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Near Horizon Boundary Conditions for Spin-3 Gravity in Flat Space

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Betreuer
Daniel Grumiller

Mitbetreuer
Stefan Prohazka

durch
Raphaela Wutte
Matr.Nr.: 1126042
rwutte@hep.itp.tuwien.ac.at

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Student

Abstract

The holographic principle proposes a solution to one of the most prominent problems of our time – the search for a consistent, quantized theory of gravity. According to this principle a theory of gravity in $(d + 1)$ dimensions is equivalent to a quantum field theory (without gravity) in d dimensions. An important realization of this conjecture is the Anti-de-Sitter/conformal field theory (AdS/CFT) correspondence. However, since this correspondence is a strong-weak correspondence, it is hard to explicitly check the holographic principle by calculating observables on the field theory and the gravitational theory side.

Conversely, higher-spin theories lead to weak-weak dualites, which can provide useful insights into aspects of the holographic principle [1–3]. Furthermore, since calculations in three dimensions may be done in the Chern-Simons formulation and are technically less challenging than in higher dimensions, it is often useful to restrict onself to three dimensions to clear up conceptional issues and obtain a better understanding of the holographic principle. In this thesis, we construct a new set of boundary conditions for spin-3 gravity in three-dimensional flat space. This set of boundary conditions is inspired by the recent "Soft Heisenberg hair"-proposal for Einstein gravity in three-dimensional Anti-de-Sitter space [4], which has subsequently been extended to flat space [5] and higher-spin gravity in AdS space [6].

In chapter 2 we discuss the peculiarities of restricting oneself to three dimensions and review the Chern-Simons formalism and the canonical analysis. In chapter 3 we discuss boundary conditions for gravity in three-dimensional AdS space and give a review of the Brown-Henneaux boundary conditions [7] and the near horizon boundary conditions proposed in [4].

In chapter 4 we motivate the respective near horizon boundary conditions for spin-3 gravity in three-dimensional flat space and compute the canonical boundary charges and the asymptotic symmetry algebra. As in previous, related work [4–6] the boundary conditions ensure regularity of the solutions independently of the charges. The asymptotic symmetry algebra is again given by a set of $\hat{\mathfrak{u}}(1)$ current algebras. We find that the vacuum descendants generated by the charges all have the same energy as the vacuum, i.e. they are higher-spin "soft hair" in the sense of Hawking, Perry and Strominger [8]. Furthermore, we derive the entropy for solutions that are continuously connected to flat space cosmologies and find the same result as in the spin-2 case: the entropy is linear in the spin-2 zero-mode charges and independent from the spin-3 charges. Using twisted Sugawara-like constructions of the higher-spin currents we show that our simple result for entropy of higher-spin flat space cosmologies coincides precisely with the complicated earlier results expressed in terms of higher-spin zero-mode charges.

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Contents

1 Introduction

1.1 The Search for Quantum Gravity

At the beginning of the 20th century, physics underwent two grand revolutions. First, the Newtonian theory of space and time was reformed by Einstein's general relativity, which employed a new, geometrical understanding of the gravitational force. Second, it was discovered that physics on the atomic and subatomic scale was also incorrectly described by classical mechanics. The introduction of quantum mechanics enabled us to understand a vast variety of phenomena and led to multiple technical advances, while changing our understanding of nature drastically. Today, both general relativity and quantum mechanics have been very well tested individually and have experienced many successes throughout the course of the years. General relativity has been praised for its accuracy and has predicted phenomena in the cosmos, such as the possible expansion of the universe and the possible existence of black holes, long before they were experimentally accessible. The theory of quantum mechanics in turn led to the formulation of the standard model of particle physics, which manages to describe the electromagnetic, weak and strong interactions in a single model.

Throughout the history of science, a reductionist point of view appeared to be successful [9], which was affirmed by the grand success of Maxwell's electromagnetism and peaked with the success of the standard model. Nevertheless, despite many attempts to solve it, the search for a consistent quantized theory of gravity, i.e. quantum gravity remains one of the biggest physics problems to this day.

One possible theory that unifies the fundamental forces of the standard model and gravity is string theory. The fundamental objects of string theory are given by one-dimensional objects (strings) that live in a ten-dimensional spacetime. The different particles of the standard model correspond to oscillations of the string and the additional dimensions, which are needed for string theory to be consistent, are assumed to be compactified at such small scales that they are rendered unobservable. Although string theory is a beautiful theory in terms of mathematics which makes it tempting to believe in its validity, a theory which aims to describe our universe must also pass experimental tests. Therefore, string theory is often criticized for not making any testable predictions, since corrections from string theory only appear at energy scales much larger than the ones that are accessible ($\sim$ TeV) at the moment. Note that the same is true for any consistent theory of quantum gravity, since the Planck length, the scale where quantum gravity effects are certainly relevant, is too small to be accessible with current experimental technology.

1.2 AdS/CFT and the Holographic Principle

Even if string theory may not be the fundamental theory governing all laws in nature, it has already led to important advances in physics and mathematics. Among other things string theory led to the discovery of the Anti-de-Sitter/Conformal field theory (AdS/CFT) correspondence, which in its first version by Maldacena in 1997 [10] conjectured a duality (a mathematical equivalence between two theories) between a type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}_5$ and a $N = 4$ supersymmetric Yang-Mills theory.

The AdS/CFT correspondence is an important realization of a more general conjecture – the holographic principle [11]. This principle proposes a duality between a $(d+1)$ dimensional theory of gravity and a d-dimensional quantum field theory, located at the boundary of the gravitational theory. Hence, the gravitational theory in the bulk can be equivalently described by a quantum field theory living at the boundary of the spacetime in question. This in turn would allow to freely switch between a gravitational description and a quantum field theory description in one dimension lower. The name of the principle was coined by the fact that a hologram in "real life" represents a two-dimensional screen that encodes all the information necessary to reproduce a given three-dimensional object. The object can thus either be described as a three-dimensional or a two-dimensional object – both descriptions are equivalent from a physicists point of view, since no information loss occurs.
A general proof of the AdS/CFT correspondence does not exist, since this would require a full understanding of string theory in a curved background, which is lacking at the moment [12]. Nevertheless, numerous non-trivial tests of the conjectured duality are possible by calculating observables on both sides and checking whether they are in agreement (for a slightly outdated review see [13]). Although there are many mathematical tests which suggest that a correspondence exists (for instance integrability provides numerous checks, see [14] for a review), the question naturally arises whether there are also "real life" realizations of the holographic principle that confirm that our world is in fact a hologram. The most prominent example for such a realization in nature is given by the entropy of a black hole. Initially, black holes were assumed to be objects of zero entropy. Bekenstein found that this assumption would violate the second law of thermodynamics, since in this case one could decrease the amount of entropy in the universe by throwing some object with a certain amount of entropy into the black hole. Thus, one immediately runs into contradictions if one demands that black hole thermodynamics be in accordance with the laws of thermodynamics as we know them from statistical mechanics. This led Bekenstein to think that black holes are indeed non-zero entropy objects. In fact, they possess the maximum amount of entropy possible for a certain region of spacetime. An intuitive way to think about this is to imagine adding more and more mass to a given system until it collapses to a black hole, endowed with the same (or more) amout of entropy than before. More precisely, Bekenstein found that there exists an upper bound for the entropy S of a certain volume V that is proportional to the area of the black hole horizon

$$S \leq \frac{\text{Area}(\partial V)}{4G_N} = S_{BH}\,, \tag{1.1}$$

where G_N denotes Newton's constant and ∂V is the boundary of V. This comes as a surprise, since the statistical mechanics definition of entropy as the logarithm of the number of microstates compatible with a certain macrostate would rather indicate that the entropy is proportional to the volume of a given system. Due to the fact that the entropy is a measure of information, the Bekenstein bound can be interpreted as the maximal amount of information that may be put into a region V of spacetime. Hence, employing a holographic interpretation this suggests that (broadly speaking) the information might as well be described via a field theory living at the boundary. This and many similar arguments have encouraged countless amount of research into this direction.
Although it is interesting to work on, the standard AdS/CFT duality, which deals with gravitational theories in AdS space (space with *negative* cosmological constant), can not be directly applied to our universe, which is endowed with a very small, *positive* cosmological constant. Therefore, different types of non-AdS holography have been studied throughout the last years.
One line of research deals with flat space holography, which is particularly interesting, since the cosmological constant measured in our universe is incredibly small, making flat space a very good approximation for most purposes in physics.
To this day, the generality of the holographic principle is still unclear and many open questions remain. This Master thesis serves as a modest step towards a more thorough understanding of flat space holography.

1.3 Higher-Spin Gauge Theories

One extensively studied aspect of (AdS) holography are higher-spin symmetries, which were first studied by Klebanov and Polyakov [1] and Sezgin and Sundell [2], who conjectured a duality between the large N limit of the critical $3d$ $O(N)$ model and the minimal bosonic higher-spin theory in AdS_4. On the gravitational side, these higher-spin symmetries may be considered as generalizations of local coordinate transformations, while on the field theory side, they correspond to fields with spin[1] $s > 2$. On the gravitational side "higher-spin" is equivalent to higher-rank gauge field, i.e. spin-1, spin-2 and spin-3 fields are is denoted by Φ_μ, $\Phi_{(\mu\nu)}$ and $\Phi_{(\mu\nu\lambda)}$ (where the parantheses denote total symmetrization). The metic $g_{\mu\nu}$, the fundamental field of Einstein gravity, is precisely such a spin-2 field.

Higher-spin excitations appear naturally in string theory, where additionally to the massless modes of spin $s \leq 2$ an infinite tower of massive modes appears. These modes are very heavy and thus unobservable at currently accessible energy scales. Through the standard model we are acquainted with the fact that particles usually aquire mass through some kind of spontaneous symmetry breaking. Therefore, it is reasonable to ask if string theory is just a broken phase of a superior gauge theory equipped with additional higher-spin symmetries. If this was the case, one could view string tension generation as a mechanism of symmetry breaking, since the higher-spin modes become massless in the tensionless limit. Another interesting feature of higher-spin theories is that they lead to weak-weak dualities in contrast to the usual AdS/CFT correspondence that relates a strongly coupled field theory to a weakly coupled gravitational theory (or vice versa). This strong-weak duality can be used to tackle problems like singularities in general relativity or non-abelian plasma, such as the strongly coupled quark gluon plasma in quantum chromodynamics, since one may use the duality to switch to "the other side" whenever one runs into technical problems. However, at the same time this perk is a drawback, since a weak-strong duality makes calculations on the gravitational, as well as on the field theory side at least difficult, if not impossible. Since these calculations are necessary for explicit checks of the holographic principle, weak-weak dualities are very useful.

Still, while writing down a theory of free higher-spin fields is unproblematic, coupling them to gravity is not an easy endeavor and was even believed to be impossible due to various No-Go results (see [15] for a summary). In particular, at the end of the 1960s Weinberg [16] and Coleman and Mandula [17] argued that higher-spin symmetries cannot be realized in a nontrivial field theory in flat space. Fradkin and Vasiliev [18] showed that this problem may be circumvented by considering higher-spin gauge theories involving gravity on curved background. Nevertheless, these theories come at the price of needing an infinite tower of massless fields to be consistent.

The entire situation changes drastically when considering the special case of three dimensions, where interesting higher-spin theories involving gravity in flat space indeed exist [19], [20]. Only in three dimensions it is known to date that it is possible to write down a consistent theory considering only excitations up to a certain spin n and thus truncate the otherwise infinite tower of higher-spin fields.

[1] In this context a notion of spin s can be defined through the transformation properties of a field under Lorentz transformations. On the conformal field theory side this transformation behaviour is determined by the conformal weight $h = s$. Therefore, the terms spin and conformal weight will be used interchangable. More information on conformal weights is provided in appendix C.

2 Prerequisites

2.1 Gravity in three Dimensions

In (2+1) dimensions, gravity has no local propagating degrees of freedom, i.e. gravitational waves do not exist. In fact, one may show that the curvature of spacetime is completely determined in terms of the Einstein tensor. To make this more explicit, we first revisit Einstein gravity. The field equations describing the relation between the curvature of spacetime and the matter and energy distribution are given by

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G_N T_{\alpha\beta}\,, \tag{2.1}$$

where $T_{\alpha\beta}$ is the stress-energy tensor, G_N is Newton's constant, Λ is the cosmological constant, $g_{\alpha\beta}$ is the metric tensor and the Einstein tensor $G_{\alpha\beta}$ is defined as

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R\,. \tag{2.2}$$

The Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar R are contractions of the Riemann curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}$

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}\,, \qquad R = g^{\alpha\beta} R_{\alpha\beta} \tag{2.3}$$

with

$$R^{\gamma}{}_{\delta\alpha\beta} v^{\delta} = \left[\nabla_\alpha, \nabla_\beta\right] v^{\gamma}\,. \tag{2.4}$$

Note that in writing down this we have already used torsion freedom of the covariant derivative, i.e. $\left[\nabla_\alpha, \nabla_\beta\right] f = 0$ for arbitrary functions f. The covariant derivative ∇_a

$$\nabla_\alpha v^\beta = \partial_\alpha v^\beta + \Gamma^{\beta}{}_{\alpha\gamma} v^\gamma \tag{2.5}$$

consists of the ordinary partial derivative $\partial/\partial x^\alpha$ usually denoted as ∂_α plus an additional term $\Gamma^{\beta}{}_{\alpha\gamma}$, which defines an affine connection on our manifold. This correction term equips us with a notion of parallel transport between different points of the tangent space of our manifold.
To define a unique connection and thus, a unique covariant derivative we need torsion freedom (which we already employed above) and metric compatibility $\nabla_\gamma\, g_{\alpha\beta} = 0$[2]. This determines the connection coefficients $\Gamma^{\beta}{}_{\alpha\gamma}$, also called Christoffel symbols, in the context of general relativity, to be

$$\Gamma^{\gamma}{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\beta g_{\delta\alpha} + \partial_\alpha g_{\delta\beta} - \partial_\delta g_{\alpha\beta}\right)\,. \tag{2.6}$$

Equations (2.1), (2.2) and (2.4) give an explicit relation between the curvature of space encoded in the Einstein tensor $G_{\alpha\beta}$ and the matter content of the theory $T_{\alpha\beta}$. In the words of the famous physicist John Archibald Wheeler: "*Spacetime tells matter how to move and matter tells spacetime how to curve.*"

[2]Metric compatibility physically means that angles are preserved under parallel transport.

Now the consequences of restricting ourselves to three dimensions will be considered. Since the Einstein tensor is a symmetric tensor, the number of independent components are given by $\frac{1}{2}d(d+1)$. Furthermore, the symmetries of the Riemann tensor restrict the number of independent components to $\frac{1}{12}d^2(d^2-1)$. A derivation of this statement is recalled in appendix D.1. Thus, we find that the Riemann and the Einstein tensor have the same number of degrees of freedom in three dimensions

$$\#_R = \frac{1}{12}9 \cdot (9-1) = 6 = \frac{1}{2}3 \cdot (3+1) = \#_G \,. \tag{2.7}$$

Since the Einstein and the Riemann tensor are related through (2.2) and (2.3), this means that the curvature of spacetime is completely determined by the matter content of our theory and the cosmological constant. In the absence of local matter sources ($T_{\alpha\beta} = 0$) we have no dynamical gravitational degrees of freedom - there are no gravitational waves/gravitons in three dimensions. However, theories other than Einstein gravity in three dimensions allow for (typically) massive gravitons [21].
This suggests that gravity in three dimensions might not be a worthwhile topic to study. Fortunately, global effects of the manifold allow for interesting solutions, in particular for black hole solutions. The most famous example is the BTZ black hole solution found by Bañados, Teitelboim and Zanelli [22]. Although the BTZ solution is locally AdS_3, it differs from global AdS by conserved charges at the boundary of AdS spacetime. Furthermore, the causal structure of the BTZ black hole differs from global AdS: the former has two Killing horizons and singularities. Brown and Henneaux showed that these global charges generate two copies of the Virasoro algebra [7]. In fact, the mass and angular momentum of the BTZ black hole are the zero-modes of these conserved global charges. Subsequently, this led to the holographic conjecture that AdS in three dimensions may be described equivalently by a two-dimensional conformal field theory living at the boundary, see [23] and references therein..

2.2 Introduction to the Chern-Simons Formalism

We have seen in the previous section that gravity in three dimensions does not possess any local degrees of freedom, and thus is a purely topological theory. This becomes more evident when using a suitable formulation of gravity – the Chern-Simons formulation. In this section we give a brief introduction to the first order formulation of gravity in order to pave the way for the following discussion of the Chern-Simons theory. We will furthermore expand on the Hamiltonian analysis which is used to construct the canonical charges and hence, the asymptotic symmetries of our theory.

2.2.1 Non-Coordinate Basis

The Einstein-Hilbert action is given by

$$I_{EH}[g] = \frac{1}{16\pi G_N} \int_{\mathcal{M}} \mathrm{d}^3x \sqrt{-g}\,(R - 2\Lambda)\,, \tag{2.8}$$

where G_N is Newton's constant and $g = \det(g_{\mu\nu})$. The equations of motion (2.1) may be obtained from (2.8) by using the principle of stationary action and neglecting boundary terms. The metric $g_{\mu\nu}$ is the fundamental dynamical field of (2.8) and acts as a symmetric bilinear form on the tangent space of the manifold $\mathcal{M}$. In general relativity one usually chooses a basis for the tangent space with respect to some (in principle arbitrary) coordinates, the so-called coordinate basis. At a point p of the manifold the tangent space T_p is given by all tangent vectors at this point. It is common practice to take the set of derivatives $\{e_\mu\} = \{\partial_\mu\}$ along the set of chosen coordinates as basis of the tangent space. The dual space of our tangent space is called the cotangent space. It is given by the set of all maps $f : T_p \to \mathbb{R}$ and the corresponding basis elements – either called dual vectors, cotangent vectors or one-forms mathematically speaking – consist of all $\{\mathrm{d}x^\mu\}$, since $\langle \mathrm{d}x^\mu, \partial_\nu \rangle = \delta^\mu_\nu$. However, for many purposes, it is advantageous to employ a first order formulation of general relativity instead of the better known second order formulation we have already discussed[3]. First order formulations often strongly reduce the computational difficulty of a problem. Furthermore, the first order formulation of (2.8), the so-called Einstein-Hilbert-Palatini action, see (2.15) below, may be rewritten as a Chern-Simons action, see (2.18) below. Since all calculations in this thesis are done in the Chern-Simons formulation we show explicitly that the Chern-Simons action is (up to boundary terms) classically equivalent to the Einstein-Hilbert action (2.8), see appendix D.2.

In the first order formulation, the basis of the tangent space consists of an orthonormal set of vectors $\{e_a\}$, which are not related to the coordinates (i.e. non-coordinate basis). We demand that the inner product of these basis vectors is given by

$$g_{\mu\nu}\, e_\mu{}^a e_\nu{}^b = \eta_{ab}, \tag{2.9}$$

where η_{ab} is the Minkowski metric. The frame fields $\{e_a\}$ are often referred to as "vielbein", which means "many legs" in German, or in three dimensions "dreibein". The coordinate basis $\{e_\mu\}$ may be expressed in terms of the orthonormal basis $\{e_a\}$ as

$$e_\mu = e_\mu{}^a e_a\,. \tag{2.10}$$

In this context, the components of the vielbein e^a_μ are conventionally referred to as the vielbein itself. One advantage of choosing the tangent space this way is that one can easily promote objects from a flat (orthonormal basis) to a curved spacetime (coordinate basis). For instance, for a vector v^a this means

$$v^\mu = e^\mu{}_a v^a\,. \tag{2.11}$$

The introduction of the non-coordinate basis leads to an additional gauge freedom, since we may change basis vectors independently of our choice of coordinates. In fact, each basis that satisfies condition (2.9) is equally good. The transformations that connect all valid bases are the well-known Lorentz transformations. This local Lorentz invariance at each point of our manifold $\mathcal{M}$ suggests that an associated gauge field exists and indeed, this gauge field is given by the spin connection $\omega^{ab} = \omega^{ab}{}_\mu \mathrm{d}x^\mu$. We are now able to compute the action of the covariant derivative on objects with both coordinate and non-coordinate indices, since the spin connection provides us with an affine connection with respect to the flat spacetime. The covariant derivative on a generalized tensor carrying both Greek and Latin indices is defined in analogy to (2.5)

$$\nabla_\mu v^a{}_\nu = \partial_\mu v^a{}_\nu + \omega^a{}_{b\mu} v^b{}_\nu - \Gamma^\sigma{}_{\nu\mu} v^a{}_\sigma\,. \tag{2.12}$$

2.2.2 Chern-Simons Formulation of Gravity

Having introduced a non-coordinate basis, we now review the first order and subsequently also the Chern-Simons formulation of gravity. In the following, we restrict ourselves to three dimensions.

We can use the epsilon symbol to write the antisymmetric spin connection with just one index instead of two[4]

$$\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}\,. \tag{2.13}$$

Formally, this is referred to as the Hodge dualization (see appendix B) of the spin connection. The dualized Riemann curvature can now be defined as

$$R^a = \frac{1}{2}\epsilon^{abc}R_{bc} = \mathrm{d}\omega^a + \frac{1}{2}\epsilon^{abc}\omega^b \wedge \omega^c\,, \tag{2.14}$$

[3]First order or second order formulation refers to the number of derivatives of the fundamental fields appearing in the action. The number of derivatives of the metric appearing in (2.8) is two, because the curvature tensor is given in terms of the covariant derivative, see (2.4), which contains the Christoffel symbols (2.6) that in turn contain first derivatives of the metric.

[4]Hereafter we omit the form indices of the tensors in question $\omega^{ab}{}_\mu \to \omega^{ab}$ and $R^a{}_{b\mu\nu} \to R^a{}_b$.

where $\wedge$ denotes the wedge product defined in appendix B. Now, we are finally equipped with the necessary tools to rewrite (2.8) in terms of the dualized curvature two-form R^a and the dreibein e^a

$$I_{EHP}[e,\omega] = \frac{1}{8\pi G_N}\int_{\mathcal{M}} \left\{ e_a \wedge R^a - \frac{\Lambda}{6}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right\}, \tag{2.15}$$

where Λ denotes the cosmological constant. This form of the action is referred to as the Einstein-Hilbert-Palatini action. It is indeed a first order formulation of the theory of gravity, since the curvature two-form contains only first derivatives of the dualized spin connection. The field equations are then obtained by varying the action with respect to the two independent fields e^a and ω^a

$$R^a = \mathrm{d}\omega^a + \frac{1}{2}\epsilon^{abc}\omega^b \wedge \omega^c = \frac{\Lambda}{2}\epsilon^a{}_{bc} e^b \wedge e^c\,, \tag{2.16}$$

$$T^a = \mathrm{d}e^a + \epsilon^a{}_{bc}\omega^b \wedge e^c = 0\,. \tag{2.17}$$

In order to formulate general relativity as a gauge theory, the vielbein e_a and the spin connection ω_a need to be combined into a gauge field. In three dimensions, we are equipped with the necessary tools to do so, since both ω_a and the e_a have the same index structure and thus can be linearly combined into a single Lie algebra valued gauge field. The structure of (2.15) looks very similar to the Chern-Simons action in three dimensions, a topological gauge theory

$$I_{CS}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \langle A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \rangle\ . \tag{2.18}$$

Due to the fact that both theories are topological theories and have a very similar structure, the question arises naturally whether one can rewrite the Einstein-Hilbert-Palatini action in three dimensions as a Chern-Simons action. This is indeed possible and was explicitly shown by Witten in [24, 25]. The appropriate combination of dreibein and spin connection into a gauge field is

$$A = e^a \mathtt{P}_a + \omega^a \mathtt{J}_a\,, \tag{2.19}$$

where $\mathtt{P}_a$ and $\mathtt{J}_a$ generate the following Lie algebra

$$[\mathtt{P}_a, \mathtt{P}_b] = -\Lambda\epsilon_{abc}\mathtt{J}^c, \qquad [\mathtt{J}_a, \mathtt{J}_b] = \epsilon_{abc}\mathtt{J}^c, \qquad [\mathtt{J}_a, \mathtt{P}_b] = \epsilon_{abc}\mathtt{P}^c\,. \tag{2.20}$$

For different values of the cosmological constant the gauge algebra is then given by

- $\Lambda > 0$ (de Sitter): $\mathfrak{so}(3,1)$,
- $\Lambda = 0$ (Minkowski): $\mathfrak{isl}(2,\mathbb{R}) := \mathfrak{iso}(2,1) \sim \mathfrak{sl}(2,\mathbb{R}) \bigoplus_s \mathbb{R}^3$,
- $\Lambda < 0$ (Anti-de-Sitter): $\mathfrak{so}(2,2) \sim \mathfrak{sl}(2,\mathbb{R}) \bigoplus \mathfrak{sl}(2,\mathbb{R})$,

where $\bigoplus_s$ denotes the semidirect sum, $\bigoplus$ denotes the direct sum and "$\sim$" means "isomorphic to". For Anti-de-Sitter space the gauge algebra is $\mathfrak{so}(2,2)$, the global conformal algebra of $\mathbb{R}^{1,1}$, which may be interpreted as a hint for the conjectured AdS$_3$/CFT$_2$ correspondence. The final ingredient needed to rewrite (2.18) as a Chern-Simons action is a non-degenerate[5], invariant[6] bilinear form $\langle\,,\rangle$. In general, such a non-degenerate bilinear form may not be found in any number of dimensions [24], which is another reason why $d = 3$ is so special.

The invariant, non-degenerate bilinear form of (2.20) is given by

$$\langle \mathtt{J}_a, \mathtt{P}_b \rangle = \eta_{ab}, \qquad \langle \mathtt{J}_a, \mathtt{J}_b \rangle = \langle \mathtt{P}_a, \mathtt{P}_b \rangle = 0\,. \tag{2.21}$$

[5] Def: Let $\langle\,,\rangle$ be a bilinear form on a Lie algebra $\mathfrak{g}$ with $x, y \in \mathfrak{g}$. The bilinear form $\langle\,,\rangle$ is called non-degenerate, if $\langle x, y\rangle = 0\,\forall x \in \mathfrak{g} \Rightarrow y = 0$.

[6] Def: Let $\mathfrak{g}$ be a Lie algebra and $x, y, z \in \mathfrak{g}$. A bilinear form $\langle\,,\rangle$ is called invariant, if $\langle [x,y], z\rangle = \langle x, [y,z]\rangle$.

In (2.18) the bilinear over the wedge product of three gauge fields is a shorthand way of denoting

$$\langle A \wedge A \wedge A\rangle = \frac{1}{2} f_{ce}{}^{a} A^c A^e A^b \gamma_{ab}\,, \tag{2.22}$$

where γ_{ab} denotes the bilinear form and $f_{ce}{}^{a}$ are the structure constants of the Lie algebra. The Chern-Simons action (2.18) together with the generators (2.20) and the bilinear form (2.21) is up to boundary terms equivalent to the Einstein-Hilbert-Palatini action(2.15) provided that the CS-level is taken to be

$$k = \frac{1}{4G_N}\,. \tag{2.23}$$

We provide the explicit calculation in appendix D.2. The equations of motion (EOM) are obtained by varying (2.18) with respect to the gauge field A

$$F = \mathrm{d}A + A \wedge A = 0\,. \tag{2.24}$$

By inserting the expression for the gauge field in terms of the dreibein and the spin connection (2.19) into (2.24), we recover the EOM for the torsion and the curvature (2.16). Eq. (2.24) implies, at least locally (on some open set in $\mathcal{M}$), that A is a gauge transformation of the trivial connection $A = 0$, i.e.

$$A = g^{-1}\,\mathrm{d}g \qquad (\text{locally})\,. \tag{2.25}$$

Moving from the metric to the Chern-Simons formalism brings the technical advantage that one may now use well-known techniques from gauge theories on gravitational problems, such as using a gauge transformation to switch to a better suited gauge. Nevertheless, one has to be careful when doing so, since the gauge transformations that we are considering are usually finite and therefore, will generally not leave the Chern-Simons action invariant, i.e.

$$\tilde{A} = g^{-1}(A + \mathrm{d})g\,, \tag{2.26a}$$

$$S_{CS}[\tilde{A}] = S_{CS}[A] + \delta S_{CS}[A]\,. \tag{2.26b}$$

We see that an additional term $\delta S_{CS}[A]$ with respect to $S_{CS}[A]$ appears that is given by [26]

$$\delta S_{CS}[A] = \frac{k}{4\pi}\int_{\partial\mathcal{M}} \langle g^{-1}\,\mathrm{d}g A\rangle - \frac{k}{12\pi}\int_{\mathcal{M}} \langle g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\rangle\ . \tag{2.26c}$$

This additional term $\delta S_{CS}[A]$ vanishes, if

- $g \to 1$ sufficiently fast when approaching the boundary of $\mathcal{M}$
- gauge transformations are topologically trivial (such as infinitesimal gauge transformations).

However, $\delta S_{CS}[A]$ does not vanish for general finite gauge transformations. One interesting feature which will come up again in the following chapters is the connection between diffeomorphisms and infinitesimal gauge transformations. An infinitesimal gauge transformation generated by a gauge parameter ϵ is given by

$$\delta_\epsilon A = \mathrm{d}\epsilon + [A, \epsilon]\,. \tag{2.27}$$

If we now consider a special form of the gauge parameter $\epsilon = \xi^\nu A_\nu$ we get

$$\delta_{\xi^\nu A_\nu} = \partial_\mu \xi^\nu A_\nu + \xi^\nu \partial_\mu A_\nu + \xi^\nu [A_\mu, A_\nu]\,. \tag{2.28}$$

Addition of the vanishing term $\epsilon^\nu\left(\partial_\nu A_\mu - \partial_\nu A_\mu\right)$ yields after rearrangement

$$\delta_{\xi^\nu A_\nu} = \underbrace{\xi^\nu\partial_\mu A_\nu + \partial_\mu\xi^\nu A_\nu}_{\mathcal{L}_\xi A_\mu} + \underbrace{\epsilon^\nu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \epsilon^\nu[A_\mu, A_\nu]}_{\epsilon^\nu F_{\mu\nu}} = \mathcal{L}_\xi A_\mu + \epsilon^\nu F_{\mu\nu}\,. \tag{2.29}$$

Hence, we see that on-shell ($F = 0$) diffeomorphisms for gravity in three dimensions correspond to infinitesimal gauge transformations generated by parameters

$$\epsilon = \xi^\nu A_\nu\,. \tag{2.30}$$

2.3 Canonical Analysis

In this section, we review the canonical analysis for three-dimensional Einstein gravity in the Chern-Simons formulation. Since our theory has no local dynamical degrees of freedom, the investigation of the Chern-Simons action (2.18) in the Hamiltonian formalism will give us constraints on the phase space Γ. The review follows [26] and [21] closely. We find a very practical way of computing the canonical charges and the asymptotic symmetry algebra, which are of great importance in the following chapters. We start by writing down the action (2.18) in index notation

$$I_{CS}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^3x\,\epsilon^{\mu\nu\lambda}\gamma_{ab}\left(A^a{}_\mu\partial_\nu A^b{}_\lambda + \frac{1}{3}f^a{}_{cd}A^c{}_\mu A^d{}_\nu A^b{}_\lambda\right)\,, \tag{2.31}$$

where γ_{ab} denotes the bilinear form on the Lie algebra and the generators $\mathtt{T}_a$ satisfy

$$[\mathtt{T}_a, \mathtt{T}_b] = if^c{}_{ab}\mathtt{T}_c\,. \tag{2.32}$$

The gauge field A is a Lie algebra valued one-form

$$A = A^a{}_\mu\,\mathrm{d}x^\mu\mathtt{T}_a\,. \tag{2.33}$$

In the following we assume that our manifold $\mathcal{M}$ has the topology of a cylinder $\mathcal{M} = \Sigma\times\mathbb{R}$ parametrized by $x = (t, \rho, \varphi)$. As a consequence Σ is topologically a disk that we parametrize by φ and ρ with $\varphi\sim\varphi + 2\pi$. The coordinates split accordingly

$$\mathcal{M} \;\rightarrow\; \Sigma\times\mathbb{R} \tag{2.34}$$

$$x^\mu = \begin{pmatrix} t\\ \rho\\ \varphi\end{pmatrix} \;\rightarrow\; t,\; x^{\bar\mu} = \begin{pmatrix}\rho\\ \varphi\end{pmatrix}\,. \tag{2.35}$$

The boundary of the disk is located at some $\rho = \rho_0$. This 2+1 decomposition of (2.31) leads to

$$I_{CS}[A] = \frac{k}{4\pi}\int_{\mathbb{R}}\mathrm{d}t\int_\Sigma \mathrm{d}^2x\,\epsilon^{\bar\mu\bar\nu}\gamma_{ab}\left(\dot A^a{}_{\bar\mu}A^b{}_{\bar\nu} + A^a{}_0F^b{}_{\bar\mu\bar\nu} + \partial_{\bar\nu}\left(A^a{}_{\bar\mu}A^b{}_0\right)\right)\,, \tag{2.36}$$

with $F^a_{\bar\mu\bar\nu} = \partial_{\bar\mu}A^a{}_{\bar\nu} - \partial_{\bar\nu}A^a{}_{\bar\mu} + f^a{}_{bc}A^b{}_{\bar\mu}A^c{}_{\bar\nu}$ and $\epsilon^{\bar\mu\bar\nu} = \epsilon^{t\mu\nu}$. The Lagrangian reads

$$\mathcal{L}[A, \dot A] = \frac{k}{4\pi}\epsilon^{\bar\mu\bar\nu}\gamma_{ab}\left(\dot A^a{}_{\bar\mu}A^b{}_{\bar\nu} + A^a{}_0F^b{}_{\bar\mu\bar\nu} + \partial_{\bar\nu}\left(A^a{}_{\bar\mu}A^b{}_0\right)\right)\,. \tag{2.37}$$

Computing the canonical momenta from (2.37) gives

$$\phi_a{}^{\bar\mu} := \pi_a{}^{\bar\mu} - \frac{k}{4\pi}\epsilon^{\bar\mu\bar\nu}\gamma_{ab}A^b{}_{\bar\nu} \approx 0\,,$$

$$\phi_a{}^0 := \pi_a{}^0 \approx 0\,. \tag{2.38}$$

The fact that the spatial canonical momenta are proportional to the fields themselves means that introducing canonical momenta is redundant, as we can just describe our entire phase space by using the gauge fields themselves. Furthermore, the canonical momentum of the temporal component of the gauge field vanishes. This means that the fields $A^a{}_0$ are not dynamical fields, since a kinetic term for them in the Lagrangian is missing. All physical fields that we know of bring about kinetic terms. Thus, it seems natural to exclude $A^a{}_0$ from the physical phase space of our theory. Instead $A^a{}_0$ may be interpreted as Lagrange multipliers enforcing the field equations $F_{\bar{\mu}\bar{\nu}} = 0$.
If one goes from the Lagrangian formalism to the Hamiltonian formalism, the time derivatives of the canonical coordinates $\dot{q}$ are replaced by their canonical momenta p

$$\{q, \dot{q}\} \quad \rightarrow \quad \{q, p\}\ , \tag{2.39}$$
$$\mathcal{L}\,(q, \dot{q}) \quad \rightarrow \quad \mathcal{H}\,(q, p)\ , \tag{2.40}$$

where in our case the role of the canonical coordinates is played by $A^a{}_\mu$ and the the canonical momenta are denoted by $\pi_a{}^\mu$. However, if one would simply replace the velocities $\dot{A}^a{}_\mu$ by the canonical momenta (2.38) one would effectively lose degrees of freedom, since the canonical momenta only depend on $A^a{}_{\bar{\mu}}$. This tells us that there exist constraints $\phi_m(A^a{}_\mu, \dot{A}^a{}_\mu) = 0$, which restrict our phase space Γ to the physical subspace Γ_1. In our case these constraints are given by (2.38).
This also explains why we wrote "$\approx$" instead of "$=$" in (2.38). From now on we will have to distinguish between weak ($\approx$) and strong ($=$) equalities. Two functions f, g in phase space are weakly equal ($f \approx g$), if they are equal under restriction to a constraint surface. In our case this constraint surface has to include Γ_1. If f and g are equal independently of the constraints, they are called strongly equal ($f = g$).
The Poisson brackets of the canonical variables are defined as

$$\{A^a{}_\mu(\mathbf{x}), \pi_b{}^\nu(\mathbf{y})\} = \delta^a{}_b \delta_\mu{}^\nu \delta^2(\mathbf{x} - \mathbf{y})\,, \tag{2.41a}$$
$$\{A^a{}_\mu, A^b{}_\nu\} = \{\pi_a{}^\mu, \pi_b{}^\nu\} = 0\,. \tag{2.41b}$$

After this discussion we return to further development of the Hamiltonian formalism. Let us consider the canonical Hamiltonian density $\mathcal{H}_c$, which may be obtained from the Lagrangian density (2.37) via a Legendre transformation

$$\mathcal{H}_c = \pi^\mu_a \dot{A}^a_\mu - \mathcal{L} \approx -\frac{k}{4\pi} \epsilon^{\bar{\mu}\bar{\nu}} \gamma_{ab} \left(A^a{}_0 F^b{}_{\bar{\mu}\bar{\nu}} + \partial_{\bar{\nu}} \left(A^a{}_{\bar{\mu}} A^b{}_0 \right) \right)\ , \tag{2.42}$$

where we already used (2.38). Throughout calculations involving $\mathcal{H}_c$, we must be consistent with the constraints of our system. Instead of enforcing the constraints by hand each time, it is useful to introduce the total Hamiltonian

$$\mathcal{H}_T = \mathcal{H}_c + u^a{}_\mu \phi_a{}^\mu\,, \tag{2.43}$$

where the constraints (2.38) are added, using arbitrary multipliers $u^a{}_\mu$. From this one may infer the following relations (also see appendix D.3)

$$\frac{\partial H_T}{\partial \pi_a{}^\mu} = \frac{\partial H_c}{\partial \pi_a{}^\mu} + u^b{}_\nu \frac{\partial \phi_b{}^\nu}{\partial \pi_a{}^\mu} = \dot{A}_\mu{}^a\,, \tag{2.44a}$$
$$\frac{\partial H_T}{\partial A_a{}^\mu} = \frac{\partial H_c}{\partial A_a{}^\mu} + u^b{}_\nu \frac{\partial \phi_b{}^\nu}{\partial A^\mu_a} = -\frac{\partial \mathcal{L}}{\partial A_a{}^\mu} \approx -\dot{\pi}^\mu_a\,, \tag{2.44b}$$

which together with the constraints

$$\phi_a{}^\mu \approx 0\,, \tag{2.44c}$$

give exactly the EOM that can be derived from the variational principle in the Lagrange formalism

$$\delta \int \mathrm{d}^3x \mathcal{L} = \int \mathrm{d}^3x \left(\pi_a^\mu \dot{A}_\mu^a - \mathcal{H}_T + u^a{}_\mu \phi_a{}^\mu\right) . \tag{2.45}$$

Using the definition of the Poisson bracket (2.41a) and the Hamiltonian EOM (2.44) we may deduce for an arbitrary function $g = g\left(A_a{}^\mu, \pi_a{}^\mu\right)$ that

$$\{g, \mathcal{H}_T\} = \frac{\partial g}{\partial A_a{}^\mu}\frac{\partial H_T}{\partial \pi_a{}^\mu} - \frac{\partial g}{\partial \pi_a{}^\mu}\frac{\partial H_T}{\partial A_a{}^\mu} \approx \frac{\partial g}{\partial A_a{}^\mu}\dot{A}_a{}^\mu + \frac{\partial g}{\partial \pi_a{}^\mu}\dot{\pi}_a{}^\mu = \dot{g}\,. \tag{2.46}$$

2.3.1 Consistency Equations

Since we are dealing with a constrained system, it is natural to demand that time evolution takes place on the constraint surface. This may be regarded as a consistency requirement. Hence, we require that the primary constraints are conserved during the time evolution of the system, i.e.

$$\dot{\phi}_a{}^\mu = \{\phi_a{}^\mu, \mathcal{H}_T\} \approx 0\,. \tag{2.47}$$

This leads to the conditions

$$\mathcal{K}_a \equiv -\frac{k}{4\pi}\epsilon^{\bar{\mu}\bar{\nu}}\gamma_{ab}F^b{}_{\bar{\mu}\bar{\nu}} \approx 0\,, \tag{2.48}$$

$$\nabla_{\bar{\mu}}A^a{}_0 - u^a{}_{\bar{\mu}} \approx 0\,, \tag{2.49}$$

where $\nabla_{\bar{\mu}}X^a = \partial_{\bar{\mu}}X^a + f^a{}_{bc}A^b{}_{\bar{\mu}}X^c$ is the covariant derivative. Equation (2.48) leads to a secondary constraint, while (2.49) determines $u^a{}_{\bar{\mu}}$. At last we may employ the Hamiltonian EOM (2.44) to simplify (2.49). Due to the fact that canonical momenta only appear within the constraints, differentiation with respect to them just gives us the Lagrange multipliers

$$\dot{A}^a{}_{\bar{\mu}} = \frac{\partial \mathcal{H}_T}{\partial \pi_a{}^{\bar{\mu}}} = u^a_{\bar{\mu}}\,. \tag{2.50}$$

This allows us to rewrite (2.49) as

$$\nabla_{\bar{\mu}}A^a{}_0 - u^a{}_{\bar{\mu}} = \nabla_{\bar{\mu}}A^a{}_0 - \partial_0 A^a{}_{\bar{\mu}} = \partial_{\bar{\mu}}A^a{}_0 - \partial_0 A^a{}_{\bar{\mu}} + f^a{}_{bc}A^b{}_{\bar{\mu}}A^c{}_0 = F^a{}_{\bar{\mu}0} \approx 0\,. \tag{2.51}$$

Plugging (2.48) and (2.49) into (2.43) we may write the total Hamiltonian as

$$\mathcal{H}_T = A^a{}_0\bar{\mathcal{K}}_a + u^a{}_0\phi_a{}^0 + \partial_{\bar{\mu}}\left(A^a{}_0\pi_a{}^{\bar{\mu}}\right) \tag{2.52}$$

with

$$\bar{\mathcal{K}}_a = \mathcal{K}_a - \nabla_{\bar{\mu}}\phi_a^{\bar{\mu}}\,. \tag{2.53}$$

Up to now we were differentiating between the constraints in terms of primary and secondary constraints. However, the discrimination that is physically relevant is the one between first *class* and second *class* quantities. While first class constraints generate gauge transformations, second class constraints do not. Instead they restrict the phase space.

A variable $f = f(p,q)$ is called first class if it has weakly vanishing Poisson brackets with all other constraints in the theory. If this does not hold, the variable is called second class. We compute the Poisson brackets of the constraints $\phi_a{}^\mu, \bar{\mathcal{K}}_a$ by employing the commutation relations (2.41a) and find that

$$\{\phi_a{}^{\bar{\mu}}(\mathbf{x}), \phi_b{}^{\bar{\nu}}(\mathbf{y})\} = -\frac{k}{2\pi}\epsilon^{\bar{\mu}\bar{\nu}}\gamma_{ab}\delta^2(\mathbf{x}-\mathbf{y})\,, \tag{2.54a}$$

$$\{\phi_a{}^{\bar{\mu}}(\mathbf{x}), \bar{\mathcal{K}}_a(\mathbf{y})\} = -f_{ab}{}^c \phi_c{}^{\bar{\mu}} \delta^2(\mathbf{x}-\mathbf{y})\,, \tag{2.54b}$$

$$\{\bar{\mathcal{K}}_a(\mathbf{x}), \bar{\mathcal{K}}_b(\mathbf{y})\} = -f_{ab}{}^c \bar{\mathcal{K}}_c \delta^2(\mathbf{x}-\mathbf{y})\,. \tag{2.54c}$$

All Poisson brackets involving $\phi_a{}^0$ vanish identically. From (2.54) we see that only $\phi_a{}^0$ and $\bar{\mathcal{K}}_a$ vanish weakly with every other constraint and are thus first class constraints, while $\phi_a{}^{\bar{\mu}}$ are second class constraints. Since $\phi_a{}^{\bar{\mu}}$ are not generators of gauge transformations and hence effectively restrict our phase space, we would like to encode this information not only in the Hamiltonian (as we already did by $\mathcal{H}_c \to \mathcal{H}_T$), but also in the time evolution of our system, i.e. the Poisson brackets. Inclusion of the second class constraints promotes the Poisson brackets to Dirac brackets. For the case at hand this means treating the second class constraints $\phi_a{}^{\bar{\mu}} \approx 0$ as strong equalities, i.e. $\phi_a{}^{\bar{\mu}} = 0$. This way we obtain the Dirac brackets

$$\{A^a{}_{\bar{\mu}}, A^b{}_{\bar{\nu}}\}_{DB} = \frac{2\pi}{k} h^{ab} \epsilon_{\bar{\mu}\bar{\nu}} \delta^2(\mathbf{x}-\mathbf{y}), \tag{2.55}$$

where $\epsilon_{\bar{\mu}\bar{\nu}}$ is obtained via $\epsilon_{\bar{\mu}\bar{\alpha}}\epsilon^{\bar{\alpha}\bar{\nu}} = \delta^{\bar{\mu}}{}_{\bar{\nu}}$. Now we are finally able to verify that counting the degrees of freedom per spacetime point $\mathbf{x}$ in the Chern-Simons formalism gives in fact zero – as expected. The number of degrees of freedom # is given by

$$\# = N - (2N_1 + N_2)\,, \tag{2.56}$$

where N denotes the dimension of our phase space, N_1 is the number of first class constraints and N_2 is the number of second class constraints. Note that each first class constraint comes with exactly one related gauge condition due to the fact that first class constraints generate gauge transformations. After fixing the gauge the gauge conditions (first class constraints) may be treated the same way as second class constraints. Hence, these constraints have to be taken into account into the Dirac bracket. Note that since no first class constraints appear in (2.55), the relation does not change. The phase space at a point $\mathbf{x}$ is spanned by the fields $A^a{}_{\bar{\mu}}(\mathbf{x})$ and their canonical momenta $\pi^a{}_{\bar{\mu}}(\mathbf{x})$ and has the dimension $6\mathfrak{D}$, where $\mathfrak{D}$ denotes the dimension of our Lie algebra. The number of first class constraints ($\phi_a{}^0$, $\bar{\mathcal{K}}_a$) is $N_1 = 2\mathfrak{D}$, the number of second class constraints ($\phi_a{}^{\bar{\mu}}$) is $N_2 = 2\mathfrak{D}$. Hence, we indeed find that

$$\# = 6\mathfrak{D} - (4\mathfrak{D} + 2\mathfrak{D}) \equiv 0 \qquad (\text{per point}). \tag{2.57}$$

This counting is valid at each point and thus gives $\infty - \infty$ in total. Hence, it could yield something finite or infinite of "lower degree" (read: "zero per point"). It is therefore no contradiction with later statements that a physical state space in three-dimensional gravity indeed exists due to boundary states.

2.3.2 Generators of Gauge Transformation and Canonical Boundary Charges

In the previous subsection we have already mentioned that first class constraints generate gauge transformations. We now explicitly construct their gauge generator $\mathcal{G}$. Per definition, $\mathcal{G}$ generates gauge transformations by acting on the dynamical variables via Dirac brackets[7]. In 1982 Castellani developed an algorithm for the construction of all gauge generators of a constrained Hamiltonian system [27]. Generally speaking, the procedure goes as follows. We start by taking the subsequent ansatz for the gauge generator

$$\mathcal{G}[\epsilon(t)] = \epsilon^{(k)}(t)\mathcal{G}_k + \epsilon^{(k-1)}(t)\mathcal{G}_{k-1} + \ldots + \epsilon(t)\mathcal{G}_0\,, \tag{2.58}$$

where G_k has to be a primary first class constraint and all other $G_n (n < k)$ must be first class constraints that satisfy

$$\mathcal{G}_k = C_{PFC}\,, \tag{2.59a}$$

[7]Note that the entire procedure is insensitive to the existence of second class constraints, since their only effect is to reduce the phase-space of the theory.

$$\mathcal{G}_{k-1} + \{\mathcal{G}_k, H_t\} = C_{PFC}\,, \tag{2.59b}$$
$$\dots \qquad \dots$$
$$\{\mathcal{G}_0, H_T\} = C_{PFC}\,. \tag{2.59c}$$

Hence, one starts with a primary first class constraint and calculates its Dirac bracket with the total Hamiltonian to determine the gauge generator up to primary first class constraints. The procedure stops when we obtain the constraint G_0, for which the Dirac bracket with H_T again gives a primary first class constraint. Note that one must at each step try to add suitable primary first class constraints in order to find the basic gauge generators $\mathcal{G}_k$, which cannot be split into a sum of independent gauge generators $\sum_{m\neq k}\mathcal{G}_m$. One finds that for the case in question the minimal chain is given by

$$\tilde{\mathcal{G}} = \epsilon^a \bar{K}_a{}^0 + D_0\epsilon^a\pi_a{}^0\,. \tag{2.60}$$

For our purpose it is useful to define the so-called smeared generator, which is obtained by integrating $\tilde{\mathcal{G}}$ over the spatial surface Σ

$$\mathcal{G} = \int_\Sigma \mathrm{d}^2x\,\tilde{\mathcal{G}} = \int_\Sigma \mathrm{d}^2x\left(\epsilon^a\bar{K}_a{}^0 + D_0\epsilon^a\pi_a{}^0\right)\,. \tag{2.61}$$

One can show that the gauge generator creates the following gauge transformations via $\delta_\epsilon\bullet := \{\bullet, \mathcal{G}[\epsilon]\}$[8]

$$\delta_\epsilon A^a{}_0 = D_0\epsilon^a\,, \tag{2.62a}$$
$$\delta_\epsilon A^a{}_{\bar{\mu}} = D_{\bar{\mu}}\epsilon^a\,, \tag{2.62b}$$
$$\delta_\epsilon \pi_a{}^0 = -f_{ab}^{\ \ c}\epsilon^b\pi_c{}^0\,, \tag{2.62c}$$
$$\delta_\epsilon \pi_a{}^{\bar{\mu}} = \frac{k}{4\pi}\epsilon^{\bar{\mu}\bar{\nu}}\gamma_{ab}\partial_{\bar{\nu}}\epsilon^b - f_{ab}^{\ \ c}\epsilon^b\pi_c{}^{\bar{\mu}}\,, \tag{2.62d}$$
$$\delta_\epsilon \phi_a{}^{\bar{\mu}} = -f_{ab}^{\ \ c}\epsilon^b\pi_c{}^{\bar{\mu}}\,. \tag{2.62e}$$

In principle, this would be the end result for our generators. However, the fact that we are considering a theory with a boundary leads to respective boundary terms, which render the generator $\mathcal{G}$ non-functionally differentiable. This we wish to avoid that at all costs, since the generators act on dynamic variables via Poisson brackets and thus should have well-defined derivatives. Let us consider a variation of the gauge generator $\mathcal{G}$ in field space for field independent gauge parameters, i.e. $\delta\epsilon = 0$

$$\begin{aligned}\delta\mathcal{G}[\epsilon] = \int_\Sigma\left(\delta\left(D_0\epsilon^a\pi_a{}^0\right) + \epsilon^a\delta\bar{K}_a\right) = \int_\Sigma \mathrm{d}^2x\Bigg(& f^a_{\ bc}\epsilon^c\pi_a{}^\mu\delta A^b{}_\mu + D_\mu\epsilon^a\delta\pi_a{}^\mu + \frac{k}{4\pi}\epsilon^{\bar{\mu}\bar{\nu}}\gamma_{ab}\partial_{\bar{\mu}}\epsilon^a\delta A^b{}_{\bar{\nu}} \\ & - \partial_{\bar{\mu}}\left(\frac{k}{4\pi}\epsilon^{\bar{\mu}\bar{\nu}}\gamma_{ab}\epsilon^a\delta A^b{}_{\bar{\nu}} + \epsilon^a\delta\pi_a{}^{\bar{\mu}}\right)\Bigg)\,.\end{aligned} \tag{2.63}$$

The first three terms are bulk terms and do not spoil functional differentiability, but the last term – a boundary term – does. Since we require our canonical generators to be functionally differentiable, we improve their form by adding another boundary term that precisely cancels the boundary term in (2.63), i.e.

$$\delta\mathcal{G}_{\text{diff}}[\epsilon] = \delta\mathcal{G}[\epsilon] + \delta\mathcal{Q}[\epsilon]\,, \tag{2.64}$$

where the variation of the canonical boundary charge is given by

$$\delta\mathcal{Q}[\epsilon] = \int_\Sigma d^2x\,\partial_{\bar{\mu}}\left(\frac{k}{4\pi}\epsilon^{\bar{\mu}\bar{\nu}}\gamma_{ab}\epsilon^a\delta A^b{}_{\bar{\nu}} + \epsilon^a\delta\pi_a{}^{\bar{\mu}}\right)\,. \tag{2.65}$$

[8]Throughout the entire thesis we differentiate carefully between δ_ϵ, which is an abbreviation for the action of a gauge generator on a variable $\bullet$ via $\delta_\epsilon\bullet := \{\bullet, \mathcal{G}[\epsilon]\}$, and δ, which stands for the variation in field space.

This expression may be further simplified by setting $\phi_a{}^{\bar{\mu}} = 0$ and hence, going to the reduced phase space.

$$\begin{aligned}\delta\mathcal{Q}[\epsilon] &= \int_\Sigma \mathrm{d}^2x\,\partial_{\bar{\mu}}\left(\frac{k}{4\pi}\epsilon^{\bar{\mu}\bar{\nu}}\gamma_{ab}\epsilon^a\delta A^b{}_{\bar{\nu}} + \epsilon^a\delta\left(\frac{k}{4\pi}\epsilon^{\bar{\mu}\bar{\nu}}\gamma_{ab}A^b{}_{\bar{\nu}}\right)\right) = \frac{k}{2\pi}\int_\Sigma \mathrm{d}^2x\,\partial_{\bar{\mu}}\left(\epsilon^{\bar{\mu}\bar{\nu}}\gamma_{ab}\epsilon^a\delta A^b{}_{\bar{\nu}}\right)\\ &= \frac{k}{2\pi}\int_\Sigma \mathrm{d}\rho\,\mathrm{d}\varphi\,\gamma_{ab}\,\epsilon^a\left(\partial_\rho(\underbrace{\epsilon^{\rho\varphi}}_{=1}\delta A^b{}_\varphi) + \partial_\varphi(\underbrace{\epsilon^{\varphi\rho}}_{=-1}\delta A^b{}_\rho)\right) = \frac{k}{2\pi}\int_\Sigma \mathrm{d}\left(\gamma_{ab}\epsilon^a\delta A^b\right)\end{aligned} \tag{2.66}$$

The use of Stokes' Theorem[9] leads to the following expression for the variation of the boundary charge

$$\delta\mathcal{Q}[\epsilon] = \frac{k}{2\pi}\int_{\partial\Sigma}\mathrm{d}\varphi\,\gamma_{ab}\epsilon^a\delta A^b_\varphi = \frac{k}{2\pi}\int_{\partial\Sigma}\mathrm{d}\varphi\,\langle\epsilon\delta A_\varphi\rangle\,. \tag{2.67}$$

Whether this expression is functionally integrable depends on the explicit form of the gauge parameter ϵ and thus on the theory under consideration. For constant ϵ we may trivially integrate (2.67) and get

$$\mathcal{Q}[\epsilon] = \frac{k}{2\pi}\int_{\partial\Sigma}\mathrm{d}\varphi\,\langle\epsilon A_\varphi\rangle\,. \tag{2.68}$$

2.3.3 Asymptotic Symmetries

Finally, after having laid the groundwork by performing the canonical analysis, we are able to discuss the asymptotic symmetries of our theory. Asymptotic symmetries are symmetries that leave the proposed asymptotic form of the field in question – in gravitational theories this would be the metric or the Chern-Simons connection – invariant. This, however, does not mean that the solution does not change, but rather means that under the action of an asymptotic symmetry a solution is mapped to a generically physically distinct solution, which differs from the original one by global charges such as (2.67). The algebra of asymptotic symmetries is determined by the Poisson brackets of the improved gauge generator $\mathcal{G}_{\text{diff}}$, which are promoted to Dirac brackets in the context of constrained Hamiltonian systems.

$$\{\mathcal{G}_{\text{diff}}[\epsilon], \mathcal{G}_{\text{diff}}[\lambda]\} = \mathcal{G}_{\text{diff}}[\sigma(\epsilon,\lambda)] + \mathcal{Z}[\epsilon,\lambda]\,, \tag{2.69}$$

where $\mathcal{Z}[\epsilon,\lambda]$ denotes possible central terms and $\sigma(\epsilon,\lambda)$ denotes a composite gauge parameter. As already discussed in section 2.3.2, promoting Poisson to Dirac brackets in our case means that the second class constraints are strongly set to zero. Additionally, after fixing the gauge, also the first class constraints are strongly set to zero. This in turn means that the bulk part of the gauge generator, which is just the sum of first class constraints multiplied by certain parameters, see (2.60), *identically* vanishes. Thus, the asymptotic symmetry algebra may be determined by computing the Dirac brackets of the canonical boundary charges

$$\{\mathcal{Q}[\epsilon], \mathcal{Q}[\lambda]\} = \mathcal{Q}[\sigma(\epsilon,\lambda)] + \mathcal{Z}[\epsilon,\lambda]\,. \tag{2.70}$$

In general, this asymptotic symmetry algebra may be determined by "simply" evaluating the Dirac brackets using (2.55). As this is often rather cumbersome, we will employ a shortcut to determine the asymptotic symmetry algebra throughout this thesis. Given two functions $\mathcal{V},\mathcal{W}$ and a canonical boundary charge $Q[\epsilon] = \int\mathrm{d}\varphi\,\epsilon(x)\mathcal{V}(x)$ one may use the fact that this charge is the generator of infinitesimal gauge transformations

$$\delta_\epsilon\bullet = \{\bullet, Q[\epsilon]\}\,. \tag{2.71}$$

So, if one knows the transformation behaviour of $\mathcal{W}$ under gauge transformations one can determine the Dirac bracket of $\{\mathcal{V}(\varphi),\mathcal{W}(\varphi)\}$ via

$$\delta_\epsilon\mathcal{W}(\varphi) = -\{Q[\epsilon],\mathcal{W}(\varphi)\} = -\int\mathrm{d}\varphi\,\epsilon(\mathbf{x})\{\mathcal{V}(\mathbf{x}),\mathcal{W}(\mathbf{y})\}\,. \tag{2.72}$$

[9] $\int_\mathcal{M}\mathrm{d}\omega = \int_{\partial\mathcal{M}}\omega$, where d denotes the exterior derivative.

Before closing this section and thus the review of the canonical analysis, we want to demonstrate the relation between the canonical boundary charges and physical properties of the system in consideration, already advertised in section 2.1. We recall our current knowledge: First of all, we know that the algebra of gauge transformations is spanned by (2.70). Furthermore, we know that certain gauge transformations correspond to diffeomorphisms (on shell), see (2.28). The last piece of the puzzle is Noether's theorem, which states that *every continuous symmetry transformation leads to a related conservation law.* In the present formalism this theorem is realized the following way: Killing vectors ξ^μ are generators of isometries, flows that preserve the form of the metric. Thus, Killing vectors generate symmetry transformations that preserve the notion of distance on our manifold. Statements well known from classical mechanics that claim that the Hamiltonian H and the angular momentum operator J are the generators of time translations and rotations, respectively, can now be translated into the Chern-Simons formalism. The Killing vectors $\xi = \frac{\partial}{\partial t}$ and $\kappa = \frac{\partial}{\partial \varphi}$ are related to the parameter of gauge transformations through (2.30). Thus, mathematically speaking we find that

$$\mathcal{H} = \mathcal{Q}[\epsilon = \xi^\mu A_\mu = A_t] \overset{(2.68)}{=} \frac{k}{2\pi}\int_{\partial\Sigma} \mathrm{d}\varphi\ \langle A_t A_\varphi \rangle\ , \tag{2.73}$$

$$\mathcal{J} = \mathcal{Q}[\epsilon = \kappa^\mu A_\mu = A_\varphi] \overset{(2.68)}{=} \frac{k}{2\pi}\int_{\partial\Sigma} \mathrm{d}\varphi\ \langle A_\varphi A_\varphi \rangle\ . \tag{2.74}$$

2.4 Anti-de-Sitter Spacetimes

As we have seen in section 2.2, one particularly practical feature of spacetimes with negative cosmological constant is that the gauge algebra $\mathfrak{so}(2,2)$ is a direct sum of two copies of $\mathfrak{sl}(2,\mathbb{R})$. We now explicitly perform the split, since it will be used heavily throughout the entire next chapter. We start by defining the new generators as

$$\mathtt{J}_a^\pm = \frac{1}{2}\left(\mathtt{J}_a \pm l\,\mathtt{P}_a\right)\ . \tag{2.75}$$

These new generators satisfy

$$[\mathtt{J}_a^+, \mathtt{J}_b^-] = 0\,, \qquad [\mathtt{J}_a^\pm, \mathtt{J}_b^\pm] = \epsilon_{ab}{}^c \mathtt{J}_c^\pm\ . \tag{2.76}$$

The split can now be explicitly realized via

$$\mathtt{J}_a^+ = \begin{pmatrix} \mathtt{T}_a^+ & 0 \\ 0 & 0 \end{pmatrix}\ , \qquad \mathtt{J}_a^- = \begin{pmatrix} 0 & 0 \\ 0 & \mathtt{T}_a^- \end{pmatrix} \tag{2.77}$$

where both $\mathtt{T}_a^+$ and $\mathtt{T}_a^-$ satisfy an $\mathfrak{sl}(2,\mathbb{R})$ algebra. Furthermore, we deduce the following bilinear form for $\mathtt{T}_a^+$ and $\mathtt{T}_a^-$ from (2.21)

$$\langle \mathtt{T}_a^+, \mathtt{T}_b^+ \rangle = \frac{l}{2}\eta_{ab}\,, \qquad \langle \mathtt{T}_a^-, \mathtt{T}_b^- \rangle = -\frac{l}{2}\eta_{ab}\,. \tag{2.78}$$

It is now possible to rewrite the gauge field A as

$$A = \begin{pmatrix} \left(\omega^a + \frac{1}{l}e^a\right)\mathtt{T}_a^+ & 0 \\ 0 & \left(\omega^a - \frac{1}{l}e^a\right)\mathtt{T}_a^- \end{pmatrix} = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix}\ . \tag{2.79}$$

Thus, after implementation of this split the Chern-Simons action (2.18) splits into

$$S[A^+, A^-] = S_{CS}[A^+] + S_{CS}[A^-]\,, \tag{2.80}$$

with the invariant bilinear form given by (2.78). However, due to the fact that both $\mathtt{T}_a^+$ and $\mathtt{T}_a^-$ satisfy an $\mathfrak{sl}(2,\mathbb{R})$ algebra, it is convenient not to distinguish between the two generators, i.e. setting $\mathtt{T}_a = \mathtt{T}_a^+ = \mathtt{T}_a^-$.

The only issue one has to take care of when doing so is the minus sign appearing in (2.78). However, the minus sign can easily be introduced manually by taking instead of the sum the difference of the two Chern-Simons actions in (2.80), i.e.

$$S[A^+, A^-] = S_{CS}[A^+] - S_{CS}[A^-]\,. \tag{2.81}$$

Obviously, the split has to be taken care of whenever the bilinear form appears, which is also the case in the canonical boundary charges, i.e.

$$Q[\epsilon^+, \epsilon^-] = Q[\epsilon^+] - Q[\epsilon^-] \tag{2.82a}$$

with

$$Q[\epsilon^+] = \frac{k}{2\pi}\int \mathrm{d}\varphi\, \langle \epsilon^+ \delta A^+_\varphi \rangle\;, \qquad Q[\epsilon^-] = \frac{k}{2\pi}\int \mathrm{d}\varphi\, \langle \epsilon^- \delta A^-_\varphi \rangle\;. \tag{2.82b}$$

The metric may be recovered from (2.79) via

$$g_{\mu\nu} = \frac{l^2}{2}\left\langle A^+_\mu - A^-_\mu, A^+_\nu - A^-_\nu \right\rangle\;, \tag{2.83}$$

since

$$g_{\mu\nu} = \frac{l^2}{2}\left\langle A^+_\mu - A^-_\mu, A^+_\nu - A^-_\nu \right\rangle = \frac{l^2}{2}\cdot 2\frac{e^a{}_\mu}{l}\frac{e^b{}_\nu}{l}\left\langle \mathtt{T}_a, \mathtt{T}_b \right\rangle \overset{(2.78)}{=} e^a{}_\mu e^b{}_\nu \eta_{ab}\,, \tag{2.84}$$

which is just the usual expression of the metric as the contraction over the local dreibein, see (2.9).

Change of Basis

Until now it was convenient to use the generators $\mathtt{T}_a$ as a basis of the $\mathfrak{sl}(2,\mathbb{R})$ algebra given by

$$[\mathtt{T}_a, \mathtt{T}_b] = \epsilon_{ab}{}^c \mathtt{T}_c \tag{2.85}$$

and equipped with the bilinear form (2.78). However, throughout all the calculations in chapters 3 and 4 we employ a different basis for the $\mathfrak{sl}(2,\mathbb{R})$ algebra in order to be in accordance with the notation in [4–6]. The change of basis is defined by the following linear combination

$$\mathtt{L}_0 = \mathtt{T}_1\,, \tag{2.86}$$
$$\mathtt{L}_1 = \mathtt{T}_0 + \mathtt{T}_2\,, \tag{2.87}$$
$$\mathtt{L}_{-1} = \mathtt{T}_0 - \mathtt{T}_2\,. \tag{2.88}$$

Furthermore, we absorb the AdS radius l into k via

$$k = \frac{1}{4G_N} \to \frac{l}{4G_N}\,. \tag{2.89}$$

Clearly, the generators $\mathtt{L}_n$ still fulfill the $\mathfrak{sl}(2,\mathbb{R})$ algebra (after all, only the basis was changed), which may be written as

$$[\mathtt{L}_n, \mathtt{L}_m] = (n-m)\mathtt{L}_{n+m}\,, \tag{2.90}$$

where n runs from -1 to 1 and the bilinear form is given by

$$\langle \mathtt{L}_1, \mathtt{L}_{-1}\rangle = -1\,, \qquad \langle \mathtt{L}_0, \mathtt{L}_0\rangle = \frac{1}{2}\,. \tag{2.91}$$

2.5 Anti-de-Sitter Higher-Spin Gravity

As already mentioned in the introduction higher-spin gravity theories provide an interesting class of gravitational theories, exhibiting more symmetries than just diffeomorphism and local Lorentz invariance. Furthermore, we recall that only in three dimensions it is known to be possible to write down a consistent theory considering only excitations up to a certain spin n. An interesting (and potentially confusing) aspect of higher-spin theories is that the metric and associated notions such as Riemannian curvature, singularities or horizons are not gauge-invariant entities anymore. Nevertheless, field configurations exist that may be most naturally interpreted as higher-spin black holes or higher-spin cosmologies[10], i.e. solutions equipped with some characteristic gauge invariant quantity such as some characteristic temperature and entropy (see [28] for a review). Therefore, since a straightforward geometrical interpretation is not possible in the metric formalism while the extension from ordinary spin-2 gravity to higher-spin gravity in the Chern-Simons formalism is fairly straightforward, one usually employs the latter formulation when discussing higher-spin gauge symmetries. Additionally, the Chern-Simons formulation has a precise notion of gauge invariance that can be extended to higher-spin symmetries. In the Chern-Simons formalism regularity of a black hole can be defined through holonomies (see [29], [30]) of the connection A; geodesics and their proposed higher-spin extensions can be related to Wilson lines (see [31]).
The extension from spin-2 to spin-N gravity in the case of AdS can be performed by simply replacing the gauge group $\mathfrak{sl}(2,\mathbb{R})\bigoplus\mathfrak{sl}(2,\mathbb{R})$ by $\mathfrak{sl}(N,\mathbb{R})\bigoplus\mathfrak{sl}(N,\mathbb{R})$. It was shown in [32] that for $N \geq 2$ such a Chern-Simons theory describes a theory of gravity coupled to a finite tower of massless integer spin-$s \leq N$ fields. Thus, the problem of incorporating the usual spin-2 gravity into the higher-spin theory reduces to the problem on how to embed $\mathfrak{sl}(2,\mathbb{R})$ into $\mathfrak{sl}(N,\mathbb{R})$. Dependent on the embedding qualitatively different higher-spin theories are obtained. However, note that one can also use different gauge algebras such as $\mathfrak{hs}[\lambda]\bigoplus\mathfrak{hs}[\lambda]$ as a gauge algebra, which describes spin-2 gravity coupled to spin fields $s = 3, 4, ...\infty$. In this thesis we will focus on the principal embedding of $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(3,\mathbb{R})$ or $\mathfrak{isl}(2,\mathbb{R}) \hookrightarrow \mathfrak{isl}(3,\mathbb{R})$, respectively[11].
Hence, higher-spin gravity in the principal embedding for $\Lambda < 0$ can be described by a $\mathfrak{sl}(N,\mathbb{R})\bigoplus\mathfrak{sl}(N,\mathbb{R})$ Chern-Simons theory with the corresponding action given by (2.80) and (2.18) with the bilinear form denoting the trace with respect to $\mathfrak{sl}(N,\mathbb{R})$. Furthermore, the Chern-Simons level k is adjusted such that it matches the normalization of the Einstein-Hilbert action, i.e.

$$k = \frac{l}{8G_N \langle \mathtt{L}_0, \mathtt{L}_0 \rangle}\,, \tag{2.92}$$

with

$$\langle \mathtt{L}_0, \mathtt{L}_0 \rangle = \frac{N\left(N^2-1\right)}{12}\,, \tag{2.93}$$

where $\mathtt{L}_0$ is the Cartan subalgebra generator of $\mathfrak{sl}(2,\mathbb{R})$ algebra contained in $\mathfrak{sl}(N,\mathbb{R})$. The metric can be obtained from the Chern-Simons connection via [28]

$$g_{\mu\nu} = \frac{l^2}{2\langle \mathtt{L}_0, \mathtt{L}_0 \rangle} \langle A_\mu^+ - A_\mu^-, A_\nu^+ - A_\nu^- \rangle\ . \tag{2.94}$$

[10] Flat space cosmologies represent flat space analogons to black holes in three dimensions, since they exhibit a cosmological horizon and they carry a mass and an angular momentum.

[11] We recall (see section 2.2.2) that since $\mathfrak{isl}(2,\mathbb{R}) \sim \mathfrak{sl}(2,\mathbb{R})\bigoplus_s\mathbb{R}^3$, a certain embedding $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(3,\mathbb{R})$ implies an embedding $\mathfrak{isl}(2,\mathbb{R}) \hookrightarrow \mathfrak{isl}(3,\mathbb{R})$.

3 Boundary Conditions for AdS Space

In field theories the physical content of the theory is given by the field equations and the boundary conditions. While it is common practice in non-gravitational theories to demand that the fields asymptotically vanish, demanding that the metric asymptotically vanishes at the boundary of spacetime is a highly unnatural choice for gravitational theories, which results in a singularity. Nevertheless, while in other field theories vanishing of the field in question just denotes a transition to the vacuum state, a singularity in general relativity usually denotes either a problem in the theory itself or a physically very distinct point like the center of a black hole for instance. Hence, these boundary conditions would lead to highly unnatural spacetimes.

Instead, a better approach is to assume that the metric asymptotes to a certain solution of Einstein's equations. In particular, for the case of flat space one assumes that the metric g asymptotically takes the form of the Minkowski metric η. In the cases of de-Sitter and Anti-de-Sitter the most natural choice is to assume that the metric asymptotes to global de-Sitter g_{dS} or Anti-de-Sitter g_{AdS} spacetime. In the following we consider three-dimensional manifolds $\mathcal{M}$, which are taken to have the topology of a cylinder $\mathcal{M} = \Sigma \times \mathbb{R}$, parametrized by $x^i = (t, r, \varphi)$. The boundary of the cylinder is then located at $r \to \infty$.

$$g_{\Lambda=0} \overset{r\to\infty}{\longrightarrow} \eta + \text{subleading terms}\,, \tag{3.1}$$

$$g_{\Lambda<0} \overset{r\to\infty}{\longrightarrow} g_{\text{AdS}} + \text{subleading terms}\,, \tag{3.2}$$

$$g_{\Lambda>0} \overset{r\to\infty}{\longrightarrow} g_{\text{dS}} + \text{subleading terms}\,, \tag{3.3}$$

It is furthermore important to recall that global de-Sitter/Anti-de-Sitter space in global coordinates is given by

$$\mathrm{d}s^2_{\text{dS}} = -\left(1 - \frac{r^2}{l^2}\right) dt^2 + \left(1 - \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2\,\mathrm{d}\varphi^2\,, \tag{3.4}$$

$$\mathrm{d}s^2_{\text{AdS}} = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2\,\mathrm{d}\varphi^2\,, \tag{3.5}$$

with the cosmological constant given as $\Lambda = \pm 1/l^2$, respectively.

3.1 Brown-Henneaux Boundary Conditions

The objective of this section is to specify what we mean by *asymptotically Anti-de-Sitter* spacetimes, i.e. we will consider a set of metrics that approaches the metric of AdS_3 in a certain way when going to infinity. Asymptotically AdS spacetimes in the sense of Brown and Henneaux are required to fulfill the boundary conditions

$$
\begin{aligned}
g_{tt} &= -\frac{r^2}{l^2} + \mathcal{O}(1)\,, && (3.6a)\\
g_{tr} &= \mathcal{O}\left(\frac{1}{r^3}\right)\,, && (3.6b)\\
g_{t\varphi} &= \mathcal{O}(1)\,, && (3.6c)\\
g_{rr} &= \frac{l^2}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right)\,, && (3.6d)\\
g_{r\varphi} &= \mathcal{O}\left(\frac{1}{r^3}\right)\,, && (3.6e)\\
g_{\varphi\varphi} &= r^2 + \mathcal{O}(1)\,, && (3.6f)
\end{aligned}
$$

where $\mathcal{O}(r^n)$ denotes that the fluctuation of the corresponding metric component behaves *at most* proportional to r^n. These boundary conditions may *equivalently* be stated in the *Fefferman-Graham* expansion [33], where the metric is expanded in orders of r^2 and the proposed form of the metric is taken to be

$$
\mathrm{d}s^2 = \frac{l^2}{r^2}\,\mathrm{d}r^2 + \gamma_{ij}(r, \tilde{x}^k)\,\mathrm{d}\tilde{x}^i\,\mathrm{d}\tilde{x}^j \tag{3.7a}
$$

with $\tilde{\mathbf{x}} = (t/l, \varphi)$. Close to the boundary $r \to \infty$ the tensor $\gamma_{ij}(r, \tilde{x}^k)$ may be expanded in orders of r^2, i.e.

$$
\gamma_{ij} = r^2 g^{(0)}_{ij}(\tilde{x}^k) + \mathcal{O}(1)\,. \tag{3.7b}
$$

We call metrics asymptotically AdS$_3$ in the sense of Brown and Henneaux, if

$$
g^{(0)}_{ij}\,\mathrm{d}\tilde{x}^i\,\mathrm{d}\tilde{x}^j = \eta_{ij}\,\mathrm{d}\tilde{x}^i\,\mathrm{d}\tilde{x}^j\,. \tag{3.7c}
$$

Thus, the Brown-Henneaux boundary conditions may be regarded as Dirichlet boundary conditions with a flat boundary metric on the cylinder located at infinity [34]. It was shown by Bañados in [35] that the most general solution subject to the boundary conditions (3.6) and (3.7), respectively, (up to *trivial* diffeomorphisms[12]) is given by

$$
\mathrm{d}s^2 = \frac{l^2}{r^2}\,\mathrm{d}r^2 - \left(r\,\mathrm{d}x^+ + \frac{2\pi l^2}{kr}\mathcal{L}_-(x^-)\,\mathrm{d}x^-\right)\left(r\,\mathrm{d}x^- + \frac{2\pi l^2}{kr}\mathcal{L}_+(x^+)\,\mathrm{d}x^+\right)\,, \tag{3.8}
$$

where $k = l/4G_N$ is the Chern-Simons level and we employed light-cone coordinates $x^\pm = \frac{t}{l} \pm \varphi$. The solution space is parametrized by two arbitrary functions $\mathcal{L}_-(x^-)$ and $\mathcal{L}_+(x^+)$ and contains, among others, the following solutions:

- Global AdS$_3$ in global coordinates $\mathcal{L}_-(x^-) = \mathcal{L}_+(x^+) = \frac{k}{8\pi}$
- BTZ black hole for $\mathcal{L}_+(x^+) = -\frac{1}{4\pi}(Ml - J) = \text{const}$ and $\mathcal{L}_-(x^-) = -\frac{1}{4\pi}(Ml + J) = \text{const}$

The functions $\mathcal{L}_-(x^-)$ and $\mathcal{L}_+(x^+)$ are also referred to as *state-dependent* functions, since they depend on the state, i.e the particular solution of our system. Note that requiring that a given metric is part of this most general class (3.8) of solutions (up to trivial diffeomorphisms) is equivalent to prescribing the fall-off conditions (3.6), (3.7) for the metric components at large distances.
Another useful set of coordinates, which will be used in the following are the Gaussian coordinates, which

[12]By trivial diffeomorphisms we mean diffeomorphisms that do not change the canonical boundary charges.

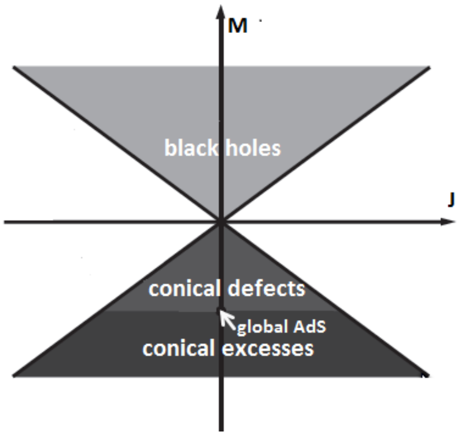

Figure 3.1: Spectrum of the Brown-Henneaux (= asymptotically AdS) boundary conditions taken from [36]. Black holes exist for $M \geq 0$, $|J| \leq Ml$. Global AdS$_3$ is separated by a gap from the continuous black hole spectrum. For $M < 0$ conical singularities appear in the spectrum: a conical defect occurs if φ has a period less then 2π as it is in the case of a cone – the result is a curvature singularity on the tip. A conical excess means that the period is greater than 2π, which also results in a singularity.

are obtained from the global coordinates by transforming the radial coordinate acoording to $r = e^{\rho}\, l$. In Gaussian coordinates (3.8) becomes

$$ds^2 = l^2 \left[\mathrm{d}\rho^2 - \frac{2\pi}{k} \left\{ \mathcal{L}_+ (\mathrm{d}x^+)^2 + \mathcal{L}_- (\mathrm{d}x^-)^2 \right\} - \left\{ e^{2\rho} + \frac{4\pi^2}{k^2} \mathcal{L}_+ \mathcal{L}_- e^{-2\rho} \right\} \mathrm{d}x^+ \mathrm{d}x^- \right], \tag{3.9}$$

where $\mathcal{L}_+ \equiv \mathcal{L}_+(x^+)$ and $\mathcal{L}_- \equiv \mathcal{L}_-(x^-)$. Following the discussion of section 2.4 we know that metric (3.9) may equally be described using two Lie algebra valued gauge fields A^+ and A^- (Chern-Simons gauge fields) that fulfill $\mathfrak{sl}(2,\mathbb{R})$ (see appendix A.1 for details on the algebra) We partially fix the gauge by choosing

$$A^\pm = b_\pm^{-1} \left(a^\pm(x^\pm) + \mathrm{d} \right) b_\pm \tag{3.10a}$$

with

$$b_\pm = b_\pm(\rho) = e^{\pm \rho L_0} \tag{3.10b}$$

and $a^\pm$, hereafter also referred to as “auxiliary connection”, being given by

$$a^\pm = \pm \left(\mathtt{L}_{\pm 1} + \frac{2\pi}{k} \mathcal{L}_\pm \mathtt{L}_{\mp 1} \right) dx^\pm \,. \tag{3.10c}$$

Different solutions parametrized by distinct and generally non-constant values of $\mathcal{L}_+$ and $\mathcal{L}_-$ may be connected through finite gauge transformations (2.26a). These improper gauge transformations are not true gauge transformations in the sense that they leave the system in question invariant. Actually, they change the canonical boundary charges which in turn – as the following computation explicitly shows – define physical properties of the system in consideration. This way a BTZ black hole parametrized by a certain mass and angular momentum may be mapped to global AdS through a gauge transformation, which changes the canonical boundary charges. These gauge transformations are precisely spanned by the asymptotic symmetry algebra, which is per definition (see section 2.3) the algebra of gauge transformations that preserves the boundary conditions given by (3.6) or (3.7) modulo proper gauge transformations. Thus, the asymptotic symmetry algebra solely contains improper gauge transformations. The spectrum of solutions is given by (3.10) or (3.9), respectively can be nicely summed up in a picture, see Fig. 3.1.

3.1.1 Canonical Boundary Charges and Asymptotic Symmetry Algebra

In the following, we determine the asymptotic symmetry algebra in accordance to the procedure we already discussed in section 2.3. We consider infinitesimal gauge transformations generated by parameters

$\tilde{\epsilon}^{\pm}$, which we choose as

$$\tilde{\epsilon}^{\pm} = b_{\pm}^{-1}\epsilon^{\pm}(x^{\pm})\, b_{\pm} \tag{3.11}$$

with $b_{\pm}$ given by (3.10b). Assuming this specific form for the gauge parameters $\tilde{\epsilon}^{+}$ and $\tilde{\epsilon}^{-}$ is smart, since this takes us to the gauge of the auxiliary connection a. Furthermore, (3.11) already implements the information that improper gauge transformations preserving the form of $a^{+} = a^{+}(x^{+})$ and $a^{-} = a^{-}(x^{-})$ should only act on the subspaces spanned by x^{+} and x^{-} respectively[13]. The asymptotic symmetry algebra corresponds to the set of infinitesimal gauge transformations

$$\delta_{\epsilon^{\pm}} a^{\pm} = \mathrm{d}\epsilon^{\pm} + [a^{\pm}, \epsilon^{\pm}]\,, \tag{3.12}$$

that preserve the form of (3.10). These transformations take the form

$$\epsilon^{\pm}(x^{\pm}) = \lambda_{\pm}\mathrm{L}_{\pm 1} \mp \lambda_{\pm}'\mathrm{L}_0 + \left(\frac{\lambda_{\pm}''}{2} + \frac{2\pi}{k}\mathcal{L}_{\pm}\lambda_{\pm}\right)\mathrm{L}_{\mp 1}\,, \tag{3.13}$$

where $\lambda_{\pm} \equiv \lambda_{\pm}(x^{\pm})$ are arbitrary parameters and prime denotes a derivative with respect to the argument of the function, i.e. $f'(x^{\pm}) = \partial_{x^{\pm}} f(x^{\pm})$. Under the transformation defined by (3.11) and (3.13) the functions $\mathcal{L}_{+}$ and $\mathcal{L}_{-}$ transform according to $\mathcal{L}_{+} \to \mathcal{L}_{+} + \delta_{\epsilon^{+}}\mathcal{L}_{+}$ and $\mathcal{L}_{-} \to \mathcal{L}_{-} + \delta_{\epsilon^{-}}\mathcal{L}_{-}$ with

$$\delta_{\epsilon^{\pm}}\mathcal{L}_{\pm} = \pm\left(\lambda_{\pm}\mathcal{L}_{\pm}' + 2\mathcal{L}_{\pm}\lambda_{\pm}' + \frac{k}{4\pi}\lambda_{\pm}'''\right)\,. \tag{3.14}$$

Thus, under a boundary condition preserving transformation g, parametrized by (3.11) and (3.13), one solution is mapped to a *physically distinct* solution. This solution is now parametrized by *different* state-dependent functions $\mathcal{L}$ and $\bar{\mathcal{L}}$, but takes the same *form*, i.e.

$$a^{\pm} = a^{\pm}(\mathcal{L}_{\pm}) \to a^{\pm} = a^{\pm}(\mathcal{L}_{\pm} + \delta_{\epsilon^{\pm}}\mathcal{L}_{\pm})\,. \tag{3.15}$$

The variation of the canonical boundary charges associated with the asymptotic symmetries spanned by (3.12) and (3.13) may be readily calculated in the Chern-Simons formalism, see (2.82) and (2.67). Furthermore, we may use the fact that the boundary charges $\mathcal{Q}[\epsilon^{+},\epsilon^{-}] = \mathcal{Q}[\epsilon^{+}] - \mathcal{Q}[\epsilon^{-}]$ stay invariant under a change of gauge from the connection A to the auxiliary connection a, since

$$\begin{aligned}\delta\mathcal{Q}^{+}[\epsilon^{+}] &= \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle\tilde{\epsilon}^{+}\delta A_{\varphi}^{+}\rangle = \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle b_{+}^{-1}\epsilon^{+}b_{+}\delta(b_{+}^{-1}(a_{\varphi}^{+} + \mathrm{d})b_{+})\rangle \\ &\overset{\delta b_{+}=0}{=} \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle b_{+}^{-1}\epsilon^{+}\delta a_{\varphi}^{+}b_{+}\rangle = \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle\epsilon^{+}\delta a_{\varphi}^{+}\rangle\,,\end{aligned} \tag{3.16}$$

where we used cyclicity of the trace in the last step[14]. Obviously, the same line of reasoning may also be followed for $\delta\mathcal{Q}^{-}[\epsilon^{-}]$ and thus, by using (3.12), (3.13) and (A.2) we obtain

$$\delta\mathcal{Q}^{+}[\epsilon^{+}] = \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle\tilde{\epsilon}^{+}\delta A_{\varphi}^{+}\rangle = \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle\epsilon^{+}\delta a_{\varphi}^{+}\rangle = \int \mathrm{d}\varphi\,\lambda_{+}\delta\mathcal{L}_{+}\,, \tag{3.17a}$$

$$\delta\mathcal{Q}^{-}[\epsilon^{-}] = \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle\tilde{\epsilon}^{-}\delta A_{\varphi}^{-}\rangle = \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle\epsilon^{-}\delta a_{\varphi}^{-}\rangle = -\int \mathrm{d}\varphi\,\lambda_{-}\delta\mathcal{L}_{-}\,. \tag{3.17b}$$

These charges are integrable, finite and conserved in time. The asymptotic symmetry algebra can then be computed via $\delta_{\epsilon_1}\mathcal{Q}[\epsilon_2] = \{\mathcal{Q}[\epsilon_2], \mathcal{Q}[\epsilon_1]\}$ by employing the usual trick (2.71). Using the transformation behaviour (3.14) and decomposing $\mathcal{L}_{+}$ and $\mathcal{L}_{-}$ into Fourier modes

$$L_m^{+} = \frac{k}{2\pi}\int e^{im\varphi}\mathcal{L}_{+}\,\mathrm{d}\varphi\,, \qquad L_m^{-} = \frac{k}{2\pi}\int e^{im\varphi}\mathcal{L}_{-}\,\mathrm{d}\varphi\,, \tag{3.18}$$

[13]This can of course also be derived by simply inserting a general function $\epsilon^{\pm}(x^{+},x^{-})$ into (3.12).

[14]Remember that the bilinear form is merely a trace over the representation of the gauge algebra.

we obtain

$$[L_m^+, L_n^+] = (m-n)L_{m+n}^+ + \frac{c}{12}m^3\delta_{n+m,0}\,, \tag{3.19a}$$

$$[L_m^-, L_n^-] = (m-n)L_{m+n}^- + \frac{c}{12}m^3\delta_{n+m,0}\,, \tag{3.19b}$$

$$[L_m^+, L_n^-] = 0\,, \tag{3.19c}$$

with the central element c given by

$$c = 6k = \frac{3l}{2G_N}\,, \tag{3.19d}$$

where we already replaced the Dirac brackets with commutators via $i\{\,,\,\} \to [\,,\,]$ to quantize the system. We see that the algebra associated to symmetry transformations that preserve asymptotically AdS_3 boundary conditions in the sense of (3.6) is given by two copies of the *Virasoro algebra* with *central charge* c (3.19d). The Virasoro algebra (3.19) is the algebra of local conformal tranformations in two dimensions and is the central extension of the *Witt algebra*. Interestingly, the algebra (3.19) is infinite-dimensional (recall $m \in \mathbb{Z}$) – a peculiarity of the algebra of conformal transformations in two dimensions. Note that the standard redefinitions $L_0^+ \to L_0^+ + \frac{c}{24}$, $L_0^- \to L_0^- + \frac{c}{24}$ change the algebra's central extensions to $\frac{c}{12}m(m^2-1)$ and therefore lead to the the Virasoro algebra with $\mathfrak{sl}(2,\mathbb{R})$-invariant subalgebra (2.90). Furthermore, note that the central extensions already appear on the algebra level before the quantization of the system by promoting the Dirac brackets to commutators.
The algebra (3.19) was first derived by Brown and Henneaux in their seminal paper [7] and gave rise to the conjecture that the holographic dual of AdS_3 is a conformal field theory in two dimensions, i.e. a CFT_2. Before closing this subsection, we explicitly calculate the angular momentum from (3.10) following the discussion at the end of subsection 2.3.3 for constant $\mathcal{L}_+$, $\mathcal{L}_-$

$$\begin{aligned}\delta M &\overset{(2.73)}{=} \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle A_t^+\delta A_\varphi^+\rangle - \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle A_t^-\delta A_\varphi^-\rangle \overset{(3.16)}{=} \frac{k}{2\pi}\int \mathrm{d}\varphi\,\left(\langle a_t^+\delta a_\varphi^+\rangle - \langle a_t^-\delta a_\varphi^-\rangle\right)\\ &= \frac{k}{2\pi}\int \mathrm{d}\varphi\left\{-\frac{2\pi}{kl}\left(\delta\mathcal{L}_+ + \delta\mathcal{L}_-\right)\right\} = -\frac{2\pi}{l}\left(\delta\mathcal{L}_+ + \delta\mathcal{L}_-\right)\,,\end{aligned} \tag{3.20}$$

$$\begin{aligned}\delta J &\overset{(2.73)}{=} \frac{k}{2\pi}\int_{\partial\Sigma} \mathrm{d}\varphi\,\langle A_\varphi^+\delta A_\varphi^+\rangle - \frac{k}{2\pi}\int_{\partial\Sigma} \mathrm{d}\varphi\,\langle A_\varphi^-\delta A_\varphi^-\rangle \overset{(3.16)}{=} \frac{k}{2\pi}\int_{\partial\Sigma} \mathrm{d}\varphi\,\left(\langle a_\varphi^+\delta a_\varphi^+\rangle - \langle a_\varphi^-\delta a_\varphi^-\rangle\right)\\ &= \frac{k}{2\pi}\int \mathrm{d}\varphi\left\{\frac{2\pi}{k}\left(\delta\mathcal{L}_+ - \delta\mathcal{L}_-\right)\right\} = 2\pi\left(\delta\mathcal{L}_+ - \delta\mathcal{L}_-\right)\,.\end{aligned} \tag{3.21}$$

This expression can be trivially functionally integrated and yields precisely the result we proposed above.

3.1.2 Addition of Chemical Potentials

We have seen that the Brown-Henneaux boundary conditions (3.10) lead to a CFT consisting of two copies of the Virasoro algebra. Hence, on the CFT side we would have two quasi-primary fields of conformal weight two. As usual in (quantum) field theory one can introduce source terms, which modify the solutions of the field equations in question. In literature these source terms are usually referred to as "chemical potentials" – we will expand on the reason for this name later on in this subsection. As in the rest of this thesis we will discuss the effect of source terms exclusively on the gravitational side – for more information consult for instance the excellent review [28]. In the case of the Brown-Henneaux boundary conditions chemical potentials are introduced by adding $\nu_\pm \mathtt{L}_\pm$ as additional terms to the temporal component $a_t^\pm$ of the auxiliary connections $a^\pm$ (3.10c) and making a general ansatz for the other components, which are then fixed by solving (2.24). By making this ansatz the $a_\varphi^\pm$ component and thus the asymptotic symmetry algebra is not modified. This leads to

$$a^{\pm} = \pm\left(\mathtt{L}_{\pm 1} + \frac{2\pi}{k}\mathcal{L}_{\pm}\mathtt{L}_{\mp 1}\right)\mathrm{d}x^{\pm} \pm \frac{1}{l}\Lambda_{\pm}(\nu_{\pm})\,\mathrm{d}t\,, \tag{3.22a}$$

with

$$\Lambda_{\pm}(\nu_{\pm}) = \nu_{\pm}\mathtt{L}_{\pm 1} \mp \nu_{\pm}'\mathtt{L}_0 + \left(\frac{\nu_{\pm}''}{2} + \frac{2\pi}{k}\mathcal{L}_{\pm}\nu_{\pm}\right)\mathtt{L}_{\mp 1}\,, \tag{3.22b}$$

where $x' = \partial_\varphi x$. *Constant* chemical potentials can be absorbed by performing a coordinate transformation and therefore, the Brown-Henneaux boundary conditions with *constant* chemical potentials are equivalent to the ones without chemical potentials. We note that (3.22b) has precisely the form of a gauge parameter generating asymptotic symmetry transformations, see (3.13). This relation between gauge parameters and chemical potentials is not a feature solely characteristic for the case at hand, but holds in general, since any connection can be transformed into any other connection through an improper gauge transformation (recall that we are dealing with a theory of locally flat connections). Hence, one would think that (3.22) is already contained in the solution space spanned by (3.10c). Yet, this is not the case. While the state-dependent functions $\mathcal{L}_{\pm}$ vary under an asymptotic symmetry transformation, the chemical potentials $\nu_{\pm}$ are – per definition – fixed, i.e. $\delta_\epsilon \nu_{\pm} \equiv 0$.
Furthermore, note that the highest weight boundary conditions (3.22) can be brought into a different form, which we will use in the following, via the redefinitions $\mu_{\pm} = (1+\nu_{\pm})/l$

$$A^{\pm} = b_{\pm}^{-1}\left(\mathrm{d} + a^{\pm}\right)b_{\pm}\,, \qquad b_{\pm} = e^{\pm\rho \mathtt{L}_0}\,, \tag{3.23a}$$

$$a_\varphi^{\pm} = \left(\mathtt{L}_{\pm 1} + \frac{2\pi}{k}\mathcal{L}_{\pm}\mathtt{L}_{\mp 1}\right)\,, \qquad a_t^{\pm} = \pm\left(\mu_{\pm}\mathtt{L}_{\pm 1} \mp \mu_{\pm}'\mathtt{L}_0 + \left(\frac{1}{2}\mu_{\pm}'' + \frac{2\pi}{k}\mathcal{L}_{\pm}\mu_{\pm}\right)\mathtt{L}_{\mp 1}\right)\,. \tag{3.23b}$$

Comment

The fact that introducing *constant* chemical potentials leads to nothing new should not surprise us. In fact, this was to be expected, since as in ordinary statistical mechanics each chemical potential comes with a connected charge. However, one chemical potential is already present in the case of (3.10c) – the temperature β, corresponding to the periodicity of imaginary time $\tau \sim \tau + \beta$.
For simplicity's sake we will restrict ourselves to constant chemical potentials $\mu_+ = \mu_- = \mu$ and constant state-dependent function $\mathcal{L}_+ = \mathcal{L}_- = \mathcal{L}$. A notion of temperature can then be defined by performing a Wick rotation and computing the holonomy around the contractible τ-cycle [30]

$$\mathcal{H}_{\mathcal{C}} = \mathcal{P}e^{\int_{\mathcal{C}} a_\tau^{\pm}\,\mathrm{d}\tau} = \mathbb{1}\,, \tag{3.24}$$

which leads to the condition

$$\beta\mu = n\pi\sqrt{\frac{k}{2\pi\mathcal{L}}}\,, \tag{3.25}$$

where n is an arbitrary integer. Note that ordinary Brown-Henneaux boundary conditions without chemical potentials can be reproduced by setting $\mu = 1/l$. In the absence of rotation the black hole partition function can then be defined as

$$Z(\beta) = \mathrm{Tr}\,\left[e^{-\beta H}\right] = \mathrm{Tr}\,\left[e^{-\alpha\mathcal{L}}\right]\,, \tag{3.26}$$

where β is given by (3.25) and H given by (3.20). After the first equal sign we absorbed all irrelevant constants via a redefinition of the chemical potential β. At this point we see that the introduction of a constant chemical potential in the spin-2 case indeed does not lead to an additional chemical potential, since (3.25) relates β and μ. In the more general case of other charges Q (for instance higher-spin charges) being present the partition function takes the form

$$Z(\beta, \gamma) = \mathrm{Tr}\left[e^{\alpha\mathcal{L}+\gamma\mathcal{Q}}\right]. \tag{3.27}$$

where γ is the chemical potential connected to the charge in question. Thus, chemical potentials appear in our theory as *Lagrange multipliers* in front of conserved charges, which is in complete analogy to the way they are introduced in ordinary statistical mechanics.

3.2 Near Horizon Boundary Conditions

In the last section we discussed boundary conditions inspired by an asymptotic perspective. In particular, we discussed the most general solution which asymptotically approaches AdS_3 spacetime. In this section we discuss boundary conditions first proposed in [4]. These boundary conditions are rather inspired by a near horizon than by an asymptotic perspective. More specifically, we discuss the most general solution subject to the condition that the metric approaches Rindler spacetime for $r \to 0$. Rindler space in Gaussian coordinates[15] is given by

$$\mathrm{d}s^2 = -a^2 r^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + \gamma^2\,\mathrm{d}\varphi^2\,, \tag{3.28}$$

where a is the Rindler acceleration and $r = 0$ stands for the location of the Rindler horizon. The angular coordinate is assumed to be periodic $\varphi \sim \varphi + 2\pi$. The horizon area is given by

$$A = \int \mathrm{d}\varphi\,\gamma\,. \tag{3.29}$$

with $\gamma > 0$ in order to always render the horizon area positive.

This setting is of interest, since spacetime geometry around non-extremal horizons is universally approximated by Rindler space [37,38]. While there exists exhaustive literature on extremal black holes [39–41] in the context of holography, non-extremal black holes are not that well investigated.

The convenient aspect of our boundary conditions is that they per definition guarantee the existence of a regular[16], non-extremal horizon and are thus very well suited when asking conditional questions like "Given a black hole, what are the scattering amplitudes in a given channel?" or "Given a black hole, can we microscopically account for the Bekenstein-Hawking entropy?" [4]. The most general solution of the Einstein equations obeying (3.28) for small r is given by [5]

$$\begin{aligned}\mathrm{d}s^2 = \mathrm{d}r^2 &- ((a^2l^2 - \Omega^2)\cosh^2(r/l) - a^2l^2)\,\mathrm{d}t^2 + 2(\gamma\Omega\cosh^2(r/l) + a\omega l^2\sinh^2(r/l))\,\mathrm{d}t\,\mathrm{d}\varphi \\ &+ (\gamma^2\cosh^2(r/l) - \omega^2 l^2\sinh^2(r/l))\,\mathrm{d}\varphi^2\,,\end{aligned} \tag{3.30}$$

where ω, γ, Ω and a are arbitrary functions subject to

$$\dot\gamma = \Omega'\,, \qquad \dot\omega = -a'\,, \tag{3.31}$$

where prime denotes ∂_φ and dot denotes ∂_t. One reason for interpreting Ω as an angular velocity is that it appears as a leading order term in the $\mathrm{d}t\,\mathrm{d}\varphi$ component of the metric and thus, changes the sign under

[15] Gaussian coordinates are often useful if there exists a non-null-hypersurface S – in our case the horizon. On every point P in S one can define a unique normal vector n to S with respect to the metric g. The geodesic, which starts at P with tangent vector n is also uniquely defined by the fundamental theorem of ODEs. One can now label each point in a sufficiently small neighbourhood of S by arbitrary coordinates on S and the affine parameter of the geodesic, which runs through this point.

[16] In this context demanding regularity of the horizon means demanding the absence of singularities, in particular conical singularities.

time or φ reversal. We choose $\Omega = 0$ for regularity, which corresponds to us adopting a corotating frame, and $a = \text{const}$. This in turn means that the functions ω and γ depend on φ only, i.e. $\omega = \omega(\varphi)$ and $\gamma = \gamma(\varphi)$. In consequence, (3.30) reduces to

$$\begin{aligned} \mathrm{d}s^2\Big|_{\Omega=0}^{a=\text{const}} &= \mathrm{d}r^2 - a^2 l^2 (\cosh^2(r/l) - 1)\,\mathrm{d}t^2 + 2a\omega l^2 \sinh^2(r/l)\,\mathrm{d}t\,\mathrm{d}\varphi \\ &+ (\gamma^2 \cosh^2(r/l) - \omega^2 l^2 \sinh^2(r/l))\,\mathrm{d}\varphi^2\,. \end{aligned} \tag{3.32}$$

The line elements (3.30) and (3.32), respectively, describe different *exact* solutions of Einstein's EOM parametrized by functions $\omega(\varphi)$ and $\gamma(\varphi)$. While these functions are referred to as *state-dependent* functions, since their value differs from solution to solution, the functions a and Ω are referred to as *chemical potentials*, meaning that they are arbitrary, but fixed functions. As we will see later, a fixed Rindler acceleration implies that all states in our theory have the same temperature

$$T = \frac{a}{2\pi}\,. \tag{3.33}$$

On the one hand this does not seem like a good choice, since different states typically have different temperatures as is the case for the BTZ black hole. On the other hand fixing the temperature is exactly the setup one is looking for when investigating a certain macrostate – a black hole with fixed temperature and angular velocity. Additionally, it is difficult to implement a Rindler acceleration that varies from solution to solution. First of all, the line element is invariant under rescalings of the Rindler acceleration with simultaneous rescaling of the coordinates[17], which means that a statement like "the Rindler acceleration is 42" has no meaning unless the scale is fixed [5]. A previous work [42] achieved this by periodically identifying advanced/retarded time with a certain length scale L that breaks the scale invariance. However, the physical interpretation of this setup and the corresponding dual field theory is rather difficult, see [42].

The geometry described by (3.32) is generically not spherically symmetric, since the different components of the metric tensor contain functions of φ, i.e.$\gamma = \gamma(\varphi)$ and $\omega = \omega(\varphi)$. However, for the case of constant γ and ω the metric (3.32) becomes spherically symmetric and reduces to the BTZ black hole, as we will see in subsection 3.2.3. In ingoing Eddington-Finkelstein coordinates

$$v = t - \frac{1}{2} a^{-1} \log\left(\frac{f(\rho)}{\rho}\right)\,, \tag{3.34}$$

$$\rho = a l^2 \left(\cosh\left(\frac{r}{l}\right) - 1\right)\,, \tag{3.35}$$

the metric (3.30) becomes

$$\mathrm{d}s^2 = -2a\rho f(\rho)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}\rho + 4\omega\rho f(\rho)\,\mathrm{d}v\,\mathrm{d}\varphi - 2\frac{\omega}{a}\,\mathrm{d}\varphi\,\mathrm{d}\rho + \left(\gamma^2 + \frac{2\rho}{al^2}\left(\gamma^2 - l^2\omega^2\right) f(\rho)\right)\mathrm{d}\varphi^2\,, \tag{3.36}$$

where $f(\rho) = 1 + \rho/(2al^2)$. Requiring the absence of closed time-like curves leads to the constraint

$$\gamma > |\omega| l\,. \tag{3.37}$$

This may be for instance seen from (3.30) by taking the large r limit and chosing an azimuthal curve $\eta = \{t = \text{const}, r = \text{const}\}$

$$\lim_{r\to\infty} \mathrm{d}s^2\Big|_{r,t=\text{const}} = \left(\gamma^2 - \omega^2 l^2\right)\frac{e^{\frac{2r}{l}}}{4}\,\mathrm{d}\varphi^2\,. \tag{3.38}$$

[17]This may be best seen by looking at the line element in Eddington-Finkelstein coordinates given below (3.36), which stays invariant under $a \to \lambda a$, $\rho \to \lambda\rho$ and $v \to \lambda/v$.

The curve (3.38) becomes a closed *timelike* curve for $\mathrm{d}s^2 < 0$ and thus we arrive at the constraint (3.37). After having discussed the motivation for our boundary conditions in the metric formalism, which is geometrically more intuitive than the Chern-Simons formalism, we now switch to the latter, which enables us to use all the machinery developed in chapter 2. The $\mathfrak{sl}(2,\mathbb{R})$-valued Chern-Simons connections $A^{\pm}$ that precisely reproduce the line element in Gaussian coordinatese via (2.83) are given by

$$A^{\pm} = b_{\pm}^{-1}(\mathrm{d} + a^{\pm})b_{\pm}\,, \tag{3.39a}$$

with

$$b_{\pm} = \exp\left(\pm\frac{r}{2l}\left(\mathtt{L}_1 - \mathtt{L}_{-1}\right)\right) \tag{3.39b}$$

and

$$a^{\pm} = \mathtt{L}_0\,\left(\pm\mathcal{J}_{\pm}\,\mathrm{d}\varphi + \zeta_{\pm}\,\mathrm{d}t\right)\,, \tag{3.39c}$$

where

$$\mathcal{J}_{\pm} = \gamma l^{-1} \pm \omega \qquad \text{and} \qquad \zeta_{\pm} = -a \pm \Omega l^{-1}\,. \tag{3.40}$$

At first, the particular form of (3.39) may look arbitrary, since it is not unambiguously defined by (3.36). And while choosing boundary conditions involves usually a lot of educated guessing, the particular form (3.39a) brings about the following advantages

- Capturing the entire radial dependence in the group element $b_{\pm}$ leads to radially independent charges and hence, a radially independent asymptotic symmetry algebra.
- Choosing the auxiliary connection proportional to $\mathtt{L}_0$, the generator that is diagonal in the fundamental representation of $sl(2,\mathbb{R})$, see (A.1), simplifies the calculation of the holonomy conditions that assure regularity of the horizon.

The field equations (2.24) yield

$$\dot{\mathcal{J}}_{\pm} = \pm\zeta_{\pm}'\,, \tag{3.41}$$

which, with the use of (3.40), reduces to (3.31) as is needed for the two formulations to be equivalent.

3.2.1 Canonical Boundary Charges and Asymptotic Symmetry Algebra

We can now determine the canonical boundary charges and the asymptotic symmetry algebra in accordance with (2.3). Again, as in the case of the Brown-Henneaux boundary conditions, see (3.11), it is useful to choose the following form of the gauge parameters

$$\tilde{\epsilon}^{\pm} = b_{\pm}^{-1}\epsilon^{\pm}(\mathbf{x})\,b_{\pm} \tag{3.42}$$

with $b_{\pm}$ given by (3.39b), which takes us to the gauge of the auxiliary connection a, and the gauge parameters $\epsilon^{\pm}$ given by

$$\epsilon^{\pm} = \epsilon^{\pm}_{-1}\,\mathtt{L}_{-1} + \epsilon^{\pm}_{0}\,\mathtt{L}_0 + \epsilon^{\pm}_{1}\,\mathtt{L}_1\,. \tag{3.43}$$

Thus, the canonical boundary charges $\mathcal{Q}[\epsilon^+,\epsilon^-] = \mathcal{Q}^+[\epsilon^+] - \mathcal{Q}^-[\epsilon^-]$ associated with the theory defined by the boundary conditions (3.39) may readily be computed via

$$\begin{aligned}\delta\mathcal{Q}^{\pm}[\epsilon^{\pm}] &= \frac{k}{2\pi}\int \mathrm{d}\varphi\;\langle\tilde{\epsilon}^{\pm},\delta A^{\pm}_{\varphi}\rangle \overset{\delta b_{\pm}=0}{=} \frac{k}{2\pi}\int \mathrm{d}\varphi\;\langle\epsilon^{\pm},\delta a^{\pm}_{\varphi}\rangle \\ &= \pm\frac{k}{2\pi}\int \mathrm{d}\varphi\,\epsilon^{\pm}_0\delta\mathcal{J}_{\pm}\,\langle\mathtt{L}_0,\mathtt{L}_0\rangle = \pm\frac{k}{4\pi}\int \mathrm{d}\varphi\,\epsilon^{\pm}_0\delta\mathcal{J}_{\pm}\,,\end{aligned} \tag{3.44}$$

where we used that $\langle\mathtt{L}_0,\mathtt{L}_1\rangle = \langle\mathtt{L}_0,\mathtt{L}_{-1}\rangle = 0$ and $\langle\mathtt{L}_0,\mathtt{L}_0\rangle = 1/2$. Therefore, we see that the parameters $\epsilon^{\pm}_{-1}$ and $\epsilon^{\pm}_1$ lead to trivial gauge transformations ($\delta\mathcal{Q}_{\pm} = 0$) and thus, will not be considered in the following. To reduce clutter and to be in accordance with [4] we rename $\epsilon_0^\pm = \eta_\pm$. We now derive the most general non-trivial gauge transformation generated by $\epsilon^\pm(\mathbf{x}) = \eta_\pm(\mathbf{x})\mathrm{L}_0$ that preserves the boundary conditions, see (3.12). Per definition $\zeta_\pm$ does not vary if we move from one solution to another, i.e.

$$\delta_{\epsilon^\pm} a^\pm \overset{\delta\zeta_\pm=0}{=} \pm\delta_{\eta^\pm}\mathcal{J}_\pm \mathrm{L}_0 \,\mathrm{d}\varphi \,. \tag{3.45a}$$

Furthermore, we find that

$$\begin{aligned} \mathrm{d}\epsilon^\pm + [a^\pm, \epsilon^\pm] &= (\partial_\rho\eta_\pm \,\mathrm{d}\rho + \partial_\varphi\eta_\pm \,\mathrm{d}\varphi + \partial_t\eta_\pm \,\mathrm{d}t)\, \mathrm{L}_0 + (\pm\mathcal{J}_\pm \,\mathrm{d}\varphi + \zeta_\pm \,\mathrm{d}t)\; \eta_\pm \underbrace{[\mathrm{L}_0, \mathrm{L}_0]}_{=0} \\ &= (\partial_\rho\eta_\pm \,\mathrm{d}\rho + \partial_\varphi\eta_\pm \,\mathrm{d}\varphi + \partial_t\eta_\pm \,\mathrm{d}t)\, \mathrm{L}_0 \,. \end{aligned} \tag{3.45b}$$

Therefore, the asymptotic symmetry algebra is spanned by all transformations, whose corresponding gauge parameters $\eta_\pm(\mathbf{x})$ satisfy

$$\delta_{\eta_\pm}\mathcal{J}_\pm = \pm\partial_\varphi\eta_\pm \equiv \pm\eta'_\pm \qquad \text{and} \qquad \partial_\rho\eta_\pm = \partial_t\eta_\pm = 0 \,. \tag{3.45c}$$

The global charges are then obtained by functionally integrating (3.44)

$$\mathcal{Q}^\pm[\eta_\pm] = \pm\frac{k}{4\pi}\int \mathrm{d}\varphi \,\eta_\pm\, \mathcal{J}_\pm \tag{3.46}$$

and turn out to be finite, integrable and conserved in (advanced) time. Now we see explicitly what we claimed above: The surface integral (3.46) does not depend on the radial coordinate r, which in turn implies that our charges and hence, our boundary analysis does not depend on whether r is chosen to be close to the horizon $r = r_0$ or close to infinity.
The asymptotic symmetry algebra is spanned by the Dirac brackets of the canonical boundary charges and can be readily determined via the relation $\delta_\kappa \mathcal{Q}[\eta] = \{\mathcal{Q}[\eta], \mathcal{Q}[\kappa]\}$, see (2.71), where

$$\delta_{\kappa_\pm}\mathcal{Q}^\pm[\eta_\pm] = \pm\frac{k}{4\pi}\int \mathrm{d}\varphi \,\eta_\pm\, \delta_{\kappa_\pm}\mathcal{J}_\pm \overset{(3.45c)}{=} +\frac{k}{4\pi}\int \mathrm{d}\varphi\, \eta_\pm\, \kappa'_\pm \tag{3.47a}$$

and

$$\{\mathcal{Q}^\pm[\eta_\pm], \mathcal{Q}^\pm[\kappa_\pm(\varphi)]\} = \frac{k^2}{16\pi^2}\int \mathrm{d}\varphi\int \mathrm{d}\tilde{\varphi}\, \eta_\pm(\varphi)\kappa_\pm(\tilde{\varphi})\{\mathcal{J}_\pm(\varphi), \mathcal{J}_\pm(\tilde{\varphi})\} \,. \tag{3.47b}$$

Expanding the state-dependent functions $\mathcal{J}^\pm$ and the gauge parameters $\eta^\pm$ into Fourier modes

$$\mathcal{J}_\pm(\varphi) = \frac{2}{k}\sum_{n=-\infty}^{\infty} J_n^\pm e^{-in\varphi} \,, \qquad J_n^\pm = \frac{k}{4\pi}\int \mathrm{d}\varphi\, e^{in\varphi}\mathcal{J}^\pm(\varphi) \,, \tag{3.48a}$$

$$\eta_\pm(\varphi) = \frac{2}{k}\sum_{n=-\infty}^{\infty} \eta_n^\pm e^{-in\varphi} \,, \qquad \eta_n^\pm = \frac{k}{4\pi}\int \mathrm{d}\varphi\, e^{in\varphi}\eta^\pm(\varphi) \tag{3.48b}$$

and inserting the Fourier expansion (3.48) into (3.47) gives

$$\begin{aligned} \delta_{\kappa_\pm}\mathcal{Q}[\eta_\pm] &= +\frac{k}{4\pi}\int \mathrm{d}\varphi \cdot \frac{4}{k^2}\sum_{n,m}\eta_n^\pm\kappa_m^\pm(-im)e^{-i(n+m)\varphi} = -\frac{1}{k\pi}\sum_{n,m} im\,(2\pi\delta_{n+m,0})\eta_n^\pm\kappa_m^\pm \\ &= -\frac{2}{k}\sum_{n,m} im\delta_{n+m,0}\eta_n^\pm\kappa_m^\pm \end{aligned} \tag{3.49a}$$

and

$$\{\mathcal{Q}^{\pm}\left[\eta_{\pm}\right], \mathcal{Q}^{\pm}\left[\kappa_{\pm}(\varphi)\right]\} = \frac{k^2}{16\pi^2}\frac{16}{k^4}\int \mathrm{d}\varphi \int \mathrm{d}\tilde{\varphi} \sum_{n,m,p,q} \eta_n^{\pm}\kappa_m^{\pm}\{J_p^{\pm}, J_q^{\pm}\}e^{-i(n+p)\varphi}e^{-i(m+q)\tilde{\varphi}}$$
$$= \frac{1}{k^2\pi^2}\sum_{n,m,p,q} \eta_n^{\pm}\kappa_m^{\pm}\{J_p^{\pm}, J_q^{\pm}\}(2\pi\delta_{n+p,0})\,(2\pi\delta_{m+q,0})$$
$$= \frac{4}{k^2}\sum_{n,m}\eta_n^{\pm}\kappa_m^{\pm}\{J_{-n}^{\pm}, J_{-m}^{\pm}\}\,, \tag{3.49b}$$

where we have repeatedly exploited that

$$\int \mathrm{d}\varphi\, e^{i(n+m)\varphi} = 2\pi\delta_{n+m,0}\,. \tag{3.50}$$

Hence, we infer the following relation from (2.71) by comparing the summands

$$\{J_{-n}^{\pm}, J_{-m}^{\pm}\} = -\frac{k}{2}\delta_{n+m,0}\, im \qquad \Rightarrow \qquad \{J_n^{\pm}, J_m^{\pm}\} = -\frac{k}{2}\delta_{n+m,0}\, in\,. \tag{3.51}$$

Furthermore, we immediately see that

$$\{J_n^{+}, J_m^{-}\} = 0\,, \tag{3.52}$$

since $\delta_{\eta^+}J^- = \delta_{\eta^-}J^+ = 0$. Thus, after quantization of our system via the replacement of the Dirac brackets by commutators $i\{,\} \to [,]$ we obtain

$$[J_n^{\pm}, J_m^{\pm}] = +\frac{k}{2}\, n\, \delta_{m+n,0} \qquad \text{and} \qquad [J_n^{\pm}, J_m^{\mp}] = 0\,. \tag{3.53}$$

The algebra (3.53) consists of two $\hat{\mathfrak{u}}\,(1)$ current[18] algebras with levels $+\frac{k}{2}$. In the context of AdS/CFT this algebra may be interpreted as the current algebra of the two-dimensional conformal field theory living at the "boundary". This algebra is nothing else than the algebra of creation and annihilation operator for a free boson. Note that the value of the levels in (3.53) is arbitrary, since a redefinition of the prefactors in (3.48) leads to different levels. The algebra (3.53) may be brought into a more promising form by linearly combining the generators according to

$$X_n = J_n^{+} - J_{-n}^{-}\,, \tag{3.54}$$

$$P_0 = J_0^{+} + J_0^{-}\,, \tag{3.55}$$

$$P_n = \frac{i}{kn}\left(J_{-n}^{+} + J_n^{-}\right) \qquad \text{if } n \neq 0\,. \tag{3.56}$$

With this change of basis (3.53) becomes

$$[X_n, X_m] = [P_n, P_m] = [X_0, P_n] = [P_0, X_n] = 0\,. \tag{3.57}$$
$$[X_n, P_m] = i\delta_{n,m} \quad \text{if } n \neq 0 \tag{3.58}$$

Thus, X_0 and P_0 are Casimir operators and all other X_n, P_n form canonical pairs, which span the *Heisenberg algebra*. This is a surprisingly simple result.

[18] In conformal field theory a current is a chiral field $j(z)$ with conformal weight $h = 1$. For more information on conformal weights consult appendix C.

3.2.2 Hamiltonian and Soft Hair

As already discussed in detail in section 2.3.3, the Hamiltonian is defined as the charge connected with the temporal Killing vector $\xi = \partial_t$. The corresponding gauge parameter is related to the Killing vector via (2.30) and thus

$$\epsilon^{\pm}\big|_{\partial_t} = \xi^{\mu} a^{\pm}_{\mu} \overset{(3.39c)}{=} \zeta_{\pm} \mathrm{L}_0 \overset{(3.40)}{=} -a\mathrm{L}_0\,. \tag{3.59}$$

Hence, we get

$$\begin{aligned}\mathcal{H} := \mathcal{Q}[\epsilon^{\pm}\big|_{\partial_t}] = \mathcal{Q}^{+}[\epsilon^{+}\big|_{\partial_t}] - \mathcal{Q}^{-}[\epsilon^{-}\big|_{\partial_t}] &\overset{(3.46)}{=} \frac{k}{4\pi}\int \mathrm{d}\varphi\,(-a)\{+\mathcal{J}_{+} + \mathcal{J}_{-}\} \\ = \frac{k}{4\pi}\int \mathrm{d}\varphi\,(-a)\{+\mathcal{J}_{+} + \mathcal{J}_{-}\}e^{in\varphi}\delta_{n,0} &\overset{(3.48)}{=} -a\left(J^{+}_0 + J^{-}_0\right) = -aP_0\,.\end{aligned} \tag{3.60}$$

Therefore, the Hamiltonian commutes with all canonical coordinates X_n, P_n.

We can now consider vacuum descendants $|\psi(\{q\})\rangle$, which are labelled by a set q of arbitrary, non-negative quantum numbers $\{q\} = \{n_i, m_i\}$. One common way to define the vacuum state in a conformal field theory is as a *highest weight state*. In quantum field theory one usually employs the convention that generators J_n with $n \geq 0$ are annihilators and the ones with $n < 0$ are creators of states. A highest weight state is then a state that is annihilated by all annihilators of our theory, i.e. in our case

$$J_n\,|\text{highest weight}\rangle = 0 \qquad\qquad \forall\, n \geq 0\,. \tag{3.61}$$

Vacuum descendants are generated by acting with the generators $J^{\pm}_n$ on the vacuum $|0\rangle$ just established, i.e.

$$|\psi(\{q\})\rangle = N(\{q\}) \prod_{n_i>0} J^{+}_{-n_i} \prod_{m_j>0} J^{-}_{-m_j}\,|0\rangle\,. \tag{3.62}$$

The normalization constant $N(\{q\})$ is chosen such that $\langle\psi(\{q\})|\psi(\{q\})\rangle = 1$. Since the Hamiltonian $\mathcal{H}$ commutes with all generators $J^{\pm}_n$ the energy of the vacuum descendants (3.62) is the same as for the vacuum itself

$$\mathcal{H}\,|0\rangle = E_{\text{vac}}\,|0\rangle \qquad \text{and} \qquad \mathcal{H}\,|\psi(\{q\})\rangle = E_{\text{vac}}\,|\psi(\{q\})\rangle\,. \tag{3.63}$$

This implies that all descendants of the vacuum have the same energy as the vacuum itself and are thus zero-energy excitations. Therefore, they may be referred to as "soft hair" in the sense of Hawking, Perry and Strominger [8].

Discussion: Soft Hair

The term "soft hair" was first coined by Hawking, Perry and Strominger in their paper "Soft Hair on Black Holes" [8]. While "soft" is a term from particle physics and just means "zero-energy", hair refers to the phrase "Black holes are bald, they have no hair" by John Archibald Wheeler. This describes the fact that black holes can be *completely* characterised by only three classical parameters mass, electric charge and angular momentum. In that sense they are bald – they have no "hair" emerging from other properties. However, this immediately gives rise to a paradox, the "black hole information paradox", emerging from the fact that physical information, for instance realized by a particle carrying a certain spin, could permanently disappear and therefore, get lost after passing the black hole horizon. Although Hawking, Perry and Strominger's paper does not propose a solution to the problem, it pinpoints a problem in the formulation of the black hole information paradox, namely the assumption that black holes have no hair. In their paper they suggest the existence of soft hair at the black hole horizon, which could possibly be excited by particles falling into the black hole. These excited states could then possibly store information of particles, which transfer this information before passing the black hole horizon. This in turn would resolve the problem of information loss. Nevertheless, these speculations are a far stretch and the problem is still far from being solved.

3.2.3 Near Horizon and Asymptotic Boundary Conditions

After having established the near horizon boundary conditions and discussed their consequences it is natural to ask how these boundary conditions relate to boundary conditions which are inspired by an asymptotic perspective. To avoid confusion we want to stress again that the form of the connection (3.39) holds *in the entire spacetime*, not just at the horizon. Therefore, it might be confusing to call the connection (3.39) a boundary condition and talk about a near horizon or an asymptotic perspective, respectively. The reason for these names is that the boundary conditions in question are most naturally obtained from requiring a specific form of the connection at the horizon and are thus inspired by a "near horizon" prespective. The standard Brown-Henneaux boundary conditions, see section 3.1, were originally obtained by requiring a certain fall-off behaviour at infinity and solving Einstein's equations under this requirement. They are not given in the *diagonal gauge*, but rather in the so-called *highest weight gauge*[19]. For a generic choice of unspecified chemical potential $\mu_\pm$ the standard boundary conditions in highest weight gauge, which we already discussed in section 3.1.2 are given by [43], [30]

$$\hat{A}^\pm = \hat{b}_\pm^{-1}\left(\mathrm{d}+\hat{a}^\pm\right)\hat{b}_\pm\,, \qquad \hat{b}_\pm = e^{\pm\rho \mathtt{L}_0}\,, \tag{3.64a}$$

$$\hat{a}^\pm_\varphi = \left(\mathtt{L}_{\pm1} - \frac{1}{2}\mathcal{L}_\pm \mathtt{L}_{\mp1}\right)\,, \qquad \hat{a}^\pm_t = \pm\left(\mu_\pm \mathtt{L}_{\pm1} \mp \mu'_\pm \mathtt{L}_0 + \left(\frac{1}{2}\mu''_\pm - \frac{1}{2}\mathcal{L}_\pm\mu_\pm\right)L_{\mp1}\right)\,, \tag{3.64b}$$

where $\mathcal{L}_\pm$ and $\mu_\pm$ are arbitrary functions of t and φ. Note that in comparison to (3.23) we have absorbed factors of $-4\pi/k$ in the state-dependent functions $\mathcal{L}_\pm$ to be in accordance with [4]. Since the connections $A^\pm$ and $\hat{A}^\pm$ are connected to $a^\pm$ and $\hat{a}^\pm$, respectively through a gauge transformation that does not change the canonical boundary charges, see (3.16), it is sufficient to map $a^\pm$ to $\hat{a}^\pm$. The respective gauge transformation is given by

$$g_\pm = \exp\left(x_\pm \mathtt{L}_{\pm1}\right)\exp\left(-\frac{1}{2}\mathcal{J}_\pm \mathtt{L}_{\mp1}\right)\,, \tag{3.65}$$

where $x = x(v,\varphi)$ is a function subject to the conditions

$$\pm\partial_v x_\pm - \zeta_\pm x_\pm = \mu_\pm\,, \qquad x'_\pm - \mathcal{J}_\pm x_\pm = 1\,. \tag{3.66}$$

Consistency of (3.66), i.e. $\partial_v\partial_\varphi x_\pm = \partial_\varphi\partial_v x_\pm$ implies that

$$\mu'_\pm - \mathcal{J}_\pm\mu_\pm = -\zeta_\pm\,. \tag{3.67}$$

Note that the "asymptotic" chemical potentials $\mu_\pm$ depend on the "near horizon" chemical potentials $\zeta_\pm$ and the "near horizon" state-dependent functions $\mathcal{J}_\pm$, which is one way to see that near horizon boundary conditions (3.39) are inequivalent to the Brown-Henneaux boundary conditions with chemical potentials (3.23). Furthermore, the asymptotic state-dependent function $\mathcal{L}_\pm$ must fulfill

$$\mathcal{L}_\pm = \pm\left(\frac{1}{2}\mathcal{J}_\pm^2 + \mathcal{J}'_\pm\right)\,. \tag{3.68}$$

The relation (3.67) for the "near horizon" and "asymptotic" chemical potentials $\zeta_\pm$ and $\mu_\pm$ implies an analogous relation for the near horizon and asymptotic gauge parameters $\eta_\pm$ and $\lambda_\pm$ as was discussed in subsection 3.1.2

$$\lambda'_\pm - \mathcal{J}_\pm\lambda_\pm = -\eta_\pm\,. \tag{3.69}$$

[19] The name *diagonal gauge* comes from the fact that the generator $\mathtt{L}_0$ is diagonal in the fundamental representation of $\mathfrak{sl}(2,\mathbb{R})$, see appendix A.1. Highest weight gauge refers to the fact the component of a_φ with respect to $\mathtt{L}_1$ (= highest weight operator) is fixed. Note that highest and lowest weight in this context are not related to the discussion of vacuum descendants given before, but refer to the lowering/raising operators well known from a spin-1/2 system in quantum mechanics, since $\mathfrak{sl}(2,\mathbb{R}) \sim \mathfrak{su}(2)$, see appendix A.1.

From (3.69), (3.68) we can derive the transformation behaviour of the state-dependent functions

$$
\begin{aligned}
\delta_{\lambda_\pm}\mathcal{L}_\pm &= \pm\left(\mathcal{J}_\pm\delta_{\eta_\pm}\mathcal{J}_\pm + \delta_{\eta_\pm}\mathcal{J}'_\pm\right) \overset{(3.45c)}{=} \pm\left(\mathcal{J}_\pm\eta'_\pm + \eta''_\pm\right) \\
&= \pm\left(-\mathcal{J}_\pm\left(\lambda''_\pm - \mathcal{J}'_\pm\lambda_\pm - \mathcal{J}_\pm\lambda'_\pm\right) - \lambda'''_\pm + \mathcal{J}''_\pm\lambda_\pm + 2\mathcal{J}'_\pm\lambda'_\pm + \mathcal{J}_\pm\lambda''_\pm\right) \\
&= \pm\left(2\mathcal{L}_\pm\lambda'_\pm + \mathcal{L}'_\pm\lambda_\pm - \lambda'''_\pm\right)
\end{aligned} \tag{3.70}
$$

Expanding (3.68) in Fourier modes yields

$$
kL^\pm_n = \pm\sum_{p\in\mathbb{Z}} J^\pm_{n-p}J^\pm_p \pm iknJ^\pm_n\,, \tag{3.71}
$$

which is the standard twisted Sugawara construction, see comment below. The generators $L^\pm_n$ now fulfill the Virasoro algebra with the Brown-Henneaux central extension, i.e.

$$
[L^\pm_n, L^\pm_m] = (n-m)L^\pm_{n+m} + \frac{1}{2}\,k\,n^3\,\delta_{n+m,0}\,. \tag{3.72}
$$

The Virasoro algebra may be brought into the standard form via the redefinition of $L^\pm_0$ discussed in subsection 3.1.1. The gauge transformation defined through (3.65) and (3.66) is a proper gauge transformation, since it does not change the global charges, which we compute now explicitly

$$
\begin{aligned}
\delta Q^\pm &= \pm\frac{k}{4\pi}\int \mathrm{d}\varphi\,\eta_\pm\delta\mathcal{J}_\pm = \mp\frac{k}{4\pi}\int\mathrm{d}\varphi\,\left(\lambda'_\pm - \mathcal{J}_\pm\lambda_\pm\right)\delta\mathcal{J}_\pm \\
&= \mp\frac{k}{4\pi}\underbrace{\left(\lambda_\pm\delta\mathcal{J}_\pm\right)\Big|_0^{2\pi}}_{=0} \pm\frac{k}{4\pi}\int\mathrm{d}\varphi\,\left(\lambda_\pm\delta\mathcal{J}'_\pm + \mathcal{J}_\pm\lambda_\pm\,\delta\mathcal{J}_\pm\right) \\
&= \pm\frac{k}{4\pi}\int\mathrm{d}\varphi\,\lambda_\pm\left(\mathcal{J}_\pm\delta\mathcal{J}_\pm + \delta\mathcal{J}'_\pm\right) \overset{(3.68)}{\equiv} \pm\frac{k}{4\pi}\int\mathrm{d}\varphi\,\lambda_\pm\,\delta\mathcal{L}_\pm\,,
\end{aligned} \tag{3.73}
$$

which precisely coincides with (3.17). Thus, the near horizon boundary conditions (3.39) can be brought into the standard highest weight gauge through a gauge transformation that does not change the boundary charges. For constant chemical potentials and constant state-dependent functions $\mathcal{L}_+$, $\mathcal{L}_-$, which by virtue of (3.68) and (3.40) means $\gamma = \text{const}$, $\omega = \text{const}$, we obtain the BTZ black hole.
Note that although the spin-2 currents fulfill the Virasoro algebra, the corresponding global charges span the $\hat{\mathfrak{u}}(1)$ algebra (3.53). This is the case, because the chemical potentials $\mu_\pm$ are not held constant as in the Brown-Henneaux case, but are allowed to vary under boundary conditions preserving gauge transformations space, see (3.67).

Comment: Sugawara Construction

The Sugawara construction is a procedure used in conformal field theory to reconstruct the Virasoro algebra spanned by the Fourier modes $\mathtt{L}_n$ of the energy-momentum tensor T from the algebra of the currents J^a. The standard Sugawara ansatz is given by

$$
T = \gamma\sum_a J^aJ^a\,, \tag{3.74}
$$

where γ is a constant depending on the dual coxeter number and the central extension, which are both defined by the current algebra. The twisted Sugawara construction is then obtained by adding to (3.74) a "twist term" of the form

$$
T = \gamma\sum_a J^aJ^a + J^{a\prime}\,. \tag{3.75}
$$

For more details consult [44] or other textbooks on conformal field theory.

3.2.4 Entropy of Black Hole Solutions

Due to the fact that Chern-Simons theory is a theory of flat connections ($F = 0$), gauge invariant information can not be found in any local quantity (locally everything appears identical). However, holonomies, which measure the extent to which a group element changes when being transported around a closed loop, give gauge invariant information. The standard trick [28, 30, 45] to obtain a notion of temperature is to use the holonomy condition of the Chern-Simons gauge field around the contractible Euclidean time cycle. The Wick rotation to Euclidean time $\tau = -it$ leads to a deformation of the manifold into a solid torus, where τ corresponds to the contractible cycle and φ to the non-contractible one. Requiring the absence of conical singularities at the horizon corresponds to the condition that the holonomy of the Chern-Simons connection around this closed loop be trivial [30], i.e.

$$\mathcal{H}_{\mathcal{C}} = \mathcal{P}e^{\int_{\mathcal{C}} a^{\pm}} \to \exp\left(\int_{\rho=0} A_{\tau}^{\pm} d\tau\right) = \mathbb{1}\,, \tag{3.76}$$

where $\mathbb{1}$ stands for the $\mathfrak{sl}(2,\mathbb{R})$ identity and the thermal cycle runs from $\tau = 0$ to $\tau = \beta$. This in turn gives us the following condition in the diagonal gauge

$$\begin{aligned} &\exp\left(\int_{\rho=0} a_{\tau}^{\pm} d\tau\right) = \exp\left(\pm\zeta_{\pm}\beta L_0\right) = \mathbb{1} \\ &\Rightarrow \zeta_{\pm} \overset{(3.40)}{=} -a = \frac{2\pi n}{\beta} \overset{n=-1}{=} -\frac{2\pi}{\beta}\,. \end{aligned} \tag{3.77}$$

Here we set $n = -1$, since this corresponds to the solution that is obtained from the same requirement, i.e. absence of conical singularities at the horizon, in the metric formalism. Note that if we had not set $\Omega = 0$, then we would have obtained the condition $-a \pm \Omega/l = -2\pi/\beta$ by virtue of (3.40), which is only fulfilled if $\Omega = 0$. The corresponding condition for μ_+ in the highest weight gauge leads to

$$\mu_+\mu_+'' - \frac{1}{2}\mu_+'^2 - \mu_+^2\mathcal{L}_+ = -\frac{2\pi^2}{\beta^2}\,, \tag{3.78}$$

which for constant chemical potential $\mu_+ = \text{const}$ reduces after redefinition of the state-dependent functions $\mathcal{L}_+$ to (3.25). Thus, because of the appearance of $\mathcal{L}_+$ in (3.78) imposing regularity in the highest weight gauge leads to restrictions on the states. Note that an analogous condition can be written down for the ($\mathcal{L}_-$, μ_-)-sector. Here, we explicitly see one advantage of our boundary conditions, namely the regularity of all solutions contained therein. Now, we can compute the entropy using the first law of thermodynamics and find

$$\delta S = -\beta\delta\mathcal{H} = \frac{2\pi}{a}a(J_0^+ + J_0^-) = 2\pi(J_0^+ + J_0^-)\,. \tag{3.79}$$

Thus, the entropy is linear in the zero-mode charges J_0^+, J_0^-.

4 Boundary Conditions for Spin-3 Gravity in Flat Space

In this chapter we discuss two sets of boundary conditions for spin-3 gravity in flat space. More precisely, we discuss spin-3 gravity exclusively in the principal embedding $\mathfrak{isl}(2,\mathbb{R}) \hookrightarrow \mathfrak{isl}(3,\mathbb{R})$, see section 2.5, which is the simplest higher-spin extension to flat space and was first discussed in [19, 20]. We show that the near horizon boundary conditions in diagonal gauge can brought into the standard highest weight form through a proper gauge transformation, see subsection 4.2.3. Furthermore, we compute the entropy for the near horizon boundary conditions and find that for solutions which are continuously connected to flat space cosmologies the entropy is linear in the spin-2 zero-mode charges and independent from the spin-3 charges, i.e.

$$S = 2\pi \left(J_0^+ + J_0^- \right). \tag{4.1}$$

Precisely the same result was found in AdS_3 Einstein gravity [4] (see also subsection 3.2.4), in AdS_3 higher-spin gravity [6], in flat space Einstein gravity [5] and higher derivative gravity [46]. This suggest a universal form of the entropy in terms of the zero-mode charges.
Before discussing the extension of the near horizon boundary conditions (3.39) to spin-3 gravity in flat space, we give a short review of the analogue of the Brown-Henneaux boundary conditions for spin-2 gravity in flat space and then proceed to discuss the highest weight boundary conditions for spin-3 gravity in flat space.

4.1 Highest Weight Boundary Conditions

4.1.1 Asymptotically Flat Boundary Conditions

In three dimensions asymptotically flat space boundary conditions and their asymptotic symmetries have been studied and the asymptotic symmetry algebra was found to be the BMS algebra [47, 48] in three dimensions [49], which has a non-trivial central extension [50]. As in the case of asymptotically AdS boundary conditions the notion of asymptotically flat space must be defined precisely. It was shown in [36] that asymptotically flat boundary conditions may be obtained from asymptotically AdS boundary conditions by taking a flat space limit. Asymptotically flat space metrics in the so-called BMS gauge are then given by [36]

$$\mathrm{d}s^2 = \mathcal{M}\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + 2\mathcal{N}\,\mathrm{d}u\,\mathrm{d}\varphi + r^2\,\mathrm{d}\varphi^2\,, \tag{4.2}$$

where $0 \leq r < \infty$ is the radial coordinate, $-\infty < u < \infty$ is the retarded time coordinate and $\varphi \sim \varphi + 2\pi$ is the angular coordinate parametrizing the boundary of our cylinder. The functions $\mathcal{M}$, $\mathcal{N}$ are *state-dependent* functions and thus describe the particular solution taken by our system. The Einstein equations (2.1) yield the on-shell conditions

$$\dot{\mathcal{M}} = 0\,, \qquad 2\dot{\mathcal{N}} = \mathcal{M}'\,, \tag{4.3}$$

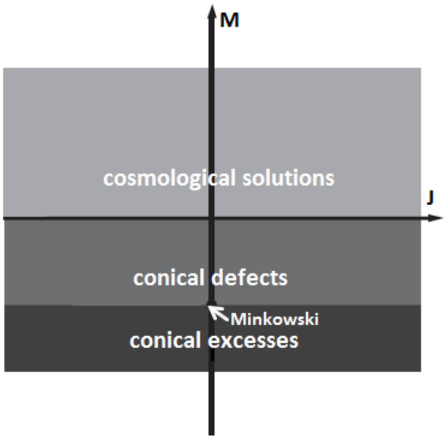

Figure 4.1: Spectrum of the asymptotically flat space boundary conditions taken from [36]. Flat space cosmologies exist for $M \geq 0$. Minkowski space is separated by a gap from the continuous flat space cosmologies spectrum. For $M < 0$ conical defects and conical excesses appear.

where $\dot{x} = \partial_u x$ and $x' = \partial_\varphi x$. These equations are only satisfied iff $\mathcal{M} = \mathcal{M}(\varphi), \mathcal{N}(u,\varphi) = \mathcal{L}(\varphi) + \frac{u}{2}\mathcal{M}'(\varphi)$ for some function $\mathcal{L} = \mathcal{L}(\varphi)$. The metric (4.2) contains the following solutions:

- For $\mathcal{M} = -1$, $\mathcal{N} = 0$ we obtain global flat space in Eddington-Finkelstein coordinates, which may be brought into the form $\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2$ via the coordinate transformation $u = t - r$.
- For generic $\mathcal{M} \geq 0$ and $\mathcal{N} \neq 0$ we obtain so-called "flat space cosmologies" [51], see Figure 4.1. Flat space cosmologies represent the flat space analogon to the BTZ black hole in AdS_3. Similar to a black hole, they carry a mass $\mathcal{M}$ and an angular momentum $\mathcal{N}$ and exhibit a cosmological horizon.

In [36] it was shown that the asymptotic symmetry algebra of these solutions is spanned by the centrally extended $\mathfrak{bms}_3$ algebra

$$[L_n, L_m] = (n-m)L_{n+m}\,, \tag{4.4a}$$
$$[L_n, M_m] = (n-m)M_{n+m} + \frac{c_M}{12}n(n^2-1)\delta_{n+m,0}\,, \tag{4.4b}$$
$$[M_n, M_m] = 0\,, \tag{4.4c}$$

where $c_M = 3/G_N$. As in the AdS case, these boundary condtions can be rewritten in the Chern-Simons formulation in terms of a gauge field A that is an element of the $\mathfrak{isl}(2,\mathbb{R})$ algebra

$$[\mathtt{L}_n, \mathtt{L}_m] = (n-m)\mathtt{L}_{n+m}\,, \tag{4.5a}$$
$$[\mathtt{L}_n, \mathtt{M}_m] = (n-m)\mathtt{M}_{n+m} \tag{4.5b}$$
$$[\mathtt{M}_n, \mathtt{M}_m] = 0 \tag{4.5c}$$

with $n, m = -1, 0, 1$. The boundary conditions are then given by

$$A = b^{-1}\,(\mathrm{d} + a)\,b \tag{4.6a}$$

with the auxiliary connection a

$$a = \left(\mathtt{M}_1 - \frac{\mathcal{M}}{4}\mathtt{M}_{-1}\right)\mathrm{d}u + \left(\mathtt{L}_1 - \frac{\mathcal{M}}{4}\mathtt{L}_{-1} - \frac{\mathcal{N}}{2}\mathtt{M}_{-1}\right)\mathrm{d}\varphi \tag{4.6b}$$

and the group element b

$$b = e^{\frac{r}{2}\mathtt{M}_{-1}}\,. \tag{4.6c}$$

The field equations (2.24) yield (4.3) as expected. The connection (4.6) exactly reproduces (4.2) via the usual contraction over the local dreibein, i.e. (2.9). Note that similarly to the corresponding boundary conditions in AdS, capturing the entire radial dependence in b and choosing b such that $\delta b = 0$ is advantageous, as this yields radially independent charges and enables the computation of the charges in the gauge of the auxiliary connection a.

4.1.2 Spin-3 Highest Weight Boundary Conditions

In this subsection we discuss the extension of the boundary conditions (4.2) to spin-3 gravity in flat space. The easiest way to obtain a $\mathfrak{isl}(3,\mathbb{R})$ valued connection A in the principal embedding. is by appropriately modifying the auxiliary connection a such that it becomes an element of the $\mathfrak{isl}(3,\mathbb{R})$ algebra, while leaving the form of the group element b, see (4.6c), invariant. The $\mathfrak{isl}(3,\mathbb{R})$ algebra is spanned by $\mathtt{L}_i, \mathtt{M}_i, \mathtt{U}_n, \mathtt{V}_n$ with $i = -1, 0, 1$ and $n = -2, -1, 0, 1, 2$

$$[\mathtt{L}_n, \mathtt{L}_m] = (n-m)\mathtt{L}_{n+m}\,, \tag{4.7a}$$
$$[\mathtt{L}_n, \mathtt{M}_m] = (n-m)\mathtt{M}_{n+m}\,, \tag{4.7b}$$
$$[\mathtt{L}_n, \mathtt{U}_m] = (2n-m)\mathtt{U}_{n+m}\,, \tag{4.7c}$$
$$[\mathtt{L}_n, \mathtt{V}_m] = (2n-m)\mathtt{V}_{n+m}\,, \tag{4.7d}$$
$$[\mathtt{U}_n, \mathtt{U}_m] = \sigma(n-m)(2n^2+2m^2-nm-8)\mathtt{L}_{n+m}\,, \tag{4.7e}$$
$$[\mathtt{U}_n, \mathtt{V}_m] = \sigma(n-m)(2n^2+2m^2-nm-8)\mathtt{M}_{n+m}\,. \tag{4.7f}$$

The generators $\mathtt{L}_i$ and $\mathtt{M}_i$ span the $\mathfrak{isl}(2,\mathbb{R})$ subalgebra. The constant σ fixes the overall normalization of the spin-3 generators $\mathtt{U}_n$ and $\mathtt{V}_n$ and can be chosen arbitrarily. In this thesis we follow the convention of [52] and choose $\sigma = -\frac{1}{3}$. Explicit expressions for the $\mathfrak{isl}(3,\mathbb{R})$ valued connections that obey asymptotically flat boundary conditions were established independently in [19] and [20] and read

$$A = b^{-1}\left(\mathrm{d} + a\right) b \tag{4.8a}$$

with the auxiliary connection a given by

$$a = a_\varphi\,\mathrm{d}\varphi + a_u\,\mathrm{d}u\,, \tag{4.8b}$$

where

$$a_\varphi = \mathtt{L}_1 - \frac{\mathcal{M}}{4}\mathtt{L}_{-1} - \frac{\mathcal{N}}{2}\mathtt{M}_{-1} + \frac{\mathcal{V}}{2}\mathtt{U}_{-2} + \mathcal{Z}\mathtt{V}_{-2}\,, \tag{4.8c}$$
$$a_u = \mathtt{M}_1 - \frac{\mathcal{M}}{4}\mathtt{L}_{-1} + \frac{\mathcal{V}}{2}\mathtt{V}_{-2}\,. \tag{4.8d}$$

The equations of motion (2.24) yield

$$\dot{\mathcal{M}} = \dot{\mathcal{V}} = 0\,, \qquad \dot{\mathcal{N}} = \tfrac{1}{2}\,\mathcal{M}'\,, \qquad \dot{\mathcal{Z}} = \tfrac{1}{2}\,\mathcal{V}'\,. \tag{4.9}$$

These constraints are solved in terms of four arbitrary functions of the angular coordinate φ [19]

$$\mathcal{M} = \mathcal{M}(\varphi)\,, \qquad \mathcal{V} = \mathcal{V}(\varphi)\,, \qquad \mathcal{N} = \mathcal{L}(\varphi) + \tfrac{u}{2}\,\mathcal{M}'(\varphi)\,, \qquad \mathcal{Z} = \mathcal{U}(\varphi) + \tfrac{u}{2}\,\mathcal{V}'(\varphi)\,. \tag{4.10}$$

By setting the spin-3 generators $\mathtt{U}_n$ and $\mathtt{V}_n$ to zero, (4.8) and (4.9) precisely reduce to (4.6) and (4.3).

4.1.3 Spin-3 Highest Weight Boundary Conditions with Chemical Potentials

In this section we generalize the discussion from section 4.1.2 to spin-3 gravity with chemical potentials for the spin-2 and the spin-3 fields μ_{M}, μ_{L}, μ_{V}, μ_{U}, which were first introduced in [52, 53]. Following the procedure of [30], chemical potentials may be introduced by leaving the form of a_φ (and therefore the charges) invariant and making a general ansatz for a_u with arbitrary coefficients. The form of the coefficients can then be determined by solving the field equations (2.24). Associating the coefficients of the highest weight components with the corresponding chemical potentials, i.e. $\alpha \mathtt{M}_1 \to \mu_{\text{M}}\mathtt{M}_1$ one obtains [21] [52]

$$a_u = a_u^{(0)} + a_u^{(\mu_{\text{M}})} + a_u^{(\mu_{\text{L}})} + a_u^{(\mu_{\text{V}})} + a_u^{(\mu_{\text{U}})}\,, \qquad a_\varphi = a_\varphi^{(0)} \tag{4.11a}$$

with $a_u^{(0)}$, $a_\varphi^{(0)}$ being the connection (4.8) in the absence of chemical potentials and

$$\begin{aligned}
a_u^{(\mu_{\text{M}})} &= \mu_{\text{M}}\, M_+ - \mu_{\text{M}}'\, M_0 + \tfrac{1}{2}\left(\mu_{\text{M}}'' - \tfrac{1}{2}\mathcal{M}\mu_{\text{M}}\right) M_- + \tfrac{1}{2}\,\mathcal{V}\,\mu_{\text{M}}\, V_{-2}\,, &(4.11\text{b})\\
a_u^{(\mu_{\text{L}})} &= a_u^{(\mu_{\text{M}})}\big|_{M\to L} - \tfrac{1}{2}\,\mathcal{N}\,\mu_{\text{L}}\, M_- + \mathcal{Z}\,\mu_{\text{L}}\, V_{-2}\,, &(4.11\text{c})\\
a_u^{(\mu_{\text{V}})} &= \mu_{\text{V}}\, V_2 - \mu_{\text{V}}'\, V_1 + \tfrac{1}{2}\left(\mu_{\text{V}}'' - \mathcal{M}\mu_{\text{V}}\right) V_0 + \tfrac{1}{6}\left(-\mu_{\text{V}}''' + \mathcal{M}'\mu_{\text{V}} + \tfrac{5}{2}\mathcal{M}\mu_{\text{V}}'\right) V_{-1}\\
&\quad + \tfrac{1}{24}\left(\mu_{\text{V}}'''' - 4\mathcal{M}\mu_{\text{V}}'' - \tfrac{7}{2}\mathcal{M}'\mu_{\text{V}}' + \tfrac{3}{2}\mathcal{M}^2\mu_{\text{V}} - \mathcal{M}''\mu_{\text{V}}\right) V_{-2} - 4\mathcal{V}\,\mu_{\text{V}}\, M_-\,, &(4.11\text{d})\\
a_u^{(\mu_{\text{U}})} &= a_u^{(\mu_{\text{V}})}\big|_{M\to L} - 8\mathcal{Z}\,\mu_{\text{U}}\, M_- - \mathcal{N}\,\mu_{\text{U}}\, V_0 + \left(\tfrac{5}{6}\mathcal{N}\mu_{\text{U}}' + \tfrac{1}{3}\mathcal{N}'\mu_{\text{U}}\right) V_{-1}\\
&\quad + \left(-\tfrac{1}{3}\mathcal{N}\mu_{\text{U}}'' - \tfrac{7}{24}\mathcal{N}'\mu_{\text{U}}' - \tfrac{1}{12}\mathcal{N}''\mu_{\text{U}} + \tfrac{1}{4}\mathcal{M}\mathcal{N}\mu_{\text{U}}\right) V_{-2}\,, &(4.11\text{e})
\end{aligned}$$

where the subscript $M \to L$ denotes that in the corresponding quantity all odd generators and chemical potentials are replaced by corresponding even ones, $M_n \to L_n$, $V_n \to U_n$, $\mu_{\text{M}} \to \mu_{\text{L}}$ and $\mu_{\text{V}} \to \mu_{\text{U}}$, i.e.

$$\begin{aligned}
a_u^{(\mu_{\text{M}})}\big|_{M\to L} &= \mu_{\text{L}}\, L_+ - \mu_{\text{L}}'\, L_0 + \tfrac{1}{2}\left(\mu_{\text{L}}'' - \tfrac{1}{2}\mathcal{M}\mu_{\text{L}}\right) L_- + \tfrac{1}{2}\,\mathcal{V}\,\mu_{\text{L}}\, U_{-2}\,, &(4.11\text{f})\\
a_u^{(\mu_{\text{V}})}\big|_{M\to L} &= \mu_{\text{U}}\, U_2 - \mu_{\text{U}}'\, U_1 + \tfrac{1}{2}\left(\mu_{\text{U}}'' - \mathcal{M}\mu_{\text{U}}\right) U_0 + \tfrac{1}{6}\left(-\mu_{\text{U}}''' + \mathcal{M}'\mu_{\text{U}} + \tfrac{5}{2}\mathcal{M}\mu_{\text{U}}'\right) U_{-1}\\
&\quad + \tfrac{1}{24}\left(\mu_{\text{U}}'''' - 4\mathcal{M}\mu_{\text{U}}'' - \tfrac{7}{2}\mathcal{M}'\mu_{\text{U}}' + \tfrac{3}{2}\mathcal{M}^2\mu_{\text{U}} - \mathcal{M}''\mu_{\text{U}}\right) U_{-2} - 4\mathcal{V}\,\mu_{\text{U}}\, L_-\,. &(4.11\text{g})
\end{aligned}$$

As before, dots (primes) denote derivatives with respect to retarded time u (angular coordinate φ). The equations of motion (2.24) impose the conditions

$$\begin{aligned}
\dot{\mathcal{M}} &= -2\mu_{\text{L}}''' + 2\mathcal{M}\mu_{\text{L}}' + \mathcal{M}'\mu_{\text{L}} + 24\mathcal{V}\mu_{\text{U}}' + 16\mathcal{V}'\mu_{\text{U}}\,, &(4.12\text{a})\\
\dot{\mathcal{N}} &= \tfrac{1}{2}\,\dot{\mathcal{M}}\big|_{L\to M} + 2\mathcal{N}\mu_{\text{L}}' + \mathcal{N}'\mu_{\text{L}} + 24\mathcal{Z}\mu_{\text{U}}' + 16\mathcal{Z}'\mu_{\text{U}}\,, &(4.12\text{b})\\
\dot{\mathcal{V}} &= \tfrac{1}{12}\,\mu_{\text{U}}''''' - \tfrac{5}{12}\,\mathcal{M}\mu_{\text{U}}''' - \tfrac{5}{8}\,\mathcal{M}'\mu_{\text{U}}'' - \tfrac{3}{8}\,\mathcal{M}''\mu_{\text{U}}' + \tfrac{1}{3}\,\mathcal{M}^2\mu_{\text{U}}'\\
&\quad - \tfrac{1}{12}\mathcal{M}'''\mu_{\text{U}} + \tfrac{1}{3}\,\mathcal{M}\mathcal{M}'\mu_{\text{U}} + 3\mathcal{V}\mu_{\text{L}}' + \mathcal{V}'\mu_{\text{L}}\,, &(4.12\text{c})\\
\dot{\mathcal{Z}} &= \tfrac{1}{2}\,\dot{\mathcal{V}}\big|_{L\to M} - \tfrac{5}{12}\,\mathcal{N}\mu_{\text{U}}''' - \tfrac{5}{8}\,\mathcal{N}'\mu_{\text{U}}'' - \tfrac{3}{8}\,\mathcal{N}''\mu_{\text{U}}' + \tfrac{2}{3}\,\mathcal{M}\mathcal{N}\mu_{\text{U}}'\\
&\quad - \tfrac{1}{12}\,\mathcal{N}'''\mu_{\text{U}} + \tfrac{1}{3}\,(\mathcal{M}\mathcal{N})'\mu_{\text{U}} + 3\mathcal{Z}\mu_{\text{L}}' + \mathcal{Z}'\mu_{\text{L}}\,, &(4.12\text{d})
\end{aligned}$$

which after applying the inverse substitution rules to above becomes

$$\begin{aligned}
\tfrac{1}{2}\,\dot{\mathcal{M}}\big|_{L\to M} &= -\mu_{\text{M}}''' + \mathcal{M}\mu_{\text{M}}' + \tfrac{1}{2}\,\mathcal{M}'(1+\mu_{\text{M}}) + 12\mathcal{V}\mu_{\text{V}}' + 8\mathcal{V}'\mu_{\text{V}}\,, &(4.12\text{e})\\
\tfrac{1}{2}\,\dot{\mathcal{V}}\big|_{L\to M} &= \tfrac{1}{24}\,\mu_{\text{V}}''''' - \tfrac{5}{24}\,\mathcal{M}\mu_{\text{V}}''' - \tfrac{5}{16}\,\mathcal{M}'\mu_{\text{V}}'' - \tfrac{3}{16}\,\mathcal{M}''\mu_{\text{V}}' + \tfrac{1}{6}\,\mathcal{M}^2\mu_{\text{V}}'\\
&\quad - \tfrac{1}{24}\,\mathcal{M}'''\mu_{\text{V}} + \tfrac{1}{6}\,\mathcal{M}\mathcal{M}'\mu_{\text{V}} + \tfrac{3}{2}\,\mathcal{V}\mu_{\text{M}}' + \tfrac{1}{2}\,\mathcal{V}'(1+\mu_{\text{M}})\,. &(4.12\text{f})
\end{aligned}$$

The chemical potentials μ_{M}, μ_{L}, μ_{V} and μ_{U} are arbitrary functions of the angular coordinate φ and the retarded time u. The most general gauge transformations $\delta_{\tilde{\varepsilon}}A = \mathrm{d}\tilde{\varepsilon} + [A, \tilde{\varepsilon}]$ that preserve the boundary conditions (4.11) are given by parameters

$$\tilde{\varepsilon} = b^{-1}\varepsilon b \tag{4.13}$$

with

$$\begin{aligned}
\varepsilon = \epsilon\, L_+ - \epsilon'\, L_0 + \tfrac{1}{2}\left(\epsilon'' - \tfrac{1}{2}\mathcal{M}\epsilon - 8\mathcal{V}\chi\right) L_- \\
+ \tau\, M_+ - \tau'\, M_0 + \tfrac{1}{2}\left(\tau'' - \tfrac{1}{2}\mathcal{M}\tau - \mathcal{N}\epsilon - 8\mathcal{V}\kappa - 16\mathcal{Z}\chi\right) M_- \\
+ \chi\, U_2 - \chi'\, U_1 + \tfrac{1}{2}\left(\chi'' - \mathcal{M}\chi\right) U_0 - \tfrac{1}{6}\left(\chi''' - \tfrac{5}{2}\mathcal{M}\chi' - \mathcal{M}'\chi\right) U_{-1} \\
+ \tfrac{1}{24}\left(\chi'''' - 4\mathcal{M}\chi'' - \tfrac{7}{2}\mathcal{M}'\chi' - \mathcal{M}''\chi + \tfrac{3}{2}\,\mathcal{M}^2\chi + 12\mathcal{V}\epsilon\right) U_{-2} \\
+ \kappa\, V_2 - \kappa'\, V_1 + \tfrac{1}{2}\left(\kappa'' - \mathcal{M}\kappa - 2\mathcal{N}\chi\right) V_0 \\
- \tfrac{1}{6}\left(\kappa''' - \tfrac{5}{2}\mathcal{M}\kappa' - \mathcal{M}'\kappa - 5\mathcal{N}\chi' - 2\mathcal{N}'\chi\right) V_{-1} + \tfrac{1}{24}\left(\kappa'''' - 4\mathcal{M}\kappa'' - \tfrac{7}{2}\mathcal{M}'\kappa' \right.\\
\left. - \mathcal{M}''\kappa + \tfrac{3}{2}\mathcal{M}^2\kappa - 8\mathcal{N}\chi'' - 7\mathcal{N}'\chi' - 2\mathcal{N}''\chi + 6\mathcal{M}\mathcal{N}\chi + 12\mathcal{V}\tau + 24\mathcal{Z}\epsilon\right) V_{-2}\,,
\end{aligned} \tag{4.14}$$

where the gauge parameters ϵ, σ, χ and ρ depend on φ only. Additionaly, we introduce new parameters $\tau = \sigma + u\epsilon'$ and $\kappa = \rho + u\chi'$. The canonical boundary charges can then be calculated using (2.67)

$$\delta Q[\varepsilon] = \frac{k}{2\pi}\int \mathrm{d}\varphi\ \langle\tilde{\varepsilon}\delta A_\varphi\rangle \overset{\delta b=0}{=} \frac{k}{2\pi}\int \mathrm{d}\varphi\ \langle\varepsilon\delta a_\varphi\rangle\ . \tag{4.15}$$

Insertion of the expressions (4.8c) and (4.14) yields

$$\delta Q[\varepsilon] = \frac{k}{2\pi}\int \mathrm{d}\varphi\left(\epsilon\,\delta\mathcal{L} + \tfrac{1}{2}\sigma\,\delta\mathcal{M} + 8\chi\,\delta\mathcal{U} + 4\rho\,\delta\mathcal{V}\right). \tag{4.16}$$

Since the parameters ϵ, σ, χ and ρ are independent of the state-dependent functions, equation (4.16) can be integrated trivially in field space and we obtain

$$Q[\epsilon,\tau,\chi,\kappa] = \frac{k}{2\pi}\int \mathrm{d}\varphi\left(\epsilon\,\mathcal{L} + \tfrac{1}{2}\sigma\,\mathcal{M} + 8\chi\,\mathcal{U} + 4\rho\,\mathcal{V}\right). \tag{4.17}$$

Thus, the canonical charges are integrable, finite and conserved in (retarded) time, $\partial_u Q = 0$. Expanding the state-dependent functions $\mathcal{L}$, $\mathcal{M}$, $\mathcal{U}$ and $\mathcal{V}$ into Fourier modes leads after a suitable shift of the zero-modes to the following relations

$$[L_n,\, L_m] = (n-m)L_{n+m}\,, \tag{4.18a}$$

$$[L_n,\, M_m] = (n-m)M_{n+m} + k\,(n^3-n)\delta_{n+m,\,0}\,, \tag{4.18b}$$

$$[L_n,\, U_m] = (2n-m)U_{n+m}\,, \tag{4.18c}$$

$$[L_n,\, V_m] = (2n-m)V_{n+m}\,, \tag{4.18d}$$

$$[M_n,\, U_m] = (2n-m)V_{n+m}\,, \tag{4.18e}$$

$$\begin{aligned}[U_n,\, U_m] &= -\tfrac{1}{3}\,(n-m)(2n^2+2m^2-nm-8)L_{n+m}\\ &\quad - \frac{16}{3k}(n-m)\Lambda_{n+m} + \frac{88}{45k^2}(n-m)\Theta_{n+m}\,,\end{aligned} \tag{4.18f}$$

$$\begin{aligned}[U_n,\, V_m] &= -\tfrac{1}{3}\,(n-m)(2n^2+2m^2-nm-8)M_{n+m}\\ &\quad - \frac{8}{3k}(n-m)\Theta_{n+m} - \frac{k}{3}\,n(n^2-1)(n^2-4)\delta_{n+m,\,0}\,,\end{aligned} \tag{4.18g}$$

with

$$\Theta_m = \sum_p M_p\, M_{m-p} \qquad \Lambda_m = \sum_p :L_p\, M_{m-p}: \, - \tfrac{3}{10}(m+2)(m+3)M_m\,. \tag{4.18h}$$

Normal ordering is defined by $:L_n\, M_m: \, = L_n\, M_m$ if $n < -1$ and $:L_n\, M_m: \, = M_m\, L_n$ otherwise. The Fourier modes of the state-dependent functions $\mathcal{L}$, $\mathcal{M}$, $\mathcal{U}$ and $\mathcal{V}$ are denoted by L_n, M_n, U_n and V_n. In these relations we have already quantized the system by replacing the Dirac brackets by commutators via $i\{,\} \to [,]$. This is an İnönü–Wigner contraction[20] of two copies of the $\mathcal{W}_3$ algebra, which is also referred to as an $\mathcal{FW}_3$ algebra.

Entropy for Spin-3 Flat Space Cosmologies

It was shown in [52, 53] that the spin-3 entropy of the branch that is continuously connected to the flat space cosmologies of Einstein gravity, i.e. [53]

$$S_{\mathrm{GR}} = 2\pi\sqrt{\frac{\pi k}{\hat{P}}}|\hat{J}|\,, \tag{4.19}$$

is given by

$$S = 2\pi\sqrt{\frac{\pi k}{\hat{P}}}\sec\left(\Phi\right)\left\{|\hat{J}|\cos\left(\frac{2\Phi}{3}\right)+\sqrt{\frac{3k}{\pi\hat{P}}}\frac{\hat{V}}{4}\sin\frac{\Phi}{3}\right\}\,, \tag{4.20}$$

with

$$\Phi = \mp\arcsin\left(\frac{3}{8}\sqrt{\frac{3k}{\pi\hat{\mathcal{P}}^3}}\hat{\mathcal{W}}\right)\,, \tag{4.21}$$

where consistency implies that the sign of $\hat{J}$ coincides with the sign of (4.21). The map between the zero-modes of the state-dependent functions in [53] $\hat{\mathcal{P}}$, $\hat{\mathcal{J}}$, $\hat{\mathcal{W}}$ and $\hat{\mathcal{V}}$ and the state-dependent functions we were using until now $\mathcal{M}$, $\mathcal{N}$, $\mathcal{V}$ and $\mathcal{Z}$ [52] is provided via

$$\hat{\mathcal{P}} = \frac{k}{4\pi}\mathcal{M}\,, \qquad \hat{\mathcal{J}} = \frac{k}{2\pi}\mathcal{L}\,, \qquad \hat{\mathcal{W}} = \frac{2k}{\pi}\mathcal{V}\,, \qquad \hat{\mathcal{V}} = \frac{4k}{\pi}\mathcal{U}. \tag{4.22}$$

For constant state-dependent functions these equations reduce to

$$\hat{P} = \frac{k}{4\pi}M\,, \qquad \hat{J} = \frac{k}{2\pi}L \overset{(4.10)}{=} \frac{k}{2\pi}N\,, \qquad \hat{W} = \frac{2k}{\pi}V\,, \qquad \hat{V} = \frac{4k}{\pi}U \overset{(4.10)}{=} \frac{4k}{\pi}Z\,. \tag{4.23}$$

where M, L N, V, U, Z and $\hat{P}$, $\hat{J}$, $\hat{W}$, $\hat{V}$ denote the zero-modes of the state-dependent functions $\mathcal{M}$, $\mathcal{L}$, $\mathcal{N}$, $\mathcal{V}$, $\mathcal{U}$, $\mathcal{Z}$ and $\hat{\mathcal{P}}$, $\hat{\mathcal{J}}$, $\hat{\mathcal{W}}$, $\hat{\mathcal{V}}$, respectively. Note that this result can also be brought into the form presented in [52]

$$S_{\mathrm{Th}} = \pi k\frac{N\left(2R-6+3P\sqrt{R}\right)}{4\sqrt{M}(R-3)\sqrt{1-\frac{3}{4\mathcal{R}}}}\,, \tag{4.24}$$

where we have introduced the dimensionless ratios

$$\frac{V}{2M^{\frac{3}{2}}} = \frac{R-1}{R^{\frac{3}{2}}}, \qquad \frac{Z}{N\sqrt{M}} = P. \tag{4.25}$$

4.2 Near Horizon Boundary Conditions for Spin-3 Gravity

In this section we discuss the extension of the near horizon boundary conditions (3.39) to spin-3 gravity in flat space[21]. The starting point is once more the Chern-Simons action given by (2.18), where the

[20] In 1953 [54] Erdal İnönü and Eugene Wigner discussed the possibility of obtaining certain groups from other groups through a limiting procedure – the İnönü–Wigner contraction, for more details refer to [21].

[21] Note that, since this chapter is based on a collaboration with Martin Ammon, Daniel Grumiller, Stefan Prohazka and Max Riegler, several parts of this chapter coincide with the forthcoming publication [55].

connection A is now an element of $\mathfrak{isl}(3,\mathbb{R})$, see appendix A.2. Furthermore, as we already discussed in section 2.5, since in higher-spin gauge theories the metric is not a higher-spin gauge invariant object, we will directly propose a form for the higher-spin analog of the near horizon boundary conditions (3.39) for the gauge field A. Considering other, prior work on near horizon boundary conditions in spin-3 AdS [6] and flat space [5], the choice of the corresponding boundary conditions for spin-3 gravity in flat space is natural. To motivate this choice, we review the already existing boundary conditions and also present their asymptotic symmetry algebras in Table 4.1. It is easiest to promote the spin-2 to spin-3 gravity boundary conditions by switching on the generators which span the higher-spin part of the gauge algebra exclusively in the auxiliary connection a, while leaving the group element b invariant. The near horizon boundary conditions are then extended most naturally by only switching on the generators, $\mathtt{U}_0$ and $\mathtt{V}_0$ of the respective Lie algebra. This way we obtain correspondent characteristic features of the ones for Einstein gravity in AdS (such as regularity of the fields regardless of the charges) also in spin-3 gravity theories. The flat space boundary conditions can be obtained in two different ways: either by directly working in flat space or by taking the $\Lambda = -\frac{1}{l} \to 0$ limit of the spin-3 AdS results [6]. Since the limiting procedure is sometimes subtle, we choose to work directly in flat space. However, taking the limit from AdS to flat space serves as a non-trivial check for our results[22]. The spin-2 boundary conditions for flat space, see Table 4.1, were already computed in [5] and can be obtained by taking $\Lambda = -\frac{1}{l} \to 0$ limit of (3.39). Now, following the discussion above these boundary conditions can be extended naturally to spin-3 gravity in flat space by switching on the generators $\mathtt{U}_0$ and $\mathtt{V}_0$, while leaving the group element b invariant. Following the notation of [55] the proposed form of the connection is given by

$$A = b^{-1}(a + \mathrm{d}\,)\,b, \tag{4.26a}$$

where the radial dependence is encoded in the group element b as

$$b = \exp\left(\frac{1}{\mu_{\mathcal{P}}}\,\mathtt{M}_1\right)\,\exp\left(+\frac{\rho}{2}\,\mathtt{M}_{-1}\right) \tag{4.26b}$$

and the auxiliary connection a reads

$$a = a_\varphi\,\mathrm{d}\varphi + a_v\,\mathrm{d}v \tag{4.26c}$$

with

$$a_\varphi = \mathcal{J}\,\mathtt{L}_0 + \mathcal{P}\,\mathtt{M}_0 + \mathcal{J}^{(3)}\,\mathtt{U}_0 + \mathcal{P}^{(3)}\,\mathtt{V}_0, \tag{4.26d}$$

$$a_v = \mu_{\mathcal{P}}\,\mathtt{L}_0 + \mu_{\mathcal{J}}\,\mathtt{M}_0 + \mu_{\mathcal{P}}^{(3)}\,\mathtt{U}_0 + \mu_{\mathcal{J}}^{(3)}\,\mathtt{V}_0. \tag{4.26e}$$

All the functions appearing in (4.26) are in principle free functions of the advanced time v and the angular coordinate φ. However, the functions μ_a will be identified as *chemical potentials* and thus will be fixed such that $\delta\mu_a = 0$ under an asymptotic symmetry transformation. The functions $\mathcal{J}$, $\mathcal{P}$, $\mathcal{J}^{(3)}$ and $\mathcal{P}^{(3)}$ are *state-dependent* functions and are thus allowed to vary in the solution space described by our boundary conditions. The equations of motion (2.24) give constraints on the functions $\mathcal{J}, \mathcal{P}$, $\mathcal{J}^{(3)}$ and $\mathcal{P}^{(3)}$:

$$\partial_v\mathcal{J} = \partial_\varphi\mu_{\mathcal{P}}, \quad \partial_v\mathcal{P} = \partial_\varphi\mu_{\mathcal{J}}, \quad \partial_v\mathcal{J}^{(3)} = \partial_\varphi\mu_{\mathcal{P}}^{(3)}, \quad \partial_v\mathcal{P}^{(3)} = \partial_\varphi\mu_{\mathcal{J}}^{(3)}. \tag{4.27}$$

4.2.1 Canonical Boundary Charges and Asymptotic Symmetry Algebra

The next step in the asymptotic symmetry analysis is to determine the gauge transformations $\delta_{\tilde\epsilon}A = \mathrm{d}\tilde\epsilon + [A, \tilde\epsilon]$ that preserve the boundary conditions (4.26). As in the case of the Brown-Henneaux

[22]In fact, this check was performed by our collaborators, Martin Ammon and Max Riegler, who instead of working directly in flat space obtained their results by taking the limit of the spin-3 AdS results [6]. Our results precisely agree with theirs.

	AdS [4]	spin-3 AdS [6]	flat space [5]	spin-3 flat space [55]
gauge algebra	$\mathfrak{sl}(2,\mathbb{R})\bigoplus\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{R})\bigoplus\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{isl}(2,\mathbb{R})$	$\mathfrak{isl}(3,\mathbb{R})$
	$\mathtt{L}_i,\ i=-1,0,1$	$\mathtt{L}_i,\ i=-1,0,1$ $\mathtt{W}_n,\ n=-2,-1,0,1,2$	$\mathtt{L}_i,\mathtt{M}_i,\ i=-1,0,1$	$\mathtt{L}_i,\mathtt{M}_i,\ i=-1,0,1$ $\mathtt{U}_i,\mathtt{V}_i,\ i=-2,-1,0,1,2$
b.c. for A	$A^\pm=b_\pm^{-1}(\mathrm{d}+a^\pm)b_\pm$ $b_\pm=\exp\left[\pm\frac{1}{l\zeta^\pm}\mathtt{L}_1\right]\left[\pm\frac{\rho}{2}\mathtt{L}_{-1}\right]$	$A^\pm=b_\pm^{-1}(\mathrm{d}+a^\pm)b_\pm$ $b_\pm=\exp\left[\pm\frac{1}{l\zeta^\pm}\mathtt{L}_1\right]\left[\pm\frac{\rho}{2}\mathtt{L}_{-1}\right]$	$A=b^{-1}(\mathrm{d}+a)b$ $b=\exp\left[\frac{1}{\mu_\mathcal{P}}\mathtt{M}_1+\frac{\rho}{2}\mathtt{M}_{-1}\right]$	$A=b^{-1}(\mathrm{d}+a)b$ $b=\exp\left[\frac{1}{\mu_\mathcal{P}}\mathtt{M}_1+\frac{\rho}{2}\mathtt{M}_{-1}\right]$
b.c. for a	$a_\varphi^\pm=\pm\mathcal{J}^\pm\mathtt{L}_0$ $a_t^\pm=\zeta^\pm\mathtt{L}_0$	$a_\varphi^\pm=\pm\mathcal{J}^\pm\mathtt{L}_0+\mathcal{J}_{(3)}^\pm\mathtt{W}_0$ $a_v^\pm=\zeta^\pm\mathtt{L}_0+\zeta_{(3)}^\pm\mathtt{W}_0$	$a_\varphi=\mathcal{J}\mathtt{L}_{(0)}+\mathcal{P}\mathtt{M}_{(0)}$ $a_v=\mu_\mathcal{P}\mathtt{L}_{(0)}+\mu_\mathcal{J}\mathtt{M}_{(0)}$	$a_\varphi=\mathcal{J}\,\mathtt{L}_0+\mathcal{P}\,\mathtt{M}_0+\mathcal{J}^{(3)}\,\mathtt{U}_0+\mathcal{P}^{(3)}\,\mathtt{V}_0$ $a_v=\mu_\mathcal{P}\,\mathtt{L}_0+\mu_\mathcal{J}\,\mathtt{M}_0+\mu_\mathcal{P}^{(3)}\,\mathtt{U}_0+\mu_\mathcal{J}^{(3)}\,\mathtt{V}_0$
a.s.a.	$[J_n^\pm,J_m^\pm]=\frac{1}{2}kn\delta_{n+m,0}$	$[J_n^\pm,J_m^\pm]=\frac{1}{2}kn\delta_{n+m,0}$ $[J_n^{(3)\pm},J_m^{(3)\pm}]=\frac{2}{3}kn\delta_{n+m,0}$	$[J_n,P_m]=kn\delta_{n+m,0}$	$[J_n,P_m]=k\,n\,\delta_{n+m,0}$ $[J_n^{(3)},P_m^{(3)}]=\frac{4k}{3}\,n\,\delta_{n+m,0}$
entropy	$S=2\pi\left(J_0^++J_0^-\right)$	$S=2\pi\left(J_0^++J_0^-\right)$	$S=2\pi P_0=2\pi\left(J_0^++J_0^-\right)$	$S=2\pi P_0=2\pi\left(J_0^++J_0^-\right)$

Table 4.1: Near horizon boundary conditions (b.c.) and their corresponding asymptotic symmetry algebra (a.s.a) for spin-2 / spin-3 gravity in flat and AdS space: We see that the entropy for all boundary conditions takes the same form in terms of the zero-modes of the state-dependent functions. Please note that in the spin-3 AdS/flat space case this statement only holds for the entropy branch continuously connected to BTZ black hole/cosmological spacetimes of Einstein gravity. Note that all connections are presented in ingoing Eddington-Finkelstein coordinates, the connection for spin-2 gravity in Gaussian coordinates is given in (3.39). For more information on the given algebras (invariant, non-degenerate bilinear form, representation) see appendix A.2.

boundary conditions, see (3.11), and also in the case of the near horizon boundary conditions in AdS space, see (3.42), it is again useful to choose the gauge parameter $\tilde{\epsilon}$ as

$$\tilde{\epsilon} = b^{-1}\epsilon\, b = b^{-1}(\epsilon_{\mathcal{P}}\mathtt{L}_0 + \epsilon_{\mathcal{J}}\mathtt{M}_0 + \epsilon_{\mathcal{P}}^{(3)}\mathtt{U}_0 + \epsilon_{\mathcal{J}}^{(3)}\mathtt{V}_0)\, b. \tag{4.28}$$

This takes the gauge parameter into the gauge of the auxiliary connection a; thus we only have to consider gauge transformations $\delta_\epsilon a = \mathrm{d}\epsilon + [a,\epsilon]$. As a consequence, the infinitesimal transformation behaviour of the state-dependent fields takes a particularly simple form

$$\delta_\epsilon \mathcal{J} = \partial_\varphi \epsilon_{\mathcal{P}}, \quad \delta_\epsilon \mathcal{P} = \partial_\varphi \epsilon_{\mathcal{J}}, \quad \delta_\epsilon \mathcal{J}^{(3)} = \partial_\varphi \epsilon_{\mathcal{P}}^{(3)}, \quad \delta_\epsilon \mathcal{P}^{(3)} = \partial_\varphi \epsilon_{\mathcal{J}}^{(3)}. \tag{4.29}$$

The conserved charges $\mathcal{Q}[\epsilon]$ associated to boundary conditions preserving transformations may then be computed using (2.67) and (A.9). Evaluating this expression for the case at hand, we obtain the following expression for the variation of the canonical boundary charge

$$\delta Q[\epsilon] = \frac{k}{2\pi}\int \mathrm{d}\varphi\, \langle \epsilon\, \delta A_\varphi \rangle = \frac{k}{2\pi}\int \mathrm{d}\varphi \left(\epsilon_{\mathcal{J}}\delta\mathcal{J} + \epsilon_{\mathcal{P}}\delta\mathcal{P} + \frac{4}{3}\epsilon_{\mathcal{J}}^{(3)}\delta\mathcal{J}^{(3)} + \frac{4}{3}\epsilon_{\mathcal{P}}^{(3)}\delta\mathcal{P}^{(3)}\right). \tag{4.30}$$

The global charges may now be obtained by functionally integrating (4.30)

$$Q[\epsilon] = \frac{k}{2\pi}\int \mathrm{d}\varphi\, \langle \epsilon\, A_\varphi \rangle = \frac{k}{2\pi}\int \mathrm{d}\varphi \left(\epsilon_{\mathcal{J}}\mathcal{J} + \epsilon_{\mathcal{P}}\mathcal{P} + \frac{4}{3}\epsilon_{\mathcal{J}}^{(3)}\mathcal{J}^{(3)} + \frac{4}{3}\epsilon_{\mathcal{P}}^{(3)}\mathcal{P}^{(3)}\right). \tag{4.31}$$

After having determined the canonical boundary charges, the Dirac bracket algebra of the charges can be calculated from the transformation behaviour under infinitesimal gauge transformations $\delta_Y Q[X] = \{Q[X], Q[Y]\}$, see section 2.3.3 for more information. After expanding the state-dependent functions in Fourier modes

$$\mathcal{J}(\varphi) = \frac{1}{k}\sum_{n\in\mathbb{Z}} J_n e^{-in\varphi}, \qquad \mathcal{P}(\varphi) = \frac{1}{k}\sum_{n\in\mathbb{Z}} P_n e^{-in\varphi}, \tag{4.32a}$$

$$\mathcal{J}^{(3)}(\varphi) = \frac{3}{4k}\sum_{n\in\mathbb{Z}} J_n^{(3)} e^{-in\varphi}, \qquad \mathcal{P}^{(3)}(\varphi) = \frac{3}{4k}\sum_{n\in\mathbb{Z}} P_n^{(3)} e^{-in\varphi}, \tag{4.32b}$$

and replacing the Dirac brackets by commutators using $i\{\cdot,\cdot\} \to [\cdot,\cdot]$ we obtain the following asymptotic symmetry algebra for the boundary conditions (4.26)

$$[J_n, P_m] = k\, n\, \delta_{n+m,0}, \qquad [J_n^{(3)}, P_m^{(3)}] = \frac{4k}{3}\, n\, \delta_{n+m,0}. \tag{4.33}$$

At this point it should also be noted that the algebra (4.33) can be brought to the same form as in [6] by making the redefinitions

$$J_n^{\pm} = \frac{1}{2}(P_n \pm J_n), \qquad J_n^{(3)\pm} = \frac{1}{2}(P_n^{(3)} \pm J_n^{(3)}). \tag{4.34}$$

The generators $J_n^{\pm}$ and $J_n^{(3)\pm}$ then satisfy

$$[J_n^{\pm}, J_m^{\pm}] = \frac{k}{2} n\delta_{n+m,0}, \qquad [J_n^{(3)\pm}, J_m^{(3)\pm}] = \frac{2k}{3} n\delta_{n+m,0}\,. \tag{4.35}$$

4.2.2 Hamiltonian and Soft Hair

In this subsection we show that the states associated to the near horizon symmetries (4.33) all have the same energy and thus correspond to higher-spin soft hair. In order to show this, we first determine the Hamiltonian in terms of near horizon variables and then proceed in building vacuum descendants using (4.33). Finally, we show that all states have the same energy as the vacuum.

As already discussed in subsection 2.3.3, the Hamiltonian is associated to the charge that generates time translations. In the metric formalism this would correspond to the Killing vector ∂_v. Since the gauge transformations (4.28) are related to the asymptotic Killing vectors ξ^μ via (2.30), the variation of the charge associated to translations in the retarded time coordinate v can be determined via

$$\begin{aligned}\delta H := \delta Q[\epsilon\big|_{\partial_v}] &= \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle \xi^\mu a_\mu\,\delta a_\varphi\rangle = \frac{k}{2\pi}\int \mathrm{d}\varphi\,\langle a_v\,\delta a_\varphi\rangle \\ &= \frac{k}{2\pi}\int \mathrm{d}\varphi\left(\mu_{\mathcal{J}}\delta\mathcal{J} + \mu_{\mathcal{P}}\delta\mathcal{P} + \frac{4}{3}\mu_{\mathcal{J}}^{(3)}\delta\mathcal{J}^{(3)} + \frac{4}{3}\mu_{\mathcal{P}}^{(3)}\delta\mathcal{P}^{(3)}\right).\end{aligned} \tag{4.36}$$

This expression can be trivially functionally integrated and yields the Hamiltonian

$$H = \frac{k}{2\pi}\int \mathrm{d}\varphi\left(\mu_{\mathcal{J}}\mathcal{J} + \mu_{\mathcal{P}}\mathcal{P} + \frac{4}{3}\mu_{\mathcal{J}}^{(3)}\mathcal{J}^{(3)} + \frac{4}{3}\mu_{\mathcal{P}}^{(3)}\mathcal{P}^{(3)}\right). \tag{4.37}$$

For constant chemical potentials μ_a and $\mu_a^{(3)}$ the Hamiltonian thus reduces to

$$H = \left(\mu_{\mathcal{J}}J_0 + \mu_{\mathcal{P}}P_0 + \frac{4}{3}\mu_{\mathcal{J}}^{(3)}J_0^{(3)} + \frac{4}{3}\mu_{\mathcal{P}}^{(3)}P_0^{(3)}\right). \tag{4.38}$$

After having determined the Hamiltonian, the next step in our analysis is to build vacuum descendants using the algebra (4.33). There are two ways of building vacuum descendants relevant to our analysis. One is via *highest weight* representations whereas the other one uses a construction similar to *induced* representations.

We first start with descendants built from *highest weight* representations of (4.33), which works analogously to the discussion in subsection 3.2.2. We again assume that the vacuum state $|0\rangle$ is a highest weight state, i.e. a state that satisfies

$$J_n|0\rangle = P_n|0\rangle = J_n^{(3)}|0\rangle = P_n^{(3)}|0\rangle = 0, \qquad \forall\, n \geq 0\,. \tag{4.39}$$

New states can then be constructed from such a vacuum state by repeated application of operators with $n < 0$, i.e.

$$|\psi(\{p\})\rangle \propto \prod_{n_i>0} J_{-n_i}\prod_{n_i^{(3)}>0} J_{-n_i^{(3)}}\prod_{m_i>0} P_{-m_i}\prod_{m_i^{(3)}>0} P_{-m_i^{(3)}}|0\rangle, \tag{4.40}$$

where $\{p\} \equiv \{n_i, n_i^{(3)}, m_i, m_i^{(3)}\}$. Since the Hamiltonian (4.38) is a linear combination of J_0, P_0, $J_0^{(3)}$ and $P_0^{(3)}$, it is evident that these operators commute with any element appearing in the asymptotic symmetry algebra (4.33). In particular, this also means that H commutes with all J_n and $J_n^{(3)}$. Thus, when acting with H on any vacuum descendant $\psi(\{p\})$, one obtains the same value for the energy as for the vacuum $|0\rangle$.

After having shown that highest weight modules constructed from (4.33) all correspond to soft excitations, we consider modules built from representations that are similar to the *induced* representations found in flat space holography, see e.g. [56, 57].

In the induced representation vacuum descendants can be built from a vacuum state $|0\rangle$ via

$$|\psi(\{q\})\rangle \sim \prod_{n_i}(J_{n_i})\prod_{n_i^{(3)}}\left(J_{n_i^{(3)}}\right)|0\rangle, \tag{4.41}$$

where $\{q\} \equiv \{n_i, n_i^{(3)}\}$. The vacuum state has to satisfy

$$P_n|0\rangle = P_n^{(3)}|0\rangle = 0, \quad \forall\, n \in \mathbb{Z}\,. \tag{4.42}$$

These conditions are similar to the ones presented in [57], where J_n and P_n span a BMS/Poincarè algebra and generate (super-)rotations and (super-)translations, respectively. In [57] the vacuum state corresponds per definition to a state with momentum $p = 0$ and "boosted" states are obtained by acting with J_n on $|0\rangle$, which motivates (4.41) and (4.42). Since in our case J_n and P_n span $\hat{\mathfrak{u}}(1)$ current algebras, the physical interpretation of this setup requires further investigation.
We can now act once more with the Hamiltonian (4.38) on all states in the module (4.41). Using the same line of argument as for the highest weight representations, one finds that all states built from induced representations also have the same energy eigenvalue and can thus be interpreted as soft excitations.

4.2.3 Near Horizon and Asymptotic Boundary Conditions

In this subsection we discuss the relationship between the boundary conditions we established in the last section, see (4.26), and the ordinary highest weight boundary conditions for spin-3 gravity in flat space with chemical potentials, see (4.8) and (4.11), introduced in [52], discussed in subsection 4.1.3. As in the spin-2 AdS, see subsection 3.2.3, the spin-2 flat [5] and the spin-3 AdS space [6] case the near horizon boundary conditions can be brought from diagonal to highest weight gauge through a proper gauge transformation. It is algebraically interesting that by performing such a gauge transformation, spin-2 and spin-3 charges of higher-spin cosmological solutions in flat space emerge as composite operators constructed from the $\hat{\mathfrak{u}}(1)$ charges (4.33).
The next step is to find an appropriate gauge transformation that maps the connection a in (4.26) to the connection $\tilde{a}$ in (4.11) via $\tilde{a} = g^{-1}(a + \mathrm{d})g$. As in the AdS case discussed in subsection 3.2.3, it is sufficient to map the auxiliary connections to one another. This is the case, since the additional gauge transformation b that takes the auxiliary connection a to the connection A is a proper gauge transformation and thus does not change the canonical boundary charges. Since the gauge algebra in the case of $\mathfrak{isl}(3,\mathbb{R})$ is 16-dimensional, this involves a fair amount of algebraic manipulation. However, this algebraic manipulation may be done in a systematic manner – as is presented in appendix D.4. The group element that provides the appropriate map is then given as $g = g^{(1)}g^{(2)}$ with

$$g^{(1)} = \exp\left[\mathfrak{l}\,\mathtt{L}_1 + \mathfrak{m}\,\mathtt{M}_1 + \mathfrak{u}_1\,\mathtt{U}_1 + \mathfrak{v}_1\,\mathtt{V}_1 + \mathfrak{u}_2\,\mathtt{U}_2 + \mathfrak{v}_2\,\mathtt{V}_2\right], \tag{4.43a}$$

$$\begin{aligned} g^{(2)} = \exp\Bigg[&-\frac{\mathcal{J}}{2}\,\mathtt{L}_{-1} - \frac{\mathcal{J}^{(3)}}{3}\,\mathtt{U}_{-1} + \frac{1}{6}\left(\mathcal{J}\mathcal{J}^{(3)} + \frac{\mathcal{J}^{(3)\prime}}{2}\right)\mathtt{U}_2 \\ &-\frac{\mathcal{J}}{2}\,\mathtt{M}_{-1} - \frac{\mathcal{P}^{(3)}}{3}\,\mathtt{V}_{-1} + \frac{1}{6}\left(\mathcal{P}\mathcal{J}^{(3)} + \mathcal{J}\mathcal{P}^{(3)} + \frac{\mathcal{P}^{(3)\prime}}{2}\right)\mathtt{V}_{-2}\Bigg]. \end{aligned} \tag{4.43b}$$

The functions $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{u}_a$ and $\mathfrak{v}_a$ depend on v and φ only and have to satisfy

$$\mathfrak{l}' = 1 + \mathfrak{l}\mathcal{J} + 2\mathfrak{u}_1\mathcal{J}^{(3)}, \tag{4.44a}$$

$$\mathfrak{m}' = \mathfrak{l}\mathcal{P} + \mathfrak{m}\mathcal{P} + 2\mathfrak{u}_1\mathcal{P}^{(3)} + 2\mathfrak{v}_1\mathcal{J}^{(3)}, \tag{4.44b}$$

$$\mathfrak{u}_1' = \mathfrak{u}_1\mathcal{J} + 2\mathfrak{l}\mathcal{J}^{(3)}, \tag{4.44c}$$

$$\mathfrak{v}_1' = \mathfrak{u}_1\mathcal{P} + \mathfrak{v}_1\mathcal{J} + 2\mathfrak{l}\mathcal{P}^{(3)} + 2\mathfrak{m}\mathcal{J}^{(3)}, \tag{4.44d}$$

$$\mathfrak{u}_2' = -\frac{\mathfrak{u}_1}{2} + 2\mathfrak{u}_2\mathcal{J}, \tag{4.44e}$$

$$\mathfrak{v}_2' = -\frac{\mathfrak{v}_1}{2} + 2\mathfrak{u}_2\mathcal{P} + 2\mathfrak{v}_2\mathcal{J}, \tag{4.44f}$$

and

$$\mu_L = \frac{4}{3}\mu_U \mathcal{J}^{(3)} - \mu_\mathcal{P}\mathfrak{l} - 2\mu_\mathcal{P}^{(3)}\mathfrak{u}_1 + \dot{\mathfrak{l}}, \tag{4.45a}$$

$$\mu_M = \frac{4}{3}\mu_U \mathcal{P}^{(3)} + \frac{4}{3}\mu_V \mathcal{J}^{(3)} - \mu_\mathcal{P}\mathfrak{m} - \mu_\mathcal{J}\mathfrak{l} - 2\mu_\mathcal{P}^{(3)}\mathfrak{v}_1 - 2\mu_\mathcal{J}^{(3)}\mathfrak{u}_1 + \dot{\mathfrak{m}}, \tag{4.45b}$$

$$\mu_U = -2\mu_\mathcal{P}\mathfrak{u}_2 + \mu_\mathcal{P}^{(3)}\mathfrak{l}^2 + \mu_\mathcal{P}^{(3)}\mathfrak{u}_1^2 + \frac{1}{2}\mathfrak{u}_1\dot{\mathfrak{l}} - \frac{1}{2}\mathfrak{l}\dot{\mathfrak{u}}_1 + \dot{\mathfrak{u}}_2, \tag{4.45c}$$

$$\begin{aligned}\mu_V &= -2\mu_\mathcal{P}\mathfrak{v}_2 - 2\mu_\mathcal{J}\mathfrak{u}_2 + 2\mathfrak{l}\mathfrak{m}\mu_\mathcal{P}^{(3)} + 2\mathfrak{u}_1\mathfrak{v}_1\mu_\mathcal{P}^{(3)} + \mu_\mathcal{J}^{(3)}\mathfrak{l}^2 - \mu_\mathcal{J}^{(3)}\mathfrak{u}_1^2 \\ &\quad + \frac{1}{2}\mathfrak{v}_1\dot{\mathfrak{l}} + \frac{1}{2}\mathfrak{u}_1\dot{\mathfrak{m}} - \frac{1}{2}\mathfrak{m}\dot{\mathfrak{u}}_1 - \frac{1}{2}\mathfrak{l}\dot{\mathfrak{v}}_1 + \dot{\mathfrak{v}}_2.\end{aligned} \tag{4.45d}$$

Consistency of (4.44) with (4.45), e.g. $\partial_v\partial_\varphi\mathfrak{l} = \partial_\varphi\partial_v\mathfrak{l}$, leads to the following relation for the "asymptotic" chemical potentials in terms of the "near horizon" variables

$$\mu_\mathcal{P} = \mu_L\mathcal{P} + \frac{8}{3}\mu_U\mathcal{J}\mathcal{J}^{(3)} + \frac{4}{3}\mu_U\mathcal{J}' - \frac{2}{3}\mu_U'\mathcal{J} - \mu_L', \tag{4.46a}$$

$$\begin{aligned}\mu_\mathcal{J} &= \mu_M\mathcal{P} + \frac{8}{3}\mu_U\mathcal{P}\mathcal{J}^{(3)} + \frac{8}{3}\mu_U\mathcal{J}\mathcal{P}^{(3)} + \frac{8}{3}\mu_V\mathcal{J}\mathcal{J}^{(3)} \\ &\quad + \frac{4}{3}\mu_U\mathcal{P}' + \frac{4}{3}\mu_V\mathcal{J}' - \frac{2}{3}\mu_U'\mathcal{P} - \frac{2}{3}\mu_V'\mathcal{J} - \mu_M',\end{aligned} \tag{4.46b}$$

$$\mu_\mathcal{P}^{(3)} = \mu_L\mathcal{J}^{(3)} + \mu_U\mathcal{J}^2 - \frac{4}{3}\mu_U\left(\mathcal{J}^{(3)}\right)^2 - \mu_U\mathcal{J}' - \frac{3}{2}\mu_U'\mathcal{J} + \frac{1}{2}\mu_U'', \tag{4.46c}$$

$$\begin{aligned}\mu_\mathcal{J}^{(3)} &= \mu_L\mathcal{P}^{(3)} + \mu_M\mathcal{J}^{(3)} + 2\mu_U\mathcal{P}\mathcal{J} + \mu_V\mathcal{J}^2 - \frac{8}{3}\mu_U\mathcal{P}^{(3)}\mathcal{J}^{(3)} - \frac{4}{3}\mu_V\left(\mathcal{J}^{(3)}\right)^2 \\ &\quad - \mu_V\mathcal{J}' - \mu_U\mathcal{P}' - \frac{3}{2}\mu_V'\mathcal{J} - \frac{3}{2}\mu_U'\mathcal{P} + \frac{1}{2}\mu_V''.\end{aligned} \tag{4.46d}$$

The gauge fields a and $\tilde{a}$ are then mapped to each other, provided that

$$\mathcal{M} = \mathcal{J}^2 + \frac{4}{3}\left(\mathcal{J}^{(3)}\right)^2 + 2\mathcal{J}', \tag{4.47a}$$

$$\mathcal{N} = \mathcal{J}\mathcal{P} + \frac{4}{3}\mathcal{J}^{(3)}\mathcal{P}^{(3)} + \mathcal{P}', \tag{4.47b}$$

$$\mathcal{V} = \frac{1}{54}\left(18\mathcal{J}^2\mathcal{J}^{(3)} - 8\left(\mathcal{J}^{(3)}\right)^3 + 9\mathcal{J}'\mathcal{J}^{(3)} + 27\mathcal{J}\mathcal{J}^{(3)'} + 9\mathcal{J}^{(3)''}\right), \tag{4.47c}$$

$$\begin{aligned}\mathcal{Z} = \frac{1}{36}&\Big(6\mathcal{J}^2\mathcal{P}^{(3)} - 8\mathcal{P}^{(3)}\left(\mathcal{J}^{(3)}\right)^2 + 3\mathcal{P}^{(3)}\mathcal{J}' + 3\mathcal{J}^{(3)}\mathcal{P}' \\ &+9\mathcal{J}\mathcal{P}^{(3)'} + 9\mathcal{P}\mathcal{J}^{(3)'} + 12\mathcal{P}\mathcal{J}\mathcal{J}^{(3)} + 3\mathcal{P}^{(3)''}\Big).\end{aligned} \tag{4.47d}$$

Additionally, it is possible to explicitly check that the equations of motion in the highest weight gauge (4.12) indeed reduce to the very simple ones given by (4.27). The relations (4.46) show that the "asymptotic chemical potentials" μ_{L}, μ_{M}, μ_{U}, μ_{V} depend not only on the "near horizon chemical potentials" $\mu_\mathcal{P}$, $\mu_\mathcal{J}$, $\mu_\mathcal{P}^{(3)}$, $\mu_\mathcal{J}^{(3)}$, but also on the state-dependent functions $\mathcal{P}$, $\mathcal{J}$, $\mathcal{P}^{(3)}$, $\mathcal{J}^{(3)}$, which is one way to see that our near horizon boundary conditions (4.26) are inequivalent to the asymptotic ones given by see (4.8) and (4.11). Moreover, the same relations directly map the corresponding gauge parameters that preserve the respective boundary conditions by replacing $\mu_{\text{L}} \to \epsilon$, $(1+\mu_{\text{M}}) \to \tau$, $\mu_{\text{U}} \to \chi$, $\mu_{\text{V}} \to \kappa$ as well as $\mu_\mathcal{J} \to \epsilon_\mathcal{J}$, $\mu_\mathcal{P} \to \epsilon_\mathcal{P}$, $\mu_\mathcal{J}^{(3)} \to \epsilon_\mathcal{J}^{(3)}$ and $\mu_\mathcal{P}^{(3)} \to \epsilon_\mathcal{P}^{(3)}$. Therefore, also the infinitesimal transformation laws for $\mathcal{N}$, $\mathcal{M}$, $\mathcal{V}$ and $\mathcal{Z}$ can be directly read off from (4.12) by replacing e.g. $\dot{\mathcal{M}}$ by $\delta_\epsilon\mathcal{M}$ as well as all occurrences of chemical potentials μ_a and $\mu_a^{(3)}$ by the corresponding gauge parameters ϵ_a and $\epsilon_a^{(3)}$, respectively. This holds, since precisely this replacement leads from the EOM for the auxiliary connection $\partial_v a_\varphi = \partial_\varphi a_v + [a_\varphi, a_v]$, inferred from (2.24), to the transformation behaviour under gauge transformation $\delta_\epsilon a_\varphi = \partial_\varphi \epsilon + [a_\varphi, \epsilon]$. Thus, one can readily see that the fields $\mathcal{N}$, $\mathcal{M}$, $\mathcal{V}$ and $\mathcal{Z}$ transform precisely the same way as would generators satisfying an $\mathcal{FW}_3$ algebra. However, their associated canonical charges still satisfy $\hat{\mathfrak{u}}(1)$ current algebras as before. This can be seen by looking at the variation of the canonical boundary charge. In particular, after using the relations between the "near horizon" and "asymptotic" gauge parameters, defined by the corresponding relations between the chemical potentials (4.46), and (4.47) we find that the variation of the "asymptotic" charges (4.17) just reduces to the one of the "near horizon" charges (4.31)

$$\begin{aligned}\delta\mathcal{Q} &= \frac{k}{2\pi}\int d\varphi\left(\epsilon\delta\mathcal{L} + \frac{1}{2}\sigma\delta\mathcal{M} + 8\chi\delta\mathcal{U} + 4\rho\delta\mathcal{V}\right)\\ &\equiv \frac{k}{2\pi}\int \mathrm{d}\varphi\left(\epsilon_{\mathcal{J}}\delta\mathcal{J} + \epsilon_{\mathcal{P}}\delta\mathcal{P} + \frac{4}{3}\epsilon_{\mathcal{J}}^{(3)}\delta\mathcal{J}^{(3)} + \frac{4}{3}\epsilon_{\mathcal{P}}^{(3)}\delta\mathcal{P}^{(3)}\right).\end{aligned} \tag{4.48}$$

with

$$\mathcal{L} = \mathcal{N} - \tfrac{v}{2}\mathcal{M}' \qquad \mathcal{U} = \mathcal{Z} - \tfrac{v}{2}\mathcal{V}', \tag{4.49a}$$

and

$$\sigma = \tau - v\,\epsilon' \qquad \rho = \kappa - v\chi'. \tag{4.49b}$$

4.2.4 $\mathcal{FW}$-Algebras from $\mathfrak{u}(1)$-Algebras[23]

Using (4.47) as well as the mode expansions (4.32) and

$$\mathcal{N}(\varphi) = \frac{1}{k}\sum_{n\in\mathbb{Z}} L_n e^{-in\varphi}, \qquad \mathcal{M}(\varphi) = \frac{2}{k}\sum_{n\in\mathbb{Z}} M_n e^{-in\varphi}, \tag{4.50a}$$

$$\mathcal{Z}(\varphi) = \frac{\sqrt{3}}{8k}\sum_{n\in\mathbb{Z}} U_n e^{-in\varphi}, \qquad \mathcal{V}(\varphi) = \frac{\sqrt{3}}{4k}\sum_{n\in\mathbb{Z}} V_n e^{-in\varphi}, \tag{4.50b}$$

$$\delta(\varphi - \bar\varphi) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-in(\varphi-\bar\varphi)}, \tag{4.50c}$$

we find that the twisted Sugawara construction for the $\mathcal{FW}_3$ algebra is given by

$$L_n = \frac{1}{k}\sum_{p\in\mathbb{Z}}\left(J_{n-p}P_p + \frac{3}{4}J_{n-p}^{(3)}P_p^{(3)}\right) - inP_n, \tag{4.51a}$$

$$M_n = \frac{1}{2k}\sum_{p\in\mathbb{Z}}\left(J_{n-p}J_p + \frac{3}{8}J_{n-p}^{(3)}J_p^{(3)}\right) - inJ_n, \tag{4.51b}$$

$$\begin{aligned}U_n = &\frac{\sqrt{3}}{k^2}\sum_{p,q\in\mathbb{Z}}\left[\left(J_{n-p-q}J_p - \frac{3}{4}J_{n-p-q}^{(3)}J_p^{(3)}\right)J_q^{(3)} + 2J_{n-p-q}J_p^{(3)}P_q\right]\\ &- \frac{\sqrt{3}i}{2k}\sum_{p\in\mathbb{Z}}\left[(3n-2p)\,J_{n-p}^{(3)}P_p + (n+2p)J_{n-p}P_p^{(3)}\right] - \frac{\sqrt{3}}{2}n^2P_n^{(3)},\end{aligned} \tag{4.51c}$$

[23] The explicit computations in this section were performed by our collaborator Max Riegler.

$$V_n = \frac{\sqrt{3}}{k^2} \sum_{p,q\in\mathbb{Z}} \left(J_{n-p-q} J_p - \frac{1}{4} J^{(3)}_{n-p-q} J^{(3)}_p \right) J^{(3)}_q - \frac{\sqrt{3}i}{2k} \sum_{p\in\mathbb{Z}} (3n-2p) J^{(3)}_{n-p} J_p - \frac{\sqrt{3}}{2} n^2 J^{(3)}_n . \tag{4.51d}$$

At this point it is important to note that we already implicitly assumed some kind of normal ordering prescription for the constituents of the operators appearing in (4.51). The ordering prescription we chose is in accordance with the ones for induced representations as shown in [57]. Computing the commutation relations of these new operators, we find that they satisfy the $\mathcal{FW}_3$ algebra, see (4.18), however, with different Λ_n and Θ_n

$$\Lambda_n = \sum_{p\in\mathbb{Z}} M_p L_{n-p}, \qquad \Theta_n = \sum_{p\in\mathbb{Z}} M_p M_{n-p} \,, \tag{4.52}$$

since the normal ordering is performed with respect to induced representations, instead of being performed with respect to highest weight representations. In addition to the commutation relations of the spin-2 and spin-3 generators, we find the following non-vanishing commutation relations with the spin-1 currents

$$[L_n, J_m] = - mJ_{n+m} - in^2 k\delta_{n+m,0}, \tag{4.53a}$$
$$[L_n, P_m] = - mP_{n+m}, \tag{4.53b}$$
$$[L_n, J^{(3)}_m] = - mJ^{(3)}_{n+m}, \tag{4.53c}$$
$$[L_n, P^{(3)}_m] = - mP^{(3)}_{n+m}, \tag{4.53d}$$
$$[M_n, P_m] = - mJ_{n+m} - in^2 k\delta_{n+m,0}, \tag{4.53e}$$
$$[M_n, P^{(3)}_m] = - mJ^{(3)}_{n+m}, \tag{4.53f}$$
$$[U_n, J_m] = - \frac{2\sqrt{3}}{k} m \sum_{p\in\mathbb{Z}} J_{n+m-p} J^{(3)}_q + \frac{\sqrt{3}i}{2} m(3n+2m) J^{(3)}_{n+m}, \tag{4.53g}$$
$$[U_n, P_m] = - \frac{2\sqrt{3}}{k} m \sum_{p\in\mathbb{Z}} \left(J_{n+m-p} P^{(3)}_q + J^{(3)}_{n+m-p} P_q \right) + \frac{\sqrt{3}i}{2} m(3n+2m) P^{(3)}_{n+m}, \tag{4.53h}$$
$$[U_n, J^{(3)}_m] = \frac{\sqrt{3}}{k} m \sum_{p\in\mathbb{Z}} J^{(3)}_{n+m-p} J^{(3)}_q + \frac{2i}{\sqrt{3}} m(3n+2m) J_{n+m} - \frac{2k}{\sqrt{3}} n^3 \delta_{n+m,0}, \tag{4.53i}$$
$$[U_n, P^{(3)}_m] = \frac{2\sqrt{3}}{k} m \sum_{p\in\mathbb{Z}} \left(J^{(3)}_{n+m-p} P^{(3)}_q - \frac{4}{3} J_{n+m-p} P_q \right) + \frac{2i}{\sqrt{3}} m(3n+2m) P_{n+m}, \tag{4.53j}$$
$$[V_n, P^{(3)}_m] = \frac{\sqrt{3}}{k} m \sum_{p\in\mathbb{Z}} J^{(3)}_{n+m-p} J^{(3)}_q + \frac{2i}{\sqrt{3}} m(3n+2m) J_{n+m} - \frac{2k}{\sqrt{3}} n^3 \delta_{n+m,0}, \tag{4.53k}$$
$$[V_n, P_m] = - \frac{2\sqrt{3}}{k} m \sum_{p\in\mathbb{Z}} J_{n+m-p} J^{(3)}_q + \frac{\sqrt{3}i}{2} m(3n+2m) J^{(3)}_{n+m}. \tag{4.53l}$$

4.2.5 Entropy of Cosmological Solutions

After having found a map between the boundary conditions in diagonal and highest weight gauge, one can show explicitly that the entropy calculated using the near horizon boundary conditions also coincides precisely with the one given in [52,53]. Note that this has to be the case, since we have already shown that our charges precisely coincide with the ones in [52,53] and the entropy is given as $S = -\beta H = -\beta Q[\epsilon|_{\partial_v}]$. As discussed in [4–6] one advantage of the near horizon boundary conditions is that regularity of the solutions holds regardless of the value of the global charges, since the corresponding holonomy conditions are solved trivially. We assume that our Euclidean manifold has the topology of a solid torus, where the

Euclidean time coordinate $\tau = iv$ with $0 \leq \tau < \beta$ corresponds to the contractible cycle.
Regularity of the solution requires the holonomy of the gauge fields around any contractible circle be trivial. Hence,

$$\mathcal{H}_{\mathcal{C}} = \mathcal{P}e^{\int_{\mathcal{C}} a} \to \exp\left(\int_{\rho=0} a_\tau \, \mathrm{d}\tau\right) = \mathbb{1} \, . \tag{4.54}$$

For our boundary conditions, which are in diagonal gauge, this holonomy condition is trivially solved by

$$\mu_{\mathcal{J}} = \mu_{\mathcal{J}}^{(3)} = 0 \, , \qquad \mu_{\mathcal{P}} = -\frac{m\pi}{\beta} \qquad \text{and} \qquad \mu_{\mathcal{P}}^{(3)} = -\frac{m\pi - 2n\pi}{2\beta} \qquad \text{with } n, m \in \mathbb{Z}. \tag{4.55}$$

Hence, as already stated above regularity holds regardless of the value of the state-dependent functions $\mathcal{J}$, $\mathcal{P}$, $\mathcal{J}^{(3)}$ and $\mathcal{P}^{(3)}$. The entropy can now be computed via

$$S = -\beta \, H = -\frac{k}{2\pi}\beta \int \mathrm{d}\varphi \, \langle A_\tau A_\varphi \rangle = -\frac{k}{2\pi}\beta \int \mathrm{d}\varphi \, \langle a_\tau a_\varphi \rangle \, . \tag{4.56}$$

Using (4.26) and (4.55) we get

$$S = \frac{k\pi}{3}\left(3mP + 2(m-2n)P^{(3)}\right) \, . \tag{4.57}$$

For simplicity's sake we restrict ourselves to zero-modes only. Since flat space cosmologies are solutions with constant state-dependent functions, this is the most important case. Thus,

$$\begin{aligned}
&\mathcal{J}(\varphi) \to J := \frac{1}{k}J_0 \, , \quad \mathcal{P}(\varphi) \to P := \frac{1}{k}P_0 \, , \quad \mathcal{J}^{(3)}(\varphi) \to J^{(3)} := \frac{3}{4k}J_0^{(3)} \, , \quad \mathcal{P}^{(3)}(\varphi) \to P^{(3)} := \frac{3}{4k}P_0^{(3)} \, , \\
&\mathcal{M}(\varphi) \to M = \text{const} \, , \quad \mathcal{V}(\varphi) \to V = \text{const} \, , \quad \mathcal{Z}(\varphi) \to Z = \text{const} \, , \quad \mathcal{N}(\varphi) \to N = \text{const} \, .
\end{aligned}$$

For the branch $m = 2$, $n = 1$ the entropy (4.57) reduces to

$$S = 2k\pi P \, . \tag{4.58}$$

We will see that it is exactly this branch that is connected continuously to the cosmological spacetimes in pure Einstein gravity and coincides with the results from [52,53]. The expression for P in terms of the asymptotic charges is given as (for details of the calculation refer to appendix D.4)

$$P = \pm\frac{N\sqrt{M}\cos\left(\frac{2x}{3}\right) - 4\sqrt{3}Z\sin\left(\frac{x}{3}\right)}{M\sqrt{1-\frac{108V^2}{M^3}}} \qquad \text{where} \qquad x = \arcsin\left(6\sqrt{3}\left(\frac{1}{M}\right)^{3/2} V\right) \, . \tag{4.59}$$

The form of (4.59) makes it easier to compare our result to the explicit expression given in [53], see also subsection 4.1.3, where a different basis was used. In terms of the asymptotic charges given in [53], see (4.23), the entropy (4.58) may be rewritten as

$$S = \pm 2\pi\sqrt{\frac{k\pi}{\hat{\mathcal{P}}}}\sec(\Phi)\left[+\hat{\mathcal{J}}\cos\left(\frac{2\Phi}{3}\right) - \sqrt{\frac{3k}{\pi\hat{P}}}\frac{\hat{V}}{4}\sin\left(\frac{\Phi}{3}\right)\right] \qquad \text{with} \tag{4.60a}$$

$$\Phi = \arcsin\left(\frac{3}{8}\sqrt{\frac{3k}{\pi\hat{P}^3}}\hat{W}\right) \, . \tag{4.60b}$$

In order to make contact with the cosmological configurations in pure Einstein gravity (4.20) we have to restrict ourselves to the branch

$$S = 2\pi\sqrt{\frac{k\pi}{\hat{\mathcal{P}}}}\sec(\Phi)\left[+|\hat{\mathcal{J}}|\cos\left(\frac{2\Phi}{3}\right) + \sqrt{\frac{3k}{\pi\hat{P}}}\frac{\hat{V}}{4}\sin\left(\frac{\Phi}{3}\right)\right] \qquad \text{with} \tag{4.61a}$$

$$\Phi = \mp \arcsin\left(\frac{3}{8}\sqrt{\frac{3k}{\pi \hat{P}^3}}\hat{W}\right) , \tag{4.61b}$$

where the sign of Φ has to be the opposite sign with respect to $\hat{J}$ such that consistency with (4.60) is guaranteed. Equation (4.61) coincides precisely with formula (65) given in [53]. Thus, we see that the branch of the entropy that is continuously connected to the cosmological spacetimes in pure Einstein gravity only depends on the $P_0 = kP$ mode, see (4.58). Equation (4.58) can be brought into a more suggestive form by linearly combining the zero-modes (4.34) such that (4.61) becomes

$$S = 2k\pi P = 2\pi P_0 = 2\pi(J_0^+ + J_0^-) , \tag{4.62}$$

which is precisely the result, already found in the spin-2 AdS [4], spin-3 AdS [6] and spin-2 flat space case [5]. This suggests a universal relation for the entropy in terms of the spin-2 (near horizon) zero-mode charges. It is surprising that the entropy does not at all depend on the spin-3 zero-modes.

5 Conclusion and Outlook

In this work, novel boundary conditions for spin-3 gravity in flat space were introduced. In chapter 3 we reviewed the asymptotically AdS (Brown-Henneaux) boundary conditions and the "near horizon" boundary conditions for spin-2 gravity in AdS. In chapter 4 we proposed new boundary conditions for spin-3 gravity in flat space with chemical potentials (4.26) and showed that they lead to finite, integrable and conserved charges. Furthermore, we showed that, similarly to the already existing cases of "near horizon" boundary conditions [4–6], the asymptotic symmetry algebra for spin-3 gravity in flat space is given by four $\hat{\mathfrak{u}}(1)$ current algebras (4.33). We also showed that all vacuum descendants of our theory are soft, in the sense of having zero energy while still being non-trivial; they thus may be regarded as higher-spin soft hair. We found that through a proper gauge transformation our "near horizon" boundary conditions can be translated from the diagonal gauge into the standard highest weight gauge. This made it possible to relate the remarkably simple entropy in terms of near horizon variables to the more complicated one in terms of asymptotic variables. We found that the branch that is continuously connected to the cosmological spacetimes of general relativity only depends on the spin-2 (near horizon) zero-modes (4.58). The result is precisely the same as was found in [4–6], which suggests a universal relation.

However, many questions remain open. First of all, it would be interesting to investigate the universality of the entropy result by generalizing our considerations to conformal gravity, supersymmetric gravity, higher dimensions and non-principally embedded higher-spin theories. To do so, it would be useful to investigate a more systematic way of proposing extensions of the near horizon boundary conditions. Furthermore, it would be interesting to explicitely construct higher-spin microstates along the lines of [58]. Our considerations have focused on the (higher-spin generalization) of future null infinity. Investigations using the full structure of asymptotically flat spacetimes [59] have provided fascinating connections between soft modes and conservation laws [60–62]. Furthermore, such investigations might even provide insights into the black hole information paradox [8].

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A Algebras and their Matrix Representations

A.1 $\mathfrak{sl}(2,\mathbb{R})$

Algebra

$$[\mathtt{L}_n, \mathtt{L}_m] = (n-m)\mathtt{L}_{n+m} \tag{A.1}$$

with $n, m = -1, 0, 1$.

Non-Degenerate Invariant Bilinear Form

$$\langle \mathtt{L}_n, \mathtt{L}_m \rangle = \begin{array}{cc} & \begin{matrix} \mathtt{L}_1 & \mathtt{L}_0 & \mathtt{L}_{-1} \end{matrix} \\ \begin{matrix} \mathtt{L}_1 \\ \mathtt{L}_0 \\ \mathtt{L}_{-1} \end{matrix} & \begin{pmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & 0 \end{pmatrix} \end{array} \tag{A.2}$$

Fundamental Representation

$$\mathtt{L}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathtt{L}_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathtt{L}_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{A.3}$$

Connection to $\mathfrak{su}(2)$

The fundamental representation of $\mathfrak{su}(2)$ is given by

$$\mathtt{S}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathtt{S}_0 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \mathtt{S}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

We know from quantum mechanics that one can define a raising and lowering operator of a spin-1/2 system via

$$\mathtt{S}_+ = \mathtt{S}_1 + i\mathtt{S}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -\mathtt{L}_{-1}\,, \qquad \mathtt{S}_- = \mathtt{S}_1 - i\mathtt{S}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathtt{L}_1 \tag{A.5}$$

which act on $\langle\downarrow| = (0,1)$ or $\langle\uparrow| = (1,0)$ via

$$\mathtt{S}_+ \left|\downarrow\right\rangle = \left|\uparrow\right\rangle, \qquad \mathtt{S}_- \left|\uparrow\right\rangle = \left|\downarrow\right\rangle, \qquad \mathtt{S}_+ \left|\uparrow\right\rangle = \mathtt{S}_+ \left|\downarrow\right\rangle = 0\,. \tag{A.6}$$

Thus, up to a sign in (A.5) we find that $\mathtt{L}_1$ and $\mathtt{L}_{-1}$ are nothing else than the raising and lowering operators of a spin-1/2 system, which in turn coins the term *highest weight* gauge in section 3.2.3.

A.2 $\mathfrak{isl}(3,\mathbb{R})$ algebra

Algebra

$$[\mathtt{L}_n, \mathtt{L}_m] = (n-m)\mathtt{L}_{n+m}\,, \tag{A.7a}$$
$$[\mathtt{L}_n, \mathtt{M}_m] = (n-m)\mathtt{M}_{n+m}\,, \tag{A.7b}$$
$$[\mathtt{L}_n, \mathtt{U}_m] = (2n-m)\mathtt{U}_{n+m}\,, \tag{A.7c}$$
$$[\mathtt{L}_n, \mathtt{V}_m] = (2n-m)\mathtt{V}_{n+m}\,, \tag{A.7d}$$
$$[\mathtt{U}_n, \mathtt{U}_m] = \sigma(n-m)(2n^2+2m^2-nm-8)\mathtt{L}_{n+m}\,, \tag{A.7e}$$
$$[\mathtt{U}_n, \mathtt{V}_m] = \sigma(n-m)(2n^2+2m^2-nm-8)\mathtt{M}_{n+m}\,, \tag{A.7f}$$

with $i=-1,0,1$ and $m=-2,-1,0,1,2$. The $\mathtt{L}_n$ generate rotations, the $\mathtt{M}_n$ generate translations and $\mathtt{U}_n, \mathtt{V}_n$ generate spin-3 transformations. The factor σ fixes the normalization of the spin-3 generators $\mathtt{U}_n$ and $\mathtt{V}_n$. We choose

$$\sigma = -\frac{1}{3}\,. \tag{A.8}$$

Non-Degenerate Invariant Bilinear Form

$$\langle \mathtt{L}_n\, \mathtt{M}_m\rangle = -2\begin{pmatrix} & \mathtt{M}_1 & \mathtt{M}_0 & \mathtt{M}_{-1} \\ \hline \mathtt{L}_1 & 0 & 0 & 1 \\ \mathtt{L}_0 & 0 & -\frac{1}{2} & 0 \\ \mathtt{L}_{-1} & 1 & 0 & 0 \end{pmatrix}, \tag{A.9a}$$

as well as

$$\langle \mathtt{U}_n\, \mathtt{V}_m\rangle = 2\begin{pmatrix} & \mathtt{V}_2 & \mathtt{V}_1 & \mathtt{V}_0 & \mathtt{V}_{-1} & \mathtt{V}_{-2} \\ \hline \mathtt{U}_2 & 0 & 0 & 0 & 0 & 4 \\ \mathtt{U}_1 & 0 & 0 & 0 & -1 & 0 \\ \mathtt{U}_0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ \mathtt{U}_{-1} & 0 & -1 & 0 & 0 & 0 \\ \mathtt{U}_{-2} & 4 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.9b}$$

All other pairings of generators inside the bilinear form $\langle\cdot,\cdot\rangle$ are zero.

(8 + 1)-Matrix Representation

Throughout this work we have used the following matrix representation of the $\mathfrak{isl}(3,\mathbb{R})$ generators $\mathtt{G}$

$$\mathtt{G} = \begin{pmatrix} \mathrm{ad}_{8\times 8} & \mathrm{odd}_{8\times 1} \\ \mathbb{O}_{1\times 8} & 0 \end{pmatrix}, \tag{A.10}$$

with $\mathrm{ad}_{8\times 8}$ being an 8×8 matrix and $\mathrm{odd}_{8\times 1}$ being an 8×1 column vector. The even generators $\mathtt{L}_n$ and $\mathtt{U}_n$ have $\mathrm{ad}\neq\mathbb{O}$, $\mathrm{odd}=\mathbb{O}$, while the odd generators $\mathtt{M}_n$ and $\mathtt{V}_n$ have $\mathrm{ad}=\mathbb{O}$, $\mathrm{odd}\neq\mathbb{O}$. If we now use the odd generators as unit basis vectors we get

$$\mathrm{odd}_{\mathtt{M}_n} = E_{n+2}\,, \qquad \mathrm{odd}_{\mathtt{V}_n} = E_{n+6} \tag{A.11}$$

with

$$E_i = (\underbrace{0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{8-i})^T, \qquad i=1\ldots 8\,. \tag{A.12}$$

The ad-parts of the even generators are then given by

$$
\mathrm{ad}_{\mathtt{L}_{-1}} = -\begin{pmatrix} 0&1&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&3&0\\ 0&0&0&0&0&0&0&4\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \qquad
\mathrm{ad}_{\mathtt{L}_{0}} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&2&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-2 \end{pmatrix},
$$

$$
\mathrm{ad}_{\mathtt{L}_{1}} = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 2&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&4&0&0&0&0\\ 0&0&0&0&3&0&0&0\\ 0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&1&0 \end{pmatrix}, \qquad
\mathrm{ad}_{\mathtt{U}_{-2}} = \begin{pmatrix} 0&0&0&0&0&0&4&0\\ 0&0&0&0&0&0&0&16\\ 0&0&0&0&0&0&0&0\\ 0&-2&0&0&0&0&0&0\\ 0&0&-4&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},
$$

$$
\mathrm{ad}_{\mathtt{U}_{-1}} = \begin{pmatrix} 0&0&0&0&0&-2&0&0\\ 0&0&0&0&0&0&-2&0\\ 0&0&0&0&0&0&0&4\\ 1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&-3&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \qquad
\mathrm{ad}_{\mathtt{U}_{0}} = \begin{pmatrix} 0&0&0&0&2&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-2&0\\ 0&0&0&0&0&0&0&0\\ 2&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&-2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},
$$

$$
\mathrm{ad}_{\mathtt{U}_{1}} = \begin{pmatrix} 0&0&0&-4&0&0&0&0\\ 0&0&0&0&2&0&0&0\\ 0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 3&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{pmatrix}, \qquad
\mathrm{ad}_{\mathtt{U}_{2}} = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&-16&0&0&0&0\\ 0&0&0&0&-4&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 4&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0 \end{pmatrix}.
$$

(6×6) representation

Another useful representation of the $\mathfrak{isl}(3,\mathbb{R})$ algebra is given in terms of 6×6 block-diagonal matrices[24]. It is convenient to write them as 3×3 matrices tensored by 2×2 diagonal matrices. The block structure is a remnant of the decomposition of the AdS algebra $\mathfrak{so}(2,2) \sim \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$ before the İnönü–Wigner

[24]Our collaborators Max Riegler and Martin Ammon used this representation to perform their calculations.

contraction. In this representation the generators are given as

$$\mathtt{L}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}\,, \qquad \mathtt{L}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}\,, \qquad \mathtt{L}_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}\,,$$
$$\mathtt{U}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}\,, \qquad \mathtt{U}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}\,, \qquad \mathtt{U}_0 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix} \otimes \mathbb{1}_{2\times 2}\,,$$
$$\mathtt{U}_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}\,, \quad \mathtt{U}_{-2} = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}\,. \tag{A.13}$$

All odd generators can be written as a product of corresponding even generators times a γ^*-matrix,

$$\mathtt{M}_n = \epsilon\,\mathtt{L}_n \times \gamma^*\,, \qquad \mathtt{V}_n = \epsilon\,\mathtt{U}_n \times \gamma^*\,, \tag{A.14}$$

where ϵ is a Grassmann parameter ($\epsilon^2 = 0$) and

$$\gamma^* = \begin{pmatrix} \mathbb{1}_{3\times 3} & \mathbb{O}_{3\times 3} \\ \mathbb{O}_{3\times 3} & -\mathbb{1}_{3\times 3} \end{pmatrix}\,. \tag{A.15}$$

B Differential Forms[25]

In this appendix we provide a short collection of useful definitions and relations for differential forms, which have been used throughout this work – we closely follow the textbook by Nakahara [64].

Def: A **differential form** of order p is a totally antisymmetric tensor of type $(0, p)$, where the entry "0" denotes the number of contravariant indices and p denotes the number of covariant indices.

Def: The **wedge product** $\wedge$ of p one-forms is defined by the totally antisymmetric tensor product

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} = \sum_{P \in S_p} \mathrm{sgn}(P)\ \mathrm{d}x^{\mu_{P(1)}} \otimes \mathrm{d}x^{\mu_{P(2)}} \otimes \cdots \otimes \mathrm{d}x^{\mu_{P(p)}}\,, \tag{B.1}$$

where P is an element of S_p, the symmetric group of order p and $\mathrm{sgn}(P)$ is $+1$ for even and -1 for odd permutations. If we denote the vector space of p-forms at a certain point m of our manifold $\mathcal{M}$ as $\Omega^p_m(\mathcal{M})$, then the set of p-forms (B.1) forms a basis of $\Omega^p_m(\mathcal{M})$. An element $\alpha \in \Omega^p_m(\mathcal{M})$ can be expanded as

$$\alpha = \frac{1}{p!}\alpha_{\mu_1\mu_2\cdots\mu_p}\,\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\,. \tag{B.2}$$

For the p-form α and the b-form β the definition of the exterior product $\wedge$ leads to a $(p+b)$-form and can be written as

$$\alpha \wedge \beta = \frac{1}{p!\,b!}\,\alpha_{\mu_1\mu_2\cdots\mu_p}\beta_{\mu_{p+1}\mu_{p+2}\cdots\mu_{p+b}}\,\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+b}}. \tag{B.3}$$

Def: The **exterior derivative** d is defined as

$$\mathrm{d}\alpha = \frac{1}{p!}\partial_\rho \alpha_{\mu_1\mu_2\cdots\mu_p}\,\mathrm{d}x^\rho \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{B.4}$$

From this one may infer other useful relations

$$\alpha \wedge \beta = (-1)^{pb}\beta \wedge \alpha\,, \tag{B.5}$$

$$\alpha \wedge \alpha = 0\,, \tag{B.6}$$

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)\,, \tag{B.7}$$

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta\,, \tag{B.8}$$

$$\mathrm{d}^2 = 0\,. \tag{B.9}$$

Def: The **Hodge dualization** of a generic antisymmetric tensor with $(n-k)$-indices is defined as

$$(*\eta)_{i_1,i_2,\ldots,i_{n-k}} = \frac{1}{k!}\eta_{j_1,j_2,\ldots,j_k}\sqrt{|\mathrm{det}g|}\,\epsilon^{j_1,\ldots,j_k}{}_{i_1,i_{n-k}}\,, \tag{B.10}$$

where g is the metric tensor.

[25] This appendix is based on a file originally written by Stefan Prohazka as part of his Master thesis [63]

C Conformal Weight

We have already mentioned in section 1.3 that a notion of spin s can be introduced through the transformation behaviour of fields under conformal transformations[26]. In conformal field theory the property, which characterizes the transformation behaviour of a conformal field is called conformal weight h. The two terms spin and conformal weight are used synonymously in this thesis, i.e. $s = h$. For more information on conformal field theory consult e.g. [44, 65].
A field $\Phi(z)$ that transforms under conformal transformations $z \to f(z)$ as

$$\Phi'(z) = \left(\frac{\partial f(z)}{\partial z}\right)^h \Phi(z)\,. \tag{C.1}$$

is called a primary field of conformal weight (or conformal dimension) h. Therefore, as already mentioned in section 1.3 higher-spin fields are equivalent to higher-rank fields. For instance, the metric, a rank two tensor, has conformal weight two.
Under infinitesimal transformations $z \to f(z) = z + \epsilon(z)$ the primary field $\Phi(z)$ transforms as

$$\delta_\epsilon \Phi = \Phi\left(z + \epsilon(z)\right) - \Phi(z) \approx \left(1 + h\partial_z \epsilon(z)\right)\left(\Phi(z) + \epsilon(z)\partial_z \Phi(z)\right) - \Phi(z) = h\frac{\partial \epsilon(z)}{\partial z}\Phi(z) + \epsilon(z)\frac{\partial \Phi(z)}{\partial z}\,. \tag{C.2}$$

Expanding $\epsilon(z)$ into Fourier modes

$$\epsilon(z) = \sum_n \epsilon_n z^{n+1} \tag{C.3}$$

leads to

$$\delta_\epsilon \Phi = z^n \epsilon_n \left((n+1)h + z\partial_z\right)\Phi(z)\,. \tag{C.4}$$

If we now assume that $\mathtt{L}_n$ is the generator of these conformal transformations and acts on the field $\Phi(z)$ via a commutator, we may rewrite (C.4) as

$$[\mathtt{L}_n, \Phi(z)] = z^n \left((n+1)h + z\partial_z\right)\Phi(z)\,. \tag{C.5}$$

Expanding the field $\Phi(z)$ into Fourier modes leads to

$$\begin{aligned}
[\mathtt{L}_n, \Phi(z)] &= \sum_m [\mathtt{L}_n, \Phi_m] z^{-m-h} = \sum_m z^n \left((n+1)h + z\partial_z\right)\Phi_m z^{-m-h} \\
&= \sum_m \left((n+1)h + (-m-h)\right)\Phi_m z^{-m+n-h} = \sum_m \left((n+1)h + (-m-n-h)\right)\Phi_{m+n} z^{-m-h} \\
&= \sum_m \left(n(h-1) - m\right)\Phi_{m+n} z^{-m-h}\,,
\end{aligned} \tag{C.6}$$

[26]Note that conformal transformations also include Lorentz transformations.

where we have performed an index shift in the second line. By comparing the summands we find that

$$[\mathtt{L}_n, \Phi_m] = (n(h-1) - m)\, \Phi_{m+n} \,. \tag{C.7}$$

Thus, for the Fourier modes of a spin-2 field $\mathtt{M}_n$ we find the relation

$$[\mathtt{L}_n, \mathtt{M}_m] = (n-m)\, \mathtt{M}_{m+n} \,, \tag{C.8}$$

while the Fourier modes of a spin-3 field $\mathtt{U}_n$ satisfy

$$[\mathtt{L}_m, \mathtt{U}_n] = (2m-n)\mathtt{U}_{m+n} \,. \tag{C.9}$$

The generators $\mathtt{L}_n$, which generate such infinitesimal conformal transformations span the Virasoro algebra

$$[\mathtt{L}_m, \mathtt{L}_n] = (m-n)\mathtt{L}_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \,, \tag{C.10}$$

where c denotes the central extension. In a conformal field theory $\mathtt{L}_n$ are the Fourier modes of the energy-momentum tensor, a quasi-primary field of conformal weight two. Equipped with this knowledge we are now able to identify the generators in e.g. (4.18) as the Fourier modes of spin-2 and spin-3 fields, respectively.

Comment

Until now we have restricted the discussion to chiral fields $\Phi = \Phi(z)$. However, note that in two-dimensional conformal field theories fields Φ can generally depend on both coordinates, i.e. $\Phi = \Phi(z, \bar{z})$. In two dimensions the conformal field theory is spanned by *two* copies of the Virasoro algebra, generated by $\mathtt{L}_n$ and $\bar{\mathtt{L}}_n$, see (3.19). In this case the Hamiltonian and the angular momentum (= spin) operator are given as the sum or the difference of the Virasoro zero-modes, i.e. $H = \mathtt{L}_0 + \bar{\mathtt{L}}_0$ and $J = S = \mathtt{L}_0 - \bar{\mathtt{L}}_0$ with eigenvalues $E = h + \bar{h}$ and $j = s = h - \bar{h}$.
A chiral quasi-primary field of conformal weight $(h, \bar{h}) = (2, 0)$ can thus be generated by acting with $\mathtt{L}_{-2}$ on the vacuum $|0\rangle$, i.e. via $\mathtt{L}_{-2}\,|0\rangle$. This excitation carries spin two and energy two. However, if we consider the vacuum descendant $\mathtt{L}_{-2}\bar{\mathtt{L}}_{-2}\,|0\rangle$, a quasi-primary field of conformal weight $(h, \bar{h}) = (2, 2)$, we readily infer that it corresponds to a spin-0 and energy-4 excitation.

D Calculations

D.1 Symmetries and DOF of the Riemann Tensor

According to equation (2.4) the Riemann tensor is antisymmetric in α and β

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}\,. \tag{D.1}$$

Now, we explicitly derive other symmetry properties of the Riemann tensor, which will impose restrictions on the number of degrees of freedom in our theory.

Antisymmetry in the second index pair

One may easily proove the antisymmetry of the Riemann tensor in the second index pair by using the metric compatibility of the covariant derivative

$$\nabla_\gamma\, g_{\alpha\beta} = 0 \qquad (\text{metric compatibility})\,, \tag{D.2}$$

$$0 = [\nabla_\alpha, \nabla_\beta] g_{\gamma\delta} = R_{\alpha\beta\gamma}{}^{\zeta} g_{\zeta\delta} + R_{\alpha\beta\delta}{}^{\zeta} g_{\gamma\zeta} = R_{\alpha\beta\gamma\delta} + R_{\alpha\beta\delta\gamma} = 0\,, \tag{D.3}$$

$$\Rightarrow R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}\,. \tag{D.4}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\beta\alpha\delta\gamma}\,. \tag{D.5}$$

A general rank 4-tensor in d dimensions has d^4 entries. Due to the antisymmetry in the first and second index pair $I = \{i,j\}$ and $J = \{k,l\}$, respectively, each one considered alone would give us $\frac{d\cdot(d-1)}{2}$ independent entries, i.e.

$$I = \{i,j\} \qquad \to \qquad \frac{d\cdot(d-1)}{2}\,, \tag{D.6}$$

$$J = \{k,l\} \qquad \to \qquad \frac{d\cdot(d-1)}{2}\,. \tag{D.7}$$

Bianchi identity

After encountering the antisymmetry of the Riemann tensor in both index pairs, one would expect the number of degrees of freedom to be

$$\#_R = \left(\frac{d\,(d-1)}{2}\right)^2\,. \tag{D.8}$$

However, the Bianchi identity restricts the number of degrees of freedom further

$$R_{\alpha[\beta\gamma\delta]} = \frac{1}{3}\left(R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma}\right) = 0\,, \tag{D.9}$$

where the square brackets denote total antisymmetrization[27]. The Bianchi identity gives another $d \cdot \binom{d}{3}$ independent equations which reduce the number of degrees of freedom to

$$\begin{aligned} \#_R &= \left(\frac{d\,(d-1)}{2}\right)^2 - \frac{d \cdot d!}{(d-3)!3!} = \frac{\left(d^2-d\right)^2}{4} - \frac{d^2\,(d-1)\,(d-2)}{6} \\ &= \frac{d^4-2d^3+d^2}{4} - \frac{d^4-3d^3+2d^2}{6} = \frac{1}{12}\left(d^4-d^2\right) . \end{aligned} \tag{D.10}$$

Additional Symmetries of the Riemann Tensor

Furthermore, the Riemann tensor is symmetric in the index pair $\{I, J\}$, i.e.

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \, . \tag{D.11}$$

For the derivation of this relation we need the first Bianchi identity. Furthermore, we use the antisymmetry of the Riemann tensor in the first and second index pair

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \, , \qquad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \, . \tag{D.12}$$

Note that this symmetry does not restrict the number of degrees any further, since this symmetry follows directly from the Bianchi identity. We start by considering

$$\begin{aligned} R_{\alpha\beta\gamma\delta} - R_{\gamma\delta\alpha\beta} &\overset{(D.9)}{=} R_{\alpha\beta\gamma\delta} - \left(-R_{\gamma\alpha\beta\delta} - R_{\gamma\beta\delta\alpha}\right) \overset{(D.12)}{=} R_{\alpha\beta\gamma\delta} - R_{\alpha\gamma\beta\delta} - R_{\beta\gamma\delta\alpha} \\ &\overset{(D.9)}{=} R_{\alpha\beta\gamma\delta} - \left(-R_{\alpha\beta\delta\gamma} - R_{\alpha\delta\gamma\beta}\right) - \left(-R_{\beta\alpha\gamma\delta} - R_{\beta\delta\alpha\gamma}\right) \\ &= \underbrace{R_{\alpha\beta\gamma\delta} + R_{\alpha\beta\delta\gamma}}_{=0} + R_{\alpha\delta\gamma\beta} + R_{\beta\alpha\gamma\delta} + R_{\beta\delta\alpha\gamma} \overset{(D.12)}{=} R_{\alpha\delta\gamma\beta} - R_{\alpha\beta\gamma\delta} + R_{\beta\delta\alpha\gamma} \\ &= \left(-R_{\alpha\delta\beta\gamma} - R_{\alpha\beta\gamma\delta}\right) + R_{\beta\delta\alpha\gamma} \overset{(D.9)}{=} R_{\alpha\gamma\delta\beta} + R_{\beta\delta\alpha\gamma} = -R_{\alpha\gamma\beta\delta} + R_{\beta\delta\alpha\gamma} \, . \end{aligned} \tag{D.13}$$

Interchanging the names $\alpha \leftrightarrow \beta$ and $\gamma \leftrightarrow \delta$ gives

$$R_{\alpha\beta\gamma\delta} - R_{\gamma\delta\alpha\beta} = -R_{\beta\delta\alpha\gamma} + R_{\alpha\gamma\beta\delta} \, . \tag{D.14}$$

Adding (D.13) to (D.14) gives $2\left(R_{\alpha\beta\gamma\delta} - R_{\gamma\delta\alpha\beta}\right) = 0$ and thus we arrive at

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \, , \tag{D.15}$$

as proposed.

D.2 Equivalence of EHP and CS action

We explicitly derive that the Einstein-Hilbert-Palatini action is equivalent to the Chern-Simons action up to a boundary term. Starting from the Chern-Simons-action

$$I_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \langle A, \mathrm{d}A \rangle + \frac{2}{3} \langle A \wedge A, A \rangle \tag{D.16}$$

[27]The Bianchi identity can be derived by using the Jacobi identity and torsion freedom of the covariant derivative.

and inserting $A = e^a \mathtt{P}_a + \omega^a \mathtt{J}_a$ yields

$$\begin{aligned}
I_{CS}[A] =& \frac{k}{4\pi} \int_{\mathcal{M}} e^a \wedge d\omega_a + de^a \wedge \omega_a \\
&+ \frac{k}{4\pi} \int_{\mathcal{M}} \frac{2}{3} \left\langle e^a \wedge e^b \left(-\Lambda \epsilon_{abc} J^c\right) + 2e^a \wedge \omega^b \left(\epsilon_{abc} P^c\right) + \omega^a \wedge \omega^b \left(\epsilon_{abc} J^c\right), e^d P_d + \omega^d J_d \right\rangle \\
=& \frac{k}{4\pi} \int_{\mathcal{M}} e^a \wedge d\omega_a + de^a \wedge \omega_a + \frac{1}{3} \left(-\Lambda \epsilon_{abc} e^a \wedge e^b \wedge e^c + \omega^a \wedge \omega^b \wedge e^c \epsilon_{abc} + 2e^a \wedge \omega^b \wedge \omega^c \epsilon_{abc}\right) \\
=& \frac{k}{4\pi} \int_{\mathcal{M}} d\left(e^a \wedge \omega_a\right) + 2e^a \wedge d\omega_a - \frac{1}{3} \Lambda \epsilon_{abc} e^a \wedge e^b \wedge e^c + \epsilon_{abc} e^a \wedge \omega^b \wedge \omega^c \,,
\end{aligned} \tag{D.17}$$

where we have used properties of the wedge product, see appendix B, the algebra relation (2.20) and the bilinear form (2.21). Here, the first term is a boundary term which shall be neglected in our discussion and the rest can be written as

$$I_{CS}[A] = \frac{k}{2\pi} \int_{\mathcal{M}} e^a \wedge R_a - \frac{\Lambda}{6} \epsilon_{abc} e^a \wedge e^b \wedge e^c \,, \tag{D.18}$$

which is exactly the Einstein-Hilbert-Palatini action (2.15) after setting k equal to $\frac{1}{4G_N}$.

D.3 EOM of a Constrained System

The following analysis is based on [66]. In a gauge system defined by the Lagrangian $L(q, \dot{q})$ the matrix $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is typically not invertible, which implies that one cannot uniquely recover the velocities $\dot{q}^i$ in terms of the momenta $p_i = \frac{\partial L}{\partial \dot{q}^i}$. In fact, non-trivial relations $\phi_m(q, p) = 0$ between the coordinates and the momenta called *primary constraints* exist. The construction of the Hamiltonian is now slightly more involved than usual. The *canonical Hamiltonian* is given by

$$H_C = p_i \dot{q}^i - L. \tag{D.19}$$

This Hamiltonian is not yet complete because it tells us nothing about the primary constraints. Varying (D.19) yields

$$\delta H_C = \delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \frac{\partial L}{\partial q^i} \delta q^i - \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i = \delta p_i \dot{q}^i - \frac{\partial L}{\partial q^i} \delta q^i \,, \tag{D.20}$$

which can be rewritten as

$$\left(\frac{\partial H_C}{\partial p_i} - \dot{q}^i\right) \delta p_i + \left(\frac{\partial H_C}{\partial q^i} + \frac{\partial L}{\partial q^i}\right) \delta q^i = 0. \tag{D.21}$$

To analyze this equation further, one has to make use of the fact that p and q are varied on the *constrained surface,* which is an embedded submanifold of dimension $2N - M$ (where N is the number of degrees of freedom and M the number of primary constraints). For the whole expression to be zero, the coefficients of δp_i and δq^i must be vectors within the complement of the tangent space of the constrained surface, which is a M-dimensional subspace of the whole tangent space of the phase space. Clearly, the M vectors $\frac{\partial \phi_m}{\partial p_i} \delta p_i$, $\frac{\partial \phi_m}{\partial q^i} \delta q^i$ also lie in this subspace. If the constraints obey certain regularity conditions, see [66], they are all linearly independent and provide a basis. Hence, the coefficients of (D.21) can be expanded in this basis, which yields

$$\dot{q}^i = \frac{\partial H_C}{\partial p_i} + u^m \frac{\partial \phi_m}{\partial p_i} \tag{D.22}$$

$$-\frac{\partial L}{\partial q^i} \quad = \quad \frac{\partial H_C}{\partial q^i} + u^m \frac{\partial \phi_m}{\partial q^i}. \tag{D.23}$$

The u^m can be thought of additional coordinates of the configuration space which guarantee that the Legendre transformation from $(q, \dot{q})$ to (q, p, u) is invertible. After using the Lagrange equation $\frac{\partial L}{\partial q^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}$ we arrive at the final form of the Hamiltonian equations

$$\dot{q}^i \quad = \quad \frac{\partial H_T}{\partial p_i} \tag{D.24}$$

$$\dot{p}^i \quad = \quad -\frac{\partial H_T}{\partial q_i} \tag{D.25}$$

with the *total Hamiltonian*

$$H_T = H_C + u^m \phi_m. \tag{D.26}$$

D.4 Sketch of the Transformation to Highest Weight Boundary Conditions

In the following we sketch how the diagonal boundary condition can be systematically transformed into the highest weight boundary conditions through a gauge transformation. We recall that the diagonal boundary conditions for spin-3 gravity are given by

$$a = a_v \, \mathrm{d}v + a_\varphi \, \mathrm{d}\varphi \tag{D.27}$$

with

$$a_\varphi = \mathcal{J}\, \mathtt{L}_0 + \mathcal{P}\, \mathtt{M}_0 + \mathcal{J}^{(3)}\, \mathtt{U}_0 + \mathcal{P}^{(3)}\, \mathtt{V}_0, \tag{D.28a}$$

$$a_v = \mu_{\mathcal{P}}\, \mathtt{L}_0 + \mu_{\mathcal{J}}\, \mathtt{M}_0 + \mu_{\mathcal{P}}^{(3)}\, \mathtt{U}_0 + \mu_{\mathcal{J}}^{(3)}\, \mathtt{V}_0. \tag{D.28b}$$

The highest weight boundary conditions are given by

$$\tilde{a}_\varphi = \mathtt{L}_1 - \frac{\mathcal{M}}{4}\mathtt{L}_{-1} - \frac{\mathcal{N}}{2}\mathtt{M}_{-1} + \frac{\mathcal{V}}{2}\mathtt{U}_{-2} + \mathcal{Z}\mathtt{V}_{-2}, \tag{D.29a}$$

$$\tilde{a}_v = a_v^{(0)} + a_v^{(\mu_{\mathrm{M}})} + a_v^{(\mu_{\mathrm{L}})} + a_v^{(\mu_{\mathrm{V}})} + a_v^{(\mu_{\mathrm{U}})}\,, \tag{D.29b}$$

where

$$a_v^{(0)} = \mathtt{M}_1 - \frac{\mathcal{M}}{4}\mathtt{M}_{-1} + \frac{\mathcal{V}}{2}\mathtt{V}_{-2}, \tag{D.30a}$$

$$a_v^{(\mu_{\mathrm{M}})} = \mu_{\mathrm{M}}\, \mathtt{M}_1 - \mu_{\mathrm{M}}'\, \mathtt{M}_0 + \tfrac{1}{2}\big(\mu_{\mathrm{M}}'' - \tfrac{1}{2}\mathcal{M}\mu_{\mathrm{M}}\big)\, \mathtt{M}_{-1} + \tfrac{1}{2}\,\mathcal{V}\,\mu_{\mathrm{M}}\, \mathtt{V}_{-2}, \tag{D.30b}$$

$$a_v^{(\mu_{\mathrm{L}})} = a_v^{(\mu_{\mathrm{M}})}\big|_{M\to L} - \tfrac{1}{2}\,\mathcal{N}\,\mu_{\mathrm{L}}\, \mathtt{M}_{-1} + \mathcal{Z}\,\mu_{\mathrm{L}}\, \mathtt{V}_{-2}, \tag{D.30c}$$

$$\begin{aligned} a_v^{(\mu_{\mathrm{V}})} &= \mu_{\mathrm{V}}\, \mathtt{V}_2 - \mu_{\mathrm{V}}'\, \mathtt{V}_1 + \tfrac{1}{2}\big(\mu_{\mathrm{V}}'' - \mathcal{M}\mu_{\mathrm{V}}\big)\, \mathtt{V}_0 + \tfrac{1}{6}\big(-\mu_{\mathrm{V}}''' + \mathcal{M}'\mu_{\mathrm{V}} + \tfrac{5}{2}\mathcal{M}\mu_{\mathrm{V}}'\big)\, \mathtt{V}_{-1} \\ &\quad + \tfrac{1}{24}\big(\mu_{\mathrm{V}}'''' - 4\mathcal{M}\mu_{\mathrm{V}}'' - \tfrac{7}{2}\mathcal{M}'\mu_{\mathrm{V}}' + \tfrac{3}{2}\mathcal{M}^2\mu_{\mathrm{V}} - \mathcal{M}''\mu_{\mathrm{V}}\big)\, \mathtt{V}_{-2} - 4\mathcal{V}\,\mu_{\mathrm{V}}\, \mathtt{M}_{-1}, \end{aligned} \tag{D.30d}$$

$$\begin{aligned} a_v^{(\mu_{\mathrm{U}})} &= a_v^{(\mu_{\mathrm{V}})}\big|_{M\to L} - 8\mathcal{Z}\,\mu_{\mathrm{U}}\, \mathtt{M}_{-1} - \mathcal{N}\,\mu_{\mathrm{U}}\, \mathtt{V}_0 + \big(\tfrac{5}{6}\mathcal{N}\mu_{\mathrm{U}}' + \tfrac{1}{3}\mathcal{N}'\mu_{\mathrm{U}}\big)\, \mathtt{V}_{-1} \\ &\quad + \big(-\tfrac{1}{3}\mathcal{N}\mu_{\mathrm{U}}'' - \tfrac{7}{24}\mathcal{N}'\mu_{\mathrm{U}}' - \tfrac{1}{12}\mathcal{N}''\mu_{\mathrm{U}} + \tfrac{1}{4}\mathcal{M}\mathcal{N}\mu_{\mathrm{U}}\big)\, \mathtt{V}_{-2}. \end{aligned} \tag{D.30e}$$

First, we concentrate on the a_φ component of the connection only and try to find a gauge transformation that maps a_φ into $\tilde{a}_\varphi$. Following this we read off the relation between the near horizon and the asymptotic state-dependent functions. Then we explicitly check that this transformation also maps the a_v components, from which we then read off the relation between near horizon and asymptotic chemical potentials.

Mapping a_φ to $\hat{a}_\varphi$

We consider a gauge transformation $g_{(1)}$ of the form

$$g_{(1)} = \exp\left[\mathfrak{l}\,\mathtt{L}_1 + \mathfrak{m}\,\mathtt{M}_1 + \mathfrak{u}_1\,\mathtt{U}_1 + \mathfrak{v}_1\,\mathtt{V}_1 + \mathfrak{u}_2\,\mathtt{U}_2 + \mathfrak{v}_2\,\mathtt{V}_2\right] . \tag{D.31}$$

After performing a gauge transformation $g_{(1)}^{-1}\,(\partial_\varphi + a_\varphi)\,g_{(1)}$ on a_φ we arrive at

$$\hat{a}_\varphi^{(1)} = \mathtt{L}_1 + \mathcal{J}\,\mathtt{L}_0 + \mathcal{P}\,\mathtt{M}_0 + \mathcal{J}^{(3)}\,\mathtt{U}_0 + \mathcal{P}^{(3)}\,\mathtt{V}_0 \tag{D.32}$$

if we demand that

$$\mathfrak{l}' = 1 + \mathfrak{l}\mathcal{J} + 2\mathfrak{u}_1\mathcal{J}^{(3)}, \tag{D.33a}$$
$$\mathfrak{m}' = \mathfrak{l}\mathcal{P} + \mathfrak{m}\mathcal{P} + 2\mathfrak{u}_1\mathcal{P}^{(3)} + 2\mathfrak{v}_1\mathcal{J}^{(3)}, \tag{D.33b}$$
$$\mathfrak{u}_1' = \mathfrak{u}_1\mathcal{J} + 2\mathfrak{l}\mathcal{J}^{(3)}, \tag{D.33c}$$
$$\mathfrak{v}_1' = \mathfrak{u}_1\mathcal{P} + \mathfrak{v}_1\mathcal{J} + 2\mathfrak{l}\mathcal{P}^{(3)} + 2\mathfrak{m}\mathcal{J}^{(3)}, \tag{D.33d}$$
$$\mathfrak{u}_2' = -\frac{\mathfrak{u}_1}{2} + 2\mathfrak{u}_2\mathcal{J}, \tag{D.33e}$$
$$\mathfrak{v}_2' = -\frac{\mathfrak{v}_1}{2} + 2\mathfrak{u}_2\mathcal{P} + 2\mathfrak{v}_2\mathcal{J} . \tag{D.33f}$$

The motivation for doing this is that we can see from the structure of the Lie algebra, see (A.7), that by acting on (D.32) with generators $\mathtt{T}_i = \{\mathtt{L}_i, \mathtt{M}_i, \mathtt{U}_n, \mathtt{V}_n | i, n < 0\}$, we only change the components of a_φ with respect to the generators $\mathtt{T}_i = \{\mathtt{L}_i, \mathtt{M}_i, \mathtt{U}_n, \mathtt{V}_n | i, n \leq 0\}$. Thus, we now act on $\hat{a}_\varphi^{(1)}$ with a gauge transformation of the form

$$g_{(2)} = \exp\left[\mathfrak{a}\,\mathtt{L}_{-1} + \mathfrak{b}\,\mathtt{U}_{-1} + \mathfrak{c}\,\mathtt{U}_2 + \mathfrak{d}\,\mathtt{M}_{-1} + \mathfrak{e}\,\mathtt{V}_{-1} + \mathfrak{f}\,\mathtt{V}_{-2}\right] . \tag{D.34}$$

Requiring that the subsequent connection $a_\varphi^{(2)}$ takes the form (D.29) leads to the relations

$$\mathfrak{a} = -\frac{\mathcal{J}}{2}\,, \tag{D.35}$$
$$\mathfrak{b} = -\frac{\mathcal{J}^{(3)}}{3}\,, \tag{D.36}$$
$$\mathfrak{c} = \frac{1}{6}\left(\mathcal{J}\mathcal{J}^{(3)} + \frac{\mathcal{J}^{(3)\prime}}{2}\right)\,, \tag{D.37}$$
$$\mathfrak{d} = -\frac{\mathcal{J}}{2}\,, \tag{D.38}$$
$$\mathfrak{e} = -\frac{\mathcal{P}^{(3)}}{3}\,, \tag{D.39}$$
$$\mathfrak{f} = +\frac{1}{6}\left(\mathcal{P}\mathcal{J}^{(3)} + \mathcal{J}\mathcal{P}^{(3)} + \frac{\mathcal{P}^{(3)\prime}}{2}\right) \tag{D.40}$$

with the identifications of the asymptotic state-dependent functions via

$$\mathcal{M} = \mathcal{J}^2 + \frac{4}{3}\left(\mathcal{J}^{(3)}\right)^2 + 2\mathcal{J}', \tag{D.41a}$$
$$\mathcal{N} = \mathcal{J}\mathcal{P} + \frac{4}{3}\mathcal{J}^{(3)}\mathcal{P}^{(3)} + \mathcal{P}', \tag{D.41b}$$

$$\mathcal{V}=\frac{1}{54}\left(18\mathcal{J}^2\mathcal{J}^{(3)}-8\left(\mathcal{J}^{(3)}\right)^3+9\mathcal{J}'\mathcal{J}^{(3)}+27\mathcal{J}\mathcal{J}^{(3)\prime}+9\mathcal{J}^{(3)\prime\prime}\right), \tag{D.41c}$$

$$\begin{aligned}\mathcal{Z}=\frac{1}{36}\Big(&6\mathcal{J}^2\mathcal{P}^{(3)}-8\mathcal{P}^{(3)}\left(\mathcal{J}^{(3)}\right)^2+3\mathcal{P}^{(3)}\mathcal{J}'+3\mathcal{J}^{(3)}\mathcal{P}'\\&+9\mathcal{J}\mathcal{P}^{(3)\prime}+9\mathcal{P}\mathcal{J}^{(3)\prime}+12\mathcal{P}\mathcal{J}\mathcal{J}^{(3)}+3\mathcal{P}^{(3)\prime\prime}\Big).\end{aligned} \tag{D.41d}$$

Now we can act on the a_v component of the gauge field with the gauge transformation $g=g_{(1)}g_{(2)}$ and find that it indeed takes the form (D.28b) if we identify the asymptotic chemical potentials with

$$\mu_L=\frac{4}{3}\mu_U\mathcal{J}^{(3)}-\mu_{\mathcal{P}}\mathfrak{l}-2\mu_{\mathcal{P}}^{(3)}\mathfrak{u}_1+\dot{\mathfrak{l}}, \tag{D.42a}$$

$$\mu_M=\frac{4}{3}\mu_U\mathcal{P}^{(3)}+\frac{4}{3}\mu_V\mathcal{J}^{(3)}-\mu_{\mathcal{P}}\mathfrak{m}-\mu_{\mathcal{J}}\mathfrak{l}-2\mu_{\mathcal{P}}^{(3)}\mathfrak{v}_1-2\mu_{\mathcal{J}}^{(3)}\mathfrak{u}_1+\dot{\mathfrak{m}}, \tag{D.42b}$$

$$\mu_U=-2\mu_{\mathcal{P}}\mathfrak{u}_2+\mu_{\mathcal{P}}^{(3)}\mathfrak{l}^2+\mu_{\mathcal{P}}^{(3)}\mathfrak{u}_1^2+\frac{1}{2}\mathfrak{u}_1\dot{\mathfrak{l}}-\frac{1}{2}\mathfrak{l}\dot{\mathfrak{u}}_1+\dot{\mathfrak{u}}_2, \tag{D.42c}$$

$$\begin{aligned}\mu_V=&-2\mu_{\mathcal{P}}\mathfrak{v}_2-2\mu_{\mathcal{J}}\mathfrak{u}_2+2\mathfrak{l}\mathfrak{m}\mu_{\mathcal{P}}^{(3)}+2\mathfrak{u}_1\mathfrak{v}_1\mu_{\mathcal{P}}^{(3)}+\mu_{\mathcal{J}}^{(3)}\mathfrak{l}^2-\mu_{\mathcal{J}}^{(3)}\mathfrak{u}_1^2\\&+\frac{1}{2}\mathfrak{v}_1\dot{\mathfrak{l}}+\frac{1}{2}\mathfrak{u}_1\dot{\mathfrak{m}}-\frac{1}{2}\mathfrak{m}\dot{\mathfrak{u}}_1-\frac{1}{2}\mathfrak{l}\dot{\mathfrak{v}}_1+\dot{\mathfrak{v}}_2.\end{aligned} \tag{D.42d}$$

The consistency requirement $\partial_v\partial_\varphi\mathfrak{y}=\partial_\varphi\partial_v\mathfrak{y}$ for all parameters $\mathfrak{y}\in\{\mathfrak{l},\mathfrak{m},\mathfrak{u}_1,\mathfrak{v}_1,\mathfrak{u}_2,\mathfrak{v}_2\}$ leads to an explicit relation between the near horizon and asymptotic chemical potentials

$$\mu_{\mathcal{P}}=\mu_L\mathcal{P}+\frac{8}{3}\mu_U\mathcal{J}\mathcal{J}^{(3)}+\frac{4}{3}\mu_U\mathcal{J}'-\frac{2}{3}\mu_U'\mathcal{J}-\mu_L', \tag{D.43a}$$

$$\begin{aligned}\mu_{\mathcal{J}}=&\mu_M\mathcal{P}+\frac{8}{3}\mu_U\mathcal{P}\mathcal{J}^{(3)}+\frac{8}{3}\mu_U\mathcal{J}\mathcal{P}^{(3)}+\frac{8}{3}\mu_V\mathcal{J}\mathcal{J}^{(3)}\\&+\frac{4}{3}\mu_U\mathcal{P}'+\frac{4}{3}\mu_V\mathcal{J}'-\frac{2}{3}\mu_U'\mathcal{P}-\frac{2}{3}\mu_V'\mathcal{J}-\mu_M',\end{aligned} \tag{D.43b}$$

$$\mu_{\mathcal{P}}^{(3)}=\mu_L\mathcal{J}^{(3)}+\mu_U\mathcal{J}^2-\frac{4}{3}\mu_U\left(\mathcal{J}^{(3)}\right)^2-\mu_U\mathcal{J}'-\frac{3}{2}\mu_U'\mathcal{J}+\frac{1}{2}\mu_U'', \tag{D.43c}$$

$$\begin{aligned}\mu_{\mathcal{J}}^{(3)}=&\mu_L\mathcal{P}^{(3)}+\mu_M\mathcal{J}^{(3)}+2\mu_U\mathcal{P}\mathcal{J}+\mu_V\mathcal{J}^2-\frac{8}{3}\mu_U\mathcal{P}^{(3)}\mathcal{J}^{(3)}-\frac{4}{3}\mu_V\left(\mathcal{J}^{(3)}\right)^2\\&-\mu_V\mathcal{J}'-\mu_U\mathcal{P}'-\frac{3}{2}\mu_V'\mathcal{J}-\frac{3}{2}\mu_U'\mathcal{P}+\frac{1}{2}\mu_V''.\end{aligned} \tag{D.43d}$$

Mapping the Entropy of Spin-3 Cosmologies

Here we solve the relations given between asymptotic and near horizon zero-mode charges, see (D.41), in terms of near horizon charges. Under restriction to constant state-dependent functions (D.41) reduces to

$$M=J^2+\frac{4}{3}\left(J^{(3)}\right)^2, \tag{D.44a}$$

$$N=JP+\frac{4}{3}J^{(3)}P^{(3)}, \tag{D.44b}$$

$$V=\frac{1}{54}\left(18J^2J^{(3)}-8\left(J^{(3)}\right)^3\right), \tag{D.44c}$$

$$Z=\frac{1}{36}\left(6J^2P^{(3)}-8P^{(3)}\left(J^{(3)}\right)^2+12PJJ^{(3)}\right). \tag{D.44d}$$

Solving (D.44a) for J^2

$$J^2 = \frac{1}{3}\left(3M - 4\left(J^{(3)}\right)^2\right) \tag{D.45}$$

and inserting the expression into (D.44c) leads to a cubic equation in terms of $J^{(3)}$

$$16(J^{(3)})^3 + 27V = 9J^{(3)}M\,. \tag{D.46}$$

The general solution of this cubic equation can be looked up, see e.g. [67], and is given by

$$J^{(3)} = \frac{1}{2}\sqrt{3}\sqrt{M}\sin\left(\frac{1}{3}\arcsin\left(6\sqrt{3}\left(\frac{1}{M}\right)^{3/2}V - \frac{2\pi g}{3}\right)\right)\,, \tag{D.47}$$

where the three roots are labelled by integers $g = 0, 1, 2$. We choose $g = 0$ since this is the branch connected to the cosmological solutions of Einstein gravity (4.19). Solving (D.44b) in terms of JP

$$JP = \frac{1}{3}\left(3N - 4J^{(3)}P^{(3)}\right) \tag{D.48}$$

and inserting (D.45) and (D.48) into (D.44d) gives

$$P^{(3)} = -\frac{6\left(3Z - J^{(3)}N\right)}{16\left(J^{(3)}\right)^2 - 3M}\,. \tag{D.49}$$

This in turn can be reinserted into (D.44b) such that (D.44b) can be solved in terms of P

$$P = \frac{-8\left(J^{(3)}\right)^2 N - 24J^{(3)}Z + 3NM}{J\left(3M - 16\left(J^{(3)}\right)^2\right)}\,. \tag{D.50}$$

Inserting (D.47) into (D.50) and simplifying the expression leads to

$$P = \pm\frac{N\sqrt{M}\cos\left(\frac{2x}{3}\right) - 4\sqrt{3}Z\sin\left(\frac{x}{3}\right)}{M\sqrt{1 - \frac{108V^2}{M^3}}} \qquad \text{with} \qquad x = \arcsin\left(6\sqrt{3}\left(\frac{1}{M}\right)^{3/2}V\right)\,. \tag{D.51}$$

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