MERIDIAN ARCHITECTONICS 2

Elias Zafiris

Mathematical Thinking

An Involution for Architects

TU Wien
Academic Press

MATHEMATICAL THINKING Elias Zafiris

MERIDIAN ARCHITECTONICS
Volume 2

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MERIDIAN

ARCHITECTONICS

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H. WEYL

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N. A. SALINGAROS

Foreword:
Nature in Nuptials, Orphean Quests and the Portico of the Arts

by Vera Bühlmann

The rose is without "why"; it flowers because it flowers.

ANGELUS SILESIUS

Do you want the flower to open only once? The unveiling of the opening would then belong to you. The beauty or truth of the opening would be your discovery. Proposed and exposed in one definitive blossoming. The nightly closing of the flower, its folding back into itself would not take place.

L. IRIGARAY, *ELEMENTAL PASSIONS (1980)*

Mathematical Thinking, An Involution for Architects (2025), these are lecture notes composed by Elias Zafiris who has been teaching our architecture theory students in mathematical thinking for years now. Students are often puzzled by what they are shown in his classes: *it is not about solving a problem, it is about becoming friends with an obstacle,* he likes to say, and *it is about figuring out how to tune better with what appears to block the way.* This goes without compromising either the ratiocination[1] nor a personal and wholistic world approach. *Mathematical thinking is embodied thinking, and yes, the ideas it articulates are timeless. This is not a logical contradiction—it is a miracle!* Zafiris will exclaim. A theorem (any theorem) is inexhaustible in its capacity—and yet it can be "discovered". Who would have said the last word on Thales's or Pythagoras's theorems, or those associated with Gauß, Riemann, or Grothendjik? We can speak of a theorem as "remarkable" and "noticeable" in an evaluative sense, of course. But we can do so also in a very practical sense, expressing astonishment *that* such witnessing is possible and evidenced, *that* theorems can be taken note of, and *that* these notes can be shared, circulated in communication, and help others to begin "witnessing" too.

*

A theorem, to Elias Zafiris, is a surging well, mathematics the art of a cornucopian minimalism. Its discovery and its appreciation involves inventiveness, interrogative cunning, generational respect and attentive witnessing for which it takes the humbleness of thinking not in one's personal name, but anonymously.

Mathematical theorems bound and tile an impredicative domain of iconicity, they install an inside-outside distinction to which it is categorical. In the more recent logics (of roughly the past one hundred years) we have tended to overpaint in continuous white color, so to speak, this discretionary, colorful and mosaic

1 We use the notion of "ratiocination" to indicate that a theorisation involves rationalisation, in the circular and recursive sense that also the rationalisation already involves a theorisation.

nature that is in constellatory play on the inner firmament of the starry skies which we can study when performing abstractions. One writes in architectonic orthography, ways or styles of spelling that articulate *words of witnessing* in the notational codes of cryptographic syntaxis. One relates to this domain of iconicity through vision, but the images are neither properly representations nor icons. They are *suggestive* of being iconic. They affect the one who enters their spell(ing), but inversely to the singularising address that is peculiar to an icon. The iconicity at stake here is visionary in that its ideations are composed of what Elias Zafiris calls "schematic tilings". While an icon lets one know *you have been meant, it is* you *that is being addressed here*, the schematically patterned "appearances" on the other hand let us know: *it does not matter that you are you. What appears to you here can be contemplated by anyone at all*. They also caution whoever contemplates them by whisphering: *don't think that you have seen it, gotten it all; surrender to the absurd depth in the presence of this experience by facing up to it*.

Mathematical thinking is anonymous, but it is active and embodied; miraculous or not, there is an irreducible aspect of experience that propels and triggers its vivacity. Ideas of mathesic iconicity are aesthetic and motivic, they are ideas that "comport" themselves in things that embody them. They grasp how for example the universal reign of gravity all throughout the natural world is receptive to being "charmed" in a great variety of interplays—arches embody such "enticements" of physical "sympathies" perhaps more than anything else. It is such anonymous embodiments of ideas that are remembered or forgotten, respected or belittled by the arts in general—ideas in a *mathesic iconicity* for which it matters not who one is, individually, when one contemplates them; this is in inverse fashion to what religious icons are appraised for. But here too, there is appreciation for a quickness to thought, an autonomy even for which it is not a contradiction to say that it needs figuring out. This is an artistic and a realist stance. Thoughts and ideas are what they are, but they are always also subject to formation. Their clarity is incandescent as well as caustic, it is not something self-evident. This is the realism of a mechanic. It is a realism that embraces this quickness as it may begin to take root and inhabit the mind.

*

The notion of the schema plays a role in such mathesic iconicity with respect to how such embodied ideas have history and maintain kinship relations. It is in schemata that they mingle and "socialise," so to speak. Things interiorate schemata in their embodiments, and an embodied idea engenders others. There is an objective *display* of ideas in the skills and knowing-how to handle things—schemata are objectively transcendental in their performed *rationcinations*, in their intelligibility. They are how embodied ideas of mathesic iconicity participate in a generational dynamics. The ordering-play of this dynamics happens in circles, its swells and ebbs in phases, with rhythms and intellectual force. It manifests in periods, but the periods are "leaky". The orderings that are displayed in the compositions of notational markings percolate through one another more like chords do in music than how letters line up in syntax. Generational ordering-plays do not proceed by linear unfolding or graphic fixation into representative images—they are instrumental for theoretic formation, but they are not "theories". Any architectonic articulation that involves mathesic iconicity is endowed with inclinations, characterised by declination and demarcated by indexical prepositions, voice, tempi. The anonymous and objective display of ideas constitutes *syntaxis* as an action word, a verb that "happens" like the weather does between orthography and grammar: architectonic articulation involves syntaxis as an action word.

The dynamics sought when thinking about how mathematical ideas are born and prosper and age (even though they are motivic and timeless) is not only one of intergenerationality, it is also one of "respect". Respect emphasises the vernacular and moral aspects that are irreducible. Respect does not reduce to the attribution of epistemological values and social protocols. The kind of respect that is being communicated, shared and exchanged through mathesic iconicity is a veritable *physics of respect*. It is moral and yet it follows an autonomy, i.e. as optics and physics of light.

*

Intergenerational respect lets one rarely think straight. It urges the study of obstacles, as Zafiris does in these lecture notes. If the study of problems comes with a certain conquest mentality, the study of obstacles comes with a stance of what I want to call *considerate resignation*.[2] Resignation is addressed here in its power to intervene and "lend" time where a course of events appears to speed up, to escalate. It does not amount to surrendering to impotency, as is often the case when we speak of resignation as a mental state rather than as an active gesturing. For a geographer too, it is not about the conquest of territories when he sets out to draw maps. His is a holisitc approach to the world, his aspiration is to embrace the whole world in every one of his maps. Resignation is the gesture that alone can give adequacy to such aspiration. A *stance of considerate resignation* then is one where one embraces obstacles like the geographer is to embrace the whole world, namely with "arched arms, with elbows that are not rigid" and "fingers

2 I translate hereby the proposal by Franco Farinelli, a geographer who thinks of resignation as a gesturing rather than as a mental state. This invites to think of resignation spatially. Cf. Franco Farinelli, *The Blinding of Polyphemus, Geography and the Models of the World,* Seagull Books, New York 2018 [2003].

that point forwards to continue along the axis of the forearm"[3]. The cultivation of intergenerational respect too asks for such a reaching out, in difference and embrace.

If we consider the casting of such stances as taking part in what we call mathematical thinking, then there is an atomist physics of respect intrinsic to mathematical thinking. Abstraction opens up into domains that are metaphysical. The practice of intergenerational respect as an atomist physics keeps suspended a categorical separation between that which can be thought and demarcated (annotated), and that which is thinkable in an absolute sense; the power of considerate resignation is a power only when kept on hold, remaining undecided with respect to the scales in which to get familiar with the obstacle. With this categorical separation, mathematical thinking can accommodate cyclicity, it can form by recursion through involutions and evolutions. Through stances of considerate resignation mathematical thinking distinguishes itself intrinsically, even though it is an anonymous mode of thought.

Mathesic iconicity lives from the pantomime-like aspiration to embrace the whole world. This is the desire of mathematical thinking in the portico of the arts, this is what propels an obstacle-oriented stance instead of an object-centric or problem-centric one. It is hence not concepts (in German *Begriffe*) that are sought, inferentially formed and achieved without a getting intimate with what they are to keep contained. What mathematical thinking seeks to achieve in the portico of the arts is to mobilise negatives as forms; to cast moulds that facilitate the display and displacement, the transposition and translation of ideas across and through the formality of its conceptions.

3 Ibid.

*

Mathematical thinking then does not compete with philosophy, it cohabits with it. There is a place where philosophy is at home with it, we could say. In the optical physics of intergenerational respect, this place is a universal house of cosmic decorum. Here, philosophy is both at once at its acutest and at its most moderate capacities. That is to say, it resigns from its office but not from its practice. Mathematical thinking helps philosophy to remember the crucial role of respect for such a place which it shares with all forms of embodied thought—the universal house is a "home" that is not properly human. It accommodates a universal *oikos* that is larger than what it can "picture".

For philosophy at home with itself like this, concerns with form and formality are not a question of choice between elements (form) *or* atoms (matter), as the discursive positions of epistemology versus positive natural science would like to make us think. Rather, such concern revolves around how they both are capable of co-accommodating each other. It would then be a question of how the *partitif* (the notational means of discretion, logically speaking: axiomatics as a discipline) exists within the *impartitif* (a predicative domain of impartiality, logically speaking: existence or universality). The ethical "choice" ultimately is not an act of decision at all, but one of figuring out a stance that manifests exterior to one's own act of assuming it—stances as negative forms or plastic moulds to grow into. Such casting of stances manifests in the mathematics with which one articulates the situation that asks for a *ratiocination* in *logoi* (in words and graphism).

In the sense of a cosmic decorum, such a universal home is a place where it would be proper (house rule) for any of thought's individuated instances to dwell not by claiming a "proper" place but one where they can temporarily *resign from their political and economical affairs*—this would be philosophy as an involved and situative as well as abstract and rational life form. With a home place where philosophy itself is not fully in charge, philosophy would be capable of teaching *artistically* and *architectonically*, not only reflectively or dogmatically. Such teaching, one can imagine, would be guided by the following recursive rapport: mathematical thinking proceeds methodically, and it always entails philosophical stances; philosophical thinking on the other hand is embodied, and it always entails mathematical casts. Like this, architectonics would put in place a self-governing principle (and its mechanics) of common moderateness.

*

With editing these lecture notes as one of the first volumes in the Meridian Architectonics Books Series, our ambition is also to reconnect with what Elias Zafiris has placed as a kind of chord in the beginning of the book:

> Historically, architecture was part of mathematics, and in many periods of the past, the two disciplines were indistinguishable. In the ancient world, mathematicians were architects, whose constructions—the pyramids, ziggurats, temples, stadia, and irrigation projects—we marvel at today.[4]

There is a sense of datedness to a statement like this. But why is it, that this sounds out of time to our ears? Was not the passion of the modernists also what we are talking about here, namely to relate art, including architecture, more strongly with mathematics? What especially the Bauhaus movement sought was arguably not a "home" place for architecture and design, but primarily a place *among* and *amidst* the institutionalised power of

4 Nikos A. Salingaros, "Architecture, Patterns, and Mathematics", in *Nexus Network Journal*, 1(1): 75–86, 1999.

the sciences. Mathematics counted to them as that which can endow things with universality. But it had been assumed as a gratuitous guarantee, a back up for the outlook upon how the arts could work together with the sciences in building a global culture. The Bauhaus movement's ambition was to offer a non-suspicious (modern, neutral) catalog of architectural elements. Those elements were to be systematised through technical and economical *standards* rather than through decorum orders of style and canons of well-proportioning.

Erwin Panofsky, the art historian, has captured already in the 1930s a certain spirit of resentment against the association of mathematics with decorum through the question of proportion. It is a resentment that perhaps prevails even more so today than it did then:

> Considerations about questions of proportion are usually received with scepticism but without any particular interest. Neither is surprising. The mistrust is based on the observation that research into proportions in particular is all too often subject to the temptation to read something out of things that it has put into them itself; the indifference is explained by the modern subjectivist view that an artistic achievement is something irrational. A modern viewer, with his still essentially romantic conception of art, finds it downright embarrassing, or at least uninteresting, when the historian tells him that this or that representation is based on a rational law of proportion or even a certain geometric scheme.[5]

Endowed with a certain licentiousness of how the irrational *cannot-but-will-have-to-be* accommodated within the rational, the promise was as grand as it was forceful: it absolved the individual

5 Erwin Panofsky, "Die Entwicklung der Proportionslehre als Abbild der Stilentwicklung", *Monatshefte für Kunstwissenschaft*, November 1921/22, Vol.14, Nr. 2, Deutscher Kunstverlag GmbH, 188–219. Here and in the following my own translation.

from taking an ethical stance as a person while it promised that as standards, such mathematatised and catalogued "elements" of architecture were to be continually refined and optimised not only collectively, but formostly also pragmatically. The role of mathematics in such a pragmatic program was meant as one capable of reconnecting knowledge practices with an unbiased neutral mindset. As an aspirtion, Umberto Eco has gently but also wittily portrayed this in an essay that made a strong case for a necessary critique of such "mass culture". Eco portrayed the driving momentum as the split mindset of being apocalyptic and integrated.[6] In it, Eco aptly captured the twisted gesture that animates such a notion of pragmatic and moral moderateness—with its mass-culture confidence in personally riskless generosity in questions of morality. There was an implicit certainty at work here, believing that nobody in her or his "right" mind could question the *sense* (and that is, the *direction*) of such collective movement forwards—at least when matters of public concern are at stake, so this certainty suggests, one ought to be able to count on everyone thinking "reasonably".

This indeed is also what inspired Panofsky when he thought to focus anew on the canons of proportioning. He writes:

> Nevertheless, it is by no means unrewarding for scientific research in the arts (provided that it confines itself entirely to the established facts and is prepared to work with meagre rather than dubious material) to concern itself with the history of proportion studies [...].[7]

There is a clear conspiracy with the modernist spirit Panofsky had addressed initially in his text (the sense of embarrassment

6 Umberto Eco, *Apocalittici e integrati: comunicazioni di massa e teorie della cultura di massa*, Bompiani, Milano 1964.

7 Ibid.

or at least boredom when mathematical precision in art is being emphasised), as he promises a reward: "It is not *unrewarding* [my emphasis] to attend to", as he says. It is not unrewarding because contemplating the "grip" that held the study of proportion through historical developments, this will strengthen also Panofsky's modernist spirit; it will make it stronger not weaker. He elaborates thus:

> If we seek to recognise the various systems of proportion of which we are aware, not by their appearance but by their meaning, i.e. if we consider not the solution given in them but rather the question contained in them, they will reveal themselves to us as expressions of the same artistic volition [*das Kunstwollen*] that was realised in the buildings, sculptures and paintings of the same period or by the same master: the history of the theory of proportion is the image of the history of style, and given the unambiguity with which we can communicate with each other in the mathematical field, it may even be regarded as an image that often surpasses its archetype in clarity. One could argue that the theory of proportion expresses the volition of art [*das Kunstwollen*], which is often not easy to conceptualise, in a clearer or at least more definable form than the works of art.[8]

The emphasis is exactly on such "unambiguity with which we can communicate with each other in the mathematical field" that excludes intergenerational respect in the atomist manner of a physics of light.[9] It propagates an attention to mathematical ideas not as embodied thought, animated by an own and ultimately

8 Ibid.

9 Panofsky's "strategy" here (to hide an "agenda" behind the "plainness" of mathematical unambiguity) is—in miniature gesture perhaps, but nevertheless in essence—not unlike that which happens on broadest scale today in the rule and reign of algorithms, which inevitably hide and embody social and moral bias and administrative agendas that are likely to introduce great unfairness and injustice, cf. Cathy O'Neil, *Weapons of Math Destruction. How Big Data Increases Inequality and Threatens Democracy*, Crown, New York 2016.

miraculous "quickness" as it had intimately affected the *mechané* tradition (ars/techné) in pre-modern institutionalisations of *scienzia*. Instead, the open and opening interplay of such quickness was to be *fixed and bound up* in new (modern) authoritative representations. History as practice in the service of the modernist aspirations was to authorise depictions of the collective "Kunstwollen", as the terminology popular at the time phrased its attention to the anonymous quickness that animates the lifes and deaths of mathematical ideas through time.

In order to grasp the restrictive agenda at work here, we should remember that around the same time other great minds like Alfred Einstein or earlier in the eighteenth century also Immanuel Kant were not shy in publicly marvelling why it is—as if miraculously so—that one can apparently *understand* the world through mathematics. With it, one can master processes reliably, repetitively, concerning things as different as the course of the stars, the velocity of rocks or projectiles or the damming of water masses, the circulation of money and the expectations of regional shortages in supplies and even the dis/pleasing effects of sounds in music or of shapes in volumes or placements of colours on canvases, or the casting of figures in sculpture.

Why indeed is it, that mathematics seems needed whenever it is about recognising things *aptly*?

*

In recent decades, voices of such marvelling have largely silenced. While the application of mathematical thinking gives great stability, thinking about the status of mathematics with respect to our thinking does not. Rationalists and empiricists hold very different views in this matter—views which nourish the love and conviction that scientists feel for science; over this matter, storms are stirred in the hearts of beliefs most intimate to the soul. This is iconically captured with the second motive that Elias Zafiris has placed as a motivic key to his lecure notes. It is a famous citation by the mathematician Hermann Weyl, one of the greatest polymaths in the 20th century, who observed plainly:

> In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics.[10]

Among the modern conquest of the global scale through networks of logistics, not only individual persons were imagined to live in a continuous fight for their souls (the ancient quest for living an ethical life) but also the different disciplines of the sciences. Among them are even those to which we usually (and a bit thoughtlessly) refer to as if there were 'one' in the sense of a systematic order, namely the disciplines of mathematics themselves.

As much as 20th century philosophy (of mathematics, technology and communication/computation) passionately engages in fights for one over the other, calculus over geometry (or the reverse), algebra over logics (or the reverse), Elias Zafiris' approach presented here in these lecture notes is not continuing this struggle.[11] To him, mathematics, like music or architecture, does not represent anything—it does not *represent,* neither nature nor mind, neither word nor idea nor things or events. But it indispensible for the display (*Darstellung* in German) of their representation; it is indispensible for giving attention and taking notice of all of the above in the first place. This is also why we speak allegorically of the place where the different disciplines distinguish themselves through mingling and collaborating and joining forces

10 Hermann Weyl, "Invariants", *Duke Mathematical Journal*, 5(3): 489–502, 1939.

11 His approach is elaborated elsewhere more fully and discursively: *Natural Communication. The Obstacle-Embracing Art of Abstract Gnomonics*, Birkhäuser, Vienna 2021.

with one another neither as competitive fields of expertise nor as a common domain of inter- or transdisciplinarity, but more specifically as the *portico of the arts*—a place of anonymous display and disinterested study, a place of outlook and encounter that neither belongs properly to the orders of the house (oikos, economy) nor those of the institutions (the public and the sacred), but more like a mingling place or *foyer* prior ot any of these orders—vibrant with outlooks beyond the familiar and pulsating with unsettled and exuberant forefeeling and foresight that keep reserved about turning into forecasts and predictions. They are embodied acts of abstraction and intellection that attend to the whole world in summational manners, collecting themselves and keeping composed in maintaining vivid *Lichtblickbeziehungen* rather than choosing a "focus", targeting a direction; *Lichtblickbeziehungen* is a poetic German word that circumscribes something like *rapports of radiance and brightness*, logistic "traffic" relations of crossing sight with appearance, active gaze with lightrays. Can we reconnect with the generalist tradition of "studies" as known from natural philosophy and the architectonics of "arts" that are as "fine" as they are "mechanical", arts that are liberal, *frei* in German?

Zafiris rejects the often predominant aversion against tautology (a logos that refers to itself) and instead urges us to learn inventing cosmic *scales* for tempering such self-referential circuitry: Like music *as* music, mathematics *as* mathematics can be experienced in distinctive ways through a physics of respect that studies intergenerational convivality: one cannot but cohabit in abstractions when keen on figuring out how to embody a thinking that is not one's own, that comes from elsewhere and that begins to make sense and take hold only when engaging with it affirmatively. The portico of the arts is a generic place, but it is a place that celebrates universality and distinctiveness. It is is full of stasis and motion. A dynamics of considerate resignation accommodates and houses plays of recognition in this portico—a dynamics like that of the global climate which accommodates and houses the local weathers.

Such a notion of abstraction does not seek to realize them, nor to build foundations for some of them, rather it collects in its un-achievable summations the un-fathomable depth of reality. The domain of abstraction is absurd but it is also cornucopian—if one does not fix it to any one ideal reference in particular. If mathematics itself gathers, collects and organizes different disciplines in its organon (its body, its *instrument* of *pulsation* as Zafiris calls it), then this organon must not only host and co-accommodate different coding instruments that "sound the same breath" mathematically—instruments like rhetorics, poetics, or architecture; it also owes its own living breath at least partly to the wholistic interplay between them.

The predominant question then is this: How to think of such *active* summation, the abounding totality and the fertile absoluteness of mathematics's cornucopian *Organon*? Art has always been more unsettling than just the expression of aesthetic sentiments of individual persons—like science and like technology, art provides mirrors in time and acts as catalyser for inquiring quests into the not firmly settled and known. The arts and the sciences have roots in the same grounds: they share the thrust to explore the unfamiliar and to transgress established borders, to shift boundaries and challenge us to view the world as a whole anew such as to accommodate the ceaseless surging of unheard possibilities.

*

We are well used to thinking that rhetorics, and to a certain degree even poetics, work with mathematical thinking insofar as the certain mechanisms that objectify and teach their crafts involve, like all mechanisms do, a certain "mechanics", i.e. mathematics. To consider that same relation also inversely in the other direction, this feels much more unsettling. And yet it is what Elias Zafiris proposes: not only does figurative, metaphoric speech involve a certain kind of mathematical thinking; but mathematical thinking in reverse involves acts of figurative, metaphoric thought. He follows the natural philosophy commitment that embraces the stance of the "mechanic", literally the one who is committed to truth but who also thinks resourcefully when searching it (*mechané* once meant "resourceful" in Greek). The stance of the mechanic will take pride in knowing as precisely as it can where the boundaries of its empowerment are; this stance humbly trusts experience above all else—it stands for an embodied way of *knowing with accomplishedness* what it can dare to take up as a challenge. The stance of the mechanic involves an embodied *Können*, as the German language puts it. Like any skill that needs to be acquired through exercise, instruction and experience, such interiorised *Können* is only partially formalisable. That is because all formalisation (in this proposed place of a portico of the arts) proceeds by analogous reasoning in the service of invariances that cannot fully be positivised, that are ultimately bound to remain cryptic. Respectfully to this, analogous reasoning weighs actively in each instanc how to proportion things. It maintains a vernacular but cosmic approach to schemata that make cases and mediate between them. It is in this sense that Zafiris speaks of *schematic logoi*.

*

To say that an equation, a theorem, a proof, a system (which all accommodate a certain "mechanics") in turn "work" only because they employ *vessels of transport* (metaphora) and *vessels of collection* (amphora), this introduces conditions of possibility to notational devices of demonstration. This is very significant, because it multiplies domains of objectivity. The challenge to proportion what is only *like* makes a three-body problem out of an analog polarity (adding the means or scale employed for its rationalisation as a transparent yet constitutive "third").[12] And this is because in a three-body problem objects keep balance in a tripod way. Such *active weighing* activates or animates objective thought with a transcendental motion, and while this motion is what notational means capture, it ultimately remains elusive in the full quickness by which this motion is animated and propelled. With regard to language we may be less uncomfortable in seeing it this way—words are "spirited", in usage and communication they hardly stay put entirely in the confines of semantic definitions. This motion, to Zafiris, manifests actively (yet never immediately) not only in language but in any act of coding: notational devices of demonstration (equations, proofs, demonstrations etc.) work only because they involve a transformation of *transposed* schemata of sense making. Inspired by Proclus's commentary on Plato's *Timeaus*, Zafiris speaks therefore of such formal *vessels* (of transport, or collection) as *schematic logoi*. Viewed in this way, the metaphoric trans-positioning of sense is no less substantial for the logics of an articulation than that triggered by figurative speech, speech that displays itself in analogies, metaphors, metonymies.

*

If one makes peace with these proposals, even if only for the duration of an encounter with a certain face, in which mathematics introduces itself here, then one can concede: Mathematics may be

12 The domain in which one proceeds by analogy and in the stance of a mechanic opens up through inflation, so to speak, as an expanse that results from a point that pulsates, that fills and empties. We can think of it perhaps as libration points (also called Lagrange points), which refer to rare places in the outerspace where two large masses, like the sun and the moon, are in such constellation that the gravitational forces act upon each other constantly so as to cancel each other out. "Libration" comes from Latin "librare" and means "to keep balance". Cf. https://en.wikipedia.org/wiki/Lagrange_point (accessed November 13, 2024)

entirely abstract, unaccountable in where it results in or originates from; but like a "living" (spoken) language, it is to be respected as a living "thing". There are letters spelled out and cast into words, yes, but there is also breath circulating and articulated through mathematical thinking as there is breath circulating and articulated through words.[13]

This is not only to say that mathematical thinking is *like* a thinking in words—it also says that thinking in words is *like* thinking in mathematics. There is no hierarchy between the two poles of such principle *likeness*. Ineradicably there is *the presence of an absence* in all notations in code—we will discuss a proposal by Luce Irigaray of how to include *mourning* as a second principle alongside to that of *likeness* in a moment. For if it is code that constitutes the notations in play, then the relation of likeness plays in a *formal* but also *cryptographic* domain of *conveying ways of addressing* as much as in the proposed *address* (the attribution of name and proper place in a larger ordering) itself. Code hence works logistically and tautologically—notations of mathematics and of language mingle together like the elements do in nature. They render multiple embodiments of custom and discipline both same and distinctive. It is not despite but *because* notations in code involve circularity that code can be resourceful in what it unravels: code, like all mechanisms, renders things *apparently* present. Such *rendering* is a two way relation: forth and back, inside-out and outside-in, involving the motions of evolution and of involution.

A stasis (a resting pose, a stand-still, a well-achieved proportioning of multiple involved factors) is never simply a given for the one who articulates in such cryptographic coding. Cryptography in the portico of the arts is what the pantomime does: there are

13 Cf. my elaborations in Vera Bühlmann, *Information and Mathematics in the Philosophy of Michel Serres*, Bloomsbury Academic, London 2020.

“sources” but no “source text” that would exists apart from the copious attention given to the world at large and kept in notations that want to teach and convey anonymously what has been found through personal achievement.

To the mechanically-philosophically reasoning mind, this is no news. Mechanics involves an ethical stance, the zero point as a fulcrum for architectonic coding helps to balance schemata in constellations. Like this, one can appreciate that mathematics literally means “all that pertains to learning”, from *mathemata*, Greek for “what can be learnt”. Code provides a fulcrum, a conversion-point between what can be said and pointed out (indexed) in speech or graphisms and what can be shown and contemplated, demonstrated, in acts of mathesic iconicity.

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But whence-from comes code, one might ask? Zafiris shares a belief in ancient wisdom at this point. Notations in code gather what is *not* contained intrinsically to their notation. They gather things into an interplay by *marking up*, by indicating extrinsically to the things themselves how they may be put into more or less harmonic constellations. Notations can do this in manners that feel forceful or even forced, or with ease and lightness of touch, to the degree that they render themselves almost transparent. Notations can stir up things as much as they can harmonise the world—they can bring things in consonance or dissonance, they can find euphonies (what sounds pleasing) or they can result in noise.

Through the eminent role played by notations in both mathematics and philosophy, there is an overall concern with how things are not immediately graspable, accessible. It is why natural philosophy has always placed mechanics at its center. In antiquity (and also still throughout Renaissance) this was characteristic of an epistemic and mental stance that people used to call *architectonics*.

Achitectonics centers around optics as a particular branch of mechanics—the study of the behaviour and the properties of light.[14] We can think of the relation between natural philosophy and architectonics like that between categories and metaphysics: categories indicate marks of distinction that are (to be) universally valid, and whose arrangement into a coherent whole (a system) is to achieved by metaphysics; architectonics indicates marks of distinction that are to be valid with respect to a plurality of scales that may not be commensurate with one another immediately neither locally nor globally—scales for which notations matter, scales with respect to what we have called a fulcrum provided by code. Mechanics quite literally acts rhetorically from this point of view, and sometimes even poetically: it is capable to translate the complexity of motions from one constellatory interplay to another. Such translation needs to be invented. But such invention involves less an act of *authoring* than one of *discovering and demonstrating*. The centrality of mechanics—literally a mode of thought that proceeds *resourcefully*, never strictly necessary in any one path (method) opted for—results from this capacity to translate physical motions by encrypting and deciphering them notationally, such as to act together in a great variety of constellations. This is where mechanics and rhetorics are kin to each other: rhetorics too works with a mechanics, as much as mechanics works with a rhetorics.

Certain approaches to metaphysics, especially in the modern traditions, have distinguished themselves from the natural philosophy "trunk" through eclipsing the constitutional role of code, schemata, and technical notions for the way they convey signification (meaning and sense making) in their teachings.

14 Johann Heinrich Lambert, *Anlage zur Architectonic, oder Theorie des Einfachen und des Ersten in der philosophischen und mathematischen Erkenntniß*, Hartknoch, Riga 1771.

Architectonics in the natural philosophy tradition works with metaphysical *gestures*; it involves irreducibly an embodied mode of thought. It is an embodied mode not despite but because of the constitutional role played by mathematics for it. For natural philosophy stances, the (metaphysical) categories and their (metanomic) architectonics[15] are not motionless frames extrinsic to natural things themselves, supposed to be holding them together and putting them into place. Categories and their architectonics are instead interiorised as an organon—a framing or skeleton. They can be inhabited projectively and anonymously by anyone through mastering the dispositional play they facilitate, like a mechanism can be acquired projectively and anonymously by anyone. But like in the case of artistic instruments, the scopes of potential are rendered determinable while remaining indefinite: no artist ever exhausts the full potential of an instrument. For embodied thought, mechanics involves an artistic kind of discipline.

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No scale, no mechanics of a fulcrum, works in the vacuum as a play of two, between binary dualities. Scales are embodied. This domain of conveying insights that is established by notational code results from how such embodiments of schematic logoi encrypt and decipher motions in their gestures of thinking, when they "prepare" those motions and "hand them over" for mechanical translation work.

15 Michel Serres has pointed out the necessity to double the "meta" for such architectonic axiology. Ancient librarians gave the title *Metaphysics* to the books of Aristotle that followed or preceded the latter's book titled Physics, he elaborates and continues: "it would have been a fitting addition to this happy inspiration if they had called *Metanomics* the possible books that could have followed those on the constitution of Athens, on ethics or rhetoric," in short "on those formidable rudiments of what went to the humanities". Cf. Michel Serres, *Branches, A Philosophy of Time, Event and Advent*, trans. Randolph Burks, Bloomsbury, London 2019, 48.

We can think of the two poles of a scale (positive, negative) hence as living in a larger domain. Both of them co-constitute it while this domain still transcends each of them alone, in respectfully different ways—Zafiris calls this *the domain informatics.* It is where one deals with notations, ciphering, and coding. His lecture notes do not promise to provide foundations that would work uniformally, but it does offer a stabilisation device: *a tripod*. A three legged stool. It is the most stable construction principle for chairs to sit on, to provide for taking a break, for finding some rest. Designers and engineers know this this characteristic metaphorics well when they entrust so many of their calculations to the *Dreibein-Logik* supported mechanically by the mobile platform on a well-joint plane of stasis, a *Stativ* in German, or in English a *tripod stand*. Not only material calculations work with their support, also conceptual and immaterial ones do—among the idioms of engineers there is an entire vernacular vocabulary that speaks of space as "snuggling" with respect to tripodic logics in differential geometry, as *Schmiegeraum* in German; space as it "nestles" itself respective to certain scalar orderings of embodied spatial articulations.

Zafiris commits himself to the natural philosophy tradition when he proposes to put an epistemic tripod in place, to find some *Muße*—laxness, receptiveness for inspirations—while contemplating intensely the architectonic interrelations between mathematics, philosophy, and informatics. It is indeed a "euphonic" association that Zafiris proposes here. It locates resting places on various degrees of height in abstraction. When attempting to grasp something holistically (in depth, by attempting to sound something more fully) one cannot but learn to inhabit and become familiar with such abstract places. Such resting places, they are places of considerate resignation where philosophy is at home;

they are places where room for intergenerational respect is made, reserved, and kept common (inappropriable).

But one should not forget that this promise of an epistemic resting place depends upon a *pact* between the one who issues the foresight, who fabricates the forecast of a promise and the one who picks it up and seeks to accommodate herself within it: if writing in the stance of providing lecture notes through encryption involves such a making of a promise, then reading those lecture notes involves a certain conspiration with the composer—like a musician who relies on the scores of a composer when interpreting a work of composition. In order to learn from the notes, the reader must actively decipher what is gathered and arranged for demonstration by them. The notes index and talk of something that has an existence autonomous from the one who makes annotations (code gathers things extrinsic to what is contained in its particular notation). Such cryptography works like the deciphering of the Rosetta Stone which was at the same time the reconstructive encryption of the hieroglyph script: it involves intertextuality and mechanics of translation across documents. In the case of mathematics, such "intertextuality" involves also "documents" in various "media" (embodiments). It involves assuming a resignation stance that embraces the absence of the sought reference text in any plain, immediately evident or apparently naked presence. In other words: the tripod will only provide a resting place if the reader finds in it the place to actively cultivate respect. Whenever one gets dizzy and confused at the edges of how two of the domains relate (mathematics, architectonics, informatics), one can find stability by turning to the third. Neither one of them can do without the others, none of them can, hence, dominate the others.

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With the address of this treatise to architects, who is meant are not specifically students or professionals of architecture. The invitation is to explore anew the relation between arts and mathematics as irreducible and integral to science at large. But why does Elias Zafiris propagate such exploration as an "involution" stance? Attending to mathematics as a *praxis* of thought involves a (trans) formation that happens inwardly. While unfolding its insights outwardly into more generally valid applications, Zafiris lecture notes also ask of the reader to explore an interiority to thought itself while doing so. In this, it resembles what is often called "design thinking". The latter too brings our attention to a certain kind of material agency of things, a kind of interiority of objects that manifests itself in their inclinations, tempers, circumstantialities. But unlike design thinking, which tends to proceed empirically and economically, the mathematical thinking at stake here proceeds more along the paths of the rationalists: it's explorations dive inwardly into what Zafiris calls "the depths and heights of abstractions".

How do they differ? Abstractive thought takes artefacts not as solutions to problems. Rather it wants to celebrate the enthusiasm for the ordinary and common that is necessary to view any one artefact in particular as "remarkable" in the first place. It appears to be drawn to destabilise what feels most secure, but this is not out of a revolutionary commitment. Rather, what is most ordinary—almost transparent and unnoticable—is studied as instances of greatest unlikeliness. Wherever a problem seems to present itself, abstractive thought takes the problem into sight as an obstacle and relates to it as a means to defamiliarise mentally with the landscape in which the problem appears to present itself *as* a problem. Hence the emotion of wondering is key for abstraction, but this wonder is not that of a principle sceptic; it is also not that of the cautious methodist—it is no methodical doubt for which doubt were to play on the scale of moral consciousness. Rather, it is something like an existentialist spiritual stance, the stance of someone who seeks to transcend the composed self not by longing for the extraordinary, hoped to be found in a beyond to “this” place.[16] It is the stance of someone who ponders the apparently self-evident in its depths. Abstractive thought then is thinking about the ordinary—beyond a principle distinction between artificial and natural—as resulting from “making kin”, as Donna Haraway says with respect to animals and all kinds of beings other-than-human.[17] But this making-kin involves an alienation of the self that involves generic recollection, it cannot be a giving-up of the self into a common will or something such. It involves assuming a stance of resignation such as to think holisitically. Abstraction involves gestures of disarming, a putting oneself at risk. This is how I understand Elias Zafiris's call for an involution: an inevitable willingness to *recompose* through processes of interiorization that involve both a faith and a forgetting. It is similar to how learning a novel bodily skill—like swimming, gymnastics, riding a bike, or even playing ball do. One needs, in short, a willingness to mingle with the world.

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Mathematical comprehension is being explained, at least since Plato, with a forgetting called *anamnesis*: when we learn, in this strict sense of thinking mathematically, it is because we actually *recollect* what each and anyone of us implicitly already knows but has “forgotten” when being born. Recollection and remembering

16 Wherever 'this' place would be for the reader; the preposition is here used as an indexical, not demonstrative preposition.

17 Donna J. Haraway, *Staying with the Trouble: Making Kin in the Chthulucene*, Duke University Press, Durham USA 2016.

on the scales of an individual's historical identity cannot be addressed in the same way. The respective philosophical and ethical implications of such forgetting and remembering are radically different. With respect to *anamnesis*, learning has to do with this disarming gesture of involution, with the willingness to mingle with the world, a readiness for inward recomposition und a kind of self-recognition that forms instrinsically, calibrated with the cultivation of intergenerational respect. With regard to *remembering*, on the other hand, learning has most often to do with the opposite, it involves defensive and aggressive gestures that seek recognition from the outside, with the intent to ward inner recomposition und recognition off.

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This role of forgetting that is to be recollected through mathematics—it involves a notion of witnessing that is related to the kind of pact elaborated earlier, between the one who "talks" in code about something whose existence remains autonomous from how it is talked about. Evocation of the world as cosmos arguably know about this pact—what a cosmographer seek is attempting to create "world models that account for the maximum of phenomena", as Zafiris puts it. The notion of cosmos not only addresses the world as an animate being with a receptive interiority (a soul), it also conveys that "the cosmos" only shows itself in decorums, that is with adornment and in the coatings of a certain "cosmetics".

With the notion of the cosmos, mathematics can be regarded as something vibrant, quick and alive. We are more used to attribute this to language perhaps, at least in the common sense relation maintained by anyone who loves literature or witty dialogs, songs and lyrics; anyone who makes promises shares power and builds lifes upon an oath—not only marriage, but professional oaths too; anyone who admires or is scandalised by the rhetorical skills at work in the speeches of some, and the healing power that "having words" can unfold—for coping with what traumatises when feeling the pain of *wanting to do something but not knowing how to* (for example, or also to adequately remember someone who has passed).

What one can learn generally from the cosmographer then is that the recollection in anamnesis is to be grasped best by a kind of mechanics, involving a play of signification but not exhausting its meaning in any such plays in particular. In physics too, we are well used that mechanics brings about paradigms that pertain to different elemental scalarities: such as wind or water or fire or air, as in times before the unification of their *Stofflichkeit* into a notion of uniform matter via the calculus of forces and the thnking of energy primarily in terms of resources. But even now, the mechanics of heat in thermodynamics is called a mechanics, as the one of light in optics is in nuclear and quantum physics. They are and are not, strictly speaking "one"—similar perhaps to how certain feminists emphasise an ethics of difference rather than a moral discourse on the universal nature of humankind, when they maintain that the "sex" indeed is natural and universal, but not "one".

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Luce Irigaray is a philosopher who thinks of the world in this manner as a *cosmic home*—the cosmos is a home as the place of co-habitation, where relations are elemental and constitutive but also rapportive, not hierarchically fixed, as she develops it in her book *Elemental Passions* (2000). Elemental interplays involves the soul, passions as much as formality—to the degree that home to Irigaray is where relations are not settled orderly in customs and habits but need to be actively "maintained" in how they can accommodate everything that enters from the outside. Her ideas involve a cosmic notion of elements that recollects their meteorological aspects. It propels a thinking of elemental relations as relations of being wedded, relations that consummate marital acts in an unending cosmic marriage between heaven and earth. Through acts of elemental passion the cosmos communicates itself in miniature throughout all the "homes" inhabited by beings.

Marriage is a pact that was not reserved for the arrangements for two persons to share a life together; it was evoked in a broad sense for *Festtage* at large, ceremonial days. It denotes relations that are enacted again and again when the marital acts (think of a poetic extension here, a notion of sexual intercourse that is not reduced to reproduction) are consummated—*to consummate*, this means literally "to complete, perfect, carried to the utmost extent or degree," from Latin *consummatus* "perfected, complete", past participle of *consummare* for "to sum up, complete", from *con* for "together, with" + *summa* for "sum, total", in turn from *summus* for "highest". Irigaray writes:

> The home—the couple or family—should be a locus for the singular and universal for both sexes, as should the life of a citizen as well. This means that the order of cultural identity, not only natural identity, must exist within the couple, the family, the state. Without a cultural identity suited to the natural identity of each sex, nature and the universal are parted, like heaven and earth; within an infinite distance between them, they marry no more.[18]

What can such proposed spiritualisation of the body as body and the earth as earth mean, how to think of it? Let us listen again to how Irigaray puts it:

> [...] the body is cultivated to become both more spiritual and more carnal at the same time. A range of movements and nutritional practices, attentiveness to breath in respiration, respect for the

18 Luce Irigaray, *I love to You— Sketch of a Felicity within History*, Routledge, London 1996, 23.

> rhythms of day and night, for the seasons and years as the calendar of the flesh, for the world and for History, the training of the senses for accurate, rewarding and concentrated perception—all these gradually bring the body to rebirth, to give birth to itself, carnally and spiritually, at each moment of every day. [19]

The universal then provides for the possibility for an intimacy rather than enforcing estrangement:

> By training the senses in concentration we can integrate multiplicity and remedy the fragmentation associated with singularity and the distraction of desiring all that is perceived, encountered, or produced.[20]

The poetic question—the epic, comic and tragic, not the *Trauerspiel*[21] one—that pertains to relational ontologies is how can there be association without one part appropriating the encountered other, without a resulting reduction to sameness (and silencing or suppressing otherness). Irigaray developed an elemental-existentialist philosophy in response to this question where she recognises the logos of *likeness* as a codified principle in the form of a mechanism of association. But she counters this mechanism with a second one, that of *mourning*, and posits both in the framework of analogous proportioning. For *mourning*, the universal consists not in submission to death (as it does in the mechanics of likeness, where "nothing that is alive is not subject to death. For Irigaray, relations invovle death in how they are constitutive for life; they need to be contracted in an ethical dimension of pact. It is about

19 Ibid.

20 Ibid.

21 Cf. Walter Benjamin, "Erkenntniskritische Vorrede" in *Ursprung des deutschen Trauerspiels*, Suhrkamp, Frankfurt a. Main 1993 [1925].

striving for fulfilment of life through accommodating with how the universal involves death.

For Irigaray, the poles of such mechanics of association are cosmic notions of She and He (at stake here are not the social roles of the biological sexes, but metaphysical difference as it is at work in a sexed nature of Being. By doing this—by mechanising the incommensurable instead of moralising it—she can propose this second mechanism next to the one that has foregrounded likeness (and hence the reign of He): The mechanics of mourning applies to She, that of likeness to He. Both are equally existential, there cannot be an order between the two—this is why the "home" needs to be a cosmic house built such as to accommodate "a maximum of phenomena". Such mechanics of *mourning* concerns coping with the absence of what is lost, cannot be retrieved, has died. There can be association through shared respect for the mourning of the other, just as there can be through shared respect for aspects of likeness between one another.

Irigaray speaks of a kind of gaze that is not an inattentive nor a predatory gaze (in her example that of gazing at a flower), nor the decline of the speculative into flesh. She sees in this gazing an act in both material and spiritual contemplation, the furnishing of thought with "an already sublimated energy".[22] What does one apprehend from such gazing? The flower can actually provide us with a model, Irigaray elaborates:

> Between us, we can train ourselves to be both contemplative regard and the beauty appropriate to our matter, the spiritual and carnal fulfilment of the forms of our body. Pursuing this simultaneously natural and spiritual meditation... I'd say that a flower usually has a pleasant scent. It sways with the wind, without rigidity. It also evolves within itself; it grows, blossoms, grows back. Some of them, those I

22 Irigaray, 1996, 24.

> find most engaging, open with the rising sun and close up with the evening. There are flowers for every season. The most hardy among them, those least cultivated by man, come froth while preserving their roots; they are constantly moving between the appearance of their forms and the earths resources. They survive bad weather and winters. These are the ones, perhaps, that might best serve us as a spiritual model.[23]

For understanding the role of such *models*, important is also a saying by Angelus Silesius—namely that "The rose is without 'why'; it flowers because it flowers". It is in response to this famous saying that Irigaray asks in suggestive and metaphorical manner:

> Do you want the flower to open only once? The unveiling of the opening would then belong to you. The beauty or truth of the opening would be your discovery. Proposed and exposed in one definitive blossoming. The nightly closing of the flower, its folding back into itself would not take place.[24]

How to study something lovingly, such as not to appropriate its fragility and beauty? Such self-less and attentive gaze that marvels at the world is what can be trained by mathematical thinking and the anamnesis it performs. It is not the gaze of the analytical detective who is inevitably guided by a mission—because there is crime that has to be "made right". The power of *ratiocination* here is dedicated to cosmic modelling that could accommodate, through abstractive reductions that involve a certain spiritualisation of the body, a maximum of phenomena and experiences.

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Attending to the universal with such gaze involves an attention that starts out from listening, not from imagining. Another con-

23 Ibid.

24 Luce Irigaray, *Elemental Passions*, Routledge, London 1999 [1980].

temporary philosopher, Michel Serres, therefore proposes to think of it through the mythical figure of Orpheus, who delves into the lands of the dead in search for the woman he loves.[25] In Serres's tale, Orpheus can train himself with the help of nine Muses, daughters of Mother Memory. She is portrayed as "an old witch, evil and brilliant, full of knowledge and resentment"[26], gifted and cursed at the same time because being unable to forget, she is incapable of recollection. Her malice comes from a tormented body, bursting with the mere aggregates of disparate collections. She tries to gather all the memories of the world, "namely, stars and crystals; and the memories of bodies and living things, namely, folds and fossils; and finally, the memories of societies, namely, lies and archives"[27], but since there is no plainness to memory, memories cannot exhaustively capture the fullness of events. Hence the absence of consonance and symphony among the disparate collections which she picks up. It is only by her giftedness for witchcraft that she brings "order" to the noisy sounds of the prophetic voices.

Young Orpheus travels the Mediterranean, in Serres's tale. He encounters the Sibyls, Bacchantes and Pythia and asks himself: why all this noise? He sets out to interrogate the oracles about the reason of their noisy spectacle. But he does not try to read their messages well, he does not try to decipher their meaning. He asks not the oracles but himself for the reason of the noise; it is a turning inwards that moves cyclically and is broken up with interceptions, in response to Orpheus' listening to what the oracles's prophetic speech says *rhythmically*. Orpheus listens

25 Michel Serres, *Musique*, Le Pommier, Paris 2011. Here and in the following my own translation.

26 Ibid., 14.

27 Ibid.

to the motions embodied by the syllables, those units of poetic meter, wherein voices and silences are in play together, where the voiced vowels hold back or run forth or ahead to furnish and prepare accommodation places for the silent consonants to arrive and present themselves into a versed or phrased accentuation of a composition. All the while he attends to the overall motion that comprehends them all. Orpheus shares no common language with the prophetic oracles, he interrogates them through gesturing. There is a faith he follows hereby. It is that the orphic voices do not want to explain and conspire with him, but to instruct him how to "act" in "witnessing"—how to attend to and take note of what is *not* him. Serres writes:

> If you want to learn to speak, or later even find your profession in it as an actor, lawyer, teacher or speaker; if you desire to sing, to carry your voice out of your body to fill a room up to the opposite wall; if you wish to bring forth from your throat a vibrating column, like a blaze of haunting sound and vocal dexterity, know that even before speech carries a meaning and even before song elicits a feeling, the voice comes from the body, from its stance, from its base, from its upright posture, from its centre of gravity, from the animal connectedness with the earth through the ground contact of the soles of the feet, the firm anchoring of the toes with deep roots; that every life-giving source flows from the chthonic stream, ascending the bones and muscles, legs, buttocks, abdomen, mediastinum to the shoulder girdle; that your voice will speak, will signify, if its deep inspiration is grounded in this grounding.[28]

What then is Orpheus attentive to, what does he listen to when he interrogates the prophetic women through his gesturing for the reason of their noisy spectacle? Silence, Serres maintains. Listening in search for the musicality of these voices, his faith is that there must be tunings between them and the universal nature of

28 Serres, *Musique*, 10.

this ceaselessly originary and "chthonic stream" of messages—tunings that make themselves (re)markable through undoing loudness, through producing silence.

Such silence is perhaps what Irigaray has in mind for the mechanics we associated with mourning. Like this, music is the mediator between mathematics and the study of nature. Music is an art in this canonic sense not because it can bring forth and perform harmonious compositions, but because it is capable of harmonising the noisy world through silent notations and scales in an indefinite amount of compositional plays. What does one hear behind those languages of the sciences if they speak mathematically, and if mathematics does not represent anything in particular? Let us turn to Serres once more:

> I will tell you, exactly what the Sibyls have taught you: the stochastic noise of the world and—you don't know it yet—its music sum. Like Aphrodite, mother of all beauty, born from foam and surf, rises suddenly from the chaotic sea of noise: music. It smoothes out the thorns and integrates the signals. The grand récit flows in a grand rhapsody.[29]

The music sum, what a word! What kind of summation, what kind of arithmetics are we talking about here? That of information, Serres maintains. Information encodes sequences of compositions numerically, forgetting all about the euphonic wholeness of any composition in particular. They are remembered through rhapsodic speech that conveys the gestures through which one can learn to listen for the *presenting* of silences. There is then a mechanics of rhapsodic speech—it entails embodying a cosmic maximum of moves, stances, motions, that allow for tunings with the maximum of experiences which a cosmic model is capable of preserving. Science remembers herself in rhapsody which Michel

29 Serres, *Musique*, 10.

Serres calls *Le Grand Récit*. Different than memory who cannot forget, science does forget. The numerical encodings embody such forgetting. Science needs to forget, this is its relation to the universal—without which there would be only history, no cosmos; only gravity and no grace.

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Such an approach to science (as a rhapsody, as a *Récit*) involves a kind of transcendental communication not unlike what we perhaps more privately engage in in prayers, but as a collective or communal "communication"[30]—a narrative, a recounting and accounting, a story telling, legends and history all at once. This is why the Grand Récit as an anonymous rhapsode, embodied by science, requires what has been called a quantum literacy[31] to tune with the enthusiastic speech for which—like for the prophetic talk of the Sibyls, Bacchantes and Pythia—the full presence of the sounding word matters only insofar as it gives cues of how to listen for silence. Listening for silences amidst the rhythms of the spoken syllables, this is how to sound the truth of an *epiphania*, the appearance of phenomena for experience. Such phenomena can be witnessed only, with no full identification of the events, nor one's role in them. Such witnessing itself is acting, in recognising the absence of truth in whatever it is one remembers from the

30 Cf. also Paul Ricoeur, T*emps et Récit I-III*, Éditions du Seuil, Paris 1985.

31 Cf. Felicity Colman, Vera Bühlmann, Iris van der Tuin, Aislinn O'Donnell, *Ethics of Coding: A Report on the Algorithmic Condition*, EU Horizon 2020 Project 732407 (2018); Vera Bühlmann, Felicity Colman, Iris van der Tuin, "Introduction to New Materialist Genealogies: New Materialisms, Novel Mentalities, Quantum Literacy", in the *minnesota review*, issue 88, 2022; Vera Bühlmann, Felicity Colman, Iris van der Tuin, "Introduction: New Materialisms: Quantum Ideation across Dissonance" in Felicity Colman and Iris van der Tuin (Eds.), *Methods and Genealogies of New Materialisms*, Edinburgh University Press, Edinburgh 2023; Vera Bühlmann, "Quantum Literacy" in *Mathematics and Information in the Philosophy of Michel Serres*, Bloomsbury, London 2020.

experience. Orphean recollection can be shared only in motions that cannot be conceived, only marked upon by notations and in consensual abstinence from seeking to identify semantic representations of meaning and sense. Consensus by such "canonic" sharing of recollective motions that sing rhapsodically of a physics for which sound, voice, language, sense emerge out of noise—the chthonic stream that is the source of life, and whose quantum eventualities celebrate the inconclusive *Hochzeit* of the elements in every act of *connaissance*: literally the being-born together.

The first muses are not language muses, like in the classical Trivium (Grammar, Rhetorics and Dialectics). The first muses for such Orphean quests are musical muses. As the daughters of Memory they know how to incorporate language. The forgetting in this anamnesis is not the default that comes with having a body (when being born), it comes with acquiring a body (through learning to think). Thought's relation to the universal is a spiritualised but also an embodied relation, like we saw with Irigaray. It is a body that weds Heaven and Earth in its love for life and in its strive for fulfilment that travels through silence and absence, the lands of the dead. Musical summation means to consummate relations—striving for all poles to flower, to bloom for no reason but that it is capable of blooming.

This trivium here, hence, is dedicated not to language but to Pantomime, the miming of everything (pan). Pantomime is silently eloquent. Such a trivium teaches (re)collective thinking in embodied modes, in modes that are to be capable of forgetting in all the senses (directions) that language is capable of establishing: The body of the Pantomime is a restless body that needs to actively comprehend, accommodate and interiorise all it attends to. As an art, she is to embody memory in the silent talks of the quadrivium rhapsodies.

The first of the muses who teaches Orpheus in her art is Polyhymnia. She holds keys to such scientific wisdom in that she knows how to sing silently the song of songs. Imitation as pantomime, the miming of all as an art, takes no privileged object, no "object of desire" to identify and melt with. Like the young Orpheus does instinctively, Polyhymnia can teach how to interrogate the clamouring presences through gestures.

> The quiet, supple, feline, studious first muse, fascinated by imitation, begins the work of the nine sisters by first of all inventing rhythm, whose repetitions can only line up and whose thrusts only begin again and whose beats only continue ... when they take place in a presence, in the face, in a specular image. Reflection, then doubling. Facing everything, Polyhymnia's body duplicates people and everything else, imitates them or, better still, becomes all the things of the world: on the lookout for signs it can reproduce. Simulation in space produces simultaneity in duration. The first brings the second with it and immediately displaces it. First two gestures in the same bar, then one and the same gesture in two bars. Double, imitate; double, repeat; one foot, then two feet. Then for successful imitation: start again from the beginning. Imitate, then reproduce. Placed vis-à-vis everything, Polyhymnia's body polycopies beings and others, counterfeits, better still, becomes all things of the World: watches for signs in order to reproduce them.[32]

Polyhymnia says nothing, yet carries everything—in rhythms. Her moved interiority finds expression through the second muse, Terpsichore, who delights herself in dance.

> Trained by the rhythm, the second muse starts dancing just as smoothly and nimbly. She no longer reproduces something like her sister, but discovers and invents the body, humanises it. In dance, she throws herself into unimaginable, unexpected and new positions, movements, contortions, tensions, leaps and gestures that neither

32 Serres, *Musique*, 22.

walking, running, hunting, nor any other vital function would require. By freeing life from its native prison, Terpsichore creates a life even fuller and more colourful than the courtship dance of titmice, the round dance of bees, or the vagabonding of whales in the ocean depths sending out calls of attraction. Yes, the dance collects a bundle, a repertoire, a reservoir of bodily forms.[33]

These bodily forms may be of no immediate use, but they bring plasticity and resilience. Terpsichore's body knows how to adapt, because

the choreography teaches her an almost universal sum of hundreds of figures and thousands of movements, because it gives her a new human body—white, like the sum of human gestures and white, like the sum of all colours.[34]

To spiritualise one's body as body, this means for us, being humans, to acquire a *human* body. With the first two muses, the human body is a white trove of motions. They do not invent motion and emotion, but they interiorise and embody the anamnesic acquisition of them. Polyhymnia mimes and traces le Grand Récit of the world by inventing rhythm, and Terpsichore transcends the human body and continues its hominescence. Both is not possible without interiorizing and then exteriorising the laws of the universe – mechanics, light, gravity, consonance and dissonance, but first of all, one's own pulse. Rhythm cuts but it also connects, a cadence interrupts but through doing that, it also provides flow, as Serres puts it: *both muses hear time and render it back*. Divided in rhythmical elements, minutes and seconds, hours and days, centuries and millennia, time flows like a river and unravels itself: this is how rhythm smoothes the thorns of noise and clamour. Discontinuously and yet continuously, the two first muses inter-

33 Ibid., 24.

34 Ibid.

twine tact and bond. They throw themselves a million times back into chaos, they expose themselves to it and bring back rich stocks of useless treasures which will impact the evolutionary course.

Serres's tale goes on with seven further muses, but for our context here, Elias Zafiris' lecture notes on mathematical thinking and what they convey, these first two muses are more relevant. They speak of encountering obstacles through Orphean quests – they too seek to be accommodative to a maximum of experiences and phenomena. They present a body of thinking that can be acquired by anyone. But since the body of thinking they manifests is such a rich organon that has learnt literally from the sum of existent physics, what one needs to "forget" is also a lot.

*

Perhaps it is not a bad disposition to be rather a novice with respect to logarithms, groups, functions and functors, symmetries and homologies, harmonics, optics and so on if one is to make the most of this course –in that case, one can learn not in order to master, but such as to then spiritualise what one has learnt by forgetting its "sense" and "semantics", its ordered models that—let us remember how we started out—have thrown mathematical thinking as an art into antagonisms of missionary hatred and unreasonably passionate hopes and expectations. What the student of this treatise can learn, perhaps, is forgetting the hostility and agonism in favour of a notion of "mathematical communication" where human bodies of thinking can build homes in Luce Irigaray's sense—homes where cohabitation is characterised by striving not to consume, but to *consummate* the fabrics of relations that make a home a home. It involves a mathematical stance that is like the bodies of the two muses: it is not to be applied and useful immediately. A rich treasure of motions, moves, tensions, stances etc. which one has some familiarity with—as allies to make kin in an ethics of difference that agrees to seek the consummation – perfection, completion, accomplishment in the sense of affirming a spiritual marriage, a conscious awareness to live in co-dependency and a pact not to proceed by dominance and suppression.

Models of cosmic architectonics provide homes for *Nature in Nuptials*, places of convivial celebrations where thinking is a feast that consists in exhausting oneself on striving not to consume any of the abundant dishes one finds set on the table. Mathematical thinking, in the beautiful legends and tracts delivered here by Elias Zafiris, forms a constitutive component of such convivality.

Vera Bühlmann,
Buti (Italy), Summer 2024

1.
Wondering and Wandering around the Vortex

Abstraction and Diachronicity

The two most predominant characteristics of mathematical thinking are abstraction and diachronic validity. By the former we understand a process of percolation, which allows the filtering out of all irrelevant details pertaining to a particular problem, so that the invariants are eventually revealed. It is precisely the latter that enunciate the diachronic validity of mathematical thinking.

What is peculiar with mathematical thinking is that it is not based on a sequential concatenation of facts proved in the past for its evolution. Rather, what proves always to be of the ultimate value is the method to conceive of and establish theorems in the context of certain axiomatic frameworks, which are themselves variable.

This immediately annihilates the persistent illusion that mathematical thinking is about assemblages of theorems becoming more and more complex on the basis of some formal axiomatic background. At the same time, the emphasis on the method guided by abstraction, unveils the fact that the thinking in mathematics cannot be separated in any possible way from its bonding with philosophical ideas permeating the conceptions of its objects together with the architectonics of relations weaving these objects consistently.

The intricacy and beauty of mathematical thinking can be appreciated in its universality only by studying the works of the great mathematicians and figuring out their elaborate patterns of navigating in the unknown, as well as the connectivity among all these different patterns. Then, what is actually emerging is a panorama of spectra elucidating under similar or different hypotheses a vast array of obstacles.

All these spectra cannot be assembled together spatially, but they can harmonize non-locally in historical time giving rise to a symphony. This symphony is enacted by all angles of meaning, or all senses of orientation and circulation around obstacles that make up these spectra. In turn, the above is precisely what characterizes a method to obtain a spectrum. It is not accidental that the great mathematical master Archimedes called his most important work "The Method".

Simultaneously, this constitutes the division between pure and applied mathematics superfluous and against the essence of mathematical thinking. Ironically, this is considered as an important methodological division, although it is against the principles of mathematical thinking. Rather, a dangerous form of exploitation views the organism of mathematical thinking as a dead body to be dissected for the promised value of its parts in pre-targeted problem-solving in the so called applied sciences. Unfortunately, this attitude leads with precision to mathematical illiteracy and a certain form of fear for mathematics.

According to Nicholas Bourbaki, a collective pseudonym for a group of renowned French mathematicians focussing especially on the articulation of mathematical structure:

"From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms—the mathematical structures; and it so happens-without our knowing why-that certain aspects of empirical reality fit themselves into these forms, as if through a kind of pre-adaptation. Of course, it cannot be denied that most of these forms had originally a very definite intuitive content; but, it is exactly by deliberately throwing out this content, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations and for the development of their full power.

It is only in this sense of the word "form" that one can call the axiomatic method a "formalism". The unity which it gives to mathematics is not the armor of formal logic, the unity of a lifeless skeleton; it is the nutritive fluid of an organism at the height of its development, the supple and fertile research instrument to which all the great mathematical thinkers since Gauss have contributed, all those who, in the words of Lejeune-Dirichlet, have always labored to "substitute ideas for calculations".

Bridges in Time: Metaphora and Modular Substitution

The primordial act of the human mind under the influence of the senses is the act of wondering. "Wonder is the only beginning of philosophy", Plato has Socrates say at 155d of the "Theaetetus". And at 982b of the "Metaphysics" Aristotle says, "it is owing to their wonder that men both now begin and at first began to philosophize"."Wonder", called "thaumazein" in Ancient Greek, is intricate, since it both opens up our senses wide, and simultaneously plunges us into the dark, into the directly inaccessible, or even inconceivable.

The primal realization following the act of wondering is that a certain architectonics of relations, bearing the power to eventually give rise to a "theoria", stabilized diachronically out of envisioning, experimenting, and essentially, communicating with Nature (*Physis*), is indispensable in all cases, where a direct accessibility to sharply distinguishable domains of objects, and their behaviour, is not feasible, due to obstacles of any particular type.

The most characteristic of these cases pertains to objects of a different scale, like the microcosmic, or the macroscopic one, evading direct observability and individuality. Out of these domains, objects manifest in foamy or cloudy patterns, and they are characterized by plasticity, emergent properties, and generically probabilistic attributes.

The fundamental idea that marked the beginning of Natural Philosophy in general, and Mathematical Thinking in particular, pertains precisely to the architectonic modelling of not directly accessible, or broadly-speaking, obstacle-laden domains.

In a nutshell, it is the following: Instead of addressing them in terms of absolute constituents bearing pre-defined properties, adjoin to them other adequately-understood or directly accessible domains, which can provide pointers and open up communication channels with the former ones. (Figure 1.1)

The architectonics of adjoining should not be ad-hoc and should not depend on artificial choices, but it should always follow from a "Logos", meaning the elaborate articulation of a "why" question according to a cause of Nature that can be communicated through a spectrum of recognizable distinctions. Such a spectrum enunciates a rhetorical topos that communication takes place.

In this manner, the architectonics of adjoining gives rise to a process of modular substitution of the inaccessible domain by the accessible spectrum. The suitability of the spectrum is determined by its capability of modulation by the fluid architecture of the enacted rhetorical topos. The depth of the spectrum, its resolution capacity and adaptability under modulation, qualify and quantify the objective chance of congruence or consonance with the directly inaccessible domain.

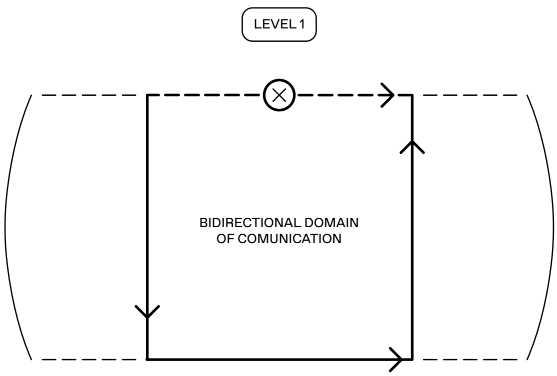

FIGURE 1.1
Bidirectional domain of communication

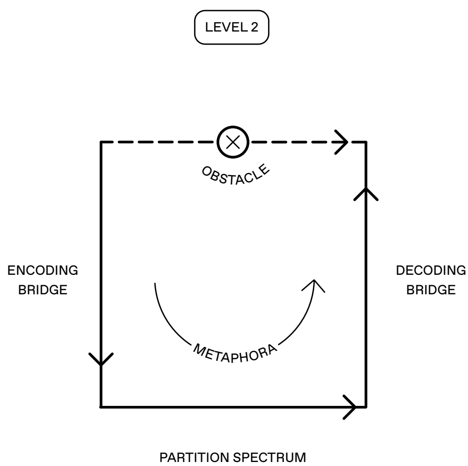

FIGURE 1.2
The architectonics of adjoining gives rise to a process of modular substitution of the inaccessible domain by the accessible spectrum

This process of modular substitution, carried through conjugation by means of the pertinent architectonics of adjoining, has been expressed by the philosophical term "analogia" in the ancient literature, the seeds of which mark the beginning of abstract mathematical thinking. This takes place, both in the context of music involving the acoustic harmonics of sound by Pythagoras, and in the context of visual light-geometry involving the shadows by Thales and Anaximander.

In the first case, the architectonics of adjoining amounts to the plucking of a vibrating string on a resonator, giving rise to the "monochord", whence in the second case, it amounts to the adjunction of a sundial on a stick placed orthogonally to the ground surface, giving rise to the "gnomon". The abstractions achieved by both of these methods is remarkable, in terms of their ingenuity, universality, and diachronic stability. The term "method" in mathematical thinking should not be confused by what currently is called "methodology".

For instance, the modern conception and elaboration of measurement in quantum mechanics, would not be possible without the inter-temporal bond among harmonics, geometry, and spectroscopy under variable modulations of the pertinent architectonic scaffolding. A *method*, in the Archimedean elaborate sense of the term, which marks the genesis of mechanics, calculus, and computing, is symphonic, and thus, effective of congruence with the directly inexplicable, if it accomplishes a metaphor of structure from the former to another spectrally accessible domain where it becomes explicable. We prefer to employ the ancient term "metaphora" instead of metaphor, since the latter is currently loaded with such a broad and multi-faceted linguistic connotation that is not congruent to the original term.

The metaphora of structure may be thought of as a circulation around the obstacle that undermines the means of direct accessibility to a domain. Circulation is the essence of the rhetorical topos opened up as a place of communication between the former obstacle-laden domain and the spectrum devised for its explication. Circulation around the obstacle without stasis eludes the concept of information. It is only stasis that allows in-formation to appear spectrally as "anadyomene", that is, emerge via another spectrally qualified domain evading the directly inexplicable one.

This is the old, but diachronic, motif of in-formation as anadyomene stochastically from the foam, personified by Goddess Aphrodite, in the poetry of Hesiodus and Homer. It is the same motif that resonated in the mind of the young mathematician Galois to resolve once and for all the problem of solvability of polynomial equations, through a metaphora that opened up and revealed the spectrum of equivalences under the action of permutations, the crucial move to conceptualize roots of equations structurally and invariantly through the notion of a group, which traces the origin of abstract algebra.

FIGURE 1.3
Venus rising from the sea ('Venus Anadyomene')

('The Birth of Aphrodite', George Cruikshank, 1860, Cruikshank collection)

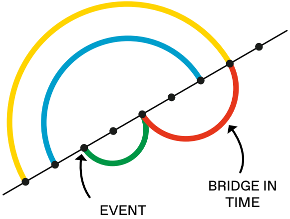

FIGURE 1.4
Event and bridge in time

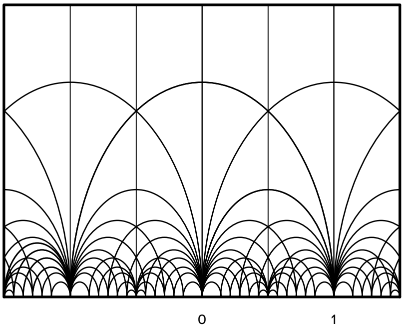

FIGURE 1.5
The gnosis (knowledge) in each of these three domains can only be evaluated and refined if each one of them is enunciated and evaluated in the context of the other two.

Mathematical thinking is non-linear because it has the capability of building bridges among events independently of their temporal distance. These bridges induce a connectivity in time, which is beyond the linear sequential ordering of events in terms of their chronology.

The Threefold: Philosophy—Mathematics—Architectonics

The gnosis (knowledge) in each of these three domains can only be evaluated and refined if each one of them is enunciated and evaluated in the context of the other two.

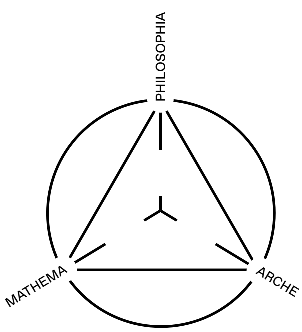

FIGURE 1.6
The threefold: philosophia, mathema, arche

These three domains are continuously influencing each other, in the sense that there exists a continuous feedback loop that pertains and runs through these three subjects of gnosis.

The underlying reason is that in the domains of Nature and Life, where they are applied, there is intrinsic objective indistinguishability, giving rise to uncertainty and probabilistics. Equivalently, uncertainty is not an artifact on knowledge that you do not have due to subjective ignorance. Consequently, dualistic true-false distinctions do not operate under conditions of objective indistinctness. The threefold is conceptually the way out of dualism on a par with the opening up of a communication domain.

Domain of Communication

If you consider any two of the three available domains of the threefold Philosophy/Mathematics/Architectonics as different layers or levels, the third domain acts as a bridge between the other two. Bridges are always bidirectional, such that a domain of communication opens up. What is bounded by the bridges is a rhetorical topos where communication takes place.

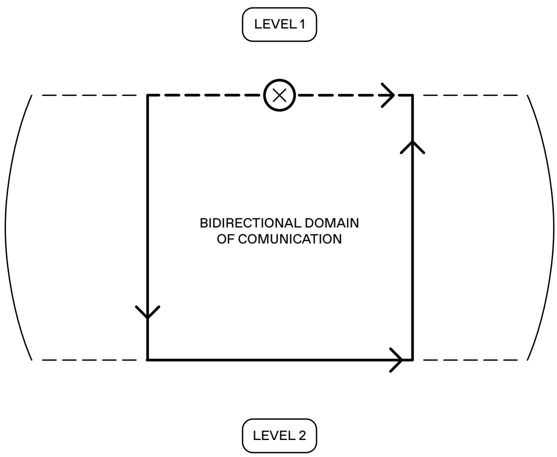

FIGURE 1.7
Bidirectional domain of communication

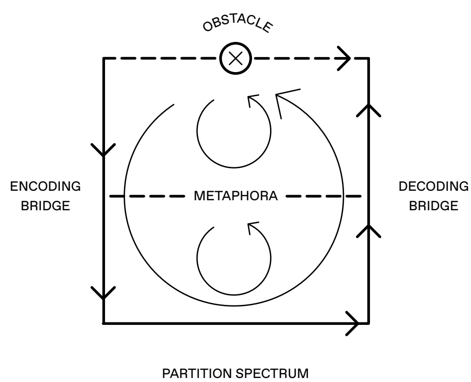

FIGURE 1.8
The schema of metaphora

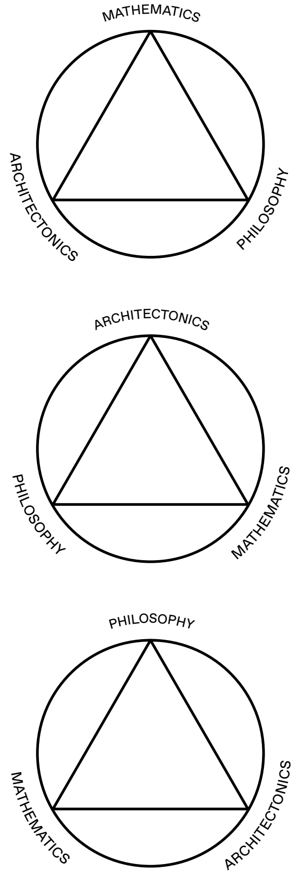

FIGURE 1.9
Permutation invariance of the threefold

The Central Obstacle—The Inconceivable

In the center of the communication topos is the directly inaccessible, the inconceivable. The inconceivable exerts an attraction when you wonder about something out of pure curiosity.

Wondering in admiration for what cannot be accessed or conceived directly is the precondition for creativity as well as for any type of original thinking. The attraction exerted transforms wondering to wandering. Circulation initiates around the inconceivable.

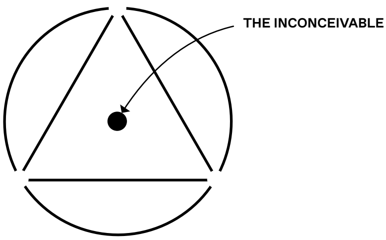

FIGURE 1.10
The inconceivable as the center of the circulation

Utopia—Doxa—Paradox

The purpose of the circulation around the inconceivable is to convert the inconceivable into an utopia. The transition from the inconceivable to utopia requires the making of a viable hypothesis. Mathematical Thinking always starts with the making of a viable hypothesis, called doxa. What is called paradox, literally means "para doxa", that is, parallel to doxa.

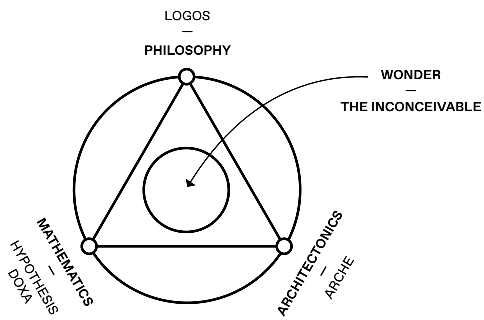

FIGURE 1.11
The transition from the inconceivable to utopia

What is parallel to doxa is the inconceivable. Paradox is the name given to the inconceivable, when instead of starting from wondering and wandering to circulate the inconceivable, we attempt to reach it in a linear way that is not distinguishing between different levels.

Logos—Utopia—Praxis

Logos in the ancient Greek language pertains to wondering about the cause of the inconceivable. When the inconceivable becomes an utopia through doxa, logos obtains the connotation of speech, that is, of spoken language where a rhetorical topos for communication may open up, so that Philosophy by means of dialogos can be practiced.

This is possible only if an arche is specified, both in the sense of a starting point, and in the sense of bridging bidirectionally the initial inconceivable logos with the pertinent hypothesis. Wandering around the inconceivable, that is, circulating in terms of dialogos, becomes effective when a metaphora is accomplished that transforms the utopia to praxis. Praxis embraces the initial obstacle of the inconceivable, and makes logos conceivable through analogia.

Arche—Architectonics—Architecture

The root of the terms Architecture and Architectonics is Arche which denotes the principle of beginning in both of the senses attributed above. Architectonics is what makes communication possible through metaphora. In Mathematical Thinking, architectonics determines both the starting point and the encoding/decoding bridges from a problem located in an obstacle-laden domain to another obstacle-free spectral domain under a viable hypothesis.

The resolution of the problem requires that a metaphora takes place between these two domains communicating architectonically to each other via the bridges. The circulation bounds the topos of communication between the directly inaccessible, or initially inconceivable, and the spectrum obtained under the hypothesis. The topos of communication bounds a geometric space opened up in this manner, whose geometric imprint gives rise to architecture. In mathematics, this corresponds to the architecture of the theoria (theory).

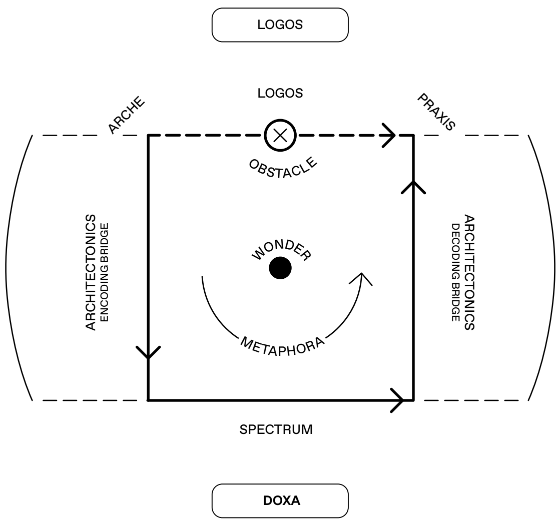

FIGURE 1.12
Architectonics of the logos-doxa metaphora and genesis of theoria

Abduction—Induction—Deduction

In the above transcription, the viable hypotheses become the axioms of the theory, whence the evaluated hypotheses give rise to theorems. The theorems constitute the praxis of the theory. The evaluation always requires a spectrum. The inverse architectonic bridge from an evaluated hypothesis to a theorem, that is the completion of the praxis of the metaphora, is synthetic. The validity of a theorem requires a corresponding proof based on the axioms.

Although the proof is usually presented deductively, as the most economic manner of demonstrating the validity of a theorem under the axioms, the conception and formulation of the theorem rarely follows a strict deductive pathway. Since metaphora is the method of praxis, the conception is either synthetically abductive or inductive, and only in quite simple cases deductive.

Dissociation—Metaphora—Association

Metaphora in mathematical thinking involves at least two different domains and the architectonics of encoding/decoding bridges bringing these domain into a topos of communication.

A counter-intuitive aspect of metaphora is that in order to grasp something on the higher level, where some obstacle resides, it is necessary to dissociate from this level, that is, the context of the obstacle. The encoding architectonic bridge directs away from this level into another level that is either obstacle-free or more tractable. Thus, instead of moving horizontally towards the obstacle, we move vertically away from it, in order to be able to start the circulation that encompasses the metaphora. Only when a spectrum is eventually retained as an encoding of the obstacle-laden domain, it becomes possible to start reverting back towards that domain by decoding the evaluated hypotheses.

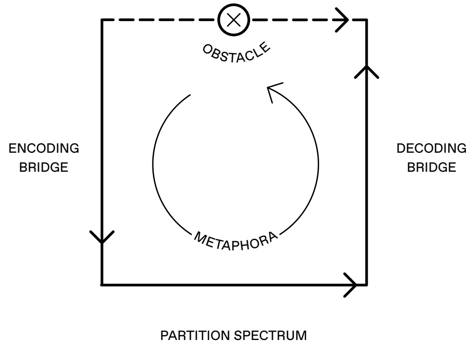

FIGURE 1.13
Dissociation from the obstacle—metaphora around the obstacle—association with the obstacle

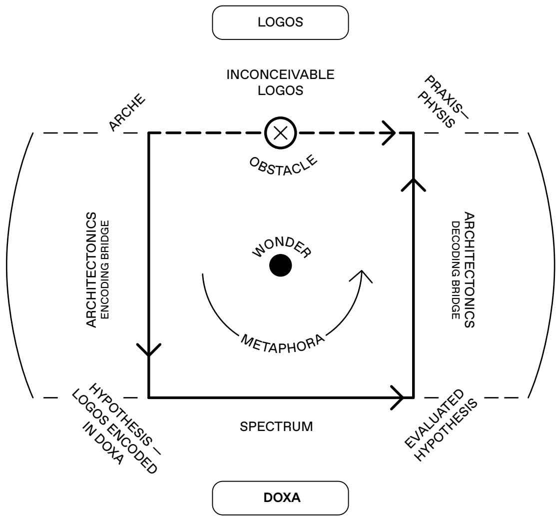

FIGURE 1.14
Architectonic topos of communication and the bounding of the inconceivable logos

Abstraction—Percolation—Invariance

The method of resolving a problem in a domain by means of metaphora through another domain in communication with the former is inseparable from what abstraction really is in mathematical thinking. The important insight in this respect is that the resolution of the problem rests on figuring out the invariant characteristic inherited on this domain by the obstacle underlying the problem.

Concisely put, figuring out the invariants and modelling them in terms of an algebraic structure (a group, or a module defined over some ring of scalars) is the essence of every metaphora irrespectively of its depth in terms of more and more refined resolution spectra. As soon as the invariants are unveiled at some spectral level in communication with the level of the original problem, they are adequate to determine completely the resolution of this problem by circulation.

This fact constitutes all other details and descriptions or complications expressed at the same level where the obstacle resides, essentially irrelevant to the resolution of the problem. At best, they serve at a phenomenological level, but they are not capable of reaching the root without encoding/decoding and partitioning until the invariants become manifest in the spectra.

Therefore, without metaphora and modular substitution via the spectra, all these details and phenomenological descriptions, instead of assisting and guiding the resolution of the problem, they obstruct it, because they hide, overcrowd, and do not allow the natural emergence of the invariants as expected by natural communication. This is precisely the objective of abstraction, to filter out all the irrelevant and obstructing details, so that the invariants can emerge spectrally. We call the emergence of invariants under filtering out all irrelevant details percolation.

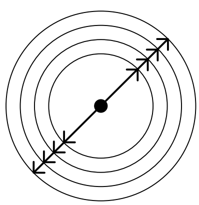

FIGURE 1.15
Zoom out
- filter
- percolation

In this manner, abstraction operates as follows architectonically: Instead of focussing on individual cases residing in particular contexts the aim is to find out criteria of equivalence grouping together a lot of these individual cases in equivalence classes.

This is like zooming out of the context of an individual case, which in turn, becomes modularly substituted by a whole equivalence class of cases under the adopted criteria. Such an equivalence class may be considered as a block of a partition spectrum obtained by turning our focus from the particular to everything else that is spectrally equivalent to it, and not to the general. In this sense, abstraction is different from generalization.

This becomes more clear, if we proceed to the next objective of abstraction, which operates by zooming in again as follows: After, the grouping together of individual cases under variable, but controllable, criteria of equivalence, abstraction aims to instantiate an optimal terminus (local or even global—depending on the complexity of the problem) of resolution. The economy of this terminus rests on the idea that everything more refined than this terminus is not only irrelevant to figuring out what remains spectrally invariant, but instead it obfuscates this task.

In this way, abstraction operates by setting up a sieve whose openings are variable, depending on the criteria of equivalence, with the purpose to filter out eventually everything else except the invariants pertaining to a problem. In order that the sieve captures the invariants, an optimal terminus of resolution is required so that percolation becomes effective, which is the most difficult aspect of abstraction.

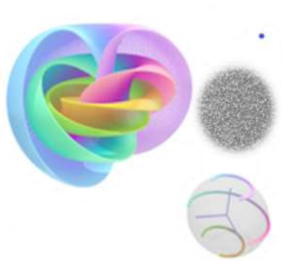

FIGURE 1.16
Sieve of a fibration

Therefore, the notion of a terminus cannot be abstract itself, but it has to be concrete. In a nutshell, abstraction works with concrete universals and participation. It is precisely these termini that give rise to the algebraic ciphers that the encoding/decoding bridges utilize through metaphora and modular substitution. These ciphers specify the criteria of structural identity, which is characterized in this manner algebraically by its neutrality to variation due to the encapsulation of an invariant character.

Stability in Vorticity: Static Tripod

In the cultural discourse, mathematical thinking as pertaining to metaphora for the circulation around obstacles, symbols play the major role for the exemplification of architectonic encoding/

decoding bridges from one domain to another. In particular, artifacts bear symbolic meaning, which is very interesting to articulate.

FIGURE 1.17
Static tripod

(Elias Zafiris, 2024, CC BY-SA)

We will consider the Static Tripod of the Oracle at Delphi. It consisted of three pillars that exposed themselves as three intertwined snakes. The bodies of the snakes where hidden underground.

Whence each pillar corresponds to a circle, the tripod is composed of three components which are intertwined in a specific way. More concretely, the connectivity of the tripod is such that none of the three components is directly linked with any of the other two. The connectivity is such that the linkage of any two of them takes place through the third, and such that if any of the components is removed the other two fall apart.

In other words, the stability of the tripod is not due to the direct linkage of the three components taken in pairs, but in contrast, owes to the indirect linkage of any two of them through the third. Therefore, the stability of the Delphic tripod refers to the stability of a rhetorical topos that is gained in time if any of these components manifests architectonically as a bidirectional bridge for the indirect linkage, and thus, metaphora between the other two.

The topos bounded by the bidirectional bridges when any two of the components are indirectly linked through metaphora by the third delineate a state of stasis of the configuration. The whole configuration is intrinsically dynamic, since any of the components is capable of serving bridges for a metaphora between the other two. Thus, the whole is innately in a state of homeorhesis.

Whenever a topos is bounded where mataphora takes place—through the bridges afforded by any of the components for the linkage of the other two—we are in a state of homeostasis. The homeotic aspect refers to the fact that any of the three components can afford bridges and thus initiate a metaphora.

The notion of stasis originates from harmonics. A state of stasis is precisely what distinguishes the harmonic frequencies in an acoustic flow. We conjecture that this is how the use of the vowels has been conceived in the ancient Greek language. Vowels are the states of stasis in speech, so that consonants can sound together in harmonic symphony.

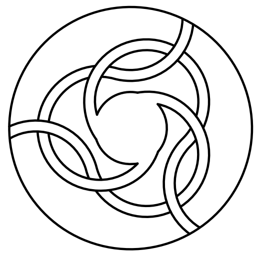

FIGURE 1.18
Homeostasis and metaphora

2.
Mathematical Weaving of the Cosmos

Congruence and Invariance

The diachronic character, intricacy, and value of mathematical thinking, especially in correlation with its communicative articulation, requires a substantially more rich thinking about the notion of time, not only pertaining to the linear ordering of events, but bearing the capacity to unravel the fibers of the weave making up a good mathematical theory, by which we mean what gives the character of abstraction, as well as the character of the historically diachronic and persistent to such a theory.

A characteristic instance is Gödel's first incompleteness theorem, which can be summarized in the assertion that if a formal system containing arithmetic is consistent, then it contains undecidable propositions, namely statements whose truth or falsity cannot be expressed within the language of this formal system.

Although Gödel's theorem bears the tag of a "metamathematical proposition", the inertia against incorporating any viable temporal notion in the so called foundations by certain schools of purists has been so prevailing that the "meta" characterization is devoid of any temporal connotation despite the meaning of this term, which, as a consequence, is interpreted at a purely formal skeletal logical level. From our perspective, this is a case par excellence of the conflation and confusion between Logos and Formal Logic in the continuity of Mathematical Thinking.

Instead of abolishing time completely from Mathematical Thinking, it is worth exploring the implications of a richer conception of time in relation to the genesis and growth of mathematical concepts. This is a fundamental aspect of the body of Mathematical Thinking that constitutes its integrity, its communicative capability, and its overall functionality as a living temporal entity, thus, certainly not as a formal skeleton.

The objective is to comprehend the conditions of partial congruence among abstractions and modular substitutions due to different types of obstacles in the course of unfolding of historical time, irrespectively of the temporal distance among events, which is actually the decisive factor for both, the qualification of diachronic validity, and the success of abstract thinking.

This richer conception of time is always implicit in the metaphora of structure from some domain to another one via the architectonics of the process of circulation around the pertinent obstacle, and in essence pertains to the distinguishable harmonics of the "Logos", the characteristic frequencies of resonance in the topos where communication takes place. It is these directly elusive harmonics extending much beyond the acoustic range that engulf the invariants.

The unveiling of the invariants, literally making them appear in light, called "epiphaneia", in its philosophical and technical meaning as a spectrum of distinguishable appearances following the analysis of distinctions incident to the architectonics of adjoining and modular substitution, requires for its articulation a canon of transcription to the visual domain of color, called "chroma". The "canonics" of this metaphora requires tempering within uniformly-partitioned chromatic intervals, underlying in turn the notion of a uniform probability distribution in relation to a directly inaccessible domain, which is the cornerstone of stochastics. In this way, a spectrum is subordinate to a specific partition, although the invariants are independent of the partition employed.

In general, the role of a partition spectrum is the instantiation of distinct blocks or cells consisting of equivalent elements with respect to some relation. The notion of an element is not that of a constituent part, but it refers to the observable distinction that is capable of imprinting on the concomitant spectrum. Each block of a partition consists of all those elements imprinting the same distinction, being thus equivalent to each other, and therefore indistinguishable from the perspective of the imprinted distinctions.

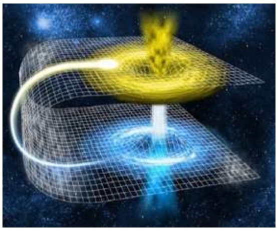

FIGURE 2.1
Curved partition spectrum

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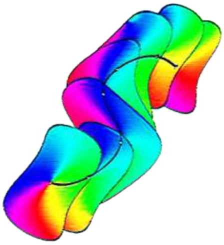


FIGURE 2.2
Continuation of partition spectrum along a path

The artifact of a partition spectrum is that it provides the means to reduce the complexity of an obstacle-laden domain through an analysis pertaining only to a finite, or countably infinite, number of blocks. Each block, since it contains indistinguishable elements, requires a single representative to be grasped. Every other element equivalent to this representative will be surely located within the same block of the partition.

An economy principle is at work here, which is operationally reductive, but not reductionistic, since it gives rise to a manageable quotient, preserving all the distinctions afforded by the spectrum. This is the conceptual core of the modularity characteristic of a partition spectrum. Of course, the actual utility of a spectrum rests on its resolution capacity in relation to the invariants.

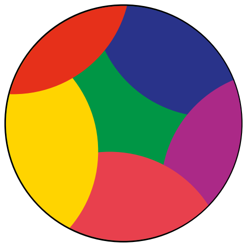


FIGURE 2.3
Circular partition spectrum

Recall that the invariants of the obstacle-laden domain are independent of the partition spectrum devised for its articulation, but simultaneously, a partition spectrum offers indirectly and metaphorically through communication the only possible way to gain access and grasp these invariants.

This makes the notion of a uniform partition, equipartition, and uniform distribution, especially important in the unveiling of the invariants. A uniform distribution amounts to a well-defined condition of neutrality at this level, pertaining essentially to averages. The idea is that in the absence of an obstacle of any particular type everything would behave uniformly, setting in this manner the standards of comparison and congruence that are not existent *ab initio*.

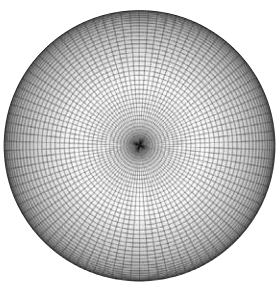


FIGURE 2.4
Weaving of a circular partition spectrum

The specification of the appropriate means suitable to a directly inaccessible domain amounts to the metronomy applied to this domain. It is the metronomy that underlies the architectonic techne of adjunction and conjugation giving rise to the modular technics of a tempered distribution over a spectrum. Concisely put, the technics of weaving a grid in relation to a partition spectrum is the explication of the underlying metronomy in the presence of obstacles.

For example, the deviation of the spectrum of black-body radiation from the average expected according to the standard of equipartition—known as the ultraviolet catastrophe—led Max Planck to the hypothesis that energy should be quantized, a hypothesis that pertains to the metronomy of energy in terms of harmonic frequencies, which marks the beginning of quantum mechanics.

Therefore, we should follow the thread that qualifies an obstacle as a source of invariance. Invariance can be operationally recognized only through action directed initially away from the level or context of the obstacle.

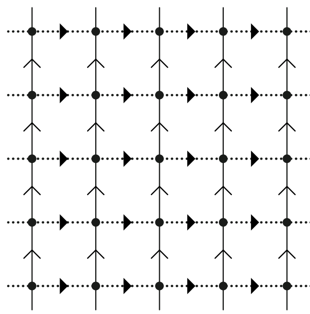


FIGURE 2.5
Weaved rectangular partition spectrum and flow

The purpose of action is to initiate a stream flow that is capable of retracting the inaccessibility or obstruction imposed by the obstacle to some generic situation at another level through which a passage becomes viable, and then re-direct the flow back toward the initial level, so that the obstacle can be embraced.

Embracing an obstacle successfully always leaves a residue, to be thought of in terms of "countable quanta" emerging through periodicity. These quanta are spectral quantities, entering into rhythmic arrangements within regular temporal cycles, thought of as harmonic frequencies. Most important, these quanta encode the invariance of the obstacle they refer to with respect to all possible embracing circular flows initiated by temporal actions.

Architectonics of Natural Communication

The method proposed by the model of "Natural Communication", which is used to explicate mathematical thinking, is implemented briefly as follows:

We consider a problem in the context of a domain whose objects and relations are directly inaccessible. We may think of this domain as a particular level in a universe accommodating other possible levels as well. First, we move out of the context of the problem, formulated at the level of the inaccessible domain, by adjoining to it architectonically another accessible domain, through which a partition spectrum can be built regarding the former under some viable hypotheses.

In order to accomplish this, we have to set-up an encoding bridge from the level of the inaccessible domain to the level of the accessible domain, such that a partial or local congruence can be established between these two domains entering into communication. The partition spectrum is the outcome of this congruence and amounts to the evaluation of the hypotheses.

The process is completed by setting up an inverse decoding bridge from the level of the accessible domain back to the level of the directly inaccessible one.

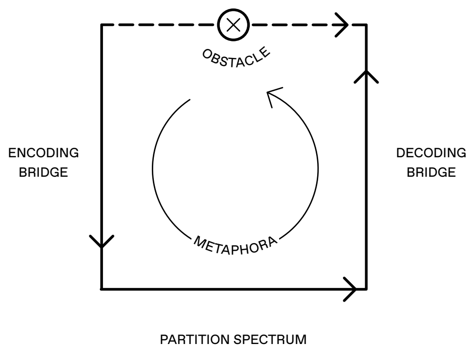


FIGURE 2.6
Metaphora through a pair of encoding/decoding bridges

In this way, the available means and knowledge pertaining to the accessible domain can be lifted at the initial context of the problem. This process accomplishes a metaphora, which is usually pertaining to structure. Therefore, the problem can be effectively resolved in the context of its initial formulation by the embracing of the obstacle it engulfs via the communication channels opened up through the encoding/decoding bridges with the other domain.

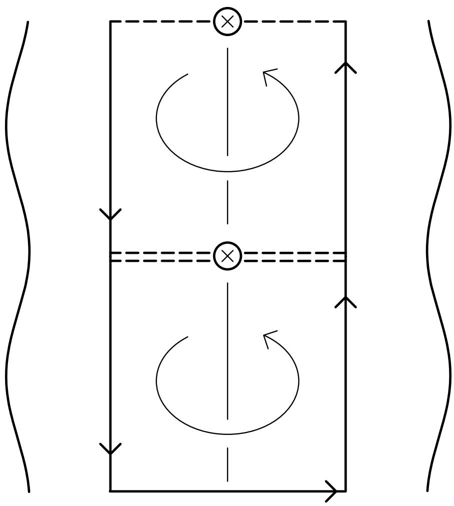
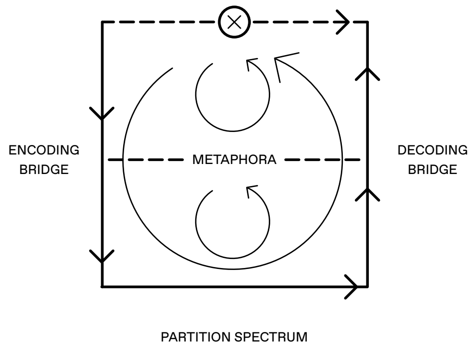


FIGURE 2.7
Iteration of a metaphora

Metaphora can be iterated through the adjunction of more than one controllable domains adjoined in succession to the inaccessible domain. Therefore, the circulation achieved through metaphora is capable of resolving the problem spectrally under the adjunction of one or more levels in communication via the bridges. Finally, the resolution of the initial problem translates to the qualification and quantification of the invariants associated with the obstacle imposing direct inaccessibility, expressed in terms of the congruence relations of the spectra.

FIGURE 2.8
Extension of a metaphora in depth



Since the central issue is the notion of an obstacle, and the metaphora devised to circulate around it via different domains capable of entering into communication and establish congruence relations with the domain where this obstacle is located, the notion of a rigid and absolute foundation for all mathematical objects is not applicable.

Instead, what is crucial always is the conception and effectuation of an architectonic scaffolding that is potentially able to bridge bidirectionally, bound, and bond together these domains so that the metaphora can be performed successfully unveiling the invariants that underpin the congruences. The invariants hold the *mnemosyne* (living memory) of the embraced obstacle in the body of mathematical thinking.

The synthesis of the invariants is not dependent on the sequential ordering of events, that is, it is not dependent on their chronological ordering. Rather, it requires tempering, balancing through the means, and a critical temporal state of attunement, what was called Kairos in the ancient literature.

From this viewpoint, the metaphora enacting the circulation through another spectral domain may be though of as a motif that can be iterated, and as such, it guides mathematical thinking by opening up more and more elaborate communication channels, depending on the ingenuity of the architectonics. Due to tempering and attunement, the motif energetically transforms to a motivic key that bears the capacity to unravel the invariants.

Harmonic Analysis and Synthesis

Harmonics originates from the domain of music, poetry, and spoken language. In mathematical thinking it is the field called Harmonic Analysis and Synthesis. Harmonics does not pertain to the domain of space, but it pertains to the domain of time that lies beyond the sequential ordering of events.

Analysis in space requires fragmentation into parts, which can be subject to geometric transformation in space, and then assembled appropriately together. In contrast, harmonics does not require any fragmentation into separate spatial parts. It pertains to different wholes and examines how these wholes may be bridged together in time by entering into a rhythm.

The crucial aspect of time that is targeted by Harmonics is the aspect of periodicity, and the fact that different wholes display different frequencies in their periodic behavior. Frequency quantifies how fast or how slow periodicity takes place in time. Harmonic synthesis studies how wholes with different frequencies may be cast together into a rhythm.

The domain of frequency is complementary to the domain of ordering in time. Different wholes may vibrate in different characteristic frequencies or exhibit periodic patterns unfolding in different characteristic speeds. The basic principle underlying a characteristic frequency is that it is invariant with respect to ordering in time.

This means that exact information pertaining to time in its ordered aspect bears zero information pertaining to time in its periodic/frequential aspect, and conversely. Usually, the first is identified with the domain of time, since the notion of ordering in time is the prevailing one in the current scientific thinking, whereas the second one is identified with the domain of frequency. The essence is that time in its ordered aspect is inversely correlated to time in its periodic/frequential aspect.

FIGURE 2.9
Non-local bridge in time

(Touch up by Elias Zafiris based on Figure 2.1, 2024, CC BY-SA)

This is a fundamental notion whose recurring traces are intact in the history of mathematical thinking from the Pythagorean theory of music and the Hipparchus/Ptolemy theory of astronomy, to the Fourier theory of heat and vibrations, the theory of signals, and modern quantum mechanics.

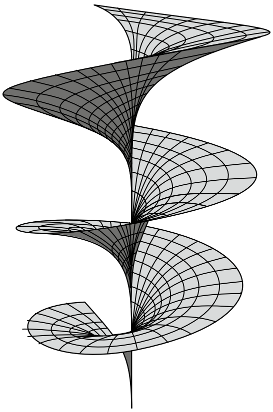


FIGURE 2.10
Ordering and periodicity

We may provisionally use the term uncertainty principle to refer to the correlation between these two aspects of time. Only within continuous spans in the ordering domain does it become possible to gain partial access to the frequency domain through tempering and probability theory. Inversely, only within continuous spans in the frequency domain via modulation does it become possible to gain partial access to the ordering domain.

A characteristic frequency in its role as an invariant of a whole is always hidden from direct access, although it may perceived sensorially within certain ranges. In this manner, a characteristic frequency is the outcome of percolation, it emerges through filtering out and resonating with it.

The term resonance pertains to harmonic synchronization independent of its ordering in time.

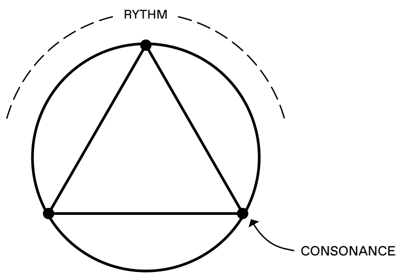


FIGURE 2.11
Cycle of symphony through a rhythm

It is what allows different wholes bearing different characteristic frequencies to enter into a rhythmic arrangement jointly, that is, to synchronize harmonically together in a circulation without losing their invariant characteristic.

Harmonic synchronization is a cycle of symphony of different periodic wholes in consonance when communicating to each other. It opens up the domain of living and tempering time in complementary relation to the domain of ordering time.

In this domain, the wholes in a cycle of symphony co-participate with their own characteristic frequencies, without mixing, but in consonance. This is only possible if any single whole functions as a bidirectional bridge of synchronization in time between any two of the others. As a consequence, if any whole of this cyclic constellation is removed or dissociated, then the cycle itself falls apart and dissonance prevails.

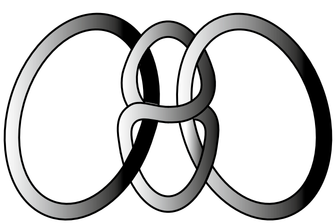


FIGURE 2.12
A whole functioning as a bidirectional bridge of synchronization between two other wholes

It is incorrect to think of a literal transcription of cycles of symphony in spatial terms. Philosophically, it is a category mistake between the distinct domains of harmonics and geometry. If, for instance, we think of harmonic frequencies and consonances produced by musical instruments in an orchestra that can be perceived in the acoustic range of frequencies, the ear can hear not only the compound tone, but it can separate all the different characteristic frequencies sounding together in consonance.

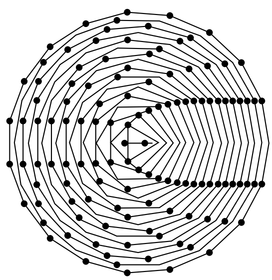


FIGURE 2.13
Harmonic analysis and synthesis

Thus, it hears each one of them as emerging from a different whole separately, but it also hears the compound consonant tone simultaneously, performing in this sense both harmonic analysis and harmonic synthesis. This is not the case pertaining to the perception of the eye.

At a first stage, we may think of geometry in visual and spatial terms, so that geometry refers to the forms stabilized through time in space under the action of light. Note that these forms are not necessarily rigid, they may be transient and fluid as well.

We do not think of a geometric object as something that is illuminated in light, but as an obstacle to the unobstructed course of light.

As such, an object opaque to light has a shadow that is intrinsically transient and fluid. A shadow is a topological and projective entity. Geometric objects have been conceptualized out of metaphora from their shadows.

A shadow is topological because its shape changes continuously depending on the time observed in the ordering chain of events during a day. Thus, a shadow depends on the ordering aspect of time.

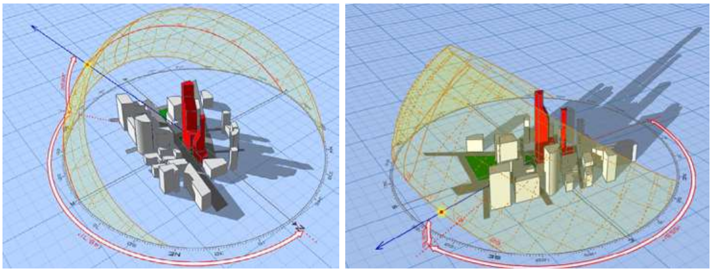


FIGURE 2.14 & 2.15
Projective shadows at different times

(Elias Zafiris, 2024, CC BY-SA)

It becomes independent of it only if it is frozen in time, meaning that it is fixed at a particular time of the day. A shadow is projective because it always appears on an epiphaneia, the outlook or contour of a shadow (peri-gramma) always appears on a surface. For a frozen shadow the contour bounds an area, the geometric magnitude that Pythagoras' theorem refers to.

Is a shadow independent of the periodic/frequential aspect of time? Definitely not, because of the fact that shadows do have colors. The area of a frozen shadow is a colored area. Color is a spectral quantity; it is characteristic of the range of frequencies of light.

If light propagates directly from a single source, then the color of the frozen shadow will be black. If light is also reflected from a variety of other sources, then although the contour is imprinted projectively from the main source, the area inscribed in this contour bears the color of the reflected light. It is enough to recall the red colored area appearing at the total eclipse of the moon, when light from the sun is reflected by the shadow of the earth, making the moon that is in the earth's shadow appear red.

FIGURE 2.16
The red moon

(Re-elaboration based on photo by Robert Jay GaBany, 2014, Wikipedia, CC BY-SA)

But there is a fundamental difference between harmonic frequencies and colors, at least to the extent certified by sense perception, through the ear and the eye correspondingly.

Although harmonic frequencies keep their characteristic invariance and can be separated in a symphony cycle while simultaneously being in consonance to each other and synthetically sound all together, colors do not bear such an invariance.

Colors tend to overlap, superimpose, and mix continuously. Mixing is the major aspect of the frequency domain pertaining to light. All colors can be reproduced out of the mixture of three basic colors.

At the ideal extremes we have the white and the black colors. Both of them pertain to the isonomic mixture of all colors, the first in relation to the emission of light, and the second in relation to the absorbtion of light.

A black shadow is the state of complete isonomy of all absorbed colors mixed together under the metronomy of light. Therefore, the periodic/frequential aspect of time is different from its spectral imprint in space in terms of colored areas and mixtures.

A modular substitution of cycles of symphony in time with an entity of a spatial nature composed of colored areas requires a metaphora from the domain of harmonics to the domain of geometry, a certain canonics of transcribing from one domain to the other. Paradoxes are again generated by a literal transcription without the intervention of a topos where communication through metaphora takes place from one domain to the other.

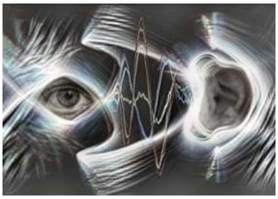


FIGURE 2.17
Metaphora from the domain of harmonics to the domain of spectral geometry

These domains are innately different. The argument is that space opens up architectonically and stabilizes via a topos of communication between these domains, a place called chora emanating from chorus. Since cycles of symphony are generated out of invariant characteristic frequencies of different wholes, which are independent of ordering, they can link together events independently of their distance on their temporal ordering chain.

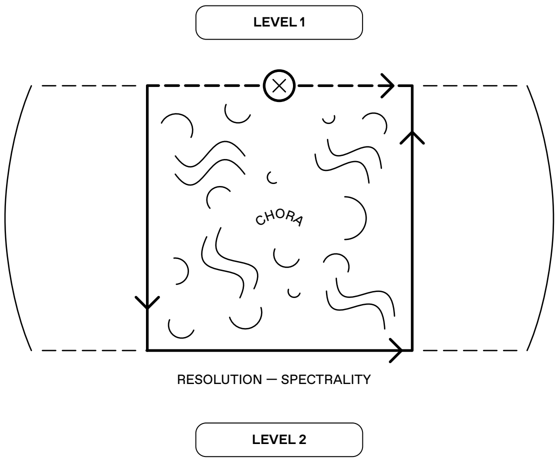


FIGURE 2.18
The chora

Still this temporal ordering chain is imprinted in space as a part-whole or local-global relation. If we follow Euclid and think of a geometric point as that which does not have any parts, then a point in space corresponds to all invariant harmonic frequencies.

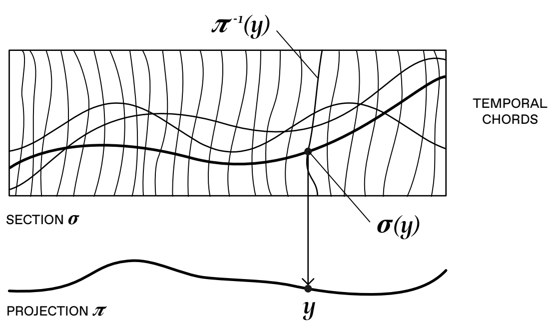


FIGURE 2.19
Sections of a fibration as temporal chords

But, since the colors in space pertain to areas and not to points, and since the mixing refers to areas and not to points as well, geometric space analysis cannot be elaborated in terms of points, but in terms of areas that span both the ordering aspect and the periodic/frequential aspect of time.

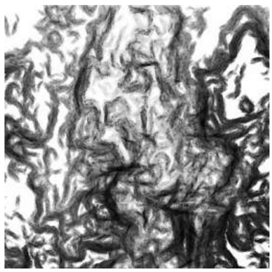


FIGURE 2.20
Geometric space analysis in terms of areas

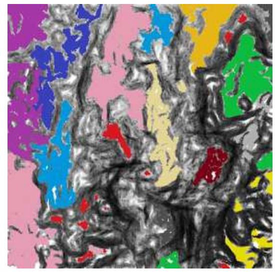


FIGURE 2.21
Mixing of colors pertaining to areas

Armonia-Arithmos-Arche

The mathematical field of Arithmetics traces back from the term “Arithmos”, what we call number presently, which interestingly enough shares the same origin with the term “rhythm”. Not only this, but most important is the fact that all the terms “Arithmos”, “Armonia”—from which Harmonics emerges—and “Arche”—from which Architectonics and Architecture emerge—all have grown and shared etymologically, obtaining their meaning from the same root.

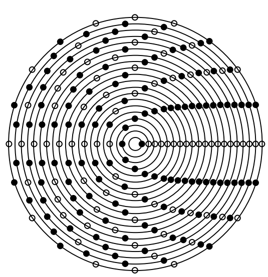


FIGURE 2.22
Constellations of wholes according to the periodic/frequential aspect of time

This root does not pertain to the domain of space, but it pertains to the domain of time that lies beyond its standard ordering aspect. It is the periodic/frequential aspect that pertains to this root, which does not require any fragmentation into separate spatial parts. It refers to different wholes and examines how these wholes may be bridged together in time within a symphony cycle, what we have called harmonic synchronization of wholes in a rhythmic circular constellation.

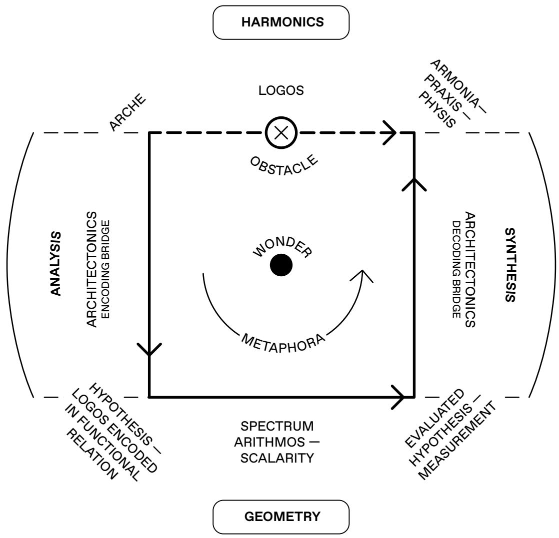


FIGURE 2.23
Natural communication scheme from the domain of harmonics to the domain of geometry

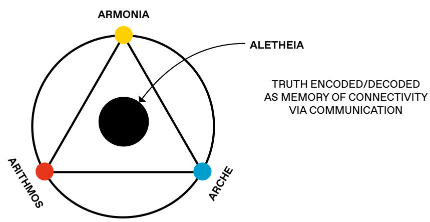
Synchronization and Unveiling

The central point of harmonic synchronization should not be thought of as an origin point in space. Its connotation is different and bears the name "omphalos". The omphalos keeps implicitly the living memory of the invariants characterizing the wholes that enter in a rhythmic arrangement, and thus the memory of the implicit "Logos" that makes the harmonic synchronization possible.

Although the umbilical cord is cut off at the time of birth, the memory of harmonic synchronization and connectivity is retained in the form of a trace imprinted on the body that is called the umbilicus (navel). The circulation around the omphalos after the separation of wholes has taken place in space is the process of uncovering, disclosing, and unveiling the "Logos" that makes a symphony cycle of these wholes possible.

This is what is called "Aletheia", which literally means unveiling from "lethe" (forgetfulness). The connotation of this term is quite different from what we call "Truth" within a logical framework. The direct inaccessibility to this center of harmonic synchronization is an obstacle to be embraced by bringing these separate wholes into communication through metaphora, and thus circulation in the paradigm of the Static Tripod.

FIGURE 2.24
Aletheia: unveiling from lethe



Meander and Rhythm

In the process of initiating a metaphora taking place by means of bidirectional architectonic bridges bearing temporal depth, the role of artifacts is fundamental. An artifact is a symbol of connectivity persisting in time, whose role is to encode in the most economic way the knowledge of a stage of a certain depth during a multi-level metaphoric circulation involving more than one spectra, which can be decoded and retrieved effectively. Daedalus, considered as the first architect, is famous for the production of artifacts. The concept of Daedalus was that the task of Architecture is not to fill in space, but to open up space in rhythm with the domain of living time. The most prominent artifacts of Daedalus are the "Meanders".

The meanders cannot be imprinted in space through a single line. There must be a recurring periodically winding pattern formed by two oppositely facing and intertwining assemblage of lines in right angular relation to each other, which imprint in space the impression of a synchronization rhythm.

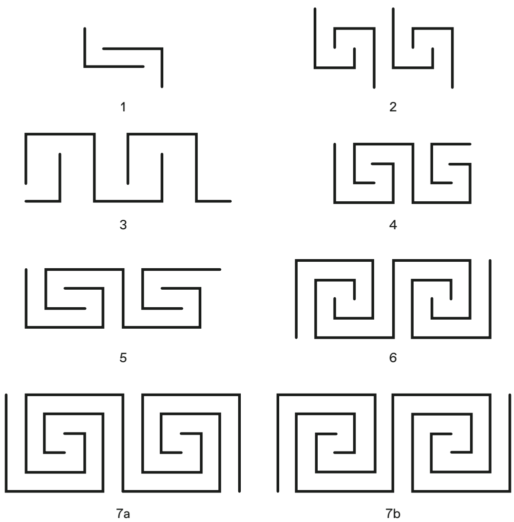


FIGURE 2.25
Types of meanders and synchronization rhythms

This rhythm resembles what we would call in physical terms a standing or stationary wave, from where the harmonic frequencies emerge. From this perspective, a meander is like a motif, a motivic key that imprints on the homogenous void the trace of the harmonics it carries, producing space according to its rhythm through time.

The conception of the "Labyrinth" abstracts from the notion of a meander. It employs together with a meander its mirror image reflection as well with respect to a line, which becomes an implicit diameter of a circle, where a circulation is imprinted—with respect to a directly inaccessible center—that has to wind around it in many different scales following in-between the bridges formed by a meander and its mirror at every scale.

FIGURE 2.26
Notion of labyrinth

Architectonic Scaffolding

An architectonic artifact in its symbolic function can be thought of as a scaffolding that delineates the topos where communication takes place, according to the preceding. In this sense, the chora is metaphorically equivalent to a theatrical stage, the epiphaneia where metaphora manifests by adjoining and lifting the audience from spectators to participants in the praxis of the actors, according to consonance or dissonance in relation to their actions.

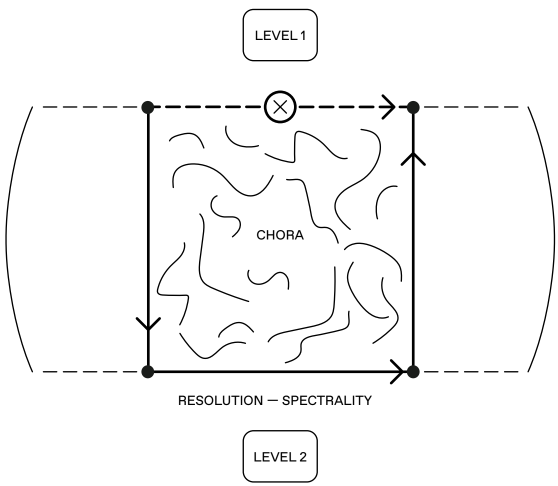


FIGURE 2.27
Chora as the theatrical stage

The architectonic scaffolding of the theatrical stage plays the equivalent role of a lever with respect to a fulcrum, a fundamental notion in mathematical thinking conceived and devised by Archimedes.

From this perspective, abstraction always uses a lever with respect to an imaginary fulcrum, based on hypothesis that can be spectrally evaluated. Its uniqueness and effectiveness—something that characterizes the most important theorems and the elegance of their proofs in mathematics—is due to the fact that, although a lever is always employed in relation to a scaffolding, it becomes implicit by internalizing its function while retaining the imaginary fulcrum.

This might be considered as an automation through harmonic synchronization guided by this imaginary fulcrum in the totality of its stages, where the completion of each stage internalizes the function of a lever. The imaginary fulcrum is the spatial imprint of the center of the cycle of symphony accomplished through abstraction by modular substitution of this internalized series of levers.

The substitution amounts to the internalization of each lever in the series in terms of the invariant symplectic area that generates through the fulcrum, that is, its spatial areal shadow. The fulcrum does not have any parts, but it is the accumulation point of all imaginary diameters meeting at this point.

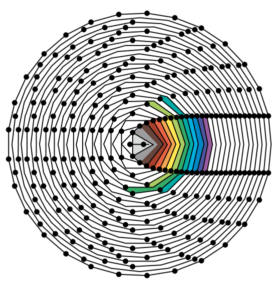


FIGURE 2.28
Modular substitution via an internalized series of levers

Since, each one of them is an integral whole in a cycle of symphony—pertaining to its periodic/frequential temporal aspect in synchronization with the others—and each one of them has an areal shadow; it can only be thought of as an imaginary dimensionality axis orthogonal to the shadows.

The generic case involves a single imaginary axis through which the spatial representation of the complex numbers and the complex plane is generated from. In contrast to the usual identification of the real two-dimensional space with the complex Gaussian plane, the above is concordant with Gauss' idea to think of the imaginary unit(y) as inferrable from a "shadow of shadows".

The historical convoluted evolution of mathematical thinking pertaining both to abstraction and diachronic stability may be though of as an intricate imaginary weave. The bridges in time out of all cycles of symphony and the elaborate architectonic network where metaphora takes place from node to node ascribe to this weave two peculiar characteristics.

FIGURE 2.29
Imaginary weave of the omphalos

The first is that the weave bears such a temporal depth and multiple-connectivity that makes it impossible to consider mathematical ideas of Arithmetics, Algebra, Topology, Analysis, Probabilistics, Combinatorics, Geometry, and Harmonics in separation from each other without an integral synthetic imaginary fulcrum.

The second is that the crossings of this weave are knotted nodes, where the "truth of a theorem" is preserved and stabilized as the encapsulated memory of all communication topoi that led to it through metaphora.

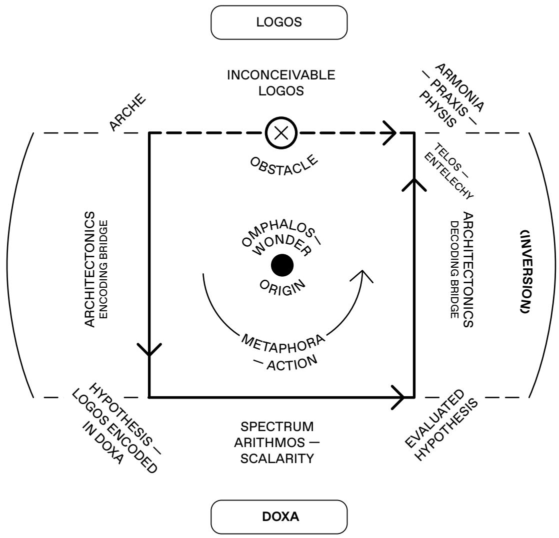


FIGURE 2.30
Metaphora through the architectonic weave and circulation

Bounding the Rhetorical Topos

We consider the threefold Philosophy/Mathematics/Architectonics in its capacity to give rise to a symphony cycle, involving their harmonic synchronization around an initially inconceivable, or directly inaccessible, common center according to a Logos.

We recall that if you consider any two of these domaions as different layers or levels, the third domain acts as a bridge between the other two. Bridges are always bidirectional, such that a domain of communication opens up. What is bounded by the bridges is a rhetorical topos where communication takes place.

Our present objective is to bound the notion of Logos to the notion of Arithmos through metaphora with respect to a series of architectonic bidirectional bridges according to the metronomy specified. At each stage, the spectral evaluation of a hypothesis in relation to the inexplicable Logos resolves the spectrum according to a spectroscopic scale.

The scale together with its numerical/elemental and structural specification gives rise to an arithmetic Cosmos that can be communicated, such that Logos can be bound to Arithmos. The arithmetic Cosmos bound to the Logos metaphorically defines the notion of scalarity that pertains to the way that the Logos is encapsulated and linked to the corresponding arithmetic Cosmos.

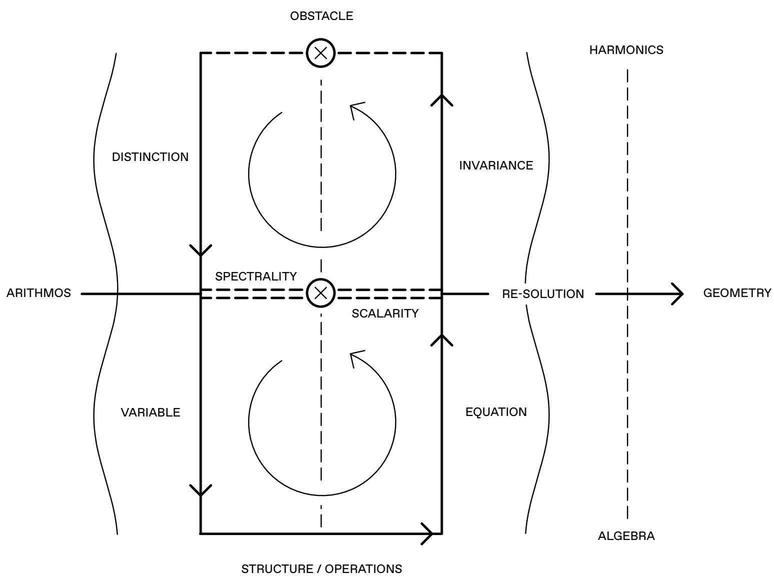


FIGURE 2.31
Metaphora via geometry and algebra: spectrality and scalarity

The Gnomon: Logos Bound to Proportion as Ratio

The gnomon is literally speaking the part of the sundial that casts the shadow.

The metronomy of the gnomon is intended for the measurement of the absolute value of directly inaccessible integer geometric magnitudes. An integer geometric magnitude is conceived with respect to a monad 1, that is, a unity that cannot be broken into parts. As such, an integer geometric magnitude does not express by itself any source of invariance, but under multiplication with other integer magnitudes it can give rise to an invariant relation.

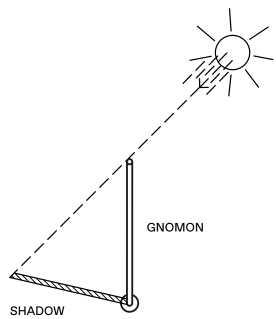


FIGURE 2.32
The gnomon and its shadow

In turn, this invariant relation specifying the metronomy of the gnomon instantiates a partition spectrum through which an inaccessible magnitude can be indirectly determined. In this setting, the invariant relation is the relation of proportionality, and the partition spectrum is the spectrum of ratios.

The objective is the bounding of Logos to *Analogia* as Ratio, and as such, the grasping of Ratio (Portion) from the invariance of Proportionality among four integer geometric magnitudes. In this context, we realize that the obstacle preventing the direct access to a geometric magnitude is

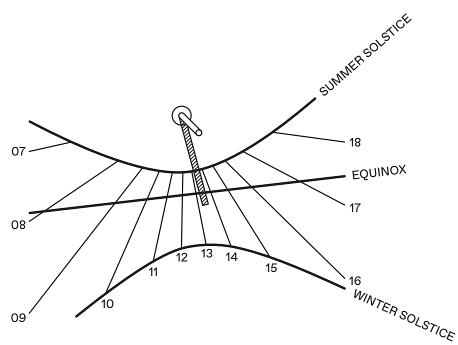


FIGURE 2.33
The shadow of the gnomon traces out a hyperbola throughout a day

actually a modular source of invariance, expressed by the invariant relation of proportionality (*analogia*).

The above becomes concrete if we examine its original context of instantiation. This pertains to the magnitude measurement of the height of a pyramid by Thales using a vertically placed measuring stick as a gnomon, architectonically bridging bidirectionally together the level of rigid geometric objects with the level of their shadows.

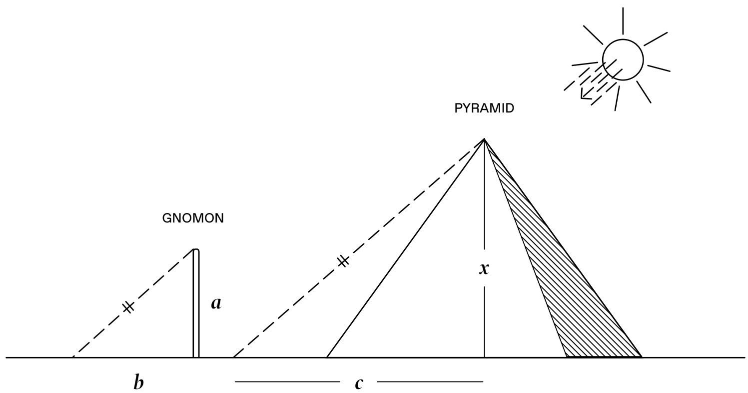


FIGURE 2.34
Gnomon measurement of the height of a pyramid by means of proportionality

This takes place by means of proportionality of integer magnitudes, which geometrically translates to the invariance of angle under parallelism. The proportionality relation involves the frozen shadows, since the natural communication bridge of sunlight is considered fixed at the same time of the day in order to gain this invariance. In turn, the same invariance gives rise to the geometric theory of Homeothesis.

Michel Serres, describes the gnomon in a beautiful way as follows: "This discovery has ancient letters: neuter in gender, the word "gnomon", which in the Greek language designated the sundial's axis, signified "that which understands, decides, judges, distinguishes, interprets, yes, that which knows"; as if a thing, already, knew. Intercepting the sunlight, its shadow writes, on the dial itself, a few events of the sky and of the Earth, the solstice, the equinox and the latitude of the site. It functions automatically. "Automatic" means: without the intervention of intention, which is subjective and cognitive.

It can be said of the gnomon that it knows the way it is said that it rains. The gnomon looks like a stylus, but no one holds it in their hand. Some things of the world give themselves to be seen to an object that shows them: entirely objective, theory does without any subject. A thing, the gnomon, intervenes in the world, and the world reads on itself the writing drawn by it. This type of intrahardware software conditions our cognitive performances, like a kind of objective transcendental.

Let us pay a closer attention to Thales' theory of Homeothesis. The objective of Thales was to find the directly inaccessible height x of a pyramid, given the length c of its accessible shadow cast by projection, as well as the height a, and the shadow length b, of the gnomon placed orthogonally to the ground.

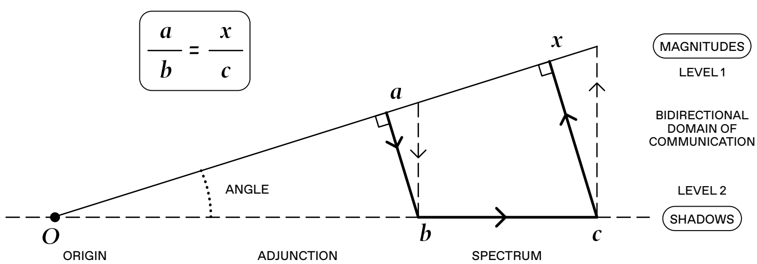


FIGURE 2.35
Analogia of Thales

The invariant relation devised by Thales, called *analogia*, is based on the idea that light rays coming from the sun induce a natural bridge projectively between the level of heights and the level of shadows, which if fixed for each specific time of the day gives rise to an equivalence relation among the four magnitudes. The *analogia* of homeothesis is expressed symbolically as follows:

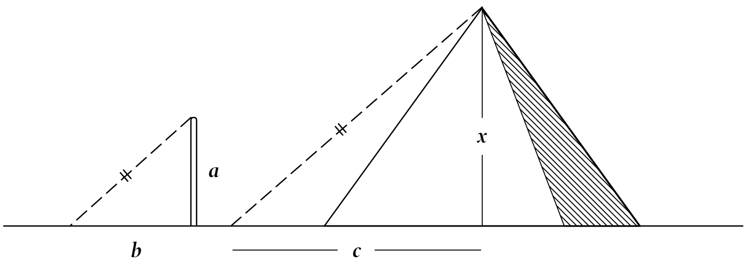


FIGURE 2.36
Proportion between magnitudes

(*a* to *b*) is as (*x* to *c*)

Note that the four terms of this proportion between magnitudes are arranged into two distinct levels according to a qualifying characteristic, that is, *a* and *x* occupy one level as vertical heights, whereas *b* and *c* occupy the other level as horizontal shadows.

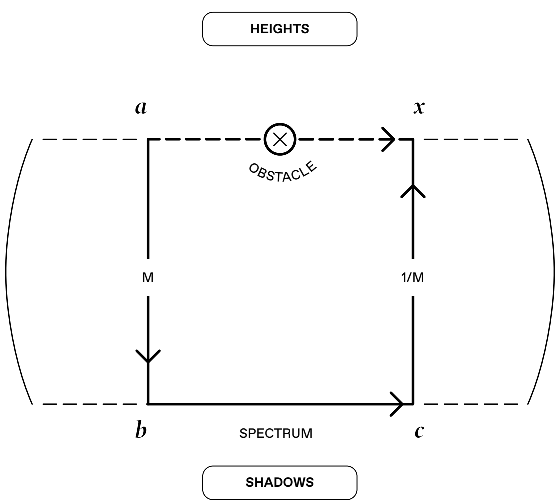


FIGURE 2.37
Analogia in the set up of the natural communication scheme

We will explain later, when we examine the proportionality invariant relation from the structural perspective of the algebraic cosmos of the Rationals, that the solution of the corresponding simple equation

$$a / b = x / c$$

in terms of the unknown x, involves the group-theoretic operations of multiplication and, inversely, division of positive integer magnitudes.

Thus, from the viewpoint of natural communication, the geometric theory of homeothesis, contains all the seeds of abstraction leading to the conception of the modern algebraic structure of the multiplicative group of the rationals.

In a suggestive manner, we can rewrite the solution of this equation as follows

$$x = M_a c M_b^{-1}$$

meaning that to obtain the not directly accessible magnitude x, "multiply by a" (denoted by M_a) the magnitude c, and then, divide by b (denoted by M_b^{-1}).

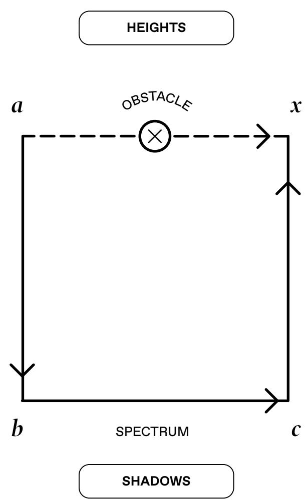


FIGURE 2.38
The metaphora between the levels of the heights and the shadows

From the simple natural communication diagram above, we immediately see that a proportionality relation is a case of metaphora taking place between the two delineated levels.

This is also in concordance with Aristotle's qualifying statement of metaphora in *Poetics*, according to which: "Metaphora is the substitution of the name of something else, and this may take place from genus to species, or from species to genus, or from species to species, or according to proportion."

The Monochord: Logos Bound to Harmonics as Integer

In the Thalesian geometric setting of the gnomon an integer geometric magnitude does not express by itself any source of invariance, but under multiplication with other integer magnitudes gives rise to an invariant relation: the proportionality relation. A natural question is if an integer itself can be thought of as a source of invariance going beyond the context of rigid geometric objects and their shadows. We will see that this task requires to delve deeper in the domain of Harmonics.

Leaving the domain of rigid objects, we enter into the domain of a magma. At a first approach, a magma is something that cannot be characterized in terms of directly distinguishable constituents. It displays a stochastic behavior as a whole at some spectroscopic scale of observation. Thus, it is not amenable to a part-whole relation in space, but it has to be thought of in terms of a distribution in time involving its potential periodic/frequential aspects.

In other words it has to be considered as a whole in the domain of Harmonics. From a philosophical standpoint, we may say that a magma does not have any discernible ontology, at least without further qualifications. But, what a magma has is what Aristotle calls "entelechy", a kind of teleonomy that is manifested through its periodic/frequential temporal behavior.

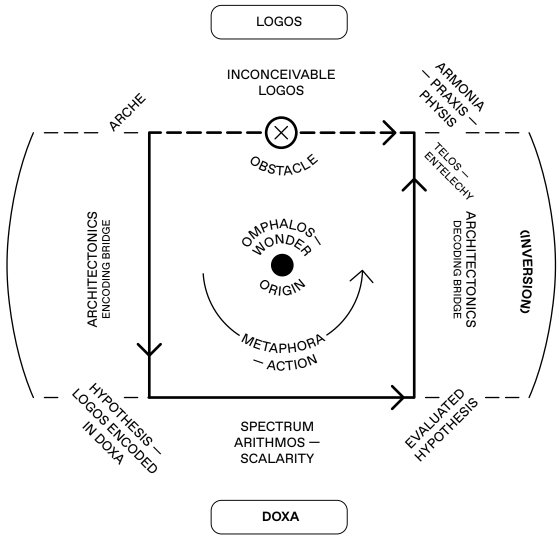


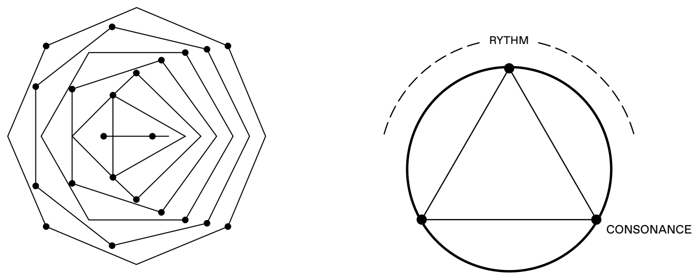
FIGURE 2.39
From arche to entelechy of the magma

This teleonomy may be thought of in terms of a Choreography out of which a certain rhythm becomes apparent pertaining to a symphony cycle originating from an imaginary synchronization center, which brings us to the domain of Harmonics. If we represent such a cycle in terms of a circle, but not confusing it with a geometric circle drawn by a compass since its center is imaginary, then a rhythm may be represented in terms of a polygon inscribed within this circle (Figure 2.40).

If the nodes of a polygon correspond to the characteristic frequencies, called the harmonic frequencies or simply harmonics, for which a synchronization of wholes takes place in such a symphony cycle, then the ratios of arcs would correspond to the consonances or dissonances produced.

FIGURE 2.40

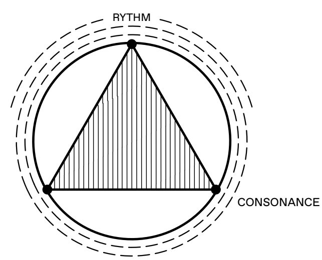
Rhythms as polygons inscribed in a circle



It is important to specify what type of scalarity should be ascribed to the harmonics. If the cycle of symphony corresponds to the joint unity in time, then the harmonics effect a cyclotomy expressed by the countable number of windings of the corresponding node per the joint unity of time. The number of windings per this unity, which is represented by the whole circle, expresses the periodic/frequential aspect we seek for. The greater the number of windings of a node, the greater its frequency, thus the greater its speed on the circle.

FIGURE 2.41

Nodes of a polygon as the roots of unity



We call the nodes of a polygon inscribed in the circle the roots of its unity in time. The idea is that a root raised to its harmonic frequency, thus adjoining the circulation with its characteristic speed of winding equals the unity. In this sense, a harmonic corresponds to the power that the root has to be raised such that unity emerges.

We will deal with the intricate aspects of powers and roots later on. For the time being, we aim to retain only the countable integer aspect of the windings, since this is an invariant aspect of a whole in a symphony cycle. The important thing to retain at this stage is that integers do pertain to an invariant relation in the domain of Harmonics.

In this sense, the ratios of arcs on the circle correspond precisely to the ratios between countable windings of different speeds. Of course, all these windings need to have the same orientation according to a choreography. Needless to say, there exist two possible orientations on the circle, the clock-wise and the anti-clock-wise.

In a choreography both of these orientations pertain, so it is important to realize their significance. It will turn out soon that it is due to this bivalent aspect of orientation that characteristic frequencies—harmonics emerge for the invariant depiction of a whole in a symphony cycle, that is, the "entelechy" of the magma.

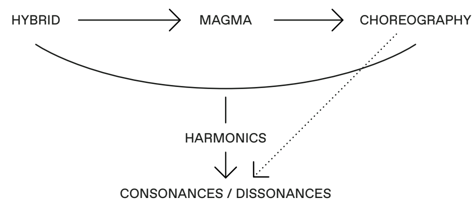


FIGURE 2.42
Choreography of magma and harmonics

We think of orientation as a digital bivalent gnomon in the domain of Harmonics. The basic idea is that a magma exhibits self-interference when bounded, which emanates from the interference of one orientation with the opposite, which we call the conjugate one.

This self-interference gives rise to nodes of stasis in the magma that delineate the harmonics. Self-interference is the major characteristic of quantum behavior, therefore the harmonics bear the role of quanta with respect to which cyclotomy takes place.

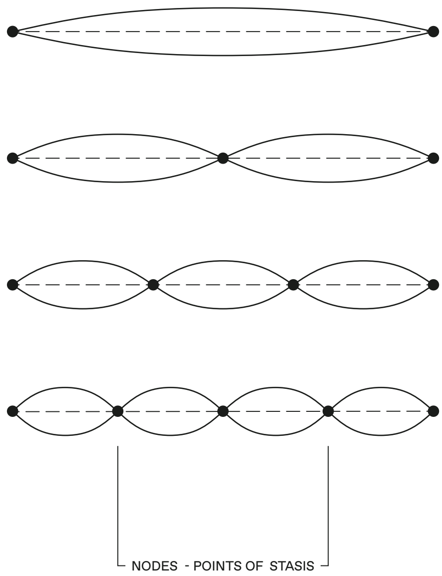


FIGURE 2.43
Harmonics and points of stasis

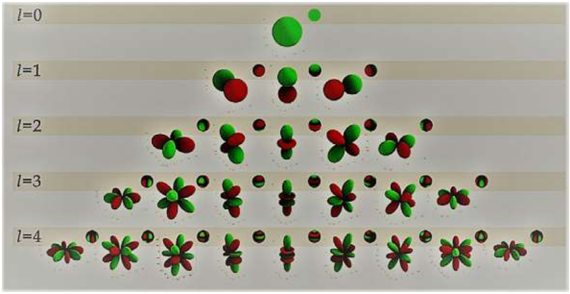


FIGURE 2.44
The spherical harmonics

(Re-elaboration based on graphic by © Bruno Masiero, 2011, ResearchGate)

Quanta are not constituent particles of the magma, they pertain to harmonic synchronization of the sections/arcs obtained through cyclotomy with respect to the time unity of the cycle. Quanta are the eigenfrequencies of the magma and define its discrete partition spectrum.

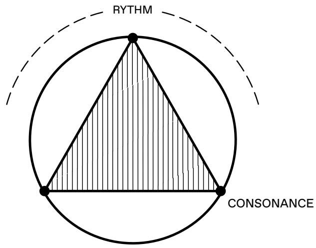


FIGURE 2.45
Quanta as the Eigenfrequencies of the magma

This is the major qualitative difference between the domain of Harmonics and the domain of Geometry. In the first domain, an entity can co-exist and interact with its conjugate one, without cancelling each other, but producing a spectrum of discrete nodes of stasis out of self-interference.

The self in this context is constituted teleonomically out of the interference of two oppositely oriented cycles, that is, out of the digital gnomon of orientation.

This is clearly not possible in the domain of Geometry. It is impossible for a geometric object to interact and interfere with its mirror image. The two copies, the original and the mirror image, retain their separability.

However, there is an inherited characteristic in the transcription from the first domain to the second. This is the aspect of chirality, the original and the mirror image are enantiomorphs of each other. In this manner, geometric areas are signed, that is, they are directed, and the duplication of signs owes to the digital gnomon of orientation.

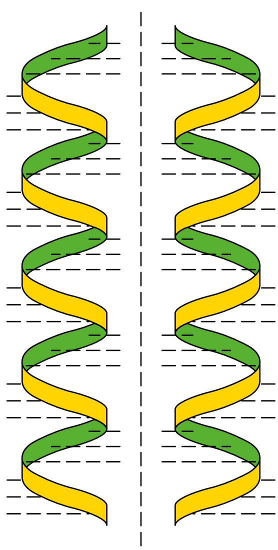


FIGURE 2.46
Mirror images and chirality

It is worth pointing out that Harmonic analysis and synthesis makes the distinction between the continuous and the discrete look superficial, pertaining only to a topological/geometric context. In Harmonics, the continuous, the discrete, and the bivalent co-exist without any contradiction. The choreography and the motion of the magma is clearly continuous.

Due to the absence of a predefined phenomenological ontology the magma does not have any parts; it is not subject to the part-whole relation. It is a whole that is subject to a certain type of periodic behaviour that gives it the chance to enter into a symphony cycle and synchronize in terms of harmonics. These are discrete and countable. They emerge out of self-interference, owing to the bivalency of orientation, as quanta delineated by nodes of stasis. These nodes pertain to a rhythm, which leads to cyclotomy, and the instantiation of consonances or dissonances expressed through ratios.

The Harmonic Scale: Genesis of Incommensurability

The notion of a magma subject to Harmonic analysis and synthesis obtains a concreteness via thinking in terms of the simpler case of sound, the subject of Pythagorean harmonics that inheres with music. In this case, a consonance or a dissonance, as well as a rhythm can be sensed acoustically through the ear within its perception range of frequencies.

The obstacle which Pythagoras was facing was the Logos pertaining to sound, that is, why the ear perceives consonances and dissonances and out of this directly inconceivable cause, how it is possible to compose and synthesize music in terms of ratios corresponding to consonances.

Ratios was the type of scalarity that Pythagoras employed to make his partition spectrum, of course in relation to the temporal periodic/frequential nature of acoustic sound, meaning that these ratios are not ratios of magnitudes, but they are ratios of frequencies (pitch). Surely, it is very difficult to construct the partition spectrum just by the acoustics of the ear.

For this purpose, he devised the monochord, an instrument consisted of a single vibrating string—a chord—plucked to the cavity of this instrument acting as a resonator, on which by attunement he intended to make a scale consisting of consonant ratios of frequencies.

It is evident that the string can vibrate—within certain bounds—only if it is excited under the action of a bow . The coupling of a plucked string with a bow gives rise to a hybrid entity, a whole that is not separable in parts, since such a separation eliminates sound.

Thus, the bow and the string—bound architectonically by the bridges of the resonator—refers to a whole, a kind of magma, according to the preceding. The idea of Pythagoras was to study the consonant ratios through moving the bridges where the string is attached, thus altering the length of vibration of the string. This is his method of attunement and it deserves special attention. Note that it is sufficient to keep one end of the string fixed and vary the other.

The underlying reason is that Pythagoras actually devised the monochord in its role as a canon on which a scale—built with respect to intervals of string length—can be imprinted as a modular substitute of a ratio of acoustic frequencies.

This is the essential aspect of the Pythagorean method, which aspires to make manifest the harmonic partition spectrum of consonant ratios through the geometric partition spectrum of concordant intervals of length on the monochord, that is, through metaphora from the domain of harmonics to the domain of geometry. Concordant intervals of length under this bidirectional correspondence—modular substitution—give rise to a partition spectrum at the geometric level, which are directly accessible on the canon, the length geometric spectrum of the monochord. Is is precisely this kind of metaphora that is called canonics.

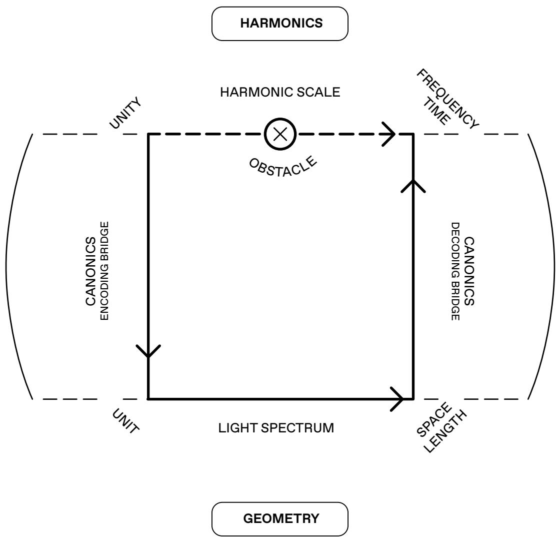


FIGURE 2.47
Metaphora between harmonics and geometry—the canonics of the architectonic scaffolding

The invariant relation that makes the bidirectional bridge between the harmonic and the geometric domain possible is that frequency and string length are inversely correlated to each other. This means that their product is always a constant, hence it is invariant. The spectrum afforded by the canon is a manifestation of this invariance.

The crucial point to observe in the canonics from the harmonic to the geometric, and inversely, is that ratios of harmonic frequencies are transcribed to intervals of length, which are evidently made out of differences in length.

Thus, the bridge from the harmonic to the geometric should be able to transform the operation of division to the operation of subtraction, where the first pertains to the harmonic and the second to the geometric.

Inversely, the bridge from the geometric to the harmonic should be able to transform the operation of substraction to the operation of division, so that from an interval we can obtain a ratio.

There is a unique function that bears the property required to be satisfied from the harmonic to the geometric, and it is the logarithmic function. Thus, the images of harmonic ratios on the canon are actually intervals on a logarithmic scale.

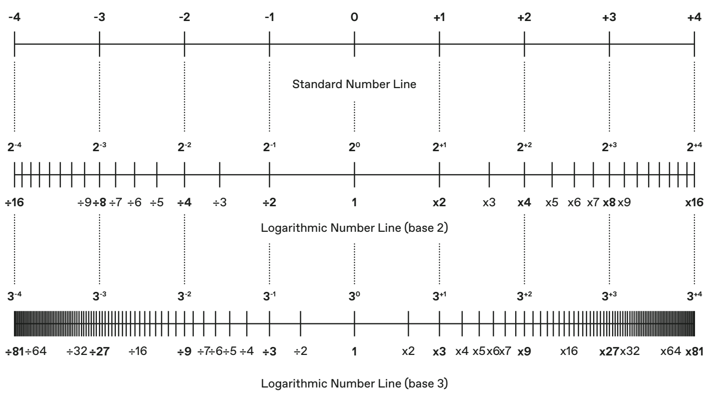


FIGURE 2.48
The logarithmic scales in base 2 and base 3 respectively

Since the value of the logarithmic function is, in general, not an integer, nor a ratio, the transcription from the harmonic to the geometric requires the extension of the arithmetic cosmos to the irrationals, which incorporate the topological assumption of continuity. Thus, the canon bounds the Logos to the Arithmos in terms of the continuous logarithmic function, whose values are logarithms.

Moreover, since ratios of harmonic frequencies pertain to the periodic/frequential aspect of time, we conclude that this aspect of time is transcribed logarithmically in space. We will see in the sequel, how this logarithmic transcription is expressed through the notion of geometric area and color of shadows.

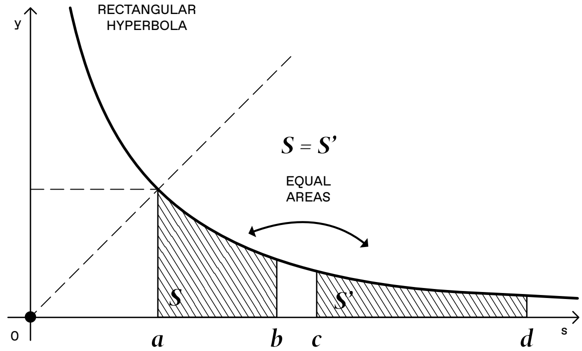


FIGURE 2.49

Proportionality and logarithmic equal areas under the rectangular hyperbola

Inversely, there is a unique function that bears the property required to be satisfied from the geometric to the harmonic, and it is the exponential function. It transforms the logarithmic scale of intervals in space, i.e. on the canon, to a rational scale of harmonic frequencies in time.

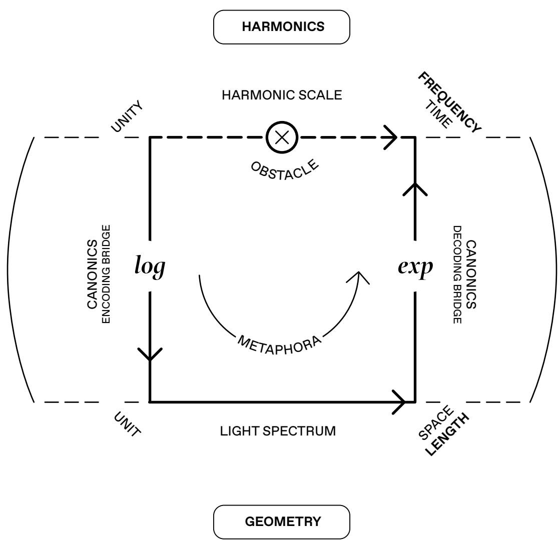


FIGURE 2.50

The Log/Exp canonics of the architectonic scaffolding

In order to grasp the notion of a harmonic frequency, it is instructive to consider briefly the physical notion of a stationary, or standing wave. A standing wave is a wave that is bound in space, i.e. between two walls, and therefore, since it cannot propagate beyond the walls, it gets reflected and interferes with itself. It is bound in space, and thus its recurring oscillatory behavior, as an outcome of self-interference, pertains to the periodic/frequential aspect of time.

A stationary wave has particular frequencies at which the amplitude of the wave is maximized. The wave is characterized by its amplitude which makes the sound higher or lower, its frequency, and its phase.

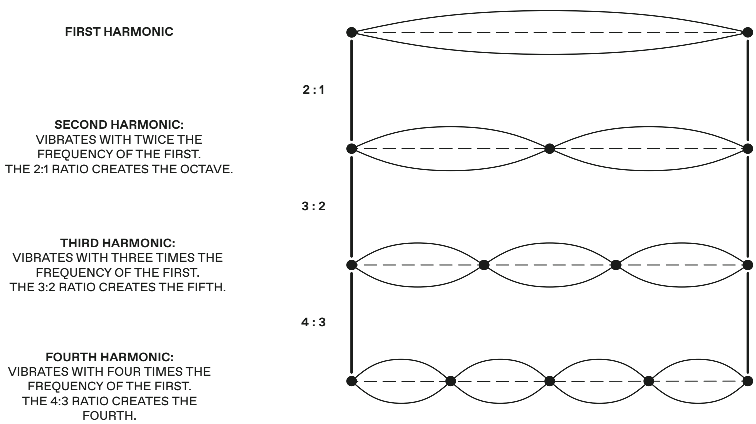
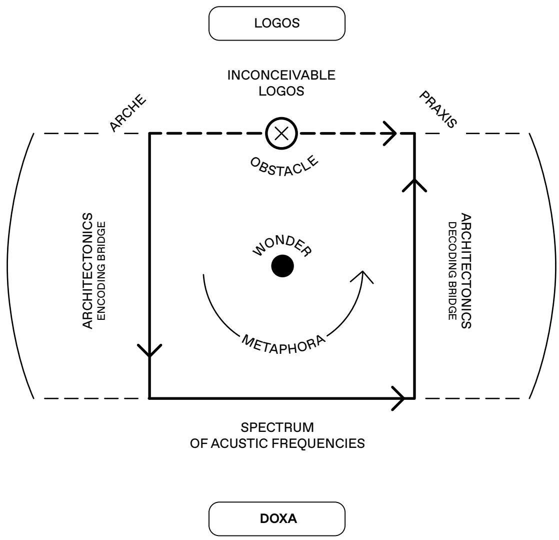


FIGURE 2.51
The harmonics and their consonant ratios

Harmonics refer to the range of integer frequencies where the amplitude becomes maximal. This is based on the principle of minimization of energy, an economy principle of Nature, which in Physics originates and is explained by the Principle of Least Action. The minimization of energy is implemented by the maximization of the amplitude. Thus, the harmonic frequencies are those with a minimal energy and a maximal amplitude (highest sound).

FIGURE 2.52

Percolation of the harmonics through the acoustic frequency spectrum



Based on the principle of minimization of energy, Nature filters out all frequencies except those that maximize the amplitude. Therefore, Nature abstracts and percolates the harmonics through sound, achieving in this way minimization of energy, something that qualifies sound as a natural communication bridge.

The ear distinguishes the harmonics via resonance, and can perceive them separately in the acoustic range it affords, although they sound together. Within the range of all the harmonic frequencies the fundamental frequency, called the first harmonic, is the first where this phenomenon takes place.

In the descriptive terms of standing waves, all the harmonics are the outcome of self-interference. They are distinguished in terms of nodes and anti-nodes. Nodes are the states of stasis, where the amplitude is null, whereas anti-nodes are the states where the amplitude is maximal. Clearly at the bounds the amplitude is null.

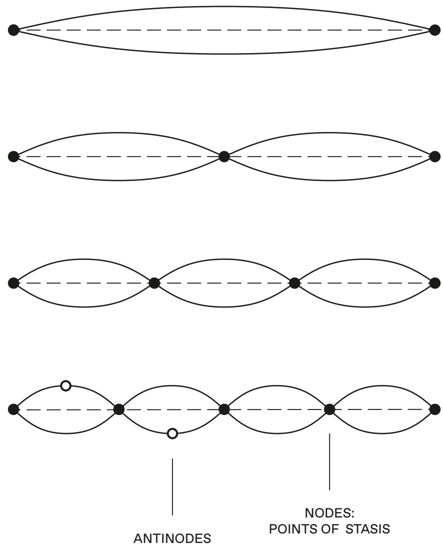


FIGURE 2.53
Nodes and Antinodes

Consonances manifest when different harmonics sounding together are in symphony, and dissonances when they are not. Consonances are expressed in terms of ratios of harmonics in symphony. It is consonances that give rise to the partition spectrum of sound.

For this purpose, a scale is needed effecting cyclotomy of the unity. The canon transcribes this scale to a logarithmic scale of length intervals—what is called musical intervals—in such a way that the whole length of the bounded string corresponds—in space—to the first, or fundamental, harmonic—in time. This is the unison attunement of the monochord.

If a dividing bridge is placed in the middle of the string, then only half of it can vibrate. You can hear that the frequency is higher than when the whole string vibrates, according to the ratio of *2/1*. Thus, the harmonic frequency doubles by cutting the length of the string in half, leaving their product invariant. This frequency, called the second harmonic, is in symphony with the first harmonic and a consonance is produced giving rise to the consonant ratio of $2/1$, called the octave. In an analogous manner, we may think of all higher harmonics and their transcription to string length on the canon. The most consonant ratios of harmonics, beyond the octave, is the ratio $3/2$, and the ratio $4/3$, called the interval of the perfect fifth, and the interval of the perfect forth, after their transcription on the canon.

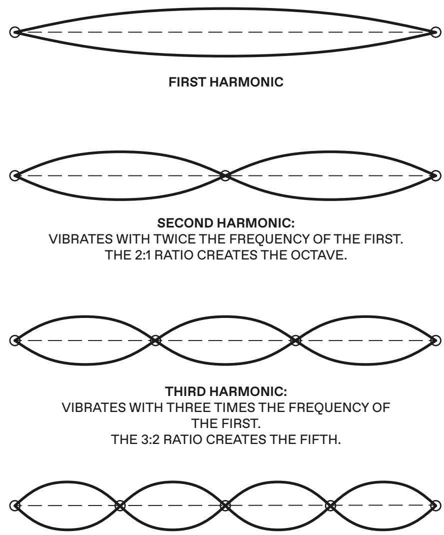


FIGURE 2.54
The consonant ratios of harmonics

These consonant ratios of harmonics, intended to be used for setting up a scale of musical intervals, may be obtained naturally through the method of the means. It is quite significant to focus our attention on the meaning of this method and its unifying power. According to a surviving fragment of the work of Archytas of Tarentum, the person who resolved the Delian problem in antiquity: "There are three means in music: one is the arithmetic, the second is the geometric, and the third is the subcontrary, which they call harmonic".

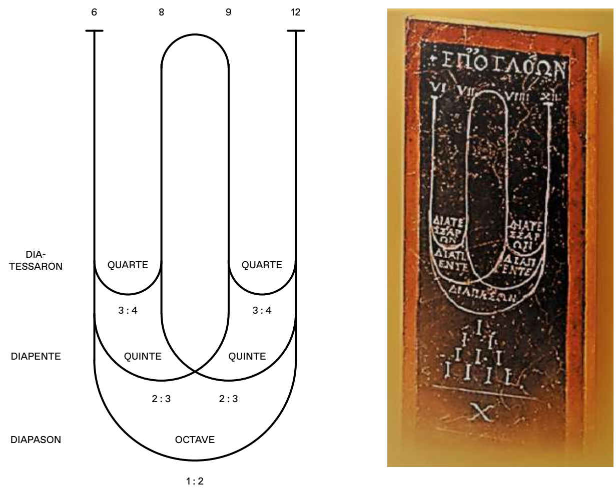


FIGURE 2.55
The consonant harmonic ratios and the pythagorean tetractys

(Re-elaboration based on photo by © unknown creator, Tetraktys)

Implicit in the notion of means is the notion of progressions in time. This pertains to the ordering aspect of time, which in sound and music pertains to the ordering of the musical intervals on the canon. In this manner, the ordering aspect of time, what we may think of as the melodic, corresponds in space to the ordering of the intervals on the canon. The basic idea here is that harmonics are ordered under following some progression. Since harmonics are integers their ordering should naturally follow the arithmetic progression.

But recall the digital bivalent gnomon of orientation lurking in the domain of Harmonics. For every progression following the arithmetic progression there exists a conjugate progression that reverses the orientation, and thus the former ordering, thus following the contrary to the arithmetic progression.

This is what is called the subcontrary, or harmonic progression by Archytas. Since harmonics are not geometric magnitudes, but they are invariant integer powers, their reciprocals, which are the roots on the cycle of symphony that they apply to, so that they can harmonically synchronize, and thus be in consonance to each other, should naturally follow the subcontrary progression, i.e. the subjected contrary to the progression of the powers.

To every type of progression there corresponds a mean respecting this progression. The arithmetic mean, defined by

$$AM = (a + b)/2$$

for any two consecutive elements a and b of this progression respects the progression of the harmonic frequencies.

The subcontrary mean, defined by

$$HM = 2ab/(a + b)$$

for any two consecutive elements $1/b$ and $1/a$ of this progression respects the order-reversing progression of the reciprocals of the former ones. Note that in terms of the canon, changing the orientation though reciprocals and their progression amounts to reversing the ordering of musical intervals in space.

This complicates the construction of a scale, because matters of ordering should be settled appropriately in the geometric transcription to the canon, so that the scale should be invariant under reversing the ordering.

The resolution of this issue comes from the geometric mean, which corresponds to a third type of progression, called the geometric progression, which pertains to the metaphora from the harmonic to the geometric domain.

The geometric mean of two positive elements a and b is the element $GM = p$ whose square equals the product $a \cdot b$ (elements pertain to geometric magnitudes in this case):

$$p^2 = a \cdot b$$

The geometric mean answers the following question: given a rectangle with sides a and b, find the side of the square whose area equals that of the rectangle. Euclid calls it the mean proportional p between a and b, which are in geometric progression, according to the invariant proportion relation:

$$a/p = p/b$$

The fundamental property of the geometric mean GM of two positive elements a and b is that its square equals the product of their arithmetic mean (considered in arithmetic progression) with their harmonic mean (considered in harmonic progression through their reciprocals):

$$GM^2 = AM \cdot HM$$

Thus, the area of the square made through the geometric mean is equal geometrically with the area made by the product of the arithmetic with the subcontrary mean of two harmonics, from which the geometric length provided on the canon as a unit is the square root of this product. But, this is an irrational, implicating that cyclotomy in terms of consonant ratios of harmonic frequencies, and thus a scale in terms of ratios, will always be incomplete, there always going to be a residue.

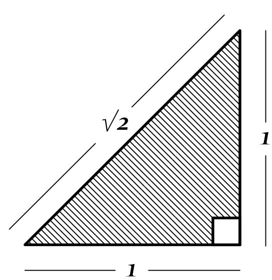


FIGURE 2.56
The incommensurability of the side with the diagonal of an orthogonal triangle

In the Pythagorean setting of consonant ratios the scale is constructed as follows: Since the scale is a scale made of ratios, the absolute values of the harmonics are not relevant. The integer

6 is the first perfect number, meaning that it is invariant under addition and multiplication of its factors. Clearly, $6 = 1 + 2 + 3$, and $6 = 1 \cdot 2 \cdot 3$. In terms of this, the consonant ratio $2/1$ is expressible equivalently as the ratio $12/6$.

The arithmetic mean of 6 and 12 is

$$AM = (6 + 12)/2 = 9$$

The harmonic mean of 6 and 12 is

$$HM = (2 \cdot 6 \cdot 12)/(6 + 12) = 8$$

The product of the arithmetic with the harmonic mean of 6 and 12 is:

$$AM \cdot HM = 9 \cdot 8 = 72$$

which is equal to the square of their geometric mean.

But, this product also equals the product of 6 and 12, therefore:

$$9 \cdot 8 = 12 \cdot 6$$

from which, we obtain the equality of ratios $9/6 = 12/8$ and $8/6 = 12/9$. The first of these ratios is the consonant ratio $3/2$, and the second is the consonant ratio $4/3$.

We conclude that the musical intervals of the [*Octave, Fifth, Fourth*] have their corresponding logos in the harmonic ratios [$2:1$, $3:2$, $4:3$]. The composition of two musical intervals corresponds to the multiplication product of their respective ratios.

We have now three small ratios corresponding to consonant musical intervals: $2/1$, $3/2$ and $4/3$. The question is if there is a common multiplicative measure among them, so that the scale can be completed with reference to this measure. It is easy to see that such a common measure does not exist, verifying in this way, the conclusion reached also previously.

$$2/1 = 3/2 \cdot 4/3$$

$$3/2 = [4/3][9/8]$$

$9/8$ corresponds to a new musical interval called a “tone”. Is the “tone” the required common measure?

$$4/3 = [9/8][32/27]$$

But, $[32/27] > [9/8]$, hence we obtain:

$$32/27 = [9/8][256/243]$$

$256/243$, corresponds to a new musical interval called a “diesis”, or “leimma”, from which:

$$4/3 = [9/8]^2 \ [256/243]$$

The diesis and the tone are not commensurable to each other, thus we end up with two incommensurable measures for making the scale; the tone, and the diesis. In more detail, we see that

$$Tone = [Diesis]^2 \cdot [Comma]$$

where the $[Comma]$ is an even smaller interval, showing that the procedure can be continued ad infinitum without finding a common measure.

The gain of this exercise is to figure out that:

$$[\,\textit{Comma}\,] = 3^{12} / 2^{19}$$

which encodes the following:

$$[\,12 \;\; \textit{Intervals of the Fifth}\,] \;\; = \;\; [\,7 \;\; \textit{Intervals of the Octave}\,] \;\; \textit{modulo} \;\; [\,\textit{Comma}\,]\,.$$

The [*Comma*] is the residue of the analysis, certifying the incommensurability, thus the presence of the irrational in music. Note that since *3* is odd and *2* is even it is impossible to be commensurable under raising to powers. It is because of this fact that musical composition allows an infinity of possibilities.

In terms of the two incommensurable measures at disposal, we have:

$$\textit{Octave} = [\,2/1\,] = [\,\textit{Tone}\,]^{5/1} \cdot [\,\textit{Diesis}\,]^{2/1}$$

In this manner, the scale is defined, although it bears a residue, as a trace of the irrationality lurking in the background, from which the octachord (8 chords) in the so called diatonic genus emerges out of the disjunction of two tetrachords (4 chords).

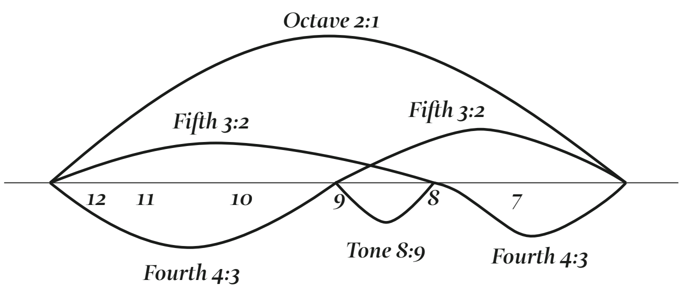


FIGURE 2.57
The tone and the incompleteness in the rational division of the harmonic scale

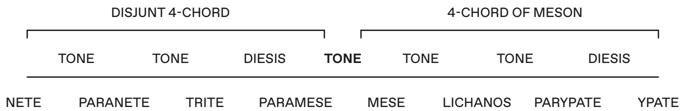


FIGURE 2.58
The disjunction of two tetrachords

The outcome of this method of making the scale, based on the arithmetic and the subcontrary mean, is a partition spectrum consisting only of ratios, and therefore, corresponds to the notion of pure tuning. Note that starting from the unison, and proceeding eight steps (intervals) from it, we reach the octave that the harmonic frequency doubles.

On top of this scale, we may now start from the octave and proceeding again eight steps along it we reach the double octave where the harmonic frequency quadruples, and so on. Therefore, the ladder of scales follows the powers of 2 . This may be though of as a helix that unfolds according to the powers of 2 .

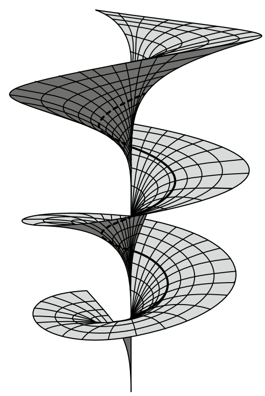


FIGURE 2.59
The ladder of scales

From the definition of the residue, that is, the [*Comma*] ;

$$[\,Comma\,] = 3^{12}/2^{19}$$

we see that, since it is very small, and thus almost imperceptible, we may approximate 2^{19} in the ladder of scales with 3^{12} following the powers of 3 instead. This is what usually is called the circle of fifths, although it is not a circle, but a spiral of fifths, showing that the harmonic domain

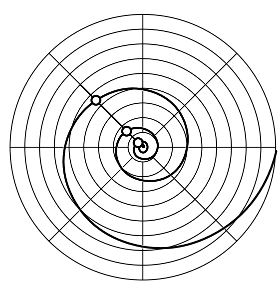


FIGURE 2.60
The never-ending spiral due to the comma

cannot be transcribed only in terms of ratios to the geometric, without a trace, the residue or memory of this transcription through the canon.

Observe that, since a fifth refers to the ratio $3/2$, we may employ the fifth as the unit of the scale, such that going up the ladder by the 12th power we expect to be able to descend back through 7 octaves, considering literally that $3^{12} = 2^{19}$, i.e. forgetting or neglecting the comma completely.

Of course, this is not possible, and the best we can do is to obtain a never-ending spiral of fifths. It is impossible to close the circle, that is to achieve harmonic cyclotomy with pure ratios, and the manifestation of the comma is to remind us of exactly this.

From the perspective of natural communication, there emerges the question of how it is possible to employ the canon as an actual geometric scale bearing a unit and respecting at the same time the harmonic domain, that is, encoding appropriately the irrationality inherent in the transcription.

At the moment, the canon bearing the metronomy of pure ratios, for instance the pure fifth, requires two different incommensurable measures, such that any attempt to suppress the comma is in vain, meaning that the partition spectrum of the canon cannot be operable and scalable by geometric means. This does not mean that making music is impossible as the ingenuity and the variety of scales devised and used throughout history is impressive.

The Tempering Screen: Logos Bound to Spectrum as Chroma

The incommensurability of the canon means the seed of the harmonic in the geometric domain cannot be encoded through pure ratios, which opens up, in turn, architectonically the geometric domain beyond the grasp of ratios and frozen shadows, to the grasp of the irrational.

This is only possible by substituting the fixed origins of invariant angles of sight on the ground that give rise to the pure ratios geometrically by roots that are hidden underground and thus their positioning is not directly accessible to vision.

But still, they can be envisioned through their uplifting—via imagination—on an epiphaneia, a boundary screen between the harmonic and the rigid geometric.

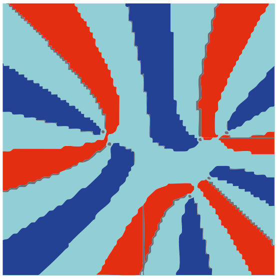


FIGURE 2.61
Envisioning the roots

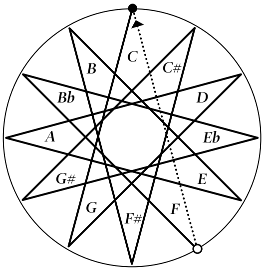


FIGURE 2.62
Uplifting the roots on an epiphaneia

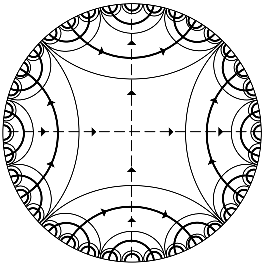


FIGURE 2.63
Tempering and uniform probabilistic distribution of the residue/comma on the epiphaneia

This is an epiphaneia where continuity and variability prevails, and where differentiability, stochastics, and probabilistics take place. The role of this epiphaneia is to distribute probabilistically the residue of the harmonic domain, the dark inaccessible shadow area of the comma on the rigid geometric canon, all around the epiphaneia, in a uniform manner. This is what we call tempering.

This implies a stochastic equipartition of the epiphaneia through a uniform probability distribution pertaining to areas. Put simply, the harmonic residue—the obstacle from a rigid geometric perspective—is distributed evenly all around the disk epiphaneia bounded by the circle whose cyclotomy is impossible geometrically through ratios.

The artifact of this metaphora through the epiphaneia is the topological metamorphosis of the dark residue to light, which is evenly distributed probabilistically all around the epiphaneia. This is not a probability that originates from ignorance, nor randomness, but it is intrinsic and objective chance.

The dark residue is not visible geometrically, it has to be neutralized through differentiation, then spread out probabilistically and uniformly on the epiphaneia, so that it can be finally integrated back from the equipartitioned spectrum of colors distinguishing the equally-spaced areas on the disk.

The obstacle is substituted by a circularly distributed diffraction pattern of light whose angularity is not rigid but it is invariant under scaling. Invariance under scaling is what makes possible the grasping of the residue on the epiphaneia geometrically.

The epiphaneia as an in-between boundary refracts the diffracted light—encapsulating the residue—giving rise to a uniform probabilistic distribution of equally-spaced areas on the disk, hence dividing it isonomically—that is in an equiareal way. Each of these areas acquires a color through refraction that distinguishes it from the rest.

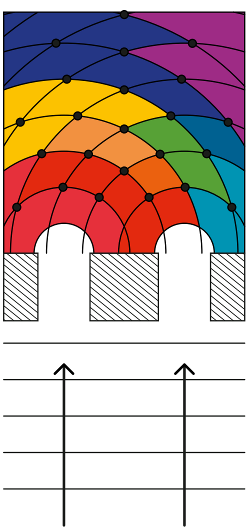


FIGURE 2.64 Circularly distributed diffraction pattern of light

For instance, the equally-tempered scale referring to such an epiphaneia, called the chromatic scale—where chroma is the ancient name for color—is the refraction of the circularly distributed diffraction of the comma—the dark residue from the harmonic domain—in the twelve colors of the different musical keys under tempering equally, thus democratically.

This completes the metaphora from the harmonic to the geometric domain, where the architecture of the latter is now empowered by the refracting epiphaneia it encapsulates, allowing geometry to be operative through tempering, i.e. probabilistically, stochastically, and topologically.

The bounded vibrating temporal chord bearing the harmonics is thus bonded to the spatial geometric domain via an elastic, but equally tempering boundary epiphaneia, spectrally distinguishable through the colors of the light it refracts. Each key of the equally-tempered scale corresponds to a different color of this democratic light spectrum.

The tempered canon achieves the modular substitution through metaphora of the temporal harmonic cord to a spatially distributed—via circular polarization—optical fiber.

Consequently, the equally-tempered scale arises from detuning the pure harmonics through tempering equivariantly. The chromatic scale employs an equivariant tempering of 12 equally-sized steps. It is possible to initiate musical composition starting from any key, any color of this spectrum. The only requirement is the preservation of the octaves, the powers of 2, as we proceed geometrically.

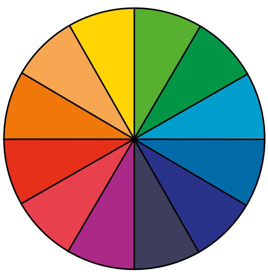


FIGURE 2.65
The chromatically tempered canon

The fact that the progression is geometric signifies that we should use the geometric mean to progress along the scale between any two equally spaced marks on the periphery of the disk until we arrive at the octave. Since the color cyclotomy is in terms of 12 equal intervals, if we start from the unison we should be able to reach the octave in 12 equally-sized steps, where the unit making this step is the geometric mean between its initial and terminal mark on the scale.

Thus, the cyclotomy of the equally-tempered scale is not in terms of pure ratios, but in terms of the single irrational unit $\sqrt[12]{2}$ that proceeds geometrically from any chosen mark in 12 steps until it completes an octave. In this way each of the corresponding 12 blocks of the partition of the disk corresponds to a chroma, a color of the spectrum.

The geometric progression from the unison to the octave takes place, according to the above, as follows:

$$1 = (\sqrt[12]{2})^0, (\sqrt[12]{2})^1, (\sqrt[12]{2})^2, \dots (\sqrt[12]{2})^{12} = 2$$

This is a logarithm system with the base $\sqrt[12]{2}$, which stands for the unit of the geometric progression along the scale, where the logarithms are the exponents $1, 2, 3, 4, \dots, 12$. Therefore stepping from one tone to the next on the equally-tempered canon—according to the geometric progression—corresponds to logarithmization.

Independently of music, the set up of tables of logarithms to simplify calculations performed by hand may be thought of as a kind of artificial memory to cope with the irrational domain of numbers. A table of logarithms is constructed by the successive calculation of a series of logarithms.

In this sense, we arrive at an interesting idea that instantiates a bridge in time through metaphora. The construction of a table of logarithms in arithmetic is equivalent to tuning a musical instrument according to the equally-tempered scale, where the metaphora from one domain to the other takes place through the geometric progression, or equivalently through a spectrum of colors.

Conclusively, the discrete integer temporal harmonics exert their invariance only via exponentiation, that is, via their inscription through powers, and manifest in space, thus recognized as quanta of light, via logarithmization.

This pertains both, to the series expansion of the transcendental exponential function, and to its inversion by the equally transcendental logarithm function that recognizes the exponents. The exponential function enciphers by elevating the power of the in-scripted harmonic invariants from the implicit roots to the exponent, which in turn, is stochastically recognizable through tempering, and thus decipherable on the chromatic epiphaneia, by the logarithm.

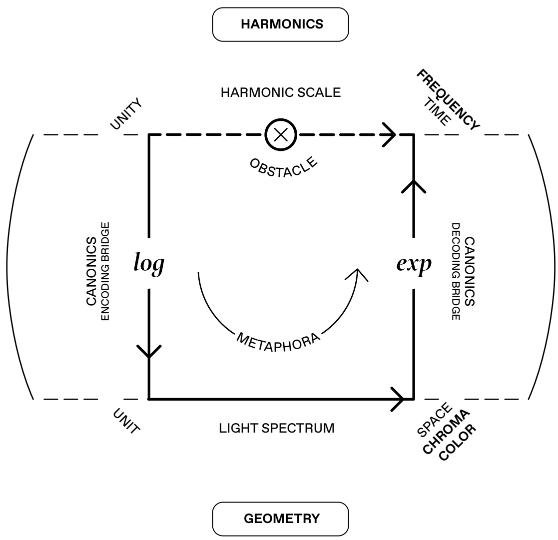


FIGURE 2.66
The universal Log/Exp architectonic scaffolding from the domain of harmonics to the domain of spectral geometry

The geometric progression that takes place along the equally-tempered scale can be grasped geometrically in terms of a logarithmic spiral extending outwards from the center of the disk.

Since the progression preserves the octaves, the radius of the logarithmic spiral grows according to $2^{(\theta/2\pi)}$, where θ denotes the corresponding angle, that is:

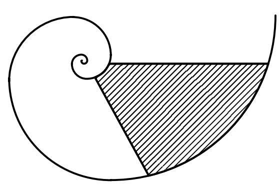


FIGURE 2.67
Geometric progression via the logarithmic spiral

$$R = 2^{(\theta / 2\pi)}$$

where R is the extending radius of the spiral across the disk. Thus, progressing geometrically to the first octave, denoting the first turn of the spiral, means that $\theta / 2\pi = 1$, thus $\theta = 2\pi$, $R = 2$, and so on.

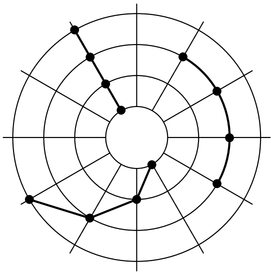


FIGURE 2.68
Epiphaneia as the web of a spider

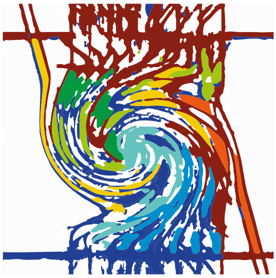


FIGURE 2.69
Architecture of the arachne

The important thing to note is that upon completion of each octave in the ascent of the spiral, corresponding to the geometric progression, we can descend back towards the center following the exponents of 2, that is, backwards along the stairs of the octaves following the arithmetic progression this time pertaining to the logarithms.

In this manner, we accomplish the architectural manifestation of the epiphaneia of the disk as the web of a spider—called *arachne*—which pertains to the dual understanding of the polar grid under progression, that is, the geometric progression outwards following the spiral radius of expansion, and the arithmetic progression inwards following the logarithmic stairs.

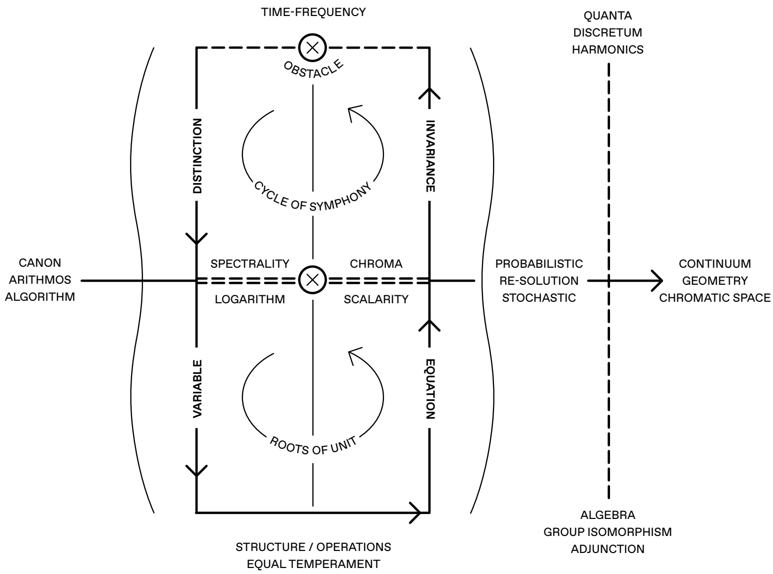


FIGURE 2.70

Natural communication scheme of quanta (discrete harmonics) via the equally tempered chromatic spectrality and the logarithmic scalarity

FIGURE 2.71

Equally tempered architectonic weave

3.
The Devil of Algebra: The Time Art of Adjoining and Inverting

Structure and Symbol

Algebra pertains to the structural enunciation of the "obstacle-embracing", and communicative process of metaphora between any levels. This enunciation is formulated operationally in terms of symbolic algebraic structures like groups, rings, modules, and categories. The notion of an algebraic symbol does not bear, neither the temporal connotation of a "symbolon", nor the spatial connotation of a "sign". In this manner, an algebraic structure maintains an independence from both the harmonic (symbolon), and the geometric (sign) connotation of its symbols, although it may properly mediate between them and abstract from both of them.

What constitutes a structure is defined in terms of elemental closure with respect to certain operations—like addition and multiplication—that can be, in principle, inverted within the same type of structure. The necessity of inverting operations comes from the origin of Algebra, which is the capability of unveiling the roots of equations.

An algebraic equation involves un unknown variable that satisfies this equation formulated in terms of operations, for instance, the well-known polynomial equations. The resolution of an equation amounts to working backwards until a root emerges, which fixes the variable that satisfies the equation. In turn, working backwards consists in the ability to move terms from one side of the equation to the other, and thus invert operations. The inversion of operations is the biggest stumbling block.

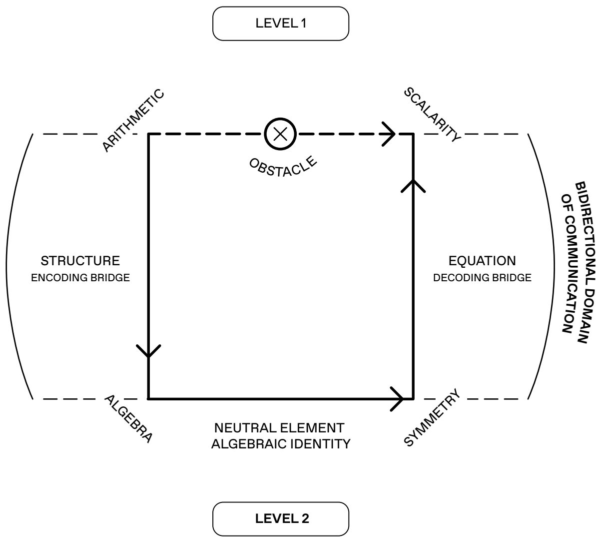


FIGURE 3.1
Modular substitution of the obstacle by a neutral element through the algebraic scaffolding of structure/equation

In the majority of the cases, inversion cannot be performed directly. In these cases, the type of algebraic scalarity has to be extended into a more rich structural domain by means of an adjunction. The purpose of an adjunction is to adjoin new elements that make the inversion of operations possible in the new augmented structural domain. The adjunction should not be ad hoc, but structure-respecting, in the sense that the old structure is not demolished, but it is embedded inside the new, such that the restriction of the new to the old agrees with the old.

The art of inverting operations cannot be performed without the existence of neutral elements. Every algebraic structure bears a neutral element with respect to a corresponding operation. The notion of neutral element characterizes the identity of an algebraic structure with respect to the operations it carries. In other words, the identity of an algebraic structure is enciphered in its neutral element. Conversely, the criterion of equivalence that constitutes the identity of an algebraic structure is deciphered by it neutral element.

The simplest way of presentation of an algebraic structure is by means of a set of elements—on which certain operations are applied—satisfying the condition of closure. The notion of a set is used as an architectonic scaffolding for expressing the operations of a structure as operations applied to the elements of the scaffolding. The notion of structure is independent from the scaffolding of sets.

Conjugation: The Algebra of Metaphora

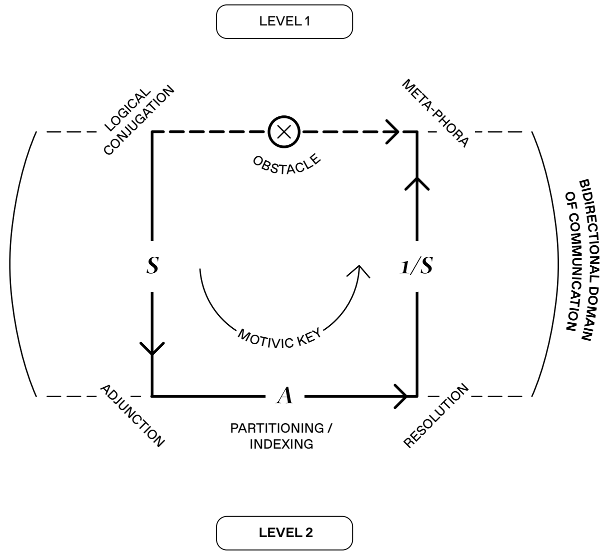
In general terms, the method of metaphora from a directly inaccessible, or obstacle-laden, domain X, to another domain A architectonically adjoined to the former, and entering into communication with it, is represented symbolically as follows:

$$X = SAS^{-1}$$

This relation defines X to be conjugate to A under S, where S^{-1} is considered to be the conceptual inverse of S.

FIGURE 3.2

The algebraic conjugation method via a motivic key



The algebraic expression $X = SAS^{-1}$ consists of two basic organic structural parts: The first part is delineated by the two conceptually inverse vertically displayed arrows S and S^{-1}, forming the outer, or architectonic part, which specifies the boundary of the metaphora.

It consists of a bidirectional bridge of encoding/decoding between two different levels entering into a communication to each other. The second part is constituted by the horizontally displayed arrow A, forming the inner part of the metaphora, which gives rise to a partition spectrum of the obstacle-laden domain. if the inner part A is absent, then the outer part simply collapses since it cancels out. Based on this fact, we can formulate the basic properties of this logical conjugation scheme, which models metaphora—by means of a motivic key—as follows:

(1) Horizontal Extension of Metaphora in Length: This expresses the property of juxtaposition, that is, if a metaphora shares the same encoding/decoding bridges with another metaphora, then they can be juxtaposed horizontally as follows:

$$\text{If } X_1 = SA_1S^{-1} \text{ and } X_2 = SA_2S^{-1}$$
$$\text{then } X_1X_2 = SA_1A_2S^{-1}$$

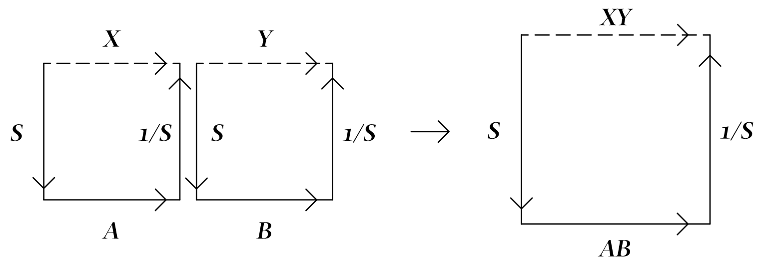


FIGURE 3.3
Horizontal extension of metaphora in length

(2) Vertical Extension of Metaphora in Depth: This expresses the stacking of two metaphoras arising from the substitution of the inner part of a metaphora by another metaphora, such that, the initial metaphora can be accomplished via a splitting into a deeper level, and so on, as follows:

$$\text{If } X = SAS^{-1} \text{ and } A = TBT^{-1} \text{ so that } X = STBT^{-1}\ S^{-1}$$
$$\text{then } X = (ST)\,B\,(ST)^{-1}$$

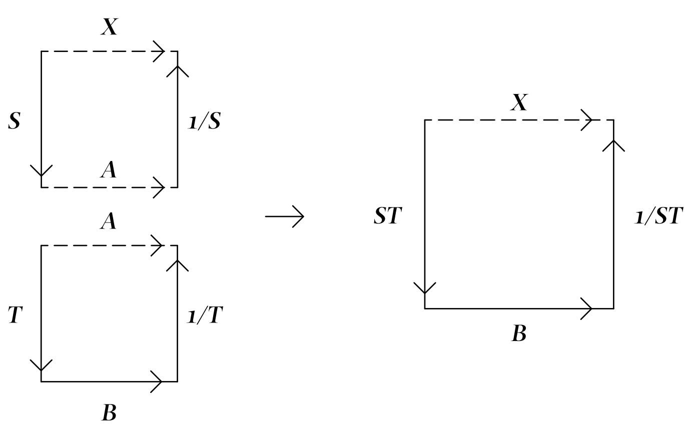


FIGURE 3.4
Vertical extension of metaphora on depth

(3) Inversion of Metaphora: This means that if a process X is conjugate to a process A at another level under the action of a bridge S, then A is conjugate to X under S^{-1}, as follows:

$$\text{If } X = SAS^{-1}, \quad \text{then } A = S^{-1}XS.$$

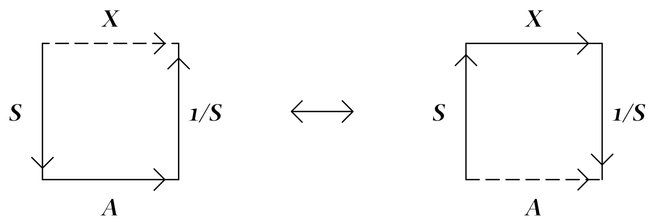


FIGURE 3.5
Inversion of metaphora

Monoids of Natural Numbers

A monoid is the simplest type of algebraic structure. It involves a set of elements, which satisfy the condition of closure upon application of the operations. For instance, in case the operation is addition, then the sum of two elements must also be an element of the same monoid.

We consider the set of Natural Numbers *1 , 2 , 3 , 4 ...* —until countable infinity—as abstracted from a spectrum of harmonic frequencies, denoted by $\mathbb{N}$.

Cardinality and Ordinality in $\mathbb{N}$

Cardinality pertains to the notion of counting, while ordinality pertains to the notion of ordering. Cardinality is though of as a measure of the size of a set. The naturals is a countably infinite set. From Ordinality derives the relation of order. There exist two basic types of order, partial order and total order.

A binary relation R on a set A is a partial order if and only if it is:

(1) reflexive,

(2) antisymmetric, and

(3) transitive.

In more detail, for all a, b and c in P, the following hold:

$a \leq a$ (reflexivity)

if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry)

if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

The ordered pair $< A, R >$ is called a poset (partially ordered set) when R is a partial order.

For example, the subset relation on the power set of a set, say $\{1, 2\}$, is also a partial order, and the set $\{1, 2\}$ with the subset relation is a poset.

Total Ordering and Well Ordering

A binary relation R on a set A is a total order if and only if it is:

(1) a partial order, and

(2) for any pair of elements a and b of A, $< a, b > \in R$ or $< b, a > \in R$.

We simplify the above writing $a \leq b$, or $b \leq a$. That is, every element is related with every element one way or the other. A total order is also called a linear order.

The Natural numbers are totally ordered with respect to the less-than-or-equal-to relation ($\leq$). Thus, the set of the naturals is a totally ordered set.

The well-ordering principle of natural numbers is expressed as follows:

Every non-empty subset of the set of all Natural numbers contains a least element.

Induction Principle

Suppose that S is a statement meaningful for each natural number, and suppose moreover that both a) S is true of the number 1, and b) whenever S is true of the number n, then S is also true of the number $n+1$. Then S is true for each natural number.

Operations in $\mathbb{N}$

Closure of Addition $+$ and Multiplication $\cdot$ in $\mathbb{N}$:

$$\{a, b\} \in \mathbb{N} : (a+b) \in \mathbb{N}$$

$$\{a, b\} \in \mathbb{N} : (a \cdot b) \in \mathbb{N}$$

Addition is a linear operation: $1 + 2 + 3 + \cdots$.

Multiplication is a bilinear operation—it should be thought of as operating on rows and columns.

Powers in $\mathbb{N}$

$$\{a, b\} \in \mathbb{N} : a^b \in \mathbb{N}$$

The operation of raising to a natural power consists of the operations of multiplication and recursion.

Dependency of Operations

$(+) < (\times) < (exp)$: Multiplication can be implemented via recursion upon addition and exponentiation (raising to a power) can be implemented via recursion upon multiplication.

Spectrum Quanta

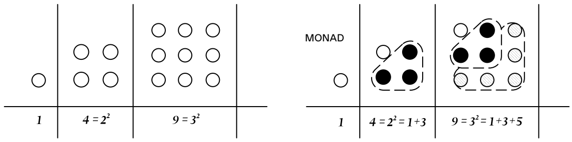
The Natural numbers are Quanta that build up a spectrum of invariance. The discrete spectrum of $\mathbb{N}$ subsumes four things: the spectrum is well-ordered; it admits addition, multiplication and exponentiation.

Digital Gnomons: Primes and Fundamental Theorem

Primes are numbers that do not have any parts: They are the atoms of the arithmetic cosmos because they cannot be partitioned in $\mathbb{N}$.

Squares can be expressed as sums of primes, as it can be easily seen below.

FIGURE 3.6
Monads, primes, and right angles in arithmetics



If you add a gnomon to a square, you produce again a square. The prime numbers appear as right angles in the Natural numbers, revealing the hidden Geometry inside $\mathbb{N}$.

Fundamental Theorem of Arithmetic

If m is a composite natural number, then m is the product of primes; that is,

$$m = p_1 \; p_2 \; p_3 \cdots \; p_n$$

where p_i is prime for $1 \leqq i \leqq n$.

If m is a natural number other than 1, then m can be factored into the product of primes, and this factorization is unique (apart from the order of the prime factors).

Equivalently, we obtain the following version of the fundamental theorem of Arithmetic:

If m is a natural number other than 1, then there is one and only one way of expressing m as the product

$$m = p_1^{n_1} \; p_2^{n_2} \; p_3^{n_3} \cdots \; p_k^{n_K}$$

where the exponents $n_1, n_2, \; \ldots \; n_k$ are natural numbers, and $p_1, p_2, \; \ldots \; p_k$ are primes arranged so that $p_1 < p_2 < \cdots < p_k$

Language of Algebra

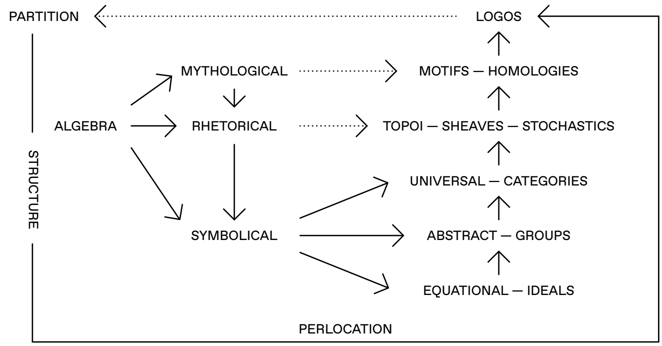


FIGURE 3.7
The unfolding of the algebraic logos through its structural invariants

Algebra is the mathematical domain pertaining to structure—defined by means of operations—and the solution to equations. The language of algebra is multi-faceted and provides the most refined way of expressing the notions of equivalence and invariance in structural terms.

Algebraic language permeates through the whole history of human thinking in relation to its expression in symbolic terms. In this manner, we distinguish among mythological, rhetorical, and symbolic algebra. The latter is further refined in terms of equational, abstract, and universal—or categorical—algebras. A broad scheme of qualifying this proposition appears in the following diagram:

For the algebraic resolution of equations it is necessary to have a precise symbolic language consisting of a syntax and a grammar. Only in this case the notion of structure—based on the operations and expressed by means of a scaffolding—acquires an invariant meaning. The symbols—including the variables and the coefficients—and the algebraic signs participate in the syntax. The operations—the rules of closure -, and the equilibria—

the equality between the two parts of an equation -, participate in the grammar.

Let us suppose that an equation involves the operations of addition and multiplication. In order to proceed to its resolution we need neutral elements with respect to both operations. Otherwise, it becomes impossible to move terms from one side of the equation to the other until a root of the equation emerges. The existence of neutral elements allow to perform inversion of the operations appropriately. Inversion is what abstractly underlies the process of moving back and forth in resolving an equation. This may be though as a process of breathing, which if repeated twice leads us back to the same side.

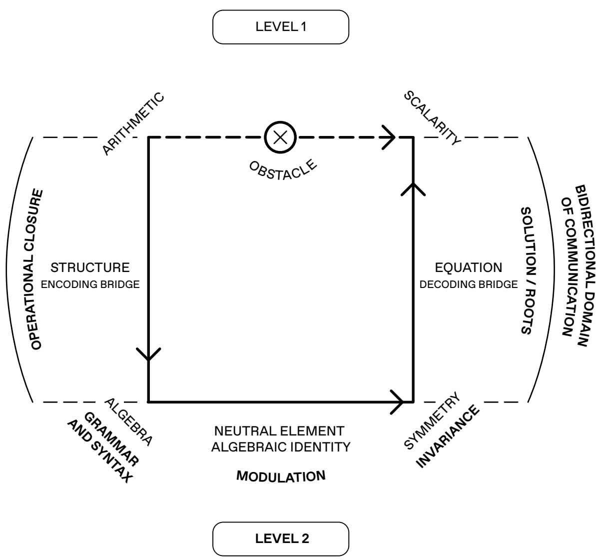


FIGURE 3.8
The algebraic breathing modulation of the identity

Group Structure on the Integers

The inverse operation to addition, that is, the operation of subtraction cannot always been performed for natural numbers.

Indeed, the difference $1 - 2$ is not a natural number. Let us consider this operation in some details.

By definition, $a - b$ is a unknown number x such that $x + b = a$. In other words, $a - b$ is a solution to the equation

$$x + b = a$$

and the absence of a natural number $1 - 2$ translates to the absence of solutions for $x + 2 = 1$.

We construct the group of the integers as an algebraic extension of the monoid of the naturals—under the operation of addition—in order to be able to express the solution to equations of the above simple form. Note that this is not only an extension of our arithmetic domain, but it also a structural extension, since starting from a monoid we accomplish the solution through the structure of a group.

Simply put, the group structure is more powerful because the inversion of addition—required for the solution of the above equation—that is, subtraction can be carried out in the integers, meaning that the integers are closed under subtraction.

This is the major characteristic of every group structure. Namely, the operation defining a group can be inverted, such that the result is located within the same group. This is called the closure algebraic property. The feasibility of this depends on the existence of a neutral element, such that operating with the neutral element on every other element it leaves the latter invariant.

In this sense, the neutral element with respect to the operation of the group, constitutes the algebraic identity of the group. Furthermore, the existence of the neutral element secures that for every element exists an inverse element, such that their composition under the operation yields the neutral element. Currently, we restrict our attention to the notion of a commutative group.

For instance, the sum of two integers is the same irrespective of the order in which they are added together.

We focus again to the solution to the simple equation

$$x + b = a$$

stressing again that the absence of the natural number $1-2$ indicates that the equation $x+2=1$ cannot be solved within the additive monoid of the naturals.

But, it is easy to figure out that the solution to the equation $x+2=1$ is the same as the solution to the equation $x+3=2$, $x+4=3$, and so on. This is a clear indication that we have to consider all these equations as equivalent, in the sense that their solution is the same.

Equivalence Relation and Partition Classes

For this purpose, we need to able to express the relation of equivalence using the architectonic scaffolding of set theory, since our elements are represented through the notion of a set. An equivalence relation $[\sim]$: on a set of elements—independently of what these elements stand for—is defined in terms three properties: Reflexivity, symmetry, and transitivity.

Reflexivity: $a \sim a$ (everything is equivalent to itself);

Symmetry: $a \sim b \Rightarrow b \sim a$ (if a is equivalent to b, then b is equivalent to a);

Transitivity: $(a \sim b, b \sim c) \Rightarrow a \sim c$ (if a is equivalent to b and if b is equivalent to c, then a is equivalent to c).

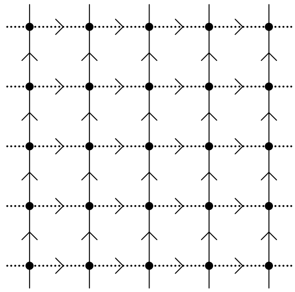


FIGURE 3.9
The cells or blocks of a partition spectrum via an equivalence relation

An equivalence relation partitions our initial sets of elements into equivalence classes. Each equivalence class defines a block or cell of this partition. Imagine a set that has infinite elements. An equivalence relation is able to partition the set into a finite number of blocks. All elements that are equivalent to each other with respect to this relation are located within the same block.

The economy of a partition rests on the fact that knowledge of a single element in a block is sufficient to characterize the whole block. All the blocks of the partition are mutually exclusive and jointly exhaustive.

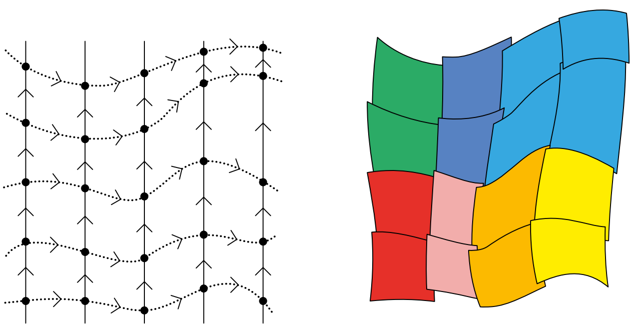


FIGURE 3.10
Mutually exclusive and jointly exhaustive cells of a partition spectrum

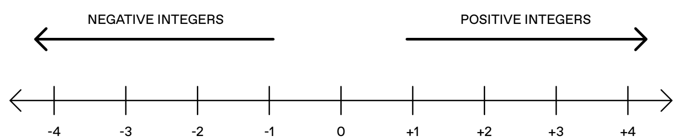
Each block of a partition, or each equivalence class, is completely characterized by means of a single representative. This is the case because every other element located within the same block of this partition is equivalent to this single representative. Note that every element of a partition block can play the role of a representative of this block. This is precisely the idea that we are going to apply in order to figure out the solution to all equivalent equations of the form $x + b = a$, for instance, $x + 2 = 1$, $x + 3 = 2$, $x + 4 = 3$, and so on.

Construction of the Integers from the Naturals

Every such equation is determined by an ordered pair of natural numbers (a, b). We cannot, however, think of an integer as such a pair, because different equations may have the same solution, as we already pointed out. Thus an integer should be a set of these equations which have the same solution, or, equivalently, a set of ordered pairs (a, b). This consideration motivates our construction.

We consider the set $\mathbb{N} \times \mathbb{N} \equiv \mathbb{N} \otimes \mathbb{N}$ of ordered pairs of natural numbers. Consider the relation $\sim$ on this set defined by $(m, n) \sim (p, q)$, if and only if:

FIGURE 3.11
Adjunction of dublicate mirror copy of the naturals with respect to the cipher 0



$$m + q = n + p$$

The relation $\sim$ is an equivalence relation. We define the set of integers as follows:

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim$$

We think of the pair (m, n) as the "difference $m - n$" that solves the equation:

$$x + n = m$$

The solution is an equivalence class of the partition defined by the above equivalence relation. Thus, an integer is an equivalence class of ordered pairs of natural numbers, which we denote by the symbol $[(m, n)]_{\sim}$.

By ordering the Naturals along a line directed from left to right, we devise a mirror placed in such a position in the extension of this ordering line to the left such that every Natural acquires a mirror image reflection and the order is inverted.

The mirror specifies the position of the neutral element 0—the cipher under addition—such that, all Naturals are duplicated, and each copy bears a minus signs. In this sense, the Integers include the 0, the positive, and the negative Naturals.

The adjunction of the duplicate copy of the Naturals to the original is described by the direct sum of these copies $\mathbb{N} \oplus \mathbb{N}$, which is identified, in this case, with their product $\mathbb{N} \otimes \mathbb{N}$.

All integers generate a set of elements, but we should be able to extend the operations of addition and multiplication to this new set in order to preserve the algebraic structure.

First, note that an integer $n \in \mathbb{N}$ is identified with the equivalence class $[(n, 0)]_{\sim}$. Indeed, n is the solution of $x + 0 = n$. This identification allows us to consider $\mathbb{N}$ as a subset of $\mathbb{Z}$.

The zero 0 is the cipher that will play the role of the neutral element in the set of the integers—endowed with the operation of addition—which can be inverted within the same set. In this sense, the set of the integers is closed with respect to the operation of addition, and its inverse, the operation of subtraction. Furthermore, for each integer, there exists a unique inverse integer with respect to the neutral element 0.

Under these conditions the integers is not only a set, but bears the algebraic structure of a commutative group under the operation of addition, and its inversion, the operation of subtraction. This group is called a commutative group because addition is a commutative operation, in the sense that the order in which two integers are added can be reverted giving precisely the same sum.

If we use the representation of integers as equivalence classes, according to the above construction, then addition is defined by:

$$[(m,n)]_\sim + [(p,q)]_\sim = [(m+p, n+q)]_\sim$$

Clearly, we may use the same representation of the integers to define the operation of multiplication. Recall that this is a bilinear operation, which is defined as follows:

$$[(m,n)]_\sim \, [(p,q)]_\sim = [(mp+nq, np+mq)]_\sim$$

It is easy to assure that both of them are well-defined operations.

Furthermore, the operations of addition and multiplication defined on the integers $\mathbb{Z}$ coincide with the usual operations of addition and multiplication under restriction to the naturals $\mathbb{N} \subset \mathbb{Z}$.

Distributive law

There is a relation between addition and multiplication for natural numbers, namely the identity:

$$a(b+c) = ab + ac$$

holds true for any $a, \; b, \; c$ in $\mathbb{N}$. The same identity, called the distributive law between addition and multiplication, holds true for any $a, \; b, \; c$ in $\mathbb{Z}$.

Integer Partition Spectrum

Recall that the purpose of extending the scalars from the natural numbers to the integers was the ability to solve equations of the form

$$x + b = a$$

Let us now verify that this is accomplished in the integers. Indeed, for any integers:

$$a = [(m, n)]_{\sim} \in \mathbb{Z} \qquad \text{and} \qquad b = [(p, q)]_{\sim} \in \mathbb{Z}$$

the above equation has solution:

$$x = [(m + q, n + p)]_{\sim} \in \mathbb{Z}$$

We may elaborate the above in more detail as follows:

$$x + b = [(m + q, n + p)]_{\sim} + [(p, q)]_{\sim} =$$
$$[(m + q + p, n + p + q)]_{\sim} = [(m, n)]_{\sim}$$

where the last equality is implied by $(m + q + p, n + p + q) \sim (m, n)$. This allows us to consider the subtraction of the integers:

$$[(m, n)]_{\sim} - [(p, q)]_{\sim} = [(m + q, n + p)]_{\sim}$$

for any naturals $m, n, p, q \in \mathbb{N}$ as the inverse operation to addition, and simply write:

$$x = a - b$$

for the solution of $x + b = a$.

Every integer number can be represented by infinitely many pairs under the addition of a natural with the mirror image (inverse) of another natural, for instance:

$$-2 = 5 - 7$$

$$-2 = 3 - 5$$

All right sides of these equations must be equivalent because they all give the same result. In this way, we have to think of every integer as the representative of a block spectrally identified by all such pairs that through this equation give the same integer result.

We grasp every block through a single representative of this block, identified as such by an integer. All the blocks are independent from each other but the union of all the blocks gives rise to the whole partition spectrum.

Notion of Quotient Structure

Having reflected all the Natural numbers across the cipher 0, and after adjoining the duplicate copy to the original, that is $\mathbb{N} \oplus \mathbb{N}$, the partition spectrum of the integers—blocks of the partition—is expressed by the quotient of $\mathbb{N} \oplus \mathbb{N}$ by the equivalence relation $\sim$.

This amounts to the modular substitution of each pair as above with the unique representative integer they give rise to, expressed as follows:

$$(\mathbb{N} \oplus \mathbb{N})/\sim\, = \mathbb{Z}$$

where $\mathbb{Z}$ is the spectrum of the integers.

The important issue here is that in this new arithmetic spectrum $\mathbb{Z}$ we are able not only to perform addition, but also to invert it, that is perform subtraction, remaining within the same arithmetic domain. This is due to the existence of the cipher 0, and consequently, the existence of an additive inverse for each integer.

Note that the set $(\mathbb{N} \oplus \mathbb{N})/\sim\, = \mathbb{Z}$ has to be re-instated as a new entity in the domain of algebraic structures. This new entity extending the monoid of the Naturals under addition is the group of the Integers, which is closed under both the operation of addition and its inversion, the operation of subtraction: $(\mathbb{Z}, +, -)$.

The group of the integers subsumes algebraically, and thus structurally, the invariants of the praxis of counting multitudes.

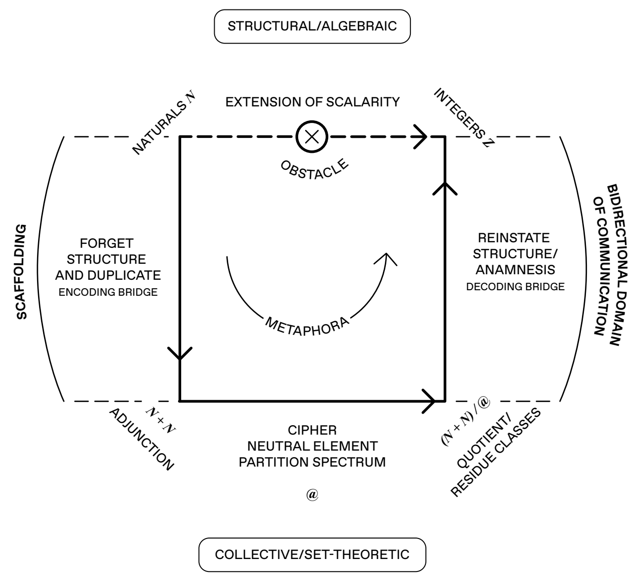


FIGURE 3.12

Metaphora for the extension of scalarity from the naturals to the integers via the architectonic scaffolding of lethe/ anamnesis

A Group is a structure that is expressed in terms of closure with respect to an operation (to be though of in terms of addition in the present case). This operation can be inverted (to be thought of in terms of subtraction in the present case) due to the existence of a group identity element (a cipher called the neutral element of the group). The solution of an equation involves and requires the group-theoretic structure in an essential way.

Commutativity of Addition $(+)$ defines a Commutative or Abelian Group: $a+b=b+a$ for every a, b. Note that not all possible operations giving rise to a group structure have the commutativity property.

Neutral Element $0: a+0=0+a=a$.

Existence of Inverses: $a+(-a)=-a+a=0$ for every a.

The notion of a group-theoretic structure has been conceived by Galois. The major idea is that underneath every equation there is always a pertinent algebraic group structure that determines its roots/solutions.

Commutative Action of a Group and Symmetry

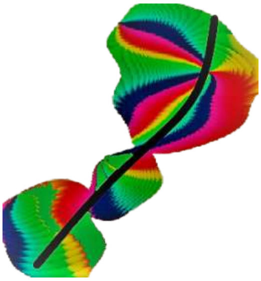


FIGURE 3.13
The seed of a spectral partition effected by the action of an algebraic group

The algebraic structure of a group should be thought of in an energetic way. A group is a structure which is acting on a set, or more generally, on a space, and its action gives rise to a partition.

In this sense, the notion of a partition is emergent from the action of a group structure. Therefore, the blocks of the partition under this action are identified with the orbits of the corresponding group action.

We may formulate it concisely as follows: The idea of an algebraic group is the seed of a spectral partition effected by the symmetry action of this group.

The notion of a group action elaborates the notion of an equivalence relation, since equivalence classes of this relation are identified with the orbits of the corresponding group action. Thus, we obtain a threefold conceptual identification among equivalence classes, partition blocks, and orbits of a group action.

Finally, orbits of a free and transitive group action on a space may be thought of in terms of the fibers of a fibered geometric structure, called a fiber bundle, which locally is expressed as a product, according to the following diagram:

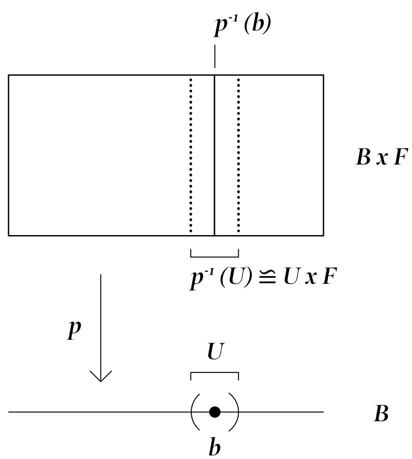


FIGURE 3.14
Free and transitive action of a group gives rise to a fiber bundle

The notion of a group action constitutes an expression of the notion of symmetry, in the sense that all elements in the orbit of a group action are symmetric to each other. Reciprocally, we may say that a partition block, or an equivalence class, is represented by a single element—the representative—because all other elements in the same block, or in the same class, are symmetric with the representative.

Ambiguity— Objective Probability— Information

We revert our perspective and imagine that we start from a partition under an unknown group action. The partition consists of mutually exclusive and jointly exhaustive blocks. Each block of the partition manifests a state of ambiguity, since the elements of the block are not distinguishable from each other.

The notion of indeterminacy, or indistinguishability, pertaining to a partition block formed under an unknown group action leads to the notion of objective probability.

Imagine a block of a partition spectrum as a cloud. Ambiguity in constituency amounts to objective probability. The latter is totally different notion from the one that refers to subjective probability, which in turn, pertains to a mere lack of knowledge. This means that, in principle, the elements of a partition block are distinguishable, but due to various reasons, like combinatorial complexity, there is lack of exact knowledge.

In the above sense, the structural action of a group on a bare set of elements, under which a partition spectrum may be obtained has two consequences:

The first is that the elements of this set are subordinate to a symmetry condition. The orbits of the group action express precisely this symmetry condition. All the elements in an orbit, or a block of the induced partition, are symmetric to each other;

The second is that all symmetric elements under a group action are indistinguishable to each other. Thus a block of a partition subsumes an ambiguity structure that is objectively indeterminate constitutively. Consequently, an orbit of an unknown group action is amenable to an objective probabilistic treatment.

This leads to the rather peculiar relation between symmetry and information. According to the above, complete symmetry amounts to null information, since everything is indistinguishable from anything else. Reciprocally, complete lack of symmetry amounts to maximal information, since all elements are sharply distinguishable.

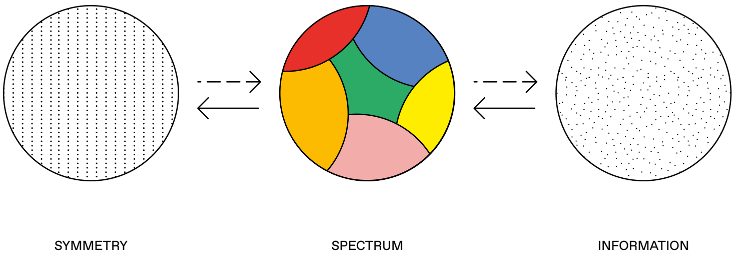


FIGURE 3.15
From complete symmetry to information via the distinguishability of a partition spectrum

As a consequence, the notion of a set, as a scaffolding, engulfs in its conception the assumption of sharp distinguishability of its elements, thus starts from the assumption of maximal information and zero symmetry. On the antipode is the notion of the totally indiscrete partition. This is a state of maximal symmetry, since all elements are symmetric and indistinguishable to each other, and therefore null information.

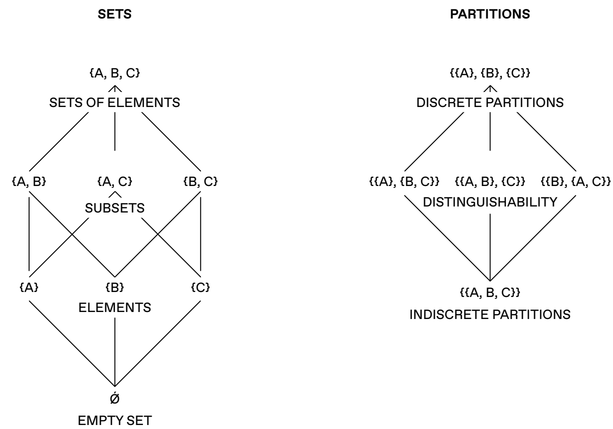


FIGURE 3.16
The duality between sets and partitions with respect to symmetry and information

Integer Modular Arithmetic

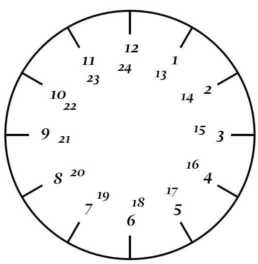


FIGURE 3.17
Integer modular spectrum of a clock

On keeping the time during a day on a clock with 12 hours division, the indexes 3 and 15 are in the same equivalence class:

$$15 mod 12 = 3$$

The modularity is imposed by the 12 hours partition.

The Euclidean algorithm of division runs as follows:

$$A = k \cdot Z + B$$

where Z is the modulus, B is the residue, and k is an integer that counts how many times the modulus fits within A. For example:

$$25 = 2 \cdot 12 + 1$$

The algorithm says that A is equivalent, or congruent, with the residue B, with respect to the modulus Z.

In this sense, modularity in integer arithmetic pertains to the relation that finds out how many countable times the modulus fits inside a greater number and what remains. The given number is congruent to the residue with respect to the modulus.

Modularity is expressed as a quotient with respect to the respective relation of equivalence which inherits and preserves, both the group structure of the integers—under addition and its inversion—and the monoid structure of the naturals under multiplication.

Extension of Scalarity from the Integers to the Rationals

Multiplication is an essential operation that can be performed on the integers endowing them with the closed structure of a multiplicative monoid. Division, the inverse operation to multiplication, is nevertheless not a total operation on integers, meaning that applying this operation the result, in general, lies outside the integers.

The objective is to extend the resolution of our spectral scalarity into a novel domain—including the integers—in which division, and thus the capability to form ratios, can be achieved. This novel domain should be closed with respect to both the operations of addition and multiplication as well as their inversions, that is subtraction, and division.

Therefore, the objective is to construct the algebraic field of the rational numbers, which consists of two separate group structures, the fist under addition, and the second under multiplication, such that multiplication is distributive over addition. If this is accomplished, then the novel domain of numbers, the Rationals $\mathbb{Q}$, should be a field.

A field constitutes an algebraic body of knowledge, thus the field-structure of the Rationals gives rise to a rational body of knowledge. In turn, this determines the depth of our spectral scalarity in unraveling relations, and invariants, in the Arithmetic Cosmos.

The Field of the Rationals should be thought of in terms of the following two group structures:

$$\text{Group } (\mathbb{Q}; +, -) \oplus \text{ Group } (\mathbb{Q}; \times, :) \Rightarrow \text{ Field } (\mathbb{Q}; +, -, \times, :)$$

The resolution of the problem of making the operation of division total requires the extension of this multiplicative domain into a new domain of numbers, where the required inverse operation can be always implemented. This means that we seek an appropriate extension of the initial closed structure (Integers) with respect to the operation of multiplication into a new structure (Rationals) being closed with respect to both multiplication and its inverse operation of division.

For this purpose, it is necessary to devise the architectonic encoding/decoding bridges and elaborate the corresponding partition spectrum, according to the general pattern characterizing a metaphora, for the construction of the rationals from the integers, or equivalently, for the algebraic extension of the scalars from the integers to the rationals. We remind that the rationals constitutes the set of all fractions a/b, where a and b are integers, and $b \neq 0$, obeying the relation:

$$a / b \equiv c / d$$

if the following holds:

$$a \cdot d = b \cdot c$$

which makes invertible every non-zero element of the integers.

The crucial insight guiding the architectonics of the metaphora we look for is that the set of the non-zero elements of the integers is multiplicatively closed. Therefore, the task of this structural metaphora is to make every element of the multiplicative closed subset of non-zero integers invertible, such that the new structure of numbers obtained in this manner, fulfills the following objectives:

First, it bears a structural similarity to the initial domain of numbers, meaning that it is also a commutative ring with respect to the operations of addition and multiplication;

Second, the operation of division (inverse to multiplication) can be performed by the inverses of the non-zero integers. Third, the initial domain of numbers together with their arithmetic can be embedded in the new one.

We consider the commutative and unital ring of integers $\mathbb{Z}$ and let $S \subseteq \mathbb{Z}$ be the multiplicative closed subset of non-zero integers. The first step is to set up a directed bridge from the level of commutative unital rings to the level of sets, encoding the process of extending the underlying set-theoretic domain of integers $\mathbb{Z}$ into a new domain formed by the cartesian product of sets $\mathbb{Z} \times S$.

Note that the ordered pairs of integers (a, s) with $s \neq 0$, are not supposed to have any a priori structure, since their existence is required at the level of sets by means of the encoding directed bridge connecting the involved structural levels.

In the set $\mathbb{Z} \times S$ we define the following binary relation:

$$(a, s) \diamond (b, t)$$

if and only if there exists $v \in S$ such that:

$$v(at - bs) = 0$$

The relation $\diamond$ is an equivalence relation, partitioning the set $\mathbb{Z} \times S$ into equivalence classes.

This gives rise to a partition spectrum whose blocks or cells are these equivalence classes. We will denote the partition spectrum, that is the quotient set by $\mathbb{Z}_S$, and the equivalence class of (a, s) by the fraction symbol a/s. Thus, the quotient set $\mathbb{Z}_S$ contains elements which can be interpreted as fractions, bearing the semantics of numbers allowing division by non-zero integers.

The structural metaphora is completed by setting up an inversely directed decoding bridge from the level of sets to the level of commutative unital rings (where the unit 1 would play the role of the multiplicative cipher), effectuating the circulation around the obstacle of division as follows: We set

$$a/s + b/t := ((ta + sb))/st$$

$$a/s \cdot b/t = ab/st$$

for every $a/s,\ b/t \in \mathbb{Z}_S$. The operations are well defined and endow $\mathbb{Z}_S$ with the structure of a ring. The zero and unit elements are, respectively, $0/s$ and s/s, for every $s \in S$.

Finally, we define the canonical homomorphism (structure-preserving mapping) of rings

$$h : \mathbb{Z} \to \mathbb{Z}_S$$

given by $h(a) = a/1$, for every $a \in \mathbb{Z}$.

Note that for any $s \in S$ we have that $1/s$ is the inverse of $h(s)$ in $\mathbb{Z}_S$. Hence, $\mathbb{Z}_S$ is the smallest ring containing $\mathbb{Z}$, in which every element of the multiplicative closed subset of non-zero integers S is invertible.

Thus, the extension of scalarity of the commutative and unital ring of the integers $\mathbb{Z}$ with respect to the multiplicative closed subset of non-zero integers, is accomplished by means of this structural algebraic metaphora.

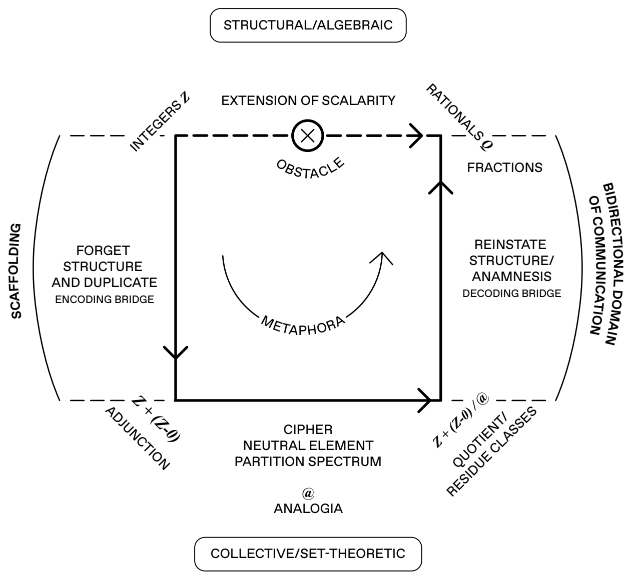


FIGURE 3.18
Extension of scalarity from the integers to the rationals and the partition spectrum of an analogia

Structural Metaphora: Adjunction—Partition—Quotient

First, we observe that the encoding of the underlying set-theoretic domain of $\mathbb{Z}$, into the new extended domain $\mathbb{Z} \times S := \mathbb{Z} \oplus S$, takes place by means of extending the scalars of $\mathbb{Z}$ with respect to the scalars of the multiplicative closed subset S of $\mathbb{Z}$, that is, by injecting $\mathbb{Z}$ into $\mathbb{Z} \times S$.

Second, the level of sets should be thought of in terms of an underlying architectonic scaffolding adjoined to the upper structural level, such that a partition spectrum can be obtained. More precisely, at the level of sets the operational role of the distinguished part S of $\mathbb{Z}$ appears through the equivalence relation defined on the set-theoretic domain $\mathbb{Z} \times S$.

Any suitable criterion of indistinguishability R must lead to a partition of $\mathbb{Z} \times S$ into disjoint classes, that is, the blocks of the partition, bearing the relation R, and hence R must be an equivalence relation.

In this manner, an equivalence class modulo R, consists of all the elements of $\mathbb{Z} \times S$, indistinguishable with respect to R, and thus equivalent.

More specifically, the indistinguishability relation R imposed on $\mathbb{Z} \times S$, requires that the pair of integers (va, vs) should be equivalent as (a, s) for any non-zero integer v, under the interpretation of the equivalence class (partition block) of the representative (a, s) by the fraction symbol a/s.

Note that the equivalence classes (a, s) are metaphorically interpreted as fractions a/s, being now elements of the partition spectrum, by which we mean the quotient set $\mathbb{Z}_S$. It is important to notice that consequent to the transition from $\mathbb{Z} \times S$ to $\mathbb{Z}_S$ is the replacement of equivalence modulo R by equality (isonomy and identification due to indiscernibility of the resolution spectrum) in the quotient $\mathbb{Z}_S$.

Third, the structural metaphora is completed by means of the inversely directing bridge from the level of sets back to the initial level of commutative and unital rings.

The semantic aspect of this bridge amounts to a re-casting of the elements of the quotient set $\mathbb{Z}_S$, as elements of a new ring, viz. as elements of the same closed structural genus as the initial one $\mathbb{Z}$, where the obstacle of division resides. This is accomplished by extending architectonically the addition and multiplication operations referring to fractions.

The extension takes place according to the principle that the new operations should incorporate and reproduce the effect of the old ones, when restricted to the old elements.

In turn, this guarantees the naturality of the established communication. It neither destroys, nor de-constructs the algebraic structure of the integers, as an act of barbarism, but it extends it to the more rich algebraic structure of the rationals, where division can be performed without obstruction, and in which the former structure of the integers is preserved via embedding.

The Bridges of Forgetfulness and Remembrance

The most important characteristic of the process of extension of an algebraic structure by means of metaphora—through the architectonic scaffolding of sets—is that the encoding and decoding bridges bear some particular meaning that is worth focussing on. More concretely, the encoding bridge is always a forgetful bridge—referring to the initial algebraic structure, whereas the decoding bridge is a remembrance (anamnesis) bridge that re-establishes the algebraic structure at the initial level.

The reason is that the algebraic criterion of identity, meaning the neutral element, or cipher, through which inversion becomes possible is established in terms of a regular equivalence relation at the level of a partition spectrum, and then, the algebraic structure is synthesized in a suitable way. In this way, the extended—through adjunction—algebraic structure, incorporates the initial one and does not discard it. This means that the restriction of the extended structure at the initial one agrees with it.

Moreover, the extension of an algebraic structure in the above manner, always emerges from the necessity to solve an algebraic equation. For instance, the extension necessitated by the requirement to determine the algebraic structure, where the inverse operation to multiplication becomes a feasible operation, emanates from the necessity to solve an equation by means of division.

From this perspective, a structural algebraic metaphora provides the means to evade the obstacle preventing the solution of an algebraic equation by means of inverting the operation in the initially specified algebraic domain.

Modular Substitution of Neutral Element: The Evasion of Self-Reference

From the perspective of modular substitution, the most important issue that arises in the algebraic context of thinking is the following: How it becomes possible for a metaphora to take place within the same algebraic structural genus—for example, groups or rings—such that no self-referential paradox arises? Actually, it is precisely the possibility of metaphora within the same algebraic genus that can evade self-reference indirectly.

For this purpose, we will make use of the framework of bare sets, not in its role as a foundation, but in its role as an architectonic scaffolding. Up to present, it has become evident that a set can be related to a distinguished part of it by the imposition of a relation on their jointly formed cartesian product with respect to a characteristic of equivalence that leads to a partition spectrum.

The metaphora taking place for the realization of this spectrum may be abstracted as follows:

Initially, we assume that a set of elements—considered as an object within the genus—or category—of sets (characterized by the membership relation), can relate to itself by separation of a well-defined part of it, that is, a subset bearing some particular characteristic for the initiation of a partition.

In turn, a criterion of equivalence is applied to the extended set obtained from the initial one—by adjoining the distinguished part—which delineates equivalence classes. Finally, using the quotient construction, we collapse the extended set into a new set that bears the modularity of the imposed relation.

This is possible if the following conditions are met:

First, if the initial set can split its substance between two internal levels, which we call hypostases, within the same genus;

Second, if the imposition of a similarity relation on the extended set partitions it into equivalence classes, forcing in this way an indiscernibility relation pertaining to the blocks of the corresponding partition.

Third, if the equivalence classes of the partition can be re-interpreted as elements of a new set, namely the quotient whose elements are the blocks of the partition.

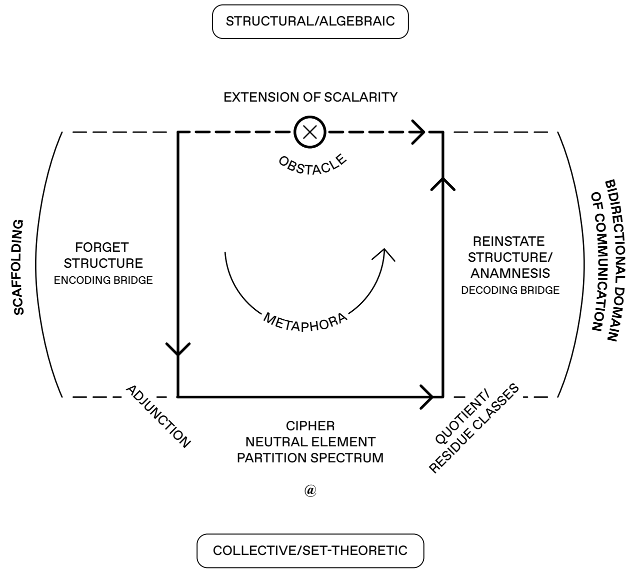


FIGURE 3.19

Indirect evasion of self reference via the neutral element and notion of quotient structure

This is significant because a metaphora taking place within the same structural genus—through the scaffolding of the category of sets—is equivalent to an indirect self-referential relation within this genus. Most interestingly, this relation is not paradoxical, since it pertains algebraically to the criterion of identity of an algebraic structure—that is, to the specification of the neutral element that enciphers a modular criterion of equivalence.

The crucial idea is that an equivalence relation giving rise to a partition spectrum is transcribed to equality in the quotient. In this way, it gives rise to a new neutral element that embodies the criterion of equivalence. Thus, structures within the same algebraic genus communicate homomorphically in a structure-preserving way by embodying distinct neutral elements. This is what characterizes the economy of the corresponding genus.

Consider two inverse internal bridges operating within the same algebraic genus as follows: The first bridge carries out the extension process of an object of this genus to another level of hypostasis, which takes place by adjoining to it a distinguished part of itself delineated by means of some criterion of similarity.

At the new level, an appropriate equivalence relation on the extended object implements this criterion of similarity. As a result, we end up with a partition spectrum consisting of equivalence classes. These classes are the blocks of the partition, and each block contains all elements indiscernible with respect to the imposed criterion.

Finally, an inverse bridge performs the transition back to the initial level, qualifying the equivalence relation into an equality (identity) of elements in the quotient set, which appears at the initial level.

From the perspective of the set-theoretic scaffolding employed for this type of metaphora, if two elements α and β of the augmented—by adjunction—set are equivalent with respect to an equivalence relation R, that is, $\alpha R \beta$, then their images inside the quotient set, interpreted as new elements—identified with the partition blocks of this relation—are identical, that is:

$$[\alpha]_R = [\beta]_R$$

Homomorphism and Modulation of Neutrality by Ideals

The minimum requirements for an algebraic structure, employing a set-theoretic scaffolding for its description, is the existence of a set S with an equality relation, endowed with a binary law of composition, that is, a single-valued function of pairs α, β such that $\alpha\beta$ is in S for α, β in S. Using this composition law, we can define a criterion of similarity, which gives rise to an equivalence relation R on S.

The appropriate use of this scaffolding in the case of algebraic structures, for instance in the case of groups, requires that an operation $\odot$ —like addition or multiplication—is defined on Σ based upon the composition operation in S. If this is feasible, we call the equivalence relation R on S, a regular one.

The idea is that a regular equivalence relation applied to the scaffolding will provide the means of specification of the corresponding neutral element of the quotient structure. At the algebraic level of structure, the quotient will be a natural homomorphic image of the original, incorporating the modularity imposed by the equivalence as a new neutral element.

Thus, the process of modulating a group S with the aid of a regular equivalence R produces a homomorphic—that is, structure preserving—image Σ of S that is also a group.

This is how the structural specification of the algebraic genus is preserved under modulation. Conversely, given a homomorphic image Σ of S, there is defined a partition, and therefore, an equivalence relation R on S. Moreover, the homomorphism property implies that R is a regular equivalence relation.

In a nutshell, we conclude that in the case of groups, the problem of finding all homomorphic images of S reduces to that of finding all regular equivalence relations over S.

We consider the general case of a non-commutative group. This means that the result of its operation on elements of the underlying set is not preserved by exchanging the order of the elements, that is:

$$ab \neq ba$$

for any elements a, b.

The modular substitution under metaphora in this case, that is what makes possible the quotient construction is a subgroup N, called a normal subgroup of S, satisfying:

$$xN = Nx$$

for all x in S. Thus, a regular equivalence relation R in S stems from a normal subgroup N of S, that is a subgroup remaining invariant under conjugation, meaning that $N = xNx^{-1}$ for all x in S.

Conversely, a normal subgroup of S defines a regular equivalence relation on S. Now, if N is a normal subgroup of S, then the blocks xN form a group with the following composition rule of closure:

$$\alpha N \odot \beta N = \alpha\beta N$$

The resulting quotient $\Sigma = S/N$ is a group homomorphic to S and constitutes that group, which collapses the normal subgroup N of S to the identity of Σ.

In other words, the neutral element of Σ is the whole normal subgroup N. It enciphers the modularity of Σ with respect to S through its neutral element—identified with N.

Conversely, every homomorphic image of S can be duplicated by such a quotient group, thus becoming isomorphic to it. This means that its structure is indistinguishable from the structure of the latter, meaning that these groups are equal and indiscernible within their genus.

In this manner, indirect self-referential metaphora between algebraic structures of the same genus is not only feasible, but it reveals the process that gives rise to the neutral element of such a structure.

More concretely, the metaphora in this case transcribes a regular equivalence relation on the underlying set scaffolding to a structural comparison morphism—a homomorphism—that is either an isomorphism or it is a quotient with respect to a kernel, identified as a normal subgroup—in the case of groups. This kernel emerging in a structure-preserving way defines the neutral element of the quotient.

In particular, referring to the algebraic genus of groups, we have the following:

Let S and T be groups and let ϕ be a group homomorphism from S to T. If e_T is the neutral element of T, then the kernel of ϕ is the subset of S consisting of all those elements of S which are being mapped by ϕ to the element e_T:

$$Ker(\phi) = \{x \in S : \phi(x) = e_T\}$$

Since a group homomorphism preserves neutral elements, the neutral element e_S of S must belong to $Ker(\phi)$.

By the preceding analysis, it turns out that $Ker(\phi)$ is actually a normal subgroup of S. Thus, we can form the quotient group $S/Ker(\phi)$, which is naturally isomorphic to $Im(\phi)$, that is, the image of ϕ, which is a subgroup of T.

Analogously, in the case of rings with a unit element we have the following:

Let S and T be rings and let ϕ be a ring homomorphism from S to T. If 0_T is the zero element of T, then the kernel of ϕ is the subset of S consisting of all those elements of S which are being mapped by ϕ to the element 0_T:

$$Ker(\phi) = \{x \in S : \phi(x) = 0_T\}$$

Since a ring homomorphism preserves zero elements, the zero element 0_S of S must belong to the kernel.

It turns out that, although $Ker(\phi)$ is generally not a subring of S, since it may not contain the multiplicative identity, it is nevertheless a two-sided ideal of S. Thus, we can form the quotient ring $S/Ker(\phi)$, which is naturally isomorphic to $Im(\phi)$, that is, the image of ϕ, which is a subring of T.

The notion of ideal in the theory of algebraic rings has been introduced by Dedekind. If we think of the two operations that a ring carries as addition and (non-commutative) multiplication, then a two-sided ideal is a subgroup with respect to addition, such that each of its elements absorbs—under multiplication from the left and from the right—all the elements of the ring within the ideal.

Powers and Double Invertibility: Extraction of Roots and Logarithms

It is well known that the notion of raising to a power is defined by recursion on the operation of multiplication. The complexity in the notion of a power is that it involves two numerical entities assuming different operational roles.

More concretely, we have the base of the power and the power itself, such that the operation of raising the base to a power is not a commutative operation, meaning the result is not invariant under exchanging the roles of bases and powers.

In this sense, the non-commutativity appearing for the first time algebraically in the procedure of raising a base to a power, necessitates the consideration of two distinct inverses, i.e. one referring to the base, and the other referring to the power.

If we call this non-commutative operation with respect to the base and the power as the operation of exponentiation, then its inversion is twofold: Inverting with respect to the base is the procedure called root extraction, whereas inverting with respect to the power is the procedure called logarithmization.

This should be contrasted with the commutativity of multiplying numerical entities and the uniquely defined inversion of the operation of multiplication, giving rise to the operation of division, and the extension of the integers to the rationals.

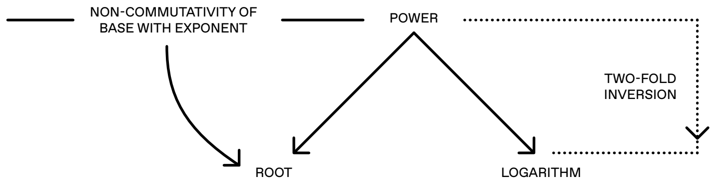


FIGURE 3.20
Two-fold inversion of exponentiation

Therefore, two distinct types of metaphora are needed in order to invert exponentiation:

The first, referring to the powers with respect to a base necessitates the extension of the field structure of the rationals to the field structure of the reals. In this manner, logarithmization becomes a total operation in the domain of real numbers.

The second, referring to the roots, necessitates the extension of the field structure of the rationals to the field structure of the complex numbers, if we include the roots of negative numbers.

Both of these inversions are unified in the field domain of the complex numbers under the notion of the complex logarithm. It is important to highlight that both of these inversions are not purely algebraic, but necessitate topological arguments for the effectuation of the respective metaphoras. The first requires an argument of continuity, whereas the second requires additionally a topological argument in relation to the obstacle of non-simple connectivity.

First, we consider real logarithmization in functional and algebraic terms. If we consider that b is any positive natural base different from the unit 1, then the exponentiation equation:

$$b^y = x$$

where $x > 0$, is solved in terms of y by logarithmization, that is:

$$y = log_b x$$

Equivalently, the power y is expressed as the real logarithm of x in the base or root b.

It is clearly not allowed to take the real logarithm of zero or a negative number.

If we think of b^y as a function of y, then this function is a continuous (and differentiable) function of the variable y, whose inverse is the continuous (and differentiable) real logarithm function:

$$x \mapsto log_b x = y$$

It is significant to highlight this necessary topological qualification for accomplishing this inversion. In turn, this requires the structural extension of the scalars from the rationals to the reals, through the incorporation of the irrationals, interpreting for instance the meaningful convergence to limits of sequences (convergence through exhaustion) pertaining to real analysis.

The real logarithm function is characterized as the unique monotonically increasing function from the domain of the positive non-zero reals to the codomain of all the reals, such that

$$log_b b = 1$$

$$log_b (y_1 \cdot y_2) = log_b y_1 + log_b y_2$$

The important fact here is that the real logarithm function converts multiplication of positive non-zero reals to addition of reals and it is order preserving. This provides the insight that the real logarithm plays the role of an architectonic bridge for the communication of the multiplicative group of the positive non-zero reals with the additive group of all the reals.

A natural question emerging in grasping the real logarithm function is how to express the procedure of raising to a power independently of the base employed. For this purpose, we define the exponential function

$$exp : x \mapsto exp (x)$$

from the reals to the positive non-zero reals, that is, the value of the exponential function is never zero and never negative, characterized by the property that

$$exp (x_1 + x_2) = exp (x_1) \, exp (x_2)$$

We conclude that the real exponential function converts the operation of addition of reals to the operation of multiplication of positive non-zero reals.

In turn, this provides the insight that the real exponential plays the role of an architectonic bridge for the communication of the additive group of all the reals with the multiplicative group of the positive non-zero reals. This is clearly a bridge inverse to the bridge of the real logarithm. Therefore, if one of them is an encoding bridge, then the other is a decoding bridge between the additive and multiplicative group structures.

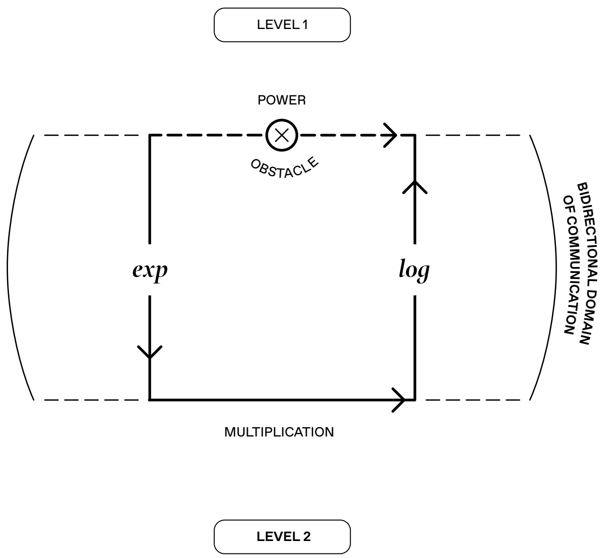


FIGURE 3.21
Conjugation of a power by multiplication via the Exp/Log architectonic bridges

Then, the problem of raising to any power a with respect to a base ϱ, where ϱ is thought of as a variable, is resolved by regarding the exponential and logarithm functions as inverse bridges between the group theoretic domains of the positive non-zero reals with respect to multiplication and the reals with respect to addition.

More concretely, these inverse bridges, are inverse bijective (injective and surjective) homomorphisms between these two groups, and thus, constitute the group of the reals under addition isomorphic to the group of the positive non-zero reals under multiplication.

In other words, the real exponential function and the real logarithm function are not only inverse functions, but more important, they are inverse group homomorphisms.

Therefore from a structural algebraic perspective, we may define the real logarithm architectonic bridge (Functor)

$$log : \mathbb{R}^{+} \rightarrow \mathbb{R}$$

as the group homomorphism from the multiplicative group of positive non-zero reals $(\mathbb{R}^{+}, \cdot)$ to the additive group of all reals $(\mathbb{R} , +)$ since the property:

$$log (y_1 \cdot y_2) = log\, y_1 + log\, y_2$$

is satisfied for any positive non-zero reals y_1 and y_2.

Inversely, the real exponential architectonic bridge (Adjoint Functor)

$$exp : \mathbb{R} \rightarrow \mathbb{R}^{+}$$

is a group homomorphism from the additive group $(\mathbb{R} , +)$ to the multiplicative group $(\mathbb{R}^{+}, \cdot)$ satisfying the property:

$$exp (x_1 + x_2) = exp (x_1) \cdot exp (x_2)$$

As such these two group homomorphisms are bijective and inverse to each other, meaning that they establish an isomorphism between these two different group structures on the reals. In other words, this means that architectonically these two structural worlds are totally equivalent under their encoding/decoding bridges such that each one of them provides a non-further refinable resolution spectrum for the other.

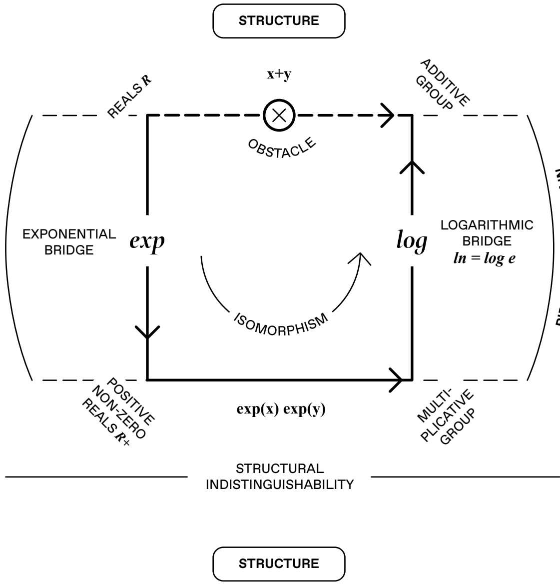


FIGURE 3.22
The algebraic isomorphism from the additive group of the reals to the multiplicative group of non-zero positive reals via the Exp/Log inverse bridges

The most important consequence of this isomorphism is that the additive group structure of all real numbers, i.e. of the values of the logarithm under addition, is indistinguishable from the multiplicative group structure of the positive reals, i.e. of the values of the exponential function under multiplication.

Consequently, the difficult operation of raising to a power can be conjugated to the easy operation of multiplication by metaphora from the additive group of the reals to the multiplicative group of the positive non-zero reals, where exp and log play the role of the inverse bridges.

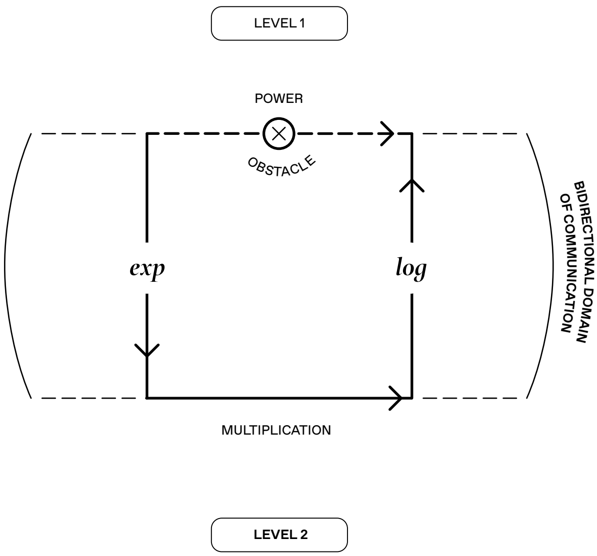


FIGURE 3.23
Raising to a power is modularly substituted by multiplication via the Exp/Log inverse architectonic bridges

Symbolically and equationally, the above diagram reads as follows:

$$\varrho^a = exp\,[\,a\,]\,log(\,\varrho\,)$$

$$\varrho^a = exp\,[\,a\,]\,exp^{-1}(\,\varrho\,)$$

Conversely, the above metaphora solving the problem of raising to a power by conjugating it to multiplication is equivalent to the group isomorphism induced by the inverse bridges identified with the real exponential and the real logarithm function.

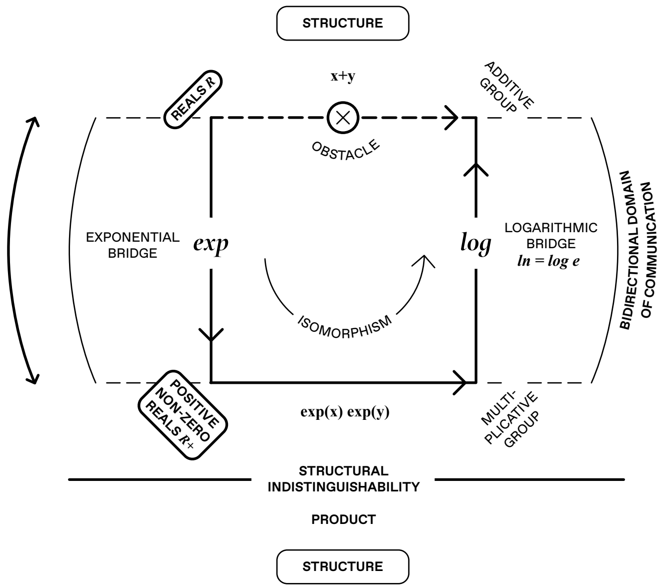


FIGURE 3.24

The isomorphic natural communication scheme between the additive group structure of the reals and the multiplicative group structure of positive reals

4.
The Transcendental Realm of Eternity: Abduction of Space from Time

The Irrationals: Cuts on the Arithmetic Line

The integer numbers can be placed on a line extending to infinity in both directions with respect to the additive cipher 0. The positive integers lie on the right of the line as they are extending from 0, whereas the negative integers lie of the left as they are extending from 0. The negative integers are the mirror images of the positive ones, if we imagine a mirror placed at 0. The multiplicative cipher 1 is the monad, and all positive integers are replicating this monad.

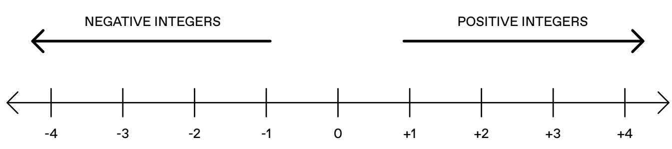


FIGURE 4.1
The integer number line

As soon as we accept, that the monad is divisible—something that was unacceptable to the ancient mathematical world-view—we can also place all the rational numbers on the same line. The rational numbers are infinite, and they are dense on the line, but they do not fill it. They leave holes on the line that they have to be properly filled in, if we demand completeness of the line.

Note that, in general, roots and logarithms of positive numbers are not rational numbers. The idea is that this new species of numbers, called the irrational numbers, also infinite in cardinality, but of a higher order of infinity in comparison to the rationals, if they can be located on the line, then they can fill the holes left from the rationals, leading to the completeness of the line.

The arithmetic line may be though of as a line in space. The basic characteristic of a line is that entities placed upon it can be totally ordered. The line extending infinitely in both directions is also continuous. Note that continuity has not be employed at all in the case of the integers and the rationals.

But continuity requires that there no holes or gaps on the line. These are essentially the holes left from the placement of the rationals on the line. Thus, we may think of the irrationals as involving the idea of continuity essentially in their conception, in case that all holes left from the rationals can be filled in.

If this is possible, then adjoining the irrationals together with the rationals on the line, continuity and completeness may be accomplished, such that the new arithmetic domain of numbers, called the real numbers are identified constitutively with the line. In this sense, our whole arithmetic cosmos, up to the real numbers, can be represented by means of a line, called the real line.

The idea of Dedekind was to provide an arithmetic grasping of continuity based on the notion of order. In other words, he elevated the notion of order as the crucial one for locating the irrationals on the line, such that they fill in the holes left from the rationals.

Recall, that the rationals $\mathbb{Q}$ is not only a set, but it bears the algebraic structure of a field. Thus, the extension from the rationals to the real numbers, that would be obtained by adjoining appropriately the irrationals to them, should also bear the field algebraic, structure, meaning that it should constitute a real and complete body of knowledge. Since the ordering relation is considered as the essential one for grasping the irrationals, according to Dedekind, the objective is the following:

First, we should devise a forgetful bridge, that is, a bridge that forgets the algebraic structure of the field of the rationals, such that we can descend to the underlying set-theoretic scaffolding. In this context, we should be able to consider a set F such that:

(a) $Q \subset F$;

(b) F is an ordered set: $(F, <)$;

(c) F should have a least upper bound.

If this is fulfilled, then F should be equipped with a sum $+$ and a product $\times$, such that $(F, +, \times)$ becomes structurally a field, which is compatible with the ordering relation. In this manner, we will be able to re-instate the field algebraic structure in the new extended domain of numbers F, such that their restriction to the rationals $\mathbb{Q}$, is identical with the field structure we already have in the rationals.

The idea of Dedekind relies on the observation that every real number α is completely determined by all the rational numbers that are less than α, and all the rational numbers that are greater or equal than α. This is how the ordering relation should be employed to identify the irrationals, i.e. the holes or gaps left from the placement of the rationals on the line.

We start with the obvious statement that $\forall r \in \mathbb{Q}$, r divides the set $\mathbb{Q}$ into two subsets:

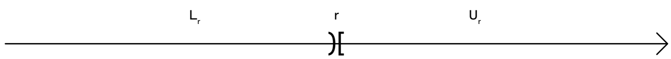
$$L_r := \{ q \in \mathbb{Q} \mid q < r \}$$

$$U_r := \{ q \in \mathbb{Q} \mid q \geq r \}$$

For each $r \in Q$, we call a special Dedekind cut the pair (L_r, U_r).

Note that L_r does not have a maximum (rational) number, while U_r does have a minimum (rational) number and it is precisely the number r.

FIGURE 4.2
Dedekind cut



Then, the mapping defined by:

$$D : \mathbb{Q} \to \{ \text{Dedekind cuts} \}$$

$$r \mapsto (L_r, U_r)$$

is a bijection, meaning that it is $1 - 1$ and onto.

A Dedekind cut $\alpha = (L_\alpha, U_\alpha)$ is a subdivision of $\mathbb{Q}$ into two non-empty subsets L_α (lower interval) and U_α (upper interval) such that:

(a) L_α has no maximum;

(b) $\forall x \in L_\alpha, y \in U_\alpha$ we have $x < y$;

Note that U_α could have a minimum, or not. In particular, if U_α has a minimum, then α is a rational number, $\alpha \in \mathbb{Q}((L_\alpha, U_\alpha))$ is a special Dedekind cut, as defined previously. If U_α does not have a minimum, then we say that α is an irrational.

For instance, consider $\alpha = (L_\alpha, U_\alpha)$ defined as

$$L_\alpha := \{ q \in \mathbb{Q} \mid q < 0 \} \cup \{ q \in \mathbb{Q} \mid q > 0, q^2 < 2 \}$$

$$U_\alpha := \{ q \in \mathbb{Q} + \mid q^2 \geq 2 \}$$

Then, α is a Dedekind cut, the minimum of U_α does not exist in $\mathbb{Q}$. By post-anticipation, we can say that α is actually the real number $\sqrt{2}$.

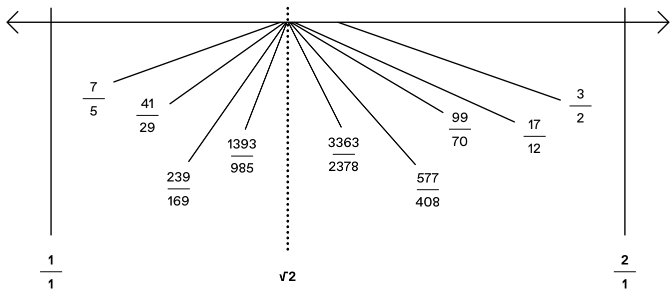


FIGURE 4.3
The square root of 2

Then, we define the set of real numbers $\mathbb{R}$ as follows:

$$\mathbb{R} := \{ \text{Dedekind cuts} \quad \alpha = (L_\alpha, U_\alpha) \}$$

equipped with the ordering relation:

$$\alpha < \beta \text{ for any } \alpha, \beta \in R \quad \Leftrightarrow \quad L_\alpha \subset L_\beta$$

We conclude that $(\mathbb{R}, <)$ is an ordered set that contains Q.

Indeed, $\mathbb{Q} \leftrightarrow \{ \text{special Dedekind cuts } r = (L_r, U_r) \} \hookrightarrow$

$\hookrightarrow \{ \text{Dedekind cuts} \vdash \alpha = (L_\alpha, U_\alpha) \} = \mathbb{R}$

where, $\leftrightarrow$ means that the map is a bijection and $\hookrightarrow$ means that the map is injective (it is an inclusion *map*).

It remains to verify that $\mathbb{R}$ bears the least-upper-bound property, meaning that for any subset $S \subseteq \mathbb{R}$, bounded from above, there exists a supremum in $\mathbb{R}$, which is indeed the case. This establishes the completeness of $\mathbb{R}$.

Then, from the set-theoretic level, where we have constructed R as a complete totally ordered set, we need to return to the algebraic structural level of fields. Since the rationals constitute a field themselves, it is necessary to establish the field algebraic structure for the extension of the rationals by the irrationals.

The necessity is due to the fact that the metaphora should respect, and not demolish, the already existing structure of the rationals. We can show that the algebraic operations of addition, and multiplication—together with their inverses—as well as their neutral elements, and the distributive property, as previously, can be re-instated in $(\mathbb{R}, <)$, such that the latter is structurally an ordered field.

We conclude that $(\mathbb{R}, +, \times; <)$ is a complete ordered field, called the field of the reals, constructed by the adjunction of the irrationals—as Dedekind cuts—to the rationals. This constructions fills in all the holes on the line, and the completeness amounts to the continuity of the line. The success of calculus over the real numbers is based on this property.

Actually, it can be shown that $\mathbb{R}$ is the unique field with all these properties, up to isomorphism (a continuous bijection that preserves the sum, product and ordering of the elements).

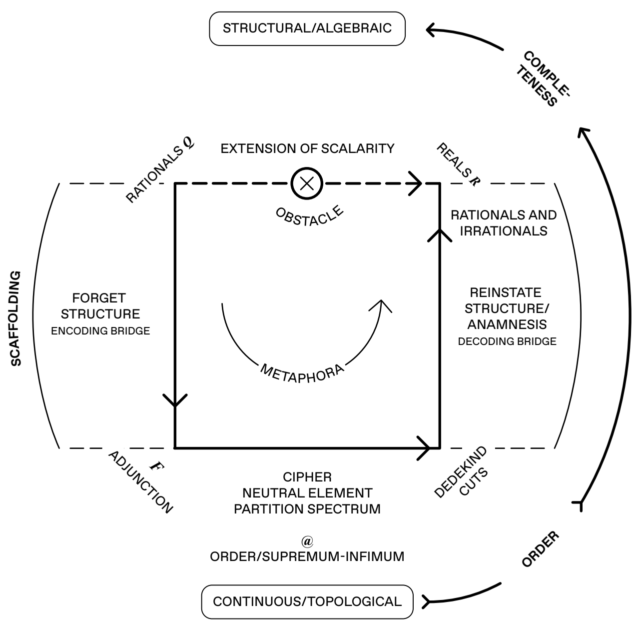


FIGURE 4.4
The extension of scalarity from the rationals to the reals

Abduction of Space from Eternity via Ordering

The method of Dedekind to recognize the irrationals—by means of cuts that utilize the notion of order—may be thought of as a panopticon view that completes the arithmetic line from the realm of eternity. Then, the completed, and perfect, arithmetic line is generating space along its extension in a continuous matter.

It is interesting to note that the notion of a cut has been conceived already in antiquity by Eudoxus, as it is presented in the fifth book of Euclid. Eudoxus' idea was to say that a general length magnitude is determined by those rational lengths less than it and those rational lengths greater than it.

But the notion was not targeting the construction of the real numbers, since ratios were not accepted as numbers. In other words, in the ancient Greek arithmetical cosmos every ratio is meaningful only in the context of a proportionality relation (analogia), and ratios do not have the status of numbers themselves. The underlying reason for this was that the division of the monad *1* should be unacceptable.

In this sense, we may say that, in the ancient view, the *1* was not a unit in space, but a unity in time, which should be indivisible in the status of a whole not dissected into parts. Thus, from that perspective, the notion of completeness is irrelevant. But, from the spatially-oriented perspective of Dedekind cuts, the objective is precisely the completeness property that constitutes space continuously along the extension of the line.

The abduction of space—extending continuously along the infinite extension of the arithmetic line—from the ideal realm of eternity, through the relation of order, is what the notion of a cut accomplishes. The abduction of space from eternity rests on the fact that time is conceived only in its sum in this context. The ordering relation along the line—that leads to completeness—pertains to space, and completeness is a spatial property. The filling in of a hole left unoccupied by the rationals takes place in a single stroke, the location of the corresponding cut on the line. This leaves the notion of an irrational number itself inexplicable directly, although graspable from above, via the ordering relation.

A great advantage of this method is that the ordering relation makes invisible the apparent contradiction emerging, for instance, in the case of an irrational square root, if approached from a temporal perspective. The issue is that a square root appears simultaneously under a positive and a negative manifestation. Thus, if the extraction of a square root is though of as a temporal process, then the simultaneous appearance of both the positive and the negative root cannot be comprehended at once. Only the ideal realm of eternity allows a non-contradictory consideration—in logical terms—of both roots simultaneously.

The Method of Exhaustion: Bounding and Converging to the Limit

Despite this advantage, and since the notion of an irrational seems elusive in its core from the perspective of flowing time, the question is if there is an alternative method to grasp its essence from within, without traversing its ideal sum. This is possible by the method of the so called Cauchy sequences, which is nevertheless rooted in the ancient method of exhaustion, developed by Eudoxus and perfected by Archimedes.

This is an essentially kinematical method, which is based on the stochastic idea of best approximation. The best approximation, in the modern phraseology of real Analysis, is expressed by the notion of convergence to a limit, where the idea of convergence is the objective pertaining to the temporal sequence targeting an irrational. In turn, this irrational is identified with the limit of the corresponding sequence converging to it.

The method of exhaustion involves in a crucial manner the notion of marching to an irrational by a balanced multi-stage act of approximation. Since the actual obstacle is the uncountable infinity incorporated to the essence of an irrational, something that reciprocally is the artifact of the continuity in the temporal sequencing process itself, the marching towards a convergence limit amounts to an eventual bounding of infinity from above and from below, such that the best approximation emerges.

In this sense, from the perspective of the unfolding temporal sequence seeking convergence, which is its ultimate purpose, the marching towards a limit—the pertinent irrational—is an act of will primarily, combined with ingenuity in devising bounds in an infinite domain.

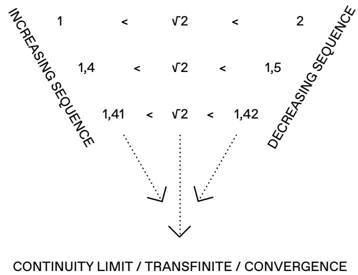
Consider the square root of 2:

$$\sqrt{2} = 1.4142 \quad \cdots$$

The decimal form continues to infinity. There is no ratio capable of capturing the whole sequence. Although the target is out of our reach with ratios, we can march from two directions towards it. Exhaustion amounts to develop a strategy to approach it from every possible direction (Figure 4.5).

We know that the square root of 2 should be localized between 1 and 2. The strategy involves the directions to approach it. One needs to develop a method for approaching it from below—in a increasing way—and from above—in a decreasing way—such that the sought target can be bounded appropriately—from above and from below—via a "zooming in" series of rational ap-

FIGURE 4.5
Method of exhaustion



proximations, which are graspable. The success of the method relies on the emergence of a limit where the sequence paths from above and from below converge. The design of this method is equivalent to the design of a choreography.

Consider the square root of 2 and try to bound it—from above and below—with rationals. The first obvious bounds are 1 and 2.

$$1 < \sqrt{2} = 1.41421 \cdots < 2$$

If we continue "zooming in" from both directions by tightening the rational bounds, then the resolution towards the square root of 2 is increased. The sequence of lower bounds, and the sequence of upper bounds, are called Cauchy sequences:

$$1 < \sqrt{2} < 2$$
$$1.4 < \sqrt{2} < 1.5$$
$$1.41 < \sqrt{2} < 1.42$$

Grasping the irrational root of 2 amounts to convergence from the two anti-diametric directions of zooming in towards it, in the present case. Convergence is subordinate to the condition of continuity, which is of topological nature. It grants that there are no gaps preventing the gradual unfolding in a series of successive steps towards a convergent limit.

This is the pre-condition for tuning in to the rhythm required for converging to a limit through a series of lower bounds, and a series of upper bounds, which—in essence—bound the totally indeterminate infinite.

The convergence aiming towards attaining a limit—in the case of the irrationals—generates the necessity to move kinematically in time using an intrinsically continuous scaffolding, i.e. what is needed is the topological plasticity of a continuum, and not the rigidity of a discretum. Then, the zooming in series of rational approximations bounding the infinity of the irrational—for instance, in its decimal representation—more and more efficiently, is modulated by the existence of the attained limit, such that convergence is accomplished.

Therefore, whereas completeness was the major objective of the method of cuts, continuity in the process of converging to the limit is the major objective of the method of exhaustion in terms of bounding sequences.

The key idea here is that a sequence of rational approximations is converging to a number that is not rational, although it can be located eventually on the arithmetic line. But, this shows precisely that rational numbers do not fill in the line, meaning that there are holes among them. Therefore, convergence is based on the conception that, as soon as the line becomes complete, there would surely exist a limit.

In other words, completeness is transcribed to filled in holes, and hence, to the continuity of the line that guarantees that convergence to a limit is attainable. Reciprocally, order completeness of the real line is secured by the existence of a least upper bound and a greatest lower bound in the sequences marching anti-diametrically towards an irrational, which is identified eventually with the convergent limit of these sequences from above and from below.

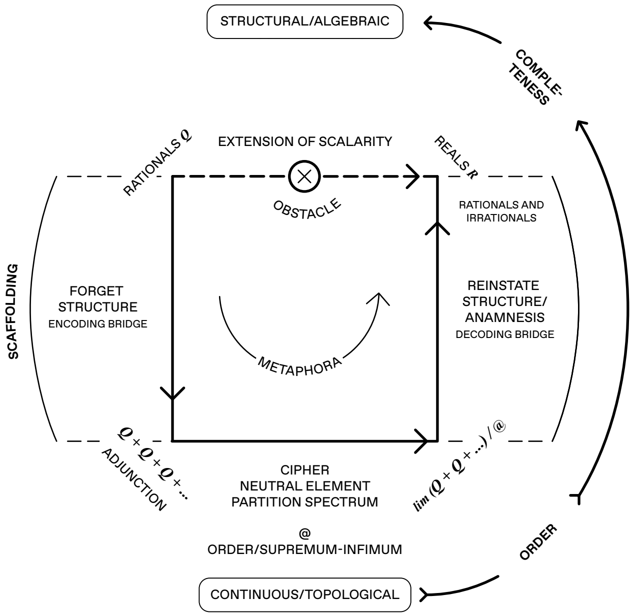


FIGURE 4.6

Extension of scalarity from the rationals to the reals by the method of continuous convergence

Completing the Arithmetic Cosmos: Extension by Imaginaries

In the domain of the real numbers, we are able to extract the roots of equations involving powers, whose bases is a positive number. But, if we consider the simple equation:

$$x^2 = -1$$

we need to be able to extract the square root of the negative -1. The result cannot be an irrational because we do not have the means to locate it on the real arithmetic line. The latter is already completely filled in by the adjunction of the irrationals to the rationals, and it is important to devise other means pertaining to the comprehension of roots of negative numbers. This is how the extension from the real numbers $\mathbb{R}$ to the complex numbers $\mathbb{C}$ may be justified algebraically.

Like every square root, the $\sqrt{(-1)}$ appears in two avatars, which are mirror images of each other, namely the positive and the negative square root with respect to the additive cipher 0 — that is, $+\sqrt{(-1)}$, and $-\sqrt{(-1)}$. Recall that in the case of real numbers this issue has been essentially made invisible, either because linear space is abducted from eternity in its completion by the method of cuts, or because sequences of rational approximations converge uniquely to one or the other root.

Nevertheless, in the case of the square root of a negative number, this issue has to be confronted by considering seriously the notion of duplication involved in the extraction of square roots.

Let us symbolize the positive square root of -1 by i, that is, $i \equiv \sqrt{(-1)}$. The idea is that i provides a new unit, called the imaginary unit, for the duplication of the real line orthogonally to itself.

In this way, we duplicate the domain of the real numbers, $\mathbb{R} \oplus i\,\mathbb{R}$, such that the copy represented orthogonally to the original bears the imaginary unit i as its unit of progression in a continuous way.

There is a subtle topological assumption involved in this duplication, which—beyond continuity—pertains to the notion of connectivity on the plane that we are going to deal with at a later stage.

The imaginary unit i is though of simultaneously with its complex conjugate $i^{\star}$, such that:

$$i \cdot i^{\star} = i^{2} = -1$$

Note that $i^\star$ stands for the negative square root of the negative -1. The consequence of the above is the following:

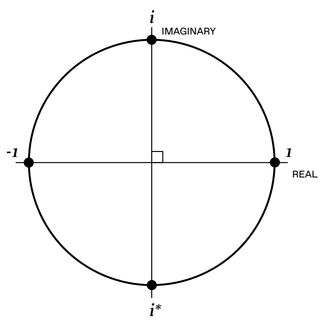
$$i^4 = i^2 \cdot i^2 = 1$$

that is, the imaginary unit is the positive forth root of the unity 1. This justifies the placement of the imaginary axis orthogonally to the real axis, extending in the upward direction along it. The complex conjugate $i^\star$ is the mirror image of i with respect to the real axis, such that the imaginary axis extends in the downward direction along it simultaneously.

There is no real temporal interval intervening between the positive and negative square root of -1, thus the imaginary axis should be thought of as synchronically extending—vertically with respect to the real axis—in both directions towards infinity. There is though a phase difference between these two roots, which is represented, in the space opened up on the plane through them, by the phase difference of a binary rotation involving two right angles in succession.

FIGURE 4.7

Real and imaginary axes: the unit circle on the complex plane



Note that the same prescription works now for the square root of every negative number, since the imaginary unit—together with its complex conjugate—allows to grasp all of them at once in the same spatial setting, identified with the complex plane.

This is the criterion to qualify the complex plane as our spectral epiphaneia, where roots of negative numbers appear, and are visible along the imaginary axis as above.

In this sense, a new domain of numbers emerges, which extends on the plane, and which contains two copies of the real numbers extending orthogonally to each other, the first horizontally along the multiplicative unit 1 in the positive direction, and the other vertically along the imaginary multiplicative unit $1 \cdot i$, giving rise to a screen where every complex number is represented.

Therefore, the metaphora that underlies the extension of the real numbers to the complex numbers—to address and circulate around the obstacle posed by the roots of negative numbers—uses the imaginary unit as an encoding vertical bridge and its complex conjugate as a decoding vertical bridge to open up the topos of communication between these two domains of numbers.

On the complex plane there are four units extending along both the horizontal and vertical directions in the positive and negative sense correspondingly, $1,\ -1,\ i,\ i^{\star}$. The positioning of these units on the plane gives rise to the unit circle on the complex plane, which intersects the real axis and the imaginary axis on their respective units. Hence, the unit circle is what constitutes the unity on the screen offered by the complex plane.

It can be easily shown, that since the complex numbers are constructed in terms of two duplicate copies of the real numbers extending orthogonally to each other via the action of the imaginary unit, all the algebraic operations pertaining to each copy can be extended to their product, or direct sum, thus preserving their validity if extended to the plane. In this way, the field algebraic structure of the reals with respect to addition and multiplication is extended to the field structure of the complex numbers under the same operations.

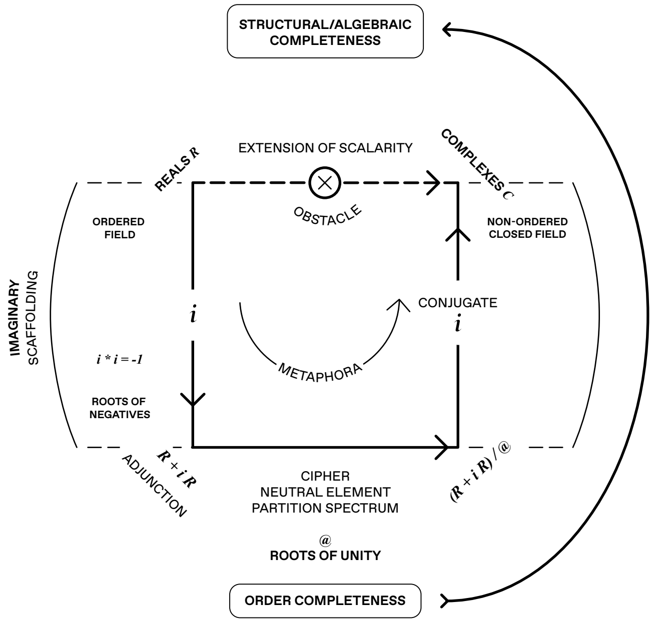


FIGURE 4.8

Extension of scalarity from the reals to the complexes and algebraic completeness

There is a considerable price to be paid on for the extension of our arithmetic cosmos from the real to the complex numbers. The price is that the relation of total order that led to the completeness of the real line, is not attained to the complex plane any more. This means that adjoining the imaginary copy of the arithmetic line orthogonally to the former real copy results in loss of total order on the plane that led to completeness of the line in the first place.

This loss of order completeness on the complex plane is compensated by a higher order of completeness, called algebraic completeness. According to the fundamental theorem of Algebra, which has been proved by the prince of mathematics, Gauss, every single-variable polynomial with complex coefficients has at least one complex root.

Equivalently, every non-zero, single-variable, degree n polynomial with complex coefficients has precisely n complex roots, counted with multiplicity. Thus, the complex numbers constitutes the algebraic closure of the real numbers, which makes it an algebraically closed field, according to the theorem above.

It is interesting to point out that, although there exist many different proofs of the fundamental theorem of Algebra, the one which captures the best the essence of the theorem is based on the topological notion of the winding number, to which we will come later on. In this sense, the expression of the theorem is subordinate to the connectivity properties of the plane, in comparison to the simple connectivity of the line.

Vectorial Representation of Complex Numbers

Each complex number $z = x + i\, y$ is represented by means of a vector on the complex plane whose base is ay the origin of the complex plane and its tip is at this number. A vector is characterized, either by its projections on the horizontal and vertical axis—giving the real and the imaginary part of the complex number respectively -, or by its modulus, that is, its length measured in terms of the real positive distance from the origin, and its argument, which is the angle with respect to the real horizontal axis.

Note that the length of the above vector, determining the modulus of the corresponding complex number on the plane, is calculated by means of the Pythagorean theorem as the diagonal of the orthogonal triangle formed from the vector and the two orthogonal axes—the real and the imaginary one. Thus, the modulus of $z = x + i\, y$ is given by:

$$r^2 = x^2 + y^2$$
$$r = \sqrt{(x^2 + y^2)}$$

Polar Representation and the Complex Exponential

Euler exploited the polar form of a complex number in an ingenious way. The issue pertains to the multi-valency of the angle under complete rotations around the origin of the complex plane. Namely, a complex number of unit modulus represented by a point on the unit circle—and reached via the vector of unit modulus from the origin 0 of the complex plane—stands simultaneously for all other complex numbers of the same unit modulus whose argument differs by an integer number of complete rotations around the origin.

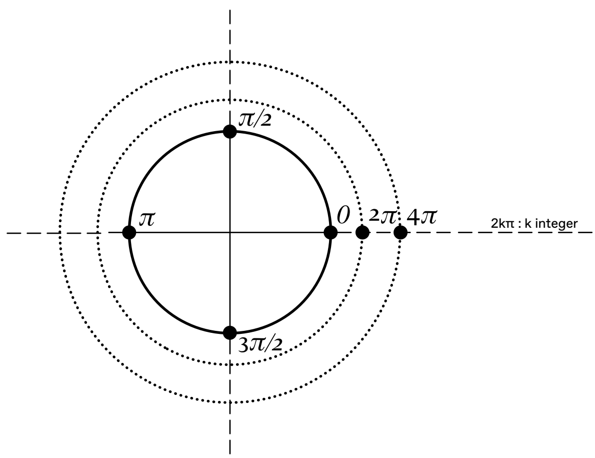


FIGURE 4.9
A complex number on the unit circle is defined modulo 2kπ: k integer

This is the issue of multi-valency of the notion of polar angle with respect to the real horizontal axis. To embrace this obstacle, Euler devised the expression of all these points of unit modulus through a complex phase, that is, a phase having the same representing value for all integer complete rotations around the origin ending up at the same point of the unit circle.

In other words, a complex phase, represents the whole equivalence class of all angles differing by an integer number of complete rotations around the origin and ending up at the same point of the unit circle. This amounts to an angular power partition of the complex plane along all possible radii on the unit circle, such that the single representative of each block of this partition is a complex phase.

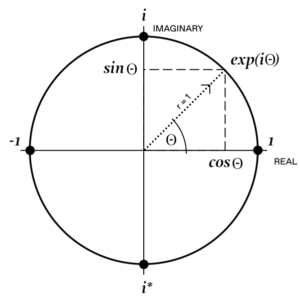


FIGURE 4.10
The notion of phase of a complex number

The latter is formally expressed via the extension of the exponential function from the real line to the unit circle on the complex plane, that we are going to treat in detail, regarding its function and implications later on.

For the time being, the notion of a complex phase, denoted by $exp\,(i\,\theta) := e^{i \cdot \theta}$, where θ is the angle between the vector radius and the horizontal axis, is enciphering the multiple-connectivity of the complex plane, due to the multi-valency of the notion of angle in its differences by an integer number of complete rotations. Thus, the polar form of a complex number, through its Eulerian representation, reads simply as follows:

$$z = r \cdot e^{i \cdot \theta}$$

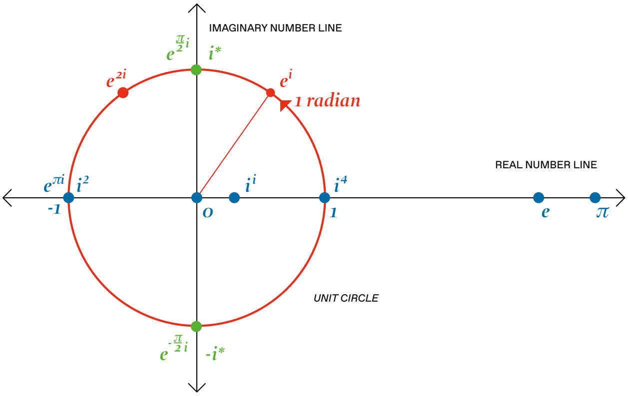


FIGURE 4.11
Polar representation of complex numbers

Note that in the polar representation of a complex number, which captures the underlying multiply-connected topology of the plane implicated by the multi-valency of angles, the multiplication of two complex numbers amounts to the multiplication of their moduli, but it pertains to the addition of their respective angles in the exponent.

The angles are real—expressed in radians—but their expression on the exponent of the exponential functions is mediated by the imaginary unit. Due to the fundamental property of the exponential function the product of two exponentials is transcribed to addition of the respective arguments in the resultant exponent.

In brief, two complex numbers are multiplied by multiplying their moduli, and adding their angles. In this sense, the moduli and the angles appear under different guises in the complex domain, the first as radii, whereas the second as exponential phases that essentially are conceptualized exclusively through the unit circle on the complex plane.

Squaring a complex number, amounts to squaring its modulus-radius, and doubling its angle:

$$z^2 = r^2 \cdot e^{i2\theta}$$

$r^2 = r \cdot r \quad \longrightarrow \quad$ Multiplication

$2\theta = \theta + \theta \quad \longrightarrow \quad$ Addition

Similarly, in the multiplication of two complex numbers, the radii are multiplied and the angles are added on the exponent:

$$z_1 \cdot z_2 = (r_1 \cdot r_2) \cdot e^{i(\theta 1 + \theta 2)}$$

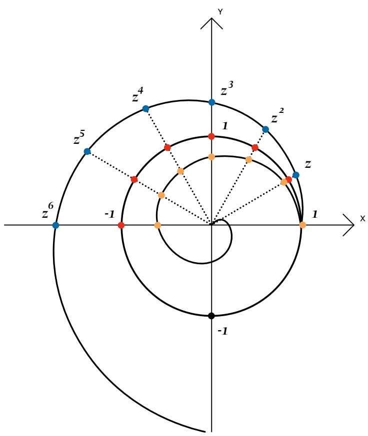


FIGURE 4.12
Spiral of multiplication

Quantum Neutrality: Abduction of Space from Eternity by Phasing

The set of all possible phases on the unit circle gives rise to a multiplicative group structure on the unit circle, where multiplication is simply expressed by addition of the corresponding angles. Since phases remain invariant under any integer number of complete rotations, the group structure is well-defined in the sense that the multiplication of any two phases gives a phase defined also on the unit circle.

Note that the neutral element of this multiplicative group, that is, its unit algebraic identity under multiplication, involves the modulation of all angles differing by an integer number of complete rotations, i.e. in radians $0, 2\pi, 4\pi$, and so on. This is expressed as $2\kappa\pi$, where κ is an integer. All these angles are indistinguishable to each other, represented in the same partition block, by the complex phase:

$$e^{i2\kappa\pi} = 1$$

This is very important because it establishes the relation of the additive cipher—as pertaining to the angles—and the multiplicative cipher of the complex numbers in their representation on the complex plane—thus, captures its essence as an algebraic field of numbers.

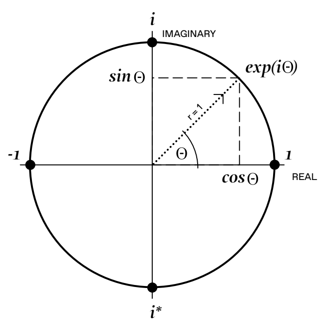


FIGURE 4.13
The trigonometric functions as axes projections of the complex exponential function

Algebraically thinking, the discrete group of the integers under addition, appears as the kernel of the group homomorphism from the additive group of the real numbers to the multiplicative group of complex phases on the unit circle. Thus, it is identified as an ideal in the field of complex numbers, which absorbs the whole group structure of the integers within its action, in order to modulate the neutral element of the multiplicative group of its complex phases residing on the unit(y) circle of the complex plane.

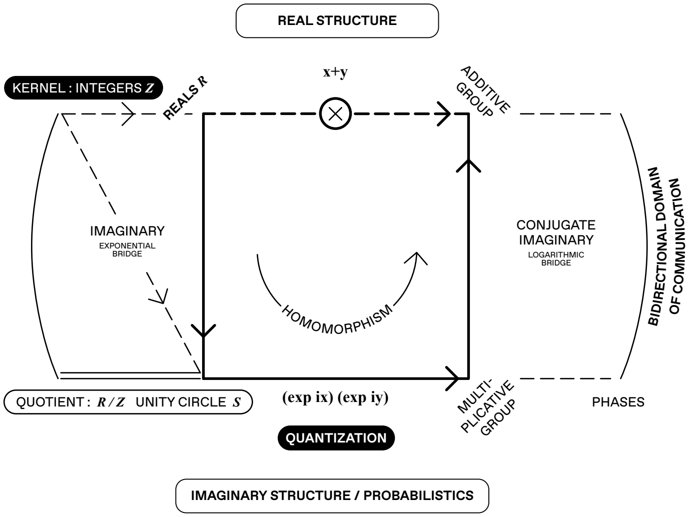


FIGURE 4.14
Complex architectonic scaffolding: the imaginary exponential bridge and its conjugate imaginary logarithmic bridge

Since the discrete group structure of the integers is the ideal in the field of complex numbers that determines the neutral element of its group of phases, and since the integers emanate originally from the domain of harmonics, it would be incorrect to consider the polar representation of the complex numbers as the geometric form of the complex numbers, as it usually displayed.

The notion of a screen, or epiphaneia, is not geometric by itself. It rather intervenes between the domain of harmonics and the domain of geometry as a mediator, it is a bridge of communication making it possible that these two different domains talk to each other. It is the rhetorical topos of communication opening up architectonically between the domain of harmonics and the domain of geometry that this screen accomplishes by incorporating the strength of the whole of our arithmetic cosmos, not only in its elemental substance, but in its structural elaboration as well.

The modulation of the whole discrete spectrum of harmonics by the neutral element of the group of phases, accomplishes something deeper in the action of this ideal, in comparison to the abduction of space from the realm of eternity, in the case of the irrationals. Given the uncertainty principle between time and frequency, this ideal neutralizes time in its frquential/periodic aspect algebraically as a whole in a single stroke, and gives rise to quantization.

Thus, although the construction of the irrationals led to the abduction of space from eternity through the neutralization of the ordered aspect of time, this was applicable only along the extension of space along a continuous line, leaving the frequential/periodic aspect of time completely unsettled. This is exactly what is accomplished by this "quantization ideal" of the complex number field.

It abducts space from the realm of eternity by neutralizing its periodic/frequential aspect, such that space extends not only along a single line, but in every possible branching direction. In turn, this is what makes possible the communication between the domain of geometry—generated by elements of the plane—and the domain of harmonics—generated invariantly by the discrete harmonic frequencies in their pertinent role as quanta.

Since, this is accomplished by means of the technology of the exponential function it is suitable to think of this function as a kind of a transcendental gnomon, and focus again, at the first place, at the real domain to examine its power as a bridge between the domain of harmonics and the domain of geometry. This is what constitutes the essence of the architectonic form and function in its entirety from our viewpoint.

The Spiral and the Catenary Natural Bridges

Both the exponential bridge and its inverse logarithmic bridge are characterized by self-similarity. Thus, they can be conceived in gnomonic terms. More concretely, since both of them are transcendental functions they act as inverse bridges between the harmonic domain and the geometric domain, i.e. the exponential is a bridge from the geometric to the harmonic, and inversely, the logarithm is a bridge from the harmonic to the geometric domain.

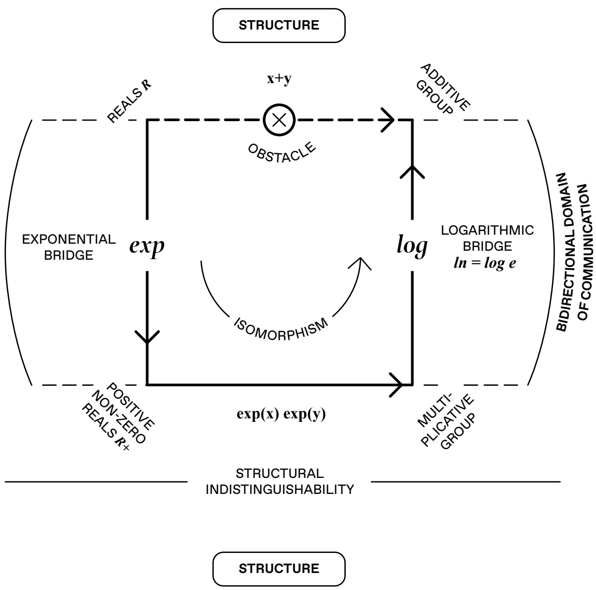
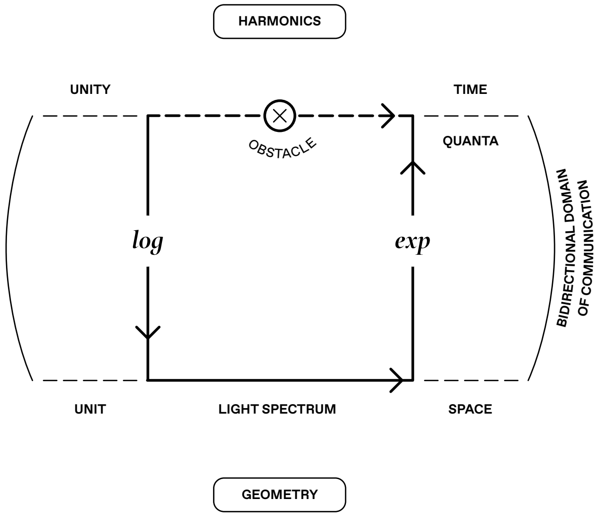


FIGURE 4.15
The metaphora from the reals with addition to the positive non-zero reals via the Exp/Log architectonic scaffolding is not only a homomorphism but an isomorphism.

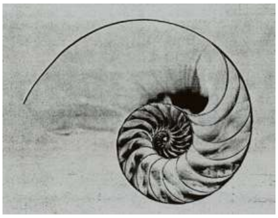
FIGURE 4.16
The metaphora from the reals with addition to the positive non-zero reals via the Exp/Log architectonic scaffolding is not only a homomorphism but an isomorphism.



If we consider the well-known example of the logarithmic spiral, it clearly provides an example of gnomonic growth, which is encountered in the natural world, for instance in the case of the Nautilus shell.

FIGURE 4.17
Gnomonic growth of logarithmic spiral

(Onofrio Scaduto, 1998, Wikimedia Commons, CC BY-SA)



The logarithmic or equiangular spiral differs from the arithmetic spiral in the sense that the distances between successive windings are not constant, but they increase in geometric progression. It has been first discovered by Descartes—in its role as an equiangular spiral -, and studied extensively by Jacob Bernoulli, who called it "spira mirabilis". In particular, this spiral can be grasped through four equivalent perspectives:

Equiangular spiral: A spiral whose radius vector cuts the curve at a constant angle;

Geometrical spiral: A spiral whose radius increases in geometrical progression as its polar angle increases in arithmetical progression.

Proportional spiral: A spiral in which the lengths of the segments of the curve cut by a fixed radial ray are in continued geometric proportion. Equivalently, the segments are scaled versions of each other, where the scaling ratios between successive pairs are equal;

Logarithmic spiral: A spiral having a linear radius of curvature as a function of arc-length.

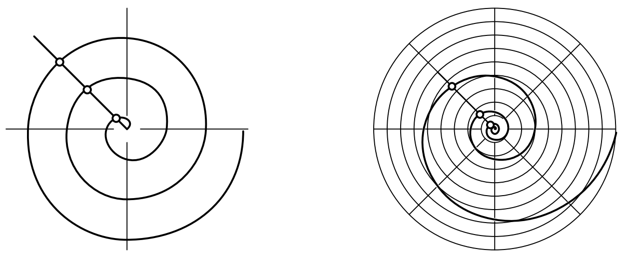


FIGURE 4.18
The archimedean spiral and the logarithmic spiral

The most interesting aspect of these transcendental gnomonic curves, which is absent from the initial rational conception of the notion of a gnomon, is the appearance of curvature. Moreover, the pattern of gnomonic growth is not a linear trapezium as in the former case, but an angular trapezoidal sector, as they are depicted for comparison below.

Regarding the exponential bridge, if we consider the arithmetic mean of the exponentials $exp(x)$ and $exp(-x)$:

$$AM = (exp(x) + exp(-x))/2$$

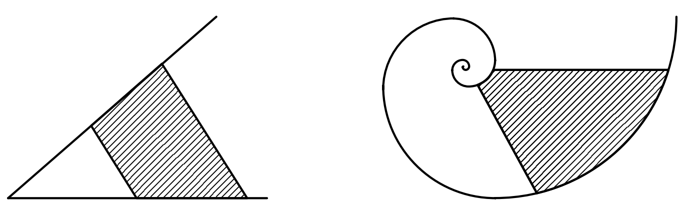


FIGURE 4.19
Pattern of gnomonic growth: from a linear trapezium to an angular trapezoidal sector

then we obtain a well-known curve, called the catenary curve.

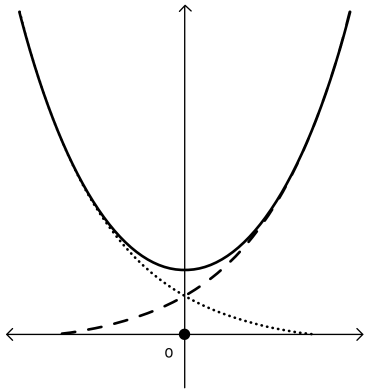


FIGURE 4.20
The catenary curve as the arithmetic mean of real exponentials with opposite exponents

The origin of this curve is physical, and more precisely, it is the solution to the least action problem referring to a chain in a gravitational field. Put simply, the catenary curve composed by the arithmetic mean of two exponential bridges according to the above, is the natural shape of a hanging chain under the pull of gravity.

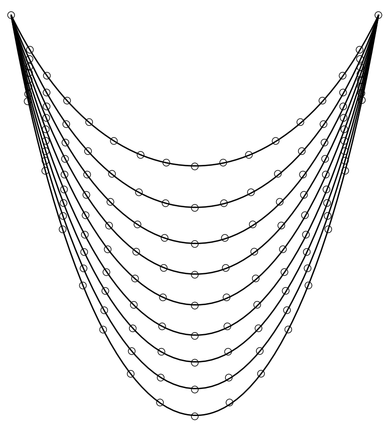


FIGURE 4.21
Catenary as the natural shape of a hanging chain

Leibniz realized that the catenary curve—emerging naturally via the least action principle pertaining to gravity—provides the physical means to grasp the notion of the real-valued logarithm function. This played an instrumental role in Leibniz's conception of real infinitesimal calculus. This can be demonstrated as follows:

(i) Consider the suspension of a chain from two horizontally aligned nails, and draw the horizontal through the endpoints as well as the vertical axis through the lowest point;

(ii) Put a third nail through the lowest point and extend one half of the catenary horizontally;

(iii) Connect the endpoint to the midpoint of the horizontal, and then bisect this line segment. Drop the perpendicular through this point, and then draw the horizontal axis through the point where the perpendicular intersects the vertical axis. After this, take the distance from the origin of the coordinate system to the lowest point of the catenary to be the unit length. The catenary curve is now characterized by the equation $y = (e^{x} + e^{-x})/2$ in this coordinate system.

(iv) To locate the real-valued logarithm $log(Y)$, find $(Y + 1/Y)/2$ on the y-axis, and measure the corresponding x-value on the original catenary curve. This assumes that $Y > 1$. To locate the logarithms of negative values, use the fact that $log(1/Y) = -log(Y)$.

The above steps, pertaining to Leibniz's natural representation of real-valued logarithms through the catenary curve, may be depicted schematically as follows:

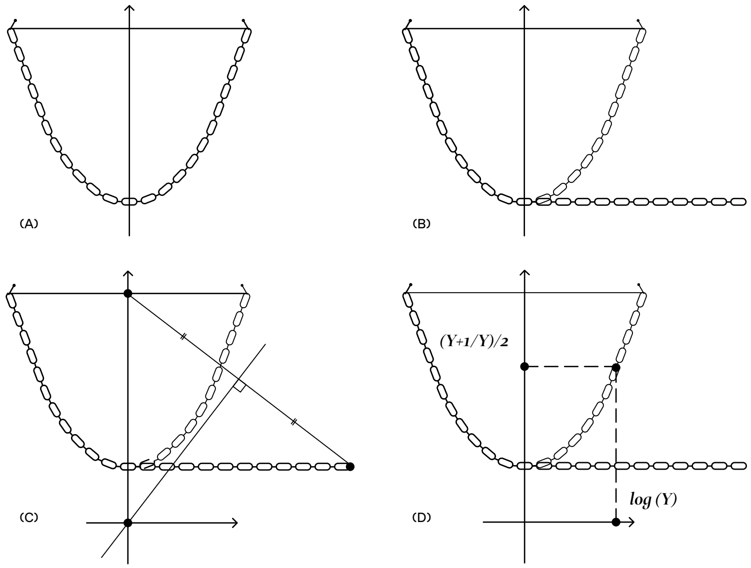


FIGURE 4.22
Leibniz's representation of the real logarithm via the catenary

The Meteoron of the Catenary Arch

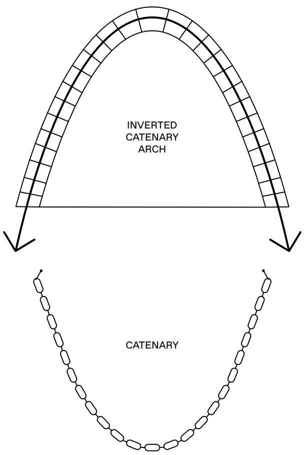


FIGURE 4.23
The catenary arch via the inversion of the catenary

The inverted shape of the catenary is the well-known catenary arch in architectonics with myriad of applications. The catenary arch by its specification through the real exponential bridges stands by itself without any support, defying in a sense the pull of gravity as the inverse of the shape assumed by a hanging chain.

The catenary-arch is the only type of arch that stands solely under its own weight. Note that in the case of the hanging chain the tension is equally distributed along its curved extension from one side to the other, where it is bounded. By inverting the hanging chain, tension is transcribed and substituted by contraction which, in a real sense, neutralizes the pull of gravity.

The central stone at the top of the catenary arch is called the key stone. It hyper-supports the whole of the arc from above—it does not support it from below, like the foundation of a column. The key stands in the vacuum solely out of compression without any under-support. In this sense, the key is the realization of the neutral element of the structure in the geometric world.

This is what pertains to the function of a "meteoron", which neutralizes in this case the ordering pull of gravity, making the geometric curvature manifesting as a consequence of this structural neutralization. Note that the key defines the geometric position where the inverse exponential curves meet each other.

The key is the geometric way to represent the communication between the additive structural world of the reals with the multiplicative structural world of the positive reals, through the compatibility of their ciphers, established by means of the real exponential bridge and the real logarithmic bridge in the transcription from harmonics to geometry, and inversely. Furthermore, it provides the means to conceptualize geometrically what abduction stands for, in the sense that the space opened up through catenary arches postpones ad infinitum the temporally ordering pull of gravity.

Surfaces of Revolution and Curvature

The different types of local curvature on the screen are already evident by considering the catenary curve. The geometric way of detecting the local curvature, according to Gauss who formulated this notion intrinsically, involves the consideration of the tangent and the normal at a point. The normal may be thought of as the radius of a circle at the specified point—called the osculating circle—whereas the tangent is the orthogonal to the normal, identified with the tangent of the circle at this point.

We now imagine another curve that bears the inverse specification of tangents and normals, meaning that the former tangents are the normals of the new curve and the former normals are the tangents of the new curve. Then, we obtain a geometric inversion—with respect to the local curvature—referring to these two curves. If we apply this to the case of the catenary, then we obtain another curve called the tractrix as depicted below.

The tractrix is a curve with constant tangent, called for this reason, the equi-tangential curve. It has been studied by Huygens thoroughly, who gave the name tractrix to this curve. It can be physically thought of in two ways:

(i) The orbit of the back wheels of a vehicle the front wheels of which describe a line;

(ii) The orbit of an object under friction, when pulled on a horizontal plane by a line segment attached to a tractor point, which moves at a right angle to the initial line between the object and the tractor.

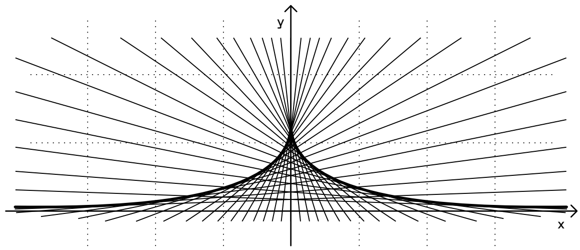


FIGURE 4.24
The tractrix

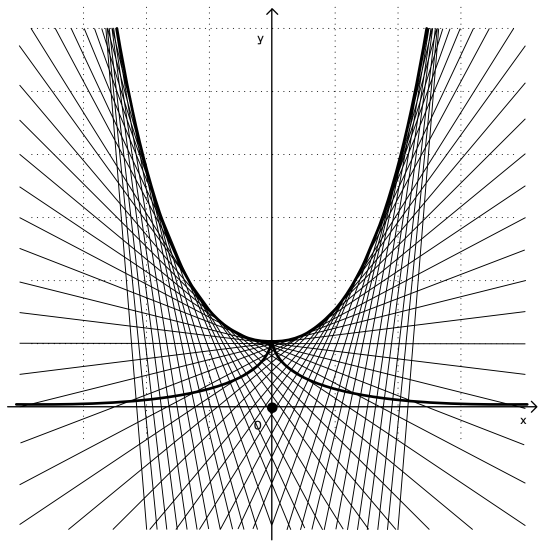


FIGURE 4.25
The tractrix as the geometric inversion of the catenary

The surface of revolution emerging by rotating the tractrix about its asymptote is a pseudo-sphere, that is, a surface with constant negative intrinsic curvature, characterized geometrically as a hyperbolic surface.

The analogy with the sphere comes from the fact that a sphere has constant positive curvature $1/R^2$, where R is the radius of the sphere, whereas the pseudo-sphere has constant negative curvature.

They can be treated on an equal footing by considering the radius of the pseudo-sphere as an imaginary radius, that is, iR, such that its curvature becomes the negative magnitude $-1/R^2$ as follows:

$$K = 1/((iR)^2) = -1/R^2$$

This is a non-trivial step that requires an imaginary metaphora between the harmonic and the geometric domain culminating in the role of the imaginary unit in relation to the understanding of constant geometric curvature on the screen, which gives rise to a partition spectrum consisting of three sole blocks: the spherical, the flat, and the hyperbolic. These three cases of constant geometric curvature can be localized appropriately in the context of the differential geometry of smooth manifolds, i.e. geometric spaces that are only locally—in the infinitesimal vicinity of a point—flat.

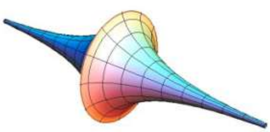


FIGURE 4.26
The pseudo-sphere as an imaginary radius sphere

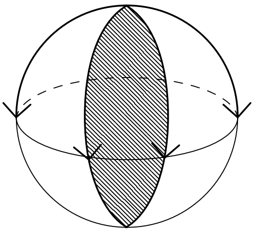


FIGURE 4.27
Positive curvature

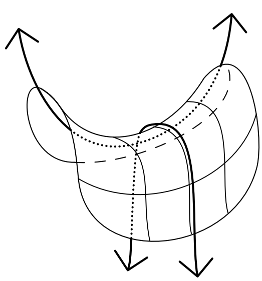


FIGURE 4.28
Negative curvature

In more detail, the sphere has the property that the curves obtained by cutting it by normal planes at a point P all have their radii of curvature on the same side, and in particular, they are ending at the center of the sphere.

In contrast, a saddle-shaped surface has sections whose radii ρ_1 and ρ_2 of curvature are on opposite sides. Note that the radius of curvature at a point P is the radius of the osculating circle, that is, the circle which most closely approximates the curve section at P.) Moreover, the curvature of a section curve of radius ρ is measured by the reciprocal $1/\rho$.

The appropriate measure of the synthesized curvature of sections curves of radii ρ_1, ρ_2 is $1/(\rho_1\rho_2)$, where the radii are signed in order to distinguish their directions. In particular, according to Gauss, the section curves should be chosen so that ρ_1, ρ_2 take the maximum and minimum signed values at the point P, meaning that they are principal curvatures. Then, the Gaussian curvature at P is given by:

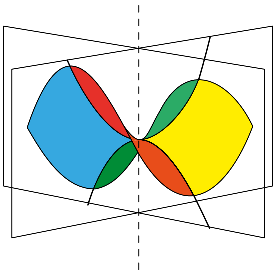


FIGURE 4.29
Radius of curvature at a point is the radius of the osculating circle at this point

$$\kappa = 1/(\rho_1\rho_2)$$

In this sense, the Gaussian curvature of a sphere of radius ρ is the positive constant $1/\rho^2$, whereas the Gaussian curvature of a saddle-shaped piece of surface is negative at all points.

According to Gauss' Egregium theorem, the product of the principal curvatures is intrinsic to the surface, that is, it is totally independent from any embedding in space. In other words, the curvature $\kappa = 1/(\rho_1\rho_2)$ is an invariant characterizing intrinsically a surface.

If we consider the pseudo-sphere, then PQ and QR are the radii of curvature at Q of the two normal section curves of the pseudo-sphere corresponding to the maximum and minimum radii of curvature. Moreover, PQ and QR are reciprocal to each other, and bear opposite signs. Therefore we have:

$$\kappa = [(-1)/PQ] \cdot [1/QR] = -1$$

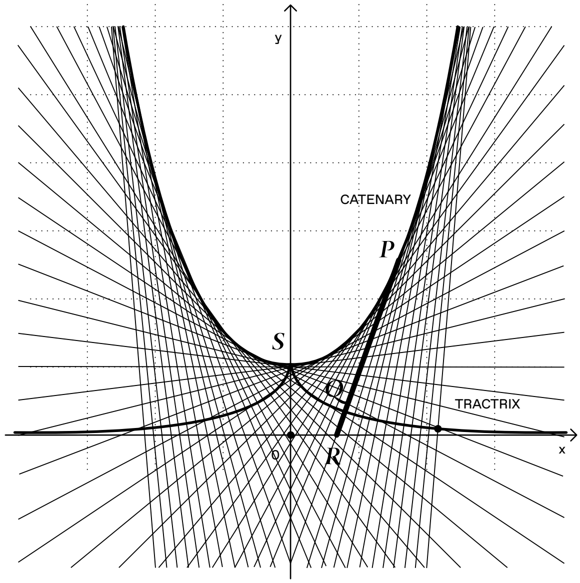


FIGURE 4.30
Generation of the nagative curvature of the pseudo-sphere

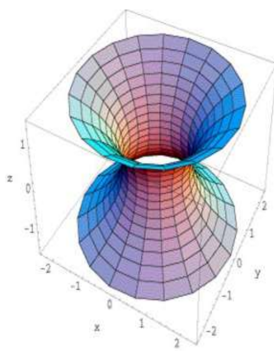


FIGURE 4.31
The catenoid as a surface of revolution

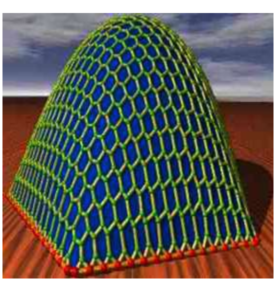


FIGURE 4.32
The catenary dome

(© Tim Tyler, HexDome)

Furthermore, the revolution of the catenary around an axis can be performed in two ways, that is, in a concave and in a convex way. The surface of revolution obtained in the first case is a catenoid, whence in the second it is a catenary dome.

The catenoid is a minimal surface, that is, it occupies the least area when bounded from above and below, e.g. by two circular rings. Because of this fact, it has mean curvature zero everywhere. As such it should be thought of as the curved abstraction of the plane, which is also a minimal surface considered as a surface of revolution.

The catenary dome should be thought of as the optimal correction to the shape of an ideally symmetric spherical dome when acceleration due to gravity is in force.

Helicoid: The Minimal Surface Bridge from Harmonics to Geometry

From a topological perspective, the catenoid is non-simply connected due to the hole it bears in the middle. If we make a cut, then it is transfigured periodically to a simply-connected helicoid, which is also a minimal surface, although not a surface of revolution. In particular, it occupies the least area when bounded sideways by two helices.

In this way, the catenoid becomes locally isometric to the helicoid, or equivalently, they both have the same local Gaussian curvature. A two-dimensional entity could not distinguish locally the catenoid from the helicoid. The fact that this locally isometric transfiguration exists is a strong motive to explore the implications of the exponential and logarithmic bridges when extended to the imaginary and complex number domains.

The crucial observation is that after half a period a mirror image of the same helicoidal surface arises, which may be grasped topologically as the twisting of a band. For example, we may think of a belt as a toy model whose two sides are coloured differently. The closed belt is an approximation to the region around the equator of the catenoid. If we open the belt and move the left end up and the right end down we have an approximate model of a helicoid. On the other side, if we move the left end down and the right end up we obtain the mirror or twisted image of the former helicoid.

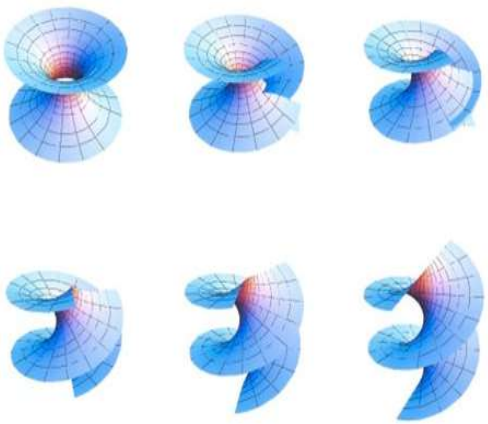


FIGURE 4.33
From the catenoid to the helicoid of the same local Gaussian curvature

Note that the rotation axis of both the helicoid and its mirror is orthogonal to the equator of the catenoid, i.e. there is a $\pi/2$ rotation counterclockwise, or clockwise, in relation to the equator. This is a strong indication about the role of the imaginary unit from a transcendental viewpoint.

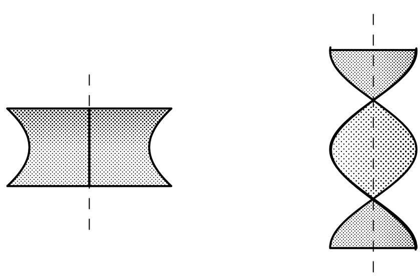


FIGURE 4.34
Rotation axis of the helicoid is orthogonal to the equator of the catenoid

The extension of the real exponential function to the imaginary domain takes place via the circle-valued exponential function:

$$exp : R \rightarrow S^1$$

where S^1 denotes the unit circle, whose elements are described via Euler's formula:

$$e^{i\theta} = cos\ \theta + i\ sin\ \theta$$

Notice the appearance of the imaginary unit, which is interpreted geometrically as a rotation by $\pi/2$ radians, making the imaginary axis orthogonal to the real axis in the plane of the complex numbers.

Together with the imaginary unit, there always exists its mirror image, described as its complex conjugate. Since the unit circle is coordinatized exponentially via the imaginary unit, we think of this circle as an imaginary ring. We stress that it should not to be interpreted geometrically, as it is usually the case.

The manifestation on the screen of the complex plane is an artifact between the harmonic and the geometric domain. To make this distinction explicit, we may call the imaginary ring—bearing the algebraic structure of the imaginary unit—a harmonic ring, in the sense that it descends, not from the domain of geometry, but from the transcendental domain of harmonics via the exponential function.

Notwithstanding this fact, the image of the ring in the geometric domain of forms may be visualised as a circle, more precisely, as a circular shadow of a harmonic entity. The latter is what is analytically expressed via the circle-valued—or complex—exponential function whose value is a phase on the unit circle.

For the consistency of this metaphora from the harmonic to the geometric domain it is necessary to qualify this harmonic entity, as well as its expression as an imaginary power, that is, a power whose exponent is imaginary. The intuition comes from the consideration of the helicoid together with its mirror image in its function to express this entity from the harmonic domain. Note that the helicoid unfolds continuously by parallel translation of its tangent planes, and after half a period of rotation, a mirror image of the same helicoidal surface arises.

We may think of the helicoid together with its mirror image as helical waves propagating in opposite directions such that the mirror image is the reflection of the first if bounded from above and from below. This is possible if these helical waves are bounded from above and below for temporal length of one period so as to give rise to a helical standing wave.

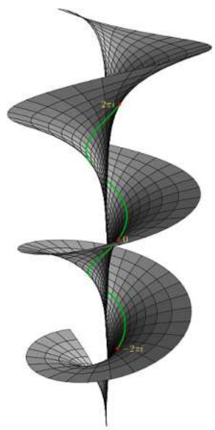


FIGURE 4.35
Helical standing wave

Here, this condition is equivalent to the requirement that within this bounded interval the helical wave is in unison with its reflection, i.e. its mirror image. Being in unison means that they are consonant in the fundamental harmonic frequency corresponding to the frequency ratio $1:1$, which in turn, would correspond to an angular temporal interval of one whole period 2π.

Self-Interference: The Spectral Resolution of Time

Let us focus on the distinctive difference between a vibrating straight chord whose length is spatial, and a vibrating helical chord whose length is temporal, according to the above. The unison ratio in the former case corresponds to a zero length spatial interval, whereas it corresponds to a 2π temporal interval in the latter case.

Notwithstanding this fact, we are able to establish the whole harmonic series in the helicoidal case, such that there is an inverse relationship between frequency and temporal extent as this is specified in terms of angles. The visual imaginary ring in this context, that is, the unit circle descending from the harmonic domain of relations—between a variably bounded helicoid and its mirror image—into the visible geometric domain making up its observable shadow, spatializes temporal extents via the imaginary unit and its conjugate, to allow for twofold directionality.

The spatialization amounts to the architectonic opening up of new space unraveled by temporal extents in a twofold imaginary axis—qualified via a positive and a negative direction as usual—as simultaneously, or synchronically extended imaginary spatial lengthes, at the present of the shadow.

Equivalently, these spatialized extents can be viewed as angular sectors of the imaginary unit circle via the complex exponential function. In this manner, being in unison in the harmonic context of a helical standing wave has a shadow in the visible geometric domain quantified by the imaginary spatial length $2\pi i$, which is identical to the period of the complex exponential function. Alternatively, via the complex exponential function, being in unison corresponds to the whole 2π angular sector of the circumference of the imaginary ring.

It is worth pondering again on some specific characteristics of the harmonic domain that make it different from the visible geometric one. If we think ontologically in terms of substances, then, in the harmonic domain, the twisted or mirror image, or simply the reflection, is of the same substance as the original, since it can interact and interfere with it to produce a standing helical wave bounded from above and below. This is actually the explication of self-interference, which physically is described in terms of quantum theory.

Note that a standing or stationary helical wave is not traveling in space at all. In contrast, it resolves time in terms of the harmonics series and the concomitant harmonic ratios of frequencies. As such a standing helical wave—in the context of its resonating environment—is not an ontological entity in physical space, although it has a shadow quantified through the imaginary ring.

What is crucial with respect to it, is that it resolves time periodically in terms of the harmonics, in a manner that time and frequency are reciprocally correlated. Thus, in the same way that time is spatialized via the imaginary axis, to give an imaginary length, frequency is spatialized orthogonally to the former as speed or momentum.

What really matters is the orthogonal placement of frequency and spatialized temporal extent via the intervention of the imaginary unit. As such, the opposite convention of indexing frequencies as imaginary quantities and spatialized temporal intervals as real is also valid and acceptable. Keeping the former convention, we identify the stairs of any bounded portion of the helical wave unfolding orthogonally to the imaginary ring that constitutes its synchronized shadow on the screen of the complex plane, with the harmonic series, which is able to induce any harmonic ratio.

The harmonics in this manner are qualified in terms of powers for the manifestation of consonances and dissonances. The negative harmonics, setting up the whole frequency spectrum, correspond to the harmonic series of the reflection, which is necessary for self-interference.

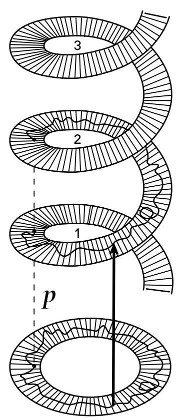


FIGURE 4.36
Resolution of time in terms of the harmonic series

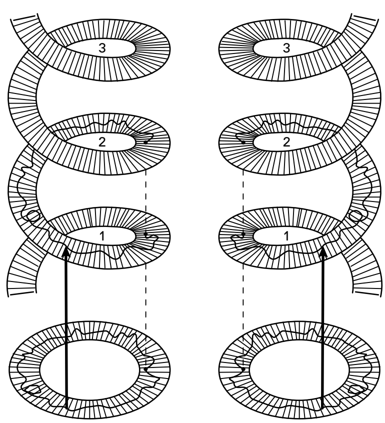


FIGURE 4.37
Helical mirror copies

Logarithmic Branching

From the structural algebraic viewpoint, the complex exponential is a group homomorphism from the additive group $(\mathbb{R}, +)$ to the multiplicative group $(S^1, \cdot)$ satisfying:

$$exp\,(i\,(\theta_1 + \theta_2)) = exp\,(i\,\theta_1) \cdot exp\,(i\,\theta_2)$$

The ideal modulating the neutral element in the image of this homomorphism, that is, on the unit circle, enciphers the criterion of spectral recognition on the unit circle in terms of complex phases. In turn, this is what constitutes the group-theoretic identity of the circle expressed by the neutral element of the group of phases.

In this manner, spectral recognition on the unit circle works on the basis of a criterion of identity that encapsulates all angles differing by an integer number of complete rotations in the block indexed by a single phase, their representative. We call this criterion, the "homeotic" criterion of identity on the unit circle of the complex plane, which gives rise to the pertinent neutrality condition.

Equivalently, algebraic identity of the group of phases on the unit circle is enciphered by the kernel of the group homomorphism $(\mathbb{R}, +) \rightarrow (S^1, \cdot)$, which is $2\pi\mathbb{Z}$.

Note that the homeotic criterion of identity is established in terms of the angular temporal interval in radians of one whole period 2π times the harmonic series, identified to the discrete group of all the integers $\mathbb{Z}$ under addition. In this way, a single moment of time on the extension of the real line is resolvable by the whole spectrum of discrete harmonics, or quanta.

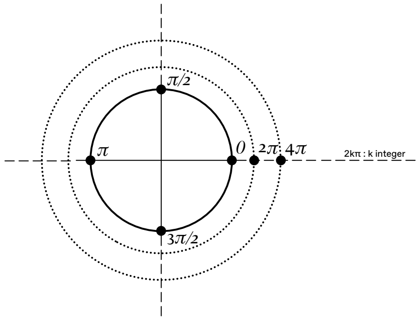


FIGURE 4.38
Resolution in terms of discrete harmonics

The existence of this homeotic kernel $2\pi\mathbb{Z}$ of the complex exponential group homomorphism:

$$e^{i}:(\mathbb{R},+)\rightarrow(S^{1},\cdot)$$

which amounts to the fact that the neutral element of the group of phases incorporates all the structure of the integers has a price. The price is that the complex exponential is not globally invertible, but only locally. This leads us to the significance of the domain of sheaves pertaining to the notion of topological localization of information, which we will briefly examine later on.

At present, the fact that it is not possible to have a well-defined global notion of a complex logarithm by inverting the complex exponential—as in the corresponding case of the real-valued logarithm—entails the novel phenomenon of branching. In other words, the projection from the helix to the circle, although it bears well defined local sections inverting exponentiation locally, it does not possess a global inverse.

We assert that branching is the geometric way to encounter the issue of multiple-connectivity—originating from the multi-valency of the notion of angle—in the harmonic domain. Branching is the geometric way to evade multi-connectivity by a process of cutting, bounding, and unfolding, until everything unravels and becomes simply connected. Reciprocally, branching is opening up new space architectonically that can be navigated geometrically.

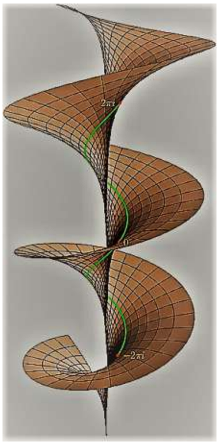


FIGURE 4.39
The branching structure and simple connectivity

Considering the complex logarithm, we figure out that an inverse homomorphism from the multiplicative group $(S^1, \cdot)$ to the additive group $(\mathbb{R}, +)$ can be defined only locally, i.e. by restricting the values of the angle within a period, i.e. from $-\pi$ to $+\pi$, $-\pi < \theta \leq \pi$, or from 0 to 2π, $0 \leq \theta < 2\pi$, which depicts a branch by cutting.

The meaning of the branch is simply that the complex logarithm is single-valued within this branch. Recall again that branching arises from the multi-valency of the angle, i.e. a complex phase on the unit circle is exactly the same for angle θ, and $\theta + 2k\pi$, where k is an integer. This is precisely what is encapsulated in the neutral element of the group of phases on the unit circle, and unraveled by means of the helicoid.

Concisely put, we assert the following conclusion:

Whereas the harmonic domain is associated with multiplexing and knotting, the geometric domain is associated with branching and weaving. Topologically, the main theme here is connectivity, and the metaphora pertains to the complete unraveling of harmonic multi-connectivity into geometric simple-connectivity, achieved by the concatenation of all logarithmic branches.

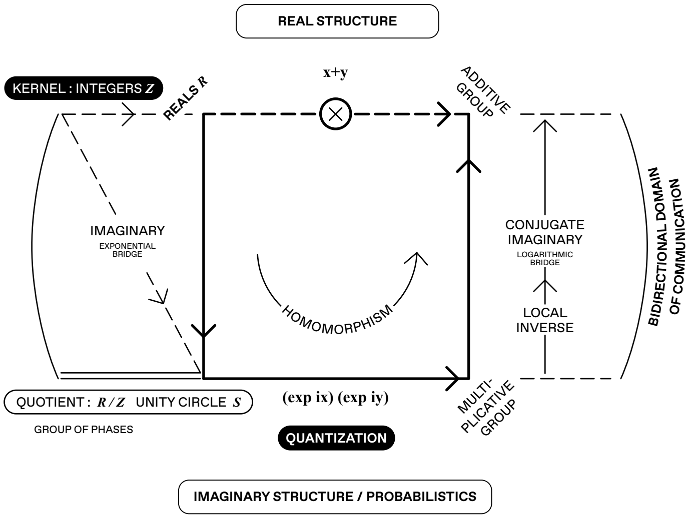


FIGURE 4.40
The conjugate logarithmic bridge is the local inverse of the exponential bridge in the complex domain

Canon of Metamorphosis and Modular Substitution

Since the harmonic and the geometric domain incorporate different principles of organization, we may consider the transcendental exponential and logarithmic functions from the perspective of canonics. The notion of a canon establishes the means of metamorphosis and modular substitution—via metaphora—between two different structurally domain in communication to each other.

Recall that the notion of canonics traces all the historical way back to the Pythagorean monochord. The canon sets up the means of transcription from an acoustic chord to an optical in terms of a scale that translates between acoustic frequency ratios and visual length intervals. The conception of frequency ratios as powers that can be perceived by the ear in the acoustic spectrum implies that the bridge from the harmonic to the geometric is of a logarithmic nature.

In other words, the logarithm function transcribes frequency ratios to length intervals, since it converts division to subtraction. Inversely, the metamorphosis from the geometric to the harmonic domain is of an exponential nature. This fact is at the root of the impossibility to set up a rational scale of musical intervals.

In turn, this has the consequence that the first historical encounter with the irrationals does not originate from the geometric domain, as it is usually thought of in relation to the incommensurability of the diagonal with the sides of an orthogonal triangle, but descends from harmonics. The notion of an equally tempered scale, based on the equipartition of musical intervals, and ending up on the chromatic scale gives rise to a screen that radiates in 12 directions, i.e. the number of the semi-tones of the scale.

Along these lines, the complex exponential is an encoding bridge from the geometric to the harmonic domain, whereas the complex logarithm is a decoding bridge from the harmonic to the geometric domain. The logarithmic bridge inverts the exponential only locally in this case, giving rise to the phenomenon of branching.

From a structural algebraic viewpoint, the complex exponential defined in terms of a group homomorphism from the additive group $(\mathbb{R}, +)$ to the multiplicative group $(S^1, \cdot)$ is finally extended to a group homomorphism from the additive group of complex numbers $(\mathbb{C}, +)$ to the multiplicative group of non-zero complex numbers $(\mathbb{C}^{\sim}, \cdot)$.

Considering the complex logarithm, an inverse homomorphism from the multiplicative group $(\mathbb{C}^{\sim}, \cdot)$ to the additive group $(\mathbb{C}, +)$ can be defined only locally, i.e. by restricting the values of the angle within a period, i.e. from $-\pi$ to $+\pi$, $-\pi < \theta \leq \pi$, or from 0 to 2π, $0 \le \theta < 2\pi$, which depicts a branch where the complex logarithm is continuous and single-valued.

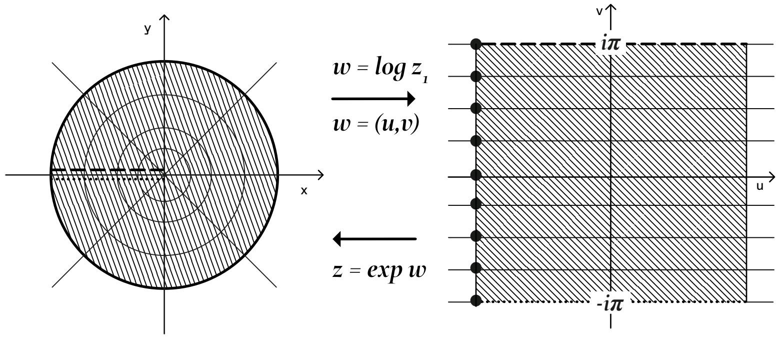


FIGURE 4.41
Complex Exp/Log architectonic scaffolding along a branch

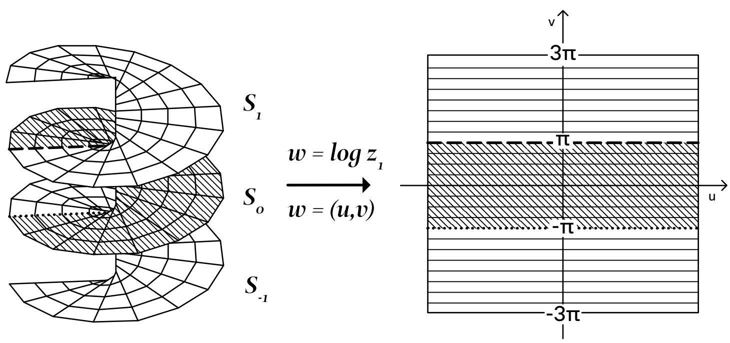
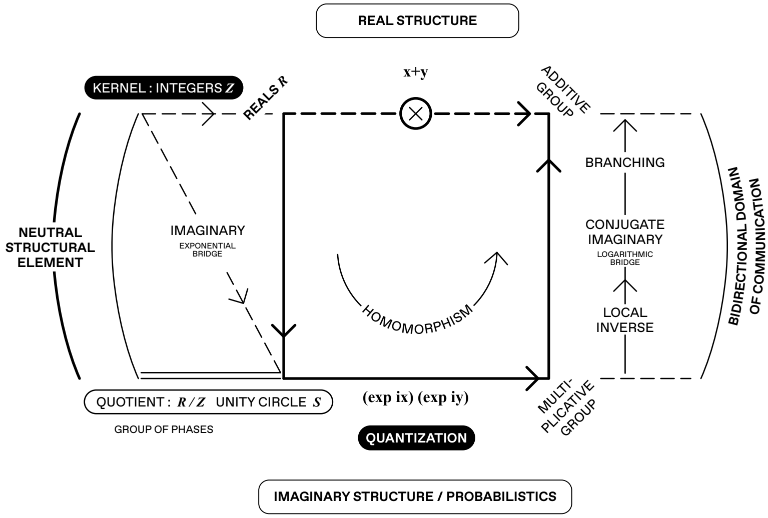


FIGURE 4.42
Decoding complex logarithmic bridge extending from branch to branch

It is quite remarkable that the invariance of the harmonic domain is encapsulated in the homeotic kernel $2\pi\mathbb{Z}$ of the exponential group homomorphism from the real line to the unit circle, which specifies the neutral element of the group of complex phases coordinating the unit circle, on the screen of the complex plane.

FIGURE 4.43
The imaginary logarithmic bridge inverts the imaginary exponential bridge locally along a branch giving rise to bidirectional domain of communication between the additive group of the reals and the multiplicative group of complex phase



The Archimedean Spiral Bridge of Circle Rectification

The major problem of ancient Greek mathematical antiquity has been the problem of squaring the circle. We view this problem as a problem of natural communication, and for this reason, the method proposed by Archimedes to address it bears great significance.

We call the above Archimede's metaphora, because the Archimedean method does not supply a constructible solution to this problem by straightedge and compass. Having realized that a constructible solution does not exist, Archimedes invents a metaphora from the circular to the linear domain. It is this metaphora that deserves a proper emphasis and appreciation.

The problem of squaring the circle refers to the instantiation of a square that has the same area with the area of the disk bounded by a circle. In the first stage, Archimedes considers an isomorphic problem. Namely, the problem of geometrically straightening

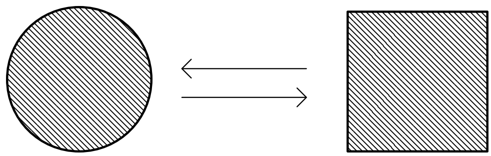


FIGURE 4.44
The problem of squaring the circle

the perimeter of a circle to a linear length. This more fundamental problem, conceptually can be cast isomorphic to the original as follows:

If the geometric straightening of the perimeter of a circle to a linear length is possible, then the area of a circle can be made equivalent to the area of an orthogonal triangle whose sides are given by the radius of the circle and the perimeter of the circle. The main problem arises from the irrationality of π, which is a transcendental irrational number, that is, it is not the solution of any algebraic equation.

For every conceivable circle of some radius, π is an invariant characterizing the perimeter through the radius. The incommensurability of the circular domain with the linear domain is precisely captured by the irrationality of π.

In the "Measurement of the Circle" Archimedes devised an ingenious approximation to the perimeter of the circle involving the method of exhaustion by means of inscribed and superscribed polygons, i.e. approximating and grasping the perimeter both from the inside and the outside using polygons involving up to 96 sides.

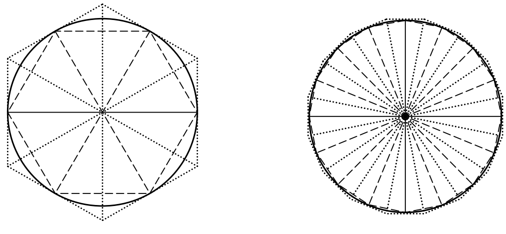


FIGURE 4.45
Polygonal approximation of the circle from both the inside and the outside

In relation to the pertinent problem of squaring the circle, Archimedes devised the means of metaphora from the circular domain to the linear domain in terms of the spiral. In other words, the Archimedean spiral provides the means of communication between these incommensurable domains, or equivalently, it is a bridge of metaphora from the circular to the linear and inversely.

The spiral is conceived in physical terms by Archimedes. He considers a point particle, located initially at the centre of the circle, which starts to move uniformly from the centre to the periphery of the circle along the radius. Simultaneously, Archimedes considers that the radius rotates uniformly counterclockwise around the centre of the circle. Thus, the particle movers according to the composition of these two uniform motions, the first linear, and the second circular. The composition of these two uniform motions is a non-uniform motion, which describes the trajectory of motion of the considered particle.

It is this trajectory, or orbit, that bears the geometric form of the Archimedean spiral. The spiral provides the natural means of transcription of the perimeter of the circle into a measurable linear length. This is accomplished by realizing the tangent to the spiral after one turn, i.e. at the point of intersecting the circle after one turn.

Archimedes showed that the tangent line to the spiral at this point crosses the vertical axis at a point whose distance from the origin is exactly $2\pi r$, where r is the radius of the circle. In this manner, the tangent to the spiral at the point of its intersection with the circle corresponding to a 2π rotation, accomplishes the metamorphosis of the circular perimeter to a linear length, which is expressed by the distance of the point of intersection—of the tangent with the vertical axis—from the origin.

The important thing to point out is that the abduction of this linear length—from the tangent to the spiral—corresponds to the time needed by the particle to complete one turn of its spiral trajectory, i.e. the perimeter of the circle becomes a spatial distance through a temporal extent. The underlying means of temporal unfolding is periodic and can be analogously recaptured for all higher turns of the spiral. Note that the radius of the spiral at each point of the trajectory of the particle is determined by the angle with respect to the horizontal axis.

According to the above, the perimeter of the circle is encoded by means of the linear length $2\pi r$ from the origin. In turn, this is the length of the vertical side of an orthogonal triangle whose horizontal side is the radius r of the circle.

Then, the area of this triangle is half of the area of the parallelogram having the same sides, which is clearly $2\pi r^2$. Thus, the area of the circle is the same as the area of the above triangle, i.e. πr^2.

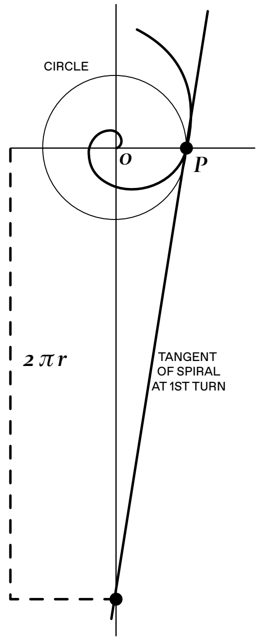


FIGURE 4.46
Unrolling the perimeter of the circle by means of the tangent to the spiral

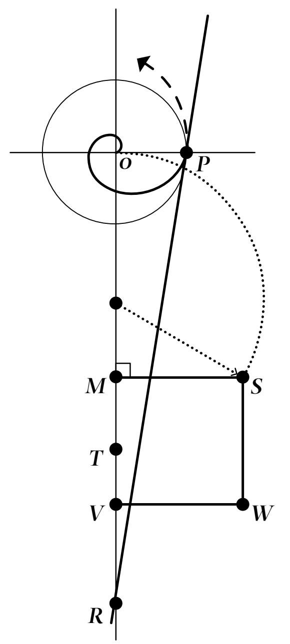


FIGURE 4.47
Genesis of the square having the same area as the disk bounded by the circle

The Helix and the Imaginary Axis of Archimedes

Let us think of the possible interpretation of Archimedes's method in terms of the unit circle in the complex plane. Then, the spatially two-dimensional Archimedean spiral is the bridge that unfolds the unit circle into the imaginary axis of the complex plane.

The question is how we qualify the imaginary axis in this setting. The key lies on the periodicity that is implicit in the successive turns of the spiral. More precisely, Archimedes' method allows the unfolding of the perimeter of the circle multiple times—recorded by the turns of the spiral—i.e. the spiral is poly-strophic and not only mono-strophic. This idea forces the conception of time as a helix—in three dimensions—unfolding continuously and orthogonally to the screen of the complex pane, such that its projection is the unit circle on the complex plane.

This is nothing else than the exponential group homomorphism $exp : \mathbb{R} \to S^1$, whose kernel is $2\pi\mathbb{Z}$. Thus, the discrete group of the integers—in its role as the neutral element of the group of complex phases—corresponds to the winding number of the helix. It simply counts the integer number of turns around the origin of the complex plane, which is invisible through the domain of values of the exponential.

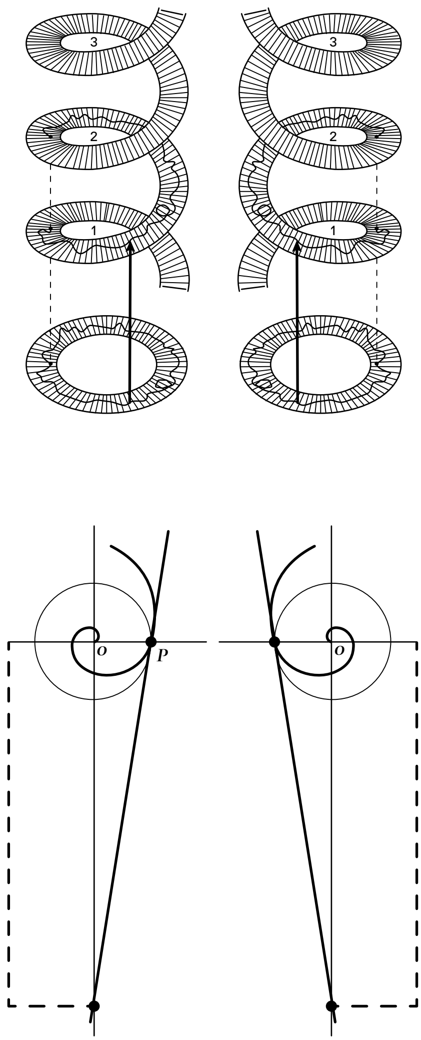


FIGURE 4.48

The spiral of Archimedes with its distinctive number of turns as an encoding and decoding bridge with respect to orientation of the metaphora between the circular and the bi-linear domain

If we consider Archimedes's spiral as the means of metaphora from the circular to the linear domain, the counterclockwise oriented spiral is the encoding bridge, whereas the—inversely oriented—clockwise spiral is the decoding bridge. Thus, we may invoke harmonic considerations in our setting, as pertaining to the helical conception of time, according to the above. Note that this requires to think of the helix in terms of a self-interfering helical stationary wave that is characterized by its harmonic series.

The leading idea again is that a single phase of the unit circle on the screen of the complex plane, is resolvable by the whole spectrum of harmonics, giving a precise meaning to the poly-strophic quality of the Archimedean spiral unfolding there in. The effect of this manoeuvre through the domain of harmonics is that we can now qualify the integers as a quantum spectrum of frequencies on the imaginary axis of the complex plane via branches of the complex logarithm.

We emphasize that this is viable due to the fact that both the complex exponential and its local inversion—in terms of a branch of the complex logarithm—preserve oriented angles. In this manner, the homeotic criterion of identity can be imprinted on the imaginary axis in terms of the angular temporal interval of one whole period 2π times the harmonic series.

The aftermath of the periodic resolution of time in terms of the harmonics is that time and frequency become reciprocally correlated, and represented orthogonally to each other. Time in the form of the helix in three dimensions unfolds orthogonally to the complex plane. The helix considered together with its mirror image give rise to a helical stationary wave bounded by temporal intervals of integer periods.

The latter projects down to the complex plane—excluding its origin, since the exponential can never obtain the value 0—on an annular strip of the polar grid. Applying a corresponding branch of the complex logarithm transforms this strip in an angle-preserving way to a rectangular region on the complex plane (Figure 4.49).

The imaginary axis is marked in this way by the harmonic frequencies corresponding to the integer number of cycles per unit of time, where the latter is taken to correspond to the tem-

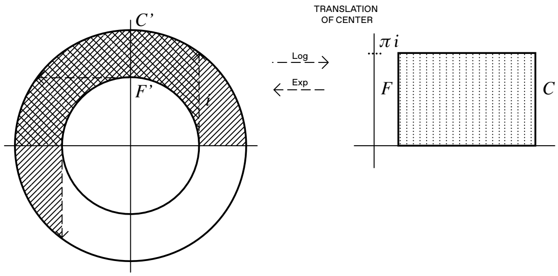


FIGURE 4.49
Conformal transformation of an annular strip to a rectangular region and inversely

poral length of one whole period 2π. Henceforth, we note that the process of bounding—a necessary condition for unraveling the harmonics in the complex analytic setting—takes place via logarithmic branch cutting, which originates from the fact that there does not exist a global complex logarithm function to invert the complex exponential function.

Notwithstanding this fact, the restriction to branches preserves oriented angles, such that annular strips are transcribed to rectangular regions on the complex plane, and inversely.

As a side remark, which will become more explicit soon, we grasp the finite topological coverings of the circle by itself—corresponding to the whole range of discrete integer powers—as bounded restrictions pertaining to whole angular periods of the universal covering of the circle by the helix. Therefore, the topological notion of the winding number—which encapsulates the homological invariance of the helix—physically descends from the domain of harmonics.

In the setting of the universal covering of the circle by the helix, the windings are qualified in terms of integers powers for the manifestation of consonances, that is, consonant harmonic ratios. Finally, it is the action of the logarithm through its single-valued branches that transforms these ratios into spatialized spectral intervals measured along the imaginary axis.

Harmonics of the Screw: The Area-Preserving Projection of the Sphere

A mechanical model that encapsulates all of the aspects discussed in the previous section is the Archimedean screw.

If we consider a finite portion of the screw, then the projection of the screw onto the plane thought of as perpendicular to the central axis of the screw, depicts an annular strip of the polar grid on the complex plane (excluding the origin, as usual in our approach).

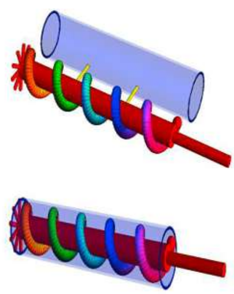


FIGURE 4.50
The Archimedean screw

If we apply the complex logarithm, this strip is transformed to a rectangular region on the complex plane leaving all angles invariant, that is, preserving all angles. We imagine the bounding of such a finite portion of the screw by an open cylinder. This is essential to understand in relation to Archimedes's method of calculating the surface of a sphere, as an application of the spiral bridge from the circular to the linear, which unravels the perimeter of a circle into a linear length.

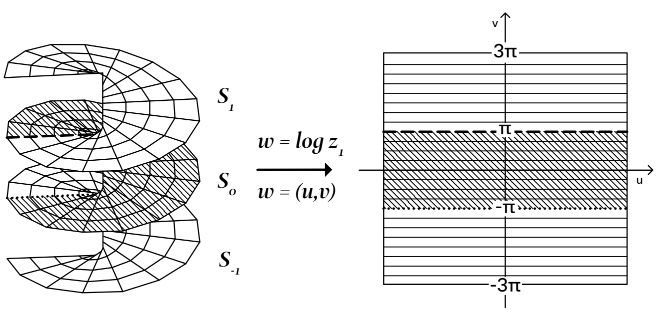


FIGURE 4.51
Angle-preserving pransformation of a portion of the screw to a rectangular region along a branch of the complex logarithm

We recall that the Archimedean metaphora in terms of the spiral leads to the conclusion that, the area of a circle is equal to the area of an orthogonal triangle, whose perpendicular sides are equal to the radius of the circle, and the perimeter of the circle, respectively.

Since the area of the sphere is $4\pi r^2$, this area is the same as the area bounded by four circles of the same radius r, or equivalently, the same as the area of four orthogonal triangles fitting compatibly together, whose big side is $2\pi r$ and small side is r, where r is identified with the radius of the sphere.

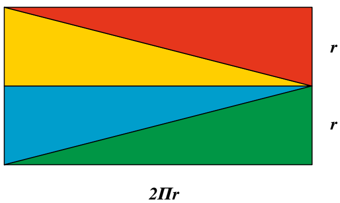


FIGURE 4.52
Archimedean unraveling of the area of the sphere to the area of four orthogonal triangles fitting compatibly together

It is necessary that these circles should be considered as great circles of the sphere passing through the North and South pole, such that their radius is equal to the radius of the sphere. The unraveling of any such circle into a linear length equals the equatorial length of the sphere, i.e. the length of the perimeter of the equator, given by $2\pi r$.

Thus, all four orthogonal triangles should have a big side equal to the equatorial length of the sphere $2\pi r$, and small side equal to the radius of the sphere. It is immediate to realize that these four triangles fit together in a plane region divided in two equal halves by the horizontal equatorial line of length $2\pi r$, such that the small side of each triangle r equals the vertical side of a half of this plane region.

Conclusively, this plane region has a horizontal side equal to the equatorial length of the sphere $2\pi r$ and a vertical side equal to $2r$. Each horizontally conceived half divided by the equatorial line has sides $2\pi r$ and r respectively. In each half there fit two orthogonal triangles sharing the same diagonal of sides $2\pi r$ and r, respectively. Thus, the area of the sphere equals the area of these four orthogonal triangles, each one of which equals πr^2.

In order to obtain a proper insight on the Archimedean method of determining the area of a sphere using the spiral as a bridge, it is significant to clarify what it implies. Precisely speaking, it implies that there exists an equi-areal projection of a sphere onto a cylinder, which we call the Archimedean projection. Note that this projection of the sphere does not preserve angles, but it preserves areas.

It is realized as a horizontal radial projection emerging by placing the sphere within an open cylinder, which is touching the sphere along the equator. Topologically, we may easily see that if we cut the sphere along a meridian passing through both the North and South pole of the sphere, we unwrap the sphere onto an open cylinder of height $2r$.

The area of this cylinder equals the area of a plane region on which it can roll for the temporal length corresponding to one rotation, i.e. $2\pi r$, identified with the equatorial length of the sphere. Hence, the sphere excluding its North and South pole can be projected in an area-preserving manner onto a open cylinder, and inversely. Rolling the cylinder as above, we obtain an equi-areal projection of the sphere on a planar region whose horizontal side is the equatorial length of the sphere $2\pi r$ and whose vertical side is $2r$, together with a rectangular and straight weaving grid of meridians and parallels.

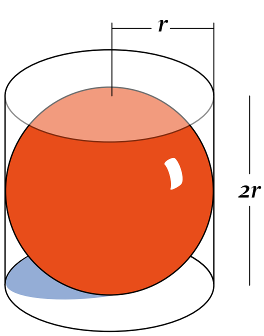


FIGURE 4.53
Equiareal projection of the sphere on an open cylinder

The Resonating Screen: Equatorial Cyclotomy

Since the sphere—in terms of its area-preserving projection to a cylinder—encodes the spiral metaphora from the circular to the linear, it is natural to wonder how the harmonics—in terms of the windings—manifest on our screen. The spectral recognition of the harmonics on the screen of the complex plane qualifies the screen as a resonator, that is, it makes it capable to resonate with the whole spectrum of spherical harmonics.

The necessary condition for this is the identification of the equator of the sphere with the unit circle on the complex plane, or equivalently, the multiplicative group of complex phases coordinating it.

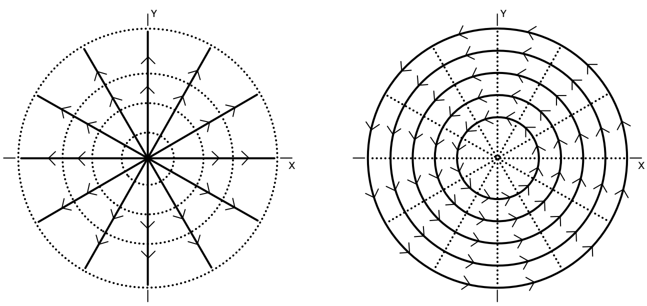


FIGURE 4.54
Radial and angular weaving grid of the equatorial disk

The idea is the following: Given a certain harmonic, devise the positioning—on the unit circle—of the phases equaling the unity, if raised to the power specified by this harmonic.

Conceptually, the positioning of these phases on the unit circle symbolizes the specific way of cyclotomy encapsulated by a certain harmonic. For this reason, the positions of these phases mark the roots of unity with respect to a certain harmonic. In a nutshell, the roots of unity invert the harmonics in relation to the unity, so that the latter manifest on our screen.

Conclusively, the harmonics manifest through inversion as the positions of the complex roots of unity on the equator of the sphere. There are always n different complex n-th roots of unity, i.e. complex numbers whose n-th power equals to unity, equally spaced around the perimeter of the unit circle in the complex plane. Since they are equally spaced, they constitute an equally-tempered scale on the screen of the complex plane. It remains to qualify how they are understood geometrically.

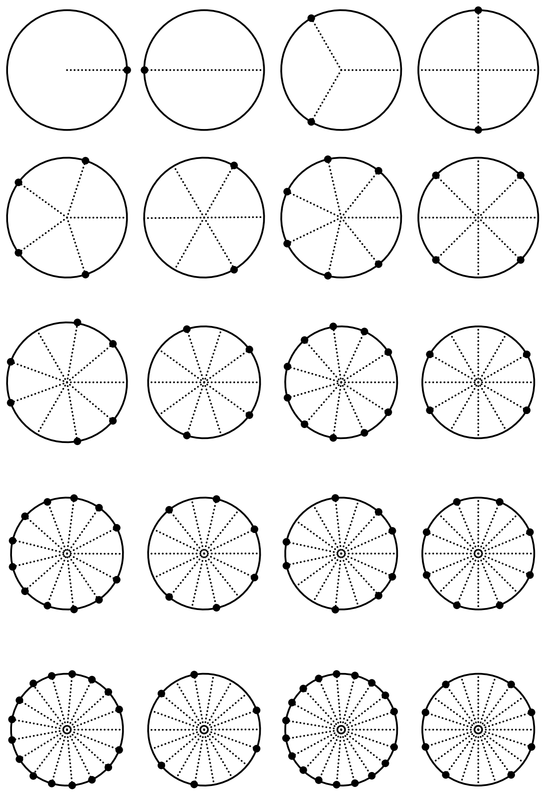


FIGURE 4.55

The tomographic process of cyclotomy: from the first to the twentieth root of unity. primitive roots correspond to dots

PRESENT - EPIPHANEIA - EQUAL TEMPERAMENT

The positioning of the roots of unity on the unit circle takes place geometrically through radii, which like cords bind the origin with the periphery. Depending on the harmonic, the positions of the roots on the periphery activated with respect to this harmonic are connected through linear segments, which always gives rise to a polygon inscribed in the unit circle. Hence, the roots of unity are manifested geometrically as the vertices of a regular polygon that binds them together with respect to a corresponding harmonic.

Of particular importance are the primitive roots of unity. More precisely, on the unit circle with n equally spaced rays, there is now a mark on ray k, denoting a primitive root of unity, if and only if k and n are relatively prime, i.e. they have no common divisors other than 1.

An equally-tempered equatorial scale constitutes the means of cyclotomy, which becomes manifest geometrically in terms of regular polygons inscribed in the equatorial circle. Thus, cyclotomy is what lies underneath the spectral generation of regular polygons. The deeper the resolution of the cyclotomy is, the higher the number of vertices appearing equi-distantly on the unit circle, and thus, the higher the number of sides of the inscribed regular polygon.

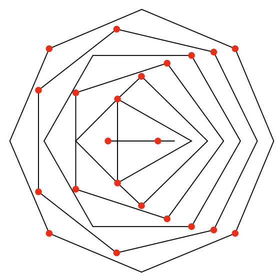


FIGURE 4.56
Polygonal unfolding of the roots of unity

5. The Mediator of Analysis Situs: Continuity and Connectivity

Topological Plasticity: The Erasure of Distance

Euler's resolution of the famous Königsberg bridges problem is the first work to address Leibniz's conception of "Analysis Situs". It is claimed that the citizens of Königsberg used to spend their Sunday afternoons walking around their beautiful city. The city itself consisted of four land areas separated by branches of the city river over which there were seven bridges, as illustrated in the figure of Euler's paper.

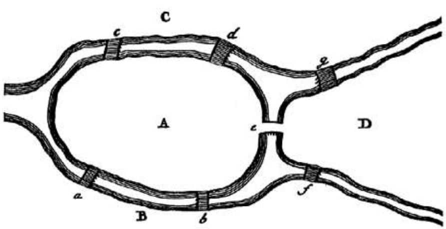


FIGURE 5.1
The bridges of Königsberg

(Sarapuig, 2015, Wikimedia Commons, CC BY-SA)

The problem that the citizens set themselves was to walk around the city, crossing each of the seven bridges exactly once and, if possible, returning to their starting point. The abstraction of this problem reads as follows: Given any division of a river into branches and any arrangement of bridges, is there a general method for determining whether such a route exists?

Euler poses the problem as follows:

"In addition to that branch of geometry which is concerned with distances, and which has always received the greatest attention, there is another branch, hitherto almost unknown, which Leibniz first mentioned, calling it the geometry of position [Geometriam situs]. This branch is concerned only with the determination of position and its properties; it does not involve distances, nor calculations made with them. It has not yet been satisfactorily determined what kinds of problem are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position—especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this problem, as an example of the geometry of position."

The attribution to Leibniz of the conception of geometry of position appears in a letter Leibniz wrote to Huygens, where he remarks the following:

"I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitudes ... ."

Leibniz introduced the term "Analysis Situs" to characterize the domain of plasticity, what we call presently topology. Analysis Situs pertains to the analysis of position or situation, where the notions of distance, length, angle, and area, are erased. What remains after this forgetting, this erasure, is only the notions of continuity and connectivity.

The domain of plasticity mediates in the metaphora from harmonics to geometry and inversely. Mediation always requires an act of forgetting, in the first place. The mediation from geometry to harmonics—the encoding bridge—consists in the forgetting of distance and angle. Then, the spectrum becomes non-rigid and malleable.

What constitutes this spectrum is continuity—where local positioning takes place—and connectivity—where obstacles act as sources of invariance. Topological invariants usually pertain to a structural—group-theoretic characterization -, but the invariance itself can be captured arithmetically, like in the case of the Euler characteristic. The decoding bridge consists in the translation of harmonics to topological invariants pertaining especially to connectivity. Then, the notion of a universal covering space provides the bridge back to geometry, as we will shortly explain.

The Euler Characteristic Invariant of Shape

In the case of the Euler invariant of the plasticity domain, it pertains to the characteristic of various classes of geometric figures, involving only the relationship between the numbers of vertices (V), edges (E), and faces (F), of a geometric figure. This invariant number, given by:

$$C = V - E + F$$

is the same for all figures whose boundaries are composed of the same number of connected parts. For all simple polygons the Euler characteristic equals to unity.

This can be shown through a process called triangulation of a figure. We draw lines that connect vertices, such that the region of the figure is subdivided into triangles. The triangles are then erased, one at a time, starting from the outside and moving inwards until only one remains, whose Euler invariant equals one.

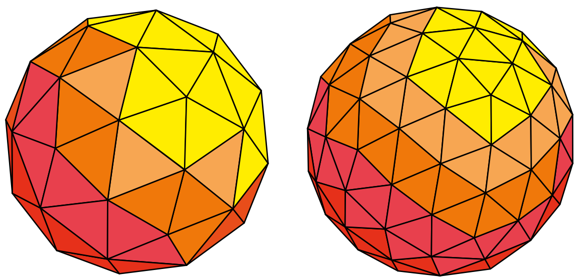


FIGURE 5.2
Cellular decomposition of a sphere

Note that this process of adjoining and erasing lines does not alter the Euler characteristic of the original figure—since it is a topological invariant—and so it must also equal one.

For instance, in the case of a square, we have $V=4$, $E=4$, $F=1$, so the Euler characteristic is 1. Equivalently, we can partition it into two triangles and get $V=4$, $E=5$, $F=2$.

Further, the Euler characteristic is used to demonstrate that there are only five regular polyhedra, namely the Platonic solids.

More concretely, a convex polyhedron whose surface comprises of V vertices, E edges, and F faces, satisfies:

$$V-E+F=2$$

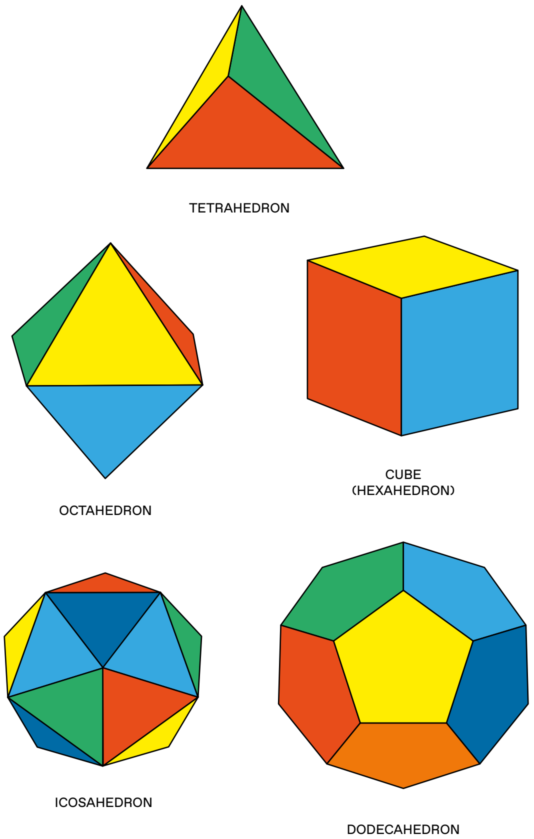


FIGURE 5.3
The five regular platonic polyhedra

Being convex, the polyhedron can be enclosed in a sphere and its surface projected bijectively to the surface of the sphere. This gives a partition of the sphere's surface into polygons, which is called a cellular decomposition. For any two cellular decompositions P and P', we say that P' is a refinement of P if all vertices and edges of P are also in P.

Since any two cellular decompositions have a common refinement, if P' is a refinement of P, then they have the same Euler characteristic.

Moreover, the Euler characteristic is constant for any cellular decomposition of a suitably nice surface. For example, on the surface of a torus, the following decomposition gives—after unfolding—four faces of rectangles.

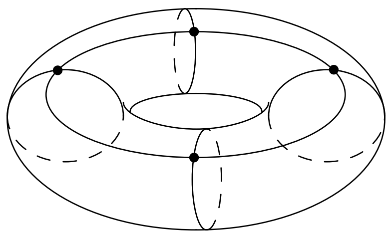


FIGURE 5.4
Cellular decomposition of a torus

Hence, in the case of the torus $V = 4$, $E = 8$, $F = 4$, and $V - E + F = 0$.

The Euler characteristic invariant V-E+F pertains to the global topological property of the shape. If we partition the surface of a two-holed torus, we find that its Euler characteristic is -2.

Abstracting from the above cases, the Euler characteristic of the surface of a g-holed torus is $2 - 2g$, where g is the genus. Put simply, the genus is equal to the number of holes in the topological shape. It is expressed by an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant shape disconnected.

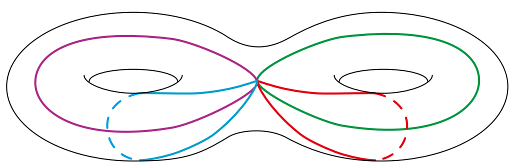


FIGURE 5.5
The double torus

Continuity and Information: Localization and Sheaves

The domain of plasticity is introduced through the distinction between the local and the global, and it is of an information-theoretic character. The idea is that the global is not directly graspable, and thus, global information should be constituted by local means, together with the appropriate gluing conditions from the local to the global.

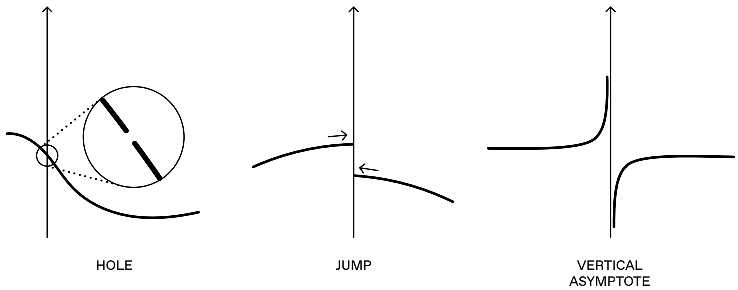
The obstacle here is objective indistinguishability, intrinsic randomness, or indeterminateness that is not due to subjective conditions of lack of knowledge. The abstract idea characterizing the domain of plasticity is continuity, and its local manifestation in terms of continuous functions. This is precisely what leads to the notion of a topology on space.

In the conception of a topological space, the notion of a topology provides the scaffolding that enables to model the notion of a continuous function. It is enough, at the present stage, to consider the domain of values as the real line.

A topology essentially specifies how we cover the space, for instance, in terms of open sets. In this case, we consider a topology consisting of open covers distributed all around and intersecting each other. The condition is that an arbitrary union of these covers, as well as, a finite only intersection of any of them, should also qualify in the topology.

A function is said to be continuous if and only if the inverse image of every open cover of the range is an open cover of the domain topological space. This is an attempt to capture the intuition that there are no breaks or separations in a continuous function.

FIGURE 5.6
Notions of hole, jump, and vertical asymptote



We stress the following:

(i) The definition of a topology on a space is solely used as a scaffolding to express what a continuous function is on that space;

(ii) the continuity of a function is a property which is determined locally, that is, only referring to the open covers of a space.

This means that due to the plasticity of an open cover, the property of continuity of a function should respect the restriction of this function to smaller open covers, as well as, its extension to bigger and bigger open covers in the topology up to covering the whole space.

Regarding the second property, a continuous function defined globally over the whole space, and not merely over an open set, is subordinate to the condition that its extension to bigger and bigger open covers takes place in a unique way, otherwise gluing the local pieces together is not feasible.

In that case, a globally defined continuous function can be restricted consistently to all open subcovers of any open cover in the topology, and inversely, extended by gluing uniquely together all its local restrictions. This is the crucial conceptual insight—in relation to the local conception of continuity—that is encapsulated in the notion of a sheaf.

The above insight may be generalized in two directions:

Firstly, instead of open covers of a topological space we may consider generalized covers under the constraint that they collectively obey topological closure conditions analogous to the ones used for open covers.

Secondly, instead of functions varying continuously over local covers, we may consider generalized functional relations, to be thought of as local information carriers.

These carriers encode functional relations relatively to a local cover, such that the information obtained by restriction to a subcover is exactly the same as the local information carrier over this cover. Of course, the objective is that local information carriers can be uniquely extended from the local to the global.

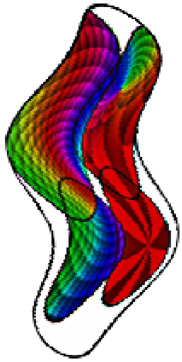


FIGURE 5.7
Gluing along compatible intersections in terms of color

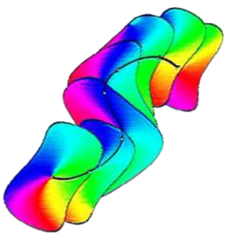


FIGURE 5.8
Analytic continuation along a path

This happens only if they are compatible under their pairwise intersections, which constitute the bridges of extending these information carriers to bigger and bigger covers. If this is accomplished, we obtain an information structure called a sheaf, consisting of the totality of its information carriers, called the sections of the sheaf.

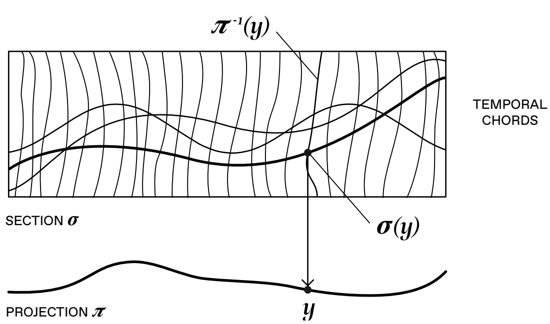
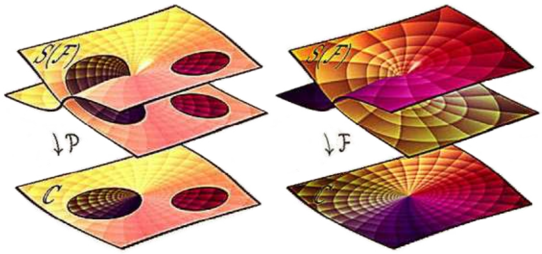


FIGURE 5.9
Sections of a sheaf as temporal chords

FIGURE 5.10
Sheaf-theoretic notion of a Riemann surface and branching structure



The latter condition is not always fulfilled. Then, we obtain a weaker structure called a presheaf, which satisfies only the restriction property. In case that the compatibility property of sections under extension is also satisfied, but not uniquely, the preheaf is characterized as a separated one. Only in case that extension results in a unique gluing of sections, that is, local information carriers can be uniquely glued together, a separated presheaf becomes a sheaf.

Consequently, there will always be more local sections than global ones, since not all local sections can be extended to global ones.

In more detail, the information structures—pertaining to the local conception of continuity—are defined as follows, for the case of a topological space:

A presheaf $\mathbb{F}$ of sets on a topological space X, is constituted as an information structure in relation to this space, as follows:

I) For every open set U of X, there is defined a set of elements denoted by $\mathbb{F}(U)$;

and II) For every inclusion $V \hookrightarrow U$ of open sets of X, there is defined a restriction morphism of sets in the opposite direction:

$$r(U \mid V) : \mathbb{F}(U) \to \mathbb{F}(V) \qquad (1)$$

such that:

a) $r(U \mid U) =$ identity at $\mathbb{F}(U)$ for all open sets U of X; and

b) $r(V \mid W) \circ r(U \mid V) = r(U \mid W)$ for all open sets $W \hookrightarrow V \hookrightarrow U$.

Usually, the following simplifying notation is used: $r(U \mid V)(s) := s \mid_V$.

A presheaf $\mathbb{F}$ of sets on a topological space X, is defined to be a sheaf if it satisfies the following two conditions, for every family V_a, $a \in I$, of local open covers of V, where V open set in X, such that $V = \cup_a V_a$:

I) Local identity axiom of a sheaf: Given $s, t \in \mathbb{F}(V)$ with $s \mid_{Va} = t \mid_{Va}$ for all $a \in I$, then $s = t$; and

II) Gluing axiom of a sheaf: Given $s_a \in \mathbb{F}(V_a)$, $s_b \in \mathbb{F}(V_b)$, $a, b \in I$, such that:

$$s_a \mid_{(Va \cap Vb)} = s_b \mid_{(Va \cap Vb)} \qquad (2)$$

for all $a, b \in I$, then there exists a unique $s \in \mathbb{F}(V)$, such that: $s \mid_{Va} = s_a \in \mathbb{F}(V_a)$ and $s \mid_{Vb} = s_b \in \mathbb{F}(V_b)$.

We note that the above definitions hold if instead of presheaves/sheaves of sets, we consider presheaves/sheaves of algebraic structures, for example groups, algebras, or modules.

As the most basic example, if $\mathbb{F}$ denotes the presheaf that assigns to each open set $U \subset X$, the commutative algebra of all real-valued continuous functions on U, then $\mathbb{F}$ is actually a sheaf.

This is due to the fact that the specification of a topology on X, is solely used as an architectonic scaffolding for grasping the notion of a continuous function on X.

Thus, the continuity of each function can be determined locally. This means that continuity respects the operation of restriction to open sets, and moreover, that continuous functions can be collated together uniquely, as it is required for the satisfaction of the sheaf condition.

In the case of abstract topological figures the homeotic criterion of local identity, that is, the criterion of local congruence, is expressed by the notion of a local homeomorphism. This notion pertains to the local preservation of shape by means of a continuous function. For instance, from the perspective of the domain of plasticity, the exponential function from the real line to the circle is indeed a local homeomorphism of topological spaces, i.e. it preserves locally the shape, but not globally.

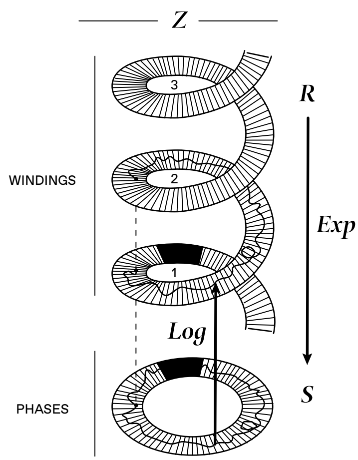


FIGURE 5.11
Local homeomorphism from the real line to the circle

It is natural to wonder how the notion of the gnomon can be thought of in the topological domain. In other words, if we erase the length of the gnomon and the right angle of its placement, what remains of this notion?

For this purpose, it is interesting to focus on the abstraction of this notion, conceived by the great mathematician Heron of Alexandria, who gave the following formulation: A gnomon is that entity which, if it is adjoined to some other originally given unknown entity, it results in a new augmented entity, which is partially, or locally congruent, with the original one.

Recall that in the standard context of the Thalesian theory of homeothesis, the gnomon is, literally speaking, the part of the sundial that casts the shadow.

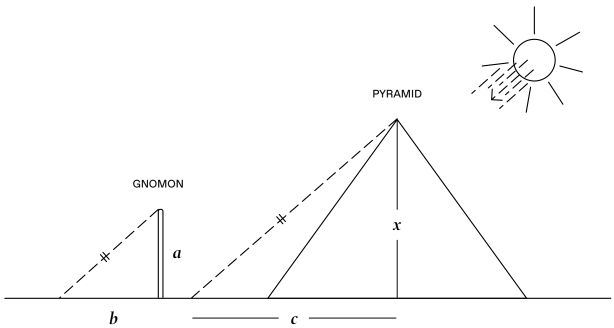


FIGURE 5.12
Gnomon of Thales

In this sense, it is the adjunction of the gnomon to the pyramid, which induces a homeothetic congruence between the level of objects and the level of their shadows—with reference to their magnitudes at the same time of the day—and consequently, makes the metaphora feasible from one level to the other, leading to the determination of the height of the pyramid in terms of proportion.

In its simplest possible form the general process of adjoining a gnomon in order to obtain a relation of homeothesis may be visualized as follows:

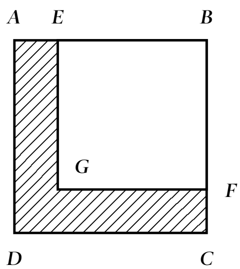


FIGURE 5.13
Adjoining a gnomon and the notion of homeothesis

Formally, the relation of homeothesis is an equivalence relation, and thus induces a partition spectrum consisting of equivalence classes standing for the blocks or cells of this partition. The quotient structure obtained by factoring out this equivalence relation incorporates a new homeotic criterion of identity in comparison to the initial one, which is precisely characterized in terms of the gnomon.

In other words, the notion of logical identity is relativized with respect to the gnomon, such that the neutral element of the quotient structure expresses equivalence modulo the gnomon. This is exactly the notion of modular substitution we encountered before.

From the rather rigid case of homeothesis—proportionality of magnitudes—, the metaphora may be abstracted to a deeper level involving the continuous recursive, or periodic, adjunction of a gnomon. This leads naturally to the dynamical notions of gnomonic growth or gnomonic unfolding, and reciprocally, gnomonic subdivision or gnomonic folding.

An example of the continuous recursive adjunction of a gnomon, is the case of growth manifesting by the so called "golden mean" spiral, depicted graphically as follows:

FIGURE 5.14
The golden mean construction

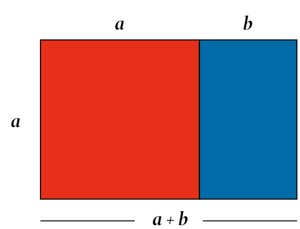
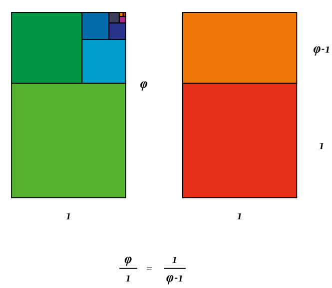


FIGURE 5.15
Iteration of the golden mean construction



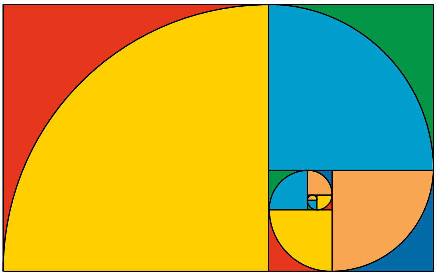


FIGURE 5.16
The golden mean spiral

Nevertheless, we need an even deeper level of abstraction to enter into the domain of plasticity equipped with a gnomon adapted to this context, where distances and angles are intrinsically non-viable as means of comparison and congruence. For this purpose, we have to enter the phantasms of shadows themselves, which are continuously altering their shapes topologically.

At this level, the abstraction consists in thinking of a gnomon as a means to indicate position according to Analysis Situs, or discern, and distinguish locally. The adjunction of a gnomon in this domain should provide the means of metaphora and modular substitution, through a partition spectrum of locally congruent figures.

For instance, in the case of an abstract topological figure, called a manifold, the gnomon to be adjoined is a local covering patch of the figure that is locally homeomorphic to a standard space, like a Euclidean space. As a consequence, locally, the means of indicating position through the gnomon takes place in terms of the standard space.

In a similar way to what we have already discussed, the global topological figure may be grasped as a sheaf-theoretic information structure, whose modularity type is expressed by the gluing conditions of all local patches adjoined to the figure so as to cover it entirely.

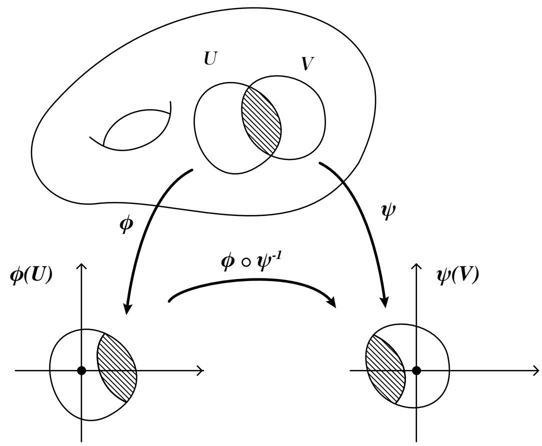


FIGURE 5.17
Gluing condition on overlapping local patches of a manifold

Finally, it is worth explicating very briefly, from our perspective, the notion of gnomon, employed in homology theory, as it is appears in the field of algebraic topology. We think of homology as the algebraic-topological abstraction of homeothesis. The key idea pertaining to topological connectivity—not merely to continuity—is that the notion of boundary seems to be the most appropriate for the topological metamorphosis of the gnomon.

This already makes perfect sense in relation to shadows, under certain precautions. Since what matters here is connectivity in a continuous stratum, the congruence condition for building the spectrum should refer to this criterion. In general, if we think of a topological obstacle, all bidirectional bridges of connectivity around it—which we call connectivity chains—can be distinguished in a binary way: Either they are cycles, or they are boundaries.

Intuitively, a boundary at some dimension is a bounding chain of a higher dimensional topological form, whereas a cycle stands for a non-bounding chain. Visually, non-bounding chains may be thought of in terms of holes or punctures or higher dimensional cavities, whereas boundaries may be thought of in terms of filled in, and thus bounding chains.

The basic idea of a boundary as a gnomon, establishing the criterion of congruence in homology, is that adjoining a boundary to a cycle gives a topologically similar or homologous cycle. Thus, two cycles differing by a boundary belong to the same homology equivalence class as it is depicted visually below.

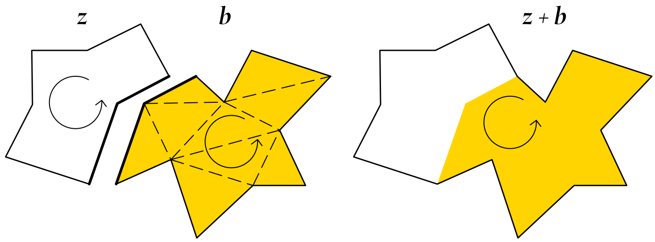


FIGURE 5.18
Boundary as a homological gnomon in algebraic topology

In this sense, homology equivalence classes, which are actually commutative groups due to the algebraic operations involved in composing chains and orienting boundaries, encode the invariant information of holes and cavities of topological forms.

The dual theory, called cohomology theory, is based correspondingly on the notion of cochains of connectivity, that is, the evaluations of chains on some group of coefficients, like the integers or the reals. In turn, cochains are classified into cocycles and coboundaries respectively.

In this case, two cocycles are cohomologically equivalent if they differ by a coboundary. For example, in the case of de Rham cohomology theory, the cocycles are represented as closed differential forms and the coboundaries as exact differential forms.

One important difference between these two dual theories is that the notion of boundary in homology pertains to the global, whereas the notion of coboundary in cohomology pertains to the local. The whole of calculus may be interpreted topologically via homology and cohomology theory.

The law of variation that activates and enables differential calculus in the setting of these theories can be formulated in the following simple manner: The boundary of a boundary is zero, and the coboundary of a coboundary is zero. Every discrepancy with respect to this law is expressed by means of a topological invariant that characterizes globally the topological figure.

Multi-Connectivity: The Fundamental Group of Contraction Invariance

It is the renowned mathematician Poincaré, who first attempted to probe the connectivity problem of a topological space by using paths, and in particular, loops based at a point of this space. This approach gave rise to homotopy theory, and led to the notion of the fundamental group of a topological space.

The fundamental group at a point of a space is defined in terms of the set of based loops at this point modulo homotopies. The notion of homotopy establishes a homeotic criterion of identity of based loops, which gives rise to a partition spectrum that is based on the notion of invariance under continuous contraction. Contraction is thought of in terms of continuous distortion of based loops and shrinking. Intuitively, if there is no topological obstacle we expect the all based loops at a point are contractible to this point.

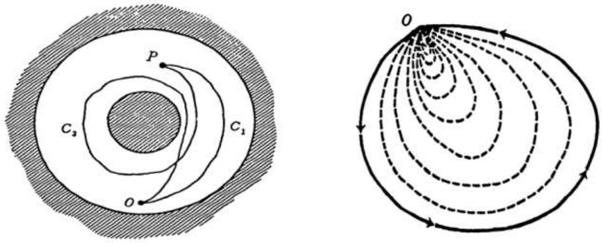


FIGURE 5.19
The notion of contractibility of based loops

The most basic example is the fundamental group of the circle S^1. Note the difference between the circle and the closed disk. The closed disk has a boundary, which is clearly a circle. But, in the absence of the disk the circle stands topologically for a hole. It is equivalent to the erasure of a single point from the plane.

If we think of based loops winding around the circle, then the criterion of their homological congruence depends only on the number of times they are winding around. In this case, the winding criterion serves also as a homotopy criterion, since based loops winding around the same number of times can be continuously contracted to each other—this is the essence of the domain of plasticity. This is not possible for based loops winding around a different number of times, they cannot be neither homologically, nor homotopically equivalent to each other.

Thus, the fundamental group of the circle is the additive group of the integers $\mathbb{Z}$, corresponding to the countable number of the windings. This presents a case where homotopy is abducted from homology, but, in general this is not the case.

For instance, in case there exist two punctures on the topological screen, then the fundamental group is not reducible to the (first) homology group, like in the case of the circle. As a general rule, at first order, homology is the commutative image of homotopy.

In general, the set of equivalence classes of based loops with respect to continuous contraction can be always endowed with a multiplicative group structure, under the operation of composition of paths.

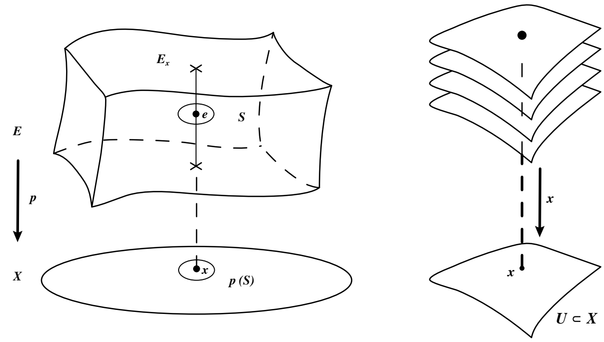
If the topological space is path-connected, that is, if any two points can be joined by a path, then the isomorphic class of the fundamental group does not depend on the selection of the base point, since the respective fundamental groups at two different base points can be made isomorphic. A path-connected space is always connected. The crucial thing is that it is simply-connected, and thus a geometric space, if it has a trivial fundamental group.

The Universal Covering of the Circle by the Helix

A covering space of a topological space—called the base topological space—is a local homeomorphism, such that for each point on the base space, the inverse image of an open set containing this point is a disjoint union of open sets in the covering space lying over the base, each of which is mapped homeomorphically on this open set, as it is displayed schematically below.

In particular, if the base space X is connected, then the fibers of the covering space projection $p : Y \twoheadrightarrow X$ are all homeo-

FIGURE 5.20
Notion of a covering space



morphic to the same discrete space I, such that locally, Y is isomorphic with $X \times I$.

The main idea pertaining to the notion of covering space is that, both paths and loops (belonging to a homotopy class of the fundamental group) on the base space can be lifted uniquely from the base to the fibers of a covering space. In particular, if we consider a based loop at a marked point of the base space, then its unique lift on a covering space is not necessarily a loop, but it is always going to be a path whose starting and ending point belong to the same fiber of the covering space, and which projects to the marked point of the base, where the loop is based.

Let us now imagine the real line $\mathbb{R}$ in the domain of plasticity. The idea is to interpret the exponential $exp: \mathbb{R} \twoheadrightarrow S^1$ as a local homeomorphism of topological spaces. In this setting, the topological real line is thought of as a helix that is continuously covering the circle in each of its windings.

The local homeomorphism $exp: \mathbb{R} \twoheadrightarrow S^1$ is surjective, and the circle is universally covered by the helix, in the sense that, every possible covering of the circle is a restriction of the helical covering. Intuitively, every finite covering of the circle emerges from the helical one, by restricting on a certain integer number of windings.

Every finite such covering is actually a covering of the circle by itself, whereas only the universal one is the covering of the circle by the helix—taking into account the countably infinite number of windings of the helix.

It is important to point out the different interpretations of the exponential (valued on the circle) acquires, depending on our viewpoint. If the viewpoint is topological then, according to the above, the topological real line—though of as a helix—is locally homeomorphic, and covering universally the topological circle.

If the viewpoint is algebraic, and thus structural, then the exponential is the homomorphism from the group structure of the real arithmetic line (under addition) to the group structure of the phases (under multiplication).

These viewpoints can be considered jointly without any contradiction. It is an example of how the topological sheds light on the algebraic, and conversely, how the algebraic structures the topological. This is a recurring theme in algebraic topology.

Conclusively, the exponential map wraps the real line continuously (anti-clockwise) around the circle . Note that the real line R is simply connected, whereas the circle is multiply-connected. This is the objective of every universally covering space, that is, it should be simply-connected.

Finally, let us consider the symmetries of the real line $\mathbb{R}$ leaving the circle S^1 invariant. They are simply expressed by the map $t \mapsto t + k$, where t in $\mathbb{R}$, and k in $\mathbb{Z}$ is the winding number. Thus, we obtain the group isomorphism $S^1 \cong \mathbb{R} / \mathbb{Z}$, which identifies structurally the circle as the group of phases.

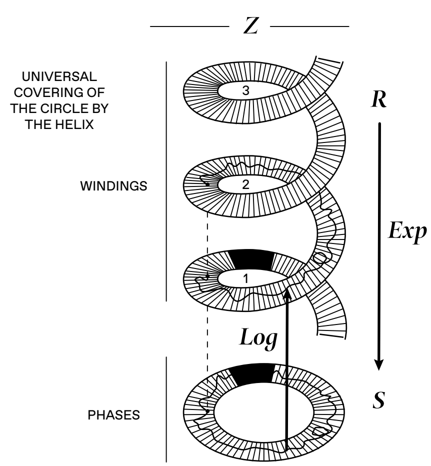


FIGURE 5.21
Exponential covering projection and the universal covering of the circle

6.
The Angel of Geometry:
The Space Art of Angles and Areas

Metronomy of Euclidean Elements: Geometric Topos

The obstacle that Euclid addressed through the development of the Elements (Stoicheia) is the possibility of obtaining measurable geometric magnitudes out of the qualification of an epiphaneia in terms of a metronomy, which amounts to a geometric construction using the ruler (straightedge without division markings) and the compass.

This metronomy is based on the idea that a completed construction in Geometry via the ruler and the compass should correspond either to a consonance in Harmonics, or to a rhythm fitting into a synchronization cycle.

The implementation of this metronomy is based on the notion of Elements, which should be though of as an alphabet that is capable of giving rise to a language that grasps geometric magnitudes out of certain grammatical and syntactical rules pertaining to the interactions and synthesis of these elements under varying combinations on the epiphaneia.

The grasping corresponds precisely to a geometric construction with the ruler and the compass that respects these rules. The grammatical rules are about the morphology of the elements in combinations and synthesis, whereas the syntactical rules are about the ordering of the steps of synthesis giving rise to a geometric construction.

The axioms and the definitions refer to the grammatical morphology emanating from some reasonable hypotheses. The successive ordering of the actions of synthesis that in the Euclidean setting completes a geometric construction refer to the syntactic arrangement of these actions. They are reflected in the ordering of the propositions that give rise to the proof of a theorem expressed deductively. A theorem corresponds to a completed geometric construction that is possible to be orderly synthesized out of the analysis conducted by the elements under the grammatical rules.

The elements that Euclid introduced for this purpose are the following: The point, the straight line/straight line segment, and the circle. A straight line is drawn by the ruler, and a circle is drawn by the compass. The notion of a point as an element is holistic and synthetic. It is holistic because it does not correspond to a dot on the epiphaneia, but to something that does not have any parts, thus it is not amenable to analysis, but only to synthesis.

A point is the meeting point of all lines passing through this point, and any two points define a line passing through them, bounding a linear segment on this line. A point is also the implicit center of a circle drawn by the compass, being implicit because the center does not belong to the circle. The center together with a point on the periphery of the circle bound a straight line, identified as a radius of the circle, which also does not belong to the circle.

The elements of Euclid should not be identified with the objects of sense perception. The latter are not points, straight lines, and circles. They belong to different levels—if the objects of sense perception are on the ground, then the elements are on the imaginary epiphaneia together with their grammatical and syntactical rules.

The purpose is the architectonic bridging of these levels through metaphora, in the sense that the shadows on the ground are lifted up to the epiphaneia through the elements, becoming in this way amenable to geometric construction, which in turn, allows their measurement in terms of geometric magnitudes, that is, areas, lengths, and angles.

After the comprehension of all the possible constructions on an epiphaneia, we have at our disposal all the available means to enter into the domain of solid geometry, conceived as the study of crystallization in space emerging out of the combinations and interactions of these bounding surfaces.

The geometry made out of the elements is neither a kinematical geometry, nor a mechanics. Since the objective is construction through the ruler and the compass, kinematical notions are not allowed to be employed as a means of proof in the context of the Elements. At the same time, this pertains to the eventual limit of the allowable constructions in this context.

For instance, the big unresolved problems of antiquity involving only the use of the ruler and the compass, that is; the squaring of the circle, the trisection of the angle, and the doubling of the cube, determine the limits of this metronomy. The kinematical solutions to these problems gave rise to mechanics, which essentially comprehends a geometric magnitude out of its variation—expressed as relative motion—in relation to other magnitudes.

We distinguish two main notions that underlie the morphology of the relations in the setting of the geometry of Euclid:

The first is the notion of parallelism, that is, the notion of parallel transport of a unit of measurement along the ruler. Recall that the ruler is not marked and not divided ab initio, thus as soon as a unit is chosen, the means of transporting this unit along the ruler are needed.

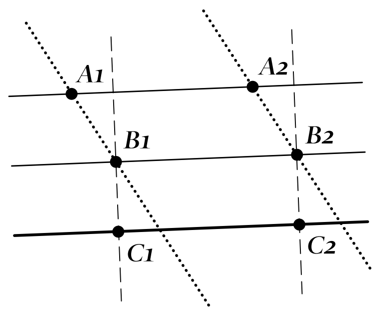


FIGURE 6.1
Notion of parallelism of lines

This requirement gives rise to the axiom of parallelism, according to which parallel lines never meet on the epiphaneia. This is what makes possible the transference of a unit along the ruler by means of a geometric construction. The metaphora of the unit along the ruler through parallelism transfuses the affinity property to the ruler.

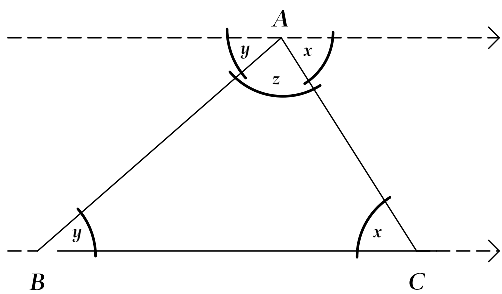


FIGURE 6.2
Notion of angle and equality of angles

The second is the notion of orthogonality, that is, the notion of a right angle between two lines. This morphological qualification of an angle should be thought of in relation to the perpendicular placement of the gnomon. According to the definition 10 in the first book of the elements, orthogonality indeed is lifted up as a relation from the gnomon: "When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands".

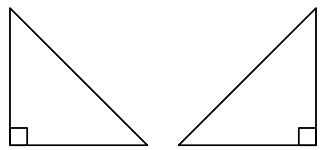


FIGURE 6.3
Notion of right angle and orthogonality

Parallelism—Orthogonality—Homeothesis

We are ready now to present the geometric construction of Thales, which gives rise to homeothesis, interpreted as a proportionality relation among four geometric magnitudes:

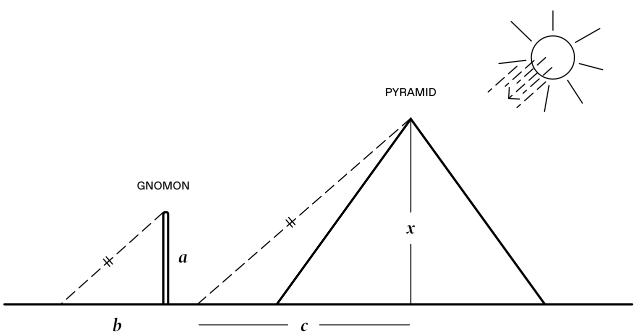


FIGURE 6.4
Gnomon of Thales and notion of analogia

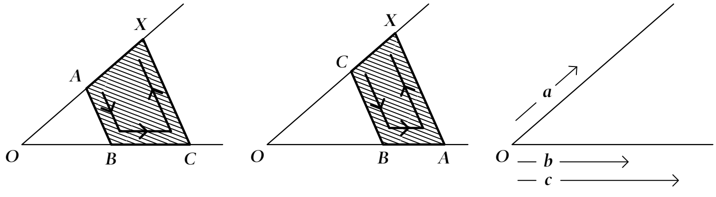


FIGURE 6.5
Proportionality of magnitudes via metaphora

Consider two straight lines intersecting at O. On the sloping line a point A is specified such that $OA = a$. On the horizontal line two points $B, \; C$ are specified such that: $OB = b$, and $OC = c$. Somewhere on the sloping line there exists a point X, such that $OX = x$. The Homeothesis theorem is concisely expressed as follows:

$$a \text{ to } b \text{ is as } x \text{ to } c$$

The above gives rise to the following proportionality relation:

$$a / b = x / c$$

holds, if and only if CX is parallel to BA. The point X is at the intersection of the sloping line with the straight line through C, which is parallel to AB.

The geometric construction is completed through metaphora in three steps: First draw the straight line segment AB, then transfer from B to C, and finally draw the straight line segment CX parallel to AB. We conclude that homeothesis is equivalent to the fact that the triangles OAB and OXC are similar meaning that all corresponding angles are equal to each other, and thus, as a consequence all corresponding sides are in the same ratio.

Henceforth, the geometric construction shows that it is the invariance of the angle formed between the two straight lines intersecting at O, which together with parallelism, captures the proportion among the four magnitudes. According to the original geometric conception, every ratio emerges out of a proportion, the notion of a ratio independently of the proportion that gives rise to it is not meaningful. This contrasts the modern conceptualization that considers ratios as a separate domain of numbers, the rational numbers.

The main difference between these two views boils down to the conceptual difference that a ratio acquires with respect to the notions of unit and unity. In particular, in the ancient conception

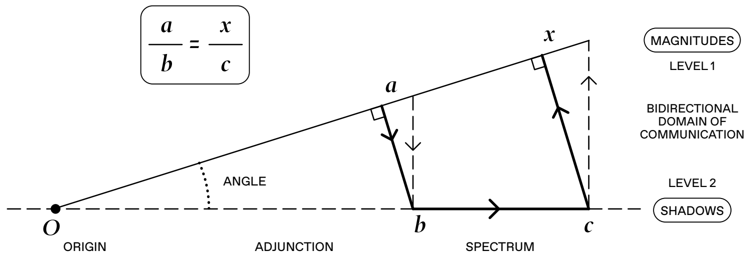


FIGURE 6.6
Proportionality of geometric magnitudes in the form of a natural communication diagram

units are indivisible, they are monads, and thus, the division of units is not permissible. Units can be only transported along lines, and can also enter into relations with other magnitudes.

What is permissible geometrically is the rational division of unity, though of in terms of the circle, with respect to lines intersecting at its center, and as such opening up angles, distinguishing in this way parts of the unity. Therefore, a proportion delineates parts of the unity that share the same angle, and because of this invariance, equality of ratios formed by magnitudes along these intersecting lines can be established through parallelism.

The ancient geometric view is interesting, not only because it implies that ratios are inconceivable independently of an invariant angle that distinguishes only a part of the unity as the prerequisite for a proportion, but because it reveals that rationality is impossible without the existence of an origin *O* - the centre of the unity circle and the meeting point of the intersecting lines forming an invariant angle.

The implication is that such an origin signifies a point—something without parts—in its capacity to break the geometric extension of a line at this point by opening up an angle that reveals part of the unity. The breaking of a line at an origin-point into two linear segments meeting at this point, and forming an angle as the preconditioning of proportionality, means that the origin breaks the ordered continuous progression of time at this point—it freezes time at the origin as the necessary condition for proportionality.

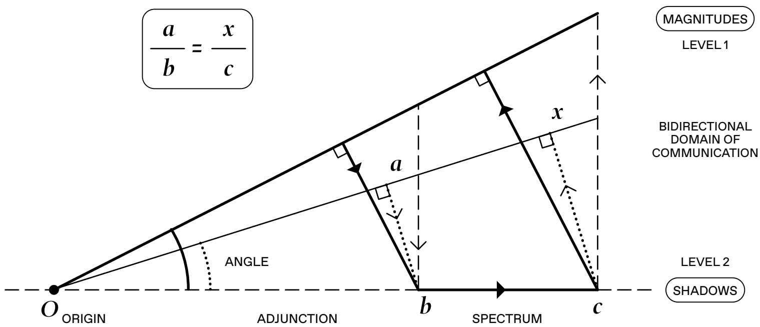


FIGURE 6.7
Notion of origin and fixed angle as geometric conditions for establishing proportionality

How else can the shadows of the gnomon and the pyramid in the Thalesian setting be rationally compared? These shadows have to be considered necessarily at the same time of the day, otherwise they are not rationally comparable. The fixation of a certain time of the day, the intentional freezing of time at a certain instance is what gives rise to an origin.

Recall that the shadows are under a continuous variation, their shapes undergo a continuous metamorphosis during a day. How can a shadow be lifted up on the epiphaneia geometrically, that is, in terms of a geometric magnitude in the metronomy of Euclid? Only if it is a frozen shadow does it become possible to make it measurable by a magnitude, and only if frozen shadows refer to the same time, that is, share a common origin, they become comparable.

Invariance of Angles and Areas

The previous analysis has shown that although motion is not the way to think of in the context of the Elements, since a theorem amounts to a geometric construction, the means of variation are not absent. More concretely, the means of variation are expressed in terms of a varying angle with respect to an origin—conceived as the center of the unity.

The invariance of this angle, under the fixation of two lines intersecting at the origin, is the pre-condition for expressing proportions between length magnitudes on these lines through parallelism. Thus, length magnitudes, being integer-valued, can change, in proportion to each other, only through the change of the invariant angle, only rationally. In this sense, the notion of a length magnitude is subordinate to the notion of an invariant angle delineating a part of the unity.

But, this is not the only means of variation in the context of the Elements, as it is certified by the Pythagorean theorem, which involves areas. We will see that invariant areas offer a complementary way to think about homeothesis and proportion in comparison to invariant angles. Put differently, areas also express means of variation on a par with angles delineating a part of the unity, such that again the notion of a length magnitude is subordinate to the notion of an invariant area.

Grasping this fact lies at the heart of the Pythagorean theorem and makes its interpretation clear. To post-anticipate what follows, the basic idea lies on the abductive mechanical observation that on the epiphaneia an area depends only on rotation, and not on translation. In other words, parallel translation of a line segment along another line does not enclose any area if this segment does not rotate simultaneously.

Thus, under parallel translation and rotation the only area that can be enclosed is the one enclosed only by rotation. In this way, the geometric magnitude of area—through its dependence only on rotation—allows the metaphora of all translated linear segments -independently of their distance- to a single center that synchronizes all of them. The area enclosed depends only on the angle between the initial and the final segment.

In this way, from an areal perspective, the origin of the unity is not only a point where time is frozen for the expression of a proportion, according to the preceding, but it also serves as a center of synchronization of length magnitudes—in terms of the enclosed area magnitude—independently of their distance under translation. Thus, there exists a subtle encoding/decoding bridge between area and angle underlying the potency of geometry. From this viewpoint, the significance of the ancient problem of squaring the circle can be properly appreciated.

Archimedes showed that this bidirectional bridge between area and angle is nothing else than the spiral, which translates from one domain to the other and inversely. But, unfortunately, it is not constructible by means of the ruler and the compass. Thus, the architectonic bridge of the spiral lies beyond the spectra that can be grasped in terms of the Elements.

To be fair, angles and areas can be perfectly grasped, but the architecture and mechanics of this higher order communication between areas and angles that makes Geometry powerful lies beyond geometric constructibility. The domains of angles and areas are separate, despite of the fact that the seed of their bidirectional communication bridge lies in the theorem of Pythagoras.

But, at the same time, it opens up the domain of mechanics in a spectacular manner. This is the essence of the method of Archimedes. The spiral is not understood as a figure in space, but as a bridge for the metaphora between the circular domain of angles and the bilinear domain of areas.

Let us first consider how the notion of an invariant area emerges out of a proportionality relation. We stress the fact that a proportion involves four geometric magnitudes distributed in 2 levels: a and x are vertical heights, whereas b and c are horizontal shadows. Note that, due to commutativity in the multiplication of integer magnitudes, the following holds:

$$a / b = x / c \Leftrightarrow c / b = x / a$$

If we commute the factors, we again obtain four magnitudes distributed in 2 levels: $b, \; a$ refer to the gnomon, whereas $c, \; x$ refer to non-directly accessible object.

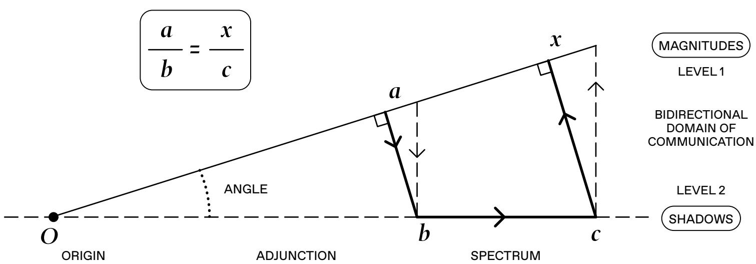


FIGURE 6.8
The domain of shadows constitutes the partition spectrum of geometry

The invariant bilinear product $A = a \cdot c = x \cdot b$ describes geometrically the areas of two rectangles with sides $a, \; c$; and $x, \; b$ respectively, which have the same area due to the proportionality relation. Considering this invariance of area in reverse, we conclude that the equality of ratios in proportion as above, is the expression of leaving some area invariant, identified as the area of equal rectangles. Note the diagonal formation of these rectangles—the vertical height of the gnomon is paired with the shadow of the pyramid, whereas the vertical height of the pyramid is paired with the shadow of the gnomon.

Geometic Mean and the Pythagorean Theorem

We recall now our previous discussion focussing on the notion of the geometric mean. This notion has a meaning only in the context of a geometric progression of magnitudes, or else, in the context of a continued proportion among magnitudes.

The geometric mean of two integer magnitudes k and l in continued proportion is the magnitude $GM = p$ whose square equals the product $k \cdot l$:

$$p^2 = k \cdot l$$

Thus, given a rectangle with sides k and l, the geometric mean determines a square whose area equals that of the rectangle. Thus, the geometric progression is addressed in terms of areas by employing the mean proportional p between k and l, according to the invariant proportion:

$$k / p = p / l$$

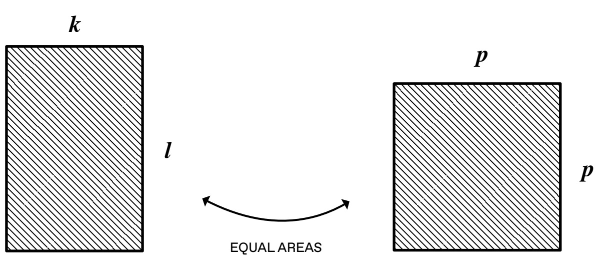


FIGURE 6.9
Equiareal conception of the geometric mean

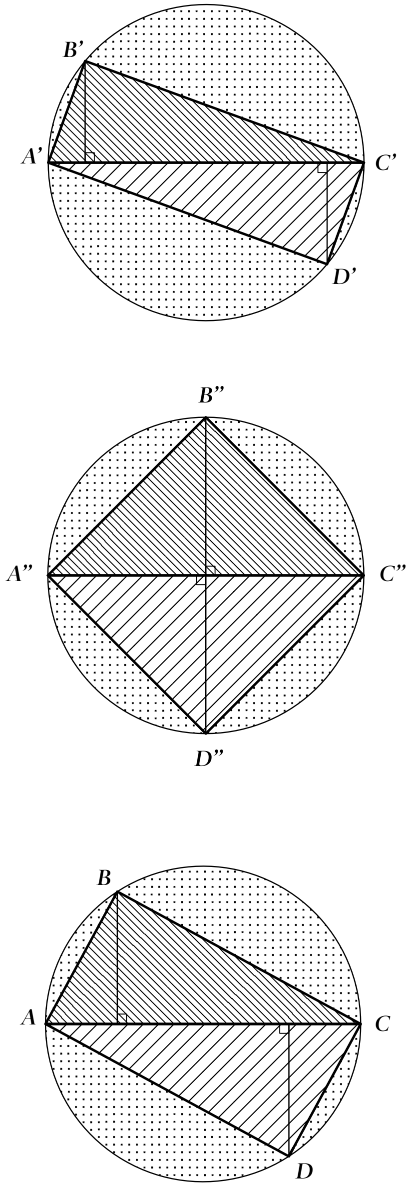


FIGURE 6.10

Invariance of the area of the inscribed rectangle and the theorem of Thales

The relation that the geometric mean engulfs is grasped by means of a rectangle, inscribed in the unity circle, which keeps its area invariant, according to the following figures in succession:

The vertical straight line segment from any point on the periphery of the circle, orthogonal to the horizontal diameter, is the geometric mean of the line segments lying on the left and on the right of it along the diameter.

In this guise, the length of the perpendicular segment emerges—through the geometric mean—as the square root of the area of the rectangle formed by the left and right segments along the diameter in continued proportion, that is, in geometric progression. Thus, the notion of length obtained out of segments in continued geometric progression is an irrational square root.

It is worth thinking about the implications of this irrational square root in relation to the gnomon. Let us start from the symmetric figure in the middle, and consider the vertical line segment as the gnomon and the line segments on the left and right of the diameter as the shadows obtained through a cone of light rays. The unity semi-circle would be the course of the sun in the sky during a day. Then, the geometric mean answers the question how the length of the gnomon should vary in continued proportion with the shadows, such that the area of the rectangle above remains invariant.

Furthermore, since the construction refers to a rectangle inscribed in the unity circle, which keeps its area invariant, the angle formed between the line segments connecting the vertical line of the geometric mean, from any point on the periphery of the circle, with the endpoints of the diameter, is always a right angle.

All right angles are equal, thus due to orthogonality, we immediately see that we obtain three similar orthogonal triangles having all their angles equal. Namely, the big triangle whose diagonal is the whole diameter and the two smaller triangles lying on the left and on the right of the vertical line whose diagonals are the above-mentioned line segments. Thus, from a rectangle, inscribed in the circle, which keeps its area invariant, we always end up with three similar orthogonal triangles, as we see in the following figure:

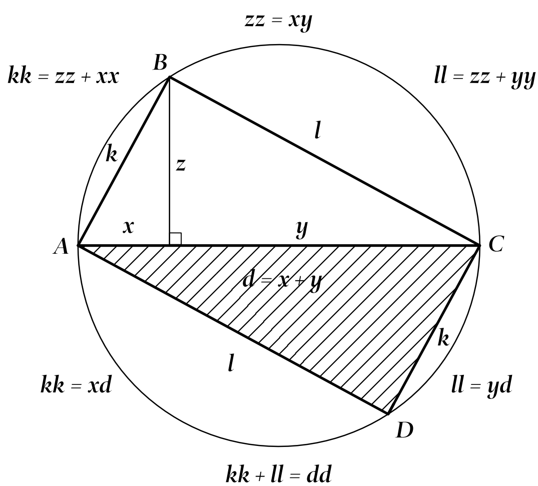


FIGURE 6.11
Derivation of the Pythagorean theorem via the notion of geometric mean

From the similarity of the three triangles, we obtain three geometric means in this configuration. The first is the vertical straight line segment from any point on the periphery of the circle that meets orthogonally the horizontal diameter, as we have seen previously. This is the geometric mean of the line segments lying on the left and on the right of it along the diameter. The second and the third are due to the fact that the angle formed between the line segments connecting the vertical line segment from any point on the periphery of the circle, with the endpoints of the diameter, is always a right angle.

In particular, we have the following: $Z^2 = X \cdot Y$, from the fact that Z is the geometric mean between X and Y. The sum of X and

Y gives the diameter D, i.e. $D = X + Y$. From the similarity of the small triangle on the left with the big triangle, $K^2 = X \cdot D$, thus, K is the geometric mean of X and D. From the similarity of the small triangle on the right with the big triangle, $L^2 = Y \cdot D$, thus, L is the geometric mean of Y and D. Summing up, these two geometric means, we obtain $K^2 + L^2 = X \cdot D + Y \cdot D$, thus, $K^2 + L^2 = D^2$. Moreover, since $K^2 = X \cdot D = X \cdot (X + Y) = X^2 + Z^2$, and similarly, $L^2 = Y^2 + Z^2$.

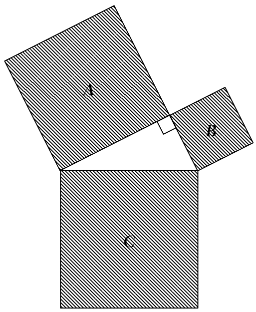


FIGURE 6.12
The Pythagorean theorem as the equality of areas A = B + C

We have shown the Pythagorean theorem applying to all orthogonal triangles. The square of the hypotenuse equals the sum of the squares of the adjacent sides. This relation can be interpreted in terms of areas or in terms of lengths. In terms of areas, if we think of each of the terms as the area of a square, we conclude that the area of the square on the hypotenuse equals the sum of the areas of the squares on the two adjacent sides. This is a commensurability relation between these areas, since the first is expressed by the addition of the two others.

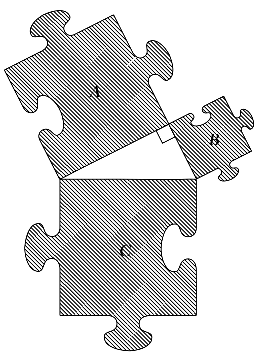


FIGURE 6.13
Invariance incorporated in the Pythagorean theorem

We also immediately observe that we may scale the two sides of the equation by the same factor, which anyway cancels out, so that these areas do not necessarily refer to areas of squares. They could be areas of triangles, or rectangles, or even irregular figures, similarly scaled and proportioned, leaving the relation invariant.

This is a clear indication that the roots of validity of the Pythagorean theorem lie in a deeper domain which pertains essentially to the nature of areas. The method of the geometric mean grasping magnitudes in continued proportion through areas of squares, which made the proof of the theorem quite easy, is just a bridge to express the result in the geometric domain, meaning to make it constructible by ruler and compass in the context of the Elements.

This is particularly important if we interpret the Pythagorean relation in terms of lengths. It says that the length of the hypotenuse is the square root of the sum of the squared lengths of the adjacent sides. Thus, in terms of lengths, the length of the hypotenuse is incommensurable to the lengths of the adjacent sides, since it is a square root, and thus, irrational. But, according to our analysis, this is something to be expected since the notion of length obtained out of segments in continued geometric progression is an irrational square root. Notwithstanding this fact, the square root is constructible by means of the ruler and the compass.

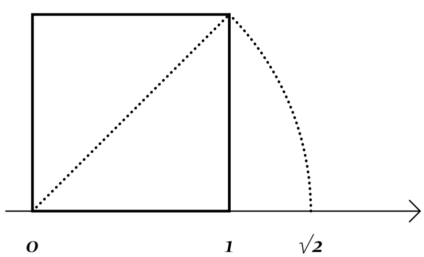


FIGURE 6.14
Constructibility of the square root

Translation-Rotation-Area: Space Synchronization

We need a higher layer of abstraction to grasp the essence of the Pythagorean theorem in relation to the notion of area in geometry. This is justified by the fact that the Pythagorean theorem permeates through all branches of advanced mathematical thinking, including physics, and its applications.

For this purpose, we will ascend the ladder from geometry to mechanics in the attempt to percolate the truth of the theorem from the higher domain through metaphora. The idea is that if we manage to grasp the type of invariance engulfed in this theorem from the domain of mechanics, then we will be able to bridge it back to the geometric domain through the already understood notion of the geometric mean. If the metaphora is successful, then the gain will be the understanding of the geometric notion of area in terms of the invariance it encapsulates, which is of quite significant value especially in architecture.

We consider an orthogonal triangle and identify the point lying on the intersection of the hypotenuse with one of its adjacent sides, as in the figure below, with the center of a circle whose radius is the hypotenuse. Geometrically, the center appears as the origin of this circle, and the intersection of these two line segments opens up an angle. Mechanically, we start to circulate the whole triangle around this center by rotating the hypotenuse until it returns to its initial position. We consider that the rotation takes place at a constant rate. We observe that the hypotenuse traces a full circle being the radius R of this circle. Beyond this, the horizontal side of the triangle, adjacent to the hypotenuse, will also trace another circle of smaller radius r having the same center as the former, and is identified with this radius (Figure 6.15 and Figure 6.16).

In contrast to the hypotenuse R, and the horizontal adjacent side r of the triangle, the other vertical side k, will start sliding along the periphery of the smaller circle of radius r, while rotating with the whole triangle at the same time, until it comes back to its initial position. Note that k is tangent to the circle of radius r at all times while its tip touches the circle of radius R from inside, like a velocity vector of constant length k. The interesting thing here is that while the hypotenuse R, and side r, will sweep out the

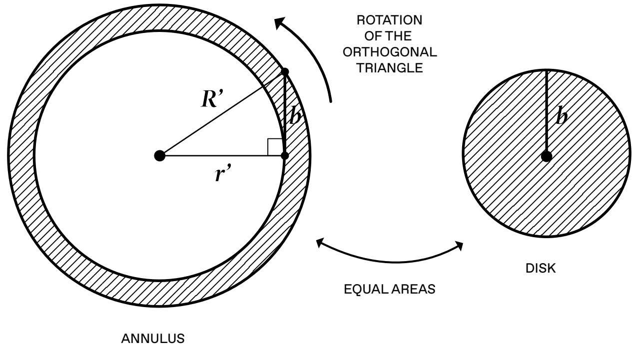


FIGURE 6.15

Mechanical method of equiareal transformation from the annulus to the disk via a Pythagorean orthogonal triangle

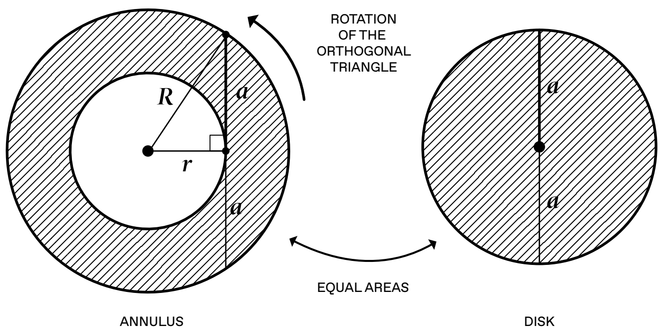


FIGURE 6.16

Sweeping area through rotation and the role of translation

area of a disk, of radius R and r correspondingly, the side k will sweep out the area of the annulus formed between these two disks.

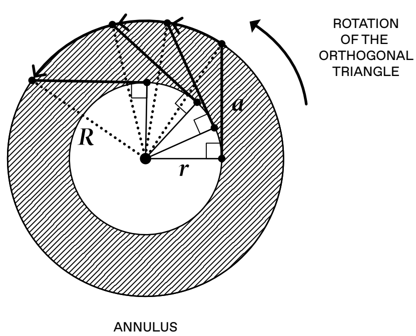


FIGURE 6.17

Tangent segment to the inner circle rotates and translates at a constant rate

If we accept knowledge of the area of a disk, the larger one will have area $\pi \cdot R^2$, while the smaller will have area πr^2. Thus, the area of the annulus will be $\pi (R^2 - r^2)$. Since the triangle is orthogonal, the Pythagorean theorem says that $r^2 + k^2 = R^2$, thus, the area of the annulus will be equal to πk^2.

This result about the area of the annulus swept out by the side k is interesting because it says that the area of the annulus is precisely equal to the area of a disk of radius k.

Note that the annulus is swept out by the combination of two mechanical motions: The first is translation along the periphery of the circle of radius r in the direction of the tangent at each point, and the second is rotation around the point of tangency. The result shows that all the area enclosed by k translating and rotating at a constant rate is exactly the same with the area that would be obtained solely by rotation with respect to a center, that is, with the area enclosed in a circle of radius k.

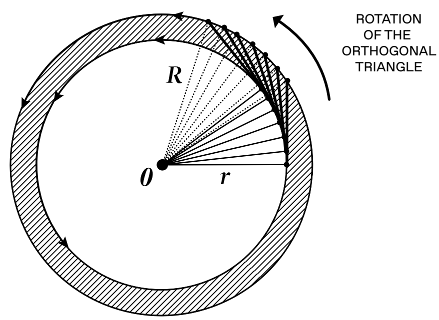


FIGURE 6.18
Translation does not affect the rate of enclosing area through rotation

The invariance embodied in the concept of area is that all area enclosed by combined translation in the direction of motion and simultaneous rotation is the same as the area enclosed only through rotation synchronized with respect to a center.

In this way, area pertains to circulation and synchronization with respect to a center—not an origin in the geometric sense of the term—and is independent of translation in the ordered progression of time.

Simply put, this amounts to synchronization of all tangent vectors (velocities) of constant length k at a joint center, being in resonance with all of them.

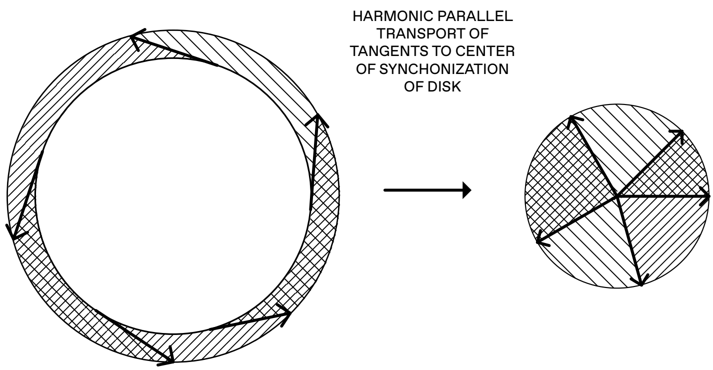


FIGURE 6.19

Harmonic parallel transport to the center of the synchronization disk in momentum space

We call this process harmonic parallel transport. Note that harmonic parallel transport does not require any time in its ordered progression to take place. The harmonic center is in resonance with all tangents in the periphery, and thus synchronizes them all together spontaneously.

Then, the invariant area covered by all these synchronized tangents of the same length at their joint center, that is, the area enclosed in a circle of radius k , equals the area of the annulus swept out during combined translation and rotation of these tangents, according to the above.

Since this notion abstracts something significant about the nature of area, we need now to use it in reverse, and examine if the proof of the Pythagorean theorem rests precisely on this fact. If this is true, the validity of the Pythagorean theorem is the artifact symbolizing and encapsulating geometrically the synchronization—or resonance—aspect of harmonic parallel transport that determines geometrically the nature of area.

Harmonic Parallel Transport

All the difficulty intrinsic to the proof of the Pythagorean theorem disappears if we think about its truth from this higher level in communication with the geometric through metaphora. Recall that initially we had to use the Pythagorean theorem to derive that the area of the annulus will be equal to the area of a disk of radius πk^2. This is the analytic stage. If we remain there we gain no insight in why this is so, why the area of the annulus emerges from the synchronized area pertaining to the resolution spectrum of the disk of its tangents. We can calculate the area of the annulus, but we cannot grasp the invariance encapsulated in the concept of area.

Thus, we need to evaluate the spectrum, and then work in reverse, that is, synthesize the truth of the Pythagorean theorem by the type of invariance engulfed in the nature of area. This is the architectonic bridge back to Geometry that completes the metaphora.

This bridge is enunciated by harmonic parallel transport, which through synchronization with respect to a center of unity, sheds light on why all area enclosed mechanically by combined translation in the direction of motion and simultaneous rotation is the same as the area enclosed only through rotation.

Thus, making use of this argument in reverse, and erecting the bridge back to the geometric domain, that is, by grasping the area of the annulus via the area of the disk through harmonic parallel transport, the Pythagorean theorem emerges as a simple consequence. The synthesis rest on the fact that harmonic parallel transport of all the tangents with respect to the center of their synchronization gives by spanning the area of a disk of radius k.

Thus, since the area of the annulus equals $\pi(R^2 - r^2)$, and also equals—through harmonic parallel transport—the area πk^2 of a disk of radius k, then the Pythagorean theorem follows from the equality:

$$\pi(R^2 - r^2) = \pi k^2$$

which by cancelling out the common scaling factor π, gives:

$$R^2 = r^2 + k^2$$

This transfers an extra layer of sophistication onto our equally-tempered epiphaneia of the disk. The idea is that this epiphaneia is capable of recording area synchronically and invariantly with respect to a center that is not explicit in advance. Since the tangential component in the direction of motion following the ordering aspect of time does not have any impact on the rate of change of area, it all depends mechanically upon rotation with respect to this center, as well as upon the rate of this rotation, that is, how fast the winding takes place.

Thus, what matters referring to invariant area is a variable angle, and the rate of change, i.e. the differential of this angle, independently of translation. Concomitantly, the rate of change of rotation per unit area gives rise to the notion of curvature.

This realization opens up three new pathways to qualify and strengthen the capacity of our epiphaneia, which we will navigate in what follows. The first is differential calculus leading to differential geometry; the second is the concept of imaginary numbers that in combination with the first leads to complex analysis and complex geometry; and the third is mechanics based on the invariance of area. These pathways are not straight and independent. Rather they intermingle and facilitate the encoding and decoding bridges required for different types of metaphora between the harmonic and the geometric domain.

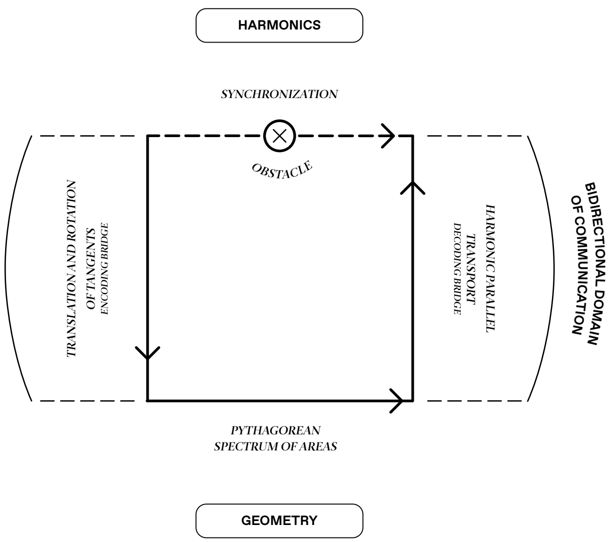


FIGURE 6.20
The architectonic scaffolding of the metaphora between the domain of harmonics to the domain of geometry established by the Pythagorean theorem

Conic Sections

Consider our epiphaneia as a geometric one, equipped with the gnomon placed orthogonally to it, as in the original setting of an upright stick bearing a sundial. We would like to think of the gnomon as a measure of solar time. During the course of a day, the shadow of the gnomon is moving following the sun. Consider the sun's orbit as a semi-circle during the course of a day. Then, as the sun rises and sets, a branch of a hyperbola is traced out on the epiphaneia by the tip of the shadow of the gnomon, according to the figure:

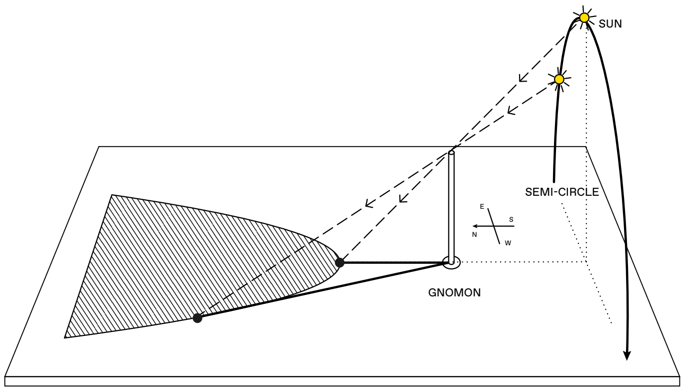


FIGURE 6.21
The shadow of the gnomon traces out a hyperbola on the epiphaneia

The study of the conic sections originates in the geometric understanding of the shadow of the gnomon. The idea is that a unified treatment can be obtained if our epiphaneia is thought of as a cutting plane on a cone. After the masterly work of Apollonius on conic sections the cone should be always considered as a double cone.

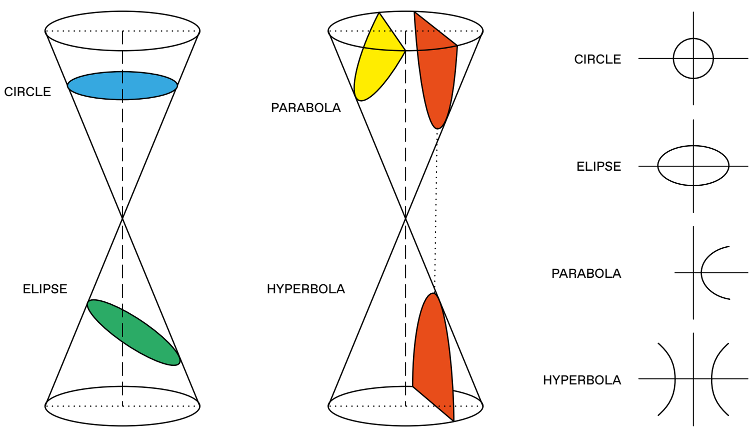


FIGURE 6.22
The conic sections

The pointing tip of the gnomon is considered as the vertex of the cone. The double cone arises by the idea of harmonically conjugating the simple cone. This may be thought of as a mirror image, obtained by reflection of the simple cone with respect to its vertex. In the double cone, the cutting epiphaneia delineates both branches of the hyperbola, and symmetrizes the semi-circular orbit of the sun to a full circle.

An important question pertains to the nature of the cone, and this is related with the propagation of light rays—the cone of light—as well as the transport of the source from the sun to the tip of the gnomon, serving as the vertex of the double cone. It is with respect to this vertex that the shadows are qualified geometrically on our epiphaneia—the cutting plane.

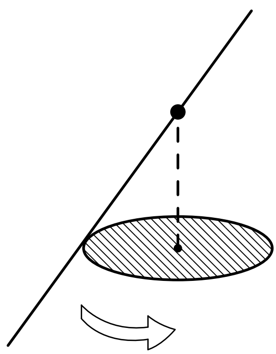


FIGURE 6.23
Mechanical generation of the double cone

Geometrically, the double cone is generated as follows: We imagine a horizontal geometric disk and an inclined line segment—called the generatrix—being tangent to the disk. Then, if this line segment rotates around the disk while it holds fast at the point lying exactly above the center of the disk, the rotation will generate a cone above and below this point, identified as the vertex of the double cone. In this function, the rotating inclined line is called the directrix.

If we intersect the double cone with our cutting-through screen in various positions the conic sections emerge as follows:

Conic Sections	*Name*
Ortogonal to the central axis of the cone	Circle

Conic Sections	*Name*
Between right angles to the central axis of the cone and parallel to the generatrix	Ellipse
Parallel to the generatrix	Parabola
Between a plane parallel to the generatrix and a plane parallel to the central axis	Hyperbola

The geometric topos of points on our screen/epiphaneia, for each conic section, is specified as follows:

Geometric Topos of Points	*Conic Section*
Same distance to the center	Circle
Distances to two given foci have a constant sum	Ellipse
Same distance to a focus and the directrix	Parabola
Distances to two given foci have a constant difference	Hyperbola

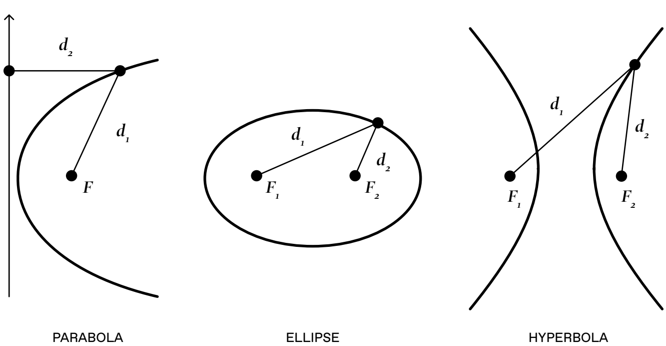


FIGURE 6.24
Characterization of the conic sections

An ellipse is a simple closed curve and it can be traced by a point moving in a plane in such a way that the sum of its distances from two other fixed points—called the foci of the ellipse—is constant. Thus, the sum of the distances F_1P and F_2P is invariant in the case of the ellipse.

The parabola is the geometric topos of points P such that the distance from the directrix to P is equal to the distance from P to a fixed point F—called the focus of the parabola.

The hyperbola is the geometric topos of points whose distances to the foci have an invariant difference. As it is clear, the hyperbola has two symmetric, equal, and infinite branches.

The conic sections, after Apollonius, are also graspable in terms of an invariant ratio, called the eccentricity of the conic section.

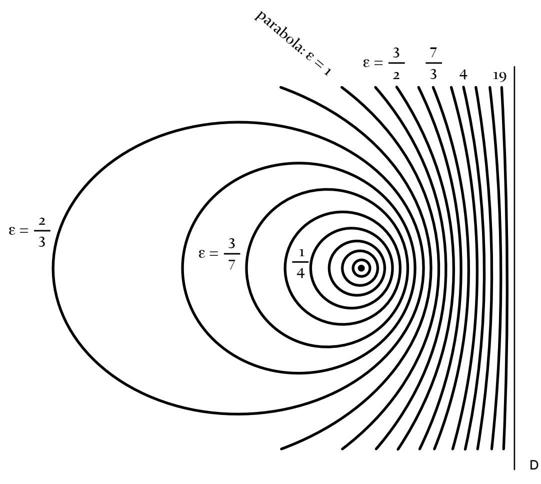


FIGURE 6.25
The notion of eccentricity in the characterization of conic sections according to Apollonius

Given the focus, F, the directrix d, and a point on the conic P, eccentricity e is defined by the following ratio:

$$e = r/a$$

where $r = PF$ and $a = PD$.

For a parabola, since $r = a$, then $e = 1$. For an ellipse, since $r < a$, we have that $e < 1$. For the hyperbola, since $r > a$, we have that $e > 1$. Clearly, the circle has zero eccentricity. It is this eccentricity that provides the etymological ground for the respective names of these three conic sections, in relation to the act of throwing that characterizes their respective symptoms:

e	*Conic Section*	*Throwing /Symptom*
$e = 1$	Parabola	Equal to
$0 < e < 1$	Ellipse	To fall short of
$e > 1$	Hyperbola	In excess of

Trapezium Invariance of Double Cone and Equiangular Spiral

Let us focus our attention now from the conic sections to the double cone. As we pointed out already, the vertex of the double cone is identified with the tip of the gnomon that throws the shadows.

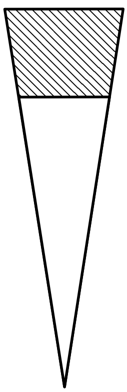


FIGURE 6.26
Trapezium increment of growth on a cone and invariance of shape

What is of interest is the modular substitution of the sun by the tip of the gnomon—by means of parallel transport of the rays at the tip—such that the spectrally qualified shadows on the epiphaneia give rise to the family of conics. Recall that the velocity of light is constant in all directions.

What is special with the cone is that its geometric shape remains invariant under incremental addition, as depicted in the adjacent figure:

Of course, the same holds symmetrically for the double cone. The increment of growth on the cone is a trapezium. This means that the cone grows retaining its shape.

In other words, it remains always self-similar, and thus invariant, under adjoining to it a trapezium. This naturally leads to the notion of gnomonic growth, where the trapezium is conceived as an extension or contraction gnomon for the cone itself.

In this sense, the notion of the gnomon—as a right angle—with respect to the geometrically-cutting epiphaneia/screen, if transcribed in terms of the double cone where the cutting takes place, gives rise to the notion of the gnomon of growth of the cone itself. This is what characterizes the invariance of the double cone.

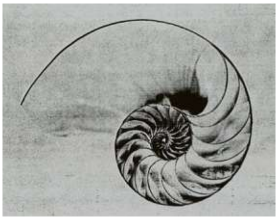


FIGURE 6.27
Modular substitution of gnomonic invariance from the cone to the logarithmic spiral

(Onofrio Scaduto, 1998, Wikimedia Commons, CC BY-SA)

A further interesting metaphora pertains to the modular substitution of the gnomonic invariance of the cone by the gnomonic invariance of the logarithmic spiral pertaining to the fact that it grows retaining its shape invariant. The logarithmic spiral may be simply obtained by first bending, and then coiling a cone. The same transcription of gnomonic growth from the cone to the spiral takes place, in case that the growth of the cone is scaled by an invariant ratio on one of its sides in comparison to the other.

Optical and Acoustic Symptoms of the Conics

The conic sections may be also understood from an optical viewpoint pertaining to the reflection of light on the epiphaneia. This approach, based on the notion that the angle of reflection equals the angle of incidence of a light ray, leads to an understanding of the conics in relation to their capacity to concentrate or reflect light rays or sound beams. The great Archimedes, according to the legend, devised a parabolic mirror to concentrate the rays of the Sun at the focus of the mirror.

Conic Section	*Reflection of Rays from the Focus*
Circle	Back to the center of the circle
Ellipse	Into the focus
Parabola	As a parallel outgoing ray
Hyperbola	As if coming from the other focus

Because of the reflective property of parabolas, they are used in automobile headlights or searchlights. Note that the mirror in each headlight has a curved surface formed by rotating a parabola about its axis of symmetry.

If we place a light at the focus of the mirror, it is reflected in rays parallel to the axis. In this way, a straight beam of light is formed. The opposite principle is used in the giant mirrors of reflecting telescopes, radar transmitters, solar furnaces, sound reflectors, and in satellite or radio wave dishes. With these instruments the beam comes toward the parabolic surface and is brought into focus at the focus point.

The reflective property of an ellipse is used in the creation of what are called "whispering galleries". If you are at one focal point in a room shaped in the size of an ellipse—imagine an ellipsoidal roof—you will be able to hear the whispers of a person located at the other focal point, even in the case that other conversations are taking place in the same room.

The hyperbola is the curve that a shock wave gives rise to. When an airplane flies faster than the speed of sound, it induces a shock wave, which is heard at the screen as a "sonic boom". The shock wave has the shape of a cone with its apex located at the front of the airplane. It intersects the screen in the shape of the hyperbola.

Kepler Laws in Momentum Space and Unification of Conics

The invariance embodied in the concept of area is that all area enclosed by combined translation in the direction of motion and simultaneous rotation is the same as the area enclosed only through rotation synchronized with respect to a center.

In this way, area pertains to circulation and synchronization with respect to a center—not an origin in the geometric sense of the term—and is independent of translation in the ordered progression of time. Simply put, this amounts to synchronization of all tangent vectors—instantaneous velocities—of constant length k at a joint center, being in resonance with all of them (Figure 6.28).

We have seen that harmonic parallel transport allows the synchronization of all tangents at the center. In other words, the center is in resonance with all tangents in the periphery, and thus

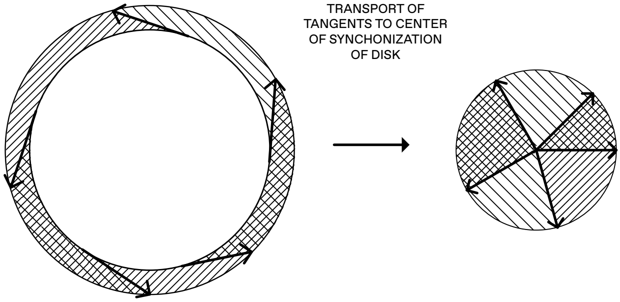


FIGURE 6.28
Synchronization of tangents in momentum space through harmonic parallel transport to the center of the disk of momenta

synchronizes them all together spontaneously. Then, the invariant area that is covered by all the synchronized at this center is precisely the area enclosed in a circle of radius k . In turn, it equals the area of the annulus swept out during combined translation and rotation of these tangents.

In the above setting, we have assumed that all tangents have the same length. Let us think from the perspective of mechanics and consider the tangents as velocity vectors. A natural question is what are the implications of harmonic parallel transport in the case of velocities that change their magnitude, that is, in the case of varying length of the tangents according to some condition. This is of importance because it demonstrates that the harmonic center of synchronization is not by necessity the center of a geometric circle or a geometric disk.

For this purpose, we consider Kepler's first two laws of orbital motion of the planets around the sun.
According to these laws:

(a) Each planet moves in an elliptical orbit with the sun at one focus;

(b) The line segment from the sun to any particular planet sweeps out equal areas in equal times;

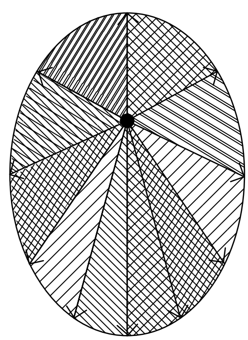


FIGURE 6.29
Elliptical Keplerian orbit and the invariant law of equal areas at equal times

From the first law, the orbit of a planet is an ellipse, where the sun is located at one focus of this ellipse. The elliptical orbit is a closed orbit, whose particular geometric shape is obtained as a consequence of the second law, that is, the line connecting the sun with a planet covers equal areas at equal times. This is the condition that we intend to focus on.

The condition implied by the second law is that the rate of change of area is constant. We think again of the possibility that the area can be actually be swept out by velocity vectors if synchronized with respect to a center.

Consider a velocity vector directed tangentially to the orbit, so that its direction changes continuously. Since the orbit is elliptical with the sun at one focus, it is clear that the magnitudes of these velocities cannot be constant. The question is if all these velocity vectors can be parallel transported to a center, such that they generate all the area by rotation around this center.

This is exactly the fact that it is vindicated by the condition that equal areas are swept out in equal times, as it can be seen in the following figure:

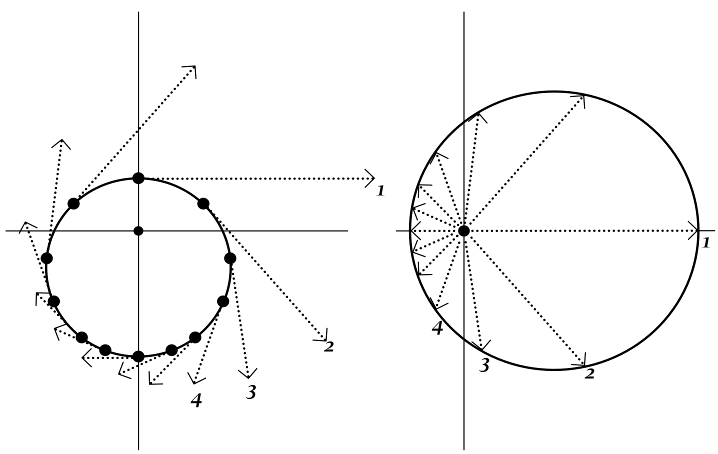


FIGURE 6.30
Harmonic parallel transport of tangent velocities at the ellipse to the synchronization center of a disk whose actual center is translated to the synchronization center

The important thing to point out is that all velocity vectors are synchronized with respect to a common center by harmonic parallel transport, but this center is not the center of a geometric circle.

It is located within the closed curve that is generated by the lengths of the velocities, which is actually a circle, but the center of this circle is not the center of synchronization. Having grasped this fact, we may use the argument in reverse. It shows that in case that the velocity vectors sweep out a circle synchronized with respect to a center inside this circle, but different from its center, then the orbit to which they are tangent is an ellipse.

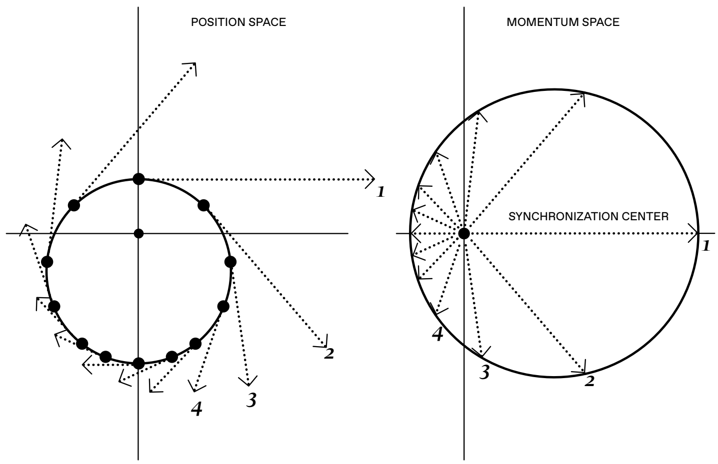
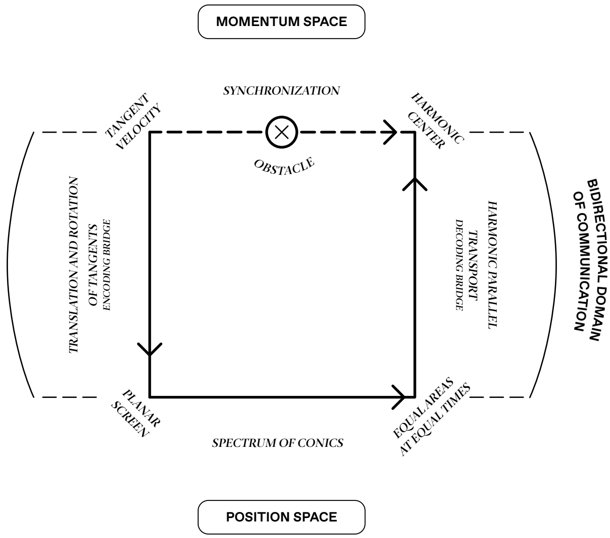


FIGURE 6.31
The position space is an ellipse, whereas the momentum space is a circle whose center is translated to the synchronization center of momenta

We call the topos of velocity vectors, where they synchronize all together in a circle, with respect to their center of synchronization, the momentum space. Therefore, the condition that the rate of change of area is constant is equivalent to a circular orbit in momentum space, whereas it is an ellipse in position space. The subtle point is that the center of synchronization in momentum space is not the center of the circle in this space.

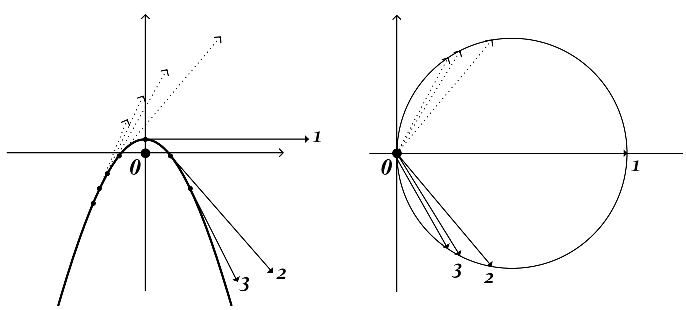
FIGURE 6.32
Architectonic scaffolding for the metaphora between position and momentum space established by the Keplerian laws



Thus, harmonic parallel transport of directed tangent linear segments—the instantaneous velocities—works equally well in case that the lengths of these segments are varying according to the law that the rate of change of area is invariant—equal areas at equal times. Inversely, this law is equivalent to a circular orbit in momentum space, where all the tangent segments synchronize at a center that is different from the center of this circle.

This fact is important, because it shows that the metaphora from position space to momentum space allows the unified treatment of all conic sections.

FIGURE 6.33
Harmonic parallel transport of momenta and synchronization in the case of a parabola



Consider the case of an open parabolic orbit where the rate of change of area is invariant—for instance the orbit of a satellite. From the figure below, it is clear that it can be thought of as the limiting case of an elliptical orbit whose apogee recedes to infinity, such that the length of the tangent velocity at apogee tends to zero.

In momentum space, the orbit is a closed circle whose center of synchronization is not inside, but on the periphery of this circle. Thus, in momentum space the orbit is a circle, where the center of synchronization of all the velocity vectors lies on the periphery of this circle, whereas in position space the orbit is a parabola.

This is a strong indication that we may also treat the case of a hyperbolic orbit in a similar unifying manner, from the perspective of harmonic parallel transport that realizes what we called the momentum space. We expect that in this case, the condition that the rate of change of area is invariant is equivalent to a circular orbit in momentum space, such that the center of synchronization lies outside the circle.

This is very interesting because it demonstrates that the center of synchronization, not only is not restricted to be inside or at the periphery of the synchronizing circle of tangent velocities, but it can also be at the outside of it.

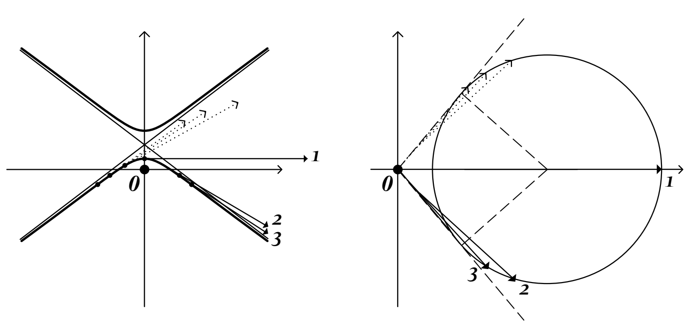


FIGURE 6.34
Harmonic parallel transport and synchronization of momenta in the case of a hyperbola

This pertains to the significance of the architectural inside/outside distinction with respect to the notion of a center of synchronization of velocities in relation to grasping the distinctive role of the conic sections in position—geometric—space. More concretely, the area enclosed as a magnitude, is swept out by velocity vectors that are synchronized with respect to a center—according to the law of equal areas at equal times.

This metaphora shows that harmonic parallel transport gives rise to a circular orbit in momentum space, whose center of synchronization can be either inside the circle, or outside the circle, or even at the periphery. Then, each of these cases gives rise to a different conic section, observed in geometric space in an area-preserving way, thus allowing us to understand how they emerge and how they are unified all together under the notion of harmonic parallel transport.

It is worth reflecting on the power of this method, which essentially abstracts the notion of area from its underpinning in geometric space in an invariant way. Not only it provides an immediate intuitive proof of the Pythagorean theorem, but it leads to a deep understanding of the conic sections under the requirement of equal areas at equal times.

Moreover, it shows that the notion of a center of synchronization is not always fixed at the center of the corresponding disk, but it is a variable and floating notion, that can be—depending on the conditions—within the disk, at the periphery of the disk, or even outside the disk.

Mechanics: Fulcrum and Archimedean Law of the Lever

The notion of a fulcrum originates in Mechanics and traces back to the Archimedean conception of an equilibrium condition achieved by kinematical means.

There are two basic ideas involved here: The first is that the equilibrium condition is derived kinematically as a consequence of area preservation with respect to the fulcrum; The second is that the equilibrium condition is derived in terms of the product of two variables, which are inversely co-related to each other, that is, they are subject to the uncertainty principle.

The notion of the lever based on a fulcrum, which constitutes the mechanical instantiation of the above idea pertaining to a fulcrum, provides the means to realize the equilibrium condition kinematically through the circle. This takes place if we think of the lever as a diameter of a rotating disk whose center is allowed to float along the diameter due to "weights" attached at its endpoints. The bigger the weight at one of the endpoints, the smaller the length of this endpoint of the lever-diameter from the fulcrum, such that the area made of their product is preserved.

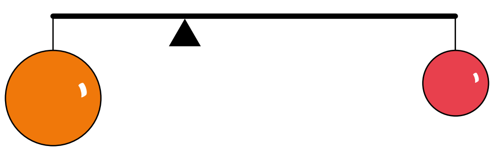


FIGURE 6.35
Mechanical notion of lever based on a fulcrum

From this viewpoint, the combined role of a lever balancing on a fulcrum amounts to area preservation under proportionate re-scaling of the inversely correlated variables, that is, the weights and the lengths.

There is a further qualification in relation to the notion of a weight, that can be described as follows mechanically: Imagine a homogeneous bar of length $2m + 2n$, which is placed symmetrically on a balance line. Therefore, a "weight" $m + n$ is distributed to the left of the fulcrum, and an equal weight is distributed to the right.

At the next stage, consider a slicing of the bar, which splits it into two bars of weights $2m$ —on the left—and $2n$ —on the right -. Replace each of those bars by single weights $2m$ and $2n$ concentrated at their respective centers of gravity.

Then the concentrated weight weighing $w = 2m$ is of distance $d = n$ to the left of the fulcrum, while the concentrated weight weighing $w' = 2n$ is of distance $d' = m$ to the right of the fulcrum. By the postulates of Archimedes, we obtain that this is in equilibrium:

$$w \cdot d = w' \cdot d'$$

The two weights $w, \ w'$ on either side of the fulcrum being at distances $d, \ d'$ from the fulcrum respectively, are in equilibrium if and only if the above condition holds. But, this means that the area defined by the corresponding pairs of $w, \ d$, remains invariant.

Moreover, since it is the product that is preserved at the equilibrium condition, this invariant relation corresponds geometrically to a rectangular hyperbola, where the preserved bounded areas under variation of $w = y$ and $d = s$ are areas under the hyperbola $y = 1/s$, which remain invariant under proportionate stretching and contraction by the same scaling factor (Figure 6.37).

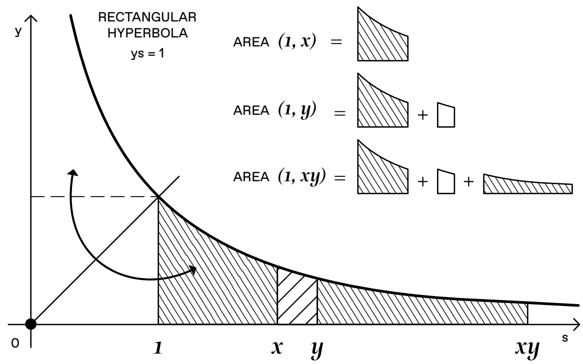


FIGURE 6.37

The invariance of the law of the lever is expressed geometrically through the areas under the rectangular hyperbola

We re-express the law of the lever—the equilibrium condition—with respect to a fulcrum as follows:

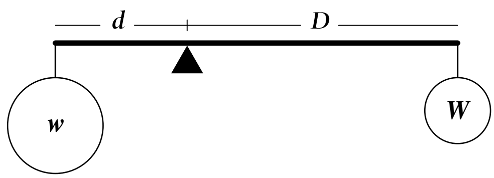


FIGURE 6.38

$$W \cdot d = w \cdot D$$

Reciprocals and Invariance of Area

If we abstract from the standard mechanical context, the law of the lever—through the preservation of area—can shed light on any situation that an equilibrium condition—as the neutrality condition pertaining to the uncertainty principle governing two reciprocally related variables—is established through a fulcrum—a center of "mass"—thought of as a center of synchronization. For this purpose, a necessary condition is that the reciprocally correlated variables commute at least infinitesimally.

It is worth explaining in more detail the uncertainty principle referring to two reciprocally correlated variables α and ω in terms of areas displayed under the rectangular hyperbola $\alpha \cdot \omega = 1$, where $\alpha = s$ is considered to vary on the horizontal axis, and $\omega = y$ is considered to vary on the vertical axis.

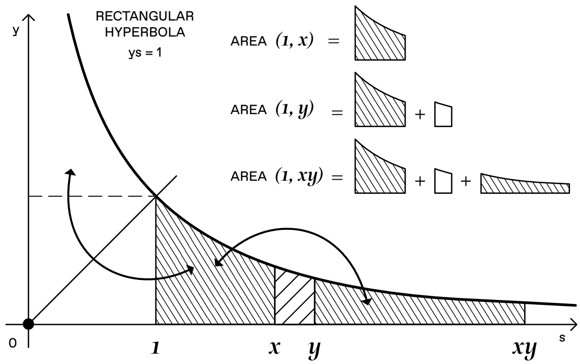


FIGURE 6.39
The uncertainty principle for two reciprocally correlated variables

Consider an area-preserving map, which contracts in the horizontal direction and extends in the vertical direction in proportion. The contraction in the horizontal direction is though of as gaining information about α since its range of possible values is more narrow in comparison to the initial situation.

Since, area is preserved the narrowing in the range of α should be equivalent to a proportionate stretching of the range of values of ω, that is, losing information about ω. The more the range of possible values of α shrinks the more the range of possible values of ω expands, always in proportion to each other.

Consider now the area-preservation as a mapping:

$$\psi : (\alpha, \omega) \rightarrow (A, \Omega)$$

expressing the law of the lever with respect to a center of synchronization, such that the preserved area is expressed via the following integrals:

$$\oint \omega \, d\alpha = \oint \Omega \, dA$$

where the differential pertains to the horizontal variables α , A , and the integration contour is a simple closed curve in both (α , ω) and (A , Ω) , denoted by c , C , as pertaining to the differentials $d\alpha$, and dA , respectively.

Then, each of these integral represents the work done by ω , and Ω , in a circuit of variation of $d\alpha$ and dA respectively, such that $C = \psi (c)$. Each of the contour integrals gives an area that is preserved by ψ .

These areas are the areas enclosed within the circuits c , C , respectively, which are equal to each other. In turn, since the corresponding amounts of work are equal, and the sum of these works is zero, we obtain the principle of energy conservation by the preservation of area referring to these processes.

In this case, the area-preservation mapping:

$$\psi : (\alpha , \omega) \rightarrow (A , \Omega)$$

is called a symplectic mapping.

In this sense, the law of the lever is abstracted in the notion of a symplectic mapping that preserves areas in (α , ω) , and (A , Ω) , according to:

$$\oint \omega \, d\alpha = \oint \Omega \, dA$$

The Uncertainty Principle of Light: Symplectic Optics

We consider any optical device, irrespectively of its particular function, as a black box. This device can be a telescope, or a microscope, or even, a binocular—its precise specification is not relevant to the argument. The idea is how we may abstract from the particular details of optical devices—treating them as black boxes—by thinking of them in terms of the law of the lever, or equivalently, in terms of a symplectic mapping that preserves areas.

For this purpose, we have to consider a ray of light that passes through such a black box. Let α, $\hat{A}$ denote two parallel vertical axes, standing for the vertical axes of intersection of a light ray before and after entering the black box.

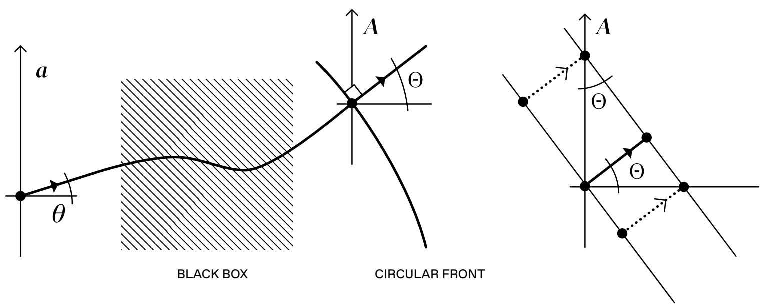


FIGURE 6.40
A ray of light passing through a black box and the symplectic area invariance

The time elapsed from the entrance to the exit of a light ray from the black box is denoted by $T(\alpha, A)$. If we consider the constant velocity of light c to be unity, then the length $L(\alpha, A)$ equals $c \cdot T(\alpha, A)$. Taking the velocity of light 1, we have that $L(\alpha, A) = T(\alpha, A)$.

Let θ be the angle that the light ray makes with the horizontal axis at its entrance point characterized through its sine function, that is $\omega = sin\,\theta$. Similarly, Θ is the angle that the light ray makes with the horizontal axis at its exit point from the black box, characterized by $\Omega = sin\,\Theta$.

The idea is that the mapping induced by the elapsing time between the entry and exit of a light ray from the black box, that is:

$$(\alpha, \omega) \mapsto (A, \Omega)$$

$$(\alpha, sin\,\theta) \mapsto (A, sin\,\Theta)$$

is a symplectic mapping, meaning that it is area-preserving.

Every optical device can be characterized by this area-preservation property. Let us imagine the front of all light rays originating from α. Then, this front is a circle upon exiting from the black box, such that each light ray is orthogonal—as a radius—to the circular front.

Light rays propagate with the same velocity in all directions, thus, if we consider that α is the center, all points on the periphery of this circle are equidistant to α. Essentially, α is thought of a center of synchronization for all points in its periphery.

If $c\,dT$ denotes an infinitesimal translation of the circular front, then upon meeting the $\hat{A}$ axis is displaced by dA, where:

$$dA = c \cdot dT / sin\,\Theta$$

Thus, we obtain:

$$c \cdot dT / dA = dL / dA = sin\,\Theta = \Omega$$

Consider the following figure, where the function of a magnifying optical device is displayed, from the perspective of a symplectic area-preserving mapping as above. The magnification is explained by means of the uncertainty principle governing the variables co-related symplectically.

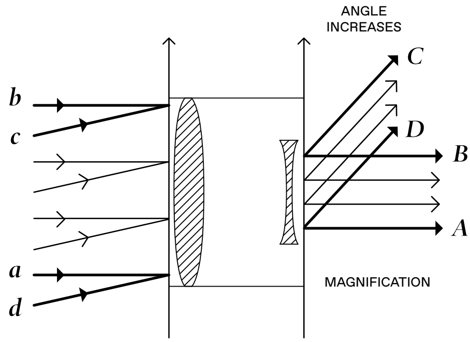


FIGURE 6.41
Symplectic area invariance for a magnifying optical device

If a beam of rays enters the device and upon exiting it gets more narrow, this means that the angle between the rays increases, that is, we have the phenomenon of magnification. Thus, by the area-preservation property, whenever a parallel beam of rays narrows, we have magnification on the other side.

It is quite clear to see that this pertains to contraction in the $\hat{A}$ direction followed by extension in the Ω direction—such that area is preserved—which is precisely the magnification.

Differential Calculus: Tangent Slope and Integration Area

As soon as we become aware of the fact that the notion of differentiation of a smooth function with respect to time pertains to its instantaneous rate of change called the derivative, that is, to its tangent slope or tangent vector; whereas the notion of integration pertains to the area enclosed under the curve of the function within the boundaries of integration; it becomes clear that these constitute inverse bridges between the domain of angles (though of as infinitesimally changing) and the domain of areas.

In this manner, we may conceptualize a mapping (natural transformation) of the Pythagorean or the Archimedean setting to the abstract setting of differential calculus.

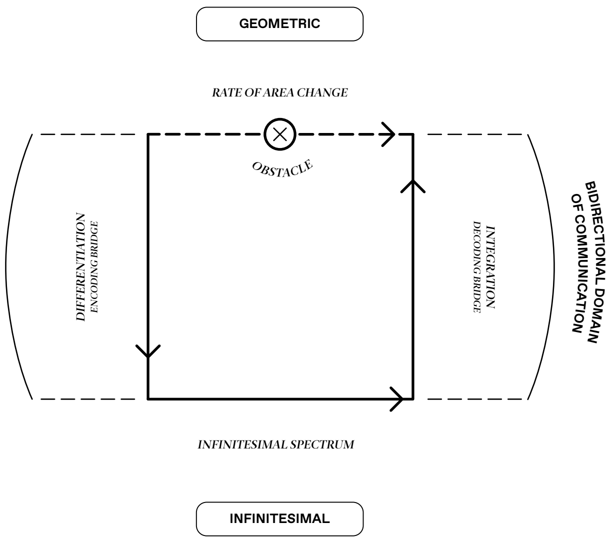


FIGURE 6.42
Metaphora from the geometric domain of areas to the infinitesimal domain via the architectonic scaffolding of differentiation/integration

The fundamental theorem of calculus for one real variable, expressing precisely the fact that differentiation and integration are inversely co-related encoding and decoding bridges, where the first performs the analysis and the second the synthesis for what is spectrally observable from the behaviour of a smooth function, is formulated as follows:

$$d/dt \int_a^t f(s)\,ds = f(t)$$

Consider the figure below:

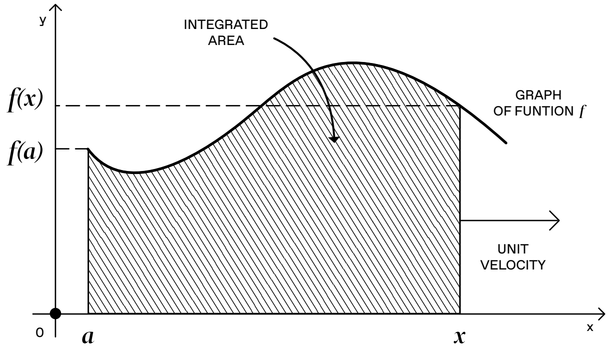


FIGURE 6.43
Geometric interpretation of the fundamental theorem of calculus

It displays a function segment, which we think of that is moving with unit velocity in the direction orthogonal to itself. The theorem says that the rate of change of the area enclosed equals the altitude $f(x)$ times its velocity. Thus, the function segment displayed sweeps the area at a rate equal to its length $f(x)$ times its velocity. We think in this setting of the variable x as time t, pertaining to the continuous and ordered progression of time along the real line.

In reverse, we conclude that the definite integral of a real valued function gives the sum-total of accumulated area change, where the rate of change of area is given by the function being integrated.

Neutral Geometric Element: The Real Exponential Function

The fundamental theorem of calculus—viewed through the encoding and decoding bridges of differentiations and integration respectively—leads to the design of a function whose tangent slope equals the altitude at all times, namely the real-valued exponential function:

$$t \mapsto y(t) = exp(t) := e^t$$

where e, the base of the function, is Euler's constant, whose meaning will be explored from a novel perspective shortly.

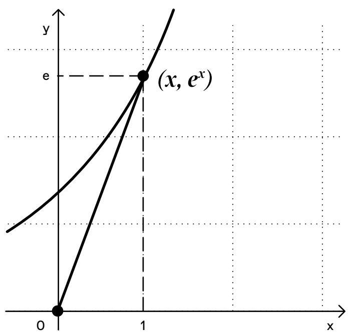


FIGURE 6.44
The graph of the real exponential function

This function bears the unique property—among all real-valued functions—that its tangent slope, i.e. its derivative, equals the altitude $y(t)$ of the graph of the function at all times. In other words, the exponential functions remains invariant under differentiation:

$$d/dt \;\; exp(t) = exp(t)$$

If we think of the tangent slope as the instantaneous velocity, then the above means that the velocity equals the function at all times. Equivalently, the rate of growth of the function at any time equals the value of the function at this time.

In terms of a differential equation, the exponential function is the solution to the following equation, in the notation of Leibniz:

$$d y / d t = y$$

Moreover, the exponential function remains invariant under integration. Let us see what this means geometrically in terms of the invariance of area under translation, that is, the fact that translation does not have any impact on the rate of change of accumulated area.

Consider the following figure that shows the graph of the exponential function, displaying its tangent segments at various times:

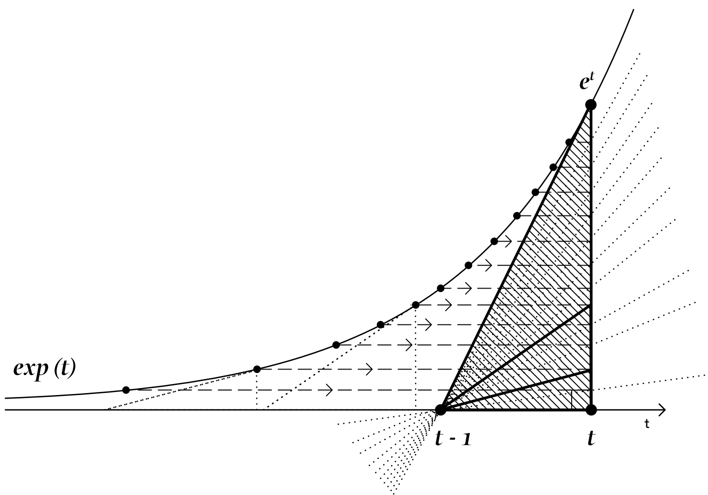


FIGURE 6.45
Harmonic parallel transport and synchronization of all tangents of the real exponential function

Note that the tangent line segment at $(t, e t)$ passes through the point $(t-1, 0)$ on the t-axis. We call 1 the subtangent of the exponential function on the t-axis, since it is a constant for any tangent segment crossing the t-axis.

This is a key observation, because it allows to perform a very simple translation that leaves the area under the exponential curve invariant. More concretely, we parallel translate all tangents inside the Pythagorean right triangle formed by the tangent at t, the t-axis, and the y-axis.

This is clearly an orthogonal triangle whose hypotenuse is the tangent segment at t, its horizontal side is 1 —the constant length of the subtangent, and its vertical side is e^t, the altitude of the graph of the exponential function at t.

Since all possible area that can be accumulated under the curve is by means of area swept through all the tangent segments for all t, at any t we can parallel translate all of them inside this orthogonal triangle formed at t, re-filling in this way homogenously its enclosed area. This is again another instance of the principle of harmonic synchronization of the tangents at their common center identified as the point $t - 1$ on the t-axis for every t.

We conclude that the total area enclosed under the exponential curve at t is two times the area of this orthogonal triangle, that is, the area of a rectangle of base 1 and height e^t, hence equal to e^t. This demonstrates the invariance of the real exponential function under integration as pertaining to the invariant accumulated area by means of harmonic parallel transport of the tangents.

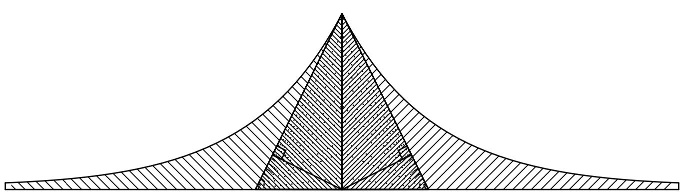


FIGURE 6.46
Invariance of the real exponential function under integration

The invariance of the real-valued exponential function under these inverse processes defines the neutrality condition in the enhanced spectrum of the metaphora conducted through the inverse bridges of differentiation and integration.

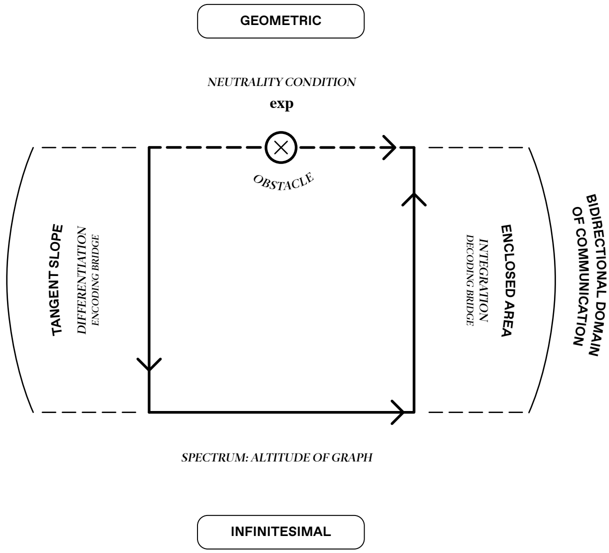


FIGURE 6.47
The exponential gives rise to the neutrality condition in the metaphora conducted by the architectonic scaffolding of differentiation/ integration

As such it should bear a structural role, which will be explored again in more detail at a later stage. Intuitively, the real-valued exponential function takes any real number and gives a positive non-negative real as its value. Due to its fundamental property:

$$exp(t+s) = exp(t) \cdot exp(s)$$

it converts the addition of real numbers in its domain to the multiplication of powers (positive non-zero reals) in its co-domain where it takes its values.

Henceforth, it is an architectonic bridge connecting the additive structural world of the reals with the multiplicative structural world of the powers, i.e. the positive non-zero reals. Astonishingly, this takes place in symphony with its role as a neutral element in the metaphora conducted by the inverse analytic/synthetic bridges of differentiation and integration.

The zero-th power is the multiplicative unity, that is:

$$exp(0) = 1$$

Furthermore, since the domain is all real numbers under addition, we may consider for each real its inverse with respect to addition. This means that for each t, we consider $-t$, which amounts to $exp(-t) = 1/exp(t)$, i.e. reversing the order of time. Since, the exponential function is invariant under integration, and integration gives the accumulated area between two boundary values, it is reasonable to consider the arithmetic mean of $exp(t)$ and $exp(-t)$, since areas are additive. This functional arithmetic mean gives rise again to the catenary curve as a manifestation of the explicated neutrality condition in the world of geometry and architecture.

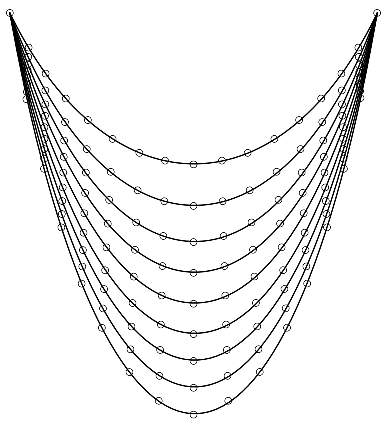


FIGURE 6.48
Derivation of the catenary in terms of the neutrality condition with respect to differentiation/integration

Integration Areas and Color: The Natural Logarithm Function

Having grasped the real-valued exponential bridge, we should be able to invert it, such that a metaphora can take place in terms of it. The main issue is the following: The process of taking a power whose exponent is a real number (not merely an integer) with respect to an arbitrary real-valued base presents an obstacle.

The real exponential function provides an encoding bridge to resolve the problem.

Note that it operates with the choice of a special base, the base provided by Euler's constant. This constant is not only an irrational, but it is a transcendental real number, bearing the same status as π. The purpose of its use in terms of the exponential function is the following: Every process of exponential growth, irrespectively of the base chosen, is characterized by the property that the rate of growth is proportional to the power itself. Among all of them, the Euler transcendental is the unique one such that the rate of growth equals exactly the power itself.

Therefore, if we transfer every power to this natural base so that we become able to follow it without obstruction, then due the property of the real exponential function above, the process of addition in the exponents is transmuted to the process of multiplying positive powers, and hence the spectrum becomes transparent under evaluation.

Then, we need to invert the exponential function (in the Euler base) that we used as a ladder, to be able to ascend to the domain of general powers (in arbitrary real base). This inversion amounts to designing the natural logarithm function as a decoding bridge back to the level of the initial problem. This is what is expressed in terms of the identity:

$$B^{\Phi} = [exp][\Phi][log_e B] = [exp][\Phi][ln B]$$

where B is an arbitrary base, and Φ is an arbitrary real-valued exponent.

This completes the metaphora without any residue, since in the domain of the reals the exponential and the logarithm func-

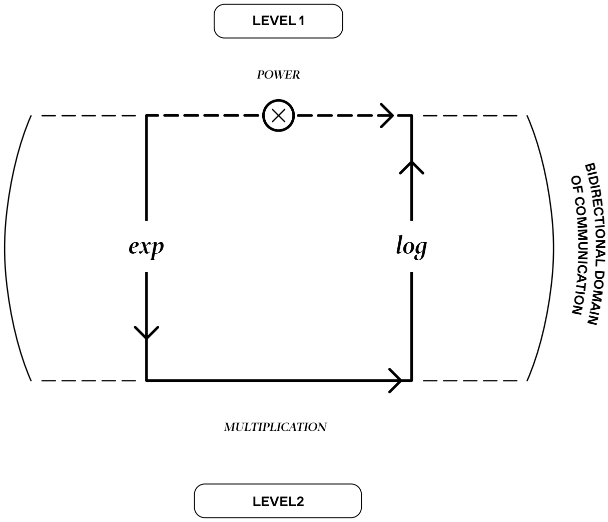


FIGURE 6.49
The conjugation of the process of raising to a power by means of multiplication via the Exp/Log inverse bridges

tion are exact inverses of each other, which make the additive structural world of the reals (as exponents) isomorphic—and thus structurally indistinguishable—with the multiplicative structural world of the positive reals (as powers). What is gained through this metaphora is that the process of raising to a power applies to the whole additive world of the reals (including the irrationals) and is communicated simply through multiplication at the spectrum.

Let us now delve into the natural logarithm function constituting the inverse architectonic bridge in the setting above. This is important because the outcome of this scrutiny will lead to grasping the spectrum of our tempered epiphaneia in terms of colors, something which has not been articulated to its full extent up to the present stage (Figure 6.50).

The motivic key is the association of the notion of an invariant area—which due to its invariance can be distinguished via a color—with the inverse of the real exponential function, that is, the (natural) logarithm function. Since, the domain of this function includes only the positive non-zero real numbers (as multiplicative powers), and its co-domain includes are all real

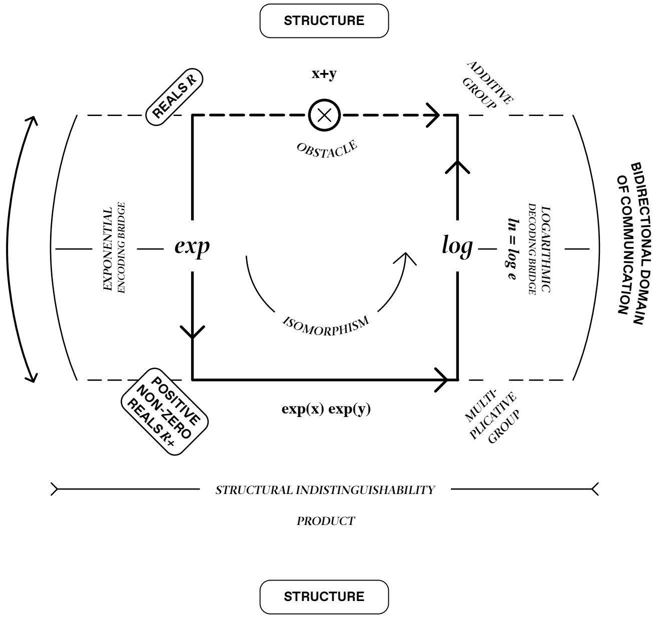


FIGURE 6.50

The Isomorphism between the additive group of the reals with the multiplicative group of non-zero positive reals conducted via the real Exp/Log architectonic scaffolding of inverse bridges

numbers (as additive exponents), we will think of the values of the logarithm function as colors in a continuous spectrum of frequencies of light that can be mixed and superimposed to each other, due to the additive closure property of the exponents. This pertains to the superposition principle of colors.

Note that the exponent variable of the real exponential function has been interpreted as time in its ordered aspect, by which we mean in its role as a continuously progressing real variable. In turn, the values of the logarithm function, which are also real additive exponents, are interpreted as a continuous spectrum of frequencies, the colors of the spectrum, which pertain to the periodic/frequential aspect of time.

How is such a dual reading of the exponents possible? The answer, as we already know lies in the invariance of area under translation, but the technology of real numbers under addition does not allow to capture the analytic/synthetic functional metaphora, which is required to explicate the periodic/frequential aspect of time in its becoming through rotation.

For this, we need to extend the resolving spectrum of the exponential function to the circle—which is anyhow essential since our epiphaneia is designed as a disk—that is, we need to make it capable of recognizing angles even in a multi-valent way. Nevertheless, the recognition of angles necessitates the extension of our notion of numbers to the imaginary ones by the adjunction of a single angular unit in the real world (called the imaginary unit—extending our domain of numbers to the complex numbers), which always comes together with its conjugate, since rotation can be performed clockwise or anticlockwise.

From our perspective, this capability means that we need to make the epiphaneia/screen of our disk capable to rotate—out of which we can comprehend why the tempered colors of our light spectrum are mixing. Since what we comprehend geometrically are areas—which if distinguished in the spectrum are invariant admitting a color—and since areas are invariant under progressing in time, i.e. their rate of change is not affected by translating linearly in time, the following conclusion emerges:

Architecture pertains to discerning and crystallizing these colors/tempered continuous intervals of light frequency in space by means of area that is enclosed through rotation.

From this viewpoint, we may also come to appreciate the close association of architecture with mechanics in the abstractions percolating through these seemingly different domains.

Mechanics has been considered as the art and science of the rolling wheel since its inception. In its abstraction, it is the capability of our disk—the equally-tempered epiphaneia—to roll over the ground, that is, to roll as a resolving screen over the domain of the shadows. But rolling is nothing else than the synthesis of translation and rotation, thus in terms of area invariance, architecture and mechanics are naturally equivalent.

What can be translated by rolling in mechanics can be encapsulated in an equi-areal manner and crystallized geometrically in architecture purely by rotation. This crystallization is natural only through the discernibility of the colors—the area sectors of the continuous equally-tempered frequency spectrum—that are not distinguishable ab initio due to mixing.

After this interlude, whose purpose was to clarify the concepts that usually remain in the shadows of the symbols and the signs, we come back to the study of the real logarithm functional bridge. We recall that the domain of this function includes only the positive non-zero real numbers (as multiplicative powers), and its co-domain includes are all real numbers (as additive exponents).

We think of the values of the logarithm function as colors in a continuous spectrum of frequencies of light that can be mixed and superimposed to each other. How is the association of colors with areas possible, in the first place?

We recall that the exponential function is the solution to the following differential equation:

$$dy/dt = y$$

If we invert the above, i.e. consider y as a variable, and t as the function, we obtain:

$$dt/dy = 1/y$$

that is, the differentiation of t —as a function of the variable y — with respect to y, equals the reciprocal of y.

In other words, differentiation of t in the additive world of the reals is equivalent to the reciprocation of the power y —in which it intrudes as exponent—in the multiplicative world of the powers.

The solution of the above equation by means of integration gives the following:

$$t = \int dy / y$$

$$t = \int d\,ln\,y$$

We start the integration at $y = 1$, i.e. the zero-th exponent power which is the multiplicative unity—the neutral element in the world of powers—such that the cipher $ln\,1 = log_e\ 1 = 0$ emerges in the additive world, determining its neutral element as well.

This explains the role of Euler's constant from a multiplicative viewpoint. It is the unique real constant, through which by natural logarithmization we obtain the neutral element, that is, the group identity of the additive real world.

Through the introduction of a dummy variable to be integrated, we write the above as a definite integral in order to obtain its qualification by means of area, as follows:

$$t = \int_1^y ds / s = ln\,y$$

where the integrated variable pertains to spans of mixtures in the continuous tempered real frequency spectrum.

Since the definite integral calculates areas under the curve of the reciprocal 1/s of s that is being integrated, we denote by $A(1, y)$, the area:

$$A(1, y) = \int_1^y ds / s = ln\,y$$

In this sense, by variation of y over its continuous real spectrum, the evaluation of the area at $y = K$, gives the color of the region enclosed under the graph of $1/s$ between 1 and K. In turn, this color is quantified by the value of the logarithm function at K, that is by $ln\,K$.

Note that the graph of the function $\tau = 1/s$ —corresponding to the action of reciprocation—is a rectangular hyperbola. It is immediate to see that:

$$\tau \cdot s = 1$$

meaning that the product of τ with s is always invariant expressed by the constant 1.

This is reminiscent of the inverse co-relation between linear length and frequency, that we have termed the uncertainty principle, since complete knowledge of the first leaves the second completely undetermined and conversely. In turn, that is exactly what is represented by the rectangular hyperbola $\tau \cdot s = 1$. It is worth emphasizing that grasping the uncertainty principle in this vain emerges from the infinitesimal reciprocal action required to invert, and thus decode, the real exponential function.

After this clarification in relation to the shape of the graph of the function $\tau = 1/s'$, we may abduct all the properties of the logarithm function by means of colored areas of the spectrum, that is, by means of areas under the hyperbola, according to the Figure 6.51.

The fundamental property of the logarithm function is that it transforms commutative multiplication to addition, and as an outcome of this, division to subtraction:

$$ln\,(y_1 \cdot y_2) = ln\,(y_2 \cdot y_1) = ln\,y_1 + ln\,y_2$$

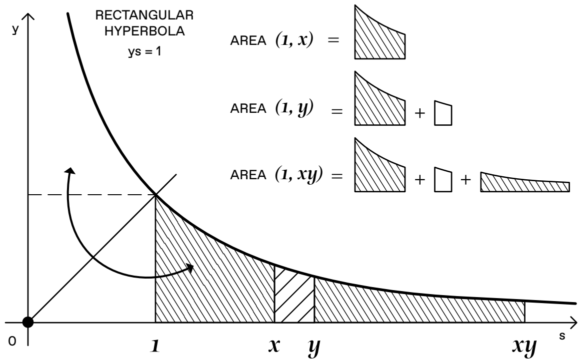


FIGURE 6.51
The areas under the rectangular hyperbola are the natural logarithms

$$ln(y_1/y_2) = ln(y_1) - ln(y_2)$$

This property is easily established in terms of colored areas, according to the additive area relations under the hyperbola, obtained by the above figure, as follows:

$$A(1, y_1 \cdot y_2) = A(1, y_1) + A(1, y_2)$$

Note that what is obvious from the figure is:

$$A(1, y_1 \cdot y_2) = A(1, y_2) + A(y_2, y_1 \cdot y_2)$$

By subtracting $A(1, y2)$ from both sides, what has to be shown is that:

$$A(y_2, y_1 \cdot y_2) = A(1, y_1)$$

This means that the area under the hyperbola from 1 to y_1 equals the area under the hyperbola from y_2 to $y_1 \cdot y_2$. In other words these regions should bear the same color. We understand this equality of areas if we focus on the invariance of the hyperbola, namely that the product of τ with s is always invariant, expressed by the constant 1.

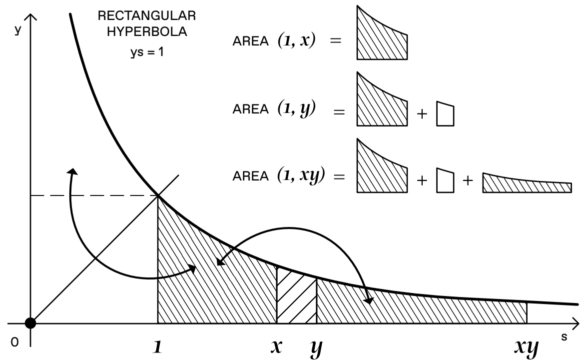


FIGURE 6.52
The triple correspondence between areas under the rectangular hyperbolas, natural logarithms, and colors

In particular, since $\tau \cdot s = 1$, if we extend horizontally by the constant factor λ —multiplying s by λ —and then contract vertically by the same constant factor λ —multiplying τ by $1/\lambda$ — then the hyperbola is preserved:

$$\tau / \lambda \cdot s \lambda = 1$$

This means that by proportionate re-scaling of both s and τ through a constant factor λ, the invariance remains intact. Thus, areas are calculated under the same hyperbola $\tau = 1/s$.

As a consequence, it becomes clear why the area under the hyperbola from 1 to y_1 equals the area under the hyperbola from y_2 to $y_1 \cdot y_2$. Starting from the first area, we extend horizontally by the factor y_2, and then contract vertically by the same factor y_2.

Note that, although the first area is squeezed via the inverse processes of extending and contracting, it remains invariant and fits precisely under the hyperbola, so that it is equal in magnitude to the second area. Thus, these two areas actually bear the same colour, which is identified in magnitude with $ln\, y_1$, since it holds:

$$ln\ (y_1 \cdot y_2)/y_2 = ln\,y_1$$

We also note the following relations:

$$ln\ e = 1$$

$$ln\,y = A(1, y) = -A(y, 1) = -ln\,y$$

$$ln\,1 = 0$$

The first of the above relations provides an interpretation of Euler's constant from the perspective of area. Euler's constant is the unique real number whose natural logarithm equals the neutral element, that is, the group identity of the additive world of the reals.

Due to the above association of logarithms with colors corresponding to areas under the rectangular hyperbola, we conclude that Euler's constant is the unique real number such that the area contained under the hyperbola between 1 and e equals the unit area. Note that the cipher $ln\,1 = 0$ is the logarithmic equivalent of the exponential cipher $exp(0) = 1$.

Finally, from the second of the above relations we see that areas are signed magnitudes in the sense that:

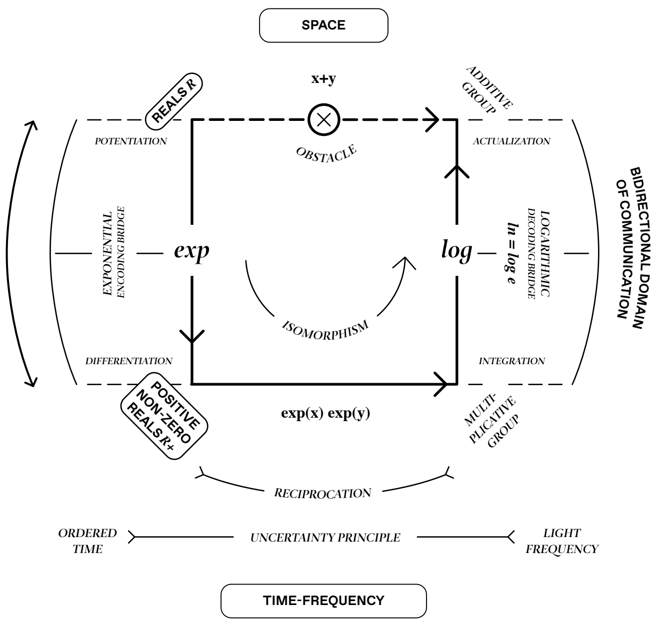
$$A(1, y) = -A(y, 1)$$

which explains in terms of area why the values of the logarithm function are both positive and negative real numbers.

The main conclusion is that the notion of area bears geometric plasticity, in the sense that areas can be stretched and contracted retaining their invariance. This is what is revealed by modelling the uncertainty principle by means of invariant areas, which are identified as colors under the rectangular hyperbola. Moreover, all the properties of areas under superposition and mixing are prescribed analytically in terms of the natural logarithm function, which inverts the real exponential function.

FIGURE 6.53

The natural communication scheme of the metaphora from space to time—and inversely—through the combined architectonic scaffoldings of Exp/Log and differentiation/integration



7. The Pneuma of Stochastics: Quantum Phase and Thermal Spectrum

Helicoidal Universal Covering and Branch Cutting

The task we face now is how to extend the resolving spectrum of the exponential function to the circle. As we already mentioned above this is of essential value since our epiphaneia is designed as a disk. Thus, we have to be able to think of this disk as capable of rotating with a variety of frequencies. In terms of real area invariance the rotating disk could be a rolling wheel, since the rate of change of its area is unaffected by translation.

The issue is how we should envision the exponential function in its capacity to recognize and encode angles. This is not straightforward for many reasons. The most important of them has to do with the multi-valency of the notion of an angle. Recall that geometrically an angle is considered with respect to the origin of a circle, and all right angles are equal, in the sense that they share the same measure of congruence.

It is the notion of orthogonality that together with the ability to translate in parallel fashion—the affine property—characterizes the geometry of the Euclidean elements. If we start thinking mechanically, the notion of an angle is made operational through rotation, but an integer number of whole rotations (clock-wise or counter-clock-wise) makes an angle look exactly the same geometrically with respect to the geometric origin. In this way, rotation affords a multi-valued encoding of an angle, that is, all angles differing by an integer number of full circulations are cast equivalent in the spectrum, they are recognized as the same block of the partition.

From a topological viewpoint, this is referred as the issue of multiple-connectivity, and it is an obstacle. The obstacle rests on the fact that geometry requires simple-connectivity to be effective and operational in its constructible/synthetic role, it cannot handle the seeming redundancy—the gauge freedom—inherited by the multi-valency of angles.

This problem has been resolved by Riemann, who conceived of a principle to deal with multiple connectivity, called the covering principle, which led to the notion of a universal covering epiphaneia.

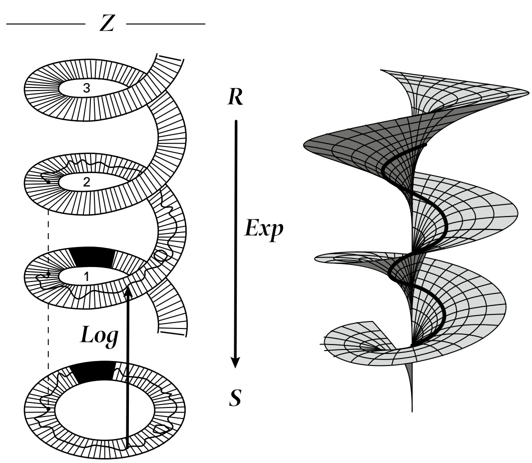


FIGURE 7.1
Universal covering principle via the complex Exp/Log architectonic scaffolding

The basic abstraction of Riemann was the following: Wherever a phenomenon of multiple-connectivity appears through circulation, make a cut, and continue after the cut on a different vertical layer extending over the initial level. This procedure continues until all multi-valency is resolved in a number of successive vertical layers covering surjectively the initial one.

All these different layers are bonded together through a common axis. In the case of the multi-valency of the angle we are interested in, it is clear that the number of layers should be countably infinite, equipotent with all the integers.

We think of each layer as a different branch of the universal covering epiphaneia extending over the disk. To be consistent, the disk bears a dark spot in the middle, thought of topologically as a puncture, since it is the locus of the vertical axis, where all branches are bonded to each other over the disk.

The universal covering is like a helicoid covering the underlying disk. The helical boundary, is universally covering the circle surrounding the disk. The helix is nothing else than the real line of temporal progression, but bearing an imprint of integer angularity through integral rotations—called windings—that inherits from the binding cord of all the branches.

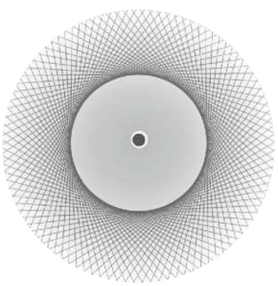


FIGURE 7.2
The underlying disk bearing a dark spot in the center, together with its tangent caustic envelope

Imaginary Potentiation of Angles through Phases

What the covering principle achieves is the universal covering of the circle by the helix, so that the underlying circle is viewed from the simply-connected perspective of the helix as the real line $\mathbb{R}$ modulo the integers $\mathbb{Z}$, that is, the circle $S^1 := S$ is made isomorphic to the spectrum of the reals modulo the integers:

$$S \cong \mathbb{R} / \mathbb{Z}$$

This is remarkable, since the circle appears topologically as a spectrum that emerges out of the modulation of the real line of progression by the discrete integer windings of the helix.

Nevertheless, the integers are undisclosed from the perspective of the circle, each point of the circle carries with it an invisible fiber containing all the discrete integers, all the windings. In this sense, each point of the circle indexes a block of the partition spectrum, expressed by the quotient $\mathbb{R}/\mathbb{Z}$. Equivalently, each point of the circle indexes a fiber or orbit of the transitive translation action of the discrete integers on the continuous real line.

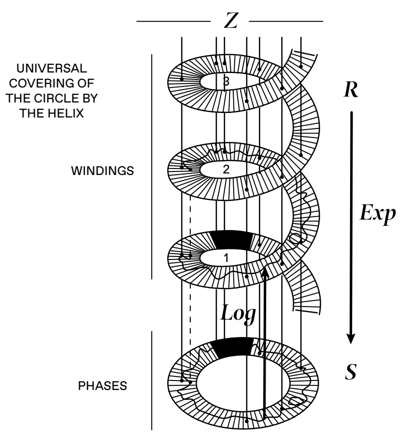


FIGURE 7.3
The relation between windings and phases in the universal covering of the circle by the helix conducted through the complex Exp/Log architectonic scaffolding

This topological spectrum is valuable, but we should be able to have a calculus, like in the case we deal with real-valued functions. This is the extension of real analysis to complex analysis and the theory of Riemann surfaces. Our objective presently is more humble than going all the way through, but we should scrutinize what makes the abstraction leading to these fields of mathematical thinking possible.

We should envision how the exponential function bears the capacity to recognize and encode angles. We already know that the real-valued exponential function takes the reals under addition and transmutes them to non-zero positive reals under multiplication, such that these two worlds are made structurally isomorphic.

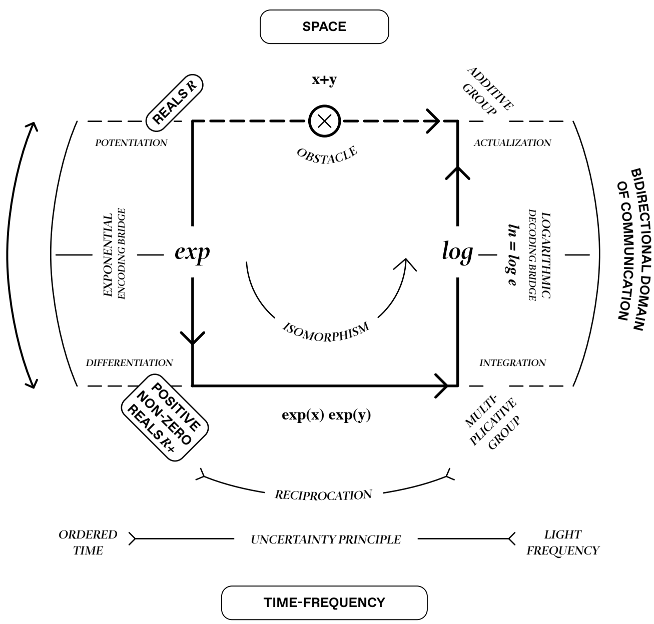


FIGURE 7.4
Architectonics of the metaphora from potentiation to actualization

Note that the non-zero positive reals are powers, thus the efficacy of the real-valued exponential function is based on the potentiation of all the reals as powers.

We need to do the same for angles; the idea is that the extension of the exponential function to the circle should be able to potentiate angles. We call phases those envisioned potentiated angles.

Intuitively, if we think mechanically in terms of rotation, angles become additive. Thus, they fit in the plan if we manage to express them as real numbers. Correspondingly phases, that is potentiated angles, should be multiplicative and defined on the circle. In other words, the circle should be coordinated in terms of multiplicative phases via the exponential function resolving the circle.

The caveat in this case is that the world of additive angles and multiplicative phases cannot be made isomorphic, for reasons we have already discussed, i.e. multi-valence and multiple connectivity of rotation. Despite this fact, this is actually not a caveat, because the situation can be turned upside down.

Instead of viewing this as a redundancy, we consider it as a modular way to encode an underlying invariance. Recall that the invariants of the harmonic domain are integers—pertaining to the periodic/frequential aspect of time—and currently, the gauge freedom of rotation in terms of integer whole turns presents an opportunity to encapsulate these invariants through rotation (Figure 7.5).

The structural idea is that the extension of the exponential function from the reals to the circle—and not only to the positive non-zero reals—should determine the identity of the circle through this unresolvable kernel, that is, precisely this kernel is what should characterize structurally the circle.

Recall that the identity of the positive non-zero reals under multiplication, is the $1 = e^0$, functioning as the neutral element in this world. In the case of the circle, the identity of the circular world under multiplication of phases should be constituted in terms of all the integers. This means that all the integers, the

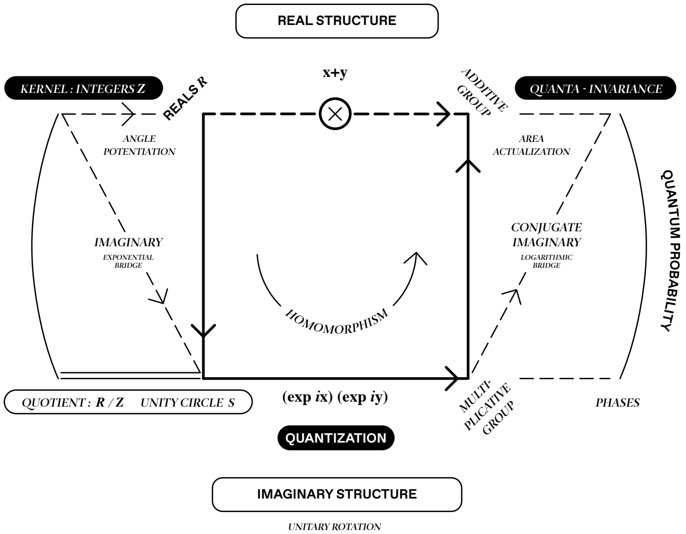


FIGURE 7.5

The metaphora from angle potentiation to area actualization in terms of quanta, conducted through the extension of the Exp/Log architectonic scaffolding to the complex numbers

invisible fiber of all discrete integers, at each of its points, is the neutral element in the circular world.

Concomitantly, the evaluation of the power on the circle at each integer should give the unity 1 in this world. What this means mechanically is simply that every integer number of whole rotations with respect to any point gives the unity to the world of phases, specifying in this way its neutral element.

This is the perplexing issue we have to deal with if we wish to grasp the essence of extending the exponential function to the circle that makes possible the strengthening and efficacy of our calculus. Although we prepared the ground to make the required metaphora from the positive reals to the phases as multiplicative powers in the domain of values of the exponential function, we still need to think of how this recognition procedure should take place in relation to angles.

Recall that the domain of the exponential function are all real numbers. If we hope to extend the power of our calculus with phases on the circle, we should be able, first of all, to offer the means of recognizing angles as real numbers, something not evident geometrically.

Our rescue is again at the domain of transcendental numbers like with the choice of basis for the exponential function, the Euler constant. Since the measurement of the circle by Archimedes, we know geometrically the following:

> In any circle, the ratio of the perimeter of the circle over its diameter is an irrational constant, which is equal to the ratio of the area of the disk it encloses over the square of its radius. This constant is the transcendental number pi, which cannot be identified as the root of any polynomial equation with rational coefficients.

This offers a natural real– valued measure of an angle in terms of the arc it opens up on the circle. The measure of the angle equals the length of the arc subtended by it on the unit circle (the circle with radius 1). A circle has 2π radians in total, which is also its circumference $2\pi\rho$ divided by its radius ρ. Thus, the measure of an angle in terms of the reals is expressed through π as a length in terms of radians, such that the whole circle has real– valued length 2π.

Therefore, angles expressed as irrational lengths under the invariant of any circle π enter into the domain of the real exponential function as real entities amenable to addition of length. The evaluation of an angle θ under the real exponential function gives rise to an angular multiplicative power, that is, the positive non– zero real e^{θ}.

These powers—although not constructible geometrically—can be comprehended mechanically through the combination of

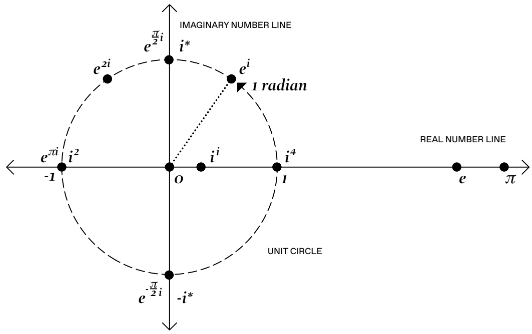


FIGURE 7.6
The notion of complex phases on the unit circle via the complex exponential function

rotation and translation, for instance, the non– uniform motion expressed by a radius ϱ —with respect to a center—that varies according to e^{θ}, the well-known logarithmic spiral $\varrho = e^{\theta}$.

A further more radical step is needed to potentiate the notion of an angle, that is, to extend the exponential function to the circle, resolving its points in terms of phases. For this purpose, we need to adjoin a new angular unit to the world of the reals under addition, the imaginary unit i, which comes inseparably from its conjugate i^*.

This refers to the well-known identity:

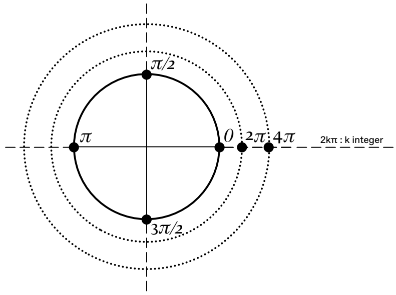
$$i^2 = i \cdot i^* = -1$$

which identifies the imaginary unit as the extracted square root of the negative unity—*1* of the reals. We usually think of the imaginary unit in terms of an anti-clock-wise rotation by angle of real length measure $\pi/2$, such that, its conjugate corresponds to a clock-wise rotation by the same measure of angle.

In this manner, the composition of two consecutive rotations by $\pi/2$ gives rise to aπ rotation that transforms from the unity *1* to the negative unity—*1* of the reals.

FIGURE 7.7

The imaginary unit of rotation by angle of real length measure $\pi/2$



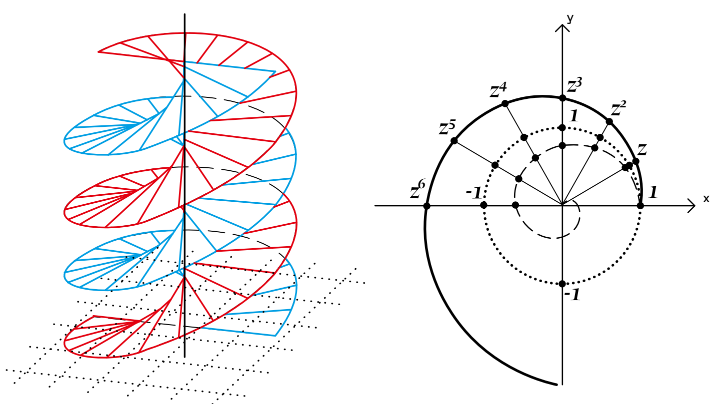
Imaginary Geometry of Quantum Phases

There are two issues with this standard approach that need to be clarified before proceeding. The first is why we identify a new unit—that we call imaginary—with an anti-clock-wise rotation by angle of real length measure $\pi / 2$, and the second, is how the extension from the real numbers to the complex numbers takes place by the introduction of this unit.

The immediate identification of the complex numbers with the two-dimensional space of real numbers, and the concomitant representation of the exponential function in terms of the trigonometric functions is not satisfactory from our perspective,

FIGURE 7.8

Helical standing wave in the bounded covering space of the circle and spiral of complex powers



because it hides under the carpet the structural significance of extending the exponential function from the positive reals to the circle. It is this extension that allows the trigonometric reduction of the exponential function, rather than the other way round.

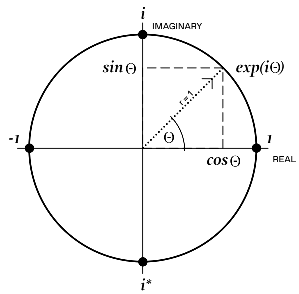


FIGURE 7.9

The trigonometric functions are the orthogonal projections of the complex exponential function on the unit circle

For this purpose, we will try to explicate, first of all, what the imaginary unit stands for. Let us recall the invariance of the real exponential function under integration as pertaining to the invariant accumulated area:

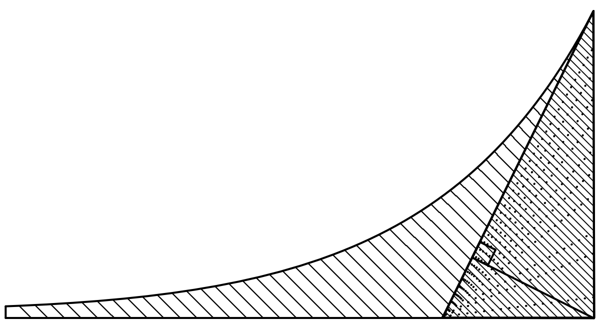


FIGURE 7.10

Invariance of the real exponential function under integration

The fact that all the area can be calculated in terms of the displayed Pythagorean orthogonal triangle for any t bears the major significance here. If we wish to extend the exponential function to the circle, we should be able to keep this invariance. Note that by rotating orthogonally this triangle four times it retains its initial geometric configuration without accumulating new area. This means that it is possible to realize our objective.

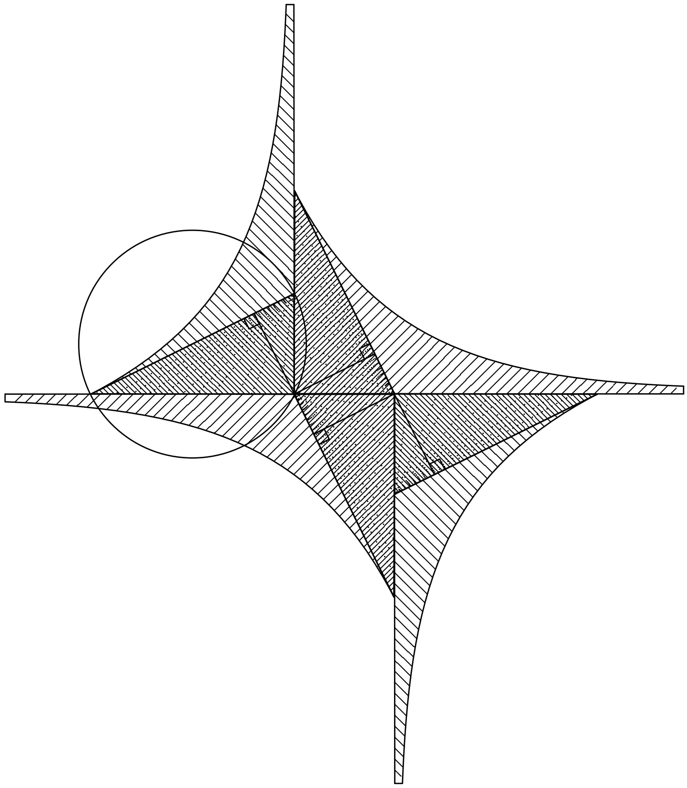


FIGURE 7.11

Invariance of area under fourfold imaginary rotation conducted by the imaginary unit.

Additionally, we obtain the following symmetries under reflections of the Pythagorean orthogonal triangle for any t:

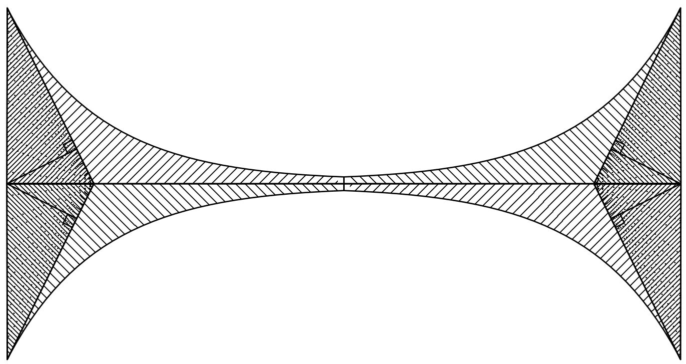


FIGURE 7.12
Symmetry 1

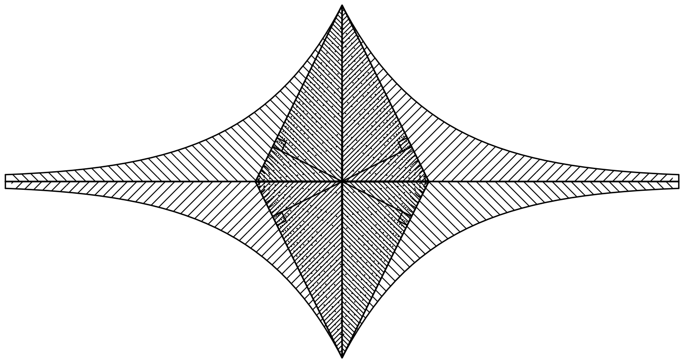


FIGURE 7.13
Symmetry 2

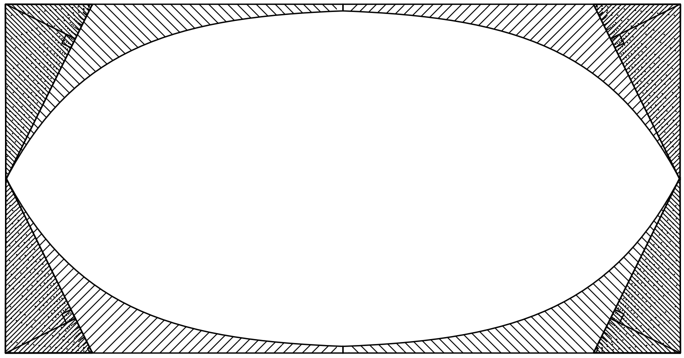


FIGURE 7.14
Symmetry 3

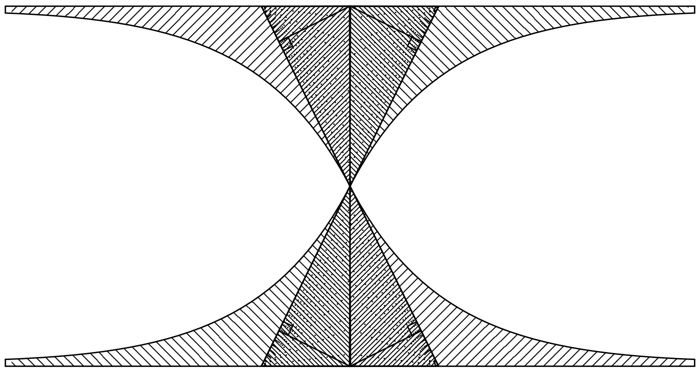


FIGURE 7.15
Symmetry 4

We consider an imaginary rotation ξ by means of an angle of real length measure ϕ in radians, conducted by means of a unit i, to be specified shortly, such that it takes the real angle ϕ to a phase on the circle S, denoted by $e^{i\phi}$, according to:

$$\xi : \phi \mapsto e^{i\phi}$$

where we have the identification:

$$\xi(\pi/2) := e^{(i\cdot\pi/2)} = i$$

and such that, the following condition is satisfied:

$$e^{2k\pi i} = 1$$

for any integer k, specifying in this manner the algebraic identity of the group of phases, that is, the neutral element on the circle S under the operation of multiplication of phases (Figure 7.16).

Thus, the imaginary unit is the image of a right angle, which is measured as $\pi/2$ length in radians, under the extension of the exponential function to the circle. This means that the potentia-

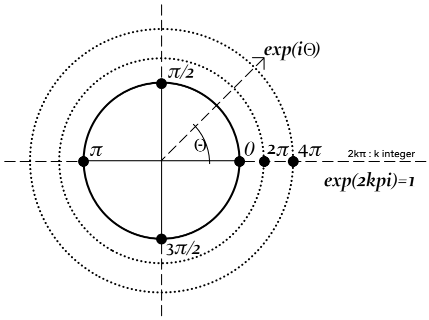


FIGURE 7.16

Algebraic identity of the multiplicative group of complex phases

tion of the right angle—through its real length measure $\pi / 2$ —as a phase on the circle gives rise to, and is identified analytically with the imaginary unit i :

$$\xi (\pi / 2) = i$$

This is in symphony with the universal covering principle of the circle by the helix, according to which the circle is viewed from the simply– connected perspective of the helix as the real line $\mathbb{R}$ modulo the integers $\mathbb{Z}$, that is, the circle S is made isomorphic to the spectrum of the reals modulo the integers:

$$exp: \mathbb{R} \mapsto S \cong \mathbb{R} / \mathbb{Z}$$

$$\phi \mapsto exp(i \phi) = e^{i \phi}$$

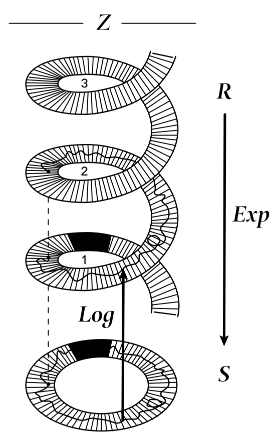


FIGURE 7.17

The circle is isomorphic with the spectrum of the reals modulo the integers

Neutrality Condition: The Universal Equation of Ciphers

The neutral element on the circle under multiplication of phases constitutes the universal equation of ciphers that our calculus is based on, that is:

$$e^{2k\pi i} = 1$$

By squaring the identity

$$e^{(i \cdot \pi/2)} = i$$

we obtain the identity relation on ciphers, called the Euler identity, in the form:

$$e^{i\pi} + 1 = 0$$

Note that this is derived without employing the trigonometric functions cosinus and sinus as projections on the horizontal and vertical axis of the two– dimensional plane. The trigonometric functions are derived from the exponential function on the circle, under projection, and not the other way round. Let us see how this takes place.

Every point of the circle (whose radius is considered one—called the unit circle) is parameterized surjectively by a phase $e^{i\phi}$, which is the potentiation of the real length ϕ that measures the angle in the domain of the exponential function. This happens through the imaginary unit i that characterizes the corresponding rotation as an imaginary rotation.

An imaginary rotation preserves the area contained within the unit circle. If we consider the disk bounded by this circle it preserves its area under the structure inflicted upon it by the imaginary unit. Each point of the perimeter, rotates with the same velocity under imaginary rotation of the disk, which is expressed by the derivative of the phase $e^{i\phi}$ with respect to the real length ϕ of the angle, that is, $i \cdot e^{i\phi}$

This means that the velocity of each point on the perimeter is tangential to the circle and orthogonal to its radius at this point, due to the relation:

$$\xi(\pi/2) := e^{(i \cdot \pi/2)} = i$$

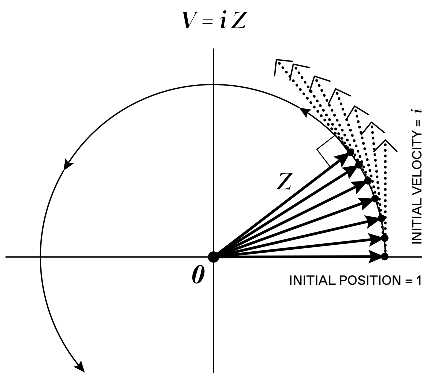


FIGURE 7.18
The complex structure: if the initial position is real, then the initial velocity is imaginary

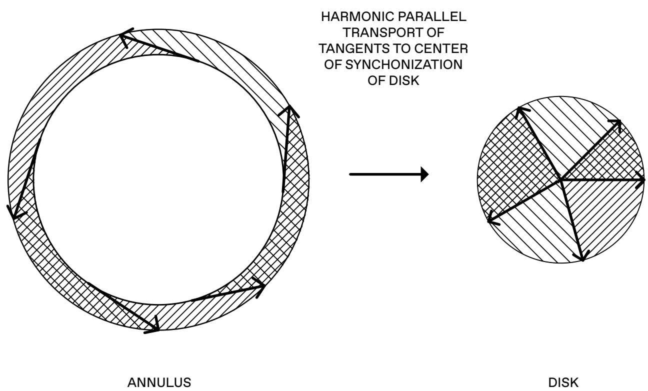
$$d(e^{i\varphi})/d\varphi = i \cdot e^{i\varphi}$$

$$d/d\varphi \quad \mathbb{Z} = i \cdot \mathbb{Z}$$

Thus, the imaginary unit can be considered—in geometric terms—as orthogonal to the radius at each point on the circle, forcing in this way the instantaneous tangency of the velocity at this point, for all points. Hence, differentiation is expressed in the analytic terms of the exponential function on the circle, in the most economic terms, simply as multiplication by the imaginary unit of the corresponding phase. This is what underlies the strength of this function.

Nevertheless, the imaginary unit does not pertain to the circumference of the circle—it only allows the expression of the points as phases, as well as the expression of the velocities at each point of the circle as multiplication by the imaginary unit. However, it pertains to the center of the circle, it emanates from the center, not the circumference. In order to understand this subtle issue, we need to recall the principle of synchronization with respect to the center by means of harmonic parallel transport—the reason for the validity of the Pythagorean theorem.

FIGURE 7.19
The principle of synchronization



The center of the disk is the unique point that is in synchrony with all the points at the circumference. This is what justifies the harmonic parallel transport of all tangent vectors to the center.

For the calculation of the area of the disk, since all tangents at all points bear the same length, it is clear that their lengths will fill in the whole disk when synchronized at the center after transport. Thus, the area of the disk is the same with the area being swept by the length of any single one after rotation by 2π.

Note that it is the length that sweeps the area and the length is always real. Thus, all the contribution to the measure of area emerges by the positive non– zero reals. Clearly, in the case of unit length of the tangent, that is unit velocity, the swept out area is π.

Principle of the Imaginary Rolling Wheel

In the case of an imaginary rotation, there is no contribution to new area at all, since any length that can sweep out area is a positive non– zero real. A phase does not have any contribution to area whatsoever. The imaginary velocities can be considered simultaneously synchronized all together being uniformly distributed all around the center.

But this synchronization refers to their directions only; this is why it is area– preserving. Hence, during a whole imaginary rotation all these directions cancel out, their net average is precisely zero. The same happens for any integer number of complete imaginary rotations. This is what is expressed by the null value of the definite integral of $e^{i\phi}$, integrated over the real ϕ, from 0 to 2π:

$$\int_0^{2\pi} e^{i\phi} \, d\phi = 0$$

for any complete imaginary rotation. It is an area– preservation condition under complete imaginary rotations.

Henceforth, the extension of the exponential function on the circle preserves all area—accumulated through sweeping by synchronized positive lengths—under complete imaginary rotations.

Then, since the accumulated area remains invariant in this case, we abduct the conclusion that an arbitrary imaginary rotation is actually the same as the translation of the center of spontaneous synchronization along a straight line axis with net average velocity—corresponding to the average of all velocities during this phase—bearing the i as unit, which we call an imaginary translation.

Equivalently put, in the imaginary domain, change of phase through rotation—at the circumference of the unit circle—and imaginary translation of the center with an average velocity are entirely equivalent, thus identified.

What happens is that an imaginary rotation corresponding to some phase different from a complete rotation has a net average of velocities whose area span—in comparison to the whole area preserved during a complete rotation—is transfigured to the imaginary linear translation of the center with this velocity. We call this the principle of the rolling imaginary wheel.

The linear translation of the center takes place along the imaginary axis, being thus measurable through the unit i by means of a real number that expresses net average velocity during the corresponding phase. This identifies the imaginary axis as a $i\mathbb{R}-$ axis.

Moreover, since $\xi(\pi/2) := e^{i\pi/2} = i$, meaning that the imaginary unit is a unit corresponding to a right angle, we conclude that the imaginary axis should be orthogonal to the real axis, arriving thus, to the standard representation of the complex plane. An immediate outcome is the derivation of the trigonometric expression of a complex number consisting of a real and an imaginary part through projection to these two orthogonal axis:

$$z = r e^{i\phi} := r(\cos\phi + i \sin\phi)$$

However, we are now able to appreciate the strength of the calculus based on the complex numbers through grasping the notion of an imaginary rotation. This is the actual content of the metaphora needed for the extension of the exponential function from the reals to the circle.

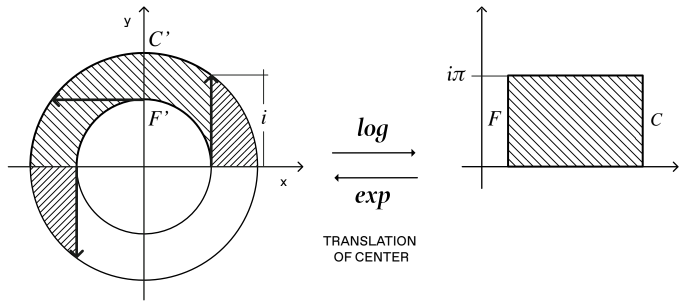


FIGURE 7.20
Imaginary rotation as imaginary translation of the center of synchronization via the complex Exp/ Log architectonic scaffolding

Through the calculus of the imaginary unit, the area span of the velocities-average corresponding to a phase difference on the circle—under imaginary rotation—is transformed to the linear translation of the center along the imaginary axis bearing this velocity average. This would be impossible without the area preservation under complete imaginary rotations, and without the spontaneous synchronization of all these velocities at the center.

Imaginary Rotation and Real Tempering: The π -Tuning

Since we have accomplished the extension of the exponential function from the positive non-zero reals to the circle, and since this extension has led to the domain of complex numbers by the adjunction of the imaginary axis orthogonally to the real axis, we can consider the action of the exponential on the whole complex plane screen.

The domain of the complex exponential function will be the complex numbers $\mathbb{C}$ under addition—to serve as the exponents of the exponential—and the codomain will be the non-zero complex numbers $\mathbb{C}^{\times}$ —to serve as the powers under multiplication. Note that the powers in the image of the exponential can never be zero.

From a topological viewpoint, the zero on the complex plane is thought of as a puncture. It is the dark spot of our topological disk in the middle, the center of synchronization, since it is the locus of the helicoidal axis where all branches are bonded to each other over the topological disk.

From a structural algebraic viewpoint, the complex exponential is not only a function from $\mathbb{C}$ to $\mathbb{C}^{\times}$, but it is a structural morphism between groups—the first in the domain under addition, and the second in the codomain under multiplication. This is of the utmost significance because it qualifies the role of the neutral element in the punctured complex plane, though of as the action space of complex powers—obtained through the multiplication of a real power with an imaginary power, as follows:

Consider the complex exponential mapping,

$$exp: \mathbb{C} \mapsto \mathbb{C}^{\times}, \quad z \mapsto exp\, z$$

as a homomorphism of the additive group $\mathbb{C}$ into the multiplicative group $\mathbb{C}^{\times}$.

As in the case of a general group homomorphism $\sigma : \mapsto \mathbb{H}$ we should consider the image group $\sigma (\boldsymbol{G}) := exp(\mathbb{C})$, and the kernel that qualifies the neutral element of the codomain group:

$$Ker\, \sigma := \{ g \in \boldsymbol{G} : \sigma (g) = \text{neutral element of } \mathbb{H} \}$$

For the exponential group homomorphism we obtain:

$$exp (\mathbb{C}) = \mathbb{C}^{\times}, \quad Ker\, (exp) = 2\, \pi\, i\, \mathbb{Z}\, .$$

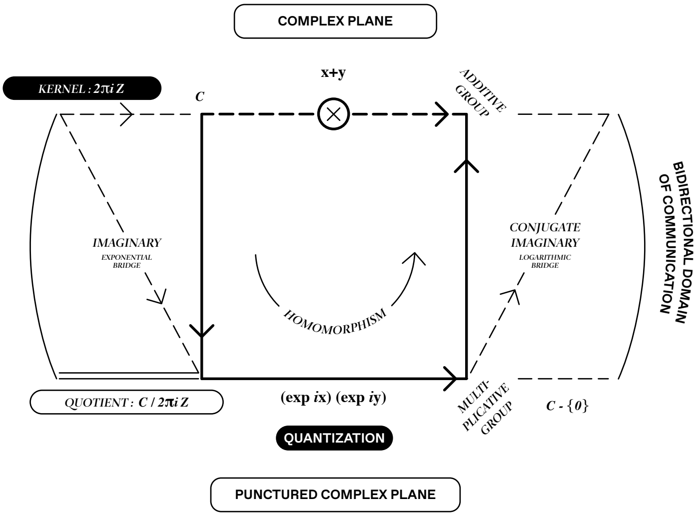


FIGURE 7.21
Metaphora between the complex plane and the punctured complex plane under the structural cipher of the integers

This is the structural cipher that bridges together the additive world of the whole complex plane with the multiplicative world of the punctured complex plane bearing the center of synchronization. This cipher is structural—and not only arithmetical—because it involves the group structure of the discrete integers $\mathbb{Z}$ under addition.

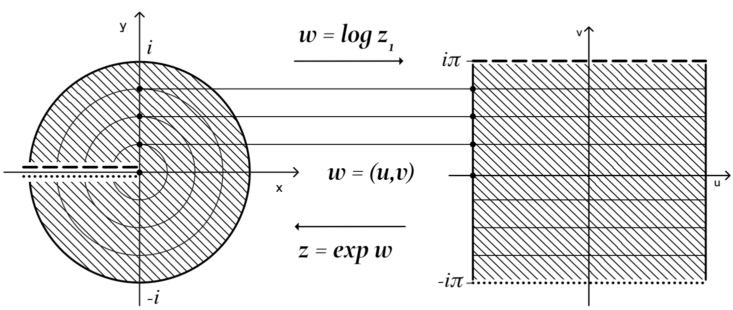


FIGURE 7.22
Imaginary translations are quantized in discrete quanta of length $2\pi i$

The latter is instrumental for the consistency of imaginary translations. It shows that they should be quantized in discrete quanta of length $2\pi i$. In this sense, the complex exponential accomplishes the resolution of the harmonic invariants, which were hidden from the resolving spectrum of the positive reals.

As the most important consequence of quantization, we can grasp the underlying reason of appearance of the transcendental number π in the geometry of the circle, the invariant that determines both its perimeter and its area. It emerges from the quantization cipher engulfed in the complex exponential as follows:

There exists a uniquely defined real number $\pi > 0$, such that the numbers $2\pi i k$, $k \in \mathbb{Z}$, constitute the set of numbers mapped on to 1, the multiplicative identity of the punctured complex plane—its neutral element that pertains to the center of synchronization—by the complex exponential mapping *exp*.

Equivalently there is a unique tuning real number π with the property that:

$$\{ w \in \mathbb{C} : exp\, w = 1 \} = 2\pi i \mathbb{Z}$$

The above property characterizes π uniquely, thus it amounts to the definition of π.

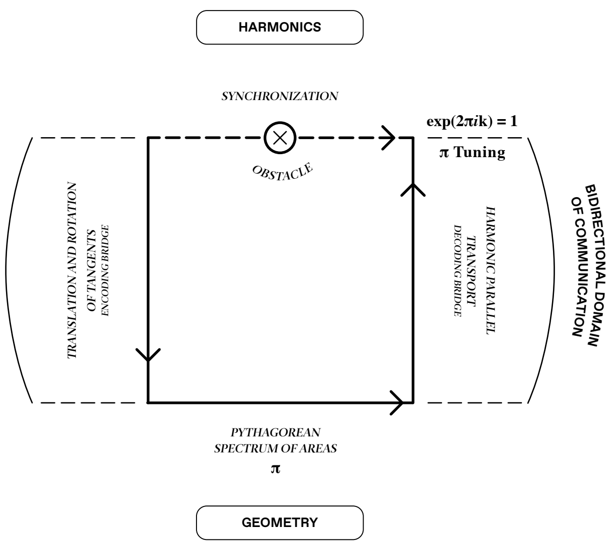


FIGURE 7.23
The transcendental π-tuning in the metaphora from the domain of harmonics to geometry

In this sense, the unique real number π is the invariant that allows the effectuation of our equally– tempered scale as a light spectrum of colours, by specifying uniquely and invariantly the irrational tuning parameter that allows the natural communication between the harmonic domain and the geometric domain without any *ad-hoc* assumptions.

We have arrived to the point that our disk epiphaneia equipped with the action of the complex exponential is capable to decipher the discrete harmonic invariants through quantization.

Therefore, it bears the capacity of the polar spider web, not only pertaining to real rotations and translations, but also to imaginary rotations and translations as well as to their subtle interplay—the type of intricate weaving that takes place with complex numbers.

Note that the above exponential group homomorphism morphism:

$$exp(\mathbb{C}) = \mathbb{C}^{\times}, \qquad Ker(exp) = 2\pi i \mathbb{Z},$$

is clearly an epimorphism, that is, surjective, but it is not injective. Using the fact that:

$$|\, exp\, z \,| = 1$$

holds if and only if $z \in \mathbb{R} \cdot i$, we derive that $exp(i \cdot \mathbb{R}) = S^1$, where S^1 denotes the multiplicative circle group, that is, the circle viewed structurally as a group under multiplication of (quantum) phases. Anyhow, this is already what has been accomplished by the

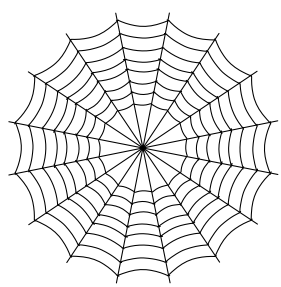


FIGURE 7.24
The polar spider web—arachne unveiled

potentiation of angles as phases parameterizing the points of the circle.

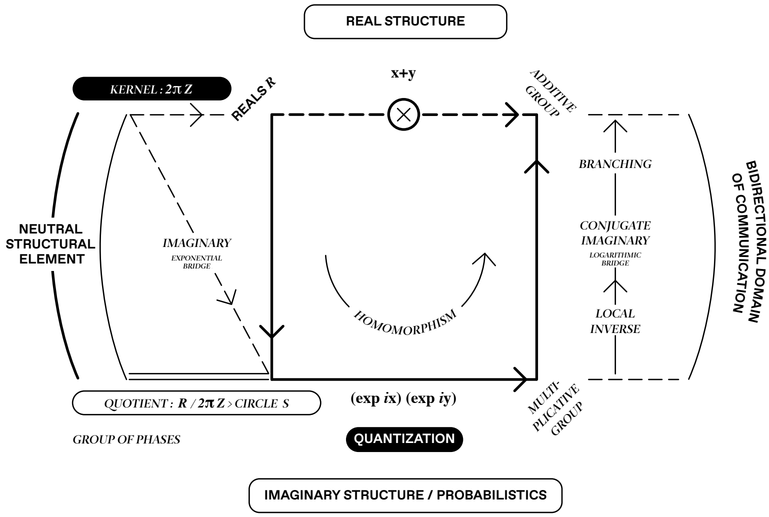


FIGURE 7.25
The architectonic scaffolding of the imaginary polar spider web and the notion of quantization

Hence, the imaginary polar spider web epimorphism of our epiphaneia is expressed as follows:

The following homomorphism of groups:

$$p : \mathbb{R} \mapsto S^1$$

$$\varphi \mapsto e^{i\varphi}$$

is a group epimorphism whose kernel is the group 2πZ such that: p(π/2)=i. The important thing to emphasize here is that this kernel ideal specifies the structural identity of the circle, that is, the algebraic neutral element of the circle bearing the structure of the multiplicative group of (quantum) phases.

Note that for $\varphi, \psi \in \mathbb{R}$, we have:

$$p(\varphi + \psi) = exp(i\varphi + i\psi) = (exp\, i\varphi)\cdot(exp\, i\psi) = p(\varphi)\cdot p(\psi)$$

Thus, p is an epimorphism, since:

$$p(\mathbb{R}) = exp(i\cdot\mathbb{R}) = S^1 := S$$

Furthermore, since:

$$Ker\,(exp) = 2\,\pi\, i\,\mathbb{Z}$$

we obtain that:

$$Ker\, p = \{ t \in \mathbb{R} : i\, t \in Ker\,(exp)\} = \{ t \in \mathbb{R} : t \in 2\,\pi\,\mathbb{Z} \}$$

This is in complete accordance with our previous discussion, regarding the nature of the circle invariant π as a real transcendental tuning parameter, elucidating the conception of our screen as a polar spider web capable of deciphering and resonating with the discrete invariants of the harmonic domain.

Therefore, although our equally– tempered scale is intrinsically probabilistic, it is not because of any type of subjective ignorance, but it is so due to the irrational nature of the universal tuning parameter π that makes the visible spectrum intrinsically continuous, and tempered according to objective chance.

Imaginary Polar Spider Web Cyclotomy: Windings and Quanta

The imaginary polar spider web, based on the technology of the exponential function on the unit circle, allows the extraction of the roots of the multiplicative unity 1 of the world of powers on the circle—that is, of the world of quantum phases.

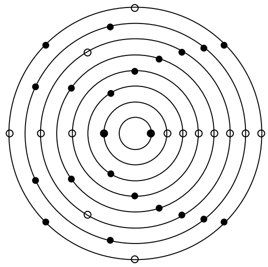


FIGURE 7.26
Cyclotomy and the extraction of the roots of unity

In this manner, we are able to perform cyclotomy in terms of the extracted roots. The idea is that a root of unity is a complex number whose power living on the circle equals the unity, that is, the unity phase—the neutral element on the circle.

By the fundamental theorem of algebra, there are always n different complex n-th roots of unity, that is complex numbers z, whose n-th power equals the unity:

$$z^n = 1$$

The roots of unity are equally spaced around the periphery of the unit circle in the complex plane.

Since they are equally spaced they constitute an equally-tempered scale on the epiphaneia of the disk. Roots of unity are manifested geometrically as the vertices of a regular polygon that binds them together.

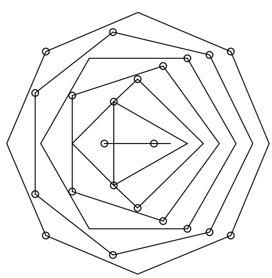


FIGURE 7.27
Polygonal constellations of roots of unity

In this sense, the roots are abducted from the unity phase through cyclotomy by the algebraic means of root extraction. The crucial idea is that complex root extraction from the unity—through cyclotomy—reciprocates the respective exponents, resulting in the neutralization of the powers.

Of particular importance are the primitive roots of unity. More precisely, on the unit circle with n equally spaced rays, there is a mark on the ray k, denoting a primitive root of unity, if and

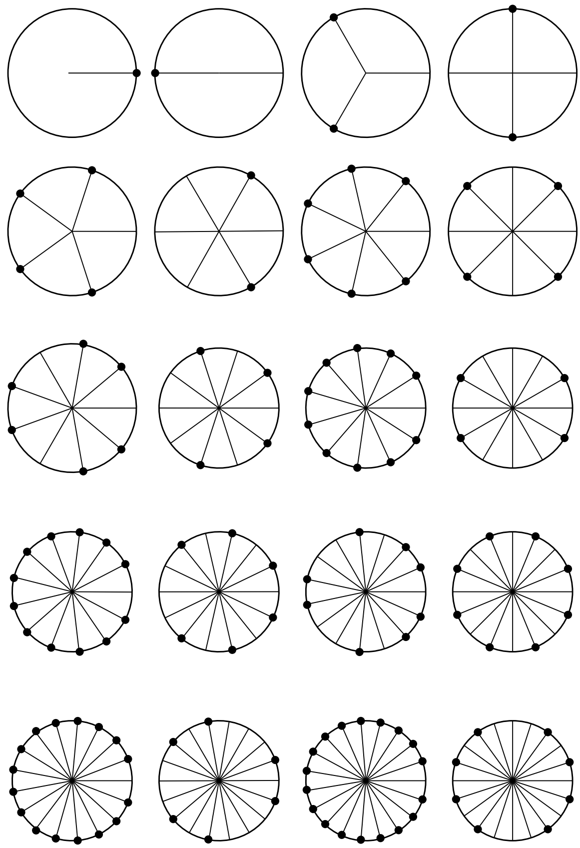


FIGURE 7.28
The extraction of the primitive roots of unity

only if k and n are relatively prime, meaning that they share no common divisors other than the unity *1* .

An equally-tempered scale marking the unit circle on the complex plane enforces cyclotomy, which is manifested geometrically on the screen in terms of regular polygons inscribed in the unit circle. Thus, cyclotomy corresponds spectrally to the generation of regular polygons. The deeper the resolution of the cyclotomy is, the higher the number of vertices appearing equally-spaced on the unit circle, and thus, the higher the number of sides of the inscribed regular polygon.

Note that the square power in relation to the unit circle corresponds to doubling the angle, and so on for all higher integer powers. Since the imaginary unit corresponds to a right angle, it is qualified as a fourth root of unity, since it has to be raised to the forth power to equal the unity, i.e. $i^4 = 1$, from which it is immediate that $i^2 = -1$.

From this perspective, the complex plane is set up according to the fourth roots of unity, where the vertices i, i^2, i^3, i^4, effect cyclotomy by means of a square inscribed in the unit circle, as follows:

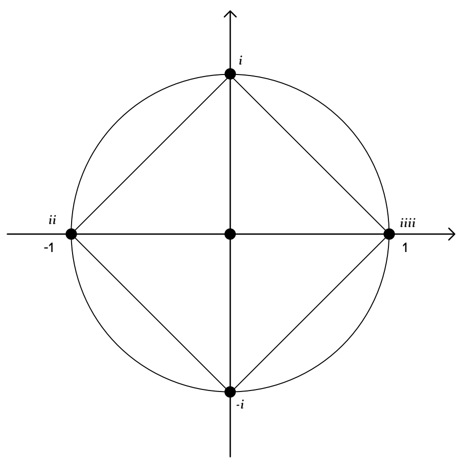


FIGURE 7.29
The fourth roots of unity and the constitution of the real and imaginary axes

Let us consider now the helicoidal universal covering epiphaneia extending over the disk included in the unit circle. Recall that the disk bears a dark spot in the middle, that is, a topological puncture, since it is the undisclosed resonance locus of the helicoidal axis bonding together all the branches.

The helical boundary, is universally covering the circle surrounding the disk. According to the universal covering principle of the circle by the helix, the circle S is made isomorphic to the spectrum of the reals modulo the integers:

$$S \cong \mathbb{R} / \mathbb{Z}$$

We recall that due to the above isomorphism, the circle appears as the partition spectrum emerging out of the modulation of the real line by the discrete integer windings of the helix.

Under the action of the polar spider web, the circle is identified with $i\mathbb{R}$, the imaginary axis of translation. Further, due to the quantization condition, qualifying the neutral element—the multiplicative unity on the circle—in terms of the discrete integers, we conclude that the windings of the helix are identical with the integer exponents of the powers that force the equality with the unity on the circle.

Therefore, we obtain a triple correspondence—a static tripod—which consists of the following: The roots of unity, the quanta characterized by the integrality condition, and the windings of the universally covering helix.

Recall that the integers are undisclosed from the perspective of the circle, each point of the circle carries with it an invisible fiber containing all the discrete integers—all the windings—such that, due to the cipher of quantization, is tuned to the harmonic domain via the invariant π that pertains both to the real length of its perimeter, and to its area.

This leads to the conclusion that the discrete integers with their additive structure, should be identified with both, the invariant harmonics, and with the exponents of the phases that enforce the unity on the circle. Concomitantly, the abducted roots of the scale are the bases of the corresponding powers effecting cyclotomy—under their extraction from the neutral unity—by means of their integer winding number.

Henceforth, any finite bounded portion of the helix qualify these windings in terms of harmonics, corresponding to finite covering spaces of the circle, and expressed as powers in the complex analytic setting. Recall again that the square power in relation to the unit circle corresponds to doubling the angle, and so on for all higher integer powers.

Let us consider the finite double covering of the circle by the circle. This corresponds spectrally to doubling the velocity, and thus the frequency, therefore dividing the unit circle into half. Similarly, if we consider the finite triple covering, it corresponds to tripling the frequency, and thus dividing the circle in three parts. Analogously, we treat all higher integer powers, and by inversion, that is, in terms of the extracted roots of unity, we are able to accomplish cyclotomy of any resolving depth, in an invariant manner.

Area Homology: Complex Logarithm and Residue Calculus

The winding number bears major significance because it allows to express homological relations pertaining to areas bounded under circulation around the center of synchronization in the punctured complex plane, independently of their location in geometric space.

All contours around the center sharing the same winding number are homologically equivalent, in the sense that they specify under integration the same quantum length of imaginary translation measured in terms of $2 \pi i \kappa$, where κ counts the number of quanta, identified in this way with the winding number.

The analytical tool to achieve this identification is the complex logarithm function, that is the inverse of the complex exponential function, where this inversion can be accomplished only locally, i.e. only in relation to branches of this function owing to the multi– valence of the angle.

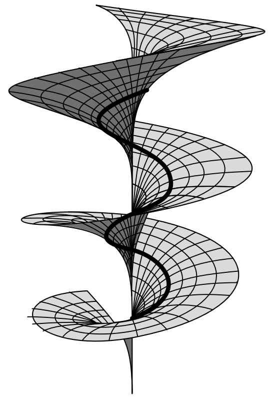


FIGURE 7.30
Branching structure of the complex logarithm

Nevertheless, the local inversion is conformal, that is, it respects and preserves all angles and their orientation within the respective branches.

Henceforth, the complex logarithm—in its operational role under contour integration around the center of synchronization—acts like an extremely high precision surgery instrument, which cuts through the continuum of the reals—continuous light spectrum of colours—and recognizes the quanta, the discrete harmonic invariants under any circulation around the center. For the complex logarithm the light spectrum is transparent, it delineates the harmonics via winding around, thus establishing the basic simple law of our screen/epiphaneia by means of unveiling and recognition of the discrete invariants.

In turn, these invariants are characterized as co– homological, meaning that they are revealed by means of contour integration around the center, which takes place along any circular chain surrounding the center.

As we shall see, this is expressed analytically in the simplest possible manner, through the reciprocation of the complex variable z, i.e. $1/z$, which is defined everywhere except the center $z = 0$ —thus living in the punctured plane excluding the center—by contour integration around the center.

It is absolutely clear in this manner that contour integration around any other point except the center will give a null result, something that determines the criterion of analytic behavior of any other function defined on the complex plane, or in a region of it, with respect to integration.

In other words, analyticity of a continuous function of a complex variable, making it holomorphic, amounts to giving a null result under contour integration for any closed chain contained in this region. Thus, a continuous complex-valued function f is analytic, if and only if:

$$\int_{\gamma} f(z)\, dz = 0$$

that is, the integral $\int_{\gamma} f(z)\, dz$ around any closed path is 0, or equivalently, the integral $\int_{\gamma} f(z)\, dz$ is path independent.

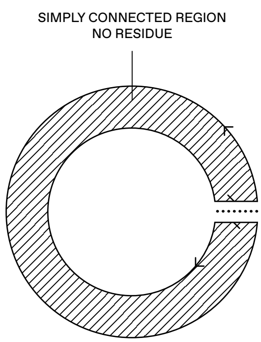


FIGURE 7.31
Notion of simple connectivity

It is this fact underlying the globally non-holomorphic branching behavior of the complex logarithm, which gives rise to the Riemann helicoid, consisting of an infinite number of branches bonded together around the axis of the helicoid. Locally, in the vicinity of a single branch, the holomorphic behavior is retained. The process of gluing those branches together, or equivalently, extending a single branch locally along another branch, is called continuous analytic continuation.

Let us consider in more detail the inversion of the complex exponential function, that is, the complex logarithm. Due to multiple connectivity, the complex exponential function is not globally invertible. It can be only locally inverted giving rise to a branch of the complex logarithm. Thus, we have:

$$log(r e^{i\theta}) = \{ log\, r + i(\theta + 2\pi k) : k \in \mathbb{Z} \}$$

which shows the multi- valency of the angle under integer complete circulations around the center of synchronization.

The complex logarithm can be also consistently defined through contour integration, corresponding to a complete circulation around the center of synchronization:

$$log(z) = \{ \int_{\gamma} (1/z)\, dz = \int_{\gamma} d\, log(z) : \gamma \text{ is a contour from 1 to } z \}$$

From the latter, we obtain:

$$\int_S dz/z = \int_S d\, log(z) = 2\pi i$$

In more detail, $z(\varphi) := e^{i\varphi},\ 0 \le \varphi \le 2\pi$, and we note that:

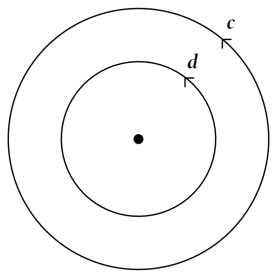
$$dz/d\varphi := z'(\varphi) = i\, z(\varphi)$$

such that:

$$\int_{S1} dz/z = \int_0^{2\pi} (z'(\varphi)/z(\varphi))\, d\varphi = \int_0^{2\pi} i\, d\varphi = 2\pi i$$

This means that the undisclosed center of synchronization is qualified—by means of the complex logarithm—in terms of the residue $2\pi i$ for one complete circulation around the center on the punctured complex plane, which specifies one quantum, according to the preceding.

FIGURE 7.32
Specification of one quantum



BOTH SIMPLE CLOSED CURVES *C* AND *D* SURROUNDING ONCE THE CENTER OF SYNCHRONIZATION LEAVE THE RESIDUE 2πi UNDER INTEGRATION

Equivalently, complex residue calculus spreads the undisclosed center of synchronization uniformly around the contour of integration by means of tempering through the tuning real parameter π. Of course, the number of circulations around the center is the integer winding number that determines the homology of the punctured disk screen.

We conclude that if γ is a finite– length path in $\mathbb{C} \setminus \{0\}$ from 1 to $z = r e^{i\theta}$, then there is a $k \in \mathbb{Z}$ such that:

$$\int_{\gamma} 1/w \ \ dw = \log r + i(\theta + 2\pi k)$$

where the integer k is the winding number, or else, the net number of times that the path crosses the positive real axis from the fourth quadrant.

Therefore, we obtain the following relation:

$$\begin{aligned} e^{\int_\gamma \frac{1}{w} dw} &= e^{\log(r) + i(\theta + 2\pi k)} \\ &= \underbrace{r\, e^{i\theta}}_{z} \ \underbrace{e^{i2\pi k}}_{=1} \\ &= z \end{aligned}$$

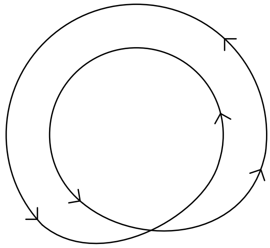


FIGURE 7.33
Specification of the winding number

The above certifies the consistency of the complex logarithm in relation to these two equivalent perspectives, that is, as the (local) inverse of the complex exponential effecting the branching behavior, and as the primitive—or potential—of the reciprocal of a complex variable accomplishing the contour integration for any circulation surrounding the undisclosed center.

Area Plasticity and the Architectonic Inside-Outside Distinction

From the elaboration of the notion of the complex logarithm function through contour integration in the punctured complex plane, we have concluded that the undisclosed center of synchronization is qualified topologically and analytically in terms of the factor $2\pi i$ for one complete circulation around the center in the punctured complex plane.

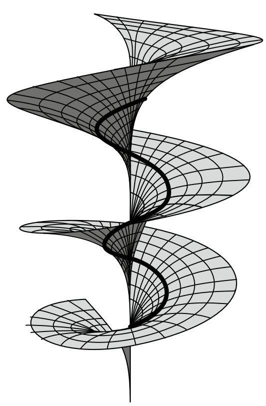


FIGURE 7.34
Simple connectivity of the universal covering helicoid

Note that the punctured complex plane is not a geometric space, since it is not simply-connected. But, the universal covering helicoidal epiphaneia is a geometric space, since it is simply-connected—not bearing any puncture.

There is something important encoded in the helicoidal shape of this universal covering epiphaneia that allows the homological parallel transport of area from layer to layer in terms of the additive group of the discrete integers, the winding numbers.

It is this fact which makes the notion of area essentially independent of its positioning in space, although it manifests geometrically as a magnitude. In other words, homological parallel transport abducts area from space. This gives a certain type of plasticity in the notion of area—abstracted from its geometric context—that needs to be explored further. A related question regarding this fact pertains to the role of the negative winding numbers, as well as their interrelation with the positive winding numbers.

We start this investigation by observing first of all that the shape of the contour of integration on the punctured complex plane circulating once around the center of synchronization does not matter. Any simple closed curve that circulates once around the center could be used as a contour of integration. Thus, we may simply consider a closed curve in the shape of a circle that circulates once around the center.

Topologically, we have a punctured disk whose boundary is a circle with a certain orientation—taken anti-clock-wise as a rule. This circle separates what is inside the contour and what is outside the contour. The distinction between the inside and the outside is very important in architecture and instrumental for establishing homological relations referring to area.

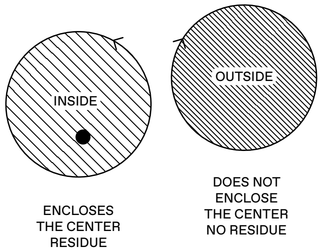


FIGURE 7.35
The fundamental architectonic inside/outside distinction

Let us consider the outside of this circle, or equivalently, the outside of the corresponding punctured disk, where the latter is identified as a unit disk with its center excluded topologically. The interesting thing that happens outside is that integration with respect to any contour, that is, with respect to any simple closed curve, returns a null result.

The reason is that the contour does not surround the center in this case, and as such the enclosed region outside is simply-connected. Consequently, any region bounded by a simple closed curve in the simply-connected outside is possible to be mapped analytically and conformally to the inside of another unit disk. The bounding circle of this unit disk is oriented oppositely to the one referring to the inside unit disk.

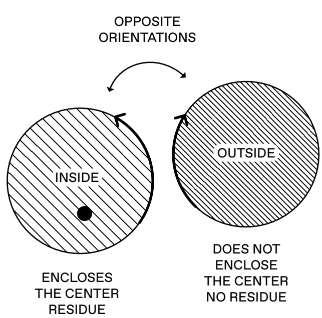


FIGURE 7.36
The opposite orientability distinguishing the inside from the outside

At the inside/outside cyclical boundary especial care is needed since the inside circle bears the opposite orientation from the one outside. If we glue them together we obtain a circular boundary that bounds a Möbius band formed by the antipodal gluing of these two disks.

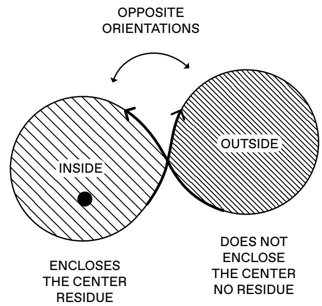


FIGURE 7.37
Gluing of the inside with the outside in the form of the analemma by antipodal tuning

We may employ the covering principle again, and consider the orientable double cover, where we take two copies, each of which corresponds to a different orientation. The interesting thing here is that, as a consequence, the real radial length of imaginary translation doubles although the boundary remains circular.

But, this is exactly what underlies the result of the integration $2\pi i$ corresponding to a circulation around the center. Recall that the inside area of the unit disk that can be unfolded to imaginary length by means of an imaginary translation is worth of π .

However, the imaginary length of the quantum between any two windings is $2\pi i$, meaning that we should consider the contribution of the outside disk as well. It is clear that in the orientable double cover the length doubles from the gluing of the inside with the outside disk. This is the reason that underlies the significance of the 2:1 ratio for our epiphaneia that becomes aware about the distinction between the inside and the outside.

Let us examine now the above in terms of the integrand f(z)=1/z, i.e. the reciprocal of the complex variable z, that appears as the derivative of the complex logarithm, in the sense that:

$$\int_S dz/z = 2\pi i$$

It is easy to see that the reciprocal function $f(z) = 1/z$ effects a metamorphosis of the inside of the unit disk to the outside and conversely. If the complex plane is viewed as a Cartesian plane, then the center of the inside disk is distributed uniformly and synchronically everywhere on the bounding circle of the simply-connected outside disk. In this sense, the unity of the outside disk is a distributive or dispersed unity.

Note that the radius of every point in the outside disk is the reciprocal of the corresponding point on the inside disk. However, their arguments are not the same, but they are complex conjugate to each other, something that explains why the imaginary unit always comes together with its conjugate.

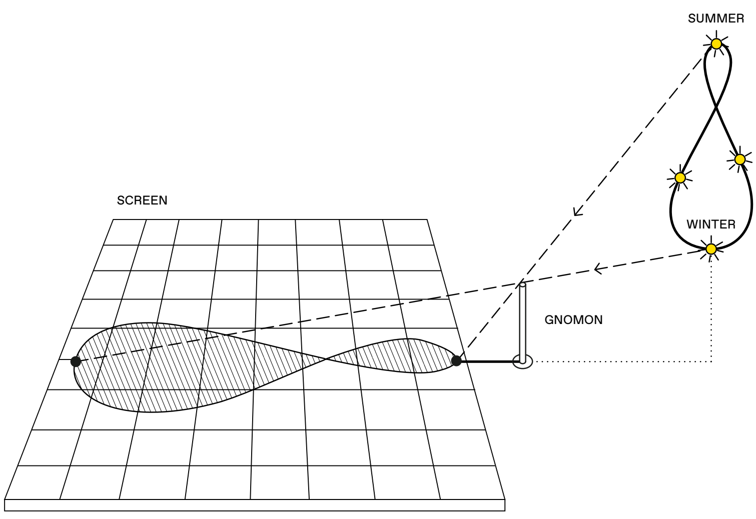


FIGURE 7.38
The geometric form of the analemma projected on the screen by a gnomon

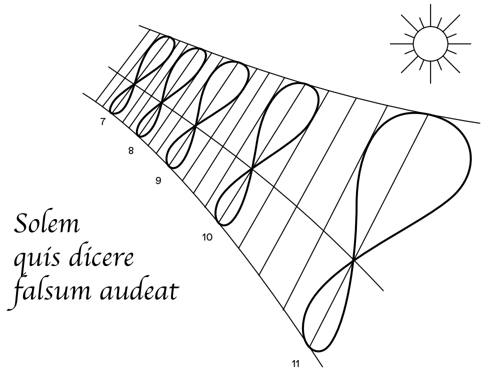


FIGURE 7.39
Horological device

Digitalization: Probability and the Abduction of Area from Space

The next problem we have to tackle is how it is possible to revert back from the complex to the real domain, such that the metaphora can be completed. Recall that the metaphora has been initiated by the bridge of extension of the real exponential function to the circle, which needed the technology of the imaginary unit, and the residue calculus based on it, to be accomplished. Further, the exponential function has been extended to the punctured disk screen, which is topologically homeomorphic with the complex plane under exclusion of the synchronization center.

In algebraic terms, if we consider the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers, it is isomorphic with the product of the circle with the positive real line, that is, $S^1 \times \mathbb{R}_+$. In other words, the former is structurally indistinguishable from the latter:

$$S^1 \times \mathbb{R}_+ \mapsto \mathbb{C}^\times$$

$$(\theta, r) \quad \mapsto r\,exp(i\,\theta)$$

Consider the complex exponential mapping:

$$exp : \mathbb{C} \mapsto \mathbb{C}^{\times}, \qquad z \mapsto exp\ z$$

as a homomorphism of the additive group of complex numbers $\mathbb{C}$ into the multiplicative group of non– zero complex numbers $\mathbb{C}^{\times} \cong S^1 \times \mathbb{R}_+$:

$$exp(\mathbb{C}) = \mathbb{C}^{\times}, \qquad Ker\,(exp) = 2\,\pi\,i\,\mathbb{Z}\,.$$

The complex exponential mapping bridges bidirectionally the additive world of the whole complex plane with the multiplicative world of the punctured complex plane bearing the center of synchronization.

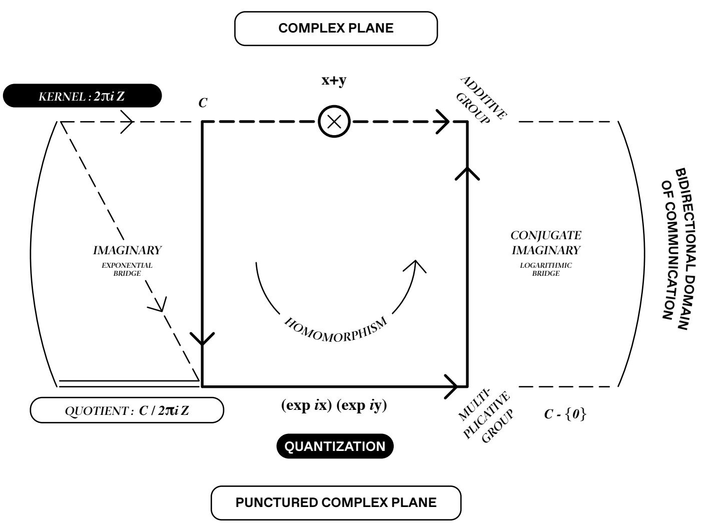


FIGURE 7.40
Architectonic scaffolding of the homomorphism from the additive group of complex numbers to the multiplicative group of non-zero complex numbers

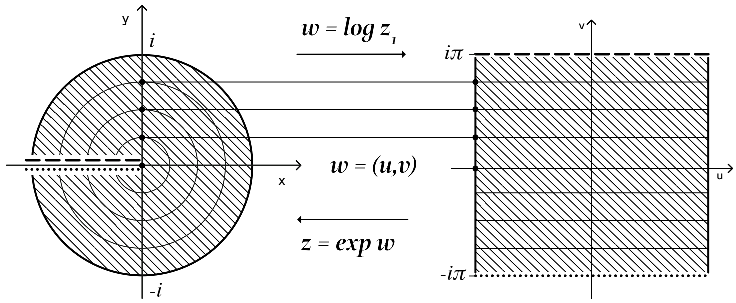


FIGURE 7.41
Spectral unrolling of a quantum from the harmonic to the geometric domain through a branch of the complex logarithm

This culminated in the key role of the complex logarithm, through which we managed to build—by contour integration—a spectrum, which is so sensitive and refined that bypasses the continuum of the reals to gain access to the harmonic invariants, revealing in this manner, the discrete quanta of length $2 \pi i$, where π plays the role of the tuning parameter in the tempering of the spectrum.

In this sense, the complex exponential encoding accomplishes the resolution of the harmonic invariants, which were hidden from the resolving spectrum of the positive reals, whereas the complex logarithm decoding—through the residue calculus—spreads the undisclosed center of synchronization uniformly around the contour of integration by means of tempering through the tuning real parameterπ, such that the number of circulations around the center is the integer winding number that determines the homology of the punctured disk screen.

The objective now is to examine how we can revert back from the complex domain to the real domain. This essentially amounts to qualifying amplitudes, i.e. imaginary translations, in terms of some type of transmutation of real geometric area. In this sense, we think of transmutation as a process that underlies the possibility of abducting the notion of area from its positioning in space, that is, what is encoded in the homology by means of the winding number.

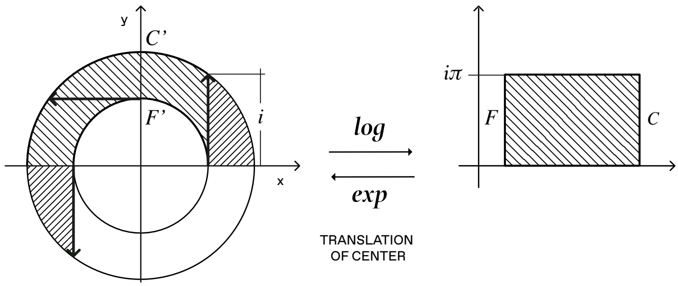


FIGURE 7.42
Homological aduction of area from its positioning in space by means of winding

Recall that the area span corresponding to a phase difference on the circle—under imaginary rotation—is transformed to the linear translation of the center of synchronization along the imaginary axis. This is necessarily pre– conditioned on the area preservation under complete imaginary rotations, as well as on the spontaneous synchronization of all the velocities at the undisclosed center of the disk.

Since imaginary translations pertain to quanta of length $2 \pi i$, and since the real geometric continuum is directly insensitive to the fine structure of the discrete invariants, the homological abduction of area from geometric space is possible only probabilistically, that is, in terms of objective chance.

Firstly, we recall that, since the kernel of the complex exponential is $2 \pi i \mathbb{Z}$, that is:

$$Ker\,(exp) = 2 \pi i \mathbb{Z}$$

the restriction of the complex exponential to the circle S, i.e.

$$p = exp: \mathbb{R} \mapsto S$$

is such that:

$$Ker\ p = Ker\,(exp) = \{\phi \in \mathbb{R} : i\,t \in Ker\,(exp)\} = \{\phi \in \mathbb{R} : \phi \in 2\pi\mathbb{Z}\}$$

Secondly, the simplest way to obtain a positive definite real– valued function in $\mathbb{R}_+$ — to be interpreted as a quantum probability density—from an amplitude in $\mathbb{C}^\times \cong S^1 \times \mathbb{R}_+$ is by squaring, and subsequently, taking the absolute value of the squared amplitude.

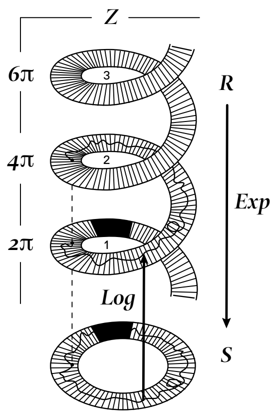


FIGURE 7.43
Homological notion of winding number

In this way, we obtain the interpretation of a quantum probability density $\psi(\phi)$, whose integration—with respect to the variable ϕ —over a 2π-interval is normalized to unity:

$$\int_{-\pi}^{+\pi} |\psi(\phi)|^2 \, d\phi = 1$$

Therefore, from the real-valued perspective, a quantum probability density, obtained in this manner, is what underlies the transmutation of the notion of area from geometric space.

The idea is that the synchronized area corresponding to an amplitude is totally undifferentiated in terms of space. But, we already know that it can be qualified in terms of a color. Recall that color, identified with the value of the real logarithm function, emerges from the area integration of the reciprocal of a real variable, which we think of as frequency in a continuous spectrum.

In this sense, the probability density over this continuous frequency spectrum should pertain to some quality of color that is not apparent yet, since it emerges from the imaginary domain, and is also amenable to quantum behavior.

What should be evident is that the quality we are looking for should be independent of classical geometric space. It rather opens up architectonically digital information space in a de-territorialized way by implementing the homological uplifting of the notion of area from geometric space by means of quantum probability.

Imaginary Time Arc: Potentiation of Angles on the Hyperbola

Let us start from the consideration of the exponential function on the circle:

$$\phi \mapsto exp(i\phi) = e^{i\phi}$$

Recall that every point of the circle (whose radius is considered one—called the unit circle) is parameterized surjectively by a phase $e^{i\phi}$, which is the potentiation of the real length ϕ that measures the angle in the domain of the exponential function. This happens through the imaginary unit i that characterizes the corresponding rotation as an imaginary rotation. An imaginary rotation preserves the area contained within the unit circle.

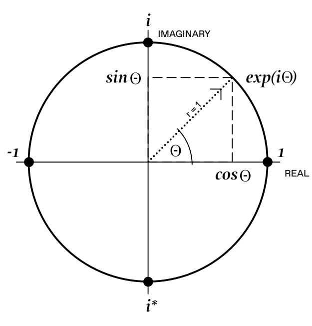


FIGURE 7.44
The complex exponential function valued on the unit circle on the complex plane

Note that what is real– valued in this setting is the arc length ϕ, which characterizes the angle in radians.

What happens if we consider the notion of an imaginary arc length? Is this notion meaningful at all? The reason behind posing this question is that if we consider that $\phi = i\,\varphi$, where φ is real, then by substitution, we obtain:

$$e^{i\phi} = e^{ii}\varphi = e^{-\varphi}$$

that is, we revert the exponential function in the real domain, but with the difference that the exponent bears a negative sign.

This is equivalent to the reciprocation of the positive real power e^{φ}, since the following identity holds:

$$1/e^{\varphi} = e^{-\varphi}$$

This is promising, but we have to make sense of what the operation of making the initial real length ϕ imaginary, that is, $i\,\varphi$, where φ is real, actually encodes.

For this purpose, we recall that the operation of the imaginary unit on a real variable χ amounts to a $\pi/2$ turn, such that $i\chi$ lies orthogonally with respect to χ. Thus, if χ runs around the unit circle on the complex plane, then $i\chi$ should run orthogonally to it. Nevertheless, synchronization of all velocities $i\chi$ at their common center, can be thought of equivalently in terms of an imaginary translation of the center, according to the principle of the imaginary rolling wheel.

We observe that if the real variable χ becomes imaginary, then:

$$e^{ii\chi} = e^{-\chi}$$

that is, if the real χ becomes imaginary, then, applying the exponential function upon it, we obtain a reciprocated positive real power. The result is in the positive real domain, and as such, it should be amenable to a qualification according to the real logarithm function, that is, in terms of colors under integration.

Recall that the colors identify areas under the rectangular hyperbola in an area-preserving manner. This is precisely what is expressed by the properties of the real logarithm function.

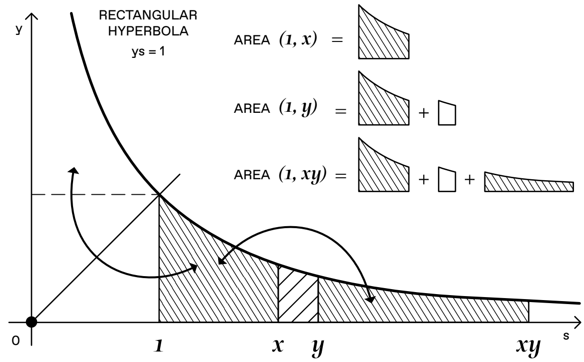


FIGURE 7.45
Areas under the rectangular hyperbola bearing a specific color correspond to real natural logarithms

Therefore, referring to the exponential function, the rectangular hyperbola should play in the real domain the role that the circle plays in the imaginary domain. This is the metaphora that we have to consider carefully. In the same way, that $e^{i\theta}$ potentiates the angle θ as a phase on the circle, we expect that $e^{ii\theta} = e^{-\theta}$ potentiates the angle θ on the rectangular hyperbola (Figure 7.45).

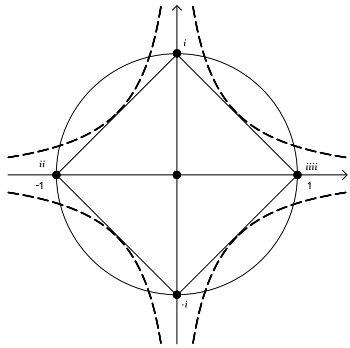


FIGURE 7.46
Potentiation of the angle on the rectangular hyperbola

Real Reciprocal Phases and Duplication of Areas

What type of angle potentiation is the one referring to the rectangular hyperbola? We may call the positive non-zero real, obtained in this manner, a reciprocal phase on the hyperbola, expressed as follows:

$$e^{i(i\theta)} = e^{-\theta}$$

We bear in mind that a hyperbola is characterized by two branches. Notwithstanding this fact, areas under the hyperbola retain exactly the same meaning and magnitude for both branches.

If we think of the arc length θ in terms of elapsing time on the circle to proceed anti-clock-wise from zero, then we expect that $i\,\theta$ would be the elapsing time on the hyperbola to proceed upwards—orthogonally to the circle—on the up branch, whereas $i^*\,\theta$ would be the elapsing time on the hyperbola to proceed downwards—orthogonally to the circle—on the lower branch.

In both cases, the orientation is such that areas are positive, and mirror each other precisely with respect to the symmetry axis of the hyperbola. Hence, the only invariant way to think of

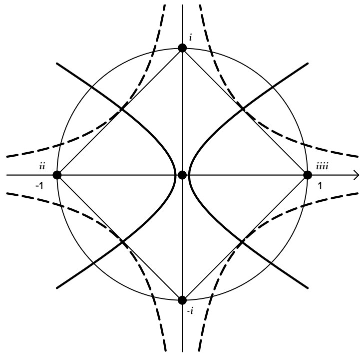


FIGURE 7.47
The imaginary arc length of the hyperbola and its relation to area

the elapsing time on the hyperbola is in terms of the area of the sector that is delineated on the hyperbola by the corresponding imaginary arc length.

In this sense, the complex conjugate of *i* amounts to time-reversal considered with respect to the lower branch. But, since areas are synchronized, it seems that each area under the upper branch of the hyperbola bears an exact copy identified under the lower branch. There is a duplication which is not apparent in space, that is, if we consider only the upper branch of the hyperbola and interpret the areas under it as geometric areas. This duplication should be the key that unlocks the association of areas with colors, and underlies as such, the quality of color.

From this perspective, the upper and the lower branch of the hyperbola, which seem disconnected in space, although they are placed symmetrically to each other, they acquire connectivity to each other via the circle. In particular, the connectivity of the two branches takes place through synchronizing imaginary translations, which are area-preserving by default.

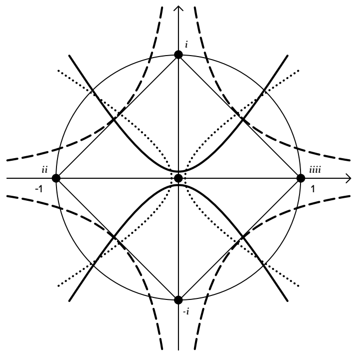


FIGURE 7.48
Indirect connectivity of the upper and lower branches of the hyperbolas and synchronization of areas

Arithmetic Mean Synchronization of Branches

We have to think now what this synchronization amounts to in relation to the notion of elapsing time. The crucial thing is that if we think of arc length—real on the circle, and imaginary on the hyperbola—in terms of elapsing time we have to consider how fast or slow this arc is traversed.

This is a general rule that pertains to the periodic/frequential aspect of time. The idea is that if we parameterize a line by a temporal parameter, then this parameter expresses continuously the total ordering of time along the line. But, if we parameterize an arc by a temporal parameter, then, because of multi– valency of the corresponding angle, the total ordering is lost, and we need to take into account how fast or slow this arc is traversed—expressing in this way the periodic/frequential aspect of time.

In our case, this is precisely what happens. Thus, if arc length θ is parameterized by elapsed time τ, it should be expressed as the product $\nu \ \tau$, where ν is the frequency of traversing the arc, and τ is the elapsed time. Clearly, the same arc can be traversed faster or slower, such that ν and τ are reciprocally co- related and their product is invariant. Note also that the arc remains invariant if both ν and τ are negative. In other words, this is again an expression of the uncertainty principle.

In the case of the hyperbola, parameterized by imaginary arc length, the task we face regarding synchronization amounts to matching each interval of ordered time t with an imaginary arc length $i\theta$ with respect to a whole spectrum of frequencies. But, since $i\theta$ parameterizes the upper branch, whereas $i^*\theta$ parameterizes the lower branch which are equipotent in terms of real area, t should be synchronized with both of them simultaneously. We will examine when this can be accomplished in what follows.

First, we consider the case, where t is real. Since t is real, and both, $i\theta$ and $i^*\theta$ — pertaining to the upper and lower branch of the hyperbola—are imaginary, it is not feasible directly. Plugging in the imaginary arcs in the exponential function on the circle, we obtain: For $i\theta$, the factor $e^{-\theta}$, and for $i^*\theta$, the factor $e^{+\theta}$.

We resolve the issue of synchronization with both branches by taking their arithmetic average, that is, $(e^{-\theta} + e^{+\theta})/2$.

This means that ordered time t runs in proportion to the area under the graph of the function $(e^{-\theta} + e^{+\theta})/2$. But, this is the famous catenary curve, or equivalently, the catenary arch, which is characterized precisely by this property. Thus, we obtain, from first principles, the major characteristic that exemplifies all the properties of the catenary.

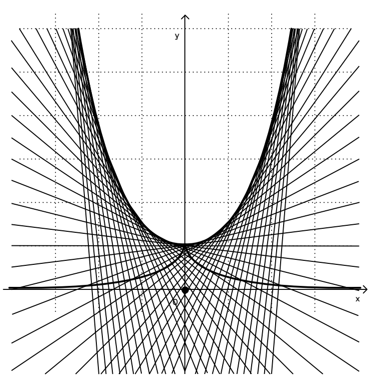


FIGURE 7.49
Ordered time runs in proportion to the area under the graph of the catenary

Imaginary Time and Constant Light Velocity: Special Relativity

Next, we consider the case of imaginary time arc, that is, $i\,t = i\,\theta = i\,\nu \cdot \tau$, where both ν and τ are reciprocally co- related and their product is invariant.

If time is imaginary, the issue of synchronization is resolved by default since we have both it and its complex conjugate i^* t simultaneously. We still have to be able to figure out what imaginary time means.

For this purpose, we have to distinguish two further possibilities. Interestingly enough, the first of them leads to relativity theory. As we will see in the next section, the second one leads to the bridge between quantum mechanics and thermal radiation. But, these different looking theories appear now as emerging from the same root. We proceed as follows:

First, we consider the case that time is imaginary, according to $i\,t = i\,\theta = i\,\nu \cdot \tau$, but we do not allow ν to vary ab initio. The only way that this can be feasible is by expressing spatial length in terms of temporal length through a constant, identified with the maximal velocity that the arc $i\,\theta$ can be traversed.

The idea is that this constant is provided by the velocity of light in vacuum, under the principle that light propagates with the same velocity for all observers independently of their state of motion. In other words, light through its velocity of propagation in vacuum effects the sought after synchronization.

The effect of this maneuver is to think of spatial distance in terms of imaginary temporal distance through the intervention of the universal constant velocity of light c, normalized to unity. Since, the distance in space is calculated by means of the Pythagorean theorem, that is, in terms of its square, by squaring $i\,t = i\,\theta = i\,\tau$, we obtain the term $-\tau^2$, interpreted spatially, but bearing a negative sign in comparison to ordinary spatial geometric distance. The crucial idea is that space and time should be considered as a unity, and the minus squared term changes the geometry of unified space– time from Euclidean to pseudo– Euclidean.

In space– time, there is no unique time– direction, just as there is no unique space direction. Specification of the velocity of an object—characterized by its inertial frame—is equivalent to the specification of its "time– direction", that is, the direction of its path through space– time, called the world– line.

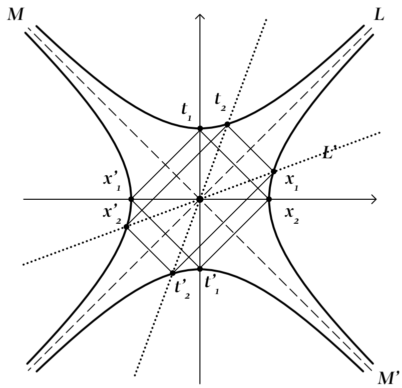


FIGURE 7.50
Invariance of rectangular areas in spacetime

In the above diagram, the lines $L'\ OL,\ M'\ OM$, which represent the paths of light on the light cone, are at right angles for light going in opposite directions, and in this case, the time– direction $T'\ OT$ of an object and its corresponding space– direction $X'\ OX$ are equally inclined to the light line $L'\ OL$ between them. Motion of an object involves rotating its time– direction and its space– direction through equal angles $T_1\ OT_2,\ X_1\ OX_2$ towards the light line which goes in the direction of motion.

The invariant connecting two events in space– time emerges as follows: Since space coordinates and time coordinates manifest under opposite signs when squared—according to the Pythagorean theorem—we need to consider both, the screen that cuts the light cone in a circle, and the orthogonal screen that cuts the light cone at the two branches of the hyperbola.

The idea is that the upper and the lower branch of the hyperbola, which seem disconnected in space, although they are placed symmetrically to each other, they acquire connectivity to each other via the circle in space-time. In particular, the connectivity of the two branches takes place through synchronizing imaginary translations, which are area-preserving by default.

In this way, the invariant connecting two events in space-time is given by:

$$I^2 = T^2 - X^2 - Y^2$$

where the spatial distances bear a minus sign, since they are expressed through the corresponding squared imaginary temporal distances. It is clear that if we adjoin a third spatial coordinate Z —contributing the term—Z^2 to the invariant formula—the validity of the argument is not affected.

The curve of constant radius and uniform curvature in a space– time screen instead of being a circle, as in a classical screen, is a rectangular hyperbola having a pair of light lines as asymptotes. Thus, in the diagram, $OT_1 = OT_2$, and $OX_1 = OX_2$. Unlike the circle, the rectangular hyperbola has a circumference of infinite length.

Since the velocity of an object gives the direction of its world– line path in space– time, acceleration corresponds to the curvature of its path, and is to be measured by the hyperbolic angle—imaginary arc length—through which the path turns per unit of its length, that is, per unit of time as measured by the object.

Instead of the relative velocity between objects in motion in space– time, we should consider the hyperbolic angle—imaginary arc length—between the world-lines of these objects, which is additive for objects going in the same direction.

Note that hyperbolic rotation pertaining to the above diagram involves rotating the space-direction and the corresponding time-direction by equal amounts in opposite senses.

Recall that $i\,\theta$ parameterizes the upper branch, whereas $i * \theta$ parameterizes the lower branch which are equipotent in terms of real area. Thus, i should be synchronized with both of them simultaneously.

Consider that the relative velocity between two objects in the same direction is v, where $-1 < v < 1$, where 1 is the speed of light in vacuum. We would like to express the relative velocity v in terms of the hyperbolic phases e^{θ}, and $e^{-\theta}$, where the first refers to the upper branch, and the second to the lower branch, synchronized with the invariant space– time interval.

The hyperbolic phases e^{θ}, and $e^{-\theta}$, are subject to the additive group structure of the reals. The sought after homomorphism should allow us to re-scale the additive group structure of the reals to the additive group structure of relative velocities in the interval $(-1, 1)$, since they are bounded by the constant speed of light.

The following function accomplishes the above objective:

$$v = \zeta(\theta) = (e^{\theta} - e^{-\theta}) / (e^{\theta} + {}^{-\theta}) : \mathbb{R} \mapsto (-1, 1)$$

This function is a bijection from R to (– 1,1), with inverse:

$$\theta = \zeta^{(-1)}\ (v) := \varpi(v) = 1/2$$

$$log((1 + v)/(1 - v)) = log\sqrt{((1 + v)/(1 - v))}$$

which is equivalent to the following expression of the hyperbolic phase e^{θ}:

$$e^{\theta} = ((1 + v)/(1 - v))^{1/2} = \sqrt{((1 + v)/(1 - v))}$$

The fact that ζ is a homomorphism of groups is established by means of the following:

$$\zeta(\theta + \delta) = \zeta(\theta) \oplus \zeta(\delta)$$

meaning that ζ transforms ordinary addition + in $\mathbb{R}$ to addition $\oplus$ restricted within the interval $(-1, 1)$. Inversely π transforms addition $\oplus$ in in the interval $(-1, 1)$ to ordinary addition + in $\mathbb{R}$, that is:

$$\varpi(v \oplus w) = \varpi(v) + \varpi(w)$$

where $v \oplus w$, that is the addition of relative velocities v and w in the interval $(-1, 1)$ is expressed as follows:

$$v \oplus w = (v + w)/(1 + v \cdot w)$$

in terms of ordinary addition and multiplication.

The underlying concept is that ordinary addition of relative velocities is not a closed operation within the interval $(-1, 1)$ bounded by the velocity of light. Thus, the additive group structure can be preserved in the interval $(-1, 1)$—referring to the structure of relative velocities—only if the additive group law is modified according to the above.

In a nutshell, the above bijection establishes a group isomorphism between the additive group of the real numbers—expressing real arc length—under the ordinary addition law +, and the additive group of the real numbers restricted in the interval $(-1, 1)$—expressing relative velocity—under the modified addition law $\oplus$.

We note that the function ζ that accomplishes the above bijective mapping—and structural isomorphism of additive groups—from $\mathbb{R}$ to $(-1, 1)$ is the hyperbolic tangent function, which is the natural sigmoid function, denoted by:

$$v = \zeta(\theta) = (e^{\theta} - e^{-\theta}) / (e^{\theta} + e^{-\theta}) := \tanh(\theta)$$

Its inverse function expresses 2θ in terms of the area under the graph of the hyperbola, which is characterized analytically in terms of the color identified by the real logarithm function, as follows:

$$2\theta = \log((1+v)/(1-v))$$

Since the curve of constant radius—the speed of light normalized to unity—and uniform curvature on the space– time screen is a rectangular hyperbola having a pair of light lines as asymptotes, the uncertainty principle holds, as an equivalent relation expressing the space– time invariance in terms of the hyperbola.

If we make use of the uncertainty principle, we can identify the blue– shift and red– shift factors in the relativistic Doppler effect in terms of the hyperbolic phases e^{θ} and $e^{-\theta}$ correspondingly.

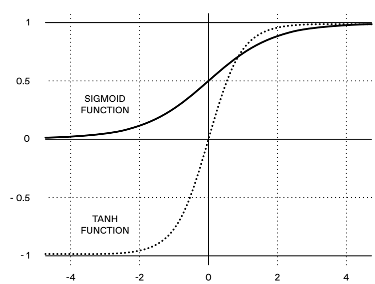


FIGURE 7.50
The graphs of the sigmoid and the hyperbolic tangent functions

Imaginary Time and Heat of Color: Quantization and Temperature

We consider that time is imaginary, according to the relation

$$i\,t = i\,\theta = i\,\nu\cdot\tau$$

but we allow ν to vary ab initio, interpreted genuinely as frequency with reference to a spectrum that has to be resolved appropriately for its meaning.

The variation takes place such that both ν and τ are reciprocally co-related and their product is invariant for any θ. Plugging in the imaginary time in the exponential function on the circle, we obtain the factor $e^{(-\nu\cdot\tau)}$, that is, a reciprocal phase on the rectangular hyperbola, where ν is on the horizontal axis and τ is on the vertical axis.

But, since from the perspective of the circle time is real, and $i\,\nu\cdot\tau$ is an imaginary translation preserving the area instantiated by the product $\nu\cdot\tau$ for any arc θ, it means that it is quantized in integer multiples of $2\,\pi\,i$.

Let $h/2\,\pi$ be the quantization proportionality constant. Since τ is real, the corresponding continuum frequencies ν should be made equivalent within classes indexed by the integers. But this is a further qualification of the spectrum, which was not apparent before.

How should we think of these blocks of the real partition spectrum, which are indexed by the integers, such that $\nu\cdot\tau$ is invariant under varying both ν and τ for a fixed arc, and $i\,\nu\cdot\tau$ is quantized in integer multiples of $2\,\pi\,i$?

Since the integer-indexed equivalence classes pertain to frequencies, while τ is real, and since, we keep thinking of the corresponding areas as colors, there must be some characteristic of color that is qualified here.

This characteristic is the heat of a color, leading to the conclusion that the blocks—that is, the equivalent areas—should each correspond to a certain temperature in equilibrium.

In other words, reciprocal phases $e^{-\nu\cdot\tau}$ on the hyperbola are equilibrium states corresponding to a certain temperature T with respect to the quantization constant of proportionality.

Each reciprocal phase corresponds—via this quantization constant—to a spatially undifferentiated macro-state in equilibrium, at a certain temperature T, described as a probability density function over the underlying indistinguishable microstates. Note that each of these phases keeps the real area $\nu\cdot\tau$ invariant for any considered arc.

Thermal Quantum Spectrum of Hyperbola and Incandescence

Starting from the assumption of imaginary time, according to the preceding, we obtain a thermal spectrum identified in terms of invariant areas under the hyperbola, where each block corresponds to a certain temperature T, and described as a reciprocal phase:

$$e^{-(h/2\pi)\cdot\nu\cdot 1/T}$$

on the hyperbola with respect to the quantization constant $h/2\pi$, which assures that $i\nu\cdot\tau$ is quantized in integer multiples of $2\pi i$.

In this manner, a real thermal radiation spectrum has been obtained by the dissociation of the notion of area from geometric space. This dissociation captures the essential property of color, which was veiled up to present. Namely, color is associated with temperature through thermal radiation.

In other words, our spectral epiphaneia not only absorbs light, but it emits light as radiation when heated at a certain temperature. The color of light radiated depends on the temperature. Thus, the real spectrum we have obtained in terms of reciprocal phases on the hyperbola is a spectrum of thermal radiation that is characterized by a specific color depending on the temperature, and thus, on heat.

Recall that this spectrum has been revealed through metaphora from the real domain to the imaginary and complex domains and back, where the major role has been played by the exponential function and its interaction with the circle and the hyperbola.

The thermal spectrum of color temperature requires both, quantization, and the abstraction of the notion of area from its spatial geometric association. Area can be transferred homologically and invariantly independently of space and throughout time by means of thermal radiation, which is made visible through color temperature.

Incandescence is the emission of light by an area that has been heated at a high temperature until it starts to radiate light. It is incandescence that makes heat visible with a particular color, which is independent of location in space. It depends only on temperature, which as we have seen corresponds to making time imaginary, and inducing a quantization condition on the spectrum in terms of the discrete integers. The imaginary qualification of the arc length on the hyperbola is the necessary condition for transcribing the quantization condition to the real thermal radiation spectrum.

In essence, our epiphaneia has been uplifted from the geometric to the harmonic domain, since it is capable not only to recognize the discrete harmonic invariants, but to make them visible macroscopically in a space-independent way through thermal radiation, and most important, the temperature of color.

What would be a proper scale for the radiating screen/epiphaneia?

We already know that our canon is logarithmic, and the logarithms quantify color under the hyperbola. Especially, the transcription of the complex logarithm that recognizes quanta in the real macroscopic spectrum of colors is based on the homology of area, understood invariantly via the rectangular hyperbola—including both of its branches—and reciprocal phases depending on temperature.

Macroscopically, our radiating color wheel can resonate independently of space– time distance in terms of temperature. This takes place through a probability density—based not on ignorance, but on objective indistinguishability and objective chance—which at the macroscopic equilibrium phases folds the continuum in countable partition blocks of the same color temperature. However, this is accomplished by the homology of area that is synchronized, which is at the root of the quantization condition.

But, recall that the quanta of arc length, refer to the imaginary translation of some undisclosed center of synchronization, the dark spot of the disk epiphaneia, the puncture on the topological complex plane.

This is where the logarithmic scale refers to in the thermal spectrum, to the radiation of the dark spot, the so called quantum black body radiation, which absorbs and then radiates everything.

But notice that the dark spot—as the center of synchronization—is excluded from the values of the complex exponential function. We have to revert back to the macroscopic real spectral domain of heat, where at very high temperature, the undisclosed dark harmonic center becomes visible reciprocally as the incandescent white light.

It is now possible to transmute both the gnomon—in the center of the disk— and its shadow—under the hyperbola— from space through metaphora that takes place architectonically between the harmonic and the geometric domain in natural communication to each other.

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