{5.20b}$$

In particular, the inf-sup stability (5.20a) implies that

$$\|v\|_\star := \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{|a(v,w)|}{\|w\|_{H^1(\Omega)}} \quad \text{for all } v \in H_0^1(\Omega) \tag{5.21}$$

defines a norm on $H_0^1(\Omega)$ that is equivalent to the H^1-norm. Let

$$H(\operatorname{div};\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^d : \operatorname{div}\boldsymbol{\tau} \in L^2(\Omega)\}$$

denote the space of $[L^2(\Omega)]^d$-functions whose divergence exists in a weak sense and belongs to $L^2(\Omega)$ equipped with the weighted graph norm

$$\|\boldsymbol{\tau}\|_{H(\operatorname{div};\Omega)} := \big(\|\boldsymbol{\tau}\|_{L^2(\Omega)}^2 + \operatorname{diam}(\Omega)^2 \,\|\operatorname{div}\boldsymbol{\tau}\|_{L^2(\Omega)}^2\big)^{1/2}.$$

For some $v \in H_0^1(\Omega)$, suppose that we have a function $\boldsymbol{\sigma}[v] \in H(\operatorname{div};\Omega)$ at our disposal satisfying

$$\operatorname{div}\boldsymbol{\sigma}[v] = g[v] := f - \boldsymbol{b}\cdot\nabla v - c\,v \quad \text{in } \Omega. \tag{5.22}$$

We will refer to $\boldsymbol{\sigma}[v]$ as the *flux* of v. Then, the inf-sup stability (5.20a), the fact that $u^\star$ is the solution of (2.3), integration by parts, and the Cauchy–Schwarz inequality verify

$$\begin{aligned}
\alpha^{-1}\,\|u^\star - v\|_{H^1(\Omega)} &\overset{(5.20a)}{\leq} \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{|a(u^\star - v, w)|}{\|w\|_{H^1(\Omega)}} \overset{(5.21)}{=} \|u^\star - v\|_\star \\
&\overset{(2.3)}{=} \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{|\langle f, w\rangle_{L^2(\Omega)} + \langle \boldsymbol{f}, \nabla w\rangle_{L^2(\Omega)} - a(v,w)|}{\|w\|_{H^1(\Omega)}} \\
&\overset{(5.22)}{=} \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{|\langle g[v], w\rangle_{L^2(\Omega)} - \langle \boldsymbol{A}\nabla v - \boldsymbol{f}, \nabla w\rangle_{L^2(\Omega)}|}{\|w\|_{H^1(\Omega)}} \\
&\overset{(5.22)}{=} \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{|\langle \boldsymbol{\sigma}[v] + \boldsymbol{A}\nabla v - \boldsymbol{f}, \nabla w\rangle_{L^2(\Omega)}|}{\|w\|_{H^1(\Omega)}} \\
&\leq \|\boldsymbol{\sigma}[v] + \boldsymbol{A}\nabla v - \boldsymbol{f}\|_{L^2(\Omega)}.
\end{aligned} \tag{5.23}$$

Assuming that the function $\boldsymbol{\sigma}[v]$ is computable, the expression $\|\boldsymbol{\sigma}[v] + \boldsymbol{A}\nabla v - \boldsymbol{f}\|_{L^2(\Omega)}$ is thus a guaranteed upper bound for the error $\|u^\star - v\|_\star$ (with known constant 1) and a reliable error estimator for $\|u^\star - v\|_{H^1(\Omega)}$ with explicit reliability constant α. Moreover, since $u^\star$ solves (2.4), the choice $\boldsymbol{\sigma}[u^\star] := \boldsymbol{f} - \boldsymbol{A}\nabla u^\star$ satisfies

$$\begin{aligned}
\langle \boldsymbol{\sigma}[u^\star], \nabla w\rangle_{L^2(\Omega)} &= \langle \boldsymbol{f} - \boldsymbol{A}\nabla u^\star, \nabla w\rangle_{L^2(\Omega)} \\
&\overset{(2.4)}{=} \langle \boldsymbol{f}, \nabla w\rangle_{L^2(\Omega)} + \langle \boldsymbol{b}\cdot\nabla u^\star + c\,u^\star, w\rangle_{L^2(\Omega)} - a(u^\star, w) \\
&\overset{(2.3)}{=} \langle \boldsymbol{b}\cdot\nabla u^\star + c\,u^\star, w\rangle_{L^2(\Omega)} - \langle f, w\rangle_{L^2(\Omega)} \\
&\overset{(5.22)}{=} -\langle g[u^\star], w\rangle_{L^2(\Omega)} \quad \text{for all } w \in H_0^1(\Omega).
\end{aligned}$$

Therefore, $\boldsymbol{\sigma}[u^\star]$ solves (5.22) and is thus the solution of the minimization problem

$$\boldsymbol{\sigma}[u^\star] = \operatorname*{argmin}_{\substack{\boldsymbol{\tau}\in H(\mathrm{div};\Omega)\\ \mathrm{div}\,\boldsymbol{\tau}=g[u^\star]}} \|\boldsymbol{\tau} + \boldsymbol{A}\nabla u^\star - \boldsymbol{f}\|_{L^2(\Omega)}, \tag{5.24}$$

with

$$\min_{\substack{\boldsymbol{\tau}\in H(\mathrm{div};\Omega)\\ \mathrm{div}\,\boldsymbol{\tau}=g[u^\star]}} \|\boldsymbol{\tau} + \boldsymbol{A}\nabla u^\star - \boldsymbol{f}\|_{L^2(\Omega)} = \|\boldsymbol{\sigma}[u^\star] + \boldsymbol{A}\nabla u^\star - \boldsymbol{f}\|_{L^2(\Omega)} = 0.$$

In particular, we note that $\boldsymbol{\sigma}[u^\star]$ is indeed the unique minimizer to (5.24). For a discrete function $v_H \in \mathcal{X}_H$, one might thus aim to compute $\boldsymbol{\sigma}[v_H]$ as the minimizer of $\|\boldsymbol{\sigma}[v_H] + \boldsymbol{A}\nabla u - \boldsymbol{f}\|_{L^2(\Omega)}$ in a finite-dimensional subspace of $H(\mathrm{div};\Omega)$ with side constraint (5.22) in order to obtain a computable error estimator for $\|u^\star - v_H\|_{H^1(\Omega)}$, which hopefully still has an explicit reliability constant. The obvious disadvantage of this approach is that the computation of $\boldsymbol{\sigma}[v_H]$ requires the solution of global problem.

Therefore, we will first introduce a local version of $\boldsymbol{\sigma}[v_H]$ before providing a discretization of the corresponding local problems. This is the content of the next subsection. After that, we will show that the equilibrated flux estimator constructed in this way is weakly equivalent to the residual-based estimator. Finally, we will prove weak stability of the local equilibrated flux estimator, which will allow us to conclude full R-linear convergence and optimal complexity of Algorithm B steered by the local equilibrated flux estimator.

5.2.1 Construction of the equilibrated flux estimator

Let $\mathcal{T}_H \in \mathbb{T}$ be a triangulation of Ω. For every $z \in \mathcal{V}_H$, let $\phi_z \in \mathcal{S}^1(\mathcal{T}_H)$ denote the nodal basis function characterized by $\phi_z(z) = 1$ and $\phi_z(z') = 0$ for all $z' \in \mathcal{V}_H$ with $z' \neq z$. Again, we consider $\boldsymbol{\sigma}[u^\star] = \boldsymbol{f} - \boldsymbol{A}\nabla u^\star$ and define the *local fluxes* $\boldsymbol{\sigma}_z[u^\star] := \phi_z\,\boldsymbol{\sigma}[u^\star]$. Since the nodal basis forms a partition of unity, it holds

$$\boldsymbol{\sigma}[u^\star] = \Big(\sum_{z\in\mathcal{V}_H} \phi_z\Big)\,\boldsymbol{\sigma}[u^\star] = \sum_{z\in\mathcal{V}_H} \boldsymbol{\sigma}_z[u^\star].$$

Let $\boldsymbol{n}$ denote the unit normal vector on the edges $\bigcup\mathcal{E}_H$ of $\mathcal{T}_H$ (with arbitrary but fixed orientation). By definition, $\boldsymbol{\sigma}_z[u^\star]$ (restricted to $\Omega_H[z]$) is contained in

$$H_0(\mathrm{div};\Omega_H[z]) := \begin{cases} \{\boldsymbol{\tau}\in H(\mathrm{div};\Omega_H[z]) : \boldsymbol{\tau}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega_H[z]\} & \text{if } z \in \mathcal{V}_H\cap\Omega,\\ \{\boldsymbol{\tau}\in H(\mathrm{div};\Omega_H[z]) : \boldsymbol{\tau}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega_H[z]\setminus\partial\Omega\} & \text{if } z \in \mathcal{V}_H\cap\partial\Omega. \end{cases}$$

The divergence of $\boldsymbol{\sigma}_z[u^\star]$ is given by

$$\mathrm{div}\,\boldsymbol{\sigma}_z[u^\star] = \phi_z\,\mathrm{div}\,\boldsymbol{\sigma}[u^\star] + \nabla\phi_z\cdot\boldsymbol{\sigma}[u^\star] \overset{(5.22)}{=} \phi_z\, g[u^\star] - \nabla\phi_z\cdot(\boldsymbol{A}\nabla u^\star - \boldsymbol{f}) =: g_z[u^\star]. \tag{5.25}$$

Therefore, the local fluxes $\boldsymbol{\sigma}_z[u^\star] = \phi_z\,\boldsymbol{\sigma}_z[u^\star] = \phi_z(\boldsymbol{f} - \boldsymbol{A}\nabla u^\star)$ are the solutions of the minimization problems

$$\boldsymbol{\sigma}_z[u^\star] = \operatorname*{argmin}_{\substack{\boldsymbol{\tau}\in H_0(\mathrm{div};\Omega_H[z])\\ \mathrm{div}\,\boldsymbol{\tau}=g_z[u^\star]}} \|\boldsymbol{\tau} + \phi_z\,(\boldsymbol{A}\nabla u^\star - \boldsymbol{f})\|_{L^2(\Omega_H[z])}, \tag{5.26}$$

with

$$\min_{\substack{\boldsymbol{\tau}\in H_0(\operatorname{div};\Omega_H[z])\\ \operatorname{div}\boldsymbol{\tau}=g_z[u^\star]}} \|\boldsymbol{\tau} + \phi_z\,(\boldsymbol{A}\nabla u^\star - \boldsymbol{f})\|_{L^2(\Omega_H[z])} = \|\boldsymbol{\sigma}_z[u^\star] + \phi_z\,(\boldsymbol{A}\nabla u^\star - \boldsymbol{f})\|_{L^2(\Omega_H[z])} = 0,$$

which are localized versions of the minimization problem (5.24). In particular, we note that $\boldsymbol{\sigma}_z[u^\star]$ is indeed the unique minimizer to (5.26). Since we have

$$\langle \operatorname{div}\boldsymbol{\tau}, 1\rangle_{L^2(\Omega_H[z])} = \langle \boldsymbol{\tau}, \nabla 1\rangle_{L^2(\Omega_H[z])} = 0 \quad \text{for all } \boldsymbol{\tau}\in H_0(\operatorname{div};\Omega_H[z]) \text{ with } z\in\mathcal{V}_H\cap\Omega,$$

the range of the divergence operator applied to $H_0(\operatorname{div};\Omega_H[z])$ is contained in

$$L^2_\star(\Omega_H[z]) := \begin{cases}\{q\in L^2(\Omega_H[z]) : \langle q,1\rangle_{L^2(\Omega_H[z])} = 0\} & \text{if } z\in\mathcal{V}_H\cap\Omega,\\ L^2(\Omega_H[z]) & \text{if } z\in\mathcal{V}_H\cap\partial\Omega.\end{cases}$$

The next lemma shows that the divergence operator is even surjective. Although this result will only be needed for a discrete setting, the subsequent construction will also be used in the proof of the discrete analogue.

Lemma 5.8 (divergence is surjective). *The operator* $\operatorname{div}\colon H_0(\operatorname{div};\Omega_H[z]) \to L^2_\star(\Omega_H[z])$ *is surjective for all* $z\in\mathcal{V}_H$*. Moreover, there exists a constant* $\beta_\sigma > 0$ *depending only on uniform* σ*-shape regularity of* $\mathcal{T}_H\in\mathbb{T}$ *such that*

$$\inf_{q\in L^2_\star(\Omega_H[z])}\ \sup_{\boldsymbol{\tau}\in H_0(\operatorname{div};\Omega_H[z])} \frac{\langle \operatorname{div}\boldsymbol{\tau}, q\rangle_{L^2(\Omega_H[z])}}{\|\boldsymbol{\tau}\|_{H(\operatorname{div};\Omega_H[z])}\|q\|_{L^2(\Omega_H[z])}} \geq \operatorname{diam}(\Omega_H[z])^{-1}\,\beta_\sigma > 0. \tag{5.27}$$

Proof. Let $z\in\mathcal{V}_H\cap\Omega$ and $q\in L^2_\star(\Omega_H[z])$ be arbitrary. Consider the homogeneous Neumann problem

$$\begin{aligned} -\Delta u &= q \quad \text{in } \Omega_H[z],\\ \nabla u\cdot\boldsymbol{n} &= 0 \quad \text{on } \partial\Omega_H[z], \end{aligned}$$

with corresponding variational formulation

$$\langle \nabla u, \nabla v\rangle_{L^2(\Omega_H[z])} = \langle q, v\rangle_{L^2(\Omega_H[z])} \quad \text{for all } v\in H^1(\Omega_H[z]). \tag{5.28}$$

Since the Neumann compatibility condition $\langle q,1\rangle_{L^2(\Omega_H[z])} = 0$ is satisfied, there exists a unique solution $u\in H^1_\star(\Omega_H[z]) := \{v\in H^1(\Omega_H[z]) : \langle v,1\rangle_{L^2(\Omega_H[z])} = 0\}$ of (5.28). By (5.28), q is the weak divergence of $\boldsymbol{\zeta} := -\nabla u$. Moreover, (5.28) implies $\boldsymbol{\zeta}\cdot\boldsymbol{n} = 0$ on $\partial\Omega_H[z]$ in the sense of traces. Thus, we have found a function $\boldsymbol{\zeta}\in H_0(\operatorname{div};\Omega_H[z])$ with $\operatorname{div}\boldsymbol{\zeta} = q$. The Cauchy–Schwarz inequality and the Poincaré inequality (see [EG21a, Lemma 3.24]) prove

$$\begin{aligned} \|\nabla u\|^2_{L^2(\Omega_H[z])} &= \langle \nabla u, \nabla u\rangle_{L^2(\Omega_H[z])} \overset{(5.28)}{=} \langle q, u\rangle_{L^2(\Omega_H[z])}\\ &\leq \|q\|_{L^2(\Omega_H[z])}\,\|u\|_{L^2(\Omega_H[z])} \leq C_P\,\operatorname{diam}(\Omega_H[z])\,\|q\|_{L^2(\Omega_H[z])}\,\|\nabla u\|_{L^2(\Omega_H[z])}, \end{aligned}$$

i.e., it holds $\|\nabla u\|_{L^2(\Omega_H[z])} \leq C_{\mathrm{P}}\,\operatorname{diam}(\Omega_H[z])\,\|q\|_{L^2(\Omega_H[z])}$. The Poincaré constant $C_{\mathrm{P}} > 0$ depends only on the shape of $\Omega_H[z]$ and thus only on uniform σ-shape regularity of $\mathcal{T}_H\in\mathbb{T}$. Hence, the definitions of the $H(\operatorname{div})$-norm and $\boldsymbol{\zeta}$ yield

$$\begin{aligned} \|\boldsymbol{\zeta}\|^2_{H(\operatorname{div};\Omega_H[z])} &= \|\boldsymbol{\zeta}\|^2_{L^2(\Omega_H[z])} + \operatorname{diam}(\Omega_H[z])^2\,\|\operatorname{div}\boldsymbol{\zeta}\|^2_{L^2(\Omega_H[z])}\\ &\leq \operatorname{diam}(\Omega_H[z])^2\,(C_{\mathrm{P}}^2 + 1)\,\|q\|^2_{L^2(\Omega_H[z])} \end{aligned}$$

With $\beta_\sigma := (1 + C_{\mathrm{P}}^2)^{-1/2}$, we therefore have

$$\sup_{\boldsymbol{\tau} \in H_0(\mathrm{div};\Omega_H[z])} \frac{\langle \mathrm{div}\, \boldsymbol{\tau}, q \rangle_{L^2(\Omega_H[z])}}{\|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega_H[z])} \|q\|_{L^2(\Omega_H[z])}} \geq \frac{\langle \mathrm{div}\, \boldsymbol{\zeta}, q \rangle_{L^2(\Omega_H[z])}}{\|\boldsymbol{\zeta}\|_{H(\mathrm{div};\Omega_H[z])} \|q\|_{L^2(\Omega_H[z])}}$$
$$= \frac{\|q\|_{L^2(\Omega_H[z])}}{\|\boldsymbol{\zeta}\|_{H(\mathrm{div};\Omega_H[z])}} \geq \mathrm{diam}(\Omega_H[z])^{-1} \beta_\sigma.$$

Taking the infimum over functions in $L^2_\star(\Omega_H[z])$ shows (5.27) for $z \in \mathcal{V}_H \cap \Omega$. The case $z \in \mathcal{V}_H \cap \partial\Omega$ can be treated analogously by considering the Poisson problem with mixed boundary conditions and the Friedrichs inequality. This concludes the proof. □

As the next step, we will discretize the local problems (5.26) in order to obtain computable local fluxes $\boldsymbol{\sigma}_z[v_H]$ for any discrete function $v_H \in \mathcal{X}_H$. To this end, we introduce *local Raviart–Thomas spaces* $\mathcal{RT}^q_0(\mathcal{T}_H[z])$ of order $q \in \mathbb{N}_0$ on the patches $\Omega_H[z]$, which are defined as

$$\mathcal{RT}^q_0(\mathcal{T}_H[z]) := \{\boldsymbol{\tau}_H \in H_0(\mathrm{div};\Omega_H[z]) : \boldsymbol{\tau}_H|_T \in [\mathcal{P}^q(T)]^d + x\,\mathcal{P}^q(T) \text{ for all } T \in \mathcal{T}_H[z]\}.$$

Here, the notation $\boldsymbol{\tau} \in [\mathcal{P}^q(T)]^d + x\,\mathcal{P}^q(T)$ means that there exist polynomials $p_1 \in \mathcal{P}^q(T)$ and $\boldsymbol{p}_2 \in [\mathcal{P}^q(T)]^d$ such that $\boldsymbol{\tau}(x) = \boldsymbol{p}_2(x) + x\,p_1(x)$ for all $x \in T$. This will be the discrete counterpart of the space $H_0(\mathrm{div};\Omega_H[z])$, in which the *local equilibrated fluxes* $\boldsymbol{\sigma}_{H,z}[v_H]$ for discrete functions $v_H \in \mathcal{X}_H$ will be sought. By definition, the range of the divergence operator applied to $\mathcal{RT}^q_0(\mathcal{T}_H[z])$ is contained in

$$L^2_\star(\Omega_H[z]) \cap \mathcal{P}^q(\mathcal{T}_H[z]) =: \mathcal{P}^q_\star(\mathcal{T}_H[z]).$$

Let $\Pi^\star_{H,z} \colon L^2(\Omega_H[z]) \to \mathcal{P}^q_\star(\mathcal{T}_H[z])$ denote the L^2-orthogonal projection onto $\mathcal{P}^q_\star(\mathcal{T}_H[z])$. The straightforward approach to define the constraint for the discrete minimization problems would be to simply replace the exact solution $u^\star$ in (5.25) with the discrete function v_H, i.e.,

$$\mathrm{div}\, \boldsymbol{\sigma}_{H,z}[v_H] = g_z[v_H] := \phi_z\, g[v_H] - \nabla\phi_z \cdot (\boldsymbol{A}\nabla v_H - \boldsymbol{f}). \tag{5.29}$$

However, a discrete $\boldsymbol{\sigma}_{H,z}[v_H] \in \mathcal{RT}^q_0(\mathcal{T}_H[z])$ can in general no longer satisfy that constraint. Instead, we have to project $g_z[v_H]$ onto $\mathcal{P}^q_\star(\mathcal{T}_H[z])$, which leads to the discrete local minimization problems

$$\boldsymbol{\sigma}_{H,z}[v_H] := \operatorname*{argmin}_{\substack{\boldsymbol{\tau}_H \in \mathcal{RT}^q_0(\mathcal{T}_H[z]) \\ \mathrm{div}\,\boldsymbol{\tau}_H = \Pi^\star_{H,z}(g_z[v_H])}} \|\boldsymbol{\tau}_H + \phi_z\,(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\|_{L^2(\Omega_H[z])}. \tag{5.30}$$

Let $\Pi^\star_z \colon L^2(\Omega_H[z]) \to L^2_\star(\Omega_H[z])$ denote the L^2-orthogonal projection onto $L^2_\star(\Omega_H[z])$, which, for all $v \in L^2(\Omega_H[z])$, is given by

$$\Pi^\star_z(v) = \begin{cases} v - \frac{1}{|\Omega_H[z]|}\,\langle v, 1\rangle_{L^2(\Omega_H[z])} & \text{if } z \in \mathcal{V}_H \cap \Omega, \\ v & \text{if } z \in \mathcal{V}_H \cap \partial\Omega. \end{cases} \tag{5.31}$$

Furthermore, we write $\Pi_H \colon L^2(\Omega) \to \mathcal{P}^q(\mathcal{T}_H)$ for the L^2-orthogonal projection onto $\mathcal{P}^q(\mathcal{T}_H)$. The following lemma shows that the minimization problems (5.30) indeed admit unique solutions, which can be computed by solving local saddle-point problems.

Lemma 5.9 (local flux equilibration). *For every $\mathcal{T}_H \in \mathbb{T}$ and $z \in \mathcal{V}_H$, the minimization problem (5.30) admits a unique solution $\boldsymbol{\sigma}_{H,z}[v_H] \in \mathcal{RT}_0^q(\mathcal{T}_H[z])$. With*

$$\begin{aligned} \mathsf{a}(\boldsymbol{\tau}_H, \widetilde{\boldsymbol{\tau}}_H) &:= \langle \boldsymbol{\tau}_H, \widetilde{\boldsymbol{\tau}}_H \rangle_{L^2(\Omega_H[z])} && \text{for all } \boldsymbol{\tau}_H, \widetilde{\boldsymbol{\tau}}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[z]), \\ \mathsf{b}(\boldsymbol{\tau}_H, q_H) &:= \langle \operatorname{div} \boldsymbol{\tau}_H, q_H \rangle_{L^2(\Omega_H[z])} && \text{for all } \boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[z]),\ q_H \in \mathcal{P}^q(\mathcal{T}_H[z]), \end{aligned} \tag{5.32}$$

and

$$\begin{aligned} F(\boldsymbol{\tau}_H) &:= -\langle \boldsymbol{\tau}_H, \phi_z\,(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) \rangle_{L^2(\Omega_H[z])} && \text{for all } \boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[z]), \\ G(q_H) &:= \langle \Pi_z^\star(g_z[v_H]), q_H \rangle_{L^2(\Omega_H[z])} && \text{for all } q_H \in \mathcal{P}^q(\mathcal{T}_H[z]), \end{aligned} \tag{5.33}$$

this solution is also the first component of the unique solution $(\boldsymbol{\sigma}_{H,z}[v_H], r_{H,z}[v_H]) \in \mathcal{RT}_0^q(\mathcal{T}_H[z]) \times \mathcal{P}_\star^q(\mathcal{T}_H[z])$ of the saddle-point problem

$$\mathsf{a}(\boldsymbol{\sigma}_{H,z}[v_H], \boldsymbol{\tau}_H) + \mathsf{b}(\boldsymbol{\tau}_H, r_{H,z}[v_H]) = F(\boldsymbol{\tau}_H) \quad \text{for all } \boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[z]), \tag{5.34a}$$

$$\mathsf{b}(\boldsymbol{\sigma}_{H,z}[v_H], q_H) = G(q_H) \quad \text{for all } q_H \in \mathcal{P}^q(\mathcal{T}_H[z]). \tag{5.34b}$$

Moreover, there exists a constant $\widetilde{\beta}_\sigma > 0$ depending only on uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$ and the polynomial degree q such that

$$\|\boldsymbol{\sigma}_{H,z}[v_H]\|_{H(\operatorname{div};\Omega_H[z])} \le \|F\|_{H(\operatorname{div};\Omega_H[z])'} + 2\,\operatorname{diam}(\Omega_H[z])\,\widetilde{\beta}_\sigma^{-1}\,\|G\|_{L^2(\Omega_H[z])'}. \tag{5.35}$$

For the proof we need Brezzi's theorem from [Bre74]. The following formulation of Brezzi's theorem is taken from [EG21b, Theorem 49.13].

Theorem 5.10 (Brezzi). *Let X and Y be reflexive Banach spaces. Let $\mathsf{a}\colon X \times X$ and $\mathsf{b}\colon X \times Y$ be continuous bilinear forms. Define the operator $B \in L(X, Y')$ by $Bx := \mathsf{b}(x, \cdot)$, i.e., $\langle Bx, y\rangle_{Y',Y} = \mathsf{b}(x,y)$ for all $x \in X$ and $y \in Y$. Then, for any $F \in X'$ and $G \in Y'$, there exists a unique solution $(\widehat{x}, \widehat{y}) \in X \times Y$ of the saddle-point problem*

$$\begin{aligned} \mathsf{a}(\widehat{x}, x) + \mathsf{b}(x, \widehat{y}) &= F(x) && \text{for all } x \in X, \\ \mathsf{b}(\widehat{x}, y) &= G(y) && \text{for all } y \in Y. \end{aligned} \tag{5.36}$$

if and only if $\mathsf{a}(\cdot,\cdot)$ satisfies the inf-sup condition and the non-degeneracy condition on the kernel of B, i.e.,

$$\begin{cases} \displaystyle\inf_{x \in \ker(B)\setminus\{0\}} \sup_{\widetilde{x} \in X\setminus\{0\}} \frac{\mathsf{a}(x, \widetilde{x})}{\|x\|_X \|\widetilde{x}\|_X} =: \boldsymbol{\alpha} > 0, \\ \forall \widetilde{x} \in \ker(B)\setminus\{0\}\ \exists x \in \ker(B):\ |\mathsf{a}(x, \widetilde{x})| > 0, \end{cases} \tag{5.37}$$

and $\mathsf{b}(\cdot,\cdot)$ satisfies the inf-sup condition

$$\inf_{y \in Y} \sup_{x \in X} \frac{|\mathsf{b}(x,y)|}{\|x\|_X \|y\|_Y} =: \boldsymbol{\beta} > 0. \tag{5.38}$$

In this case, we have the following a priori estimates for the solution $(\widehat{x}, \widehat{y})$:

$$\|\widehat{x}\|_X \le \frac{1}{\boldsymbol{\alpha}}\,\|F\|_{X'} + \frac{1}{\boldsymbol{\beta}}\left(1 + \frac{\|\mathsf{a}\|}{\boldsymbol{\alpha}}\right)\|G\|_{Y'}, \tag{5.39a}$$

$$\|\widehat{y}\|_Y \le \frac{1}{\boldsymbol{\beta}}\left(1 + \frac{\|\mathsf{a}\|}{\boldsymbol{\alpha}}\right)\|F\|_{X'} + \frac{\|\mathsf{a}\|}{\boldsymbol{\beta}^2}\left(1 + \frac{\|\mathsf{a}\|}{\boldsymbol{\alpha}}\right)\|G\|_{Y'}. \tag{5.39b}$$

The condition (5.37) *is automatically satisfied if* $\mathsf{a}(\cdot,\cdot)$ *is elliptic on the kernel of* B*, whereas the inf-sup condition* (5.38) *is equivalent to the surjectivity of* B*.* □

Furthermore, we need the following proposition, which links the saddle-point problem (5.36) to a minimization problem.

Proposition 5.11 (equivalent minimization problem for saddle-point problem). *Let* X *and* Y *be reflexive Banach spaces. Let* $\mathsf{a}\colon X \times X$ *and* $\mathsf{b}\colon X \times Y$ *be continuous bilinear forms that satisfy* (5.37) *and* (5.38)*. Furthermore, suppose that* $\mathsf{a}(\cdot,\cdot)$ *is symmetric and positive semidefinite, i.e.,*

$$\mathsf{a}(x,\widetilde{x}) = \mathsf{a}(\widetilde{x},x) \quad \textit{and} \quad \mathsf{a}(x,x) \geq 0 \quad \textit{for all } x,\widetilde{x} \in X.$$

Then, for any $F \in X'$ *and* $G \in Y'$*,* $(\widehat{x},\widehat{y}) \in X \times Y$ *is a solution of the saddle-point problem* (5.36) *if and only if the Lagrange functional*

$$L(x,y) \coloneqq \frac{1}{2}\mathsf{a}(x,x) - F(x) + \mathsf{b}(x,y) - G(y) \quad \textit{for all } (x,y) \in X \times Y$$

satisfies

$$L(\widehat{x},y) \leq L(x,y) \leq L(x,\widehat{y}) \quad \textit{for all } (x,y) \in X \times Y,$$

i.e., $(\widehat{x},\widehat{y})$ *is a saddle-point of* L*. In that case, the first component* $\widehat{x}$ *is the unique solution of the minimization problem*

$$E(\widehat{x}) = \min_{\substack{x\in X\\ \mathsf{b}(x,\cdot)=G}} \left(\frac{1}{2}\mathsf{a}(x,x) - F(x)\right).$$ □

For a proof, we refer to [EG21b, Proposition 49.11]. Finally, we need a discrete version of Lemma 5.8.

Lemma 5.12 (discrete divergence is surjective). *For all* $z \in \mathcal{V}_H$*, the operator* $\operatorname{div}\colon \mathcal{RT}_0^q(\mathcal{T}_H[z]) \to \mathcal{P}_\star^q(\mathcal{T}_H[z])$ *is surjective. Moreover, there exists a constant* $\widetilde{\beta}_\sigma > 0$ *depending only on uniform* σ*-shape regularity of* $\mathcal{T}_H \in \mathbb{T}$ *and the polynomial degree* q *such that*

$$\begin{aligned}\inf_{q_H\in\mathcal{P}_\star^q(\mathcal{T}_H[z])} \sup_{\boldsymbol{\tau}_H\in\mathcal{RT}_0^q(\mathcal{T}_H[z])} &\frac{\langle \operatorname{div}\boldsymbol{\tau}_H, q_H\rangle_{L^2(\Omega_H[z])}}{\|\boldsymbol{\tau}_H\|_{H(\operatorname{div};\Omega_H[z])}\|q_H\|_{L^2(\Omega_H[z])}} \\ &\geq \operatorname{diam}(\Omega_H[z])^{-1}\,\widetilde{\beta}_\sigma > 0.\end{aligned} \tag{5.40}$$

Proof. Let $q_H \in \mathcal{P}_\star^q(\mathcal{T}_H[z])$ be arbitrary. In Lemma 5.8, we have already shown that $\operatorname{div}\colon H_0(\operatorname{div};\Omega_H[z]) \to L^2_\star(\Omega_H[z])$ is surjective for all $z \in \mathcal{V}_H$. Verbatim as in the proof of Lemma 5.8, we first construct a function $\boldsymbol{\zeta} \in H_0(\operatorname{div};\Omega_H[z])$ with

$$\operatorname{div}\boldsymbol{\zeta} = q_H \quad \text{and} \quad \|\boldsymbol{\zeta}\|_{L^2(\Omega_H[z])} \leq C_{\mathrm{P}}\operatorname{diam}(\Omega_H[z])\,\|q_H\|_{L^2(\Omega_H[z])}. \tag{5.41}$$

Let $\Pi_z\colon L^2(\Omega_H[z]) \to \mathcal{P}^q(\mathcal{T}_H[z])$ denote the L^2-orthogonal projection onto $\mathcal{P}^q(\mathcal{T}_H[z])$. [EGSV22, Theorem 3.2] proves the existence of an interpolation operator

$$\mathcal{J}_H : H_0(\operatorname{div};\Omega_H[z]) \to \mathcal{RT}_0^q(\mathcal{T}_H[z])$$

that satisfies the commuting property

$$\operatorname{div} \mathcal{J}_H(\boldsymbol{\tau}) = \Pi_z(\operatorname{div} \boldsymbol{\tau}) \quad \text{for all } \boldsymbol{\tau} \in H_0(\operatorname{div}; \Omega_H[\,z\,]). \tag{5.42}$$

Moreover, $\mathcal{J}_H$ is uniformly L^2-stable with respect to the mesh size H up to data oscillations of the divergence, i.e., there exists a constant $C_{\mathcal{J}} > 0$ depending only on uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$ and the polynomial degree q such that, for all $\boldsymbol{\tau} \in H_0(\operatorname{div}; \Omega_H[\,z\,])$,

$$\|\mathcal{J}_H(\boldsymbol{\tau})\|^2_{L^2(\Omega_H[\,z\,])} \leq C^2_{\mathcal{J}} \bigg(\|\boldsymbol{\tau}\|^2_{L^2(\Omega_H[\,z\,])} + \sum_{T \in \Omega_H[\,z\,]} \Big[\frac{\operatorname{diam}(T)}{q+1} \|\operatorname{div} \boldsymbol{\tau} - \Pi_z(\operatorname{div} \boldsymbol{\tau})\|_{L^2(T)} \Big]^2 \bigg).$$

In particular, since $\operatorname{div} \boldsymbol{\zeta} = q_H \in \mathcal{P}^q(\mathcal{T}_H[\,z\,])$ and Π_z is a projection, we have

$$\|\mathcal{J}_H(\boldsymbol{\zeta})\|_{L^2(\Omega_H[\,z\,])} \leq C_{\mathcal{J}} \, \|\boldsymbol{\zeta}\|_{L^2(\Omega_H[\,z\,])}. \tag{5.43}$$

By defining $\boldsymbol{\zeta}_H \coloneqq \mathcal{J}_H(\boldsymbol{\zeta})$, we thus obtain a function $\boldsymbol{\zeta}_H \in \mathcal{RT}^q_0(\mathcal{T}_H[\,z\,])$ with

$$\begin{aligned} \operatorname{div} \boldsymbol{\zeta}_H &= \operatorname{div} \mathcal{J}_H(\boldsymbol{\zeta}) \overset{(5.42)}{=} \Pi_z(\operatorname{div} \boldsymbol{\zeta}) \overset{(5.41)}{=} \Pi_z(q_H) = q_H, \quad \text{and} \\ \|\boldsymbol{\zeta}_H\|_{L^2(\Omega_H[\,z\,])} &\overset{(5.43)}{\leq} C_{\mathcal{J}} \, \|\boldsymbol{\zeta}\|_{L^2(\Omega_H[\,z\,])} \overset{(5.41)}{\leq} C_{\mathrm{P}} \, C_{\mathcal{J}} \operatorname{diam}(\Omega_H[\,z\,]) \, \|q_H\|_{L^2(\Omega_H[\,z\,])}. \end{aligned}$$

Proceeding as in the proof of Lemma 5.8, we define $\widetilde{\beta}_\sigma \coloneqq (1 + C^2_{\mathrm{P}} \, C^2_{\mathcal{J}})^{-1/2}$ and obtain

$$\inf_{q_H \in \mathcal{P}^q_\star(\mathcal{T}_H[\,z\,])} \sup_{\boldsymbol{\tau}_H \in \mathcal{RT}^q_0(\mathcal{T}_H[\,z\,])} \frac{\langle \operatorname{div} \boldsymbol{\tau}_H, q_H \rangle_{L^2(\Omega_H[\,z\,])}}{\|\boldsymbol{\tau}_H\|_{H(\operatorname{div};\Omega_H[\,z\,])} \|q_H\|_{L^2(\Omega_H[\,z\,])}} \geq \operatorname{diam}(\Omega_H[\,z\,])^{-1} \, \widetilde{\beta}_\sigma > 0,$$

where the constant $\widetilde{\beta}_\sigma$ depends only on the uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$ and the polynomial degree q. This concludes the proof. □

Now we are ready to prove Lemma 5.9.

Proof of Lemma 5.9. The proof is split into four steps.

Step 1 (construction of $\boldsymbol{\sigma}_{H,z}[v_H]$ and $r_{H,z}[v_H]$). Let $z \in \mathcal{V}_H$ be arbitrary. In order to apply Theorem 5.10, we set $X \coloneqq \mathcal{RT}^q_0(\mathcal{T}_H[\,z\,])$, $Y \coloneqq \mathcal{P}^q_\star(\mathcal{T}_H[\,z\,])$ and, as in the Brezzi theorem, define the operator $B \in L(X, Y')$ via $B\boldsymbol{\tau}_H \coloneqq \langle \operatorname{div} \boldsymbol{\tau}_H, \cdot \rangle_{L^2(\Omega_H[\,z\,])}$. Since $\operatorname{div} \boldsymbol{\tau}_H \in \mathcal{P}^q_\star(\mathcal{T}_H[\,z\,])$ for all $\boldsymbol{\tau}_H \in \mathcal{RT}^q_0(\mathcal{T}_H[\,z\,])$, the kernel of B is then given by

$$\ker(B) = \{\boldsymbol{\tau}_H \in \mathcal{RT}^q_0(\mathcal{T}_H[\,z\,]) : \operatorname{div} \boldsymbol{\tau}_H = 0\},$$

i.e., the subspace of divergence-free functions in $\mathcal{RT}^q_0(\mathcal{T}_H[\,z\,])$. By definition of the $H(\operatorname{div})$-norm, it therefore holds

$$\begin{aligned} \mathsf{a}(\boldsymbol{\tau}_H, \boldsymbol{\tau}_H) &= \langle \boldsymbol{\tau}_H, \boldsymbol{\tau}_H \rangle_{L^2(\Omega_H[\,z\,])} \\ &= \|\boldsymbol{\tau}_H\|^2_{L^2(\Omega_H[\,z\,])} = \|\boldsymbol{\tau}_H\|^2_{H(\operatorname{div};\Omega_H[\,z\,])} \quad \text{for all } \boldsymbol{\tau}_H \in \ker(B). \end{aligned} \tag{5.44}$$

Thus, the bilinear form $\mathsf{a}(\cdot,\cdot)$ is elliptic on $\ker(B)$, which implies the condition (5.37). Moreover, for all $\boldsymbol{\tau}_H \in \ker(B)\setminus\{0\}$, it holds that

$$\sup_{\widetilde{\boldsymbol{\tau}}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[\,z\,])\setminus\{0\}} \frac{\mathsf{a}(\boldsymbol{\tau}_H, \widetilde{\boldsymbol{\tau}}_H)}{\|\boldsymbol{\tau}_H\|_{H(\mathrm{div};\Omega_H[\,z\,])}\|\widetilde{\boldsymbol{\tau}}_H\|_{H(\mathrm{div};\Omega_H[\,z\,])}} \geq \frac{\mathsf{a}(\boldsymbol{\tau}_H, \boldsymbol{\tau}_H)}{\|\boldsymbol{\tau}_H\|^2_{H(\mathrm{div};\Omega_H[\,z\,])}} = 1.$$

Thus, the inf-sup condition of (5.37) is even satisfied with $\boldsymbol{\alpha} = 1$. Lemma 5.12 provides the inf-sup condition (5.38) with $\boldsymbol{\beta} := \mathrm{diam}(\Omega_H[\,z\,])^{-1}\,\widetilde{\beta}_\sigma$. Clearly, the bilinear forms $\mathsf{a}(\cdot,\cdot)$ and $\mathsf{b}(\cdot,\cdot)$ are continuous. In particular, the Cauchy–Schwarz inequality and (5.44) show $\|\mathsf{a}\| = 1$. Overall, all assumptions of the Brezzi theorem 5.10 are satisfied. This yields the existence and uniqueness of the solution $(\boldsymbol{\sigma}_{H,z}[v_H], r_{H,z}[v_H]) \in \mathcal{RT}_0^q(\mathcal{T}_H[\,z\,]) \times \mathcal{P}_\star^q(\mathcal{T}_H[\,z\,])$ of the saddle-point problem (5.34) with $\mathcal{P}^q(\mathcal{T}_H[\,z\,])$ replaced by $\mathcal{P}_\star^q(\mathcal{T}_H[\,z\,])$ in the second equation (5.34b). Using $\boldsymbol{\alpha} = 1$, $\|a\| = 1$, and $\boldsymbol{\beta} = \mathrm{diam}(\Omega_H[\,z\,])^{-1}\,\widetilde{\beta}_\sigma$, we obtain (5.35) from the *a priori* estimate (5.39a) of the Brezzi theorem 5.10.

Step 2 (solution of saddle-point problem (5.34)). So far, everything has been proved for the saddle-point problem (5.34) with $\mathcal{P}^q(\mathcal{T}_H[\,z\,])$ replaced by its subspace $\mathcal{P}_\star^q(\mathcal{T}_H[\,z\,])$. In case that $z \in \mathcal{V}_H \cap \partial\Omega$, we have $\mathcal{P}^q(\mathcal{T}_H[\,z\,]) = \mathcal{P}_\star^q(\mathcal{T}_H[\,z\,])$, which already implies that $(\boldsymbol{\sigma}_{H,z}[v_H], r_{H,z}[v_H])$ is indeed the unique solution of the saddle-point problem (5.34). From now on, we thus suppose that $z \in \mathcal{V}_H \cap \Omega$. It holds

$$\mathcal{P}^q(\mathcal{T}_H[\,z\,]) = \mathcal{P}_\star^q(\mathcal{T}_H[\,z\,]) + \mathrm{span}\{1\}.$$

Therefore, it remains to show that

$$\mathsf{b}(\boldsymbol{\sigma}_{H,z}[v_H], 1) \overset{(5.32)}{=} \langle \mathrm{div}\,\boldsymbol{\sigma}_{H,z}[v_H], 1\rangle_{L^2(\Omega_H[\,z\,])} \overset{!}{=} \langle \Pi_z^\star(g_z[v_H]), 1\rangle_{L^2(\Omega_H[\,z\,])} \overset{(5.33)}{=} G(1). \quad (5.45)$$

Integration by parts with vanishing boundary term $(\boldsymbol{\sigma}_{H,z}[v_H]\cdot\boldsymbol{n})|_{\partial\Omega_H[\,z\,]} = 0$ shows for the left-hand side of (5.45) that

$$\langle \mathrm{div}\,\boldsymbol{\sigma}_{H,z}[v_H], 1\rangle_{L^2(\Omega_H[\,z\,])} = -\langle \boldsymbol{\sigma}_{H,z}[v_H], \nabla 1\rangle_{L^2(\Omega_H[\,z\,])} = 0.$$

For the right-hand side of (5.45), the fact that $\Pi_z^\star$ is the L^2-orthogonal projection onto $L^2_\star(\Omega_H[\,z\,])$ already implies

$$\langle \Pi_z^\star(g_z[v_H]), 1\rangle_{L^2(\Omega_H[\,z\,])} = 0.$$

Therefore, both sides of (5.45) vanish, which proves that $(\boldsymbol{\sigma}_{H,z}[v_H], r_{H,z}[v_H])$ is indeed the unique solution of the saddle-point problem (5.34).

Step 3 ($\boldsymbol{\Pi_{H,z}^\star = \Pi_H}$ on $\boldsymbol{L^2_\star(\Omega_H[\,z\,])}$). By extending the functions in $L^2(\Omega_H[\,z\,])$ and $\mathcal{P}_\star^q(\mathcal{T}_H[\,z\,])$ by zero, we have the inclusions

$$L^2(\Omega_H[\,z\,]) \subseteq L^2(\Omega) \quad \text{and} \quad \mathcal{P}_\star^q(\mathcal{T}_H[\,z\,]) \subseteq \mathcal{P}^q(\mathcal{T}_H).$$

In particular, we can consider both $\Pi_{H,z}^\star$ and Π_H as mappings from $L^2(\Omega_H[\,z\,])$ to $\mathcal{P}^q(\mathcal{T}_H)$. For $v \in L^2(\Omega_H[\,z\,])$, these two projections are characterized by

$$\begin{aligned} \langle \Pi_{H,z}^\star\, v, q_H\rangle_{L^2(\Omega_H[\,z\,])} &= \langle v, q_H\rangle_{L^2(\Omega_H[\,z\,])} \quad \text{for all } q_H \in \mathcal{P}_\star^q(\mathcal{T}_H[\,z\,]), \\ \langle \Pi_H\, v, q_H\rangle_{L^2(\Omega_H[\,z\,])} &= \langle v, q_H\rangle_{L^2(\Omega_H[\,z\,])} \quad \text{for all } q_H \in \mathcal{P}^q(\mathcal{T}_H[\,z\,]). \end{aligned} \quad (5.46)$$

If $z \in \mathcal{V}_H \cap \partial\Omega$, $\Pi_z^\star$ is the identity and it holds $\mathcal{P}_\star^q(\mathcal{T}_H[z]) = \mathcal{P}^q(\mathcal{T}_H[z])$ by definition, which implies $\Pi_{H,z}^\star = \Pi_H \circ \Pi_z^\star$. For $z \in \mathcal{V}_H \cap \Omega$, the characterization (5.46) shows that

$$\Pi_{H,z}^\star v = (\Pi_H \circ \Pi_z^\star)\, v \quad \text{for all } v \in L^2(\Omega_H[z]).$$

In any case, we obtain that $\Pi_{H,z}^\star = \Pi_H \circ \Pi_z^\star$. In particular, the projections $\Pi_{H,z}^\star$ and Π_H coincide on $L_\star^2(\Omega_H[z])$.

Step 4 ($\boldsymbol{\sigma}_{H,z}[v_H]$ is minimizer of (5.30)). It remains to show that $\boldsymbol{\sigma}_{H,z}[v_H]$ is the unique solution of the minimization problem (5.30). Since $\Pi_{H,z}^\star = \Pi_H \circ \Pi_z^\star$ by Step 3 and Π_H is the L^2-orthogonal projection onto $\mathcal{P}^q(\mathcal{T}_H) \supseteq \mathcal{P}_\star^q(\mathcal{T}_H[z])$, we have

$$\langle \Pi_{H,z}^\star(g_z[v_H]), q_H\rangle_{L^2(\Omega_H[z])} = \langle \Pi_z^\star(g_z[v_H]), q_H\rangle_{L^2(\Omega_H[z])} = G(q_h) \quad \text{for all } q_H \in \mathcal{P}_\star^q(\mathcal{T}_H[z]).$$

This shows that the constraint $\operatorname{div} \boldsymbol{\tau}_H = \Pi_{H,z}^\star(g_z[v_H])$ in (5.30) is equivalent to the constraint $\operatorname{div} \boldsymbol{\tau}_H = \Pi_z^\star(g_z[v_H])$. For any $\boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[z])$, we rewrite the minimized functional in (5.30) as

$$\begin{aligned}\|\boldsymbol{\tau}_H + &\phi_z\,(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\|_{L^2(\Omega_H[z])}^2 \\ &= \|\boldsymbol{\tau}_H\|_{L^2(\Omega_H[z])}^2 + 2\,\langle \boldsymbol{\tau}_H, \phi_z\,(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\rangle_{L^2(\Omega_H[z])} + \|\phi_z\,(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\|_{L^2(\Omega_H[z])}^2\end{aligned}$$

Therefore, with

$$E(\boldsymbol{\tau}_H) \coloneqq \frac{1}{2}\|\boldsymbol{\tau}_H\|_{L^2(\Omega_H[z])}^2 + \langle \boldsymbol{\tau}_H, \phi_z\,(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\rangle_{L^2(\Omega_H[z])} = \frac{1}{2}\,\mathsf{a}(\boldsymbol{\tau}_H, \boldsymbol{\tau}_H) - F(\boldsymbol{\tau}_H) \quad \text{for all } \boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[z]),$$

the minimization problem (5.30) is equivalent to the minimization problem

$$\boldsymbol{\sigma}_{H,z}[v_H] = \operatorname*{argmin}_{\substack{\boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[z]) \\ \operatorname{div} \boldsymbol{\tau}_H = \Pi_z^\star(g_z[v_H])}} E(\boldsymbol{\tau}_H). \tag{5.47}$$

Since $\mathsf{a}(\cdot,\cdot)$ is symmetric and positive semidefinite, Proposition 5.11 guarantees that $\boldsymbol{\sigma}_{H,z}[v_H]$ is the unique solution of the minimization problem (5.47) and thus also of the minimization problem (5.30). This concludes the proof. □

By extending functions in $H_0(\operatorname{div};\Omega_H[z])$ by zero to $\Omega \setminus \Omega_H[z]$, we have the inclusion

$$H_0(\operatorname{div};\Omega_H[z]) \subset H(\operatorname{div};\Omega) \quad \text{for all } z \in \mathcal{V}_H.$$

Hence, we obtain that the *global equilibrated flux*

$$\boldsymbol{\sigma}_H[v_H] \coloneqq \sum_{z \in \mathcal{V}_H} \boldsymbol{\sigma}_{H,z}[v_H] \tag{5.48}$$

is contained in the global Raviart-Thomas space, i.e.,

$$\boldsymbol{\sigma}_H[v_H] \in \mathcal{RT}^q(\mathcal{T}_H) \coloneqq \{\boldsymbol{\tau}_H \in H(\operatorname{div};\Omega) : \boldsymbol{\tau}_H|_T \in [\mathcal{P}^q(T)]^d + x\,\mathcal{P}^q(T) \text{ for all } T \in \mathcal{T}_H\}.$$

Moreover, the following corollary shows that the global equilibrated flux $\boldsymbol{\sigma}_H[u_H^\star]$ for the Galerkin solution $u_H^\star$ satisfies the constraint (5.22) up to the application of the L^2-orthogonal projection Π_H.

Corollary 5.13 (global flux). *The global equilibrated flux $\boldsymbol{\sigma}_H[u_H^\star]$ for the Galerkin solution $u_H^\star$ satisfies*

$$\operatorname{div} \boldsymbol{\sigma}_H[u_H^\star] = \Pi_H(g[u_H^\star]). \tag{5.49}$$

Proof. The proof is split into two steps.

Step 1 ($g_z[u_H^\star] \in L^2_\star(\Omega_H[z])$). The definition of the Galerkin solution (2.3) implies

$$\begin{aligned}\langle g_z[u_H^\star], 1\rangle_{L^2(\Omega_H[z])} &\overset{(5.29)}{=} \langle \phi_z\, g[u_H^\star] - \nabla\phi_z \cdot (\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}), 1\rangle_{L^2(\Omega_H[z])}\\ &= \langle g[u_H^\star], \phi_z\rangle_{L^2(\Omega_H[z])} + \langle \boldsymbol{f}, \nabla\phi_z\rangle_{L^2(\Omega_H[z])} - \langle \boldsymbol{A}\nabla u_H^\star, \nabla\phi_z\rangle_{L^2(\Omega_H[z])}\\ &\overset{(5.22)}{=} \langle f, \phi_z\rangle_{L^2(\Omega_H[z])} + \langle \boldsymbol{f}, \nabla\phi_z\rangle_{L^2(\Omega_H[z])} - a(u_H^\star, \phi_z) \overset{(2.3)}{=} 0.\end{aligned}$$

Hence, the Galerkin solution $u_H^\star$ satisfies

$$g_z[u_H^\star] \overset{(5.29)}{=} \phi_z\, g[u_H^\star] - \nabla\phi_z \cdot (\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) \in L^2_\star(\Omega_H[z]) \quad \text{for all } z \in \mathcal{V}_H. \tag{5.50}$$

In particular, we have $\Pi_z^\star(g_z[u_H^\star]) = g_z[u_H^\star]$.

Step 2 ($\operatorname{div} \boldsymbol{\sigma}_H[u_H^\star] = \Pi_H(g[u_H^\star])$). Since $g_z[u_H^\star] \in L^2_\star(\Omega_H[z])$, Step 3 of the proof of Lemma 5.9 shows $\Pi^\star_{H,z}(g_z[u_H^\star]) = \Pi_H(g_z[u_H^\star])$ for all $z \in \mathcal{V}_H$. Thus, the definition (5.30) of the local equilibrated fluxes $\boldsymbol{\sigma}_{H,z}[u_H^\star]$ provides

$$\begin{aligned}\operatorname{div} \boldsymbol{\sigma}_H[u_H^\star] = \sum_{z\in\mathcal{V}_H} \operatorname{div} \boldsymbol{\sigma}_{H,z}[u_H^\star] &\overset{(5.30)}{=} \sum_{z\in\mathcal{V}_H} \Pi^\star_{H,z}(g_z[u_H^\star])\\ &\overset{(5.50)}{=} \sum_{z\in\mathcal{V}_H} \Pi_H(g_z[u_H^\star]) = \Pi_H\Big(\sum_{z\in\mathcal{V}_H} g_z[u_H^\star]\Big).\end{aligned}$$

Moreover, the fact that ϕ_z is a partition of unity implies

$$\sum_{z\in\mathcal{V}_H} g_z[u_H^\star] \overset{(5.29)}{=} \sum_{z\in\mathcal{V}_H} \big(\phi_z\, g[u_H^\star] - \nabla\phi_z \cdot (\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f})\big) = g[u_H^\star] - \nabla 1 \cdot (\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) = g[u_H^\star].$$

Combining these two equations yields the desired result (5.49), which concludes the proof. □

In the spirit of (5.23), we now define the *equilibrated flux estimator* μ_H as in (2.12) with refinement indicators

$$\mu_H(T, v_H) := \|\boldsymbol{\sigma}_H[v_H] + \boldsymbol{A}\nabla v_H - \boldsymbol{f}\|_{L^2(T)} + \frac{\operatorname{diam}(T)}{\pi}\, \|(1-\Pi_H) g[v_H]\|_{L^2(T)} \qquad \text{for all } T \in \mathcal{T}_H,\ v_H \in \mathcal{X}_H. \tag{5.51}$$

We finish this subsection by proving reliability (A3) of the equilibrated flux estimator (5.51). Proving reliability of μ_H is, in theory, not necessary, since it already follows from the weak equivalence (2.23) of the equilibrated flux estimator and the residual-based estimator, which will be shown in the next subsection. However, as one of the main properties of the equilibrated flux estimator is the known reliability constant, it is indeed worth proving reliability of μ_H.

Theorem 5.14 (reliability of the equilibrated flux estimator). *For $\mathcal{T}_H \in \mathbb{T}$, let μ_H be the equilibrated flux estimator (5.51). Let $\alpha > 0$ denote the inf-sup constant (5.20a). Then, it holds*

$$\alpha^{-1} \|u^\star - u_H^\star\|_{H^1(\Omega)} \leq \|u^\star - u_H^\star\|_\star \leq \mu_H(u_H^\star), \tag{5.52}$$

i.e. the equilibrated flux estimator μ_H is reliable in the sense of (A3) with reliability constant $C_{\mathrm{rel}} := 1$ if the discretization error is measured with respect to the dual norm $\|\cdot\|_\star$, and with reliability constant $C_{\mathrm{rel}} := \alpha$ if the discretization error is measured with respect to the H^1-norm.

Proof. Inf-sup stability (5.20a), Corollary 5.13 and integration by parts show

$$\begin{aligned}
\alpha^{-1} \|u^\star - u_H^\star\|_{H^1(\Omega)} &\leq \sup_{w \in H_0^1 \setminus \{0\}} \frac{|a(u^\star - u_H^\star, w)|}{\|w\|_{H^1(\Omega)}} \\
&\overset{(2.3)}{=} \sup_{w \in H_0^1(\Omega) \setminus \{0\}} \frac{|\langle f, w\rangle_{L^2(\Omega)} + \langle \boldsymbol{f}, \nabla w\rangle_{L^2(\Omega)} - a(u_H^\star, w)|}{\|w\|_{H^1(\Omega)}} \\
&\overset{(5.22)}{=} \sup_{w \in H_0^1(\Omega) \setminus \{0\}} \frac{|\langle g[u_H^\star], w\rangle_{L^2(\Omega)} - \langle \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}, \nabla w\rangle_{L^2(\Omega)}|}{\|w\|_{H^1(\Omega)}} \\
&\overset{(5.49)}{=} \sup_{w \in H_0^1(\Omega) \setminus \{0\}} \frac{|\langle \operatorname{div} \boldsymbol{\sigma}_H[u_H^\star] + (1 - \Pi_H) g[u_H^\star], w\rangle_{L^2(\Omega)} - \langle \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}, \nabla w\rangle_{L^2(\Omega)}|}{\|w\|_{H^1(\Omega)}} \\
&= \sup_{w \in H_0^1(\Omega) \setminus \{0\}} \frac{|\langle \boldsymbol{\sigma}_H[u_H^\star] + \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}, \nabla w\rangle_{L^2(\Omega)} + \langle (1 - \Pi_H) g[u_H^\star], w\rangle_{L^2(\Omega)}|}{\|w\|_{H^1(\Omega)}}
\end{aligned} \tag{5.53}$$

Splitting the nominator element-wise and using the triangle inequality leads to

$$\begin{aligned}
&\big|\langle \boldsymbol{\sigma}_H[u_H^\star] + \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}, \nabla w\rangle_{L^2(\Omega)} + \langle (1 - \Pi_H) g[u_H^\star], w\rangle_{L^2(\Omega)}\big| \\
&\quad \leq \sum_{T \in \mathcal{T}_H} \Big(\big|\langle \boldsymbol{\sigma}_H[u_H^\star] + \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}, \nabla w\rangle_{L^2(T)}\big| + \big|\langle (1 - \Pi_H) g[u_H^\star], w\rangle_{L^2(T)}\big| \Big).
\end{aligned} \tag{5.54}$$

For the second term in the sum, the fact that Π_H an L^2-orthogonal projection, the Cauchy–Schwarz inequality, and the Poincaré inequality on the convex domain T with known constant $C_{\mathrm{F}} = \pi^{-1}$ (cf. [EG21a, Lemma 3.27]) show

$$\begin{aligned}
\big|\langle (1 - \Pi_H) g[u_H^\star], w\rangle_{L^2(T)}\big| &\leq \big|\langle (1 - \Pi_H) g[u_H^\star], (1 - \Pi_H) w\rangle_{L^2(T)}\big| \\
&\leq \|(1 - \Pi_H) g[u_H^\star]\|_{L^2(T)} \frac{\operatorname{diam}(T)}{\pi} \|\nabla w\|_{L^2(T)}.
\end{aligned} \tag{5.55}$$

Hence, after applying the Cauchy–Schwarz inequality to the first term in the sum, a combination of the previous inequalities together with the Cauchy–Schwarz inequality for sums results in

$$\begin{aligned}
&\big|\langle \boldsymbol{\sigma}_H[u_H^\star] + \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}, \nabla w\rangle_{L^2(\Omega)} + \langle (1 - \Pi_H) g[u_H^\star], w\rangle_{L^2(\Omega)}\big| \\
&\overset{(5.54),(5.55)}{\leq} \sum_{T \in \mathcal{T}_H} \Big(\|\boldsymbol{\sigma}_H[u_H^\star] + \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}\|_{L^2(T)} + \frac{\operatorname{diam}(T)}{\pi} \|(1 - \Pi_H) g[u_H^\star]\|_{L^2(T)} \Big) \|\nabla w\|_{L^2(T)} \\
&\overset{(5.51)}{=} \sum_{T \in \mathcal{T}_H} \mu_H(T, u_H^\star) \|\nabla w\|_{L^2(T)} \leq \mu_H(u_H^\star) \|\nabla w\|_{L^2(\Omega)} \leq \mu_H(u_H^\star) \|w\|_{H^1(\Omega)}.
\end{aligned}$$

Together with (5.53), this proves the reliability estimate (5.52), which concludes the proof. □

Remark 5.15 (pure diffusion problems). For pure diffusion problems, i.e. $\boldsymbol{b} = 0$ and $c = 0$ in (2.4), it is possible to achieve the reliability constant $C_{\mathrm{rel}} = 1$ by measuring the discretization error in the energy norm $|||\cdot|||^2 = \langle \boldsymbol{A}\nabla(\cdot), \nabla(\cdot)\rangle_{L^2(\Omega)}$. To this end, one uses the estimate

$$|||u^\star - v||| \leq \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{\langle \boldsymbol{A}\nabla(u^\star - v), \nabla w\rangle_{L^2(\Omega)}}{|||w|||}$$

instead of the inf-sup stability (5.20a) in (5.23), which leads to

$$|||u^\star - v||| \leq \|A^{-1/2}(\boldsymbol{\sigma}[v] - \boldsymbol{f}) + A^{1/2}\nabla v\|_{L^2(\Omega)}.$$

By appropriately altering the minimization problem (5.30) for $\boldsymbol{\sigma}_{H,z}[v_H]$ to

$$\boldsymbol{\sigma}_{H,z}[v_H] = \operatorname*{argmin}_{\substack{\boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[z]) \\ \operatorname{div} \boldsymbol{\tau}_H = \Pi_z^\star(g_z[v_H])}} \|\boldsymbol{A}^{-1/2}\boldsymbol{\tau}_H + \phi_z\,(\boldsymbol{A}^{1/2}\nabla v_H - \boldsymbol{A}^{-1/2}\boldsymbol{f})\|_{L^2(\Omega_H[z])},$$

the resulting equilibrated flux estimator $\widetilde{\mu}_H$ is reliable with constant $C_{\mathrm{rel}} = 1$ in the energy norm, i.e., it holds $|||u^\star - u_H^\star||| \leq \widetilde{\mu}_H(u_H^\star)$.

5.2.2 Weak equivalence of the equilibrated flux estimator and the residual-based estimator

In the following proposition, we show that the equilibrated flux estimator μ_H from (5.51) and the residual-based estimator η_H from (2.14) are weakly equivalent in the sense of (2.23).

Proposition 5.16 (equilibrated flux estimator and residual-based estimator are weakly equivalent). *Suppose $q \geq p$ and define $s := q - p \geq 0$. Furthermore, suppose that $\boldsymbol{A} \in [\mathcal{P}^s(\mathcal{T}_0)]^{d\times d}$, $\boldsymbol{b} \in [\mathcal{P}^{s+1}(\mathcal{T}_0)]^d$, $c \in \mathcal{P}^s(\mathcal{T}_0)$, $f \in \mathcal{P}^q(\mathcal{T}_0)$, and $\boldsymbol{f} \in [\mathcal{P}^{q-1}(\mathcal{T}_0)]^d$. Then, the equilibrated flux estimator μ_H from (5.51) is weakly equivalent to the residual-based estimator η_H from (2.14) in the sense of (2.23) with $m = 1$. The equivalence constant C_{eq} depends only on the polynomial degrees p and q, the initial triangulation $\mathcal{T}_0$, and the use of newest vertex bisection.*

Proof. The proof is split into two steps, corresponding to the two bounds (2.23a) and (2.23b) of the weak equivalence (2.23).

Step 1 (proof of (2.23*b*)). For $\mathcal{T}_H \in \mathbb{T}$, let $T \in \mathcal{T}_H$ be arbitrary. An inverse estimate [EG21a, Lemma 12.1] and $H(T) \simeq \operatorname{diam}(T)$ show for the local volume residual term (2.15) of η_H that

$$\begin{aligned}
&H(T)^2\,\|-\operatorname{div}(\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) + \boldsymbol{b}\cdot\nabla u_H^\star + c\,u_H^\star - f\|_{L^2(T)}^2 \\
&\quad\overset{(5.22)}{=} H(T)^2\,\|g[u_H^\star] + \operatorname{div}(\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f})\|_{L^2(T)}^2 \\
&\quad\overset{(5.49)}{=} H(T)^2\,\|\operatorname{div}(\sigma_H[u_H^\star] + \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) + (1 - \Pi_H)g[u_H^\star]\|_{L^2(T)}^2 \qquad (5.56)\\
&\quad\lesssim \|\sigma_H[u_H^\star] + \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}\|_{L^2(T)}^2 + \operatorname{diam}(T)^2\,\|(1 - \Pi_H)g[u_H^\star]\|_{L^2(T)}^2 \\
&\quad\overset{(5.51)}{\lesssim} \mu_H(T, u_H^\star)^2.
\end{aligned}$$

Here, the hidden constant depends only on uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$ and the polynomial degree q. Since $\boldsymbol{\sigma}_H[u_H^\star] \in H(\mathrm{div};\Omega)$, its normal jumps $[\![\boldsymbol{\sigma}_H[u_H^\star] \cdot \boldsymbol{n}_E]\!]$ vanish across any face $E \in \mathcal{E}_H^\Omega$. Therefore, the trace inequality [DE12, Lemma 1.49] and an inverse estimate [EG21a, Lemma 12.1] show for the local jump term of η_H that

$$
\begin{aligned}
H(T)\, &\|[\![(\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) \cdot \boldsymbol{n}]\!]\|_{L^2(\partial T \cap \Omega)}^2 \\
&= H(T)\, \|[\![(\boldsymbol{\sigma}_H[u_H^\star] + \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) \cdot \boldsymbol{n}]\!]\|_{L^2(\partial T \cap \Omega)}^2 \\
&\lesssim \|\boldsymbol{\sigma}_H[u_H^\star] + \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}\|_{L^2(\Omega_H[T])}^2 \\
&\le \mu_H(\mathcal{T}_H[T], u_H^\star)^2.
\end{aligned}
\tag{5.57}
$$

Again, the hidden constant depends only on uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$ and the polynomial degree q. The combination of (5.56) and (5.57) provides

$$
\eta_H(T, u_H^\star)^2 \lesssim \mu_H(\mathcal{T}_H[T], u_H^\star)^2 \quad \text{for all } T \in \mathcal{T}_H.
$$

This concludes the proof of (2.23b).

Step 2 (proof of (2.23*a*)). For $\mathcal{T}_H \in \mathbb{T}$, let $T \in \mathcal{T}_H$ be arbitrary. Let $u_H^\star(f, \boldsymbol{f})$ be the Galerkin solution of the model problem (2.4) with data $f \in \mathcal{P}^q(\mathcal{T}_H)$ and $\boldsymbol{f} \in [\mathcal{P}^{q-1}(\mathcal{T}_H)]^d$. By (2.3), $u_H^\star(f, \boldsymbol{f})$ depends linearly on $(f, \boldsymbol{f}) \in \mathcal{P}^q(\mathcal{T}_H) \times [\mathcal{P}^{q-1}(\mathcal{T}_H)]^d =: V$. Hence, the mapping $|(f, \boldsymbol{f})|_2 := \eta_H(\mathcal{T}_H[T], u_H^\star(f, \boldsymbol{f}))$ defines a seminorm on V.

Since the right-hand in the saddle-point problem (5.34) is linear in f, $\boldsymbol{f}$, and $u_H^\star(f, \boldsymbol{f})$, the local equilibrated flux $\boldsymbol{\sigma}_{H,z}[u_H^\star(f, \boldsymbol{f})]$ also depends linearly on $(f, \boldsymbol{f})$ for all $z \in \mathcal{V}_H$. Therefore, also

$$
|(f, \boldsymbol{f})|_1 := \sum_{z \in \mathcal{V}_H \cap T} \|\boldsymbol{\sigma}_{H,z}[u_H^\star(f, \boldsymbol{f})] + \phi_z \left(\boldsymbol{A}\nabla u_H^\star(f, \boldsymbol{f}) - \boldsymbol{f}\right)\|_{L^2(\Omega_H[z])} \tag{5.58}
$$

defines a seminorm on V. Since the nodal basis functions ϕ_z form a partition of unity, the triangle inequality shows

$$
\begin{aligned}
\|\boldsymbol{\sigma}_H[u_H^\star] &+ \boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}\|_{L^2(T)} \\
&\le \sum_{z \in \mathcal{V}_H \cap T} \|\boldsymbol{\sigma}_{H,z}[u_H^\star] + \phi_z \left(\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}\right)\|_{L^2(\Omega_H[z])} = |(f, \boldsymbol{f})|_1.
\end{aligned}
\tag{5.59}
$$

Since V is finite-dimensional, Lemma 5.4 guarantees that there exists a constant $C_{\mathrm{eq}} > 0$ such that for all $(f, \boldsymbol{f}) \in V$

$$
\mu_H(T, u_H^\star(f, \boldsymbol{f})) \overset{(5.59)}{\le} |(f, \boldsymbol{f})|_1 \le C_{\mathrm{eq}}\, |(f, \boldsymbol{f})|_2 = C_{\mathrm{eq}}\, \eta_H(\mathcal{T}_H[T], u_H^\star(f, \boldsymbol{f}))
$$

if and only it holds

$$
|(f, \boldsymbol{f})|_2 = 0 \implies |(f, \boldsymbol{f})|_1 = 0 \quad \text{for all } (f, \boldsymbol{f}) \in V. \tag{5.60}
$$

The constant C_{eq} obtained in this way initially depends on the shape of the patch $\mathcal{T}_H[T]$ and the polynomial degrees p and q. However, since newest vertex bisection leads to finitely many patch shapes, it follows that C_{eq} depends only on the use of newest vertex bisection, the initial triangulation $\mathcal{T}_0$, and the polynomial degrees p and q. It thus only remains to prove (5.60).

To this end, let $(f, \boldsymbol{f}) \in V$ be arbitrary with $|(f, \boldsymbol{f})|_2 = 0$. For the sake of readability, denote $u_H^\star(f, \boldsymbol{f})$ by $u_H^\star$. By the definition of the residual-based estimator η_H from (2.14) and $g[u_H^\star]$ from (5.22), the condition $\eta_H(\mathcal{T}_H[T], u_H^\star) = 0$ implies that

$$\llbracket (\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) \cdot \boldsymbol{n} \rrbracket = 0 \qquad \text{for all } E \in \mathcal{E}_H^\Omega \cap \Omega_H[T] \tag{5.61a}$$

$$-\operatorname{div}(\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) = g[u_H^\star] \quad \text{on all } T \in \mathcal{T}_H[T]. \tag{5.61b}$$

From (5.61a), it follows that $\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}$ is continuous across all faces $E \in \mathcal{E}_H^\Omega \cap \Omega_H[T]$, which means that (5.61b) even holds on $\Omega_H[T]$. For arbitrary $z \in \mathcal{V}_H$ with $z \in T$, we define $\boldsymbol{\tau}_{H,z} := -\phi_z\,(\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) \in [\mathcal{P}^q(\mathcal{T}_H[z])]^d$. By the previous observations, $\boldsymbol{\tau}_{H,z}$ is a $H(\operatorname{div}; \Omega_H[z])$-function with divergence

$$\operatorname{div} \boldsymbol{\tau}_{H,z} \overset{(5.61b)}{=} \phi_z\, g[u_H^\star] - \nabla\phi_z \cdot (\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) \overset{(5.29)}{=} g_z[u_H^\star] \in \mathcal{P}^q(\mathcal{T}_H[z]).$$

The last inclusion follows from the assumptions on the polynomial degrees of $\boldsymbol{A}$, b, c, f, and $\boldsymbol{f}$. By (5.50), we even have $g_z[u_H^\star] \in L^2_\star(\Omega_H[z])$ and therefore

$$\operatorname{div} \boldsymbol{\tau}_{H,z} = g_z[u_h^\star] = \Pi_{H,z}^\star(g_z[u_H^\star]).$$

In the case that $z \in \mathcal{V}_H \cap T \cap \partial\Omega$, we already have $\boldsymbol{\tau}_{H,z} \in H_0(\operatorname{div}; \Omega_H[z])$. If $z \in \mathcal{V}_H \cap T \cap \Omega$, it holds $\boldsymbol{\tau}_{H,z} \cdot \boldsymbol{n} = \phi_z((\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f}) \cdot \boldsymbol{n}) = 0$ on $\partial\Omega_H[z]$ by the definition of ϕ_z, which verifies $\boldsymbol{\tau}_{H,z} \in H_0(\operatorname{div}; \Omega_H[z])$ for all $z \in \mathcal{V}_H \cap T$. Thus, it holds

$$\boldsymbol{\tau}_{H,z} \in H_0(\operatorname{div}; \Omega_H[z]) \cap [\mathcal{P}^q(\mathcal{T}_H[z])]^d \subset \mathcal{RT}_0^q(\mathcal{T}_H[z]).$$

Overall, we have shown that $\boldsymbol{\tau}_{H,z}$ is an admissible function in the local minimization problem (5.30). Since $\boldsymbol{\tau}_{H,z}$ is, by definition, clearly the minimizer of this problem, we have $\boldsymbol{\sigma}_{H,z}[u_H^\star] = \boldsymbol{\tau}_{H,z} = -\phi_z\,(\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f})$. By definition (5.58), it therefore holds $|(f, \boldsymbol{f})|_1 = 0$. This concludes the proof of (5.60) and thus the proof of the weak equivalence (2.23). □

Remark 5.17. We stress that Step 1 of the previous proof uses only that the global flux $\boldsymbol{\sigma}_H[u_H^\star]$ satisfies (5.49). The restrictions on the coefficients and the data are only necessary for Step 2. However, there is a different proof for Step 2 that, instead of a seminorm argument, relies on the non-trivial estimate

$$\begin{aligned}\min_{\substack{\boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[z]) \\ \operatorname{div} \boldsymbol{\tau}_H = \Pi_{H,z}^\star(g_z[u_H^\star])}} &\big\| \boldsymbol{\tau}_H + \boldsymbol{\Pi}_{H,z}\big[\phi_z\,(\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f})\big] \big\|_{L^2(\Omega_H[z])} \\ &\lesssim \min_{\substack{\boldsymbol{\tau} \in H_0(\operatorname{div}; \Omega_H[z]) \\ \operatorname{div} \boldsymbol{\tau} = \Pi_{H,z}^\star(g_z[u_H^\star])}} \big\| \boldsymbol{\tau} + \boldsymbol{\Pi}_{H,z}\big[\phi_z\,(\boldsymbol{A}\nabla u_H^\star - \boldsymbol{f})\big] \big\|_{L^2(\Omega_H[z])},\end{aligned} \tag{5.62}$$

where $\boldsymbol{\Pi}_{H,z} \colon [L^2(\Omega_H[z])]^d \to [\mathcal{P}^q(\mathcal{T}_H[z])]^d + x\mathcal{P}^q(\mathcal{T}_H[z])$ is the L^2-orthogonal projection onto the space of piecewise Raviart–Thomas functions (see [BPS09, Theorem 7] for $d = 2$, and [EV20, Theorem 2.5, Corollary 3.3] for $d = 3$). Although this proof is a lot more involved, it has the advantage that it only requires $\boldsymbol{A}$ and $\boldsymbol{f}$ to be piecewise polynomial and that the resulting equivalence constant is independent of the polynomial degrees p and q. We note that the converse estimate of (5.62) holds with known constant 1 as $\mathcal{RT}_0^q(\mathcal{T}_H[z]) \subset H_0(\operatorname{div}; \Omega_H[z])$ and hence the minimum on the right-hand side is taken over a much larger space.

5.2.3 Weak stability of the equilibrated flux estimator

In order to fulfill the requirements of Theorem 3.14 and Theorem 4.3, it only remains to prove weak stability (W1) for the equilibrated flux estimator (5.51). This is the content of the following proposition.

Proposition 5.18 (weak stability (W1) of the equilibrated flux estimator). *The equilibrated flux estimator μ_H from (5.51) satisfies weak stability (W1) with $r = 0$. The constant $\widetilde{C}_{\text{stab}}$ depends only on the uniform σ-shape regularity of $\mathcal{T}_H \in \mathbb{T}$, the polynomial degree q, the bounds $\|\boldsymbol{A}\|_{L^\infty}$, $\|\boldsymbol{b}\|_{L^\infty}$, $\|c\|_{L^\infty}$, and the ellipticity constant $C_{\text{ell}} > 0$.*

Proof. The proof consists of three steps.

Step 1 (first stability estimates). For $\mathcal{T}_H \in \mathbb{T}$, let μ_H be the equilibrated flux estimator (5.51). Let $v_H, w_H \in \mathcal{X}_H$ and $\mathcal{U}_H \subseteq \mathcal{T}_H$ be arbitrary. The reverse triangle inequalities on the sequence space ℓ^2 and the Lebesgue space L^2 show

$$\begin{aligned} |\mu_H(\mathcal{U}_H, v_H) - \mu_H(\mathcal{U}_H, w_H)|^2 &\le \sum_{T \in \mathcal{U}_H} |\mu_H(T, v_H) - \mu_H(T, w_H)|^2 \\ &\overset{(5.51)}{\underset{(3.14)}{\lesssim}} \sum_{T \in \mathcal{U}_H} \Big(\|\boldsymbol{\sigma}_H[v_H] - \boldsymbol{\sigma}_H[w_H]\|^2_{L^2(T)} + \|\boldsymbol{A}\nabla(v_H - w_H)\|^2_{L^2(T)} \\ &\qquad + \frac{\operatorname{diam}(T)^2}{\pi^2} \|(1 - \Pi_H)(g[v_H] - g[w_H])\|^2_{L^2(T)} \Big). \end{aligned} \tag{5.63}$$

As in the proof of stability (A1) in Proposition 3.10, we want to bound the right-hand side of (5.63) by $|||v_H - w_H|||^2$ up to a constant. To this end, we estimate the terms in the sum separately. The second term in the sum can be estimated using the assumption $\boldsymbol{A} \in [L^\infty(\Omega)]^{d \times d}_{\text{sym}}$, i.e.,

$$\|\boldsymbol{A}\nabla(v_H - w_H)\|_{L^2(T)} \le \|\boldsymbol{A}\|_{L^\infty} \|v_H - w_H\|_{H^1(T)} \quad \text{for all } T \in \mathcal{U}_H. \tag{5.64}$$

Similarly, since Π_H is an L^2-orthogonal projection, $\boldsymbol{b} \in [L^\infty(\Omega)]^d$ and $c \in L^\infty(\Omega)$ imply for the third term in the sum

$$\begin{aligned} \frac{\operatorname{diam}(T)}{\pi} \|(1 - \Pi_H)(g[v_H] - g[w_H])\|_{L^2(T)} &\le \frac{\operatorname{diam}(T)}{\pi} \|g[v_H] - g[w_H]\|_{L^2(T)} \\ &\overset{(5.22)}{=} \frac{\operatorname{diam}(T)}{\pi} \|\boldsymbol{b} \cdot \nabla(v_H - w_H) + c\,(v_H - w_H)\|_{L^2(T)} \\ &\le \frac{\operatorname{diam}(\Omega)}{\pi} \big(\|\boldsymbol{b}\|_{L^\infty} + \|c\|_{L^\infty}\big) \|v_H - w_H\|_{H^1(T)} \quad \text{for all } T \in \mathcal{U}_H. \end{aligned} \tag{5.65}$$

It thus only remains to estimate the first term $\|\boldsymbol{\sigma}_H[v_H] - \boldsymbol{\sigma}_H[w_H]\|_{L^2(T)}$ in the sum (5.63).

Step 2 (estimate of local flux difference). The triangle inequality and the definition of the global equilibrated flux (5.48) yield

$$\|\boldsymbol{\sigma}_H[v_H] - \boldsymbol{\sigma}_H[w_H]\|_{L^2(T)} \le \sum_{\substack{z \in \mathcal{V}_H \\ z \in T}} \|\boldsymbol{\sigma}_{H,z}[v_H] - \boldsymbol{\sigma}_{H,z}[w_H]\|_{L^2(\Omega[z])}. \tag{5.66}$$

Subtraction of the saddle-point problems (5.34) for $\boldsymbol{\sigma}_{H,z}[v_H]$ and $\boldsymbol{\sigma}_{H,z}[w_H]$ shows

$$\begin{aligned} &\mathsf{a}(\boldsymbol{\sigma}_{H,z}[v_H] - \boldsymbol{\sigma}_{H,z}[w_H], \boldsymbol{\tau}_H) + \mathsf{b}(\boldsymbol{\tau}_H, r_{H,z}[v_H] - r_{H,z}[w_H]) \\ &\qquad = -\langle \boldsymbol{\tau}_H, \phi_z\,(\boldsymbol{A}\nabla(v_H - w_H))\rangle_{L^2(\Omega[\,z\,])} \quad \text{for all } \boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[\,z\,]), \\ &\mathsf{b}(\boldsymbol{\sigma}_{H,z}[v_H] - \boldsymbol{\sigma}_{H,z}[w_H], q_H) \\ &\qquad = \langle \Pi_z^\star(g_z[v_H] - g_z[w_H]), q_H\rangle_{L^2(\Omega[\,z\,])} \quad \text{for all } q_H \in \mathcal{P}_\star^q(\mathcal{T}_H[\,z\,]). \end{aligned} \tag{5.67}$$

Therefore, the *a priori* estimate (5.35) provides

$$\begin{aligned} \|\boldsymbol{\sigma}_{H,z}[v_H] - \boldsymbol{\sigma}_{H,z}[w_H]\|_{H(\mathrm{div};\Omega[\,z\,])} &\leq \big\|\langle \phi_z\,(\boldsymbol{A}\nabla(v_H - w_H)), \cdot\rangle_{L^2(\Omega[\,z\,])}\big\|_{H(\mathrm{div};\Omega[\,z\,])'} \\ &\quad + 2\,\mathrm{diam}(\Omega[\,z\,])\,\widetilde{\beta}_\sigma^{-1}\,\big\|\langle \Pi_z^\star(g_z[v_H] - g_z[w_H]), \cdot\rangle_{L^2(\Omega[\,z\,])}\big\|_{L^2(\Omega[\,z\,])'}. \end{aligned} \tag{5.68}$$

The Cauchy–Schwarz inequality, the fact that $|\phi_z| \leq 1$, and $\boldsymbol{A} \in [L^\infty(\Omega)]^{d\times d}_{\mathrm{sym}}$ imply for all $\boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[\,z\,])$ that

$$|\langle \phi_z\,(\boldsymbol{A}\nabla(v_H - w_H)), \boldsymbol{\tau}_H\rangle_{L^2(\Omega[\,z\,])}| \leq \|\boldsymbol{A}\|_{L^\infty}\,\|\nabla(v_H - w_H)\|_{L^2(\Omega[\,z\,])}\,\|\boldsymbol{\tau}_H\|_{L^2(\Omega[\,z\,])}.$$

Hence, the first term of the right-hand side in (5.68) can be estimated by

$$\big\|\langle \phi_z\,(\boldsymbol{A}\nabla(v_H - w_H)), \cdot\rangle_{L^2(\Omega[\,z\,])}\big\|_{H(\mathrm{div};\Omega[\,z\,])'} \leq \|\boldsymbol{A}\|_{L^\infty}\,\|v_H - w_H\|_{H^1(\Omega[\,z\,])}. \tag{5.69}$$

For the second term in (5.68), plugging in the definitions of g_z and g and using the Cauchy–Schwarz inequality leads, for all $q_H \in \mathcal{P}_\star^q(\mathcal{T}_H[\,z\,])$, to

$$\begin{aligned} &|\langle \Pi_z^\star(g_z[v_H] - g_z[w_H]), q_H\rangle_{L^2(\Omega[\,z\,])}| \\ &\quad \overset{(5.29)}{\leq} \big\|\Pi_z^\star\big(\phi_z(g[v_H] - g[w_H]) - \nabla\phi_z \cdot (\boldsymbol{A}\nabla(v_H - w_H))\big)\big\|_{L^2(\Omega[\,z\,])}\|q_H\|_{L^2(\Omega[\,z\,])} \end{aligned}$$

Since $\Pi_z^\star$ is an L^2-orthogonal projection, we can further estimate the first term in the product by

$$\begin{aligned} &\big\|\Pi_z^\star\big(\phi_z(g[v_H] - g[w_H]) - \nabla\phi_z \cdot (\boldsymbol{A}\nabla(v_H - w_H))\big)\big\|_{L^2(\Omega[\,z\,])} \\ &\quad \leq \|\phi_z(g[v_H] - g[w_H]) - \nabla\phi_z \cdot (\boldsymbol{A}\nabla(v_H - w_H))\|_{L^2(\Omega[\,z\,])} \\ &\quad \overset{(5.22)}{=} \|\phi_z(\boldsymbol{b}\cdot\nabla(v_H - w_H) + c\,(v_H - w_H)) + \nabla\phi_z \cdot (\boldsymbol{A}\nabla(v_H - w_H))\|_{L^2(\Omega[\,z\,])} \\ &\quad \leq \|\phi_z(\boldsymbol{b}\cdot\nabla(v_H - w_H) + c\,(v_H - w_H))\|_{L^2(\Omega[\,z\,])} + \|\nabla\phi_z \cdot (\boldsymbol{A}\nabla(v_H - w_H))\|_{L^2(\Omega[\,z\,])}. \end{aligned}$$

The fact that $|\phi_z| \leq 1$, $\boldsymbol{b} \in [L^\infty(\Omega)]^d$, and $c \in L^\infty(\Omega)$ imply

$$\|\phi_z(\boldsymbol{b}\cdot\nabla(v_H - w_H) + c\,(v_H - w_H))\|_{L^2(\Omega[\,z\,])} \leq (\|\boldsymbol{b}\|_{L^\infty} + \|c\|_{L^\infty})\,\|v_H - w_H\|_{H^1(\Omega[\,z\,])}.$$

The scaling $\|\nabla\phi_z\|_{L^\infty(\Omega)} \lesssim \mathrm{diam}(\Omega[\,z\,])^{-1}$ and the assumption $\boldsymbol{A} \in [L^\infty(\Omega)]^{d\times d}_{\mathrm{sym}}$ show

$$\begin{aligned} \|\nabla\phi_z \cdot (\boldsymbol{A}\nabla(v_H - w_H))\|_{L^2(\Omega[\,z\,])} &\lesssim \mathrm{diam}(\Omega[\,z\,])^{-1}\,\|\boldsymbol{A}\nabla(v_H - w_H)\|_{L^2(\Omega[\,z\,])} \\ &\leq \mathrm{diam}(\Omega[\,z\,])^{-1}\,\|\boldsymbol{A}\|_{L^\infty}\,\|v_H - w_H\|_{H^1(\Omega[\,z\,])}. \end{aligned}$$

Altogether, combining the previous four estimates we obtain

$$\big\| \langle \Pi_z^\star (g_z[v_H] - g_z[w_H]), \cdot \rangle_{L^2(\Omega[z])} \big\|_{L^2(\Omega[z])'} \lesssim \operatorname{diam}(\Omega[z])^{-1} \, \|v_H - w_H\|_{H^1(\Omega[z])}. \tag{5.70}$$

Overall, a combination of the estimates (5.66), (5.68), (5.69), and (5.70) shows

$$\|\boldsymbol{\sigma}_H[v_H] - \boldsymbol{\sigma}_H[w_H]\|_{L^2(T)} \lesssim \sum_{\substack{z \in \mathcal{V}_H \\ z \in T}} \|v_H - w_H\|_{H^1(\Omega[z])} \tag{5.71}$$

Step 3 (combination of local estimates). Using (5.71), (5.64), and (5.65), we can further estimate the right-hand side of (5.63) by

$$|\mu_H(\mathcal{U}_H, v_H) - \mu_H(\mathcal{U}_H, w_H)|^2 \lesssim \sum_{T \in \mathcal{U}_H} \|v_H - w_H\|^2_{H^1(T)} + \sum_{T \in \mathcal{U}_H} \sum_{\substack{z \in \mathcal{V}_H \\ z \in T}} \|v_H - w_H\|^2_{H^1(\Omega[z])}. \tag{5.72}$$

Therefore, uniform σ-shape regularity and uniform ellipticity of $a(\cdot,\cdot)$ imply

$$|\mu_H(\mathcal{U}_H, v_H) - \mu_H(\mathcal{U}_H, w_H)|^2 \lesssim |||v_H - w_H|||^2.$$

This concludes the proof of weak stability (W1). □

Finally, we have verified all conditions of Theorem 3.14 and Theorem 4.3 and can therefore conclude full R-linear convergence and optimal complexity of Algorithm B steered by the equilibrated flux estimator.

Corollary 5.19. *Let $0 \le \theta \le 1$, $C_{\mathrm{mark}} \ge 1$, $\lambda > 0$, and $u_0^0 \in \mathcal{X}_0$ be arbitrary. Suppose $q \ge p$ and define $s := q - p \ge 0$. Furthermore, suppose that $\boldsymbol{A} \in [\mathcal{P}^s(\mathcal{T}_0)]^{d \times d}$, $\boldsymbol{b} \in [\mathcal{P}^{s+1}(\mathcal{T}_0)]^d$, $c \in \mathcal{P}^s(\mathcal{T}_0)$, $f \in \mathcal{P}^q(\mathcal{T}_0)$, and $\boldsymbol{f} \in [\mathcal{P}^{q-1}(\mathcal{T}_0)]^d$ in the model problem (2.4). Let Algorithm B be steered by the equilibrated flux estimator μ_ℓ defined in (5.51). Then, Theorem 3.14 guarantees full R-linear convergence of the quasi-error (3.27) and Theorem 4.3 ensures optimal complexity of Algorithm B.*

6 Numerical experiments

In this chapter, we examine the numerical performance of AFEM with inexact solver steered by equivalent estimators (Algorithm B). Our primary focus lies on demonstrating optimal convergence rates with respect to the overall computational time, as this is the key result of this thesis (Theorem 4.3). To this end, we consider the ZZ-estimator (5.6) and the equilibrated flux estimator (5.51) from Chapter 5, for which we have shown that they satisfy the assumptions of Theorem 4.3 and thus guarantee optimal complexity of Algorithm B provided that the adaptivity parameters are sufficiently small (Corollary 5.7 and Corollary 5.19). We also present comparisons of both estimators with the standard residual-based estimator (2.14), demonstrating their potential as practical alternatives in adaptive finite element methods. All experiments in this chapter employ the MATLAB software package `MooAFEM` from [IP23].

6.1 Experiments with the ZZ-estimator

In this section, we first provide a brief overview of the implementational details for the ZZ-estimator, followed by numerical experiments for the Poisson model problem (6.3), a nonsymmetric second-order PDE (6.6), and a diffusion problem (6.9).

6.1.1 Implementational aspects for the ZZ-estimator

Recall the ZZ-estimator (5.6), which we defined in Section 5.1 for the PDE (5.1). In the following, we consider a weighted version of the ZZ-estimator (5.6), where the patch contributions in the oscillation term are scaled by the inverse of the number of patch-elements, i.e.,

$$\mu_H(T, v_H)^2 \coloneqq \mu_H^{\mathrm{rec}}(T, v_H)^2 + \mu_H^{\mathrm{osc}}(T, v_H)^2, \tag{6.1}$$

where the recovery term $\mu_H^{\mathrm{rec}}(T, v_H)$ and the oscillation term $\mu_H^{\mathrm{osc}}(T, v_H)$ are defined as

$$\begin{aligned} \mu_H^{\mathrm{rec}}(T, v_H)^2 &\coloneqq \|\alpha^{1/2}(1 - G_H)\nabla v_H\|_{L^2(T)}^2 \\ \mu_H^{\mathrm{osc}}(T, v_H)^2 &\coloneqq \sum_{\substack{z \in \mathcal{V}_H \cap \Omega \\ z \in T}} \frac{H(z)^2}{\#\mathcal{T}_H[z]} \|R_H(v_H) - r_{H,z}(v_H)\|_{L^2(\Omega_H[z])}^2. \end{aligned} \tag{6.2}$$

Analogously to (2.12), we write

$$\mu_H^{\mathrm{rec}}(v_H)^2 \coloneqq \sum_{T \in \mathcal{T}_H} \mu_H^{\mathrm{rec}}(T, v_H)^2 \quad \text{and} \quad \mu_H^{\mathrm{osc}}(v_H)^2 \coloneqq \sum_{T \in \mathcal{T}_H} \mu_H^{\mathrm{osc}}(T, v_H)^2.$$

The weighting in (6.2) is introduced to lower the influence of the oscillation terms $\mu_H^{\mathrm{osc}}(T, v_H)$ in the global error estimate $\mu_H(v_H)$. Since the number of elements in a vertex patch $\#\mathcal{T}_H[z]$

is uniformly bounded by the constant $C_{\mathrm{patch}} \geq 1$ from Remark 2.12, the analysis in Section 5.1 also holds for the weighted ZZ-estimator (6.1) with slightly different constants.

For ease of implementation, we consider only the case of lowest-order approximation $r_{H,z}(v_H) \in \mathcal{P}^0(\Omega_H[z])$ of the residual $R_H(v_H)$, i.e., $q = 0$ in (5.5). For $q \geq 1$, the implementation becomes more involved, as one needs to introduce a patch-based basis for the spaces $\mathcal{P}^q(\Omega_H[z])$.

In Section 5.1, we have defined the averaging operator G_H in terms of the patch averaging operator (5.2) (in the case $p = 1$) and the Scott–Zhang projection (5.4) (for $p \geq 1$). In contrast to the patch averaging operator (5.2), the implementation of the Scott–Zhang projection (5.4) might seem more involved at first glance. However, as it only needs to be evaluated for discrete functions $\nabla v_H \in [\mathcal{P}^{p-1}(\mathcal{T}_H)]^d$, property (5.3) of the dual basis $\{\psi_j\}_{j=1}^J$ guarantees

$$(\psi_j, \nabla v_H)_{L^2(S_j)} \overset{(5.3)}{=} \nabla v_H|_{S_j}(a_j) \quad \forall j = 1, \dots, J,$$

i.e., the coefficients of the Scott–Zhang projection (5.4) of ∇v_H are determined by point evaluations of the gradient ∇v_H in the nodes $\{a_j\}_{j=1}^J$ of the nodal basis $\{\phi_j\}_{j=1}^J$. While ∇v_H is not continuous across the element boundaries, the associated element $S_j \in \mathcal{T}_H$ for each node a_j can be selected arbitrarily according to the definition of the Scott–Zhang projection (5.4), allowing to choose any of the values ∇v_H takes at a node a_j as the corresponding coefficient in the Scott–Zhang projection (5.4).

Overall, the effort required to implement the ZZ-estimator (6.1) is at least comparable, if not less, than that needed for the residual-based estimator (2.14).

6.1.2 AFEM with the ZZ-estimator for the Poisson problem

In order to verify Corollary 5.7, which is a consequence of Theorem 4.3, we first consider the Poisson model problem on the L-shaped domain, i.e.,

$$-\Delta u^\star = f \quad \text{in } \Omega := (-1,1)^2 \setminus [0,1]^2, \quad u^\star = 0 \quad \text{on } \partial\Omega. \tag{6.3}$$

The right-hand side f is chosen such that the exact solution $u^\star \in H_0^1(\Omega)$, for polar coordinates $(r, \varphi) \in \mathbb{R}_0^+ \times [0, 2\pi)$, is given by

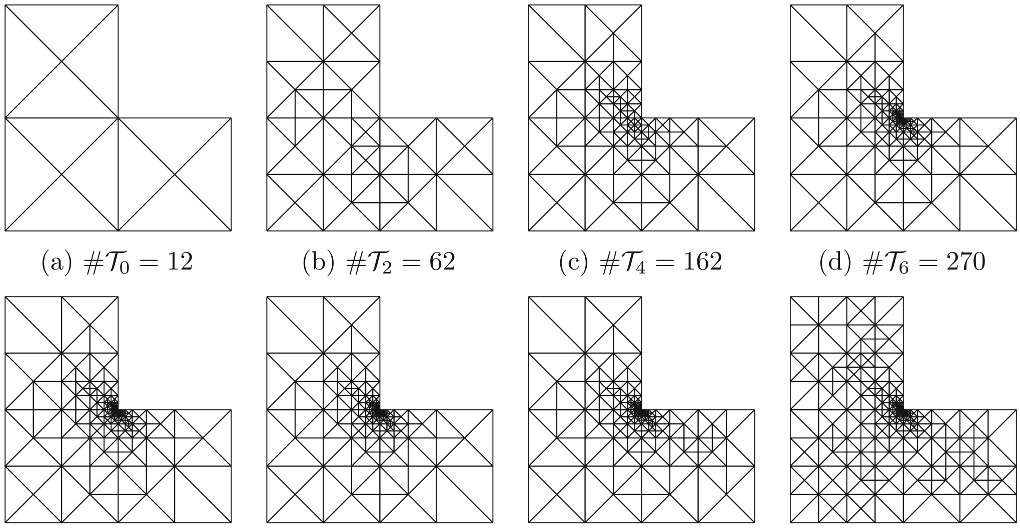
$$u^\star(r,\varphi) = r^{2/3} \sin\left(\frac{2}{3}\left(\varphi - \frac{\pi}{2}\right)\right)\left(1 - r^2\sin^2(\varphi)\right)\left(1 - r^2\cos^2(\varphi)\right). \tag{6.4}$$

Problem (6.3) is well-known to exhibit a singularity at the re-entrant corner $(0,0)$ of the L-shaped domain Ω, making it a suitable test case for adaptive finite element methods.

Figure 6 visualizes the initial mesh $\mathcal{T}_0$ and the adaptively generated meshes $\mathcal{T}_2, \mathcal{T}_4 \dots \mathcal{T}_{14}$, computed by Algorithm B using the ZZ-estimator (6.1). We observe that the algorithm captures the singularity at $(0,0)$ by refining the mesh in the vicinity of the origin.

Figure 7 and Figure 8 illustrate the convergence of the ZZ-estimator $\mu_\ell(u_\ell^k)$ in Algorithm B. In Figure 7, we use the Scott–Zhang projection (5.4) for the averaging operator G_H, while in Figure 8, we use the patch averaging operator (5.2). In either case, we see that Algorithm B leads to optimal convergence rates $-p/2$ with respect to the theoretical complexity

$$\texttt{complexity}(\ell) := \sum_{\substack{(\ell',k') \in \mathcal{Q} \\ |\ell',k'| \leq |\ell,\underline{k}|}} \#\mathcal{T}_{\ell'} \tag{6.5}$$



(a) $\#\mathcal{T}_0 = 12$ (b) $\#\mathcal{T}_2 = 62$ (c) $\#\mathcal{T}_4 = 162$ (d) $\#\mathcal{T}_6 = 270$

(e) $\#\mathcal{T}_8 = 386$ (f) $\#\mathcal{T}_{10} = 514$ (g) $\#\mathcal{T}_{12} = 687$ (h) $\#\mathcal{T}_{14} = 941$

Figure 6: Sequence of meshes $\mathcal{T}_\ell$ generated by Algorithm B using the ZZ-estimator (6.1) with the Scott–Zhang projection (5.4). The algorithm is applied to problem (6.3) with f corresponding to the exact solution (6.4), parameters $\theta = 0.5$, $\lambda = 0.1$, and polynomial degree $p = 4$.

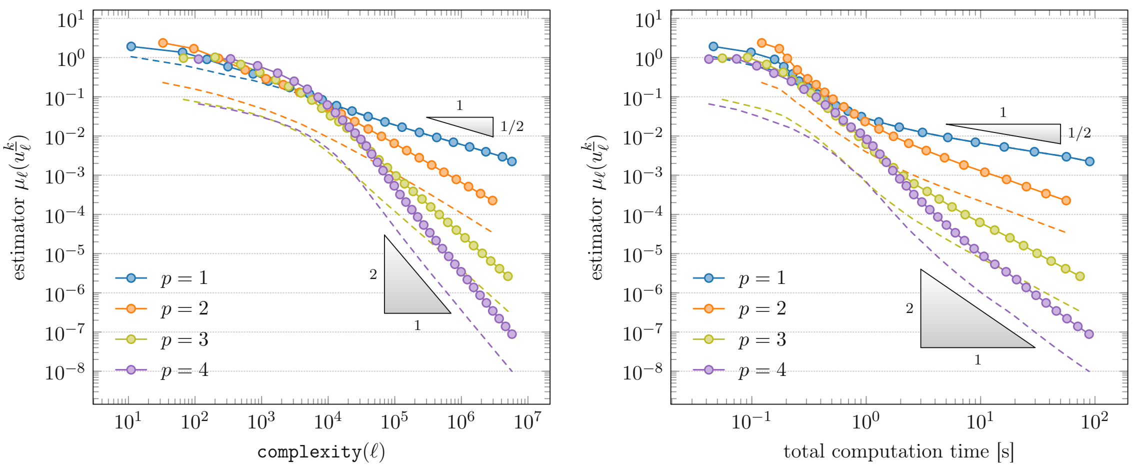


Figure 7: Convergence history plots of Algorithm B using the ZZ-estimator μ_ℓ (6.1) with the Scott–Zhang projection (5.4). The algorithm is applied to problem (6.3) with f corresponding to the exact solution (6.4), initial mesh $\mathcal{T}_0$, depicted in Figure 6a, fixed parameters $\theta = 0.5$ and $\lambda = 0.1$, and polynomial degrees $p = 1, 2, 3, 4$. The convergence of $\mu_\ell(u_\ell^k)$ (solid lines) and the corresponding total error $|||u^\star - u_\ell^k|||$ (dashed lines) is presented with respect to the theoretical complexity (6.5) (left) and the total computation time (right). For $p = 1$, the ZZ-estimator appears to be asymptotically exact.

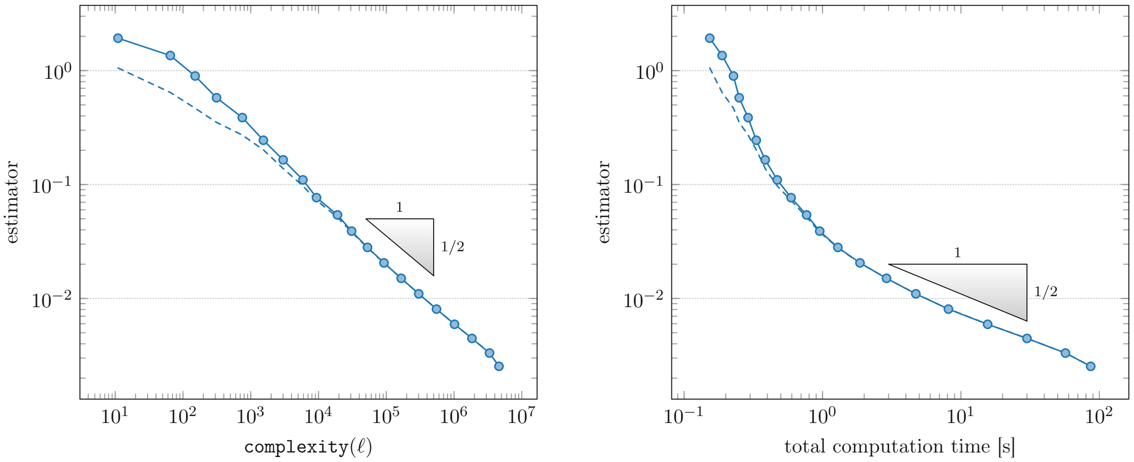


Figure 8: Convergence history plots of Algorithm B using the ZZ-estimator μ_ℓ (6.1) with patch averaging (5.2). The algorithm is applied to problem (6.3) with f corresponding to the exact solution (6.4), initial mesh $\mathcal{T}_0$, depicted in Figure 6a, and fixed parameters $\theta = 0.5$ and $\lambda = 0.1$. The convergence of $\mu_\ell(u_\ell^{\underline{k}})$ (solid line) and the corresponding total error $|||u^\star - u_\ell^{\underline{k}}|||$ (dashed line) is presented with respect to the theoretical complexity (6.5) (left) and the total computation time (right). Again, this variant of the ZZ-estimator appears to be asymptotically exact.

and, more importantly, with respect to the total computation time. Hence, the observed convergence rates are consistent with the theoretical results presented in Theorem 4.3 and Corollary 5.7.

In Figure 9, we compare the convergence of the contributions $\mu_\ell^{\mathrm{rec}}(u_\ell^{\underline{k}})$ and $\mu_\ell^{\mathrm{osc}}(u_\ell^{\underline{k}})$ (6.2) of the ZZ-estimator (6.1) with the residual-based estimator $\eta_\ell(u_\ell^{\underline{k}})$ (2.14) and the total error $|||u^\star - u_\ell^{\underline{k}}|||$. The ZZ-estimator is computed using the Scott–Zhang projection (5.4) as the averaging operator G_H. The results are presented for the polynomial degrees $p = 1$ (left) and $p = 4$ (right). In the case of $p = 1$, we observe that the oscillation part $\mu_\ell^{\mathrm{osc}}(u_\ell^{\underline{k}})$ is of higher order, since it converges with rate -1. Hence, one might consider to drop the oscillation term $\mu_\ell^{\mathrm{osc}}(u_\ell^{\underline{k}})$ in the ZZ-estimator (6.1) for this case. While the residual-based estimator $\eta_\ell(u_\ell^{\underline{k}})$ overestimates the total error $|||u^\star - u_\ell^{\underline{k}}|||$, the ZZ-estimator $\mu_\ell(u_\ell^{\underline{k}})$ gives a very accurate error estimate. In contrast, for $p = 4$, the oscillation term $\mu_\ell^{\mathrm{osc}}(u_\ell^{\underline{k}})$ dominates the recovery term $\mu_\ell^{\mathrm{rec}}(u_\ell^{\underline{k}})$ approximately by a factor of 10, which is likely due to the simplifying choice of lowest-order approximation $q = 0$ in (5.5). Because of this, the ZZ-estimator $\mu_\ell(u_\ell^{\underline{k}})$ overestimates the total error $|||u^\star - u_\ell^{\underline{k}}|||$ more than the residual-based estimator $\eta_\ell(u_\ell^{\underline{k}})$. However, the recovery term $\mu_\ell^{\mathrm{rec}}(u_\ell^{\underline{k}})$ still provides a good approximation of the total error $|||u^\star - u_\ell^{\underline{k}}|||$. Moreover, even though the oscillation term $\mu_\ell^{\mathrm{osc}}(u_\ell^{\underline{k}})$ is not of higher order for $p \geq 2$, using only the recovery term $\mu_\ell^{\mathrm{rec}}(u_\ell^{\underline{k}})$ for steering Algorithm B still led to optimal convergence rates in our testing.

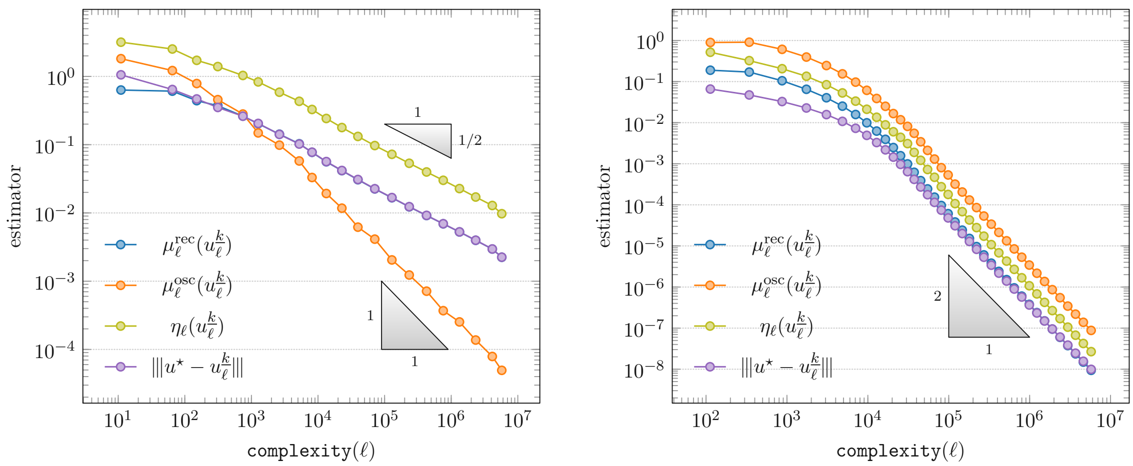


Figure 9: Convergence history plots of Algorithm B using the ZZ-estimator (6.1) with the Scott–Zhang projection (5.4). The algorithm is applied to problem (6.3) with f corresponding to the exact solution (6.4), initial mesh $\mathcal{T}_0$, depicted in Figure 6a, fixed parameters $\theta = 0.5$ and $\lambda = 0.1$, and polynomial degrees $p = 1$ (left) and $p = 4$ (right). The convergence of the estimator contributions $\mu_\ell^{\mathrm{rec}}(u_\ell^{\underline{k}})$ and $\mu_\ell^{\mathrm{osc}}(u_\ell^{\underline{k}})$, the residual-based estimator $\eta_\ell(u_\ell^{\underline{k}})$ (2.14), and the total error $|||u^\star - u_\ell^{\underline{k}}|||$ is presented with respect to the theoretical complexity (6.5).

6.1.3 AFEM with the ZZ-estimator for a nonsymmetric second-order PDE

Since the analysis in Section 5.1 covers more general problems than the Poisson problem (6.3), we now consider the nonsymmetric second-order PDE

$$-\operatorname{div}(\alpha \nabla u^\star) + \boldsymbol{b} \cdot \nabla u^\star + u^\star = 1 \quad \text{in } \Omega := (-1,1)^2 \setminus [0,1]^2, \quad u^\star = 0 \quad \text{on } \partial\Omega \tag{6.6}$$

with $\alpha(x) = 10 \cdot e^{-\|x-y\|_2}$ for $y := (0.5, -0.5)^\top$ and $\boldsymbol{b}(x) = 1 - x$. This problem is nonsymmetric due to the presence of the convection term $\boldsymbol{b} \cdot \nabla u^\star$. We deal with the nonsymmetry as described in Section 2.4.2, where we showed how to derive a contractive solver for nonsymmetric problems from contractive solvers for symmetric problems by means of the Zarantonello iteration (2.36). More specifically, we proved that for a sufficiently small parameter $\delta > 0$, there exists a lower bound $j_0 > 0$, such that $\underline{j} \geq j_0$ solver iterations applied to the symmetric problem of the Zarantonello iteration (2.36) correspond to one step of a contractive solver for the nonsymmetric problem (6.6). In contrast to the Poisson problem (6.3), the diffusion in (6.6) is not constant. The diffusion coefficient $\alpha \in C(\overline{\Omega})$ chosen in (6.6) peaks at the point $y = (0.5, -0.5)^\top$ and decays exponentially. Its effect can be observed in the illustration of the computed solution $u_\ell^{\underline{k}}$ in Figure 10. The non-constant diffusion is an additional challenge for the adaptive algorithm, apart from the singularity at the re-entrant corner $(0,0)$ of the L-shaped domain Ω.

Figure 11 illustrates a mesh $\mathcal{T}_\ell$ generated by Algorithm B using the ZZ-estimator (6.1) with the Scott–Zhang projection (5.4). We can clearly observe that the algorithm captures both the singularity at $(0,0)$ and the diffusion peak at $(0.5, -0.5)$ by refining the mesh in the vicinity of these points.

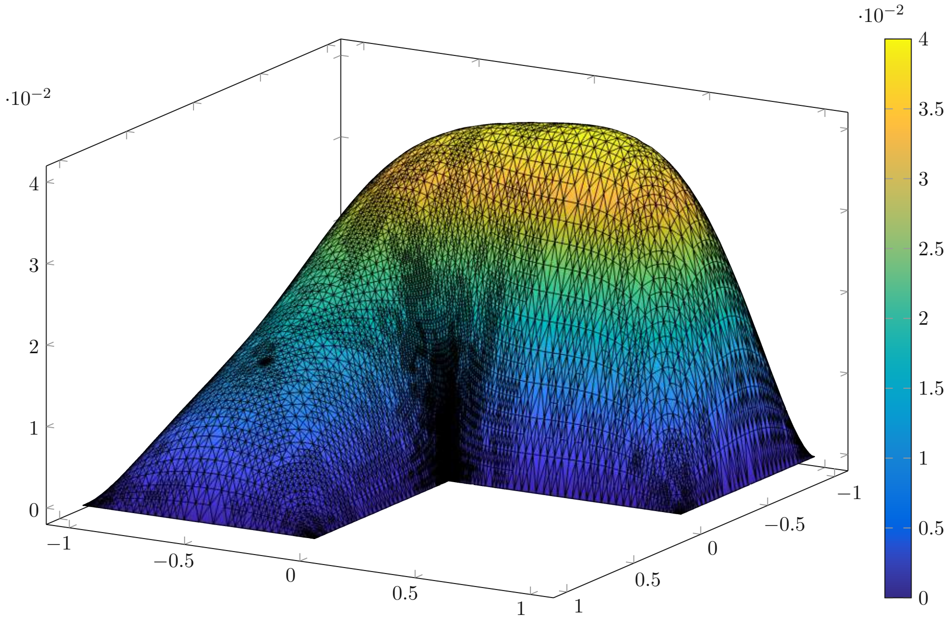


Figure 10: Illustration of the computed solution u_{20}^{k} for problem (6.6). The solution is obtained from Algorithm B using the ZZ-estimator (6.1) with the Scott–Zhang projection (5.4). In the algorithm, we use $\mathcal{T}_0$, depicted in Figure 6a, as initial mesh, fixed parameters $\theta = 0.5$, $\lambda = 0.1$, $\delta = 0.5$, $\underline{j} = 5$, and polynomial degree $p = 3$.

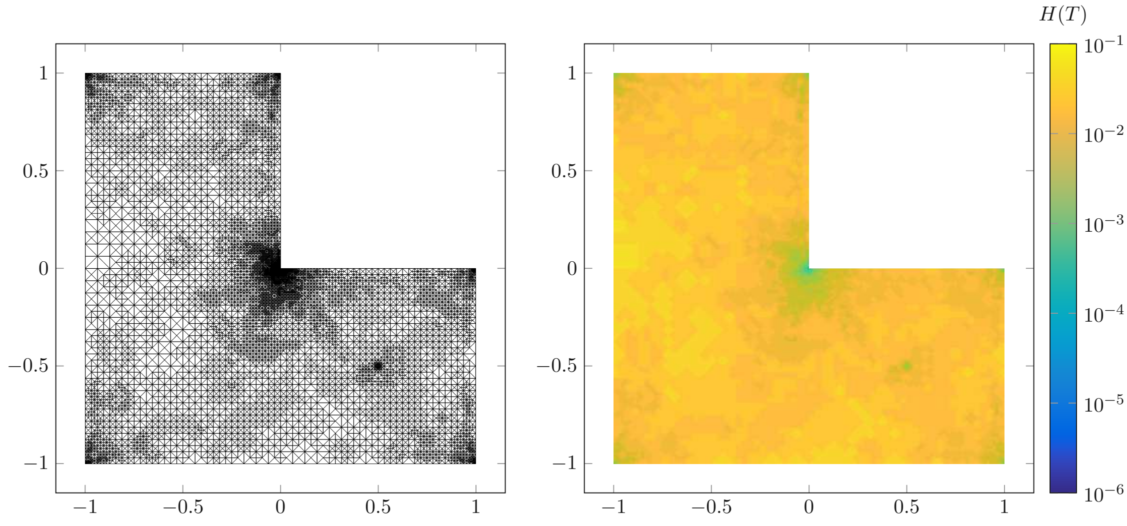


Figure 11: Mesh generated by Algorithm B using the ZZ-estimator μ_ℓ (6.1) with the Scott–Zhang projection (5.4). The algorithm is applied to problem (6.6) with initial mesh $\mathcal{T}_0$, depicted in Figure 6a, fixed parameters $\theta = 0.5$, $\lambda = 0.1$, $\delta = 0.5$, $\underline{j} = 5$, and polynomial degree $p = 3$. We illustrate the mesh $\mathcal{T}_{20}$, consisting of 20291 elements. In addition to displaying the mesh (left), we also visualize the corresponding mesh-size function $H(T) = |T|^{1/2}$ (right).

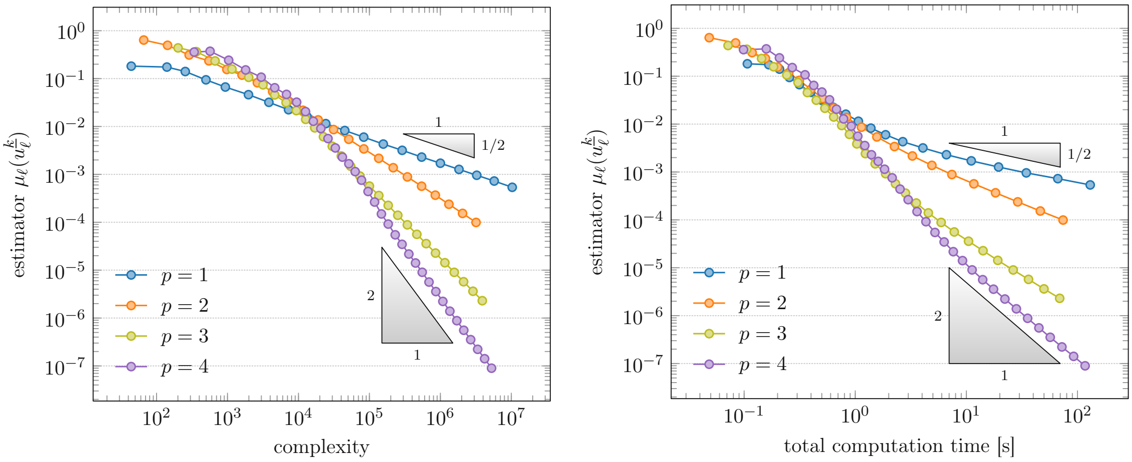


Figure 12: Convergence history plots of Algorithm B using the ZZ-estimator μ_ℓ (6.1) with the Scott–Zhang projection (5.4). The algorithm is applied to problem (6.6) with initial mesh $\mathcal{T}_0$, depicted in Figure 6a, fixed parameters $\theta = 0.5$, $\lambda = 0.1$, $\delta = 0.5$, $\underline{j} = 5$, and polynomial degrees $p = 1, 2, 3, 4$. The convergence of $\mu_\ell(u_\ell^{\underline{k}})$ is presented with respect to the theoretical complexity (6.5) (left) and the total computation time (right).

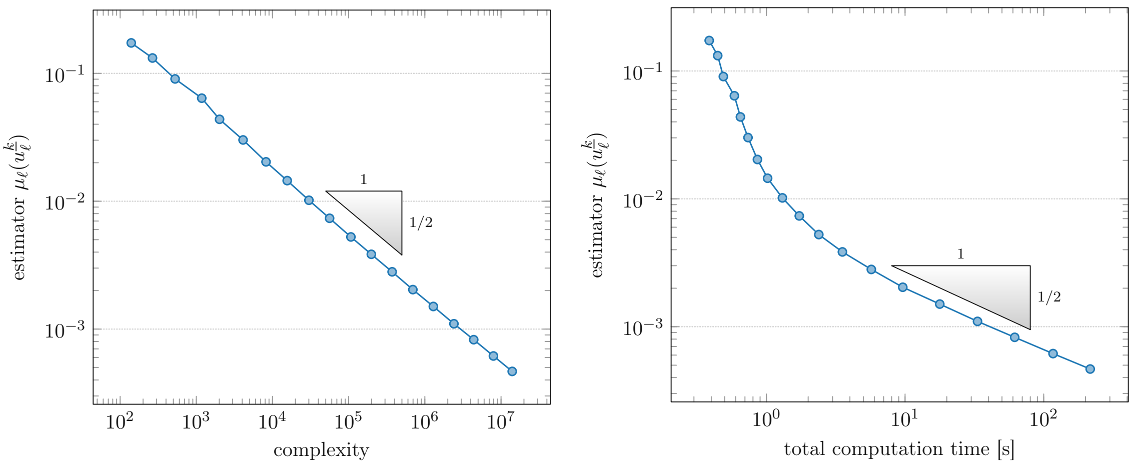


Figure 13: Convergence history plots of Algorithm B using the ZZ-estimator μ_ℓ (6.1) with patch averaging (5.2). The algorithm is applied to problem (6.6) with initial mesh $\mathcal{T}_0$, depicted in Figure 6a, and fixed parameters $\theta = 0.5$, $\lambda = 0.1$, $\delta = 0.5$, and $\underline{j} = 5$. The convergence of $\mu_\ell(u_\ell^{\underline{k}})$ is presented with respect to the theoretical complexity (6.5) (left) and the total computation time (right).

Figure 12 and Figure 13 show the convergence of the ZZ-estimator $\mu_\ell(u_\ell^{\underline{k}})$ in Algorithm B for the nonsymmetric problem (6.6). In Figure 12, we use the Scott–Zhang projection (5.4) for the averaging operator G_H, while in Figure 13, we use the patch averaging operator (5.2). In either case, we observe that Algorithm B leads to optimal convergence rates $-p/2$ both with respect to the theoretical complexity (6.5) and the total computation time. Thus, the observed convergence rates further confirm the theoretical results presented in Theorem 4.3 and Corollary 5.7.

Finally, Table 1 and Table 2 present experimental contraction factors for q_{alg} (2.38) and q_{ctr} (2.40). For an index pair (k, j), the experimental contraction factor $\overline{q}_{\mathrm{alg}}$ is computed as

$$\overline{q}_{\mathrm{alg}} := \frac{|||u_H^{k,\star} - u_H^{k,j}|||}{|||u_H^{k,\star} - u_H^{k,j-1}|||}. \tag{6.7}$$

Similarly, for a solver index k, the experimental contraction factor $\overline{q}_{\mathrm{ctr}}$ is defined as

$$\overline{q}_{\mathrm{ctr}} := \frac{|||u_H^{\star} - u_H^{k,\underline{j}}|||}{|||u_H^{\star} - u_H^{k-1,\underline{j}}|||}. \tag{6.8}$$

Table 1 and Table 2 present the mean value and standard deviation of the experimental contraction factors $\overline{q}_{\mathrm{ctr}}$ and $\overline{q}_{\mathrm{alg}}$ computed in different runs of Algorithm B applied to the nonsymmetric problem (6.6). In each run, we varied the polynomial degree $p = 1, 2, 3, 4$ and the number of solver iterations $\underline{j} = 1, 2, 3, 4, 5$ while keeping all other parameters fixed. We observe that regardless of the polynomial degree p and the number of solver iterations $\underline{j}$, both $\overline{q}_{\mathrm{alg}}$ and $\overline{q}_{\mathrm{ctr}}$ are mostly below 0.7, which is considered a good contraction factor. While this is to be expected for $\overline{q}_{\mathrm{alg}}$, it is surprising that $\overline{q}_{\mathrm{ctr}}$ is low even for $\underline{j} = 1$.

	$p = 1$	$p = 2$	$p = 3$	$p = 4$
$\underline{j} = 1$	0.6124 ± 0.0701	0.7624 ± 0.0732	0.7721 ± 0.0732	0.7804 ± 0.0658
$\underline{j} = 2$	0.5468 ± 0.0346	0.6461 ± 0.0535	0.6623 ± 0.0531	0.6654 ± 0.0489
$\underline{j} = 3$	0.5132 ± 0.0128	0.5802 ± 0.0344	0.5933 ± 0.0350	0.5966 ± 0.0329
$\underline{j} = 4$	0.5065 ± 0.0065	0.5522 ± 0.0214	0.5634 ± 0.0228	0.5627 ± 0.0207
$\underline{j} = 5$	0.5023 ± 0.0026	0.5277 ± 0.0120	0.5357 ± 0.0133	0.5330 ± 0.0112

Table 1: Mean value and standard deviation of the experimental contraction factor $\overline{q}_{\mathrm{ctr}}$ (6.8) in different runs of Algorithm B applied to the nonsymmetric problem (6.6). In each run, we used the ZZ-estimator (6.1) with the Scott–Zhang projection (5.4), the same initial mesh $\mathcal{T}_0$, depicted in Figure 6a, and the same parameters $\theta = 0.5$, $\lambda = 0.1$, $\delta = 0.5$, but different polynomial degrees $p = 1, 2, 3, 4$ and different numbers of solver iterations $\underline{j} = 1, 2, 3, 4, 5$. The algorithm was run until the number of degrees of freedom exceeded 10^6.

6.1.4 AFEM with the ZZ-estimator for a diffusion problem

In this section, we want to confirm that $\alpha^{1/2}$ is indeed the right scaling in the recovery term μ_H^{rec} (6.2) of the ZZ-estimator (6.1). To this end, we consider the symmetric diffusion problem

$$-\operatorname{div}(\alpha \nabla u^\star) = 1 \quad \text{in } \Omega := (-1, 1)^2 \setminus [0, 1]^2, \quad u^\star = 0 \quad \text{on } \partial\Omega \tag{6.9}$$

with $\alpha(x) = 1000 \cdot e^{-\|x-y\|_2}$ for $y := (0.5, -0.5)^\top$. As in the nonsymmetric problem (6.6), the diffusion coefficient $\alpha \in C(\overline{\Omega})$ chosen in (6.9) peaks at the point $y = (0.5, -0.5)^\top$ and decays

	$p=1$	$p=2$	$p=3$	$p=4$
$\underline{j}=1$	0.3378 ± 0.1256	0.5550 ± 0.1075	0.5667 ± 0.1200	0.5763 ± 0.1083
$\underline{j}=2$	0.3993 ± 0.1458	0.5818 ± 0.1189	0.6034 ± 0.1350	0.6029 ± 0.1250
$\underline{j}=3$	0.4229 ± 0.1479	0.6065 ± 0.1254	0.6324 ± 0.1378	0.6317 ± 0.1269
$\underline{j}=4$	0.4473 ± 0.1459	0.6239 ± 0.1244	0.6533 ± 0.1335	0.6481 ± 0.1249
$\underline{j}=5$	0.4656 ± 0.1453	0.6340 ± 0.1236	0.6683 ± 0.1294	0.6627 ± 0.1224

Table 2: Mean value and standard deviation of the experimental contraction factor $\overline{q}_{\mathrm{alg}}$ (6.7) in different runs of Algorithm B applied to the nonsymmetric problem (6.6). In each run, we used the ZZ-estimator (6.1) with the Scott–Zhang projection (5.4), the same initial mesh $\mathcal{T}_0$, depicted in Figure 6a, and the same parameters $\theta = 0.5$, $\lambda = 0.1$, $\delta = 0.5$, but different polynomial degrees $p = 1, 2, 3, 4$ and different numbers of solver iterations $\underline{j} = 1, 2, 3, 4, 5$. The algorithm was run until the number of degrees of freedom exceeded 10^6.

exponentially. While the exact solution $u^\star$ of the diffusion problem (6.9) is unavailable, Galerkin orthogonality (2.6) implies that the total error $|||u^\star - u_\ell^{\underline{k}}|||$ is equal to

$$|||u^\star - u_\ell^{\underline{k}}|||^2 \overset{(2.6)}{=} |||u^\star - u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^{\underline{k}}|||^2 \overset{(2.6)}{=} |||u^\star|||^2 - |||u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^{\underline{k}}|||^2. \tag{6.10}$$

By estimating or guessing the value $|||u^\star|||^2$ (e.g., by first computing $|||u_H^\star|||^2$ on a very fine mesh $\mathcal{T}_H$), formula (6.10) allows us to compute the total error $|||u^\star - u_\ell^{\underline{k}}|||$ even without knowing the exact solution $u^\star$.

Figure 14 shows the convergence of the recovery term $\mu_\ell^{\mathrm{rec}}(u_\ell^{\underline{k}})$ with different scalings in comparison to the total error $|||u^\star - u_\ell^{\underline{k}}|||$. Apart from the scaling $\alpha^{1/2}$, we also consider the scalings α and 1 (i.e., no scaling). We observe that the recovery term $\mu_\ell^{\mathrm{rec}}(u_\ell^{\underline{k}})$ with the scaling $\alpha^{1/2}$ gives the best approximation of the total error $|||u^\star - u_\ell^{\underline{k}}|||$. Hence, we conclude that the scaling $\alpha^{1/2}$ in (6.2) is indeed the right choice.

6.2 Experiments with the equilibrated flux estimator

As in the previous section, we first present implementational details for the equilibrated flux estimator, and then proceed to numerical experiments. For simplicity, we will restrict ourselves to to the Poisson model problem (6.3) in this section.

6.2.1 Implementational aspects for the equilibrated flux estimator

Recall the equilibrated flux estimator (5.51) from Section 5.2. For the Poisson problem (6.3), the refinement indicators of the estimator are given by

$$\mu_H(T, v_H) := \|\boldsymbol{\sigma}_H[v_H] + \nabla v_H\|_{L^2(T)} + \frac{\mathrm{diam}(T)}{\pi} \|(1 - \Pi_H) f\|_{L^2(T)}, \tag{6.11}$$

where Π_H is the L^2-projection onto the space $\mathcal{P}^q(\mathcal{T}_H)$ and $\boldsymbol{\sigma}_H[v_H] \in \mathcal{RT}_0^q(\mathcal{T}_H[z])$ is the global equilibrated flux defined in (5.48). The latter is constructed as the sum of local

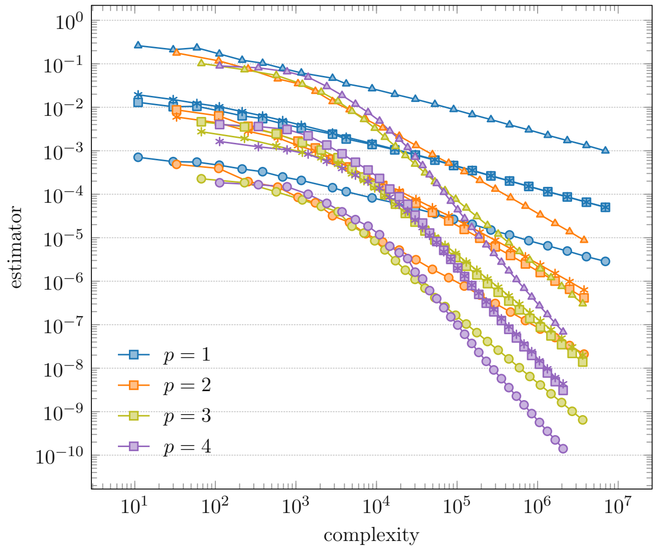


Figure 14: Convergence history plots of Algorithm B using the ZZ-estimator μ_ℓ (6.1) with the Scott–Zhang projection (5.4). The algorithm is applied to problem (6.9), initial mesh $\mathcal{T}_0$, depicted in Figure 6a, fixed parameters $\theta = 0.5$ and $\lambda = 0.1$, and polynomial degrees $p = 1, 2, 3, 4$. The convergence behavior of the recovery term $\mu_\ell^{\mathrm{rec}}(u_\ell^{\underline{k}})$ is shown for three different scalings: $\alpha^{1/2}$ (square markers —■—), α (triangle markers —▲—), and without scaling (circle markers —●—). For comparison, the corresponding total error $|||u^\star - u_\ell^{\underline{k}}|||$ is also shown (asterisk markers —✱—), which is computed using (6.10). The results are presented with respect to the theoretical complexity (6.5). For $p = 1$, we observe asymptotic exactness of the ZZ-estimator.

fluxes $\boldsymbol{\sigma}_{H,z}[v_H]$, which, according to Lemma 5.9, are the first component of the unique solution $(\boldsymbol{\sigma}_{H,z}[v_H], r_{H,z}[v_H]) \in \mathcal{RT}_0^q(\mathcal{T}_H[z]) \times \mathcal{P}_\star^q(\mathcal{T}_H[z])$ of the local saddle-point problems

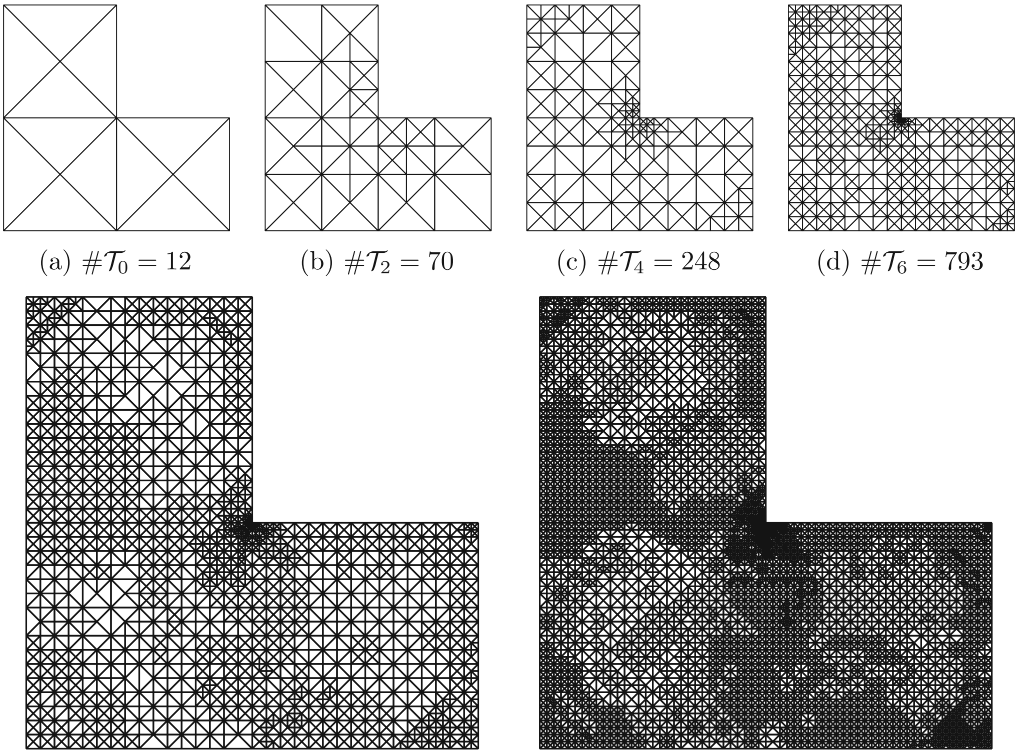
$$\begin{aligned} \langle \boldsymbol{\sigma}_{H,z}[v_H], \boldsymbol{\tau}_H \rangle_{L^2(\Omega_H[z])} + \langle \operatorname{div} \boldsymbol{\tau}_H, r_{H,z}[v_H] \rangle_{L^2(\Omega_H[z])} &= -\langle \phi_z \nabla v_H, \boldsymbol{\tau}_H \rangle_{L^2(\Omega_H[z])} \\ \langle \operatorname{div} \boldsymbol{\sigma}_{H,z}[v_H], q_H \rangle_{L^2(\Omega_H[z])} &= \langle \Pi_z^\star(\phi_z f - \nabla \phi_z \cdot \nabla v_H), q_H \rangle_{L^2(\Omega_H[z])} \\ \text{for all } \boldsymbol{\tau}_H \in \mathcal{RT}_0^q(\mathcal{T}_H[z]) &\text{ and all } q_H \in \mathcal{P}^q(\mathcal{T}_H[z]). \end{aligned} \tag{6.12}$$

Here, $\Pi_z^\star$ denotes the L^2-orthogonal projection (5.31) onto $L^2_\star(\Omega_H[z])$ and $\phi_z \in \mathcal{S}^1(\mathcal{T}_H)$ is the hat function associated with the vertex $z \in \mathcal{V}_H$.

Since `MooAFEM` does not provide a built-in implementation of Raviart–Thomas elements, we use the computational basis derived in [Erv12]. For simplicity, we restrict ourselves to $p = 1$ and Raviart–Thomas elements of order $q = 1$. Initially, we implemented lowest-order Raviart–Thomas elements, i.e., $q = 0$, but we did not observe optimal convergence rates of the equilibrated flux estimator (6.11) in that case. While this is not a contradiction to Corollary 5.19, which requires $q \geq p$ and therefore does not cover the case $p = 1$ and $q = 0$, it makes the implementation of the equilibrated flux estimator (6.11) more laborious in contrast to, e.g., the residual-based estimator (2.14) or the ZZ-estimator (6.1).

6.2.2 AFEM with the equilibrated flux estimator for the Poisson problem

In order to verify Corollary 5.19, we consider the Poisson model problem (6.3) with the right-hand side $f \equiv 1$ on the L-shaped domain $\Omega := (-1,1)^2 \setminus [0,1]^2$. As discussed in Section 6.1.2, this is a well-suited test case for the adaptive algorithm due to the singularity at the re-entrant corner $(0,0)$. In Figure 15, we visualize the initial mesh $\mathcal{T}_0$ and the adaptively generated meshes $\mathcal{T}_2, \mathcal{T}_4, \dots, \mathcal{T}_{10}$, computed by Algorithm B using the equilibrated flux estimator (6.11). We see that the algorithm captures the singularity at $(0,0)$, since it refines the mesh particularly in the vicinity of the origin.



(a) $\#\mathcal{T}_0 = 12$ (b) $\#\mathcal{T}_2 = 70$ (c) $\#\mathcal{T}_4 = 248$ (d) $\#\mathcal{T}_6 = 793$

(e) $\#\mathcal{T}_8 = 2771$ (f) $\#\mathcal{T}_{10} = 9864$

Figure 15: Sequence of meshes $\mathcal{T}_\ell$ generated by Algorithm B using the equilibrated flux estimator (6.11). The algorithm is applied to problem (6.3) with $f \equiv 1$, fixed parameters $\theta = 0.5$ and $\lambda = 0.1$, and polynomial degrees $p = 1$ and $q = 1$.

Figure 16 shows the convergence of the equilibrated flux estimator $\mu_\ell(u_\ell^k)$ in Algorithm B. We observe that the algorithm leads to the optimal convergence rate $-1/2$ both with respect to the theoretical complexity (6.5) and the total computation time, which confirms the theoretical results presented in Theorem 4.3 and Corollary 5.19.

Finally, we want to consider a non-constant right-hand side f in the Poisson problem (6.3). To this end, we choose the right-hand side f such that the exact solution $u^\star \in H_0^1(\Omega)$ is given by (6.4). This enables us to compare the equilibrated flux estimator $\mu_\ell(u_\ell^k)$ and the residual-based estimator $\eta_\ell(u_\ell^k)$ (2.14) with the actual total error $|||u^\star - u_\ell^k|||$. Such a

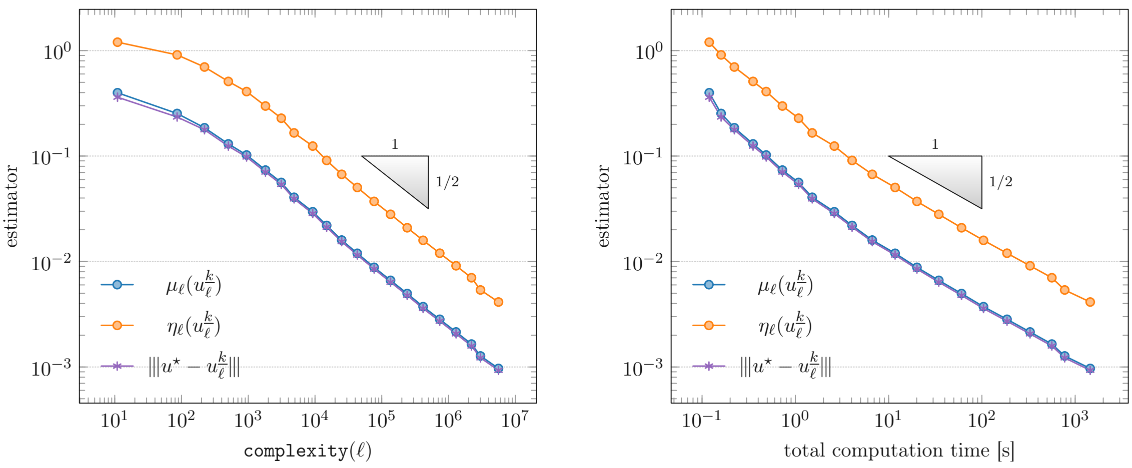


Figure 16: Convergence history plots of Algorithm B steered by the equilibrated flux estimator μ_ℓ (6.11). The algorithm is applied to problem (6.3) with $f \equiv 1$, initial mesh $\mathcal{T}_0$, depicted in Figure 15a, fixed parameters $\theta = 0.5$ and $\lambda = 0.1$, and fixed polynomial degrees $p = 1$ and $q = 1$. The convergence of $\mu_\ell(u_\ell^k)$, the residual-based estimator $\eta_\ell(u_\ell^k)$ (2.14), and the total error $|||u^\star - u_\ell^k|||$ (computed using (6.10)) is presented with respect to the theoretical complexity (6.5) (left) and the total computation time (right).

comparison is illustrated in Figure 17 (left), in which the components

$$\begin{aligned} \mu_\ell^{\text{flux}}(u_\ell^k) &:= \Big(\sum_{T \in \mathcal{T}_\ell} \|\boldsymbol{\sigma}_H[u_\ell^k] + \nabla u_\ell^k\|_{L^2(T)}^2 \Big)^{1/2} \quad \text{and} \\ \mu_\ell^{\text{osc}} &:= \Big(\sum_{T \in \mathcal{T}_\ell} \frac{\operatorname{diam}(T)^2}{\pi^2} \|(1 - \Pi_H) f\|_{L^2(T)}^2 \Big)^{1/2} \end{aligned} \tag{6.13}$$

of the equilibrated flux estimator $\mu_\ell(u_\ell^k)$ are plotted next to $\eta_\ell(u_\ell^k)$ and $|||u^\star - u_\ell^k|||$. We can see that the flux term $\mu_\ell^{\text{flux}}(u_\ell^k)$, the residual-based estimator $\eta_\ell(u_\ell^k)$, and the total error $|||u^\star - u_\ell^k|||$ all converge with the optimal rate $-1/2$, while the oscillation term μ_ℓ^{osc} converges with the rate $-3/2$. Thus, the effect of the oscillation term μ_ℓ^{osc} on the equilibrated flux estimator $\mu_\ell(u_\ell^k)$ is almost negligible in this experiment. Theorem 5.14 and Remark 5.15 show that the equilibrated flux estimator (6.11) is a guaranteed upper bound for the total error $|||u^\star - u_\ell^k|||$, which can also be observed in Figure 17 (left). Moreover, we see that the flux term $\mu_\ell^{\text{flux}}(u_\ell^k)$ provides a much better estimate of the total error $|||u^\star - u_\ell^k|||$ than the residual-based estimator $\eta_\ell(u_\ell^k)$. This is illustrated in Figure 17 (right), which shows the respective experimental C_{rel} constants, i.e., the ratio of the total error $|||u^\star - u_\ell^k|||$ to the respective estimator. While the experimental C_{rel} constant of the flux term $\mu_\ell^{\text{flux}}(u_\ell^k)$ is very close to 1, meaning that it provides a very accurate estimate of the total error $|||u^\star - u_\ell^k|||$, the residual-based estimator $\eta_\ell(u_\ell^k)$ has an experimental C_{rel} constant of approximately 0.2, i.e., it overestimates the total error by a factor of around 5.

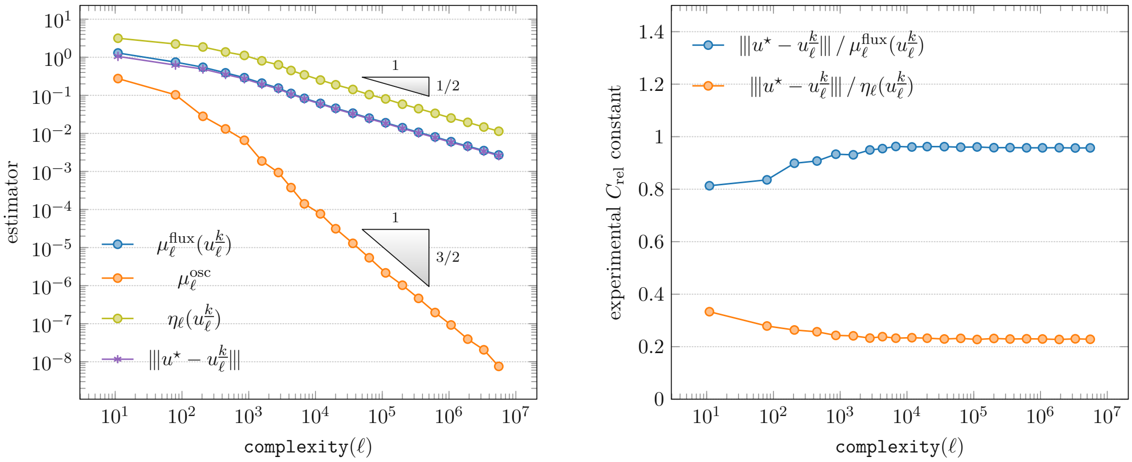


Figure 17: Convergence history plots and experimental C_{rel} constants for Algorithm B steered by the equilibrated flux estimator μ_ℓ (6.11). The algorithm is applied to problem (6.3) with f corresponding to the exact solution (6.4), initial mesh $\mathcal{T}_0$, depicted in Figure 15a, fixed parameters $\theta = 0.5$ and $\lambda = 0.1$, and fixed polynomial degrees $p = 1$ and $q = 1$. On the left, the convergence of the estimator contributions $\mu_\ell^{\mathrm{flux}}(u_\ell^k)$ and μ_ℓ^{osc} (6.13), the residual-based estimator $\eta_\ell(u_\ell^k)$ (2.14), and the total error $|||u^\star - u_\ell^k|||$ is presented with respect to the theoretical complexity (6.5). The right plot shows the experimental C_{rel} constants for $\mu_\ell^{\mathrm{flux}}(u_\ell^k)$ and $\eta_\ell(u_\ell^k)$, which are computed as the ratio of the total error to the respective estimator.

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