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Joint Functional Calculus for Definitizable Tuples of Self-Adjoint Krein Space Operators

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Abstract

The indefiniteness of Krein spaces gives rise to substantial complications. For instance, bounded self-adjoint linear Krein space operators are not well-behaved enough to allow for an appropriate analogue of the Spectral Theorem. To overcome this, classical literature imposes the additional assumption of definitizability. In the present work, we extend the notion of definitizability to tuples of pairwise commuting bounded self-adjoint operators and formulate the Spectral Theorem, expressed as a joint functional calculus, for definitizable tuples of Krein space operators. The definitizability of a tuple is a significantly weaker assumption than requiring each operator in the tuple to be definitizable.

The constructed functional calculus will produce the zero operator if applied to a function that vanishes on the joint spectrum of the respective operator tuple. Moreover, while the construction of the functional calculus is based on the choice of generators of the smallest ideal containing all definitizing polynomials of the respective operator tuple, it will be shown that the resulting functional calculus is not affected by that choice. Finally, the functional calculus will be compatible with the functional calculus of subtuples via the canonical projection.

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Eidesstattliche Erklärung

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Introduction

The Spectral Theorem for bounded linear self-adjoint operators on a Hilbert space is a well-known result, which has been generalized to finite tuples of pairwise commuting self-adjoint operators; see [Sch12] or Theorem 1.7.7. On the other hand, self-adjointness does not suffice in the Krein space setting to formulate a Spectral Theorem. Classical Spectral Theorems for Krein space operators mostly assume definitizability in addition to boundedness and self-adjointness of the operator; see for example [KP15, SK20]. A self-adjoint $A \in L_b(\mathcal{K})$ for a Krein space $(\mathcal{K}, [\cdot, \cdot])$ is called *definitizable* if there exists a so-called *definitizing polynomial* $p \in \mathbb{R}[z] \setminus \{0\}$ of A , such that $[p(A)x, x] \geq 0$ for all $x \in \mathcal{K}$.

In the present thesis, we introduce a joint functional calculus for a finite tuple of pairwise commuting bounded self-adjoint Krein space operators inspired by [Kal17] and [SK20]. While [SK20] also presented a joint functional calculus, the construction was built upon the assumption that each operator in the tuple is definitizable. However, we relax this condition and work with a *definitizable tuple* $A_1, \dots, A_n$ of pairwise commuting self-adjoint operators, that is, there exists $p \in \mathbb{R}[x_1, \dots, x_n] \setminus \{0\}$ such that $[p(A_1, \dots, A_n)x, x] \geq 0$ for all $x \in \mathcal{K}$. This definition of definitizability is a natural extension of the definitizability of normal operators introduced in [Kal17]. In fact, the main result of the present thesis is a generalization of the functional calculus for 2-tuples, decoded as a normal operator, as developed in [Kal17] for n -tuples.

We fix a definitizable tuple of pairwise commuting bounded self-adjoint operators $\mathbf{A} := (A_j)_{j=1}^n \in L_b(\mathcal{K})^n$, where $\mathcal{K}$ is a Krein space. Moreover, we denote by $\mathcal{I} \subseteq \mathbb{C}[x_1, \dots, x_n]$ the smallest ideal containing all definitizing polynomials of $\mathbf{A}$. Due to the Ascending Chain Condition 1.1.5, there exist definitizing polynomials $p_1, \dots, p_m \in \mathbb{R}[x_1, \dots, x_n]$ that generate the ideal $\mathcal{I}$. With the help of $p_1, \dots, p_m$ we will define several embeddings such as Θ and Ξ in Section 2.1, which have been first studied in [KP15]. Θ is a bounded $*$ -homomorphism defined on a certain subalgebra of $L_b(\mathcal{K})$ mapping into $L_b(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and $\Xi : L_b(\mathcal{H}) \rightarrow L_b(\mathcal{K})$ is an injective bounded linear operator. These embeddings allow us to move back and forth between Krein and Hilbert space setting. An essential part of our joint functional calculus will be the Hilbert space functional calculus of the tuple $\Theta[\mathbf{A}] := (\Theta(A_j))_{j=1}^n \in L_b(\mathcal{H})^n$.

In Section 2.4, we define a function algebra $\mathcal{F}_{\mathbf{A}}$, which will be the domain of the joint functional calculus. Functions belonging to $\mathcal{F}_{\mathbf{A}}$ are defined on $\sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I}) \subseteq \mathbb{C}^n$, where $V(\mathcal{I})$ is the variety of $\mathcal{I}$ and $\sigma(\Theta[\mathbf{A}])$ the *joint spectrum* of $\Theta[\mathbf{A}]$; see Definition 1.4.6. A crucial property of $\phi \in \mathcal{F}_{\mathbf{A}}$ is that it admits a decomposition (r, f) , where $r \in \mathbb{C}[x_1, \dots, x_n]$

and $f : \sigma(\Theta[\mathbf{A}]) \rightarrow \mathbb{C}$ is a bounded measurable function vanishing on $\sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$. Furthermore, polynomials can be embedded naturally into $\mathcal{F}_{\mathbf{A}}$ via $p \mapsto p_{\mathbf{A}}$; see Proposition 2.4.16.

For $\phi \in \mathcal{F}_{\mathbf{A}}$ admitting the decomposition (r, f) , we define the joint functional calculus by

$$\phi(\mathbf{A}) := r(\mathbf{A}) + \Xi \left(\int f dE \right),$$

where E denotes the *joint spectral measure* of $\Theta[\mathbf{A}] \in L_b(\mathcal{H})^n$; see Theorem 1.7.7. It will be verified that the choice of the decomposition of ϕ does not affect the functional calculus. Moreover, we will show that $\phi \mapsto \phi(\mathbf{A})$ constitutes a $*$ -homomorphism satisfying $p(\mathbf{A}) = p_{\mathbf{A}}(\mathbf{A})$ for all $p \in \mathbb{C}[x_1, \dots, x_n]$.

The main contribution of the present thesis is a significant generalization of the joint functional calculus developed in [Kal17]. In [Kal17], the functional calculus was constructed for an operator such that the ideal $\mathcal{I} \subseteq \mathbb{C}[x, y]$ generated by its definitizing polynomials had a finite codimension, i.e., $\dim \mathbb{C}[x_1, \dots, x_n]/\mathcal{I} < \infty$. This assumption implies that the variety $V(\mathcal{I})$ is finite. In contrast to this strong assumption, we only require the irreducible components of $V(\mathcal{I})$ to be pairwise disjoint; see Assumption 2.3.1 and Remark 2.3.2.

Moreover, we will see in Section 3.3 that the resulting functional calculus is not affected by the choice of the generators $p_1, \dots, p_m$ of $\mathcal{I}$. This robustness is remarkable since the embeddings Θ and Ξ , which are central to our construction, vary depending on the choice of $p_1, \dots, p_m$.

Chapter 1

Preliminaries

1.1 Ideals and Varieties

Throughout this section $\mathbb{K}$ shall be an arbitrary field.

Definition 1.1.1. Let $\mathcal{R}$ be a nonempty set. If $+, \cdot : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ are two binary operations on $\mathcal{R}$ such that

- (i) $(\mathcal{R}, +)$ constitutes an abelian group,
- (ii) $(\mathcal{R}, \cdot)$ constitutes a semigroup,
- (iii) $\cdot$ distributes over $+$,

then $(\mathcal{R}, +, \cdot)$ is called a *ring*. We call

- an element $e \in \mathcal{R}$ *unity* if $ex = xe = x$. If $\mathcal{R}$ contains a unity, $(\mathcal{R}, +, \cdot)$ is *unital*.
- an element $x \in \mathcal{R}$ in a unital ring $(\mathcal{R}, +, \cdot)$ *invertible* if there exists an element $y \in \mathcal{R}$ such that $xy = yx = e$, where e is the unity. The set of all invertible elements of $\mathcal{R}$ will be denoted by $\text{Inv}(\mathcal{R})$.
- $(\mathcal{R}, +, \cdot)$ *commutative* if $\cdot$ is commutative.

If the operations are clear from context, we will write $\mathcal{R}$ instead of $(\mathcal{R}, +, \cdot)$.

Definition 1.1.2. Let $(\mathcal{R}, +, \cdot)$ be a ring. A nonempty subset $\mathcal{I} \subseteq \mathcal{R}$ is an *ideal* if $(\mathcal{I}, +)$ is a subgroup of $(\mathcal{R}, +)$ and if $x \in \mathcal{I}, y \in \mathcal{R}$ implies $xy, yx \in \mathcal{I}$. An ideal $\mathcal{I} \subseteq \mathcal{R}$ is called

- *proper* if $\mathcal{I} \neq \mathcal{R}$.
- *maximal* if $\mathcal{I}$ is proper and $\mathcal{I} \subseteq \mathcal{I}'$ for some proper ideal $\mathcal{I}'$ implies $\mathcal{I} = \mathcal{I}'$.

Definition 1.1.3. By $\mathbb{K}[x_1, \dots, x_n]$ we denote the ring of polynomials in n variables with coefficients in $\mathbb{K}$. For $\mathbf{a} = (a_i)_{i=1}^n \in \mathbb{K}^n$ and $p \in \mathbb{K}[x_1, \dots, x_n]$, we set $p(\mathbf{a}) := p(a_1, \dots, a_n)$.

Lemma 1.1.4. If $\mathcal{R}$ is a commutative ring and $p_1, \dots, p_m \in \mathcal{R}$, then the set

$$\langle p_1, \dots, p_m \rangle := \left\{ \sum_{j=1}^m h_j p_j : h_1, \dots, h_m \in \mathcal{R} \right\}$$

constitutes an ideal in $\mathcal{R}$. We refer to $p_1, \dots, p_m$ as generators of $\langle p_1, \dots, p_m \rangle$, which is the smallest ideal containing $p_1, \dots, p_m$.

Proof. Given $f, g \in \langle p_1, \dots, p_m \rangle$ there exist $h_1, \dots, h_m, h'_1, \dots, h'_m \in \mathcal{R}$ such that

$$f = \sum_{j=1}^m h_j p_j, \quad g = \sum_{j=1}^m h'_j p_j.$$

Therefore,

$$f + g = \sum_{j=1}^m (h_j + h'_j) p_j \in \langle p_1, \dots, p_m \rangle.$$

Given $h \in \mathcal{R}$ we have

$$fh = \sum_{j=1}^m (h_j h) p_j \in \langle p_1, \dots, p_m \rangle.$$

The minimality is clear. □

The following result will be crucial throughout this thesis. A proof can be found in [CLO07, p.79].

Theorem 1.1.5 (Ascending Chain Condition). *Let $(\mathcal{I}_k)_{k \in \mathbb{N}}$ be a sequence of ideals in $\mathbb{K}[x_1, \dots, x_n]$. If $\mathcal{I}_k \subseteq \mathcal{I}_{k+1}$ for all $k \in \mathbb{N}$, then there exists $N \in \mathbb{N}$ such that $\mathcal{I}_{N+j} = \mathcal{I}_N$ for all $j \in \mathbb{N}$.*

Clearly, the Ascending Chain Condition implies the following result.

Theorem 1.1.6 (Hilbert's Basis Theorem). *Every ideal $\mathcal{I} \subseteq \mathbb{K}[x_1, \dots, x_n]$ has a finite set of generators.*

Definition 1.1.7. Given $S \subseteq \mathbb{K}[x_1, \dots, x_n]$ we define

$$V(S) = \{z \in \mathbb{K}^n : p(z) = 0 \text{ for all } p \in S\},$$

the *variety* of S . Clearly, $V(\mathbb{K}[x_1, \dots, x_n]) = \emptyset$.

Notation 1.1.8. For $n, m \in \mathbb{Z}$ with $n < m$, we set $[n, m]_{\mathbb{Z}} := \{n, n+1, \dots, m-1, m\}$.

The following lemma provides a simple way to analyze the variety of an ideal.

Lemma 1.1.9. *Given $p_1, \dots, p_m \in \mathbb{K}[x_1, \dots, x_n]$ we have*

$$V(\langle p_1, \dots, p_m \rangle) = \{z \in \mathbb{K}^n : p_j(z) = 0, \quad j \in [1, m]_{\mathbb{Z}}\}.$$

Proof. For $N := \{z \in \mathbb{K}^n : p_j(z) = 0, \quad j \in [1, m]_{\mathbb{Z}}\}$ we have

$$\sum_{j=1}^m h_j(z) p_j(z) = 0, \quad h_1, \dots, h_m \in \mathbb{K}[x_1, \dots, x_n], \quad z \in N,$$

which means $N \subseteq V(\langle p_1, \dots, p_m \rangle)$. The converse inclusion is obvious. □

Definition 1.1.10. Let $\mathcal{I}, \mathcal{J} \subseteq \mathbb{K}[x_1, \dots, x_n]$ be ideals. We define the sum and product of $\mathcal{I}$ and $\mathcal{J}$ as

$$\begin{aligned}\mathcal{I} + \mathcal{J} &:= \{f + g : f \in \mathcal{I}, g \in \mathcal{J}\}, \\ \mathcal{I} \cdot \mathcal{J} &:= \left\{ \sum_{i=1}^m f_i g_i : m \in \mathbb{N}, f_1, \dots, f_m \in \mathcal{I}, g_1, \dots, g_m \in \mathcal{J} \right\}.\end{aligned}$$

Lemma 1.1.11. If $\mathcal{I} = \langle f_1, \dots, f_{m_1} \rangle$, $\mathcal{J} = \langle g_1, \dots, g_{m_2} \rangle \subseteq \mathbb{K}[x_1, \dots, x_n]$ are ideals, then $\mathcal{I} + \mathcal{J}$ constitutes an ideal satisfying

$$\mathcal{I} + \mathcal{J} = \langle f_1, \dots, f_{m_1}, g_1, \dots, g_{m_2} \rangle.$$

Moreover, $V(\mathcal{I} + \mathcal{J}) = V(\mathcal{I}) \cap V(\mathcal{J})$.

Proof. By definition $\mathcal{I} + \mathcal{J}$ is an additive subgroup of $\mathbb{K}[x_1, \dots, x_n]$. Given $f \in \mathcal{I}$, $g \in \mathcal{J}$, and $h \in \mathbb{K}[x_1, \dots, x_n]$, we have $fh \in \mathcal{I}$ and $gh \in \mathcal{J}$ as $\mathcal{I}$ and $\mathcal{J}$ are ideals. Hence, $(f + g)h = fh + gh \in \mathcal{I} + \mathcal{J}$. Consequently, $\mathcal{I} + \mathcal{J}$ is an ideal.

Since $\langle f_1, \dots, f_{m_1}, g_1, \dots, g_{m_2} \rangle$ is an ideal that contains both $\mathcal{I}$ and $\mathcal{J}$, we have $\mathcal{I} + \mathcal{J} \subseteq \langle f_1, \dots, f_{m_1}, g_1, \dots, g_{m_2} \rangle$. The other inclusion follows from Lemma 1.1.4.

By Lemma 1.1.9 we have

$$\begin{aligned}V(\mathcal{I} + \mathcal{J}) &= V(\langle f_1, \dots, f_{m_1}, g_1, \dots, g_{m_2} \rangle) \\ &= \{z \in \mathbb{K}^n : f_i(z) = g_j(z) = 0, i \in [1, m_1]_{\mathbb{Z}}, j \in [1, m_2]_{\mathbb{Z}}\} \\ &= \{z \in \mathbb{K}^n : f_i(z) = 0, i \in [1, m_1]_{\mathbb{Z}}\} \cap \{z \in \mathbb{K}^n : g_j(z) = 0, j \in [1, m_2]_{\mathbb{Z}}\} \\ &= V(\mathcal{I}) \cap V(\mathcal{J}).\end{aligned}$$

□

Lemma 1.1.12. If $\mathcal{I} = \langle f_1, \dots, f_{m_1} \rangle$, $\mathcal{J} = \langle g_1, \dots, g_{m_2} \rangle \subseteq \mathbb{K}[x_1, \dots, x_n]$ are ideals then $\mathcal{I} \cdot \mathcal{J}$ constitutes an ideal satisfying

$$\mathcal{I} \cdot \mathcal{J} = \langle \{f_i g_j : i \in [1, m_1]_{\mathbb{Z}}, j \in [1, m_2]_{\mathbb{Z}}\} \rangle.$$

Moreover, $V(\mathcal{I} \cdot \mathcal{J}) = V(\mathcal{I}) \cup V(\mathcal{J})$.

Proof. We have $0 = 0 \cdot 0 \in \mathcal{I} \cdot \mathcal{J}$. Given $l_1, l_2 \in \mathbb{N}$, $p_1, \dots, p_{l_1}, r_1, \dots, r_{l_2} \in \mathcal{I}$, and $q_1, \dots, q_{l_1}, s_1, \dots, s_{l_2} \in \mathcal{J}$, we have

$$\sum_{i=1}^{l_1} p_i q_i + \sum_{i=1}^{l_2} r_i s_i \in \mathcal{I} \cdot \mathcal{J}.$$

Additionally, given $h \in \mathbb{K}[x_1, \dots, x_n]$, we have

$$h \sum_{i=1}^{l_1} p_i q_i = \sum_{i=1}^{l_1} \underbrace{h p_i}_{\in \mathcal{I}} q_i \in \mathcal{I} \cdot \mathcal{J}.$$

Thus $\mathcal{I} \cdot \mathcal{J}$ constitutes an ideal.

For $i \in [1, l_1]_{\mathbb{Z}}$ there exist $h_{i,1}, \dots, h_{i,m_1}, h'_{i,1}, \dots, h'_{i,m_2} \in \mathbb{K}[x_1, \dots, x_n]$ such that

$$p_i = \sum_{j=1}^{m_1} h_{i,j} f_j, \quad q_i = \sum_{j=1}^{m_2} h'_{i,j} g_j$$

implying

$$p_i q_i = \sum_{j=1}^{m_1} \sum_{k=1}^{m_2} (h_{i,j} h'_{i,k}) f_j g_k.$$

Hence,

$$\sum_{i=1}^{l_1} p_i q_i = \sum_{j=1}^{m_1} \sum_{k=1}^{m_2} \left(\sum_{i=1}^{l_1} h_{i,j} h'_{i,k} \right) f_j g_k$$

belongs to $\langle \{f_i g_j : i \in [1, m_1]_{\mathbb{Z}}, j \in [1, m_2]_{\mathbb{Z}}\} \rangle$. We have shown that $\mathcal{I} \cdot \mathcal{J}$ is a subset of $\langle \{f_i g_j : i \in [1, m_1]_{\mathbb{Z}}, j \in [1, m_2]_{\mathbb{Z}}\} \rangle$. The converse inclusion follows from Lemma 1.1.4.

Finally, by Lemma 1.1.9

$$\begin{aligned} V(\mathcal{I} \cdot \mathcal{J}) &= V(\langle \{f_i g_j : i \in [1, m_1]_{\mathbb{Z}}, j \in [1, m_2]_{\mathbb{Z}}\} \rangle) \\ &= \{z \in \mathbb{K}^n : f_i(z) g_j(z) = 0, \quad i \in [1, m_1]_{\mathbb{Z}}, \quad j \in [1, m_2]_{\mathbb{Z}}\} \\ &= \{z \in \mathbb{K}^n : f_i(z) = 0 \vee g_j(z) = 0, \quad i \in [1, m_1]_{\mathbb{Z}}, \quad j \in [1, m_2]_{\mathbb{Z}}\} \\ &= \{z \in \mathbb{K}^n : f_i(z) = 0, \quad i \in [1, m_1]_{\mathbb{Z}}\} \cup \{z \in \mathbb{K}^n : g_j(z) = 0, \quad j \in [1, m_2]_{\mathbb{Z}}\} \\ &= V(\mathcal{I}) \cup V(\mathcal{J}). \end{aligned} \quad \square$$

Remark 1.1.13. It is easy to check that the intersection of ideals from a ring again constitutes an ideal. Moreover, $\mathcal{I} \cdot \mathcal{J} \subseteq \mathcal{I} \cap \mathcal{J}$ for ideals $\mathcal{I}$ and $\mathcal{J}$.

Definition 1.1.14. Let $\mathcal{R}$ be a unital ring. Two ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{R}$ satisfying $\mathcal{I} + \mathcal{J} = \mathcal{R}$ are called *comaximal*.

Notation 1.1.15. Let $\mathcal{I}$ be an ideal in $\mathbb{K}[x_1, \dots, x_n]$. We are going to denote the elements from the factor algebra $\mathbb{K}[x_1, \dots, x_n]/\mathcal{I}$ by

$$[p]_{\mathcal{I}} := p + \mathcal{I}, \quad p \in \mathbb{K}[x_1, \dots, x_n].$$

A proof for the following result can be found in [GW22].

Theorem 1.1.16. (Chinese Remainder Theorem) *Let $\mathcal{R}$ be a unital ring and $\mathcal{I}_1, \dots, \mathcal{I}_m \subseteq \mathcal{R}$ pairwise comaximal ideals. For $\mathcal{I} := \bigcap_{j=1}^m \mathcal{I}_j$, the mapping*

$$\varphi : \begin{cases} \mathcal{R}/\mathcal{I} & \rightarrow \mathcal{R}/\mathcal{I}_1 \times \dots \times \mathcal{R}/\mathcal{I}_m, \\ [r]_{\mathcal{I}} & \mapsto ([r]_{\mathcal{I}_1}, \dots, [r]_{\mathcal{I}_m}), \end{cases}$$

constitutes a ring-isomorphism.

Lemma 1.1.17. *If we have $\mathcal{R} = \mathbb{K}[x_1, \dots, x_n]$ in the setting of Theorem 1.1.16, then φ is linear.*

Proof. Since φ is a ring homomorphism, it is additive with respect to the ring addition, which coincides with the vector space addition in $\mathbb{K}[x_1, \dots, x_n]$. In order to show the homogeneity, let $p \in \mathbb{K}[x_1, \dots, x_n]$ and $\lambda \in \mathbb{K}$. Writing the multiplication in a ring as $\cdot$ and the scalar multiplication without a symbol, we have

$$\lambda \cdot p = \lambda p,$$

for $\lambda \in \mathbb{K}$ and $p \in \mathbb{K}[x_1, \dots, x_n]$ if we interpret λ as an element of $\mathbb{K}[x_1, \dots, x_n]$. This implies

$$\begin{aligned} \varphi(\lambda [p]_{\mathcal{I}}) &= \varphi([\lambda \cdot p]_{\mathcal{I}}) = \varphi([\lambda]_{\mathcal{I}}) \cdot \varphi([p]_{\mathcal{I}}) = ([\lambda]_{\mathcal{I}_j})_{j=1}^m \cdot ([p]_{\mathcal{I}_j})_{j=1}^m \\ &= ([\lambda \cdot p]_{\mathcal{I}_j})_{j=1}^m = \lambda ([p]_{\mathcal{I}_j})_{j=1}^m = \lambda \varphi([p]_{\mathcal{I}}). \end{aligned} \quad \square$$

Lemma 1.1.18. *Let $\mathcal{R}$ be a unital commutative ring. If $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{R}$ are comaximal ideals, then $\mathcal{I}_1 \cap \mathcal{I}_2 = \mathcal{I}_1 \cdot \mathcal{I}_2$.*

Proof. The inclusion $\mathcal{I}_1 \cap \mathcal{I}_2 \supseteq \mathcal{I}_1 \cdot \mathcal{I}_2$ follows immediately from the definition of ideals; see Remark 1.1.13. For the other inclusion, suppose $x \in \mathcal{I}_1 \cap \mathcal{I}_2$ and let $a \in \mathcal{I}_1, b \in \mathcal{I}_2$ be such that $a + b = 1$. We derive $x = xa + xb \in \mathcal{I}_1 \cdot \mathcal{I}_2$. $\square$

The following theorem is often referred to as the weak Nullstellensatz. A proof can be found in [CLO07, p.170].

Theorem 1.1.19. *Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{I} \subseteq \mathbb{K}[x_1, \dots, x_n]$ an ideal. If $V(\mathcal{I}) = \emptyset$, then $\mathcal{I} = \mathbb{K}[x_1, \dots, x_n]$.*

Lemma 1.1.20. *Ideals $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathbb{C}[x_1, \dots, x_n]$ have disjoint varieties if and only if they are comaximal.*

Proof. According to Lemma 1.1.11, we have $V(\mathcal{I}_1 + \mathcal{I}_2) = V(\mathcal{I}_1) \cap V(\mathcal{I}_2)$. Hence, by Theorem 1.1.19, $V(\mathcal{I}_1) \cap V(\mathcal{I}_2) = \emptyset$ yields $\mathcal{I}_1 + \mathcal{I}_2 = \mathbb{C}[x_1, \dots, x_n]$. On the other hand, if $\mathcal{I}_1, \mathcal{I}_2$ are comaximal, then $V(\mathcal{I}_1 + \mathcal{I}_2) = V(\mathbb{C}[x_1, \dots, x_n]) = \emptyset$. $\square$

Notation 1.1.21. Given $N \in \mathbb{N}$ and ideals $\mathcal{R}_1, \dots, \mathcal{R}_N \subseteq \mathbb{C}[x_1, \dots, x_n]$, we write

$$\prod_{j=1}^N \mathcal{R}_j := (((\mathcal{R}_1 \cdot \mathcal{R}_2) \cdot \mathcal{R}_3) \cdots) \cdot \mathcal{R}_N$$

for the iterative product of the ideals $\mathcal{R}_1, \dots, \mathcal{R}_N$ in the sense of Definition 1.1.10. This notation makes sense as multiplication of ideals is associative, which is evident from Lemma 1.1.12.

Lemma 1.1.22. *Let $N \in \mathbb{N}$ and $\mathcal{R}_1, \dots, \mathcal{R}_N \subseteq \mathbb{C}[x_1, \dots, x_n]$ be ideals such that their varieties are pairwise disjoint. For every $k \in [1, N]_{\mathbb{Z}}$ we conclude*

$$\bigcap_{j=1}^k \mathcal{R}_j = \prod_{j=1}^k \mathcal{R}_j, \quad (1.1)$$

$$V\left(\prod_{j=1}^k \mathcal{R}_j\right) = \bigcup_{j=1}^k V(\mathcal{R}_j). \quad (1.2)$$

Proof. The case $N = 1$ is trivial. We show (1.1) and (1.2) for all $k \in [2, N]_{\mathbb{Z}}$ by induction. The case $k = 2$ is covered by Lemma 1.1.18 and Lemma 1.1.12. Assuming $N > 2$ suppose (1.1) and (1.2) hold true for some $k \in [2, N - 1]_{\mathbb{Z}}$. We conclude

$$V(\mathcal{R}_{k+1}) \cap V\left(\prod_{j=1}^k \mathcal{R}_j\right) = V(\mathcal{R}_{k+1}) \cap \bigcup_{j=1}^k V(\mathcal{R}_j) = \emptyset.$$

By Lemma 1.1.20, $\mathcal{R}_{k+1}$ and $\prod_{j=1}^k \mathcal{R}_j$ are comaximal. By Lemma 1.1.18,

$$\bigcap_{j=1}^{k+1} \mathcal{R}_j \stackrel{(1.1)}{=} \left(\prod_{j=1}^k \mathcal{R}_j\right) \cap \mathcal{R}_{k+1} = \prod_{j=1}^{k+1} \mathcal{R}_j.$$

Finally, Lemma 1.1.12 yields

$$V\left(\prod_{j=1}^{k+1} \mathcal{R}_j\right) = V\left(\prod_{j=1}^k \mathcal{R}_j\right) \cup V(\mathcal{R}_{k+1}) \stackrel{(1.2)}{=} \bigcup_{j=1}^{k+1} V(\mathcal{R}_j).$$

□

1.2 Primary Decomposition of an Ideal

Definition 1.2.1. Let $\mathcal{I} \subseteq \mathcal{R}$ be an ideal in a commutative unital ring $\mathcal{R}$ and $x, y \in \mathcal{R}$.

- If $\mathcal{I}$ is proper and $xy \in \mathcal{I} \wedge x \notin \mathcal{I}$ implies $y \in \mathcal{I}$, then $\mathcal{I}$ is called *prime*.
- If $\mathcal{I}$ is proper and $xy \in \mathcal{I} \wedge x \notin \mathcal{I}$ implies $y^k \in \mathcal{I}$ for some $k \in \mathbb{N}$, then $\mathcal{I}$ is called *primary*.
- If $\mathcal{I}$ is proper and $x^k \in \mathcal{I}$ for some $k \in \mathbb{N}$ implies $x \in \mathcal{I}$, then $\mathcal{I}$ is called *radical*.
- The *radical* of $\mathcal{I}$, denoted $\sqrt{\mathcal{I}}$, is the set

$$\left\{x : x^k \in \mathcal{I} \text{ for some } k \in \mathbb{N}\right\}.$$

Clearly, $\mathcal{I} \subseteq \sqrt{\mathcal{I}}$.

Lemma 1.2.2. If $\mathcal{I} \subseteq \mathbb{K}[x_1, \dots, x_n]$ is an ideal, then $\sqrt{\mathcal{I}}$ constitutes an ideal.

Proof. Given $f, g \in \sqrt{\mathcal{I}}$ there exist $k, l \in \mathbb{N}$ such that $f^k, g^l \in \mathcal{I}$. For every term in the binomial expansion

$$(f + g)^{k+l} = \sum_{i=0}^{k+l} \binom{k+l}{i} f^i g^{k+l-i},$$

either $i \geq k$ or $k + l - i \geq l$. Hence, $(f + g)^{k+l} \in \mathcal{I}$ and, in turn, $f + g \in \sqrt{\mathcal{I}}$. If $h \in \mathbb{K}[x_1, \dots, x_n]$, then $(hf)^k = h^k f^k \in \mathcal{I}$. Hence, $hf \in \sqrt{\mathcal{I}}$. □

Lemma 1.2.3. If $\mathcal{I} \subseteq \mathbb{K}[x_1, \dots, x_n]$ is a primary ideal, then $\sqrt{\mathcal{I}}$ is a prime ideal.

Proof. Let $\mathcal{I}$ be primary and $f, g \in \mathbb{K}[x_1, \dots, x_n]$ be such that $fg \in \sqrt{\mathcal{I}}$ and $f \notin \sqrt{\mathcal{I}}$. By the definition of the radical, there exists $k \in \mathbb{N}$ such that $(fg)^k \in \mathcal{I}$. Since $f^k \notin \mathcal{I}$ and $\mathcal{I}$ is primary, there exists $l \in \mathbb{N}$ such that $g^{kl} \in \mathcal{I}$. We conclude $g \in \sqrt{\mathcal{I}}$. □

Recall the following facts; see for example [GW22, 3.3.2.4].

Facts 1.2.4. Let $\mathcal{R}$ be a commutative unital ring.

- (i) An ideal $\mathcal{I} \subseteq \mathcal{R}$ is maximal if and only if $\mathcal{R}/\mathcal{I}$ constitutes a field.
- (ii) Every maximal ideal is prime.

Lemma 1.2.5. If $\mathcal{I} \subseteq \mathbb{K}[x_1, \dots, x_n]$ is an ideal such that $\sqrt{\mathcal{I}}$ is maximal, then $\mathcal{I}$ is primary.

Proof. For $f, g \in \mathbb{K}[x_1, \dots, x_n]$ we assume $fg \in \mathcal{I}$ and $f \notin \mathcal{I}$. In order to show that $\mathcal{I}$ is primary, it suffices to demonstrate that $[g]_{\mathcal{I}}$ is nilpotent in $\mathcal{R} := \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}$. Note that $\mathbb{K}[x_1, \dots, x_n]$ and $\mathcal{R}$ are commutative unital rings. Since $\sqrt{\mathcal{I}}$ is maximal,

$$\mathbb{K}[x_1, \dots, x_n]/\sqrt{\mathcal{I}} \cong (\mathbb{K}[x_1, \dots, x_n]/\mathcal{I})/(\sqrt{\mathcal{I}}/\mathcal{I}) = \mathcal{R}/\sqrt{\{0\}}$$

constitutes a field. Thus the ideal $\sqrt{\{0\}}$ of all nilpotent elements is a maximal ideal in $\mathcal{R}$. Since every prime ideal in $\mathcal{R}$ contains $\sqrt{\{0\}}$, $\sqrt{\{0\}}$ is the only prime ideal in $\mathcal{R}$. As every maximal ideal is prime, $\sqrt{\{0\}}$ is the only maximal ideal in $\mathcal{R}$. Since $[g]_{\mathcal{I}}$ is a zero divisor, $\langle [g]_{\mathcal{I}} \rangle$ must be a proper ideal. Hence, $\langle [g]_{\mathcal{I}} \rangle \subseteq \sqrt{\{0\}}$. $\square$

Proposition 1.2.6. $V(\mathcal{I}) = V(\sqrt{\mathcal{I}})$ holds true for every ideal $\mathcal{I} \subseteq \mathbb{K}[x_1, \dots, x_n]$.

Proof. The inclusion $V(\mathcal{I}) \supseteq V(\sqrt{\mathcal{I}})$ is a consequence of $\mathcal{I} \subseteq \sqrt{\mathcal{I}}$. If $(a_i)_{i=1}^n \notin V(\sqrt{\mathcal{I}})$, then there exists $p \in \sqrt{\mathcal{I}}$ such that $p(a_1, \dots, a_n) \neq 0$. By the definition of a radical, $p^k \in \mathcal{I}$ for some $k \in \mathbb{N}$. Thus, $(a_i)_{i=1}^n \notin V(\mathcal{I})$. $\square$

Definition 1.2.7. A *primary decomposition* of an ideal $\mathcal{I} \subsetneq \mathbb{K}[x_1, \dots, x_n]$ is an expression of $\mathcal{I}$ as a finite intersection of primary ideals $\mathcal{Q}_1, \dots, \mathcal{Q}_m$. We call

- $\mathcal{Q}_1, \dots, \mathcal{Q}_m$ the *primary components*.
- $\sqrt{\mathcal{Q}_j}$ the associated prime of $\mathcal{Q}_j$, $j \in [1, m]_{\mathbb{Z}}$.

A primary decomposition is called *minimal* if the following conditions are met.

- (i) The associated primes $\sqrt{\mathcal{Q}_1}, \dots, \sqrt{\mathcal{Q}_m}$ are pairwise distinct.
- (ii) The primary components satisfy $\mathcal{Q}_j \not\supseteq \bigcap_{k \neq j}^m \mathcal{Q}_k$ for $j \in [1, m]_{\mathbb{Z}}$.

A proof of the following theorem is provided in [CLO07, p.211].

Theorem 1.2.8 (Lasker-Noether). Every ideal $\mathcal{I} \subsetneq \mathbb{K}[x_1, \dots, x_n]$ admits a minimal primary decomposition.

The following theorem is one of the two results concerning uniqueness of minimal primary decompositions. A proof appears in [BWK93, p.362] and [Ati16, p.52].

Theorem 1.2.9. Let $\mathcal{I} \subsetneq \mathbb{K}[x_1, \dots, x_n]$ be an ideal. Any two minimal primary decompositions of $\mathcal{I}$ have the same number of primary components and the same set of associated primes.

Definition 1.2.10. Let $\mathcal{I} \subsetneq \mathbb{K}[x_1, \dots, x_n]$ be an ideal. We refer to the unique associated primes of a minimal primary decomposition of $\mathcal{I}$ as primes *belonging to* $\mathcal{I}$. We say that a prime $\mathcal{P}$ belonging to $\mathcal{I}$ is

- *isolated* if $\mathcal{P}' \subseteq \mathcal{P}$ implies $\mathcal{P} = \mathcal{P}'$ for any $\mathcal{P}'$ belonging to $\mathcal{I}$.
- *embedded* if $\mathcal{P}$ is not isolated.

We cite the following result in Example 1.2.12. A proof using polynomial division in $\mathbb{K}[x_1, \dots, x_n]$ can be found in [CLO07, p.201].

Proposition 1.2.11. For $(a_j)_{j=1}^n \in \mathbb{K}^n$ the ideal

$$\langle x_1 - a_1, \dots, x_n - a_n \rangle \subseteq \mathbb{K}[x_1, \dots, x_n]$$

is maximal.

The example below demonstrates that the minimal primary decomposition is not unique in general.

Example 1.2.12. Consider the ideal $\langle x^2, xy \rangle \subseteq \mathbb{K}[x, y]$. We have

$$\langle x^2, xy \rangle = \langle x \rangle \cap \langle x^2, y \rangle = \langle x \rangle \cap \langle x^2, xy, y^2 \rangle.$$

The ideal $\langle x \rangle$ is prime and thus primary. Both $\langle x^2, y \rangle$ and $\langle x^2, xy, y^2 \rangle$ have the radical $\langle x, y \rangle$, which is maximal by Proposition 1.2.11. By Lemma 1.2.5, $\langle x^2, y \rangle$ and $\langle x^2, xy, y^2 \rangle$ are primary. Thus the ideal $\langle x^2, xy \rangle$ has two distinct minimal primary decompositions.

Notice that in Example 1.2.12, we could vary the primary component with an embedded associated prime to obtain a different minimal primary decomposition. The following result is the second uniqueness theorem regarding minimal primary decompositions and states that the primary component with an isolated associated prime is unique. A proof can be found in [BWK93, p.364], [Ati16, p.54], and [BGS24, p.242, T.7.12]

Theorem 1.2.13. If $\mathcal{I} \subsetneq \mathbb{K}[x_1, \dots, x_n]$ is an ideal and $\mathcal{P}$ is an isolated prime belonging to $\mathcal{I}$, then there exists a primary ideal $\mathcal{Q}$ such that in any minimal primary decomposition of $\mathcal{I}$ the primary component with the associated prime $\mathcal{P}$ is $\mathcal{Q}$.

Corollary 1.2.14. Let $\mathcal{I} \subsetneq \mathbb{K}[x_1, \dots, x_n]$ be an ideal. If the primes belonging to $\mathcal{I}$ have pairwise disjoint varieties, then the minimal primary decomposition of $\mathcal{I}$ is unique.

Proof. Note that none of the primes belonging to $\mathcal{I}$ has an empty variety due to Theorem 1.1.19. Since the primes belonging to $\mathcal{I}$ have pairwise disjoint varieties, they are pairwise incomparable with respect to $\subseteq$. Hence each prime belonging to $\mathcal{I}$ is isolated. The uniqueness of the minimal primary decomposition follows from Theorem 1.2.13. $\square$

The minimal primary decomposition of an ideal generated by a univariate polynomial corresponds to the prime factorization of the polynomial. Since $\mathbb{C}[x]$ is a principal ideal domain, finding a minimal primary decomposition of an ideal in $\mathbb{C}[x]$ boils down to finding its greatest common divisor and factorizing it.

Proposition 1.2.15. For $f \in \mathbb{C}[x]$ let $a_1, \dots, a_l \in \mathbb{C}$ and $m_1, \dots, m_l \in \mathbb{N}$ be such that $f = \prod_{i=1}^l (x - a_i)^{m_i}$ is the prime factorization of f , where $a_1, \dots, a_l$ are pairwise distinct. Then

$$\langle f \rangle = \bigcap_{i=1}^l \langle (x - a_i)^{m_i} \rangle$$

is the unique minimal primary decomposition of $\langle f \rangle$.

Proof. First, note that $V(\langle (x - a_i)^{m_i} \rangle) = \{a_i\}$; see Lemma 1.1.9. Since $a_1, \dots, a_l$ are pairwise distinct, the varieties of the ideals $\langle (x - a_i)^{m_i} \rangle, i \in [1, l]_{\mathbb{Z}}$, are pairwise disjoint. Thus we can apply Lemma 1.1.22 and obtain

$$\bigcap_{i=1}^l \langle (x - a_i)^{m_i} \rangle = \prod_{i=1}^l \langle (x - a_i)^{m_i} \rangle.$$

We know from Lemma 1.1.12 that the ideal $\prod_{i=1}^l \langle (x - a_i)^{m_i} \rangle$ is generated by the product of the generators of each ideal, which is $f = \prod_{i=1}^l (x - a_i)^{m_i}$.

For a fixed $i \in [1, l]_{\mathbb{Z}}$ the radical of $\langle (x - a_i)^{m_i} \rangle$ is $\langle x - a_i \rangle$, which is maximal according to Proposition 1.2.11. By Lemma 1.2.5, $\langle (x - a_i)^{m_i} \rangle$ is primary. Finally, we obtain the minimality of this primary decomposition from the fact that $a_1, \dots, a_l$ are pairwise distinct and the uniqueness from Corollary 1.2.14. $\square$

1.3 Joint Spectrum in Commutative Unital Algebras

In the present section, we will define the joint spectrum of a tuple of elements of a commutative unital algebra. The goal is to show that the joint spectrum is nonempty and compact for tuples of elements of a commutative unital Banach algebra.

Definition 1.3.1. Let $\mathcal{A} \neq \{0\}$ be a vector space over $\mathbb{C}$. If $\mathcal{A}$ is endowed with an associative bilinear map

$$\cdot : \begin{cases} \mathcal{A} \times \mathcal{A} & \rightarrow \mathcal{A}, \\ (a, b) & \mapsto a \cdot b, \end{cases}$$

then $\mathcal{A}$ together with $\cdot$ is called *algebra*. Note that an algebra is a ring with a vector space structure. Thus we will employ the terms defined in Definition 1.1.1.

- We call a linear subspace B of an algebra $\mathcal{A}$ a *subalgebra* if B is closed under the multiplication.
- If $\mathcal{A}$ is endowed with a norm $\|\cdot\|$ which satisfies

$$\|xy\| \leq \|x\| \|y\| \text{ for all } x, y \in \mathcal{A},$$

then we call $\mathcal{A}$ a *normed algebra*. If in addition $(\mathcal{A}, \|\cdot\|)$ is a Banach space, $\mathcal{A}$ is called a *Banach algebra*.

- If a normed algebra $\mathcal{A}$ contains a unity e , then we say e is *normed* if $\|e\| = 1$. If a normed algebra contains a normed unity, then we call $\mathcal{A}$ a *unital normed algebra*. If additionally $\mathcal{A}$ is a Banach algebra, $\mathcal{A}$ is called *unital Banach algebra*.

Definition 1.3.2. If an algebra $\mathcal{A}$ is additionally endowed with a mapping $.^* : \mathcal{A} \rightarrow \mathcal{A}$ such that

- (i) $(x^*)^* = x$ for all $x \in \mathcal{A}$,
- (ii) $(\lambda x + y)^* = \bar{\lambda}x^* + y^*$ for all $x, y \in \mathcal{A}$, $\lambda \in \mathbb{C}$,
- (iii) $(xy)^* = y^*x^*$ for all $x, y \in \mathcal{A}$,

then $\mathcal{A}$ is called **-algebra*. If $\mathcal{A}$ is also a Banach algebra and $.^*$ satisfies

$$\|a\| = \|a^*\| \quad \text{and} \quad \|aa^*\| = \|a\|^2, \quad a \in \mathcal{A},$$

then $\mathcal{A}$ is called *C*-algebra*. An element a in a *-algebra $\mathcal{A}$ is called

- *normal* if $a^*a = aa^*$.
- *self-adjoint* if $a = a^*$.
- *unitary* if $\mathcal{A}$ is unital and $aa^* = a^*a = e$.

Lemma 1.3.3. Let $\mathcal{A}$ be a normed algebra. If we endow $\mathcal{A} \times \mathcal{A}$ with the product topology, the multiplication is continuous.

Proof. Let $(a_i, b_i)_{i \in \mathbb{N}}$ be a sequence in $\mathcal{A} \times \mathcal{A}$ that converges to (a, b) with respect to the norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$ for $x, y \in \mathcal{A}$. One easily derives $a_i \rightarrow a$ and $b_i \rightarrow b$. In particular, $(a_i)_{i \in \mathbb{N}}$ is bounded and therefore

$$\|a_i b_i - ab\| = \|a_i(b_i - b) + (a_i - a)b\| \leq \|a_i\| \|b_i - b\| + \|a_i - a\| \|b\|$$

yields $\|a_i b_i - ab\| \rightarrow 0$. □

Definition 1.3.4. Let $\mathcal{A}$ be an algebra.

- A subalgebra I of $\mathcal{A}$ is called *ideal* if $ai, ia \in I$ for all $a \in \mathcal{A}$ and $i \in I$.
- An ideal I is called *proper* if $I \neq \mathcal{A}$.
- A proper ideal I is called *maximal* if $I \subseteq J$ implies $I = J$ for a proper ideal J .

Remark 1.3.5. Note that while an algebra is a ring, the definition of a ring-ideal in Definition 1.1.2 and algebra-ideal in Definition 1.3.4 do not coincide in general. While an algebra-ideal is always a ring-ideal, the converse is not universally true. This is because an algebra-ideal is a subalgebra and therefore additionally requires the closedness under scalar multiplication. In this section, we will use the term ideal to refer to algebra-ideals as in Definition 1.3.4.

A special case where these two definitions coincide is when $\mathcal{A}$ is a unital algebra. Indeed, consider a ring-ideal I in a unital algebra $\mathcal{A}$ with unity e . Due to the bilinearity of the multiplication, we have

$$\alpha i = \alpha(e \cdot i) = (\alpha e) \cdot i \in I, \quad i \in I, \quad \alpha \in \mathbb{C}.$$

Thus I is closed under scalar multiplication.

Example 1.3.6. The ring of polynomials $\mathbb{C}[x_1, \dots, x_n]$ is an algebra. Endowed with the mapping $p \mapsto p^\#$, where $p^\#(x_1, \dots, x_n) := \overline{p(\overline{x_1}, \dots, \overline{x_n})}$, $\mathbb{C}[x_1, \dots, x_n]$ constitutes a $*$ -algebra. Since it is unital, the two notions of ideals coincide for $\mathbb{C}[x_1, \dots, x_n]$.

Definition 1.3.7. For every x in a unital algebra $\mathcal{A}$ with unity e , we denote by

$$\rho_{\mathcal{A}}(x) := \{\lambda \in \mathbb{C} : (x - \lambda e) \in \text{Inv}(\mathcal{A})\}$$

the *resolvent set* of x and by

$$\sigma_{\mathcal{A}}(x) := \mathbb{C} \setminus \rho_{\mathcal{A}}(x) = \{\lambda \in \mathbb{C} : (x - \lambda e) \notin \text{Inv}(\mathcal{A})\}$$

the *spectrum* of x . If the algebra is clear from context, we will write $\rho(x)$ and $\sigma(x)$.

See [WKB25, 6.4] for a verification of the following facts.

Facts 1.3.8. Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then

- (i) $\text{Inv}(\mathcal{A}) \subseteq \mathcal{A}$ is open.
- (ii) $\sigma(a) \subseteq \mathbb{C}$ is compact.
- (iii) $\rho(a) \subseteq \mathbb{C}$ is open.

A proof of the following result is given in [Kal23, Satz 1.1.16].

Theorem 1.3.9 (Gelfand-Mazur). *For a unital Banach algebra $\mathcal{A}$ such that $\text{Inv}(\mathcal{A}) = \mathcal{A} \setminus \{0\}$, we have $\dim \mathcal{A} = 1$.*

Lemma 1.3.10. *In a unital algebra $\mathcal{A}$ any proper ideal I satisfies $I \cap \text{Inv}(\mathcal{A}) = \emptyset$. If $\mathcal{A}$ is a commutative unital algebra, then $a \in \mathcal{A}$ is invertible if and only if $a \notin I$ for any proper ideal I .*

Proof. If $a \in \text{Inv}(\mathcal{A})$, then there exists $b \in \mathcal{A}$ such that $ab = e$, where e is the unity. Thus for any ideal I with $a \in I$, we have $e = ab \in I$, which means $I = \mathcal{A}$.

If $\mathcal{A}$ is commutative and $a \notin \text{Inv}(\mathcal{A})$, then $a\mathcal{A} = \{ax : x \in \mathcal{A}\}$ constitutes a proper ideal. □

Lemma 1.3.11. *If $\mathcal{A}$ is a unital Banach algebra, then the closure of a proper ideal is a proper ideal.*

Proof. Let I be a proper ideal and $a \in \mathcal{A}$. The closure of a subspace is also a subspace. Since the multiplication by a is continuous, we have $a \cdot \text{cl}(I) \subseteq \text{cl}(aI) \subseteq \text{cl}(I)$ and, similarly, $\text{cl}(I) \cdot a \subseteq \text{cl}(I)$. Hence I constitutes an ideal.

Since I is proper, we obtain $I \subseteq \mathcal{A} \setminus \text{Inv}(\mathcal{A})$ from Lemma 1.3.10. $\text{Inv}(\mathcal{A})$ being open yields $\text{cl}(I) \subseteq \text{cl}(\mathcal{A} \setminus \text{Inv}(\mathcal{A})) = \mathcal{A} \setminus \text{Inv}(\mathcal{A})$. Since $\text{Inv}(\mathcal{A}) \ni e$ is nonempty, $\text{cl}(I)$ is proper. □

Lemma 1.3.12. *Let $\mathcal{A}$ be a normed algebra and I a closed proper ideal. $\mathcal{A}/I$ endowed with the quotient norm*

$$\|a + I\| := \inf_{i \in I} \|a + i\|$$

constitutes a normed algebra. If in addition $\mathcal{A}$ is a unital normed algebra (Banach algebra), so is $\mathcal{A}/I$. Moreover, the canonical projection

$$\pi_{\mathcal{A}/I} : \begin{cases} \mathcal{A} & \rightarrow \mathcal{A}/I, \\ a & \mapsto a + I, \end{cases}$$

constitutes an algebra homomorphism.

Proof. As I is closed, $\mathcal{A}/I$ constitutes a normed space with the quotient norm; see [WKB25, 2.4.9]. If $\mathcal{A}$ is a Banach space, so is $\mathcal{A}/I$; see [WKB25, 2.4.9]. Given $a, b \in \mathcal{A}$ and $i, j \in I$, we have

$$\|a + i\| \|b + j\| \geq \|(a + i)(b + j)\| = \|ab + \underbrace{aj + ib + ij}_{\in I}\| \geq \|ab + I\|.$$

Thus, $\|ab + I\| \leq \|a + I\| \|b + I\|$.

If $\mathcal{A}$ is a unital normed algebra, we have

$$0 \neq \|e + I\| \leq \|e + I\| \|e + I\|,$$

implying $1 \leq \|e + I\|$. On the other hand, we have $\|e + I\| \leq \|e + 0\| = 1$. By the definition of the operations on $\mathcal{A}/I$, $\pi_{\mathcal{A}/I}$ constitutes an algebra homomorphism. $\square$

Proposition 1.3.13. *If $\mathcal{A}$ is a commutative unital Banach algebra, then every maximal ideal in $\mathcal{A}$ has codimension one.*

Proof. Let $I \subseteq \mathcal{A}$ be a maximal ideal. By Lemma 1.3.11, I is closed. According to Lemma 1.3.12, $\mathcal{A}/I$ constitutes a unital Banach algebra. Since $\mathcal{A}$ is a commutative unital ring, we identify $\mathcal{A}/I$ as a field. Hence, $\text{Inv}(\mathcal{A}/I) = (\mathcal{A}/I) \setminus \{0\}$. By Theorem 1.3.9 the dimension of $\mathcal{A}/I$, which equals the codimension of I , is one. $\square$

Definition 1.3.14. Let $\mathcal{A}, \mathcal{B}$ be algebras. We call a mapping $\phi : \mathcal{A} \rightarrow \mathcal{B}$ an *algebra homomorphism* if it is linear and satisfies $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{A}$.

- If an algebra homomorphism is bijective, it is called an *algebra isomorphism*.
- If $\mathcal{A}, \mathcal{B}$ are $*$ -algebras and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is an algebra homomorphism satisfying $\phi(x^*) = \phi(x)^*$ for all $x \in \mathcal{A}$, then ϕ is called a *$*$ -homomorphism*.

Definition 1.3.15. Let $\mathcal{A}$ be an algebra. A linear functional $m : \mathcal{A} \rightarrow \mathbb{C}$ is *multiplicative* if $m \neq 0$ and

$$m(xy) = m(x)m(y) \text{ for all } x, y \in \mathcal{A}.$$

We denote by $M_{\mathcal{A}}$ the *Gelfand space* of $\mathcal{A}$, which is defined to be the set of all multiplicative functionals on $\mathcal{A}$.

Remark 1.3.16. A multiplicative linear functional is nothing but a non-trivial algebra homomorphism into $\mathbb{C}$.

Lemma 1.3.17. If $\mathcal{A}$ is a unital algebra and $m \in M_{\mathcal{A}}$, then $m(e) = 1$.

Proof. First of all, $m(e) \neq 0$ because $m(e) = 0$ would imply $m(x) = m(x)m(e) = 0$ for all $x \in \mathcal{A}$. From $m(e) = m(e)m(e)$ we infer $m(e) = 1$. $\square$

Proposition 1.3.18. Let $\mathcal{A}$ be a commutative unital Banach algebra. If I is a maximal ideal, then there exists a multiplicative functional $m_I \in M_{\mathcal{A}}$ such that $\ker m_I = I$.

Proof. By Proposition 1.3.13, $\mathcal{A}/I$ has codimension one and $e + I \neq 0$. Therefore, the mapping $\phi_I : \lambda e + I \mapsto \lambda$ constitutes a linear bijection from $\mathcal{A}/I$ onto $\mathbb{C}$. Furthermore, it is multiplicative. Hence ϕ_I is an algebra isomorphism.

We set $m_I := \phi_I \circ \pi_{\mathcal{A}/I}$, where $\pi_{\mathcal{A}/I}$ is as in Lemma 1.3.12. As a composition of two algebra homomorphisms, m_I is also one. $\ker \phi_I = \{0\}$ together with $\ker \pi_{\mathcal{A}/I} = I$ implies $\ker m_I = I$. In particular, $m_I \neq 0$. $\square$

Corollary 1.3.19. The Gelfand space of a commutative unital Banach algebra is non-empty.

Proof. Let $\mathcal{A}$ be a commutative unital Banach algebra. Since $\{0\}$ is a proper ideal, there exists a maximal ideal I in $\mathcal{A}$ due to Zorn's lemma. Proposition 1.3.18 then guarantees the nonemptiness of $M_{\mathcal{A}}$. $\square$

Definition 1.3.20. Let $\mathcal{A}$ be a commutative unital algebra with unity e . Moreover, let $\mathbf{x} = (x_i)_{i=1}^n \in \mathcal{A}^n$.

- $\mathbf{x}$ is called *invertible in $\mathcal{A}^n$* if there exists $\mathbf{y} \in \mathcal{A}^n$ such that

$$\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^n x_i y_i = e.$$

The set of all invertible elements of $\mathcal{A}^n$ will be denoted by $\mathbf{Inv}(\mathcal{A}^n)$.

- Interpreting $\boldsymbol{\lambda} \in \mathbb{C}^n$ as an element of $\mathcal{A}^n$ by $\boldsymbol{\lambda} = (\lambda_i e)_{i=1}^n \in \mathcal{A}^n$, the set

$$\rho_{\mathcal{A}}(\mathbf{x}) := \{\boldsymbol{\lambda} \in \mathbb{C}^n : (\mathbf{x} - \boldsymbol{\lambda}) \in \mathbf{Inv}(\mathcal{A}^n)\}$$

is called *joint resolvent set* of $\mathbf{x}$, where the subtraction of tuples is to be interpreted as a pointwise subtraction. The set

$$\sigma_{\mathcal{A}}(\mathbf{x}) := \mathbb{C}^n \setminus \rho_{\mathcal{A}}(\mathbf{x}) = \{\boldsymbol{\lambda} \in \mathbb{C}^n : (\mathbf{x} - \boldsymbol{\lambda}) \notin \mathbf{Inv}(\mathcal{A}^n)\}$$

is called *joint spectrum* of $\mathbf{x}$. If the algebra is clear from context, we will write $\rho(\mathbf{x})$ and $\sigma(\mathbf{x})$.

Remark 1.3.21. Let $\mathcal{A}$ be a commutative unital algebra and $\mathbf{x} = (x_i)_{i=1}^n \in \mathcal{A}^n$. If there exists $j \in [1, n]_{\mathbb{Z}}$ such that x_j is invertible in $\mathcal{A}$ in the classical sense, then $\mathbf{x}$ is invertible. Indeed, $\mathbf{y} = (y_i)_{i=1}^n$ with $y_i = 0$ for $i \neq j$ and $y_j = x_j^{-1}$ satisfies $\mathbf{x} \cdot \mathbf{y} = x_j x_j^{-1} = e$.

Notation 1.3.22. If $\mathcal{A}, \mathcal{B}$ are algebras and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ an algebra homomorphism, then for $\mathbf{x} \in \mathcal{A}^n$ we will employ the abbreviation

$$\phi[\mathbf{x}] := (\phi(x_i))_{i=1}^n \quad (\in \mathcal{B}^n).$$

Proposition 1.3.23. Let $\mathcal{A}$ be a commutative unital Banach algebra. For $\mathbf{a} = (a_i)_{i=1}^n \in \mathcal{A}^n$ and $\boldsymbol{\lambda} \in \mathbb{C}^n$ the following statements are equivalent.

- (i) $\boldsymbol{\lambda} \in \sigma(\mathbf{a})$.
- (ii) $I := \{(\mathbf{a} - \boldsymbol{\lambda}) \cdot \mathbf{b} : \mathbf{b} \in \mathcal{A}^n\}$ is a proper ideal in $\mathcal{A}$.
- (iii) There exists $m \in M_{\mathcal{A}}$ such that $m[\mathbf{a}] = \boldsymbol{\lambda}$.

Proof. It is clear that, in any case, I is the ideal generated by $a_i - \lambda_i$, $i \in [1, n]_{\mathbb{Z}}$.

(i) $\Leftrightarrow$ (ii): The ideal I is proper if and only if $e \notin I$. This is equivalent to $(\mathbf{a} - \boldsymbol{\lambda}) \cdot \mathbf{b} \neq e$ for all $\mathbf{b} \in \mathcal{A}^n$, which is the definition of $\mathbf{a} - \boldsymbol{\lambda}$ not being invertible and, in turn, equivalent to $\boldsymbol{\lambda} \in \sigma(\mathbf{a})$.

(ii) $\Rightarrow$ (iii): If $J \supseteq I$ is a maximal ideal, the mapping $m_J \in M_{\mathcal{A}}$ as in Proposition 1.3.18 satisfies $I \subseteq J = \ker m_J$. For $k \in [1, n]_{\mathbb{Z}}$ and $\mathbf{b}_k := (\delta_{i,k}e)_{i=1}^n$ we have

$$m_J(a_k) - \lambda_k = m_J(a_k - \lambda_k e) = m_J(\underbrace{(\mathbf{a} - \boldsymbol{\lambda}) \cdot \mathbf{b}_k}_{\in I}) = 0.$$

Consequently, $m_J[\mathbf{a}] = \boldsymbol{\lambda}$.

(ii) $\Leftarrow$ (iii): Let $m \in M_{\mathcal{A}}$ be such that $m[\mathbf{a}] = \boldsymbol{\lambda}$. Given $\mathbf{b} \in \mathcal{A}^n$ we have

$$m((\mathbf{a} - \boldsymbol{\lambda}) \cdot \mathbf{b}) = \sum_{i=1}^n \underbrace{(m(a_i) - m(\lambda_i e))}_{=0} m(b_i) = 0,$$

showing that $I \subseteq \ker m \subsetneq \mathcal{A}$. □

Corollary 1.3.24. Let $\mathcal{A}$ be a commutative unital Banach algebra. Given $\mathbf{a} \in \mathcal{A}^n$ the joint spectrum $\sigma(\mathbf{a})$ is nonempty.

Proof. Let $m \in M_{\mathcal{A}} \neq \emptyset$; see Corollary 1.3.19. By Proposition 1.3.23, $m[\mathbf{a}]$ is an element of the joint spectrum $\sigma(\mathbf{a})$. □

Lemma 1.3.25. Let $\mathcal{A}$ be a commutative unital $*$ -algebra. If $\mathbf{x} = (x_i)_{i=1}^n \in \mathcal{A}^n$ is a tuple of self-adjoint elements, then the joint spectrum $\sigma(\mathbf{x})$ is invariant under the component-wise complex conjugation.

Proof. Given $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^n \in \rho(\mathbf{x})$ there exists $\mathbf{y} \in \mathcal{A}^n$ such that

$$\sum_{i=1}^n (x_i - \lambda_i) y_i = e.$$

Applying the involution yields

$$e = \left(\sum_{i=1}^n (x_i - \lambda_i) y_i \right)^* = \sum_{i=1}^n (x_i - \lambda_i)^* y_i^* = \sum_{i=1}^n (x_i - \bar{\lambda}_i) y_i^*,$$

which demonstrates that $\bar{\boldsymbol{\lambda}}$ belongs to $\rho(\mathbf{x})$. We conclude that $\rho(\mathbf{x})$, and in turn $\sigma(\mathbf{x})$, is invariant under $z \mapsto \bar{z}$. □

Lemma 1.3.26. *Let $\mathcal{A}$ be a commutative unital normed algebra. Given $\mathbf{x}, \mathbf{y} \in \mathcal{A}^n$ the mapping $\mathbb{C}^n \ni \boldsymbol{\lambda} \mapsto (\mathbf{x} - \boldsymbol{\lambda})\mathbf{y} \in \mathcal{A}$ is continuous.*

Proof. Without loss of generality, we assume $y_i \neq 0$ for some $i \in [1, n]_{\mathbb{Z}}$. Given an arbitrary $\epsilon > 0$ let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{C}^n$ be such that

$$\max_{i \in [1, n]_{\mathbb{Z}}} |\mu_i - \lambda_i| < \delta := \epsilon \left(\sum_{i=1}^n \|y_i\| \right)^{-1}.$$

Then we have

$$\begin{aligned} \|(\mathbf{x} - \boldsymbol{\lambda})\mathbf{y} - (\mathbf{x} - \boldsymbol{\mu})\mathbf{y}\| &= \left\| \sum_{i=1}^n (x_i - \lambda_i)y_i - \sum_{i=1}^n (x_i - \mu_i)y_i \right\| \\ &= \left\| \sum_{i=1}^n (\mu_i - \lambda_i)y_i \right\| \\ &\leq \sum_{i=1}^n |\mu_i - \lambda_i| \|y_i\| \\ &\leq \max_{i \in [1, n]_{\mathbb{Z}}} |\mu_i - \lambda_i| \cdot \sum_{i=1}^n \|y_i\| < \epsilon. \end{aligned}$$

□

Proposition 1.3.27. *Let $\mathcal{A}$ be a commutative unital Banach algebra. If $\mathbf{x} = (x_i)_{i=1}^n \in \mathcal{A}^n$, then the joint spectrum $\sigma(\mathbf{x})$ is compact and satisfies $\sigma(\mathbf{x}) \subseteq \sigma(x_1) \times \cdots \times \sigma(x_n)$.*

Proof. Suppose $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^n \notin \sigma(x_1) \times \cdots \times \sigma(x_n)$. Without loss of generality, we assume $\lambda_1 \in \rho(x_1)$ and conclude $x_1 - \lambda_1 \in \text{Inv}(\mathcal{A})$. Hence $(\mathbf{x} - \boldsymbol{\lambda}) \in \text{Inv}(\mathcal{A}^n)$ and $\boldsymbol{\lambda} \in \rho(\mathbf{x})$. We derived $\sigma(\mathbf{x}) \subseteq \sigma(x_1) \times \cdots \times \sigma(x_n)$. Due to this inclusion, it suffices to demonstrate that $\rho(\mathbf{x})$ is open in $\mathbb{C}^n$ in order to prove the compactness of $\sigma(\mathbf{x})$. Given $\boldsymbol{\lambda} \in \rho(\mathbf{x})$ there exists $\mathbf{y} \in \mathcal{A}^n$ such that $(\mathbf{x} - \boldsymbol{\lambda})\mathbf{y} = e$. Since $\boldsymbol{\mu} \mapsto (\mathbf{x} - \boldsymbol{\mu})\mathbf{y}$ is continuous according to Lemma 1.3.26, there exists $\delta > 0$ such that for any $\boldsymbol{\mu} \in \mathbb{C}^n$ with $\max_{i \in [1, n]_{\mathbb{Z}}} |\mu_i - \lambda_i| < \delta$ we have

$$\|e - (\mathbf{x} - \boldsymbol{\mu})\mathbf{y}\| < 1.$$

This implies absolute convergence of

$$\sum_{k=0}^{\infty} (e - (\mathbf{x} - \boldsymbol{\mu})\mathbf{y})^k =: z \in \mathcal{A}.$$

From

$$\begin{aligned} (e - (\mathbf{x} - \boldsymbol{\mu})\mathbf{y})z &= (e - (\mathbf{x} - \boldsymbol{\mu})\mathbf{y}) \sum_{k=0}^{\infty} (e - (\mathbf{x} - \boldsymbol{\mu})\mathbf{y})^k \\ &= \sum_{k=1}^{\infty} (e - (\mathbf{x} - \boldsymbol{\mu})\mathbf{y})^k = z - e, \end{aligned}$$

we obtain

$$e = ((\mathbf{x} - \boldsymbol{\mu})\mathbf{y})z = \sum_{i=1}^n (x_i - \mu_i)y_i z$$

meaning that $\boldsymbol{\mu}$ belongs to $\rho(\mathbf{x})$. Consequently, the open δ -ball around $\boldsymbol{\lambda}$ with respect to the maximum metric is contained in $\rho(\mathbf{x})$, which implies the openness of $\rho(\mathbf{x})$. □

1.4 Joint Spectrum of Operators

Given a Banach space X , $L_b(X)$ constitutes a unital Banach algebra, which is only commutative for $\dim X = 1$. We will make use of commutants to define the joint spectrum of a tuple in $L_b(X)$.

Definition 1.4.1. Let $\mathcal{A}$ be an algebra and $C \subseteq \mathcal{A}$. The *commutant* C' of C is defined as

$$C' := \{x \in \mathcal{A} : xc = cx \text{ for all } c \in C\}.$$

For $\mathbf{x} \in \mathcal{A}^n$ we set $\mathbf{x}' := \{x_i : i \in [1, n]_{\mathbb{Z}}\}'$. In particular, $x' = \{x\}'$ for all $x \in \mathcal{A}$. The set $C'' := (C')'$ is called the *bicommutant* of C .

Facts 1.4.2. Let $\mathcal{A}$ be an algebra. If $C, C_1, C_2 \subseteq \mathcal{A}$, then

- (i) $C \subseteq C''$,
- (ii) $C_1 \subseteq C_2$ implies $C'_1 \supseteq C'_2$,
- (iii) $C' = C'''$,

as can be seen for example in [Kal23, 1.2].

Lemma 1.4.3. Let $\mathcal{A}$ be an algebra. For any $C \subseteq \mathcal{A}$ the set C' constitutes a subalgebra. If additionally $\mathcal{A}$ is a normed algebra, then C' is closed.

Proof. For $c \in C$ we define the linear

$$\psi_c : \begin{cases} \mathcal{A} & \rightarrow \mathcal{A}, \\ x & \mapsto xc - cx. \end{cases}$$

Since we can express C' as

$$C' = \bigcap_{c \in C} \ker \psi_c, \tag{1.3}$$

it is a linear subspace. Given $a, b \in C'$ and $c \in C$, $abc = acb = cab$ shows $ab \in C'$. We conclude that C' is a subalgebra.

If $\mathcal{A}$ is a normed algebra,

$$\|\psi_c(x)\| = \|xc - cx\| \leq 2\|c\|\|x\|, \quad c \in C,$$

verifies that ψ_c is continuous. Due to (1.3), C' is closed. $\square$

Lemma 1.4.4. Let $\mathcal{A}$ be a $*$ -algebra. If $C \subseteq \mathcal{A}$ is a subset that only contains self-adjoint elements, then C' constitutes a $*$ -subalgebra.

Proof. We know from Lemma 1.4.3 that C' constitutes a subalgebra. Given $x \in C'$ we have $x^*c^* = (cx)^* = (xc)^* = c^*x^*$ for all $c \in C$. Since every element in C is self-adjoint, $x^* \in C'$. $\square$

Proposition 1.4.5. Let $\mathcal{A}$ be a unital Banach algebra. If $C \subseteq \mathcal{A}$ satisfies $C \subseteq C'$, then C'' is a commutative unital Banach algebra. Moreover, $\text{Inv}(C'') = \text{Inv}(\mathcal{A}) \cap C''$ and $\sigma_{C''}(x) = \sigma_{\mathcal{A}}(x)$ for all $x \in C''$.

Proof. By Lemma 1.4.3, C'' is a Banach algebra. From $C \subseteq C'$ we infer $C''' = C' \supseteq C''$ meaning that C'' is a commutative Banach algebra. Since the unity commutes with every element, C'' is also unital.

Given $c \in \text{Inv}(\mathcal{A}) \cap C''$ and $x \in C'''$, $xc = cx$ implies $c^{-1}x = xc^{-1}$. Hence, $c^{-1} \in C''' = C''$. We obtain $\text{Inv}(C'') = \text{Inv}(\mathcal{A}) \cap C''$ and, in turn, $\sigma_{C''}(x) = \sigma_{\mathcal{A}}(x)$ for all $x \in C''$. $\square$

Definition 1.4.6. Let X be a Banach space and $\mathbf{A} = (A_i)_{i=1}^n \in L_b(X)^n$ a tuple of pairwise commuting operators.

- The tuple $\mathbf{A}$ is called *invertible* if $\mathbf{A}$ is invertible as an element of $(\mathbf{A}'')^n$ in the sense of Definition 1.3.20.
- We define the *joint resolvent set* of $\mathbf{A}$ as $\rho_{L_b(X)}(\mathbf{A}) := \rho_{\mathbf{A}''}(\mathbf{A})$ and the *joint spectrum* of $\mathbf{A}$ as $\sigma_{L_b(X)}(\mathbf{A}) := \sigma_{\mathbf{A}''}(\mathbf{A})$; see Definition 1.3.20.

The following result is an immediate consequence of Corollary 1.3.24 and Proposition 1.3.27

Corollary 1.4.7. Let X be a Banach space. If $\mathbf{A} = (A_i)_{i=1}^n \in L_b(X)^n$ is a tuple of pairwise commuting operators, then the joint spectrum $\sigma(\mathbf{A}) \subseteq \mathbb{C}^n$ is nonempty and compact.

1.5 Krein Space

Definition 1.5.1. Let X be a vector space over $\mathbb{C}$. If the mapping $[\cdot, \cdot] : X \times X \rightarrow \mathbb{C}$ satisfies

- (i) $[\lambda x + y, z] = \lambda[x, z] + [y, z]$, (Linearity in the first argument)
- (ii) $[x, y] = \overline{[y, x]}$, (Hermitian)

for $x, y, z \in X$ and $\lambda \in \mathbb{C}$, then it is called an *inner product* and $(X, [\cdot, \cdot])$ an *inner product space*. The inner product $[\cdot, \cdot]$ is called

- *positive semidefinite* if $[x, x] \geq 0$ for all $x \in X$ and *negative semidefinite* if $-[\cdot, \cdot]$ is positive semidefinite.
- *positive definite* if $[x, x] > 0$ for all $x \in X \setminus \{0\}$ and *negative definite* if $-[\cdot, \cdot]$ is positive definite.
- *indefinite* if it is neither positive nor negative semidefinite.

Furthermore,

- we call $x, y \in X$ *orthogonal*, denoted by $x \perp y$, if $[x, y] = 0$.
- we call $A, B \subseteq X$ *orthogonal*, denoted by $A \perp B$, if $a \perp b$ for all $a \in A$ and $b \in B$.
- for $A \subseteq X$ we denote by $A^\perp := \{x \in X : [x, a] = 0 \text{ for all } a \in A\}$ the *orthogonal companion* of A .

- we call $x \in X$ *isotropic* if $\{x\} \perp X$. By $(X, [.,.])^\circ$ we denote the set of all isotropic elements, which we call the *isotropic part* of $(X, [.,.])$.
- we call $(X, [.,.])$ *nondegenerate* if $(X, [.,.])^\circ = \{0\}$ and *degenerate* otherwise. Correspondingly we call the inner product $[.,.]$ *nondegenerate/degenerate*.

The following theorem will be used to characterize Krein spaces.

Theorem 1.5.2. *Given an inner product space $(X, [.,.])$, the following statements are equivalent.*

- There exist two linear subspaces X_+ and X_- such that $X_+ \perp_{[.,.]} X_-$, $X = X_+ + X_-$, and both $(X_+, [.,.])$ and $(X_-, -[.,.])$ are Hilbert spaces.*
- There exists a linear mapping $J : X \rightarrow X$ satisfying $J^2 = I$ such that X provided with*

$$\langle x, y \rangle_J := [Jx, y], \quad x, y \in X,$$

constitutes a Hilbert space.

- There exists a positive definite inner product $\langle ., . \rangle$ on X such that $(X, \langle ., . \rangle)$ is a Hilbert space and the associated norm $\|.\|$ satisfies*

$$\|x\| = \sup_{\|y\| \leq 1} |[x, y]|, \quad x \in X.$$

We prepare the proof of this theorem with several auxiliary results.

Lemma 1.5.3. *Let $(X, [.,.])$ be an inner product space. If $J : X \rightarrow X$ is the mapping satisfying (b) of Theorem 1.5.2, then*

- $[Jx, y] = [x, Jy]$ for all $x, y \in X$.
- J is self-adjoint and unitary with respect to the Hilbert space $(X, \langle ., . \rangle_J)$.

Proof. $\langle ., . \rangle_J$ and $[.,.]$ being Hermitian, we obtain

$$[Jx, y] = \langle x, y \rangle_J = \overline{\langle y, x \rangle_J} = \overline{[Jy, x]} = [x, Jy], \quad x, y \in X,$$

and hence

$$\langle Jx, y \rangle_J = [J^2x, y] = [Jx, Jy] = \langle x, Jy \rangle_J, \quad x, y \in X,$$

showing that J is self-adjoint with respect to the Hilbert space $(X, \langle ., . \rangle_J)$. The existence of J^{-1} and the equality $J = J^{-1}$ are direct consequences of $J^2 = I$. Therefore, $J^* = J = J^{-1}$, which means that J is unitary and self-adjoint. $\square$

Lemma 1.5.4. *Let $(X, [.,.])$ be an inner product space. If there exists a norm $\|.\|$ on X satisfying*

$$\|x\| = \sup_{\|y\| \leq 1} |[x, y]|, \quad x \in X, \tag{1.4}$$

then $(X, [.,.])$ is nondegenerate and

$$|[x, y]| \leq \|x\| \|y\|, \quad x, y \in X.$$

Proof. If $(X, [\cdot, \cdot])$ were degenerate, there would exist $x_0 \in X \setminus \{0\}$ such that $[x_0, y] = 0$ for all $y \in X$, implying the contradiction

$$0 \neq \|x_0\| = \sup_{\|y\| \leq 1} |[x_0, y]| = 0.$$

If $\|\cdot\|$ satisfies (1.4), then

$$|[x, y]| = \|y\| \left\| \left[x, \frac{y}{\|y\|} \right] \right\| \leq \|y\| \sup_{\|z\| \leq 1} |[x, z]| = \|y\| \|x\|, \quad x, y \in X \setminus \{0\}.$$

□

Finally, for the proof of Theorem 1.5.2, we will employ the following result. A proof can be found in [WKB25, 3.2.6 Satz].

Theorem 1.5.5 (Lax-Milgram). *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space. If $[\cdot, \cdot]$ is a sesquilinear form on $\mathcal{H}$ satisfying $|[x, y]| \leq C\|x\|\|y\|$, $x, y \in \mathcal{H}$, for some $C \geq 0$, then there exists a unique bounded linear operator $G \in L_b(\mathcal{H})$ with $\|G\| \leq C$ such that*

$$[x, y] = \langle Gx, y \rangle, \quad x, y \in \mathcal{H}.$$

Proof of Theorem 1.5.2. (a) $\Rightarrow$ (b). Given subspaces X_+, X_- satisfying (a), we have $X_+ \cap X_- = \{0\}$. We denote by P_+, P_- the projections onto X_+ along X_- and vice versa. $J := P_+ - P_-$ satisfies

$$J^2 = P_+^2 - P_+P_- - P_-P_+ + P_-^2 = P_+^2 + P_-^2 = P_+ + P_- = I.$$

Given $x, y \in X$ we have

$$\begin{aligned} \langle x, y \rangle_J &= [Jx, y] = [P_+x - P_-x, P_+y + P_-y] \\ &= [P_+x, P_+y] - [P_-x, P_-y]. \end{aligned}$$

Thus $\langle \cdot, \cdot \rangle_J$ is positive definite on X and coincides with the inner product on the direct sum of the Hilbert spaces $(X_+, [\cdot, \cdot])$ and $(X_-, -[\cdot, \cdot])$.

(b) $\Rightarrow$ (a). Given J with the properties described in (b), we set

$$J_+ := \frac{1}{2}(I + J), \quad J_- := \frac{1}{2}(I - J).$$

We conclude

$$J_+ = I - J_- \tag{1.5}$$

and

$$JJ_+ = \frac{1}{2}J(I + J) = \frac{1}{2}(J + I) = J_+, \tag{1.6}$$

$$JJ_- = \frac{1}{2}J(I - J) = \frac{1}{2}(J - I) = -J_-. \tag{1.7}$$

Because of

$$J_+^2 = \frac{1}{4}(I + J)(I + J) = \frac{1}{4}(I + 2J + J^2) = \frac{1}{2}(I + J) = J_+$$

the operator J_+ constitutes a projection. Since J is self-adjoint with respect to $(X, \langle \cdot, \cdot \rangle_J)$ according to Lemma 1.5.3, $J_+^* = 1/2(I+J)^* = 1/2(I+J) = J_+$. Hence, J_+ is an orthogonal projection with respect to $(X, \langle \cdot, \cdot \rangle_J)$. Due to (1.5), $X_+ := J_+X$ and $X_- := J_-X$ are closed subspaces of the Hilbert space $(X, \langle \cdot, \cdot \rangle_J)$ satisfying $X = X_+ + X_-$ and $X_+ \perp_{\langle \cdot, \cdot \rangle_J} X_-$. Given $x \in X_+$ and $y \in X_-$ we have

$$0 = \langle x, y \rangle_J = [Jx, y] \stackrel{(1.6)}{=} [x, y]$$

showing $X_+ \perp_{[\cdot, \cdot]} X_-$.

Given $x, y \in X_-$ and employing $J^* = J = J^2$, we obtain

$$\begin{aligned} -[x, y] &= -[J_-x, J_-y] \\ &= \frac{1}{4} (-[x, y] + [Jx, y] + [x, Jy] - [Jx, Jy]) \\ &= \frac{1}{4} (-\langle Jx, y \rangle_J + \langle x, y \rangle_J + \langle Jx, Jy \rangle_J - \langle x, Jy \rangle_J) \\ &= \frac{1}{2} (-\langle Jx, y \rangle_J + \langle x, y \rangle_J) \\ &\stackrel{(1.7)}{=} \frac{1}{2} (\langle x, y \rangle_J + \langle x, y \rangle_J) = \langle x, y \rangle_J. \end{aligned}$$

Since X_- is a closed subspace of $(X, \langle \cdot, \cdot \rangle_J)$, $(X_-, -[\cdot, \cdot])$ constitutes a Hilbert space. Similarly, using (1.6) instead of (1.7), we can show that $(X_+, [\cdot, \cdot])$ is a Hilbert space.

(b) $\Rightarrow$ (c). Let $J : X \rightarrow X$ satisfy the conditions in (b). Due to the Cauchy-Schwarz inequality, the norm $\|x\|_J := \sqrt{\langle x, x \rangle_J}$ satisfies

$$|\langle Jx, y \rangle_J| \leq \|Jx\|_J, \quad x, y \in X, \quad \|y\|_J = 1,$$

where equality prevails if Jx and y are linearly dependent. We obtain

$$\|Jx\|_J = \sup_{\|y\|_J \leq 1} |\langle Jx, y \rangle_J|.$$

Since J is unitary according to Lemma 1.5.3, we have $\|x\|_J = \|Jx\|_J$. Therefore,

$$\|x\|_J = \|Jx\|_J = \sup_{\|y\|_J \leq 1} |\langle Jx, y \rangle_J| = \sup_{\|y\|_J \leq 1} |[J^2x, y]| = \sup_{\|y\|_J \leq 1} |[x, y]|.$$

(c) $\Rightarrow$ (b). Let $\langle \cdot, \cdot \rangle$ be the positive definite inner product as in (c). According to Lemma 1.5.4 the inner product $[\cdot, \cdot]$ is a sesquilinear form that satisfies the requirements of Theorem 1.5.5 (Lax-Milgram) for the Hilbert space $(X, \langle \cdot, \cdot \rangle)$. Therefore, there exists a bounded linear operator J on $(X, \langle \cdot, \cdot \rangle)$ with $\|J\| \leq 1$ such that

$$[x, y] = \langle Jx, y \rangle, \quad x, y \in X.$$

By assumption

$$\|x\| = \sup_{\|y\| \leq 1} |[x, y]| = \sup_{\|y\| \leq 1} |\langle Jx, y \rangle| \leq \|Jx\| \leq \|x\|, \quad x \in X,$$

where the first inequality is a consequence of the Cauchy-Schwarz inequality. Hence, J is isometric with respect to $\|\cdot\|$ and, in consequence, with respect to $\langle \cdot, \cdot \rangle$, which implies $J^*J = I$. Moreover, $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ being Hermitian yields

$$\langle Jx, y \rangle = [x, y] = \overline{[y, x]} = \overline{\langle Jy, x \rangle} = \langle x, Jy \rangle.$$

Thus J is self-adjoint, and we obtain $J^2 = J^*J = I$. Finally, we have

$$\langle x, y \rangle = \langle J^2x, y \rangle = [Jx, y], \quad x, y \in X.$$

□

Definition 1.5.6. An inner product space $(X, [\cdot, \cdot])$ satisfying one of the equivalent statements in Theorem 1.5.2 is called *Krein space*. If $(X, [\cdot, \cdot])$ is a Krein space, then

- a pair of subspaces X_+ and X_- with the properties in Theorem 1.5.2 (a) is called *fundamental decomposition*;
- a linear mapping $J : X \rightarrow X$ satisfying the condition in Theorem 1.5.2 (b) is called *fundamental symmetry*;
- a norm $\|\cdot\|$ on X with the properties in Theorem 1.5.2 (c) is called *fundamental norm*.

Remark 1.5.7. The proof of Theorem 1.5.2 shows that the class of fundamental decompositions, the class of fundamental symmetries, and the class of fundamental norms of a Krein space are in bijective correspondences. In particular, for any fundamental norm $\|\cdot\|$ on a Krein space $(X, [\cdot, \cdot])$, there exists a fundamental symmetry J such that

$$\|x\| = \sqrt{\langle x, x \rangle_J} = \sqrt{[Jx, x]}, \quad x \in X,$$

and for any fundamental symmetry J , there exists a fundamental decomposition X_+, X_- with associated projections P_+, P_- such that $J = P_+ - P_-$.

Remark 1.5.8. Every Hilbert space is a Krein space with the identity as fundamental symmetry. In fact, the identity is the only fundamental symmetry since, by Remark 1.5.7, the class of fundamental decompositions and the class of fundamental symmetries are in bijective correspondence and a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ only has the trivial fundamental decomposition $\mathcal{H}, \{0\}$.

A somewhat intuitive way to understand Krein spaces is to think of them as the complex and arbitrarily dimensional analogues of the Minkowski space. The Minkowski space is a four-dimensional pseudo-Euclidean space that serves as a description of spacetime in the absence of gravity, for example in Special Relativity.

Example 1.5.9. Consider the complex analogue $(\mathbb{C}^4, [\cdot, \cdot])$ of the Minkowski space, where

$$[x, y] := x^T A \bar{y}, \quad x, y \in \mathbb{C}^4, \quad A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The linear $J := A$ satisfies $J^2 = I$ and

$$\langle x, y \rangle_J := [Jx, y] = x^T J^T A \bar{y} = x^T I \bar{y}, \quad x, y \in \mathbb{C}^4,$$

defines a positive definite inner product. Clearly, $(\mathbb{C}^4, \langle \cdot, \cdot \rangle_J)$ constitutes a Hilbert space. Thus, J is a fundamental symmetry which witnesses that $(\mathbb{C}^4, [\cdot, \cdot])$ is a Krein space.

Theorem 1.5.10. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two complete norms on the nondegenerate inner product space $(X, [\cdot, \cdot])$, such that $[\cdot, \cdot]$ is continuous with respect to each norm. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.*

Proof. We will show that

$$\|x\| := \max_{j=1,2} \|x\|_j, \quad x \in X, \quad (1.8)$$

is equivalent to $\|\cdot\|_j$, $j = 1, 2$. If $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X with respect to $\|\cdot\|$, then, due to $\|x\| \geq \|x\|_j$, $x \in X$, $j = 1, 2$, it is also a Cauchy sequence with respect to $\|\cdot\|_j$, $j = 1, 2$. Let y_j be the limit of $(x_n)_{n \in \mathbb{N}}$ with respect to $\|\cdot\|_j$, $j = 1, 2$. Since the inner product is continuous with respect to both $\|\cdot\|_1$ and $\|\cdot\|_2$, we have

$$\lim_{n \rightarrow \infty} [x_n, z] = [y_j, z], \quad z \in X, \quad j = 1, 2.$$

$(X, [\cdot, \cdot])$ being nondegenerate, we conclude from $[y_1 - y_2, z] = 0$ for all $z \in X$ that $y_1 = y_2$. By (1.8), y_1 is also the limit with respect to $\|\cdot\|$. In particular, $\|\cdot\|$ is complete.

For a fixed $j = 1, 2$ this means that for every sequence $(x_n)_{n \in \mathbb{N}}$ in X , such that there exists a limit with respect to both $\|\cdot\|_j$ and $\|\cdot\|$, these limits coincide. This statement is equivalent to the closedness of the graph of the identity mapping $\text{id}_j : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|_j)$. By the closed graph theorem $\text{id}_j : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|_j)$ is a homeomorphism, showing that $\|\cdot\|$ and $\|\cdot\|_j$ are equivalent. $\square$

Corollary 1.5.11. *Any two fundamental norms on a Krein space are equivalent.*

Proof. As a result of Lemma 1.5.4, property (c) of Theorem 1.5.2 implies the continuity of the inner product with respect to any fundamental norm together with its nondegeneracy. Applying Theorem 1.5.10 to any two fundamental norms completes the proof. $\square$

From now on, we will equip every Krein space with the norm topology of a fundamental norm.

1.6 Operators on Krein Spaces

Definition 1.6.1. Let $(\mathcal{K}_1, [\cdot, \cdot]_1)$ and $(\mathcal{K}_2, [\cdot, \cdot]_2)$ be Krein spaces and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be fundamental norms on each Krein space, respectively. For a linear operator $A : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ we define the operator norm with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$ as

$$\|A\| := \sup_{x \in \mathcal{K}_1 \setminus \{0\}} \frac{\|Ax\|_2}{\|x\|_1} \in [0, +\infty]$$

and say that A is *bounded* if $\|A\| < +\infty$. By $L_b(\mathcal{K}_1, \mathcal{K}_2)$ we denote the space of all bounded linear operators from $\mathcal{K}_1$ to $\mathcal{K}_2$. In case $\mathcal{K}_1 = \mathcal{K}_2$, we will write $L_b(\mathcal{K}_1)$.

Since by Corollary 1.5.11 a different choice of fundamental norms results in an equivalent operator norm, the boundedness of a linear $A : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ is independent of the choice of fundamental norms.

Proposition 1.6.2. *Let $(\mathcal{K}_1, [\cdot, \cdot]_1)$, $(\mathcal{K}_2, [\cdot, \cdot]_2)$ be Krein spaces. Given $A \in L_b(\mathcal{K}_1, \mathcal{K}_2)$ there exists a unique operator $A^+ \in L_b(\mathcal{K}_2, \mathcal{K}_1)$ satisfying*

$$[Ax, y]_2 = [x, A^+y]_1, \quad x \in \mathcal{K}_1, y \in \mathcal{K}_2. \quad (1.9)$$

Moreover, we have $\|A\| = \|A^+\|$, where the operator norms in $L_b(\mathcal{K}_1, \mathcal{K}_2)$ and $L_b(\mathcal{K}_2, \mathcal{K}_1)$ are constructed with respect to fixed fundamental norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $(\mathcal{K}_1, [\cdot, \cdot]_1)$ and $(\mathcal{K}_2, [\cdot, \cdot]_2)$, respectively.

Proof. Let J_1 and J_2 be fundamental symmetries of $(\mathcal{K}_1, [\cdot, \cdot]_1)$ and $(\mathcal{K}_2, [\cdot, \cdot]_2)$, respectively. Let A^* be the Hilbert space adjoint of A with respect to the Hilbert spaces $(\mathcal{K}_1, \langle \cdot, \cdot \rangle_{J_1})$, $(\mathcal{K}_2, \langle \cdot, \cdot \rangle_{J_2})$. $A^+ := J_1 A^* J_2$ satisfies

$$[Ax, y]_2 = \langle J_2 Ax, y \rangle_{J_2} = \langle x, A^* J_2 y \rangle_{J_1} = [J_1 x, A^* J_2 y]_1 = [x, A^+ y]_1, \quad x \in \mathcal{K}_1, y \in \mathcal{K}_2,$$

and, in turn, (1.9). The uniqueness of A^+ follows from the nondegeneracy of $(\mathcal{K}_1, [\cdot, \cdot]_1)$.

From $\|A\| = \|A^*\|$ and the fact that J_1 and J_2 are unitary with respect to $(\mathcal{K}_1, \langle \cdot, \cdot \rangle_{J_1})$ and $(\mathcal{K}_2, \langle \cdot, \cdot \rangle_{J_2})$, respectively, we obtain

$$\|A^+\| = \|J_1 A^* J_2\| \leq \|J_1\| \|A^*\| \|J_2\| = \|A\|. \quad (1.10)$$

Furthermore, nondegeneracy of $(\mathcal{K}_2, [\cdot, \cdot]_2)$ yields $A = A^{++}$. Thus (1.10) applied to A^+ implies $\|A\| = \|A^{++}\| \leq \|A^+\|$. By Remark 1.5.7 any fundamental norm is associated with a fundamental symmetry. Thus, $\|A\| = \|A^+\|$ for any pairs of fundamental norms on $(\mathcal{K}_1, [\cdot, \cdot]_1)$ and $(\mathcal{K}_2, [\cdot, \cdot]_2)$. $\square$

Definition 1.6.3. Let $(\mathcal{K}_1, [\cdot, \cdot]_1)$ and $(\mathcal{K}_2, [\cdot, \cdot]_2)$ be Krein spaces and $A \in L_b(\mathcal{K}_1, \mathcal{K}_2)$. The unique bounded operator $A^+ \in L_b(\mathcal{K}_2, \mathcal{K}_1)$ satisfying (1.9) is called the *Krein space adjoint* of A .

Remark 1.6.4. In the case that $(\mathcal{K}_1, [\cdot, \cdot]_1)$ and $(\mathcal{K}_2, [\cdot, \cdot]_2)$ are Hilbert spaces, the Hilbert space adjoint and the Krein space adjoint of $A \in L_b(\mathcal{K}_1, \mathcal{K}_2)$ coincide according to (1.9).

Definition 1.6.5. Let $(\mathcal{K}, [\cdot, \cdot])$ be a Krein space and $A \in L_b(\mathcal{K})$. We call A

- *normal* if A commutes with A^+ .
- *self-adjoint* if $A = A^+$.
- *positive* if A is self-adjoint and satisfies $[Ax, x] \geq 0$ for all $x \in \mathcal{K}$.

Facts 1.6.6. Let $(\mathcal{K}_1, [\cdot, \cdot]_1)$, $(\mathcal{K}_2, [\cdot, \cdot]_2)$, and $(\mathcal{K}_3, [\cdot, \cdot]_3)$ be Krein spaces. Given $A, B \in L_b(\mathcal{K}_1, \mathcal{K}_2)$, $C \in L_b(\mathcal{K}_2, \mathcal{K}_3)$, and $\lambda \in \mathbb{C}$, we have

- $A^{++} = A$, $I^+ = I$,
- $(A + \lambda B)^+ = A^+ + \bar{\lambda} B^+$,

$$\bullet (CA)^+ = A^+C^+.$$

Lemma 1.6.7. *Let $(\mathcal{K}_1, [\cdot, \cdot]_1)$ and $(\mathcal{K}_2, [\cdot, \cdot]_2)$ be Krein spaces. For every $A \in L_b(\mathcal{K}_1, \mathcal{K}_2)$*

$$(\text{ran } A)^\perp = \ker A^+.$$

Proof. From the definition of the orthogonal companion and the nondegeneracy of a Krein space we obtain

$$\begin{aligned} (\text{ran } A)^\perp &= \{x \in \mathcal{K}_2 : [x, Ay]_2 = 0, y \in \mathcal{K}_1\} \\ &= \{x \in \mathcal{K}_2 : [A^+x, y]_1 = 0, y \in \mathcal{K}_1\} \\ &= \{x \in \mathcal{K}_2 : A^+x = 0\} = \ker A^+. \end{aligned}$$

□

In [Kal17] a new notion of definitizability for normal operators was introduced. We extend this definition to finite tuples of pairwise commuting self-adjoint Krein space operators.

Definition 1.6.8. Let $(\mathcal{K}, [\cdot, \cdot])$ be a Krein space. A tuple $\mathbf{A} = (A_j)_{j=1}^n \in L_b(\mathcal{K})^n$ of pairwise commuting self-adjoint operators is called *definitizable* if there exists a polynomial $p \in \mathbb{C}[x_1, \dots, x_n] \setminus \{0\}$ such that $p(A_1, \dots, A_n) \in L_b(\mathcal{K})$ is a positive operator. Such a polynomial p is called *definitizing polynomial* of $\mathbf{A}$.

The joint spectral theorem for finite tuples of pairwise commuting definitizable self-adjoint operators was formulated in [SK20], where a *definitizable operator* is an operator that is definitizable as a 1-tuple in the sense of Definition 1.6.8. Definition 1.6.8 however does not exclude the possibility of a definitizable tuple $(A_j)_{j=1}^n$ of self-adjoint operators, where a subtuple $(A_j)_{j \in N}$, $N \subseteq [1, n]_{\mathbb{Z}}$, is not definitizable. Hence considering a “definitizable tuple” as in Definition 1.6.8 rather than a “tuple of definitizable operators” enables us to work in a more general setting.

Proposition 1.6.9. *Let $(\mathcal{K}, [\cdot, \cdot])$ be a Krein space and $\mathbf{A} = (A_j)_{j=1}^n \in L_b(\mathcal{K})^n$ a definitizable tuple of operators. If we denote by $\mathcal{I} \subseteq \mathbb{C}[x_1, \dots, x_n]$ the smallest ideal that contains all definitizing polynomials of $\mathbf{A}$, then there exist finitely many definitizing polynomials $q_1, \dots, q_l \in \mathbb{R}[x_1, \dots, x_n]$ of $\mathbf{A}$ such that $\mathcal{I} = \langle q_1, \dots, q_l \rangle$. In particular, the variety $V(\mathcal{I}) \subseteq \mathbb{C}^n$ is invariant under the componentwise complex conjugation.*

Proof. We construct an ascending chain of ideals contained in $\mathcal{I}$ inductively. For some definitizing polynomial p_1 of $\mathbf{A}$, we define $\mathcal{I}_1 := \langle p_1 \rangle$. Given $\mathcal{I}_n$ we choose p_{n+1} as a definitizing polynomial of $\mathbf{A}$ such that $p_{n+1} \notin \mathcal{I}_n$ if such a polynomial exists and set

$$\mathcal{I}_{n+1} := \langle p_1, \dots, p_{n+1} \rangle.$$

If such p_{n+1} does not exist, $\mathcal{I}_{n+1} := \mathcal{I}_n$. According to the Ascending Chain Condition 1.1.5 there exists $N \in \mathbb{N}$ such that $\langle p_1, \dots, p_N \rangle = \mathcal{I}_N = \mathcal{I}$.

Recall that $p^\#(x_1, \dots, x_n) = \overline{p(\overline{x_1}, \dots, \overline{x_n})}$, $p \in \mathbb{C}[x_1, \dots, x_n]$. Given $p \in \mathbb{C}[x_1, \dots, x_n]$,

$$p_{\text{Re}} := \frac{p + p^\#}{2}, \quad p_{\text{Im}} := \frac{p - p^\#}{2i},$$

are elements of $\mathbb{R}[x_1, \dots, x_n]$ such that

$$p = p_{\text{Re}} + ip_{\text{Im}}. \quad (1.11)$$

$\mathbf{A}$ being a tuple of self-adjoint operators implies

$$p^\#(\mathbf{A}) = p(A_1^+, \dots, A_n^+)^+ = p(A_1, \dots, A_n)^+ = p(\mathbf{A})^+. \quad (1.12)$$

If p is a definitizing polynomial of $\mathbf{A}$, $p(\mathbf{A})$ is self-adjoint. Therefore, (1.12) yields $p^\#(\mathbf{A}) = p(\mathbf{A})$ and, in turn, $p_{\text{Re}}(\mathbf{A}) = p(\mathbf{A})$ and $p_{\text{Im}}(\mathbf{A}) = 0$. Hence, if $p_{\text{Re}}, p_{\text{Im}} \neq 0$, they are definitizing polynomials of $\mathbf{A}$; see Definition 1.6.8. Because of (1.11),

$$G := \{p_{j_{\text{Re}}} : j \in [1, N]_{\mathbb{Z}}\} \cup \{p_{j_{\text{Im}}} : j \in [1, N]_{\mathbb{Z}}\} \subseteq \mathbb{R}[x_1, \dots, x_n]$$

generates $\mathcal{I}$. Therefore, $G \setminus \{0\} \neq \emptyset$ is a set of real definitizing polynomials that generate $\mathcal{I}$.

Since $\mathcal{I}$ has a set of generators in $\mathbb{R}[x_1, \dots, x_n]$, we derive from Lemma 1.1.9 that $V(\mathcal{I})$ is invariant under componentwise complex conjugation. $\square$

Lemma 1.6.10. *Let $(\mathcal{K}, [\cdot, \cdot])$ be a Krein space and $P \in L_b(\mathcal{K})$ a positive operator. Then there exists a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and an injective $T \in L_b(\mathcal{H}, \mathcal{K})$ such that $TT^+ = P$.*

Proof. We define a positive semidefinite inner product on $\mathcal{K}$ by $[x, y]_P := [Px, y]$. Factorizing $\mathcal{K}$ by its isotropic part $N := (\mathcal{K}, [\cdot, \cdot]_P)^\circ$ yields the pre-Hilbert space $(\mathcal{K}/N, \langle \cdot, \cdot \rangle)$, where $\langle x + N, y + N \rangle := [x, y]_P$ is well-defined.

Denoting by $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ the Hilbert space completion of $(\mathcal{K}/N, \langle \cdot, \cdot \rangle)$, we can interpret the canonical projection

$$\pi : \begin{cases} \mathcal{K} & \rightarrow \mathcal{K}/N, \\ x & \mapsto x + N, \end{cases}$$

as a mapping into $\mathcal{H}$. Given a fundamental norm $\|\cdot\|_{\mathcal{K}}$ on $\mathcal{K}$ we have

$$\|\pi x\|_{\mathcal{H}}^2 = \langle \pi x, \pi x \rangle = [Px, x] \stackrel{\star}{\leq} \|Px\|_{\mathcal{K}} \|x\|_{\mathcal{K}} \leq \|P\| \|x\|_{\mathcal{K}}^2,$$

where $\star$ holds true due to Lemma 1.5.4, proving $\pi \in L_b(\mathcal{K}, \mathcal{H})$. We set $T := \pi^+ \in L_b(\mathcal{H}, \mathcal{K})$ and derive from the continuity of $\langle \cdot, \cdot \rangle$

$$\ker T = (\text{ran } \pi)^\perp = (\overline{\text{ran } \pi})^\perp = \mathcal{H}^\perp = \{0\}.$$

Hence, T is injective. The nondegeneracy of a Krein space and

$$[TT^+x, y] = \langle T^+x, T^+y \rangle = \langle x + N, y + N \rangle = [Px, y], \quad x, y \in \mathcal{K},$$

yield $TT^+ = P$. $\square$

1.7 Spectral Theory in Hilbert Spaces

The main result here will be Theorem 1.7.7, the joint spectral theorem for pairwise commuting tuples of self-adjoint Hilbert space operators. It motivates the main result of the present thesis and plays a key role in its proof.

Definition 1.7.1. We denote by $\mathfrak{B}^{\mathfrak{S}}(\Omega)$ the C^* -algebra of complex-valued bounded $\mathfrak{S}$ -measurable functions on Ω endowed with pointwise algebraic operations and the norm $\|\phi\|_{\infty} := \sup_{x \in \Omega} |\phi(x)|$. If the σ -algebra is the Borel σ -algebra, we will write $\mathfrak{B}(\Omega)$.

Furthermore, for compact Ω we denote by $C(\Omega)$ the C^* -algebra of continuous complex-valued functions on Ω endowed with pointwise operations and $\|\cdot\|_{\infty}$.

Definition 1.7.2. Let Ω be a set endowed with a σ -algebra $\mathfrak{S}$ and $\mathcal{H}$ a Hilbert space. If $E : \mathfrak{S} \rightarrow L_b(\mathcal{H})$ is such that

- (i) $E(\Delta)$ is an orthogonal projection for all $\Delta \in \mathfrak{S}$,
- (ii) $E(\emptyset) = 0$ and $E(\Omega) = I$,
- (iii) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for $\Delta_1, \Delta_2 \in \mathfrak{S}$, and
- (iv) for pairwise disjoint $\Delta_n \in \mathfrak{S}$ we have

$$E\left(\bigcup_{n \in \mathbb{N}} \Delta_n\right) = \sum_{n \in \mathbb{N}} E(\Delta_n)$$

in the sense of pointwise convergence,

then E is called *spectral measure* for $(\Omega, \mathfrak{S}, \mathcal{H})$.

We first recall the spectral theorem of a bounded self-adjoint operator on a Hilbert space; see [WKB25, Chapter 7].

Theorem 1.7.3. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $A \in L_b(\mathcal{H})$ self-adjoint. Denoting by $\mathcal{B}$ the Borel σ -algebra on $\sigma(A)$, there exists a unique spectral measure $E : \mathcal{B} \rightarrow L_b(\mathcal{H})$, called the spectral measure of A , such that

$$A = \int t \, dE(t).$$

Moreover, given $T \in L_b(\mathcal{H})$, the following statements are equivalent.

- (i) $TA = AT$.
- (ii) $TE(\Delta) = E(\Delta)T$ for all $\Delta \in \mathcal{B}$.
- (iii) $T(\int \phi \, dE) = (\int \phi \, dE)T$ for all $\phi \in C(\sigma(A))$.
- (iv) $T(\int \phi \, dE) = (\int \phi \, dE)T$ for all $\phi \in \mathfrak{B}(\sigma(A))$.

Definition 1.7.4. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and Ω_1, Ω_2 be sets endowed with σ -algebras $\mathfrak{S}_1, \mathfrak{S}_2$, respectively. We say that the spectral measures $E_i : \mathfrak{S}_i \rightarrow L_b(\mathcal{H})$, $i = 1, 2$ commute if

$$E_2(\Delta_2)E_1(\Delta_1) = E_1(\Delta_1)E_2(\Delta_2), \quad \Delta_1 \in \mathfrak{S}_1, \Delta_2 \in \mathfrak{S}_2.$$

Lemma 1.7.5. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space. If $(A_k)_{k=1}^n \in L_b(\mathcal{H})^n$ is a tuple of pairwise commuting self-adjoint operators, then the tuple $(E_k)_{k=1}^n$ of corresponding spectral measures commute pairwise.

Proof. Let $\mathcal{B}_k$ denote the Borel σ -algebra on $\sigma(A_k)$. Given $\Delta_j \in \mathcal{B}_j$, $A_i A_j = A_j A_i$ yields $A_i E_j(\Delta_j) = E_j(\Delta_j) A_i$ which, in turn, implies $E_i(\Delta_i) E_j(\Delta_j) = E_j(\Delta_j) E_i(\Delta_i)$ for any $\Delta_j \in \mathcal{B}_j$, $\Delta_i \in \mathcal{B}_i$; see Theorem 1.7.3. $\square$

The following theorem from [Sch12, Theorem 4.10] defines a product spectral measure of pairwise commuting spectral measures.

Theorem 1.7.6. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space. For $k \in [1, n]_{\mathbb{Z}}$ let Ω_k be a locally compact Hausdorff space which has a countable base of open sets and $E_k : \mathcal{B}_k \rightarrow L_b(\mathcal{H})$ a spectral measure defined on the Borel σ -algebra $\mathcal{B}_k$ of Ω_k . If the spectral measures $(E_k)_{k=1}^n$ commute pairwise, then there exists a unique spectral measure E on the Borel σ -algebra $\mathcal{B}$ of the product space $\Omega_1 \times \cdots \times \Omega_n$ satisfying*

$$E(\Delta_1 \times \cdots \times \Delta_n) = E_1(\Delta_1) \cdots E_n(\Delta_n), \quad \Delta_k \in \mathcal{B}_k, \quad k \in [1, n]_{\mathbb{Z}}.$$

We will refer to E as the product spectral measure of $(E_k)_{k=1}^n$.

We are going to formulate a result analogous to Theorem 1.7.3 for a tuple of pairwise commuting self-adjoint operators. For a (spectral) measure μ and a measurable function T , we will refer to the (spectral) measure $\Delta \mapsto \mu(T^{-1}(\Delta))$ as $\mu \circ T^{-1}$.

Theorem 1.7.7. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space. If $\mathbf{A} = (A_k)_{k=1}^n \in L_b(\mathcal{H})^n$ is a tuple of pairwise commuting self-adjoint operators, then there exists a unique spectral measure E defined on the Borel σ -algebra $\mathcal{B}$ of $\Omega := \sigma(A_1) \times \cdots \times \sigma(A_n)$ such that*

$$A_k = \int_{\Omega} \pi_k dE, \quad k \in [1, n]_{\mathbb{Z}}, \quad (1.13)$$

where π_k denotes the projection onto the k -th coordinate. In addition, E satisfies

$$\int_{\sigma(A_k)} \phi dE_k = \int_{\Omega} \phi \circ \pi_k dE, \quad k \in [1, n]_{\mathbb{Z}}, \quad \phi \in \mathfrak{B}(\sigma(A_k)), \quad (1.14)$$

where E_k denotes the spectral measure of A_k . Moreover, for $T \in L_b(\mathcal{H})$, the following statements are equivalent.

- (i) $TA_k = A_k T$ for all $k \in [1, n]_{\mathbb{Z}}$.
- (ii) $TE(\Delta) = E(\Delta)T$ for all $\Delta \in \mathcal{B}$.
- (iii) $T(\int \phi dE) = (\int \phi dE)T$ for all $\phi \in C(\Omega)$.
- (iv) $T(\int \phi dE) = (\int \phi dE)T$ for all $\phi \in \mathfrak{B}(\Omega)$.

We will refer to E as joint spectral measure of $\mathbf{A}$.

Proof. By Lemma 1.7.5, $(E_k)_{k=1}^n$ commute pairwise. According to Theorem 1.7.6 there exists a unique product spectral measure E of $(E_k)_{k=1}^n$. Without loss of generality we will verify (1.14) for $k = 1$. Let $f \in \mathcal{H}$. Note that $\langle E(\cdot)f, f \rangle$ and $\langle E_1(\cdot)f, f \rangle$ constitute non-negative measures. Denoting by $\mathcal{B}_1$ the Borel σ -algebra on $\sigma(A_1)$,

$$E(\Delta \times \sigma(A_2) \times \cdots \times \sigma(A_n)) = E_1(\Delta), \quad \Delta \in \mathcal{B}_1,$$

yields $E \circ \pi_1^{-1} = E_1$. Thus, for $\phi \in \mathfrak{B}(\sigma(A_1))$ we obtain

$$\begin{aligned} \left\langle \left(\int_{\sigma(A_1)} \phi dE_1 \right) f, f \right\rangle &= \int_{\sigma(A_1)} \phi d\langle E_1(\cdot)f, f \rangle \\ &= \int_{\sigma(A_1)} \phi d(\langle E(\cdot)f, f \rangle \circ \pi_1^{-1}) \\ &= \int_{\Omega} \phi \circ \pi_1 d\langle E(\cdot)f, f \rangle = \left\langle \left(\int_{\Omega} \phi \circ \pi_1 dE \right) f, f \right\rangle. \end{aligned}$$

Due to the polarization identity in $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, we obtain

$$\int_{\sigma(A_1)} \phi dE_1 = \int_{\Omega} \phi \circ \pi_1 dE$$

and, in turn, (1.13) by choosing $\phi(t) = t$.

Concerning uniqueness, let E' be a spectral measure on Ω that satisfies (1.13). For $k \in [1, n]_{\mathbb{Z}}$ it is straightforward to check that $E' \circ \pi_k^{-1}$ is a spectral measure on $\sigma(A_k)$. From $\langle E' \circ \pi_k^{-1}(\cdot)g, h \rangle = \langle E'(\cdot)g, h \rangle \circ \pi_k^{-1}$, $g, h \in \mathcal{H}$, we infer

$$\int_{\sigma(A_k)} t d(E' \circ \pi_k^{-1})(t) = \int_{\Omega} \pi_k dE' = A_k = \int_{\sigma(A_k)} t dE_k(t).$$

From Theorem 1.7.3 we conclude $E_k = E' \circ \pi_k^{-1}$ for $k \in [1, n]_{\mathbb{Z}}$. Given $\Delta_k \in \mathcal{B}_k$, $k \in [1, n]_{\mathbb{Z}}$, we have

$$\begin{aligned} E'(\Delta_1 \times \cdots \times \Delta_n) &= \int_{\Omega} \mathbb{1}_{\Delta_1 \times \cdots \times \Delta_n} dE' \\ &= \int_{\Omega} \prod_{k=1}^n \mathbb{1}_{\pi_k^{-1}(\Delta_k)} dE' \\ &= \prod_{k=1}^n E'(\pi_k^{-1}(\Delta_k)) = \prod_{k=1}^n E_k(\Delta_k). \end{aligned}$$

According to Theorem 1.7.6, E' must be the product spectral measure of $(E_k)_{k=1}^n$.

The statements (ii), (iii), and (iv) are equivalent for any spectral measure defined on a Borel σ -algebra of a compact Hausdorff space; see [WKB25, 7.1.13]. (i) is a trivial conclusion from (iv). From (i) we obtain that $T \in L_b(\mathcal{H})$ commutes with $p(\mathbf{A}) = \int p dE$ for all $p \in \mathbb{C}[x_1, \dots, x_n]$. Note that $\mathbb{C}[x_1, \dots, x_n]$ as a set of functions on Ω is a point separating and nowhere vanishing subalgebra of $C(\Omega)$. Furthermore, as $\Omega \subseteq \mathbb{R}^n$, $\mathbb{C}[x_1, \dots, x_n]$ is closed under $f \mapsto \bar{f}$. By the Stone-Weierstrass theorem $\mathbb{C}[x_1, \dots, x_n]$ as a set of functions on Ω is dense in $C(\Omega)$. Since the bounded linear mapping

$$\psi : \begin{cases} C(\Omega) & \rightarrow L_b(\mathcal{H}), \\ f & \mapsto T \left(\int_{\Omega} f dE \right) - \left(\int_{\Omega} f dE \right) T, \end{cases}$$

satisfies $\psi|_{\mathbb{C}[x_1, \dots, x_n]} \equiv 0$, we obtain $\psi \equiv 0$. Thus, (i) implies (iii). $\square$

Corollary 1.7.8. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbf{A} = (A_k)_{k=1}^n \in L_b(\mathcal{H})^n$ a tuple of pairwise commuting self-adjoint operators, and $\mathbf{A}_N = (A_k)_{k \in N}$, $N \subseteq [1, n]_{\mathbb{Z}}$, a subtuple of $\mathbf{A}$. We denote by $\pi_N : \mathbb{R}^n \rightarrow \mathbb{R}^N$ the canonical projection $(x_k)_{k=1}^n \mapsto (x_k)_{k \in N}$. If E and E_N are the joint spectral measures of $\mathbf{A}$ and $\mathbf{A}_N$, respectively, then we have $E \circ \pi_N^{-1} = E_N$.*

Proof. Let $\Delta_k \subseteq \sigma(A_k)$, $k \in [1, n]_{\mathbb{Z}}$, be Borel subsets such that $\Delta_k = \sigma(A_k)$ for $k \in [1, n]_{\mathbb{Z}} \setminus N$. Denoting by $|N|$ the cardinality of N , let $m : [1, |N|]_{\mathbb{Z}} \rightarrow N$ be an increasing enumeration of N . Because of

$$\begin{aligned} E \circ \pi_N^{-1}(\Delta_{m(1)} \times \cdots \times \Delta_{m(|N|)}) &= E(\Delta_1 \times \cdots \times \Delta_n) \\ &= E(\Delta_1) \cdots E(\Delta_n) \\ &= E(\Delta_{m(1)}) \cdots E(\Delta_{m(|N|)}) \\ &= E_N(\Delta_{m(1)} \times \cdots \times \Delta_{m(|N|)}), \end{aligned}$$

the uniqueness of product spectral measures guarantees $E_N = E \circ \pi_N^{-1}$. $\square$

Definition 1.7.9. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and E a spectral measure on the Borel σ -algebra $\mathcal{B}$ of a topological space Ω . The *support* of the spectral measure E is defined as

$$\text{supp } E := \{x \in \Omega : E(U) \neq 0 \text{ for all open neighbourhood } U \text{ of } x\}.$$

Remark 1.7.10. If the topology on Ω has a countable basis, then $\Omega \setminus \text{supp } E$ is the largest open subset Q satisfying $E(Q) = 0$; see Definition 1.7.2, (iv).

Remark 1.7.11. We can extend a joint spectral measure E of a pairwise commuting tuple of self-adjoint operators $\mathbf{A} \in L_b(\mathcal{H})^n$ to the Borel σ -algebra of $\mathbb{R}^n$ by setting

$$\tilde{E}(\Delta) := E(\Delta \cap \text{supp } E).$$

Hence we will often interpret the joint spectral measure to be defined on the Borel σ -algebra on $\mathbb{R}^n$.

The next result establishes a connection between the joint spectrum of operators and the spectral measure.

Proposition 1.7.12. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathbf{A} = (A_k)_{k=1}^n \in L_b(\mathcal{H})^n$ a tuple of pairwise commuting self-adjoint operators. Then the joint spectral measure E of $\mathbf{A}$ satisfies

$$\text{supp } E = \sigma(\mathbf{A}).$$

Proof. Let $\lambda \in \text{supp } E$ and assume $\lambda \in \rho(\mathbf{A})$. We find $\mathbf{B} \in L_b(\mathcal{H})^n$ such that $\mathbf{B}(\mathbf{A} - \lambda) = I$. If $0 < \epsilon < (\sum_{k=1}^n \|B_k\|)^{-1}$, then $\lambda \in \text{supp } E$ implies $E(B_\epsilon(\lambda)) \neq 0$. Hence, there exists $0 \neq f \in \text{ran } E(B_\epsilon(\lambda)) \subseteq \mathcal{H}$. Given $\Delta \in \mathbb{R}^n$ such that $\Delta \cap B_\epsilon(\lambda) = \emptyset$ we have

$$\langle E(\Delta)f, f \rangle = \underbrace{\langle E(\Delta)E(B_\epsilon(\lambda))f, f \rangle}_{=0} = 0.$$

Hence, for $k \in [1, n]_{\mathbb{Z}}$ we conclude

$$\begin{aligned} \|(A_k - \lambda_k)f\|^2 &= \int |x_k - \lambda_k|^2 d\langle E(\mathbf{x})f, f \rangle \\ &= \int_{B_\epsilon(\lambda)} |x_k - \lambda_k|^2 d\langle E(\mathbf{x})f, f \rangle \leq \epsilon^2 \langle f, f \rangle, \end{aligned}$$

which yields the contradiction

$$\|f\| = \|\underbrace{\mathbf{B}(\mathbf{A} - \boldsymbol{\lambda})}_{=I} f\| \leq \sum_{k=1}^n \|B_k\| \|(A_k - \lambda_k)f\| \leq \|f\| \underbrace{\epsilon \sum_{k=1}^n \|B_k\|}_{<1} < \|f\|.$$

For the converse, let $\boldsymbol{\lambda} \in \mathbb{R}^n \setminus \text{supp } E$. For $k \in [1, n]_{\mathbb{Z}}$,

$$\mathbf{x} \mapsto \mathbb{1}_{\text{supp } E}(\mathbf{x}) \frac{x_k - \lambda_k}{\|\mathbf{x} - \boldsymbol{\lambda}\|_2^2}$$

constitutes a bounded measurable function on $\mathbb{R}^n$. If we define $\mathbf{B} = (B_k)_{k=1}^n$ by

$$B_k := \int \mathbb{1}_{\text{supp } E}(\mathbf{x}) \frac{x_k - \lambda_k}{\|\mathbf{x} - \boldsymbol{\lambda}\|_2^2} dE(\mathbf{x}), \quad k \in [1, n]_{\mathbb{Z}},$$

then Remark 1.7.10 implies

$$\begin{aligned} (\mathbf{A} - \boldsymbol{\lambda})\mathbf{B} &= \sum_{k=1}^n (A_k - \lambda_k)B_k \\ &= \sum_{k=1}^n \int (x_k - \lambda_k) dE(\mathbf{x}) \int \mathbb{1}_{\text{supp } E}(\mathbf{x}) \frac{x_k - \lambda_k}{\|\mathbf{x} - \boldsymbol{\lambda}\|_2^2} dE(\mathbf{x}) \\ &= \int \mathbb{1}_{\text{supp } E}(\mathbf{x}) \sum_{k=1}^n \frac{(x_k - \lambda_k)^2}{\|\mathbf{x} - \boldsymbol{\lambda}\|_2^2} dE(\mathbf{x}) \\ &= \int \mathbb{1}_{\text{supp } E} dE = \int \mathbb{1} dE = I. \end{aligned}$$

Hence, $\boldsymbol{\lambda} \in \rho(\mathbf{A})$. □

1.8 Diagonal Transform on Krein Spaces

In the present section, we are going to introduce a $*$ -homomorphism that maps Krein space operators to Hilbert space operators. Any vector space mentioned in this section will be over $\mathbb{C}$.

Definition 1.8.1. Let X, Y be vector spaces. We call a subspace $T \leq X \times Y$ *linear relation* between X and Y . Given a linear relation T between X and Y , we define

- the *domain* of T by $\text{dom } T := \{x \in X : (x; y) \in T \text{ for some } y \in Y\}$,
- the *range* of T by $\text{ran } T := \{y \in Y : (x; y) \in T \text{ for some } x \in X\}$,
- the *kernel* of T by $\text{ker } T := \{x \in X : (x; 0) \in T\}$,
- the *multi-valued-part* of T by $\text{mul } T := \{y \in Y : (0; y) \in T\}$.

A linear relation between X and X will be called a linear relation on X .

Remark 1.8.2. The graph of any linear operator is a linear relation. Any linear relation T with $\text{mul } T = \{0\}$ is the graph of a linear operator defined on $\text{dom } T$.

Definition 1.8.3. Given vector spaces X, Y, Z , linear relations $S, T \leq X \times Y$, $R \leq Y \times Z$, and $\alpha \in \mathbb{C}$, we define

- $S + T := \{(x; y_1 + y_2) \in X \times Y : (x; y_1) \in S, (x; y_2) \in T\}.$
- $\alpha T := \{(x; \alpha y) \in X \times Y : (x; y) \in T\}.$
- $T^{-1} := \{(y; x) \in Y \times X : (x; y) \in T\}.$
- $RS := \{(x; z) \in X \times Z : (x; y) \in S, (y; z) \in R, \text{ for some } y \in Y\}.$

Simple calculations show that these sets are linear relations.

Definition 1.8.4. Let X, Y be vector spaces and $T : X \rightarrow Y$ a linear operator. We define the mapping

$$T \times T : \begin{cases} X \times X & \rightarrow Y \times Y, \\ (a; b) & \mapsto (Ta; Tb). \end{cases}$$

Facts 1.8.5. Let X, Y be vector spaces and $A \leq X \times X$, $B \leq Y \times Y$. If $T : X \rightarrow Y$ is a linear operator, then

- (i) $T \times T$ constitutes a linear operator.
- (ii) $(T \times T)(A) = \{(Ta; Tb) : (a; b) \in A\} \leq Y \times Y.$
- (iii) $(T \times T)^{-1}(B) = \{(a; b) : (Ta; Tb) \in B\} \leq X \times X.$

Lemma 1.8.6. Let X, Y be vector spaces and $A \leq X \times X$, $B \leq Y \times Y$. If $T : X \rightarrow Y$ is a linear operator, then

$$(T \times T)(A) = TAT^{-1}, \quad (T \times T)^{-1}(B) = T^{-1}BT,$$

where the products on the right-hand sides are relational products as in Definition 1.8.3 with T being interpreted as its graph. In particular, T need not be invertible as an operator.

Proof. We will show the second equality, as the first equality can be shown in a similar manner. If $(a; b) \in (T \times T)^{-1}(B)$, then there exists $(x; y) \in B$ such that $(Ta; Tb) = (x; y)$. Hence $(a; x) \in T$, $(x; y) \in B$, and $(y; b) \in T^{-1}$, which yields $(a; b) \in T^{-1}BT$.

If $(a; b) \in T^{-1}BT$, there exist $x, y \in Y$ such that $(a; x) \in T$, $(x; y) \in B$, and $(y; b) \in T^{-1}$. Hence $(Ta; Tb) = (x; y) \in B$, which implies $(a; b) \in (T \times T)^{-1}(B)$. $\square$

The following theorem from [KP15, Theorem 5.8] enables us to construct a $*$ -homomorphism from a subalgebra of $L_b(\mathcal{K})$ into a subalgebra of $L_b(\mathcal{H})$, where $\mathcal{K}$ is a Krein space and $\mathcal{H}$ a Hilbert space.

Theorem 1.8.7. Let $(\mathcal{H}, (.,.))$ be a Hilbert space and $(\mathcal{K}, [.,.])$ a Krein space. If $T \in L_b(\mathcal{H}, \mathcal{K})$ is an injective operator, then

$$\Theta : \begin{cases} (TT^+)' & \rightarrow (T^+T)', \\ C & \mapsto (T \times T)^{-1}(C), \end{cases}$$

constitutes a bounded $*$ -homomorphism. Hereby, $\Theta(I) = I$, $\Theta(TT^+) = T^+T$, and

$$\ker \Theta = \{C \in (TT^+)' : \text{ran } C \subseteq \ker T^+\}.$$

$(TT^+)' \subseteq L_b(\mathcal{K})$ and $(T^+T)' \subseteq L_b(\mathcal{H})$ denote the commutant of TT^+ and T^+T , respectively.

The next lemma from [KP15, Lemma 5.11] introduces an operator that maps Hilbert space operators to Krein space operators.

Lemma 1.8.8. *Let $(\mathcal{H}, (\cdot, \cdot))$ be a Hilbert space and $(\mathcal{K}, [\cdot, \cdot])$ a Krein space. If $T \in L_b(\mathcal{H}, \mathcal{K})$ is an injective operator, then*

$$\Xi : \begin{cases} L_b(\mathcal{H}) & \rightarrow L_b(\mathcal{K}), \\ D & \mapsto TDT^+, \end{cases}$$

constitutes an injective bounded linear operator. Moreover, Ξ maps $(T^+T)' \subseteq L_b(\mathcal{H})$ into $(TT^+)' \subseteq L_b(\mathcal{K})$. Given $C \in (TT^+)'$ and $D, D_1, D_2 \in (T^+T)'$, we have

- (i) $\Xi(D^*) = \Xi(D)^+$,
- (ii) $\Xi(D\Theta(C)) = \Xi(D)C$,
- (iii) $\Xi(\Theta(C)D) = C\Xi(D)$,
- (iv) $\Xi(T^+TD_1D_2) = \Xi(D_1)\Xi(D_2)$,
- (v) $\Xi \circ \Theta(C) = TT^+C = CTT^+$,

where Θ is defined as in Theorem 1.8.7 with the same operator T . Furthermore, $\Xi(D)$ commutes with all operators in $(TT^+)'$ if D commutes with all operators in $(T^+T)'$, i.e. $\Xi((T^+T)'' \cap (T^+T)') \subseteq (TT^+)'' \cap (TT^+)'$.

Chapter 2

Construction Framework

In this chapter, we will prepare the framework necessary for the construction and definition of the joint functional calculus of a definitizable tuple of self-adjoint Krein space operators. Throughout the chapter, we will employ the following notation.

- $(\mathcal{K}, [\cdot, \cdot])$ is a Krein space.
- $\mathbf{A} = (A_j)_{j=1}^n \in L_b(\mathcal{K})^n$ is a definitizable tuple of pairwise commuting self-adjoint operators.
- $\mathcal{I}$ denotes the smallest ideal in $\mathbb{C}[x_1, \dots, x_n]$ that contains all definitizing polynomials of $\mathbf{A}$.

2.1 Embeddings

For the whole section, $p_1, \dots, p_m \in \mathbb{C}[x_1, \dots, x_n]$ shall be some definitizing polynomials of $\mathbf{A}$.

Definition 2.1.1. Let $j \in [1, m]_{\mathbb{Z}}$. We denote by $(\mathcal{H}_j, \langle \cdot, \cdot \rangle_j)$ and $T_j \in L_b(\mathcal{H}_j, \mathcal{K})$ the Hilbert space and the injective operator that results from applying Lemma 1.6.10 to the positive operator $p_j(\mathbf{A})$. Moreover, we fix the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and the injective $T \in L_b(\mathcal{H}, \mathcal{K})$ that we obtain from Lemma 1.6.10 for the positive operator $\sum_{k=1}^m p_k(\mathbf{A})$. We have

$$T_j T_j^+ = p_j(\mathbf{A}) \quad \text{and} \quad T T^+ = \sum_{k=1}^m p_k(\mathbf{A}) = \sum_{k=1}^m T_k T_k^+. \quad (2.1)$$

Note that if $p_j(\mathbf{A}) = 0$, then $\mathcal{H}_j = \{0\}$ and T_j is the zero operator.

Lemma 2.1.2. *We have*

$$\mathbf{A}' \subseteq \bigcap_{k=1}^m (T_k T_k^+)' \subseteq (T T^+)'.$$

Proof. It follows immediately from (2.1). □

Lemma 2.1.3. *Given $j \in [1, m]$ there exists a unique injective contraction $R_j \in L_b(\mathcal{H}_j, \mathcal{H})$ such that $T_j = T R_j$. These contractions satisfy $\sum_{k=1}^m R_k R_k^* = I \in L_b(\mathcal{H})$.*

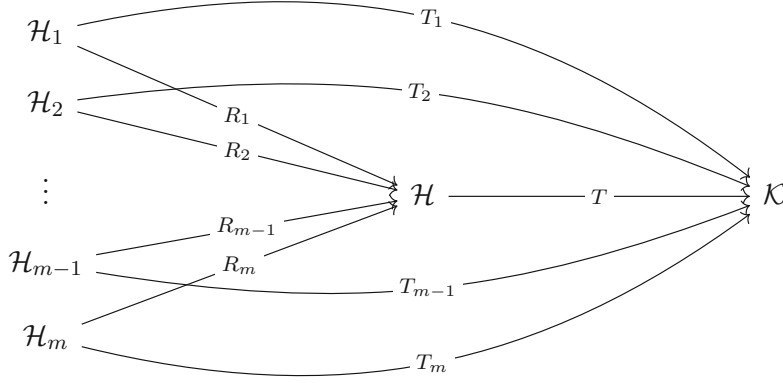


Figure 2.1: Commutative diagram of the mappings involved in Lemma 2.1.3.

Proof. For $x \in \mathcal{K}$ and $j \in [1, m]_{\mathbb{Z}}$ we have

$$\langle T^+x, T^+x \rangle = [TT^+x, x] = \sum_{k=1}^m [T_k T_k^+ x, x] = \sum_{k=1}^m \langle T_k^+ x, T_k^+ x \rangle_k \geq \langle T_j^+ x, T_j^+ x \rangle_j.$$

Thus

$$B_j : \begin{cases} \text{ran } T^+ & \rightarrow \text{ran } T_j^+, \\ T^+x & \mapsto T_j^+x, \end{cases}$$

constitutes a well-defined linear contraction. Since T as well as T_j is injective, we have $(\text{ran } T^+)^\perp = \ker T = \{0\}$ and $(\text{ran } T_j^+)^\perp = \ker T_j = \{0\}$. Hence, $\text{ran } T^+ \subseteq \mathcal{H}$ and $\text{ran } T_j^+ \subseteq \mathcal{H}_j$ are dense. In consequence B_j has a unique continuous extension defined on $\mathcal{H}$ with a dense range in $\mathcal{H}_j$. We denote by R_j the adjoint map of the extension of B_j . R_j satisfies $\|R_j\| = \|R_j^*\| = \|B_j\| \leq 1$ and $\ker R_j = (\text{ran } R_j^*)^\perp = \{0\}$. Moreover, $T_j^+ = B_j T^+$ yields $T_j^+ = R_j^* T^+$ implying $T_j = T R_j$. The uniqueness of R_j is a consequence of the injectivity of T . Furthermore,

$$TIT^+ = TT^+ = \sum_{k=1}^m T_k T_k^+ = \sum_{k=1}^m T R_k R_k^* T^+ = T \left(\sum_{k=1}^m R_k R_k^* \right) T^+$$

implies $I = \sum_{k=1}^m R_k R_k^*$ due to the density of $\text{ran } T^+$ and the injectivity of T . $\square$

Definition 2.1.4. Let $j \in [1, m]_{\mathbb{Z}}$. We denote by

$$\Theta : \underbrace{(TT^+)' \rightarrow (T^+T)'}_{\subseteq L_b(\mathcal{K})} \quad C \mapsto T^{-1}CT,$$

$$\Theta_j : \underbrace{(T_j T_j^+)' \rightarrow (T_j^+ T_j)'}_{\subseteq L_b(\mathcal{K})} \quad C \mapsto T_j^{-1}CT_j,$$

$$\Gamma_j : \underbrace{(R_j R_j^*)' \rightarrow (R_j^* R_j)'}_{\subseteq L_b(\mathcal{H})} \quad C \mapsto R_j^{-1}CR_j,$$

the $*$ -homomorphisms obtained by applying Theorem 1.8.7 to T , T_j , and R_j as defined by Lemma 2.1.3.

Lemma 2.1.5. For $j \in [1, m]_{\mathbb{Z}}$ we have $\Theta((T_j T_j^+)' \cap (TT^+)') \subseteq (R_j R_j^*)' \cap (T^+ T)'$, where in fact

$$\Theta(C) R_j R_j^* = R_j \Theta_j(C) R_j^* = R_j R_j^* \Theta(C), \quad C \in (T_j T_j^+)' \cap (TT^+). \quad (2.2)$$

Moreover,

$$\Theta_j(C) = \Gamma_j \circ \Theta(C), \quad C \in (T_j T_j^+)' \cap (TT^+). \quad (2.3)$$

Proof. Let $C \in (T_j T_j^+)' \cap (TT^+)'$. By Lemma 1.8.6 we have

$$\Theta_j(C) T_j^+ = T_j^{-1} C T_j T_j^+ = T_j^{-1} T_j T_j^+ C = T_j^+ C.$$

Similarly, $\Theta(C) T^+ = T^+ C$. Therefore,

$$T R_j \Theta_j(C) R_j^* T^+ = T_j \Theta_j(C) T_j^+ = T_j T_j^+ C = T R_j R_j^* T^+ C = T R_j R_j^* \Theta(C) T^+.$$

Since T is injective and $\text{ran } T^+$ is dense, we obtain $R_j \Theta_j(C) R_j^* = R_j R_j^* \Theta(C)$. Applying this equation to C^+ and taking adjoints yields $R_j \Theta_j(C) R_j^* = \Theta(C) R_j R_j^*$. In particular, $\Theta(C)$ belongs to $(R_j R_j^*)'$. Hence, we can apply Γ_j to $\Theta(C)$ and obtain

$$\Gamma_j \circ \Theta(C) = R_j^{-1} T^{-1} C T R_j = T_j^{-1} C T_j = \Theta_j(C).$$

□

Lemma 2.1.6. Given $j \in [1, m]_{\mathbb{Z}}$ we have $R_j R_j^* \in (T^+ T)'$ and $R_j^* R_j \in (T_j^+ T_j)'$. Moreover,

$$\Theta(T_j T_j^+) = R_j R_j^* T^+ T = T^+ T R_j R_j^*. \quad (2.4)$$

In particular,

$$p_j(\Theta[\mathbf{A}]) = R_j R_j^* \sum_{k=1}^m p_k(\Theta[\mathbf{A}]) = \sum_{k=1}^m p_k(\Theta[\mathbf{A}]) R_j R_j^*. \quad (2.5)$$

Proof. According to Theorem 1.8.7 we have $\Theta(TT^+) = T^+ T$. Thus, by (2.2), $R_j R_j^*$ commutes with $T^+ T$ implying

$$T_j^+ T_j R_j^* R_j = R_j^* (T^+ T R_j R_j^*) R_j = R_j^* (R_j R_j^* T^+ T) R_j = R_j^* R_j T_j^+ T_j,$$

i.e. $R_j^* R_j \in (T_j^+ T_j)'$. Moreover,

$$\Theta(T_j T_j^+) = T^{-1} T_j T_j^+ T = T^{-1} T R_j R_j^* T^+ T = R_j R_j^* T^+ T.$$

Since Θ is an algebra homomorphism, we have

$$p_j(\Theta[\mathbf{A}]) = \Theta(p_j(\mathbf{A})) = \Theta(T_j T_j^+)$$

and

$$\sum_{k=1}^m p_k(\Theta[\mathbf{A}]) = \Theta\left(\sum_{k=1}^m T_k T_k^+\right) = \Theta(TT^+) = T^+ T.$$

Considering these facts, we obtain (2.5) from (2.4). □

The following theorem will play a key role in the proof of Proposition 2.1.8; see for example [Rud87, Theorem 6.19]. We say a function $f : X \rightarrow \mathbb{C}$ vanishes at infinity, if for any $\epsilon > 0$ there exists a compact set C such that $|f(x)| < \epsilon$ for all $x \in X \setminus C$.

Theorem 2.1.7 (Riesz-Markov-Kakutani). *Let X be a locally compact Hausdorff space. We denote by $C_0(X)$ the space of all continuous complex-valued functions that vanish at infinity. If Φ is a bounded linear functional on $C_0(X)$, then there exists a unique regular complex Borel measure μ on X such that*

$$\Phi(f) = \int_X f d\mu, \quad f \in C_0(X).$$

For the next proposition, keep in mind that as Θ and Θ_j are $*$ -homomorphisms, the tuples $\Theta[\mathbf{A}] = (\Theta(A_i))_{i=1}^n \in L_b(\mathcal{H})^n$ and $\Theta_j[\mathbf{A}] = (\Theta_j(A_i))_{i=1}^n \in L_b(\mathcal{H}_j)^n$ are tuples of pairwise commuting self-adjoint operators.

Proposition 2.1.8. *For $j \in [1, m]_{\mathbb{Z}}$ we denote by E and E^j the joint spectral measures of $\Theta[\mathbf{A}] \in L_b(\mathcal{H})^n$ and $\Theta_j[\mathbf{A}] \in L_b(\mathcal{H}_j)^n$, respectively, defined on the Borel subsets of $\mathbb{R}^n$; see Remark 1.7.11. The following assertions hold true.*

- (i) $E(\Delta) \in (T^+T)' \cap \bigcap_{k=1}^m (R_k R_k^*)'$ for all Borel subsets $\Delta \subseteq \mathbb{R}^n$.
- (ii) $E^j(\Delta) = \Gamma_j(E(\Delta)) \in (T_j^+T_j)' \cap (R_j^* R_j)'$ for all Borel subsets $\Delta \subseteq \mathbb{R}^n$.
- (iii) $\int \phi dE \in (T^+T)' \cap \bigcap_{k=1}^m (R_k R_k^*)'$ for $\phi \in \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$.
- (iv) $\int \phi dE^j = \Gamma_j(\int \phi dE) \in (T_j^+T_j)' \cap (R_j^* R_j)'$ for $\phi \in \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$.

Note that $\mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$ is the C^* -algebra of complex-valued bounded Borel-measurable functions on $\sigma(\Theta[\mathbf{A}])$; see Definition 1.7.1.

Proof. (i) and (iii): Let $i \in [1, n]_{\mathbb{Z}}$ and $j \in [1, m]_{\mathbb{Z}}$. From Lemma 2.1.2 and Lemma 2.1.5, we derive $\Theta(A_i) \in (R_j R_j^*)' \cap (T^+T)'$. This implies $R_j R_j^*, T^+T \in \Theta[\mathbf{A}]'$, $j \in [1, m]_{\mathbb{Z}}$, and the equivalent statements (ii) and (iv) in Theorem 1.7.7 imply (i) and (iii) of the present proposition.

(ii): By Definition 2.1.4,

$$\Gamma_j(C) R_j^* = R_j^{-1} C R_j R_j^* = R_j^{-1} R_j R_j^* C = R_j^* C, \quad C \in (R_j R_j^*)'. \quad (2.6)$$

For a Borel subset $\Delta \subseteq \mathbb{R}^n$ we conclude $E(\Delta) \in (R_j R_j^*)'$ from (i). Thus,

$$\langle \Gamma_j(E(\Delta)) R_j^* g, h \rangle_j = \langle R_j^* E(\Delta) g, h \rangle_j = \langle E(\Delta) g, R_j h \rangle, \quad g \in \mathcal{H}, h \in \mathcal{H}_j. \quad (2.7)$$

Let $s \in \mathbb{C}[x_1, \dots, x_n]$. In the following, keep in mind that for any algebras $\mathcal{A}, \mathcal{B}$, an algebra homomorphism ψ , and $\mathbf{x} = (x_j)_{j=1}^n \in \mathcal{A}^n$, we have $\psi(s(\mathbf{x})) = s(\psi[\mathbf{x}])$. Lemma 2.1.2 and Lemma 2.1.5 show

$$s(\Theta[\mathbf{A}]) = \Theta(s(\mathbf{A})) \in \Theta(\mathbf{A}') \subseteq (R_j R_j^*)' \cap (T^+T)',$$

implying

$$R_j^* s(\Theta[\mathbf{A}]) \stackrel{(2.6)}{=} \Gamma_j(s(\Theta[\mathbf{A}])) R_j^* = s(\Gamma_j \circ \Theta[\mathbf{A}]) R_j^* \stackrel{(2.3)}{=} s(\Theta_j[\mathbf{A}]) R_j^*.$$

Let $g \in \mathcal{H}, h \in \mathcal{H}_j$. From (2.7) we infer

$$\begin{aligned}
 \int_{\mathbb{R}^n} s(\mathbf{x}) d\langle \Gamma_j(E(\mathbf{x})) R_j^* g, h \rangle_j &= \int_{\mathbb{R}^n} s(\mathbf{x}) d\langle E(\mathbf{x}) g, R_j h \rangle \\
 &= \langle s(\Theta[\mathbf{A}]) g, R_j h \rangle \\
 &= \langle R_j^* s(\Theta[\mathbf{A}]) g, h \rangle_j \\
 &= \langle s(\Theta_j[\mathbf{A}]) R_j^* g, h \rangle_j = \int_{\mathbb{R}^n} s(\mathbf{x}) d\langle E^j(\mathbf{x}) R_j^* g, h \rangle_j.
 \end{aligned} \tag{2.8}$$

We will show $\langle \Gamma_j(E(\Delta)) R_j^* g, h \rangle_j = \langle E^j(\Delta) R_j^* g, h \rangle_j$ for Borel subsets $\Delta \subseteq \mathbb{R}^n$. First observe that

$$\begin{aligned}
 \text{supp } \langle \Gamma_j(E) R_j^* g, h \rangle_j &\subseteq \text{supp } E = \sigma(\Theta[\mathbf{A}]), \\
 \text{supp } \langle E^j R_j^* g, h \rangle_j &\subseteq \text{supp } E^j = \sigma(\Theta_j[\mathbf{A}]) \stackrel{\star}{\subseteq} \sigma(\Theta[\mathbf{A}]),
 \end{aligned}$$

where $\star$ follows from $\Theta_j[\mathbf{A}] = \Gamma_j \circ \Theta[\mathbf{A}]$. Since $\mathbb{C}[x_1, \dots, x_n]$ as a set of functions on $\sigma(\Theta[\mathbf{A}]) \subseteq \mathbb{R}^n$ is dense in $C(\sigma(\Theta[\mathbf{A}]))$ due to Stone-Weierstrass theorem, (2.8) implies

$$\int_{\sigma(\Theta[\mathbf{A}])} f(\mathbf{x}) d\langle \Gamma_j(E(\mathbf{x})) R_j^* g, h \rangle_j = \int_{\sigma(\Theta[\mathbf{A}])} f(\mathbf{x}) d\langle E^j(\mathbf{x}) R_j^* g, h \rangle_j, \quad f \in C(\sigma(\Theta[\mathbf{A}])).$$

Consequently, Theorem 2.1.7 implies $\langle \Gamma_j(E(\Delta)) R_j^* g, h \rangle_j = \langle E^j(\Delta) R_j^* g, h \rangle_j$ for Borel subsets $\Delta \subseteq \mathbb{R}^n$. Since g, h were arbitrary, we obtain

$$(\Gamma_j(E(\Delta)) - E^j(\Delta)) R_j^* = 0, \quad \Delta \subseteq \mathbb{R}^n \text{ Borel subset},$$

which, due to $(\text{ran } R_j^*)^\perp = \ker R_j = \{0\}$, implies

$$E^j(\Delta) = \Gamma_j(E(\Delta)) \in (R_j^* R_j)'.$$

$\Theta_j(A_i) \in (T_j^+ T_j)'$ for all $i \in [1, n]_{\mathbb{Z}}$ shows $E^j(\Delta) \in (T_j^+ T_j)'$; see Theorem 1.7.7.

(iv): Given $\phi \in \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$ its restriction to $\sigma(\Theta_j[\mathbf{A}])$ belongs to $\mathfrak{B}(\sigma(\Theta_j[\mathbf{A}]))$. Let $g \in \mathcal{H}, h \in \mathcal{H}_j$. Due to $E_j(\Delta) R_j^* = \Gamma_j(E(\Delta)) R_j^* = R_j^* E(\Delta)$, we obtain

$$\begin{aligned}
 \left\langle \Gamma_j \left(\int \phi dE \right) R_j^* g, h \right\rangle_j &\stackrel{(2.6)}{=} \left\langle R_j^* \int \phi dE g, h \right\rangle_j \\
 &= \left\langle \int \phi dE g, R_j h \right\rangle \\
 &= \int \phi(\mathbf{x}) d\langle E(\mathbf{x}) g, R_j h \rangle \\
 &= \int \phi(\mathbf{x}) d\langle R_j^* E(\mathbf{x}) g, h \rangle_j \\
 &= \int \phi(\mathbf{x}) d\langle E_j(\mathbf{x}) R_j^* g, h \rangle_j = \left\langle \int \phi dE_j R_j^* g, h \right\rangle_j.
 \end{aligned}$$

We conclude

$$\left(\Gamma_j \left(\int \phi dE \right) - \int \phi dE_j \right) R_j^* = 0,$$

which by the density of $\text{ran } R_j^*$ implies

$$\Gamma_j \left(\int \phi dE \right) = \int \phi dE_j.$$

Finally, $\int \phi dE_j \in (T_j^+ T_j)' \cap (R_j^* R_j)'$ is a direct consequence of (ii); see Theorem 1.7.7. $\square$

Definition 2.1.9. Let $j \in [1, m]_{\mathbb{Z}}$. We denote by

$$\begin{aligned}\Xi &: L_b(\mathcal{H}) \rightarrow L_b(\mathcal{K}), \quad D \mapsto TDT^+, \\ \Xi_j &: L_b(\mathcal{H}_j) \rightarrow L_b(\mathcal{K}), \quad D \mapsto T_jDT_j^+, \\ \Lambda_j &: L_b(\mathcal{H}_j) \rightarrow L_b(\mathcal{H}), \quad D \mapsto R_jDR_j^*,\end{aligned}$$

the injective bounded linear operators obtained by applying Lemma 1.8.8 to T , T_j , and R_j .

Lemma 2.1.10. Given $j \in [1, m]_{\mathbb{Z}}$ we have $\Xi_j = \Xi \circ \Lambda_j$. Moreover, if E and E^j are the joint spectral measures of $\Theta[\mathbf{A}]$ and $\Theta_j[\mathbf{A}]$, respectively, and $\phi \in \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$, then

$$\Xi_j \left(\int \phi dE^j \right) = \Xi \circ \Lambda_j \circ \Gamma_j \left(\int \phi dE \right) = \Xi \left(R_j R_j^* \int \phi dE \right). \quad (2.9)$$

Finally,

$$p_j(\mathbf{A})u(\mathbf{A}) = \Xi_j(u(\Theta_j[\mathbf{A}])) = \Xi(R_j R_j^* u(\Theta[\mathbf{A}])), \quad u \in \mathbb{C}[x_1, \dots, x_n]. \quad (2.10)$$

Proof. From $T_j = TR_j$ we derive $\Xi_j = \Xi \circ \Lambda_j$.

(2.9): The first equality is a consequence of Proposition 2.1.8 and $\Xi_j = \Xi \circ \Lambda_j$. For the second equality note that Γ_j and Λ_j were constructed by applying Theorem 1.8.7 and Lemma 1.8.8 to the operator R_j , respectively, and Lemma 1.8.8 (v) implies $\Lambda_j \circ \Gamma_j \left(\int \phi dE \right) = R_j R_j^* \left(\int \phi dE \right)$.

(2.10): By (2.1) we have $p_j(\mathbf{A})u(\mathbf{A}) = T_j T_j^+ u(\mathbf{A})$. Since Θ_j and Ξ_j were constructed by applying Theorem 1.8.7 and Lemma 1.8.8 to the operator T_j , respectively, Lemma 1.8.8 (v) yields

$$T_j T_j^+ u(\mathbf{A}) = \Xi_j \circ \Theta_j(u(\mathbf{A})) = \Xi_j(u(\Theta_j[\mathbf{A}])) \stackrel{(2.9)}{=} \Xi(R_j R_j^* u(\Theta[\mathbf{A}])).$$

□

Lemma 2.1.11. For $j \in [1, m]_{\mathbb{Z}}$ we have

$$\left\{ \mathbf{z} \in \mathbb{C}^n : |p_j(\mathbf{z})| > \|R_j R_j^*\| \cdot \left| \sum_{k=1}^m p_k(\mathbf{z}) \right| \right\} \subseteq \rho(\Theta[\mathbf{A}]).$$

Proof. For $N \in \mathbb{N}$ we set

$$\Delta_N := \left\{ \mathbf{z} \in \mathbb{C}^n : |p_j(\mathbf{z})|^2 > \frac{1}{N} + \|R_j R_j^*\|^2 \cdot \left| \sum_{k=1}^m p_k(\mathbf{z}) \right|^2 \right\}.$$

Let E be the spectral measure of $\Theta[\mathbf{A}]$. In the following, we will regard E to be defined on $\mathbb{C}^n$ by extending it canonically. For $x \in \text{ran } E(\Delta_N)$ we have

$$\begin{aligned}\|p_j(\Theta[\mathbf{A}])x\|^2 &= \|p_j(\Theta[\mathbf{A}])E(\Delta_N)x\|^2 = \int_{\Delta_N} |p_j(\mathbf{z})|^2 d\langle E(\mathbf{z})x, x \rangle \\ &\geq \int_{\Delta_N} \frac{1}{N} d\langle E(\mathbf{z})x, x \rangle + \|R_j R_j^*\|^2 \int_{\Delta_N} \left| \sum_{k=1}^m p_k(\mathbf{z}) \right|^2 d\langle E(\mathbf{z})x, x \rangle \\ &\geq \frac{1}{N} \|x\|^2 + \left\| R_j R_j^* \sum_{k=1}^m p_k(\Theta[\mathbf{A}])x \right\|^2 \\ &\stackrel{(2.5)}{=} \frac{1}{N} \|x\|^2 + \|p_j(\Theta[\mathbf{A}])x\|^2.\end{aligned}$$

Obviously, these inequalities can only be true if $x = 0$. Hence, $E(\Delta_N) = 0$. Since Δ_N is open, we have $\Delta_N \subseteq \mathbb{C}^n \setminus (\text{supp } E) = \mathbb{C}^n \setminus \sigma(\Theta(A)) \subseteq \rho(\Theta[A])$. We conclude

$$\left\{ z \in \mathbb{C}^n : |p_j(z)| > \|R_j R_j^*\| \cdot \left| \sum_{k=1}^m p_k(z) \right| \right\} = \bigcup_{N \in \mathbb{N}} \Delta_N \subseteq \rho(\Theta[A]).$$

□

Corollary 2.1.12. *The zeros of $\sum_{k=1}^m p_k$ are contained in $\rho(\Theta[A]) \cup V(\langle p_1, \dots, p_m \rangle)$; see Lemma 1.1.4 and Definition 1.1.7.*

Proof. Let $z \in \mathbb{C}^n$ such that $\sum_{k=1}^m p_k(z) = 0$ and $z \notin V(\langle p_1, \dots, p_m \rangle)$. By Lemma 1.1.9 there exists $i \in [1, m]_{\mathbb{Z}}$ such that $|p_i(z)| > 0 = \|R_i R_i^*\| \cdot |\sum_{k=1}^m p_k(z)|$. From Lemma 2.1.11 we conclude $z \in \rho(\Theta[A])$. □

Lemma 2.1.13. *If $M \subseteq \mathbb{R}^n$ is a Borel subset such that $\sum_{j=1}^m p_j(z) \neq 0$ for all $z \in M$, then*

$$R_j R_j^* E(M) = E(M) R_j R_j^* = \int_M \frac{p_j}{\sum_{i=1}^m p_i} dE, \quad j \in [1, m]_{\mathbb{Z}},$$

where E denotes the spectral measure of $\Theta[A]$.

Proof. The first equality is known from Proposition 2.1.8. Note that the integral on the right-hand side is well-defined as the integrand is a bounded measurable function on $\sigma(\Theta[A]) \cap M$ according to Lemma 2.1.11.

As the concerned operators clearly vanish on $\text{ran } E(\mathbb{R}^n \setminus M)$, we will show the equality for the restriction to $\text{ran } E(M)$. The operator

$$\int_M \sum_{k=1}^m p_k dE = E(M) \Theta \left(\sum_{k=1}^m p_k(A) \right) \quad (2.11)$$

is self-adjoint as $\sum_{k=1}^m p_k(A)$ is self-adjoint and Θ is a $*$ -homomorphism. Given $0 \neq x \in \text{ran } E(M)$, $\langle E(\cdot)x, x \rangle$ is not the zero-measure because of $\langle E(M)x, x \rangle = \|x\|^2$ and hence

$$\left\| \int_M \sum_{k=1}^m p_k dE x \right\|^2 = \int_M \underbrace{\left| \sum_{k=1}^m p_k(z) \right|^2}_{>0} d\langle E(z)x, x \rangle > 0.$$

Since (2.11) vanishes on $\text{ran } E(\mathbb{R}^n \setminus M)$, we have

$$\left(\text{ran } \int_M \sum_{k=1}^m p_k dE \right)^\perp = \ker \int_M \sum_{k=1}^m p_k dE = \text{ran } E(\mathbb{R}^n \setminus M).$$

If y is in the range of $\int_M \sum_{k=1}^m p_k dE$, then there exists $x \in \text{ran } E(M)$ such that $y = \int_M \sum_{k=1}^m p_k dE x$. Consequently,

$$\begin{aligned} \int_M \frac{p_j}{\sum_{i=1}^m p_i} dE y &= \int_M p_j dE x = p_j(\Theta[A]) E(M)x \\ &\stackrel{(2.5)}{=} R_j R_j^* \int_M \sum_{k=1}^m p_k dE E(M)x = R_j R_j^* y. \end{aligned}$$

Due to the density of the range of $\int_M \sum_{k=1}^m p_k dE$ in $\text{ran } E(M)$, the above equality holds true for all $y \in \text{ran } E(M)$. □

2.2 Abstract Functional Calculus

The joint functional calculus for $\mathbf{A}$ will be defined for functions that can be represented by elements in $\mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$. In the present section, we are going to construct an abstract functional calculus, i.e. a $*$ -homomorphism from $\mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$ to $\mathbf{A}''$, which will be utilized in Section 3.1 to construct the joint functional calculus.

Let $\mathcal{I}$ be the ideal generated by all definitizing polynomials of $\mathbf{A}$ and $p_1, \dots, p_m \in \mathbb{R}[x_1, \dots, x_n]$ be definitizing polynomials of $\mathbf{A}$ such that $\mathcal{I} = \langle p_1, \dots, p_m \rangle$ according to Proposition 1.6.9. Moreover, we will import the notations and definitions from Section 2.1 such as Θ and Ξ , which are constructed using $p_1, \dots, p_m$, and denote by E the joint spectral measure of the tuple $\Theta[\mathbf{A}] \in L_b(\mathcal{H})^n$.

Lemma 2.2.1. *For $f \in \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$ and Ξ as in Definition 2.1.9, the operator $\Xi(\int f dE)$ belongs to $\mathbf{A}''$.*

Proof. Let $C \in \mathbf{A}' \subseteq (TT^+)'$; see Lemma 2.1.2. Keep in mind that by Proposition 2.1.8, $\int f dE$ belongs to $(T^+T)'$. According to Lemma 1.8.8, (ii) and (iii), we have

$$C\Xi\left(\int f dE\right) = \Xi\left(\Theta(C) \int f dE\right), \quad (2.12)$$

$$\Xi\left(\int f dE\right)C = \Xi\left(\int f dE \Theta(C)\right). \quad (2.13)$$

Since Θ is a $*$ -homomorphism and $C \in \mathbf{A}'$, we have $\Theta(C) \in \Theta[\mathbf{A}]'$. By Theorem 1.7.7, $\Theta(C)$ commutes with $\int f dE$, which implies the equality of the right-hand side of (2.12) and (2.13). Consequently, $\Xi(\int f dE)$ commutes with $C \in \mathbf{A}'$. $\square$

Definition 2.2.2. We define $\Psi : \mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}])) \rightarrow \mathbf{A}'' \subseteq L_b(\mathcal{K})$ by

$$\Psi(r, f) = r(\mathbf{A}) + \Xi\left(\int f dE\right).$$

By $\mathcal{N}$ we denote the set of all $(r, f) \in \mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$ such that

(i) for all $z \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$,

$$r(z) + f(z) \sum_{k=1}^m p_k(z) = 0.$$

(ii) $f(z) = 0$ for all $z \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$.

(iii) there exist $u_1, \dots, u_m \in \mathbb{C}[x_1, \dots, x_n]$ such that $r = \sum_{k=1}^m u_k p_k$ and $u_k(z) = 0$ for all $k \in [1, m]_{\mathbb{Z}}$ and $z \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$.

Recall that $p^\#(x_1, \dots, x_n) = \overline{p(\overline{x_1}, \dots, \overline{x_n})}$ for $p \in \mathbb{C}[x_1, \dots, x_n]$; see Example 1.3.6.

Proposition 2.2.3. *If we endow $\mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$ with*

- $(r, f) + (s, g) := (r + s, f + g)$,
- $\lambda(r, f) := (\lambda r, \lambda f)$,

- $(r, f) \cdot (s, g) := (rs, rg + sf + fg \sum_{j=1}^m p_j),$
- $(r, f)^* := (r^\#, \bar{f}),$

for $(r, f), (s, g) \in \mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}])))$ and $\lambda \in \mathbb{C}$, then $\mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}])))$ constitutes a commutative unital $*$ -algebra, where $(1, 0)$ is the unity.

Proof. Clearly, $\mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}])))$ turns into a vector space with the defined addition and scalar multiplication.

Let $(r, f), (s, g), (q, h) \in \mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}])))$ and $\lambda \in \mathbb{C}$. $\sigma(\Theta[\mathbf{A}])$ being compact, the second component of $(r, f) \cdot (s, g)$ is bounded and measurable. As the multiplication is clearly commutative, bilinearity follows from

$$\begin{aligned} ((r, f) + (s, g)) \cdot (q, h) &= (r + s, f + g) \cdot (q, h) \\ &= \left(rq + sq, rh + sh + qf + qg + (fh + gh) \sum_{j=1}^m p_j \right) \\ &= \left(rq, rh + qf + fh \sum_{j=1}^m p_j \right) + \left(sq, sh + qg + gh \sum_{j=1}^m p_j \right) \\ &= (r, f) \cdot (q, h) + (s, g) \cdot (q, h), \end{aligned}$$

$$(\lambda r, \lambda f)(s, g) = \left(\lambda rs, \lambda rg + \lambda sf + \lambda fg \sum_{j=1}^m p_j \right) = \lambda((r, f) \cdot (s, g)).$$

Furthermore, the multiplication is associative because of

$$\begin{aligned} ((r, f) \cdot (s, g)) \cdot (q, h) &= \left(rs, rg + sf + fg \sum_{j=1}^m p_j \right) \cdot (q, h) \\ &= \left(rsq, rsh + q(rg + sf + fg \sum_{j=1}^m p_j) \right. \\ &\quad \left. + (rg + sf + fg \sum_{j=1}^m p_j)h \sum_{j=1}^m p_j \right) \\ &= \left(rsq, sqf + r(sh + qg + gh \sum_{j=1}^m p_j) \right. \\ &\quad \left. + (sh + qg + gh \sum_{j=1}^m p_j)f \sum_{j=1}^m p_j \right) \\ &= (r, f) \cdot \left(sq, sh + qg + gh \sum_{j=1}^m p_j \right) = (r, f) \cdot ((s, g) \cdot (q, h)). \end{aligned}$$

It is straightforward to check that $(1, 0)$ is the unity.

As $^\#$ and $\bar{\cdot}$ are conjugate linear involutions on the respective spaces, so is * . Note that for $p \in \mathbb{C}[x_1, \dots, x_n]$ we have $p^\#(z) = \overline{p(\bar{z})} = \overline{p(z)}$, $z \in \sigma(\Theta[\mathbf{A}]) \subseteq \mathbb{R}^n$. $p_j^\# = p_j \in \mathbb{R}[x_1, \dots, x_n]$ for all $j \in [1, m]$ implies

$$\begin{aligned} ((r, f) \cdot (s, g))^* &= (rs, rg + sf + fg \sum_{j=1}^m p_j)^* \\ &= (s^\# r^\#, \overline{rg + sf + fg \sum_{j=1}^m p_j}) \\ &= (s^\# r^\#, r^\# \bar{g} + s^\# \bar{f} + \bar{f} g \sum_{j=1}^m p_j) \\ &= (s^\#, \bar{g}) \cdot (r^\#, \bar{f}) = (s, g)^* \cdot (r, f)^*, \end{aligned}$$

demonstrating the compatibility of * with multiplication. □

Lemma 2.2.4. For $r \in \mathbb{C}[x_1, \dots, x_n]$ and $f \in \mathfrak{B}(\sigma(\Theta[\mathbf{A}])))$ we have

$$r(\mathbf{A}) \Xi \left(\int f dE \right) = \Xi \left(\int f dE \right) r(\mathbf{A}) = \Xi \left(\int r f dE \right). \quad (2.14)$$

Proof. By Lemma 2.1.2 we have $r(\mathbf{A}) \in (TT^+)'$ and, therefore, Lemma 1.8.8 yields

$$\begin{aligned}\Xi(D)r(\mathbf{A}) &= \Xi(D\Theta(r(\mathbf{A}))) = \Xi(Dr(\Theta[\mathbf{A}])), \\ r(\mathbf{A})\Xi(D) &= \Xi(\Theta(r(\mathbf{A}))D) = \Xi(r(\Theta[\mathbf{A}])D),\end{aligned}$$

for all $D \in (T^+T)'$. By Proposition 2.1.8 we have $\int f dE \in (T^+T)'$. Applying the previous equality to $D = \int f dE$, we obtain (2.14) because $\int f dE$ commutes with $r(\Theta[\mathbf{A}]) = \int r dE$. $\square$

Lemma 2.2.5. *The map Ψ introduced in Definition 2.2.2 constitutes a unit-preserving $*$ -homomorphism.*

Proof. By Lemma 1.8.8, Ξ is linear and compatible with taking adjoints. As both the spectral integral and the evaluation homomorphism are linear and compatible with taking adjoints, so is Ψ . Let $(r, f), (s, g) \in \mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$. From Lemma 1.8.8 and Lemma 2.2.4 we infer

$$\begin{aligned}\Psi(r, f) \Psi(s, g) &= \left(r(\mathbf{A}) + \Xi \left(\int f dE \right) \right) \left(s(\mathbf{A}) + \Xi \left(\int g dE \right) \right) \\ &= r(\mathbf{A})s(\mathbf{A}) + r(\mathbf{A})\Xi \left(\int g dE \right) \\ &\quad + \Xi \left(\int f dE \right) s(\mathbf{A}) + \Xi \left(\int f dE \right) \Xi \left(\int g dE \right) \\ &= (r \cdot s)(\mathbf{A}) + \Xi \left(\int rg dE \right) + \Xi \left(\int sf dE \right) + \Xi \left(T^+T \int fg dE \right).\end{aligned}$$

By Theorem 1.8.7 we have $T^+T = \Theta(TT^+)$ and conclude

$$T^+T = \Theta(TT^+) = \Theta \left(\sum_{j=1}^m p_j(\mathbf{A}) \right) = \sum_{j=1}^m p_j(\Theta[\mathbf{A}]) = \int \sum_{j=1}^m p_j dE.$$

Hence,

$$\begin{aligned}\Psi(r, f) \Psi(s, g) &= (r \cdot s)(\mathbf{A}) + \Xi \left(\int \left(rg + sf + fg \sum_{j=1}^m p_j \right) dE \right) \\ &= \Psi \left(r \cdot s, rg + sf + fg \sum_{j=1}^m p_j \right) = \Psi((r, f) \cdot (s, g)),\end{aligned}$$

and Ψ is compatible with multiplication. Because of $\Psi(1, 0) = I + \Xi(\int 0 dE) = I$, Ψ is unit-preserving. $\square$

Lemma 2.2.6. *$\mathcal{N}$ constitutes an ideal satisfying $\mathcal{N} = \mathcal{N}^*$.*

Proof. $\mathcal{N}$ is clearly a subspace. Let $(r, f) \in \mathcal{N}$ and $(s, g) \in \mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$. We will show $(r, f) \cdot (s, g) = (rs, rg + sf + fg \sum_{j=1}^m p_j) \in \mathcal{N}$. Given $z \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$

we have

$$\begin{aligned} r(\mathbf{z})s(\mathbf{z}) + \left(r(\mathbf{z})g(\mathbf{z}) + s(\mathbf{z})f(\mathbf{z}) + f(\mathbf{z})g(\mathbf{z}) \sum_{j=1}^m p_j(\mathbf{z}) \right) \sum_{j=1}^m p_j(\mathbf{z}) \\ = s(\mathbf{z}) \underbrace{\left(r(\mathbf{z}) + f(\mathbf{z}) \sum_{j=1}^m p_j(\mathbf{z}) \right)}_{=0} + g(\mathbf{z}) \sum_{j=1}^m p_j(\mathbf{z}) \underbrace{\left(r(\mathbf{z}) + f(\mathbf{z}) \sum_{j=1}^m p_j(\mathbf{z}) \right)}_{=0} = 0. \end{aligned}$$

$(r, f) \in \mathcal{N}$ implies the existence of $u_1, \dots, u_m \in \mathbb{C}[x_1, \dots, x_n]$ with $r = \sum_{j=1}^m u_j p_j$ and $u_j(\mathbf{z}) = 0$ for $j \in [1, m]_{\mathbb{Z}}$ and $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$. Hence,

$$rs = s \sum_{j=1}^m u_j p_j = \sum_{j=1}^m (su_j) p_j,$$

such that $s(\mathbf{z})u_j(\mathbf{z}) = 0$ for $j \in [1, m]_{\mathbb{Z}}$ and $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$. Finally, for $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$ we have

$$\underbrace{r(\mathbf{z})g(\mathbf{z})}_{=0} + s(\mathbf{z}) \underbrace{f(\mathbf{z})}_{=0} + \underbrace{f(\mathbf{z})g(\mathbf{z})}_{=0} \sum_{j=1}^m p_j(\mathbf{z}) = 0.$$

$(r, f)^* = (r^\#, \bar{f}) \in \mathcal{N}$ follows in a straightforward manner from $\sigma(\Theta[\mathbf{A}]) \subseteq \mathbb{R}^n$ and $p_j^\# = p_j$, $j \in [1, m]_{\mathbb{Z}}$. Hence, $\mathcal{N}^* \subseteq \mathcal{N}$ and, in consequence, $\mathcal{N}^* = \mathcal{N}$. $\square$

Lemma 2.2.7. *If $(r, f) \in \mathcal{N}$, then $\Psi(r, f) = 0$.*

Proof. We have $r = \sum_{j=1}^m u_j p_j$ for some $u_j \in \mathbb{C}[x_1, \dots, x_n]$, $j \in [1, m]_{\mathbb{Z}}$, that vanish on $V(\mathcal{I}) \cap \sigma(\Theta[\mathbf{A}])$. From (2.10) we infer

$$r(\mathbf{A}) = \sum_{j=1}^m u_j(\mathbf{A}) p_j(\mathbf{A}) = \sum_{j=1}^m \Xi(R_j R_j^* u_j(\Theta[\mathbf{A}])) = \Xi \left(\sum_{j=1}^m R_j R_j^* \int u_j dE \right).$$

Using $\sum_{j=1}^m R_j R_j^* = I \in L_b(\mathcal{H})$ from Lemma 2.1.3, we obtain

$$\begin{aligned} \Psi(r, f) &= \Xi \left(\sum_{j=1}^m R_j R_j^* \int u_j dE \right) + \Xi \left(\sum_{j=1}^m R_j R_j^* \int f dE \right) \\ &= \Xi \left(\sum_{j=1}^m R_j R_j^* \int (u_j + f) dE \right). \end{aligned}$$

As $u_j + f$ vanishes on $\sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$ for $j \in [1, m]_{\mathbb{Z}}$, the spectral integral in the last term can also be written as a spectral integral over $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$. Finally, applying

Lemma 2.1.13 to $M = \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$ yields

$$\begin{aligned} \Psi(r, f) &= \Xi \left(\sum_{j=1}^m R_j R_j^* \int_{\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})} (u_j + f) dE \right) \\ &= \Xi \left(\sum_{j=1}^m \int_{\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})} \frac{(u_j + f)p_j}{\sum_{i=1}^m p_i} dE \right) \\ &= \Xi \left(\int_{\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})} \overbrace{\frac{r + f \sum_{j=1}^m p_j}{\sum_{i=1}^m p_i}}^{\equiv 0} dE \right) = 0. \end{aligned}$$

□

Summarizing all the statements proven in this section, we obtain the following result.

Theorem 2.2.8. *The map $\Psi : \mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}])) \rightarrow \mathbf{A}''$ introduced in Definition 2.2.2 constitutes a unit-preserving $*$ -homomorphism with $\ker \Psi \supseteq \mathcal{N}$.*

2.3 Algebra Corresponding to the Variety

Let $\mathcal{I}$ be the ideal generated by all definitizing polynomials of $\mathbf{A}$ and assume $\mathcal{I} \subsetneq \mathbb{C}[x_1, \dots, x_n]$. We also fix definitizing polynomials $p_1, \dots, p_m \in \mathbb{R}[x_1, \dots, x_n]$ of $\mathbf{A}$ with $\mathcal{I} = \langle p_1, \dots, p_m \rangle$ according to Proposition 1.6.9.

Assumption 2.3.1. Let

$$\mathcal{I} = \bigcap_{j=1}^l \mathcal{Q}_j$$

be a minimal primary decomposition of the ideal $\mathcal{I}$ and set $\mathcal{P}_j := \sqrt{\mathcal{Q}_j}$ for $j \in [1, l]_{\mathbb{Z}}$; see Theorem 1.2.8. We will assume that the varieties $V(\mathcal{Q}_j)$, $j \in [1, l]_{\mathbb{Z}}$, are pairwise disjoint, which establishes the uniqueness of the minimal primary decomposition of $\mathcal{I}$ by Corollary 1.2.14.

Remark 2.3.2. We refer to $V \subseteq \mathbb{C}^n$ as *variety* if $V(S) = V$ for some $S \subseteq \mathbb{C}[x_1, \dots, x_n]$. A variety V is called *irreducible* if $V = V_1 \cup V_2$ for varieties V_1, V_2 implies $V = V_1$ or $V = V_2$. Any variety can be expressed as a finite union of irreducible varieties; see [CLO07, p.204]. In the notation of Assumption 2.3.1, $V(\mathcal{Q}_j)$, $j \in [1, l]_{\mathbb{Z}}$, are precisely those irreducible components of $V(\mathcal{I})$; see for example [CLO07, p.214].

Note that none of the varieties $V(\mathcal{Q}_j)$, $j \in [1, l]_{\mathbb{Z}}$, is empty since otherwise the corresponding primary component would be $\mathbb{C}[x_1, \dots, x_n]$ according to Theorem 1.1.19, which contradicts the primariness. Moreover, since the varieties $V(\mathcal{Q}_j)$, $j \in [1, l]_{\mathbb{Z}}$, are pairwise disjoint, they constitute a partition of the variety $V(\mathcal{I})$; see Lemma 1.1.22. Hence the relation $\sim$ on $V(\mathcal{I})$ defined by

$$\mathbf{a} \sim \mathbf{b} :\iff \exists j \in [1, l]_{\mathbb{Z}} : \mathbf{a}, \mathbf{b} \in V(\mathcal{Q}_j)$$

constitutes an equivalence relation. We set $\mathbb{V}(\mathcal{I}) := V(\mathcal{I})/\sim$ and

$$\mathbb{V}_{\mathbb{R}}(\mathcal{I}) := \{[\mathbf{a}] \in \mathbb{V}(\mathcal{I}) : \mathbf{a} \in V(\mathcal{I}) \cap \mathbb{R}^n\},$$

where $[\mathbf{a}]$ denotes the equivalence class of $\mathbf{a} \in V(\mathcal{I})$ with respect to $\sim$.

Definition 2.3.3. For $[\mathbf{a}] \in \mathbb{V}(\mathcal{I})$ and $j \in [1, l]_{\mathbb{Z}}$ such that $\mathbf{a} \in V(\mathcal{Q}_j)$, we employ the following notation:

$$\mathcal{Q}_{[\mathbf{a}]} := \mathcal{Q}_j, \quad \mathcal{P}_{[\mathbf{a}]} := \mathcal{P}_j.$$

Moreover, we define the algebras

$$\begin{aligned} \mathcal{A}_{[\mathbf{a}]} &:= \mathbb{C}[x_1, \dots, x_n] / (\mathcal{P}_{[\mathbf{a}]} \cdot \mathcal{Q}_{[\mathbf{a}]}), \\ \mathcal{B}_{[\mathbf{a}]} &:= \mathbb{C}[x_1, \dots, x_n] / \mathcal{Q}_{[\mathbf{a}]} \end{aligned}$$

Lemma 2.3.4. Given $[\mathbf{a}] \in \mathbb{V}(\mathcal{I})$ we have $V(\mathcal{P}_{[\mathbf{a}]} \cdot \mathcal{Q}_{[\mathbf{a}]}) = V(\mathcal{P}_{[\mathbf{a}]}) = V(\mathcal{Q}_{[\mathbf{a}]})$.

Proof. The second equality follows from $\mathcal{P}_{[\mathbf{a}]} = \sqrt{\mathcal{Q}_{[\mathbf{a}]}}$; see Proposition 1.2.6. Let $p \in \mathcal{P}_{[\mathbf{a}]}$. Since $\mathcal{P}_{[\mathbf{a}]}$ is the radical of $\mathcal{Q}_{[\mathbf{a}]}$, we have $p^k \in \mathcal{Q}_{[\mathbf{a}]}$ for some $k \in \mathbb{N}$, which implies $p^{k+1} \in \mathcal{P}_{[\mathbf{a}]} \cdot \mathcal{Q}_{[\mathbf{a}]}$. Consequently, $\mathcal{P}_{[\mathbf{a}]} \subseteq \sqrt{\mathcal{P}_{[\mathbf{a}]} \cdot \mathcal{Q}_{[\mathbf{a}]}}$. The inclusion $\sqrt{\mathcal{P}_{[\mathbf{a}]} \cdot \mathcal{Q}_{[\mathbf{a}]}} \subseteq \mathcal{P}_{[\mathbf{a}]}$ is trivial. Again, Proposition 1.2.6 yields the first equality. $\square$

For $N \in \mathbb{N}$ and algebras $\mathcal{A}_1, \dots, \mathcal{A}_N$ we denote by $\times_{j=1}^N \mathcal{A}_j$ their Cartesian product.

Proposition 2.3.5. The ideal

$$\mathcal{J} := \bigcap_{[\mathbf{a}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} (\mathcal{P}_{[\mathbf{a}]} \cdot \mathcal{Q}_{[\mathbf{a}]}) \cap \bigcap_{[\mathbf{a}] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{Q}_{[\mathbf{a}]}$$

satisfies $\mathcal{J} \subseteq \mathcal{I}$ and

$$\psi : \begin{cases} \mathbb{C}[x_1, \dots, x_n] / \mathcal{J} & \rightarrow \times_{[\mathbf{a}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{A}_{[\mathbf{a}]} \times \times_{[\mathbf{a}] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{B}_{[\mathbf{a}]}, \\ [p]_{\mathcal{J}} & \mapsto \left(\left([p]_{\mathcal{P}_{[\mathbf{a}]} \cdot \mathcal{Q}_{[\mathbf{a}]}} \right)_{[\mathbf{a}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}, \left([p]_{\mathcal{Q}_{[\mathbf{a}]}} \right)_{[\mathbf{a}] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \right), \end{cases} \quad (2.15)$$

constitutes an algebra isomorphism. Moreover, for any $s \in \mathcal{J}$ there exist $u_1, \dots, u_m \in \mathbb{C}[x_1, \dots, x_n]$ satisfying

$$(i) \quad u_1(\mathbf{z}) = \dots = u_m(\mathbf{z}) = 0 \text{ for } \mathbf{z} \in V(\mathcal{I}) \text{ with } [\mathbf{z}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I}),$$

$$(ii) \quad s = \sum_{i=1}^m u_i p_i; \text{ recall } \mathcal{I} = \langle p_1, \dots, p_m \rangle.$$

Proof. Since the varieties $V(\mathcal{Q}_j)$, $j \in [1, l]_{\mathbb{Z}}$, are pairwise disjoint, the varieties of

$$\mathcal{P}_{[\mathbf{a}]} \cdot \mathcal{Q}_{[\mathbf{a}]}, [\mathbf{a}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I}), \quad \mathcal{Q}_{[\mathbf{a}]}, [\mathbf{a}] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I}), \quad (2.16)$$

are pairwise disjoint according to Lemma 2.3.4. By Lemma 1.1.20 the ideals are pairwise comaximal. Thus, we can apply the Chinese Remainder Theorem 1.1.16 showing that (2.15) constitutes a ring-isomorphism. Due to Lemma 1.1.17, ψ is in fact an algebra isomorphism.

Keep in mind that since none of the varieties $V(\mathcal{Q}_j)$, $j \in [1, l]_{\mathbb{Z}}$, is empty, we have

$$\{\mathcal{Q}_1, \dots, \mathcal{Q}_l\} = \{\mathcal{Q}_{[a]} : [a] \in \mathbb{V}(\mathcal{I})\}. \quad (2.17)$$

Applying Lemma 1.1.22 to the ideals in (2.16), (2.17), and $(\mathcal{P}_{[a]})_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}$ yields

$$\begin{aligned} \mathcal{J} &= \prod_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} (\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}) \cdot \prod_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{Q}_{[a]} = \prod_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{P}_{[a]} \cdot \prod_{[a] \in \mathbb{V}(\mathcal{I})} \mathcal{Q}_{[a]} \\ &= \left(\prod_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{P}_{[a]} \right) \cdot \mathcal{I} = \left(\bigcap_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{P}_{[a]} \right) \cdot \langle p_1, \dots, p_m \rangle. \end{aligned}$$

Let $q_1, \dots, q_{m'}$ be the generators of $\bigcap_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{P}_{[a]}$. Given $s \in \mathcal{J}$, Lemma 1.1.12 implies the existence of $r_{ij} \in \mathbb{C}[x_1, \dots, x_n]$, $i \in [1, m]_{\mathbb{Z}}$, $j \in [1, m']_{\mathbb{Z}}$ such that

$$s = \sum_{i=1}^m \sum_{j=1}^{m'} r_{ij} p_i q_j = \sum_{i=1}^m \underbrace{\left(\sum_{j=1}^{m'} r_{ij} q_j \right)}_{=: u_i} p_i = \sum_{i=1}^m u_i p_i.$$

$u_1, \dots, u_m \in \bigcap_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{P}_{[a]}$ implies $u_i(\mathbf{z}) = 0$, $i \in [1, m]_{\mathbb{Z}}$, for $\mathbf{z} \in V(\mathcal{I})$ with $[\mathbf{z}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$. $\square$

Remark 2.3.6. Actually, with the same proof, Proposition 2.3.5 stays valid if you replace $\mathbb{V}_{\mathbb{R}}(\mathcal{I})$ by an arbitrary subset $W \subseteq \mathbb{V}(\mathcal{I})$. We have formulated it this way as it suffices for our causes, and it simplifies the notation and referencing.

For the remainder of the present section, we are going to construct involutions on the algebras $\mathbb{C}[x_1, \dots, x_n]/\mathcal{J}$ and $\times_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{A}_{[a]} \times \times_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{B}_{[a]}$ such that they constitute $*$ -algebras. We will also show that, with respect to the newly defined involutions, ψ from Proposition 2.3.5 constitutes a $*$ -isomorphism.

Given $\mathbf{z} \in \mathbb{C}^n$ we denote by $\bar{\mathbf{z}}$ the componentwise complex conjugation. Also, recall that $p^\#(\mathbf{z}) = \overline{p(\bar{\mathbf{z}})}$, $p \in \mathbb{C}[x_1, \dots, x_n]$, $\mathbf{z} \in \mathbb{C}^n$.

Lemma 2.3.7. *If $\mathcal{L} \subseteq \mathbb{C}[x_1, \dots, x_n]$ is an ideal, then $\mathcal{L}^\#$ is an ideal satisfying*

$$V(\mathcal{L}) = \overline{V(\mathcal{L}^\#)}.$$

Moreover, $\mathcal{L}$ is primary (prime) if and only if $\mathcal{L}^\#$ is primary (prime).

Proof. Given $f, g \in \mathcal{L}^\#$, $(f+g)^\# = f^\# + g^\# \in \mathcal{L}$. Hence, $f+g \in \mathcal{L}^\#$. If $h \in \mathbb{C}[x_1, \dots, x_n]$, then $(hf)^\# = h^\# f^\# \in \mathcal{L}$ and, therefore, $hf \in \mathcal{L}^\#$. Moreover,

$$\begin{aligned} V(\mathcal{L}) &= \{\mathbf{z} \in \mathbb{C}^n : f(\mathbf{z}) = 0, f \in \mathcal{L}\} \\ &= \left\{ \bar{\mathbf{z}} \in \mathbb{C}^n : f^\#(\mathbf{z}) = 0, f \in \mathcal{L} \right\} = \overline{V(\mathcal{L}^\#)}. \end{aligned}$$

Let $\mathcal{L}$ be primary and $f, g \in \mathbb{C}[x_1, \dots, x_n]$ be such that $fg \in \mathcal{L}^\#$ and $f \notin \mathcal{L}^\#$. Then we have $f^\# g^\# = (fg)^\# \in \mathcal{L}$ and $f^\# \notin \mathcal{L}$. Since $\mathcal{L}$ is primary, $(g^k)^\# = (g^\#)^k \in \mathcal{L}$ and, in consequence, $g^k \in \mathcal{L}^\#$ for some $k \in \mathbb{N}$. The converse is clear. The respective statement about prime ideals can be shown by substituting $k = 1$ in the present proof. $\square$

Lemma 2.3.8. *The ideal $\mathcal{I}$ is invariant under $p \mapsto p^\#$.*

Proof. The claim follows from the fact that $\mathcal{I}$ is generated by real polynomials, which are the self-adjoint elements of $\mathbb{C}[x_1, \dots, x_n]$ with respect to $\cdot^\#$. $\square$

Lemma 2.3.9. *We have $\mathcal{Q}_{[\mathbf{a}]}^\# = \mathcal{Q}_{[\bar{\mathbf{a}}]}$ and $\mathcal{P}_{[\mathbf{a}]}^\# = \mathcal{P}_{[\bar{\mathbf{a}}]}$ for any $[\mathbf{a}] \in \mathbb{V}(\mathcal{I})$.*

Proof. By Lemma 2.3.7,

$$\mathcal{I}^\# = \bigcap_{[\mathbf{a}] \in \mathbb{V}(\mathcal{I})} \mathcal{Q}_{[\mathbf{a}]}^\#$$

constitutes a primary decomposition of $\mathcal{I}^\#$. Moreover, it is a minimal primary decomposition because the varieties of the primary components are nonempty and pairwise disjoint. From $\mathcal{I} = \mathcal{I}^\#$ and the uniqueness of the minimal primary decomposition of $\mathcal{I}$ given by Corollary 1.2.14, we conclude

$$\{\mathcal{Q}_{[\mathbf{a}]} : [\mathbf{a}] \in \mathbb{V}(\mathcal{I})\} = \{\mathcal{Q}_{[\mathbf{a}]}^\# : [\mathbf{a}] \in \mathbb{V}(\mathcal{I})\}. \quad (2.18)$$

Keep in mind that due to $\mathcal{I} = \mathcal{I}^\#$, we have $V(\mathcal{I}) = \overline{V(\mathcal{I})}$. Let $\mathbf{a} \in V(\mathcal{I})$. By definition we have $\mathbf{a} \in V(\mathcal{Q}_{[\mathbf{a}]})$ and $\bar{\mathbf{a}} \in V(\mathcal{Q}_{[\bar{\mathbf{a}}]})$. On the other hand, by Lemma 2.3.7,

$$\bar{\mathbf{a}} \in \overline{V(\mathcal{Q}_{[\mathbf{a}]})} = V(\mathcal{Q}_{[\mathbf{a}]}^\#).$$

Since the varieties of ideals in (2.18) are pairwise disjoint, we conclude $\mathcal{Q}_{[\mathbf{a}]}^\# = \mathcal{Q}_{[\bar{\mathbf{a}}]}$, and from $\mathcal{P}_{[\mathbf{a}]}^\# = \sqrt{\mathcal{Q}_{[\mathbf{a}]}^\#}$ we derive $\mathcal{P}_{[\mathbf{a}]}^\# = \mathcal{P}_{[\bar{\mathbf{a}}]}$. $\square$

Remark 2.3.10. For $\mathbf{a}, \mathbf{b} \in V(\mathcal{I})$ with $\mathbf{a} \sim \mathbf{b}$, we have $\bar{\mathbf{a}} \sim \bar{\mathbf{b}}$, which is evident from

$$\mathcal{Q}_{[\bar{\mathbf{a}}]} = \mathcal{Q}_{[\mathbf{a}]}^\# = \mathcal{Q}_{[\mathbf{b}]}^\# = \mathcal{Q}_{[\bar{\mathbf{b}}]}.$$

Therefore, complex conjugation is well-defined on $\mathbb{V}(\mathcal{I})$. Moreover, we have $[\mathbf{a}] = \overline{[\mathbf{a}]}$ for $[\mathbf{a}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$.

Lemma 2.3.11. *If $\mathcal{L} \subseteq \mathbb{C}[x_1, \dots, x_n]$ is an ideal, then*

$$\cdot^* : \begin{cases} \mathbb{C}[x_1, \dots, x_n]/\mathcal{L} & \rightarrow \mathbb{C}[x_1, \dots, x_n]/\mathcal{L}^\#, \\ [p]_{\mathcal{L}} & \mapsto [p^\#]_{\mathcal{L}^\#}, \end{cases}$$

constitutes a conjugate linear bijection which is compatible with multiplication, i.e. $([p]_{\mathcal{L}} \cdot [q]_{\mathcal{L}})^ = [p]_{\mathcal{L}}^* [q]_{\mathcal{L}}^*$, $p, q \in \mathbb{C}[x_1, \dots, x_n]$. Furthermore, $\cdot^* : \mathbb{C}[x_1, \dots, x_n]/\mathcal{L}^\# \rightarrow \mathbb{C}[x_1, \dots, x_n]/\mathcal{L}$ is the inverse of $\cdot^* : \mathbb{C}[x_1, \dots, x_n]/\mathcal{L} \rightarrow \mathbb{C}[x_1, \dots, x_n]/\mathcal{L}^\#$.*

Proof. Given $u \in \mathcal{L}$ we have $u^\# \in \mathcal{L}^\#$. Hence $\cdot^*$ is well-defined. The conjugate linearity and the compatibility with multiplication is inherited from $\cdot^\#$ to $\cdot^*$. Since $\cdot^\#$ is an involution, $\cdot^* : \mathbb{C}[x_1, \dots, x_n]/\mathcal{L}^\# \rightarrow \mathbb{C}[x_1, \dots, x_n]/\mathcal{L}$ is the inverse mapping of $\cdot^* : \mathbb{C}[x_1, \dots, x_n]/\mathcal{L} \rightarrow \mathbb{C}[x_1, \dots, x_n]/\mathcal{L}^\#$. Hence $\cdot^* : \mathbb{C}[x_1, \dots, x_n]/\mathcal{L} \rightarrow \mathbb{C}[x_1, \dots, x_n]/\mathcal{L}^\#$ is bijective. $\square$

Lemma 2.3.12. *The ideal $\mathcal{J}$ from Proposition 2.3.5 is invariant under $\cdot^\#$. In particular, $\mathbb{C}[x_1, \dots, x_n]/\mathcal{J}$ endowed with $\cdot^*$ as in Lemma 2.3.11 constitutes a $*$ -algebra.*

Proof. From Lemma 2.3.9 we derive that $\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}$, $[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$, is invariant under $\cdot^\#$. Furthermore, exploiting the invariance of $\mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})$ under $\bar{\cdot}$, we obtain

$$\bigcap_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{Q}_{[a]}^\# = \bigcap_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{Q}_{[\bar{a}]} = \bigcap_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{Q}_{[a]}.$$

This implies

$$\mathcal{J}^\# = \bigcap_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} (\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]})^\# \cap \bigcap_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{Q}_{[a]}^\# = \mathcal{J}.$$

As a result, the mapping $\cdot^* : \mathbb{C}[x_1, \dots, x_n]/\mathcal{J} \rightarrow \mathbb{C}[x_1, \dots, x_n]/\mathcal{J}$ from Lemma 2.3.11 is an involution that turns $\mathbb{C}[x_1, \dots, x_n]/\mathcal{J}$ into a $*$ -algebra. $\square$

Lemma 2.3.13. *Endowed with the mapping $\cdot^\dagger$ on $\times_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{A}_{[a]} \times \times_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{B}_{[a]}$ defined by*

$$\begin{aligned} & \left(\left([p_{[a]}]_{\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}, \left([p_{[a]}]_{\mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \right)^\dagger \\ & := \left(\left([p_{[a]}^\#]_{\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}, \left([p_{[\bar{a}]}^\#]_{\mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \right), \end{aligned}$$

$\times_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{A}_{[a]} \times \times_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{B}_{[a]}$ constitutes a $*$ -algebra.

Proof. First note that $\cdot^*$ on $\mathcal{A}_{[a]} = \mathbb{C}[x_1, \dots, x_n]/(\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]})$, $[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$, is an involution as $\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}$ is invariant under $\cdot^\#$. We have

$$\begin{aligned} & \left(\left([p_{[a]}]_{\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}, \left([p_{[a]}]_{\mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \right)^\dagger \\ & = \left(\left([p_{[a]}]_{\mathcal{P}_{[a]}^\star \cdot \mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}, \left([p_{[\bar{a}]}]_{\mathcal{Q}_{[a]}^\star} \right)_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \right), \end{aligned}$$

where $\cdot^\star$ is the mapping defined in Lemma 2.3.11. The conjugate linearity and the compatibility with multiplication is inherited from $\cdot^*$ to $\cdot^\dagger$. Furthermore, Lemma 2.3.11 yields

$$\begin{aligned} & \left(\left([p_{[a]}]_{\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}, \left([p_{[a]}]_{\mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \right)^{\dagger\dagger} \\ & = \left(\left([p_{[a]}]_{\mathcal{P}_{[a]}^{\star\star} \cdot \mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}, \left([p_{[\bar{a}]}]_{\mathcal{Q}_{[a]}^{\star\star}} \right)_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \right) \\ & = \left(\left([p_{[a]}]_{\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}, \left([p_{[a]}]_{\mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \right), \end{aligned}$$

which demonstrates that $\cdot^\dagger$ is an involution. $\square$

Proposition 2.3.14. *The mapping ψ in Proposition 2.3.5 constitutes a $*$ -isomorphism if the algebras are equipped with the involutions introduced in Lemma 2.3.12 and 2.3.13, respectively.*

Proof. Given $p \in \mathbb{C}[x_1, \dots, x_n]$ we have

$$\begin{aligned} \psi([p^\#]_{\mathcal{J}}) &= \left(\left([p^\#]_{\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}, \left([p^\#]_{\mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \right) \\ &= \left(\left([p]_{\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})}, \left([p]_{\mathcal{Q}_{[a]}} \right)_{[a] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \right)^\dagger \\ &= \psi([p]_{\mathcal{J}})^\dagger. \end{aligned}$$

Since we already identified ψ as an algebra isomorphism, ψ constitutes a $*$ -isomorphism. $\square$

2.4 Function Spaces

This section will be devoted to the construction of a space of functions for which the functional calculus of the tuple of operators $\mathbf{A}$ can be defined.

We will continue to denote by $\mathcal{I}$ the ideal generated by all definitizing polynomials of $\mathbf{A}$ and fix definitizing polynomials $p_1, \dots, p_m \in \mathbb{R}[x_1, \dots, x_n]$ of $\mathbf{A}$ such that $\mathcal{I} = \langle p_1, \dots, p_m \rangle$ according to Proposition 1.6.9.

Definition 2.4.1. We assume $\mathcal{I} \subsetneq \mathbb{C}[x_1, \dots, x_n]$ and Assumption 2.3.1. We interpret the product $*$ -algebra

$$\mathcal{M}_{\mathbf{A}} := \mathbb{C}^{\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})} \times \bigtimes_{[\mathbf{z}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{A}_{[\mathbf{z}]} \times \bigtimes_{[\mathbf{z}] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})} \mathcal{B}_{[\mathbf{z}]}$$

as a $*$ -algebra of functions ϕ defined on $\sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I}) \subseteq \mathbb{C}^n$ satisfying

- $\phi(\mathbf{z}) \in \mathbb{C}$ for $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$,
- $\phi(\mathbf{z}) \in \mathcal{A}_{[\mathbf{z}]}$ for $\mathbf{z} \in V(\mathcal{I})$ such that $[\mathbf{z}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$,
- $\phi(\mathbf{z}) \in \mathcal{B}_{[\mathbf{z}]}$ for $\mathbf{z} \in V(\mathcal{I})$ such that $[\mathbf{z}] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})$,
- $\phi(\mathbf{z}) = \phi(\mathbf{w})$ for $\mathbf{z}, \mathbf{w} \in V(\mathcal{I})$ such that $[\mathbf{z}] = [\mathbf{w}]$.

We denote by $\cdot^\#$ the involution on $\mathcal{M}_{\mathbf{A}}$ defined by

- $\phi^\#(\mathbf{z}) := \overline{\phi(\mathbf{z})}$ for $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$,
- $\phi^\#(\mathbf{z}) := \phi(\mathbf{z})^*$ for $\mathbf{z} \in V(\mathcal{I})$ such that $[\mathbf{z}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$,
- $\phi^\#(\mathbf{z}) := \phi(\bar{\mathbf{z}})^*$ for $\mathbf{z} \in V(\mathcal{I})$ such that $[\mathbf{z}] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})$.

The mapping $\cdot^*$ is defined in Lemma 2.3.11.

We can meaningfully extend Definition 2.4.1 to the case $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$. As $V(\mathbb{C}[x_1, \dots, x_n]) = \emptyset$, it is natural to set $\mathbb{V}(\mathbb{C}[x_1, \dots, x_n]) := \emptyset$; see page 47. A straightforward way to define $\mathcal{M}_{\mathbf{A}}$ for $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$ would be to set it as

$$\mathbb{C}^{\sigma(\Theta[\mathbf{A}])} \times \bigtimes_{[\mathbf{a}] \in \emptyset} \mathcal{A}_{[\mathbf{a}]} \times \bigtimes_{[\mathbf{a}] \in \emptyset} \mathcal{B}_{[\mathbf{a}]} = \mathbb{C}^{\sigma(\Theta[\mathbf{A}])} \times \{\emptyset\} \times \{\emptyset\} \cong \mathbb{C}^{\sigma(\Theta[\mathbf{A}])}.$$

Definition 2.4.2. If $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$, we set $\mathcal{M}_{\mathbf{A}} := \mathbb{C}^{\sigma(\Theta[\mathbf{A}])}$, which constitutes a $*$ -algebra with pointwise operations in $\mathbb{C}$.

Until the end of the present section, we allow $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$. While the majority of the proofs in this section will cite results from Section 2.3, where $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$ was forbidden, those parts of the proofs can be simply ignored in the case of $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$. This arises from the fact that there are no “algebras corresponding to the variety” to be considered in the degenerate case of $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$ due to $\mathbb{V}(\mathbb{C}[x_1, \dots, x_n]) = \emptyset$.

For $\mathcal{I} \subsetneq \mathbb{C}[x_1, \dots, x_n]$ we will continue to employ Assumption 2.3.1, so that the results from Section 2.3 can be applied.

Definition 2.4.3. By $\mathfrak{S}_A$ we denote the set of functions $f : \text{dom } f \rightarrow \mathbb{C}$ such that

- (i) $\sigma(\Theta[A]) \cup V(\mathcal{I}) \subseteq \text{dom } f \subseteq \mathbb{C}^n$.
- (ii) $\text{dom } f$ is open and invariant under the componentwise complex conjugation.
- (iii) f is Borel measurable and bounded on $\sigma(\Theta[A]) \setminus V(\mathcal{I})$.
- (iv) for $\mathbf{a} \in V(\mathcal{I})$, there exist $p_{\mathbf{a}} \in \mathbb{C}[x_1, \dots, x_n]$ and a neighbourhood $U_{\mathbf{a}} \subseteq \mathbb{C}^n$ of $\mathbf{a}$ such that $f(\mathbf{z}) = p_{\mathbf{a}}(\mathbf{z})$ for $\mathbf{z} \in U_{\mathbf{a}}$.
- (v) for $\mathbf{a}, \mathbf{b} \in V(\mathcal{I})$ such that $[\mathbf{a}] = [\mathbf{b}]$, the polynomial $p_{\mathbf{a}} - p_{\mathbf{b}}$ belongs to $\mathcal{P}_{[\mathbf{a}]} \cdot \mathcal{Q}_{[\mathbf{a}]}$ if $[\mathbf{a}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$ and to $\mathcal{Q}_{[\mathbf{a}]}$ otherwise.

We can consider such f as an element $f_A \in \mathcal{M}_A$ by setting

- $f_A(\mathbf{z}) := f(\mathbf{z})$ for $\mathbf{z} \in \sigma(\Theta[A]) \setminus V(\mathcal{I})$,
- $f_A(\mathbf{z}) := [p_{\mathbf{z}}]_{\mathcal{P}_{[\mathbf{z}]} \cdot \mathcal{Q}_{[\mathbf{z}]}}$ for $\mathbf{z} \in V(\mathcal{I})$ such that $[\mathbf{z}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$,
- $f_A(\mathbf{z}) := [p_{\mathbf{z}}]_{\mathcal{Q}_{[\mathbf{z}]}}$ for $\mathbf{z} \in V(\mathcal{I})$ such that $[\mathbf{z}] \in \mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})$.

Remark 2.4.4. $\sigma(\Theta[A])$ being compact, every polynomial is bounded on $\sigma(\Theta[A]) \setminus V(\mathcal{I})$. In fact, every polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ is an element of $\mathfrak{S}_A$ since we can choose $p_{\mathbf{a}} := p$ for all $\mathbf{a} \in V(\mathcal{I})$.

Lemma 2.4.5. If for $f, g \in \mathfrak{S}_A$ we define the functions $f + g$, $f \cdot g$ with $\text{dom } f + g = \text{dom } f \cdot g = \text{dom } f \cap \text{dom } g$ by

- $(f + g)(\mathbf{z}) := f(\mathbf{z}) + g(\mathbf{z})$ for all $\mathbf{z} \in \text{dom } f + g$ and
- $(f \cdot g)(\mathbf{z}) := f(\mathbf{z}) \cdot g(\mathbf{z})$ for all $\mathbf{z} \in \text{dom } f \cdot g$,

then $f + g, f \cdot g \in \mathfrak{S}_A$.

Proof. As $\text{dom } f, \text{dom } g \supseteq \sigma(\Theta[A]) \cup V(\mathcal{I})$ are invariant under complex conjugation, so is $\text{dom } f + g \supseteq \sigma(\Theta[A]) \cup V(\mathcal{I})$. Measurability and boundedness on $\sigma(\Theta[A]) \setminus V(\mathcal{I})$ is inherited from f and g to $f + g$.

Given $\mathbf{w} \in V(\mathcal{I})$, let $U_f, U_g \subseteq \mathbb{C}^n$ be neighbourhoods of $\mathbf{w}$ and $p_f, p_g \in \mathbb{C}[x_1, \dots, x_n]$ such that $f|_{U_f} = p_f|_{U_f}$ and $g|_{U_g} = p_g|_{U_g}$. For $U_{f+g} := U_f \cap U_g$ we have $(f + g)|_{U_{f+g}} = (p_f + p_g)|_{U_{f+g}}$. Hence $f + g$ satisfies Definition 2.4.3, (iv). The property (v) of Definition 2.4.3 follows from the fact that f and g satisfy (v). Hence, $f + g \in \mathfrak{S}_A$. By swapping $+$ with $\cdot$ in the proof above, we obtain $f \cdot g \in \mathfrak{S}_A$. $\square$

Remark 2.4.6. The set $\mathfrak{S}_A$ endowed with addition from Lemma 2.4.5 and pointwise scalar multiplication does not form a vector space. Indeed, the neutral element with respect to addition is the 0 function on $\mathbb{C}^n$, but $f + (-1) \cdot f = 0|_{\text{dom } f}$, where $\text{dom } f \neq \mathbb{C}^n$ in general.

We will factorize $\mathfrak{S}_A$ by the equivalence relation that identifies two functions defining the same germ around $\sigma(\Theta[A]) \cup V(\mathcal{I})$ and obtain a vector space.

Definition 2.4.7. We define a relation $\asymp$ on $\mathfrak{S}_{\mathbf{A}}$ by writing $f \asymp g$ if and only if there exists an open neighbourhood U of $\sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I})$ contained in $\text{dom } f \cap \text{dom } g$ with $f|_U = g|_U$. We set $\mathcal{S}_{\mathbf{A}} := \mathfrak{S}_{\mathbf{A}}/\asymp$ and denote by $[f]_{\asymp}$ the equivalence class of $f \in \mathfrak{S}_{\mathbf{A}}$.

Lemma 2.4.8. $\mathcal{S}_{\mathbf{A}}$ constitutes a commutative unital $*$ -algebra with operations induced by the following $\mathfrak{S}_{\mathbf{A}}$ -operations:

- addition and multiplication from Lemma 2.4.5,
- pointwise scalar multiplication,
- involution $f^{\#}(z) := \overline{f(\bar{z})}$.

Proof. Let $f, g, h \in \mathfrak{S}_{\mathbf{A}}$ be such that $f \asymp g$, where U is the open neighbourhood of $\sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I})$ with $f|_U = g|_U$. Due to $(f + h)(z) = (g + h)(z)$ for $z \in U \cap \text{dom } h$, which is also an open neighbourhood of $\sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I})$, we conclude $(f + h) \asymp (g + h)$. Thus $[f]_{\asymp} + [h]_{\asymp} := [f + h]_{\asymp}$ is well-defined. As the addition in $\mathfrak{S}_{\mathbf{A}}$ is commutative, so is the induced addition. Analogously, the induced multiplication is well-defined and commutative. It is clear that for $\lambda \in \mathbb{C}$ the scalar multiplication $\lambda \cdot [f]_{\asymp} := [\lambda f]_{\asymp}$ is well-defined. Because of

$$[f]_{\asymp} + (-1) \cdot [f]_{\asymp} = [0|_{\text{dom } f}]_{\asymp} = [0|_{\mathbb{C}^n}]_{\asymp},$$

the operations on $\mathcal{S}_{\mathbf{A}}$ guarantee the existence of an additively inverse element. It is straightforward to check the validity of the rest of the vector space axioms. Hence, $\mathcal{S}_{\mathbf{A}}$ constitutes a vector space. From the distributivity of multiplication with respect to addition in $\mathbb{C}$, we infer the bilinearity of multiplication in $\mathcal{S}_{\mathbf{A}}$. Since $\sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I})$ is invariant under complex conjugation, $\bar{U}$ is also an open neighbourhood of $\sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I})$. Due to $f|_U = g|_U$, we have $f^{\#}|_{\bar{U}} = g^{\#}|_{\bar{U}}$, implying $f^{\#} \asymp g^{\#}$. We conclude that $[f]_{\asymp}^{\#} := [f^{\#}]_{\asymp}$ is well-defined. Simple calculations show that $^{\#}$ in $\mathcal{S}_{\mathbf{A}}$ is a conjugate linear involution. Due to

$$([f]_{\asymp} \cdot [h]_{\asymp})^{\#} = [(fh)^{\#}]_{\asymp} = [f^{\#}h^{\#}]_{\asymp} = [f]_{\asymp}^{\#} \cdot [h]_{\asymp}^{\#},$$

the set $\mathcal{S}_{\mathbf{A}}$ constitutes a $*$ -algebra. $[1_{\mathbb{C}^n}]_{\asymp}$ is the unity. $\square$

Lemma 2.4.9. Given $f, g \in \mathfrak{S}_{\mathbf{A}}$, $f \asymp g$ implies $f_{\mathbf{A}} = g_{\mathbf{A}}$.

Proof. Since $f|_U = g|_U$ for some neighbourhood U of $\sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I})$, we have $f_{\mathbf{A}}(z) = f(z) = g(z) = g_{\mathbf{A}}(z)$ for $z \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$.

For $z \in V(\mathcal{I})$ there exist a sufficiently small neighbourhood $U_z \subseteq U$ of z and polynomials p, q such that $p|_{U_z} = f|_{U_z} = g|_{U_z} = q|_{U_z}$. Thus, $f_{\mathbf{A}}(z) = [p]_X = g_{\mathbf{A}}(z)$, where $X = \mathcal{A}_{[z]}$ if $[z] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$ and $X = \mathcal{B}_{[z]}$ otherwise. $\square$

Proposition 2.4.10. $[f]_{\asymp} \mapsto f_{\mathbf{A}}$ constitutes a $*$ -homomorphism from $\mathcal{S}_{\mathbf{A}}$ to $\mathcal{M}_{\mathbf{A}}$.

Proof. The mapping $[f]_{\asymp} \mapsto f_{\mathbf{A}}$ is well-defined on $\mathcal{S}_{\mathbf{A}}$ due to Lemma 2.4.9. The compatibility of the mapping $_{\mathbf{A}} : \mathfrak{S}_{\mathbf{A}} \rightarrow \mathcal{M}_{\mathbf{A}}$ with addition, multiplication, and scalar multiplication obviously holds true.

For $f \in \mathfrak{S}_{\mathbf{A}}$ we are going to verify $(f^\#)_{\mathbf{A}} = (f_{\mathbf{A}})^\#$. By definition, for $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}) \subseteq \mathbb{R}^n$ we have

$$(f^\#)_{\mathbf{A}}(\mathbf{z}) = f^\#(\mathbf{z}) = \overline{f(\overline{\mathbf{z}})} = \overline{f(\mathbf{z})} = \overline{f_{\mathbf{A}}(\mathbf{z})} = (f_{\mathbf{A}})^\#(\mathbf{z}).$$

Given $\mathbf{z} \in V(\mathcal{I})$ let $p_{\mathbf{z}} \in \mathbb{C}[x_1, \dots, x_n]$ and $U_{\mathbf{z}}$ be a sufficiently small neighbourhood of $\mathbf{z}$ such that $f|_{U_{\mathbf{z}}} = p_{\mathbf{z}}|_{U_{\mathbf{z}}}$. Since $\text{dom } f$ is invariant under complex conjugation, we have $f^\#|_{\overline{U_{\mathbf{z}}}} = p_{\mathbf{z}}^\#|_{\overline{U_{\mathbf{z}}}}$. If $[\mathbf{z}] \notin \mathbb{V}_{\mathbb{R}}(\mathcal{I})$, then

$$(f^\#)_{\mathbf{A}}(\overline{\mathbf{z}}) = [p_{\mathbf{z}}^\#]_{\mathcal{B}(\overline{\mathbf{z}})} = [p_{\mathbf{z}}]_{\mathcal{B}(\mathbf{z})}^\star = f_{\mathbf{A}}(\mathbf{z})^\star = (f_{\mathbf{A}})^\#(\overline{\mathbf{z}}).$$

Since $\mathbb{V}(\mathcal{I}) \setminus \mathbb{V}_{\mathbb{R}}(\mathcal{I})$ is invariant under the conjugation, we obtain $(f^\#)_{\mathbf{A}}(\mathbf{z}) = (f_{\mathbf{A}})^\#(\mathbf{z})$; see Remark 2.3.10. In case $[\mathbf{z}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$, we obtain

$$(f^\#)_{\mathbf{A}}(\mathbf{z}) = [p_{\mathbf{z}}^\#]_{\mathcal{A}[\mathbf{z}]} = [p_{\mathbf{z}}]_{\mathcal{A}[\mathbf{z}]}^\star = f_{\mathbf{A}}(\mathbf{z})^\star = (f_{\mathbf{A}})^\#(\mathbf{z}).$$

□

Notation 2.4.11. For functions $f, g : D (\subseteq \mathbb{R}^n) \rightarrow \mathbb{C}$ such that $\text{ran } g \subseteq (0, +\infty)$ and an accumulation point $\mathbf{w} \in \mathbb{R}^n$ of D , we write

$$f(\mathbf{z}) = O(g(\mathbf{z})) \text{ as } \mathbf{z} \rightarrow \mathbf{w} \text{ if } \limsup_{\mathbf{z} \rightarrow \mathbf{w}} \frac{|f(\mathbf{z})|}{g(\mathbf{z})} < +\infty.$$

Proposition 2.4.12. If $\mathbf{a} \in V(\mathcal{I}) \cap \sigma(\Theta[\mathbf{A}])$ is an accumulation point of $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$ and $h \in \mathcal{Q}_{[\mathbf{a}]}$, then we have

$$h(\mathbf{z}) = O\left(\left|\sum_{j=1}^m p_j(\mathbf{z})\right|\right) \text{ as } \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}) \ni \mathbf{z} \rightarrow \mathbf{a}.$$

Proof. Since $\mathcal{Q}_{[\mathbf{b}]}$, $[\mathbf{b}] \in \mathbb{V}(\mathcal{I})$, stem from a minimal primary decomposition of $\mathcal{I}$ with pairwise disjoint varieties, every primary component $\mathcal{Q}_{[\mathbf{b}]}$, $[\mathbf{b}] \neq [\mathbf{a}]$, contains a polynomial $s_{[\mathbf{b}]}$ that does not vanish at $\mathbf{a}$. We set

$$g := h \cdot \prod_{\substack{[\mathbf{b}] \in \mathbb{V}(\mathcal{I}) \\ [\mathbf{a}] \neq [\mathbf{b}]}} s_{[\mathbf{b}]} \in \mathcal{I}$$

and choose $u_1, \dots, u_m \in \mathbb{C}[x_1, \dots, x_n]$ such that $g = \sum_{j=1}^m u_j p_j$. Because of

$$\limsup_{\mathbf{z} \rightarrow \mathbf{a}} \frac{\left|\sum_{j=1}^m u_j(\mathbf{z}) p_j(\mathbf{z})\right|}{\max_{j \in [1, m]_{\mathbb{Z}}} |p_j(\mathbf{z})|} \leq \limsup_{\mathbf{z} \rightarrow \mathbf{a}} \frac{\max_{j \in [1, m]_{\mathbb{Z}}} |p_j(\mathbf{z})| \sum_{j=1}^m |u_j(\mathbf{z})|}{\max_{j \in [1, m]_{\mathbb{Z}}} |p_j(\mathbf{z})|} = \sum_{j=1}^m |u_j(\mathbf{a})| < +\infty$$

we have $g(\mathbf{z}) = O(\max_{j \in [1, m]_{\mathbb{Z}}} |p_j(\mathbf{z})|)$ as $\mathbf{z} \rightarrow \mathbf{a}$. From $s_{[\mathbf{b}]}(\mathbf{a}) \neq 0$, $[\mathbf{b}] \in \mathbb{V}(\mathcal{I})$, $[\mathbf{a}] \neq [\mathbf{b}]$, we infer $h(\mathbf{z}) = O(\max_{j \in [1, m]_{\mathbb{Z}}} |p_j(\mathbf{z})|)$ as $\mathbf{z} \rightarrow \mathbf{a}$, which by Lemma 2.1.11 implies $h(\mathbf{z}) = O(|\sum_{j=1}^m p_j(\mathbf{z})|)$ as $\mathbf{z} \rightarrow \mathbf{a}$. □

Remark 2.4.13. Let $\mathbf{w} \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$ be an accumulation point of $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$. First note that for $\phi \in \mathcal{M}_{\mathbf{A}}$, $\phi(\mathbf{w}) = [r]_{\mathcal{P}_{[\mathbf{w}]} \cdot \mathcal{Q}_{[\mathbf{w}]}} = r + (\mathcal{P}_{[\mathbf{w}]} \cdot \mathcal{Q}_{[\mathbf{w}]})$ for some $r \in \mathbb{C}[x_1, \dots, x_n]$. Hence $\phi(\mathbf{w})$ can be interpreted as an affine subspace of $\mathbb{C}[x_1, \dots, x_n]$. Because the difference $p - q$ of $p, q \in \phi(\mathbf{w})$ belongs to $\mathcal{P}_{[\mathbf{w}]} \cdot \mathcal{Q}_{[\mathbf{w}]} \subseteq \mathcal{Q}_{[\mathbf{w}]}$, Proposition 2.4.12 yields $p(\mathbf{z}) - q(\mathbf{z}) = O(|\sum_{j=1}^m p_j(\mathbf{z})|)$ as $\mathbf{z} \rightarrow \mathbf{w}$. Therefore, the following statements are equivalent:

- (i) $\forall p \in \phi(\mathbf{w}) : \phi(\mathbf{z}) - p(\mathbf{z}) = O(|\sum_{j=1}^m p_j(\mathbf{z})|)$ as $\mathbf{z} \rightarrow \mathbf{w}$,
- (ii) $\exists p \in \phi(\mathbf{w}) : \phi(\mathbf{z}) - p(\mathbf{z}) = O(|\sum_{j=1}^m p_j(\mathbf{z})|)$ as $\mathbf{z} \rightarrow \mathbf{w}$.

Definition 2.4.14. By $\mathcal{F}_{\mathbf{A}}$ we denote the set of functions $\phi \in \mathcal{M}_{\mathbf{A}}$ such that

- (i) $\phi|_{\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})}$ is Borel measurable and bounded.
- (ii) for any accumulation point $\mathbf{w} \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$ of $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$ there exists a polynomial $p \in \phi(\mathbf{w})$ satisfying

$$\phi(\mathbf{z}) - p(\mathbf{z}) = O\left(\left|\sum_{j=1}^m p_j(\mathbf{z})\right|\right) \text{ as } \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}) \ni \mathbf{z} \rightarrow \mathbf{w}. \quad (2.19)$$

Remark 2.4.15. In case $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$, (ii) of Definition 2.4.14 is redundant and, hence, $\mathcal{F}_{\mathbf{A}} = \mathfrak{B}(\sigma(\Theta[\mathbf{A}]));$ see Definition 1.7.1.

Proposition 2.4.16. For $f \in \mathfrak{S}_{\mathbf{A}}$ the function $f_{\mathbf{A}}$ belongs to $\mathcal{F}_{\mathbf{A}}$.

Proof. Measurability and boundedness of $f_{\mathbf{A}}$ on $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$ is guaranteed by Definition 2.4.3. If $\mathbf{w} \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$ is an accumulation point of $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$, then there exists a neighbourhood U of $\mathbf{w}$ and a polynomial p such that $f|_U = p|_U$ and $f_{\mathbf{A}}(\mathbf{w}) = [p]_{\mathcal{P}_{[\mathbf{w}]}, \mathcal{Q}_{[\mathbf{w}]}}$. For $\mathbf{z} \in U \cap (\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}))$ we have

$$f_{\mathbf{A}}(\mathbf{z}) - p(\mathbf{z}) = 0 = O\left(\left|\sum_{j=1}^m p_j(\mathbf{z})\right|\right) \text{ as } \mathbf{z} \rightarrow \mathbf{w}.$$

□

In Chapter 3, the joint functional calculus will be defined for functions belonging to $\mathcal{F}_{\mathbf{A}}$. Due to Proposition 2.4.16 every polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ can be interpreted as an element $p_{\mathbf{A}} \in \mathcal{F}_{\mathbf{A}}$. This is important as we want to test the meaningfulness of our joint functional calculus by checking if “ $p_{\mathbf{A}}(\mathbf{A})$ ” matches the already well-defined concept $p(\mathbf{A})$. As a functional calculus shall be a $*$ -homomorphism, we will first make sure that $\mathcal{F}_{\mathbf{A}}$ constitutes a $*$ -algebra.

Lemma 2.4.17. $\mathcal{F}_{\mathbf{A}}$ constitutes a unital sub $*$ -algebra of $\mathcal{M}_{\mathbf{A}}$.

Proof. It is straightforward to check that $\mathcal{F}_{\mathbf{A}}$ is a subspace. If $\phi_1, \phi_2 \in \mathcal{F}_{\mathbf{A}}$, then their product is clearly bounded and measurable on $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$. Given an accumulation point $\mathbf{w} \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$ of $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$, $p_1 \in \phi_1(\mathbf{w})$, and $p_2 \in \phi_2(\mathbf{w})$, we have $p_1 p_2 \in \phi_1(\mathbf{w}) \phi_2(\mathbf{w}) = (\phi_1 \phi_2)(\mathbf{w})$. As $\phi_1, \phi_2 \in \mathcal{F}_{\mathbf{A}}$ and $\sigma(\Theta[\mathbf{A}])$ is compact, we obtain

$$\begin{aligned} \limsup_{\mathbf{z} \rightarrow \mathbf{w}} \frac{|(\phi_1 \phi_2)(\mathbf{z}) - (p_1 p_2)(\mathbf{z})|}{\left|\sum_{j=1}^m p_j(\mathbf{z})\right|} &= \limsup_{\mathbf{z} \rightarrow \mathbf{w}} \frac{|\phi_1(\mathbf{z})| |\phi_2(\mathbf{z}) - p_2(\mathbf{z})| + |p_2(\mathbf{z})| |\phi_1(\mathbf{z}) - p_1(\mathbf{z})|}{\left|\sum_{j=1}^m p_j(\mathbf{z})\right|} \\ &\leq \limsup_{\mathbf{z} \rightarrow \mathbf{w}} |\phi_1(\mathbf{z})| \cdot \limsup_{\mathbf{z} \rightarrow \mathbf{w}} \frac{|\phi_2(\mathbf{z}) - p_2(\mathbf{z})|}{\left|\sum_{j=1}^m p_j(\mathbf{z})\right|} \\ &\quad + \limsup_{\mathbf{z} \rightarrow \mathbf{w}} |p_2(\mathbf{z})| \cdot \limsup_{\mathbf{z} \rightarrow \mathbf{w}} \frac{|\phi_1(\mathbf{z}) - p_1(\mathbf{z})|}{\left|\sum_{j=1}^m p_j(\mathbf{z})\right|} < +\infty. \end{aligned}$$

Hence, $\mathcal{F}_{\mathbf{A}}$ is closed under multiplication and therefore constitutes a subalgebra of $\mathcal{M}_{\mathbf{A}}$.

Given $\phi \in \mathcal{F}_A$, also $\phi^\#$ is bounded and measurable on $\sigma(\Theta[A]) \setminus V(\mathcal{I})$. For an accumulation point $\mathbf{w} \in \sigma(\Theta[A]) \cap V(\mathcal{I})$ of $\sigma(\Theta[A]) \setminus V(\mathcal{I})$ we have $\mathbf{w} \in \mathbb{R}^n$ and, in consequence, $[\mathbf{w}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$. For $p \in \phi(\mathbf{w})$ we have $\phi^\#(\mathbf{w}) = \phi(\mathbf{w})^* = [p]_{\mathcal{P}_{[\mathbf{w}]} \cdot \mathcal{Q}_{[\mathbf{w}]}}^* = [p^\#]_{\mathcal{P}_{[\mathbf{w}]} \cdot \mathcal{Q}_{[\mathbf{w}]}}$ showing $p^\# \in \phi^\#(\mathbf{w})$; see Definition 2.4.1.

$$|\phi^\#(z) - p^\#(z)| = |\overline{\phi(z)} - \overline{p(z)}| = |\phi(z) - p(z)|, \quad z \in \sigma(\Theta[A]) \setminus V(\mathcal{I}),$$

yields $\phi^\#(z) - p^\#(z) = O(|\sum_{j=1}^m p_j(z)|)$ as $z \rightarrow \mathbf{w}$. Therefore, $\mathcal{F}_A$ is closed under $^\#$. Finally, for the constant one-function $\mathbb{1}$, $\mathbb{1}_A$ belongs to $\mathcal{F}_A$ due to Proposition 2.4.16 and constitutes the unity. $\square$

Lemma 2.4.18. *Let $\phi \in \mathcal{F}_A$ and $\mathbf{w} \in V(\mathcal{I})$. If $\phi(\mathbf{w})$ is invertible in $\mathcal{A}_{[\mathbf{w}]}$, then $p(\mathbf{w}) \neq 0$ for all $p \in \phi(\mathbf{w})$.*

Proof. For $p \in \phi(\mathbf{w})$ and $q \in \phi(\mathbf{w})^{-1}$, we have $1 = pq + r$ with some $r \in \mathcal{P}_{[\mathbf{w}]} \cdot \mathcal{Q}_{[\mathbf{w}]}$. $r(\mathbf{w}) = 0$ yields $p(\mathbf{w})q(\mathbf{w}) = 1$. $\square$

Lemma 2.4.19. *Let $\mathbf{w} \in \sigma(\Theta[A]) \cap V(\mathcal{I})$ be an accumulation point of $\sigma(\Theta[A]) \setminus V(\mathcal{I})$. If $\phi \in \mathcal{F}_A$ is such that*

- (i) $\phi(\mathbf{w})$ is invertible in $\mathcal{A}_{[\mathbf{w}]}$ and
- (ii) *there exists a sufficiently small neighbourhood U of $\mathbf{w}$ in $\sigma(\Theta[A])$ such that 0 is not contained in the closure of $\phi(U \setminus V(\mathcal{I})) \subseteq \mathbb{C}$,*

then

$$\frac{1}{\phi(z)} - p(z) = O\left(\left|\sum_{j=1}^m p_j(z)\right|\right) \text{ as } z \rightarrow \mathbf{w}, \quad p \in \phi(\mathbf{w})^{-1}.$$

Proof. For $p \in \phi(\mathbf{w})^{-1}$, $q \in \phi(\mathbf{w})$, and $z \in U \setminus V(\mathcal{I})$, we have

$$\begin{aligned} \frac{1}{\phi(z)} - p(z) &= \left(\frac{1}{\phi(z)} - \frac{1}{q(z)}\right) + \left(\frac{1}{q(z)} - p(z)\right) \\ &= -\underbrace{\left(\frac{\phi(z) - q(z)}{\phi(z) \cdot q(z)}\right)}_{=: \alpha(z)} - \underbrace{\left(\frac{q(z) \cdot p(z) - 1}{q(z)}\right)}_{=: \beta(z)}. \end{aligned}$$

As $\phi(\mathbf{w})$ is invertible, Lemma 2.4.18 yields $q(\mathbf{w}) \neq 0$. Since 0 is not an accumulation point of $\phi(U \setminus V(\mathcal{I}))$, we have $|\phi(z) \cdot q(z)| > \varepsilon$ for all $z \in W \setminus V(\mathcal{I})$ for some $\varepsilon > 0$, where W is a sufficiently small neighbourhood of $\mathbf{w}$. We conclude $\alpha(z) = O(\phi(z) - q(z))$ as $z \rightarrow \mathbf{w}$. From $\phi(z) - q(z) = O(|\sum_{j=1}^m p_j(z)|)$ as $z \rightarrow \mathbf{w}$, we conclude $\alpha(z) = O(|\sum_{j=1}^m p_j(z)|)$ as $z \rightarrow \mathbf{w}$.

Because of $\phi(\mathbf{w}) \cdot \phi(\mathbf{w})^{-1} = [1]_{\mathcal{P}_{[\mathbf{w}]} \cdot \mathcal{Q}_{[\mathbf{w}]}}$, the numerator of $\beta(z)$ belongs to $\mathcal{P}_{[\mathbf{w}]} \cdot \mathcal{Q}_{[\mathbf{w}]} \subseteq \mathcal{Q}_{[\mathbf{w}]}$. As $q(\mathbf{w}) \neq 0$, Proposition 2.4.12 yields $\beta(z) = O(|\sum_{j=1}^m p_j(z)|)$ as $z \rightarrow \mathbf{w}$. $\square$

Clearly, a tuple of functions $(\phi_j)_{j=1}^n \in \mathcal{F}_A^n$ is invertible in the sense of Definition 1.3.20 if there exists $j_0 \in [1, n]_{\mathbb{Z}}$ such that ϕ_{j_0} is invertible in $\mathcal{F}_A$. In the following lemma, we will give a sufficient condition for invertibility of members in $\mathcal{F}_A^n$ that covers more than the trivial case. Specifically, the invertibility of the tuple can be obtained if there exists a measurable partition $(P_i)_{i \in I}$ of the domain $\sigma(\Theta[A]) \cup V(\mathcal{I})$ such that for every $i \in I$ there exists some ϕ_{j_i} , $j_i \in [1, n]_{\mathbb{Z}}$, that is pointwise invertible on P_i . In a sense, if the tuple is locally invertible everywhere, it is invertible in $\mathcal{F}_A^n$.

Lemma 2.4.20. *Let $\phi = (\phi_j)_{j=1}^n \in \mathcal{F}_A^n$ be such that there exist an arbitrary set I , a partition $(P_i)_{i \in I}$ of $\sigma(\Theta[A]) \cup V(\mathcal{I})$, and $(j_i)_{i \in I} \in ([1, n]_{\mathbb{Z}})^I$ such that*

- (i) P_i is measurable for all $i \in I$.
- (ii) $\mathbf{w} \in P_i$ implies $V(\mathcal{Q}_{[\mathbf{w}]}) \subseteq P_i$ for every $\mathbf{w} \in V(\mathcal{I})$.
- (iii) 0 is not contained in the closure of $\phi_{j_i}(P_i \setminus V(\mathcal{I})) \subseteq \mathbb{C}$ for all $i \in I$.
- (iv) $\phi_{j_i}(\mathbf{z})$ is invertible in $\mathcal{A}_{[\mathbf{z}]}$ or $\mathcal{B}_{[\mathbf{z}]}$, accordingly, for $\mathbf{z} \in V(\mathcal{I}) \cap P_i$.
- (v) for every accumulation point $\mathbf{w} \in \sigma(\Theta[A]) \cap V(\mathcal{I})$ of $\sigma(\Theta[A]) \setminus V(\mathcal{I})$, $\mathbf{w} \in P_i$ implies that $\mathbf{w}$ is an interior point of $P_i \cap \sigma(\Theta[A])$.

Then the tuple ϕ is invertible in $\mathcal{F}_A^n$ in the sense of Definition 1.3.20.

Proof. For $j \in [1, n]_{\mathbb{Z}}$ we define

$$\beta_j(\mathbf{z}) := \begin{cases} \phi_j(\mathbf{z})^{-1}, & \exists i \in I : \mathbf{z} \in P_i \text{ and } j_i = j, \\ 0, & \text{else,} \end{cases}$$

where 0 is in $\mathcal{A}_{[\mathbf{z}]}$, $\mathcal{B}_{[\mathbf{z}]}$, or $\mathbb{C}$, respectively. (ii) implies $\beta_j \in \mathcal{M}_A$. We will verify $\beta_j \in \mathcal{F}_A$. First note that

$$\beta_j^{-1}(\{0_{\mathbb{C}}\}) = \sigma(\Theta[A]) \setminus V(\mathcal{I}) \cap \bigcup_{\substack{i \in I \\ j_i \neq j}} P_i \quad (2.20)$$

is measurable. We have $\beta_j(\mathbf{z}) = \frac{1}{\phi_j(\mathbf{z})}$ for $\mathbf{z} \in (\sigma(\Theta[A]) \setminus V(\mathcal{I})) \setminus \beta_j^{-1}(\{0_{\mathbb{C}}\})$, which is measurable. Hence β_j is measurable on $\sigma(\Theta[A]) \setminus V(\mathcal{I})$ and bounded on $\sigma(\Theta[A]) \setminus V(\mathcal{I})$ because of (iii).

Let $\mathbf{w} \in \sigma(\Theta[A]) \cap V(\mathcal{I})$ be an accumulation point of $\sigma(\Theta[A]) \setminus V(\mathcal{I})$. If $\mathbf{w} \in P_i$ for some i such that $j_i \neq j$, then because of (v) we have $\beta_j(\mathbf{z}) = 0$ for all $\mathbf{z}$ in a sufficiently small neighbourhood of $\mathbf{w}$. If $\mathbf{w} \in P_i$ for some i such that $j_i = j$, then we apply Lemma 2.4.19 to $\mathbf{w}$, $\phi = \phi_j$, and U being the neighbourhood of $\mathbf{w}$ that is contained in P_i , which exists due to (v). In any case, (2.19) is satisfied.

We denote by $\mathbb{1}_B \in \mathcal{M}_A$ for $B \subseteq \sigma(\Theta[A]) \cup V(\mathcal{I})$ the function such that $\mathbb{1}_B(\mathbf{z})$ is the unity in $\mathbb{C}$, $\mathcal{A}_{[\mathbf{z}]}$, or $\mathcal{B}_{[\mathbf{z}]}$, respectively, if $\mathbf{z} \in B$, and 0 otherwise. Multiplying ϕ and $\beta := (\beta_j)_{j=1}^n$ results in

$$\phi \cdot \beta = \sum_{j=1}^n \phi_j \cdot \beta_j = \sum_{i \in I} \mathbb{1}_{P_i} = \mathbb{1}_{\sigma(\Theta[A]) \cup V(\mathcal{I})}.$$

Since $\mathbb{1}_{\sigma(\Theta[A]) \cup V(\mathcal{I})}$ is the unity in $\mathcal{F}_A$, ϕ is invertible in $\mathcal{F}_A^n$. $\square$

Lemma 2.4.21. *If $\phi \in \mathcal{F}_A$ is such that $\phi(\mathbf{z})$ is invertible in $\mathcal{A}_{[\mathbf{z}]}$ and $\mathcal{B}_{[\mathbf{z}]}$, respectively, for all $\mathbf{z} \in V(\mathcal{I})$ and $0 \in \mathbb{C}$ is not an accumulation point of $\phi(\sigma(\Theta[A]) \setminus V(\mathcal{I}))$, then ϕ is invertible in $\mathcal{F}_A$, where $\phi^{-1}(\mathbf{z}) := \phi(\mathbf{z})^{-1}$ constitutes the inverse element.*

Proof. Viewing ϕ as a tuple with one element, it satisfies the requirements in Lemma 2.4.20 with $j_i = 1$ for all $i \in I$. Since the invertibility as in Definition 1.3.20 agrees with the usual invertibility for $\mathcal{F}_A^1 = \mathcal{F}_A$, ϕ is invertible. Therefore, the function ϕ^{-1} is an element of $\mathcal{F}_A$. $\square$

Chapter 3

Joint Functional Calculus

Having established a suitable framework for the functional calculus in the previous chapter, we now proceed to define it. The joint functional calculus defined in the present thesis is a proper generalization of the joint functional calculus in the Hilbert space setting. In fact, as we will see, it coincides with the Hilbert space joint functional calculus when $(\mathcal{K}, [.,.])$ is a Hilbert space.

3.1 Assembling the Joint Functional Calculus

We will continue to work with the notations and assumptions from Section 2.4. In particular, $\mathcal{I}$ denotes the ideal generated by all definitizing polynomials of $\mathbf{A}$, and $p_1, \dots, p_m \in \mathbb{R}[x_1, \dots, x_n]$ are definitizing polynomials of $\mathbf{A}$ such that $\mathcal{I} = \langle p_1, \dots, p_m \rangle$ according to Proposition 1.6.9. Furthermore, $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$ will be allowed and if $\mathcal{I} \subsetneq \mathbb{C}[x_1, \dots, x_n]$, $\mathcal{I}$ shall satisfy Assumption 2.3.1.

Definition 3.1.1. Given $\phi \in \mathcal{F}_{\mathbf{A}}$ a pair $(r, f) \in \mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$ is called a *decomposition* of ϕ if it satisfies

- (i) $\phi(\mathbf{z}) = r_{\mathbf{A}}(\mathbf{z})$ for $\mathbf{z} \in V(\mathcal{I})$,
- (ii) $\phi(\mathbf{z}) = r(\mathbf{z}) + f(\mathbf{z}) \sum_{j=1}^m p_j(\mathbf{z})$ for all $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$,
- (iii) $f(\mathbf{z}) = 0$ for $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$.

Moreover, we define $\Phi \subseteq \mathcal{F}_{\mathbf{A}} \times \mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$ as

$$\Phi := \{(\phi; (r, f)) : (r, f) \text{ is decomposition of } \phi\}.$$

Lemma 3.1.2. *The set Φ is a linear relation between $\mathcal{F}_{\mathbf{A}}$ and $\mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[\mathbf{A}]))$.*

Proof. Let $\phi_1, \phi_2 \in \mathcal{F}_{\mathbf{A}}$ admit decompositions $(r, f), (s, g)$, respectively, and $\lambda \in \mathbb{C}$. $f + \lambda g$ clearly vanishes on $\sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$. Moreover, $(r + \lambda s)_{\mathbf{A}}(\mathbf{z}) = r_{\mathbf{A}}(\mathbf{z}) + \lambda s_{\mathbf{A}}(\mathbf{z}) = \phi_1(\mathbf{z}) + \lambda \phi_2(\mathbf{z})$ for $\mathbf{z} \in V(\mathcal{I})$. Lastly,

$$\begin{aligned} \phi_1(\mathbf{z}) + \lambda \phi_2(\mathbf{z}) &= r(\mathbf{z}) + f(\mathbf{z}) \sum_{j=1}^m p_j(\mathbf{z}) + \lambda s(\mathbf{z}) + \lambda g(\mathbf{z}) \sum_{j=1}^m p_j(\mathbf{z}) \\ &= (r + \lambda s)(\mathbf{z}) + (f + \lambda g)(\mathbf{z}) \sum_{j=1}^m p_j(\mathbf{z}), \quad \mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}), \end{aligned}$$

identifies $(r + \lambda s, f + \lambda g)$ as a decomposition of $\phi_1 + \lambda \phi_2$. $\square$

Proposition 3.1.3. *Every function in $\mathcal{F}_A$ admits a decomposition. In particular, we have $\text{dom } \Phi = \mathcal{F}_A$.*

Proof. By definition, for $\phi \in \mathcal{F}_A$ we have

$$\phi|_{V(I)} := (\phi(z))_{[z] \in V(I)} \in \prod_{[a] \in V_{\mathbb{R}}(I)} \mathcal{A}_{[a]} \times \prod_{[a] \in V(I) \setminus V_{\mathbb{R}}(I)} \mathcal{B}_{[a]}.$$

According to Proposition 2.3.5

$$\psi : \begin{cases} \mathbb{C}[x_1, \dots, x_n]/\mathcal{J} & \rightarrow \prod_{[a] \in V_{\mathbb{R}}(I)} \mathcal{A}_{[a]} \times \prod_{[a] \in V(I) \setminus V_{\mathbb{R}}(I)} \mathcal{B}_{[a]}, \\ [p]/\mathcal{J} & \mapsto \left(([p]_{\mathcal{P}_{[a]} \cdot \mathcal{Q}_{[a]}})_{[a] \in V_{\mathbb{R}}(I)}, ([p]_{\mathcal{Q}_{[a]}})_{[a] \in V(I) \setminus V_{\mathbb{R}}(I)} \right), \end{cases}$$

constitutes an isomorphism. Hence $r \in \psi^{-1}(\phi|_{V(I)})$ is a polynomial satisfying

$$r_A(z) = \begin{cases} [r]_{\mathcal{P}_{[z]} \cdot \mathcal{Q}_{[z]}}, & [z] \in V_{\mathbb{R}}(I) \\ [r]_{\mathcal{Q}_{[z]}}, & [z] \notin V_{\mathbb{R}}(I) \end{cases} = \phi(z), \quad z \in V(I).$$

We define $f : \sigma(\Theta[A]) \rightarrow \mathbb{C}$ by

$$f(z) := \frac{\phi(z) - r(z)}{\sum_{j=1}^m p_j(z)}, \quad z \in \sigma(\Theta[A]) \setminus V(I),$$

and $f(z) := 0$ for $z \in \sigma(\Theta[A]) \cap V(I)$. By Corollary 2.1.12 the zeros of $\sum_{j=1}^m p_j$ in $\sigma(\Theta[A])$ are indeed contained in $\sigma(\Theta[A]) \cap V(I)$. Hence f is well-defined and satisfies

$$\phi(z) = r(z) + f(z) \sum_{j=1}^m p_j(z), \quad z \in \sigma(\Theta[A]) \setminus V(I).$$

It remains to show that f is measurable and bounded. Note that $V(I)$ is measurable as it is a finite intersection of the preimage of $\{0\}$ under polynomials, which are measurable functions. Thus the measurability of f is a consequence of the measurability of ϕ on $\sigma(\Theta[A]) \setminus V(I)$.

As $\phi - r$ is bounded on $\sigma(\Theta[A]) \setminus V(I)$, it suffices to check the boundedness of f in the neighbourhood of every $w \in \sigma(\Theta[A]) \cap V(I)$ which is an accumulation point of $\sigma(\Theta[A]) \setminus V(I)$. Since $r \in \phi(w)$, we have $\phi(z) - r(z) = O(|\sum_{j=1}^m p_j(z)|)$ as $z \rightarrow w$; see Remark 2.4.13. Thus,

$$f(z) = \frac{\phi(z) - r(z)}{\sum_{j=1}^m p_j(z)} = O(1) \quad \text{as } z \rightarrow w.$$

$\square$

For the following recall that $\mathbb{C}[x_1, \dots, x_n] \times \mathfrak{B}(\sigma(\Theta[A]))$ carries a multiplication and an involution turning this space into a $*$ -algebra according to Proposition 2.2.3.

Lemma 3.1.4. *Given $(\phi_1; (r, f)), (\phi_2; (s, g)) \in \Phi$ we have $(\phi_1 \phi_2; (r, f) \cdot (s, g)) \in \Phi$.*

Proof. According to Proposition 2.2.3, $(r, f) \cdot (s, g) = (rs, rg + sf + fg \sum_{j=1}^m p_j)$. We have $(rs)_A(z) = r_A(z)s_A(z) = \phi_1(z)\phi_2(z)$ for $z \in V(\mathcal{I})$ due to Proposition 2.4.10. For $z \in \sigma(\Theta[A]) \setminus V(\mathcal{I})$ we calculate

$$\begin{aligned} \phi_1(z)\phi_2(z) &= \left(r(z) + f(z) \sum_{j=1}^m p_j(z) \right) \left(s(z) + g(z) \sum_{j=1}^m p_j(z) \right) \\ &= r(z)s(z) + \left(r(z)g(z) + f(z)s(z) + f(z)g(z) \sum_{j=1}^m p_j(z) \right) \sum_{j=1}^m p_j(z). \end{aligned}$$

Finally,

$$r(z) \underbrace{g(z)}_{=0} + \underbrace{f(z)}_{=0} s(z) + \underbrace{f(z)g(z)}_{=0} \sum_{j=1}^m p_j(z) = 0, \quad z \in \sigma(\Theta[A]) \cap V(\mathcal{I}).$$

□

Lemma 3.1.5. $(\phi; (r, f)) \in \Phi$ implies $(\phi^\#; (r, f)^*) \in \Phi$.

Proof. Note that $(r, f)^* = (r^\#, \bar{f})$ and $(r^\#)_A = (r_A)^\#$; see Proposition 2.4.10. Thus, $(r^\#)_A(z) = (r_A)^\#(z) = \phi^\#(z)$ for $z \in V(\mathcal{I})$. Given $z \in \sigma(\Theta[A]) \setminus V(\mathcal{I}) \subseteq \mathbb{R}^n$, the fact $p_j^\# = p_j$, $j \in [1, m]_{\mathbb{Z}}$, yields

$$\phi^\#(z) = r(z) + f(z) \sum_{j=1}^m p_j(z) = r^\#(z) + \bar{f}(z) \sum_{j=1}^m p_j(z).$$

Because $\bar{f}$ clearly vanishes on $\sigma(\Theta[A]) \cap V(\mathcal{I})$, we have $(\phi^\#; (r, f)^*) \in \Phi$. □

Lemma 3.1.6. $\text{mul } \Phi$ is contained in $\mathcal{N}$.

Proof. If (r, f) is a decomposition of the zero function, then (i) and (ii) from Definition 2.2.2 clearly hold true. As $r_A(z) = 0_A(z)$ for all $z \in V(\mathcal{I})$, Proposition 2.3.5 yields $r \in \mathcal{J}$ and the existence of $u_1, \dots, u_m \in \mathbb{C}[x_1, \dots, x_n]$ that vanish on $\sigma(\Theta[A]) \cap V(\mathcal{I})$ satisfying $r = \sum_{j=1}^m u_j p_j$. Consequently, $(r, f) \in \mathcal{N}$. □

Theorem 3.1.7. If Ψ is the $*$ -homomorphism introduced in Definition 2.2.2, then the relational product $\Psi\Phi$ constitutes a $*$ -homomorphism from $\mathcal{F}_A$ to A'' .

Proof. $\text{mul } \Phi \subseteq \mathcal{N} \subseteq \ker \Psi$ implies $\text{mul } \Psi\Phi = \{0\}$. Since $\text{dom } \Phi = \mathcal{F}_A$, $\Psi\Phi$ is a linear operator from $\mathcal{F}_A$ to A'' . Lemma 3.1.4, Lemma 3.1.5, and the fact that Ψ is a $*$ -homomorphism implies that $\Psi\Phi$ is a $*$ -homomorphism. □

Definition 3.1.8. For $\phi \in \mathcal{F}_A$ we set $\phi(A) := \Psi\Phi(\phi)$ and refer to the $*$ -homomorphism

$$\phi \mapsto \phi(A)$$

as the *joint functional calculus* of A .

Proposition 3.1.9. *Given $p \in \mathbb{C}[x_1, \dots, x_n]$ we have*

$$p(\mathbf{A}) = p_{\mathbf{A}}(\mathbf{A}).$$

In particular, $\pi_j(\mathbf{A}) = A_j$ for the polynomial $\pi_j(\mathbf{x}) = x_j$.

Proof. Since $(p, 0)$ is a decomposition of $p_{\mathbf{A}}$, we obtain $p_{\mathbf{A}}(\mathbf{A}) = \Psi\Phi(p_{\mathbf{A}}) = \Psi(p, 0) = p(\mathbf{A})$. $\square$

Remark 3.1.10. We want to summarize the requirements we have posed on $\mathbf{A}$ throughout the present thesis. The tuple $\mathbf{A} \in L_b(\mathcal{K})^n$ needs to satisfy the following statements so that Definition 3.1.8 is possible.

- (i) $\mathbf{A}$ is definitizable; see Definition 1.6.8. In particular, the members of the tuple $\mathbf{A}$ are self-adjoint and commute pairwise.
- (ii) The ideal $\mathcal{I} \subseteq \mathbb{C}[x_1, \dots, x_n]$ generated by all definitizing polynomials of $\mathbf{A}$ satisfies either of the following.
 - (a) $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$.
 - (b) $\mathcal{I} \subsetneq \mathbb{C}[x_1, \dots, x_n]$ and admits a minimal primary decomposition such that the varieties of the primary components are pairwise disjoint; see Assumption 2.3.1.

3.2 Spectrum

In Proposition 3.2.2, we will see that $\sigma(\mathbf{A}) = \sigma(\Theta[\mathbf{A}]) \cup (V(\mathcal{I}) \cap \sigma(\mathbf{A}))$. While the functions in $\mathcal{F}_{\mathbf{A}}$ are defined on $\sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I})$, it will be shown that the functional calculus only depends on the function values on $\sigma(\mathbf{A})$, as one expects from a functional calculus. Moreover, we will provide a weak spectral mapping theorem for continuous functions in $\mathfrak{S}_{\mathbf{A}}$.

We will be working with the same assumptions as in Section 3.1. In particular, $\mathcal{I}$ shall denote the ideal generated by all definitizing polynomials of $\mathbf{A}$, which in case $\mathcal{I} \subsetneq \mathbb{C}[x_1, \dots, x_n]$ shall satisfy Assumption 2.3.1. Furthermore, we fix definitizing polynomials $p_1, \dots, p_m \in \mathbb{R}[x_1, \dots, x_n]$ of $\mathbf{A}$ with $\mathcal{I} = \langle p_1, \dots, p_m \rangle$ according to Proposition 1.6.9.

Lemma 3.2.1. *Let $[w] \in \mathbb{V}(\mathcal{I})$, $X \in \{\mathcal{A}_{[w]}, \mathcal{B}_{[w]}\}$, and $N \in \mathbb{N}$. If the tuple $(s_j)_{j=1}^N \in (\mathbb{C}[x_1, \dots, x_n])^N$ satisfies*

$$V(\langle s_1, \dots, s_N \rangle) \cap V(\mathcal{Q}_{[w]}) = \emptyset,$$

then $([s_j]_X)_{j=1}^N$ is invertible in X^N in the sense of Definition 1.3.20.

Proof. Let $X = \mathcal{A}_{[w]}$. Since $V(\mathcal{P}_{[w]} \cdot \mathcal{Q}_{[w]}) = V(\mathcal{Q}_{[w]})$, we have $V(\langle s_1, \dots, s_N \rangle) \cap V(\mathcal{P}_{[w]} \cdot \mathcal{Q}_{[w]}) = \emptyset$; see Lemma 2.3.4. By Lemma 1.1.20, $\langle s_1, \dots, s_N \rangle$ and $\mathcal{P}_{[w]} \cdot \mathcal{Q}_{[w]}$ are comaximal. Let $h_1, \dots, h_k$ generate $\mathcal{P}_{[w]} \cdot \mathcal{Q}_{[w]}$. As the given ideals are comaximal, there exist $u_1, \dots, u_k, u_{k+1}, \dots, u_{k+N} \in \mathbb{C}[x_1, \dots, x_n]$ such that

$$1 = \underbrace{\sum_{i=1}^k u_i h_i}_{\in \mathcal{P}_{[w]} \cdot \mathcal{Q}_{[w]}} + \sum_{j=1}^N u_{k+j} s_j.$$

Thus $([s_j]_{\mathcal{A}[\mathbf{w}]})_{j=1}^N$ is invertible in $\mathcal{A}[\mathbf{w}]^N$, where $([u_{k+j}]_{\mathcal{A}[\mathbf{w}]})_{j=1}^N$ is its inverse. As $\mathcal{P}[\mathbf{w}] \cdot \mathcal{Q}[\mathbf{w}] \subseteq \mathcal{Q}[\mathbf{w}]$, the invertibility in $\mathcal{A}[\mathbf{w}]$ implies the invertibility in $\mathcal{B}[\mathbf{w}]$. $\square$

Proposition 3.2.2. *We have*

$$\sigma(\mathbf{A}) = \sigma(\Theta[\mathbf{A}]) \cup (V(\mathcal{I}) \cap \sigma(\mathbf{A})).$$

In particular, $\sigma(\mathbf{A}) \setminus V(\mathcal{I}) = \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$ does not depend on the choice of the generators $p_1, \dots, p_m$ of the ideal $\mathcal{I}$.

Proof. Since Θ is a non-trivial $*$ -homomorphism, $\sigma(\Theta[\mathbf{A}]) \subseteq \sigma(\mathbf{A})$. Hence, it suffices to verify $\sigma(\mathbf{A}) \subseteq \sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I})$. Let $\lambda \in \mathbb{C}^n \setminus (\sigma(\Theta[\mathbf{A}]) \cup V(\mathcal{I}))$ and set $s_j(\mathbf{x}) := x_j - \lambda_j$, $j \in [1, n]_{\mathbb{Z}}$, so that $V(\langle s_1, \dots, s_n \rangle) = \{\lambda\}$. We will show that $(s_j \mathbf{A})_{j=1}^n \in \mathcal{F}_{\mathbf{A}}^n$ is invertible by constructing $(\beta_j)_{j=1}^n \in \mathcal{F}_{\mathbf{A}}^n$ such that $\sum_{j=1}^n s_j \mathbf{A} \beta_j = 1_{\mathbf{A}}$.

Given $[\mathbf{w}] \in \mathbb{V}(\mathcal{I})$ we know from Lemma 3.2.1 that there exist $(b_j^{[\mathbf{w}]})_{j=1}^n \in \mathbb{C}[x_1, \dots, x_n]^n$ and $h^{[\mathbf{w}]} \in \mathcal{P}[\mathbf{w}] \cdot \mathcal{Q}[\mathbf{w}]$ such that

$$\sum_{j=1}^n s_j b_j^{[\mathbf{w}]} = 1 + h^{[\mathbf{w}]}.$$
 (3.1)

For $\mathbf{z} \in V(\mathcal{Q}[\mathbf{w}])$ we set $\beta_j(\mathbf{z}) := [b_j^{[\mathbf{w}]}]_X$, $j \in [1, n]_{\mathbb{Z}}$, where $X = \mathcal{A}[\mathbf{w}]$ if $[\mathbf{w}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$ and $X = \mathcal{B}[\mathbf{w}]$ otherwise. We obtain

$$\sum_{j=1}^n \beta_j(\mathbf{z}) s_j \mathbf{A}(\mathbf{z}) = \sum_{j=1}^n [b_j^{[\mathbf{w}]}]_X [s_j]_X = [1]_X.$$

Since $h^{[\mathbf{w}]}$ vanishes on $V(\mathcal{Q}[\mathbf{w}])$, there exists $\epsilon^{[\mathbf{w}]} > 0$ such that

$$|h^{[\mathbf{w}]}(\mathbf{z})| < \frac{1}{2}, \quad \mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \cap \left(V(\mathcal{Q}[\mathbf{w}]) + B_{\epsilon^{[\mathbf{w}]}}(0) \right),$$

where $B_{\epsilon^{[\mathbf{w}]}}(0)$ denotes the open ball centered at 0 with radius $\epsilon^{[\mathbf{w}]}$.

Moreover, since the sets $\sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{Q}[\mathbf{w}])$, $[\mathbf{w}] \in \mathbb{V}(\mathcal{I})$, are compact and pairwise disjoint, there exists $\tilde{\epsilon} > 0$ such that $\sigma(\Theta[\mathbf{A}]) \cap (V(\mathcal{Q}[\mathbf{w}]) + B_{\tilde{\epsilon}}(0))$, $[\mathbf{w}] \in \mathbb{V}(\mathcal{I})$, are pairwise disjoint. We set $\epsilon := \min \{ \tilde{\epsilon}, \min_{[\mathbf{w}] \in \mathbb{V}(\mathcal{I})} \epsilon^{[\mathbf{w}]} \}$ and

$$U_{[\mathbf{w}]} := \left(\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}) \right) \cap \left(V(\mathcal{Q}[\mathbf{w}]) + B_{\epsilon}(0) \right), \quad [\mathbf{w}] \in \mathbb{V}(\mathcal{I}),$$
 (3.2)

which are pairwise disjoint sets. For $\mathbf{z} \in U_{[\mathbf{w}]}$, $[\mathbf{w}] \in \mathbb{V}(\mathcal{I})$, we define

$$\beta_j(\mathbf{z}) := \frac{b_j^{[\mathbf{w}]}(\mathbf{z})}{1 + h^{[\mathbf{w}]}(\mathbf{z})}, \quad j \in [1, n]_{\mathbb{Z}},$$

and obtain

$$\sum_{j=1}^n \beta_j(\mathbf{z}) s_j \mathbf{A}(\mathbf{z}) = \frac{1}{1 + h^{[\mathbf{w}]}(\mathbf{z})} \sum_{j=1}^n b_j^{[\mathbf{w}]}(\mathbf{z}) s_j(\mathbf{z}) \stackrel{(3.1)}{=} 1.$$

We are going to define β_j , $j \in [1, n]_{\mathbb{Z}}$, on

$$C := \sigma(\Theta[\mathbf{A}]) \setminus \left(V(\mathcal{I}) \cup \bigcup_{[\mathbf{w}] \in \mathbb{V}(\mathcal{I})} U_{[\mathbf{w}]} \right) = \sigma(\Theta[\mathbf{A}]) \setminus (V(\mathcal{I}) + B_{\epsilon}(0)).$$

Due to $\lambda \notin \sigma(\Theta[\mathbf{A}])$, given $\mathbf{z} \in C$, there exist $\delta_{\mathbf{z}} > 0$ and $j_{\mathbf{z}} \in [1, n]_{\mathbb{Z}}$ such that $s_{j_{\mathbf{z}}}$ does not vanish on $\text{cl}(B_{\delta_{\mathbf{z}}}(\mathbf{z}))$. As C is compact, there exist $\mathbf{z}_1, \dots, \mathbf{z}_N$ such that $B_{\delta_{\mathbf{z}_i}}(\mathbf{z}_i)$, $i \in [1, N]_{\mathbb{Z}}$, is a finite open covering of C . We construct a partition of C inductively by

$$P_i := \left(B_{\delta_{\mathbf{z}_i}}(\mathbf{z}_i) \cap C \right) \setminus \bigcup_{k=1}^{i-1} P_k, \quad i \in [1, N]_{\mathbb{Z}}. \quad (3.3)$$

Moreover, we set $j_i := j_{\mathbf{z}_i} \in [1, n]_{\mathbb{Z}}$, which indicates that s_{j_i} does not vanish on P_i . For $\mathbf{z} \in P_i$, $i \in [1, N]_{\mathbb{Z}}$, we define β_j , $j \in [1, n]_{\mathbb{Z}}$, by

$$\beta_j(\mathbf{z}) := \begin{cases} \frac{1}{s_j(\mathbf{z})}, & j = j_i \\ 0, & \text{otherwise,} \end{cases}$$

and obtain

$$\sum_{j=1}^n \beta_j(\mathbf{z}) s_{j_{\mathbf{A}}}(\mathbf{z}) = \frac{1}{s_{j_i}(\mathbf{z})} s_{j_i}(\mathbf{z}) = 1, \quad \mathbf{z} \in P_i.$$

Thus we have constructed $\beta_j \in \mathcal{M}_{\mathbf{A}}$, $j \in [1, n]_{\mathbb{Z}}$, such that

$$\sum_{j=1}^n \beta_j(\mathbf{z}) s_{j_{\mathbf{A}}}(\mathbf{z}) = 1_{\mathbf{A}}.$$

At last, we verify $\beta_j \in \mathcal{F}_{\mathbf{A}}$, $j \in [1, n]_{\mathbb{Z}}$; see Definition 2.4.14. The measurability on $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$ is clear, as β_j is a rational function on each of the measurable sets defined in (3.2) and (3.3). Due to the choice of ϵ , $|1 + h^{[w]}|$ is bounded from below by $1/2$ on $U_{[w]}$ for any $[w] \in \mathbb{V}(\mathcal{I})$ and thus β_j is bounded on $U_{[w]}$, $[w] \in \mathbb{V}(\mathcal{I})$. The boundedness of β_j on P_i is ensured by the fact that s_{j_i} does not vanish on $\text{cl}(B_{\delta_{\mathbf{z}_i}}(\mathbf{z}))$, $i \in [1, N]_{\mathbb{Z}}$.

Let $[w] \in \mathbb{V}(\mathcal{I})$ be such that there exists $\mathbf{a} \in V(\mathcal{Q}_{[w]})$ which is an accumulation point of $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$. For $\mathbf{z} \in U_{[w]}$ and $j \in [1, n]_{\mathbb{Z}}$ we have

$$\beta_j(\mathbf{z}) - b_j^{[w]}(\mathbf{z}) = \frac{b_j^{[w]}(\mathbf{z})}{1 + h^{[w]}(\mathbf{z})} - b_j^{[w]}(\mathbf{z}) = \frac{-h^{[w]}(\mathbf{z}) b_j^{[w]}(\mathbf{z})}{1 + h^{[w]}(\mathbf{z})}.$$

Since $h^{[w]} b_j^{[w]} \in \mathcal{Q}_{[w]}$ and $\lim_{\mathbf{z} \rightarrow \mathbf{a}} 1 + h^{[w]}(\mathbf{z}) = 1$, Proposition 2.4.12 yields $\beta_j(\mathbf{z}) - b_j^{[w]}(\mathbf{z}) = O\left(\left|\sum_{j=1}^m p_j(\mathbf{z})\right|\right)$ as $\mathbf{z} \rightarrow \mathbf{a}$. Hence $\beta_j \in \mathcal{F}_{\mathbf{A}}$, $j \in [1, n]_{\mathbb{Z}}$, and we conclude that $(s_{j_{\mathbf{A}}})_{j=1}^n$ is invertible in $\mathcal{F}_{\mathbf{A}}^n$. As a consequence, we obtain the invertibility of

$$(s_{j_{\mathbf{A}}}(\mathbf{A}))_{j=1}^n = (A_j - \lambda_j)_{j=1}^n = \mathbf{A} - \lambda$$

implying $\lambda \in \rho(\mathbf{A})$. □

Corollary 3.2.3. *The function space $\mathcal{F}_{\mathbf{A}}$ does not depend on the choice of the generators $p_1, \dots, p_m$ of the ideal $\mathcal{I}$.*

Proof. According to Proposition 3.2.2 the set $\mathcal{M}_{\mathbf{A}}$ does not depend on the choice of generators. In fact, it only depends on the ideal $\mathcal{I}$ and its unique minimal primary decomposition. Assume that also $q_1, \dots, q_l$ generate $\mathcal{I}$. For any $\mathbf{a} \in V(\mathcal{I})$ we have $q_1, \dots, q_l \in \mathcal{I} \subseteq \mathcal{Q}(\mathbf{a})$. Thus by Proposition 2.4.12

$$\sum_{j=1}^l q_j(\mathbf{z}) = O\left(\left|\sum_{j=1}^m p_j(\mathbf{z})\right|\right) \text{ as } \mathbf{z} \rightarrow \mathbf{a}.$$

By swapping the roles of $q_1, \dots, q_l$ and $p_1, \dots, p_m$, we see that (2.19) is indeed a statement independent of the choice of the generators of $\mathcal{I}$. □

Definition 3.2.4. Let $[w] \in \mathbb{V}(\mathcal{I})$ be such that $V(\mathcal{Q}_{[w]}) \cap \sigma(\Theta[\mathbf{A}]) = \emptyset$. We define $\xi\delta_{[w]} \in \mathcal{M}_{\mathbf{A}}$ by

$$\xi\delta_{[w]}(z) := \begin{cases} \xi, & z \in V(\mathcal{Q}_{[w]}), \\ 0, & \text{otherwise,} \end{cases}$$

where ξ belong to $\mathcal{A}_{[w]}$ if $[w] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$ and to $\mathcal{B}_{[w]}$ otherwise.

Lemma 3.2.5. Let $[w] \in \mathbb{V}(\mathcal{I})$ be such that $V(\mathcal{Q}_{[w]}) \cap \sigma(\Theta[\mathbf{A}]) = \emptyset$. If we denote by e the unity in $\mathcal{A}_{[w]}$ and $\mathcal{B}_{[w]}$, then

- (i) $e\delta_{[w]} \in \mathcal{F}_{\mathbf{A}}$.
- (ii) $e\delta_{[w]}(\mathbf{A})$ constitutes a projection.
- (iii) $\mathbf{A}|_{\text{ran } e\delta_{[w]}(\mathbf{A})} := (A_j|_{\text{ran } e\delta_{[w]}(\mathbf{A})})_{j=1}^n \in L_b(\text{ran } e\delta_{[w]}(\mathbf{A}))^n$.
- (iv) $\sigma(\mathbf{A}|_{\text{ran } e\delta_{[w]}(\mathbf{A})}) \subseteq V(\mathcal{Q}_{[w]})$.

Proof. (i): It is straightforward to check $e\delta_{[w]} \in \mathcal{F}_{\mathbf{A}}$; see Definition 2.4.14.

(ii): From $(e\delta_{[w]})^2 = e\delta_{[w]}$, we infer that $e\delta_{[w]}(\mathbf{A})$ constitutes a projection.

(iii): Given $j \in [1, n]_{\mathbb{Z}}$ we have

$$A_j|_{\text{ran } e\delta_{[w]}(\mathbf{A})} = (x_j)_{\mathbf{A}}(\mathbf{A}) e\delta_{[w]}(\mathbf{A})|_{\text{ran } e\delta_{[w]}(\mathbf{A})} = e\delta_{[w]}(\mathbf{A}) (x_j)_{\mathbf{A}}(\mathbf{A})|_{\text{ran } e\delta_{[w]}(\mathbf{A})}.$$

Thus, $\text{ran } A_j|_{\text{ran } e\delta_{[w]}(\mathbf{A})} \subseteq \text{ran } e\delta_{[w]}(\mathbf{A})$.

(iv): We assume $[w] \notin \mathbb{V}_{\mathbb{R}}(\mathcal{I})$. For $[w] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I})$ simply replace every instance of $\mathcal{B}_{[w]}$ in the remaining part of the proof with $\mathcal{A}_{[w]}$.

For $\lambda \in \mathbb{C}^n \setminus V(\mathcal{Q}_{[w]})$ we set $s_j(\mathbf{x}) := x_j - \lambda_j$, $j \in [1, n]_{\mathbb{Z}}$. Because of $V(\langle s_1, \dots, s_n \rangle) = \{\lambda\}$, Lemma 3.2.1 yields the existence of $(b_j)_{j=1}^n \in \mathcal{B}_{[w]}$ such that $\sum_{j=1}^n [s_j]_{\mathcal{B}_{[w]}} b_j = e$ implying

$$\sum_{j=1}^n s_j \mathbf{A} \cdot b_j \delta_{[w]} = \left(\sum_{j=1}^n [s_j]_{\mathcal{B}_{[w]}} b_j \right) \delta_{[w]} = e\delta_{[w]}.$$

We conclude that the tuple

$$\left(A_j|_{\text{ran } e\delta_{[w]}(\mathbf{A})} - \lambda_j \right)_{j=1}^n = \left(s_j \mathbf{A}(\mathbf{A})|_{\text{ran } e\delta_{[w]}(\mathbf{A})} \right)_{j=1}^n \in L_b(\text{ran } e\delta_{[w]}(\mathbf{A}))^n$$

is invertible, which means $\lambda \in \rho(\mathbf{A}|_{\text{ran } e\delta_{[w]}(\mathbf{A})})$. □

Proposition 3.2.6. If $\phi \in \mathcal{F}_{\mathbf{A}}$ vanishes on $\sigma(\mathbf{A})$, then $\phi(\mathbf{A}) = 0$.

Proof. Let $w \in V(\mathcal{I})$ be such that $\phi(w) \neq 0$. ϕ being constant on $V(\mathcal{Q}_{[w]})$, $\phi|_{\sigma(\mathbf{A})} \equiv 0$ implies

$$V(\mathcal{Q}_{[w]}) \cap \underbrace{(\sigma(\Theta[\mathbf{A}]) \cup (V(\mathcal{I}) \cap \sigma(\mathbf{A})))}_{=\sigma(\mathbf{A})} = \emptyset. \quad (3.4)$$

Hence, $V(\mathcal{Q}_{[w]}) \cap \sigma(\Theta[\mathbf{A}]) = \emptyset$ and, in turn, $e\delta_{[w]}(\mathbf{A})$ constitutes a projection; see Lemma 3.2.5.

According to (3.4), $\lambda \in V(\mathcal{Q}_{[w]}) \subseteq V(\mathcal{I})$ implies $\lambda \in \rho(\mathbf{A})$. Hence $\mathbf{A} - \lambda$ and, in turn, $\mathbf{A}|_{\text{ran } e\delta_{[w]}(\mathbf{A})} - \lambda$ is invertible. From Lemma 3.2.5 we conclude $\sigma(\mathbf{A}|_{\text{ran } e\delta_{[w]}(\mathbf{A})}) = \emptyset$, which is only possible if $\text{ran } e\delta_{[w]}(\mathbf{A}) = \{0\}$. Hence, $e\delta_{[w]}(\mathbf{A}) = 0$ and

$$\phi(\mathbf{A}) = \left(\sum_{\substack{[\mathbf{a}] \in \mathbb{V}(\mathcal{I}) \\ V(\mathcal{Q}_{[\mathbf{a}]}) \cap \sigma(\mathbf{A}) = \emptyset}} \phi \cdot e\delta_{[\mathbf{a}]} \right) (\mathbf{A}) = \sum_{\substack{[\mathbf{a}] \in \mathbb{V}(\mathcal{I}) \\ V(\mathcal{Q}_{[\mathbf{a}]}) \cap \sigma(\mathbf{A}) = \emptyset}} \phi(\mathbf{A}) \cdot \underbrace{e\delta_{[\mathbf{a}]}(\mathbf{A})}_{=0} = 0.$$

□

The next result was motivated by the classical spectral mapping theorem, which states

$$\sigma(p(a)) = p(\sigma(a)), \quad p \in \mathbb{C}[z],$$

for a unital algebra A and $a \in A$.

Proposition 3.2.7. *We set $V := \{z \in V(\mathcal{I}) : V(\mathcal{Q}_{[z]}) \cap \sigma(\mathbf{A}) \neq \emptyset\}$. If $\phi \in \mathfrak{S}_{\mathbf{A}}$ is continuous, then*

$$\sigma(\phi_{\mathbf{A}}(\mathbf{A})) \subseteq \phi(\sigma(\mathbf{A}) \cup V).$$

Proof. If $\lambda \notin \phi(\sigma(\mathbf{A}) \cup V)$, then $\phi_{\mathbf{A}}(z) = \phi(z) \neq \lambda$ for all $z \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$. Since $\sigma(\mathbf{A})$ is compact and ϕ is continuous, $\phi(\sigma(\mathbf{A}))$ is compact. Hence,

$$\lambda \notin \phi(\sigma(\mathbf{A})) \supseteq \text{cl}(\phi(\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}))).$$

Given $z \in V$ let $p \in \mathbb{C}[x_1, \dots, x_n]$ and $U \subseteq \mathbb{C}^n$ be a neighbourhood of z such that $p|_U = \phi|_U$. For any $w \in V(\mathcal{Q}_{[z]})$ there exists $h^w \in \mathcal{Q}_{[z]}$ such that $(p + h^w)|_W = \phi|_W$, where W is some neighbourhood of w ; see Definition 2.4.3. $h^w(w) = 0$ implies $p(w) = p(w) + h^w(w) = \phi(w) \neq \lambda$. Therefore, the polynomial $p - \lambda$ does not vanish on $V(\mathcal{Q}_{[z]})$. Employing Lemma 3.2.1 for $N = 1$, we conclude that $(\phi - \lambda)_{\mathbf{A}}(z) = (p - \lambda)_{\mathbf{A}}(z)$ is invertible in $\mathcal{A}_{[z]}$ or $\mathcal{B}_{[z]}$, respectively. According to Proposition 3.2.6 the function values for $z \in V(\mathcal{I}) \setminus V$ do not affect the functional calculus. Hence we can assume $\phi_{\mathbf{A}}(z) = e$, $z \in V(\mathcal{I}) \setminus V$. Consequently, $(\phi - \lambda)_{\mathbf{A}}$ satisfies the assumptions of Lemma 2.4.21, resulting in its invertibility. Thus $\phi_{\mathbf{A}}(\mathbf{A}) - \lambda$ is an invertible operator. □

3.3 Independence of the Choice of Generators

Although the functional calculus was constructed using concepts that depend on the choice of the generators of $\mathcal{I}$, we are going to show in the present section that the resulting joint functional calculus is independent of this choice. As an application, we will demonstrate that the Hilbert space joint functional calculus is a special case of the Krein space joint functional calculus.

We fix two families of generators $\mathbf{s} := (s_j)_{j=1}^m$ and $\mathbf{t} := (t_i)_{i=1}^l$ of the ideal $\mathcal{I}$. In order to avoid confusion, we will mark objects with the generators used to construct them. For example, we will denote by $\phi(\mathbf{A})_{\mathbf{s}}$ the functional calculus constructed with respect to $\mathbf{s}$. Henceforth, our goal in this section is to prove that

$$\phi(\mathbf{A})_{\mathbf{s}} = \phi(\mathbf{A})_{\mathbf{t}}, \quad \phi \in \mathcal{F}_{\mathbf{A}}. \quad (3.5)$$

We consider the family of generators $\mathbf{s} \cup \mathbf{t} = (s_1, \dots, s_m, t_1, \dots, t_l)$ and show

$$\phi(\mathbf{A})_{\mathbf{s}} = \phi(\mathbf{A})_{\mathbf{s} \cup \mathbf{t}}, \quad \phi \in \mathcal{F}_{\mathbf{A}}.$$

We then obtain (3.5) by symmetry.

Recall that $T_{\mathbf{s}} \in L_b(\mathcal{H}_{\mathbf{s}}, \mathcal{K})$ and $T_{\mathbf{t}} \in L_b(\mathcal{H}_{\mathbf{t}}, \mathcal{K})$ are operators satisfying $T_{\mathbf{s}} T_{\mathbf{s}}^+ = \sum_{j=1}^m s_j(\mathbf{A})$ and $T_{\mathbf{t}} T_{\mathbf{t}}^+ = \sum_{i=1}^l t_i(\mathbf{A})$; see $T \in L_b(\mathcal{H}, \mathcal{K})$ in Definition 2.1.1. Accordingly, the operator $T_{\mathbf{s} \cup \mathbf{t}} \in L_b(\mathcal{H}_{\mathbf{s} \cup \mathbf{t}}, \mathcal{K})$ is such that

$$T_{\mathbf{s} \cup \mathbf{t}} T_{\mathbf{s} \cup \mathbf{t}}^+ = \sum_{j=1}^m s_j(\mathbf{A}) + \sum_{i=1}^l t_i(\mathbf{A}) = T_{\mathbf{s}} T_{\mathbf{s}}^+ + T_{\mathbf{t}} T_{\mathbf{t}}^+.$$

Applying Lemma 2.1.3 to the definitizing polynomials $p_1 := \sum_{j=1}^m s_j$ and $p_2 := \sum_{i=1}^l t_i$ of $\mathbf{A}$ yields the existence of unique injective contractions $R_{\mathbf{s}} \in L_b(\mathcal{H}_{\mathbf{s}}, \mathcal{H}_{\mathbf{s} \cup \mathbf{t}})$, $R_{\mathbf{t}} \in L_b(\mathcal{H}_{\mathbf{t}}, \mathcal{H}_{\mathbf{s} \cup \mathbf{t}})$ such that $T_{\mathbf{s}} = T_{\mathbf{s} \cup \mathbf{t}} R_{\mathbf{s}}$, $T_{\mathbf{t}} = T_{\mathbf{s} \cup \mathbf{t}} R_{\mathbf{t}}$, and $R_{\mathbf{s}} R_{\mathbf{s}}^* + R_{\mathbf{t}} R_{\mathbf{t}}^* = I$.

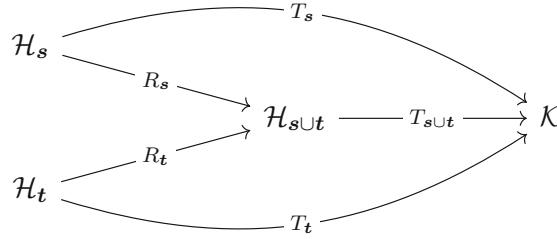


Figure 3.1: Commutative diagram of operators introduced in Section 3.3.

Recall that we have already established that the function space $\mathcal{F}_{\mathbf{A}}$ does not depend on the choice of generators; see Corollary 3.2.3.

Lemma 3.3.1. *Let $\phi \in \mathcal{F}_{\mathbf{A}}$. If $(r, f)_{\mathbf{s}}$ is a decomposition of ϕ with respect to $\mathbf{s}$, then $(r, F)_{\mathbf{s} \cup \mathbf{t}}$ constitutes a decomposition of ϕ with respect to $\mathbf{s} \cup \mathbf{t}$, where*

$$F := \mathbb{1}_{\sigma(\mathbf{A}) \setminus V(\mathcal{I})} \frac{f \sum_{j=1}^m s_j}{\sum_{j=1}^m s_j + \sum_{i=1}^l t_i} \in \mathfrak{B}(\sigma(\Theta_{\mathbf{s} \cup \mathbf{t}}[\mathbf{A}])).$$

Proof. Applying Proposition 2.4.12 for the generators $\mathbf{s} \cup \mathbf{t}$, we obtain $\sum_{j=1}^m s_j(\mathbf{z}) = O(|\sum_{j=1}^m s_j(\mathbf{z}) + \sum_{i=1}^l t_i(\mathbf{z})|)$ as $\mathbf{z} \rightarrow \mathbf{w} \in V(\mathcal{I})$. In consequence, F is bounded. f being measurable implies measurability of F . $(r, F)_{\mathbf{s} \cup \mathbf{t}}$ clearly satisfies (iii) of Definition 3.1.1 for $\Theta = \Theta_{\mathbf{s} \cup \mathbf{t}}$ and (i) since $(r, f)_{\mathbf{s}}$ is a decomposition of ϕ with respect to $\mathbf{s}$. Note that $\sigma(\Theta_{\mathbf{s} \cup \mathbf{t}}[\mathbf{A}]) \setminus V(\mathcal{I}) = \sigma(\mathbf{A}) \setminus V(\mathcal{I}) = \sigma(\Theta_{\mathbf{s}}[\mathbf{A}]) \setminus V(\mathcal{I})$ according to Proposition 3.2.2. For $\mathbf{z} \in \sigma(\mathbf{A}) \setminus V(\mathcal{I})$ we have

$$r(\mathbf{z}) + F(\mathbf{z}) \left(\sum_{j=1}^m s_j(\mathbf{z}) + \sum_{i=1}^l t_i(\mathbf{z}) \right) = r(\mathbf{z}) + f(\mathbf{z}) \sum_{j=1}^m s_j(\mathbf{z}) = \phi(\mathbf{z}).$$

Thus, Definition 3.1.1, (ii) is also satisfied. $\square$

Lemma 3.3.2. *If $f \in \mathfrak{B}(\sigma(\Theta_{s \cup t}[\mathbf{A}]))$ satisfies $f|_{\sigma(\Theta_{s \cup t}[\mathbf{A}]) \cap V(\mathcal{I})} = 0$, then*

$$\Xi_s \left(\int f dE_s \right) = \Xi_{s \cup t} \left(\int \frac{f \sum_{j=1}^m s_j}{\sum_{j=1}^m s_j + \sum_{i=1}^l t_i} dE_{s \cup t} \right),$$

where $E_s, E_{s \cup t}$ are the joint spectral measures of $\Theta_s[\mathbf{A}], \Theta_{s \cup t}[\mathbf{A}]$, respectively.

Proof. We apply Lemma 2.1.10 to $p_1 = \sum_{j=1}^m s_j$, $p_2 = \sum_{i=1}^l t_i$ and translate the notation into that of the present section by

$$R_1 = R_s, \Xi_1 = \Xi_s, \Xi = \Xi_{s \cup t}, \Theta_1 = \Theta_s, \Theta = \Theta_{s \cup t}, E^1 = E_s, E = E_{s \cup t}.$$

By (2.9) applied to our situation,

$$\begin{aligned} \Xi_s \left(\int f dE_s \right) &= \Xi_{s \cup t} \left(R_s R_s^* \int f dE_{s \cup t} \right) \\ &= \Xi_{s \cup t} \left(R_s R_s^* E_{s \cup t}(\sigma(\Theta_{s \cup t}[\mathbf{A}]) \setminus V(\mathcal{I})) \int f dE_{s \cup t} \right). \end{aligned}$$

According to Corollary 2.1.12, $\sum_{j=1}^m s_j + \sum_{i=1}^l t_i$ does not vanish on $\sigma(\Theta_{s \cup t}[\mathbf{A}]) \setminus V(\mathcal{I})$ because $s \cup t$ generates $\mathcal{I}$. Applying Lemma 2.1.13 to $p_1 = \sum_{j=1}^m s_j$, $p_2 = \sum_{i=1}^l t_i$, and $M = \sigma(\Theta_{s \cup t}[\mathbf{A}]) \setminus V(\mathcal{I})$ yields

$$R_s R_s^* E_{s \cup t}(\sigma(\Theta_{s \cup t}[\mathbf{A}]) \setminus V(\mathcal{I})) = \int_{\sigma(\Theta_{s \cup t}[\mathbf{A}]) \setminus V(\mathcal{I})} \frac{\sum_{j=1}^m s_j}{\sum_{j=1}^m s_j + \sum_{i=1}^l t_i} dE_{s \cup t}$$

and, in turn,

$$\begin{aligned} \Xi_s \left(\int f dE_s \right) &= \Xi_{s \cup t} \left(\int_{\sigma(\Theta_{s \cup t}[\mathbf{A}]) \setminus V(\mathcal{I})} \frac{\sum_{j=1}^m s_j}{\sum_{j=1}^m s_j + \sum_{i=1}^l t_i} dE_{s \cup t} \int f dE_{s \cup t} \right) \\ &= \Xi_{s \cup t} \left(\int \frac{f \sum_{j=1}^m s_j}{\sum_{j=1}^m s_j + \sum_{i=1}^l t_i} dE_{s \cup t} \right). \end{aligned}$$

□

We have gathered all the necessary tools in order to prove the main result of this section.

Theorem 3.3.3. *Given $\phi \in \mathcal{F}_A$ we have*

$$\phi(\mathbf{A})_s = \phi(\mathbf{A})_{s \cup t}.$$

Proof. Let $(r, f)_s$ be a decomposition of ϕ with respect to s . By Lemma 3.3.1, $(r, F)_{s \cup t}$ with

$$F := \mathbb{1}_{\sigma(\mathbf{A}) \setminus V(\mathcal{I})} \frac{f \sum_{j=1}^m s_j}{\sum_{j=1}^m s_j + \sum_{i=1}^l t_i},$$

is a decomposition of ϕ with respect to $s \cup t$. Employing Lemma 3.3.2 we conclude

$$\begin{aligned} \phi(\mathbf{A})_{s \cup t} &= r(\mathbf{A}) + \Xi_{s \cup t} \left(\int F dE_{s \cup t} \right) \\ &= r(\mathbf{A}) + \Xi_s \left(\int f dE_s \right) = \phi(\mathbf{A})_s. \end{aligned}$$

□

Corollary 3.3.4. *The joint functional calculus constructed in the present thesis does not depend on the choice of generators of $\mathcal{I}$, i.e., for two families of generators $\mathbf{s}, \mathbf{t}$ of $\mathcal{I}$ we have*

$$\phi(\mathbf{A})_{\mathbf{s}} = \phi(\mathbf{A})_{\mathbf{t}}, \quad \phi \in \mathcal{F}_{\mathbf{A}}.$$

Proof. By Theorem 3.3.3 we have $\phi(\mathbf{A})_{\mathbf{s}} = \phi(\mathbf{A})_{\mathbf{s} \cup \mathbf{t}} = \phi(\mathbf{A})_{\mathbf{t}}$ for $\phi \in \mathcal{F}_{\mathbf{A}}$. $\square$

The joint functional calculus indeed generalizes the joint functional calculus for tuples of pairwise commuting bounded self-adjoint operators on Hilbert spaces, as we will see in the following theorem.

Theorem 3.3.5. *If $(\mathcal{K}, [\cdot, \cdot])$ is a Hilbert space, then any tuple $\mathbf{A} \in L_b(\mathcal{K})^n$ of pairwise commuting self-adjoint operators is definitizable. Moreover, the above developed functional calculus is applicable on $\mathcal{F}_{\mathbf{A}} = \mathfrak{B}(\sigma(\mathbf{A}))$ and*

$$\phi(\mathbf{A}) = \int \phi dE, \quad \phi \in \mathfrak{B}(\sigma(\mathbf{A})),$$

where E is the joint spectral measure of $\mathbf{A}$.

Proof. In a Hilbert space the identity operator $I \in L_b(\mathcal{K})$ is a positive operator. Therefore, $1 \in \mathbb{C}[x_1, \dots, x_n]$ definitizes any tuple $\mathbf{A}$ of pairwise commuting self-adjoint operators. This implies $\mathcal{I} = \mathbb{C}[x_1, \dots, x_n]$. By Remark 3.1.10 the Krein space joint functional calculus can be constructed.

Because of Corollary 3.3.4, we can choose $p_1 = 1$ as the generator of $\mathcal{I}$ for constructing the joint functional calculus implying $\mathcal{H} = \mathcal{H}_1$ and $T = T_1$; see Definition 2.1.1. Moreover, when we take a look at how we constructed $\mathcal{H}$ and $T : \mathcal{H} \rightarrow \mathcal{K}$ in the proof of Lemma 1.6.10, we obtain $\mathcal{H} = \mathcal{K}$ and $T = I$. As a result, the operators $\Theta = \Theta_1$ and $\Xi = \Xi_1$ both coincide with the identity operator on $L_b(\mathcal{K}) = (II^+)' = (I^+I)'$. Furthermore, since $V(\mathcal{I})$ is empty, the function space $\mathcal{F}_{\mathbf{A}}$ equals $\mathfrak{B}(\sigma(\Theta[\mathbf{A}])) = \mathfrak{B}(\sigma(\mathbf{A}))$; see Remark 2.4.15.

Let $\phi \in \mathcal{F}_{\mathbf{A}} = \mathfrak{B}(\sigma(\mathbf{A}))$. Because of $V(\mathcal{I}) = \emptyset$, $\sum_{j=1}^m p_j(\mathbf{z}) = p_1(\mathbf{z}) = 1$, and $\phi \in \mathfrak{B}(\sigma(\mathbf{A}))$, the pair $(0, \phi)$ is a decomposition of ϕ ; see Definition 3.1.1. We obtain

$$\phi(\mathbf{A}) = 0 + \Xi \left(\int \phi dE \right) = \int \phi dE.$$

$\square$

3.4 Compatibility with Subtuples

For a definitizable tuple $\mathbf{A}$ of pairwise commuting self-adjoint operators, let $\mathbf{A}_N := (A_j)_{j \in N} \in L_b(\mathcal{K})^N$, $N \subseteq [1, n]_{\mathbb{Z}}$, be a subtuple. If we denote by $\pi_N : \mathbb{C}^n \rightarrow \mathbb{C}^N$ the canonical projection $(z_j)_{j=1}^n \mapsto (z_j)_{j \in N}$, then for any polynomial $p \in \mathbb{C}[x_1, \dots, x_{|N|}]$,

$$p(\mathbf{A}_N) = p \circ \pi_N(\mathbf{A}). \quad (3.6)$$

Assuming that the joint functional calculus can be defined for both $\mathbf{A}$ and $\mathbf{A}_N$, our goal in the present section is to generalize (3.6) and to show

$$\phi(\mathbf{A}_N) = \phi \circ \pi_N(\mathbf{A}), \quad \phi \in \mathcal{F}_{\mathbf{A}_N},$$

where the term on the right-hand side has yet to be defined.

For objects related to the construction of the joint functional calculus of $\mathbf{A}$, we will continue to employ the notation used since Chapter 2. In order to avoid confusion, anything related to $\mathbf{A}_N$ will be indicated by the subscript $_N$. For example, we will denote by $\mathcal{I}_N$ the ideal generated by all definitizing polynomials of $\mathbf{A}_N$.

Lemma 3.4.1. *If we denote by $\iota : \mathbb{C}[x_1, \dots, x_{|N|}] \rightarrow \mathbb{C}[x_1, \dots, x_n]$ the mapping $p \mapsto p \circ \pi_N$, then $\iota(\mathcal{I}_N) \subseteq \mathcal{I}$ and $\pi_N(V(\mathcal{I})) \subseteq V(\mathcal{I}_N)$.*

Proof. If $p \in \mathbb{C}[x_1, \dots, x_{|N|}]$ is a definitizing polynomial for $\mathbf{A}_N$, then, clearly, $p \circ \pi_N$ is a definitizing polynomial for $\mathbf{A}$. Hence, $\iota(\mathcal{I}_N) \subseteq \mathcal{I}$ and, in consequence, $\pi_N(V(\mathcal{I})) \subseteq \pi_N(V(\iota(\mathcal{I}_N))) = V(\mathcal{I}_N)$. $\square$

Let $q_1, \dots, q_l \in \mathbb{C}[x_1, \dots, x_{|N|}]$ be the generators of $\mathcal{I}_N$ used for constructing the functional calculus of $\mathbf{A}_N$. Moreover, we set $p_1 = \sum_{i=1}^l q_i \circ \pi_N \in \iota(\mathcal{I}_N) \subseteq \mathcal{I}$ and extend it to a family $p_1, \dots, p_m$ of generators of $\mathcal{I}$, which will be used to construct the joint functional calculus in Section 2.1. This choice indeed does not affect the joint functional calculus of $\mathbf{A}$, as we have seen in Corollary 3.3.4.

Definition 3.4.2. For $\phi \in \mathcal{F}_{\mathbf{A}_N}$ and $r \in \mathbb{C}[x_1, \dots, x_{|N|}]$ such that $\phi|_{V(\mathcal{I}_N)} = r_{\mathbf{A}_N}|_{V(\mathcal{I}_N)}$, we define $(\phi \circ \pi_N)_r \in \mathcal{M}_{\mathbf{A}}$ as

$$(\phi \circ \pi_N)_r(\mathbf{z}) := \begin{cases} \phi(\pi_N(\mathbf{z})), & \mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}), \pi_N(\mathbf{z}) \in \sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N), \\ r(\pi_N(\mathbf{z})), & \mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}), \pi_N(\mathbf{z}) \notin \sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N), \\ [r \circ \pi_N]_{\mathcal{P}_{[\mathbf{z}]} \cdot \mathcal{Q}_{[\mathbf{z}]}}}, & \mathbf{z} \in V(\mathcal{I}), [\mathbf{z}] \in \mathbb{V}_{\mathbb{R}}(\mathcal{I}), \\ [r \circ \pi_N]_{\mathcal{Q}_{[\mathbf{z}]}}}, & \mathbf{z} \in V(\mathcal{I}), [\mathbf{z}] \notin \mathbb{V}_{\mathbb{R}}(\mathcal{I}). \end{cases}$$

Lemma 3.4.3. *With the notation and assumption from Definition 3.4.2, the function $(\phi \circ \pi_N)_r$ belongs to $\mathcal{F}_{\mathbf{A}}$.*

Proof. The measurability of $(\phi \circ \pi_N)_r$ on $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$ is a consequence of the measurability of the function $\phi \circ \pi_N$ and the sets $V(\mathcal{I}), V(\mathcal{I}_N), \sigma(\Theta[\mathbf{A}]), \sigma(\Theta_N[\mathbf{A}_N])$. Since $\sigma(\Theta[\mathbf{A}])$ is compact and ϕ is bounded on $\sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N)$, $(\phi \circ \pi_N)_r$ is bounded on $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$.

If $\mathbf{w} \in V(\mathcal{I})$ is an accumulation point of $\sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$, then $\mathbf{w} \in \mathbb{R}^n$ and $(\phi \circ \pi_N)_r(\mathbf{w}) = [r \circ \pi_N]_{\mathcal{P}_{[\mathbf{w}]} \cdot \mathcal{Q}_{[\mathbf{w}]}}$. Furthermore, due to $\pi_N(V(\mathcal{I})) \subseteq V(\mathcal{I}_N)$, $\pi_N(\mathbf{w}) \in V(\mathcal{I}_N)$ and $\phi(\pi_N(\mathbf{w})) = [r]_{\mathcal{P}_{[\pi_N(\mathbf{w})]} \cdot \mathcal{Q}_{[\pi_N(\mathbf{w})}]}$.

We are going to consider two cases that do not exclude each other. First assume that $\pi_N(\mathbf{w})$ is an accumulation point of $\sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N)$. Due to $\phi \in \mathcal{F}_{\mathbf{A}_N}$,

$$\phi(\pi_N(\mathbf{z})) - r(\pi_N(\mathbf{z})) = O\left(\left|\sum_{q=1}^l q_i \circ \pi_N(\mathbf{z})\right|\right) \quad \text{as } \mathbf{z} \rightarrow \mathbf{w},$$

where $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$ such that $\pi_N(\mathbf{z}) \in \sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N)$. Moreover, $\sum_{q=1}^l q_i \circ \pi_N \in \mathcal{I}$ implies

$$\phi(\pi_N(\mathbf{z})) - r(\pi_N(\mathbf{z})) = O\left(\left|\sum_{j=1}^m p_j(\mathbf{z})\right|\right) \quad \text{as } \mathbf{z} \rightarrow \mathbf{w};$$

see Proposition 2.4.12.

If $\pi_N(\mathbf{w})$ is an accumulation point of $\mathbb{R}^N \setminus (\sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N))$, then for $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$ such that $\pi_N(\mathbf{z}) \notin \sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N)$, we have

$$(\phi \circ \pi_N)_r(\mathbf{z}) - r(\pi_N(\mathbf{z})) = 0 = O\left(\left|\sum_{j=1}^m p_j(\mathbf{z})\right|\right) \quad \text{as } \mathbf{z} \rightarrow \mathbf{w}.$$

Since the growth condition is satisfied in both cases, we conclude

$$(\phi \circ \pi_N)_r(\mathbf{z}) - r(\pi_N(\mathbf{z})) = O\left(\left|\sum_{j=1}^m p_j(\mathbf{z})\right|\right), \quad \text{as } \mathbf{z} \rightarrow \mathbf{w}.$$

Hence $(\phi \circ \pi_N)_r$ belongs to $\mathcal{F}_{\mathbf{A}}$. $\square$

Lemma 3.4.4. *Given $\phi \in \mathcal{F}_{\mathbf{A}_N}$ and its decomposition $(r, f)_N$, the tuple $(r \circ \pi_N, F)$ constitutes a decomposition of $(\phi \circ \pi_N)_r$, where*

$$F(\mathbf{z}) := \begin{cases} \frac{f(\pi_N(\mathbf{z})) \sum_{i=1}^l q_i(\pi_N(\mathbf{z}))}{\sum_{j=1}^m p_j(\mathbf{z})}, & \mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}), \pi_N(\mathbf{z}) \in \sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N), \\ 0, & \mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I}), \pi_N(\mathbf{z}) \notin \sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N), \\ 0, & \mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I}). \end{cases}$$

Proof. Given $\mathbf{z} \in \sigma(\Theta[\mathbf{A}]) \setminus V(\mathcal{I})$ we have

$$\begin{aligned} r(\pi_N(\mathbf{z})) + F(\mathbf{z}) \sum_{j=1}^m p_j(\mathbf{z}) &= \begin{cases} r(\pi_N(\mathbf{z})) + f(\pi_N(\mathbf{z})) \sum_{i=1}^l q_i(\pi_N(\mathbf{z})), & \pi_N(\mathbf{z}) \in \sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N), \\ r(\pi_N(\mathbf{z})), & \pi_N(\mathbf{z}) \notin \sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N), \end{cases} \\ &= (\phi \circ \pi_N)_r(\mathbf{z}). \end{aligned}$$

Clearly, F vanishes on $\sigma(\Theta[\mathbf{A}]) \cap V(\mathcal{I})$ and $(r \circ \pi_N)_{\mathbf{A}}|_{V(\mathcal{I})} = (\phi \circ \pi_N)_r|_{V(\mathcal{I})}$.

The set $\sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N)$ and f being measurable implies the measurability of F . From $\sum_{i=1}^l q_i \circ \pi_N \in \iota(\mathcal{I}_N) \subseteq \mathcal{I}$, we obtain $q_i(\pi_N(\mathbf{z})) = O(|\sum_{j=1}^m p_j(\mathbf{z})|)$ as $\mathbf{z} \rightarrow \mathbf{w} \in V(\mathcal{I})$; see Proposition 2.4.12. Since f is bounded, we derive the boundedness of F . Hence $(r \circ \pi_N, F)$ constitutes a decomposition of $(\phi \circ \pi_N)_r$; see Definition 3.1.1. $\square$

Theorem 3.4.5. *If $\phi \in \mathcal{F}_{\mathbf{A}_N}$ and $r \in \mathbb{C}[x_1, \dots, x_{|N|}]$ satisfy $\phi|_{V(\mathcal{I}_N)} = r_{\mathbf{A}_N}|_{V(\mathcal{I}_N)}$, then*

$$\phi(\mathbf{A}_N) = (\phi \circ \pi_N)_r(\mathbf{A}).$$

Proof. Let $(r, f)_N$ be a decomposition of ϕ and $(r \circ \pi_N, F)$ a decomposition of $(\phi \circ \pi_N)_r$ as defined in Lemma 3.4.4. We set

$$D := \pi_N^{-1}(\sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N)).$$

Due to $\pi_N(V(\mathcal{I})) \subseteq V(\mathcal{I}_N)$, we have $D \cap V(\mathcal{I}) = \emptyset$. If E denotes the spectral measure of $\Theta[\mathbf{A}]$, then

$$\begin{aligned} (\phi \circ \pi_N)_r(\mathbf{A}) &= r \circ \pi_N(\mathbf{A}) + \Xi \left(\int F dE \right) \\ &= r(\mathbf{A}_N) + \Xi \left(\int_D \frac{f \circ \pi_N \sum_{i=1}^l q_i \circ \pi_N}{\sum_{j=1}^m p_j} dE \right) \\ &= r(\mathbf{A}_N) + \Xi \left(\int_D f \circ \pi_N dE \int_D \frac{p_1}{\sum_{j=1}^m p_j} dE \right). \end{aligned}$$

By applying Lemma 2.1.13 to $M = D \cap \sigma(\Theta[\mathbf{A}])$ we see that the last term can be written as

$$r(\mathbf{A}_N) + \Xi \left(\left(\int_D f \circ \pi_N dE \right) R_1 R_1^* E(D \cap \sigma(\Theta[\mathbf{A}])) \right).$$

Since, by Proposition 2.1.8, $R_1 R_1^*$ commutes with $\int_D f \circ \pi_N dE$, this is equal to

$$r(\mathbf{A}_N) + \Xi \left(R_1 R_1^* \int_D f \circ \pi_N dE \right),$$

which according to (2.9) can be rearranged to

$$r(\mathbf{A}_N) + \Xi_1 \left(\int_D f \circ \pi_N dE^1 \right),$$

where E^1 denotes the joint spectral measure of $\Theta_1[\mathbf{A}]$; see Definition 2.1.4. $p_1 = \sum_{i=1}^l q_i \circ \pi_N$ implies $T_1 T_1^+ = p_1(\mathbf{A}) = \sum_{i=1}^l q_i(\mathbf{A}_N) = T_N T_N^+$, which allows us to choose $T_1 = T_N$. Consequently, $\Xi_1 = \Xi_N$ and $\Theta_1 = \Theta_N$. Since $\Theta_N[\mathbf{A}_N]$ is a subtuple of $\Theta_1[\mathbf{A}]$, Corollary 1.7.8 yields $E^1 \circ \pi_N^{-1} = E_N$, where E_N is the joint spectral measure of $\Theta_N[\mathbf{A}_N]$. Taking these facts into consideration, we conclude

$$\begin{aligned} (\phi \circ \pi_N)_r(\mathbf{A}) &= r(\mathbf{A}_N) + \Xi_1 \left(\int_D f \circ \pi_N dE^1 \right) \\ &= r(\mathbf{A}_N) + \Xi_N \left(\int_{\pi_N(D)} f d(E^1 \circ \pi_N^{-1}) \right) \\ &= r(\mathbf{A}_N) + \Xi_N \left(\int_{\sigma(\Theta_N[\mathbf{A}_N]) \setminus V(\mathcal{I}_N)} f dE_N \right) = \phi(\mathbf{A}_N). \end{aligned}$$

□

Due to Theorem 3.4.5, $\phi \circ \pi_N(\mathbf{A})$ is well-defined in the following sense.

Definition 3.4.6. For $\phi \in \mathcal{F}_{\mathbf{A}_N}$ and $r \in \mathbb{C}[x_1, \dots, x_{|N|}]$ such that $\phi|_{V(\mathcal{I}_N)} = r_{\mathbf{A}_N}|_{V(\mathcal{I}_N)}$, we set

$$\phi \circ \pi_N(\mathbf{A}) := (\phi \circ \pi_N)_r(\mathbf{A}).$$

Corollary 3.4.7. For $p \in \mathbb{C}[x_1, \dots, x_{|N|}]$ we have

$$p_{\mathbf{A}_N} \circ \pi_N(\mathbf{A}) = p_{\mathbf{A}_N}(\mathbf{A}_N) = p(\mathbf{A}_N) = p \circ \pi_N(\mathbf{A}) = (p \circ \pi_N)_{\mathbf{A}}(\mathbf{A}).$$

Proof. The first equality holds due to Theorem 3.4.5. The second and the fourth equality is true according to Proposition 3.1.9. □

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