DIPLOMARBEIT

Sharp-Interface Grenzwert für gewichtete Van-der-Waals-Cahn-Hilliard-Phasenübergänge

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unter der Anleitung von

Univ.Prof.in Elisa Davoli, Phd
Projektass.in(FWF) Dott.ssa mag. Anna Kubin, PhD

von

Tilman Baus

Matrikelnummer: 12230455

Wien, 2025

[redacted]
Univ.Prof.in Elisa Davoli, Phd

[redacted]
Tilman Baus

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Sharp-interface limit for weighted Van der Waals-Cahn-Hilliard phase transitions

submitted in partial fulfilment of the requirements for the degree of

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Univ.Prof.in Elisa Davoli, Phd
Projektass.in(FWF) Dott.ssa mag. Anna Kubin, PhD

by

Tilman Baus

Matrikelnummer: 12230455

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[redacted]
Univ.Prof.in Elisa Davoli, Phd

[redacted]
Tilman Baus

Abstract

This thesis focuses on a weighted formulation of the Van-der-Waals-Cahn-Hilliard model for liquid-liquid phase transitions. We focus on the setting of degenerate weights, namely functions ω that are not bounded by positive constants from below, but can, instead, attain any non-negative value. Our main result is that minimizers of the weighted Van-der-Waals-Cahn-Hilliard functionals converge, as the phase-transition parameter vanishes, to minimizers of a sharp-interface perimeter functional.

Similar to the classical model, the convergence result is obtained using Γ-convergence. This means that we proceed in three steps. First, we show compactness of the functional domain, second we establish the existence of a lower bound, third, we prove its optimality. The main difference induced by the weaker assumption $\omega > 0$ is that the functionals are finite on a bigger subspace than in other classical and weighted formulations. For this reason, an adaptation of both the Liminf and Limsup proof strategies is required.

Kurzfassung

Diese Arbeit befasst sich mit einer gewichteten Formulierung des Van-der-Waals-Cahn-Hilliard Modells. Wir betrachten den Fall von degenerierten Gewichten, also Funktionen ω, die keine positive untere Schranke haben, sondern jeden positiven Wert annehmen können. Das Hauptresultat der Arbeit ist, dass die Minimierer des gewichteten Van-der-Waals-Cahn-Hilliard Funktionals gegen ein Sharp-Interface Perimeter Funktional konvergieren, wenn der Parameter, der den Phasenübergang beschreibt, gegen Null konvergiert.

Für den Beweis dieser Aussage wird wie im klassischen Fall Γ-Konvergenz verwendet, die Vorgehensweise erfolgt damit in drei Schritten. Zuerst zeigen wir die Kompaktheit des Definitionsraums, als Zweites wird die Existenz einer unteren Schranke bewiesen und als Drittes zeigen wir die Optimalität eben jener Schranke.

Der Hauptunterschied, der durch die schwächere Bedingung $\omega > 0$ entsteht, ist, dass der Funktionenraum auf dem die Funktionale endlich sind im Vergleich zu anderen klassischen und gewichteten Formulierungen größer ist. Deshalb müssen Die Beweisstrategien der Liminf und Limsup Ungleichung angepasst werden.

Declaration of authorship

I certify that this thesis is my own work and that information which was taken from other sources is declared as such. It has neither been submitted for a degree or examation at any other university nor has it been published.

Vienna, August 31th 2025

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Contents

Introduction

Originally, the Van-der-Waals–Cahn–Hilliard theory is considered to be a diffuse description of phase transition occurring between immiscible fluids. In other worlds, the two pure phases associated to the two fluids are recovered via a variational procedure, in which sharp interfaces are replaced by weighted hessian penalizations.
In the past few years, such models have been employed also in the setting of image segmentation. Here, the solution of a discrete problem is obtained as the limit of a sequence of continuous ones. By studying the Van-der-Waals-Cahn-Hilliard model in this context, we essentially add a small diffusive term to the segmentation process and look at the limit when this term tends to zero. This idea has the advantage that one can use the variational method, which works in a continuous setting, to tackle discrete problems.
In [7], the authors adapt this approach to supervised segmentation by adding a weight function, capturing the training data, into the model. Since they want to use it for regular weight, they can impose the relatively strong assumption that the weight ω is bounded from below by a positive number. Nevertheless, it is interesting to know if this assumption can be weaken to $\omega > 0$ or even $\omega \geq 0$.
In this thesis, we focus on the case $\omega > 0$ and show that the resulting weighted functionals still approximate a limiting sharp-interface model.

Mathematically speaking, the van-der-Waals-Cahn-Hilliard model describes the behavior of a fluid with mass m contained in a domain $\Omega \subset \mathbb{R}^n$. The fluid consists of two phases, characterized by the densities a and b. We consider all admissible density distributions, so all measurable functions $u : \Omega \to \mathbb{R}$ satisfying the mass constraint

$$\int_\Omega u \, dx = m.$$

For these density distributions, the total energy required to maintain a given configuration is given by

$$u \to \int_\Omega W(u(x)) \, dx.$$

Here, $W : \mathbb{R} \to [0, \infty)$ is the so-called energy per unit volume. For our situation, we require W to be a double-well potential, which means that $W(t) = 0$ if and only if $t \in \{a, b\}$. This condition ensures that, at almost every point, the fluid tends to adopt one of the two densities a or b, avoiding intermediate values that would cause an energy penalty.
A first natural attempt to determine an energetically favorable configuration of the fluid is to solve the minimization problem

$$\min\Big\{\int_\Omega W(u(x)) \, dx \;\Big|\; \int_\Omega u(x) \, dx = m\Big\}. \tag{P_0}$$

Unfortunately, this formulation fails to produce physically meaningful solutions. If we assume $\mathcal{L}^n(\Omega) = 1$ and $a < m < b$, every measurable set E fulfilling $\mathcal{L}^n(E) = \frac{m-a}{b-a}$ leads to a minimizing function of P_0 by defining

$$u_E(x) = b\chi_E + a\chi_{\Omega \setminus E}.$$

Consequently, the interface ∂E between the two phases can be arbitrarily complex, a phenomenon that does not appear in nature and is undesirable in our model. From a mathematical viewpoint, our model should take the configuration that solves P_0 and

additionally minimizes the surface area of $\partial E \cap \Omega$ of all solutions of P_0.
To address this, the van der Waals–Cahn–Hilliard approach introduces an additional term in the energy functional that penalizes rapid density variations and thereby complex interfaces. Since the total energy should remain unchanged, this second term depends on a parameter $\epsilon > 0$ and we consider the family of minimization problems

$$\min\Big\{\int_\Omega W(u(x)) + \epsilon^2|\nabla u|^2 \ \ dx \ \Big| \ \int_\Omega u(x)\ dx = m\Big\} \tag{P_ϵ}$$

as $\epsilon \to 0$. In [8], it is shown that the solutions u_ϵ of these perturbed problems converge to the minimizer of P_0 with the minimal interface area. The technique to study such asymptotic behavior, is that of performing the Γ-convergence in the strong L^1-topology for the rescaled functionals

$$\mathcal{F}_\epsilon : W^{1,2}(\Omega) \to [0,\infty)\ ,\ \mathcal{F}_\epsilon(u) = \int_\Omega \frac{1}{\epsilon}W(u(x)) + \epsilon|\nabla u|^2\ dx.$$

In this thesis, we will prove a generalized version of this result. We introduce an additional weight function into the integral, consider the functionals

$$\tilde{F}_\epsilon(u) = \frac{1}{\epsilon}\int_\Omega \omega\ W(u)\ dx + \epsilon\int_\Omega \omega\ |\nabla u|^2\ dx,$$

and characterize their limiting behavior as before. This was done in [7] for weight functions ω with the additional property that there is a lower bound $0 < c \le \omega$. The main result of this thesis is to generalize the statement of [7] to weights only fulfilling $\omega > 0$.

The thesis is organized in the following way:
The first section is dedicated to Gamma-convergence since this is the notion of convergence used to prove the main result. After stating the definition, we give some additional results so that the reader can get some intuition.
In the next section, we recall the seminal Gamma-convergence proof of Gurtin's conjecture for the classical Van-der-Waals-Cahn-Hilliard Model. Normally when proving Gamma-convergence, one has to perform 3 steps. First, one has to show compactness of the domain. Then, one proves that the functional has a lower bound (step 2). Finally, such lower bound is shown to be optimal in the sense that there is a sequence on which such value is attained (step 3). Since we are more interested in the weighted setting, we just perform the first step and refer to the literature for the other ones.
In the third section we introduce the weighted BV-space. We will give an intuition of how this space differs from the "normal" one and generalize results of the standard BV-theory. For example, we clarify what it means to be a set of finite weighed perimeter and give an approximation result. We also prove additional results as the weighted Coarea Formula and discuss variations on lines in the weighted setting. Some of these results will be needed to prove our main result, some are stated to give an overview on the possibilities of the generalized setting.
Finally, we prove the main result, the Gamma-convergence in the weighted setting, in the fourth section of this thesis. Here, we perform all three steps rigorously. After adapting the compactness result, we generalize the arguments of the Liminf and Limsup inequalities to the situation where we just assume $\omega > 0$. We conclude by showing convergence of minimizers, thus establishing the weighted Van-der-Waals-Cahn-Hilliard model as a variational approximation of the perimeter functional.

1 Gamma-convergence

The concept of Γ-convergence is a notion of convergence for functionals and was introduced by De Giorgi in the 1970s. Since then it has become one of the most significant notions of convergence in variational problems. Its strength lies in its flexibility, as it does not rely on any predetermined assumptions about the form of minimizers and is therefore not restricted to a specific setting. For this reason, it finds applications not only within the calculus of variations and partial differential equations but also beyond.
There are primarily two types of applications for Γ-convergence:

Description of the Asymptotic Behavior of Sequences of Minimization Problems:
In various mathematical contexts—ranging from physics and industrial applications to abstract mathematics—parameters appear that make these problems increasingly complex and degenerate. Nevertheless, one can often observe a limiting behavior as these parameters vary. The aim of Γ-convergence is to replace the original problem with a simpler one that still captures the essential features of the original problem. More precisely, one considers a sequence of problems $(P_\epsilon) = \min\{F_\epsilon(u) \mid u \in X_\epsilon\}$ for $\epsilon > 0$ and seeks a limiting problem $(P_0) = \min\{F(u) \mid u \in X\}$, which no longer depends on ϵ but preserves the key characteristics of the minimizers as $\epsilon \to 0$.

Approximation of Complicated Problems by Simpler Ones:
Given a minimization problem $(P) = \min\{F(u) \mid u \in X\}$, the objective is to find a sequence of functionals $\{F_\epsilon\}$ such that $\min\{F_\epsilon(u) \mid u \in X_\epsilon\} \to \min\{F(u) \mid u \in X\}$, and that the corresponding solutions converge to those of the limiting problem as $\epsilon \to 0$. This approach is, for example, employed in the Ambrosio-Tortorelli approximation and the Mumford-Shah functional, where the task is to "clean" a signal corrupted by noise.

Γ-convergence is also used in fields such as homogenization (studying the behavior of solutions to increasingly oscillatory problems), dimension reduction (e.g., modeling thin elastic bodies using lower-dimensional models), and, as will be discussed in more detail later in this work, the gradient theory of phase transitions (e.g., how two immiscible and incompressible fluids distribute themselves within a container).
The effectiveness of Γ-convergence crucially depends on choosing a topology that ensures compactness, thereby guaranteeing the existence of a limit. If the chosen notion of convergence is too strong, minimizers may fail to converge making it impossible to apply the fundamental property of Γ-convergence.
We begin by defining lower semicontinuity, an essential tool for establishing the existence of minimizers. From now on, let X be a metric space.

Definition 1.1: *(lower semicontinuity)*
Let $f : X \to \mathbb{R} \cup \{\infty\}$. We say that f is lower semicontinuous (l.s.c.) at a point $x \in X$ if, for every sequence $\{x_j\}$ converging to x, the following inequality holds:

$$f(x) \leq \liminf_j f(x_j),$$

or, equivalently (see [2], Def. 1.2), if

$$f(x) = \min\{\liminf_j f(x_j) \mid x_j \to x\}.$$

A function is called lower semicontinuous on X if it is l.s.c. at every point $x \in X$.

Remark 1.2: *The following conditions are equivalent:*

(i) f is lower semicontinuous on X,

(ii) For all $x \in X$, it holds that $f(x) = \liminf_{y \to x} f(y)$,

(iii) For every $t \in \mathbb{R}$, the sublevel set $\{f \leq t\}$ is closed.

Remark 1.3: *Let $\{f_j \mid j \in I\}$ be a family of l.s.c. functions. Then the function*

$$f(x) := \sup_j f_j(x)$$

is also lower semicontinuous.

Proof. Let $\{x_k\}$ be a sequence converging to the limit x in X. By the definition of f, we have $f(x_k) \geq f_j(x_k)$ for all $j \in \mathbb{N}$. Since all f_j's are lower semicontinuous, it holds

$$\liminf_k f(x_k) \geq \liminf_k f_j(x_k) \geq f_j(x)$$

The Lemma follows by taking the supremum of the last inequality over all j. □

Definition 1.4: *(Gamma-Convergence)*
We say that a sequence $f_j : X \to \bar{\mathbb{R}}$ converges in sense of Γ-convergence in X to $f_\infty : X \to \bar{\mathbb{R}}$ if the following two conditions are satisfied:
Liminf Inequality: *For every sequence $\{x_j\}$ converging to x,*

$$\liminf_{j \to \infty} f_j(x_j) \geq f_\infty(x).$$

Limsup Inequality: *For every $x \in X$, there is a sequence $\{x_j\}$ such that*

$$\limsup_{j \to \infty} f_j(x_j) \leq f_\infty(x).$$

The function f_∞ is referred to as the Γ-limit of the sequence $\{f_j\}$. If (X, d) is the metric used for the convergence of the sequences in the definition and we want to highlight its role, we will write Γ(d)-limit.
In applications, our energies typically depend on a continuous parameter $\epsilon > 0$, yielding a family of functions $f_\epsilon : X \to \bar{\mathbb{R}}$. In this setting, we say that f_ϵ Γ-converge to f_0 if for all sequences ϵ_j converging to 0, we have $\Gamma\text{-}\lim_j f_{\epsilon_j} = f_0$

Normally, the main steps in proving Γ-convergence are: finding the right rescaling of f_j and an appropriate metric d, identifying the Γ-limit and then proving both inequalities. Next, we present some remarks on the definition of Gamma-convergence which are taken from [2] to give some intuition on Γ-convergence.
We begin with a useful property about the behaviour of Γ-convergence with respect to perturbations.

Remark 1.5: *(Stability under continuous perturbation)*
The Gamma-limit is stable under continuous perturbation. If $\{f_j\}$ $\Gamma(d)$-converge to f_∞ and g is d-continuous, then $\{f_j + g\}$ Γ-converge to $f_\infty + g$. This property follows directly from the definition.
However, if additionally $\{g_j\}$ Γ-converge to g_∞, it does not hold in general that $\{g_j + f_j\}$ Γ-converge to $\{f_\infty + g_\infty\}$. For example, consider $X = \mathbb{R}$, $f_j = \sin^2(jx)$ and $g_j = \cos^2(jx)$. Both sequences converge in the sense of Γ-convergence to 0, yet their sum $f_j + g_j \equiv 1$ converges to $g \equiv 1$ and not to 0.

Now, we define the lower semicontinuous envelope.

Definition 1.6: *(Lower Semicontinuous Envelope)*
Let $f : X \to \bar{\mathbb{R}}$ be a function. The lower semicontinuous envelope, denoted by scf, is defined as the greatest lower semicontinuous function that is not greater than f. Explicitly, for every $x \in X$

$$scf(x) = \sup\{g(x) \mid g \text{ l.s.c. }, g \leq f\}.$$

scf is l.s.c. by Remark 1.3 . Additionally, if $f_1 \leq f_2$, then $scf_1 \leq scf_2$.

The next result is taken from [3], Thm. 2.1.2 .

Remark 1.7: *We can equivalently define the lower semicontinuous envelope building the supremum over all continuous function g with $g \leq f$ or over all Lipschitz functions g with $g \leq f$.*

The next statement is an interesting fact that distinguishes Γ-convergence from many other notions of convergence.

Remark 1.8: *(Gamma-limit of constant sequence):*
Let $f_j = f$ for all $j \in \mathbb{N}$. The Liminf Inequality yields

$$\liminf_{j \to \infty} f_j(x_j) \geq f(x)$$

for every sequence converging to x. If f is not lower semicontinuous at a point $\bar{x} \in X$, we have

$$\liminf_j f(x_j) < f(\bar{x}).$$

Consequently, the Gamma-limit of the constant sequence $f_j = f$ can be different from f itself. In fact, one can show that the lower semicontinuous envelope coincides with the Γ-limit.

$$scf(x) = \Gamma\text{-}\lim_j f(x)$$

Therefore, the Γ-limit of a constant sequence equals the original function only if the function is lower semicontinuous.

The notion of Γ-convergence strongly depends on the choice of the metric used for the Liminf and Limsup inequalities. The next Remark clarifies how different metrics can influence Γ-convergence and its limit.

Remark 1.9: *(Dependence on the metric d):*
In general, even if two distances d and d' are conparable, i.e.

$$\lim_j d'(x_j, x) = 0 \quad \Rightarrow \quad \lim_j d(x_j, x) = 0, \tag{1}$$

the existence of the Γ-limit with respect to one metric does not imply its existence with respect to the other one. However, if the Γ-limit exists for both metrics, the following inequality holds

$$\Gamma(d)\text{-}\lim_j f_j \leq \Gamma(d')\text{-}\lim_j f_j.$$

Next, we want to compare Γ-convergence to other notions of convergence and point out the relation between these different definitions.

Remark 1.10: *(Comparison with point wise and uniform limit):*
If d' denotes the discrete topology, the Γ-limit coincides with the pointwise limit. For any other metric d, condition (1) holds trivially and therefore

$$\Gamma\text{-}\lim_j f_j \leq \lim_j f_j.$$

If f_j converges uniformly to f on an open set U and f is lower semicontinuous, then f_j Γ-converge to f. The Limsup Inequality is fulfilled by the constant sequence, for the Liminf Inequality we see that if $x_j \to x$ in U, then $x_j \in U$ for j large enough, and thereby

$$\liminf_j f_j(x_j) = \lim_j (f_j(x_j) - f(x_j)) + \liminf_j f(x_j) \geq f(x).$$

The following definition is needed for the fundamental theorem of Γ-convergence below.

Definition 1.11: *(Coercive functional)*
A functional $F : X \to \bar{\mathbb{R}}$ is called coercive if for all $t \in \mathbb{R}$ the set $\{x \in X \mid F(x) \leq t\}$ is precompact, that is, if its closure is compact.
A sequence of functionals $\{F_j\}$ is called equi-coercive if for all $t \in \mathbb{R}$ there exists a compact set $K_t \subset X$ such that

$$\{x \in X \mid F_j(x) \leq t\} \subset K_t \text{ for all } j \in \mathbb{N}.$$

A sequence of functionals $\{F_j\}$ is called equi-mildly-coercive if there exists a compact set $K \subset X$, $K \neq \emptyset$ such that

$$\inf_X F_j = \inf_K F_j \text{ for all } j \in \mathbb{N}.$$

Remark 1.12: *Equi-coercive implies Equi-mildly-coercive*

Our goal in most applications is to find a minimum or maximum of a given functional. The fundamental theorem of Γ-convergence states why Γ-convergence is a powerful tool for this job.

Theorem 1.13: *(Fundamental theorem of Γ-convergence)*
Let (X,d) be a metric space, let $\{F_j\}$ be a sequence of functionals $F_j : X \to \bar{\mathbb{R}}$ and let $F : X \to \bar{\mathbb{R}}$ be its Γ-limit. Assume that the functionals $\{F_j\}$ are equi-mildly coercive. Then,

$$\inf_X F = \min_X F = \lim_{j\to\infty} \Big(\inf_X F_j\Big).$$

Additionally, if $\{x_j\}$ is a relatively compact minimizing sequence, i.e.

$$\lim_j F_j(x_j) = \liminf_j \Big(\inf_X F_j\Big)$$

then every limit point of $\{x_j\}$ is a minimizer of F.

Proof. Since $\{F_j\}$ is equi-mildly-coercive, there is a compact set K, such that $\inf_X F_j = \inf_K F_j$. Hence we can find a sequence $\{x_j\} \in K$ with

$$F_j(x_j) \leq (\inf_X F_j) + \frac{1}{j}.$$

Due to the compactness of K, there exists $\bar{x} \in X$ and a subsequence $\{x_{j_m}\}$ such that $x_{j_m} \to \bar{x} \in K$ and

$$\liminf_{j\to\infty} F_j(x_j) = \lim_{j_m\to\infty} F_{j_m}(x_{j_m}).$$

Define a sequence $\{x_j\}$ by

$$x_j := \begin{cases} x_{j_m} & \text{if } j = j_m \\ \bar{x} & \text{otherwise} \end{cases}.$$

Then $x_j \to \bar{x}$ and we compute using the Liminf-inequality of Γ-convergence and afterwards the calculations above,

$$F(\bar{x}) \leq \liminf_{j\to\infty} F_j(x_j) \leq \liminf_{j_m\to\infty} F_{j_m}(x_{j_m})$$
$$= \lim_{j_m\to\infty} F_{j_m}(x_{j_m}) = \lim_{j_m\to\infty}\big(\inf_X F_{j_m}\big) = \liminf_{j\to\infty}\big(\inf_X F_j\big).$$

By the Limsup-inequality of Gamma convergence, for every $x \in X$ there is a sequence $\{x_j\} \subset X$ such that $x_j \to x$ and

$$F(x) \geq \limsup F_j(x_j) \geq \limsup_{j\to\infty}\big(\inf_X F_j\big).$$

Hence

$$\inf_X F \geq \limsup_{j\to\infty}\big(\inf_X F_j\big).$$

Combining these results, we obtain

$$\liminf_{j\to\infty}\big(\inf_X F_j\big) \geq F(\bar{x}) \geq \inf_X F \geq \limsup_{j\to\infty}\big(\inf_X F_j\big).$$

Thus we get the existence of the limit of $\{\inf_X F_j\}$ and

$$\lim_{j\to\infty}\big(\inf_X F_j\big) = \inf_X F.$$

For the second part of the claim we can assume that there is a sequence $\{x_j\} \subset X$, with $\lim_j F_j(x_j) = \lim_j\big(\inf_X F_j\big)$ and that there is a converging subsequence $x_{j_m} \to x$. We want to show $F(x) = \inf_X F$. By setting

$$x_j := \begin{cases} x_{j_m} & \text{if } j = j_m \\ x & \text{otherwise} \end{cases}$$

we have a sequence $x_j \to x$ defined for all j and, once again using the Liminf-inequality in the first step,

$$F(x) \leq \liminf_{j\to\infty} F_j(x_j) \leq \lim_{j_m\to\infty} F_{j_m}(x_{j_m}) = \lim_{j_m}\big(\inf_X F_{j_m}\big) = \inf_X F.$$

We conclude that $x \in \operatorname{argmin}(F)$. $\square$

Remark 1.14: *In general, not every minimizer of a Γ-limit F can be approximated by a sequence of minimizers of F_j:*
For instance, consider $X = \mathbb{R}^n$ with the standard metric and $F_j := \frac{|x|}{j}$, then $F_j \to F \equiv 0$ in the sense of Γ-convergence because for all converging sequences $x_j \to x$

$$F(x) = 0 \leq \liminf_j F_j(x_j).$$

For every $x \in X$, the constant sequence satisfies

$$F_j(x_j) = \frac{|x|}{j} \to 0 = F(x),$$

which is the Limsup-inequality.
However, since all F_j attain their minimum at $x = 0$, we cannot approximate all minimizers of F, e.g. $\mathbb{R}^n$.

Remark 1.15: *Γ-convergence does not generally imply convergence of local minimizers. For example, let $X = \mathbb{R}$ and $F_j(x) = x^2 + \sin(jx)$. This sequence converges in the sense of Γ-convergence to $F(x) = x^2 - 1$. To see this, one can use Remark 1.5 and show both inequalities for $\bar{F}_j(x) = \sin(jx)$ and $\bar{F}(x) = -1$. F has a unique local minimizer at $x = 0$. $\{F_j\}$ in contrast, has a sequence of local minimizers $\{x_j\}$ converging to a point different from 0, for example $x_j = 2\pi + \frac{3}{2j}\pi \to 2\pi$.*

We close this section with a statement about the existence of Γ-limits.

Proposition 1.16: *Let (X,d) be a separable metric space and $f_j : x \to \bar{\mathbb{R}}$, $j \in \mathbb{N}$ a sequence of functions. Then there is a subsequence f_{j_k} such that the $\Gamma\text{-}\lim_k f_{j_k}(x)$ exists for all $x \in X$.*

2 Van-der-Waals-Cahn-Hilliard Model

In this section, we apply Γ-convergence to the Van-der-Waals-Cahn-Hilliard Model. This theory of phase transitions will be adapted in the main result of this thesis. We investigate the asymptotic behavior of a sequence of energy minimization problems depending on a parameter $\epsilon > 0$ and demonstrate the existence of a Γ-limit.
We want to prove the following:

Theorem 2.1: *Gurtin's Conjecture (1983)*
If u_ϵ are solutions of P_ϵ and we consider the limit $\epsilon \to 0$, then $u_\epsilon \to u_0$, where u_0 is a solution of P_0 with minimal surface area. Moreover:

$$\int_\Omega W(u_\epsilon(x)) + \epsilon^2 |\nabla u_\epsilon|^2 \, dx \quad \sim \quad \epsilon \cdot \textit{surface area of } E_0$$

Gurtin's Conjecture 2.1 was established independently by [9] and [10] by proving Gamma-convergence of the rescaled functional

$$\mathcal{F}_\epsilon : W^{1,2} \to [0, \infty) \ , \ \mathcal{F}_\epsilon(u) = \int_\Omega \frac{1}{\epsilon} W(u(x)) + \epsilon |\nabla u|^2 \, dx.$$

Consequently, our first step is to find a suitable topology for this convergence.

Theorem 2.2: *Compactness*
Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure. Assume that the double-well potential W satisfies the conditions

$$W \textit{ is continuous, } W(t) = 0 \Leftrightarrow t \in \{a, b\} \textit{ for some } a, b \in \mathbb{R} \textit{ with } a < b, \tag{H1}$$

and

$$\textit{there exists } L > 0 \textit{ and } T > 0 \textit{ such that } W(t) \geq L|t| \quad \forall t \in \mathbb{R} : |t| \geq T. \tag{H2}$$

Let $\epsilon_n \to 0^+$ and assume $\{u_n\} \subset W^{1,2}(\Omega)$ satisfies

$$M := \sup_n \mathcal{F}_{\epsilon_n}(u_n) < \infty.$$

Then

$$\exists \textit{ subsequence } \{u_{n_k}\} \textit{ of } \{u_n\} \textit{ and } u \in BV(\Omega, \{a, b\}) \textit{ such that } u_{n_k} \to u \textit{ in } L^1(\Omega).$$

This proof is taken from [8]. Nevertheless, we give the whole argument because we will adapt it in the next section.

Proof. We start with showing that $\{u_n\}$ is bounded in $L^1(\Omega)$. By (H2),

$$\int_{\{|u_n| \geq T\}} |u_n| \leq \frac{1}{L} \int_{\{|u_n| \geq T\}} W(u_n(x)) \leq \frac{M}{L} \epsilon_n.$$

Since Ω has finite measure,

$$L \int_\Omega |u_n| = L \int_{\{|u_n| \geq T\}} |u_n| + L \int_{\{|u_n| < T\}} |u_n|$$

$$\leq \int_{\{|u_n| \geq T\}} W(u_n) + LT\mathcal{L}^n(\Omega) \leq M\epsilon_n + LT\mathcal{L}^n(\Omega).$$

Thus, $\{u_n\}$ is bounded in $L^1(\Omega)$. Next, we show the equi-integrability of $\{u_n\}$. Fix $\gamma > 0$ and find $N_\epsilon \in \mathbb{N}$ large enough that $\frac{M}{L}\epsilon_n \leq \frac{1}{2}\gamma$ for all $n \geq N_\epsilon$. Then, for such n,

$$\int_{\{|u_n|\geq T\}} |u_n| \leq \frac{1}{2}\gamma.$$

If $E \subset \Omega$ is measurable and $\mathcal{L}^n(E) \leq \frac{1}{2T}\gamma$,

$$\int_{E\cap\{|u_n|\leq T\}} |u_n| \leq T\mathcal{L}^n(E) \leq \frac{1}{2}\gamma.$$

Combining the last two inequalities yields for all $n \geq N_\epsilon$

$$\int_E |u_n| \leq \int_{E\cap\{|u_n|\geq T\}} |u_n| + \int_{E\cap\{|u_n|<T\}} |u_n| \leq \frac{1}{2}\gamma + \frac{1}{2}\gamma.$$

Finally, since the finite family $\{u_1, ..., u_n\}$ is equi-integrable, there exists $\delta_1 > 0$ such that

$$\int_E |u_n| \leq \gamma$$

for all $n \leq N_\epsilon$ and for every measurable set $E \subset \Omega$ with $\mathcal{L}^n(E) \leq \delta_1$. Setting $\delta :=$ $\min\{\delta_1, \frac{1}{2T}\gamma\}$ implies that $\{u_n\}$ is equi-integrable.
We claim that in our case it's enough to prove pointwise convergence of a subsequence to obtain strong convergence. By Egorov's Theorem, if $u_n \to u$ pointwise on Ω, then there is a measurable set $B \subset \Omega$ such that $\mathcal{L}^n(B) < \epsilon$ and $u_n \to u$ uniformly on $\Omega \backslash B$ for every $\epsilon > 0$. This implies that u_n converge in measure to u on Ω as well. And since we know that $\{u_n\}$ is equi-integrable, we can use Vitali's Convergence Theorem to conclude that $u_n \to u$ in $L^1(\Omega)$.
Hence we will study pointwise convergence of $\{u_n\}$. For $K > 0$ define

$$W_1(t) := \min\{W(t), K\}, \ t \in \mathbb{R}$$

and

$$f(t) := 2\int_a^t \sqrt{W_1(s)}\ ds, \ t \in \mathbb{R}.$$

Since $0 \leq W_1 \leq W$, for every $n \in \mathbb{N}$, we have, using the chain rule in $W^{1,2}(\Omega)$, that

$$\mathcal{F}_n(u_n) \geq 2\int_\Omega \sqrt{W_1(u_n(x))}|\nabla u_n(x)|\ dx = \int_\Omega |\nabla(f \circ u_n)(x)|\ dx.$$

Note that the function f is Lipschitz continuous ($f'(t) = 2\sqrt{W_1(t)}$). Then, by assumption

$$\sup_n \int_\Omega |\nabla(f \circ u_n)|\ dx \leq M.$$

Moreover, since $\mathrm{Lip}(f) \leq 2\sqrt{K}$ and $f(a) = 0$,

$$|f(u_n(x))| = |f(u_n(x)) - f(a)| \leq 2\sqrt{K}|u_n(x) - a|$$

for $\mathcal{L}^n$-a.e. $x \in \Omega$ and for all $n \in \mathbb{N}$. Since $\{u_n\}$ is bounded in $L^1(\Omega)$, it follows that the sequence $\{f \circ u_n\}$ is bounded in $L^1(\Omega)$. By Rellich-Kondrachov theorem, there exists a subsequence of $\{u_n\}$ (not relabeled) and a function $\omega \in L^1(\Omega)$ such that

$$\omega_n := f \circ u_n \to \omega \text{ in } L^1_{loc}(\Omega).$$

Since the total variation of Radon measures is lower semicontinuous with respect to the strong L^1_{loc}-convergence,

$$\|D\omega\|_{M_b(\Omega,\mathbb{R}^n)} = \int_\Omega |\nabla\omega| \leq \liminf_n \int_\Omega |\nabla\omega_n| \leq M$$

and $\omega \in BV(\Omega)$. By taking a further subsequence if necessary, we may assume that $\omega_n(x) \to \omega(x)$ and that $W(u_n(x)) \to 0$ for $\mathcal{L}^n$-a.e $x \in \Omega$ (because $\int_\Omega \frac{1}{\epsilon_n} W(u_n(x)) \leq \mathcal{F}_{\epsilon_n}(u_n) < M$). Since the function $W_1(t) > 0$ for all $t \notin \{a, b\}$, it follows from the definition of f that f is strictly increasing and continuous. Thus, its inverse f^{-1} is continuous and for $\mathcal{L}^n$-a.e. $x \in \Omega$

$$u_{n_k}(x) = f^{-1}(\omega_k(x)) \to f^{-1}(\omega(x)) =: u(x).$$

It follows by (H1) and the fact that $W(u_{n_k}) \to 0$ for $\mathcal{L}^n$-a.e. $x \in \Omega$, that $u(x) \in \{a, b\}$ for $\mathcal{L}^n$-a.e. $x \in \Omega$.
In turn, $\omega(x) \in \{f(a), f(b)\} = \{0, f(b)\}$ for $\mathcal{L}^n$-a.e. $x \in \Omega$, and so we may write

$$\omega = f(b)\chi_E$$

for a set $E \subset \Omega$. Since $\omega \in BV(\Omega)$ and Ω has finite measure, we have $\chi_E \in BV(\Omega)$ and

$$u = b\chi_E + a(1 - \chi_E)$$

belongs to $BV(\Omega)$. □

Thus the natural metric for our problem is induced by the L^1-norm. Accordingly, we extend our functional $\mathcal{F}_\epsilon$ to $L^1(\Omega)$ in the following way:

$$\mathcal{F}_\epsilon(u) = \begin{cases} \int_\Omega \frac{1}{\epsilon} W(u(x)) + \epsilon|\nabla u|^2 \, dx & \text{if } u \in W^{1,2}(\Omega) \text{ and } \int_\Omega u \, dx = m, \\ +\infty & \text{otherwise } L^1(\Omega). \end{cases} \tag{2}$$

We will show that $\mathcal{F}_\epsilon$ converges in the sense of Gamma-convergence to the functional

$$\mathcal{F}_0(u) = \begin{cases} c_w \; P(E, \Omega) & \text{if } u \in \mathrm{BV}(\Omega, \{a, b\}) \text{ and } \int_\Omega u \, dx = m, \\ +\infty & \text{otherwise in } L^1 \end{cases}$$

with respect to the strong-L^1-topology. Here $c_w := 2\int_a^b \sqrt{W(t)} \, dt$ and $E := \{x \in \Omega \mid u(x) = b\}$.

Theorem 2.3: (Γ-*convergence of* $\mathcal{F}_\epsilon$ *to* $\mathcal{F}_0$) :
Assume the double-well potential W satisfies conditions (H1) and (H2) and let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. Then $\mathcal{F}_\epsilon \to \mathcal{F}_0$ in the sense of Gamma-convergence with respect to the strong-L^1-topology, i.e. we have:
For all sequences $\{u_n\}$ with $u_n \to u$ where $u \in L^1(\Omega)$ and all sequences $\epsilon_n \to 0^+$, we have

$$\mathcal{F}_0(u) \leq \liminf_{\epsilon_n \to 0} \mathcal{F}_{\epsilon_n}(u_n),$$

and for all $u \in L^1(\Omega)$ and all sequences $\epsilon_n \to 0^+$, there exists a sequences $\{u_n\} \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and

$$\limsup_{n \to \infty} \mathcal{F}_0(u) \leq \mathcal{F}_0(u).$$

The detailed proof of this theorem, as well as the one for Gurtin's conjecture as a corollary, can be found in [8]. This establishes the desired Γ-convergence result for the Van der Waals–Cahn–Hilliard model.

3 Weighted BV-space

In the fourth section, we want to manipulate the model by a weight function as done in [7]. There, they can avoid a detailled study of the weighted spaces due to the assumption $\omega > c \geq 0$. As we will see later in Remark 3.6, due to this assumption, they basically work with BV-functions. Since we consider more general assumptions, we have to state some general results about the weighted setting. We will start by defining the weighted BV space and the weighted total variation. For all what follows, let Ω be a bounded open subset of $\mathbb{R}^n$.

3.1 Definitions and first remarks

Definition 3.1: *We define a weight ω as a function in $C(\Omega) \cap L^\infty(\Omega)$ with $\omega > 0$.*

For some of the statements in this section, we will require some different conditions on ω Whenever not specified, we will work under the above assumptions. We define the weight ω like this because we will need these assumptions for our main result (section 4).

Definition 3.2: *The associated weighted BV space $BV_\omega(\Omega)$ is defined as the set of all functions in $L^1(\Omega, \omega)$ with finite weighted total variation:*

$$TV_\omega(u) \ \big(= TV_\omega(u,\Omega)\big) = \sup\left\{ \int_\Omega u \ div(\psi) \ \middle| \ \psi \in C_c^1(\Omega), |\psi| \leq \omega \ everywhere \right\} < \ \infty.$$

Here, the space $L^1(\Omega, \omega)$ is defined by the set of all functions which are integrable with respect to the measure $\omega(x)dx$.

The only difference between this definition and the one used in [7] is that, in our case, the weight function can be 0 at the boundary of Ω. Away from the boundary, the continuity of ω ensures a positive lower bound on the (now) compact subset of Ω.
The next lemma gives us a tool to approximate the weighted total variation of Ω by the one of domains approximating Ω.

Lemma 3.3: *Let $\{V_n\}$ be a sequence of open sets such that $V_1 \subset V_2 \subset ... \subset V_n \subset ...$, $\Omega = \bigcup_{n=1}^\infty V_n$ and $u \in BV_\omega(\Omega)$. Then*

$$\lim_{n\to\infty} TV_\omega(u, V_n) = TV_\omega(u, \Omega)$$

Proof. $\mathrm{TV}_\omega(u, V_n) \leq TV_\omega(u, \Omega)$ holds trivially because $V_n \subset \Omega$ implying that all admissible test functions for V_n are also admissible for Ω.
For the reverse inequality, we take a maximizing sequence $\{\psi_k\}$ of $\mathrm{TV}_\omega(u, \Omega)$, i.e.

$$\mathrm{TV}_\omega(u, \Omega) = \int_\Omega u \ \mathrm{div}(\psi_k).$$

Taking k big enough for every n, we can find a subsequence such that $\mathrm{supp}(\psi_{k_n}) \subset V_n$, yielding

$$\mathrm{TV}_\omega(u, \Omega) = \lim_{k\to\infty} \int_\Omega u \ \mathrm{div}(\psi_k) = \lim_{k\to\infty} \int_{V_n} u \ \mathrm{div}(\psi_k) \leq \lim_{k\to\infty} \mathrm{TV}_\omega(u, V_n)$$

□

The next statement, Remark 1 from [7], will be useful in calculations later on.

Remark 3.4: *In the above definition, the set of admissible test functions can be expanded to Lipschitz functions with compact support without changing the value of the total weighted variation, i.e.*

$$TV_\omega(u) \;= sup\Big\{\int_\Omega u\ div(\psi)\ \Big|\ \psi\ Lipschitz,\ supp(\psi)\subset\subset\Omega, |\psi|\le\omega\ everywhere\Big\}.$$

The next result is taken from [3], Prop. 2.1.1 .

Remark 3.5: *Let u be a measurable function and $\omega \geq 0$ an arbitrary weight function. Then*

$$TV_\omega(u) = TV_{sc\ \omega}(u).$$

Since the lower semicontinuous envelope of a l.s.c. function is the function itself, it is natural to assume ω to be lower semicontinuous when working with $TV_\omega(u)$.

Proof. We use Remark 1.7 and the fact that $|\psi| \le sc\ \omega$ when ψ Lipschitz and $|\psi| \le \omega$ to obtain

$$TV_\omega(u) = \sup\Big\{\int_\Omega u\ div(\psi)\ \Big|\ \psi \text{ Lipschitz},\ supp(\psi)\subset\subset\Omega, |\psi|\le\omega\Big\}$$

$$= \sup\Big\{\int_\Omega u\ div(\psi)\ \Big|\ \psi \text{ Lipschitz},\ supp(\psi)\subset\subset\Omega, |\psi|\le sc\ \omega\Big\} = TV_{sc\ \omega}(u)$$

□

In the next Remark, we will see how the weaker assumptions on ω can change $BV_\omega(\Omega)$ in comparison to $BV(\Omega)$ and thereby to the situation of [7].

Remark 3.6: *(Comparison between BV and weighted BV)*
We always have

$$BV(\Omega) \subset BV_\omega(\Omega).$$

Indeed, for $u \in BV(\Omega)$ we have

$$\int_\Omega \omega\ |u| \le \|\omega\|_\infty\ \|u\|_1$$

because $\omega \in L^\infty(\Omega)$. Thus, by the definition of the total variation

$$TV_\omega(u) \le TV(u)\ \|\omega\|_\infty.$$

If we additionally assume $\omega \geq c > 0$, then for $u \in BV_\omega(\Omega)$

$$\int_\Omega \omega\ |u| \ge c\ \int_\Omega |u|$$

and for every $\psi \in C_c^1(\Omega)$ with $\psi \le 1$,

$$\int_\Omega u\ div(\psi) = \int_\Omega u\ div(\frac{1}{c}\ c\ \psi) = \frac{1}{c}\int_\Omega u\ div(c\ \psi) \le \frac{1}{c}\ TV_\omega(\Omega).$$

Hence $BV(\Omega) = BV_\omega(\Omega)$.
However , $BV(\Omega)$ and $BV_\omega(\Omega)$ do not coincide in the case in which ω is only assumed to be positive in Ω.
For simplicity, assume $n = 1$ and $\Omega = (0,1)$. If we consider the function $u(x) = \frac{1}{x}$,

we know that $u \notin L^1(\Omega)$ *and therefore* $u \notin BV(\Omega)$*. But if we define* $\omega(x) = x^2$*, we immediately see that* $u \in L^1(\Omega, \omega)$ *and for a test function* $\psi \in C_c^1(\Omega)$ *with* $|\psi| \leq \omega$*, we can compute:*

$$\int_\Omega u\ div(\psi) = -\int_\Omega div(u)\psi + \big[u\ \psi\big]_0^1 \leq 1 + \int_\Omega \frac{1}{x^2}\ x^2 = 2 < \infty.$$

Thus $TV_\omega(u) < \infty$ *and* $u \in BV_\omega(\Omega)$*.*
But even if $u \in L^1(\Omega)$*, the following example shows that it's possible for* u *to be in* $BV_\omega(\Omega)$ *but not in* $BV(\Omega)$*. We define*

$$\omega(x) = \frac{1}{3}\Big(\sin\Big(\frac{1}{x}\Big)+1 + x^2\Big),$$

an oscillating function between 0 and 1 with a sequence of local minima x_k *with accumulation point* $x = 0$*. The local minima are*

$$\frac{1}{x_k} = (2k+1)\pi + \frac{\pi}{2} \Leftrightarrow x_k = \frac{1}{(2k+1)\pi + \frac{\pi}{2}}$$

and we have $\omega(x_k) = x_k^2 \to 0$ *for* $k \to \infty$*. Next, we define a function* u *jumping at these points* x_k *between 0 and 1 by*

$$u(x) := \sum_{i=1}^{\infty} \chi_{[x_{2i}, x_{2i+1}]}.$$

Since $u(x) \leq 1$*, we have* $u \in L^1(\Omega)$ *but*

$$TV(u) = \#\big\{jump\ points\ of\ u\big\} = \infty$$

and thereby $u \notin BV(\Omega)$*. For the calculation of* $TV_\omega(u)$*, we have to include the weight function into the sum of the jump points. We obtain*

$$TV_\omega(\Omega) = \sum_{i=1}^{\infty} \omega(x_i) = \frac{1}{3}\sum_{i=1}^{\infty} \frac{1}{((2i+1)\pi + \frac{1}{2})^2} < \infty$$

and thereby $u \in BV_\omega(\Omega)$*.*

The next statement is taken from [3], section 2.3 .

Remark 3.7: *We can extend* $TV_\omega(u)$ *to general subsets* E *of* Ω *by*

$$TV_\omega(u)(E) = \inf\Big\{TV_\omega(u, V),\ V\ open, E \subset V\Big\}.$$

Then, by Remark 2.3.1 and Theorem 2.3.2 of [3], TV_ω *is a Borel regular outer measure.*

We conclude this section with a result taken from [3] about the case with the even weaker assumption $\omega \geq 0$. We study under which assumptions the norm TV_ω (as an outer measure) actually acts on the set Ω_0 defined below. Note that by Remark 3.5, we can assume without loss of generality that ω is l.s.c., so that the set Ω_0 below will be open.

Lemma 3.8: *Let* $\omega \geq 0$ *and*

$$\Omega_0 = \big\{x \in \Omega\ \big|\ \omega(x) > 0\big\}.$$

If $u \in L^1_{loc}(\Omega)$*, then*

$$TV_\omega(u, \Omega) = TV_\omega(u, \Omega_0).$$

Proof. We follow the proof of Lemma 3.1.8 of [3]. Let $u \in L^1_{loc}(\Omega)$ and ψ be a Lipschitz function with compact support in Ω such that $|\psi| \leq \omega$, and define

$$\Omega_\psi = \{x \in \Omega \mid |\psi(x)| > 0\}$$

and

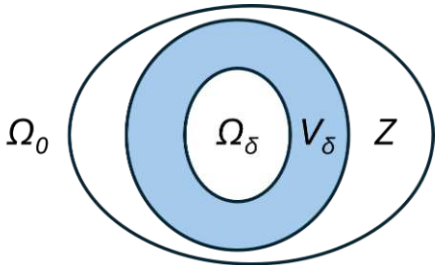
$$Z = \Omega \backslash \Omega_\psi = \{x \in \Omega \mid |\psi(x)| = 0\}.$$

Observe that Z is closed in Ω, the set Ω_ψ is open and $\Omega_\psi \subset \Omega_0$. Define for $\delta > 0$

$$\Omega_\delta = \{x \in \Omega_0 \mid \operatorname{dist}(x, Z) > \delta\}$$

and

$$V_\delta = \Omega_\psi \backslash \Omega_\delta.$$



It holds $V_\delta \subset \Omega_\psi \subset \operatorname{supp}(\psi) \subset\subset \Omega$ and hence that u is integrable on V_δ. It follows that $\lim_{\delta \to 0} \int_{V_\delta} |u| \; dx = 0$.
For every $\delta > 0$, there is a Lipschitz map η_δ mapping from Ω to $[0, 1]$ with Lipschitz constant $\frac{2}{\delta}$ and $\eta_\delta \equiv 1$ on a neighborhood of Ω_δ and $\eta_\delta \equiv 0$ on a neighborhood of Z. Also, $\operatorname{supp}(\nabla \eta_\delta) \subset V_\delta \subset \Omega_\psi$ and $|\psi| \leq L\delta$ on V_δ, where L is the Lipschitz constant of ψ. Additionally

$$div(\eta_\delta \psi) = \nabla \eta_\delta \cdot \psi + \eta_\delta \; div(\psi).$$

Using also Lemma 3.1.6 of [3] which states that

$$\int_\Omega u \; div(\psi) \; dx = \int_{\Omega_\psi} u \; div(\psi) \; dx,$$

we can compute:

$$\begin{aligned}
\int_{\Omega_\psi} u \; div(\eta_\delta \psi) \; dx &= \int_{\Omega_\psi} \eta_\delta u \; div(\psi) \; dx + \int_{\Omega_\psi} u \; (\nabla \eta_\delta \cdot \psi) \; dx \\
&= \int_{\Omega_\psi} u \; div(\psi) \; dx + \int_{V_\delta} (\eta_\delta - 1) u \; div(\psi) \; dx + \int_{V_\delta} u \; (\nabla \eta_\delta \cdot \psi) \; dx \\
&\geq \int_{\Omega_\psi} u \; div(\psi) \; dx - nL \int_{V_\delta} |u| \; dx - \frac{2}{\delta} L\delta \int_{V_\delta} |u| \; dx \\
&= \int_{\Omega_\psi} u \; div(\psi) \; dx - (2+n)L \int_{V_\delta} |u| \; dx \\
&= \int_\Omega u \; div(\psi) \; dx - (2+n)L \int_{V_\delta} |u| \; dx.
\end{aligned}$$

Hence, it follows

$$\begin{aligned}\int_\Omega u\ div(\psi)\ dx &= \lim_{\delta\to 0}\Big(\int_{\Omega_\psi} u\ div(\psi) + (2+n)L\int_{V_\delta}|u|\ dx\Big)\\ &\le TV_\omega(u,\Omega_\psi) + (2+n)L\lim_{\delta\to 0}\int_{V_\delta}|u|\ dx\\ &= TV_\omega(u,\Omega_\psi)\le TV_\omega(u,\Omega_0)\end{aligned}$$

We obtain

$$TV_\omega(u,\Omega)\le TV_\omega(u,\Omega_0)$$

by taking the supremum over all Lipschitz functions ψ with compact support. Since the other inequality holds trivially, we are done.

□

3.2 Sets of finite weighted perimeter

In this section we try to find a criteria to prove when a set is of finite weighted perimeter. The first Lemma is an important formula for the weighted total variation which is not only true in the setting of [7], but also in our case. The next Lemma is taken from [7].

Lemma 3.9: *For $E\neq\Omega$, let $E\subset\Omega$ be a regular bounded open set with C^2-boundary. Then*

$$TV_\omega(\chi_E) = \int_{\partial E}\omega\ d\mathcal{H}^{n-1}.$$

Proof. We follow the proof of Thm. 3 of [7]. We show both inequalities starting with "$\le$". Let $g\in C^1_c(\Omega,\mathbb{R}^n)$. Then

$$\int_\Omega \chi_E\ \mathrm{div}(g) = \int_{\partial E} g\cdot\overrightarrow{n}\ d\mathcal{H}^{n-1}.$$

And if $|g(x)|\le\omega$, we have that

$$\int_{\partial E} g\cdot\overrightarrow{n}\ d\mathcal{H}^{n-1}\le\int_{\partial E}\omega\ d\mathcal{H}^{n-1}$$

implying

$$\mathrm{TV}_\omega(\chi_E)\le\int_{\partial E}\omega\ d\mathcal{H}^{n-1}.$$

For the reverse inequality, we extend $\overrightarrow{n}$, the outer unit normal defined on ∂E which fulfills $|\overrightarrow{n}|=1$, to a function $\overrightarrow{N}\in C^1(\mathbb{R}^n,\mathbb{R}^n)$ such that $|\overrightarrow{N}|\le 1$ (see [6], page 5). Next, we set for $\eta\in C^1_c(\Omega)$ with $|\eta|\le\omega$, $g=\eta\overrightarrow{N}$ and compute

$$\int_\Omega\chi_E\ \mathrm{div}(g) = \int_{\partial E}\eta\ d\mathcal{H}^{n-1}.$$

Thus

$$\mathrm{TV}_\omega(\chi_E)\ge\sup\Big\{\int_{\partial E}\eta\ d\mathcal{H}^{n-1}\ \Big|\ \eta\in C^1_c(\Omega),|\eta|\le\omega\Big\}.$$

We can approximate ω monotonically from below by a sequence η_k in $C^1_c(\Omega)$. We conclude using the monotone convergence theorem

$$\mathrm{TV}_\omega(\chi_E)\ge\sup\Big\{\int_{\partial E}\eta\ d\mathcal{H}^{n-1}\ \Big|\ \eta\in C^1_c(\Omega),|\eta|\le\omega\Big\}=\int_{\partial E}\omega\ d\mathcal{H}^{n-1}.$$

□

The rest of this section is taken from [3], section 2.2 .

Remark 3.10: *If E is not a regular bounded open set with C^2-boundary, we can use the measure theoretic boundary $\partial_* E$ of E to get*

$$TV_\omega(\chi_E) = \int_{\partial_* E \ \cap \ \Omega} \omega \ d\mathcal{H}^{n-1}.$$

$\partial_* E$ is derived from the Lebesque measure but since ω is locally bounded away from zero and locally bounded from above, we can equivalently define (check Remark 2.2.1 of [3])

$$\partial_* E = \left\{ x \in \Omega \ \middle| \limsup_{r \to 0} \frac{\int_{B(x,r)\cap E} \omega \ dx}{\int_{B(x,r)} \omega \ dx} > 0, \ \limsup_{r \to 0} \frac{\int_{B(x,r)\setminus E} \omega \ dx}{\int_{B(x,r)} \omega \ dx} > 0 \right\}.$$

Here the densities are taken with respect to the measure $\omega \ dx$ which matches better to our situation.
This gives us a nice characterization: E is a set of finite weighted perimeter if $\int_{\partial_* E} \omega \ d\mathcal{H}^{n-1} < \infty$.

In the case without weights, a set E is of finite perimeter if and only if $\mathcal{H}^{n-1}(\partial_* E) < \infty$. In the last remark, we only saw a criterion where we integrate with respect to the measure $\omega \ d\mathcal{H}^{n-1}$. To improve this, we define

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(E).$$

Here, the right-hand side is defined as

$$\mathcal{H}^h_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} \frac{\mathcal{L}^n(B(x_i, r_i))}{r_i} \ \middle| \ E \subset \bigcup^{\infty} B(x_i, r_i), \ r_i < \delta \right\}.$$

One can show that this defines a Borel regular outer measure on Ω (compare Prop. 2.2.3 of [3]). Since we want to obtain a result similar to the case without weights we compare $\mathcal{H}^h$ and $\omega \ d\mathcal{H}^{n-1}$:

Lemma 3.11: *Let ω be continuous and $E \subset \Omega$. Then*

$$\mathcal{H}^h(E) = c_n \int_E \omega \ d\mathcal{H}^{n-1}$$

where $c_n = \mathcal{L}^n(B(0,1))$.

Proof. First let us assume that we can find a $\delta > 0$ such that the δ-neighborhood of E, denoted by E_δ, is contained in a compact subset of Ω. Under this premise, we can find positive constants a, b with $0 < a < \omega < b$ on E_δ.
We begin by showing that $\mathcal{H}^h$ and $\omega \ d\mathcal{H}^{n-1}$ are comparable on E_δ, i.e. that for $A \subset E_\delta$

$$a \ \mathcal{H}^{n-1}(A) \leq \frac{1}{c_n} \mathcal{H}^h(A) \leq b \ \mathcal{H}^{n-1}(A).$$

Take a covering of E_δ of balls $B(x_i, r_i)$ with radii less than δ. We can estimate

$$a \sum_{i=1}^{\infty} r_i^{n-1} = \frac{a}{c_n} \sum_{i=1}^{\infty} \frac{\mathcal{L}^n(B(x_i, r_i))}{r_i} \leq \frac{1}{c_n} \sum_{i=1}^{\infty} \frac{\int_{B(x_i,r_i)} \omega \ d\mathcal{L}^n}{r_i}$$

$$\leq \frac{b}{c_n} \sum_{i=1}^{\infty} \frac{\mathcal{L}^n(B(x_i, r_i))}{r_i} = b \sum_{i=1}^{\infty} r_i^{n-1}.$$

This holds for all admissible covers yielding

$$a\ \mathcal{H}^{n-1}_\delta(A) \leq \frac{1}{c_n}\mathcal{H}^h_\delta(A) \leq b\ \mathcal{H}^{n-1}_\delta(A).$$

Hence we can obtain the comparability by passing to the limit $\delta \to 0$.
With that at hand, since

$$\mathcal{H}^{n-1}(E) = \infty \Leftrightarrow \int_E \omega\ \mathcal{H}^{n-1} = \infty$$

and

$$\mathcal{H}^{n-1}(E) = 0 \Leftrightarrow \int_E \omega\ \mathcal{H}^{n-1} = 0,$$

it follows that

$$\mathcal{H}^h(E) = \int_E \omega\ \mathcal{H}^{n-1}.$$

In fact, consider the case when $\mathcal{H}^{n-1}(E) < \infty$. One can show that $\mathcal{H}^{n-1}$ is a Radon measure on $\mathbb{R}^n$, thus we can use Vitali's covering theorem.
We define for $t < \frac{a}{2}$

$$\mathcal{F}_t = \left\{ \bar{B}(x,r) \;\middle|\; x \in E,\ r < \delta,\ |\omega - \omega(x)| < t \text{ on } \bar{B}(x,r) \right\}.$$

Due to the continuity of ω, $\mathcal{F}_t$ is a fine cover for each t, i.e. for all $x \in E$

$$\inf\left\{ r > 0 \;\middle|\; \bar{B}(x,r) \in \mathcal{F}_t \right\} = 0.$$

By Vitali's covering theorem, there is a disjoint collection of balls B_i such that

$$\mathcal{H}^{n-1}\left(E \backslash \bigcup_i B_i \right) = 0$$

and hence

$$\mathcal{H}^h\left(E \backslash \bigcup_i B_i \right) = 0.$$

Now, we estimate

$$\begin{aligned}
\mathcal{H}^h(E) &= \sum_i \mathcal{H}^h(E \cap B_i) \leq c_n \sum_i \big(\omega(x_i) + t\big)\mathcal{H}^{n-1}(E \cap B_i) \\
&\leq c_n \sum_i \left(\frac{\omega(x_i) + t}{\omega(x_i) - t} \int_{E\cap B_i} \omega\ d\mathcal{H}^{n-1} \right) \\
&= c_n \sum_i \left(1 + \frac{2t}{\omega(x_i) - t} \right) \int_{E\cap B_i} \omega\ d\mathcal{H}^{n-1} \\
&\leq c_n \left(1 + \frac{2t}{a-t} \right) \int_E \omega\ d\mathcal{H}^{n-1}.
\end{aligned}$$

Here we know that $\omega(x_i) - t > 0$ because $\omega > a > 2t$. With an analogous argument, we can obtain

$$c_n \int_E \omega\ d\mathcal{H}^{n-1} \leq \left(1 + \frac{2t}{a-t} \right) \mathcal{H}^h(E).$$

Since these inequalities hold for all $t < \frac{a}{2}$, we can pass to the limit and obtain the desired equality

$$\mathcal{H}^h(E) = c_n \int_E \omega \; d\mathcal{H}^{n-1}.$$

For the general setting, we approximate Ω by a sequence of sets $F_1 \subset\subset F_2 \subset\subset ... \subset\subset F_k ...$ such that $E \cap F_k$ satisfies the assumptions from before. Then

$$\mathcal{H}^h(E) = \lim_{k\to\infty} \mathcal{H}^h(E \cap F_k) = \lim_{k\to\infty} c_n \int_{E \cap F_k} \omega \; d\mathcal{H}^{n-1} = c_n \int_E \omega \; d\mathcal{H}^{n-1}.$$

Applying Remark 3.10 yields

$$\mathrm{Per}_\omega(E) = \frac{1}{c_n}\mathcal{H}^h(\partial_* E \cap \Omega).$$

Hence E is a set of finite weighted perimeter if $\mathcal{H}^h(\partial_* E \cap \Omega) < \infty$.

□

3.3 Smooth approximation

The next statement gives a second characterization for the weighted total variation which also implies that we can approximate the weighted total variation of functions in weighted BV by the one of smooth functions. It is taken from [3], section 3.2 . This is useful because smooth functions are, compared to weighted BV, well understood.

Theorem 3.12: *Let $\omega > 0$ be continuous and $u \in L^1_{loc}(\Omega)$. Then*

$$TV_\omega(u) = \inf\left\{ \liminf_{k\to\infty} \int_\Omega |\nabla u_k| \; \omega \; dx \;\middle|\; (u_k - u) \to 0 \; in \; L^1(\Omega, \omega), \; u_k \in Lip_{loc}(\Omega) \right\} =: \|D_\omega u\|_M(\Omega).$$

Here, the space $Lip_{loc}(\Omega)$ is defined by

$$Lip_{loc}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \Omega \textit{ is covered by open sets in which } u \textit{ is Lipschitz} \right\}.$$

Proof. We follow the arguments of [3], Prop 3.2.1, Lemma 3.2.2 and Theorem 3.2.3 . First, we will show that

$$TV_\omega(u) \le \|D_\omega u\|_M(\Omega).$$

Let $u_k \in \mathrm{Lip}_{loc}(\Omega)$ be a sequence such that $(u_k - u) \to 0$ in $L^1(\Omega, \omega)$ and ψ be a Lipschitz function with compact support and $|\psi| \le \omega$. We know, that also $div(\psi)$ has compact support and since $\omega > c$ on compact subsets of Ω, we also have this inequality on $\mathrm{supp}(\psi)$. Thus, we can obtain

$$(u_k - u) \to 0 \text{ in } L^1(\mathrm{supp}(div(\psi)))$$

and since ψ is Lipschitz, $div(\psi)$ is bounded. Hence,

$$\begin{aligned} \int_\Omega u \; div(\psi) \; dx &= \lim_{k\to\infty} \int_\Omega u_k \; div(\psi) \; dx \\ = - \lim_{k\to\infty} \int_\Omega (\nabla u_k \cdot \psi) \; dx &\le \liminf_{k\to\infty} \int_\Omega |\nabla u_k||\psi| \; dx \\ &\le \liminf_{k\to\infty} \int_\Omega |\nabla u_k| \; \omega \; dx. \end{aligned}$$

By taking the supremum over all Lipschitz functions with compact support, we obtain

$$TV_\omega(u) \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k| \ \omega \ dx.$$

This holds for all sequences $\{u_k\}$, thus we can conclude

$$TV_\omega(u) \leq \|D_\omega u\|_M(\Omega).$$

Next, we prove equality for the special case where $u \in BV(\Omega)$.
Since $u \in BV(\Omega)$, we know that $\|Du\|_{M_b}(\Omega)$ is finite. Also, we can find a sequence $\{u_k\}$ of smooth functions, such that $u_k \to u$ in $L^1(\Omega)$, $\|Du_k\|(\Omega) \to \|Du\|(\Omega)$ and $\|Du_k\| \rightharpoonup \|Du\|$ weakly-$\star$ in $M_b(\Omega; \mathbb{R}^n)$. Let $V \subset\subset \Omega$ be an arbitrary open set. Because of Theorem 3.16, it holds

$$TV_\omega(u, V) = \int_V \omega \ d\|Du\|$$

And thus by the weak-$\star$ convergence of $\|Du_k\|$ to $\|Du\|$,

$$TV_\omega(u, V) = \int_V \omega \ d\|Du\| = \lim_{k\to\infty} \int_V \omega \ d\|Du_k\| = \lim_{k\to\infty} \int_V |\nabla u_k| \ \omega \ dx.$$

We know that $u_k \to u$ in $L^1(\Omega)$ and since ω is bounded, it follows that $u_k \to u$ in $L^1(\Omega, \omega)$ as well, yielding

$$\|D_\omega u\|_M(V) \leq \lim_{k\to\infty} \int_V |\nabla u_k| \ \omega \ dx.$$

We obtain

$$\|D_\omega u\|_M(\Omega) \leq \lim_{k\to\infty} \int_\Omega |\nabla u_k| \ \omega \ dx$$

by approximating Ω from within with open sets with compact closure. Together with the first part of the proof, we obtain equality for the special case where $u \in BV(\Omega)$.
Finally, we want to prove the claim in the general setting. If $TV_\omega(u) = \infty$, the claim is trivially true due to the inequality proven in the first part of the proof. If $TV_\omega(u) < \infty$, we obtain with Theorem 3.16 that $u \in BV_{loc}(\Omega)$. Since ω is continuous, we can once again approximate Ω by a sequence of open sets V_n. On V_n, $u \in BV(V_n)$ and we use the proof for this special case, Lemma 3.3 and Lemma 3.13 to compute

$$TV_\omega(\Omega) = \lim_{n\to\infty} TV_\omega(V_n) = \lim_{n\to\infty} \|D_\omega u\|_M(V_n) = \|D_\omega u\|_M(\Omega)$$

□

Similar to Lemma 3.3, we can approximate $\|D_\omega u\|_M(\Omega)$.

Lemma 3.13: *Let $\{V_n\}$ be a sequence of open sets such that $V_1 \subset V_2 \subset ... \subset V_n \subset ...$, $\Omega = \bigcup_{n=1}^\infty V_n$ and $u \in BV_\omega(\Omega)$. Then*

$$\lim_{n\to\infty} \|D_\omega u\|_M(V_n) = \|D_\omega u\|_M(\Omega)$$

Proof. If $\psi \in \mathrm{Lip}_{loc}(\Omega)$, we know that $\psi_{|V_n} \in \mathrm{Lip}_{loc}(V_n)$ and hence

$$\|D_\omega u\|_M(\Omega) \geq \lim_{n\to\infty} \|D_\omega u\|_M(V_n).$$

For the reverse inequality, we take for all n a minimizing sequence $\{\psi_{n,k}\}$ and want to use a diagonalization argument. Since $\psi_{n,k}$ is only defined in V_n, we have to expand all these test functions to locally Lipschitz functions in Ω in such a way that $\int_{\Omega\setminus V_n} |\nabla\psi_{k,n}|$ is bounded independently of n and k. Then

$$\|D_\omega u\|_M(\Omega) \leq \liminf_{n\to\infty} \int_\Omega |\nabla\psi_{n,k(n)}|\ \omega = \lim_{n\to\infty} \|D_\omega u\|_M(V_n).$$

□

The last statement gave us the existence of a smooth approximation, but since the proof was rather abstract, we don't have any idea how these smooth functions may look like. For this reason we give an explicit construction of smooth approximating function in the setting of Lipschitz weights using the standard mollification. The statement is taken from [3], section 4.1.

Theorem 3.14: *Let $\Omega \subset \mathbb{R}^n$ be open, $u \in L^1_{loc}(\Omega, \omega)$ and ω a Lipschitz continuous weight function. Then there is for every $\epsilon > 0$ a function $u^\epsilon \in C^\infty(\Omega)$ such that*

$$\|u^\epsilon - u\|_{L^1(\Omega,\omega)} \to 0 \text{ as } \epsilon \to 0$$

and

$$TV_\omega(u^\epsilon) \to TV_\omega(u).$$

If additionally $u \in W^{1,1}_{loc}(\Omega, \omega)$, then we can find u^ϵ such that

$$\|\nabla u^\epsilon - \nabla u\|_{L^1(\Omega,\omega)} \to 0 \text{ as } \epsilon \to 0.$$

Proof. We just construct the approximating sequence, details and the rest of the proof can be found in Theorem 4.1.6 of [3]. We define the mollification kernel $\eta : \mathbb{R}^n \to \mathbb{R}$ by

$$\eta(x) = \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}.$$

Here, c is chosen in such a way that $\int_{\mathbb{R}^n} \eta(x) = 1$. Then, we define for $\epsilon > 0$

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{n}\right).$$

We define for $k \in \mathbb{N}$

$$U_k = \left\{x \in \Omega\ \cap\ B(0,k)\ \middle|\ \frac{1}{k} < \omega(x) < k,\ d(x,\partial\Omega) > \frac{1}{k}\right\},$$

then $\Omega = \bigcup_{k=1}^\infty U_k$. There exists a partition of unity $\{\xi_k\}_{k=1}^\infty$ satisfying

$$\forall k \in \mathbb{N}:\ 0 \leq \xi_k \leq 1,$$

$$\forall k \in \mathbb{N}:\ \exists j_k \in \mathbb{N},\ \text{s.t. } \xi_k \in C_c^\infty(U_{j_k}),$$

$$\sum_{k=1}^\infty \xi_k \equiv 1 \text{ on } \bigcup_{k=1}^\infty U_k = \Omega$$

and

$$\forall x \in \Omega,\ \exists \text{ a neighborhood } U_x \text{ s.t. } \{k \in \mathbb{N} \mid \text{supp}(\xi_k)\ \cap\ U_x \neq \emptyset\} \text{ is finite.}$$

Fix $\epsilon > 0$ and let L be the Lipschitz constant of ω. For each $k \in \mathbb{N}$, we choose a small enough ϵ_k such that

$$\lim_{k\to\infty} \epsilon_k = 0,$$
$$\operatorname{supp}(\eta_{\epsilon_k} * (u\ \xi_k)) \subset U_{j_k},$$
$$\int_\Omega |\eta_{\epsilon_k} * (u\ \xi_k) - u\ \xi_k|\ \omega\ dx < \frac{\epsilon}{2^k},$$
$$L\epsilon_k j_k < \epsilon$$

and

$$\int_\Omega |\eta_{\epsilon_k} * (u\ \nabla\xi_k) - u\ \nabla\xi_k|\ \omega\ dx < \frac{\epsilon}{2^k}.$$

Finally, we set

$$u^\epsilon = \sum_{k=1}^{\infty} \eta_{\epsilon_k} * (u\ \xi_k).$$

Then $u^\epsilon \in C^\infty(\Omega)$ and since $u = \sum_{k=1}^\infty u\xi_k$,

$$\|u^\epsilon - u\|_{L^1(\Omega,\omega)} \leq \sum_{k=1}^{\infty} \int_\Omega |\eta_{\epsilon_k} * (u\ \xi_k) - u\ \xi_k|\ \omega\ dx < \epsilon$$

and hence $(u^\epsilon - u) \to 0$ in $L^1(\Omega, \omega)$ as $\epsilon \to 0$.

□

3.4 Properties of the weighted BV-space

We now summarize key properties of the weighted BV-space, most of which extend naturally from the classical theory. The first result is taken from [3].

Lemma 3.15: *Fix a measurable function u. Then there is a sequence of Lipschitz weights $\{\omega_k\}$ such that $\omega_k \to \omega$ pointwise from below and*

$$TV_{\omega_k}(u) \to TV_\omega(u)$$

Proof. We use the argument of Lemma 2.1.4 of [3]. We define

$$\tilde\omega_k(x) = \inf\big\{\omega(y) + k\ d(x,y)\ |\ y \in \Omega\big\}.$$

This definition is taken from Thm. 2.1.2 of [3]. Since ω is l.s.c., the theorem states that $\tilde\omega_k \to \omega$ pointwise from below and that all $\tilde\omega_k$ are Lipschitz. Let ψ_k be a sequence of Lipschitz functions, $|\psi_k| \leq \omega$, such that

$$\lim_{k\to\infty} \int_\Omega u\ \operatorname{div}(\psi_k) = TV_\omega(u)$$

and define $\omega_k = \max\{\tilde\omega_k, \psi_1, \ldots, \psi_k\}$. Then all ω_k are Lipschitz and since $\{\omega_k\}$ is an increasing sequence, TV_{ω_k} is increasing as well, thus $\lim_{k\to\infty} TV_{\omega_k}(u)$ exists. Since $|\psi_k| \leq \omega_k$, we can conclude that

$$TV_\omega(u) = \lim_{k\to\infty} \int_\Omega u\ div(\psi_k) \leq \lim_{k\to\infty} TV_{\omega_k}(u) \leq TV_\omega(u).$$

□

The next statement is taken from [1], Section 2.2 . It is is needed to prove the Coarea formula in the weighted setting. The proof and the following remark are taken from [3].

Lemma 3.16: *If $u \in L^1(\Omega)$, then*

$$TV_\omega(u) < \infty \Leftrightarrow u \in BV_{loc}(\Omega) \text{ and } \omega \text{ is } \|Du\|\text{-integrable.}$$

When these later conditions are true, we also have

$$TV_\omega(u) = \int_\Omega \omega \; d\|Du\|.$$

Proof. We begin with "$\Leftarrow$".
Let $u \in BV_{loc}(\Omega)$, implying that the corresponding vector-valued measure exists. We compute for $\psi \in C_c^1(\Omega)$ with $|\psi| \leq \omega$:

$$\int_\Omega u \; div(\psi) \; dx = -\int_\Omega \psi \; dDu \leq \int_\Omega \omega \; d\|Du\|.$$

Taking the supremum over all $\psi \in C_c^1(\Omega)$ yields

$$\mathrm{TV}_\omega(u) \leq \int_\Omega \omega \; d\|Du\|.$$

Since we assumed ω to be $\|Du\|$-integrable, we obtain

$$\mathrm{TV}_\omega(u) < \infty.$$

For "$\Rightarrow$", let $\mathrm{TV}_\omega(u) < \infty$ and $V \subset\subset \Omega$. Since ω is continuous and positive, we can find a $c > 0$, such that $\omega > c$ on V. We compute for $\psi \in C_c^1(V)$ with $|\psi| \leq 1$

$$\int_V u \; div(\psi) \; dx = \frac{1}{c}\int_V u \; div(c\psi) \; dx \leq \frac{1}{c}\mathrm{TV}_\omega(u, V) \leq \frac{1}{c}\mathrm{TV}_\omega(u, \Omega) < \infty.$$

This yields that $u \in BV_{loc}(\Omega)$ and

$$\mathrm{TV}_\omega(u) \leq \int_\Omega \omega \; d\|Du\|,$$

which was shown in "$\Leftarrow$"-direction.
It remains to prove the $\|Du\|$-integrability of ω. First, we assume $\omega \in L^1_{loc}(\Omega, \|Du\|)$ and consider the measure $\omega \; dDu$ on Ω:

$$\int_\Omega \omega \; d\|Du\| = \|\omega \; dDu\|(\Omega) = \sup\Big(\int_\Omega \psi \; \omega \; Du \;\Big|\; \psi \in C_c^1(\Omega, \mathbb{R}^n), \; |\psi| \leq 1\Big).$$

Since ω is continuous, there is a sequence $\{\omega_k\}$ of nonnegative functions in $C_c^1(\Omega)$ such that $\omega_k \to \omega$ point wise from below. Thus, for all $\psi \in C_c^1(\Omega)$ with $|\psi| \leq 1$, we get using the Dominated Convergence theorem in the first equality

$$\int_\Omega \psi \; \omega \; dDu = \lim_{k\to\infty} \int_\Omega \psi \; \omega_k \; dDu = \lim_{k\to\infty} -\int_\Omega u \; div(\psi \; \omega_k) \leq \mathrm{TV}_\omega(u)$$

In the last inequality, we used that $\psi \; \omega_k$ is an admissible test function for $\mathrm{TV}_\omega(u)$. Hence

$$\int_\Omega \omega \; d\|Du\| \leq \mathrm{TV}_\omega(u)$$

and ω is $\|Du\|$-integrable. Combining both inequalities we also obtain the second statement of the Lemma saying that

$$\mathrm{TV}_\omega(u) = \int_\Omega \omega \; d\|Du\|.$$

Now we want to remove the additional assumption $\omega \in L^1_{loc}(\Omega, \|Du\|)$. Thm 2.1.2 of [3] gives us a sequence $\{\hat{\omega}_k\}$ of nonnegative Lipschitz functions such that $\hat{\omega}_k \to \omega$ pointwise from below. With the argument of Lemma 3.15, we know that we can approximate the total weighted variation with respect to ω by the one with respect to ω_k, where all ω_k are Lipschitz.
And since Lipschitz functions are locally $\|Du\|$-integrable, we can compute using the Monotone Convergence theorem

$$\mathrm{TV}_\omega(u) = \lim_{k\to\infty} \mathrm{TV}_{\omega_k}(u) = \lim_{k\to\infty} \int_\Omega \omega_k \; d\|Du\| = \int_\Omega \omega \; d\|Du\|.$$

Hence $\omega \in L^1(\Omega, \|Du\|)$ and the statement is proven.

□

Remark 3.17: *An important consequence of the last Lemma is that $TV_\omega(u)$ responds to variations of ω on sets with positive $\|Du\|$-measure. Thus, if $\|Du\|$ has a nonzero singular component, changes of the weight function on Lebesgue nullsets can change the value of $TV_\omega(u)$. Therefore it is important to define the weight function ω for all points in Ω.*

The next statement is a generalization of the Coarea formula for BV-functions to the setting of weighted BV. It will be used later in the proof of the Limsup inequality Thm. 4.6 when we deal with the case of non-smooth boundaries. The theorem and the following remark is taken from [3], Thm. 3.1.13 and following.

Theorem 3.18: *(Coarea Formula)*
Let $u \in L^1(\Omega)$ and define $E_t := \left\{ x \in \mathbb{R} \;\middle|\; u(x) > t \right\}$ for all $t \in \mathbb{R}$. Then,

$$TV_\omega(\chi_{E_t}) < \infty \text{ for a.e. } t \in \mathbb{R}$$

and

$$TV_\omega(u) = \int_\Omega \omega \; \|Du\| = \int_{-\infty}^{\infty} \Big(\int_{\partial^* E_t} \omega \; d\mathcal{H}^{n-1} \Big) \; dt = \int_{-\infty}^{\infty} TV_\omega(\chi_{E_t}).$$

Proof. First, we show that χ_{E_t} is in $BV_{loc}(\Omega)$. Take $\psi \in C^1_c(\Omega)$ with $|\psi| \leq 1$, then

$$\int \chi_{E_t} \; div(\psi) \; dx = \int_{E_t} div(\psi) = \int_{\partial E_t} \psi \; \nu \; dS \leq \mathcal{H}^{n-1}(\partial E_t)$$

and since

$$\mathcal{L}^n(\Omega) \geq \int_0^\infty \mathcal{H}^{n-1}(\partial E_t),$$

almost every E_t must have finite perimeter, i.e.

$$\mathcal{H}^{n-1}(\partial E_t) < \infty \text{ a.e. } t \in \mathbb{R}.$$

Varying ψ yields

$$\mathrm{TV}(\chi_{E_t}) < \infty.$$

For $s \geq 0$, define

$$\Omega_s := \{x \in \Omega \mid \omega(x) > s\}.$$

Ω_s is open because ω is continuous. Using Lemma 3.16, the Cavalieri's Principle and the classical Coarea formula we see that

$$\mathrm{TV}_\omega(u) = \int_\Omega \omega \; d\|Du\| = \int_0^\infty \|Du\|(\Omega_s) \; ds = \int_0^\infty \int_{-\infty}^\infty \|D\chi_{E_t}\|(\Omega_s) \; dtds.$$

Applying Tonelli's theorem, once again the Cavalieri's Principle and Lemma 3.16, we get

$$= \int_{-\infty}^\infty \int_0^\infty \|D\chi_{E_t}\|(\Omega_s) \; dtds = \int_{-\infty}^\infty \int_\Omega \omega \; d\|D\chi_{E_t}\| \; dt = \int_{-\infty}^\infty \mathrm{TV}_\omega(\chi_{E_t}).$$

□

Remark 3.19: *A consequence of the Coarea formuala in the weighted setting is that we can approximate the weighted total variation of a function $u \in L^1_{loc}$ by the weighted total variation of truncations of u:*
Define

$$u_k = \max\big\{-k, \min\{u, k\}\big\},$$

then

$$TV_\omega(u_k) \to TV_\omega(u).$$

Proof. If we observe

$$\big\{x \in \Omega \;\big|\; u_k(x) > t\big\} = \Omega \quad \text{for } t < -k$$

and

$$\big\{x \in \Omega \;\big|\; u_k(x) > t\big\} = \emptyset \quad \text{for } t > k$$

and

$$\mathrm{Per}_\omega(\emptyset) = \mathrm{Per}_\omega(\Omega) = 0,$$

then the statement follows directly from the Coarea formula because

$$TV_\omega(u) = \int_{-\infty}^\infty \left(\int_{\partial_* E_t} \omega \; d\mathcal{H}^{n-1} \right) dt = \lim_{k \to \infty} \int_{-k}^k \left(\int_{\partial_* E_t} \omega \; d\mathcal{H}^{n-1} \right) dt = \lim_{k \to \infty} TV_\omega(u_k).$$

□

We close this section by developing a structure theorem for weighted BV functions. In order to do that, we need the following Lemma. Everything that follows in this section is taken from [3], section 5.1 and 5.2 .

Lemma 3.20: *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $\omega \geq 0$ on Ω. If Ω_0 is open, ω is locally bounded away from zero on Ω_0 and if additionally $TV_\omega(u) < \infty$, then*

$$u \in BV_{loc}(\Omega_0).$$

Proof. Let $V \subset\subset \Omega_0$ be open. Because ω is locally bounded away from zero on $\bar{V}$, there is a constant $c > 0$ such that $\omega \geq c$ on V. We compute for $\psi \in Lip_c(V, \mathbb{R}^n)$ with $|\psi| \leq 1$

$$\int_V u \; div(\psi) \; dx \leq \frac{1}{c} \int_V u \; div(c\psi) \; dx \leq \frac{1}{c} TV_\omega(u)(V) \leq \frac{1}{c} TV_\omega(u) < \infty.$$

Hence $u \in BV_{loc}(\Omega)$. □

Theorem 3.21: *Let $\omega \geq 0$ be Lipschitz, $\Omega \subset \mathbb{R}^n$ be open and $u \in BV(\Omega, \omega)$. Then there exists a n-dimensional Radon measure $D_\omega u$ on Ω such that*

$$-\int_\Omega u \; div(\omega\psi) \; dx = \int_\Omega \psi \; dD_\omega u \text{ for all } \psi \in C_c(\Omega, \mathbb{R}^n),$$

$$\|D_\omega u\|(V) = TV_\omega(u)(V) \text{ for open } V \subset \Omega$$

and

$$D_\omega u = \omega \; Du.$$

We know that Du is a Radon measure on Ω_0 due to the last Lemma.

Proof. We follow the arguments of Thm. 5.1.4 of [3]. We define the linear functional

$$L : \mathrm{Lip}_c(\Omega, \mathbb{R}^n) \to \mathbb{R}; \; L(\psi) = -\int_\Omega u \; div(\psi\omega) \; dx.$$

Here, $Lip_c(\Omega)$ is the set of all Lipschitz functions such that their support is compact in Ω. For $\psi \in Lip(\Omega, \mathbb{R}^n)$ with $|\psi| \leq 1$, $\psi\omega \in Lip_c(\Omega, \mathbb{R}^n)$ because ω is Lipschitz as well. We also have $|\psi\omega| \leq \omega$. This thanks to Thm. 3.12

$$L(\psi) = -\int_\Omega u \; div(\omega\psi) \; dx \leq TV_\omega(u) \leq \|D_\omega u\|_M < \infty.$$

By Riesz Representation Theorem there exists a n-dimensional vector measure $D_\omega u$ on ω with

$$L(\psi) = \int_\Omega \psi \; dD_\omega u \; \forall \psi \in C_c(\Omega, \mathbb{R}^n).$$

For open subsets V of Ω, we can compute using Lemma 3.8

$$\begin{aligned}
\|D_\omega u\|(V) &= \sup\Big\{-\int_V u \; div(\psi\omega) \; dx \;\Big|\; \psi \in \mathrm{Lip}_c(V, \mathbb{R}^n), \; |\psi| \leq 1\Big\} \\
&\leq \sup\Big\{\int_V u \; div(\psi) \; dx \;\Big|\; \psi \in \mathrm{Lip}_c(V, \mathbb{R}^n), \; |\psi| \leq \omega\Big\} \\
&= TV_\omega(u)(V) = TV_\omega(u)(V \cap \Omega_0) \\
&= \sup\Big\{\int_{V \cap \Omega_0} u \; div(\psi) \; dx \;\Big|\; \psi \in \mathrm{Lip}_c(V \cap \Omega_0, \mathbb{R}^n), \; |\psi| \leq \omega\Big\} \\
&\leq \sup\Big\{-\int_V u \; div(\psi\omega) \; dx \;\Big|\; \psi \in \mathrm{Lip}_c(V, \mathbb{R}^n), \; |\psi| \leq 1\Big\} = \|D_\omega u\|(V).
\end{aligned}$$

For $\psi \in C_c(\Omega_0, \mathbb{R}^n)$,

$$\int_{\Omega_0} \psi \; dD_\omega u = -\int_{\Omega_0} u \; div(\psi\omega) \; dx = \int_{\Omega_0} (\psi\omega) \; dDu = \int_{\Omega_0} \psi(\omega \; dDu).$$

Thus

$$D_\omega u = \omega \; Du \quad \text{as measures on } \Omega_0.$$

By the weighted Coarea formula, Thm. 3.18, we see that $\|D_\omega u\|(\Omega \backslash \Omega_0) = 0$ implying

$$D_\omega u = 0 \quad \text{as a measure on } \Omega \backslash \Omega_0 :$$

And $\omega \; Du$ is the product of zero with something possibly undefined on $\Omega \backslash \Omega_0$, which we set to be zero.

□

Now, we work with weight functions in $BV(\Omega)$. We begin by studying the divergence measure. We take $\omega \in BV(\Omega)$ and $\psi \in \mathrm{Lip}_c(\Omega, \mathbb{R}^n)$. Then

$$div(\psi\omega) = \sum^{n} D_i(\psi^i\omega) = \sum^{n}(\bar{\omega}\ D_i\psi^i + \psi^i\ D_i\omega) = \omega\ div(\psi) + \psi\cdot\ D\omega.$$

Here, $\bar{\omega}$ is the approximated limit of ω defined by

$$\bar{\omega}(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} \omega(s)\ ds$$

and since $\mathcal{L}^n$-a.e. points of Ω are Lebesgue points of ω, $\bar{\omega} = \omega$ for a.e. $x \in \Omega$.
If additionally $\omega > 0$ is l.s.c. and $TV_\omega(u) < \infty$, then Du is a Radon measure due to Thm. 3.16 and by Thm. 5.1.6 of [3], it holds for all $\psi \in Lip_c(\Omega, \mathbb{R}^n)$ that

$$-\int_\Omega \bar{u}\ div(\psi\omega)\ dx = \int_\Omega \psi\cdot\bar{\omega}\ dDu.$$

If $\|Du\|$-a.e point of Ω is a point of continuity for ω and hence $\bar{\omega} = \omega$ on these points, then

$$-\int_\Omega \bar{u}\ div(\psi\omega)\ dx = \int_\Omega \psi\cdot\omega\ dDu.$$

In general, we have $\bar{\omega} \geq \omega$ implying

$$TV_\omega(u) = \int_\Omega \omega\ d\|Du\| \leq \int_\Omega \bar{\omega}\ d\|Du\| = \|\bar{\omega}\ dDu\|.$$

This means that the total variation of $\bar{\omega}\ Du$ can be infinite even though $TV_\omega(u) < \infty$. For this reason, we consider a function $u \in L^1(\Omega, \omega)$ such that

$$TV_\omega(u) < \|D_\omega u\|_M(\Omega) < \infty.$$

Theorem 3.22: *Let $\omega \in BV_{loc}(\Omega)$ be locally bounded away from zero in Ω, $u \in L^\infty(\Omega)$ with $\|D_\omega u\|_M(\Omega) < \infty$. Then there exists a sequence $\{u_n\}$ of functions in $Lip_{loc}(\Omega)$ such that*

$$k \in \mathbb{N},\ \|u_k\|_{L^\infty(\Omega)} < \|u\|_{L^\infty(\Omega)},$$

$$\|D_\omega u\|_M(\Omega) = \lim_{k\to\infty}\int_\Omega |\nabla u_k|\ dx$$

and that there exist vector-valued Radon measures v_u^Ω and α_u^Ω such that

$$\omega\nabla u_k \rightharpoonup v_u^\Omega \text{ and } u_k\ D_s\omega \rightharpoonup \alpha_u^\Omega.$$

These measures may not be unique but satisfy

$$v_u^\Omega + \alpha_u^\Omega = \bar{\omega}\ Du + \bar{u}\ D_s\omega.$$

If $\omega \in W^{1,1}_{loc}(\Omega)$, then $D_s\omega$ and α_u^Ω are the zero measure and

$$v_u^\Omega = \bar{\omega}\ Du.$$

Proof. We sketch the proof of Thm. 5.2.1 of [3].

By Theorem 3.12, there exists a sequence $\{u_k\}$ in $\mathrm{Lip}_{loc}(\Omega)$ such that $u_k \to u$ in $L^1(\Omega,\omega)$ and $\|D_\omega u\|_M(\Omega) = \lim_{k\to\infty}\int_\Omega \omega\ |\nabla u_k|\ dx$. Because $\{\omega\ \nabla u_k\}$ is bounded, there is a vector valued Radon measure ν_u^Ω such that $\omega\ \nabla u_k \rightharpoonup \nu_u^\Omega$. If even $u \in W^{1,1}(\Omega,\omega)$, we select u_k such that $\omega\ \nabla u_k \rightharpoonup \omega\ \nabla u$ and therefore $\omega\ \nabla u = \nu_u^\Omega$.
Now, we will derive a formula for ν_u^Ω. For all $\psi \in \mathrm{Lip}_{loc}(\Omega,\mathbb{R}^n)$, we obtain using integration by parts and the fact that $div(\omega\psi) = \omega\ div(\psi) + \psi\cdot D\omega$,

$$\int_\Omega \psi\cdot\ d\nu_u^\Omega = \lim_{k\to\infty}\int_\Omega (\nabla u_k\cdot\psi)\ \omega\ dx$$
$$= \lim_{k\to\infty} -\int_\Omega u_k\ d(div(\omega\psi))\ dx = -\lim_{k\to\infty}\int_\Omega u_k\ \omega\ div(\psi)\ dx - \lim_{k\to\infty}\int_\Omega u_k\psi\cdot\ dD\omega.$$

Because $u_k \to u$ in $L^1(\Omega,\omega)$ and ω is l.s.c., u_k converges also in $L^1_{loc}(\Omega)$ and hence

$$\int_\Omega \psi\cdot d\nu_u^\Omega = -\int_\Omega u\ \omega\ div(\psi)\ dx - \lim_{k\to\infty}\int_\Omega u_k\psi\cdot\ dD\omega.$$

Since $u \in L^\infty(\Omega)$, it follows by Thm. 5.1.5 of [3] that

$$\begin{aligned}
&= \int_\Omega \psi\ dD(u\omega) - \lim_{k\to\infty}\int_\Omega u_k\psi\ dD\omega\\
&= \int_\Omega \bar{u}\psi\cdot\ dD\omega + \int_\Omega \bar{\omega}\psi\cdot dDu - \lim_{k\to\infty}\int_\Omega u_k\psi\cdot dD\omega\\
&= \int_\Omega \bar{\omega}\psi\cdot\ dDu + \lim_{k\to\infty}\int_\Omega (\bar{u}-u_k)\psi\cdot\ dD\omega.
\end{aligned}$$

If we decompose $D\omega$ into absolutely continuous and singular part, i.e.

$$D\omega = \nabla\omega\ dx + dD_s\omega,$$

we obtain

$$\int_\Omega \psi\cdot d\nu_u^\Omega = \int_\Omega \psi\bar{\omega}\ dDu + \lim_{k\to\infty}\int_\Omega (\bar{u}-u_k)\psi\cdot\nabla\omega\ dx + \lim_{k\to\infty}\int_\Omega (\bar{u}-u_k)\psi\cdot dD_s\omega.$$

By the Lebesgue Divergence Theorem

$$\lim_{k\to\infty}\int_\Omega (\bar{u}-u_k)\psi\cdot\nabla\omega\ dx = 0,$$

yielding

$$\int_\Omega \psi\cdot d\nu_u^\Omega = \int_\Omega \psi\bar{\omega}\ dDu + \lim_{k\to\infty}\int_\Omega (\bar{u}-u_k)\psi\cdot dD_s\omega.$$

Thus if $\omega \in W^{1,1}(\Omega)$, $D_s\omega$ is the zero measure and

$$\nu_u^\Omega = \bar{\omega}\ Du.$$

Next, we study the measures $\{u_k\ dD_s\omega\}$. Since

$$\lim_{k\to\infty}\int_\Omega \psi\cdot u_k\ dD_s\omega = \int_\Omega \psi\cdot\bar{\omega}\ dDu + \int_\Omega \psi\bar{u}\ dD_s\omega - \int_\Omega \psi\cdot d\nu_u^\Omega,$$

we know that $\{u_k\ D_s\omega\}$ is a bounded sequence of measures. This implies that there exists a converging subsequence and a vector-valued Radon measure α_u^Ω such that

$$u_k\ D_s\omega \rightharpoonup \alpha_u^\Omega.$$

We conclude by

$$\nu_u^\Omega + \alpha_u^\Omega = \bar{\omega}\ Du + \bar{u}\ D_s\omega.$$

□

3.5 Variation on lines for weighted BV

This section, which is taken from [3], section 4.2, considers restrictions of weighted BV functions to lines. We will show that for a given direction, the restriction to $\mathcal{L}^{n-1}$-a.e. line with this direction is a weighted BV function itself. Conversely, if a function fulfills this condition and its total variation is finite, then the function is in the weighted BV space. Note that there are also similar results in the unweighted setting (check [3], Thm. 4.2.1 and Prop 4.2.2).
First, we introduce some notation. Let $\nu \in S^{n-1}$ be a direction and π_ν the corresponding hyperplane, i.e. an orthogonal hyperplane on ν. We call the projection of Ω onto π_ν

$$\Omega^\nu = \left\{ y \in \pi_n u \;\middle|\; \exists t \in \mathbb{R},\; y + t\nu \in \Omega \right\}.$$

The line trough y in direction ν intersected with Ω is denoted by

$$\Omega^\nu_y = \left\{ t \in \mathbb{R} \;\middle|\; y + t\nu \in \Omega \right\}$$

and the variation of a function $u \in L^1_{loc}(\Omega)$ along $\nu \in S^{n-1}$ is defined by

$$V_\nu(u, \Omega, \omega) = \sup \left\{ \int_\Omega u \frac{\partial \psi}{\partial \nu} \;\middle|\; \psi \in Lip_c(\Omega),\; |\psi| \leq \omega \right\}.$$

Theorem 3.23: *Let $\nu_1, ..., \nu_n$ be linear independent unit vectors in $\mathbb{R}^n$, then $TV_\omega(u) < \infty$ if and only if $V_{\nu_i}(u, \Omega, \omega) < \infty$ for all $i \in \{1, .., n\}$.*

Proof. "$\Rightarrow$":
for $j \in \{1, .., n\}$ and $\psi \in \mathrm{Lip}_c(\Omega)$ with $|\psi| \leq \omega$,

$$\int_\Omega u \frac{\partial \psi}{\partial \nu_j}\, dx = \int_\Omega u\; div(\psi \nu_j)\; dx \leq TV_\omega(u).$$

Thus, if we take the supremum over all admissible ψ, we obtain,

$$V_{\nu_j}(u, \Omega, \omega) < TV_\omega(u) < \infty$$

For the "$\Leftarrow$"-direction we take a $\psi \in Lip_c(\Omega, \mathbb{R}^n)$ with $|\psi| \leq \omega$ and estimate

$$\int_\Omega u\; div(\psi)\; dx = \sum^n \int_\Omega u \frac{\partial \psi^k}{\partial e_k}\, dx = \sum_k^n \sum_j^n \langle e_k | v_j \rangle \int_\Omega u \frac{\partial \psi^k}{\partial \nu_j}\, dx$$
$$\leq n \sum_j^n V_{\nu_j}(u, \Omega, \omega) < \infty.$$

Once again we take the supremum over all admissible ψ and obtain

$$TV_\omega(u) \leq n \sum^n V_{\nu_j}(u, \Omega, \omega) < \infty.$$

□

Similar to $TV_\omega(u)$, we can find a formula for $V_\nu(u, \Omega, \omega)$.
Theorem 3.24: *Let $\omega > 0$ be lower semicontinuous and $\nu \in S^{n-1}$. Then $V_\nu(u, \Omega, \omega) < \infty$ if and only if $V_\nu(u, V) < \infty$ for all $V \subset\subset \Omega$ and $\omega \in L^1(\Omega, \|D^\nu u\|)$. If this holds, then*

$$V_\nu(u, \Omega, \omega) = \int_\Omega d\|D^\nu u\|.$$

Here $V_\nu(u, V)$ is the variation of u along ν in V (without weights) defined by

$$V_\nu(u,\Omega) = \sup\left\{ \int_\Omega u \frac{\partial \psi}{\partial \nu}\, dx \;\middle|\; \psi \in Lip_c(\Omega),\ |\psi| \le 1 \right\}.$$

If $V_\nu(u,\Omega) < \infty$, then $V_\nu(u,\cdot)$ defines a Radon measure on Ω due to the Riesz Representation theorem. Additionally, there is a measure, which we denote by $D^\nu u$, such that

$$\|D^\nu u\|(\cdot) = V_\nu(u,\cdot)$$

and for $\psi \in Lip_c(\Omega)$,

$$\int_\Omega \psi \; dD^\nu u = -\int_\Omega u \frac{\partial \psi}{\partial \nu}\, dx.$$

Proof. We follow the proof of Thm. 4.2.4 of [3]. Let $V_\nu(u,V) < \infty$ for all $V \subset\subset \Omega$. Then the measure $D^\nu u$ exists, and for all Lipschitz functions ψ with compact support and $|\psi| \le \omega$ we have

$$\int_\Omega u \frac{\partial \psi}{\partial \nu}\, dx = -\int_\Omega \psi \; dD^\nu u \le \int_\Omega \omega \; d\|D^\nu u\|.$$

Hence,

$$V_\nu(u,\Omega,\omega) \le \int_\Omega \omega \; d\|D^\nu u\|.$$

Since $\omega \in L^1(\Omega, \|D^\nu u\|)$,

$$V_\nu(u,\Omega,\omega) < \infty.$$

For the reverse direction, we assume $V_\nu(u,\Omega,\omega) < \infty$ and take a set $V \subset\subset \Omega$. There exists a constant c such that $\omega \ge c$ on V because ω is lower semicontinuous and positive on $\bar{V}$. We estimate for $\psi \in \mathrm{Lip}_c(V)$ with $|\psi| \le 1$

$$\int_V u \frac{\partial \psi}{\partial \nu}\, dx = \frac{1}{c}\int_V u \frac{\partial (c\psi)}{\partial \nu}\, dx \le \frac{1}{c} V_\nu(u,V,\omega) < \frac{1}{c} V_\nu(u,\Omega,\omega) < \infty.$$

Hence $V_\nu(u,V) < \infty$, which implies $V_\nu(u,\Omega,\omega) \le \int_\Omega \omega \; d\|D^\nu u\|$ by the first part of the proof.
For the second part of the statement, we begin by proving the result with the additional assumption $\omega \in L^1(\Omega, \|D^\nu u\|)$. By Theorem 2.1.2 of [3], we can take a sequence $\{\omega_k\}$ of positive Lipschitz functions such that $\omega_k \to \omega$ pointwise from below. Then we have for all admissible ψ, using the Lebesgue Dominated Convergence Theorem, that

$$\int_\Omega \psi \; \omega \; dD^\nu u = \lim_{k\to\infty} \int_\Omega \psi \; \omega_k \; dD^\nu u = \lim_{k\to\infty} -\int_\Omega u \; \frac{\partial(\psi \; \omega_k)}{\partial \nu}\, dx \le V_\nu(u,\Omega,\omega).$$

If we take the supremum over all admissible Lipschitz functions ψ, we get

$$\sup\left\{ \int_\Omega \psi \; \omega \; dD^\nu u \;\middle|\; \psi \in Lip_c(\Omega),\ |\psi| \le 1 \right\} = \|\omega \; dD^\nu u\|(\Omega) = \int_\Omega \omega \; d\|D^\nu u\|.$$

This implies

$$\int_\Omega \omega \; d\|D^\nu u\| \le V_\nu(u,\Omega,\omega)$$

and thereby

$$\int_\Omega \omega \; d\|D^\nu u\| = V_\nu(u,\Omega,\omega).$$

Now, we want to remove the additional assumption. By an argument similar to the one of Lemma 3.15, we can show that there exists a sequence of Lipschitz weights $\{\omega_k\}$ that converge pointwise from below to ω, such that

$$\lim_{k\to\infty} V_\nu(u,\Omega,\omega_k) = V_\nu(u,\Omega,\omega).$$

Because Lipschitz functions are locally $\|D^\nu u\|$-integrable, we can use the Monotone convergence Theorem to get

$$V_\nu(u,\Omega,\omega) = \lim_{k\to\infty} V_\nu(u,\Omega,\omega_k) = \lim_{k\to\infty}\int_\Omega \omega_k \; d\|D^\nu u\| = \int_\Omega \omega \; d\|D^\nu u\|.$$

Thus $\omega \in L^1(\Omega, \|D^\nu u\|)$.

□

The next result states that the set $\{\omega = 0\}$ does not affect the weighted variations on lines.

Lemma 3.25: *If $u \in L^1_{loc}(\Omega)$ and $\nu \in S^{n-1}$, then*

$$V_\nu(u,\Omega,\omega) = V_\nu(u,\Omega_0,\omega).$$

Proof. The proof of this result works similar to the one of the weighted total variation (compare Lemma 3.8) and can be found in [3], Thm. 4.2.5 . □

The next theorem relates the weighted directional variation to the weighted variation along lines. A similar result holds for BV without weights, compare [3], Thm. 4.2.1 .

Theorem 3.26: *Assume that $u \in L^1_{loc}(\Omega,\omega)$, $\nu \in S^{n-1}$ and $\omega \geq 0$ is l.s.c., then*

$$V_\nu(u,\Omega,\omega) = \int_{\Omega^\nu} TV_\omega(u_y^\nu)(\Omega_y^\nu)\; dy.$$

Here $u_y^\nu : \Omega_y^\nu \to \mathbb{R};\ u_y^\nu(t) = u(y + t\nu)$.

Proof. Once again, the proof is taken from [3] and can be found in Thm. 4.2.6 . It is an extension of the result without weights (Thm. 4.2.1 of [3]), which will be needed in the argument. First, we assume $\omega > 0$. For $t > 0$, we define $\Omega_t = \{x \in \Omega \mid \omega(x) > t\}$, Ω_t^ν as the projection of Ω_t onto π_ν and $\Omega_{t,y}^\nu$ to be the slice of Ω_t by the line through y in ν-direction. If both $V_\nu(u,\Omega,\omega)$ and $\int_{\Omega^\nu} \|D_\omega u_y^\nu\|(\Omega_y^\nu)\; dy$ are infinite, then equality holds trivially.
Hence, we begin by assuming $V_\nu(u,\Omega,\omega) < \infty$. We know $L^1(\Omega,\omega) \subset L^1(\Omega)$ because ω is positive and l.s.c. . Now, we compute using the last results

$$V_\nu(u,\Omega,\omega) = \int_\Omega \omega\; d\|D^\nu u\| = \int_0^\infty \|D^\nu u\|(\Omega_t)\; dt = \int_0^\infty V_\nu(u,\Omega)\; dt$$

the case without weights (Thm. 4.2.1 of [3]) then yields

$$= \int_0^\infty \int_{\Omega_t^\nu} \|Du_y^\nu\|(\Omega_{t,y}^\nu)\; dydt = \int_0^\infty \int_{\Omega_t^\nu} \|Du_y^\nu\|(\{x \in \Omega_y^\nu \mid \omega(t) > t\})\; dydt.$$

We have for $t \in \mathbb{R}$ that $\|Du_y^\nu\|(\{x \in \Omega_y^\nu \mid \omega(x) > t\} = \|Du_y^\nu\|(\emptyset)$ if $y \in \Omega^\nu \backslash \Omega_t^\nu$, and hence

$$\int_{\Omega_t^\nu} \|Du_y^\nu\|(\{x \in \Omega_y^\nu \mid \omega(t) > t\})\; dydt = \int_{\Omega^\nu} \|Du_y^\nu\|(\{x \in \Omega_y^\nu \mid \omega(t) > t\})\; dydt.$$

By Tonelli's Theorem we infer

$$\int_0^\infty \int_{\Omega^\nu} \|Du_y^\nu\|(\{x \in \Omega_y^\nu \mid \omega(t) > t\})\ dydt = \int_{\Omega^\nu} \int_0^\infty \|Du_y^\nu\|(\{x \in \Omega_y^\nu \mid \omega(t) > t\})\ dydt.$$

With Cavalieri's Principle and 3.16, we conclude

$$\begin{aligned} V_\nu(u,\Omega,\omega) &= \int_{\Omega^\nu} \int_0^\infty \|Du_y^\nu\|(\{x \in \Omega_y^\nu \mid \omega(t) > t\})\ dydt \\ &= \int_{\Omega^\nu} \int_{\Omega_y^\nu} \omega\ d\|Du_y^\nu\| dy = \int_{\Omega^\nu} TV_\omega(u_y^\nu)(\Omega_y^\nu)\ dy. \end{aligned}$$

If we instead assume $\int_{\Omega^\nu} TV_\omega(u_y^\nu)(\Omega_y^\nu)\ dy < \infty$, we can do the same calculation the other way around and obtain the desired equality.
We will conclude by removing the additional assumption $\omega > 0$. By Lemma 3.25, the special case $\omega > 0$ and Lemma 3.8, we infer

$$V_\nu(u,\Omega,\omega) = V_\nu(u,\Omega_0,\omega) = \int_{(\Omega_0)^\nu} TV_\omega(u_y^\nu)\big((\Omega_0)_y^\nu\big)\ dy = \int_{\Omega^\nu} TV_\omega(u_y^\nu)(\Omega_y^\nu)\ dy.$$

For $y \in \Omega^\nu \backslash (\Omega_0)^\nu$, $\omega = 0$ on Ω_y^ν and hence $TV_\omega(u_y^\nu)(\Omega_y^\nu) = 0$. Thus

$$V_\nu(u,\Omega,\omega) = \int_{\Omega^\nu} TV_\omega(u_y^\nu)(\Omega_y^\nu)\ dy.$$

$\square$

Combining the last results, we obtain

Theorem 3.27: *Let $\nu_1, ..., \nu_n$ be linear independent unit vextors in $\mathbb{R}^n$, $\omega \geq 0$ a l.s.c. weight on Ω and $u \in L^1_{loc}(\Omega,\omega)$. Then the following statements are equivalent:*

1. $TV_\omega(u) < \infty$,
2. $\forall j \in \{1, .., n\},\ V_{\nu_j}(u,\Omega,\omega) < \infty$,
3. $\forall j \in \{1, .., n\}$ *u is a weighted BV function on L^{n-1}-a.e. line parallel to ν_j with* $y \to TV_\omega(u_y^{\nu_j})(\Omega_y^{\nu_j}) \in L^1(\Omega^\nu)$.

4 Weighted Van-der-Waals-Cahn-Hilliard Model

Inspired by the work in [7], we aim to extend the model by adding a weight-function $\omega(x) \in L^1_{loc}(\Omega)$ inside the functional 2. In their paper, the authors assumed $0 < c \leq \omega \leq 1$ for some constant c, ω to be Lipschitz and lower semicontinuous and they adapted the definition of the functionals in the following way:

Definition 4.1: *We assume $0 < c \leq \omega \leq 1$, W to be $\mathcal{C}^2$, nonnegative, with precisely two zero-points at a and b such that $0 < a < b$, $W'(a) = W'(b) = 0$ and $W''(a), W''(b) > 0$. Furthermore we assume that there are constants c_1, c_2, l and an integer $p \geq 2$ such that $c_1|u|^p \leq W(u) \leq c_2|u|^p$ for $|u| > l$. We define the following sequence of functionals:*

$$\tilde{F}_\epsilon(u) = \begin{cases} \frac{1}{\epsilon}\int_\Omega \ \omega \ W(u) \ dx + \epsilon \int_\Omega \ \omega \ |\nabla u|^2 \ dx & u \in H^1(\Omega) \\ \infty & otherwise \end{cases}$$

and the limit functional

$$\tilde{F}_0(u) = \begin{cases} K \ Per_{\Omega,\omega}\{u = a\} & u(x) \in \{a, b\} \ a.e. \\ \infty & otherwise \end{cases} .$$

Here,

$$K = 2 \ \inf\Big\{ \int_{-1}^1 \sqrt{W(\gamma(s))} \ |\gamma'(s)| \ ds \ \Big| \ \gamma \ Lipschitz, \ \gamma(-1) = a, \ \gamma(1) = b \Big\}$$

and $Per_{\Omega,\omega}$ is the ω-weighted perimeter defined by

$$Per_{\Omega,\omega} = TV_\omega(\chi_E) = \int_{\Omega \cap E} \omega \ \mathcal{H}^1,$$

which will be defined rigorously later. Note that the last equality only holds if E is smooth enough.

In [7], they proved that $\tilde{F}_\epsilon$ Γ-converges to $\tilde{F}_0$ under the above assumptions by adapting the proof of the case without weight. The assumption $0 < c \leq \omega$ ensures that

$$TV_\omega(u) < \infty \Leftrightarrow u \in BV(\Omega),$$

so that the setting essentially remains the classical space of functions of bounded variation. We now seek to relax the assumption on $\omega > 0$. In this case, the equivalence $\mathrm{TV}_\omega(u) < \infty \Leftrightarrow u \in BV(\Omega)$ no longer holds, leading to the strict inclusion

$$BV(\Omega) \subsetneq BV_\omega(\Omega).$$

As a result, we must work with an enlarged class of functions.
While $0 < c \leq \omega$ on compact subsets of Ω because ω is continuous, $\omega(x) \to 0$ is possible for $x \to \partial\Omega$. Thus functions that oscillate near the boundary could belong to $BV_\omega(\Omega)$, as shown in Remark 3.6. This could lead to more complex minimizers of $\tilde{F}_\epsilon$ and potentially more complicated interfaces between the phases.
Overall, the advantage of the weighted model lies in its flexibility: by the choice of the weight function ω, one can include additional information about the system into the model.

4.1 Compactness

Now, we begin to prove the main result of this thesis. Once again, the first step in establishing Γ-convergence is to study the existence of converging subsequences and to identify the appropriate topology for the convergence.

Theorem 4.2: *(Compactness result)*
Let $\Omega \subset \mathbb{R}^n$ be open and with finite measure. Assume that the potential W fulfill the conditions (H1) and (H2) and let $\{u_n\} \subset W^{1,2}(\Omega)$ be a sequence with $\epsilon_n \to 0^+$, such that

$$\sup \tilde{\mathcal{F}}(u_n) =: M < \infty.$$

Then, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in BV_\omega(\Omega, \{a,b\})$ such that

$$u_{n_k} \to \text{ in } L^1_{loc}(\Omega).$$

The idea of the proof is essentially the same as in [7]. The main difference is that we only obtain convergence in L^1_{loc} instead of L^1. Achieving L^1 convergence is problematic at the boundary, because there the weight ω can tend to 0, with the result that we cannot say much about convergence there at all.

Proof. Let $\Omega' \subset\subset \Omega$. Since ω is continuous and Ω' is compact, ω attains a positive minimum on Ω'. We use the same argument as in Remark 3.6 to reduce the problem to the case without weights and use the compactness result in this case (Thm. 2.2) to state

$$\exists \ \{u_{n_k}\}_{k\in\mathbb{N}} \text{ and } u \in \mathrm{BV}(\Omega', \{a,b\}) \text{ such that } u_{n_k} \to u \text{ in } L^1(\Omega')$$

or equivalently, after denoting the subsequence again by $\{u_n\}$, $u_n \to u$ in $L^1_{loc}(\Omega)$.
It remains to verify that the limit function u belongs to the space $BV_\omega(\Omega, \{a,b\})$. From Theorem 2.2, we know that on all compact subsets $\Omega' \subset\subset \Omega$, u takes only the values a or b. Since Ω is bounded and open, there exists for every $x \in \Omega$ such a compact subset Ω' with $x \in \Omega'$. Hence $u(x) \in \{a,b\}$ for almost every $x \in \Omega$ and $u \in L^1(\Omega, \{a,b\})$.
Similar to the case without weights, we define for some constant $K > 0$ the truncated potential

$$W_1(t) = \min\{W(t), K\}$$

and the function

$$f(t) = \int_a^t \sqrt{W_1(t)}.$$

It holds for all $n \in \mathbb{N}$

$$\begin{aligned}
&\tilde{F}_\epsilon(u_n) - 2\int_\Omega \omega \ \sqrt{W(u_n)} \ |\nabla u_n| \\
&= \int_\Omega \omega \ \left(\frac{1}{\epsilon} W(u_n) - 2 \ \sqrt{W(u_n)} \ |\nabla u_n| + \epsilon |\nabla u_n|^2 \ dx\right) \\
&= \int_\Omega \omega \ \left(\frac{1}{\sqrt{\epsilon}}\sqrt{W(u_n)} - \sqrt{\epsilon} \ |\nabla u_n|\right)^2 \geq 0.
\end{aligned}$$

Thus

$$\tilde{F}_\epsilon(u_n) \geq 2\int_\Omega \omega \ \sqrt{W(u_n)} \ |\nabla u_n|$$

and since $0 \leq W_1 \leq W$,

$$\sup_n \tilde{F}_{\epsilon_n}(u_n) \geq 2\int_\Omega \omega \sqrt{W_1(u_n(x))} |\nabla u_n(x)| = 2\int_\Omega \omega |\nabla (f \circ u_n)|.$$

We know that $\{u_n\}$ is locally bounded in L^1 (because it converges in $\Omega' \subset\subset \Omega$), and since f is Lipschitz, we can infer that $\{f \circ u_n\}$ is also locally bounded in L^1. By Rellich-Kondrachov there exists a subsequence, again denoted by $\{f \circ u_n\}$, and a function $f_\infty \in L^1_{loc}(\Omega)$ such that

$$f \circ u_n \to f_\infty \text{ in } L^1_{loc}(\Omega).$$

We actually have $f \in BV(\Omega, \omega)$ because

$$\sup_n \int_\Omega \omega \, |\nabla(f \circ u_n)| \leq \sup_n \tilde{F}_{\epsilon_n}(u_n) = M < \infty$$

and due to the lower semicontinuity of the weighted total variation

$$TV_\omega(f_\infty) \leq \liminf_n \; TV_\omega(f \circ u_n) \leq \max_\Omega \omega \; M < \infty.$$

Because f is strictly increasing, it has an inverse f^{-1}. We have (for a subsequence)

$$f^{-1}(f \circ u_n)(x) \to f^{-1}(f_\infty(x)) = u(x).$$

Thus $f_\infty(x) \in \{f(a), f(b)\} = \{0, f(b)\}$ for a.e. $x \in \Omega$ and

$$f_\infty(x) = f(b)\chi_E(x)$$

for a set $E \subset \Omega$. Since $f_\infty \in \mathrm{BV}(\Omega, \omega)$, $\chi_E \in \mathrm{BV}(\Omega, \omega)$ and we can conclude that

$$u = b\chi_E + a(1 - \chi_E) \in \mathrm{BV}(\Omega, \omega).$$

□

4.2 Liminf-Inequality

Now we want to show that $\tilde{F}_\epsilon$ Γ - converges to $\tilde{F}_0$ with respect to the L^1_{loc}- strong topology. We start with the Liminf-Inequality.

Theorem 4.3: *Let $v_\epsilon \to v_0$ in $L^1_{loc}(\Omega)$. Then we have*

$$\tilde{F}_0(v_0) \leq \liminf_{\epsilon \to 0} \tilde{F}_\epsilon(v_\epsilon).$$

The structure of the proof is similar to that given in [7], with the main difference being the use of L^1_{loc}-convergence due to the relaxed assumptions on the weight function ω. Two auxiliary lemmata are needed to complete the argument. The first one is similar to the case of [7], its proof is taken from [10] and [11].

Lemma 4.4: *Let us denote by g the auxiliary function defined by*

$$g(u) = \inf_{\substack{\gamma(-1)=\alpha \\ \gamma(1)=u}} \int_{-1}^{1} \sqrt{W(\gamma(s))} \; |\gamma'(s)| \; ds, \tag{3}$$

where we take the infimum over all Lipschitz functions. Then one can show that for all $u \in \mathbb{R}$ there exists a function $\gamma_u : [-1, 1] \to \mathbb{R}$ such that $\gamma_u(-1) = \alpha$, $\gamma(1) = u$ and

$$g(u) = \int_{-1}^{1} \sqrt{W(\gamma_u(s))} \; |\gamma_u'(s)| \; ds.$$

Moreover, we have that g is Lipschitz and

$$|g'(u)| = \sqrt{W(u)} \; a.e. \; .$$

Proof. Normally, we would like to use the direct method to obtain a minimizer, but due to a lack of compactness, we have to start by perturbing the problem by

$$\inf_{\substack{\gamma(-1)=\alpha \\ \gamma(1)=u}} \int_{-1}^{1} \left(\sqrt{W(\gamma(s))} + \delta\right) |\gamma'(s)| \; ds := d_\delta(\alpha, u) \tag{4}$$

for $\delta > 0$, which defines a metric on the plane. The first step is to prove existence of a geodesic that minimizes d_δ.

Step 1 : Existence of geodesics for d_δ
We begin by verifying that one can extend every locally defined geodesic to a geodesic defined for the whole line $\mathbb{R}$ (geodesical completeness).
A geodesic γ of d_δ is an extremal of

$$L_\delta(\gamma) := \int_{-1}^{1} \left(\sqrt{W(\gamma(s))} + \delta\right) |\gamma'(s)| \; ds.$$

Hence it must fulfill the Euler-Lagrange equation

$$|\dot{\gamma}| \; \nabla T_\delta(\gamma) = \frac{d}{dt}\left(T_\delta(\gamma)\frac{\dot{\gamma}}{|\dot{\gamma}|}\right). \tag{5}$$

As a geodesic, γ has constant speed, i.e. $|\dot{\gamma}| \equiv c > 0$. Thus solving (5) simplifies to

$$c^2 \nabla T_\delta(\gamma) = (\nabla T_\delta \;\cdot \dot{\gamma})\dot{\gamma} + T_\delta(\gamma)\ddot{\gamma}. \tag{6}$$

Now, we can transform (6) to a first order system and then apply the standard existence and extension theorem for ODE's. Thereby we obtain a solution of the Euler-Lagrange equation on $\mathbb{R}^n$. It remains to show that every solution of (6) has constant speed. In order to do that, let γ be a local solution of (6) with constant speed $c = |v|$, and consider the inner product of (6) with $\dot{\gamma}$

$$\begin{aligned} |v|^2 \nabla T_\delta(\gamma) \cdot \dot{\gamma} &= (\nabla T_\delta \;\cdot \dot{\gamma})\dot{\gamma}^2 + T_\delta(\gamma)\dot{\gamma} \;\cdot \ddot{\gamma} \\ \Leftrightarrow \frac{|v|^2 - |\dot{\gamma}|^2}{T_\delta(\gamma)} \nabla T_\delta(\gamma) \cdot \dot{\gamma} &= \frac{d}{dt}|\ddot{\gamma}|^2. \end{aligned}$$

Interpreting the last line as a determination of $|\ddot{\gamma}|^2$ with initial condition $|\ddot{\gamma}|^2(0) = |v|^2$, we get that $|\ddot{\gamma}|^2 \equiv |v|^2$ by the local uniqueness of the solution. Hence γ has constant speed and is thereby a geodesic. Now that we have established this, we can apply the Hopf-Rinow theorem and obtain the existence of geodesics for d_δ:
Given any points x_1, x_2, one can find a geodesic $\gamma : (-1, 1) \to \mathbb{R}^2$ with $\gamma(-1) = x_1$ and $\gamma(1) = x_2$, such that

$$d_\delta(x_1, x_2) = L_\delta(\gamma).$$

Next, we want to prove the existence of geodesics for the original problem by passing to the limit of minimizers $\{\gamma_\delta\}$ for $\delta \to 0$. Therefore we need to be able to obtain a subsequence which converges to a limit curve.

Step 2 : Compactness
Let s_δ be the euclidean arc-length of γ_δ. Then there is a number $c > 0$ independent of δ, such that $s_\delta < c$ for all δ.
The proof of this statement is quite lengthy and can be obtained from [10] Lemma 6 and

7, after some small adaptions to our simpler situation. In the proof, the curve is split into parts close to the zero points a and b of W and a part connecting these end-parts. Then the bound is obtained for both situations separately. The main difficulty of this step is to prove a uniform bound when γ_δ is close to the zero points. In such a situation we parametrize the curve by its arc-length and transform to local polar coordinates, as it's done in Lemma 7. Lemma 6 gives the remaining bound for the center part of the curve. Then the bounds are combined to obtain the desired statement and we are set to prove:

Step 3 : Existence of geodesics for the original problem
We will show that there exists a subsequence $\{\gamma_{\delta_j}\}$ converging uniformly to a limit $\bar{\gamma}$. $\bar{\gamma}$ is the minimizer of (3) and satisfies the Euler-Lagrange equation (5).
We begin by reparametrizing the curves γ_δ by setting $t = \frac{s}{s_\delta}$. According to step 2, we have

$$|\frac{d\gamma_\delta}{dt}| = s_\delta < c.$$

By Arzela-Ascoli, there is a subsequence $\{\gamma_{\delta_j}\}$ uniformly converging to a Lipschitz continuous limit $\bar{\gamma}$. We will show that $\bar{\gamma}$ indeed is the minimizer of (3). Let ζ be a Lipschitz curve starting in x_1 and ending in x_2. Since γ_δ minimizes (4), we have with $L(\gamma) = L_0(\gamma)$

$$L_\delta(\gamma_\delta) \leq L_\delta(\zeta) = L(\zeta) + \delta \int_{-1}^{1} |\zeta'|\, ds.$$

Hence

$$\limsup_\delta L_\delta(\gamma_\delta) \leq L(\zeta).$$

Furthermore, the uniform convergence of $\{\gamma_{\delta_j}\}$ to $\bar{\gamma}$ implies

$$\liminf_j s_{\delta_j} \geq |\dot{\bar{\gamma}}| \text{ a.e.}$$

and with Fatou's lemma it follows

$$\begin{aligned} \liminf_j L_{\delta_j}(\gamma_j) &= \liminf_j \int_{-1}^{1} \sqrt{W(\gamma_{\delta_j})}\, |\dot{\gamma_{\delta_j}}|\, dt \geq \liminf_j \int_{-1}^{1} \sqrt{W(\gamma_{\delta_j})} |s_{\delta_j}|\, dt \\ &\geq \liminf_j \int_{-1}^{1} \sqrt{W(\bar{\gamma})}\, |\dot{\bar{\gamma}}|\, dt = L(\bar{\gamma}). \end{aligned}$$

Thus $\bar{\gamma}$ is minimal. One can show that for any two points on $\bar{\gamma}$, the geodesics between these two points must coincide with $\bar{\gamma}$, yielding that the Euler-Lagrange equation must hold for $\bar{\gamma}$.

Step 4 : Proving the identity $|g'(u)| = \sqrt{W(u)}$
Let $y \in \Omega$. Thanks to Step 1, there is a sequence $\{\beta_n\}$ minimizing

$$\inf_{\substack{\gamma(-1)=\alpha \\ \gamma(1)=y}} L_\delta(\gamma).$$

One can argue by modifying the arguments of Step 2 that

$$\int_{-1}^{1} |\dot{\beta}_\delta| < c$$

is bounded independently of δ. This ensures the existence of a limiting geodesic $\beta_y(t)$. Thus for all $y \in \Omega$, there is a geodesic between α and y minimizing (3).
Assume $y_1, y_2 \in \Omega$ and let β_1 and β_2 be the corresponding geodesics. Define

$$l(t) = (1-t)y_1 + ty_2 \text{ for } t \in [0,1].$$

Suppose y_1 and y_2 are such that $l(t) \in \Omega$ for all $t \in [0,1]$. Then,

$$h(y_1) = L(\beta_1) \le L(\beta_2) + L(l) = h(y_2) + L(l).$$

We obtain with the same argument that

$$h(y_2) \le h(y_1) + L(l).$$

Thus

$$\begin{aligned} |h(y_2) - h(y_1)| \le L(l) &= \int_0^1 \sqrt{W(l(t))}|y_2 - y_1|\, dt \\ &\le \|\sqrt{W}\|_\infty |y_2 - y_1|. \end{aligned}$$

This implies that h is locally Lipschitz and thereby differentiable almost everywhere in Ω. If y is a differentiable point of h and $\{x_n\}$ is a sequence with $x_n \to y$, we get, using the above calculation and

$$l_n(t) = (1-t)y + tx_n,$$

that

$$\frac{|h(y) - h(x_n)|}{|y - x_n|} \le \int_0^1 \sqrt{W((1-t)y + tx_n)}.$$

Thus

$$\lim_{x_n \to y} \frac{|h(y) - h(x_n)|}{|y - x_n|} \le \int_0^1 \sqrt{W(y)}.$$

Now, let $\beta_y(t)$ be the corresponding geodesic to y. We define the points

$$y_n := \beta_y(1 - \frac{1}{n}).$$

It follows that $\beta_y\big|_{[0,1-\frac{1}{n}]}$ is a geodesic between α and y_n. According to the generalization of the Mean Value Theorem, there is a point $t^* \in (1-\frac{1}{n}, 1)$ such that

$$\begin{aligned} h(y) - h(y_n) &= \int_0^1 \sqrt{W(\beta_y(t))}\, |\dot\beta_y(t)|\, dt - \int_0^{1-\frac{1}{n}} \sqrt{W(\beta_y(t))}\, |\dot\beta_y(t)|\, dt \\ &= \int_{1-\frac{1}{n}}^1 \sqrt{W(\beta_y(t))}\, |\dot\beta_y(t)|\, dt = \sqrt{W(\beta_y(t^*))} \int_{1-\frac{1}{n}}^1 |\dot\beta_y(t)|\, dt. \end{aligned}$$

Thus

$$|h(y) - h(y_n)| \ge \sqrt{W(\beta_y(t^*))}|y - y_n|$$

and

$$\lim_{n\to\infty} \frac{|h(y) - h(y_n)|}{|y - y_n|} \ge \sqrt{W(y)}.$$

□

The next Lemma will be used quite often in calculations

Lemma 4.5: *The weighted total variation is lower semicontinuous with respect to $L^1_{loc}(\Omega)$ strong convergence, i.e. if $u_n \to u$ in $L^1_{loc}(\Omega)$, we have*

$$TV_\omega(u) \leq \liminf_{n\to\infty} TV_\omega(u_n).$$

Proof. $u \to TV_\omega(u)$ maps to the supremum of a family of $L^1_{loc}(\Omega)$ continuous functionals and the supremum conserves lower semicontinuity. □

Now, we are set to prove Thm. 4.3. Compared to the arguments of [7], we have to adapt the reasoning why we can restrict ourselves to the case where $v_0(x)$ is defined like below.

Proof. (of Thm. 4.3)
We can restrict ourselves to the case where

$$v_0(x) = \begin{cases} a & x \in A \\ b & x \in B \end{cases}$$

and A and B are disjoint and satisfy $A \cup B = \Omega$.
If this is not the case, then we find a compact subset $\Omega' \subset\subset \Omega$, a set $P \subset \Omega'$ of positive finite mass and a number $c > 0$, such that

$$W(v_0(x)) > c \text{ for } x \in P.$$

Because $v_\epsilon \to v_0$ in $L^1(\Omega')$, we have a subsequence of v_ϵ converging point-wise almost everywhere in Ω'. This means that for ϵ small enough,

$$W(v_\epsilon) \geq c$$

on P as well. Since our weight ω is continuous, it attains a minimum on Ω' and we have

$$\infty \leq \liminf_{\epsilon\to 0} \frac{1}{\epsilon}\int_P (\min_P \omega)\; W(v_\epsilon) \leq \liminf_{\epsilon\to 0} \frac{1}{\epsilon}\int_P \omega\; W(v_\epsilon) \leq \liminf_{\epsilon\to 0} \tilde{F}_\epsilon(v_\epsilon)$$

and the Liminf Inequality holds.
Thus it suffices to consider v_0 as stated above. We have

$$\begin{aligned}
&\tilde{F}_\epsilon(v_\epsilon) - 2\int_\Omega \omega\; \sqrt{W(v_\epsilon)}\; |\nabla v_\epsilon| \\
&= \int_\Omega \omega\; \left(\frac{1}{\epsilon} W(v_\epsilon) - 2\; \sqrt{W(v_\epsilon)}\; |\nabla v_\epsilon| + \epsilon |\nabla v_\epsilon|^2\; dx\right) \\
&= \int_\Omega \omega\; \left(\frac{1}{\sqrt{\epsilon}}\sqrt{W(v_\epsilon)} - \sqrt{\epsilon}\; |\nabla v_\epsilon|\right)^2 \geq 0
\end{aligned}$$

and

$$\tilde{F}_\epsilon(v_\epsilon) \geq 2\int_\Omega \omega\; \sqrt{W(v_\epsilon)}\; |\nabla v_\epsilon|.$$

Now, let g be the function defined in Lemma 4.4, we set $h_\epsilon(x) = g(v_\epsilon(x))$, yielding

$$|\nabla h_\epsilon(x)| = \sqrt{W(v_\epsilon)}\; |\nabla v_\epsilon|.$$

Additionally, it holds that

$$\|h_\epsilon - g(v_0)\|_{L^1(\Omega)} \leq \|g(v_\epsilon) - g(v_0)\|_{L^1(\Omega)} \leq K_g \|v_\epsilon - v_0\|_{L^1(\Omega)},$$

where K_g is the Lipschitz constant of g. Thus, we can deduce that $h_\epsilon \to g(v_0)$ in $L^1(\Omega)$. We have

$$\liminf_{\epsilon\to 0} \tilde{F}_\epsilon(v_\epsilon) \geq \liminf_{\epsilon\to 0}\ 2\int_\Omega \omega\ \sqrt{W(v_\epsilon)}\ |\nabla v_\epsilon| = \liminf_{\epsilon\to 0} 2\int_\Omega \omega|\nabla h_\epsilon(x)|\ dx.$$

Thanks to the lower semicontinuity of the weighted total variation for $L^1_{loc}(\Omega)$-strong convergence (Lemma 4.5), we get

$$\liminf_{\epsilon\to 0} 2\int_\Omega \omega|\nabla h_\epsilon(x)|\ dx = 2\ \liminf_{\epsilon\to 0} \mathrm{TV}_\omega(h_\epsilon) \geq 2\ TV_\omega(g(v_0)).$$

To conclude, recall the definition of v_0, we obtain

$$g(v_0(x)) = \begin{cases} 0 & x \in A \\ g(b) & x \in B \end{cases}$$

and

$$TV_\omega(g(v_0)) = g(b)\ \mathrm{Per}_{\Omega,\omega}\{v_0 = a\}.$$

□

4.3 Limsup-Inequality

To establish Γ-convergence, it remains to proof that there is a recovery sequence. This is done in this section.

Theorem 4.6: *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. Assume that the double-well potential W satisfies condition (H1). Then, for every $v_0 \in H^1$, we find a sequence v_ϵ fulfilling $v_\epsilon \to v_0$ in L^1 and*

$$\tilde{F}_0(v_0) \geq \limsup_{\epsilon\to 0} \tilde{F}_\epsilon(v_\epsilon).$$

For the proof of the Limsup-Inequality we need Lemma 3 of [9]:

Lemma 4.7: *Let A be an open subset of $\mathbb{R}^n$ with smooth, compact and non-empty boundary. Define $g(x) = dist(x, \partial A)$ and*

$$S_t = \{x \in A |\ g(x) = t\} \text{ for } t > 0.$$

If Ω is another open subset of $\mathbb{R}^n$ such that $\mathcal{H}^{n-1}(\partial A \cap \partial\Omega) = 0$, then

$$\lim_{t\to 0} \mathcal{H}^{n-1}(S_t \cap \Omega) = \mathcal{H}^{n-1}(\partial A \cap B).$$

Proof. First, we prove the claim for $\Omega = \mathbb{R}^n$. Let

$$V_t := \{x \in A \mid 0 < g(x) < t\}.$$

By the arguments in [4], Appendix A, there exists for small t a diffeomorphism ϕ between V_t and $\partial A \times]0, t[$, such that

$$\det(D\phi)(x) = \prod_{i=1}^{n-1}\big(1 - k_i(\hat{\phi}(x))g(x)\big).$$

Here, $k_1, ..., k_{n-1}$ are the principle curvatures of ∂A and $\hat{\phi}$ is the component of ϕ on ∂A. Additionally, g is smooth on $\bar{V}_t$ and

$$Dg(x) = -\nu(\phi(x)) \text{ for all } x \in \bar{V}_t,$$

where ν is the outer normal vector on ∂A. Let ν_t be the outer normal vector (outward with respect to V_t) to S_t, then we have

$$\nu_t(x) = Dg(x) \text{ for all } x \in S_t.$$

Using the divergence theorem, we compute

$$\begin{aligned}\int_{V_t} \Delta g(x)\, dx &= \int_{\partial V_t} Dg\ \nu\ dS \\ &= \int_{\partial A} Dg\ \nu\ d\mathcal{H}^{n-1} + \int_{S_t} Dg\ \nu_t\ d\mathcal{H}^{n-1} \\ &= \mathcal{H}^{n-1}(S_t) - \mathcal{H}^{n-1}(\partial A).\end{aligned}$$

We will show that $|V_t| \to 0$ because then the first integral of the last calculation tends to 0. This implies that the Hausdorff measure of S_t converges to the one of ∂A and that the claim is proven.

We know that ∂A is compact and smooth yielding that the principle curvatures k_i are negative at some point on ∂A (We are "looking from inside" onto ∂A). Hence, due to the formula for $\det(D\phi)$, there exists $x \in \partial A \times]0, t[$ such that

$$\det(D\phi(x)) > 0.$$

$\det(D\phi(x)) \neq 0$ holds for all $x \in \partial A \times]0, t[$ because ϕ is a diffeomorphism, implying

$$\det(D\phi(x)) > 0$$

for the whole domain. Since all k_i are smooth, they have a minimum and a maximum and we can conclude that $\det(D\phi) \geq \mu > 0$ on V_t when t is small. Hence

$$\begin{aligned}\lim_{t\to 0+} |V_t| &= \lim_{t\to 0+} \int_{\partial A} \int_0^t \left(\det D\phi^{-1}(y, s)\right)\, ds\ d\mathcal{H}^{n-1} \\ &\leq \lim_{t\to 0+} t\mu^{-1}\mathcal{H}^{n-1}(\partial A) = 0\end{aligned}$$

which proves the claim for $\Omega = \mathbb{R}^n$.

Now, we deal with the general case. It holds that $S_t = \partial(A \backslash V_t)$. Hence for t small enough, we have

$$\mathcal{H}^{n-1}(S_t \cap \Omega) = \mathrm{Per}_\Omega(A \backslash V_t).$$

Since $|V_t| \to 0$ is equivalent to $\chi_{A\backslash V_t} \to \chi_A$ in $L^1(\Omega)$, we have by the lower semicontinuity of the perimeter

$$\mathcal{H}^{n-1}(\partial A \cap \Omega) = \mathrm{Per}_\Omega(A) \leq \liminf_{t\to 0} \mathrm{Per}_\Omega(A \backslash V_t) = \liminf_{t\to 0} \mathcal{H}^{n-1}(A \cap S_t).$$

We also know

$$\mathcal{H}^{n-1}(S_t \cap \Omega) \leq \mathcal{H}^{n-1}(S_t) - \mathcal{H}^{n-1}(S_t \cap (\mathbb{R}^n \backslash \bar{\Omega}))$$

and

$$\mathcal{H}^{n-1}(\partial A \cap (\mathbb{R}^n \backslash \bar{\Omega})) \leq \liminf_{t\to 0} \mathcal{H}^{n-1}(S_t \cap (\mathbb{R}^n \backslash \bar{\Omega})).$$

Due to the assumption $\mathcal{H}^{n-1}(\partial A \cap \partial\Omega) = 0$, it follows

$$\limsup_{t\to\infty} \mathcal{H}^{n-1}(S_t \cap \Omega) \leq \mathcal{H}^{n-1}(\partial A) - \mathcal{H}^{n-1}(\partial A \cap (\mathbb{R}^n \backslash \bar{\Omega})) = \mathcal{H}^{n-1}(\partial A \cap \Omega).$$

Combining the two inequalities, we obtain the statement in the general setting. □

At the end of the proof of Thm. 4.6, we will need the following approximation of the weighted total variation of a characteristic function by the weighted total variation of smooth functions using mollification.

Lemma 4.8: *Let $A \subset \Omega$ be such that $\chi_A \in BV_\omega(\Omega)$ and let η_n be a mollifier. Then,*

$$TV_\omega(\chi_A) = \lim_{n\to\infty} TV_\omega(\eta_n * \chi_A).$$

Proof. The "$\leq$"-inequality follows from the lower semi-continuity of the weighted total variation. For "$\geq$", we compute with $\psi \in C_c^1(\Omega),\ |\psi| \leq \omega$,

$$\begin{aligned}\int_\Omega (\eta_n * \chi_A)(x)\ div(\psi(x))\ dx &= \int_\Omega \Big(\int_\Omega \eta_n(y)\chi_A(x-y)\ dy\Big)\ div(\psi(x))\ dx \\ &= \int_\Omega \eta_n(y) \underbrace{\Big(\int_\Omega \chi_A(x-y)\ div(\psi(x))\ dx\Big)}_{\leq TV_\omega(\chi_A)}\ dy \\ &\leq TV_\omega(\chi_A) \int_\Omega \eta_n(y) = TV_\omega(\chi_A)\end{aligned}$$

which gives the second inequality. □

Similar to the proof of the Liminf-inequality, we need to establish the existence of an auxiliary function for the main result.

Lemma 4.9: *Let g be the function defined in Lemma 4.4. Then, there exists a smooth increasing function $\zeta : (-\infty, \infty) \to (-1, 1)$ such that the function $\xi(\tau) = \gamma_\beta(\zeta(\tau))$ satisfies*

$$2g(\beta) = \int_{-\infty}^{\infty} W(\xi(\tau)) + |\xi'(\tau)|^2\ d\tau$$

and has the asymptotic behaviour $\lim_{\tau\to-\infty} \xi(\tau) = a$ and $\lim_{\tau\to\infty} \xi(\tau) = b$.

Proof. We follow the proof of [11]. Define ζ as the solution of

$$\zeta' = \frac{\sqrt{W(\gamma_\beta(\zeta))}}{|\gamma_\beta'(\zeta)|} \quad , \quad \zeta(0) = 0.$$

Then $\xi(\tau) := \gamma_\beta(\zeta(\tau))$ has the property

$$|\xi'(\tau)| = \sqrt{W(\xi(\tau))},$$

which leads to

$$\begin{aligned}0 &= \Big(\sqrt{W(\xi(\tau))} - |\xi'(\tau)|\Big)^2 \\ \Leftrightarrow 2\sqrt{W(\xi(\tau))}|\xi'(\tau)| &= W(\xi(\tau)) + |\xi(\tau)|^2.\end{aligned}$$

This Implies

$$2g(\beta) = 2\int_{-\infty}^{\infty} \sqrt{W(\xi(\tau))}\ |\xi'(\tau)|\ d\tau = \int_{-\infty}^{\infty} W(\xi(\tau)) + |\xi'(\tau)|^2\ d\tau.$$

Because the value of $\tilde{F}_\epsilon$ is invariant under reparametrization, ξ is a solution of

$$\inf_{\substack{\gamma(-\infty)=a\\ \gamma(\infty)=b}} \int_{-\infty}^{\infty} W(\gamma(t)) + |\gamma'(t)|^2 \; dt$$

as well. For this reason, it must fulfill the Euler-Lagrange equation

$$2\xi'' = \nabla W(\xi).$$

If we differentiate this equation we get

$$2\xi''' = \sum_{i,j=1}^{n} \frac{\partial^2 W(\xi)}{\partial x_i \partial x_j} \xi'.$$

Now, we use the fact that all second directional derivatives of W are positive at a and b because W has its minima at these points and obtain that ξ must decay at an exponential rate to 0 when $|\tau| \to \infty$.

□

Finally, we can prove the main result of this section. Once again, we stick to the arguments of [7] and just adapt it at some parts to the slightly different situation. In this case we have to argue why we still can approximate the sets A, B of v_0 by sets with smooth boundary.

Proof. (Theorem 4.6)
Let v_0 be in $BV(\Omega, \omega)$,

$$v_0(x) = \begin{cases} a & x \in A \\ b & x \in B \end{cases}$$

where A and B are disjoint sets with $A \cup B = \Omega$. All other cases can be ruled out because then $F_0(v_0) = \infty$ by definition.

First, we assume that $\Gamma = \partial A \cup \partial B$ is in $\mathcal{C}^2$. We follow the arguments by [7] and define the signed distance function

$$d(x) = \begin{cases} -dist(x, \Gamma) & x \in A \\ dist(x, \Gamma) & x \in B \end{cases} .$$

In a neighborhood of Γ, d is smooth and $|\nabla d| = 1$. We define the sequence

$$\rho_\epsilon(x) = \begin{cases} \xi(\frac{-1}{\sqrt{\epsilon}}) & d(x) \le -\sqrt{\epsilon} \\ \xi(\frac{d(x)}{\epsilon}) & |d(x)| \le \sqrt{\epsilon} \\ \xi(\frac{1}{\sqrt{\epsilon}}) & d(x) \ge \sqrt{\epsilon} \end{cases} .$$

Here, we take the function $\xi(x)$ introduced in Lemma 4.4. We know that

$$\rho_\epsilon(x) \to v_0(x)$$

in $L^1(\Omega)$. Thanks to $\lim_{\tau \to -\infty} \xi(\tau) = \alpha$, $\lim_{\tau \to \infty} \xi(\tau) = \beta$ and

$$\nabla \rho_\epsilon(x) = \frac{1}{\epsilon} \xi'(\frac{d(x)}{\epsilon}) \nabla d(x)$$

for $|d(x)| \leq \sqrt{\epsilon}$, we compute:

$$\limsup_{\epsilon\to 0} \tilde{F}_\epsilon(v_\epsilon) = \limsup_{\epsilon\to 0} \frac{1}{\epsilon}\int_{\{|d(x)|\leq\sqrt{\epsilon}\}} \omega\Big(W\Big(\xi\Big(\frac{d(x)}{\epsilon}\Big)\Big)+\Big|\xi'\Big(\frac{d(x)}{\epsilon}\Big)\Big|^2\Big)\ dx.$$

Now we use the classical Coarea Formula to say

$$= \limsup_{\epsilon\to 0} \frac{1}{\epsilon}\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}}\int_{\{d=s\}} \omega(r)\Big(W\Big(\xi\Big(\frac{s}{\epsilon}\Big)+\Big|\xi'\Big(\frac{s}{\epsilon}\Big)\Big|^2\Big)d\mathcal{H}^{n-1}(r)\ ds$$

and obtain with a change of variables ($\tau = \frac{s}{\epsilon}$)

$$= \limsup_{\epsilon\to 0}\int_{-\frac{1}{\sqrt{\epsilon}}}^{\frac{1}{\epsilon}} (W(\xi(\tau)) + |\xi'(\tau)|^2)\int_{\{d=\epsilon\tau\}} w(r)\ d\mathcal{H}^{n-1}(r)\ d\tau.$$

Due to the fact that

$$2g(\beta) = \int_{-\infty}^{\infty}(W(\xi(\tau)) + |\xi'(\tau)|^2)d\tau,$$

we get to the bound

$$\leq 2g(\beta)\limsup_{\epsilon\to 0}\max_{|t|\leq\sqrt{\epsilon}}\Big(\int_{\{d=t\}}\omega(r)d\mathcal{H}^{n-1}(r)\Big).$$

Now, consider $t<0$ and define $S_t = \{x\in A|\ d(x) = t\}$ and $V_t = \{x\in A|\ t < d(x) < 0\}$. We want to show

$$\limsup_{t\to 0} \mathrm{TV}_\omega(\chi_{A\setminus V_t}) \leq \mathrm{TV}_\omega(\chi_A) \leq \liminf_{t\to 0}\mathrm{TV}_\omega(\chi_{A\setminus V_t}) \tag{7}$$

because then

$$\limsup_{\epsilon\to 0}\max_{|t|\leq\sqrt{\epsilon}}\Big(\int_{\{d=t\}}\omega(r)d\mathcal{H}^{n-1}(r)\Big)\to \mathrm{TV}_\omega(\chi_A).$$

One can show using Lemma 4.7 that $\chi_{A\setminus V_t}\to\chi_A$ in $L^1(\Omega)$ and since $\chi_{A\setminus V_t} = \chi_A(1-\chi_{V_t})$,

$$\chi_{V_t}\to 0 \text{ in } L^1(\Omega).$$

We compute

$$\int_\Omega \chi_{V_t}\ dx = \int_0^t\int_{\{d(x)=s\}}\chi_{S_s}(x)\ d\mathcal{H}^{n-1}\ ds = \int_0^t \mathcal{H}^{n-1}(S_s)\ ds.$$

We know that the last term converges to 0 when $t\to 0$ because of Lemma 4.7 . The lower semicontinuity of the weighted total variation implies

$$\int_{\{d=0\}}\omega(r)\ d\mathcal{H}^{n-1}(r) = \mathrm{Per}_{\Omega,\omega}(A) = \mathrm{TV}_\omega(\chi_A)$$

$$\leq \liminf_{t\to 0^-}\mathrm{TV}_\omega(\chi_{A\setminus V_t}) = \liminf_{t\to 0}\int_{\{d=t\}}\omega(r)\ d\mathcal{H}^{n-1}(r).$$

Now, we want to establish the first inequality of (7). Let $\phi\in C_c^1(\Omega)$ and calculate

$$\int_{A\setminus V_t}\mathrm{div}(\phi) = \int_A \mathrm{div}(\phi) - \int_{V_t}\mathrm{div}(\phi).$$

If we take the supremum over all admissible functions ϕ, we get

$$\sup_{\phi}\Big\{\int_{A\setminus V_t} \operatorname{div}(\phi)\Big\} \leq \sup_{\phi}\Big\{\int_{A} \operatorname{div}(\phi)\Big\} + \sup_{\phi}\Big\{\int_{V_t} \operatorname{div}(\phi)\Big\}$$

which is equivalent to

$$\mathrm{TV}_\omega(\chi_{A\setminus V_t}) \leq \mathrm{TV}_\omega(\chi_A) + \mathrm{TV}_\omega(\chi_{V_t}).$$

We can show that $\mathrm{TV}_\omega(\chi_{V_t}) \to 0$, because

$$\int_{V_t} \operatorname{div}(\phi) = \int_{\partial V_t} \phi * \overrightarrow{n}\, dS = \int_{S_t} \phi * \overrightarrow{n}\, dS + \int_{\partial A} \phi * \overrightarrow{n}\, dS$$

and if $t \to 0$, we have

$$\int_{S_t} \phi * \overrightarrow{n}\, dS \to -\int_{\partial A} \phi * \overrightarrow{n}\, dS.$$

In summary, we have

$$\limsup_{t^- \to 0} \mathrm{TV}_\omega(\chi_{A\setminus V_t}) \leq \mathrm{TV}_\omega(\chi_A).$$

We get the same inequalities for $t \to 0^+$ by just slightly adapting the last argument. This implies the desired inequality.

Our final task is to prove the case where $\Gamma = \partial A \cup \partial B$ is not in $\mathcal{C}^2$. We argue as in [9], Lemma 1.15. We will prove that one can approximate the sets A and B by sets A_n, B_n with C^2-boundary and

$$\chi_{A_n} \to \chi_A,\ \mathrm{Per}_\omega(A_n, \Omega) \to \mathrm{Per}_\omega(A, \Omega) \text{ and } \mathcal{H}^{n-1}(\partial A_n \cap \partial\Omega) = 0$$

(respectively B_n and B). Therefore let $\epsilon > 0$ and η_ϵ be a mollifier. We define

$$f_n = \eta_{\epsilon_n} \star \chi_A,$$

for $\epsilon_n \to 0^+$ and see that $0 \leq f_n \leq 1$. We know $f_n \in C^\infty$ and $f_n \to \chi_A$ in $L^1(\Omega)$. Next, we consider the set

$$A_{n,t} := \{x \in \mathbb{R}^n \mid f_n(x) > t\}.$$

By Sard's theorem we have for all n and L^1 a.e. $t \in (0,1)$: If $f_n(x) = t$ for a $x \in \Omega$, then $\operatorname{rank}(Df) = n$. Thus,

$$\partial A_{n,t} = \{x \in \Omega \mid f_n(x) = t\}$$

is a C^∞-manifold of dimension $n-1$. Since $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ and the sets $\{\partial A_{n,t}\}_{t\geq 0}$ are disjoint (fixed n), we have for all but countably many $t \in (0,1)$ that

$$\mathcal{H}^{n-1}(\partial A_{n,t} \cap \partial\Omega) = 0.$$

We compute

$$\begin{aligned}
\int_\Omega |f_n - \chi_A| &\geq \int_{\Omega\cap(A_{n,t}\setminus A)} |f_n - \chi_A| + \int_{\Omega\cap(A\setminus A_{n,t})} |f_n - \chi_A| \\
&= \int_{\Omega\cap(A_{n,t}\setminus A)} f_n + \int_{\Omega\cap(A\setminus A_{n,t})} |1 - f_n| \\
&\geq t\ \mathcal{L}^n(\Omega \cap (A_{n,t}\setminus A)) + (1-t)\ \mathcal{L}^n(\Omega \cap (A\setminus A_{n,t}).
\end{aligned}$$

And since $f_n \to \chi_A$ in L^1, the first integral converges to 0 and thereby

$$\mathcal{L}^n(\Omega \cap (A_{n,t} \backslash A)), \mathcal{L}^n(\Omega \cap (A \backslash A_{n,t}) \to 0.$$

Hence $\chi_{A_{n,t}} \to \chi_A$.
By the lower semicontinuity of the weighted perimeter, it follows that

$$\liminf_{n\to\infty} \operatorname{Per}_\omega(A_{n,t}, \Omega) \geq \operatorname{Per}_\omega(A, \Omega).$$

Using Lemma 4.8, then the Coarea Formula in the weighted case 3.18 and afterwards Fatou's Lemma, we obtain

$$\operatorname{Per}_\omega(A, \Omega) = TV_\omega(\chi_A) = \lim_{n\to\infty} TV_\omega(f_n) = \lim_{n\to\infty} \int_\Omega \omega\ |\nabla f_n|\ dx$$

$$= \lim_{n\to\infty} \int_0^1 \int_\Omega \omega\ |D\chi_{A_{n,t}}| \geq \int_0^1 \liminf_{n\to\infty} \int_\Omega \omega\ |D\chi_{A_{n,t}}| = \int_0^1 \liminf_{n\to\infty} \operatorname{Per}_\omega(A_{n,t}, \Omega).$$

Consequently

$$\int_0^1 \left(\liminf_{n\to\infty} \operatorname{Per}_\omega(A_{n,t}, \Omega) - \operatorname{Per}_\omega(A, \Omega) \right)\ dt \leq 0.$$

Together with the inequality above, we conclude

$$\liminf_{n\to\infty} \operatorname{Per}_\omega(A_{n,t}, \Omega) = \operatorname{Per}_\omega(A, \Omega) \ \text{ for } L^1 \text{ a. e. } t \in (0,1).$$

We have shown that we can approximate v_0 by a sequence $\{v_{0,n}\}$,

$$v_{0,n}(x) = \begin{cases} \alpha & x \in A_n \\ \beta & x \in B_n \end{cases}$$

which fulfills the condition $\Gamma = \partial A_n \cup \partial B_n \in \mathcal{C}^2$. For these functions, we use the first part of the proof to find a recovery sequence. Finally, one constructs a recovery sequence for the original v_0 using a diagonalization argument.

□

Remark 4.10: *For our application, additionally to the* Γ*-convergence of* $\tilde{F}_\epsilon$ *to* $\tilde{F}$*, it is desirable to have convergence of minimizers. This is a simple implication of the fundamental theorem of* Γ*-convergence. However, we must first ensure the existence of minimizers, which is done using the direct method in the calculus of variations.*
Observe that each $\tilde{F}_\epsilon$ *is bounded from below by 0 because* W *is non-negative. Fix* $\epsilon > 0$ *and let* $\{u_n\} \subset L^1(\Omega)$ *be a minimizing sequence of* $\tilde{F}_\epsilon$*, then:*

Without loss of generality, we assume that $u_n \in W^{1,2}(\Omega)$ *for all* n*. We want to show that the sequence is bounded in* $W^{1,2}(\Omega)$*. Since* $\{u_n\}$ *is a minimizing sequence, there exist* $N \in \mathbb{N}$ *and* $c > 0$ *such that*

$$\tilde{F}_\epsilon(u_n) < c \text{ for all } n \geq N.$$

In particular, this implies

$$\int_\Omega |\nabla u_n|^2 < c \quad \text{and} \quad \int_\Omega W(u_n) < c.$$

To obtain boundedness in $W^{1,2}(\Omega)$, it remains to establish a bound in the L^2-norm. For this purpose, we use the growth condition of the double-well potential W. On the set $\{|x| \leq l\}$, the function

$$f(x) = (|x| - l)(|x| + l) - 1$$

is always negative, implying

$$W(x) \geq |x|^2 - (l^2 + 1)^2.$$

This is equivalent to

$$|x|^2 \leq W(x) + (l^2 + 1).$$

On $\{|x| > l\}$, the growth condition $W(x) \geq q|x|^2$ yields

$$W(x) + (l^2 + 1) \geq min\{1, q\}|x|^2.$$

Combining both cases with $\int_\Omega |W(u_n)|^2 < c$, we obtain that

$$\int_\Omega |u_n(x)|^2 \leq \int_\Omega W(u_n(x)) \; dx + |\Omega|(l^2 + 1) < \infty.$$

Hence $\{u_n\}$ is bounded in $W^{1,2}(\Omega)$. By the Banach-Alaoglu theorem, there exists a (not relabeled) subsequence and a limit $u_\infty \in W^{1,2}(\Omega)$, such that

$$u_n \rightharpoonup u_\infty \text{ weakly in } W^{1,2}(\Omega).$$

By [5], Section 8.2.4 Thm.1, $\tilde{F}_\epsilon$ is weakly lower semicontinuous and we conclude that u_∞ is a minimizer of $\tilde{F}_\epsilon$.
To apply the fundamental theorem, it remains to verify that $\{\tilde{F}_\epsilon\}$ is equi-coercive. The Rellich-Kondrachov theorem yields the compact embedding $W^{1,2}(\Omega) \subset\subset L^2(\Omega)$ and since $L^2(\Omega) \subset L^1(\Omega)$, we obtain that

$$\{u \in L^1(\Omega) \mid \tilde{F}_\epsilon(u) \leq t\} \subset W^{1,2}(\Omega)$$

the equi-coercivity.
We conclude using the fundamental theorem of Γ-convergence that minimizers converge.

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