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Dissertation

Surface Area Measures, Minkowski Endomorphisms and j-Projection Bodies

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Kurzfassung

In dieser Arbeit werden zwei Beiträge zur Theorie der konvexen Körper (kompakte und konvexe Mengen) behandelt. Der Raum der konvexen Körper in $\mathbb{R}^n$ wird mit $\mathcal{K}^n$ bezeichnet.

Der erste Teil befasst sich mit dem Konzept von Minkowski Endomorphismen. Ein Minkowski Endomorphismus ist eine stetige, $SO(n)$-equivariante und translationsinvariante Abbildung $\Phi : \mathcal{K}^n \to \mathcal{K}^n$, die additiv bezüglich der punktweisen Addition (Minkowski Addition) von konvexen Körpern ist. Als eines der wichtigsten Resultate dieser Arbeit wird gezeigt, dass alle Minkowski Endomorphismen gleichmäßig stetig sind. Dieses Resultat beantwortet eine seit mehreren Jahren offene Frage. Desweiteren wird gezeigt, dass es nichtmonotone, gerade Minkowski Endomorphismen gibt, und es werden außerdem Fragen in Bezug auf den allgemeineren Begriff von Minkowski Bewertungen beantwortet.

Im zweiten Teil dieser Arbeit wird, auf der Grundlage eines gemeinsamen Artikels mit Franz Schuster, der Begriff von j-Projektionenkörpern eingeführt. Dieser verallgmeinert wichtige bestehende Konzepte und kann als duales Gegenstück zu dem von Spezialisten vielfach untersuchten Begriff von j-Schnittkörpern gesehen werden. Als Hauptresultat wird eine fourieranalytische Charakterisierung für j-Projektionenkörper bewiesen. Darüber hinaus wird gezeigt, dass es Zonoide gibt, welche nicht zur Klasse der j-Projektionenkörper gehören.

Abstract

This thesis is composed of two contributions to the theory of convex bodies (compact and convex sets). The space of convex bodies in $\mathbb{R}^n$ will be denoted by $\mathcal{K}^n$.

We begin with an investigation of Minkowski endomorphisms. A Minkowski endomorphism is a continuous, $SO(n)$-equivariant, and translation invariant map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ that is additive with respect to point-wise addition (Minkowski addition) of convex sets. In answer to a question that has been open for several years, we prove that all Minkowski endomorphisms are uniformly continuous. Furthermore, we show that there exist non monotone, even Minkowki endomorphisms and answer a few questions regarding the more general notion of Minkowski valuations.

In the second part, based on a joint paper with Franz Schuster, the concept of j-projection bodies will be introduced. This notion generalizes important concepts from convex geometry. Moreover, it can be seen as a dual version to the well-studied notion of j-intersection bodies. As the main result, a fourier-analytic characterization of j-projection bodies is established. In another interesting result we establish the existence of zonotopes that are not j-projection bodies.

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Contents

Chapter 1

Introduction

A convex body is a compact and convex set in $\mathbb{R}^n$. In other words, a convex body is a set K (in n-dimensional space) that possess a boundary (that is part of the body) such that line segments connecting two boundary points entirely lie inside of K. Convex bodies exhibit many desirable properties that do not hold for general compact sets in $\mathbb{R}^n$. This lends to both many applications and a beautiful and incredibly rich theory of convex bodies. This "Theory of Convex Bodies" is concerned with the *foundational investigation* of these natural geometric objects. Its origin dates back to Hermann Minkowski around 1900. In this thesis two contributions to this theory will be presented.

In Chapter 2 we will review necessary background from areas such as harmonic analysis, the theory of convex bodies and valuation theory to present our results. Harmonic analysis is a field of mathematics that deals with the problem of representing an arbitrary function as a superposition of simple base functions (that is the expansion of a function into a *Fourier series*). This has numerous applications inside and outside of mathematics and will be a crucial tool for our purposes. The concept of valuations on convex bodies arises as a straightforward generalization of measures. Valuation theory has seen a surge of contributions in the past two decades and by now is heavily intertwined with the theory of convex bodies.

In Chapter 3 we present our first contribution concerning results that deal with the notion of *Minkowski endomorphisms*. Let $\mathcal{K}^n$ henceforth denote the class of all convex bodies in $\mathbb{R}^n$. The class $\mathcal{K}^n$ is naturally endowed with set addition. This addition - more commonly denoted Minkowski addition in the context of convex bodies - is defined by

$$K + L = \{x + y : x \in K,\, y \in L\}, \quad K, L \in \mathcal{K}^n.$$

A natural and important problem is that of describing the endomorphisms of $\mathcal{K}^n$.

More precisely, we are interested in maps $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ that are Minkowski-additive, i.e. that satisfy

$$\Phi(K + L) = \Phi(K) + \Phi(L), \quad K, L \in \mathcal{K}^n. \tag{1.1}$$

Surprisingly little is known about such endomorphisms. Minkowski endomorphisms are such endomorphisms satisfying a few additional properties. Most importantly, it is assumed that they are *rigid motion equivariant*. This means that for any Minkowski endomorphism Φ and rigid motion $\bar{g}$ we have

$$\Phi(\bar{g}K) = \bar{g}\Phi(K)$$

for all $K \in \mathcal{K}^n$. Here a rigid motion is a transformation consisting of a translation and a rotation. In physical terms this means that it describes a movement of the body in space that does not change the shape of the body itself (therefore a rigid motion). The second more technical property is that of continuity in the Hausdorff metric (see chapter 2.2).

Minkowski endomorphisms were first investigated by Schneider in 1974 who gave a characterization of these endomorphisms in the plane. Later, a first step towards a better understanding in higher dimensions was taken by Kiderlen, who gave a general description of Minkowski endomorphisms and characterized the weakly-monotone (see chapter 3) Minkowski endomorphisms. Despite that several questions concerning the structure and properties of Minkowski endomorphisms remained open.

In this thesis we answer several of these open questions. Perhaps most importantly, we prove that *Minkowski endomorphisms are always uniformly continuous but not necessarily weakly-monotone*. This in turn also yields a considerably stronger form of Kiderlen's description of Minkowski endomorphisms. We also answer some other questions regarding Minkowski endomorphisms, in particular in connection with Minkowski valuations, building on research by Parapatits, Schuster and Wannerer.

In Chapter 4, which is based on a joint work with Franz Schuster (see [**21**]), we introduce the concept of j-projection bodies. If K and L are origin-symmetric convex bodies and $1 \leq j \leq n-1$, then K is called the j-projection body of L if and only if

$$\mathrm{vol}_j(K|E) = \mathrm{vol}_{n-j}(L|E^\perp) \tag{1.2}$$

for every j-dimensional linear subspace E of $\mathbb{R}^n$. Here $K|E$ denotes the orthogonal projection of K onto E.

In the theory of convex bodies there seems to be a duality between central sections and projections. This mysterious duality, to date, is not fully understood. However, j-projection bodies are in part motivated by the dual notion of

j-intersections bodies. While the class of j-intersection bodies has been investigated by several authors, no systematic study of j-projection bodies, for general $1 \leq j \leq n-1$, has been undertaken to date.

As our main result we establish a Fourier-analytic characterization of condition (1.2), dual to an existing one for j-intersection bodies. This also implies other characterization using the theory of valuations. We then go on to discuss some properties and examples of j-projection bodies. An interesting example is given by the cube, which for every $1 \leq j \leq n-1$ is a j-projection body related to a dilate of itself. One interesting and not immediately obvious property is that the notion of j-projection body is an equi-affine one. This essentially means that it does not depend on the choice of coordinate system. Finally, we end our investigation by providing negative answers to a couple of questions and thereby highlighting a remarkable discontinuity in the otherwise strong analogy between j-intersection and j-projection bodies.

Chapter 2

Background

2.1 Harmonic Analysis on the Sphere and Grassmann Manifolds

In this section we will review important definitions and results from analysis on the sphere and Grassmann manifolds. In particular we will remind the reader of the notions of convolution and multiplier transform on those spaces. All measures in this chapter and the following chapters are signed finite Borel measures.

2.1.1 Analysis on Homogeneous Spaces

In the following, we will denote the unit ball and sphere in $\mathbb{R}^n$ by B^n and $\mathbb{S}^{n-1}$, respectively. The volume of B^n and the surface area of $\mathbb{S}^{n-1}$ are respectively given by

$$\kappa_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}, \quad \omega_n = n\kappa_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

By $\mathrm{Gr}_{j,n}$ we denote the Grassmannian of all j-dimensional linear subspaces of $\mathbb{R}^n$. Both the sphere and Grassmann manifolds examples of homogeneous spaces. Let $SO(n)$ denote the group of roations in $\mathbb{R}^n$. For the remainder of this thesis we fix a pole $\bar{e}$ on the sphere. This lets us identify $SO(n-1)$ with the unique subgroup of $SO(n)$, that is isomorphic to $SO(n-1)$ having $\bar{e}$ as its stabilizer. In a similar way, we can fix a j-dimensional subspace E_j, and identify $S(O(j)\times O(n-j))$ with the unique isomorphic subgroup of $SO(n)$ having E_j as its stabilizer. Throughout this thesis let us fix a flag of such subspaces

$$\operatorname{span}\{\bar{e}\} = E_1 \subset E_2 \subset \cdots \subset E_{n-1}\ .$$

The representation of the sphere and Grassmann manifolds as homogeneous spaces is now realized as

$$\mathbb{S}^{n-1} \cong \frac{SO(n)}{SO(n-1)}, \quad \mathrm{Gr}_{j,n} \cong \frac{SO(n)}{S(O(j) \times O(n-j))} \,.$$

Let M be either the sphere $\mathbb{S}^{n-1}$, any Grassmann manifold or $SO(n)$. By

$$C^\infty(M),\, C(M),\, \mathrm{L}^2(M),\, \mathrm{L}^1(M),\, \mathcal{M}(M)$$

we denote the space of smooth functions, continuous functions, integrable functions, square integrable functions and measures on M, respectively. The integral

$$\int_M f(u)\, du$$

is to be understood with respect to the Haar measure on M. The normalization is choosen in such a way that we get a probability measure on the respective space.

The space $C^{-\infty}(M)$ of distributions on M is defined as the dual space of $C^\infty(M)$ when endowed with the usual Frechet topology. Given a function in $f \in \mathrm{L}^1(M)$, we can identify it with the measure $f\,du$. On the other hand any measure $\mu \in \mathcal{M}(M)$ can be identified with the functional $l_\mu : C(M) \to \mathbb{R}$ given by

$$l_\mu(f) = \int_M f(u)\, d\mu(u) \,.$$

By Riesz representation theorem the space $\mathcal{M}(M)$ is isomorphic to the dual space of $C(M)$ via this identification. Finally, using these identifications and the fact that the dual space of $C(M)$ embeds into $C^{-\infty}(M)$ we obtain

$$C^\infty(M) \subset C(M) \subset \mathrm{L}^2(M) \subset \mathrm{L}^1(M) \subset \mathcal{M}(M) \subset C^{-\infty}(M).$$

Often it is convenient to use *cylindrical coordinates* with respect to the pole $\bar{e}$. For a j-dimensional subspace $E \subseteq \mathbb{R}^n$ let us denote $\mathbb{S}^{n-1} \cap E$ by $\mathbb{S}^{j-1}(E)$. If $f \in C(\mathbb{S}^{n-1})$ and $n \geq 2$, then

$$\int_{\mathbb{S}^{n-1}} f(u)\, du = \frac{1}{\omega_n} \int_{-1}^{1} \int_{\mathbb{S}^{n-2}(\bar{e}^\perp)} f\left(t\bar{e} + (1-t^2)^{\frac{1}{2}} v\right) dv \; (1-t^2)^{\frac{n-3}{2}}\, dt. \tag{2.1}$$

Two important integrals that one easily calculates using cylindrical coordinates are

$$\int_{\mathbb{S}^{n-1}} |\bar{e} \cdot u|\, du = \frac{2\omega_{n-1}}{(n-1)\omega_n}, \tag{2.2}$$

and

$$\int_{\mathbb{S}^{n-1}} |\bar{e}\cdot u|^2\, du = 2\omega_{n-1}\frac{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}{4\Gamma(\frac{n}{2}+2)\omega_n} = \frac{1}{n}. \tag{2.3}$$

The natural action of the group of rotations $SO(n)$ on $C^\infty(\mathbb{S}^{n-1})$ is given by

$$\theta f(u) = f(\theta^{-1}u), \qquad u \in \mathbb{S}^{n-1},\, \theta \in SO(n).$$

Via duality, this also defines the action on $C^{-\infty}(\mathbb{S}^{n-1})$. Recall that the stabilizer of $\bar{e}$ in $SO(n)$ is denoted by $SO(n-1)$. A distribution $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$ is called *zonal* if it is invariant under $SO(n-1)$. We denote the space of zonal distributions by $C^{-\infty}(\mathbb{S}^{n-1},\bar{e})$. The spaces of zonal functions and measures we denote in the same manner (e.g. $C(\mathbb{S}^{n-1},\bar{e})$). If $f \in C(\mathbb{S}^{n-1})$ is a zonal function, then its *associated function* $\tilde{f} \in C[-1,1]$ is defined by

$$\tilde{f}(t) = f(t\bar{e} + (1-t^2)^{\frac{1}{2}}v),$$

for some $v \in \bar{e}^\perp$. It is easy to check that this does not depend on the choice of $v \in \bar{e}^\perp$. Conversely, given $g \in C[-1,1]$ we obtain a zonal function $\breve{g} \in C(\mathbb{S}^{n-1})$ by

$$\breve{g}(u) = g(\bar{e}\cdot u).$$

Since this operation is inverse to the construction of the associated function, we see that there is a one-one correspondence of zonal functions on the sphere and their associated functions (see **[81]** for more information). One important thing to point out is that while we can associate a function on $[-1,1]$ to an element of $\mathrm{L}^1(\mathbb{S}^{n-1},\bar{e})$ the resulting function is not necessarily in $\mathrm{L}^1([-1,1])$.

For $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ we denote its *Radon decomposition* by $\mu = \mu_+ - \mu_-$. Then $\|\mu\|_{\mathrm{TV}} = \mu_+(\mathbb{S}^{n-1}) + \mu_-(\mathbb{S}^{n-1})$ is the total variation of μ. We also define

$$\mu_{\mathrm{ev}} = \frac{\mu+\mu^I}{2}, \qquad \mu_{\mathrm{odd}} = \frac{\mu-\mu^I}{2},$$

where $\mu^I(\omega) = \mu(-\omega)$ for every Borel set $\omega \subseteq \mathbb{S}^{n-1}$. Note that $\|\mu_{\mathrm{ev}}\|_{\mathrm{TV}}, \|\mu_{\mathrm{odd}}\|_{\mathrm{TV}} \le \|\mu\|_{\mathrm{TV}}$.

2.1.2 Representation Theory of SO(n) and Fourier Expansions

In a lot of applictions it is very useful to have an expansion of functions on a given space into a series of simple functions. For the circle S^1, this is achieved via the well known Fourier expansion of periodic functions. In that case, the system

$$\{1,\, \cos(k\arccos(u\cdot e)),\, \sin(k\arcsin(u\cdot e))\, :\, k \in \mathbb{N}\}$$

forms an orthogonal basis of $\mathrm{L}^2(\mathbb{S}^{n-1})$ and we obtain an expansion by projecting orthogonally on this basis. A similar approach works for functions on higher dimensional spheres and Grassmannians. The key problem consists in determining a suitable orthogonal basis. This issue can be tackled using representation theory of the compact Lie group $SO(n)$. We will briefly recap some parts of this theory. For more detailed information we refer to [**44**, **55**, **87**].

Since the Lie group $SO(n)$ is compact, all its irreducible representations are finite-dimensional. Moreover, the equivalence classes of irreducible complex representations of $SO(n)$ are uniquely determined by their highest weights (see, e.g., [**55**]) which, in turn, can be indexed by $\lfloor n/2 \rfloor$-tuples of integers $(\lambda_1, \lambda_2, \ldots, \lambda_{\lfloor n/2 \rfloor})$ such that

$$\begin{cases} \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\lfloor n/2 \rfloor} \geq 0 & \text{for odd } n, \\ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n/2-1} \geq |\lambda_{n/2}| & \text{for even } n. \end{cases} \tag{2.4}$$

An important notion for our purposes is that of spherical representations of $SO(n)$ with respect to $SO(n-1)$.

Definition. *Let H be a closed subgroup of $SO(n)$. A representation of $SO(n)$ on a vector space V is called* spherical *with respect to H if there exists an H-invariant non-zero $v \in V$, that is, $\vartheta v = v$ for every $\vartheta \in H$.*

The main result about spherical representations of a compact Lie group G concerns the left regular representation of G on the Hilbert space $\mathrm{L}^2(G/H)$ of square-integrable functions on the homogeneous space G/H (see, [**87**, p. 17]). However, we only require and state here the special case of this general result, where $G = SO(n)$ and $H = SO(n-1)$ and, consequently, the homogeneous space G/H is diffeomorphic to the sphere $\mathbb{S}^{n-1}$.

Theorem 2.1.1. *Every subrepresentation of the left regular representation of $SO(n)$ on $\mathrm{L}^2(\mathbb{S}^{n-1})$ is spherical with respect to $SO(n-1)$. Moreover, if V is an $SO(n)$ irreducible representation which is spherical with respect to $SO(n-1)$, then V is isomorphic to a subrepresentation of $\mathrm{L}^2(\mathbb{S}^{n-1})$ and $\dim V^{SO(n-1)} = 1$.*

These general theorems provide the basis for working out the details of Fourier expansions of functions on the sphere and Grassmannians. Since irreducible modules of $\mathrm{L}(\mathbb{S}^{n-1})$ and $\mathrm{L}(\mathrm{Gr}_{j,n})$ are pairwise orthogonal we will obtain an orthogonal basis of these spaces by determining the irreducible modules and orthogonal basis of these. Here and in the following, we denote by V^G the subspace of G-invariant vectors of a representation V of a group G. We will consider the cases of the sphere and Grassmann manifolds separately.

(a) The decomposition of $\mathrm{L}^2(\mathbb{S}^{n-1})$ into an *orthogonal* sum of $SO(n)$ irreducible subspaces is given by

$$\mathrm{L}^2(\mathbb{S}^{n-1}) = \bigoplus_{k\in\mathbb{N}} \mathcal{H}_k^n.$$

Here, $\mathcal{H}_k^n$ is the space of spherical harmonics of dimension n and degree k. The highest weights associated with the spaces $\mathcal{H}_k^n$ are the $\lfloor n/2 \rfloor$-tuples $(k, 0, \dots, 0)$, $k \in \mathbb{N}$, and, by Theorem 2.1.1, every irreducible representation of $SO(n)$ which is spherical with respect to $SO(n-1)$ is isomorphic to one of the spaces $\mathcal{H}_k^n$.

By Theorem 2.1.1, each space $\mathcal{H}_k^n$ contains a 1-dimensional subspace of zonal functions. This subspace is spanned by the function $u \mapsto P_k^n(u \cdot \bar{e})$, where $P_k^n \in C([-1,1])$ is the *Legendre polynomial* of dimension n and degree k. Letting $\pi_k : \mathrm{L}^2(\mathbb{S}^{n-1}) \to \mathcal{H}_k^n$ denote the orthogonal projection, we write

$$f \sim \sum_{k=0}^{\infty} \pi_k f \tag{2.5}$$

for the Fourier expansion of $f \in \mathrm{L}^2(\mathbb{S}^{n-1})$. Recall that the Fourier series in (2.5) converges to f in the L^2 norm and that

$$(\pi_k f)(v) = N(n,k) \int_{\mathbb{S}^{n-1}} f(u) P_k^n(u \cdot v)\, du \tag{2.6}$$

where $N(n,k) = \dim \mathcal{H}_k^n$.

Since the orthogonal projection $\pi_k : \mathrm{L}^2(\mathbb{S}^{n-1}) \to \mathcal{H}_k^n$ is self adjoint, it is consistent, by (2.6), to extend it to the space $C^{-\infty}(\mathbb{S}^{n-1})$ of distributions by

$$(\pi_k \nu)(v) = N(n,k)\, \nu\left(u \mapsto P_k^n(u \cdot v)\right).$$

It is not difficult to show that indeed $\pi_k \nu \in \mathcal{H}_k^n$ and that the Fourier expansion

$$\nu \sim \sum_{k=0}^{\infty} \pi_k \mu$$

uniquely determines the measure $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$.

(b) For $1 \le j \le n-1$, recall that

$$\mathrm{Gr}_{j,n} \cong SO(n)/\mathrm{S}(\mathrm{O}(j) \times \mathrm{O}(n-j)).$$

The space $\mathrm{L}^2(\mathrm{Gr}_{j,n})$ is a sum of orthogonal $SO(n)$ irreducible subspaces with corresponding highest weights $(\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ satisfying the following two conditions (see, e.g., [**55**, Theorem 8.49]):

$$\begin{cases} \lambda_k = 0 \text{ for all } k > \min\{j, n-j\}, \\ \lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor} \text{ are all even.} \end{cases} \tag{2.7}$$

Of particular importance for us is the subspace $\mathrm{L}^2(\mathrm{Gr}_{j,n})^{\mathrm{sph}}$ of *spherical functions* defined as the orthogonal sum of all $SO(n)$ irreducible subspaces in $\mathrm{L}^2(\mathrm{Gr}_{j,n})$ which are spherical with respect to $SO(n-1)$. By Theorem 2.1.1, (a), and (2.7),

$$\mathrm{L}^2(\mathrm{Gr}_{j,n})^{\mathrm{sph}} = \bigoplus_{k \in \mathbb{N}} \Gamma_{(2k,0,\ldots,0)},$$

where Γ_λ denotes the $SO(n)$ irreducible subspace of $\mathrm{L}^2(\mathrm{Gr}_{j,n})$ of highest weight $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$. Note that $\mathrm{L}^2(\mathrm{Gr}_{j,n})^{\mathrm{sph}}$ is isomorphic as $SO(n)$ representation to the subspace of *even* functions in $\mathrm{L}^2(\mathbb{S}^{n-1})$.

2.1.3 Multiplier Transforms and Convolutions

We now turn to convolution transforms of functions and measures on $SO(n)$ and the homogeneous spaces $\mathbb{S}^{n-1}$ and $\mathrm{Gr}_{j,n}$. Importantly these have the basic integral transforms we require, such as cosine and Radon transforms, as special cases.

The convolution $\nu * \tau$ of distributions ν, τ on $SO(n)$ is defined as the pushforward of the product distribution $\nu \otimes \tau$ by the group multiplication $m : SO(n) \times SO(n) \to SO(n)$, that is, $\nu * \tau = m_*(\nu \otimes \tau)$. For details see ([**50**,], p. 128). For the most part we will only require the notion of convolutions of measures. In this case, if $\mu, \sigma \in \mathcal{M}(SO(n))$, the previous definition is equivalent to

$$\int_{SO(n)} f(\vartheta)\, d(\mu * \sigma)(\vartheta) = \int_{SO(n)} \int_{SO(n)} f(\eta\theta)\, d\mu(\eta)\, d\sigma(\theta), \qquad f \in C(SO(n)).$$

For a measure μ on $SO(n)$, let $l_\vartheta \mu$ and $r_\vartheta \mu$ denote the pushforward of μ by the left and right translations by $\vartheta \in SO(n)$, respectively. We also often use $\vartheta\mu := l_\vartheta \mu$ for the left translation of μ. It follows from the definition of $\mu * \sigma$ that

$$(l_\vartheta \mu) * \sigma = l_\vartheta(\mu * \sigma) \qquad \text{and} \qquad \mu * (r_\vartheta \sigma) = r_\vartheta(\mu * \sigma) \tag{2.8}$$

for every $\vartheta \in G$. Moreover, the convolution of measures on $SO(n)$ is associative but in general not commutative. In fact, if μ, σ are measures on $SO(n)$, then

$$\widehat{\mu * \sigma} = \widehat{\sigma} * \widehat{\mu},$$

where $\widehat{\mu}$ denotes the pushforward of μ by the group inversion, that is,

$$\int_{SO(n)} f(\vartheta)\, d\widehat{\mu}(\vartheta) = \int_{SO(n)} f(\vartheta^{-1})\, d\mu(\vartheta), \qquad f \in C(SO(n)).$$

In order to define the convolution of measures on $\mathbb{S}^{n-1}$ and $\mathrm{Gr}_{j,n}$, we make use of the diffeomorphisms

$$\mathbb{S}^{n-1} = SO(n)/SO(n-1) \quad \text{and} \quad \mathrm{Gr}_{j,n} = SO(n)/\mathrm{S}(\mathrm{O}(j) \times \mathrm{O}(n-j)).$$

Indeed, if H is a closed subgroup of $SO(n)$, then there is a natural one-to-one correspondence between measures on $SO(n)/H$ and right H-invariant measures on $SO(n)$ (see, e.g., [**43**, **82**] for a detailed description). Using this identification, the convolution of measures on $SO(n)$ induces a convolution product of measures on $SO(n)/H$ as follows: Let $\pi : SO(n) \to SO(n)/H$ denote the canonical projection. The convolution of measures μ and σ on $SO(n)/H$ is defined by

$$\mu * \sigma = \pi_* m_*(\pi^*\mu \otimes \pi^*\sigma), \tag{2.9}$$

where π_* and π^* denote the pushforward and pullback by π, respectively. Note that, by (2.8), definition (2.9) is consistent with the identification of measures on $SO(n)/H$ with right H-invariant measures on $SO(n)$. In the same way, the convolution of measures on different homogeneous spaces can be defined: Let H_1, H_2 be two closed subgroups of $SO(n)$ and denote by $\pi_i : SO(n) \to SO(n)/H_i$, $i = 1, 2$, the respective projections. If, say, μ is a measure on $SO(n)/H_1$ and σ a measure on $SO(n)/H_2$, then

$$\mu * \sigma = \pi_{2*} m_*(\pi_1^*\mu \otimes \pi_2^*\sigma),$$

defines a measure on $SO(n)/H_2$.

Since the projection $\pi : SO(n) \to SO(n)/H$ is given by $\pi(\vartheta) = \vartheta\bar{E}$, where H is the stabilizer in G of $\bar{E} \in SO(n)/H$ (note that we write $\bar{e}$ instead of $\bar{E}$ when $H = SO(n-1)$), the convolution of a measure μ on $SO(n)$ with the Dirac measure $\delta_{\bar{E}}$ on $SO(n)/H$ yields

$$\mu * \delta_{\bar{E}} = \int_H r_\vartheta \mu \, d\vartheta \qquad \text{and} \qquad \delta_{\bar{E}} * \mu = \int_H l_\vartheta \mu \, d\vartheta. \tag{2.10}$$

Thus, $\delta_{\bar{E}} * \mu$ is left H-invariant, $\mu * \delta_{\bar{E}}$ is right H-invariant, and $\delta_{\bar{E}}$ is the unique rightneutral element for the convolution of measures on $SO(n)/H$. Generalizing the notion of zonal measures on $\mathbb{S}^{n-1}$, a left H-invariant measure on $SO(n)/H$ is called zonal. If μ and σ are measures on $SO(n)/H$, then, by (2.10),

$$\mu * \sigma = (\mu * \delta_{\bar{E}}) * \sigma = \mu * (\delta_{\bar{E}} * \sigma).$$

Consequently, for the convolution of measures on $SO(n)/H$, the right hand side measure can always assumed to be zonal.

Before we discuss important specific examples, we recall one more critical property of the convolution of measures on $\mathbb{S}^{n-1}$. Using the identification of a zonal measure μ on $\mathbb{S}^{n-1}$ with a measure on $[-1,1]$ and the Funk-Hecke Theorem, one can show (cf. [**81**]) that the Fourier expansion of $\sigma * \mu$ is given by

$$\sigma * \mu \sim \sum_{k=0}^{\infty} a_k^n[\mu]\, \pi_k \sigma, \tag{2.11}$$

where the numbers

$$a_k^n[\mu] = \omega_{n-1} \int_{-1}^{1} P_k^n(t)\,(1-t^2)^{\frac{n-3}{2}}\, d\mu(t)$$

are called the *multipliers* of the convolution transform $\sigma \mapsto \sigma * \mu$. Here, ω_{n-1} is the surface area of the $(n-1)$-dimensional Euclidean unit ball.

Example 2.1.2.

(a) Let $1 \leq j \leq n-1$ and let $|\cos(E,F)|$ denote the cosine of the angle between two subspaces $E, F \in \mathrm{Gr}_{j,n}$ (see, e.g., [**38**]). The *cosine transform* $\mathrm{C}_j\mu$ of a measure μ on $\mathrm{Gr}_{j,n}$ is the continuous function on $\mathrm{Gr}_{j,n}$ defined by

$$(\mathrm{C}_j\mu)(F) = \int_{\mathrm{Gr}_{j,n}} |\cos(E,F)|\, d\mu(E).$$

It is not difficult to show that

$$\mathrm{C}_j\mu = \mu * |\cos(\bar{E},\,\cdot\,)|, \tag{2.12}$$

where $\bar{E} \in \mathrm{Gr}_{j,n}$ again denotes the image of the identity under the projection $\pi : SO(n) \to \mathrm{Gr}_{j,n}$. In particular, the cosine transform is a linear and self-adjoint operator which is $SO(n)$ equivariant and maps smooth functions to smooth ones, that is,

$$\mathrm{C}_j : C^\infty(\mathrm{Gr}_{j,n}) \to C^\infty(\mathrm{Gr}_{j,n}).$$

Moreover, since $|\cos(E,F)| = |\cos(E^\perp, F^\perp)|$, we have

$$(\mathrm{C}_j\mu)^\perp = \mathrm{C}_{n-j}\mu^\perp \tag{2.13}$$

where $\mu^\perp := \perp_*\mu$ denotes the pushforward of μ by the orthogonal complement map $\perp : \mathrm{Gr}_{j,n} \to \mathrm{Gr}_{n-j,n}$.

The *spherical cosine transform* of $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ is defined via

$$(\mathrm{C}\mu)(u) = (\mathrm{C}\mu_{ev})(u) = \int_{\mathbb{S}^{n-1}} |v \cdot u| \, d\mu(v).$$

Upon identifying, even measures on the sphere with measures on $\mathrm{Gr}_{1,n}$ we can identify $\mathrm{C} \cong \mathrm{C}_1$.

It is a classical fact (see, e.g., [**44**, Chapter 3]) that the cosine transform C_1 is injective and, thus, by (2.13), so is C_{n-1}. Accordingly, the spherical cosine transform is injective on even measures. For $1 < j < n-1$, Goodey and Howard [**32**] first showed that the cosine transform C_j is *not* injective. A precise description of its kernel was given by Alesker and Bernstein [**8**]. However, Goodey and Zhang [**38**, Lemma 2.1] proved that the restriction of C_j to spherical functions in $\mathrm{L}^2(\mathrm{Gr}_{j,n})^{\mathrm{sph}}$ is injective and, moreover, when restricted to the subspace of smooth spherical functions

$$C^\infty(\mathrm{Gr}_{j,n})^{\mathrm{sph}} := \mathrm{cl}_{C^\infty} \bigoplus_{k \in \mathbb{N}} \Gamma_{(2k,0,\ldots,0)} \tag{2.14}$$

the cosine transform C_j is *bijective* for every $1 \leq i \leq n-1$. Here, cl_{C^∞} denotes the closure in the C^∞ topology.

(b) Let $1 \leq i \neq j \leq n-1$. For $F \in \mathrm{Gr}_{j,n}$, we write $\mathrm{Gr}_{i,n}^F$ for the submanifold of $\mathrm{Gr}_{i,n}$ which comprises of all $E \in \mathrm{Gr}_{i,n}$ that contain (respectively, are contained in) F. The *Radon transform* $\mathrm{R}_{i,j} : \mathrm{L}^2(\mathrm{Gr}_{i,n}) \to \mathrm{L}^2(\mathrm{Gr}_{j,n})$ is defined by

$$(\mathrm{R}_{i,j} f)(F) = \int_{\mathrm{Gr}_{i,n}^F} f(E) \, d\nu_i^F(E),$$

where ν_i^F is the unique invariant probability measure on $\mathrm{Gr}_{i,n}^F$. It is well known that, for $1 \leq i < j < k \leq n-1$, we have

$$\mathrm{R}_{i,k} = \mathrm{R}_{j,k} \circ \mathrm{R}_{i,j} \qquad \text{and} \qquad \mathrm{R}_{k,i} = \mathrm{R}_{j,i} \circ \mathrm{R}_{k,j}$$

and that $\mathrm{R}_{j,i}$ is the *adjoint* of $\mathrm{R}_{i,j}$. Using this latter fact, one can define the Radon transform of a measure μ on $\mathrm{Gr}_{i,n}$ by

$$\int_{\mathrm{Gr}_{j,n}} f(F) \, d(\mathrm{R}_{i,j}\mu)(F) = \int_{\mathrm{Gr}_{i,n}} (\mathrm{R}_{j,i} f)(E) \, d\mu(E), \qquad f \in C(\mathrm{Gr}_{j,n}).$$

Also the Radon transform intertwines the orthogonal complement map. More precisely,

$$(\mathrm{R}_{i,j}\mu)^\perp = \mathrm{R}_{n-i,n-j}\mu^\perp. \tag{2.15}$$

For $1 \leq i < j \leq n-1$ let $\lambda_{i,j}$ denote the probability measure on $\mathrm{Gr}_{j,n}$ which is uniformly concentrated on the submanifold

$$\{\vartheta \bar{E} \in \mathrm{Gr}_{j,n} : \vartheta \in \mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))\}.$$

It is not difficult to show (see, e.g., [**43**]) that for measures μ on $\mathrm{Gr}_{i,n}$ and ν on $\mathrm{Gr}_{j,n}$, we have

$$\mathrm{R}_{i,j}\mu = \mu * \lambda_{i,j} \qquad \text{and} \qquad \mathrm{R}_{j,i}\nu = \nu * \widehat{\lambda}_{i,j}. \tag{2.16}$$

In particular, the Radon transform is a linear $SO(n)$ equivariant operator which maps smooth functions to smooth ones, that is,

$$\mathrm{R}_{i,j} : C^\infty(\mathrm{Gr}_{i,n}) \to C^\infty(\mathrm{Gr}_{j,n}).$$

It follows from results of Grinberg [**42**] that if $1 \leq i < j \leq n-1$, then $\mathrm{R}_{i,j}$ is injective if and only if $i + j \leq n$, whereas if $i > j$, then $\mathrm{R}_{i,j}$ is injective if and only if $i + j \geq n$. Moreover, Goodey and Zhang [**38**] proved that for all $1 \leq i \neq j \leq n-1$ the restriction of the Radon transform $\mathrm{R}_{i,j}$ to spherical functions is injective and that

$$\mathrm{R}_{i,j} : C^\infty(\mathrm{Gr}_{i,n})^{\mathrm{sph}} \to C^\infty(\mathrm{Gr}_{j,n})^{\mathrm{sph}}$$

is a *bijection*.

(c) Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of complex valued, rapidly decreasing, infinitely differentiable *test functions* on $\mathbb{R}^n$ endowed with its standard topology (see, e.g., [**58**, Chapter 2.5]). We call a linear, continuous functional on $\mathcal{S}(\mathbb{R}^n)$ a *distribution* over $\mathcal{S}(\mathbb{R}^n)$. Note that any locally integrable function on $\mathbb{R}^n$ satisfying a power growth condition at infinity (cf. [**58**, p. 34]) determines a distribution acting by integration.

The *Fourier transform* $\mathrm{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is defined by

$$(\mathrm{F}\tau)(x) = \int_{\mathbb{R}^n} \tau(y) \exp(-i\, x \cdot y)\, dy.$$

It is well known that F is an $SO(n)$ equivariant (topological) isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Moreover, F is self-adjoint on $\mathcal{S}(\mathbb{R}^n)$. This motivates the definition of the Fourier transform $\mathrm{F}\nu$ of a distribution ν over $\mathcal{S}(\mathbb{R}^n)$ as the distribution acting by

$$\langle \mathrm{F}\nu, \tau \rangle = \langle \nu, \mathrm{F}\tau \rangle, \qquad \tau \in \mathcal{S}(\mathbb{R}^n).$$

A distribution ν over $\mathcal{S}(\mathbb{R}^n)$ is called *even homogeneous of degree* $p \in \mathbb{R}$ if

$$\langle \nu, \tau(\,\cdot\,/\lambda)\rangle = |\lambda|^{n+p}\langle \nu, \tau\rangle$$

for every $\tau \in \mathcal{S}(\mathbb{R}^n)$ and every $\lambda \in \mathbb{R}\backslash\{0\}$. For the rest of this article, we only consider even homogeneous distributions ν. Note that in this case,

$$\mathrm{F}^2\nu = (2\pi)^n\nu. \tag{2.17}$$

Moreover, Koldobsky [**58**, Lemma 2.21] observed the following crucial fact.

Lemma 2.1.3. *The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$.*

Now consider the space $C_e^\infty(\mathbb{S}^{n-1})$ of all real-valued *even* smooth functions on $\mathbb{S}^{n-1}$ endowed with its standard Fréchet space topology. For $p > -n$ and $f \in C_e^\infty(\mathbb{S}^{n-1})$, we denote by f_p the homogeneous extension of f of degree p to $\mathbb{R}^n\backslash\{0\}$, that is,

$$f_p(x) = \|x\|^p f\left(\frac{x}{\|x\|}\right), \qquad x \in \mathbb{R}^n\backslash\{0\}.$$

Since $p > -n$, f_p is locally integrable and determines an even homogeneous distribution of degree p acting on test functions by integration. Thus, by Lemma 2.1.3, $\mathrm{F}f_p$ is an even homogeneous distribution of degree $-n-p$. It was first noted in [**41**] that, for $-n < p < 0$, $\mathrm{F}f_p$ is, in fact, an infinitely differentiable function on $\mathbb{R}^n\backslash\{0\}$ (which is even and homogeneous of degree $-n-p$). This gives rise to an operator $\mathbf{F}_p$ on $C_e^\infty(\mathbb{S}^{n-1})$, called the *spherical Fourier transform of degree* $p \in (-n, 0)$, defined by

$$\mathbf{F}_p f = \mathrm{F}f_p|_{\mathbb{S}^{n-1}}, \qquad f \in C_e^\infty(\mathbb{S}^{n-1}).$$

Clearly, $\mathbf{F}_p$ is a linear and $SO(n)$ equivariant map. Hence, by Schur's lemma, $\mathbf{F}_p$ acts as a multiplier transformation on the spaces $\mathcal{H}_{2k}^n$, $k \in \mathbb{N}$. Its multipliers $a_{2k}^n[\mathbf{F}_p]$ were determined in [**41**] and are given by

$$a_{2k}^n[\mathbf{F}_p] = \pi^{n/2}2^{n+p}(-1)^k\frac{\Gamma\left(\frac{2k+n+p}{2}\right)}{\Gamma\left(\frac{2k-p}{2}\right)}. \tag{2.18}$$

We will give a convolution representation of $\mathbf{F}_p$ at the end of the next section. For now, we just note that (2.18) implies that for $p \in (-n, 0)$,

$$\mathbf{F}_p : C_e^\infty(\mathbb{S}^{n-1}) \to C_e^\infty(\mathbb{S}^{n-1})$$

is bijective and that, by Lemma 2.1.3, (2.17), and the definition of $\mathbf{F}_p$,

$$\mathbf{F}_{-n-p}(\mathbf{F}_p f) = (2\pi)^n f \tag{2.19}$$

holds for every $f \in C_e^\infty(\mathbb{S}^{n-1})$. Moreover, as a multiplier transformation $\mathbf{F}_p$ is self-adjoint and, hence, admits an extension to the space $C_e^{-\infty}(\mathbb{S}^{n-1})$ of continuous, linear functionals on $C_e^\infty(\mathbb{S}^{n-1})$ defined by

$$\langle \mathbf{F}_p \nu, f\rangle = \langle \nu, \mathbf{F}_p f\rangle$$

for $\nu \in C_e^{-\infty}(\mathbb{S}^{n-1})$ and $f \in C_e^\infty(\mathbb{S}^{n-1})$. By duality, $\mathbf{F}_p$ extends to even distributions and thus in particular to even measures.

Finally, we state a fundamental relation between the spherical Fourier transform and certain Radon transforms which was first observed by Koldobsky [**56**] (see also [**70**]). Here, κ_m is the m-dimensional volume of the Euclidean unit ball in $\mathbb{R}^m$.

Proposition 2.1.4. *Suppose that* $1 \leq j \leq n-1$. *Then*

$$\mathrm{R}_{1,n-j} \circ \mathbf{F}_{-j} = \frac{(2\pi)^{n-j}\, j\, \kappa_j}{(n-j)\, \kappa_{n-j}} \perp_* \circ\, \mathrm{R}_{1,j}.$$

Note that here and in the following, we identify $C_e^\infty(\mathbb{S}^{n-1})$ with $C^\infty(\mathrm{Gr}_{1,n})$ and the transform $\mathrm{R}_{i,i}$, $1 \leq i \leq n-1$, with the identity map.

2.2 Convex Bodies

2.2.1 Basic Definitions and Results

In this section we wil review fundamental facts and results from the theory of convex bodies. For a more detailed exposition confer [**78**]. We will assume throughout that $n \geq 3$. Recall that $\mathcal{K}^n$ denotes the set of convex bodies (compact and convex sets) in $\mathbb{R}^n$. Any body $K \in \mathcal{K}^n$ is uniquely determined by its support function $h(K, u) = h_K(u) = \max\{u \cdot x : x \in K\}$ for $u \in \mathbb{S}^{n-1}$. Minkowski addition is defined by

$$K + L = \{x + y : x \in K, y \in L\}$$

for any $K, L \in \mathcal{K}^n$. The reflection in the origin of a convex body $K \in \mathcal{K}^n$ is given by

$$-K = \{-x : x \in K\}.$$

A convex body is called *origin-symmetric* if $K = -K$. The Hausdorff distance $d_H(K, L)$ of two convex bodies $K, L \in \mathcal{K}^n$ is defined via

$$d_H(K, L) = \inf\{\epsilon \geq 0; K \subseteq L + \epsilon B^n \text{ and } L \subseteq K + \epsilon B^n\} .$$

We remind the reader that

$$d_H(K, L) = \|h_K - h_L\|$$

and

$$h_{K+L} = h_K + h_L.$$

A compact set L in $\mathbb{R}^n$ which is star-shaped with respect to the origin is uniquely determined by its radial function $\rho(L, u) = \rho_L(u) = \max\{\lambda \geq 0 : \lambda u \in L\}$ for $u \in \mathbb{S}^{n-1}$. If $\rho(L, \cdot)$ is positive and continuous, we call L a star body. If a convex body $K \in \mathcal{K}^n$ contains the origin in its interior, then

$$\rho_{K^*}(\cdot) = h_K(\cdot)^{-1} \qquad \text{and} \qquad h_{K^*}(\cdot) = \rho_K(\cdot)^{-1}, \tag{2.20}$$

where $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$ is the polar body of K.

A classical result of Minkowski states that the volume of a Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$, where $K_1, \ldots, K_m \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$, can be expressed as a homogeneous polynomial of degree n, that is,

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{1 \leq j_1, \ldots, j_n \leq m} V(K_{j_1}, \ldots, K_{j_n}) \lambda_{j_1} \cdots \lambda_{j_n}.$$

The symmetric coefficients $V(K_{j_1}, \ldots, K_{j_n})$ are called the *mixed volumes* of $K_{j_1}, \ldots, K_{j_n}$. For $K, L \in \mathcal{K}^n$ and $0 \leq j \leq n$, we denote the mixed volume with j copies of K and $n-j$ copies of L by $V(K[j], L[n-j])$ and we write $V_j(K)$ for the jth *intrinsic volume* of K defined by

$$\kappa_{n-j} V_j(K) = \binom{n}{j} V(K[j], B[n-j]).$$

Let us now consider more specifically convex bodies $K \in \mathcal{K}^n$ with non-empty interior and support function $h_K \in C^2(\mathbb{S}^{n-1})$. For a pair of orthogonal vectors u and v of unit length, the radii of curvature of such a K at u in direction v is given by

$$r_K(u, v) = \frac{\partial^2}{\partial v^2} (h_K)_1 (u).$$

Here f_1 denotes the 1-homogeneous extension of a function $f \in C(\mathbb{S}^{n-1})$ to $\mathbb{R}^n \backslash \{0\}$. The radius $r_K(u, v)$ is precisely the radius of the oscillating circle to $K|\mathrm{span}\{u, v\}$ at the point $u \in \mathrm{span}\{u, v\}$. We denote the class of convex bodies with support function of class C^2 and everywhere positive radii of curvature by $\mathcal{K}^2_+$. A function $h \in C^2(\mathbb{S}^{n-1})$ is the support function of a convex body $K \in \mathcal{K}^2_+$ if and only if

$$\frac{\partial^2}{\partial v^2} (h)_1 (u) > 0 \tag{2.21}$$

for all pairs of orthogonal vectors u and v (cf. [**78**, Chapter 2.5]). The eigenvalues of the Hessian $\nabla^2(h_K)_1(u)$ are the radii of curvature in the principal directions, that is, the principle radii of curvature. For $1 \leq j \leq n-1$, these are denoted by $r_j(u)$.

The *j-th projection function* of a convex body $K \in \mathcal{K}^n$ is defined by

$$K \mapsto \mathrm{vol}(K|E), \quad E \in \mathrm{Gr}_{j,n}$$

where $K|E$ denotes the orthogonal projection of K onto E. Note that, for origin-symmetric K,

$$\mathrm{vol}_1(K|\mathrm{span}\{u\}) = 2h(K, u). \tag{2.22}$$

In view of Proposition 2.1.4 and (2.24), the following result about the injectivity of Radon transforms of projection functions is important for our purpose.

Proposition 2.2.1 ([**3**, **31**]). *Suppose that $1 \leq i, j \leq n-1$ and let $K, L \in \mathcal{K}^n$ be origin-symmetric and have non-empty interior. If $\mathrm{R}_{i,j}\mathrm{vol}_i(K|\cdot) = \mathrm{R}_{i,j}\mathrm{vol}_i(L|\cdot)$ on $\mathrm{Gr}_{j,n}$, then $K = L$.*

2.2.2 Area Measures

Recall that $\mathcal{H}^j$ denotes the j-dimensional Hausdorff measure. For any Borel set $\omega \subseteq \mathbb{S}^{n-1}$, the *surface area measure* of a convex body K is defined by

$$S_{n-1}(K,\omega) = \mathcal{H}^{n-1}\{x \in \partial K : N(K,x) \cap \omega \neq \emptyset\},$$

where $N(K,x)$ denotes the normal cone of K at the boundary point x. For every $r > 0$, the surface area measure satisfies the Steiner type formula

$$S_{n-1}(K + rB^n, \cdot) = \sum_{j=0}^{n-1} r^{n-1-j} \binom{n-1}{j} S_j(K,\cdot).$$

The measure $S_j(K,\cdot)$, $1 \leq j \leq n-1$ is called the *area measure of order* j of K. It is uniquely determined by the property that

$$V(K[j], B[n-1-j], L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(L,u)\, dS_j(K,u) \tag{2.23}$$

for all $L \in \mathcal{K}^n$. If $K \in \mathcal{K}^n$ has non-empty interior, then, by a theorem of Aleksandrov-Fenchel-Jessen (see, e.g., [**78**, p. 449]), each of the measures $S_j(K,\cdot)$, $1 \leq j \leq n-1$, determines K up to translations. In particular, if K is origin-symmetric, then $S_j(K,\cdot)$ is an even measure on $\mathbb{S}^{n-1}$ and, thus, can be identified with a measure on $\mathrm{Gr}_{1,n}$ of the same total mass. Using this identification, the important *Cauchy-Kubota formula* can be stated as follows: For every $1 \leq j \leq n-1$ and origin-symmetric $K \in \mathcal{K}^n$,

$$(\perp_* \circ \mathrm{R}_{j,n-1})\mathrm{vol}_j(K|\,\cdot\,) = \frac{\kappa_j}{2\kappa_{n-1}}\, \mathrm{C}_1 S_j(K,\cdot) = \frac{\kappa_j}{\kappa_{n-1}}\, V_j(K|\,\cdot\,)^{\perp}. \tag{2.24}$$

For a body $K \in \mathcal{K}^2_+$ the area measure of order $1 \leq j \leq n-1$ is absolutely continuous with respect to the spherical Lebesgue measure. Its continuous density is given by the j-th normalized elementary symmetric function of the principal radii of curvature:

$$s_j(K,\cdot) = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} r_{i_1} \cdots r_{i_j}.$$

Another class of convex bodies for which there is a nice description of their area measures are polytopes. Let P be a polytope. Then

$$S_j(P,\cdot) = \binom{n-1}{j}^{-1} \sum_{F \in \mathcal{F}^j(P)} \mathcal{H}^{n-1-j}(N(P,F) \cap \omega)\, \mathrm{vol}_j(F), \tag{2.25}$$

where $N(P,F)$ denotes the set of all unit normal vectors of F in P and $\mathcal{F}^j(P)$ is the set of all j-dimensional faces of P. Since $N(P,F)$ lies in an $n-1-j$ dimensional great sphere, it follows that $S_j(P,\cdot)$ is concentrated on the union of finitely many such subspheres. The following converse of this observation was obtained by Goodey and Schneider.

Proposition 2.2.2 ([33]). *Suppose that $1 \leq j \leq n-1$ and let $K \in \mathcal{K}^n$ with $\dim K \geq j+1$. If the support of the area measure $S_j(K,\cdot)$ can be covered by finitely many $n-1-j$ dimensional great spheres, then K is a polytope.*

The center of mass (centroid) of every area measure of a convex body is at the origin, that is, for every $1 \leq j \leq n-1$ and $K \in \mathcal{K}^n$, we have

$$\int_{\mathbb{S}^{n-1}} u \, dS_j(K,u) = o.$$

The set $\mathcal{S}_j = \{S_j(K,\cdot) : K \in \mathcal{K}^n\}$ is dense in the set $\mathcal{M}_o^+(\mathbb{S}^{n-1})$ of all non-negative Borel measures on $\mathbb{S}^{n-1}$ with centroid at the origin if and only if $j = n-1$. However, $\mathcal{S}_j - \mathcal{S}_j$, $1 \leq j \leq n-1$, is dense in the space $\mathcal{M}_o(\mathbb{S}^{n-1})$ of *signed* Borel measures on $\mathbb{S}^{n-1}$ with centroid at the origin (cf. [**78**, p. 477]).

The general Christoffel-Minkowski problem asks for necessary and sufficient conditions for a Borel measure on $\mathbb{S}^{n-1}$ to be the j-th area measure of a convex body. The answer to the special case $j = n-1$, known as Minkowski's existence theorem, is one of the fundamental theorems in the Brunn-Minkowski theory (see [**78**, Chapter 8.2]). It states that a non-negative measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ is the surface area measure of a convex body with non empty interior if and only if μ is not concentrated on a great subsphere and has its centroid at the origin. The analogue of Minkowski's problem for the first-order area measure is known as the *Christoffel problem.* In order to describe its solution by Berg [**12**], recall that, for $K \in \mathcal{K}^n$, the measure $S_1(K,\cdot)$ and the support function $h(K,\cdot)$ are related by a *linear* differential operator $\Box_n$ in the following way

$$S_1(K,\cdot) = h(K,\cdot) + \frac{1}{n-1}\Delta_{\mathbb{S}} h(K,\cdot) =: \Box_n h(K,\cdot). \tag{2.26}$$

Here, $\Delta_{\mathbb{S}}$ denotes the Laplacian on $\mathbb{S}^{n-1}$ and equation (2.26) has to be understood in the sense of distributions. Note that for convex bodies in $\mathcal{K}^2_+$ formula (2.26) readily follows from the description of the density of $S_1(K,\cdot)$ via principle radii of curvature. Since $\Delta_{\mathbb{S}} Y_k = -k(k+n-2)\, Y_k$ for every $Y_k \in \mathcal{H}^n_k$, the definition of $\Box_n$ implies that, for $f \in C^\infty(\mathbb{S}^{n-1})$, the spherical harmonic expansion of $\Box_n f$ is given by

$$\Box_n f \sim \sum_{k=0}^{\infty} \frac{(1-k)(k+n-1)}{n-1} \pi_k f. \tag{2.27}$$

In particular, the kernel of $\Box_n$ is the space $\mathcal{H}_1^n$ consisting of the restrictions of linear functions on $\mathbb{R}^n$ to $\mathbb{S}^{n-1}$. If we let

$$C_o^\infty(\mathbb{S}^{n-1}) := \{f \in C^\infty(\mathbb{S}^{n-1}) : \pi_1 f = 0\},$$

then $\Box_n : C_o^\infty(\mathbb{S}^{n-1}) \to C_o^\infty(\mathbb{S}^{n-1})$ is an $SO(n)$ equivariant isomorphism of topological vector spaces. An explicit expression for the inverse of $\Box_n$ was obtained by Berg [**12**]. He proved that for every $n \geq 2$ there exists a uniquely determined C^∞ function g_n on $(-1, 1)$ such that the associated zonal function $\breve{g}_n(u) = g_n(u \cdot \bar{e})$ is in $\mathrm{L}^1(\mathbb{S}^{n-1})$ and

$$a_1^n[g_n] = 0, \qquad a_k^n[g_n] = \frac{n-1}{(1-k)(k+n-1)}, \quad k \neq 1. \tag{2.28}$$

It follows from (2.27), (2.11), and (2.28) that for every $f \in C_\mathrm{o}^\infty(\mathbb{S}^{n-1})$,

$$f = (\Box_n f) * \breve{g}_n. \tag{2.29}$$

From (2.29), Berg concluded that a measure $\mu \in \mathcal{M}_o^+(\mathbb{S}^{n-1})$ is the first-order area measure of a convex body in $\mathbb{R}^n$ if and only if $\mu * \breve{g}_n$ is a support function.

At the end of this section, we need the following generalization of (2.29) that follows from a recent result of Goodey and Weil [**37**, Theorem 4.3]: For every $j \in \{2, \dots, n\}$, the convolution transform

$$\mathrm{T}_{g_j} : C_\mathrm{o}^\infty(\mathbb{S}^{n-1}) \to C_\mathrm{o}^\infty(\mathbb{S}^{n-1}), \quad f \mapsto f * \breve{g}_j,$$

is an isomorphism. Let $\Box_j : C_\mathrm{o}^\infty(\mathbb{S}^{n-1}) \to C_\mathrm{o}^\infty(\mathbb{S}^{n-1})$ denote its inverse.

The problem of finding necessary and sufficient conditions for a Borel measure on $\mathbb{S}^{n-1}$ to be an intermediate area measure of a convex body is known as the *Christoffel-Minkowski problem* and has only been partially resolved (see, e.g., [**78**, Chapter 8.4]). An in-depth analysis of the problem under additional regularity assumptions was carried out by Guan et al. [**45–47**]. The following corollary to one of their results [**45**, Theorem 1.3] is of particular interest to us.

Proposition 2.2.3. *Suppose that $1 \leq j \leq n-1$. If $K \in \mathcal{K}_s^n$ is a smooth origin-symmetric convex body, then $\rho(K, \cdot)^j \in C_e^\infty(\mathbb{S}^{n-1})$ is the density of the area measure of order j of a convex body $L \in \mathcal{K}_s^n$ with non-empty interior.*

Firey [**24**] gave the following solution of the Christoffel-Minkowski problem for sufficiently regular *convex bodies of revolution*. When considering such bodies, we will always assume that they are $SO(n-1)$ invariant, that is, their axes of revolution is the line spanned by $\bar{e} \in \mathbb{S}^{n-1}$.

Theorem 2.2.4 (**[26]**). *Suppose that $1 \leq j \leq n-1$. A continuous zonal function $s(\bar{e}\cdot\,.\,)$ on $\mathbb{S}^{n-1}$ is the density of a body of revolution $K \in \mathcal{K}^2_+$ if and only if s satisfies the following conditions:*

(i) $\int_t^1 \xi\, s(\xi)(1-\xi^2)^{\frac{n-3}{2}} d\xi > 0$ *for* $t \in (-1,1)$ *and vanishes for* $t=-1$;

(ii) $s(t)(1-t^2)^{\frac{n-1}{2}} > (n-1-j)\int_t^1 \xi\, s(\xi)(1-\xi^2)^{\frac{n-3}{2}} d\xi$ *for all* $t \in (-1,1)$.

Another result by Firey that we require, concerns a concentration property of area measures. For $0 < \alpha < \frac{\pi}{2}$ let C_α denote the spherical cap given by $C_\alpha = \{u \in \mathbb{S}^{n-1} : (\bar{e}\cdot u) \geq \cos\alpha\}$.

Theorem 2.2.5 (**[25]**). *Let $K \in \mathcal{K}^n$ and $1 \leq j \leq n-1$. Then there exists a constant $A > 0$ such that*

$$S_j(K, C_\alpha) \leq A\,\frac{\sin^{n-1-j}\alpha}{\cos\alpha}\,\|h_K\|_\infty^j.$$

2.2.3 Special Classes of Convex Bodies

In this section we will review the class of zonoids and generalization thereof and discuss a few related theorems.

Recall that a body $K \in \mathcal{K}^n$ is called a *zonotope* if it is a finite Minkowski sum of line segments. A *zonoid* then is the limit of zonotops in the Hausdorff metric. Let $\mathcal{Z}^n_s$ denote the class of *origin-symmetric zonoids* in $\mathbb{R}^n$. It is well known that a convex body K belongs to the class $\mathcal{Z}^n_s$ if and only if its support function can be represented in the form

$$h_K = \mathrm{C}\mu_K \tag{2.30}$$

for some uniquely determined non-negative $\mu_K \in \mathcal{M}_e(\mathbb{S}^{n-1})$.

This inspires the following generalizaton. An origin symmetric convex body $K \in \mathcal{K}^n$ is called a *generalized zonoid* if there exists an even measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$, called the *generating measure of the convex body K*, such that

$$h_K = C\mu.$$

The next theorem, due to Weil, characterizes the cone of continuous generating functions of generalized zonoids.

Theorem 2.2.6 ([**91**])**.** *An even function $\rho \in C(\mathbb{S}^{n-1})$ is the generating function of a convex body L if and only if*

$$\int_{\mathbb{S}^{n-2}(w^\perp)} (u \cdot \tilde{w})^2 \rho(u)\, du \geq 0, \tag{2.31}$$

for all $w \perp \tilde{w} \in \mathbb{S}^{n-1}$.

By introducing cylindrical coordinates we immediately get a characterization of generating functions of bodies of revolution. Let $\chi_{(a,b)}$ denote the indicator function of the interval (a, b).

Corollary 2.2.7. *Let $\rho \in C(\mathbb{S}^{n-1})$ be even and zonal. For $0 < \alpha, \beta \leq 1$ define*

$$\psi_{\alpha,\beta}(t) := \chi_{(-\alpha,\alpha)}(t) \left(1 - \frac{t^2}{\alpha^2}\right)^{\frac{n-4}{2}} \cdot \\ \left(\frac{t^2}{\alpha^2}\beta^2 + \frac{4\omega_{n-2}}{n-2}\frac{t}{\alpha}\beta\sqrt{\left(1 - \frac{t^2}{\alpha^2}\right)(1-\beta^2)} + \frac{\omega_{n-1}}{n-1}\left(1 - \frac{t^2}{\alpha^2}\right)(1-\beta^2)\right).$$

Then ρ is the generating function of a convex body L if and only if

$$\Psi_{\alpha,\beta}(\rho) := \int_{-\alpha}^{\alpha} \tilde{\rho}(t)\psi_{\alpha,\beta}(t)\, dt \geq 0, \tag{2.32}$$

for all $0 < \alpha, \beta < 1$.

Proof. Clearly, if (2.31) holds for all $w \neq \pm\bar{e}$ it holds in general by the continuity of ρ. Let therefore $w \neq \pm\bar{e}$. We introduce cylindrical coordinates on $\mathbb{S}^{n-2}(w^\perp)$ by fixing $\bar{e}_w := \frac{\bar{e}|w^\perp}{|\bar{e}|w^\perp|}$ as the pole. Furthermore let $\alpha = \bar{e}_w \cdot \bar{e}$ and let $\tilde{w}$ be decomposed as $\beta\bar{e}_w + \sqrt{1-\beta^2}\tilde{v}$ with $\tilde{v} \in \mathbb{S}^{n-3}(w^\perp \cap \bar{e}^\perp)$. Then the integral $\int_{\mathbb{S}^{n-2}(w^\perp)}(u \cdot \tilde{w})^2\rho(u)\, du$ can be rewritten as

$$\int_{-1}^{1}\int_{\mathbb{S}^{n-3}(w^\perp \cap \bar{e}^\perp)} t^2\beta^2 + 2t\beta\sqrt{(1-t^2)(1-\beta^2)}(v \cdot \tilde{v}) + (1-t^2)(1-\beta^2)(v \cdot \tilde{v})^2\, dv \\ \tilde{\rho}(\alpha t)\,(1-t^2)^{\frac{n-4}{2}}\, dt.$$

Using (2.2) and (2.3) this is further equal to

$$\int_{-1}^{1} \tilde{\rho}(\alpha t)\left(t^2\beta^2 + \frac{4\omega_{n-2}}{n-2}\, t\beta\sqrt{(1-t^2)(1-\beta^2)} + \frac{\omega_{n-1}}{n-1}(1-t^2)(1-\beta^2)\right)(1-t^2)^{\frac{n-4}{2}}\, dt.$$

Substituting $s = \alpha t$ completes the proof. □

For $1 \leq j \leq n-1$, Weil introduced in [**90**] the class $\mathcal{K}_s^n(j)$ consisting of all origin-symmetric convex bodies $K \in \mathcal{K}^n$ for which there exists a *non-negative* Borel measure $\varrho_j(K,\cdot)$ on $\mathrm{Gr}_{j,n}$ such that

$$\mathrm{vol}_j(K|\cdot) = \mathrm{C}_j\varrho_j(K,\cdot).$$

The classes $\mathcal{K}_s^n(j)$ were subsequently investigated by Goodey and Weil [**34**] and were recently shown to play an important role in the theory of valuations by Parapatits and Wannerer [**72**]. Note that by (2.22) and (2.30), we have $\mathcal{K}_s^n(1) = \mathcal{Z}_s^n$ and that, by (2.24), $\mathcal{K}_s^n(n-1)$ coincides with the space $\mathcal{K}_s^n$ of all origin-symmetric convex bodies in $\mathbb{R}^n$. Moreover, a result of Weil [**90**] (cf. [**78**, Theorem 5.3.5]) shows that $\mathcal{Z}_s^n \subseteq \mathcal{K}_s^n(j)$ for every $1 \leq j \leq n-1$.

2.3 Valuations on Convex Bodies

Let $\mathcal{A}$ be a semi-group. A valuation is a map $\Phi : \mathcal{K}^n \to \mathcal{A}$ that satisfies the valuation property

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}^n$. In the following sections we will review the theories of valuations with $\mathcal{A} = \mathbb{R}$ or $\mathbb{C}$ and $\mathcal{A} = \mathcal{K}^n$.

2.3.1 Scalar Valuations

Scalar valuations ($\mathcal{A} = \mathbb{R}$ or $\mathbb{C}$) were probably first considered in Dehn's solution of Hilbert's third problem. As a natural and important generalization of the notion of measure they have since then played a central role in convex and discrete geometry (see [**54**] and [**78**, Chapter 6]).

Let **Val** denote the vector space of continuous, translation invariant, scalar-valued valuations. The basic structural result about **Val** is McMullen's decomposition theorem (cf. [**78**, Theorem 6.3.1]):

$$\mathbf{Val} = \bigoplus_{0 \leq j \leq n} \left(\mathbf{Val}_j^+ \oplus \mathbf{Val}_j^-\right). \tag{2.33}$$

Here, $\mathbf{Val}_j^{\pm} \subseteq \mathbf{Val}$ denote the subspaces of *even/odd* valuations (homogeneous) of degree j. Using (2.33), it is not difficult to show that the space **Val** becomes a Banach space, when endowed with the norm $\|\phi\| = \sup\{|\phi(K)| : K \subseteq B\}$. The natural continuous action of the general linear group $\mathrm{GL}(n)$ on this Banach space is defined as follows: For $A \in \mathrm{GL}(n)$ and every $K \in \mathcal{K}^n$,

$$(A\phi)(K) = \phi(A^{-1}K), \qquad \phi \in \mathbf{Val}.$$

Building on McMullens's decomposition the general structure of **Val** is described by Alesker's Irreducibility Theorem [**4**].

Theorem 2.3.1. *The decomposition of* **Val** *into $GL(n)$-irreducible modules is given by*

$$\mathbf{Val} = \bigoplus_{0 \leq j \leq n} \left(\mathbf{Val}_j^+ \oplus \mathbf{Val}_j^- \right).$$

Suppose that $1 \leq j \leq n-1$ and let $\mathcal{M}(\mathrm{Gr}_{j,n})$ denote the space of *signed* Borel measures on $\mathrm{Gr}_{j,n}$. By the Irreducibility Theorem, the map $\mathrm{Cr}_j : \mathcal{M}(\mathrm{Gr}_{j,n}) \to \mathbf{Val}_j^+$, defined by

$$(\mathrm{Cr}_j\mu)(K) = \int_{\mathrm{Gr}_{j,n}} \mathrm{vol}_j(K|E)\, d\mu(E),$$

has dense image. This motivates the following notion.

Definition. *A measure $\mu \in \mathcal{M}(\mathrm{Gr}_{j,n})$, $1 \leq j \leq n-1$, is called a* Crofton measure *for the valuation $\phi \in \mathbf{Val}_j^+$ if $\mathrm{Cr}_j\mu = \phi$.*

In order to state a more precise description of valuations admitting a Crofton measure, we also need to recall the notion of smooth valuations.

Definition. *A valuation $\phi \in$* **Val** *is called smooth if the map $\mathrm{GL}(n) \to$* **Val***, defined by $A \mapsto A\phi$, is infinitely differentiable.*

The vector space $\mathbf{Val}^\infty$ of all smooth translation invariant valuations carries a natural Fréchet space topology (see, e.g., [**85**]) which is stronger then the Banach space topology on **Val**. Let $\mathbf{Val}_j^{\pm,\infty}$ denote the subspaces of smooth valuations in $\mathbf{Val}_j^\pm$. A basic fact from representation theory implies that the spaces of smooth valuations $\mathbf{Val}_j^{\pm,\infty}$ are $\mathrm{GL}(n)$ invariant dense subspaces of $\mathbf{Val}_j^\pm$.

Suppose that $1 \leq j \leq n-1$. The *Klain map*

$$\mathrm{Kl}_j : \mathbf{Val}_j^+ \to C(\mathrm{Gr}_{j,n}), \qquad \phi \mapsto \mathrm{Kl}_j\phi,$$

is defined as follows: For $\phi \in \mathbf{Val}_j^+$ and every $E \in \mathrm{Gr}_{j,n}$, consider the restriction $\phi|_E$ of ϕ to convex bodies in E. This is a continuous, translation invariant valuation of degree j in E. Therefore, a classical result of Hadwiger (see, e.g., [**78**, Theorem

6.4.8]) implies that $\phi|_E = (\mathrm{Kl}_j\phi)(E)\,\mathrm{vol}_j$, where $(\mathrm{Kl}_j\phi)(E)$ is a constant depending only on E. The continuous function $\mathrm{Kl}_j\phi \in C(\mathrm{Gr}_{j,n})$ defined in this way is called the *Klain function* of ϕ. It is not difficult to see that the map Kl_j is $SO(n)$ equivariant and that smooth valuations are mapped to smooth functions, that is, $\mathrm{Kl}_j : \mathbf{Val}_j^{+,\infty} \to C^\infty(\mathrm{Gr}_{j,n})$. Moreover, an important result of Klain [**53**] states that the Klain map Kl_j is injective for every $j \in \{1, \dots, n-1\}$.

Let us now consider the restriction of the Crofton map Cr_j, $1 \le j \le n-1$, to smooth functions. It is not difficult to see that $\mathrm{Cr}_j f \in \mathbf{Val}_j^{+,\infty}$ for every $f \in C^\infty(\mathrm{Gr}_{j,n})$. Moreover, the Klain function of $\mathrm{Cr}_j f$ is equal to the cosine transform $\mathrm{C}_j f$ of f, that is,

$$\mathrm{Kl}_j \circ \mathrm{Cr}_j = \mathrm{C}_j. \tag{2.34}$$

From this and the main result of [**8**], Alesker [**5**, p. 73] deduced the following:

Theorem 2.3.2 ([**5**, **8**]). *Let $1 \le j \le n-1$. The image of the Klain map $\mathrm{Kl}_j : \mathbf{Val}_j^{+,\infty} \to C^\infty(\mathrm{Gr}_{j,n})$ coincides with the image of the cosine transform $\mathrm{C}_j : C^\infty(\mathrm{Gr}_{j,n}) \to C^\infty(\mathrm{Gr}_{j,n})$. Moreover, every smooth valuation $\phi \in \mathbf{Val}_j^{+,\infty}$ admits a (not necessarily unique for $2 \le j \le n-2$) smooth Crofton measure.*

Next, we recall the definition of the Alesker-Fourier transform

$$\mathbb{F} : \mathbf{Val}_j^{+,\infty} \to \mathbf{Val}_{n-j}^{+,\infty}, \qquad 1 \le j \le n-1,$$

of even valuations (for the odd case, which is much more involved and will not be needed in this article, see [**7**]): If $\phi \in \mathbf{Val}_j^{+,\infty}$, then $\mathbb{F}\phi \in \mathbf{Val}_{n-j}^{+,\infty}$ is the valuation with Klain function given by

$$\mathrm{Kl}_{n-j}(\mathbb{F}\phi) = (\mathrm{Kl}_j\phi)^\perp. \tag{2.35}$$

By (2.13) and Theorem 2.3.2, the map $\mathbb{F}$ is a well defined $SO(n)$ equivariant involution. Moreover, (2.34) implies that if $\mu \in \mathcal{M}(\mathrm{Gr}_{j,n})$ is a (smooth) Crofton measure of ϕ, then $\mu^\perp \in \mathcal{M}(\mathrm{Gr}_{n-j,n})$ is a Crofton measure of $\mathbb{F}\phi$.

In order to define the notion of spherical valuations, let us first recall the decomposition of the subspaces $\mathbf{Val}_j$ and $\mathbf{Val}_j^\infty$ of j-homogeneous valuations into $SO(n)$ irreducible subspaces.

Theorem 2.3.3 ([**9**]). *For $0 \le j \le n$, the spaces $\mathbf{Val}_j$ and $\mathbf{Val}_j^\infty$ are multiplicity free under the action of $SO(n)$. Moreover, the highest weights of the $SO(n)$ irreducible subspaces in either of them are given by the tuples $(\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ satisfying (2.4) and the following additional conditions:*

(i) $\lambda_k = 0$ *for* $k > \min\{j, n-j\}$; *(ii)* $|\lambda_k| \ne 1$ *for* $1 \le k \le \lfloor \frac{n}{2} \rfloor$; *(iii)* $|\lambda_2| \le 2$.

The notion of spherical representations with respect to $SO(n-1)$ (compare Section 2.1.2) motivates the following.

Definition *For $0 \leq j \leq n$, the spaces $\mathbf{Val}_j^{\mathrm{sph}}$ and $\mathbf{Val}_j^{\infty,\mathrm{sph}}$ of translation invariant, continuous and smooth* spherical valuations *of degree j are defined as the closures (w.r.t. the respective topologies) of the direct sum of all $SO(n)$ irreducible subspaces in $\mathbf{Val}_j$ and $\mathbf{Val}_j^{\infty}$, respectively, which are spherical with respect to $SO(n-1)$.*

Theorems 2.3.3 and 2.1.1 imply that $\mathbf{Val}_j^{\mathrm{sph}}$ and $\mathbf{Val}_j^{\infty,\mathrm{sph}}$ are the closures of the direct sum of all $SO(n)$ irreducible subspaces in $\mathbf{Val}_j$ and $\mathbf{Val}_j^{\infty}$, respectively, with highest weights $(k,0,\ldots,0)$, $k \in \mathbb{N}$. In particular, by Theorem 2.3.3, we have

$$\mathbf{Val}_1^{(\infty)} = \mathbf{Val}_1^{(\infty),\mathrm{sph}} \qquad \text{and} \qquad \mathbf{Val}_{n-1}^{(\infty)} = \mathbf{Val}_{n-1}^{(\infty),\mathrm{sph}}$$

and, by Theorem 2.1.1 (b), every $SO(n-1)$-invariant valuation in $\mathbf{Val}_j$ or $\mathbf{Val}_j^{\infty}$, $0 \leq j \leq n$, is spherical. Moreover, the following alternative description of smooth spherical valuations was established in [**85**].

Proposition 2.3.4. *For $1 \leq j \leq n-1$, the map*

$$\mathrm{E}_j : C_o^{\infty}(\mathbb{S}^{n-1}) \to \mathbf{Val}_j^{\infty,\mathrm{sph}}, \quad (\mathrm{E}_j f)(K) = \int_{\mathbb{S}^{n-1}} f(u)\, dS_j(K,u), \tag{2.36}$$

is an $SO(n)$ equivariant isomorphism of topological vector spaces.

Proposition 2.3.4 and a recent result of Bernig and Hug [**16**, Lemma 4.8] now imply the following relation between the Alesker-Fourier transform of *even* spherical valuations and certain Radon transforms of spherical functions.

Proposition 2.3.5. *Suppose that $1 \leq j \leq n-1$. If the* even *spherical valuation $\phi \in \mathbf{Val}_j^{\infty,\mathrm{sph}}$ is given by $\phi = \mathrm{E}_j f$ for $f \in C_e^{\infty}(\mathbb{S}^{n-1})$, then $\mathbb{F}\phi \in \mathbf{Val}_{n-j}^{\infty,\mathrm{sph}}$ and, for every $K \in \mathcal{K}^n$,*

$$(\mathbb{F}\phi)(K) = \frac{\kappa_{n-j}}{\kappa_j} \int_{\mathbb{S}^{n-1}} (\mathrm{R}_{1,j}^{-1} \circ \perp_* \circ \mathrm{R}_{1,n-j}) f(u)\, dS_{n-j}(K,u). \tag{2.37}$$

Note that the function under the integral in (2.37) is well defined, since $\mathrm{R}_{1,n-j} f \in C^{\infty}(\mathrm{Gr}_{n-j,n})^{\mathrm{sph}}$ for every $f \in C_e^{\infty}(\mathbb{S}^{n-1})$ and the Radon transform $\mathrm{R}_{1,j} : C^{\infty}(\mathrm{Gr}_{1,n}) \to C^{\infty}(\mathrm{Gr}_{j,n})^{\mathrm{sph}}$ is bijective (cf. Example 2.1.2 (b)).

Comparing Propositions 2.1.4 and 2.3.5, we obtain the following critical relation between the Alesker-Fourier transform of spherical valuations and the spherical Fourier transform.

Corollary 2.3.6. *Suppose that $1 \leq j \leq n-1$. If the* even *spherical valuation $\phi \in \mathbf{Val}_j^{\infty,\mathrm{sph}}$ is given by $\phi = \mathrm{E}_j f$ for $f \in C_e^{\infty}(\mathbb{S}^{n-1})$, then, for every $K \in \mathcal{K}^n$,*

$$(\mathbb{F}\phi)(K) = \frac{j}{(2\pi)^j (n-j)} \int_{\mathbb{S}^{n-1}} (\mathbf{F}_{j-n} f)(u)\, dS_{n-j}(K,u).$$

From the computation of the multipliers of the Alesker-Fourier transform of spherical valuations in [**16**] and the spherical Fourier transform in [**41**], it follows that Corollary 2.3.6 also holds without the assumption on the parity.

2.3.2 Minkowski Valuations

The name *Minkowski valuation* for valuations with values in $\mathcal{K}^n$ was first coined by Ludwig (see [**64**]). She started a line of research focusing on Minkowski valuations that intertwine the linear group (see [**1**, **2**, **49**, **63**, **64**, **66**, **83**, **88**]). In most cases, it has been proven that the Minkowski valuations under consideration could be characterized as conic combinations of fundamental and well known valuations such as the projection or difference body operators. However, for our purposes, we will concentrate on Minkowski valuations that are equivariant with respect to the group of rotations $SO(n)$. In particular we will recall results by Schuster and Wannerer and also talk about a few open questions that will be ansered in this thesis.

Let $\mathbf{MVal}$ denote the set of all continuous and translation invariant Minkowski valuations. For $1 \leq j \leq n-1$ let $\mathbf{MVal}_j$ denote the subspace of j-homogeneous valuations. The subspaces of $SO(n)$-equivariant valuations are denoted by $\mathbf{MVal}^{SO(n)}$ and $\mathbf{MVal}_j^{SO(n)}$, respectively. It is easy to see (cf. [**82**]) that if $\Phi_j \in \mathbf{MVal}_j^{SO(n)}$ then the $SO(n-1)$ invariant real valued valuation $\psi_{\Phi_j} \in \mathbf{Val}_j$, defined by

$$\psi_{\Phi_j}(K) = h(\Phi_j K, \bar{e}),$$

uniquely determines Φ_j and is called *the associated real valued valuation of* Φ_j. Motivated by this simple fact, the following definition was first given in [**82**].

Definition. *An $SO(n)$ equivariant Minkowski valuation $\Phi_j \in \mathbf{MVal}_j$, $0 \leq j \leq n$, is called* smooth *if its associated real valued valuation $\psi_{\Phi_j} \in \mathbf{Val}_j$ is smooth.*

It was recently proved in [**85**] that any $SO(n)$ equivariant Minkowski valuation in $\mathbf{MVal}_j$, $0 \leq j \leq n$, can be approximated by smooth ones.

First efforts to describe $\mathbf{MVal}_j^{SO(n)}$ for $j > 1$ go back to Schuster. Starting from a representation for the space of $(n-1)$-homogeneous valuations $\mathbf{MVal}_{n-1}^{SO(n)}$, he was later able to prove a representation result for even and smooth elements in $\mathbf{MVal}^{SO(n)}$ of arbitrary degree of homogeneity.

Theorem 2.3.7 ([**82**]). *Let $\Phi \in \mathbf{MVal}^{SO(n)}$ be even, smooth and homogeneous of degree $j \in \{1, \dots, n\}$. Then there exists a unique $(O(j) \times O(n-j))$-invariant and even function $f_\Phi \in C^\infty(\mathbb{S}^{n-1})$, called the Crofton function of Φ, such that*

$$h_{\Phi K} = \mathrm{vol}_j(K|\cdot) * f_\Phi,$$

for all $K \in \mathcal{K}^n$.

It is not to hard to see that

$$\mathrm{C}\, f = h_L, \quad L \in \mathcal{K}^n \tag{2.38}$$

is a necessary condition for a smooth function $f \in C^\infty(\mathbb{S}^{n-1})$ to be the Crofton function of a j-homogeneous, even and smooth $\Phi \in \mathbf{MVal}^{SO(n)}$ (see [**84**]). That is, the Crofton function f of Minkowski valuation has to be the density of the generating measure of a generalized zonoid L. The question whether (2.38) is a sufficient condition for a $(O(j) \times O(n-j))$-invariant function $f \in C^\infty(\mathbb{S}^{n-1})$ to be the Crofton function of a Minkowski valuation was raised by Schuster and Wannerer in [**84**]. It will be answered in Chapter 3.3.

More recently, Schuster and Wannerer were able to obtain a general Hadwiger type theorem for $\mathbf{MVal}^{SO(n)}$.

Theorem 2.3.8 ([**85**]). *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous Minkowski valuation which is translation invariant and $SO(n)$ equivariant, then there exist uniquely determined $c_0, c_n \geq 0$, $SO(n-1)$ invariant $\mu_j \in \mathcal{M}_o(\mathbb{S}^{n-1})$, for $1 \leq j \leq n-2$, and an $SO(n-1)$ invariant $f_{n-1} \in C(\mathbb{S}^{n-1})$ such that*

$$h_{\Phi K} = c_0 + \sum_{j=1}^{n-2} S_j(K, \cdot) * \mu_j + S_{n-1}(K, \cdot) * f_{n-1} + c_n \mathrm{vol}_n(K) \tag{2.39}$$

for every $K \in \mathcal{K}^n$.

Additionally, the authors remarked that in general the measures μ_j could not have densities in $\mathrm{L}^2(\mathbb{S}^{n-1})$. They, however, left it as an open problem whether the μ_j are absolutely continuous with a density in $\mathrm{L}^1(\mathbb{S}^{n-1})$. We are going to tackle this problem in chapter 3.3, giving a positive answer under the additional assumption of homogeneity.

One should point out that this is not sufficient to answer the question in general, since one would have to be able to decompose any element of $\mathbf{MVal}^{SO(n)}$ into a sum of homogenous Minkowski valuations. The problem of whether this is possible, was first raised by Schneider and Schuster (see [**79**], Section 5 and [**82**]). More generally, Parapatits and Schuster asked this question for Minkowski valuations that do not necessarily intertwine rotations (cf. [**?**]). Recently, Parapatits and Wanner proved that in this general setting such a decomposition is not possible (see [**72**]). However, the original problem of whether such a decomposition exists for $\mathbf{MVal}^{SO(n)}$ remained open. As our final result in Chapter 3.3, we will however establish that it is not possible.

Using associated real valued valuations we now extend the Alesker-Fourier transform (at least partially) to Minkowski valuations.

Definition *Let $\Phi_j \in \mathbf{MVal}_j$ and $\Psi_{n-j} \in \mathbf{MVal}_{n-j}$, $1 \leq j \leq n-1$, both be $SO(n)$ equivariant and even. We write $\Psi_{n-j} = \mathbb{F}\Phi_j$ and say Ψ_{n-j} is the Alesker-Fourier transform of Φ_j if*

$$\mathrm{Kl}_{n-j}(\psi_{\Psi_{n-j}}) = (\mathrm{Kl}_j \psi_{\Phi_j})^{\perp}.$$

Note that if Φ_j and Ψ_{n-j} are in addition smooth, then $\psi_{\Psi_{n-j}} = \mathbb{F}\psi_{\Phi_j}$ by (2.35). Moreover, in this case, by Theorems 2.3.7 and 2.3.8, both Φ_j and Ψ_{n-j} admit zonal generating functions $g_{\Phi_j}, g_{\Psi_{n-j}} \in C_e^{\infty}(\mathbb{S}^{n-1})$ *and* smooth spherical Crofton function $f_{\Phi_j}, f_{\Psi_{n-j}} \in C^{\infty}(\mathbb{S}^{n-1})$. Hence, from the definition of the convolution, it is not difficult to show (cf. [**84**, **85**]) that

$$\psi_{\Phi_j} = \mathrm{E}_j g_{\Phi_j} = \mathrm{Cr}_j \widehat{\sigma_{\Phi_j}} \qquad \text{and} \qquad \psi_{\Psi_{n-j}} = \mathrm{E}_{n-j} g_{\Psi_{n-j}} = \mathrm{Cr}_{n-j}\widehat{\sigma_{\Psi_{n-j}}},$$

where

$$\widehat{\sigma_{\Phi_j}} := \pi_{j*}\widehat{\pi^*\sigma_{\Phi_j}} \in \mathcal{M}(\mathrm{Gr}_{j,n})$$

and $\widehat{\sigma_{\Psi_{n-j}}} \in \mathcal{M}(\mathrm{Gr}_{n-j,n})$ is defined similarly. Here, $\pi : SO(n) \to \mathbb{S}^{n-1}$ and $\pi_j : SO(n) \to \mathrm{Gr}_{j,n}$ denote the canonical projections. Note that this is well defined by the invariance of σ_{Φ_j} and $\sigma_{\Psi_{n-j}}$. Therefore, letting

$$\sigma_{\Phi_j}^{\perp} := \pi_* \widehat{\pi_{n-j}^* \widehat{\sigma_{\Phi_j}}^{\perp}} \in \mathcal{M}_e(\mathbb{S}^{n-1}),$$

Corollary 2.3.6, the remark after (2.35), and a standard approximation argument imply the following.

Proposition 2.3.9. *Let $\Phi_j \in \mathbf{MVal}_j$, $1 \leq j \leq n-1$, be $SO(n)$ equivariant and even with generating measure $\mu_{\Phi_j} \in \mathcal{M}_e(\mathbb{S}^{n-1})$ and suppose that Φ_j admits a spherical Crofton measure $\sigma_{\Phi_j} \in \mathcal{M}_e(\mathbb{S}^{n-1})$. Then $\Psi_{n-j} \in \mathbf{MVal}_{n-j}$ is the Alesker-Fourier transform of Φ_j if and only if $\frac{j}{(2\pi)^j(n-j)}\mathbf{F}_{j-n}\mu_{\Phi_j}$ and $\sigma_{\Phi_j}^{\perp}$ are the generating measure and spherical Crofton measure of Ψ_{n-j}, respectively.*

The open problem whether the Alesker-Fourier transform of every smooth even Minkowski valuation which is $SO(n)$ equivariant and translation invariant is well defined (cf. [**84**]), will also be answered, in the negative, in Chapter 3.3. However, in the following example we exhibit for every $1 \leq j \leq n-1$ a pair of Minkowski valuations which are related via the Alesker-Fourier transform.

Example 2.3.10.

(a) For $1 \leq j \leq n-1$, let $\Pi_j \in \mathbf{MVal}_j$ denote the projection body map of order j, defined by

$$h(\Pi_j K, u) = V_j(K|u^\perp) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |u \cdot v| \, dS_j(K, v), \qquad u \in \mathbb{S}^{n-1}.$$

Note that each Π_j is $SO(n)$ equivariant and even but *not* smooth. Moreover, each Π_j is injective on origin-symmetric convex bodies with non-empty interior. Their continuous generating function is given by $g_{\Pi_j}(u) = \frac{1}{2}|\bar{e} \cdot u|$. It follows from (2.24) and (2.16) that

$$h(\Pi_j K, \cdot) = \frac{\kappa_{n-1}}{\kappa_j} \mathrm{R}_{n-j,1} \mathrm{vol}_j(K|\cdot)^\perp = \frac{\kappa_{n-1}}{\kappa_j} \mathrm{vol}_j(K|\cdot) * \widehat{\lambda}^\perp_{1,n-j}. \tag{2.40}$$

Thus, the measure $\frac{\kappa_{n-1}}{\kappa_j}\widehat{\lambda}^\perp_{1,n-j}$ is the spherical Crofton measure of Π_j.

(b) For $1 \leq j \leq n-1$, Goodey and Weil introduced and investigated in **[35–37]** the *normalized* mean section operator of order j, denoted by $\mathrm{M}_j \in \mathbf{MVal}_{n+1-j}$. In [**37**, Theorem 4.4] they proved that

$$h(\mathrm{M}_j K, \cdot) = q_{n,j}\, S_{n+1-j}(K, \cdot) * \breve{g}_j, \tag{2.41}$$

where

$$q_{n,j} = \frac{j-1}{2\pi(n+1-j)} \frac{\kappa_{j-1}\kappa_{j-2}\kappa_{n-j}}{\kappa_{j-3}\kappa_{n-2}}.$$

Hence, the multiple $q_{n,j}\breve{g}_j$ of the Berg function is the generating function of M_j. Note that M_j is continuous and $SO(n)$ equivariant but *not* even. Moreover, M_j determines a convex body with non-empty interior up to translations. For the even part of M_j, Goodey and Weil [**35**] proved that

$$h(\mathrm{M}_j^+ K, \cdot) = \frac{j\kappa_j\kappa_{n-1}}{2n\kappa_{j-1}\kappa_n} \mathrm{R}_{n+1-j,1} \mathrm{vol}_{n+1-j}(K|\cdot). \tag{2.42}$$

Thus, by (2.16), $\frac{j\kappa_j\kappa_{n-1}}{2n\kappa_{j-1}\kappa_n}\widehat{\lambda}_{1,n+1-j}$ is the spherical Crofton measure of M_j^+. Comparing this with Example 2.3.10 (a), it follows from Proposition 2.3.9 that the *renormalized even* mean section operator

$$\overline{\mathrm{M}}_{n-j} := \frac{2n\kappa_n}{(j+1)\kappa_{j+1}} \mathrm{M}^+_{j+1} \in \mathbf{MVal}_{n-j}$$

is the Alesker-Fourier transform of Π_j, that is,

$$\overline{\mathrm{M}}_{n-j} = \mathbb{F}\Pi_j. \tag{2.43}$$

Comparing generating functions of Π_j and $\overline{\mathrm{M}}_{n-j}$ and using Proposition 2.3.9 as well as (2.19), yields the following representation of the spherical Fourier transform.

Corollary 2.3.11. *Suppose that* $1 \leq j \leq n-1$. *Then*

$$\mathbf{F}_{-j} = \frac{(2\pi)^{n-j} j(j+1)\kappa_{j+1}}{4(n-j)n\kappa_n} q_{n,j+1}^{-1} \mathrm{C} \circ \square_{j+1}.$$

Chapter 3

Minkowski Endomorphisms

Abstract. Several open problems concerning Minkowski endomorphisms and Minkowski valuations are solved. More precisely, it is proved that all Minkowski endomorphisms are uniformly continuous but that there exist Minkowski endomorphisms that are not weakly-monotone. This answers questions posed repeatedly by Kiderlen [**52**], Schneider [**78**] and Schuster [**81**]. Furthermore, a recent representation result for Minkowski valuations by Schuster and Wannerer is improved under additional homogeneity assumptions. Also a question related to the structure of Minkowski endomorphisms by the same authors is answered. Finally, it is shown that there exists no McMullen decomposition in the class of continuous, even, $SO(n)$-equivariant and translation invariant Minkowski valuations extending a result by Parapatits and Wannerer [**72**]. The results of this chapter have been published in [**20**].

3.1 Introduction

Over the years, the investigation of structure preserving endomorphisms of $\mathcal{K}^n$ has attracted considerable attention (see e.g. [**11**, **52**, **74–76**, **81**]). In particular, in 1974, Schneider initiated a systematic study of continuous Minkowski-additive endomorphisms commuting with Euclidean motions. This class of endomorphisms, called Minkowski endomorphisms, turned out to be particularly interesting.

Definition. *A Minkowski endomorphism is a continuous, $SO(n)$-equivariant and translation invariant map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ satisfying*

$$\Phi(K + L) = \Phi(K) + \Phi(L), \qquad K, L \in \mathcal{K}^n.$$

Note that, in contrast to the original definition, we consider translation invariant instead of translation equivariant maps. However, it was pointed out by Kiderlen that these definitions lead to the same class of maps up to addition of the Steiner point map (see [**52**] for details). The main question of characterizing the (infinite dimensional) cone of Minkowski endomorphisms is a hard - yet interesting - one since it is intimately tied to the structure of $\mathcal{K}^n$. Schneider [**75**] established a complete classification in the case $n = 2$. Since then a number of authors contributed further results and generalizations (see [**52**, **81**, **82**, **84**, **85**]). In particular, Kiderlen obtained the following important convolution representation.

Theorem 3.1.1 ([**52**]). *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Minkowski endomorphism, then there exists a unique zonal distribution $\nu \in C_o^{-\infty}(\mathbb{S}^{n-1})$ of order at most 2, called the generating distribution of Φ, such that*

$$h_{\Phi K} = h_K * \nu \tag{3.1}$$

for every $K \in \mathcal{K}^n$. Moreover, Φ is uniformly continuous if and only if ν is a signed Borel measure.

While this theorem gives an explicit description of Minkowski endomorphisms, the important question of which distributions may occur as generating distributions remains open. In particular, it is not known whether all Minkowski endomorphisms are uniformly continuous. This was conjectured by several authors (see [**52**, **81**] and [**78**, Chapter 3.3]). With our first theorem, we confirm this conjecture in a slightly stronger form. By $w(K)$ we denote the mean with of $K \in \mathcal{K}^n$.

Theorem 3.1.2. *For every $n \geq 2$, there exists a constant $C_n \geq 0$ such that any Minkowski endomorphism $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is Lipschitz continuous with Lipschitz constant*

$$c_\Phi \leq C_n\, w(\Phi B^n).$$

As a consequence, we conclude that every Minkowski endomorphism is generated by a measure; providing a stronger form of Theorem 3.1.1.

An important class of endomorphisms that are completely characterized is that of weakly monotone Minkowski endomorphisms.

Definition. *A Minkowski endomporphism Φ is called* weakly monotone *if and only if it is monotone (w.r.t. set-inclusion) on the set of all convex bodies with Steiner point at the origin.*

The following theorem by Kiderlen completely characterizes weakly monotone Minkowski endomorphisms.

Theorem 3.1.3 ([**52**]). *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a Minkowski endomorphism. Then Φ is weakly monotone if and only if it is generated by a measure $\mu \in \mathcal{M}_o(\mathbb{S}^{n-1})$, that is the orthogonal projection of a non-negative measure $\nu \in \mathcal{M}(\mathbb{S}^{n-1})$ to $\mathcal{M}_o(\mathbb{S}^{n-1})$. Moreover, any such measure $\mu \in \mathcal{M}_o(\mathbb{S}^{n-1})$ generates a weakly-monotone Minkowski endomorphism.*

Interestingly, weakly monotone Minkowski endomorphisms are the only known examples of Minkowski endomorphisms so far. Also, from Schneiders characterization for $n = 2$ it follows that all endomorphisms are weakly monotone in that case. The natural question whether Minkowski endomorphisms are weakly monotone in general already implicitly appeared in [**52**]. Later it was stressed by Schneider and Schuster (see [**81**] and [**78**, Chapter 3.3]). A positive answer would clearly yield a complete characterization of Minkowski endomorphisms by Theorem 3.1.3. However, in this article we prove the following:

Theorem 3.1.4. *For every $n \geq 3$, there exist Minkowski endomorphisms $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ that are not weakly monotone.*

More recently, the investigations of Minkowski endomorphisms were extended to the cone $\mathbf{MVal}^{SO(n)}$ of continuous, translation-invariant and $SO(n)$-equivariant Minkowski valuations. These valuations directly generalize the notion of Minkowski endomorphisms. Indeed, by a result of Spiegel (see [**86**]), the cone $\mathbf{MVal}_1^{SO(n)}$ of 1-homogeneous elements in $\mathbf{MVal}^{SO(n)}$ is precisely the cone of Minkowski endomorphisms.

In the third section of this chapter we will discuss the open problems concerning Minkowkski valuations introduced in Chapter 2.4.

First of we will show the following theorem, that gives a negative answer to a question about Minkowski valuations by disproving the statment for Minkowski endomorphisms or equivalently 1-homogeneous Minkowski valuations.

Theorem 3.1.5. *For $n \geq 3$, there exists an origin symmetric strictly convex and smooth body of revolution $L \in \mathcal{K}^n$ such that its generating function is not a generating function of an even Minkowski endomorphism.*

The next theorem gives a simplified proof of Theorem 2.3.8 under additional homogeneity assumptions and also establish the conjectured extra regularity properties. This follows as a corollary from Theorem 3.1.2.

Corollary 3.1.6. *If $\Phi \in \mathbf{MVal}_j^{SO(n)}$, then there exists a zonal $f \in \mathrm{L}^1(\mathbb{S}^{n-1})$ such that*

$$h_{\Phi K} = S_j(K, \cdot) * f$$

for every $K \in \mathcal{K}^n$.

Finally, we also introduce a novel way to construct $SO(n)$-equivariant Minkowski valuations that, together with Theorem 3.1.5 and the result from [**72**], yields the following theorem.

Theorem 3.1.7. *If $n \geq 3$, then there exists a continuous, even, translation-invariant and $SO(n)$-equivariant Minkowski valuation $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ which cannot be decomposed into a sum of homogeneous Minkowski valuations.*

3.2 Minkowski Endomorphisms

In this section we will give the proofs of our main results regarding Minkowski endomorphisms. We recall, that by Theorem 3.1.1, Minkowski endomorphisms are uniquely determined by a zonal generating distribution (measure or function).

The following Lemma introduces a crucial necessary condition for generating measures of Minkowski endomorphisms.

Lemma 3.2.1. *Let $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ be the (zonal) generating measure of a Minkowski endomorphism. Then*

$$\int_{\mathbb{S}^{n-1}} s_1(K,u)\, d\mu(u) \geq 0, \tag{3.2}$$

for all $K \in \mathcal{K}^2_+$. Moreover, if μ is absolutely continuous with continuous density $g \in C(\mathbb{S}^{n-1})$, then

$$\int_{\mathbb{S}^{n-1}} g(u)\, dS_1(K,u) \geq 0, \tag{3.3}$$

for all $K \in \mathcal{K}^n$.

Proof. Let $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ be the generating measure of a Minkowski Endomorphism Φ and let $K \in \mathcal{K}^2_+$. Using (2.26) we compute

$$0 \leq s_1(\Phi(K), \bar{e}) = \Box_n(h_K * \mu)(\bar{e}) = s_1(K, \cdot) * \mu(\bar{e}) = \int_{\mathbb{S}^{n-1}} s_1(K,u)\, d\mu(u).$$

Now let μ have a continuous density $g \in C(\mathbb{S}^{n-1})$. Then

$$\int_{S^{n-1}} g(u)\, dS_1(K,u) \geq 0$$

for all $K \in \mathcal{K}^2_+$. Approximating an arbitrary $K \in \mathcal{K}^n$ by elements from $\mathcal{K}^2_+$ in the Hausdorff metric, we finally obtain

$$\int_{\mathbb{S}^{n-1}} g(u)\, dS_1(K,u) \geq 0,$$

for all $K \in \mathcal{K}^n$ by the weak convergence of area measures. $\Box$

Remark 3.2.2. It is not to hard to see that the conditions (3.2) and (3.3) are not sufficient for a measure to be the generating measure of a Minkowski endomorphism.

Before we give the proof of Theorem 3.1.2 recall that the for a cone C in some locally convex topological vectorspace X, its dual cone is defined by

$$C^* = \{f \in X^* : f(x) \geq 0,\ x \in C\}.$$

Moreover, it is a well-known consequence of the Hahn-Banach theorem that

$$C^{**} = \overline{C}, \tag{3.4}$$

when the second dual is taken with respect to the weak topology on the dual space.

Theorem 3.2.3. *For every $n \geq 2$, there exists a constant C_n such that any Minkowski endomorphism $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is Lipschitz continuous with Lipschitz constant*

$$c_\Phi \leq C_n\, w(\Phi B^n).$$

Proof. Let

$$\mathcal{S} := \{s_1(K,\cdot) : K \in \mathcal{K}^2_+,\ \ SO(n-1)\text{-invariant and}\, s(K) = 0.\} \subseteq C_o(\mathbb{S}^{n-1}, \bar{e}).$$

By Lemma 3.2.1, we know that any generating measure μ_Φ of a Minkowski endomorphism Φ satisfies $\mu_\phi \in \mathcal{S}^*$. We will therefore start by examining $\mathcal{S}^* \subseteq \mathcal{M}_o(\mathbb{S}^{n-1}, \bar{e})$ in more detail. For $-1 < \alpha, \beta < 1$, consider the zonal measures $\tau_\alpha \in \mathcal{M}(\mathbb{S}^{n-1}, \bar{e})$ and $\sigma_\beta \in \mathcal{M}(\mathbb{S}^{n-1}, \bar{e})$ given by

$$\int_{\mathbb{S}^{n-1}} f(u)\, d\tau_\alpha(u) = \int_\alpha^1 \tilde{f}(t)\, t(1-t^2)^{\frac{n-3}{2}}\, dt$$

and

$$\int_{\mathbb{S}^{n-1}} f(u)\, d\sigma_\beta(u) = (1-\beta^2)^{\frac{1}{2}} \tilde{f}(\beta) - (n-2) \int_{\mathbb{S}^{n-1}} f(u)\, d\tau_\beta(u),$$

for $f \in C(\mathbb{S}^{n-1}, \bar{e})$. Let now $C = \operatorname{cone}\{\tau_{\alpha,o}, \sigma_{\beta,o} : -1 < \alpha, \beta < 1\}$, where $\tau_{\alpha,o}$ and $\sigma_{\beta,o}$ denote the projections of τ_α and σ_β onto $\mathcal{M}_o(\mathbb{S}^{n-1}, \bar{e})$ respectively. Using Theorem 2.2.4 we immediatly see that

$$C \subseteq S^*. \tag{3.5}$$

We are now going to show that

$$C^* \subseteq \overline{S}. \tag{3.6}$$

Indeed, let $f \in C^*$ that is $\int_{\mathbb{S}^{n-1}} f(u)\, d\mu(u) \geq 0$ for all $\mu \in C$. For any μ that is a finite conic combination of the $\tau_{\alpha,o}$ and the $\sigma_{\beta,o}$ we then have

$$\int_{\mathbb{S}^{n-1}} (f+\epsilon)\,(u)\, d\mu(u) > 0,$$

for any $\epsilon > 0$ (using Theorem 2.2.4 and the fact that the constant function is the density of the first area measure of the unit ball). Using Theorem 2.2.4 again, we conclude that $f + \epsilon 1 \in S$ for every $\epsilon > 0$. Thus $f \in \overline{S}$. Now combining (3.5), (3.6) and applying (3.4) we obtain

$$S^* = C.$$

Using this, we are now going to show that for every $n \geq 2$ there exists a constant C_n, such that for $\mu \in S^*$

$$\|\mu\|_{\mathrm{TV}} \leq C_n\, \mu(\mathbb{S}^{n-1}). \tag{3.7}$$

Indeed, it is not hard to show that $\tau_{\alpha,o}$ satisfies the above equation for every $-1 < \alpha < 1$. Let $\beta \geq 0$, then $\|(\sigma_\beta)_+\|_{\mathrm{TV}} = (1-\beta^2)^{\frac{n-1}{2}}$ and

$$\|(\sigma_\beta)_-\|_{\mathrm{TV}} = (n-2) \int_\beta^1 t(1-t^2)^{\frac{n-3}{2}}\, dt = \frac{n-2}{n-1}\,(1-\beta^2)^{\frac{n-1}{2}}.$$

We see that (3.7) holds for all σ_β and thus also for $\sigma_{\beta,o}$. By the triangle inequality, it extends to all conic combinations of the $\tau_{\alpha,o}$ and $\sigma_{\beta,o}$. Let $(\mu_j)_{j\in\mathbb{N}}$ be a sequence of such conic combinations converging weakly to an arbitrary $\mu \in C$. Recall, that

$$\|\mu\|_{\mathrm{TV}} = \sup\{\int_{\mathbb{S}^{n-1}} f(u)\, d\mu(u) : f \in C(\mathbb{S}^{n-1}),\ \|f\| = 1\}.$$

Thus

$$\|\mu\|_{\mathrm{TV}} \leq \limsup_{j\to\infty} \|\mu_j\|_{\mathrm{TV}} \leq C_n\, \mu(\mathbb{S}^{n-1}).$$

Let now μ be the generating measure of a Minkowski endomorphism. Then μ satisfies (3.7) and for any $f \in C(\mathbb{S}^{n-1})$,

$$\|f * \mu\| \leq \|\mu\|_{\mathrm{TV}}\, \|f\| \leq C_n\, \mu(\mathbb{S}^{n-1})\, \|f\|.$$

Since any smooth Minkowski endomorphism has a (smooth) generating measure, we conclude that any smooth Φ is Lipschitz continuous with a Lipschitz constant $c_\Phi \leq C_n\, w(\Phi(B^n))$. Let now Φ be an arbitrary Minkowski endomorphism. Then there exists a sequence $(\Phi_j)_{j\in\mathbb{N}}$ of smooth Minkowski endomorphisms that converges to Φ uniformly on compact subsets of $\mathcal{K}^n$ (cf. [**85**, Corollary 5.4.]). Hence, for every $\epsilon > 0$, there exists $j \geq 0$ such that for any compact convex sets K, L we have

$$\begin{aligned}
\|h_{\Phi K} - h_{\Phi L}\| &\leq \|h_{\Phi K} - h_{\Phi_j K}\| + \|h_{\Phi_j K} - h_{\Phi_j L}\| + \|h_{\Phi L} - h_{\Phi_j L}\| \\
&\leq C_n\, w(\Phi_j(B^n))\, \|h_K - h_L\| + \|h_{\Phi_j K} - h_{\Phi K}\| + \|h_{\Phi L} - h_{\Phi_j L}\| \\
&\leq C_n\, w(\Phi(B^n))\, \|h_K - h_L\| + \epsilon.
\end{aligned}$$

We conclude that every Minkowski endomorphism Φ has a Lipschitz constant smaller or equal then $C_n\, V_1(\Phi(B^n))$. □

The first step in proving Theorem 3.2.5 is the following crucial Lemma.

Lemma 3.2.4. *For any $c, C > 0$, there exists a (monotone) Minkowski endomporhism with (non-negative) generating function $g \in C(\mathbb{S}^{n-1})$ such that $g(\bar{e}) \geq C$ but*

$$r_{\Phi K}(u, v) \leq c \, \|h_K\|$$

for all orthogonal pairs $u, v \in \mathbb{S}^{n-1}$ and all strictly convex and smooth bodies K.

Proof. Let $g \in C(\mathbb{S}^{n-1})$ be zonal and non-negative and let $g(\bar{e}) = C$ be its maximum. By Theorem 3.1.3, g is the generating function of a Minkowski endomorphism Φ. It remains to show that g can be chosen in such a way that we also obtain the desired bound on the radii of curvature. Therefore, note that since $r_{\Phi(\theta^{-1}K)}(u, v) = r_{\Phi K}(\theta u, \theta v)$ for $\theta \in SO(n)$ it suffices to bound $r_{\Phi K}(\bar{e}, \bar{t})$ for $\bar{t} \in \mathbb{S}^{n-1}$ orthogonal to the pole $\bar{e}$ and all strictly convex and smooth bodies K. For these $K \in \mathcal{K}^n$, we have

$$r_{\Phi K}(\bar{e}, \bar{t}) \leq (n-1) \, s_1(\Phi K, \bar{e}).$$

By (2.26) we further obtain

$$s_1(\Phi K, \bar{e}) = \square_n (h_K * g)(\bar{e}) = (S_1(K, \cdot) * g)\,(\bar{e}) = \int_{\mathbb{S}^{n-1}} g(u) \, dS_1(K, u).$$

Let us now moreover require that g is supported on the spherical cap C_α. Then by Theorem 2.2.5 (remember that the maximum of g was chosen to be C)

$$\begin{aligned} r_{\Phi K}(\bar{e}, \bar{t}) &\leq (n-1) \int_{\mathbb{S}^{n-1}} g(u) \, dS_1(K, u) \\ &\leq (n-1) C \, S_1(K, C_\alpha) \\ &\leq (n-1) AC \, \frac{\sin^{n-2} \alpha}{\cos \alpha} \, \|h_K\|. \end{aligned}$$

Choosing α small enough completes the proof. $\square$

The proof of Theorem 3.2.5 now easily follows.

Theorem 3.2.5. *For every $n \geq 3$, there exist non-monotone even Minkowski endomorphisms.*

Proof. We are going to construct the desired endomorphism as the difference of two monotone ones. Let the first endomorphism $\Phi_1 : \mathcal{K}^n \to \mathcal{K}^n$ be given by $\Phi_1(K) = w(K) \, B^n$. Its generating function is the constant 1 function. Observe that for all origin symmetric bodies K we have

$$w(K) = \|h_K\|_{\mathrm{L}^1} \geq \frac{2\omega_{n-1}}{n-1} \|h_K\|.$$

Clearly, segments satisfy the above inequality. Since a maximal subsegment I of an arbitrary origin symmetric $K \in \mathcal{K}^n$ satisfies $\|h_I\| = \|h_K\|$ but $\|h_I\|_{\mathrm{L}^1} \leq \|h_K\|_{\mathrm{L}^1}$, we see that the inequality holds in general. This now implies that

$$r_{\Phi_1 K}(u,v) \geq \frac{2\omega_{n-1}}{n-1}\|h_K\|$$

for all orthogonal pairs $u, v \in \mathbb{S}^{n-1}$. For the second endomorphism Φ_2 we take any even endomorphism from Lemma 3.2.4 with $C > 1$ and $c < \frac{2\omega_{n-1}}{n-1}$. Let g be its generating function. For all origin symmetric, strictly convex and smooth bodies K and orthogonal pairs $u, v \in \mathbb{S}^{n-1}$, we then have

$$\frac{\partial^2}{\partial v^2}\left(h_{\Phi_1 K} - h_{\Phi_2 K}\right)(u) = r_{\Phi_1 K}(u,v) - r_{\Phi_2 K}(u,v) > 0.$$

Hence, by (2.21),

$$h_{\Phi_1 K} - h_{\Phi_2 K} = h_K * (1-g)$$

is a support function for all origin symmetric, strictly convex and smooth bodies. Note that since $1-g$ is even we only need to show that origin symmetric bodies are mapped to convex bodies. Approximating an arbitrary origin symmetric $L \in \mathcal{K}^n$ by strictly convex and smooth bodies, we therefore see that $h_{\Phi K} := h_{\Phi_1 K} - h_{\Phi_2 K}$ defines a Minkowski endomorphism. Since its generating function $1-g$ attains a negative value at $\bar{e}$ it is not monotone by Theorem 3.1.3. $\square$

3.3 Minkowski Valuations

In this section we are going to prove several statments about Minkowski valuations. The first of these is stated as result about Minkowski endomorphisms.

Theorem 3.3.1. *There exists an origin symmetric strictly convex and smooth body of revolution $L \in \mathcal{K}^n$ such that its generating function ρ_L is not a generating function of an even Minkowski endomorphism.*

Proof. Let $C(\mathbb{S}^{n-1}, \bar{e}) \subseteq C(\mathbb{S}^{n-1})$ denote the subspace of zonal functions. Moreover, let $\mathcal{MG}^\infty \subseteq C^\infty(\mathbb{S}^{n-1}, \bar{e})$ denote the cone of smooth generating functions of Minkowski endomorphisms and $\mathcal{G}^\infty \subseteq C^\infty(\mathbb{S}^{n-1}, \bar{e})$ denote the cone of generating functions of smooth bodies of revolution. We want to show that

$$\mathcal{G}^\infty \not\subseteq \mathcal{MG}^\infty.$$

The respective cones are closed in $C^\infty(\mathbb{S}^{n-1}, \bar{e})$ since the cone of support functions is closed in $C(\mathbb{S}^{n-1})$. Indeed, let $g_j \in \mathcal{MG}^\infty$ and let $(g_j)_{j\in\mathbb{N}}$ converge to $g \in$

$C^\infty(\mathbb{S}^{n-1})$. Then $h_K * g_j$ is a sequence of support functions converging in the Hausdorff metric for every $K \in \mathcal{K}^n$. We conclude that $g \in \mathcal{MG}^\infty$. An analogous argument yields that $\mathcal{G}^\infty$ is closed. By (3.4), it therefore suffices to prove the relation

$$(\mathcal{MG}^\infty)^* \nsubseteq (\mathcal{G}^\infty)^*.$$

Since, by Lemma 3.2.1, we have that $S_1(K,\cdot) \in (\mathcal{MG}^\infty)^*$ for every body of revolution $K \in \mathcal{K}^n$, it indeed suffices to find a rotationally symmetric $K \in \mathcal{K}^n$ such that

$$S_1(K,\cdot) \notin (\mathcal{G}^\infty)^*.$$

We are going to show that the first area measure of the double cone defined by

$$D = \{s\bar{e} + tv : s^2 + t^2 \leq 1,\ v \in \mathbb{S}^{n-2}(\bar{e}^\perp)\}$$

has this property. It can be shown (cf. [**38**, Section 3]) that

$$\int_{\mathbb{S}^{n-1}} f(u)\, dS_1(D,u) = 2^{-\frac{n-5}{2}} \kappa_{n-1} \tilde{f}\left(\frac{1}{\sqrt{2}}\right) + (n-2) \int_0^{\frac{1}{\sqrt{2}}} \tilde{f}(t)(1-t^2)^{\frac{n-2}{2}}\, dt,$$

for any zonal $f \in C(\mathbb{S}^{n-1})$. From (3.4) and Theorem 2.2.7 it follows that

$$(\mathcal{G}^\infty)^* = \operatorname{cone}\{\Psi_{\alpha,\beta} : 0 < \alpha, \beta < 1\} \subseteq \left(C^\infty(\mathbb{S}^{n-1}, \bar{e})\right)^*,$$

where $\Psi_{\alpha,\beta}$ are the functionals defined in Theorem 2.2.7. Let $h_\epsilon \in C^\infty(\mathbb{S}^{n-1},\bar{e})$ be non-negative with $\tilde{h}_\epsilon\left(\frac{1}{\sqrt{2}}\right) = \|h_\epsilon\| = 1$ and let $\tilde{h}_\epsilon$ be supported on $\left[\frac{1}{\sqrt{2}} - \epsilon, \frac{1}{\sqrt{2}} + \epsilon\right]$. Then there exists a constant C such that

$$\begin{aligned}
\Psi_{\alpha,\beta}(h_\epsilon) &\leq \int_{\frac{1}{\sqrt{2}}-\epsilon}^{\frac{1}{\sqrt{2}}+\epsilon} \psi_{\alpha,\beta}(t)\, dt \\
&\leq C \int_{\frac{1}{\sqrt{2}}-\epsilon}^{\frac{1}{\sqrt{2}}+\epsilon} \left(1 - \frac{t^2}{\alpha^2}\right)^{-\frac{1}{2}} \chi_{(-\alpha,\alpha)}(t)\, dt. \\
&\leq C \int_{1-2\epsilon}^{1} (1-t^2)^{-\frac{1}{2}}\, dt.
\end{aligned}$$

We conclude that for every $\delta > 0$, there exists $\epsilon > 0$ such that $\Psi_{\alpha,\beta}(h_\epsilon) \leq \delta$ for all $0 < \alpha, \beta < 1$. However, since

$$\int_{\mathbb{S}^{n-1}} h_\epsilon(u)\, dS_1(D,u) \geq 2^{-\frac{n-5}{2}} \kappa_{n-1},$$

we obtain $S_1(D,\cdot) \notin (\mathcal{G}^\infty)^*$. $\square$

As a corollary we obtain that, in general, the Alesker-Fourier transform of a Minkowski valuation does not exist.

Corollary 3.3.2. *There exists an even $(n-1)$-homogeneous Minkowski valuation Φ such that there is no 1-homogeneous Minkowski valuation that is related to it via the Alesker-Fourier transform.*

Proof. It is easy to see that any support function h_L of an origin symmetric convex body $L \in \mathcal{K}^n$ defines an even Minkowski valuation via

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * h_L.$$

From Proposition (2.3.9) and Proposition (2.1.4) we know that, if it exists, its Alesker-Fourier transform would have to be given by

$$h(\mathbb{F}\Phi\, K, \cdot) = c\, S_1(K, \cdot) * \mathrm{R}^{\perp}_{1,n-1} h_L = c\, h_K * \mathrm{C}^{-1} h_L.$$

From Theorem 3.3.1 we however know that this is not possible in general. □

In this final section we will prove Corollary 3.1.6 and Theorem 3.1.7.

Lemma 3.3.3. *Let $f \in \mathrm{L}^1(\mathbb{S}^{n-1})$ be zonal and $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$. Then $\mu * f \in \mathrm{L}^1(\mathbb{S}^{n-1})$ and*

$$\mu * f\,(\theta\bar{e}) = \int_{\mathbb{S}^{n-1}} \theta f(u)\, d\mu(u) = \int_{\mathbb{S}^{n-1}} \tilde{f}(u \cdot \theta\bar{e})\, d\mu(u) \tag{3.8}$$

whenever the integral on the right-hand side exists (which is the case at almost every point).

Proof. Consider the operator $f \mapsto \mu * f$, defined on the space of continuous functions $C(\mathbb{S}^{n-1})$. It is not hard to show that

$$\int_{\mathbb{S}^{n-1}} \|\mu * f\|(u)\, du \leq \|\mu\|\, \|f\|_{\mathrm{L}^1}.$$

Thus, the convolution with μ is continuous on $C(\mathbb{S}^{n-1})$ in the L^1-norm. Let now $f \in \mathrm{L}^1(\mathbb{S}^{n-1})$ and $f_i \in C(\mathbb{S}^{n-1})$ such that $f_i \to f$ in the L^1-norm. Then $\mu * f_i$ converges in $\mathrm{L}^1(\mathbb{S}^{n-1})$ and, since the convergence in the L^1-norm also implies weak convergence, we have

$$\mu * f = \lim_{i\to\infty} \mu * f_i \in \mathrm{L}^1(\mathbb{S}^{n-1}).$$

Moreover, we know that $\mu * f_i$ converges point-wise for almost every $u \in \mathbb{S}^{n-1}$. Obviously the limit is given by the right-hand side of (3.8). □

Corollary 3.3.4. *Let* $\Phi \in \mathbf{MVal}_j^{SO(n)}$. *Then there exists a unique zonal* $f \in \mathrm{L}^1(\mathbb{S}^{n-1})$ *such that*

$$h_{\Phi K} = S_j(K, \cdot) * f$$

for every $K \in \mathcal{K}^n$. *Moreover, there exists a unique zonal measure* $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ *such that* $f = \mu * \breve{g}_n$.

Proof. From Theorem 2.3.8 it follows that $h_{\Phi K} = S_j(K, \cdot) * \nu$ for some measure $\nu \in \mathcal{M}(\mathbb{S}^{n-1})$. Let $\Lambda : \mathbf{MVal}^{SO(n)} \to \mathbf{MVal}^{SO(n)}$ denote the derivation operator (cf. [**84**]) defined by

$$h_{\Lambda\Phi(K)}(K) = \left.\frac{d}{dt}\right|_{t=0} h_{\Phi(K+tB^n)}.$$

It is then not too hard to show (see [**85**]), that, for $K \in \mathcal{K}^2_+$,

$$h_{\Lambda^{j-1}\Phi(K)} = S_1(K, \cdot) * \nu = h_K * \Box_n \nu.$$

However, since $\Lambda^{j-1}\Phi \in \mathbf{MVal}_1^{SO(n)}$, it follows from Theorem 3.1.2 that $\nu = \mu * g_n$ for some measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$. It remains to show that $\nu = f\,du$ with $f \in \mathrm{L}^1(\mathbb{S}^{n-1})$. This immediately follows from Lemma 3.3.3.

$\square$

The proof of Theorem 3.1.7 is based on the following proposition that introduces a new construction for $SO(n)$-equivariant Minkowski valuations.

Proposition 3.3.5. *Let* $\phi \in \mathbf{Val}$ *and* $L \in \mathcal{K}^n$ *and let*

$$\Psi_{L,\phi}K\,(u) := \int_{SO(n)} \phi(\theta^{-1}K)\, h(\theta L, u)\, d\theta, \tag{3.9}$$

for $u \in \mathbb{S}^{n-1}$. *Then*

(a) $\Psi_{L,\phi} : \mathcal{K}^n \to C(\mathbb{S}^{n-1})$ *is a continuous, translation-invariant and* $SO(n)$*-equivariant valuation.*

(b) Let $\phi \geq 0$. *Then* $h_{\Phi_{L,\phi}K} = \Psi_{L,\phi}K$ *defines a continuous, translation invariant and* $SO(n)$*-equivariant Minkowski valuation* $\Phi_{L,\phi} : \mathcal{K}^n \to \mathcal{K}^n$.

(c) Let $\phi \in \mathbf{Val}_1^+$ *be* $SO(n-1)$*-invariant and* L *be origin symmetric. Then*

$$\mathrm{C}\,(\Psi_{L,\phi}K) = h_K * h_L * \mathrm{Kl}_\phi. \tag{3.10}$$

Proof. For the $SO(n)$-equivariance of $\Psi_{L,\phi}$, let $\vartheta \in SO(n)$. Then

$$\Psi_{\vartheta L,\phi}K(u) = \int_{SO(n)} \phi\left(\theta^{-1}\vartheta K\right) h_{\theta L}(u)\, d\theta.$$

By substituting $\theta = \vartheta\eta$, the right-hand side is further equal to

$$\begin{aligned}\int_{SO(n)} \phi(\eta^{-1}K) h_{\vartheta\eta L}(u)\, d\eta &= \int_{SO(n)} \phi(\eta^{-1}K) h_{\eta L}(\vartheta^{-1}u)\, d\eta \\ &= \vartheta\left(\Psi_{L,\phi}K\right)(u).\end{aligned}$$

The other properties in (a) are obvious. Statement (b) immediately follows from the fact that the class of support functions is a closed convex cone in $C(\mathbb{S}^{n-1})$. For (c), let $\phi \in \mathbf{Val}_1^{+,\infty}$ be $SO(n-1)$-invariant. Then, since ϕ is smooth, there exists f_ϕ such that

$$\phi(\theta^{-1}K) = \int_{\mathbb{S}^{n-1}} h_{\theta^{-1}K}(u)\, f_\phi(u)\, du = h_K * f_\phi\,(\theta).$$

It follows that

$$\Psi_{L,\phi}K = h_K * f_\theta * h_L = h_K * h_L * f_\phi.$$

Since $\mathrm{C}f_\phi = \mathrm{Kl}_\phi$, we can finish the proof by approximation. $\square$

Theorem 3.3.6. *If $n \geq 3$, then there exists a continuous, even, translation-invariant and $SO(n)$-equivariant, Minkowski valuation $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ which cannot be decomposed into a sum of homogeneous Minkowski valuations.*

Proof. Let $\varphi_\epsilon \in \mathbf{Val}_1^+$ be $SO(n-1)$-invariant given by $\mathrm{Kl}\varphi_\epsilon = g_\epsilon$, where $g_\epsilon \in C^\infty(\mathbb{S}^{n-1})$ is even, non-negative, zonal and converges weakly to $\frac{1}{2}\left(\delta_{\bar{e}} + \delta_{-\bar{e}}\right)$. In [**72**, Lemma 5.1.], it was shown that there exist constants c_ϵ, d_ϵ such that $\phi_\epsilon := c_\epsilon + \varphi_\epsilon + d_\epsilon V_2$ is a positive valuation. Therefore, by Proposition 3.3.5 (b), we know that $h_{\Phi_{L,\phi_\epsilon}K} = \Psi_{L,\phi_\epsilon}K$ defines an $SO(n)$-equivariant Minkowski valuation Φ_{L,ϕ_ϵ} for all $L \in \mathcal{K}^n$. Clearly, the 1-homogeneous component of Ψ_{L,ϕ_ϵ} is given by $\Psi_{L,\varphi_\epsilon}$. Let us assume $\Psi_{L,\varphi_\epsilon}K$ is a support function for every $\epsilon > 0$ and $K, L \in \mathcal{K}^n$. By (3.10), we have

$$\mathrm{C}\left(\Psi_{L,\varphi_\epsilon}K\right) = h_K * h_L * \mathrm{Kl}\varphi_\epsilon.$$

Thus,

$$\mathrm{C}^{-1}h_K * h_L$$

has to be a support function for all convex bodies K and L. In particular, this implies that

$$h_K * \rho_L$$

is a support function for all $K \in \mathcal{K}^n$ and generalized zonoids $L \in \mathcal{K}^n$. Consequently, ρ_L would have to be the generating measure of a Minkowski endomorphism for every generalized zonoid $L \in \mathcal{K}^n$. By Theorem 3.3.1 this cannot be true. $\square$

Chapter 4

The Class of j-Projection Bodies

Abstract. Dual to Koldobsky's notion of j-intersection bodies, the class of j-projection bodies is introduced, generalizing Minkowski's classical notion of projection bodies of convex bodies. A Fourier analytic characterization of j-projection bodies in terms of their area measures of order j is obtained. In turn, this yields an equivalent characterization of j-projection bodies involving Alesker's Fourier type transform on translation invariant smooth spherical valuations. As applications of these results, several basic properties of j-projection bodies are established and new non-trivial examples are constructed. The results in this chapter are published in a joint work with F. Schuster (see [**21**]).

4.1 Introduction

The Busemann–Petty problem was one of the most famous problems in convex geometric analysis of the last century. It asks whether the volume of an origin-symmetric convex body K in $\mathbb{R}^n$ is smaller than that of another such body L, if all central hyperplane sections of K have smaller volume than those of L. (Here and throughout this chapter, it is assumed that $n \geq 3$.) After more than 40 years and a long list of contributions it was shown that the answer is affirmative if $n \leq 4$ and negative otherwise (see [**28**, **30**, **95**] and the references therein). The first crucial step in the final solution was taken by Lutwak [**67**] and later refined by Gardner [**27**] who showed that the answer to the Busemann–Petty problem is affirmative if and only if every origin-symmetric convex body in $\mathbb{R}^n$ is an intersection body. This class of bodies first appeared in Busemann's definition of area in Minkowski geometry and has attracted considerable attention in different subjects since the seminal paper by Lutwak (see, e.g., [**40**, **43**, **48**, **51**, **57**, **65**] and the books [**29**, **58**, **59**]).

Since its final solution, several variants of the original Busemann–Petty problem have been investigated, each of which being related to a certain generalization of the notion of intersection body in a similar way that Lutwak's intersection bodies are related to the Busemann–Petty problem (see [**58**, **59**]). Of particular interest in this paper is the following notion of j-intersection bodies introduced by Koldobsky in 1999.

Definition. *Let $1 \leq j \leq n-1$ and let D and M be origin-symmetric star bodies in $\mathbb{R}^n$. Then D is called the* j-intersection body *of M if*

$$\mathrm{vol}_j(D \cap E^{\perp}) = \mathrm{vol}_{n-j}(M \cap E)$$

for every $n-j$ dimensional subspace E of $\mathbb{R}^n$. The class of j-intersection bodies is the closure in the radial metric of all j-intersection bodies of star bodies.

When $j = 1$, the class of 1-intersection bodies coincides with the closure of Lutwak's intersection bodies. Also note that for $j > 1$, there may be star bodies M for which a corresponding j-intersection body does not exist (cf. Theorem 4.1.1 below). However, if D is a j-intersection body of M for some $1 \leq j \leq n-1$, then D is uniquely determined (see, e.g., [**29**, Corollary 7.2.7]).

Since their definition by Koldobsky, j-intersection bodies have become objects of intensive investigations due to their connections to certain problems from functional analysis ([**57**, **73**, **92**]), asymptotic geometric analysis ([**60**]), and complex geometry ([**61**]), as well as important variants of the Busemann–Petty problem ([**56**, **70**, **70**, **93**]). The fundamental result on j-intersection bodies, which serves as starting point for most subsequent investigations, is the following Fourier analytic characterization in terms of their radial functions.

Theorem 4.1.1. ([**56**, **57**]) *Let $1 \leq j \leq n-1$ and let D and M be origin-symmetric star bodies in $\mathbb{R}^n$. Then D is the j-intersection body of M if and only if*

$$\mathbf{F}_{-j}\, \rho(D,\cdot)^j = \frac{(2\pi)^{n-j} j}{n-j}\, \rho(M,\cdot)^{n-j}.$$

Recall that the operator $\mathbf{F}_{-j}$ denotes the (distributional) spherical Fourier transform of degree $-j$ originating in the work of Koldobsky (see Section 2 for details). Note that the 'if' part of Theorem 4.1.1 is usually stated in the literature only for star bodies with smooth radial functions. However, the arguments used in Section 4 of this paper show that this additional regularity assumption can be omitted.

Over the past decades, a remarkable correspondence between results about sections of star bodies through a fixed point and those concerning projections of convex bodies has asserted itself (see, e.g., the books [**29**, **58**, **78**]). Thereby, the classical Brunn-Minkowski theory of convex bodies forms the ideal framework to deal with problems about projections while the *dual* Brunn-Minkowski theory of star bodies provides the natural setting for questions concerning sections. In this sense, the notion of j-intersection bodies and Theorem 4.1.1 belong to the latter dual theory. Surprisingly, so far no analogue of j-intersection bodies has been thoroughly studied or even explicitly defined in the Brunn-Minkowski theory. In this article, we set out to remedy this neglect.

Definition. *Let $1 \leq j \leq n-1$ and let K and L be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. Then K is called the* j-projection body *of L if*

$$\mathrm{vol}_j(K|E^{\perp}) = \mathrm{vol}_{n-j}(L|E)$$

for every $n-j$ dimensional subspace E of $\mathbb{R}^n$. The class of j-projection bodies is the closure in the Hausdorff metric of all j-projection bodies of convex bodies.

When $j = 1$, the class of 1-projection bodies coincides with the closure of Minkowski's projection bodies of convex bodies which form a central notion in convex geometric analysis (see, e.g., [**2**, **43**, **63**, **83**] and the books [**29**, **78**]). We will see in Section 3 that, as in the case of j-intersection bodies, for $j > 1$, there exist convex bodies L for which a corresponding j-projection body does not exist. However, if K is a j-projection body of L for some $1 \leq j \leq n-1$, then K is uniquely determined (see, e.g., [**29**, Theorem 3.3.6]).

Although for $j > 1$, the definition of j-projection bodies has not appeared before, special cases and examples have been previously considered by several authors (see [**68**, **69**, **77**, **80**]) and we recall them in Section 3. The main goal of this article, however, is to start a systematic investigation of j-projection bodies of convex bodies. To this end, we not only establish a number of their basic properties, such as, invariance under non-degenerate linear transformations, but also obtain an array of new examples. These are based on our first main result which is the following Fourier analytic characterization.

Theorem 4.1.2. *Let $1 \leq j \leq n-1$ and let K and L be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. Then K is the j-projection body of L if and only if*

$$\mathbf{F}_{-j}\, S_j(K,\cdot) = \frac{(2\pi)^{n-j} j}{(n-j)}\, S_{n-j}(L,\cdot).$$

The Borel measures $S_j(K,\cdot)$, $1 \leq j \leq n-1$, on $\mathbb{S}^{n-1}$ are Aleksandrov's *area measures* of the convex body K (see Section 3 for details). Considering the still not fully understood correspondence between results about sections and projections, we want to emphasize the astounding analogy between Theorem 4.1.1 and Theorem 4.1.2, where in order to pass from the characterization of j-intersection bodies to that of j-projection bodies, certain powers of radial functions simply have to be replaced by their 'dual' notion of area measures of the respective orders. We also note here that the case $j = 1$ of Theorem 4.1.2 is equivalent to a previously known relation between projection functions and the Fourier transform of surface area measures (see, e.g., [**62**]).

Our second main result relates the Alesker-Fourier transform on even spherical valuations with j-projection bodies.

Theorem 4.1.3. *Let $1 \leq j \leq n-1$ and let K and L be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. Then K is the j-projection body of L if and only if*

$$\phi(K) = (\mathbb{F}\phi)(L)$$

for all even $\phi \in \mathbf{Val}_j^{\infty,\mathrm{sph}}$.

Note that Theorem 4.1.3 is much easier to prove when the subspace $\mathbf{Val}_j^{\infty,\mathrm{sph}}$ is replaced by the entire space $\mathbf{Val}_j^{\infty}$. The main point of Theorem 4.1.3, which also relates it to Theorem 4.1.2, is to consider spherical valuations only.

Instead of spherical scalar valuations let us now consider translation invariant and $SO(n)$ equivariant even Minkowski valuations. Using Theorem 4.1.2 or 4.1.3, it turns out that we can give a characterization of j-projection bodies in terms of *a single pair* of such Minkowski valuations which are injective on origin-symmetric convex bodies and related by the Alesker-Fourier transform. In the following corollary we exhibit one such pair explicitly, namely Minkowski's projection body operator of order j, $\Pi_j : \mathcal{K}^n \to \mathcal{K}^n$, and the (renormalized) mean section operator of Goodey and Weil [**35–37**], $\overline{\mathrm{M}}_{n-j} : \mathcal{K}^n \to \mathcal{K}^n$ (cf. Section 2 for definitions).

Corollary 4.1.4. *Let $1 \leq j \leq n-1$ and let K and L be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. Then K is the j-projection body of L if and only if*

$$\Pi_j K = \overline{\mathrm{M}}_{n-j} L.$$

The proofs of our representation results will be presented in the next section. In Section 3, we establish general properties of j-projection bodies, such as invariance under non-degenerate linear transformations or the fact that a polytope can only be the j-projection body of another polytope. We also review previously known examples in Section 3 and construct a large family of new ones. In the final section, we relate j-intersection bodies and j-projection bodies via a duality transform motivated by Theorems 4.1.1 and 4.1.2 and a celebrated result of Guan and Ma [**45**] on the Christoffel-Minkowski problem. We also discuss the relation of the class of

j-projection bodies to another generalization of projection bodies. In particular we prove a theorem that highlights an intriguing discontinuity to the case of j-intersection bodies.

4.2 Representation Results

We will now prove the Fourier-analytic representation result for j-projection bodies.

Theorem 4.2.1. *Let $1 \leq j \leq n-1$ and let K and L be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. Then K is the j-projection body of L if and only if*

$$\mathbf{F}_{-j}\, S_j(K,\cdot) = \frac{(2\pi)^{n-j} j}{(n-j)}\, S_{n-j}(L,\cdot).$$

Proof. First note that, by Proposition 2.2.1 and (2.15), K is the j-projection body of L if and only if

$$(\mathrm{R}_{j,n-1}\mathrm{vol}_j(K|\,\cdot\,))^{\perp} = \mathrm{R}_{n-j,1}\mathrm{vol}_{n-j}(L|\,\cdot\,). \tag{4.1}$$

Using (2.24) as well as the fact that $\mathrm{R}_{1,n-j}$ is the adjoint of $\mathrm{R}_{n-j,1}$, we see that (4.1) holds if and only if

$$\frac{\kappa_j}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} f(u) V_j(K|u^{\perp})\, du = \int_{\mathrm{Gr}_{n-j,n}} (\mathrm{R}_{1,n-j} f)(E) \mathrm{vol}_{n-j}(L|E)\, dE \tag{4.2}$$

for every $f \in C_e^{\infty}(\mathbb{S}^{n-1})$. Since $\mathrm{R}_{1,n-j} f \in C^{\infty}(\mathrm{Gr}_{n-j,n})^{\mathrm{sph}}$ and

$$\mathrm{R}_{n-j,n-1} : C^{\infty}(\mathrm{Gr}_{n-j,n})^{\mathrm{sph}} \to C^{\infty}(\mathrm{Gr}_{n-1,n})^{\mathrm{sph}}$$

is bijective, it follows that the integral on the right hand side of (4.2) is equal to

$$\int_{\mathrm{Gr}_{n-1,n}} (\mathrm{R}^{-1}_{n-1,n-j} \mathrm{R}_{1,n-j} f)(F) (\mathrm{R}_{n-j,n-1} \mathrm{vol}_{n-j}(L|\,\cdot\,))(F)\, dF.$$

Since $\perp_*$ is clearly self-adjoint, this can be further rewritten, by using again (2.24) and (2.15), to obtain

$$\frac{\kappa_{n-j}}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} (\mathrm{R}^{-1}_{1,j} \circ \perp_* \circ \mathrm{R}_{1,n-j}) f(u) V_{n-j}(L|u^{\perp})\, du.$$

Consequently, by Proposition 2.1.4, (4.1) holds if and only if

$$\int_{\mathbb{S}^{n-1}} f(u) V_j(K|u^{\perp})\, du = \frac{j}{(2\pi)^j (n-j)} \int_{\mathbb{S}^{n-1}} (\mathbf{F}_{j-n} f)(u) V_{n-j}(L|u^{\perp})\, du$$

for every $f \in C_e^\infty(\mathbb{S}^{n-1})$. Since the cosine transform C_1 is self-adjoint, it follows from (2.24) and the obvious fact that the multiplier transformations $\mathbb{F}_{j-n}$ and C_1 commute that this is equivalent to

$$\int_{\mathbb{S}^{n-1}} \mathrm{C}_1 f(u)\, dS_j(K,u) = \frac{j}{(2\pi)^j(n-j)} \int_{\mathbb{S}^{n-1}} (\mathbf{F}_{j-n}\mathrm{C}_1 f)(u)\, dS_{n-j}(L,u)$$

for every $f \in C_e^\infty(\mathbb{S}^{n-1})$. Substituting $h = \mathbf{F}_{j-n}\mathrm{C}_1 f$ and using (2.19), we finally obtain the desired relation

$$\int_{\mathbb{S}^{n-1}} \mathbf{F}_{-j} h(u)\, dS_j(K,u) = \frac{(2\pi)^{n-j} j}{(n-j)} \int_{\mathbb{S}^{n-1}} h(u)\, dS_{n-j}(L,u)$$

for every $h \in C_e^\infty(\mathbb{S}^{n-1})$ which completes the proof. $\square$

Before we continue, we include here also a short proof of Theorem 4.1.1 which underlines the dual nature of Theorems 4.1.1 and 4.1.2 and shows that for the 'if' part of the statement no additional regularity assumptions are required.

Theorem 4.2.2. ([**56**, **57**]) *Let $1 \leq j \leq n-1$ and let D and M be origin-symmetric star bodies in $\mathbb{R}^n$. Then D is the j-intersection body of M if and only if*

$$\mathbf{F}_{-j}\, \rho(D,\cdot)^j = \frac{(2\pi)^{n-j} j}{n-j}\, \rho(M,\cdot)^{n-j}.$$

Proof. By passing to polar coordinates, it follows that D is the j-intersection body of M if and only if

$$\frac{\kappa_j}{\kappa_{n-j}} \int_{\mathrm{Gr}_{n-j,n}} f(E)(\mathrm{R}_{1,j}\rho(D,\cdot)^j)^{\perp}(E)\, dE = \int_{\mathrm{Gr}_{n-j,n}} f(E)\mathrm{R}_{1,n-j}\rho(M,\cdot)^{n-j}(E)\, dE \quad (4.3)$$

for every $f \in C^\infty(\mathrm{Gr}_{n-j,n})$. Since $\perp_*$ is self-adjoint and $\mathrm{R}_{j,i}$ is the adjoint of $\mathrm{R}_{i,j}$, (4.3) is equivalent to

$$\frac{\kappa_j}{\kappa_{n-j}} \int_{\mathbb{S}^{n-1}} (\mathrm{R}_{j,1} f^{\perp})(u)\rho(D,u)^j\, du = \int_{\mathbb{S}^{n-1}} \mathrm{R}_{n-j,1} f(u)\rho(M,u)^{n-j}\, du$$

for every $f \in C^\infty(\mathrm{Gr}_{n-j,n})$. Substituting now $h = \mathrm{R}_{n-j,1} f$ and using that the Radon transform $\mathrm{R}_{n-j,1} : C^\infty(\mathrm{Gr}_{n-j,n}) \to C_e^\infty(\mathbb{S}^{n-1})$ is surjective, we see that D is the j-intersection body of M if and only if

$$\frac{\kappa_j}{\kappa_{n-j}} \int_{\mathbb{S}^{n-1}} (\mathrm{R}_{j,1} \circ \perp_* \circ \mathrm{R}_{n-j,1}^{-1}) h(u)\rho(D,u)^j\, du = \int_{\mathbb{S}^{n-1}} h(u)\rho(M,u)^{n-j}\, du \quad (4.4)$$

for all $h \in C_e^\infty(\mathbb{S}^{n-1})$. Since $\mathbf{F}_{-j}$ and $\perp_*$ are self-adjoint and $\mathrm{R}_{j,i}$ is the adjoint of $\mathrm{R}_{i,j}$, it follows from Proposition 2.1.4 that

$$\mathbf{F}_{-j} = \frac{(2\pi)^{n-j}\, j\, \kappa_j}{(n-j)\, \kappa_{n-j}} \mathrm{R}_{j,1} \circ \perp_* \circ \mathrm{R}_{n-j,1}^{-1}.$$

Hence, (4.4) is equivalent to

$$\int_{\mathbb{S}^{n-1}} \mathbf{F}_{-j} h(u) \rho(D,u)^j \, du = \frac{(2\pi)^{n-j} j}{n-j} \int_{\mathbb{S}^{n-1}} h(u) \rho(M,u)^{n-j} \, du$$

for every $h \in C_e^\infty(\mathbb{S}^{n-1})$.

$\square$

With our next result, we complete the proofs of Theorem 4.1.3 and Corollary 4.1.4.

Theorem 4.2.3. *Let $1 \leq j \leq n-1$ and let K and L be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. Then the following statements are equivalent:*

(i) *K is the j-projection body of L;*

(ii) $\mathbf{F}_{-j} S_j(K, \cdot) = \frac{(2\pi)^{n-j} j}{n-j} S_{n-j}(L, \cdot)$*;*

(iii) $\Pi_j K = \overline{\mathrm{M}}_{n-j} L$*;*

(iv) $\phi(K) = (\mathbb{F}\phi)(L)$ *for all even* $\phi \in \mathbf{Val}_j^{\infty,\mathrm{sph}}$.

Proof. We have already seen that (i) and (ii) are equivalent. In order to prove that (ii) and (iii) are equivalent, we first note that, by the definition of Π_j and $\overline{\mathrm{M}}_{n-j}$ and (2.41), (iii) is equivalent to

$$h(\Pi_j K, \cdot) = \frac{1}{2} \mathrm{C}_1 S_j(K, \cdot) = \frac{2n\kappa_n}{(j+1)\kappa_{j+1}} q_{n,j+1}\, S_{n-j}(L, \cdot) * \breve{g}_{j+1} = h(\overline{\mathrm{M}}_{n-j} L, \cdot).$$

Since convolution transforms are self-adjoint, integrating both sides yields that this is equivalent to

$$\frac{(j+1)\kappa_{j+1}}{4n\kappa_n} q_{n,j+1}^{-1} \int_{\mathbb{S}^{n-1}} \mathrm{C}_1 f(u) \, dS_j(K,u) = \int_{\mathbb{S}^{n-1}} (f * \breve{g}_{j+1})(u) \, dS_{n-j}(L,u)$$

for every $f \in C_e^\infty(\mathbb{S}^{n-1})$. Substituting $h = f * \breve{g}_{j+1}$ and using Corollary 2.3.11, it follows that this holds if and only if

$$\int_{\mathbb{S}^{n-1}} \mathbf{F}_{-j} h(u) \, dS_j(K,u) = \frac{(2\pi)^{n-j} j}{(n-j)} \int_{\mathbb{S}^{n-1}} h(u) \, dS_{n-j}(L,u) \tag{4.5}$$

for every $h \in C_e^\infty(\mathbb{S}^{n-1})$ which is precisely (ii).

Finally, in order to see that (iv) is equivalent to (ii), first note that, by Proposition 2.3.4 and Corollary 2.3.6, (iv) is equivalent to

$$\int_{\mathbb{S}^{n-1}} f(u)\, dS_j(K,u) = \frac{j}{(2\pi)^j(n-j)} \int_{\mathbb{S}^{n-1}} (\mathbf{F}_{j-n} f)(u)\, dS_{n-j}(L,u)$$

for every $f \in C_e^\infty(\mathbb{S}^{n-1})$. Substituting this time $h = \mathbf{F}_{j-n} f$ and using (2.19), it follows that this holds if and only if (4.5) holds for every $h \in C_e^\infty(\mathbb{S}^{n-1})$, which completes the proof. □

We remark that Π_j and $\overline{\mathrm{M}}_{n-j}$ can be replaced in statement (iii) by any pair of Minkowski valuations intertwining rigid motions which are related by the Alesker-Fourier transform and injective on origin-symmetric convex bodies. This follows easily from Proposition 2.3.9. Next, note that, by Theorem 2.3.2, (2.34), and (2.35), the space $\mathbf{Val}_j^{\infty,\mathrm{sph}}$ of smooth spherical valuations can be replaced in (iv) by the entire space $\mathbf{Val}_j^\infty$, leading however to a weaker statement.

Finally, we also note that Corollary 4.1.4 follows also directly from (2.40), (2.42), and Proposition 2.2.1. Together with the arguments of the first part of the proof of Theorem 4.2.3, this can be used to give an alternative proof of Theorem 4.1.2.

4.3 Properties and Examples

In this section, we first prove that the class of j-projection bodies is invariant under non-degenerate linear transformations. We then collect several examples of j-projection bodies from the literature and compute a family of new examples using Theorem 4.1.2. At the end of the section, we generalize two more well known properties of the classical 1-projection bodies to all $j > 1$.

The fact that the class of j-intersection bodies is invariant under the general linear group $\mathrm{GL}(n)$ was first observed by Milman **[70]**. The proof of the following dual counterpart is based on ideas of Schneider **[77]**, who proved it in the special case described in Example 4.3.2 (d) below.

Theorem 4.3.1. *Let $1 \leq j \leq n-1$, $A \in \mathrm{GL}(n)$, and let K and L be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. If K is the j-projection body of L, then AK is the j-projection body of*

$$|\det A|^{\frac{1}{n-j}}\, A^{-\mathrm{T}} L.$$

In particular, the class of j-projection bodies is $\mathrm{GL}(n)$ invariant.

Proof. First let $|\det A| = 1$. Using the polar decomposition of A and the fact that the statement is clearly true for orthogonal linear maps, we may assume that A is *symmetric* and *positive definite*. Thus, we have to show that

$$\mathrm{vol}_j(AK|E^\perp) = \mathrm{vol}_{n-j}(A^{-1}L|E) \tag{4.6}$$

for every $E \in \mathrm{Gr}_{n-j,n}$. To this end, let $F \in \mathrm{Gr}_{j,n}$ and $u_1, \ldots, u_j$ be a set of orthonormal vectors in F. In order to compute $\mathrm{vol}_j(AK|F)$, we may assume, using the singular value decomposition of A, that the vectors $Au_1, \ldots, Au_j$ are also orthogonal. Then,

$$\begin{aligned}
\mathrm{vol}_j(AK|F) &= \mathrm{vol}_j\left(\left\{\sum_{i=1}^{j}(Ax \cdot u_i)u_i : x \in K\right\}\right) \\
&= \left(\prod_{i=1}^{j}\|Au_i\|_2\right)\mathrm{vol}_j(K|AF) = c_j(A,F)\,\mathrm{vol}_j(K|AF),
\end{aligned}$$

where $c_j(A,F)$ depends on A and F only and not on K. Consequently, using that $AE^\perp = (A^{-1}E)^\perp$ for every $E \in \mathrm{Gr}_{n-j,n}$, we obtain

$$\begin{aligned}
\mathrm{vol}_j(AK|E^\perp) &= c_j(A,E^\perp)\mathrm{vol}_j(K|(A^{-1}E)^\perp) = c_j(A,E^\perp)\mathrm{vol}_{n-j}(L|A^{-1}E) \\
&= \frac{c_j(A,E^\perp)}{c_{n-j}(A^{-1},E)}\mathrm{vol}_{n-j}(A^{-1}L|E).
\end{aligned}$$

Choosing now K to be the unit cube in $\mathbb{R}^n$, it follows from a result of Schnell **[80]** (see Example 4.3.2 (b) below) that $c_j(A,E^\perp) = c_{n-j}(A^{-1},E)$ which yields the desired equation (4.6).

Finally, let $A \in \mathrm{GL}(n)$ be arbitrary. Then, by the first part of the proof,

$$\begin{aligned}
\mathrm{vol}_j(AK|E^\perp) &= |\det A|^{\frac{j}{n}}\mathrm{vol}_j(|\det A|^{-\frac{1}{n}}AK|E^\perp) \\
&= |\det A|^{\frac{j}{n}}\mathrm{vol}_{n-j}(|\det A|^{\frac{1}{n}}A^{-\mathrm{T}}L|E) = \mathrm{vol}_{n-j}(|\det A|^{\frac{1}{n-j}}A^{-\mathrm{T}}L|E)
\end{aligned}$$

holds for every $E \in \mathrm{Gr}_{n-j,n}$ as desired. $\square$

Theorem 4.3.1 suggests that it should be possible to define the notion of j-projection bodies in $\mathrm{SL}(n)$ invariant terms without referring to any Euclidean structure. Indeed, the author is obliged to S. Alesker for communicating such an $\mathrm{SL}(n)$ invariant definition to us which we will state in the following. It requires a basic familiarity with the notion of a line bundle over a manifold.

Let V be an n-dimensional real vector space with a fixed volume form and let V^* denote its dual space. For $1 \le j \le n-1$, we write $\mathrm{Gr}_j(V)$ for the Grassmannian of all j-dimensional subspaces of V and $\mathcal{K}(V)$ for the space of convex bodies in

V. Moreover, if W is a finite dimensional vector space, we denote by $\mathrm{Dens}(W)$ the 1-dimensional space of Lebesgue measures on W. Finally, for any measurable subset M of W, we define an element $\mathrm{ev}_M \in \mathrm{Dens}(W)^*$ by

$$\mathrm{ev}_M(\sigma) = \sigma(M), \qquad \sigma \in \mathrm{Dens}(W).$$

The jth projection function $p_j(K, \cdot)$ of a convex body $K \in \mathcal{K}(V)$ is now no longer a function on $\mathrm{Gr}_j(V)$ but the section of the line bundle

$$X_{n-j} = \{(E, l) : E \in \mathrm{Gr}_{n-j}(V),\ l \in \mathrm{Dens}(V/E)^*\},$$

given by

$$p_j(K, E) = \mathrm{ev}_{\mathrm{pr}_E(K)}.$$

Here, $\mathrm{pr}_E : V \to V/E$ denotes the natural projection. Similarly, the $(n-j)$th projection function $p_{n-j}(L, \cdot)$ of $L \in \mathcal{K}(V^*)$ is a section of the line bundle

$$X_j^* = \{(F, l) : F \in \mathrm{Gr}_j(V^*),\ l \in \mathrm{Dens}(V^*/F)^*\}.$$

Let us denote by $C(\mathrm{Gr}_{n-j}(V), X_{n-j})$ and $C(\mathrm{Gr}_j(V^*), X_j^*)$ the spaces of all (continuous) sections of the line bundles X_{n-j} and X_j^*, respectively. Note that the group $\mathrm{SL}(V)$ acts on these vector spaces naturally by left translation. Moreover, using the annihilator map $\perp : \mathrm{Gr}_{n-j}(V) \to \mathrm{Gr}_j(V^*)$, it is not difficult to show that the canonical isomorphism

$$\mathrm{Dens}(V/E)^* \cong \mathrm{Dens}(V^*/E^\perp)^*$$

induces a canonical $\mathrm{SL}(V)$ equivariant isomorphism between the spaces of sections $C(\mathrm{Gr}_{n-j}(V), X_{n-j})$ and $C(\mathrm{Gr}_j(V^*), X_j^*)$.

For origin-symmetric bodies $K \in \mathcal{K}(V)$ and $L \in \mathcal{K}(V^*)$ with non-empty interior, we may therefore call $K \in \mathcal{K}(V)$ the j-projection body of L if

$$p_j(K, \cdot) \cong p_{n-j}(L, \cdot)$$

with respect to the isomorphism described above. Clearly, this definition coincides with the one given in the introduction if we choose a Euclidean structure on V and identify V^* with V. Furthermore, this invariant formulation immediately implies that the class of j-projection bodies is $\mathrm{GL}(V)$ invariant.

We turn now to classical and new examples of j-projection bodies. In the following list, examples 4.3.2 (b) – (d) were previously considered in the literature. Example 4.3.2 (e) and (f) are new and based on our main result, Theorem 4.1.2.

Example 4.3.2.

(a) Since, for the Euclidean unit ball B in $\mathbb{R}^n$, we have $\mathrm{vol}_j(B|E^\perp) = \kappa_j$ and $\mathrm{vol}_{n-j}(B|E) = \kappa_{n-j}$ for every $1 \le j \le n-1$ and $E \in \mathrm{Gr}_{n-j,n}$, it follows that B is the j-projection body of

$$\left(\frac{\kappa_j}{\kappa_{n-j}}\right)^{\frac{1}{n-j}} B.$$

Thus, Theorem 4.3.1 and the fact that $(AB)^* = A^{-\mathrm{T}}B$ for every $A \in \mathrm{GL}(n)$ imply that if K is an origin-symmetric ellipsoid with non-empty interior in $\mathbb{R}^n$, then K is the j-projection body of the ellipsoid

$$\left(\frac{\kappa_j V_n(K)}{\kappa_{n-j}\kappa_n}\right)^{\frac{1}{n-j}} K^*.$$

(b) McMullen **[68]** first proved that the unit cube in $\mathbb{R}^n$,

$$W = \left\{x \in \mathbb{R}^n : -\tfrac{1}{2} \le x_i \le \tfrac{1}{2}, i = 1, \dots, n\right\},$$

is the j-projection body of itself for every $1 \le j \le n-1$. Based on this curious property of W, Schnell **[80]** deduced the following special case of Theorem 4.3.1: If K is an origin-symmetric parallelotope with non-empty interior in $\mathbb{R}^n$, say $K = AW$ with $A \in \mathrm{GL}(n)$, then K is the j-projection body of the parallelotope

$$V_n(K)^{\frac{1}{n-j}} A^{-\mathrm{T}} W.$$

In particular, if $V_n(K) = 1$, then $A^{-\mathrm{T}}W$ is the j-projection body of K for every $1 \le j \le n-1$.

(c) Let $K \in \mathcal{K}^n_s$ have non-empty interior. Then, by (2.22) and the definition of the projection body operator Π_{n-1} given in Example 2.3.10 (a), we have

$$\mathrm{vol}_{n-1}(K|u^\perp) = h(\Pi_{n-1}K, u) = \mathrm{vol}_1\left(\frac{1}{2}\Pi_{n-1}K \,\middle|\, \mathrm{span}\{u\}\right)$$

for every $u \in \mathbb{S}^{n-1}$, that is, K is the $(n-1)$-projection body of $\frac{1}{2}\Pi_{n-1}K$ or, equivalently, $\frac{1}{2}\Pi_{n-1}K$ is the 1-projection body of K. In particular, the class of $(n-1)$-projection bodies coincides with $\mathcal{K}^n_s$ and, by Minkowski's existence theorem and Cauchy's projection formula, the class of 1-projection bodies coincides with the class $\mathcal{Z}^n_s$ of origin-symmetric zonoids in $\mathbb{R}^n$.

(d) Let $K, L \in \mathcal{K}^n_s$ have non-empty interior. Generalizing a notion introduced by McMullen **[69]**, Schneider **[77]** calls (K, L) a *(VP)-pair* if K is the

j-projection body of L for *every* $1 \leq j \leq n-1$. By the results of McMullen and Schnell described in (b), any pair of parallelotopes $(AW, A^{-\mathrm{T}}W)$, where $A \in \mathrm{GL}(n)$ and $|\det A| = 1$, is an example of a (VP)-pair.

Schneider proved in **[77]** that if K and L are polytopes, then they are a (VP)-pair if and only if K is the j-projection body of L for $j = 1$ and $j = n-1$. Moreover, from a result of Weil **[89]**, Schneider deduced that this holds if and only if K is a direct sum of centrally symmetric polygons and segments, $V_n(K) = 1$, and $L = \frac{1}{2}\Pi_{n-1}K$.

(e) Let K be a convex body and D a star body in $\mathbb{R}^{2n}$ and let both be origin-symmetric. Motivated by the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, the bodies K and D are called *complex* if they are invariant with respect to any coordinate-wise two-dimensional rotation (see, e.g., **[61]** for details). Note that origin-symmetric complex convex bodies in $\mathbb{R}^{2n}$ correspond precisely to the unit balls of complex norms on $\mathbb{C}^n$.

For a unit vector $u \in \mathbb{C}^n$, let $H_u = \{z \in \mathbb{C}^n : \langle u, z\rangle = \sum_{k=1}^n u_k \overline{z}_k = 0\}$ denote the complex hyperplane perpendicular to u. Under the standard mapping from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$, the hyperplane H_u becomes a $2n-2$ dimensional subspace of $\mathbb{R}^{2n}$ which is orthogonal to the vectors

$$u = (u_{11}, u_{12}, \ldots, u_{n1}, u_{n2}) \qquad \text{and} \qquad u_* = (-u_{12}, u_{11}, \ldots, -u_{n2}, u_{n1}).$$

Definition. *Let D and M be origin-symmetric complex star bodies in $\mathbb{R}^{2n}$. Then D is called the* complex intersection body *of M if*

$$\mathrm{vol}_2(D \cap H_u^{\perp}) = \mathrm{vol}_{2n-2}(M \cap H_u)$$

for every $u \in \mathbb{S}^{2n-1}$. The class of complex intersection bodies is the closure in the radial metric of all complex intersection bodies of star bodies.

Koldobsky, Paouris, and Zymonopoulou **[61]** proved that the class of complex intersection bodies coincides with the class of 2-intersection bodies which are complex. Moreover, they showed that complex intersection bodies of convex bodies are also convex. Motivated by these results, we define complex projection bodies as follows (see **[2]**, for a different notion of complex projection bodies).

Definition. *Let K and L be origin-symmetric complex convex bodies in $\mathbb{R}^{2n}$. Then K is called the* complex projection body *of L if*

$$\mathrm{vol}_2(K|H_u^{\perp}) = \mathrm{vol}_{2n-2}(L|H_u)$$

for every $u \in \mathbb{S}^{2n-1}$. The class of complex projection bodies is the closure in the Hausdorff metric of all complex projection bodies of convex bodies.

Clearly, if $K \in \mathcal{K}_s^{2n}$ is a complex 2-projection body of L, then K is a the complex projection body of L. We do not know if the converse also holds. However, if $D \in \mathcal{K}_s^{2n}$ is complex and of class C_+^∞, then, by Proposition 2.2.3, the function $\rho(D, \cdot)^2 \in C_e^\infty(\mathbb{S}^{n-1})$ is the density of the area measure of order 2 of a complex convex body $K \in \mathcal{K}_s^{2n}$. Consequently, by Theorems 4.1.1 and 4.1.2 and Proposition 2.2.3, if D is a complex intersection body, then K is a complex 2-projection body which, in turn, is a complex projection body.

(f) Finally, we consider strictly convex bodies of revolution $K_\lambda \in \mathcal{K}_s^n$ whose area measures of order $1 \leq j \leq n-2$ have a density of the form

$$s_j(K_\lambda, \bar{e} \cdot .\,) = 1 + \lambda P_2^n(\bar{e} \cdot .\,) = 1 + \frac{\lambda}{n-1}(n(\bar{e} \cdot .\,)^2 - 1), \tag{4.7}$$

where P_2^n denotes the Legendre polynomial of dimension n and degree 2. In order to determine all admissible λ in (4.7), we use Theorem 2.2.4. Clearly, condition (i) of Theorem 2.2.4 is satisfied for all $\lambda \in \mathbb{R}$. However, since

$$\int_t^1 \xi\, s_j(K_\lambda, \xi)(1-\xi^2)^{\frac{n-3}{2}} d\xi = \frac{(1-t^2)^{\frac{n-1}{2}}}{n^2-1}(\lambda(nt^2+1)+n+1),$$

it is not difficult to show that conditions (ii) and (iii) of Theorem 2.2.4 are satisfied if and only if

$$\lambda \in \left(-1, \frac{j(n+1)}{2n-j}\right). \tag{4.8}$$

Now, we want to determine which of the bodies K_λ are j-projection bodies. To this end, note that, by (2.18), we have

$$\mathbf{F}_{-j} s_j(K_\lambda, \bar{e} \cdot .\,) = \frac{\pi^{\frac{n}{2}} 2^{n-j} \Gamma\left(\frac{n-j}{2}\right)}{\Gamma\left(\frac{j}{2}\right)} \left(1 - \lambda \frac{n-j}{j} P_2^n(\bar{e} \cdot .\,)\right).$$

Hence, by Theorem 4.1.2 and (4.8), K_λ is a j-projection body if and only if

$$\lambda \in \left(-1, \frac{j}{n-j}\right).$$

This shows, in particular, that for $j < n-1$ the class of j-projection bodies is a proper subset of $\mathcal{K}_s^n$.

In the final part of this section, we want to prove two more basic properties of j-projection bodies. The first one is a generalization of the well known fact that Minkowski's projection body operator Π_{n-1} maps polytopes to polytopes. Note that, by Example 4.3.2 (c), this implies that 1- and $(n-1)$-projection bodies of polytopes are polytopes. As part of the following result we extend this observation to all $j \in \{1, \dots, n-1\}$.

Theorem 4.3.3. *Let $1 \leq j \leq n-1$ and let P and Q be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. If P is the j-projection body of Q, then*

$$S_j(P, \mathbb{S}^{n-1} \cap E) = \frac{\kappa_{n-j}}{\kappa_j} S_{n-j}(Q, \mathbb{S}^{n-1} \cap E^{\perp}) \tag{4.9}$$

for every $E \in \mathrm{Gr}_{n-j,n}$. Moreover, if P is a polytope, then so is Q and P has a j-face parallel to E if and only if Q has an $(n-j)$-face parallel to $E^{\perp}$.

Proof. In order to prove (4.9), let $E \in \mathrm{Gr}_{n-j,n}$ be an arbitrary but fixed subspace. For $\varepsilon > 0$, let $f_\varepsilon \in C([0,1])$ be monotone increasing with $\operatorname{supp} f_\varepsilon \subseteq [1-2\varepsilon, 1]$ and such that $f_\varepsilon \equiv 1$ on $[1-\varepsilon, 1]$ and define $g_\varepsilon^E \in C(\mathrm{Gr}_{j,n})$, by

$$g_\varepsilon^E(F) = f_\varepsilon(|\cos(E,F)|).$$

Note that

$$(\mathrm{R}_{j,1} g_\varepsilon^E)(u) = \int_{\mathrm{Gr}_{j,n}^u} f_\varepsilon(|\cos(E,F)|)\, d\nu_j^u(F)$$

depends only on $|\cos(E,u)|$. In particular, $\mathrm{R}_{j,1} g_\varepsilon^E$ is constant on $\mathbb{S}^{n-1} \cap E$. Consequently, we can replace g_ε^E, if necessary, by a positive multiple such that $(\mathrm{R}_{j,1} g_\varepsilon^E)(u) = 1$ whenever $u \in \mathbb{S}^{n-1} \cap E$. Next, we want to show that

$$\lim_{\varepsilon \to 0} (\mathrm{R}_{j,1} g_\varepsilon^E)(u) = \begin{cases} 1 & \text{for } u \in \mathbb{S}^{n-1} \cap E, \\ 0 & \text{for } u \notin \mathbb{S}^{n-1} \cap E. \end{cases} \tag{4.10}$$

To this end, observe that $|\cos(E,F)| \leq |\cos(E,u)|$ whenever $u \in E$. Thus, by the monotonicity of f_ε, we have

$$(\mathrm{R}_{j,1} g_\varepsilon^E)(u) \leq f_\varepsilon(|\cos(E,u)|).$$

Hence, by the definition of f_ε, for every $u \notin \mathbb{S}^{n-1} \cap E$ there exists $\varepsilon_u > 0$ such that $(\mathrm{R}_{j,1} g_\varepsilon^E)(u) = 0$ for every $\varepsilon \leq \varepsilon_u$ which completes the proof of (4.10).

The same arguments used to prove (4.10) together with the fact that $SO(n)$ acts transitively on $\mathrm{Gr}_{n-j,n}$, show that there exists a positive constant $c \in \mathbb{R}$, independent of E, such that

$$\lim_{\varepsilon \to 0} (\mathrm{R}_{n-j,1} (g_\varepsilon^E)^{\perp})(u) = \begin{cases} c & \text{for } u \in \mathbb{S}^{n-1} \cap E^{\perp}, \\ 0 & \text{for } u \notin \mathbb{S}^{n-1} \cap E^{\perp}. \end{cases} \tag{4.11}$$

Now, since P is the j-projection body of Q and $\mathrm{R}_{j,i}$ is the adjoint of $\mathrm{R}_{i,j}$, it follows from Theorem 4.1.2 and Proposition 2.1.4 that

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} (\mathrm{R}_{j,1} g_\varepsilon^E)(u)\, dS_j(P,u) &= \int_{\mathrm{Gr}_{j,n}} g_\varepsilon^E(F)\, d(\mathrm{R}_{1,j} S_j(P,\cdot))(F) \\
&= \frac{\kappa_{n-j}}{\kappa_j} \int_{\mathrm{Gr}_{n-j,n}} g_\varepsilon^E(F^{\perp})\, d(\mathrm{R}_{1,n-j} S_{n-j}(Q,\cdot))(F) \\
&= \frac{\kappa_{n-j}}{\kappa_j} \int_{\mathbb{S}^{n-1}} (\mathrm{R}_{n-j,1} (g_\varepsilon^E)^{\perp})(u)\, dS_{n-j}(Q,u).
\end{aligned}$$

Letting $\varepsilon \to 0$, (4.10), (4.11), and the dominated convergence theorem yield

$$S_j(P, \mathbb{S}^{n-1} \cap E) = \frac{c\,\kappa_{n-j}}{\kappa_j} S_{n-j}(Q, \mathbb{S}^{n-1} \cap E^{\perp}),$$

where the positive constant c is the same as in (4.11) and does not depend on P and Q. Using Example 4.3.2 (b) and taking $P = W = Q$, shows, by (2.2.2), that $c = 1$ which completes the proof of (4.9).

Now assume that P is a polytope and recall that, by (2.2.2), $S_j(P, \cdot)$ is concentrated on the union of finitely many $n-1-j$ dimensional great spheres. Let $\mathcal{G}^{n-j}(P)$ denote the *finite* set of subspaces $E \in \mathrm{Gr}_{n-j,n}$ such that $S_j(P, \mathbb{S}^{n-1} \cap E) > 0$. Summing (4.9) over all $E \in \mathcal{G}^{n-j}(P)$ yields on one hand

$$\frac{\kappa_{n-j}}{\kappa_j} \sum_{E \in \mathcal{G}^{n-j}(P)} S_{n-j}(Q, \mathbb{S}^{n-1} \cap E^{\perp}) = \sum_{E \in \mathcal{G}^{n-j}(P)} S_j(P, \mathbb{S}^{n-1} \cap E) = S_j(P, \mathbb{S}^{n-1}).$$

On the other hand, since Q is the $(n-j)$-projection body of P, Theorem 4.1.2 and (2.18) imply that

$$S_j(P, \mathbb{S}^{n-1}) = \frac{j a_0^n[\mathbf{F}_{j-n}]}{(2\pi)^j (n-j)} S_{n-j}(Q, \mathbb{S}^{n-1}) = \frac{\kappa_{n-j}}{\kappa_j} S_{n-j}(Q, \mathbb{S}^{n-1}).$$

Consequently, since $S_{n-j}(Q, \cdot)$ vanishes on great spheres of dimension $d < j-1$, we have

$$S_{n-j}(Q, \mathbb{S}^{n-1}) = \sum_{E \in \mathcal{G}^{n-j}(P)} S_{n-j}(Q, \mathbb{S}^{n-1} \cap E^{\perp}) = S_{n-j}\left(Q, \bigcup_{E \in \mathcal{G}^{n-j}(P)} (\mathbb{S}^{n-1} \cap E^{\perp})\right).$$

This shows that $S_{n-j}(Q, \cdot)$ is concentrated on a finite union of $j-1$ dimensional great spheres. An application of Proposition 2.2.2 finishes the proof. □

It is well known that for every pair of convex bodies $K, L \in \mathcal{K}^n$,

$$V(K[n-1], \Pi_{n-1} L) = V(L[n-1], \Pi_{n-1} K). \tag{4.12}$$

This basic mixed volume identity for Minkowski's projection body operator Π_{n-1} and its variants for other Minkowski valuations have found numerous applications (see, e.g., [**2**, **9**, **36**, **38**, **82**]). In view of Example (c), our final result of this section provides a generalization of (4.12) in the context of j-projection bodies.

Theorem 4.3.4. *Let $1 \le j \le n-1$ and let K_i, L_i, $i = 1, 2$, be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. If K_1 is the j-projection body of L_1 and K_2 is the $(n-j)$-projection body of L_2, then*

$$V(K_1[j], K_2[n-j]) = V(L_1[n-j], L_2[j]).$$

Proof. Consider the valuations $\phi \in \mathbf{Val}_{n-j}^{+}$ and $\psi \in \mathbf{Val}_{j}^{+}$, defined by

$$\phi(K) = V(K_1[j], K[n-j]) \qquad \text{and} \qquad \psi(K) = V(L_1[n-j], K[j]).$$

Then, by a well known relation between projection functions and mixed volumes (see, e.g., [**78**, Theorem 5.3.1]), the Klain functions $\mathrm{Kl}_{n-j}\phi \in C(\mathrm{Gr}_{n-j,n})$ and $\mathrm{Kl}_j\psi \in C(\mathrm{Gr}_{j,n})$ are given by

$$\mathrm{Kl}_{n-j}\phi(E) = \binom{n}{j}^{-1} \mathrm{vol}_j(K_1|E^\perp) \quad \text{and} \quad \mathrm{Kl}_j\psi(F) = \binom{n}{j}^{-1} \mathrm{vol}_{n-j}(L_1|F^\perp).$$

Therefore, since K_1 is the j-projection body of L_1, we have

$$\mathrm{Kl}_{n-j}\phi = (\mathrm{Kl}_j\psi)^\perp. \tag{4.13}$$

Now, assume that ϕ and ψ are smooth. Then, by (4.13) and (2.35), $\psi = \mathbb{F}\phi$. Moreover, (as already explained after the proof of Theorem 4.2.3) it follows from Theorem 2.3.2 and (2.34) that

$$\phi(K_2) = V(K_1[j], K_2[n-j]) = V(L_1[n-j], L_2[j]) = \psi(L_2) \tag{4.14}$$

which is the desired relation.

If ϕ and ψ are not smooth, but merely continuous, then a recent extension of Alesker and Faifman [**10**, Propositions 4.4 and 4.5] of the Klain and Crofton maps as well as Theorem 2.3.2 and (2.34) to generalized valuations (which include, in particular, continuous valuations) implies that (4.14) still follows from (4.13). $\square$

4.4 From j-projection bodies to j-intersection bodies

In this final section, we first recall the definition of the class of convex bodies $\mathcal{K}_s^n(j)$ and their dual analogs, the class of j-Busemann-Petty star bodies (also called generalized j-intersection bodies). Then, we relate these two classes as well as the classes of j-intersection bodies and j-projection bodies via a generalization of the duality transform introduced at the end of Example 4.3.2 (e). Finally, we prove a dual analog of a recent result of Milman on the relation between the classes of j-intersection bodies and j-Busemann-Petty bodies and perhaps more remarkable disprove another analog result with respect to these classes due to Koldobsky.

For $1 \leq j \leq n-1$, let $\mathcal{P}_s^n(j)$ denote the class of all (origin-symmetric) j-projection bodies in $\mathbb{R}^n$ and recall that $\mathcal{K}_s^n(j)$ is the class of all origin-symmetric convex bodies $K \in \mathcal{K}^n$ such that

$$\mathrm{vol}_j(K|\,\cdot\,) = \mathrm{C}_j\varrho_j(K, \cdot)$$

for some *non-negative* Borel measure $\varrho_j(K,\cdot)$ on $\mathrm{Gr}_{j,n}$.

Using the Cauchy-Kubota formula (2.24) as well as (2.13) and (2.15), it follows that $K \in \mathcal{K}_s^n(j)$ if and only if

$$\mathrm{C}_1 S_j(K,\cdot) = \frac{2\kappa_{n-1}}{\kappa_j}(\mathrm{R}_{n-j,1} \circ \mathrm{C}_{n-j})\varrho_j^{\perp}(K,\cdot)$$

which, by the composition rule for Radon and cosine transforms (see, e.g., [**38**]), is equivalent to

$$\mathrm{C}_1 S_j(K,\cdot) = \binom{n}{j}^{-1} n\kappa_{n-j}(\mathrm{C}_1 \circ \mathrm{R}_{n-j,1})\varrho_j^{\perp}(K,\cdot). \tag{4.15}$$

Consequently, the injectivity of the spherical cosine transform C_1 implies that the class $\mathcal{K}_s^n(j)$ consists precisely of those origin-symmetric $K \in \mathcal{K}^n$ for which

$$S_j(K,\cdot) = \mathrm{R}_{n-j,1}\mu_j(K,\cdot) \tag{4.16}$$

for some *non-negative* Borel measure $\mu_j(K,\cdot)$ on $\mathrm{Gr}_{n-j,n}$.

The dual analog of the class $\mathcal{K}_s^n(j)$ was introduced by Zhang in 1996.

Definition. *Suppose that $1 \leq j \leq n-1$. An origin-symmetric star body D in $\mathbb{R}^n$ is called a j-Busemann-Petty body if*

$$\rho(D,\cdot)^j = \mathrm{R}_{n-j,1}\nu_j(D,\cdot) \tag{4.17}$$

for some non-negative Borel measure $\nu_j(D,\cdot)$ on $\mathrm{Gr}_{n-j,n}$.

For $1 \leq j \leq n-1$, let $\mathcal{I}_s^n(j)$ denote the class of all (origin-symmetric) j-intersection bodies in $\mathbb{R}^n$ and let $\mathcal{BP}_s^n(j)$ denote the class of (origin-symmetric) j-Busemann-Petty bodies in $\mathbb{R}^n$. From their definition and Theorem 4.1.1 it follows easily that

$$\mathcal{I}_s^n(1) = \mathcal{BP}_s^n(1) \qquad \text{and} \qquad \mathcal{I}_s^n(n-1) = \mathcal{BP}_s^n(n-1) = \mathcal{S}_s^n, \tag{4.18}$$

where here and in the following $\mathcal{S}_s^n$ denotes the class of origin-symmetric star bodies in $\mathbb{R}^n$. Recall that $\mathcal{I}_s^n(1)$ coincides with Lutwak's intersection bodies.

The discovery and importance of the class $\mathcal{BP}_s^n(j)$ is due to their connection to the j-codimensional Busemann-Petty problem which asks whether the volume of a *convex* body $K \in \mathcal{K}_s^n$ is smaller than that of another body $L \in \mathcal{K}_s^n$ if

$$\mathrm{vol}_{n-j}(K \cap E) \leq \mathrm{vol}_{n-j}(L \cap E)$$

for all $E \in \mathrm{Gr}_{n-j,n}$. Zhang [**94**] showed that a positive answer to this problem is equivalent to whether all origin-symmetric convex bodies in $\mathbb{R}^n$ belong to the class $\mathcal{BP}_s^n(j)$. Subsequently, Bourgain and Zhang [**19**] proved that the answer is negative for $j < n-3$ but the cases $j = n-3$ and $j = n-2$ remained open. This was later reproved by Koldobsky [**57**] who also first considered the relationship between the two types of generalizations of Lutwak's intersection bodies, $\mathcal{I}_s^n(j)$ and $\mathcal{BP}_s^n(j)$, and proved that for all $1 \leq j \leq n-1$,

$$\mathcal{I}_s^n(1) \subseteq \mathcal{BP}_s^n(j) \subseteq \mathcal{I}_s^n(j). \tag{4.19}$$

Koldobsky also asked whether, in fact, $\mathcal{BP}_s^n(j) = \mathcal{I}_s^n(j)$ holds not just for $j = 1$ and $j = n-1$ but for all $2 \leq j \leq n-2$ as well. If this were true, a positive answer to the j-codimensional Busemann-Petty problem for $j \geq n-3$ would follow, since $\mathcal{K}_s^n \subseteq \mathcal{I}_s^n(j)$ for those values of j. However, Milman gave the following negative answer to Koldobsky's question.

Theorem 4.4.1. ([**71**]) *Suppose that $n \geq 4$ and $2 \leq j \leq n-2$. Then there exists a smooth star body of revolution $D \in \mathcal{I}_s^n(j)$ such that $D \notin \mathcal{BP}_s^n(j)$.*

Note that Theorem 4.4.1 did not resolve the open cases of the j-codimensional Busemann-Petty problem since the body D is not necessarily convex.

Motivated by the formal analogy between Theorems 4.1.1 and 4.1.2 as well as the definitions (4.16) and (4.17), we define now two 'duality' transforms on smooth convex bodies using Proposition 2.2.3 of Guan and Ma, thereby extending the map which already appeared at the end of Example (e) in the last section. To this end, let $\mathcal{K}_s^{n,\infty}$ denote the subset of $\mathcal{K}_s^n$ consisting of convex bodies of class C_+^∞.

Definition. *For $1 \leq j \leq n-1$, the map $\mathrm{P}_j : \mathcal{K}_s^{n,\infty} \to \mathcal{K}_s^{n,\infty}$ is defined by*

$$s_j(\mathrm{P}_j K, \cdot) = \rho(K, \cdot)^j.$$

The map $\mathrm{I}_j : \mathcal{K}_s^{n,\infty} \to \mathcal{S}_s^n$ is defined by

$$\rho(\mathrm{I}_j K, \cdot)^j = s_j(K, \cdot).$$

Clearly, the map I_j is a left inverse of P_j, that is $\mathrm{I}_j \circ \mathrm{P}_j = \mathrm{id}$. Moreover, the following immediate consequences of Theorems 4.1.1 and 4.1.2 and definitions (4.16) and (4.17), show that these maps are closely related to the various notions of intersection and projection bodies.

Corollary 4.4.2. *Suppose that $1 \leq j \leq n-1$ and let $K, L \in \mathcal{K}_s^{n,\infty}$.*

(a) If K is j-intersection body of L, then $\mathrm{P}_j K$ is j-projection body of $\mathrm{P}_{n-j} L$.

(b) If K is j-projection body of L, then $\mathrm{I}_j K$ is j-intersection body of $\mathrm{I}_{n-j} L$.

(c) If $K \in \mathcal{BP}^n_s(j)$, then $\mathrm{P}_j K \in \mathcal{K}^n_s(j)$.

(d) If $K \in \mathcal{K}^n_s(j)$, then $\mathrm{I}_j K \in \mathcal{BP}^n_s(j)$.

In view of the duality relations and various analogies between results on j-intersection bodies and j-projection bodies we have encountered so far, it is natural to ask whether there is a relation similar to (4.19) between the classes $\mathcal{K}^n_s(j)$ and $\mathcal{P}^n_s(j)$. Recall fromExample (c) that

$$\mathcal{K}^n_s(1) = \mathcal{P}^n_s(1) = \mathcal{Z}^n_s \qquad \text{and} \qquad \mathcal{K}^n_s(n-1) = \mathcal{P}^n_s(n-1) = \mathcal{K}^n_s,$$

which is the dual analog of (4.18), and $\mathcal{Z}^n_s \subseteq \mathcal{K}^n_s(j)$ for all $1 \leq j \leq n-1$. The following questions now make sense:
Questions *Suppose that $2 \leq j \leq n-2$.*

(a) Is it true that $\mathcal{K}^n_s(j) \subseteq \mathcal{P}^n_s(j)$?

(b) Is it true that $\mathcal{Z}^n_s \subseteq \mathcal{P}^n_s(j)$?

For the one-parameter family of j-projection bodies K_λ constructed in Example 4.3.2 (f) in the last section, it is possible to show that both problems have a positive answer. In fact, the entire family K_λ is also contained in $\mathcal{K}^n_s(j)$.

With our final two results result we provide a partial analog of Theorem 4.4.1 and answer Questions (a) and (b) in the negative.

Theorem 4.4.3. *There exists a smooth convex body of revolution $K \in \mathcal{P}^4_s(2)$ such that $K \notin \mathcal{K}^4_s(2)$.*

Proof. For $\varepsilon > 0$ and $t \in [-1, 1]$, let

$$s_\varepsilon(t) = 1 + \varepsilon + \frac{5}{2} P^4_4(t) = \frac{3}{2} + \varepsilon - 6t^2 + 8t^4,$$

where P^4_4 denotes the Legendre polynomial of dimension 4 and degree 4. We first want to prove that for every $\varepsilon > 0$, there exists a strictly convex body of revolution $K_\varepsilon \in \mathcal{K}^4_s$ whose area measure of order 2 has a density of the form

$$s_2(K_\varepsilon, \bar{e} \cdot .\,) = s_\varepsilon(\bar{e} \cdot .\,).$$

To this end, we will show that $s_\varepsilon(\bar{e} \cdot .\,)$ satisfies conditions (i), (ii) of Theorem 2.2.4 with $j = 2$ and $n = 4$. Note that since s_ε is an *even* polynomial which is strictly positive for every $\varepsilon > 0$, condition (i) holds trivially. However, from

$$\int_t^1 \xi\, s_\varepsilon(\xi)(1 - \xi^2)^{\frac{1}{2}}\, d\xi = \frac{(1 - t^2)^{\frac{3}{2}}}{42}(13 + 14\varepsilon - 12t^2 + 48t^4)$$

it follows by a straightforward calculation that also condition (ii) is satisfied.

Next observe that, by (2.18), $a^4_{2k}[\mathbf{F}_{-2}] = (2\pi)^2(-1)^k$. Thus, by Theorem 4.1.2, K_ε is a 2-projection body of a dilate of itself. It remains to show that $K_\varepsilon \notin \mathcal{K}^4_s(2)$ for sufficiently small ε. To this end, note that there exists a *unique spherical* function $\varrho_2(K_\varepsilon, \cdot) \in C^\infty(\mathrm{Gr}_{j,n})^{\mathrm{sph}}$ such that

$$\mathrm{vol}_2(K_\varepsilon | \cdot\,) = \mathrm{C}_2 \varrho_2(K_\varepsilon, \cdot). \tag{4.20}$$

Indeed, by (4.15), the function $\varrho_2(K_\varepsilon, \cdot)$ is given by

$$\varrho_2(K_\epsilon, \cdot) = \frac{3}{2\pi}(\perp_* \circ \mathrm{R}_{2,3})^{-1} s_\varepsilon(\bar{e} \cdot\ .\,).$$

This is well defined since $s_2(K_\varepsilon, \cdot)$ is smooth and $SO(n-1)$-invariant and therefore, by definition, spherical. The uniqueness follows from the injectivity of C_2 on spherical functions. We conclude that, since K_ε is $SO(n-1)$ invariant and C_2 commutes with rotations, the $SO(n-1)$ symmetrization of any measure satisfying (4.20) must coincide with $\varrho_2(K_\varepsilon, \cdot)$. Hence, in order to prove that $K_\varepsilon \notin \mathcal{K}^4_s(2)$, it suffices to show that $(\perp_* \circ \mathrm{R}_{2,3})^{-1} s_0(\bar{e} \cdot\ .\,)$ attains negative values.

Using the spherical Radon transform $\mathrm{R} := \perp_* \circ \mathrm{R}_{1,n-1}$ and the composition rules for Radon transforms, it follows that

$$(\perp_* \circ \mathrm{R}_{2,3})^{-1} s_0(\bar{e} \cdot\ .\,) = \left(\mathrm{R}_{1,2} \circ \mathrm{R}^{-1}\right) s_0(\bar{e} \cdot\ .\,).$$

Hence, since $a^4_0[\mathrm{R}] = 1$ and $a^4_4[\mathrm{R}] = \frac{1}{5}$ (see, e.g., [**44**, Lemma 3.4.7]), we must show that

$$\left(\mathrm{R}_{1,2} \circ \mathrm{R}^{-1}\right) s_0(\bar{e} \cdot\ .\,) = \mathrm{R}_{1,2}\left(1 + \frac{25}{2} P^4_4(\bar{e} \cdot\ .\,)\right)$$

attains negative values. Now, by [**71**, Corollary 3.3], we have for $f \in C[0, 1]$,

$$\left(R_{1,2} f(\bar{e} \cdot .)\right)(E) = \frac{2}{\pi} \int_0^1 f(|\cos(E, \bar{e})|\, t)(1 - t^2)^{-\frac{1}{2}}\, dt.$$

This means that we have to find a $\xi \in [0, 1]$ such that

$$\int_0^1 \left(1 + \frac{25}{2} P^4_4(\xi t)\right)(1 - t^2)^{-\frac{1}{2}}\, dt = \frac{15\pi}{2}\xi^2(\xi^2 - 1) + \frac{7\pi}{4} < 0.$$

Clearly, one possible choice is given by $\xi = \frac{1}{\sqrt{2}}$. □

Before we prove our last result we need a few facts about zonoids and projection generating measures. Let $S_j : (\mathbb{S}^{n-1})^j \setminus N \to \mathrm{Gr}_{j,n}$ be given by

$$S_j(u_1, \dots, u_j) = \mathrm{span}\,\{u_1, \dots, u_j\},$$

where N is the subset (of measure zero) of j-tuples of linearly dependent unit vectors. Moreover, let $D_j : (\mathbb{S}^{n-1})^j \to \mathbb{R}$ be the absolute value of the determinant of a j-tuple of vectors in the subspace spanned by them. The *j-th projection generating measure* $\rho_j(K, \cdot) \in \mathcal{M}(\mathrm{Gr_{j,n}})$ of a generalized zonoid $K \in \mathcal{K}^n$ is now defined as the push forward measure

$$\rho_j(K, \cdot) = \left(D_j \, d\rho_K^j\right)^{S_j} . \tag{4.21}$$

The projection generating measure has the property that

$$\mathrm{vol}_j(K|\cdot) = \mathrm{C}_j \rho_j(K, \cdot).$$

For a zonoid that is the Minkowski sum of vectors in general position it takes a particularly simple form.

Lemma 4.4.4. *Let $Z \in \mathcal{Z}_s^n$ be the Minkowski sum of m vectors, $\alpha_j v_j$, in general position such that its generating measure is given by*

$$\rho(Z, \cdot) = \sum_{j=1}^{m} \alpha_j \delta_{v_j}.$$

Let moreover $\mathcal{I}(Z, j)$ denote the set of ordered j-tuples of the generating vectors v_i of Z. Then the j-th projection generating measure is given by

$$\rho_j(Z, \cdot) = \sum_{I \in \mathcal{I}(Z,j)} \left(\left(\prod_{i \in I} \alpha_i \right) D_j(I) \, \delta_{E_I} \right),$$

where $E_I := S_j(I)$.

Proof. It is not hard to see that $\rho_j(Z, \cdot)$ has to be an atomic measure. Furthermore if a subspace $E \in \mathrm{Gr_{j,n}}$ contains strictly less than j of the generating vectors of Z, then clearly $\rho_j(Z, \{E\}) = 0$. Let now $I \in \mathcal{I}$. Then, since Z is in general position, $v_{i_1}, \dots, v_{i_j}$ is the only j-tuple in $\mathcal{I}$, such that all vectors are contained in E_I. Therefore, we have

$$\rho_j(Z, \{E_I\}) = \left(\prod_{i \in I} \alpha_i \right) D_j(I) \, \delta_{E_I}.$$

□

The next lemma characterizes certain cases when a zonotope is a j-projection body.

Lemma 4.4.5. *Let $n \geq 5$ and $3 \leq j \leq n-1$. Then an origin symmetric zonotope Z is the j-projection body of a $Z^* \in \mathcal{K}^n$ if and only if Z^* is a zonotope which has a projection generating measure given by*

$$\rho_{n-j}(Z^*, \cdot) = \rho_j^{\perp}(Z, \cdot).$$

Proof. Z is the j-projection body of Z^* if and only if

$$\mathrm{R}_{j,1}\rho_j(Z, \cdot) = c_{j,n}\, S_{n-j}(Z^*, \cdot). \tag{4.22}$$

Since this implies that $S_{n-j}(Z^*, \cdot)$ is concentrated on a finite union of sub spheres we conclude that $Z^* \in \mathcal{P}^n$. Since $Z \in \mathcal{K}_s^n(j)$ we also know that $Z^* \in \mathcal{K}_s^n(n-j)$. Thus $Z^* \in \mathcal{P}^n \cap \mathcal{K}_s^n(n-j)$. By corollary 3.6 in [**72**], since $n - j \leq n - 3$, we can conclude that $Z^* \in \mathcal{Z}_s^n$. We also know that its projection generating measure $\rho_{n-j}(Z^*, \cdot)$ satisfies (4.22) aswell. Since, both measures are atomic measures concentrated on a finite set of $(n-j)$-dimensional subspaces, we conclude

$$\rho_{n-j}(Z^*, \cdot) = \rho_j^{\perp}(Z, \cdot).$$

□

Theorem 4.4.6. *Let $Z \in \mathcal{Z}_s^5$ be a zonotope in general position with non-empty interior. Then, Z is a parallelotope if and only if it is a 3-projection body.*

Proof. Let $Z \in \mathcal{Z}_s^5$ be the Minkowski sum of m vectors in general position. Then by Lemma 4.4.4 we know that $\rho_3^{\perp}(Z, \cdot)$ is concentrated on $\{E_I^{\perp} : I \in \mathcal{I}(Z, 3)\} \subseteq \mathrm{Gr}_2$. Let us assume, Z is a 3-projection body. Then, by Lemma 4.4.5, there exists a zonoid Z^* such that $\rho_2(Z^*, \cdot) = \rho_3^{\perp}(Z, \cdot)$. From (4.21), it follows that any pair of independent generating vectors of Z^* has to span a subspace on which $\rho_2(Z^*, \cdot)$ is concentrated. Hence, for any generating vector v of Z^*, there exist $I, J \in \mathcal{I}(Z, 3)$ such that

$$v \in E_I^{\perp} \cap E_J^{\perp} = (E_I \cup E_J)^{\perp}.$$

It is easy to see, that for $k \geq 5$ and distinct $I_1, \ldots, I_k \in \mathcal{I}(Z, 3)$

$$E_{I_1}^{\perp} \cap \cdots \cap E_{I_k}^{\perp} = (E_{I_1} \cup \cdots \cup E_{I_k})^{\perp} = \{0\}. \tag{4.23}$$

Indeed, if the intersection was not trivial then the union of the entry vectors of the I_l, $1 \leq l \leq k$, has to be a set of at most 4 vectors. However, we can choose at most $\binom{4}{3} = 4$ distinct 3-tuples from a set of 4 vectors. Since, any of the $\binom{m}{3}$ subspaces $E_I^{\perp}$, $I \in \mathcal{I}(Z, 3)$ contains at least 2 generating vectors and by (4.23) any given generating vector lies in at most 4 subspaces we conclude that there have to be at least $\frac{1}{2}\binom{m}{3}$ generating vectors of Z^*. Let us now assume that $m > 5$. Then, there are at least 10 generating vectors. Since, any pair of them had to span an $E_I^{\perp}$,

$I \in \mathcal{I}(Z,3)$, we would obtain that there is a generating vector that lies in more the 4 of those spaces. However, this can not be true by (4.23) and we conclude that $m = 5$.

$\square$

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Curriculum Vitae

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