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Operators on complemented lattices

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Abstract

The present paper deals with complemented lattices where, however, a unary operation of complementation is not explicitly assumed. This means that an element can have several complements. The mapping $^+$ assigning to each element a the set a^+ of all its complements is investigated as an operator on the given lattice. We can extend the definition of a^+ in a natural way from elements to arbitrary subsets. In particular we study the set a^+ for complemented modular lattices, and we characterize when the set a^{++} is a singleton. By means of the operator $^+$ we introduce two other operators $\to$ and $\odot$ which can be considered as implication and conjunction in a certain propositional calculus, respectively. These two logical connectives are "unsharp" which means that they assign to each pair of elements a non-empty subset. However, also these two derived operators share a lot of properties with the corresponding logical connectives in intuitionistic logic or in the logic of quantum mechanics. In particular, they form an adjoint pair. Finally, we define so-called deductive systems and we show their relationship to the mentioned operators as well as to lattice filters.

Keywords Complemented lattice · Modular lattice · Operator of complementation · Sasaki projection · Filter · Deductive system

1 Introduction

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice and $a \in L$. An element b of L is called a *complement* of a if $a \vee b = 1$ and $a \wedge b = 0$. The lattice $\mathbf{L}$ is called *complemented* if any of its elements has a complement.

Often lattices with an additional unary operation, usually denoted by $'$, are studied where for each $a \in L$ the element a' denotes its complement. In such a case this unary operation is called a *complementation*. However, in complemented lattices we do not assume the complement being unique. This is the case with our present paper.

It is worth noticing that in a distributive complemented lattice the complement is unique. However, this need not be the case in modular complemented lattices. For example, consider the lattice $\mathbf{M}_n = (M_n, \vee, \wedge, 0, 1)$ (for $n > 1$) depicted in Fig. 1:

Then for every $i, j \in \{1, \dots, n\}$ with $i \neq j$, the element a_j is a complement of a_i.

Sometimes, for lattices with complementation, we ask if this complementation is *antitone*, i.e. if $x \leq y$ implies $y' \leq x'$, or if it is an *involution*, i.e. $x'' = x$. In distributive complemented lattices the complementation turns out to be unique, antitone and an involution. In such a case the lattice is a Boolean algebra.

Within modular lattices the situation may be different. Consider the complemented modular lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ visualized in Fig. 2:

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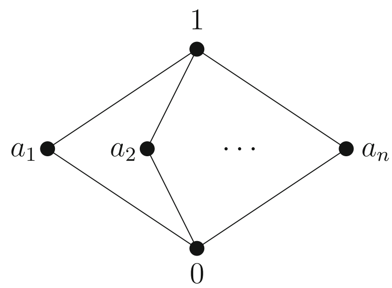


Fig. 1 The lattice $\mathbf{M}_n$

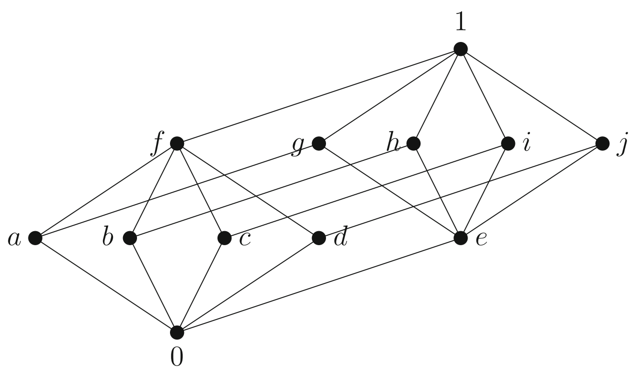


Fig. 2 Complemented modular lattice

Evidently; $\mathbf{L}$ is a complemented lattice. We have several choices for defining a complementation $'$. If we define $'$ by

x	0	a	b	c	d	e	f	g	h	i	j	1
x'	1	h	i	j	g	f	e	b	c	d	a	0

then it is not an involution. If we define $'$ by

x	0	a	b	c	d	e	f	g	h	i	j	1
x'	1	h	i	j	g	f	e	d	a	b	c	0

then it is an antitone involution and hence $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ is a so-called *orthomodular lattice* (see e.g. Beran (1985) for the definition).

Hence, not every modular lattice endowed with a complementation must be orthomodular. Of course, not every orthomodular lattice is modular (see Beran (1985)).

If $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a complemented lattice in which the complementation is not introduced in form of a unary operation then we need not distinguish between the complements of a given element a of L. Hence we will work with the whole set of complements of a. Within this paper we will use this approach.

We start by introducing some lattice-theoretical concepts.

All complemented lattices considered within this paper are assumed to be non-trivial, i.e. to have a bottom element 0 and a top element 1 with $0 \neq 1$.

Let $(L, \vee, \wedge, 0, 1)$ be a complemented lattice and $A, B \subseteq L$. We define:

$$A \vee B := \{x \vee y \mid x \in A \text{ and } y \in B\},$$
$$A \wedge B := \{x \wedge y \mid x \in A \text{ and } y \in B\},$$
$$A \leq B \text{ if } x \leq y \text{ for all } x \in A \text{ and all } y \in B,$$

$A \leq_1 B$ if for every $x \in A$ there exists some $y \in B$ with $x \leq y$,

$A \leq_2 B$ if for every $y \in B$ there exists some $x \in A$ with $x \leq y$.

2 The operator $^+$

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a complemented lattice. For $a \in L$ we define

$$a^+ := \{x \in L \mid a \vee x = 1 \text{ and } a \wedge x = 0\},$$

i.e. a^+ is the set of all complements of a. Since $\mathbf{L}$ is complemented, we have $a^+ \neq \emptyset$ for all $a \in L$. For every subset A of L we put

$$A^+ := \{x \in L \mid a \vee x = 1 \text{ and } a \wedge x = 0 \text{ for all } a \in A\}.$$

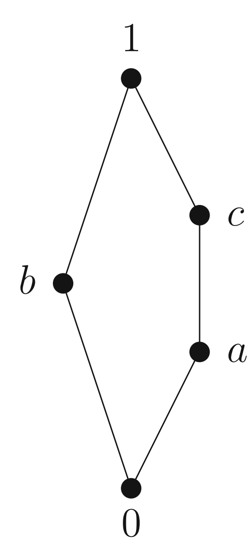
Observe that A^+ may me empty, e.g. $L^+ = \emptyset$ (and $\emptyset^+ = L$). In the following we often identify singletons with their unique element.

Example 2.1 For the lattice $\mathbf{N}_5$ depicted in Fig. 3:

we have

x	0	a	b	c	1
x^+	1	b	ac	b	0
x^{++}	0	ac	b	ac	1

Fig. 3 Non-modular lattice $\mathbf{N}_5$



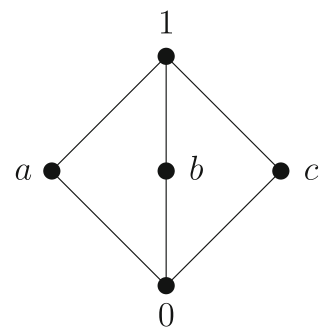


Fig. 4 Modular lattice $\mathbf{M}_3$

Here and in the following within tables we sometimes write abc instead of $\{a, b, c\}$. For the lattice $\mathbf{M}_3$ visualized in Fig. 4 we have

x	0	a	b	c	1
x^+	1	bc	ac	ab	0
x^{++}	0	a	b	c	1

Let us note that $\mathbf{M}_3$ satisfies the identity $x^{++} \approx x$.

Example 2.2 For the example from Fig. 2 we have

x	0	a	b	c	d	e	f	g	h	i	j	1
x^+	1	hij	gij	ghj	ghi	f	e	bcd	acd	abd	abc	0
x^{++}	0	a	b	c	d	e	f	g	h	i	j	1

Recall the concept of a *Galois connection* which is often used in lattices. The pair $(^+, ^+)$ is the Galois connection between $(2^L, \subseteq)$ and $(2^L, \subseteq)$ induced by the relation

$$\{(x, y) \in L^2 \mid x \vee y = 1 \text{ and } x \wedge y = 0\}.$$

From this we conclude

$$A \subseteq A^{++},$$
$$A \subseteq B \Rightarrow B^+ \subseteq A^+,$$
$$A^{+++} = A^+,$$
$$A \subseteq B^+ \Leftrightarrow B \subseteq A^+$$

for all $A, B \subseteq L$. Since $A \subseteq A^{++}$ we have that $A^{++} \neq \emptyset$ whenever $A \neq \emptyset$. A subset A of L is called *closed* if $A^{++} = A$. Let $\mathrm{Cl}(\mathbf{L})$ denote the set of all closed subsets of L. Then clearly $\mathrm{Cl}(\mathbf{L}) = \{A^+ \mid A \subseteq L\}$. Because of $A^+ \cap A^{++} = \emptyset$ for all $A \subseteq L$ we have that $\big(\mathrm{Cl}(\mathbf{L}), \subseteq, ^+, \emptyset, L\big)$ forms a complete ortholattice with

$$\bigvee_{i\in I} A_i = \Big(\bigcup_{i\in I} A_i\Big)^{++},$$
$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i$$

for all families $(A_i; i \in I)$ of closed subsets of L.

Next we describe the basic properties of the operator $^+$.

Proposition 2.3 *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a complemented lattice and $a \in L$. Then the following holds:*

(i) *$a \in a^{++}$ and $a^{+++} = a^+$,*
(ii) *$(x^+, \leq)$ is an antichain for every $x \in L$ if and only if $\mathbf{L}$ does not contain a sublattice isomorphic to $\mathbf{N}_5$ containing 0 and 1,*
(iii) *$(a^+, \leq)$ is convex,*
(iv) *if the mapping $x \mapsto x^{++}$ from L to 2^L is not injective then $\mathbf{L}$ does not satisfy the identity $x^{++} \approx x$.*

Proof (i) follows directly from above.
(ii) First assume there exists some $b \in L$ such that $(b^+, \leq)$ is not an antichain. Then $b \notin \{0, 1\}$. Now there exist $c, d \in b^+$ with $c < d$. Since $b \notin \{0, 1\}$ and $b \in c^+ \cap d^+$ we have $c, d \notin \{0, 1\}$. Because of $|L| > 1$ we have $b \notin \{c, d\}$. Hence the elements $0, b, c, d$ and 1 are pairwise distinct and form an $\mathbf{N}_5$ containing 0 and 1. If, conversely, $\mathbf{L}$ contains a sublattice $(L_1, \vee, \wedge)$ isomorphic to $\mathbf{N}_5$ and containing 0 and 1, say $L_1 = \{0, e, f, g, 1\}$ with $e < f$ then $e, f \in g^+$ and hence $(g^+, \leq)$ is not an antichain.
(iii) If $b, c \in a^+$, $d \in L$ and $b \leq d \leq c$ then $1 = a \vee b \leq a \vee d$ and $a \wedge d \leq a \wedge c = 0$ showing $d \in a^+$.
(iv) If the mapping $x \mapsto x^{++}$ is not injective then there exist $a, b \in L$ with $a \neq b$ and $a^{++} = b^{++}$ which implies $b \in b^{++} = a^{++}$ and $b \neq a$ and hence $a^{++} \neq a$ showing that $\mathbf{L}$ does not satisfy the identity $x^{++} \approx x$.

□

In the lattice $\mathbf{N}_5$ from Example 2.1 the mapping $x \mapsto x^{++}$ is not injective since $a \neq c$ and $a^{++} = c^{++}$. According to Proposition 2.3 (iv), this lattice does not satisfy the identity $x^{++} \approx x$, e.g. $a^{++} = \{a, c\} \neq a$.

Corollary 2.4 *Let $(L, \vee, \wedge, 0, 1)$ be a complemented modular lattice, $a \in L$ and A a non-empty subset of L. According to Proposition 2.3* (iii)*, $(a^+, \leq)$ is an antichain. Let $b \in A$. Then $A^+ \subseteq b^+$ and hence also $(A^+, \leq)$ is an antichain. Since a^+ is a non-empty subset of L we finally conclude that $(a^{++}, \leq)$ is an antichain, too.*

In case of finite L we can even prove the following.

Proposition 2.5 *Let $(L, \vee, \wedge, 0, 1)$ be a finite complemented lattice such that $x \mapsto x^{++}$ is injective and $a \in L$ and assume $a^{++} \neq a$. Then there exists some $b \in a^{++}$ with $b^{++} = b$.*

Proof Let $a_1 \in a^{++} \setminus \{a\}$. Then $a_1^{++} \subseteq a^{++}$. Since $a_1 \neq a$ and $x \mapsto x^{++}$ is injective we conclude $a_1^{++} \subsetneq a^{++}$. Now either $a_1^{++} = a_1$ or there exists some $a_2 \in a_1^{++} \setminus \{a_1\}$. Then

$a_2^{++} \subseteq a_1^{++}$. Since $a_2 \neq a_1$ and $x \mapsto x^{++}$ is injective we conclude $a_2^{++} \subsetneqq a_1^{++}$. Now either $a_2^{++} = a_2$ or there exists some $a_3 \in a_2^{++} \setminus \{a_2\}$. Since L is finite and $a_1^{++} \supsetneqq a_2^{++} \supsetneqq \cdots$ there exists some $n \geq 1$ with $|a_n^{++}| = 1$, i.e. $a_n^{++} = a_n$ and we have $a_n \in a_n^{++} \subseteq a_{n-1}^{++} \subseteq \cdots \subseteq a_1^{++} \subseteq a^{++}$. □

The relationship between the operator $^+$ and the partial order relation of $\mathbf{L}$ is illuminated in the following result.

Proposition 2.6 *Let $(L, \vee, \wedge, 0, 1)$ be a complemented lattice and consider the following statements:*

(i) $x^+ \vee y^+ \leq_1 (x \wedge y)^+$ *for all* $x, y \in L$,
(ii) *for all* $x, y \in L$, $x \leq y$ *implies* $y^+ \leq_1 x^+$,
(iii) $(x \vee y)^+ \leq_1 x^+ \wedge y^+$ *for all* $x, y \in L$.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof Let $a, b \in L$.
(iii) $\Rightarrow$ (ii):
$a \leq b$ implies $b^+ = (a \vee b)^+ \leq_1 a^+ \wedge b^+ \leq_1 a^+$.
(ii) $\Rightarrow$ (iii):
Because of $a, b \leq a \vee b$ we have $(a \vee b)^+ \leq_1 a^+, b^+$ which implies $(a \vee b)^+ \leq_1 a^+ \wedge b^+$.
(i) $\Rightarrow$ (ii):
$a \leq b$ implies $b^+ \leq_1 a^+ \vee b^+ \leq (a \wedge b)^+ = a^+$. □

Our next task is to characterize the property that a complemented lattice $\mathbf{L}$ satisfies the identity $x^{++} \approx x$. From Example 2.1 we know that if $\mathbf{L}$ is not modular then this identity need not hold. Hence we restrict ourselves to complemented modular lattices.

Theorem 2.7 *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a complemented modular lattice. Then the following are equivalent:*

(i) $\mathbf{L}$ *satisfies the identity* $x^{++} \approx x$,
(ii) *for every* $x \in L$ *and each* $y \in x^{++}$ *there exists some* $z \in y^+$ *satisfying either* $(x \vee y) \wedge z = 0$ *or* $(x \wedge y) \vee z = 1$.

Proof (i) $\Rightarrow$ (ii):
If $a \in L, b \in a^{++}$ and $c \in b^+$ then $b = a$ and $(a \vee b) \wedge c = b \wedge c = 0$.
(ii) $\Rightarrow$ (i):
Assume (ii). Suppose $\mathbf{L}$ not to satisfy the identity $x^{++} \approx x$. Then there exists some $a \in L$ with $a^{++} \neq a$. Let $b \in a^{++} \setminus \{a\}$. According to (ii) there exists some $c \in b^+$ satisfying either $(a \vee b) \wedge c = 0$ or $(a \wedge b) \vee c = 1$. Since a and b are different elements of a^{++} and $(a^{++}, \leq)$ is an antichain according to Corollary 2.4, we conclude $a \parallel b$. Now $(a \vee b) \wedge c = 0$ would imply

$$\begin{aligned} a \leq a \vee b &= 1 \wedge (a \vee b) = (b \vee c) \wedge (a \vee b) \\ &= b \vee \big(c \wedge (a \vee b)\big) = b \vee 0 = b, \end{aligned}$$

contradicting $a \parallel b$. On the other hand, $(a \wedge b) \vee c = 1$ would imply

$$\begin{aligned} &b = 1 \wedge b = \big((a \wedge b) \vee c\big) \wedge \\ &b = (a \wedge b) \vee (c \wedge b) = (a \wedge b) \vee 0 = a \wedge b \leq a, \end{aligned}$$

again contradicting $a \parallel b$. This shows that $\mathbf{L}$ satisfies the identity $x^{++} \approx x$. □

3 The operator →

Let $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice. Recall that the operation ϕ_x defined by $\phi_x(y) := x \wedge (x' \vee y)$ for all $x, y \in L$ was introduced by Sasaki in Sasaki (1950) and Sasaki (1952) and is called the *Sasaki projection* (see e.g. Beran (1985)) or *Sasaki hook* alias *Sasaki operation*, see Chajda and Länger (2017); its dual, i.e. the operation ψ_x defined by $\psi_x(y) := x' \vee (x \wedge y)$ for all $x, y \in L$ is then called the *dual Sasaki projection*. It was shown by the authors in Chajda and Länger (2017) that if we use these Sasaki operations in order to define

$$\begin{aligned} x \to y &:= x' \vee (x \wedge y), \\ x \cdot y &:= (x \vee y') \wedge y \end{aligned}$$

for all x, y belonging to the base set of the orthomodular lattice $\mathbf{L}$ then the operations $\to$ and $\cdot$ form an adjoint pair, i.e.

$$x \cdot y \leq z \text{ if and only if } x \leq y \to z$$

for all $x, y, z \in L$. This motivated us to introduce our next operators in a similar way where, however, instead of the element x' we use the set x^+. Hence, for a complemented lattice $(L, \vee, \wedge, 0, 1)$, $a, b \in L$ and $A, B \subseteq L$ we define

$$\begin{aligned} a \to b &:= a^+ \vee (a \wedge b), \\ A \to B &:= A^+ \vee (A \wedge B). \end{aligned}$$

Observe that $A \to B = \emptyset$ whenever $A^+ = \emptyset$.

Example 3.1 For the lattice from Fig. 2 we have e.g.

$$\begin{aligned} a \to b &= \{h, i, j\} \vee (a \wedge b) = \{h, i, j\} \vee 0 = \{h, i, j\} = a^+, \\ a \to f &= \{h, i, j\} \vee (a \wedge f) = \{h, i, j\} \vee a = 1, \\ a \to g &= \{h, i, j\} \vee (a \wedge g) = \{h, i, j\} \vee a = 1, \\ a \to h &= \{h, i, j\} \vee (a \wedge h) = \{h, i, j\} \vee 0 = \{h, i, j\} = a^+, \\ f \to e &= e \vee (f \wedge e) = e \vee 0 = e, \\ g \to h &= \{b, c, d\} \vee (g \wedge h) = \{b, c, d\} \vee e = \{h, i, j\} = a^+. \end{aligned}$$

In the following we study the relationship between $\to$ and $\wedge$.

Theorem 3.2 *Let $(L, \vee, \wedge, 0, 1)$ be a complemented modular lattice and $a, b, c \in L$. Then the following holds:*

(i) *If $a \leq_1 b \to c$ then $a \wedge b \leq c$,*
(ii) *$a \wedge b \leq c$ if and only if $a \wedge b \leq_1 b \to c$.*

Proof (i) From $a \leq_1 b \to c$ we conclude that there exists some $d \in b^+$ satisfying $a \leq d \vee (b \wedge c)$, and we obtain

$$a \wedge b \leq \big(d \vee (b \wedge c)\big) \wedge b = \big((b \wedge c) \vee d\big) \wedge b = (b \wedge c) \vee (d \wedge b) = (b \wedge c) \vee 0 = b \wedge c \leq c.$$

(ii) First assume $a \wedge b \leq c$. Let $e \in b^+$. Then

$$a \wedge b \leq e \vee (a \wedge b) = e \vee \big(b \wedge (a \wedge b)\big) \leq e \vee (b \wedge c).$$

This shows $a \wedge b \leq_1 b \to c$. Conversely, assume $a \wedge b \leq_1 b \to c$. Then there exists some $f \in b^+$ with $a \wedge b \leq f \vee (b \wedge c)$. So we get

$$a \wedge b = (a \wedge b) \wedge b \leq \big(f \vee (b \wedge c)\big) \wedge b = \big((b \wedge c) \vee f\big) \wedge b = (b \wedge c) \vee (f \wedge b) = (b \wedge c) \vee 0 = b \wedge c \leq c.$$

□

For complemented lattices, the operator $\to$ satisfies a lot of properties common in residuated structures.

Theorem 3.3 *Let $(L, \vee, \wedge, 0, 1)$ be a complemented lattice and $a, b, c \in L$. Then the following holds:*

(i) *$a \to 0 = a^+$ and $1 \to a = a$,*
(ii) *if $a \leq b$ then $a \to b = 1$,*
(iii) *$a \to b = 1$ if and only if $a \wedge b \in a^{++}$,*
(iv) *if $b \in a^+$ then $a \to b = a^+$,*
(v) *if $b \leq c$ then $a \to b \leq_i a \to c$ for $i = 1, 2$,*
(vi) *if $a \to b = a \to c = 1$ and a^{++} is closed with respect to $\wedge$ then $a \to (b \wedge c) = 1$,*
(vii) *if $a^{++} \subseteq b^{++}$ and $a \to b = 1$ then $b \to a = 1$.*

Proof (i) We have $a \to 0 = a^+ \vee (a \wedge 0) = a^+ \vee 0 = a^+$ and $1 \to a = 1^+ \vee (1 \wedge a) = 0 \vee a = a$.
(ii) If $a \leq b$ then $a \to b = a^+ \vee (a \wedge b) = a^+ \vee a = 1$.
(iii) The following are equivalent:

$$a \to b = 1,$$
$$a^+ \vee (a \wedge b) = 1,$$
$$a^+ \vee (a \wedge b) = 1 \text{ and } a^+ \wedge (a \wedge b) = 0,$$
$$a \wedge b \in a^{++}.$$

(iv) If $b \in a^+$ then $a \to b = a^+ \vee (a \wedge b) = a^+ \vee 0 = a^+$.
(v) If $b \leq c$ then $a \to b = a^+ \vee (a \wedge b) \leq_i a^+ \vee (a \wedge c) = a \to c$ for $i = 1, 2$.
(vi) Using (iii) and the assumptions we obtain $a \wedge b, a \wedge c \in a^{++}$ and hence $a \wedge (b \wedge c) = (a \wedge b) \wedge (a \wedge c) \in a^{++}$ showing $a \to (b \wedge c) = 1$.
(vii) Using (iii) and the assumptions we have $a \to b = 1$ and hence $b \wedge a = a \wedge b \in a^{++} \subseteq b^{++}$ which implies $b \to a = 1$.

□

Let us note that the converse of Theorem 3.3 (ii) does not hold in general. For example, consider the lattice $\mathbf{N}_5$ from Example 2.1. Then $c \to a = c^+ \vee (c \wedge a) = b \vee a = 1$ contrary to the fact that $c > a$. However, if x is a minimal element of x^{++} then we can prove the following.

Proposition 3.4 *Let $(L, \vee, \wedge, 0, 1)$ be a complemented lattice and $a \in L$. Then the following are equivalent:*

(i) *For all $x \in L$, $a \to x = 1$ is equivalent to $a \leq x$,*
(ii) *a is a minimal element of a^{++}.*

Proof According to (iii) of Theorem 3.3 the following are equivalent:

For all $x \in L$, $a \to x = 1$ is equivalent to $a \leq x$,
for all $x \in L$, $a \wedge x \in a^{++}$ is equivalent to $a \wedge x = a$,
for all $y \leq a$, $y \in a^{++}$ is equivalent to $y = a$,
a is a minimal element of a^{++}.

□

We are going to show how the operator $\to$ is related to the connective implication in a propositional calculus.

Theorem 3.5 *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a complemented modular lattice and $a, b \in L$. Then the following holds:*

(i) *$a \wedge (a \to b) = a \wedge b \leq b$ (Modus Ponens),*
(ii) *if $a^+ \leq b^+$ then $(a \to b) \wedge b^+ = a^+$ (Modus Tollens),*
(iii) *if $c \in a \to b$ then $a \to c = a \to b$,*
(iv) *$a \to (a \to b) = a \to b$,*
(v) *if $a^+ \leq b$ then $a \to b = b$,*

Proof (i) Using modularity of $\mathbf{L}$ we compute

$$a \wedge (a \to b) = a \wedge \big(a^+ \vee (a \wedge b)\big) = \big((a \wedge b) \vee a^+\big) \wedge a = (a \wedge b) \vee (a^+ \wedge a) = (a \wedge b) \vee 0 = a \wedge b \leq b.$$

(ii) Under the assumptions

$$(a \to b) \wedge b^+ = \big(a^+ \vee (a \wedge b)\big) \wedge$$

$$b^+ = a^+ \vee \big((a \wedge b) \wedge b^+\big) = a^+ \vee 0 = a^+.$$

(iii) If $c \in a \to b$ then there exists some $d \in a^+$ with $d \vee (a \wedge b) = c$ and hence

$$\begin{aligned} a \to c &= a^+ \vee \Big(a \wedge \big(d \vee (a \wedge b)\big)\Big) \\ &= a^+ \vee \Big(\big((a \wedge b) \vee d\big) \wedge a\Big) = \\ &= a^+ \vee \big((a \wedge b) \vee (d \wedge a)\big) = a^+ \vee \big((a \wedge b) \vee 0\big) \\ &= a^+ \vee (a \wedge b) = a \to b. \end{aligned}$$

(iv) Using (iii) we obtain

$$\begin{aligned} a \to (a \to b) = a^+ \vee \big(a \wedge (a \to b)\big) &= \bigcup_{c \in a \to b} \big(a^+ \vee (a \wedge c)\big) \\ &= \bigcup_{c \in a \to b} (a \to c) = \\ &= \bigcup_{c \in a \to b} (a \to b) = a \to b. \end{aligned}$$

(v) If $a^+ \le b$ then $a \to b = a^+ \vee (a \wedge b) = (a^+ \vee a) \wedge b = 1 \wedge b = b$.

□

Proposition 3.6 *Let $n > 1$ and $a, b, c \in M_n$. Then*

$a \wedge b \le c$ if and only if $a \le_1 b \to c$.

Proof It is easy to see that

$$a \to b = \begin{cases} 1 & a \le b, \\ b & a = 1, \\ a^+ & a \parallel b \text{ or } b = 0 \end{cases}$$

If $a = 0$ then $a \wedge b = 0 \wedge b = 0 \le c$ and $a = 0 \le_1 b \to c$. If $b \le c$ then $a \wedge b \le b \le c$ and $a \le_1 1 = b \to c$. If $b = 1$ then both $a \wedge b \le c$ and $a \le_1 b \to c$ are equivalent to $a \le c$. Hence we can assume $a \ne 0$, $b \not\le c$ and $b \ne 1$. In case $n > 2$ let a, b and c be pairwise different elements of $M_n \setminus \{0, 1\}$ and in case $n = 2$ let $M_n = \{0, a, b, 1\}$. Then the following cases remain:

x	y	z	$x \wedge y \le z$	$x \le_1 y \to z$
a	a	0	no	no
a	a	b	no	no
a	b	0	yes	yes
a	b	a	yes	yes
a	b	c	yes	yes
1	a	0	no	no
1	a	b	no	no

□

4 The operator ⊙

Similarly as it was done in Sect. 3 concerning the operator $\to$, also here we define the new operator $\odot$ by means of the generalized Sasaki projection.

For a complemented lattice $(L, \vee, \wedge, 0, 1)$, $a, b \in L$ and $A, B \subseteq L$ we define

$$\begin{aligned} a \odot b &:= b \wedge (a \vee b^+), \\ A \odot B &:= B \wedge (A \vee B^+). \end{aligned}$$

It is evident that $\odot$ need neither be commutative nor associative, but it is idempotent, i.e. it satisfies the identity $x \odot x \approx x$ (cf. Proposition 4.1 (iii)).

We list some basic properties of the operator $\odot$.

Proposition 4.1 *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ a complemented lattice and $a, b, c \in L$. Then the following holds:*

(i) *$0 \odot a = a \odot 0 = 0$,*
(ii) *$1 \odot a = a \odot 1 = a$,*
(iii) *$a \wedge b \le a \odot b \le b$ and if $b \le a$ then $a \odot b = b$,*
(iv) *if $a \le b$ then $a \odot c \le_i b \odot c$ for $i = 1, 2$,*
(v) *if $\mathbf{L}$ is modular then $a \le b$ if and only if $a \odot b = a$ and, moreover, $(a \odot b) \odot b = a \odot b$.*

Proof (i) We have $0 \odot a = a \wedge (0 \vee a^+) = a \wedge a^+ = 0$ and $a \odot 0 = 0 \wedge (a \vee 0^+) = 0$.

(ii) We have $1 \odot a = a \wedge (1 \vee a^+) = a \wedge 1 = a$ and $a \odot 1 = 1 \wedge (a \vee 1^+) = a \vee 0 = a$.

(iii) This follows from the definition of $a \odot b$.

(iv) If $a \le b$ then $a \odot c = c \wedge (a \vee c^+) \le_i c \wedge (b \vee c^+) = b \odot c$ for $i = 1, 2$.

(v) If $a \le b$ then using modularity of $\mathbf{L}$ we obtain

$$a \odot b = b \wedge (a \vee b^+) = (a \vee b^+) \wedge b = a \vee (b^+ \wedge b) = a \vee 0 = a.$$

That $a \odot b = a$ implies $a \le b$ follows from (iii). Using (iii) and modularity of $\mathbf{L}$ we obtain

$$(a \odot b) \odot b = b \wedge \big((a \odot b) \vee b^+\big) = \big((a \odot b) \vee b^+\big) \wedge b = (a \odot b) \vee (b^+ \wedge b) = (a \odot b) \vee 0 = a \odot b.$$

□

Example 4.2 The "operation tables" for $\odot$ for the lattices $\mathbf{N}_5$ and $\mathbf{M}_3$ (see Example 2.1) are as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	c	a
b	0	0	b	0	b
c	0	a	0	c	c
1	0	a	b	c	1

$\mathbf{N}_5$

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0b	0c	a
b	0	0a	b	0c	b
c	0	0a	0b	c	c
1	0	a	b	c	1

$\mathbf{M}_3$

Contrary to the relatively weak relationship between $\to$ and $\wedge$, for $\odot$ and $\to$ we can prove here a kind of adjointness.

Theorem 4.3 *Let $(L, \vee, \wedge, 0, 1)$ be a complemented modular lattice and $a, b, c \in L$. Then*

$a \odot b \leq c$ if and only if $a \leq b \to c$.

Proof If $a \odot b \leq c$ then $b \wedge (a \vee x) \leq c$ for all $x \in b^+$ and hence

$$\begin{aligned} a &\leq a \vee x = 1 \wedge (a \vee x) = (x \vee b) \wedge (a \vee x) \\ &= x \vee \big(b \wedge (a \vee x)\big) \\ &= x \vee \Big(b \wedge \big(b \wedge (a \vee x)\big)\Big) \leq x \vee (b \wedge c) \end{aligned}$$

for all $x \in b^+$ showing $a \leq b \to c$. If, conversely, $a \leq b \to c$ then $a \leq x \vee (b \wedge c)$ for all $x \in b^+$ and hence

$$\begin{aligned} b \wedge (a \vee x) &\leq b \wedge \Big(\big(x \vee (b \wedge c)\big) \vee x\Big) \\ &= b \wedge \big(x \vee (b \wedge c)\big) = \big((b \wedge c) \vee x\big) \wedge b = \\ &= (b \wedge c) \vee (x \wedge b) = (b \wedge c) \vee 0 = b \wedge c \leq c \end{aligned}$$

for all $x \in b^+$ showing $a \odot b \leq c$. □

5 Deductive systems

Deductive systems are often introduced in algebras forming an algebraic formalization of a non-classical propositional calculus. These are subsets of the algebra in question containing the logical constant 1 and representing the derivation rule Modus Ponens. Since our operator $\to$ shares a number of properties with the non-classical logical connective implication, we define this concept also for complemented lattices.

Definition 5.1 A *deductive system* of a complemented lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a subset D of L satisfying the following conditions:

- $1 \in D$,
- if $a \in D$, $b \in L$ and $a \to b \subseteq D$ then $b \in D$.

Since the intersection of deductive systems of $\mathbf{L}$ is again a deductive system of $\mathbf{L}$, the set of all deductive systems of $\mathbf{L}$ forms a complete lattice $\mathbf{Ded}\,\mathbf{L}$ with respect to inclusion with bottom element $\{1\}$ and top element L.

Example 5.2 The deductive systems of the lattice $\mathbf{M}_n$ for $n > 1$ (see Fig. 1) are given by M_n and $A \cup \{1\}$ where A is a proper subset of $\{a_1, \ldots, a_n\}$. This can be seen as follows. Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then we have

$\to$	0	a_i	a_j	1
0	1	1	1	1
a_i	a_i^+	1	a_i^+	1
a_j	a_j^+	a_j^+	1	1
1	0	a_i	a_j	1

Now let D be a deductive system of $\mathbf{M}_n$. Then the following hold:

$1 \in D$,

if $0 \in D$ then $D = M_n$,

if $\{a_1, \ldots, a_n\} \subseteq D$ then $a_1 \in D$ and $a_1 \to 0 = a_1^+ \subseteq D$ and hence $0 \in D$ which implies $D = M_n$,

if $a_i \in D$ and $\{a_1, \ldots, a_n\} \nsubseteq D$ then $a_i^+ \nsubseteq D$.

The rest follows from the table above. Moreover, $\mathbf{Ded}\,\mathbf{M}_n$ is a 2^n-element Boolean algebra since

$$A \mapsto \begin{cases} M_n & \text{if } A = \{a_1, \ldots, a_n\}, \\ A \cup \{1\} & \text{otherwise} \end{cases}$$

is an isomorphism from $(2^{\{a_1, \ldots, a_n\}}, \subseteq)$ to $\mathbf{Ded}\,\mathbf{M}_n$.

The relationship between deductive systems and filters is described in the following results.

Lemma 5.3 *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a complemented lattice and D a deductive system of $\mathbf{L}$. Then the following holds:*

(i) *D is an order filter of $\mathbf{L}$,*
(ii) *if $x \to y \subseteq D$ for all $x, y \in D$ then D is a filter of $\mathbf{L}$.*

Proof Let $a, b \in L$.

(i) If $a \in D$ and $a \leq b$ then $a \to b = 1 \in D$ and hence $b \in D$.
(ii) According to (i), D is an order filter of $\mathbf{L}$. If $a, b \in D$ then

$$\begin{aligned} a \to (a \wedge b) &= a^+ \vee \big(a \wedge (a \wedge b)\big) \\ &= a^+ \vee (a \wedge b) = a \to b \subseteq D \end{aligned}$$

showing $a \wedge b \in D$.

□

If $\mathbf{L}$ is, moreover, modular then we can prove also the following.

Proposition 5.4 *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a complemented modular lattice and* F *a filter of* $\mathbf{L}$. *Then* F *is a deductive system of* $\mathbf{L}$.

Proof If $a \in F$, $b \in L$ and $a \to b \subseteq F$ then according to Theorem 3.5 (i) we have

$$a \wedge b = a \wedge (a \to b) \subseteq F$$

and due to $a \wedge b \leq b$ we finally obtain $b \in F$. □

In the remaining part of this section we investigate when a given deductive system D may induce an equivalence relation Φ such that $D = [1]\Phi$, i.e. D being its kernel. We start with the following definition.

Definition 5.5 For every complemented lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ and every deductive system D of $\mathbf{L}$ put

$$\Theta(D) := \{(x, y) \in L^2 \mid x \to y, y \to x \subseteq D\}.$$

From Theorem 3.3 (ii) we get that $\Theta(D)$ is reflexive and, by definition, it is symmetric.

It is easy to see that every congruence on a complemented modular lattice induces a deductive system.

Proposition 5.6 *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a complemented modular lattice and* $\Phi \in \mathrm{Con}(L, \wedge)$. *Then the following holds:*

(i) $[1]\Phi$ *is a deductive system of* $\mathbf{L}$,
(ii) $\Theta([1]\Phi) \subseteq \Phi$.

Proof Let $a, b \in L$.

(i) We have $1 \in [1]\Phi$, and if $a \in [1]\Phi$ and $a \to b \subseteq [1]\Phi$ then according to Theorem 3.5 (i) we conclude

$$b = 1 \wedge b \;\Phi\; a \wedge b = a \wedge (a \to b) \subseteq [1 \wedge 1]\Phi = [1]\Phi.$$

(ii) If $(a, b) \in \Theta([1]\Phi)$ then $a \to b, b \to a \subseteq [1]\Phi$ and hence again according to Theorem 3.5 (i) we obtain

$$a = a \wedge 1 \;\Phi\; a \wedge (a \to b) = a \wedge b = b \wedge a = b \wedge (b \to a) \;\Phi\; b \wedge 1 = b$$

showing $(a, b) \in \Phi$. □

That not all deductive systems arise in the way shown in Proposition 5.6 (i) can be seen as follows: According to Example 5.2, $\{a_1, a_2, 1\}$ is a deductive system of $\mathbf{M}_3$, but there does not exist some $\Phi \in \mathrm{Con}(M_3, \wedge)$ satisfying $[1]\Phi = \{a_1, a_2, 1\}$ since this would imply $0 = a_1 \wedge a_2 \in [a_1 \wedge a_1]\Phi = [a_1]\Phi = \{a_1, a_2, 1\}$, a contradiction.

The previous proposition shows that we need a certain compatibility of the induced relation $\Theta(D)$ with the lattice operations in order to show D to be the kernel of $\Theta(D)$. For this sake, we define the following properties.

Definition 5.7 Let $(L, \vee, \wedge, 0, 1)$ be a complemented lattice and Φ an equivalence relation on L. We say that Φ has the *Substitution Property with respect to* $^+$ if

$$(a, b) \in \Phi \text{ implies } a^+ \times b^+ \subseteq \Phi,$$

and the *Substitution Property with respect to* $\to$ if

$$(a, b) \in \Phi \text{ implies } (a \to c) \times (b \to c) \subseteq \Phi \text{ for all } c \in L.$$

Such an equivalence relation Φ can be related with the equivalence relation induced by its kernel $[1]\Phi$ and, moreover, this kernel is a deductive system.

Theorem 5.8 *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a complemented lattice and* Φ *an equivalence relation on* L *having the Substitution Property with respect to* $\to$. *Then the following holds:*

(i) Φ *has the Substitution Property with respect to* $^+$,
(ii) $[1]\Phi$ *is a deductive system of* $\mathbf{L}$,
(iii) $\Phi \subseteq \Theta([1]\Phi)$.

Proof Let $a, b \in L$.

(i) According to Theorem 3.3 (i), $(a, b) \in \Phi$ implies $a^+ \times b^+ = (a \to 0) \times (b \to 0) \subseteq \Phi$.
(ii) If $a \in [1]\Phi$ and $a \to b \subseteq [1]\Phi$ then for every $x \in a \to b$ we have $(1, x) \in \Phi$ and according to Theorem 3.3 (i) also $(x, b) \in (a \to b) \times (1 \to b) \subseteq \Phi$ showing $b \in [1]\Phi$.
(iii) According to Theorem 3.3 (ii), $(a, b) \in \Phi$ implies

$$(a \to b) \times \{1\} = (a \to b) \times (b \to b) \subseteq \Phi,$$
$$(b \to a) \times \{1\} = (b \to a) \times (a \to a) \subseteq \Phi.$$

□

Now we are able to relate deductive systems with equivalence relations induced by them provided these deductive systems satisfy a certain compatibility condition defined as follows.

Definition 5.9 Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a complemented lattice and D a deductive system of $\mathbf{L}$. We call D a *compatible deductive system* of $\mathbf{L}$ if it satisfies the following two additional conditions for all $a, b, c, d \in L$:

- If $a \to b \subseteq D$ and $x \to (c \to d) \subseteq D$ for all $x \in a \to b$ then $c \to d \subseteq D$,
- if $a \to b, b \to a \subseteq D$ then $x \to (b \to c) \subseteq D$ for all $x \in a \to c$.

Since the intersection of compatible deductive systems of $\mathbf{L}$ is again a compatible deductive system of $\mathbf{L}$, the set of all compatible deductive systems of $\mathbf{L}$ forms a complete lattice with respect to inclusion with top element L.

Now we show that also conversely as in Theorem 5.8, a compatible deductive system induces an equivalence relation having the Substitution Property with respect to $\to$.

Theorem 5.10 *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a complemented lattice and D a compatible deductive system of $\mathbf{L}$. Then the following holds:*

(i) *$\Theta(D)$ is an equivalence relation on L having the Substitution Property with respect to $\to$,*
(ii) *$[1]\big(\Theta(D)\big) = D$.*

Proof Let $a, b, c, d, e, f, g \in L$.

(i) As remarked after Definition 5.5, $\Theta(D)$ is reflexive and symmetric. Now assume $(a, b), (b, c) \in \Theta(D)$. Then $b \to a, a \to b \subseteq D$ and hence $x \to (a \to c) \subseteq D$ for all $x \in b \to c$. Because of $(b, c) \in \Theta(D)$ we have $b \to c \subseteq D$ and therefore $a \to c \subseteq D$. On the other hand $b \to c, c \to b \subseteq D$ which implies $x \to (c \to a) \subseteq D$ for all $x \in b \to a$ which together with $b \to a \subseteq D$ yields $c \to a \subseteq D$. This shows $(a, c) \in \Theta(D)$, i.e. $\Theta(D)$ is transitive. Now assume $(d, e) \in \Theta(D)$. Then $d \to e, e \to d \subseteq D$ and hence $x \to (e \to f) \subseteq D$ for all $x \in d \to f$. Because of $e \to d, d \to e \subseteq D$ we have $y \to (d \to f) \subseteq D$ for all $y \in e \to f$. Since $g \to A = \bigcup_{x \in A} (g \to x)$ for all $A \subseteq L$, we have $x \to y, y \to x \subseteq D$ for all $(x, y) \in (d \to f) \times (e \to f)$ and hence $(x, y) \in \Theta(D)$ for all $(x, y) \in (d \to f) \times (e \to f)$ proving $(d \to f) \times (e \to f) \subseteq \Theta(D)$. Therefore $\Theta(D)$ has the Substitution Property with respect to $\to$.

(ii) According to Theorem 3.3 (i) and (ii) the following are equivalent: $a \in [1]\big(\Theta(D)\big)$; $a \to 1, 1 \to a \subseteq D$; $1, a \in D$; $a \in D$.

□

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