DISSERTATION

On Vector-Valued Valuations on Convex Functions

zur Erlangung des akademischen Grades eines
Doktors der technischen Wissenschaften
unter der Leitung von

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eingereicht an der Technischen Universität Wien,
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Kurzfassung

Aufbauend auf den Arbeiten von A. Colesanti, M. Ludwig und F. Mussnig, die ein funktionales Analogon des Hadwigerschen Satzes etabliert haben, das die funktionalen intrinsischen Volumina auf dem Raum der superkoerziven konvexen Funktionen charakterisiert, untersuchen wir eine neue Familie von Bewertungen mit Schwerpunkt auf vektorwertigen.

Wir führen funktionale Minkowski-Vektoren ein, das sind stetige, vektorwertige Bewertungen, die epitranslationsinvariant und rotationsäquivariant sind. Wir geben eine Charakterisierung dieser Bewertungen mittels hessischer Maße und spezieller gemischter Monge–Ampère-Maße. Sie entstehen natürlich aus einer Steiner-Formel, angewandt auf ein funktionales Analogon des Momentenvektors, das wir ebenfalls charakterisieren.

Als Anwendung in der Integralgeometrie leiten wir eine additive kinematische Formel her, die die funktionalen Minkowski-Vektoren involviert.

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Abstract

Building on the work of A. Colesanti, M. Ludwig, and F. Mussnig, who established a functional analogue of Hadwiger's theorem characterizing functional intrinsic volumes on the space of super-coercive convex functions, we investigate a new family of valuations, with a focus on vector-valued ones.

We introduce functional Minkowski vectors, which are continuous, vector-valued valuations that are epi-translation invariant and rotation equivariant. We provide a characterization of these valuations in terms of Hessian measures and special mixed Monge–Ampère measures. They arise naturally from a Steiner formula applied to a functional analogue of the moment vector, which we also characterize.

As an application in integral geometry, we derive an additive kinematic formula involving the functional Minkowski vectors.

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Acknowledgments

First and foremost, I would like to express my deep gratitude to my advisor, F. Mussnig, for introducing me to convex geometry, for his dedicated mentorship, and for his constant support and generosity with his time. I am truly proud to be his Ph.D. student and to learn from his rigor, insight, and resilience.

I would like to thank the members of the research groups on both the fifth and seventh floors for their support and collegial atmosphere. In particular, I am grateful to M. Ludwig for her insightful remarks, to J. Knoerr from the fifth floor, and to J. Ulivelli for many helpful discussions. I also thank Edith for her availability and assistance.

I'm especially grateful to my office colleagues, Fernanda, Xiaxing, and Carlos, with whom I shared many enriching experiences and wonderful memories.

My heartfelt thanks go to my parents for their unwavering care, and to my brother and sisters for their constant encouragement and support throughout this journey.

Finally, I am sincerely grateful to the university for hosting me, and to Austria for welcoming me so warmly.

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Contents

Introduction **1**

1 A Klain-Schneider Theorem for Vector-Valued Valuations on Convex Functions **4**
1.1 Introduction . . . 4
1.2 Preliminaries . . . 8
1.3 Classification of Simple Valuations . . . 9
1.4 Functional Intrinsic Moments and Minkowski Vectors . . . 15
1.5 Classification of Homogeneous Valuations . . . 17

2 The Vectorial Hadwiger Theorem on Convex Functions **22**
2.1 Introduction . . . 22
2.2 Dual Results . . . 28
2.3 Preliminaries . . . 29
2.3.1 Hessian Measures . . . 29
2.3.2 Mixed Monge–Ampère Measures . . . 31
2.3.3 Valuations on Convex Functions . . . 32
2.4 Existence . . . 35
2.4.1 Integral Transforms Between the Classes $\boldsymbol{T}_j^n$ . . . 35
2.4.2 Connecting Hessian Measures and Mixed Monge–Ampère Measures . . . 36
2.4.3 Proof of Theorem A and Theorem B . . . 41
2.4.4 Extended Representation Formulas . . . 41
2.5 Further Representations . . . 42
2.5.1 Kubota-type Formulas . . . 42
2.5.2 The Transform $\mathcal{T}$ . . . 44
2.5.3 From Convex Functions to Convex Bodies . . . 46
2.6 Characterization . . . 48
2.6.1 Retrieving the Densities . . . 48
2.6.2 The Abel Transform . . . 52
2.6.3 Classification of Smooth Valuations . . . 54
2.6.4 Approximation and Support . . . 58
2.6.5 Proof of Theorem C . . . 60

3 Additive Kinematic Formulas for Functional Minkowski Vectors **62**
3.1 Introduction . . . 62
3.2 Preliminaries . . . 65
3.3 Proof of Theorem 3.1.3 . . . 69
3.A Appendix to Chapter 3 . . . 75

Bibliography **99**

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Introduction

Let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$, that is, the set of all non-empty, closed, and convex subsets of $\mathbb{R}^n$. If a functional Z on $\mathcal{K}^n$ satisfies the following property

$$\mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) = \mathrm{Z}(K) + \mathrm{Z}(L)$$

for every $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$, then the functional Z is called *a valuation*. Originating from Dehn's solution to Hilbert's Third Problem, this notion became crucial in convex geometry. An important breakthrough was Hadwiger's characterization result of translation invariant, rotation invariant scalar-valued valuations in terms of intrinsic volumes $V_j \colon \mathcal{K}^n \to \mathbb{R}$, where $0 \leq j \leq n$, and $V_n(K)$ is the usual n-dimensional *volume* of $K \in \mathcal{K}^n$; see [38].

A few years later, Hadwiger and Schneider in [39] established a characterization result for continuous, translation covariant, and rotation equivariant vector-valued valuations in terms of intrinsic moments $\mathrm{m}_j \colon \mathcal{K}^n \to \mathbb{R}^n$, where $1 \leq j \leq n+1$, and $\mathrm{m}_{n+1}(K)$ is the *moment vector* of $K \in \mathcal{K}^n$.

In the last two decades, extensions of valuations from the space of convex bodies to function spaces have been considered [4, 7, 28, 60–62, 72, 85], and among these extensions are valuations defined on the space of (finite-valued) convex functions [20–23, 25, 26, 50, 52, 55, 56].

Let

$$\mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) = \{v : \mathbb{R}^n \to \mathbb{R} : v \text{ is convex}\}$$

denote the space of finite-valued convex functions on $\mathbb{R}^n$ and consider the topology of pointwise convergence on this space. In this setting, a functional Z on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ is called a valuation if

$$\mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v) = \mathrm{Z}(u) + \mathrm{Z}(v)$$

for every $u, v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, where $u \vee v$ denotes the pointwise maximum, and $u \wedge v$ denotes the pointwise minimum. This definition is motivated by the fact that the support function h_K of a convex body $K \in \mathcal{K}^n$, given by $h_K(y) = \max_{x \in K} \langle x, y \rangle$ for all $y \in \mathbb{R}^n$, is an element of $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, and if $L \in \mathcal{K}^n$ is such that $K \cup L \in \mathcal{K}$, then

$$h_{K \cup L} = h_K \vee h_L \quad \text{and} \quad h_{K \cap L} = h_K \wedge h_L.$$

Very recently, A. Colesanti, M. Ludwig, and F. Mussnig in [24–27] established a functional version of Hadwiger's theorem in terms of functional intrinsic volumes. This result was also obtained by J. Knoerr in [55].

Following Hadwiger and Schneider's work, the objective of this cumulative thesis is to extend valuation theory on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ to vector-valued functionals and establish integral geometric formulas for them.

In Article 1, together with my supervisor F. Mussnig, we establish a functional analog of the Klain–Schneider theorem for vector-valued valuations on convex functions, providing a classification of continuous, translation covariant, and simple valuations. Under additional rotation

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equivariance assumptions, we characterize an analytic counterpart of the moment vector, which is given by

$$\mathrm{m}^*_\xi(v) = \int_{\mathbb{R}^n} \xi(|x|)\nabla v(x) \det(\mathrm{D}^2 v(x))\,\mathrm{d}x$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$, where ξ is a continuous function with compact support on $[0,\infty)$. Here, $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$, ∇v denotes the gradient of a function v, and $\det(\mathrm{D}^2 v(x))$ is the determinant of the Hessian of v. The operator m^*_ξ generalizes the classical moment vector since

$$\mathrm{m}^*_\xi(h_K) \simeq \xi(0)\,\mathrm{m}_{n+1}(K)$$

for every $K \in \mathcal{K}^n$.

Surprisingly, applying a Steiner formula to the functional moment vector m^*_ξ results in two families of valuations, one of which represents a family of functional intrinsic moments. On the other hand, the other family that we call the family of *functional Minkowski vectors* is given by

$$\mathrm{t}^*_{j,\xi_j}(v) = \int_{\mathbb{R}^n} \xi_j(|x|)x[\mathrm{D}^2 v(x)]_j\,\mathrm{d}x$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$ and $0 \le j \le n$ where the densities ξ_j are continuous functions with bounded support on $(0,\infty)$, and may have singularities at 0^+. Here, $[\mathrm{D}^2 v(x)]_j$ denotes the jth elementary symmetric function of the eigenvalues of the Hessian matrix of v at $x \in \mathbb{R}^n$, with the convention $[\mathrm{D}^2 v(x)]_0 \equiv 1$. An important remark is that these functionals lack classical counterparts on convex bodies, as they vanish due to the Minkowski relations, that is

$$\mathrm{t}^*_{j,\xi_j}(h_K) \simeq \int_{\mathbb{S}^{n-1}} z\,\mathrm{d}S_j(K,z) = o,$$

for every $K \in \mathcal{K}^n$ and $1 \le j \le n-1$, where $S_j(K,\cdot)$ denotes the jth *area measure* of K.

In Article 2, together with F. Mussnig, we establish a complete characterization of the functional Minkowski vectors. They are continuous, dually epi-translation invariant, and rotation equivariant valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. We show the existence of these operators with respect to Hessian measures with singular densities, along with additional representations involving mixed Monge–Ampère measures, and Kubota-type formulas. We also introduce a transform that allows us to represent these functionals using area measures of higher-dimensional convex bodies.

For the characterization, we first prove our result for smooth valuations, which form a dense subset of continuous dually epi-translation invariant valuations. We then approximate the continuous valuations by smooth ones and show that their supports coincide.

One notable application of the theory of valuations is to establish integral geometric formulas. An important identity involving the intrinsic volumes in integral geometry is given in the following theorem, which is known as the *additive kinematic formula*.

Theorem ([46], Theorem 5.13)**.** *Let $0 \le j \le n$ and $K, L \in \mathcal{K}^n$, then*

$$\int_{\mathrm{SO}(n)} V_j(K+\vartheta L)\,\mathrm{d}\vartheta = \sum_{j=0}^{k} \frac{\binom{2n-j}{n-j}\kappa_{n-k}\kappa_{n+k-j}}{\binom{2n-j}{n-k}\kappa_n\kappa_{n-j}} V_k(K)V_{j-k}(L),$$

where we denote by $\mathrm{SO}(n)$ the group of special orthogonal transformations on $\mathbb{R}^n$, and $\mathrm{d}\vartheta$ denotes integration with respect to the Haar probability measure on $\mathrm{SO}(n)$. Furthermore, κ_m is the m-dimensional volume of the Euclidean unit ball B^m in $\mathbb{R}^m$, with the convention $\kappa_0 = 1$.

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An elegant proof of this result relies on Hadwiger's characterization theorem. Very recently, D. Hug, F. Mussnig, and J. Ulivelli [43] established an additive kinematic formula for the functional intrinsic volumes on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ which serves as a functional analog to the theorem above. In Article 3, we establish an additive kinematic formula for the functional Minkowski vectors in terms of mixed Monge–Ampère measures using the characterization established in Article 2.

Articles included in this thesis

Article 1 **A Klain-Schneider Theorem for Vector-Valued Valuations on Convex Functions** [69]
with Fabian Mussnig
`arXiv:2503.07287`

Article 2 **The Vectorial Hadwiger Theorem on Convex Functions** [70]
with Fabian Mussnig
`arXiv:2504.04952`

Article 3 **Additive Kinematic Formulas for Functional Minkowski Vectors** [68]
`arXiv:2508.07984`

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1 A Klain-Schneider Theorem for Vector-Valued Valuations on Convex Functions

1.1 Introduction

The Klain–Schneider Theorem Let $\mathcal{K}^n$ denote the set of non-empty, compact, convex subsets of $\mathbb{R}^n$, whose elements we call *convex bodies*. We say that a map Z on $\mathcal{K}^n$ that takes values in an Abelian semigroup $\langle \mathbb{A}, + \rangle$ is a *valuation*, whenever

$$\mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) = \mathrm{Z}(K) + \mathrm{Z}(L)$$

for every $K, L \in \mathcal{K}^n$ such that also $K \cup L \in \mathcal{K}^n$. The probably most important example of a real-valued valuation is the *n-dimensional volume* or *Lebesgue measure*, denoted by vol_n. For vector-valued valuations, taking values in $\mathbb{R}^n$, a similar role is played by the *moment vector*

$$m(K) = \int_K x \, \mathrm{d}x$$

for $K \in \mathcal{K}^n$, where integration is understood with respect to the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

Starting with Dehn's solution of Hilbert's Third Problem, the systematic study of valuations on convex bodies has since become a cornerstone of convex geometry. In line with Felix Klein's Erlangen Program, particular attention has been given to classifying valuations that remain invariant under specific group actions (see, for example, [2, 5, 9, 45, 63, 82]). An early highlight of this line of research is Hadwiger's celebrated characterization of the *intrinsic volumes* [38]. About 30 years ago, Klain found a classification of even, continuous (w.r.t. the Hausdorff metric), translation invariant, simple valuations, which allowed for an elegant new proof of Hadwiger's characterization theorem [47]. Here, a valuation $\mathrm{Z}\colon \mathcal{K}^n \to \mathbb{R}$ is *translation invariant* if $\mathrm{Z}(K + x) = \mathrm{Z}(K)$ for every $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$, and it is *simple* if it vanishes on convex bodies of dimension less than n. Shortly after Klain's work, Schneider could remove the assumption of evenness, leading to what is now known as the Klain–Schneider theorem [78, Theorem 2].

Theorem 1.1.1 (Klain–Schneider)**.** *A map $\mathrm{Z}\colon \mathcal{K}^n \to \mathbb{R}$ is a continuous, translation invariant, simple valuation, if and only if there exists a constant $c \in \mathbb{R}$ and an odd, continuous function $g\colon \mathbb{S}^{n-1} \to \mathbb{R}$ such that*

$$\mathrm{Z}(K) = c \, \mathrm{vol}_n(K) + \int_{\mathbb{S}^{n-1}} g(z) \, \mathrm{d}S_{n-1}(K, z)$$

for every $K \in \mathcal{K}^n$.

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Here, $S_{n-1}(K,\cdot)$ is the *surface area measure* of the convex body $K \in \mathcal{K}^n$. If K is full-dimensional, then $S_{n-1}(K,\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the boundary points of K that have an outer unit normal in the Borel set $\omega \subseteq \mathbb{S}^{n-1}$.

The Klain–Schneider theorem became an important asset in the development of modern valuation theory. See, for example, [2, 3, 5, 11, 12, 48, 58, 74, 81, 83]. One of its numerous consequences is the fact, that for $n \geq 2$, multiples of vol_n are the only continuous, translation and rotation invariant, simple valuations, where $\mathrm{Z}\colon \mathcal{K}^n \to \mathbb{R}$ is *rotation invariant* if $\mathrm{Z}(\vartheta K) = \mathrm{Z}(K)$ for every $K \in \mathcal{K}^n$ and $\vartheta \in \mathrm{SO}(n)$. From this fact together with the relation

$$m(K+x) = m(K) + \mathrm{vol}_n(K)x$$

for $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$, one can obtain the following characterization of the moment vector. We remark that this result independently follows from [39, Hauptsatz] and refer to [36, 59, 87] for further characterizations of the moment vector. We will use a lowercase z for vector-valued valuations throughout this article.

Corollary 1.1.2. *For $n \geq 2$, a map $\mathrm{z}\colon \mathcal{K}^n \to \mathbb{R}^n$ is a continuous, translation covariant, rotation equivariant, simple valuation, if and only if there exists a constant $c \in \mathbb{R}$ such that $\mathrm{z}(K) = c\,m(K)$ for every $K \in \mathcal{K}^n$.*

Here, we say that $\mathrm{z}\colon \mathcal{K}^n \to \mathbb{R}^n$ is *translation covariant* if there exists an associated map $\mathrm{z}^0\colon \mathcal{K}^n \to \mathbb{R}$ such that

$$\mathrm{z}(K+x) = \mathrm{z}(K) + \mathrm{z}^0(K)\,x$$

for every $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$. Furthermore, z is *rotation equivariant* if $\mathrm{z}(\vartheta K) = \vartheta\,\mathrm{z}(K)$ for every $K \in \mathcal{K}^n$ and $\vartheta \in \mathrm{SO}(n)$.

Real-Valued Valuations on Convex Functions Among the many extensions and generalizations of valuations are their extensions from convex bodies to various functions spaces [4, 7, 28, 60–62, 72, 85]. Of significant interest are convex functions which have been a particular focus of research during the last decade [20–23, 25, 26, 50, 52, 55–57]. Let

$$\mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) = \{v\colon \mathbb{R}^n \to \mathbb{R} : v \text{ is convex}\}$$

denote the space of (finite-valued) convex functions on $\mathbb{R}^n$. A map $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \langle \mathbb{A}, + \rangle$ is a *valuation* if

$$\mathrm{Z}(v \vee w) + \mathrm{Z}(v \wedge w) = \mathrm{Z}(v) + \mathrm{Z}(w)$$

for every $v, w \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ such that also their pointwise maximum $v \vee w$ and pointwise minimum $v \wedge w$ are elements of $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. In this setting, the second author, together with Colesanti and Ludwig, obtained a functional version of Hadwiger's theorem, which characterizes so-called *functional intrinsic volumes* [26]. Shortly afterward, the same authors found a new, more efficient approach to this result, based on an extension of Klain's approach to Hadwiger's theorem [25]. A critical element developed for this new proof is a Klain–Schneider theorem for valuations on convex functions. In the following we say that a map Z on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is *dually translation invariant* if

$$\mathrm{Z}(v+\ell) = \mathrm{Z}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and every linear functional $\ell\colon \mathbb{R}^n \to \mathbb{R}$. Furthermore, Z is *vertically translation invariant* if $\mathrm{Z}(v+c) = \mathrm{Z}(v)$ for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $c \in \mathbb{R}$. If Z satisfies both of these properties, then it is called *dually epi-translation invariant*, where this name comes from the fact that the composition of Z with convex conjugation is invariant under the translation

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of epi-graphs (see Section 1.3). Lastly, Z is *dually simple* when Z vanishes on functions that, up to the addition of linear functionals, depend on less than n variables (in a suitable coordinate system). In the following result, which was obtained in [25, Theorem 3.5], continuity is understood with respect to *epi-convergence* of functions, which on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ coincides with pointwise convergence.

Theorem 1.1.3 (Klain–Schneider theorem on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$)**.** *A map* $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *is a continuous, dually epi-translation invariant, dually simple valuation, if and only if there exists a function* $\zeta \in C_c(\mathbb{R}^n)$ *such that*

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n} \zeta(x)\,\mathrm{dMA}(v;x)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

Here, $C_c(\mathbb{R}^n)$ denotes the set of continuous functions with compact support on $\mathbb{R}^n$, and $\mathrm{MA}(v;\cdot)$ is the *Monge–Ampère measure* associated with $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. If in addition $v \in C^2(\mathbb{R}^n)$, then

$$\mathrm{dMA}(v;\cdot) = \det(\mathrm{D}^2 v(x))\,\mathrm{d}x, \tag{1.1.1}$$

where $\det(\mathrm{D}^2 v(x))$ denotes the determinant of the Hessian matrix of v at $x \in \mathbb{R}^n$. As a function of v, equation (1.1.1) continuously extends to $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ with respect to the topology induced by weak convergence of measures (see, for example, [31, Proposition 2.6]). At this point, let us mention that recently Theorem 1.1.3 inspired an analog of the Klain–Schneider theorem for zonal valuations on convex bodies [16]. See also [54].

Vector-Valued Valuations on Convex Functions The focus of this paper lies on vector-valued valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ that commute with rotations. We say that map $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is *dually translation covariant* if there exists an associated map $\mathrm{z}^0\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ such that

$$\mathrm{z}(v + \langle x, \cdot\rangle) = \mathrm{z}(v) + \mathrm{z}^0(v)\,x$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $x \in \mathbb{R}^n$, where $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbb{R}^n$. In the following we associate with each $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ the *Hessian measure* $\Theta_0(v;\cdot)$, which is the unique Borel measure on $\mathbb{R}^n \times \mathbb{R}^n$ such that $v \mapsto \Theta_0(v;\cdot)$ is weakly continuous and such that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \eta(x,y)\,\mathrm{d}\Theta_0(v;(x,y)) = \int_{\mathbb{R}^n} \eta(x,\nabla v(x))\det(\mathrm{D}^2 v(x))\,\mathrm{d}x$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$ and continuous $\eta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with compact support with respect to the first variable (see [22, Theorem 7.3] and Theorem 1.2.1 below). Let us point out that the Monge–Ampère measure associated with v can be retrieved as a marginal from $\Theta_0(v;\cdot)$, that is, $\mathrm{MA}(v;B) = \Theta_0(v;B\times\mathbb{R}^n)$ for every Borel set $B \subset \mathbb{R}^n$.

We establish the following vector-valued analog of the Klain–Schneider theorem for convex functions, where $C_c(\mathbb{R}^n;\mathbb{R}^n)$ denotes the set of continuous function $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$ with compact support. For the importance of the assumption of vertical translation invariance, we refer to [23] and we remark that this property will allow us to connect the appearing operators with functional intrinsic volumes (see Section 1.4).

Theorem 1.1.4. *A map* $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ *is a continuous, dually translation covariant, vertically translation invariant, dually simple valuation, if and only if there exist functions* $\psi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$ *and* $\zeta \in C_c(\mathbb{R}^n)$ *such that*

$$\mathrm{z}(v) = \int_{\mathbb{R}^n} \psi(x)\,\mathrm{dMA}(v;x) + \int_{\mathbb{R}^n\times\mathbb{R}^n} \zeta(x)y\,\mathrm{d}\Theta_0(v;(x,y)) \tag{1.1.2}$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

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Note that the moment vector on convex bodies can be retrieved from the last term (1.1.2). Indeed, it is easy to show that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \zeta(x)y \,\mathrm{d}\Theta_0(h_K;(x,y)) = \zeta(o)\, m(K) \tag{1.1.3}$$

for every convex body $K \in \mathcal{K}^n$, where o denotes the origin in $\mathbb{R}^n$ (see Lemma 1.2.2 below).

Mimicking the behavior of $K \mapsto m(K)$ on convex bodies, we will consider vector-valued valuations z on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ that are *rotation equivariant*. That is, $\mathrm{z}(v\circ\vartheta^{-1}) = \vartheta\,\mathrm{z}(v)$ for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\vartheta \in \mathrm{SO}(n)$. Similarly, we define $\mathrm{O}(n)$ *equivariance* which will be necessary for lower-dimensional cases and which we also call *reflection equivariance* when $n = 1$. Assuming such an additional property in Theorem 1.1.4 leads us to the following two operators: for a function $\alpha \in C_c([0,\infty))$ let $\mathrm{m}^*_\alpha\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ be defined as the unique continuous operator such that

$$\mathrm{m}^*_\alpha(v) = \int_{\mathbb{R}^n} \alpha(|x|)\nabla v(x)\det(\mathrm{D}^2 v(x))\,\mathrm{d}x$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$, where $|x| = \sqrt{\langle x,x\rangle}$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. The above definition continuously extends to $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, without additional C^2 assumptions, using the Hessian measure $\Theta_0(v;\cdot)$. In addition, let

$$\mathrm{t}^*_{n,\xi}(v) = \int_{\mathbb{R}^n} \xi(|x|)x \,\mathrm{dMA}(v;x)$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where ξ is a continuous function with bounded support on $(0,\infty)$, the set of which we denote by $C_b((0,\infty))$, with the additional property that $\lim_{t\to 0^+}\xi(t)t = 0$. Observe that this condition on ξ is equivalent to the fact that $x \mapsto \xi(|x|)x$ continuously extends from $\mathbb{R}^n\setminus\{o\}$ to $\mathbb{R}^n$ (see Lemma 1.3.4).

Our following main result, which is a functional analog of Corollary 1.1.2, is obtained as a consequence of Theorem 1.1.4.

Corollary 1.1.5. *For $n \geq 3$, a map $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is a continuous, dually translation covariant, vertically translation invariant, rotation equivariant, dually simple valuation, if and only if there exist functions $\alpha \in C_c([0,\infty))$ and $\xi \in C_b((0,\infty))$ with $\lim_{t\to 0^+}\xi(t)t = 0$ such that*

$$\mathrm{z}(v) = \mathrm{t}^*_{n,\xi}(v) + \mathrm{m}^*_\alpha(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. For $n \leq 2$, the same representation holds if we replace rotation equivariance with $\mathrm{O}(n)$ equivariance.

Let us stress that the new valuation $\mathrm{t}^*_{n,\xi}$ is dually epi-translation invariant and does not have a (non-trivial) counterpart in the classical theory on convex bodies. We will discuss this in more detail in Section 1.4, where we will introduce *functional intrinsic moments* and *Minkowski vectors*. Before that, however, it should be mentioned that if we replace translation covariance with translation invariance in Corollary 1.1.5, then we immediately obtain the following characterization of $\mathrm{t}^*_{n,\xi}$.

Corollary 1.1.6. *For $n \geq 3$, a map $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is a dually simple, continuous, dually epi-translation invariant, and rotation equivariant valuation, if and only if there exists a function $\xi \in C_b((0,\infty))$ with $\lim_{t\to 0^+}\xi(t)t = 0$ such that*

$$\mathrm{z}(v) = \mathrm{t}^*_{n,\xi}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. For $n \leq 2$, the same representation holds if we replace rotation equivariance with $\mathrm{O}(n)$ equivariance.

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Plan of This Article We collect basic results on convex functions, Hessian measures, and valuations in Section 1.2. In Section 1.3 we prove Theorem 1.1.4 and Corollary 1.1.5. There, we will work in the dual setting of super-coercive, convex functions. After this, we will consider a Steiner-type formula and introduce functional intrinsic moments and Minkowski vectors in Section 1.4. Finally, we present classification results under homogeneity assumptions in Section 1.5.

1.2 Preliminaries

Let us collect some basic results on convex functions, where we refer to [76, 79] as standard references. For a function $w\colon \mathbb{R}^n \to [-\infty, \infty]$, the *convex conjugate* or *Legendre–Fenchel transform* w^* is defined as

$$w^*(y) = \sup_{x\in\mathbb{R}^n} \big(\langle x, y\rangle - w(x)\big)$$

for $y \in \mathbb{R}^n$. The image of $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ under convex conjugation is

$$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) = \Big\{u\colon \mathbb{R}^n \to (-\infty,\infty] : u \text{ is l.s.c., convex, } u \not\equiv \infty,\ \lim_{|x|\to\infty} \tfrac{u(x)}{|x|} = \infty\Big\},$$

the space of proper, lower semicontinuous, super-coercive, convex functions. It is straightforward to check that $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, if and only if $u^* \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Conversely, $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, if and only if $v^* \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Furthermore, convex conjugation is a continuous involution between $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, where we equip spaces of convex functions with the topology associated with epi-convergence (see, for example, [76, Proposition 7.2] for a description). On $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, this is equivalent to pointwise convergence of functions, and on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, this is essentially equivalent to the Hausdorff convergence of level sets [21, Lemma 5].

Let us also mention that the composition of convex conjugation with building the Monge–Ampère measure of a function takes the easy form

$$\int_{\mathbb{R}^n} \zeta(x)\,\mathrm{dMA}(u^*;x) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\,\mathrm{d}x$$

for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\zeta \in C_c(\mathbb{R}^n)$. Here, $\mathrm{dom}(u) = \{x \in \mathbb{R}^n : u(x) < \infty\}$ is the *domain* of u and, by convexity, u is differentiable almost everywhere on the interior of this set. The above is a special case of the more general relation

$$\int_{\mathbb{R}^n} \zeta(x) y\,\mathrm{d}\Theta_0(u^*;(x,y)) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) x\,\mathrm{d}x \tag{1.2.1}$$

for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\zeta \in C_c(\mathbb{R}^n)$. See, for example, [22, Section 10.4].

We will use the following result, which is due to [23, Theorem 17].

Theorem 1.2.1. *If $\eta \in C(\mathbb{R}^n \times \mathbb{R}^n)$ has compact support with respect to the first variable, then*

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \eta(x,y)\,\mathrm{d}\Theta_0(v;(x,y))$$

is well-defined for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and defines a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Moreover,

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n} \eta(x,\nabla v(x))\det(\mathrm{D}^2 v(x))\,\mathrm{d}x$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$.

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Convex bodies are naturally embedded into $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ by associating with each convex body $K \in \mathcal{K}^n$ its *support function*

$$h_K(x) = \max_{y\in K}\langle x, y\rangle$$

for $x \in \mathbb{R}^n$. Taking the convex conjugate leads to

$$h_K^*(x) = \mathbf{I}_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ \infty & \text{if } x \notin K, \end{cases} \tag{1.2.2}$$

the *(convex) indicator function* of K, which is an element of $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. We close this section with the following easy result.

Lemma 1.2.2. *Let $\zeta \in C_c(\mathbb{R}^n)$. For every $K \in \mathcal{K}^n$ we have*

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \zeta(x)y \,\mathrm{d}\Theta_0(h_K;(x,y)) = \zeta(o)m(K).$$

In particular,

$$\mathrm{m}_\alpha^*(h_K) = \alpha(0)m(K)$$

for every $\alpha \in C_c([0,\infty))$ and $K \in \mathcal{K}^n$.

Proof. For $K \in \mathcal{K}^n$ it follows from Theorem 1.2.1, (1.2.1), and (1.2.2) that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \zeta(x)y \,\mathrm{d}\Theta_0(h_K;(x,y)) = \int_{\mathrm{dom}(\mathbf{I}_K)} \zeta(\nabla \mathbf{I}_K(x))x \,\mathrm{d}x = \zeta(o)\int_K x\,\mathrm{d}x = \zeta(o)\,m(K).$$

The second part of the statement immediately follows by choosing $\zeta(x) = \alpha(|x|)$. □

1.3 Classification of Simple Valuations

The purpose of this section is to prove Theorem 1.1.4 and Corollary 1.1.5. We will establish the equivalent dual version of these results on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, Theorem 1.3.6. Let us first explain this equivalence: a map $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \langle A, +\rangle$ is a valuation, if and only if $v \mapsto \mathrm{Z}^*(v) = \mathrm{Z}(v^*)$ is a valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. The operator Z is called *translation invariant*, if and only if Z^* is dually translation invariant. By the properties of the Legendre–Fenchel transform, this is equivalent to

$$\mathrm{Z}(u\circ\tau^{-1}) = \mathrm{Z}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and translation τ on $\mathbb{R}^n$. If Z is in addition vertically translation invariant, then we say that Z is *epi-translation invariant*. This means that $u \mapsto \mathrm{Z}(u)$ is invariant under translations of the *epi-graph* of u,

$$\mathrm{epi}(u) = \{(x,t) \in \mathbb{R}^n\times\mathbb{R} : u(x) \le t\},$$

in $\mathbb{R}^{n+1}$. Moreover, we say that Z is *simple* if it vanishes on functions $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ whose domain is of dimension less than n.

A vector-valued valuation z on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is *translation covariant* if there exists an associated map $\mathrm{z}^0\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ such that

$$\mathrm{z}(u\circ\tau_x^{-1}) = \mathrm{z}(u) + \mathrm{z}^0(u)x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, where τ_x denotes the translation $y \mapsto y + x$ on $\mathbb{R}^n$. Equivalently, z^* is dually translation covariant. Lastly, the definitions of rotation equivariance, $\mathrm{O}(n)$ equivariance, and reflection invariance are the same as for a valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

We start by establishing the properties of the translation covariant valuations, that we will characterize in the course of this section.

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Proposition 1.3.1. *If $\zeta \in C_c(\mathbb{R}^n)$, then*

$$u \mapsto \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))x \,\mathrm{d}x \tag{1.3.1}$$

is a continuous, translation covariant, vertically translation invariant, and simple valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Proof. Let $\zeta \in C_c(\mathbb{R}^n)$ be given. We use (1.2.1) together with Theorem 1.2.1 (applied to each coordinate of the vector-valued integral above) to see that (1.3.1) defines a continuous vector-valued valuation z on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. For $y \in \mathbb{R}^n$ and $c \in \mathbb{R}$ we have

$$\begin{aligned}
\mathrm{z}(u \circ \tau_y^{-1} + c) &= \int_{\mathrm{dom}(u\circ\tau_y^{-1}+c)} \zeta(\nabla(u \circ \tau_y^{-1} + c)(x))x \,\mathrm{d}x \\
&= \int_{\mathrm{dom}(u\circ\tau_y^{-1})} \zeta(\nabla u(x-y))x \,\mathrm{d}x \\
&= \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))(x+y) \,\mathrm{d}x \\
&= \mathrm{z}(u) + y \int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) \,\mathrm{d}x
\end{aligned}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, which shows that z is translation covariant and vertically translation invariant. Lastly, if u has lower-dimensional domain, then the use of the n-dimensional Lebesgue measure in (1.3.1) shows that $\mathrm{z}(u) = o$ or, equivalently, z is simple. □

Next, we consider the map associated with a translation covariant valuation. The following result is a functional analog of a well-known fact for translation covariant valuations on convex bodies. See, for example, [65, Theorem 10.5].

Lemma 1.3.2. *If $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is a continuous, translation covariant, vertically translation invariant valuation, then the associated map $\mathrm{z}^0\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous, epi-translation invariant valuation. In addition, if z is simple, then so is z^0.*

Proof. Let a continuous, translation covariant, vertically translation invariant valuation z on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ be given that takes values in $\mathbb{R}^n$. We first show that the associated map z^0 is a valuation. For this, let $u, w \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ be such that $u \vee w, u \wedge w \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Observe that

$$(u \circ \tau_y^{-1}) \vee (w \circ \tau_y^{-1}) = (u \vee w) \circ \tau_y^{-1} \quad \text{and} \quad (u \circ \tau_y^{-1}) \wedge (w \circ \tau_y^{-1}) = (u \wedge w) \circ \tau_y^{-1}$$

for $y \in \mathbb{R}^n$. Thus, since z is a translation covariant valuation,

$$\begin{aligned}
\mathrm{z}(u) + \mathrm{z}(w) + (\mathrm{z}^0(u) + \mathrm{z}^0(w))y &= \mathrm{z}(u \circ \tau_y^{-1}) + \mathrm{z}(w \circ \tau_y^{-1}) \\
&= \mathrm{z}((u \circ \tau_y^{-1}) \vee (w \circ \tau_y^{-1})) + \mathrm{z}((u \circ \tau_y^{-1}) \wedge (w \circ \tau_y^{-1})) \\
&= \mathrm{z}(u \vee w) + \mathrm{z}(u \wedge w) + (\mathrm{z}^0(u \vee w) + \mathrm{z}^0(u \wedge w))y \\
&= \mathrm{z}(u) + \mathrm{z}(w) + (\mathrm{z}^0(u \vee w) + \mathrm{z}^0(u \wedge w))y
\end{aligned}$$

for $y \in \mathbb{R}^n$, which shows the desired property

$$\mathrm{z}^0(u) + \mathrm{z}^0(w) = \mathrm{z}^0(u \vee w) + \mathrm{z}^0(u \wedge w).$$

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Next, for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, $x, y \in \mathbb{R}^n$, and $c \in \mathbb{R}$, it follows from the translation covariance and vertical translation invariance of z that

$$\begin{aligned}\mathrm{z}(u) + \mathrm{z}^0(u)(x+y) &= \mathrm{z}(u \circ \tau_{x+y}^{-1}) \\ &= \mathrm{z}((u+c) \circ \tau_x^{-1} \circ \tau_y^{-1}) \\ &= \mathrm{z}((u+c) \circ \tau_x^{-1}) + \mathrm{z}^0((u+c) \circ \tau_x^{-1})y \\ &= \mathrm{z}(u) + \mathrm{z}^0(u)x + \mathrm{z}^0((u+c) \circ \tau_x^{-1})y.\end{aligned}$$

Hence,

$$\mathrm{z}^0(u \circ \tau_x^{-1} + c) = \mathrm{z}^0(u),$$

which means that z^0 is epi-translation invariant. Similarly, if u_k, $k \in \mathbb{N}$, is an epi-convergent sequence in $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ with limit $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, then for every $y \in \mathbb{R}^n$, $u_k \circ \tau_y^{-1}$ epi-converges to $u \circ \tau_y^{-1}$ as $k \to \infty$. Thus, by the continuity of z,

$$\begin{aligned}\mathrm{z}(u) + \lim_{k\to\infty} \mathrm{z}^0(u_k)y &= \lim_{k\to\infty} \big(\mathrm{z}(u_k) + \mathrm{z}^0(u_k)y\big) \\ &= \lim_{k\to\infty} \mathrm{z}(u_k \circ \tau_y^{-1}) \\ &= \mathrm{z}(u \circ \tau_y^{-1}) \\ &= \mathrm{z}(u) + \mathrm{z}^0(u)y,\end{aligned}$$

which shows that z^0 is continuous. Lastly, if $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is such that the dimension of $\mathrm{dom}(u)$ is smaller than n, then so is the dimension of $\mathrm{dom}(u \circ \tau_y^{-1})$ for every $y \in \mathbb{R}^n$. Thus, if z is simple, we have

$$o = \mathrm{z}(u \circ \tau_y^{-1}) = \mathrm{z}(u) + \mathrm{z}^0(u)y = o + \mathrm{z}^0(u)y$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, and therefore also z^0 has to be simple. □

Lemma 1.3.2 indicates that studying continuous, translation covariant, vertically translation invariant, simple valuations goes hand in hand with investigating continuous, epi-translation invariant, simple, real-valued valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. The latter is the content of the Klain–Schneider theorem on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, which was proven in [25, Theorem 1.2] and which is dual to Theorem 1.1.3.

Theorem 1.3.3. *A map $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous, epi-translation invariant, simple valuation, if and only if there exists a function $\zeta \in C_c(\mathbb{R}^n)$ such that*

$$\mathrm{Z}(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

We furthermore need the following two results to deal with rotation equivariance.

Lemma 1.3.4. *Let $\xi\colon (0,\infty) \to \mathbb{R}$ and $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$ be such that*

$$\psi(x) = \xi(|x|)x$$

for $x \in \mathbb{R}^n \setminus \{o\}$. The map ψ can be chosen from $C_c(\mathbb{R}^n;\mathbb{R}^n)$, if and only if ξ is continuous with bounded support on $(0,\infty)$ such that

$$\lim_{t\to 0^+} \xi(t)t = 0. \tag{1.3.2}$$

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Proof. Suppose first that ψ is continuous with compact support on $\mathbb{R}^n$. Denoting the first coordinate of the vector-valued function ψ by ψ_1, we see that

$$\xi(t) = \xi(|te_1|) = \frac{\psi_1(te_1)}{t}$$

for $t > 0$, where e_1 is the first vector of the standard orthonormal basis of $\mathbb{R}^n$. Thus, ξ is continuous with bounded support on $(0,\infty)$. Furthermore, since ψ is continuous in o,

$$\lim_{t\to 0^+} \xi(|t|)te_1 = \lim_{t\to 0^+} \psi(te_1) = \psi(o) = \lim_{t\to 0^+} \psi(-te_1) = \lim_{t\to 0^+} \xi(|t|)(-te_1),$$

which implies (1.3.2).

Assume now that ξ is continuous with bounded support on $(0,\infty)$ and satisfies (1.3.2). Clearly, ψ is continuous with bounded support on $\mathbb{R}^n \setminus \{o\}$. In addition, for any sequence x_i, $i \in \mathbb{N}$, in $\mathbb{R}^n$ such that $\lim_{i\to\infty} x_i = o$, we have

$$\lim_{i\to\infty} |\psi(x_i)| = \lim_{i\to\infty} \left|\xi(|x_i|)x_i\right| = 0,$$

which shows that ψ continuously extends to o with $\psi(o) = 0$. □

Lemma 1.3.5. *Let $\psi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$. For $n \geq 3$, the map ψ satisfies*

$$\psi(\vartheta y) = \vartheta\psi(y) \tag{1.3.3}$$

for every $\vartheta \in \mathrm{SO}(n)$ and $y \in \mathbb{R}^n\setminus\{o\}$, if and only if there exists $\xi \in C_b((0,\infty))$ with $\lim_{t\to 0^+} \xi(t)t = 0$ such that

$$\psi(y) = \xi(|y|)y$$

for every $y \in \mathbb{R}^n \setminus \{o\}$. For $n \leq 2$, the same representation holds if ψ is assumed to commute with $\mathrm{O}(n)$ instead.

Proof. If $\xi \in C_b((0,\infty))$ is such that $\lim_{t\to 0^+} \xi(t)t = 0$, then it easily follows from Lemma 1.3.4 that $\psi(y) = \xi(|y|)y$, $y \in \mathbb{R}^n \setminus \{o\}$, has the desired properties.

Now for $n \geq 3$, let a continuous vector-valued map ψ on $\mathbb{R}^n$ be given that commutes with $\mathrm{SO}(n)$. If for $y \in \mathbb{R}^n \setminus \{o\}$ we denote

$$S_y = \{\vartheta \in \mathrm{SO}(n) : \vartheta y = y\},$$

then (1.3.3) shows that $\vartheta \in S_y$ if and only if $\vartheta \in S_{\psi(y)}$. In particular, $\psi(y)$ and y are parallel, which means that there exists $\hat{\xi}\colon (0,\infty)\times\mathbb{S}^{n-1} \to \mathbb{R}$ such that

$$\psi(rz) = \hat{\xi}(r,z)rz$$

for $r \in (0,\infty)$ and $z \in \mathbb{S}^{n-1}$. By (1.3.3) we have

$$\hat{\xi}(r,\vartheta z)r\vartheta z = \psi(r\vartheta z) = \vartheta\psi(rz) = \hat{\xi}(r,z)r\vartheta z$$

for $\vartheta \in \mathrm{SO}(n)$, and thus $\hat{\xi}(r,z)$ is independent of $z \in \mathbb{S}^{n-1}$. This means that there exists $\xi\colon (0,\infty)\to\mathbb{R}$ such that

$$\hat{\xi}(r,z) = \xi(r)$$

for $r \in (0,\infty)$ and $z \in \mathbb{S}^{n-1}$ and, therefore, $\psi(y) = \xi(|y|)y$ for $y \in \mathbb{R}^n \setminus \{o\}$. The conclusion now follows from Lemma 1.3.4.

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Next, in case $n = 2$, let ψ on $\mathbb{R}^2$ be such that $\psi(\vartheta y) = \vartheta\psi(y)$ for $\vartheta \in \mathrm{O}(2)$ and $y \in \mathbb{R}^2$. Writing $\rho_y \in \mathrm{O}(2)$ for the reflection at the line spanned by $y \in \mathbb{R}^2 \setminus \{o\}$, we have

$$\psi(y) = \psi(\rho_y y) = \rho_y \psi(y).$$

This shows that $\psi(y)$ must be parallel to y we can therefore use the same argumentation as in the case $n \geq 3$.

Lastly, for $n = 1$, we assume that the map ψ satisfies $\psi(-y) = -\psi(y)$ for $y \in \mathbb{R}$. This is equivalent to $\psi(0) = 0$ and

$$\psi(y) = \frac{\psi(|y|)}{|y|} y$$

for $y \neq 0$. The statement now follows with $\xi(t) = \psi(t)/t$ for $t \in (0, \infty)$. □

We can now prove the main result of this section, which is equivalent to Theorem 1.1.4 and Corollary 1.1.5. For $\alpha \in C_c([0, \infty))$ we consider the operator $\mathrm{m}_\alpha \colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$, which is defined as $\mathrm{m}_\alpha(u) = \mathrm{m}^*_\alpha(u^*)$, or equivalently, by (1.2.1),

$$\mathrm{m}_\alpha(u) = \int_{\mathrm{dom}(u)} \alpha(|\nabla u(x)|) x \, \mathrm{d}x$$

for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Furthermore, for $\xi \in C_b((0, \infty))$ such that $\lim_{t \to 0^+} \xi(t)t = 0$, we define $\mathrm{t}_{n,\xi} \colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ as $\mathrm{t}_{n,\xi}(u) = \mathrm{t}^*_{n,\xi}(u^*)$, which is equivalent to

$$\mathrm{t}_{n,\xi}(u) = \int_{\mathrm{dom}(u)} \xi(|\nabla u(x)|) \nabla u(x) \, \mathrm{d}x$$

for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Theorem 1.3.6. *A map $\mathrm{z} \colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is a continuous, translation covariant, vertically translation invariant, simple valuation, if and only if there exist functions $\psi \in C_c(\mathbb{R}^n; \mathbb{R}^n)$ and $\zeta \in C_c(\mathbb{R}^n)$ such that*

$$\mathrm{z}(u) = \int_{\mathrm{dom}(u)} \psi(\nabla u(x)) \, \mathrm{d}x + \int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) x \, \mathrm{d}x \tag{1.3.4}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. For $n \geq 3$, the valuation z is in addition rotation equivariant, if and only if there exist functions $\alpha \in C_c([0, \infty))$ and $\xi \in C_b((0, \infty))$ with $\lim_{t \to 0^+} \xi(t)t = 0$ such that

$$\mathrm{z}(u) = \mathrm{t}_{n,\xi}(u) + \mathrm{m}_\alpha(u) \tag{1.3.5}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$. For $n \leq 2$, the same representation holds if we replace rotation equivariance with $\mathrm{O}(n)$ equivariance.

Remark 1.3.7. If in Theorem 1.3.6 translation covariance and vertical translation invariance are replaced by epi-translation invariance, then only the valuations

$$u \mapsto \int_{\mathrm{dom}(u)} \psi(\nabla u(x)) \, \mathrm{d}x$$

and $\mathrm{t}_{n,\xi}$, respectively, remain. Furthermore, let us point out that for $n \geq 2$ the representation (1.3.5) does not hold if one merely assumes $\mathrm{SO}(n)$ equivariance. For $n = 1$, we have $\mathrm{SO}(1) = \{\mathrm{Id}\}$ and thus, also (1.3.4) is $\mathrm{SO}(1)$ equivariant. For $n = 2$, let $\Phi \colon [0, \infty) \to \mathrm{SO}(2)$ be continuous

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and let $\xi \in C_b((0,\infty))$ be such that $\lim_{t\to 0^+} \xi(t)t = 0$. Since $\mathrm{SO}(2)$ is commutative, also the valuation

$$u \mapsto \int_{\mathrm{dom}(u)} \xi(|\nabla u(x)|)\Phi(|\nabla u(x)|)\nabla u(x)\,\mathrm{d}x \tag{1.3.6}$$

is $\mathrm{SO}(2)$ equivariant and generalizes $\mathrm{t}_{n,\xi}$. In fact, one can show that every continuous, translation covariant, vertically translation invariant, $\mathrm{SO}(2)$ equivariant, simple valuation can be written as the sum of (1.3.6) and m_α with some $\alpha \in C_c([0,\infty))$.

Proof of Theorem 1.3.6. For $\psi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$ and $\zeta \in C_c(\mathbb{R}^n)$ it follows from Theorem 1.3.3, applied coordinate-wise, together with Proposition 1.3.1 that (1.3.4) defines a continuous, translation covariant, vertically translation invariant, simple valuation.

Now let a continuous, translation covariant, vertically translation invariant, simple valuation $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ be given. By Lemma 1.3.2 together with Theorem 1.3.3, there exists $\zeta \in C_c(\mathbb{R}^n)$ such that the associated map z^0 is of the form

$$\mathrm{z}^0(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\,\mathrm{d}x$$

for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. We now define $\tilde{\mathrm{z}}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ as

$$\tilde{\mathrm{z}}(u) = \mathrm{z}(u) - \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))x\,\mathrm{d}x$$

for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. For any $y \in \mathbb{R}^n$ we have

$$\begin{aligned}
&\tilde{\mathrm{z}}(u\circ\tau_y^{-1})\\
&\quad = \mathrm{z}(u) + \left(\int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\,\mathrm{d}x\right) y - \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))x\,\mathrm{d}x - \left(\int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\,\mathrm{d}x\right) y\\
&\quad = \tilde{\mathrm{z}}(u),
\end{aligned}$$

which together with Proposition 1.3.1 and the properties of z shows that $\tilde{\mathrm{z}}$ is a continuous, epi-translation invariant, simple valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Thus, another application of Theorem 1.3.3, this time applied to each of the coordinates of the vector-valued map $\tilde{\mathrm{z}}$, shows there exist functions $\psi_1,\dots,\psi_n \in C_c(\mathbb{R}^n)$ such that the ith coordinate of $\tilde{\mathrm{z}}$, $i \in \{1,\dots,n\}$, must be of the form

$$u \mapsto \int_{\mathrm{dom}(u)} \psi_i(\nabla u(x))\,\mathrm{d}x,$$

concluding the proof of the first part of the statement.

Next, let $\alpha \in C_c([0,\infty))$ and $\xi \in C_b((0,\infty))$ with $\lim_{t\to 0^+} \xi(t)t = 0$ be given, and let $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ be as in (1.3.5). We will show that z is $\mathrm{O}(n)$ equivariant. If $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\vartheta \in \mathrm{O}(n)$, then

$$\begin{aligned}
\mathrm{t}_{n,\xi}(u\circ\vartheta^{-1}) &= \int_{\mathrm{dom}(u\circ\vartheta^{-1})} \xi(|\nabla(u\circ\vartheta^{-1})(x)|)\nabla(u\circ\vartheta^{-1})(x)\,\mathrm{d}x\\
&= \vartheta \int_{\mathrm{dom}(u\circ\vartheta^{-1})} \xi(|\vartheta\nabla u(\vartheta^{-1}x)|)\nabla u(\vartheta^{-1}x)\,\mathrm{d}x\\
&= \vartheta\,\mathrm{t}_{n,\xi}(u).
\end{aligned}$$

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Similarly,

$$\begin{aligned}\mathrm{m}_\alpha(u\circ\vartheta^{-1}) &= \int_{\mathrm{dom}(u\circ\vartheta^{-1})} \alpha(|\nabla(u\circ\vartheta^{-1})(x)|)x\,\mathrm{d}x\\ &= \int_{\mathrm{dom}(u\circ\vartheta^{-1})} \alpha(|\vartheta\nabla u(\vartheta^{-1}x)|)x\,\mathrm{d}x\\ &= \vartheta\int_{\mathrm{dom}(u)} \alpha(|\nabla u(x)|)x\,\mathrm{d}x\\ &= \vartheta\,\mathrm{m}_\alpha(u),\end{aligned}$$

which shows the desired property.

Conversely, assume that z is a continuous, translation covariant, vertically translation invariant, rotation equivariant, simple valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. By the first part of the proof there exist functions $\psi\in C_c(\mathbb{R}^n;\mathbb{R}^n)$ and $\zeta\in C_c(\mathbb{R}^n)$ such that

$$\mathrm{z}(u) = \int_{\mathrm{dom}(u)}\psi(\nabla u(x))\,\mathrm{d}x + \int_{\mathrm{dom}(u)}\zeta(\nabla u(x))x\,\mathrm{d}x$$

for every $u\in\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. By the rotation equivariance of z we see that

$$\begin{aligned}o &= \mathrm{z}(u\circ\vartheta^{-1}) - \vartheta\,\mathrm{z}(u)\\ &= \int_{\mathrm{dom}(u\circ\vartheta^{-1})}\psi(\vartheta\nabla u(\vartheta^{-1}x))\,\mathrm{d}x + \int_{\mathrm{dom}(u\circ\vartheta^{-1})}\zeta(\vartheta\nabla u(\vartheta^{-1}x))x\,\mathrm{d}x\\ &\quad - \vartheta\int_{\mathrm{dom}(u)}\psi(\nabla u(x))\,\mathrm{d}x - \vartheta\int_{\mathrm{dom}(u)}\zeta(\nabla u(x))x\,\mathrm{d}x\\ &= \int_{\mathrm{dom}(u)}\psi(\vartheta\nabla u(x)) - \vartheta\psi(\nabla u(x))\,\mathrm{d}x + \vartheta\int_{\mathrm{dom}(u)}\big(\zeta(\vartheta\nabla u(x)) - \zeta(\nabla u(x))\big)x\,\mathrm{d}x\end{aligned}$$

for $u\in\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\vartheta\in\mathrm{SO}(n)$. Choosing $u=\mathbf{I}_K+\langle y,\cdot\rangle$ for some $K\in\mathcal{K}^n$ and $y\in\mathbb{R}^n$ gives

$$o = \mathrm{vol}_n(K)\big(\psi(\vartheta y) - \vartheta\psi(y)\big) + (\zeta(\vartheta y)-\zeta(y))\vartheta\, m(K).$$

Thus, considering first a convex body K with $\mathrm{vol}_n(K)>0$ and $m(K)=o$, and then a body K with $m(K)\neq 0$, shows that

$$\psi(\vartheta y) = \vartheta\psi(y) \quad\text{and}\quad \zeta(\vartheta y)=\zeta(y)$$

for every $\vartheta\in\mathrm{SO}(n)$ (or $\vartheta\in\mathrm{O}(n)$ for $n\leq 2$) and $y\in\mathbb{R}^n$. By Lemma 1.3.5 this implies that there exists $\xi\in C_b((0,\infty))$ with $\lim_{t\to 0^+}\xi(t)t=0$ such that

$$\psi(y) = \xi(|y|)y$$

for every $y\in\mathbb{R}^n\setminus\{o\}$. Furthermore, it easily follows that $\zeta(y)=\alpha(|y|)$ for some $\alpha\in C_c([0,\infty))$, where we choose $\vartheta(y)=-y$ in case $n=1$. □

1.4 Functional Intrinsic Moments and Minkowski Vectors

Let $\alpha\in C_c([0,\infty))$ and let $q(x)=|x|^2/2$. Considering a Steiner-type formula, starting with the operator m^*_α, results in

$$\mathrm{m}^*_\alpha(v+rq) = \sum_{j=0}^{n} r^{n-j}\int_{\mathbb{R}^n}\alpha(|x|)\nabla v(x)[\mathrm{D}^2v(x)]_j\,\mathrm{d}x + r^{n-j+1}\int_{\mathbb{R}^n}\alpha(|x|)x[\mathrm{D}^2v(x)]_j\,\mathrm{d}x \quad (1.4.1)$$

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for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$ and $r \geq 0$. Here, we write $[A]_j$ for the *jth elementary symmetric function* of the eigenvalues of a symmetric matrix $A \in \mathbb{R}^{n\times n}$, with the convention $[A]_0 \equiv 1$. In particular, $[\mathrm{D}^2 v(x)]_n = \det(\mathrm{D}^2 v(x))$. Using appropriate Hessian measures, the above continuously extends to general $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ (see [23, Theorem 17]).

Equation (1.4.1) gives rise to two families of operators. For $\alpha \in C_c([0,\infty))$ and $j \in \{0,\dots,n\}$, we define $\mathrm{z}^*_{j+1,\alpha}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ to be the unique continuous map such that

$$\mathrm{z}^*_{j+1,\alpha}(v) = \int_{\mathbb{R}^n} \alpha(|x|)\nabla v(x)[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$, and we call operators of this form *functional intrinsic moments*. This is motivated by the observation that when v is the support function h_K of a convex body K, then we retrieve the *intrinsic moment* $z_{j+1}(K)$, which in turn is obtained as a coefficient in a Steiner formula, starting with the moment vector $m(K)$ (we refer to [79, Section 5.4.1] for details). For the case $j = n$, this follows from (1.1.3) together with the relation $z_{n+1}(K) = m(K)$. When $j \in \{0,\dots,n-1\}$ and h_K is of class C^2, then

$$\begin{aligned}
\mathrm{z}^*_{j+1,\alpha}(h_K) &\simeq \int_{\mathbb{S}^{n-1}} \nabla h_K(z)\,[\mathrm{D}^2 h_K(z)]_j \,\mathrm{d}\mathcal{H}^{n-j}(z)\\
&\simeq \int_{\mathbb{S}^{n-1}} \nabla h_K(z)\,\mathrm{d}S_j(K,z)\\
&\simeq \int_{\mathbb{R}^n} x\,\mathrm{d}C_j(K,x)\\
&\simeq z_{j+1}(K),
\end{aligned}$$

where $\simeq$ denotes equality up to constant factors, depending on n, j, and α. Here, we write $\mathcal{H}^k$ for the k-dimensional *Hausdorff measure*, $S_j(K,\cdot)$ denotes the jth *area measure* of K, and $C_j(K,\cdot)$ is its jth *curvature measure* (see [79, p. 214] for their definitions). By continuity we now retrieve $\mathrm{z}^*_{j+1,\alpha}(h_K) \simeq z_{j+1}(K)$ for general $K \in \mathcal{K}^n$.

Similar to their counterparts on convex bodies, the behavior of functional intrinsic moments under translations is described by functional intrinsic volumes. Indeed,

$$\mathrm{z}^*_{j+1,\alpha}(v + \langle y,\cdot\rangle) = \int_{\mathbb{R}^n} \alpha(|x|)\nabla v(x)[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x + y\int_{\mathbb{R}^n} \alpha(|x|)[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x \tag{1.4.2}$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, where

$$\mathrm{V}^*_{j,\alpha}(v) = \int_{\mathbb{R}^n} \alpha(|x|)[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x$$

is one of the functional intrinsic volumes that were introduced and characterized in [26].

The second family of operators coming from (1.4.1) is of the form

$$\mathrm{t}^*_{j,\alpha}(v) = \int_{\mathbb{R}^n} \alpha(|x|)x\,[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$, $j \in \{0,\dots,n\}$, and $\alpha \in C_c([0,\infty))$. Let now $v = h_K$ with $K \in \mathcal{K}^n$. In case $j = n$, it follows from (1.2.1) and (1.2.2) that

$$\begin{aligned}
\mathrm{t}^*_{n,\alpha}(h_K) = \int_{\mathbb{R}^n} \alpha(|x|)x\,\mathrm{dMA}(h_K;x) &= \int_{\mathrm{dom}(\mathbf{I}_K)} \alpha(|\nabla \mathbf{I}_K(x)|)\nabla \mathbf{I}_K(x)\,\mathrm{d}x\\
&= \mathrm{vol}_n(K)\alpha(0)\,o = o,
\end{aligned}$$

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where we recall that $o \in \mathbb{R}^n$ denotes the origin. Furthermore, for $j \in \{0, \dots, n-1\}$, we obtain

$$\mathrm{t}^*_{j,\alpha}(h_K) \simeq \int_{\mathbb{S}^{n-1}} z \, \mathrm{d}S_j(K, z) = o, \tag{1.4.3}$$

where the last equality is a general form of the Minkowski relation (see, for example, [79, (5.30) and (5.52)]). Since (1.4.3) is a *Minkowski tensor* of rank 1 (the only non-vanishing Minkowski tensors of rank 1 are the intrinsic moments; cf. [79, Section 5.4.2]), we will refer to the valuations $\mathrm{t}^*_{j,\alpha}$ as *functional Minkowski vectors*. These operators vanish not only on support functions but also on every radially symmetric convex function, which is a consequence of the rotation equivariance of $\mathrm{t}^*_{j,\alpha}$. Of course, this raises the question of whether functions $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ exist at all, such that $\mathrm{t}^*_{j,\alpha}(v) \neq o$. Indeed,

$$\mathrm{t}^*_{0,\alpha}(v) = \int_{\mathbb{R}^n} \alpha(|x|) x \, \mathrm{d}x = o$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, but the remaining operators in this family generally do not vanish. For the case $j = n$, this easily follows from the fact that $\mathrm{MA}(h_K \circ \tau_x^{-1}; \cdot)$ is a Dirac measure concentrated at $x \in \mathbb{R}^n$ so that

$$\mathrm{t}^*_{n,\alpha}(h_K \circ \tau_x^{-1}) = \mathrm{vol}_n(K) \alpha(|x|) x$$

for every $K \in \mathcal{K}^n$. Explicit examples for the remaining cases will be presented in [70].

We remark that results for functional intrinsic volumes [26] suggest that the operators $\mathrm{z}^*_{j+1,\alpha}$ and $\mathrm{t}^*_{j,\alpha}$ may also exist for densities α with a singularity at 0^+, where the admissible singularities possibly depend on j and n. Indeed, we have already seen in Corollary 1.1.5 that $\mathrm{t}^*_{n,\xi}$ can be defined for $\xi \in C_b((0, \infty))$ such that $\lim_{t \to 0^+} \xi(t) t = 0$. A treatment of the possible singularities for the operators $\mathrm{t}^*_{j,\alpha}$ together with a complete characterization in a Hadwiger-type theorem will be the subject of a subsequent paper [70]. For the functional intrinsic moments, equation (1.4.2) indicates that similar singularities may appear as for functional intrinsic volumes. However, the fact that in the case of the latter, singular integrals exist in a Riemann sense and not in a Lebesgue sense poses an additional challenge. See also [55].

Lastly, let us point out that different functional analogs of moment vectors were previously considered, for example, for L^p-spaces [85] or log-concave functions [72].

1.5 Classification of Homogeneous Valuations

In addition to the main results presented in this article, we establish classification results for homogeneous valuations of extremal degrees. Here, $\mathrm{z} \colon \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ is *homogeneous* of degree $s \in \mathbb{R}$ if $\mathrm{z}(\lambda v) = \lambda^s \mathrm{z}(v)$ for every $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ and $\lambda > 0$. The following decomposition result for homogeneous valuations was shown in [23, Theorem 4] (and later also in [50]).

Theorem 1.5.1. *If $\mathrm{Z} \colon \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ is a continuous, dually epi-translation invariant valuation, then there exist continuous, dually epi-translation invariant valuations $\mathrm{Z}_0, \dots, \mathrm{Z}_n \colon \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ such that Z_j is homogeneous of degree j, $j \in \{0, \dots, n\}$, and $\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n$.*

Similarly to Theorem 1.5.1, it follows from recent results on polynomial valuations [57, Theorem 1.3], that every continuous, dually translation covariant, vertically translation invariant valuation can be written as a sum of homogeneous valuations with degrees ranging from 0 to $(n+1)$.

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Homogeneity of Degree 0 We start with valuations that are homogeneous of degree 0. For this we will use the following result from [23, Theorem 26]

Theorem 1.5.2. *A map $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is a continuous, dually epi-translation invariant valuation that is homogeneous of degree 0, if and only if Z is constant.*

Next, we establish an elementary result, similar to Lemma 1.3.2.

Lemma 1.5.3. *If $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is a dually translation covariant valuation that is homogeneous of degree $s+1$, then the associated map $\mathrm{z}^0\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is homogeneous of degree s.*

Proof. Let z be given. We have

$$\begin{aligned}\lambda^{s+1}\,\mathrm{z}(v) + \lambda^{s+1}\,\mathrm{z}^0(v)x &= \mathrm{z}(\lambda(v + \langle x, \cdot\rangle))\\ &= \mathrm{z}(\lambda v + \langle \lambda x, \cdot\rangle)\\ &= \mathrm{z}(\lambda v) + \mathrm{z}(\lambda v)\lambda x\\ &= \lambda^{s+1}\,\mathrm{z}(v) + \lambda\,\mathrm{z}(\lambda v)x\end{aligned}$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, $x \in \mathbb{R}^n$, and $\lambda > 0$. Therefore, z^0 is homogeneous of degree s. □

It is now straightforward to show the following classification.

Theorem 1.5.4. *A map $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is a continuous, dually translation covariant, vertically translation invariant valuation that is homogeneous of degree 0, if and only if there exists $t \in \mathbb{R}^n$ such that $\mathrm{z}(v) = t$ for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. For $n \geq 2$, the valuation z is in addition rotation equivariant, if and only if it is identically o. For $n = 1$, the same representation holds if z is reflection equivariant.*

Proof. If z is as in the statement, then Lemma 1.3.2 and Lemma 1.5.3 show that the associated map z_0 must be a continuous, dually epi-translation invariant valuation that is homogeneous of degree -1. By Theorem 1.5.1 we must have $\mathrm{z}^0 \equiv 0$, and thus, z is dually translation invariant (and therefore also dually epi-translation invariant). The result now follows from applying Theorem 1.5.2 to each coordinate of z. □

Homogeneity of Top Degrees Let us now turn our attention to valuations that are homogeneous of top degrees. As in Section 1.3, we will continue to work on the space $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ for the remainder of this section. We say that $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is *epi-homogeneous* of degree $s \in \mathbb{R}$ if $\mathrm{z}(\lambda \cdot u) = \lambda^s\,\mathrm{z}(u)$ for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\lambda > 0$. Here, $\lambda \cdot u$ is the *epi-multiplication* of u with λ, which is defined as

$$\lambda \cdot u(x) = \lambda\, u\left(\frac{x}{\lambda}\right)$$

for $x \in \mathbb{R}^n$. We remark that this definition continuously extends to $0 \cdot u = \mathbf{I}_{\{o\}}$. This is equivalent to $(\lambda \cdot u)^* = \lambda u^*$ and $u \mapsto \mathrm{z}(u)$ is epi-homogeneous, if and only if $v \mapsto \mathrm{z}(v^*)$ is homogeneous.

We start with epi-translation invariant valuations. We will use the following result from [23, Theorem 2].

Theorem 1.5.5. *A map $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous, epi-translation invariant valuation that is epi-homogeneous of degree n, if and only if there exists a function $\zeta \in C_c(\mathbb{R}^n)$ such that*

$$\mathrm{Z}(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

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We can now prove the following classification result for epi-translation invariant valuations that are epi-homogeneous of degree n.

Theorem 1.5.6. *A map* $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ *is a continuous, epi-translation invariant valuation that is epi-homogeneous of degree* n*, if and only if there exists a function* $\psi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$ *such that*

$$\mathrm{z}(u) = \int_{\mathrm{dom}(u)} \psi(\nabla u(x))\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*. For* $n \geq 3$*, the valuation* z *is in addition rotation equivariant, if and only if there exists a function* $\xi \in C_b((0,\infty))$ *with* $\lim_{t\to 0^+} \xi(t)t = 0$ *such that*

$$\mathrm{z}(u) = \mathrm{t}_{n,\xi}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*. For* $n \leq 2$*, the same representation holds if we replace rotation equivariance with* $\mathrm{O}(n)$ *equivariance.*

Proof. The first part of the statement immediately follows from Theorem 1.5.5, applied coordinatewise. Next, for $\xi \in C_b((0,\infty))$ with $\lim_{t\to 0^+} \xi(t)t = 0$, we have already seen in Theorem 1.3.6 that $\mathrm{t}_{n,\xi}$ is a continuous, epi-translation invariant valuation with the desired equivariance properties. Furthermore, Theorem 1.5.5 shows that $\mathrm{t}_{n,\xi}$ is epi-homogeneous of degree n.

If now $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is a continuous, epi-translation invariant, rotation equivariant valuation that is epi-homogeneous of degree n, then Theorem 1.5.5 shows that the ith coordinate of z must be of the form

$$u \mapsto \int_{\mathrm{dom}(u)} \zeta_i(\nabla u(x))\,\mathrm{d}x$$

for some $\zeta_i \in C_c(\mathbb{R}^n)$, $i \in \{0,\dots,n\}$. Exploiting the rotation equivariance of z (or $\mathrm{O}(n)$ equivariance for $n \leq 2$), as in the proof of Theorem 1.3.6, shows that $\mathrm{z} = \mathrm{t}_{n,\xi}$ for some $\xi \in C_b((0,\infty))$ such that $\lim_{t\to 0^+} \xi(t)t = 0$. □

Before we continue with the translation covariant valuation, we state the dual result to Theorem 1.5.6 on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

Theorem 1.5.7. *A map* $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ *is a continuous, dually epi-translation invariant valuation that is homogeneous of degree* n*, if and only if there exists a function* $\psi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$ *such that*

$$\mathrm{z}(v) = \int_{\mathbb{R}^n} \psi(x)\,\mathrm{dMA}(v;x)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$*. For* $n \geq 3$*, the valuation* z *is in addition rotation equivariant, if and only if there exists a function* $\xi \in C_b((0,\infty))$ *with* $\lim_{t\to 0^+} \xi(t)t = 0$ *such that*

$$\mathrm{z}(v) = \mathrm{t}^*_{n,\xi}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$*. For* $n \leq 2$*, the same representation holds if we replace rotation equivariance with* $\mathrm{O}(n)$ *equivariance.*

At last, we look at translation covariant valuations that are epi-homogeneous of degree $(n+1)$. We remark that the following result can be independently obtained from [57, Theorem 1.9], which uses a similar strategy as the short and direct proof that is presented here.

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Theorem 1.5.8. *A map* $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ *is a continuous, translation covariant, vertically translation invariant valuation that is epi-homogeneous of degree* $(n+1)$*, if and only if there exists a function* $\zeta \in C_c(\mathbb{R}^n)$ *such that*

$$\mathrm{z}(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) x \,\mathrm{d}x \tag{1.5.1}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*. For* $n \geq 2$*, the valuation* z *is in addition rotation equivariant, if and only if there exists a function* $\alpha \in C_c([0,\infty))$ *such that*

$$\mathrm{z}(u) = \mathrm{m}_\alpha(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*. For* $n = 1$*, the same representation holds if we replace rotation equivariance with reflection equivariance.*

Proof. For $\zeta \in C_c(\mathbb{R}^n)$ it follows from Proposition 1.3.1 that (1.5.1) defines continuous, translation covariant, vertically translation invariant valuation. Now for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\lambda > 0$ we have $\nabla(\lambda \cdot u)(x) = \nabla u(x/\lambda)$ for $x \in \mathbb{R}^n$, and therefore

$$\mathrm{z}(\lambda \cdot u) = \int_{\mathrm{dom}(\lambda \cdot u)} \zeta(\nabla u(x/\lambda)) x \,\mathrm{d}x = \lambda^{n+1} \int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) x \,\mathrm{d}x = \lambda^{n+1}\, \mathrm{z}(u),$$

which means that z is epi-homogeneous of degree $(n+1)$.

Conversely, let $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ be a continuous, translation covariant, vertically translation invariant valuation that is epi-homogeneous of degree $(n+1)$. By Lemma 1.3.2 and Lemma 1.5.3 together with Theorem 1.5.5, there exists $\zeta \in C_c(\mathbb{R}^n)$ such that the associated map z^0 is of the form

$$\mathrm{z}^0(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) \,\mathrm{d}x$$

for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. We now define $\tilde{\mathrm{z}}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ as

$$\tilde{\mathrm{z}}(u) = \mathrm{z}(u) - \int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) x \,\mathrm{d}x$$

for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. For any $y \in \mathbb{R}^n$ we have

$$\begin{aligned}
&\tilde{\mathrm{z}}(u \circ \tau_y^{-1}) \\
&\quad = \mathrm{z}(u) + \left(\int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) \,\mathrm{d}x \right) y - \int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) x \,\mathrm{d}x - \left(\int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) \,\mathrm{d}x \right) y \\
&\quad = \tilde{\mathrm{z}}(u),
\end{aligned}$$

which together with Proposition 1.3.1 and the properties of z shows that $\tilde{\mathrm{z}}$ is a continuous, epi-translation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ that is epi-homogeneous of degree $(n+1)$. Theorem 1.5.1 now implies that $\tilde{\mathrm{z}}$ vanishes identically and hence z must be as in (1.5.1).

In the rotation equivariant case, we have already shown the necessary properties of m_α in Theorem 1.3.6. Conversely, if a valuation z of the form (1.5.1) is rotation equivariant, then, as in the proof of Theorem 1.3.6, it follows from Lemma 1.3.5 that $\mathrm{z} = \mathrm{m}_\alpha$ for some $\alpha \in C_c(\mathbb{R}^n)$. □

We conclude with the dual result for valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

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Theorem 1.5.9. *A map* $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ *is a continuous, dually translation covariant, vertically translation invariant valuation that is homogeneous of degree* $(n+1)$*, if and only if there exists a function* $\zeta \in C_c(\mathbb{R}^n)$ *such that*

$$\mathrm{z}(v) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(x) y \, \mathrm{d}\Theta_0(v;(x,y))$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$*. For* $n \geq 2$*, the valuation* z *is in addition rotation equivariant, if and only if there exists a function* $\alpha \in C_c([0,\infty))$ *such that*

$$\mathrm{z}(v) = \mathrm{m}^*_\alpha(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$*. For* $n = 1$*, the same representation holds if we replace rotation equivariance with reflection equivariance.*

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2 The Vectorial Hadwiger Theorem on Convex Functions

2.1 Introduction

Minkowski Relations For a *convex body* $K \in \mathcal{K}^n$, where $\mathcal{K}^n$ is the set of non-empty, compact, convex subsets of $\mathbb{R}^n$, let $S_{n-1}(K,\cdot)$ denote its *surface area measure*, which is a Borel measure on the *Euclidean unit sphere* $\mathbb{S}^{n-1}$. When K has positive volume, this measure is given by

$$S_{n-1}(K,\omega) = \mathcal{H}^{n-1}\big(\{x \in \mathrm{bd}(K) : K \text{ has an outer unit normal at } x \text{ belonging to } \omega\}\big) \quad (2.1.1)$$

for Borel sets $\omega \subseteq \mathbb{S}^{n-1}$, where $\mathrm{bd}(K)$ is the boundary of K and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional *Hausdorff measure*. It is elementary to check that the first moment of this measure vanishes, that is

$$\int_{\mathbb{S}^{n-1}} z \,\mathrm{d}S_{n-1}(K,z) = o$$

for every $K \in \mathcal{K}^n$, with o denoting the origin in $\mathbb{R}^n$. This property essentially characterizes surface area measures and is the core of the *Minkowski problem*. Indeed, Minkowski's existence theorem [79, Theorem 8.2.2] states that a Borel measure μ on $\mathbb{S}^{n-1}$ is the surface area measure of a full-dimensional convex body K, i.e. $\mu = S_{n-1}(K,\cdot)$, if and only if μ satisfies $\int_{\mathbb{S}^{n-1}} z \,\mathrm{d}\mu(z) = o$ and is not concentrated on any great subsphere.

Generalizing the surface area measure, the *area measures* $S_j(K,\cdot)$ with $0 \le j \le n-1$ arise from the Steiner-type formula

$$S_{n-1}(K + rB^n,\cdot) = \sum_{j=0}^{n-1} \binom{n-1}{j} r^{n-1-j} S_j(K,\cdot) \quad (2.1.2)$$

for $K \in \mathcal{K}^n$ and $r > 0$, where we write B^n for *Euclidean unit ball* and

$$K + L = \{x + y : x \in K, y \in L\}$$

for the *Minkowski sum* of two convex bodies K and L. Again, the first moments of these measures vanish,

$$\int_{\mathbb{S}^{n-1}} z \,\mathrm{d}S_j(K,z) = o, \quad (2.1.3)$$

which is also referred to as the *Minkowski relations* (see, for example, [79, (5.30) and (5.52)]). While this shows that the disappearance of the first moment of a Borel measure μ on $\mathbb{S}^{n-1}$ is necessary for it to be the jth area measure of some convex body K, it is, in contrast to the case $j = n-1$, not sufficient when $j < n-1$. In particular, determining such necessary and sufficient conditions for $1 < j < n-1$, known as the *Christoffel–Minkowski problem*, is a central open problem in convex geometry and geometric analysis. See, for example, [35, 41]. We also refer to [13, 14, 17, 33, 40, 66] for related results.

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Valuations A different perspective on the Minkowski relations opens up when we look at the properties of the integral on the left side of (2.1.3), which defines a vector-valued map $\mathrm{z}\colon \mathcal{K}^n \to \mathbb{R}^n$. It is a straightforward consequence of (2.1.1) and (2.1.2) that z is *translation invariant*, meaning $\mathrm{z}(K+x) = \mathrm{z}(K)$ for every $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$, and *rotation equivariant*, that is $\mathrm{z}(\vartheta K) = \vartheta\,\mathrm{z}(K)$ for every $K \in \mathcal{K}^n$ and $\vartheta \in \mathrm{SO}(n)$. Furthermore, z is a *valuation*, which means that

$$\mathrm{z}(K) + \mathrm{z}(L) = \mathrm{z}(K \cup L) + \mathrm{z}(K \cap L) \tag{2.1.4}$$

for every $K, L \in \mathcal{K}^n$ such that also $K \cup L \in \mathcal{K}^n$. In addition, z is continuous, where we equip $\mathcal{K}^n$ with the *Hausdorff topology*. Notably, we have the following consequence of a characterization of the *intrinsic moments* by Hadwiger and Schneider [39, Hauptsatz], where we say that a map $\mathrm{z}\colon \mathcal{K}^n \to \mathbb{R}^n$ is *trivial* if $\mathrm{z} \equiv o$.

Proposition 2.1.1 (Hadwiger–Schneider)**.** *A map $\mathrm{z}\colon \mathcal{K}^n \to \mathbb{R}^n$ is a continuous, translation invariant, rotation equivariant valuation, if and only if z is trivial.*

From today's perspective, the work of Hadwiger and Schneider characterizes the rank 1 *Minkowski tensors*, a natural family of isometry covariant valuations that has received considerable attention (see, for example, [1, 11, 12, 45]). The model for such results is undoubtedly the characterization of the *intrinsic volumes* $V_j\colon \mathcal{K}^n \to \mathbb{R}$, $j \in \{0, \dots, n\}$, which are the Minkowski tensors of rank 0. For $j < n$, the jth intrinsic volumes $V_j(K)$ is proportional to the total mass $S_j(K, \mathbb{S}^{n-1})$ and $V_n(K)$ is the usual n-dimensional *volume* of $K \in \mathcal{K}^n$. While intrinsic volumes are also continuous, translation invariant valuations, they are *rotation invariant*, meaning $V_j(\vartheta K) = V_j(K)$ for every $K \in \mathcal{K}^n$ and $\vartheta \in \mathrm{SO}(n)$. We state Hadwiger's celebrated characterization theorem [38] next.

Theorem 2.1.2 (Hadwiger)**.** *A map $\mathrm{Z}\colon \mathcal{K}^n \to \mathbb{R}$ is a continuous, translation and rotation invariant valuation, if and only if there exist constants $c_0, \dots, c_n \in \mathbb{R}$ such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + \cdots + c_n V_n(K) \tag{2.1.5}$$

for every $K \in \mathcal{K}^n$.

Let us point out that it is easy to see that linear combinations of intrinsic volumes are continuous, translation and rotation invariant valuations. The strength of Hadwiger's theorem lies in establishing the highly non-trivial fact that all such valuations must be as in (2.1.5). For applications of Theorem 2.1.2, including effortless proofs of formulas in integral geometry and geometric probability, see, for example, [38, 49]. See also [2, 9, 10, 37, 63] for related results. We also mention [34] which revealed that solving the Christoffel–Minkowski problem is equivalent to characterizing a special family of *Minkowski valuations*, the so-called *mean section bodies*.

Extensions to Convex Functions Having considered convex bodies so far, we will now move to the more general space of (finite-valued) convex functions on $\mathbb{R}^n$,

$$\mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) = \{v\colon \mathbb{R}^n \to \mathbb{R} : v \text{ is convex}\}.$$

Convex bodies are naturally embedded into this space by associating with each $K \in \mathcal{K}^n$ its *support function* $h_K \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, given by

$$h_K(x) = \max_{y \in K} \langle x, y \rangle$$

for $x \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^n$. The purpose of this article is to establish a non-trivial analytic analog of Proposition 2.1.1 for operators on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$.

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For a natural extension of area measures and related quantities from convex bodies to convex functions, we first note that when h_K is of class C^2 on $\mathbb{R}^n \setminus \{o\}$, then $S_j(K,\cdot)$ is absolutely continuous with respect to $\mathcal{H}^{n-1}$ with

$$\mathrm{d}S_j(K,z) = \binom{n-1}{j}^{-1} [\mathrm{D}^2 h_K(z)]_j \,\mathrm{d}\mathcal{H}^{n-1}(z) \tag{2.1.6}$$

on $\mathbb{S}^{n-1}$ for $j \in \{0,\dots,n-1\}$. Here, $[\mathrm{D}^2 h_K(z)]_j$ denotes the jth elementary symmetric function of the eigenvalues of the Hessian matrix of h_K at $z \in \mathbb{S}^{n-1}$, with the convention $[\mathrm{D}^2 h_K(z)]_0 \equiv 1$. See, for example, [79, Corollary 2.5.3 and (4.26)]. Based on (2.1.6), functional counterparts of the intrinsic volumes on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ were introduced by Colesanti, Ludwig, and Mussnig in [26]. The so-called *functional intrinsic volumes* take the form

$$\mathrm{V}^*_{j,\zeta}(v) = \int_{\mathbb{R}^n} \zeta(|x|)[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x, \quad j \in \{0,\dots,n\}, \tag{2.1.7}$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$, where $|x|$ denotes the *Euclidean norm* of $x \in \mathbb{R}^n$ and ζ is a continuous function with bounded support on $(0,\infty)$, $\zeta \in C_b((0,\infty))$, satisfying additional assumptions at 0^+. When ζ continuously extends to 0, then it follows from earlier results [22] that $\mathrm{V}^*_{j,\zeta}$ continuously extends to all of $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where we consider spaces of convex functions together with the topology associated with *epi-convergence*, which coincides with pointwise convergence on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ (see, for example, [76, Theorem 7.17]). For this extension of (2.1.7), the measures $[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x$ are replaced by corresponding *Hessian measures* (see Section 2.3.1), which play a fundamental role in PDEs [18, 84]. Notably, when $j = n$, we retrieve the well-known *Monge–Ampère measure* $\mathrm{MA}(v;\cdot)$, which for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$ satisfies

$$\mathrm{dMA}(v;\cdot) = \det(\mathrm{D}^2 v(x)) \,\mathrm{d}x \tag{2.1.8}$$

on $\mathbb{R}^n$. Let us furthermore emphasize that $\mathrm{V}^*_{j,\zeta}(h_K)$ is a multiple of $V_j(K)$ for every $K \in \mathcal{K}^n$, which is a consequence of (2.1.6) and (2.1.7).

Functional intrinsic volumes are motivated by the extension of geometric valuation theory from convex bodies to convex functions. We mention [4, 20, 26, 50–53, 55, 71] as examples of this rapidly developing field, and refer to [16, 42, 43, 54, 56] for connections and applications with the geometry of convex bodies, as well as [28, 30, 61, 67, 73, 77] for related results. Here, a map $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is a *valuation* if

$$\mathrm{Z}(v \vee w) + \mathrm{Z}(v \wedge w) = \mathrm{Z}(v) + \mathrm{Z}(w) \tag{2.1.9}$$

for every $v, w \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ such that also their pointwise maximum $v \vee w$ and minimum $v \wedge w$ are elements of $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Since

$$h_{K\cup L} = h_K \vee h_L \quad \text{and} \quad h_{K\cap L} = h_K \wedge h_L$$

for $K, L, K \cup L \in \mathcal{K}^n$, it follows that (2.1.9) generalizes the classical valuation property (2.1.4). In addition, $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is called *dually epi-translation invariant* if $\mathrm{Z}(v+a) = \mathrm{Z}(v)$ for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and every affine function a on $\mathbb{R}^n$, and *rotation invariant* if $\mathrm{Z}(v \circ \vartheta^{-1}) = \mathrm{Z}(v)$ for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\vartheta \in \mathrm{SO}(n)$. In an effort to classify all such valuations, it was discovered that continuous extensions of (2.1.7) also exist for densities ζ with possible singularities at 0^+ [26, Theorem 1.4]. We therefore speak of *singular Hessian valuations* and denote the associated classes of possible densities by $D^n_j \subset C_b((0,\infty))$ (see (2.4.3) for a definition). Alternative proofs of this existence result were given in [27] using integral geometry and in [24] using mixed Monge–Ampère measures. Furthermore, Knoerr showed in [55, Theorem

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1.5] that $\mathrm{V}^*_{j,\zeta}$ can be expressed as a principal value integral with respect to a Hessian measure. Additional explanations for the appearance of singularities were recently given in [43, Section 5].

Analogous to Hadwiger's characterization theorem (Theorem 2.1.2), it was shown in [26, Theorem 1.5] that combinations of functional intrinsic volumes are the only continuous, dually epi-translation and rotation invariant valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. See also [23].

Theorem 2.1.3 (Hadwiger theorem on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$)**.** *For $n \geq 2$, a map $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is a continuous, dually epi-translation and rotation invariant valuation, if and only if there exist functions $\zeta_0 \in D^n_0, \dots, \zeta_n \in D^n_n$ such that*

$$\mathrm{Z}(v) = \mathrm{V}^*_{0,\zeta_0}(v) + \cdots + \mathrm{V}^*_{n,\zeta_n}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. For $n = 1$, the same representation holds if we replace rotation invariance with reflection invariance.

Here, $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is *reflection invariant* if $\mathrm{Z}(v) = \mathrm{Z}(v^-)$ for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where $v^-(x) = v(-x)$ for $x \in \mathbb{R}^n$. We remark that this is a necessary assumption in the one-dimensional case as otherwise, additional valuations may appear in the representation (see [23, Corollary 6]). Let us furthermore point out that in Theorem 2.1.3 the valuation Z uniquely determines the densities $\zeta_1, \dots, \zeta_n$ but V^*_{0,ζ_0} is constant on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and does not uniquely determine ζ_0.

Functional Minkowski Vectors Recently, the authors introduced *functional Minkowski vectors* [69] which extend the left side of the Minkowski relations (2.1.3) and take the form

$$\mathrm{t}^*_{j,\zeta}(v) = \int_{\mathbb{R}^n} \zeta(|x|)x\,[\mathrm{D}^2 v(x)]_j\,\mathrm{d}x, \quad j \in \{0,\dots,n\}, \tag{2.1.10}$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$, where ζ is a continuous function with compact support on $[0,\infty)$, $\zeta \in C_c([0,\infty))$. Using appropriate Hessian measures, we remark that (2.1.10) continuously extends to $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Given the Minkowski relations and Proposition 2.1.1, the natural question arises as to whether these integrals vanish. Indeed, $\mathrm{t}^*_{0,\zeta} \equiv o$, but the remaining operators in this family are non-trivial. It is straightforward to check that $\mathrm{t}^*_{n,\zeta}(v)$ does not vanish on translates of support functions (see [69, Section 4]) and explicit examples for the remaining cases will be given in Section 2.6.1.

In Theorem A, we show the existence of functional Minkowski vectors with singular densities. For $j \in \{1,\dots,n\}$ let

$$T^n_j = \left\{ \zeta \in C_b((0,\infty)) : \lim_{s\to 0^+} s^{n-j+1}\zeta(s) = 0 \right\}.$$

In the following we say that a vector-valued map $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is *rotation equivariant* if $\mathrm{z}(v \circ \vartheta^{-1}) = \vartheta\,\mathrm{z}(v)$ for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\vartheta \in \mathrm{SO}(n)$. Similarly, we define $\mathrm{O}(n)$ *equivariance*.

Theorem A. *For $j \in \{1,\dots,n\}$ and $\zeta \in T^n_j$, there exists a unique, continuous, dually epi-translation invariant, rotation equivariant valuation $\mathrm{t}^*_{j,\zeta}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ such that*

$$\mathrm{t}^*_{j,\zeta}(v) = \int_{\mathbb{R}^n} \zeta(|x|)x\,[\mathrm{D}^2 v(x)]_j\,\mathrm{d}x$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$. In addition, $\mathrm{t}^*_{j,\zeta}$ is $\mathrm{O}(n)$ equivariant.

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For the case $j = n$, we remark that $\zeta \in T_n^n$ if and only if $x \mapsto \zeta(|x|)x$ continuously extends to a continuous map with compact support on $\mathbb{R}^n$ (see Lemma 2.3.4 below). In particular, this case was already treated in [69].

In the course of proving Theorem A, we establish the following explicit representation of the operators $\mathrm{t}^*_{j,\zeta}$, where we use the Monge–Ampère-type measures $\mathrm{MA}_j(v;\cdot)$, which are defined through the relation

$$\mathrm{MA}(v + rh_{B^n};\cdot) = \sum_{j=0}^{n} \binom{n}{j} r^{n-j} \mathrm{MA}_j(v;\cdot) \tag{2.1.11}$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $r \geq 0$.

Theorem B. *If $j \in \{1,\dots,n\}$ and $\zeta \in T_j^n$, then*

$$\mathrm{t}^*_{j,\zeta}(v) = \int_{\mathbb{R}^n} \alpha(|x|)x \,\mathrm{dMA}_j(v;x)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where $\alpha \in T_n^n$ is given by

$$\alpha(s) = \binom{n}{j}\left(s^{n-j}\zeta(s) + (n-j)\int_s^\infty t^{n-j-1}\zeta(t)\,\mathrm{d}t\right) \tag{2.1.12}$$

for $s > 0$.

Note, that (2.1.11) and Theorem B show that $\mathrm{t}^*_{j,\zeta}(v)$ with $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\zeta \in T_j^n$ appears as a coefficient in a Steiner formula that describes the polynomial expansion of $r \mapsto \mathrm{t}^*_{n,\alpha}(v + rh_{B^n})$ with a suitable $\alpha \in T_n^n$. Conversely, since (2.1.12) is a bijection between T_j^n and T_n^n (see Lemma 2.4.2 below), all coefficients in such a Steiner formula are functional Minkowski vectors. More precisely, if $\alpha \in T_n^n$, then

$$\mathrm{t}^*_{n,\alpha}(v + rh_{B^n}) = \sum_{j=1}^{n} r^{n-j}\,\mathrm{t}^*_{j,\zeta_j}(v) \tag{2.1.13}$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $r \geq 0$, where $\zeta_j \in T_j^n$ is given by

$$\zeta_j(s) = \left(\frac{\alpha(s)}{s^{n-j}} - (n-j)\int_s^\infty \frac{\alpha(t)}{t^{n-j+1}}\,\mathrm{d}t\right)$$

for $s > 0$. In particular, every $\mathrm{t}^*_{j,\zeta_j}(v)$ with $\zeta_j \in T_j^n$ appears exactly once as a coefficient of r^{n-j} on the right side of (2.1.13) as α ranges in T_n^n.

Let us further mention that using recent integral-geometric results for the measures $\mathrm{MA}_j(v;\cdot)$ from [42], we will see in Theorem 2.5.2 that Theorem B is equivalent to a Kubota-type formula, thereby providing an additional representation of functional Minkowski vectors that avoids singular densities.

In our main result, we give a complete characterization of the vector-valued valuations $\mathrm{t}^*_{j,\zeta}$.

Theorem C. *For $n \geq 3$, a map $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is a continuous, dually epi-translation invariant, rotation equivariant valuation, if and only if there exist unique functions $\zeta_1 \in T_1^n,\dots,\zeta_n \in T_n^n$ such that*

$$\mathrm{z}(v) = \mathrm{t}^*_{1,\zeta_1}(v) + \cdots + \mathrm{t}^*_{n,\zeta_n}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. For $n \leq 2$, the same representation holds if we replace rotation equivariance with $\mathrm{O}(n)$ equivariance.

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Let us mention that for $n \leq 2$, $\mathrm{O}(n)$ equivariance is a necessary assumption, as demonstrated in [69, Remark 3.7]. We furthermore emphasize that Theorem C characterizes a new family of operators that emerge exclusively in the broader setting of convex functions. In contrast, the corresponding question for convex bodies, as addressed in Proposition 2.1.1, admits only a trivial solution.

To prove Theorem C, we will first establish a version of this result under additional smoothness assumptions in Theorem 2.6.16. For this purpose, we will also derive another representation of the functional Minkowski vectors in Section 2.5.3, which uses integrals with respect to area measures of associated higher dimensional convex bodies.

From Theorem B together with Lemma 2.3.12 below, we immediately obtain the following equivalent version of Theorem C. Here, $C_c(\mathbb{R}^n;\mathbb{R}^n)$ denotes the set of continuous functions $\xi\colon \mathbb{R}^n \to \mathbb{R}^n$ with compact support and we say that such a function is *rotation equivariant* if $\xi(\vartheta x) = \vartheta\xi(x)$ for every $x \in \mathbb{R}^n$ and $\vartheta \in \mathrm{SO}(n)$. Similarly, we define $\mathrm{O}(n)$ equivariance.

Theorem D. *For $n \geq 3$, a map $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is a continuous, dually epi-translation invariant, rotation equivariant valuation, if and only if there exist unique rotation equivariant functions $\xi_1,\dots,\xi_n \in C_c(\mathbb{R}^n;\mathbb{R}^n)$ such that*

$$\mathrm{z}(v) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} \xi_j(x)\,\mathrm{dMA}_j(v;x)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. For $n \leq 2$, the same representation holds if we replace rotation equivariance with $\mathrm{O}(n)$ equivariance.

Note that the appearance of the measures $\mathrm{MA}_j(v;\cdot)$ in Theorem D is not coincidental. As a recently established family of Kubota-type formulas from [42] shows, these measures are generated from the Monge–Ampère measure (in arbitrary dimensions) together with $\mathrm{SO}(n)$, and naturally encode the action of the rotation group (see Theorem 2.5.1). In view of Theorem B and Theorem D, one might, therefore, think that a representation of functional Minkowski vectors using Hessian measures is superfluous and that singular densities only lead to unnecessary complications. However, we will see in the classification of smooth valuations in Theorem 2.6.16 that it is natural to work with Hessian measures, and in particular, our approach relies on this representation.

The proofs of our main results build upon and extend techniques developed in [24–27], where functional intrinsic volumes were studied. While our strategy shares greater similarities with these works in proving existence, for the characterization result we incorporate several new key ideas beyond the existing framework. Notably, we use a new Klain–Schneider theorem, which we recently established in [69], and results on the supports of continuous, dually epi-translation invariant valuations due to [50]. Furthermore, compared with the scalar case, additional complexities arise in the vector-valued setting, necessitating more involved proofs. Particularly, since functional Minkowski vectors vanish on radially symmetric functions, recovering the density ζ from $\mathrm{t}^*_{j,\zeta}$ is significantly more intricate than for the operators $\mathrm{V}^*_{j,\zeta}$. In addition, restricting vector-valued valuations to a lower-dimensional setting introduces further challenges, which we address using the functional Hadwiger theorem in an equivalent form.

Overview of This Article Dual to the setting described above, we consider valuations on supercoercive convex functions in Section 2.2. After that, we collect preliminary results in Section 2.3. We then prove the existence results, Theorem A and Theorem B, in Section 2.4. Following this, we establish further representations of functional Minkowski vectors in Section 2.5, which we will then use for the proof of their characterization, Theorem C and Theorem D, in Section 2.6.

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2.2 Dual Results

For a function $w\colon \mathbb{R}^n \to [-\infty,\infty]$ we denote by

$$w^*(x) = \sup_{y\in\mathbb{R}^n} \big(\langle y,x\rangle - w(y)\big), \quad x\in\mathbb{R}^n,$$

its *convex conjugate* or *Legendre–Fenchel transform*. Taking the convex conjugates of the functions in $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ results in

$$\begin{aligned}\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) &= \{v^* : v\in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})\} \\ &= \left\{u\colon \mathbb{R}^n\to(-\infty,\infty] : u \text{ is l.s.c., convex, } u\not\equiv\infty, \lim_{|x|\to\infty}\frac{u(x)}{|x|}=\infty\right\},\end{aligned} \tag{2.2.1}$$

the space of proper, lower semicontinuous, super-coercive convex functions on $\mathbb{R}^n$. We also equip $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ with the topology associated with epi-convergence and refer to [21, Lemma 5] and [8, Theorem 3.1] for an easy description of epi-convergence on this space. Equivalently to (2.2.1), $u\in\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, if and only if $u^*\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and, in particular, convex conjugation is a continuous bijection between these two spaces (see [76, Theorem 11.34] for the continuity). Valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ are defined analogously to (2.1.9) and, by [22, Proposition 3.5], $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\to\mathbb{R}^n$ is a valuation if and only if $v\mapsto \mathrm{z}^*(v)=\mathrm{z}(v^*)$ is a valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Furthermore, z is called *epi-translation invariant* if and only if z^* is dually epi-translation invariant, or, equivalently,

$$\mathrm{z}(u\circ\tau^{-1}+c)=\mathrm{z}(u)$$

for every translation τ on $\mathbb{R}^n$, $c\in\mathbb{R}$ and $u\in\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Observe that this means that $u\mapsto \mathrm{z}(u)$ is invariant under translations of the epi-graph of u in $\mathbb{R}^{n+1}$. Lastly, rotation equivariance and $\mathrm{O}(n)$ equivariance of a valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ are defined as on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, which is compatible with the properties of convex conjugation.

Considering the above, we note that studying valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is equivalent to studying valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. In particular, it is often advantageous to switch between these two spaces, and therefore, we now state the equivalent dual results to our main results from Section 2.1. We begin with the existence results for functional Minkowski vectors on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, which are equivalent to Theorem A and Theorem B.

Theorem A*. *For $j\in\{1,\dots,n\}$ and $\zeta\in T_j^n$, there exists a unique, continuous, epi-translation invariant, rotation equivariant valuation $\mathrm{t}_{j,\zeta}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\to\mathbb{R}^n$ such that*

$$\mathrm{t}_{j,\zeta}(u)=\int_{\mathbb{R}^n}\zeta(|\nabla u(x)|)\nabla u(x)\,[\mathrm{D}^2u(x)]_{n-j}\,\mathrm{d}x$$

for every $u\in\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\cap C^2_+(\mathbb{R}^n)$. In addition, $\mathrm{t}_{j,\zeta}$ is $\mathrm{O}(n)$ equivariant.

Here, $C^2_+(\mathbb{R}^n)$ denotes the set of all functions in $C^2(\mathbb{R}^n)$ with a positive definite Hessian matrix. We remark that $u\in\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\cap C^2_+(\mathbb{R}^n)$, if and only if $u^*\in\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\cap C^2_+(\mathbb{R}^n)\subseteq\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. In particular, $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\cap C^2_+(\mathbb{R}^n)$ is, therefore, a proper subset of the convex conjugates of the elements from $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})\cap C^2(\mathbb{R}^n)$, and the equivalence of Theorem A and Theorem A* follows from the fact that $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\cap C^2_+(\mathbb{R}^n)$ is a dense subset of $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Theorem B*. *If $j\in\{1,\dots,n\}$ and $\zeta\in T_j^n$, then*

$$\mathrm{t}_{j,\zeta}(u)=\int_{\mathbb{R}^n}\alpha(|y|)y\,\mathrm{dMA}^*_j(u;x)$$

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for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*, where* $\alpha \in T_n^n$ *is given by*

$$\alpha(s) = \binom{n}{j}\left(s^{n-j}\zeta(s) + (n-j)\int_s^\infty t^{n-j-1}\zeta(t)\,\mathrm{d}t\right)$$

for $s > 0$.

Here, we use the measures $\mathrm{MA}_j^*(u;\cdot) = \mathrm{MA}_j(u^*;\cdot)$ for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $j \in \{1,\dots,n\}$. For more details, see (2.3.13). We refer to [24, Theorem 5.5] for an alternative description.

Next, we state equivalent formulations of Theorem C and Theorem D concerning the classification of vector-valued valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Theorem C*. *For* $n \geq 3$*, a map* $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ *is a continuous, epi-translation invariant, rotation equivariant valuation, if and only if there exist unique functions* $\zeta_1 \in T_1^n, \dots, \zeta_n \in T_n^n$ *such that*

$$\mathrm{z}(u) = \mathrm{t}_{1,\zeta_1}(u) + \cdots + \mathrm{t}_{n,\zeta_n}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*. For* $n \leq 2$*, the same representation holds if we replace rotation equivariance with* $\mathrm{O}(n)$ *equivariance.*

Theorem D*. *For* $n \geq 3$*, a map* $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ *is a continuous, epi-translation invariant, rotation equivariant valuation, if and only if there exist unique rotation equivariant functions* $\xi_1, \dots, \xi_n \in C_c(\mathbb{R}^n;\mathbb{R}^n)$ *such that*

$$\mathrm{z}(u) = \sum_{j=1}^n \int_{\mathbb{R}^n} \xi_j(x)\,\mathrm{dMA}_j^*(v;x)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*. For* $n \leq 2$*, the same representation holds if we replace rotation equivariance with* $\mathrm{O}(n)$ *equivariance.*

2.3 Preliminaries

In this section, we explain relevant notations and results. In addition, we prove some smaller preliminary statements. Apart from the references listed below, we refer to the monographs [76,79] as basic references. Concerning the general notation and in addition to the definitions of Section 2.1, we write e_i for the ith vector of the standard orthonormal basis of $\mathbb{R}^n$. Furthermore, κ_m is the m-dimensional volume of the Euclidean unit ball B^m in $\mathbb{R}^m$, with the convention $\kappa_0 = 1$.

2.3.1 Hessian Measures

For a convex function $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ let

$$\partial v(x) = \{y \in \mathbb{R}^n : v(z) \geq v(x) + \langle y, z-x\rangle\}$$

denote the *subdifferential* of v at $x \in \mathbb{R}^n$. Note that if v is differentiable at x with gradient $\nabla v(x)$, then $\partial v(x) = \{\nabla v(x)\}$. The Monge–Ampère measures associated with v is now defined as

$$\mathrm{MA}(v;B) = \mathcal{H}^n\big(\{y : \exists\, x \in B \text{ such that } y \in \partial v(x)\}\big) \tag{2.3.1}$$

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for every Borel set $B \subset \mathbb{R}^n$ (cf. [31, Definition 2.1]). Under C^2 assumptions on v this gives (2.1.8). More generally, we associate with each $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ the family of *Hessian measures* $\Phi_j(v;\cdot)$, $j \in \{0,\dots,n\}$, which are defined through the relation

$$\mathcal{H}^n(\{x+ry : x \in B, y \in \partial v(x)\}) = \sum_{j=0}^{n} r^j \, \Phi_j(v;B) \tag{2.3.2}$$

for $r \geq 0$, where $B \subset \mathbb{R}^n$ is again an arbitrary Borel set. In particular, this defines a family of non-negative, locally finite Borel measures on $\mathbb{R}^n$. Note, that $\Phi_n(v;\cdot) = \mathrm{MA}(v;\cdot)$. When v is of class C^2 on some open set $O \subseteq \mathbb{R}^n$, then $\Phi_j(v;\cdot)$ is absolutely continuous w.r.t. the Lebesgue measure with

$$\mathrm{d}\Phi_j(v;x) = [\mathrm{D}^2 v(x)]_j \, \mathrm{d}x \tag{2.3.3}$$

on O, which generalizes (2.1.8). Cf. [22, Section 8.5]. See also [19, 84]. Let us point out that (2.3.3) and the weak continuity of Hessian measures with respect to epi-convergence show that these measures are *locally determined*. This means that if $v_1, v_2 \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ are such that $v_1 \equiv v_2$ on some open set $O \subseteq \mathbb{R}^n$, then

$$\Phi_j(v_1; B \cap O) = \Phi_j(v_2; B \cap O) \tag{2.3.4}$$

for every relatively compact Borel set $B \subset \mathbb{R}^n$. See also [53].

If E is a k-dimensional linear subspace of $\mathbb{R}^n$, we denote by $\mathrm{Conv}(E;\mathbb{R})$ the set of all convex functions $v\colon E \to \mathbb{R}$. To avoid confusion and in case necessary, we will write $\Phi_j^{(k)}(v;\cdot)$ for $v \in \mathrm{Conv}(E;\mathbb{R})$, where the superscript indicates the dimension of the ambient space on which we consider a Hessian measure. The following result on product measures was shown in [21, Lemma 4.8]

Lemma 2.3.1. *If E and F are orthogonal, complementary subspaces of $\mathbb{R}^n$ with $k = \dim E$, then*

$$\Phi_l^{(n)}(v_E + v_F;\cdot) = \sum_{i=0\vee(l-k-n)}^{k\wedge l} \Phi_i^{(k)}(v_E;\cdot) \otimes \Phi_{l-i}^{(n-k)}(v_F;\cdot)$$

for every $v_E \in \mathrm{Conv}(E;\mathbb{R})$, $v_F \in \mathrm{Conv}(F;\mathbb{R})$, and $l \in \{0,\dots,n\}$, where $(v_E+v_F)(x_E+x_F) = v_E(x_E)+v_F(x_F)$ for $x_E \in E$, $x_F \in F$.

The following consequence of the last result was used, for example, in the proof of [25, Lemma 2.9].

Lemma 2.3.2. *Let $k \in \{1,\dots,n-1\}$, $j \in \{0,\dots,k\}$, and $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. If there exists $w \in \mathrm{Conv}(\mathbb{R}^k;\mathbb{R})$ such that*

$$v(x_1,\dots,x_n) = w(x_1,\dots,x_k)$$

for every $(x_1,\dots,x_n) \in \mathbb{R}^n$, then

$$\mathrm{d}\Phi_j(v;(x_1,\dots,x_n)) = \mathrm{d}\Phi_j^{(k)}(w;(x_1,\dots,x_k))\,\mathrm{d}x_{k+1}\cdots\mathrm{d}x_n.$$

Proof. The statement immediately follows by applying Lemma 2.3.1 with $E = \mathbb{R}^k$, $F = (\mathbb{R}^k)^\perp$, $v_E = w$,and $v_F \equiv 0$, together with the observation that $\Phi_m^{(n-k)}(v_F;\cdot) \equiv 0$ for every $m \in \{1,\dots,n-k\}$, which is an implication of (2.3.3). □

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For a super-coercive convex function $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ we consider the measures

$$\Psi_j(u;\cdot) = \Phi_j(u^*;\cdot) \tag{2.3.5}$$

with $j \in \{0,\dots,n\}$. When u is of class C^2_+, then

$$\int_{\mathbb{R}^n} b(y)\,\mathrm{d}\Psi_j(u;y) = \int_{\mathbb{R}^n} b(\nabla u(x))[\mathrm{D}^2 u(x)]_{n-j}\,\mathrm{d}x \tag{2.3.6}$$

for every measurable $b\colon \mathbb{R}^n \to \mathbb{R}$ such that the integral above exists (cf. [22, Theorem 8.2]). For the special case $j = n$ we also write $\Psi_n(u;\cdot) = \mathrm{MA}^*(u;\cdot) = \mathrm{MA}(u^*;\cdot)$, which we call the *conjugate Monge–Ampère measure* associated with u. Similar to (2.3.6), we have

$$\int_{\mathbb{R}^n} b(y)\,\mathrm{dMA}^*(u;y) = \int_{\mathrm{dom}(u)} b(\nabla u(x))\,\mathrm{d}x \tag{2.3.7}$$

for every measurable $b\colon \mathbb{R}^n \to \mathbb{R}$ and every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ without additional C^2 assumptions, where $\mathrm{dom}(u) = \{x \in \mathbb{R}^n : u(x) < \infty\}$ denotes the *domain* of u. Here, we also use the fact that every convex function is differentiable almost everywhere on the interior of its domain.

2.3.2 Mixed Monge–Ampère Measures

The *mixed Monge–Ampère measure* $\mathrm{MA}(v_1,\dots,v_n;\cdot)$ associated with $v_1,\dots,v_n \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is the unique measure that is symmetric in its entries, and that satisfies

$$\mathrm{MA}(\lambda_1 w_1 + \cdots + \lambda_m w_m;\cdot) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1}\cdots\lambda_{i_n}\mathrm{MA}(w_{i_1},\dots,w_{i_n};\cdot)$$

for every $w_1,\dots,w_m \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, $\lambda_1,\dots,\lambda_m \geq 0$, and $m \in \mathbb{N}$. This measure generalizes the Monge–Ampère measure via

$$\mathrm{MA}(v,\dots,v;\cdot) = \mathrm{MA}(v;\cdot)$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. If $v_1,\dots,v_n \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ are of class C^2 on some open set $O \subseteq \mathbb{R}^n$, then $\mathrm{MA}(v_1,\dots,v_n;\cdot)$ is absolutely continuous w.r.t. the Lesbesgue measure such that

$$\mathrm{dMA}(v_1,\dots,v_n;\cdot) = D(\mathrm{D}^2 v_1(x),\dots,\mathrm{D}^2 v_n(x))\,\mathrm{d}x \tag{2.3.8}$$

on O, where for symmetric matrices $A_1,\dots,A_n \in \mathbb{R}^{n\times n}$ we write $D(A_1,\dots,A_n)$ for their *mixed discriminant*. See [24, Theorem 4.3].

The measures $\mathrm{MA}_j(v;\cdot)$ with $j \in \{0,\dots,n\}$ and $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, which were defined in (2.1.11), are special cases of mixed Monge–Ampère measures since

$$\mathrm{MA}_j(v;\cdot) = \mathrm{MA}(v[j], h_{B^n}[n-j];\cdot). \tag{2.3.9}$$

Here, $v[j]$ means that the entry v is repeated j times and, similarly, h_{B^n} is repeated $(n-j)$ times. Furthermore, denoting the $n \times n$ identity matrix by I_n, we have

$$[A]_j = \binom{n}{j} D(A[j], I_n[n-j]) \tag{2.3.10}$$

for every symmetric $A \in \mathbb{R}^{n\times n}$ and $j \in \{0,\dots,n\}$. Thus, we also retrieve

$$\Phi_j(v;\cdot) = \binom{n}{j}\mathrm{MA}(v[j], \tfrac{1}{2}h^2_{B^n}[n-j];\cdot) \tag{2.3.11}$$

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for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where we have used that $\mathrm{D}^2\frac{1}{2}h^2_{B^n}(x) = \mathrm{D}^2\frac{1}{2}|x|^2 = I_n$ for $x \in \mathbb{R}^n$. See [24, Section 4.2].

The analog of the mixed Monge–Ampère measure on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is the *conjugate mixed Monge–Ampère measure* $\mathrm{MA}^*(u_1,\dots,u_n;\cdot)$ which we associate to $u_1,\dots,u_n \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. This is the unique measure that is symmetric in its entries such that

$$\mathrm{MA}^*((\lambda_1 \cdot w_1) \boxdot \cdots \boxdot (\lambda_m \cdot w_m);\cdot) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1}\cdots\lambda_{i_n}\mathrm{MA}^*(w_{i_1},\dots,w_{i_n};\cdot)$$

for every $w_1,\dots,w_m \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, $\lambda_1,\dots,\lambda_m \geq 0$, and $m \in \mathbb{N}$. Here, $u_1 \boxdot u_2 \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is the *infimal convolution* or *epi-sum* of $u_1, u_2 \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, that is,

$$(u_1 \boxdot u_2)(x) = \inf_{y\in\mathbb{R}^n}\big(u_1(x-y) + u_2(y)\big)$$

for $x \in \mathbb{R}^n$, which means that $u_1 \boxdot u_2 = (u_1^* + u_2^*)^*$. Furthermore, $\lambda \cdot u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ denotes the *epi-multiplication* of $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ with $\lambda \geq 0$, which can be defined as $\lambda \cdot u = (\lambda u^*)^*$. Equivalently, for positive λ we have

$$\big(\lambda \cdot u\big)(x) = \lambda\, u\left(\tfrac{x}{\lambda}\right)$$

for $x \in \mathbb{R}^n$. In particular, it is now straightforward to see that

$$\mathrm{MA}^*(u_1,\dots,u_n;\cdot) = \mathrm{MA}(u_1^*,\dots,u_n^*;\cdot)$$

for $u_1,\dots,u_n \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

The convex conjugate of the support function h_K with $K \in \mathcal{K}^n$ is the (convex) *indicator function* $\mathbf{I}_K \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ of K,

$$\mathbf{I}_K(x) = (h_K)^*(x) = \begin{cases} 0 & \text{if } x \in K \\ \infty & \text{else.} \end{cases} \tag{2.3.12}$$

Thus, by (2.1.11), the measures

$$\mathrm{MA}^*_j(u;\cdot) = \mathrm{MA}_j(u^*;\cdot) = \mathrm{MA}(u^*[j], h_{B^n}[n-j];\cdot) = \mathrm{MA}^*(u[j], \mathbf{I}_{B^n}[n-j];\cdot)$$

with $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $j \in \{0,\dots,n\}$ are equivalently obtained from the Steiner formula

$$\mathrm{MA}^*(u \boxdot (r \cdot \mathbf{I}_{B^n});\cdot) = \sum_{j=0}^{n}\binom{n}{j} r^{n-j}\mathrm{MA}^*_j(u;\cdot) \tag{2.3.13}$$

for $r \geq 0$.

2.3.3 Valuations on Convex Functions

Valuations on the spaces $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ were already introduced in Section 2.1 and Section 2.2. A valuation $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is called *epi-homogeneous* of degree $s \in \mathbb{R}$ if

$$\mathrm{z}(\lambda \cdot u) = \lambda^s\, \mathrm{z}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\lambda > 0$. Equivalently, the dual valuation z^* on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, that is $\mathrm{z}^*(v) = \mathrm{z}(v^*)$ for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, is *homogeneous* of the same degree, which means that

$$\mathrm{z}^*(\lambda v) = \lambda^s\, \mathrm{z}^*(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\lambda > 0$. Examples of homogeneous valuations are given in the next result due to [22, Proposition 4.4].

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Proposition 2.3.3. *Let $j \in \{0,\dots,n\}$. If $\beta \in C_c(\mathbb{R}^n)$ and $v_1,\dots,v_{n-j} \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, then*

$$v \mapsto \int_{\mathbb{R}^n} \beta(x)\,\mathrm{dMA}(v[j], v_1,\dots,v_{n-j};x)$$

defines a continuous, dually epi-translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ that is homogeneous of degree j.

We will use Proposition 2.3.3 together with the following consequence of [69, Lemma 3.4].

Lemma 2.3.4. *If $\alpha \in T_n^n$, then $\xi(x) = \alpha(|x|)x$ continuously extends from $\mathbb{R}^n \setminus \{o\}$ to $\mathbb{R}^n$ with $\xi(o) = o$. In particular, $\xi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$.*

We can now establish the needed properties of the valuations that are relevant to this article.

Proposition 2.3.5. *Let $j \in \{0,\dots,n\}$. If $\alpha \in T_n^n$, then*

$$v \mapsto \int_{\mathbb{R}^n} \alpha(|x|)x\ \mathrm{dMA}_j(v;x) \quad \text{and} \quad v \mapsto \int_{\mathbb{R}^n} \alpha(|x|)x\ \mathrm{d}\Phi_j(v;x) \tag{2.3.14}$$

define continuous, dually epi-translation invariant, $\mathrm{O}(n)$ equivariant, vector-valued valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ that are homogeneous of degree j.

Proof. It follows from (2.3.9) and (2.3.11) together with Proposition 2.3.3, applied coordinate-wise, as well as Lemma 2.3.4 that the maps in (2.3.14) define continuous, dually epi-translation invariant valuations that are homogeneous of degree j. Next, for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\vartheta \in \mathrm{O}(n)$ it trivially follows from (2.1.11) and (2.3.1) that

$$\mathrm{MA}_j(v\circ\vartheta^{-1};B) = \mathrm{MA}_j(v;\vartheta^{-1}B)$$

for every Borel set $B \subset \mathbb{R}^n$. Thus,

$$\int_{\mathbb{R}^n} \alpha(|x|)x\,\mathrm{dMA}_j(v\circ\vartheta^{-1};x) = \int_{\mathbb{R}^n} \alpha(|\vartheta x|)\vartheta x\,\mathrm{dMA}_j(v;x) = \vartheta\int_{\mathbb{R}^n} \alpha(|x|)x\ \mathrm{dMA}_j(v;x),$$

which shows the claimed $\mathrm{O}(n)$ equivariance of the first map in (2.3.14). The $\mathrm{O}(n)$ equivariance of the integral with respect to $\Phi_j(v;\cdot)$ is shown analogously. □

We also need an equivalent version of the last result for valuations on super-coercive convex functions.

Proposition 2.3.6. *Let $j \in \{0,\dots,n\}$. If $\alpha \in T_n^n$, then*

$$u \mapsto \int_{\mathbb{R}^n} \alpha(|y|)y\ \mathrm{dMA}_j^*(u;x) \quad \text{and} \quad u \mapsto \int_{\mathbb{R}^n} \alpha(|y|)y\ \mathrm{d}\Psi_j(u;x)$$

define continuous, epi-translation invariant, $\mathrm{O}(n)$ equivariant, vector-valued valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ that are epi-homogeneous of degree j.

The following decomposition result for real-valued valuations was shown in [23, Theorem 4] and later also in [50], where it was stated in a more general form. Let us emphasize that the version below can be trivially applied coordinate-wise to vector-valued valuations.

Theorem 2.3.7. *If $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is a continuous, dually epi-translation invariant valuation, then there exist continuous, dually epi-translation invariant valuations $\mathrm{Z}_0,\dots,\mathrm{Z}_n\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ such that Z_j is homogeneous of degree j, $j \in \{0,\dots,n\}$, and $\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n$.*

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The equivalent version on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ reads as follows.

Theorem 2.3.8. *If* $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation invariant valuation, then there exist continuous, epi-translation invariant valuations* $\mathrm{Z}_0, \dots, \mathrm{Z}_n\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *such that* Z_j *is epi-homogeneous of degree* j*,* $j \in \{0, \dots, n\}$*, and* $\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n$*.*

Classifications of vector-valued valuations of extremal degrees were established in [69]. We start with the following consequence of [69, Theorem 5.4], where the equivalent setting of valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ was considered and where we say that $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}) \to \mathbb{R}^1$ is *reflection equivariant* if z is $\mathrm{O}(1)$ equivariant.

Theorem 2.3.9. *For* $n \geq 2$*, a map* $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ *is a continuous, epi-translation invariant, rotation equivariant valuation that is epi-homogeneous of degree* 0*, if and only if it is identically* o*. For* $n = 1$*, the same representation holds if* z *is reflection equivariant.*

Next, we state [69, Theorem 5.5], which concerns valuations of top degree. Here, we also want to point out that if $\zeta \in T_n^n$, then

$$\mathrm{t}_{n,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|y|)y \,\mathrm{d}\Psi_n(u;y)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Theorem 2.3.10. *For* $n \geq 3$*, a map* $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ *is a continuous, epi-translation invariant, rotation equivariant valuation that is epi-homogeneous of degree* n*, if and only if there exists* $\zeta \in T_n^n$ *such that*

$$\mathrm{z}(u) = \mathrm{t}_{n,\zeta}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*. For* $n \leq 2$*, the same representation holds if we replace rotation equivariance with* $\mathrm{O}(n)$ *equivariance.*

The following result is a consequence of a Klain–Schneider theorem for vector-valued valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, which is due to [69, Theorem 3.6 and Remark 3.7]. Here, we say that $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is *simple* if $\mathrm{z}(u) = o$ for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ such that $\dim(\mathrm{dom}(u)) < n$.

Theorem 2.3.11. *For* $n \geq 3$*, a map* $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ *is a continuous, epi-translation invariant, rotation equivariant, simple valuation, if and only if there exists* $\zeta \in T_n^n$ *such that*

$$\mathrm{z}(u) = \mathrm{t}_{n,\zeta}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*. For* $n \leq 2$*, the same representation holds if we replace rotation equivariance with* $\mathrm{O}(n)$ *equivariance.*

Let us close this section with the following simple result due to [69, Lemma 3.5].

Lemma 2.3.12. *Let* $\xi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$*. For* $n \geq 3$*, the map* ξ *satisfies*

$$\xi(\vartheta x) = \vartheta\xi(x) \tag{2.3.15}$$

for every $\vartheta \in \mathrm{SO}(n)$ *and* $x \in \mathbb{R}^n \setminus \{o\}$*, if and only if there exists* $\alpha \in T_n^n$ *such that*

$$\xi(x) = \alpha(|x|)x$$

for every $x \in \mathbb{R}^n \setminus \{o\}$*. For* $n \leq 2$*, the same representation holds if* ξ *is assumed to commute with* $\mathrm{O}(n)$ *instead.*

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2.4 Existence

In this section, we prove Theorem A, that is, the existence of the operators $\mathrm{t}^*_{j,\zeta}$, $\zeta \in T_j^n$, in terms of singular Hessian integrals. This goes hand in hand with establishing Theorem B.

2.4.1 Integral Transforms Between the Classes T_j^n

For $\zeta \in C_b((0,\infty))$ and $s > 0$, let

$$\mathcal{R}\,\zeta(s) = s\zeta(s) + \int_s^\infty \zeta(t)\,\mathrm{d}t.$$

It is easy to see that also $\mathcal{R}\,\zeta \in C_b((0,\infty))$ and this integral transform was introduced in [27] in the context of Cauchy–Kubota formulas for functional intrinsic volumes. For $l \in \mathbb{N}$ we denote

$$\mathcal{R}^l\,\zeta = \underbrace{(\mathcal{R}\circ\cdots\circ\mathcal{R})}_{l}\zeta$$

and set $\mathcal{R}^0\,\zeta = \zeta$. It was shown in [27, Lemma 3.6] that

$$\mathcal{R}^l\,\zeta(s) = s^l\zeta(s) + l\int_s^\infty t^{l-1}\zeta(t)\,\mathrm{d}t \tag{2.4.1}$$

for $s > 0$. Using integration by parts (cf. [27, Lemma 3.8] and its proof), it is furthermore straightforward to check that the inverse operation to $\mathcal{R}^l$ is given by

$$\mathcal{R}^{-l}\,\rho(s) = (\mathcal{R}^{-1})^l\rho(s) = \frac{\rho(s)}{s^l} - l\int_s^\infty \frac{\rho(t)}{t^{l+1}}\,\mathrm{d}t \tag{2.4.2}$$

for $s > 0$, where $\rho \in C_b((0,\infty))$. Notably, for $j \in \{0,\dots,n\}$ and $l \in \{0,\dots,n-j\}$, the transform $\mathcal{R}^l$ is a bijection between D_j^n and D_j^{n-l} [27, Lemma 3.8], where

$$D_j^n = \left\{\zeta \in C_b((0,\infty)) : \lim_{s\to 0^+} s^{n-j}\zeta(s) = 0, \lim_{s\to 0}\int_s^\infty t^{n-j-1}\zeta(t)\,\mathrm{d}t \text{ exists and is finite}\right\} \tag{2.4.3}$$

for $j \in \{0,\dots,n-1\}$. In addition, D_n^n is the set of $\zeta \in C_b((0,\infty))$ such that $\zeta(0) = \lim_{s\to 0^+}\zeta(s)$ exists and is finite. In the following, we want to show that $\mathcal{R}^l$ is also a bijection between T_j^n and T_j^{n-l}. To this end, we first need the following result, the proof of which works in the same way as the proof of [27, Lemma 3.7]. Recall that for $j \in \{1,\dots,n\}$ the set T_j^n consists of all $\zeta \in C_b((0,\infty))$ such that $\lim_{s\to 0^+} s^{n-j+1}\zeta(s) = 0$.

Lemma 2.4.1. *Let $j \in \{1,\dots,n-1\}$. If $\zeta \in T_j^n$, then*

$$\lim_{s\to 0^+} s^{n-j}\int_s^\infty \zeta(t)\,\mathrm{d}t = 0. \tag{2.4.4}$$

Furthermore, if $\rho \in T_j^{n-1}$, then

$$\lim_{s\to 0^+} s^{n-j+1}\int_s^\infty \frac{\rho(t)}{t^2}\,\mathrm{d}t = 0.$$

Proof. Let $\zeta \in T_j^n$. If $\lim_{s\to 0^+}\int_s^\infty \zeta(s)\,\mathrm{d}s$ exists and is finite, then (2.4.4) is immediate. In the other case, it follows from L'Hospital's rule and the definition of T_j^n that

$$\lim_{s\to 0^+}\left|s^{n-j}\int_s^\infty \zeta(t)\,\mathrm{d}t\right| \le \lim_{s\to 0^+}\frac{\int_s^\infty |\zeta(t)|\,\mathrm{d}t}{\frac{1}{s^{n-j}}} = \lim_{s\to 0^+}\frac{|\zeta(s)|}{\frac{n-j}{s^{n-j+1}}} = \lim_{s\to 0^+}\frac{|s^{n-j+1}\zeta(s)|}{n-j} = 0.$$

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Next, let $\rho \in T_j^{n-1}$. Again, if $\lim_{s\to 0^+} \int_s^\infty \frac{\rho(t)}{t^2}\,\mathrm{d}t$ exists and is finite, then the statement is trivial. Otherwise, we argue similarly to above to obtain

$$\lim_{s\to 0^+} \left| s^{n-j+1} \int_s^\infty \frac{\rho(t)}{t^2}\,\mathrm{d}t \right| \leq \lim_{s\to 0^+} \frac{\int_s^\infty \left|\frac{\rho(t)}{t^2}\right| \mathrm{d}t}{\frac{1}{s^{n-j+1}}} = \lim_{s\to 0^+} \frac{\left|\frac{\rho(s)}{s^2}\right|}{\frac{n-j+1}{s^{n-j+2}}} = \lim_{s\to 0^+} \frac{|s^{n-j}\rho(s)|}{n-j+1} = 0.$$

□

We can now establish the relevant properties of the integral transform $\mathcal{R}$.

Lemma 2.4.2. *For $j \in \{1,\dots,n\}$ and $l \in \{0,\dots,n-j\}$, the map $\mathcal{R}^l \colon T_j^n \to T_j^{n-l}$ is a bijection.*

Proof. Since we already know that $\mathcal{R}^l$ is invertible, it remains to prove that $\mathcal{R}^l$ takes elements from T_j^n to T_j^{n-l} and, similarly, that (2.4.2) maps from T_j^{n-l} to T_j^n. Furthermore, the case $j = n$ (and thus $l = 0$) is trivial, and we therefore assume that $j \in \{1,\dots,n-1\}$.

We start by showing that $\zeta \in T_j^n$ implies $\mathcal{R}\,\zeta \in T_j^{n-1}$. First, we note that $\mathcal{R}\,\zeta$ is clearly a continuous function with bounded support. Since the definition of T_j^n and Lemma 2.4.1 show that

$$\lim_{s\to 0^+} s^{n-j}\,\mathcal{R}\,\zeta(s) = \lim_{s\to 0^+} \left(s^{n-j+1}\zeta(s) + s^{n-j} \int_s^\infty \zeta(t)\,\mathrm{d}t \right) = 0,$$

it follows that $\mathcal{R}\,\zeta \in T_j^{n-1}$.

Next, let $\rho \in T_j^{n-1}$ be given. We need to show that $\mathcal{R}^{-1}\,\rho \in T_j^n$. Again, it is straightforward to see that $\mathcal{R}^{-1}\,\rho \in C_b((0,\infty))$. Furthermore, it follows from the definition of T_j^{n-1} together with the second part of Lemma 2.4.1 that

$$\lim_{s\to 0^+} s^{n-j+1}\,\mathcal{R}^{-1}\,\rho(s) = \lim_{s\to 0^+} \left(s^{n-j}\rho(s) - s^{n-j+1} \int_s^\infty \frac{\rho(t)}{t^2}\,\mathrm{d}t \right) = 0.$$

Hence, $\mathcal{R}^{-1}\,\rho \in T_j^n$.

Finally, since the definition of T_j^n only depends on the difference $(n-j)$, it easily follows by induction that $\mathcal{R}^l\,\zeta \in T_j^{n-l}$ for $\zeta \in T_j^n$ and $l \in \{0,\dots,n-j\}$, where the case $l = 0$ is trivial. Similarly, we conclude that $\mathcal{R}^{-l}$ maps elements from T_j^{n-l} to T_j^n. □

Remark 2.4.3. While the results of this section are similar to those of [27, Section 3.2], the authors are not aware that the former follow from the latter. In this context, it should be noted that $D_j^n \subset T_j^n$ and $D_j^{n+1} \subset T_j^n$ clearly hold, and we emphasize that these inclusions are strict.

2.4.2 Connecting Hessian Measures and Mixed Monge–Ampère Measures

The purpose of this section is to show how integrals of $x \mapsto \zeta(|x|)x$ with respect to the Hessian measures $\Phi_j(v;\cdot)$ can be rewritten in terms of the mixed Monge–Ampère measures $\mathrm{MA}_j(v;\cdot)$. We note that results of this type have been established for functional intrinsic volumes in [24, Section 6] and that this section is based on the same approach.

We will frequently use the support function of the Euclidean unit ball, $h_{B^n}(x) = |x|$ for $x \in \mathbb{R}^n$. Let us remark that this function is twice continuously differentiable on $\mathbb{R}^n \setminus \{o\}$ with

$$\mathrm{D}^2 h_{B^n}(x) = \frac{1}{|x|}\left(I_n - \frac{x}{|x|} \otimes \frac{x}{|x|} \right)$$

for $x \neq o$, where $y \otimes z$ denotes the tensor product of $y, z \in \mathbb{R}^n$. For the next result, which is due to [24, Lemma 6.4], we therefore emphasize that integration with respect to the Lebesgue measure on $\mathbb{R}^n$ is considered throughout the following and that it is sufficient for integrands to be defined a.e. on $\mathbb{R}^n$.

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Lemma 2.4.4. *Let $k \in \{1, \dots, n\}$. If $\varphi \in C_b((0,\infty))$ is such that $\lim_{s\to 0^+} s^{k-1}\varphi(s)$ exists and is finite, then the integral*

$$\int_{\mathbb{R}^n} \left|\varphi(|x|)\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k])\right| \mathrm{d}x$$

is well-defined and finite for every $v_1, \dots, v_k \in C^2(\mathbb{R}^n)$.

A straightforward consequence of Lemma 2.4.4 is the following variant of this result, which shows that the relevant integrals of this section are well-defined.

Lemma 2.4.5. *Let $k \in \{1, \dots, n-1\}$. If $\psi \in C_b((0,\infty))$ is such that $\lim_{s\to 0^+} s^{k}\psi(s)$ exists and is finite, then the integral*

$$\int_{\mathbb{R}^n} \left|\psi(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k])\right| \mathrm{d}x$$

is well-defined and finite for every $v_1, \dots, v_k \in C^2(\mathbb{R}^n)$.

Proof. Let $\psi \in C_b((0,\infty))$ be as in the statement and set $\varphi(s) = s\psi(s)$ for $s > 0$. Clearly, $\lim_{s\to 0^+} s^{k-1}\varphi(s) = \lim_{s\to 0^+} s^{k}\psi(s)$ and thus, the result follows from Lemma 2.4.4. □

Next, we show that we can replace $\mathrm{D}^2 h_{B^n}$ by $\mathrm{D}^2 \frac{1}{2} h^2_{B^n} = I_n$ in the mixed discriminant, if in addition we replace the density ψ with $\mathcal{R}^{-1}\psi$. For this, we first need two lemmas. We start with [24, Lemma 6.2].

Lemma 2.4.6. *If $v_0, \dots, v_n \in C^2(\mathbb{R}^n)$ are such that at least one of the functions has compact support, then*

$$\int_{\mathbb{R}^n} v_0(x)\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_n(x))\, \mathrm{d}x = \int_{\mathbb{R}^n} v_n(x)\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_{n-1}(x), \mathrm{D}^2 v_0(x))\, \mathrm{d}x.$$

We will also use the following reformulation of [24, Lemma 6.3]. Cf. [24, (3.3)].

Lemma 2.4.7. *Let $v_1, \dots, v_{n-1} \in C^2(\mathbb{R}^n)$ and let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable vector field with compact support. If F is such that its Jacobian matrix $\mathrm{D}F$ is symmetric, then*

$$\int_{\mathbb{R}^n} D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_{n-1}(x), \mathrm{D}F(x))\, \mathrm{d}x = 0.$$

Lemma 2.4.8. *Let $k \in \{1, \dots, n-1\}$ and $\varepsilon > 0$. If $v_1, \dots, v_k \in C^2(\mathbb{R}^n)$ are such that $\mathrm{D}^2 v_1 \equiv 0$ on εB^n, then*

$$\begin{aligned}\int_{\mathbb{R}^n} \psi(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k])\, \mathrm{d}x \\ = \int_{\mathbb{R}^n} (\mathcal{R}^{-1}\psi)(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k-1], I_n)\, \mathrm{d}x\end{aligned} \tag{2.4.5}$$

for every $\psi \in C^2_b((0,\infty))$.

Proof. Since $\mathrm{D}^2 v_1 \equiv 0$ on εB^n, also the mixed discriminants in both integrals in (2.4.5) vanish on this set. Together with the compact support of ψ, this shows that both integrals in (2.4.5) are well-defined and finite. We will now prove the vector-valued equation (2.4.5) coordinate-wise. For the ith coordinate, $i \in \{1, \dots, n\}$, it follows from Lemma 2.4.6 that (2.4.5) is equivalent to

$$\begin{aligned}\int_{\mathbb{R}^n} |x|\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k-1], \mathrm{D}^2(\psi(|x|)x_i))\, \mathrm{d}x \\ = \int_{\mathbb{R}^n} \frac{|x|^2}{2}\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k-1], \mathrm{D}^2((\mathcal{R}^{-1}\psi)(|x|)x_i))\, \mathrm{d}x.\end{aligned}$$

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Here we have used that the mixed discriminants vanish on εB^n and thus, we can replace h_{B^n}, $x \mapsto \psi(|x|)x_i$, and $x \mapsto (\mathcal{R}^{-1}\psi)(|x|)x_i$ with suitable functions in $C^2(\mathbb{R}^n)$ without changing the values of the integrals. By the multilinearity of mixed discriminants, we therefore need to show that

$$\int_{\mathbb{R}^n} D\Big(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k-1], \\ |x|\,\mathrm{D}^2(\psi(|x|)x_i) - \frac{|x|^2}{2}\,\mathrm{D}^2((\mathcal{R}^{-1}\psi)(|x|)x_i)\Big)\,\mathrm{d}x = 0. \tag{2.4.6}$$

Now observe that

$$\begin{aligned}
\mathrm{D}^2(\psi(|x|)x_i) &= \left(\frac{\partial}{\partial x_m}\left(\frac{\psi'(|x|)}{|x|}x_i x_l + \delta_{il}\psi(|x|)\right)\right)_{l,m=1,\dots,n} \\
&= \left(\left(\frac{\psi''(|x|)}{|x|^2} - \frac{\psi'(|x|)}{|x|^3}\right)x_i x_l x_m + \frac{\psi'(|x|)}{|x|}(\delta_{lm}x_i + \delta_{im}x_l + \delta_{il}x_m)\right)_{l,m=1,\dots,n} \\
&= \left(\frac{\psi''(|x|)}{|x|^2} - \frac{\psi'(|x|)}{|x|^3}\right)x_i(x\otimes x) + \frac{\psi'(|x|)}{|x|}\left(x_i I_n + x\otimes e_i + e_i\otimes x\right)
\end{aligned}$$

for every $x \in \mathbb{R}^n \setminus \{o\}$, where δ_{pq} denotes the usual Kronecker delta. Since ψ has bounded support, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathcal{R}^{-1}\psi)(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\psi(s)}{s} - \int_s^\infty \frac{\psi(t)}{t^2}\,\mathrm{d}t\right) = \frac{\psi'(s)}{s}$$

for $s > 0$, and therefore, similar to the above,

$$\begin{aligned}
\mathrm{D}^2((\mathcal{R}^{-1}\psi)(|x|)x_i) &= \left(\frac{\partial}{\partial x_m}\left(\frac{\psi'(|x|)}{|x|^2}x_i x_l + \delta_{il}(\mathcal{R}^{-1}\psi)(|x|)\right)\right)_{l,m=1,\dots,n} \\
&= \left(\left(\frac{\psi''(|x|)}{|x|^3} - 2\frac{\psi'(|x|)}{|x|^4}\right)x_i x_l x_m + \frac{\psi'(|x|)}{|x|^2}(\delta_{lm}x_i + \delta_{im}x_l + \delta_{il}x_m)\right)_{l,m=1,\dots,n} \\
&= \left(\frac{\psi''(|x|)}{|x|^3} - 2\frac{\psi'(|x|)}{|x|^4}\right)x_i(x\otimes x) + \frac{\psi'(|x|)}{|x|^2}\left(x_i I_n + x\otimes e_i + e_i\otimes x\right)
\end{aligned}$$

for every $x \in \mathbb{R}^n \setminus \{o\}$. Hence,

$$\begin{aligned}
|x|\mathrm{D}^2(\psi(|x|)x_i) - \frac{|x|^2}{2}\mathrm{D}^2((\mathcal{R}^{-1}\psi)(|x|)x_i) \\
&= \frac{1}{2}\frac{\psi''(|x|)}{|x|}x_i(x\otimes x) + \frac{1}{2}\psi'(|x|)\left(x_i I_n + x\otimes e_i + e_i\otimes x\right) \\
&= \frac{1}{2}\mathrm{D}F(x)
\end{aligned}$$

for $x \in \mathbb{R}^n \setminus \{o\}$, where $F\colon \mathbb{R}^n \setminus \{o\} \to \mathbb{R}^n$ is the vector field

$$F(x) = \psi'(|x|)x_i\,x + \left(\psi(|x|)|x| + \int_{|x|}^\infty \psi(t)\right)e_i.$$

Thus, we can rewrite the left side of (2.4.6) as

$$\int_{\mathbb{R}^n} D\Big(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k-1], \tfrac{1}{2}\mathrm{D}F(x)\Big)\,\mathrm{d}x.$$

Using again that we may replace the integrands in a neighborhood of the origin, it follows from Lemma 2.4.7 that this integral vanishes, which completes the proof. □

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Now that we have shown the desired equation under additional regularity assumptions, we use standard approximation arguments to remove said assumptions.

Proposition 2.4.9. *Let $k \in \{1, \dots, n-1\}$. If $\psi \in T^n_{n-k+1}$, then*

$$\begin{aligned}\int_{\mathbb{R}^n} &\psi(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k])\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} (\mathcal{R}^{-1}\,\psi)(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k-1], I_n)\,\mathrm{d}x\end{aligned}$$

for every $v_1, \dots, v_k \in C^2(\mathbb{R}^n)$.

Proof. We start with the observation that since $\psi \in T^n_{n-k+1} \subset C_b((0,\infty))$, there exists $\gamma > 0$ such that $\psi(s) = \mathcal{R}^{-1}\,\psi(s) = 0$ for every $s \geq \gamma$. Assume first that there exists $\varepsilon > 0$ such that $\mathrm{D}^2 v_1 \equiv 0$ on εB^n. Let $\psi_\varepsilon \in C_b((0,\infty))$ be such that $\psi_\varepsilon \equiv \psi$ on $[\varepsilon, \infty)$ and $\psi_\varepsilon \equiv 0$ on $(0, \frac{\varepsilon}{2}]$. Note that this implies that also

$$\mathcal{R}^{-1}\,\psi_\varepsilon \equiv \mathcal{R}^{-1}\,\psi$$

on $[\varepsilon, \infty)$. For $\delta > 0$ we can find $\psi_{\varepsilon,\delta} \in C^2_b((0,\infty))$ such that $\psi_{\varepsilon,\delta} \equiv 0$ on $(0, \frac{\varepsilon}{2}] \cup [\gamma + \delta, \infty)$ and such that $\psi_{\varepsilon,\delta}$ converges uniformly to ψ_ε on $(\frac{\varepsilon}{2}, \gamma + \delta)$ as $\delta \to 0^+$. In particular, this gives uniform convergence of $\psi_{\varepsilon,\delta}$ to ψ_ε and furthermore of $\mathcal{R}^{-1}\,\psi_{\varepsilon,\delta}$ to $\mathcal{R}^{-1}\,\psi_\varepsilon$ on $(0,\infty)$ as $\delta \to 0^+$. Considering the assumed condition on v_1 and observing that each of the following integrands is continuous with compact support, it now follows from Lemma 2.4.8 that

$$\begin{aligned}&\int_{\mathbb{R}^n} \psi(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{Bn}(x)[n-k])\,\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^n} \psi_\varepsilon(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k])\,\mathrm{d}x \\ &\quad= \lim_{\delta\to 0^+} \int_{\mathbb{R}^n} \psi_{\varepsilon,\delta}(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k])\,\mathrm{d}x \\ &\quad= \lim_{\delta\to 0^+} \int_{\mathbb{R}^n} (\mathcal{R}^{-1}\,\psi_{\varepsilon,\delta})(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k-1], I_n)\,\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^n} (\mathcal{R}^{-1}\,\psi_\varepsilon)(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k-1], I_n)\,\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^n} (\mathcal{R}^{-1}\,\psi)(|x|)x\, D(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_k(x), \mathrm{D}^2 h_{B^n}(x)[n-k-1], I_n)\,\mathrm{d}x.\end{aligned}$$

For general $v_1 \in C^2(\mathbb{R}^n)$, we first observe that we may assume $v_1(o) = 0$ and $\nabla v_1(o) = 0$ since the equality to be shown depends only on the second derivatives of v_1. Thus, there exist $\beta, \varepsilon_0 > 0$ such that

$$|v_1(x)| \leq \beta|x|^2 \quad \text{and} \quad |\nabla v_1(x)| \leq \beta|x| \tag{2.4.7}$$

for $|x| \leq 2\varepsilon_0 B^n$. Now let $\varphi \in C^2([0,\infty))$ be such that $\varphi(t) = 0$ for $t \in [0,1]$ and $\varphi(t) = 1$ for $t \in [2,\infty)$, and define $v_{1,\varepsilon} \in C^2(\mathbb{R}^n)$ as

$$v_{1,\varepsilon}(x) = v_1(x)\,\varphi\left(\frac{|x|}{\varepsilon}\right)$$

for $x \in \mathbb{R}^n$ and $\varepsilon \in (0, \varepsilon_0)$. By the choice of φ we clearly have $\mathrm{D}^2 v_{1,\varepsilon} \equiv 0$ on εB^n for every $\varepsilon \in (0, \varepsilon_0)$ and $\mathrm{D}^2 v_{1,\varepsilon}(x) = \mathrm{D}^2 v_1(x)$ for $|x| \geq 2\varepsilon$. In particular, $\mathrm{D}^2 v_{1,\varepsilon}$ converges to $\mathrm{D}^2 v_1$

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pointwise on $\mathbb{R}^n \setminus \{o\}$ as $\varepsilon \to 0^+$. Furthermore, it is straightforward to compute that

$$\begin{aligned}\mathrm{D}^2 v_{1,\varepsilon}(x) = \varphi\left(\frac{|x|}{\varepsilon}\right)\mathrm{D}^2 v_1(x) + \frac{1}{\varepsilon|x|}\,\varphi'\left(\frac{|x|}{\varepsilon}\right)(\nabla v_1(x)\otimes x + x\otimes\nabla v_1(x) + v_1(x)\,I_n)\\ + \frac{1}{\varepsilon^2|x|^2}\,\varphi''\left(\frac{|x|}{\varepsilon}\right)v_1(x)(x\otimes x) - \frac{1}{\varepsilon|x|^3}\,\varphi'\left(\frac{|x|}{\varepsilon}\right)v_1(x)(x\otimes x)\end{aligned}$$

for $x \in \mathbb{R}^n \setminus \{o\}$. Since for $|x| < 2\varepsilon$ we have $\frac{1}{\varepsilon} < \frac{1}{2|x|}$, this shows together with (2.4.7) and the above observations that $\mathrm{D}^2 v_{1,\varepsilon}$ is uniformly bounded on γB^n for $\varepsilon \in (0,\varepsilon_0)$. Next, since $\psi \in T^n_{n-k+1}$, we have $\lim_{s\to 0^+} s^k\psi(s) = 0$. Moreover, since $T^n_{n-k+1} = T^{n-1}_{n-k}$, it follows from Lemma 2.4.2 that $\mathcal{R}^{-1}\psi \in T^n_{n-k}$ and therefore $\lim_{s\to 0^+} s^{k+1}\,\mathcal{R}^{-1}\psi(s) = 0$. Thus, by Lemma 2.4.5 and the dominated convergence theorem, together with the first part of the proof, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\psi(|x|)x\,D(\mathrm{D}^2 v_1(x),\dots,\mathrm{D}^2 v_k(x),\mathrm{D}^2 h_{B^n}(x)[n-k])\,\mathrm{d}x\\ &\quad= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}\psi(|x|)x\,D(\mathrm{D}^2 v_{1,\varepsilon}(x),\mathrm{D}^2 v_2(x),\dots,\mathrm{D}^2 v_k(x),\mathrm{D}^2 h_{B^n}(x)[n-k])\,\mathrm{d}x\\ &\quad= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}(\mathcal{R}^{-1}\psi)(|x|)x\,D(\mathrm{D}^2 v_{1,\varepsilon}(x),\mathrm{D}^2 v_2(x),\dots,\mathrm{D}^2 v_k(x),\mathrm{D}^2 h_{B^n}(x)[n-k-1],I_n)\,\mathrm{d}x\\ &\quad= \int_{\mathbb{R}^n}(\mathcal{R}^{-1}\psi)(|x|)x\,D(\mathrm{D}^2 v_1(x),\dots,\mathrm{D}^2 v_k(x),\mathrm{D}^2 h_{B^n}(x)[n-k-1],I_n)\,\mathrm{d}x,\end{aligned}$$

which concludes the proof. □

We now come to the main result of this section, which we obtain from an iterated application of the previous Proposition.

Proposition 2.4.10. *Let $j \in \{1,\dots,n-1\}$. If $\zeta \in T^n_j$, then*

$$\int_{\mathbb{R}^n}\zeta(|x|)x\,[\mathrm{D}^2 v(x)]_j\,\mathrm{d}x = \binom{n}{j}\int_{\mathbb{R}^n}(\mathcal{R}^{n-j}\zeta)(|x|)x\,D(\mathrm{D}^2 v(x)[j],\mathrm{D}^2 h_{B^n}(x)[n-j])\,\mathrm{d}x$$

for every $v \in C^2(\mathbb{R}^n)$.

Proof. Let $\zeta \in T^n_j$ be given. For every $k \in \{j,\dots,n-1\}$ it follows from Lemma 2.4.2 together with the fact that $T^k_j = T^n_{n-k+j} \subseteq T^n_{n-k+1}$, that $\mathcal{R}^{n-k}\zeta \in T^n_{n-k+1}$. Thus, applying Proposition 2.4.9 $(n-j)$ times with $\psi = \mathcal{R}^{n-k}\zeta$, $v_1\dots,v_j = v$, $v_{j+1},\dots,v_k = \frac{1}{2}h^2_{B^n}$, and $k \in \{j,\dots,n-1\}$, shows that

$$\begin{aligned}&\int_{\mathbb{R}^n}(\mathcal{R}^{n-j}\zeta)(|x|)x\,D(\mathrm{D}^2 v(x)[j],\mathrm{D}^2 h_{B^n}(x)[n-j])\,\mathrm{d}x\\ &\qquad= \int_{\mathbb{R}^n}(\mathcal{R}^{n-(j+1)}\zeta)(|x|)x\,D(\mathrm{D}^2 v(x)[j],\mathrm{D}^2 h_{B^n}(x)[n-(j+1)],I_n)\,\mathrm{d}x\\ &\qquad\vdots\\ &\qquad= \int_{\mathbb{R}^n}\zeta(|x|)x\,D(\mathrm{D}^2 v(x)[j],I_n[n-j])\,\mathrm{d}x\end{aligned}$$

for every $v \in C^2(\mathbb{R}^n)$. Together with (2.3.10) this completes the proof. □

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2.4.3 Proof of Theorem A and Theorem B

Let $j \in \{1,\dots,n\}$ and $\zeta \in T_j^n$ be given, and set $\alpha = \binom{n}{j} \mathcal{R}^{n-j}\zeta$. Since $T_j^j = T_n^n$, it follows from Lemma 2.4.2 that $\alpha \in T_n^n$. Thus, Proposition 2.3.5 shows that

$$\mathrm{z}(v) = \int_{\mathbb{R}^n} \alpha(|x|)x \,\mathrm{dMA}_j(v;x), \quad v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}),$$

defines a continuous, dually epi-translation invariant, rotation and $\mathrm{O}(n)$ equivariant valuation $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$. Furthermore, by (2.3.8) and Proposition 2.4.10 we have

$$\mathrm{z}(v) = \binom{n}{j} \int_{\mathbb{R}^n} (\mathcal{R}^{n-j}\zeta)(|x|)x \, D(\mathrm{D}^2 v(x)[j], \mathrm{D}^2 h_{B^n}(x)[n-j])\,\mathrm{d}x = \int_{\mathbb{R}^n} \zeta(|x|)x\,[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$. Since $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$ is a dense subset of $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, the valuation z is uniquely determined by the last equality and thus, Theorem A follows. Together with (2.4.1), this also shows Theorem B. □

2.4.4 Extended Representation Formulas

Let $j \in \{1,\dots,n\}$. By Theorem A and (2.3.3) we have

$$\mathrm{t}^*_{j,\zeta}(v) = \int_{\mathbb{R}^n} \zeta(|x|)x \,\mathrm{d}\Phi_j(v;x) \tag{2.4.8}$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$ and $\zeta \in T_j^n$. Furthermore, it follows from Theorem A and Proposition 2.3.5 that (2.4.8) also holds whenever $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\zeta \in T_n^n \subset T_j^n$. Similar to [26, Lemma 3.23], we want to extend this representation, for general $\zeta \in T_j^n$, to

$$\mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R}) = \{v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) : v \text{ is of class } C^2 \text{ in a neighborhood of the origin}\}.$$

Lemma 2.4.11. *Let $j \in \{1,\dots,n\}$. If $\zeta \in T_j^n$, then*

$$\mathrm{t}^*_{j,\zeta}(v) = \int_{\mathbb{R}^n} \zeta(|x|)x \,\mathrm{d}\Phi_j(v;x)$$

for every $v \in \mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$.

Proof. Let $\zeta \in T_j^n$ and $v \in \mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$ be given. By the definition of $\mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$ there exists $r_0 > 0$ such that v is of class C^2 on $r_0 B^n$. Using a standard mollification procedure, we can find a sequence $v_k \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$, $k \in \mathbb{N}$, such that v_k epi-converges to v as $k \to \infty$ and such that we have convergence in the C^2-norm on $r_0 B^n$. Since ζ is continuous on $(0,\infty)$, it follows from the weak continuity of $w \mapsto \Phi_j(w;\cdot)$ that

$$\lim_{k\to\infty} \int_{\{x:|x|\geq r\}} \zeta(|x|)x \,\mathrm{d}\Phi_j(v_k;x) = \int_{\{x:|x|\geq r\}} \zeta(|x|)x \,\mathrm{d}\Phi_j(v;x) \tag{2.4.9}$$

for every $r > 0$ such that $\Phi_j(v;\{x : |x| = r\}) = 0$, which is satisfied for a.e. $r > 0$ since $\Phi_j(v;\cdot)$ is locally finite. Furthermore, it follows from (2.3.3) together with using polar coordinates that

$$\begin{aligned}\left|\int_{\{x:|x|<r\}} \zeta(|x|)x \,\mathrm{d}\Phi_j(v_k;x)\right| &= \left|\int_{\{x:|x|<r\}} \zeta(|x|)x\,[\mathrm{D}^2 v_k(x)]_j \,\mathrm{d}x\right| \\ &\leq \int_{\{x:|x|<r\}} |\zeta(|x|)|x| \,\left|[\mathrm{D}^2 v_k(x)]_j\right| \,\mathrm{d}x \\ &\leq n\kappa_n \sup_{|x|<r} \left|[\mathrm{D}^2 v_k(x)]_j\right| \int_0^r |\zeta(s)|s^n \,\mathrm{d}s\end{aligned}$$

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for every $k \in \mathbb{N}$ and $r > 0$. Since $\lim_{s\to 0^+} \zeta(s)s^n = 0$ and since v_k converges to v in the C^2-norm on $r_0 B^n$, this shows that for every $\varepsilon > 0$ we can find $r_1 \in (0, r_0)$ such that

$$\left| \int_{\{x:|x|<r_1\}} \zeta(|x|)x \,\mathrm{d}\Phi_j(v_k; x) \right| < \varepsilon \quad \text{and} \quad \left| \int_{\{x:|x|<r_1\}} \zeta(|x|)x \,\mathrm{d}\Phi_j(v; x) \right| < \varepsilon$$

for every $k \in \mathbb{N}$. Together with (2.4.8) and (2.4.9) this gives

$$\mathrm{t}^*_{j,\zeta}(v) = \lim_{k\to\infty} \mathrm{t}^*_{j,\zeta}(v_k) = \lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta(|x|)x \,\mathrm{d}\Phi_j(v_k; x) = \int_{\mathbb{R}^n} \zeta(|x|)x \,\mathrm{d}\Phi_j(v; x),$$

where we have used the continuity of $\mathrm{t}^*_{j,\zeta}$, due to Theorem A. □

2.5 Further Representations

The purpose of this section is to find further representations of functional Minkowski vectors. First, in Section 2.5.1, we will use recent results from [42] to establish Kubota-type formulas for the operators $\mathrm{t}^*_{j,\zeta}$, which also yields another explicit representation of functional Minkowski vectors, forgoing singular integrals. After this, we represent functional Minkowski vectors in terms of integrals with respect to area measures of $(n+1)$-dimensional convex bodies in Section 2.5.3. This will be needed in the context of smooth valuations in Section 2.6.3. As an intermediate step towards this representation, we first need a result similar to Proposition 2.4.10, where one treats the case where integrals with respect to Hessian measures are transformed into integrals with respect to

$$D(\mathrm{D}^2 v(x)[j], \mathrm{D}^2 v_B(x)[n-j]) \,\mathrm{d}x,$$

where $v_B \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ is given by

$$v_B(x) = h_{B^{n+1}}(x, -1) = \sqrt{1 + |x|^2} \tag{2.5.1}$$

for $x \in \mathbb{R}^n$. While the procedure for this is largely analogous to Section 2.4, there are also essential differences. In particular, the necessary transformation of the integral densities, which we introduce in Section 2.5.2, is not a bijection between different classes T^n_j and T^{n-l}_j, but from T^n_j onto itself.

2.5.1 Kubota-type Formulas

The following Kubota-type formula for the measures $\mathrm{MA}_j(v; \cdot)$ is a consequence of [42, Theorem 1.3]. Here, for $j \in \{1, \dots, n-1\}$ we denote by $\mathrm{G}(n, j)$ the Grassmannian of j-dimensional linear subspaces of $\mathbb{R}^n$, on which we consider integration with respect to the Haar probability measure. For $E \in \mathrm{G}(n, j)$ and $w \in \mathrm{Conv}(E; \mathbb{R})$ we furthermore write $\mathrm{MA}_E(w; \cdot)$ for the Monge–Ampère measure with respect to the ambient space E.

Theorem 2.5.1. *If $j \in \{1, \dots, n-1\}$ and $\beta \colon \mathbb{R}^n \to [0, \infty)$ is measurable, then*

$$\frac{1}{\kappa_n} \int_{\mathbb{R}^n} \beta(x) \,\mathrm{dMA}_j(v; x) = \frac{1}{\kappa_j} \int_{\mathrm{G}(n,j)} \int_E \beta(x_E) \,\mathrm{dMA}_E(v|_E; x_E) \,\mathrm{d}E$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$.

We now establish Kubota-type formulas for functional intrinsic moments, where we emphasize that we consider $E \in \mathrm{G}(n, j)$ as a subspace of $\mathbb{R}^n$.

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Theorem 2.5.2. *If $j \in \{1, \dots, n\}$ and $\zeta \in T_j^n$, then*

$$\mathrm{t}^*_{j,\zeta}(v) = \int_{\mathrm{G}(n,j)} \int_E \alpha(|x_E|) x_E \, \mathrm{dMA}_E(v|_E; x_E) \, \mathrm{d}E$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, where $\alpha \in T_n^n$ is given by

$$\alpha(s) = \binom{n}{j} \frac{\kappa_n}{\kappa_j} \left(s^{n-j} \zeta(s) + (n-j) \int_s^\infty t^{n-j-1} \zeta(t) \, \mathrm{d}t \right)$$

for $s > 0$.

Proof. Let $j \in \{1, \dots, n\}$, $\zeta \in T_j^n$, and $\alpha \in T_n^n$ be given and fix some $i \in \{1, \dots, n\}$. Let

$$\beta(x) = \langle \alpha(|x|)x, e_i \rangle$$

for $x \in \mathbb{R}^n$, which by Lemma 2.3.4 defines a continuous function with compact support on $\mathbb{R}^n$. By Theorem B and Theorem 2.5.1 we now have

$$\langle \mathrm{t}^*_{j,\zeta}(v), e_i \rangle = \int_{\mathbb{R}^n} \beta(x) \, \mathrm{dMA}_j(v; x) = \int_{\mathrm{G}(n,j)} \int_E \beta(x_E) \, \mathrm{dMA}_E(v|_E; x)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, which completes the proof. □

For $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $E \in \mathrm{G}(n,j)$ we define the *projection function* $\mathrm{proj}_E u \colon E \to \mathbb{R}$ as

$$\mathrm{proj}_E u(x_E) = \min\nolimits_{z \in E^\perp} u(x_E + z), \quad x_E \in E,$$

which is again a proper, lower semicontinuous, convex function. Since

$$(\mathrm{proj}_E u)^* = u^*|_E,$$

where on the left side the convex conjugate is taken with respect to the ambient space E (see [76, Theorem 11.23]), we immediately obtain the following representation for the dual operators $\mathrm{t}_{j,\zeta}$ from Theorem 2.5.2 together with (2.3.7). Here, we write ∇_E for the gradient with respect to the ambient space E.

Theorem 2.5.3. *If $j \in \{1, \dots, n\}$ and $\zeta \in T_j^n$, then*

$$\mathrm{t}_{j,\zeta}(u) = \int_{\mathrm{G}(n,j)} \int_{\mathrm{dom}(\mathrm{proj}_E u)} \alpha(|\nabla_E \mathrm{proj}_E u(x_E)|) \nabla_E \mathrm{proj}_E u(x_E) \, \mathrm{d}x_E \, \mathrm{d}E$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, where $\alpha \in T_n^n$ is given by

$$\alpha(s) = \binom{n}{j} \frac{\kappa_n}{\kappa_j} \left(s^{n-j} \zeta(s) + (n-j) \int_s^\infty t^{n-j-1} \zeta(t) \, \mathrm{d}t \right)$$

for $s > 0$.

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2.5.2 The Transform $\mathcal{T}$

Instead of the transform $\mathcal{R}$, the transform

$$\mathcal{T}\zeta(s) = \sqrt{1+s^2}\,\zeta(s) + \int_s^\infty \frac{t\zeta(t)}{\sqrt{1+t^2}}\,\mathrm{d}t$$

with $\zeta \in C_b((0,\infty))$ and $s > 0$ needs to be considered when integrals with respect to Hessian measures are transformed into integrals with respect to the measures $\mathrm{MA}(v[j], v_B[n-j]; \cdot)$. We start with a description of $\mathcal{T}^l$ for $l \in \mathbb{N}$.

Lemma 2.5.4. *If $l \in \mathbb{N}$ and $\zeta \in C_b((0,\infty))$, then*

$$\mathcal{T}^l\zeta(s) = \underbrace{(\mathcal{T}\circ\cdots\circ\mathcal{T})}_{l}\zeta(s) = (1+s^2)^{\frac{l}{2}}\zeta(s) + l\int_s^\infty t(1+t^2)^{\frac{l}{2}-1}\zeta(t)\,\mathrm{d}t$$

for every $s > 0$.

Proof. We prove the statement by induction on l. The statement trivially holds for $l = 1$. For $l \geq 2$ and assuming that the statement holds for $l-1$, we now have

$$\begin{aligned}
\mathcal{T}^l\zeta(s) &= \mathcal{T}^{l-1}\mathcal{T}\zeta(s)\\
&= (1+s^2)^{\frac{l-1}{2}}\left(\sqrt{1+s^2}\zeta(s) + \int_s^\infty \frac{t\zeta(t)}{\sqrt{1+t^2}}\,\mathrm{d}t\right)\\
&\quad + (l-1)\int_s^\infty t(1+t^2)^{\frac{l-1}{2}-1}\left(\sqrt{1+t^2}\zeta(t) + \int_t^\infty \frac{r\zeta(r)}{\sqrt{1+r^2}}\,\mathrm{d}r\right)\mathrm{d}t \qquad (2.5.2)\\
&= (1+s^2)^{\frac{l}{2}}\zeta(s) + (l-1)\int_s^\infty t(t+t^2)^{\frac{l}{2}-1}\zeta(t)\,\mathrm{d}t\\
&\quad + (1+s^2)^{\frac{l-1}{2}}\int_s^\infty \frac{t\zeta(t)}{\sqrt{1+t^2}}\,\mathrm{d}t + (l-1)\int_s^\infty t(1+t^2)^{\frac{l-1}{2}-1}\int_t^\infty \frac{r\zeta(r)}{\sqrt{1+r^2}}\,\mathrm{d}r\,\mathrm{d}t
\end{aligned}$$

for every $s > 0$. Considering that $l \geq 2$ and that ζ has bounded support, integration by parts gives

$$\begin{aligned}
(l-1)\int_s^\infty t(1+t^2)^{\frac{l-1}{2}-1}\int_t^\infty \frac{r\zeta(r)}{\sqrt{1+r^2}}\,\mathrm{d}r\,\mathrm{d}t&\\
= -(1+s^2)^{\frac{l-1}{2}}\int_s^\infty \frac{t\zeta(t)}{\sqrt{1+t^2}}\,\mathrm{d}t + \int_s^\infty (1+t^2)^{\frac{l}{2}-1}t\zeta(t)\,\mathrm{d}t&
\end{aligned}$$

for every $s > 0$, which together with (2.5.2) gives the desired result. □

Before we consider how the transform $\mathcal{T}$ acts on the classes T_j^n, we need the following result.

Lemma 2.5.5. *Let $j \in \{1,\dots,n\}$. If $\zeta \in T_j^n$, then*

$$\lim_{s\to 0^+} s^{n-j+1}\int_s^\infty t(1+t^2)^{\frac{m}{2}-1}\zeta(t)\,\mathrm{d}t = 0$$

for every $m \in \mathbb{Z}$.

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Proof. Let $\zeta \in T_j^n$ and $m \in \mathbb{Z}$ be given. If ζ is such that $\lim_{s\to 0^+} \int_s^\infty t(1+t^2)^{\frac{m}{2}-1}\zeta(t)\,\mathrm{d}t$ exists and is finite, then the statement is trivial. In the remaining case, we use L'Hospital's rule together with the definition of T_j^n to obtain

$$\begin{aligned}\lim_{s\to 0^+}\left|s^{n-j+1}\int_s^\infty t(1+t^2)^{\frac{m}{2}-1}\zeta(t)\,\mathrm{d}t\right| &\le \lim_{s\to 0^+}\frac{\int_s^\infty |t(1+t^2)^{\frac{m}{2}-1}\zeta(t)|\,\mathrm{d}t}{\frac{1}{s^{n-j+1}}}\\ &= \lim_{s\to 0^+}\frac{|s(1+s^2)^{\frac{m}{2}-1}\zeta(s)|}{\frac{n-j+1}{s^{n-j+2}}}\\ &= \lim_{s\to 0^+}\frac{(1+s^2)^{\frac{m}{2}-1}}{n-j+1}|s^{n-j+3}\zeta(s)|\\ &= 0.\end{aligned}$$

□

Lemma 2.5.6. *Let $j \in \{1,\dots,n\}$ and $l \in \mathbb{N}$. The map $\mathcal{T}^l\colon T_j^n \to T_j^n$ is a bijection with inverse*

$$\mathcal{T}^{-l}\rho(s) = (\mathcal{T}^{-1})^l\rho(s) = \frac{\rho(s)}{(1+s^2)^{\frac{l}{2}}} - l\int_s^\infty \frac{t\rho(t)}{(1+t^2)^{\frac{l}{2}+1}}\,\mathrm{d}t \tag{2.5.3}$$

for $\rho \in T_j^n$ and $s > 0$.

Proof. Without loss of generality, let $l > 0$. We start by showing that if $\zeta \in T_j^n$, then also $\mathcal{T}^l\zeta \in T_j^n$. Note that it is enough to show this for the case $l = 1$. First, it is easy to see that $\mathcal{T}\zeta \in C_b((0,\infty))$. We now use the definition of T_j^n together with Lemma 2.5.5 to obtain

$$\lim_{s\to 0^+} s^{n-j+1}\mathcal{T}\zeta(s) = \lim_{s\to 0^+}\sqrt{1+s^2}s^{n-j+1}\zeta(s) + \lim_{s\to 0^+} s^{n-j+1}\int_s^\infty \frac{t\zeta(t)}{\sqrt{1+t^2}}\,\mathrm{d}t = 0,$$

which shows that $\mathcal{T}\zeta \in T_j^n$.

Next, we show that (2.5.3) is the inverse operation to $\mathcal{T}^l$. For $\zeta \in T_j^n$ it follows from Lemma 2.5.4 that

$$\begin{aligned}&\frac{\mathcal{T}^l\zeta(s)}{(1+s^2)^{\frac{l}{s}}} - l\int_s^\infty \frac{t\,\mathcal{T}^l\zeta(t)}{(1+t^2)^{\frac{l}{2}+1}}\,\mathrm{d}t\\ &\qquad= \zeta(s) + \frac{l}{(1+s^2)^{\frac{l}{2}}}\int_s^\infty t(1+t^2)^{\frac{l}{2}-1}\zeta(t)\,\mathrm{d}t\\ &\qquad\quad - l\int_s^\infty \frac{t\zeta(t)}{1+t^2}\,\mathrm{d}t - l^2\int_s^\infty \frac{t}{(1+t^2)^{\frac{l}{2}+1}}\int_t^\infty r(1+r^2)^{\frac{l}{2}-1}\zeta(r)\,\mathrm{d}r\,\mathrm{d}t\end{aligned} \tag{2.5.4}$$

for every $s > 0$. Taking into account that ζ has bounded support, integration by parts gives

$$\begin{aligned}&l^2\int_s^\infty \frac{t}{(1+t^2)^{\frac{l}{2}+1}}\int_t^\infty r(1+r^2)^{\frac{l}{2}-1}\zeta(r)\,\mathrm{d}r\,\mathrm{d}t\\ &\qquad= l\frac{1}{(1+s^2)^{\frac{l}{2}}}\int_s^\infty t(1+t^2)^{\frac{l}{2}-1}\zeta(t)\,\mathrm{d}t - l\int_s^\infty \frac{t\zeta(t)}{1+t^2}\,\mathrm{d}t\end{aligned}$$

for $s > 0$, which together with (2.5.4) shows that the left inverse of $\mathcal{T}^l$ is given by (2.5.3). Similarly, one shows that $\mathcal{T}^l$ is the inverse of (2.5.3).

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Lastly, let $\rho \in T_j^n$. We need to show that $\mathcal{T}^{-l}\,\rho \in T_j^n$, where it is again sufficient to consider the case $l = 1$. Clearly, $\mathcal{T}^{-1}\,\rho \in C_b((0,\infty))$. Furthermore, by the definition of T_j^n and Lemma 2.5.5,

$$\lim_{s\to 0^+} s^{n-j+1}\,\mathcal{T}^{-1}\,\rho(s) = \lim_{s\to 0^+} \frac{s^{n-j+1}\rho(s)}{\sqrt{1+s^2}} - \lim_{s\to 0^+} s^{n-j+1} \int_s^\infty \frac{t\rho(t)}{(1+t^2)^{\frac{3}{2}}}\,\mathrm{d}t = 0,$$

which shows that $\mathcal{T}^{-1}\,\rho \in T_j^n$ and completes the proof. □

2.5.3 From Convex Functions to Convex Bodies

Having established the relevant properties of the transform $\mathcal{T}$ in the previous section, we proceed analogously to Section 2.4.2. We omit the proofs as they are almost identical to those in Section 2.4.2, noting that the vector field

$$F(x) = \left(\frac{\psi'(|x|)}{|x|} + \frac{1}{2}\psi'(|x|)|x|\right) x_i\, x + \left(\psi(|x|) + \frac{1}{2}\psi(|x|)|x|^2 + \int_{|x|}^\infty \psi(t)t\,\mathrm{d}t\right) e_i$$

with $x \in \mathbb{R}^n \setminus \{o\}$ and $i \in \{1,\dots,n\}$ needs to be used in the analog to Lemma 2.4.8, which states that

$$\begin{aligned}\int_{\mathbb{R}^n} &\psi(|x|)x\, D(\mathrm{D}^2 v_1(x),\dots,\mathrm{D}^2 v_k(x), I_n[n-k])\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} (\mathcal{T}\,\psi)(|x|)x\, D(\mathrm{D}^2 v_1(x),\dots,\mathrm{D}^2 v_k(x), I_n(x)[n-k-1], \mathrm{D}^2 v_B(x))\,\mathrm{d}x\end{aligned}$$

for $k \in \{1,\dots,n-1\}$, $v_1,\dots,v_k \in C^2(\mathbb{R}^n)$ such that $\mathrm{D}^2 v_1 \equiv 0$ in a neighborhood of the origin, and $\psi \in C_b^2((0,\infty))$. Eventually, we obtain the following result.

Proposition 2.5.7. *Let $j \in \{1,\dots,n-1\}$. If $\zeta \in T_j^n$, then*

$$\int_{\mathbb{R}^n} \zeta(|x|)x\,[\mathrm{D}^2 v(x)]_j\,\mathrm{d}x = \binom{n}{j} \int_{\mathbb{R}^n} (\mathcal{T}^{n-j}\,\zeta)(|x|)x\, D(\mathrm{D}^2 v(x)[j], \mathrm{D}^2 v_B(x)[n-j])\,\mathrm{d}x$$

for every $v \in C^2(\mathbb{R}^n)$.

Remark 2.5.8. Proposition 2.5.7 shows that in many cases the functional Minkowski vectors $\mathrm{t}^*_{j,\zeta}(v)$ with $\zeta \in T_j^n$ and $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ can also be represented as integrals with respect to the measures $\mathrm{MA}(v[j], v_B[n-j];\cdot)$, for example, when $v \in \mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$ (cf. Lemma 2.4.11). However, in contrast to using the measures $\mathrm{MA}_j(v;\cdot)$, this alternative representation would still involve singular densities since $\mathcal{T}^{n-j}\,\zeta \in T_j^n$. This is not surprising insofar as a representation of the family of functional intrinsic volumes that uses continuous density functions, without singularities, is essentially only possible with mixed Monge–Ampère measures which arise from functions that are not differentiable at the origin (such as h_{B^n}). See [22, Theorem 8.2].

Next, we connect mixed Monge–Ampère measures of convex functions with mixed area measures of associated convex bodies in $\mathcal{K}^{n+1}$. Here, the *mixed area measure* $S(K_1,\dots,K_n,\cdot)$ associated to $K_1,\dots,K_n \in \mathcal{K}^{n+1}$ is the unique Borel measure on $\mathbb{S}^n$ that is symmetric in its entries such that

$$S_n(\lambda_1 L_1 + \cdots + \lambda_m L_m,\cdot) = \sum_{i_1,\dots,i_n=1}^m \lambda_{i_1}\cdots\lambda_{i_n} S(L_{i_1},\dots,L_{i_n},\cdot)$$

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for every $L_1, \dots, L_m \in \mathcal{K}^{n+1}$, $\lambda_1, \dots, \lambda_m \geq 0$, and $m \in \mathbb{N}$. The jth area measure of a given convex body $K \in \mathcal{K}^{n+1}$ is obtained as the special case

$$S_j(K, \cdot) = S(K[j], B^{n+1}[n-j], \cdot) \tag{2.5.5}$$

for $j \in \{0, \dots, n\}$. We furthermore use the *gnomonic projection* $\operatorname{gno}\colon \mathbb{S}^n_- \to \mathbb{R}^n$,

$$\operatorname{gno}(z) = \frac{(z_1, \dots, z_n)}{|z_{n+1}|}$$

for $z = (z_1, \dots, z_{n+1}) \in \mathbb{S}^n_-$, which is a C^∞ diffeomorphism between the open lower half-sphere $\mathbb{S}^n_- = \{z = (z_1, \dots, z_{n+1}) \in \mathbb{S}^n : z_{n+1} < 0\}$ and $\mathbb{R}^n$. The result below is a reformulation of [42, Corollary 4.9 and Lemma 4.10].

Proposition 2.5.9. *For every measurable $\varphi\colon \mathbb{R}^n \to [0, \infty)$ with compact support and every $v_1, \dots, v_n \in \operatorname{Conv}(\mathbb{R}^n; \mathbb{R})$ there exist convex bodies $K^{v_1}, \dots, K^{v_n} \in \mathcal{K}^{n+1}$ such that*

$$\int_{\mathbb{R}^n} \varphi(x) \operatorname{dMA}(v_1, \dots, v_n; x) = \int_{\mathbb{S}^n_-} \varphi(\operatorname{gno}(z)) |z_{n+1}| \operatorname{d}\!S(K^{v_1}, \dots, K^{v_n}; z).$$

In addition, if for $j \in \{1, \dots, n\}$ we have $v_j(x) = h_K(x, -1)$, $x \in \mathbb{R}^n$, for some $K \in \mathcal{K}^{n+1}$, then we may choose $K^{v_j} = K$.

We obtain the following alternative representation of functional Minkowski vectors with densities in T^n_n.

Theorem 2.5.10. *Let $\alpha \in T^n_n$ and $j \in \{1, \dots, n\}$. For every $v \in \operatorname{Conv}(\mathbb{R}^n; \mathbb{R})$ there exists a convex body $K^v \in \mathcal{K}^{n+1}$ such that for the ith coordinate of $\mathrm{t}^*_{j,\alpha}(v)$, $i \in \{1, \dots, n\}$, we have*

$$\big(\mathrm{t}^*_{j,\alpha}(v)\big)_i = \binom{n}{j} \int_{\mathbb{S}^n_-} (\mathcal{T}^{n-j}\, \alpha)(|\operatorname{gno}(z)|) z_i \operatorname{d}\!S_j(K^v, z).$$

In addition, if $v(x) = h_K(x, -1)$, $x \in \mathbb{R}^n$, for some $K \in \mathcal{K}^{n+1}$, then we may choose $K^v = K$.

Proof. By Theorem A, Proposition 2.5.7, and (2.3.8) we have

$$\begin{aligned} \big(\mathrm{t}^*_{j,\alpha}(v)\big)_i &= \int_{\mathbb{R}^n} \alpha(|x|) x_i [\mathrm{D}^2 v(x)]_j \operatorname{d}\!x \\ &= \binom{n}{j} \int_{\mathbb{R}^n} (\mathcal{T}^{n-j}\, \alpha)(|x|) x_i\, D(\mathrm{D}^2 v(x)[j], \mathrm{D}^2 v_B(x)[n-j]) \operatorname{d}\!x \\ &= \binom{n}{j} \int_{\mathbb{R}^n} (\mathcal{T}^{n-j}\, \alpha)(|x|) x_i \operatorname{dMA}(v[j], v_B[n-j]; x) \end{aligned} \tag{2.5.6}$$

for every $v \in \operatorname{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C^2(\mathbb{R}^n)$ and $i \in \{1, \dots, n\}$. Since $\alpha \in T^n_n$ it follows from Lemma 2.5.6 that also $\mathcal{T}^{n-j}\, \alpha \in T^n_n$ and thus, by Lemma 2.3.4, the integrand $x \mapsto (\mathcal{T}^{n-j}\, \alpha)(|x|) x_i$ is continuous with compact support on $\mathbb{R}^n$. Using the continuity of $\mathrm{t}^*_{j,\alpha}$ and Proposition 2.3.3 we may therefore continuously extend (2.5.6) from $\operatorname{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C^2(\mathbb{R}^n)$ to $\operatorname{Conv}(\mathbb{R}^n; \mathbb{R})$. The statement now follows from Proposition 2.5.9 together with (2.5.1) and (2.5.5). □

Remark 2.5.11. The observant reader might suspect at this point that Theorem 2.5.10 can also be derived directly from [25, Lemma 4.4]. However, the latter is not correct in the full generality with which it is stated in [25] and can, therefore, not be applied here.

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2.6 Characterization

2.6.1 Retrieving the Densities

Let $j \in \{1, \dots, n\}$ and let $\zeta \in D_j^n$. As was shown in [26, Lemma 2.15], it is relatively straightforward to retrieve the density ζ from the functional intrinsic volume $\mathrm{V}_{j,\zeta}^*$. This is done using the family of functions $v_s \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, $s \geq 0$, given by

$$v_s(x) = \begin{cases} 0 & \text{if } |x| \leq s, \\ |x| - s & \text{if } |x| > s, \end{cases}$$

and showing that $\mathrm{V}_{j,\zeta}^*(v_s)$ equals a multiple of $\mathcal{R}^{n-j}\,\zeta(s)$. Since $\mathcal{R}^{n-j}$ is a bijection between D_j^n and $C_c([0,\infty))$ (see [27, Lemma 3.8]), this uniquely determines ζ.

In the following, we want to establish a similar result for the operators $\mathrm{t}_{j,\zeta}^*$ with $\zeta \in T_j^n$. The difficulty here, however, is that these operators vanish on the radially symmetric functions v_s. We will, therefore, resort to a modification of the functions v_s, which entails considerably more complex calculations.

Let $n \geq 2$. For $s \geq 0$, let $w_s \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ be given by

$$w_s(x) = \begin{cases} 0 & \text{if } |x| \leq s \text{ and } x_n \geq 0, \\ |x| - s & \text{if } |x| > s \text{ and } x_n \geq 0, \\ 0 & \text{if } |(x_1, \dots, x_{n-1})| \leq s \text{ and } x_n < 0, \\ |(x_1, \dots, x_{n-1})| - s & \text{if } |(x_1, \dots, x_{n-1})| > s \text{ and } x_n < 0, \end{cases}$$

where we write $x = (x_1, \dots, x_n)$ for $x \in \mathbb{R}^n$. It is straightforward to check that w_s is of class C^2 almost everywhere on $\mathbb{R}^n$, with the subdifferential at the exceptions for $s > 0$ being

$$\partial w_s(x) = \left\{ r\frac{x}{|x|} : r \in [0,1] \right\} \tag{2.6.1}$$

if $|x| = s$ and $x_n \geq 0$, and

$$\partial w_s(x) = \left\{ r\frac{(x_1, \dots, x_{n-1}, 0)}{|(x_1, \dots, x_{n-1})|} : r \in [0,1] \right\} \tag{2.6.2}$$

when $|(x_1, \dots, x_{n-1})| = s$ and $x_n < 0$.

We start with an auxiliary result, where we note that the case $j \in \{1, \dots, n-1\}$ is a consequence of [21, Lemma 2.15] and its proof. For the case $j = n$, we will use that

$$v_s^*(x) = \mathbf{I}_{B^n}(x) + s|x| \tag{2.6.3}$$

for $x \in \mathbb{R}^n$ and $s \geq 0$, cf. [21, (2.1) and (2.2)].

Lemma 2.6.1. *If $j \in \{1, \dots, n\}$, then*

$$\Phi_j(v_s; s\mathbb{S}^{n-1}) = \kappa_n \binom{n}{j} s^{n-j}$$

for every $s \geq 0$.

Proof. As explained above, the case $j \in \{1, \dots, n-1\}$ has already been treated. It remains to consider the case $j = n$, in which it follows from (2.3.5), (2.3.7), and (2.6.3) that

$$\Phi_n(v_s; s\mathbb{S}^{n-1}) = \Psi_n(v_s^*; s\mathbb{S}^{n-1}) = \int_{B^n} \chi_{s\mathbb{S}^{n-1}}(\nabla s|x|)\,\mathrm{d}x = \kappa_n,$$

where $\chi_{s\mathbb{S}^{n-1}}$ denotes the characteristic function of $s\mathbb{S}^{n-1}$. □

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In the following, we will also use that

$$\int_{\{z\in\mathbb{S}^{n-1}:z_n\geq 0\}} z_n \,\mathrm{d}\mathcal{H}^{n-1}(z) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |\langle z, e_n\rangle| \,\mathrm{d}S_{n-1}(B^n, z) = V_{n-1}(\mathrm{proj}_{e_n^\perp} B^n) = \kappa_{n-1} \quad (2.6.4)$$

which is a consequence of Cauchy's projection formula (see [79, (5.80)] or [32, (A.45)]).

Proposition 2.6.2. *Let $n \geq 2$ and let $j \in \{1,\dots,n-1\}$. If $\zeta \in T_j^n$, then*

$$\mathrm{t}^*_{j,\zeta}(w_s) = \frac{\kappa_{n-1}}{n}\binom{n}{j} s^{n-j+1}\zeta(s)\, e_n$$

for every $s \geq 0$.

Proof. Throughout the proof, we fix an arbitrary $s > 0$, noting that it suffices to restrict to this case due to continuity. For every $\vartheta \in \mathrm{O}(n)$ such that $\vartheta e_n = e_n$ it follows from the definition of w_s together with the $\mathrm{O}(n)$ equivariance of $\mathrm{t}^*_{j,\zeta}$ that

$$\mathrm{t}^*_{j,\zeta}(w_s) = \mathrm{t}^*_{j,\zeta}(w_s \circ \vartheta^{-1}) = \vartheta\, \mathrm{t}^*_{j,\zeta}(w_s).$$

Thus, $\mathrm{t}^*_{j,\zeta}(w_s)$ must be parallel to e_n, and we therefore focus on calculating the nth coordinate of this vector. Since $w_s \in \mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$ it follows from Lemma 2.4.11 that

$$\big(\mathrm{t}^*_{j,\zeta}(w_s)\big)_n = \int_{\mathbb{R}^n} \zeta(|x|)x_n \,\mathrm{d}\Phi_j(w_s;x).$$

To calculate this integral, we split the domain $\mathbb{R}^n$ into several parts.

- Let $A_0 = \{x \in \mathbb{R}^n : x_n = 0\}$. It follows from (2.6.1) that

  $$\dim(\{x + ty : x \in A_0, y \in \partial w_s(s)\}) < n$$

  for every $t \geq 0$. Thus, by (2.3.2), we have $\Phi_j(w_s;\cdot) \equiv 0$ on A_0.

- Let $A_1 = \{x \in \mathbb{R}^n : |x| < s, x_n > 0\} \cup \{x \in \mathbb{R}^n : |(x_1,\dots,x_{n-1})| < s, x_n < 0\}$, which is an open subset of $\mathbb{R}^n$. Since $w_s \equiv 0$ on A_1, it follows from (2.3.3) that $\Phi_j(w_s;\cdot) \equiv 0$ on A_1.

- The function w_s is of class C^2 on $A_2 = \{x \in \mathbb{R}^n : |x| > s, x_n > 0\}$ and its Hessian matrix at any point x in this set has $(n-1)$ eigenvalues equal to $1/|x|$ and the last eigenvalue equal to zero. Hence,

  $$[\mathrm{D}^2 w_s(x)]_j = \binom{n-1}{j}\frac{1}{|x|^j}$$

  for every $x \in A_2$. By (2.3.3) together with using polar coordinates and (2.6.4) we now have

  $$\begin{aligned}\int_{A_2} \zeta(|x|)x_n \,\mathrm{d}\Phi_j(w_s;x) &= \binom{n-1}{j}\int_{A_2}\frac{\zeta(|x|)x_n}{|x|^j}\,\mathrm{d}x\\ &= \binom{n-1}{j}\int_s^\infty r^{n-j}\zeta(r)\,\mathrm{d}r\int_{\{z\in\mathbb{S}^{n-1}:z_n>0\}} z_n\,\mathrm{d}\mathcal{H}^{n-1}(z)\\ &= \kappa_{n-1}\binom{n-1}{j}\int_s^\infty r^{n-j}\zeta(r)\,\mathrm{d}r.\end{aligned}$$

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- The function w_s is also of class C^2 on $A_3 = \{x \in \mathbb{R}^n : |(x_1, \dots, x_{n-1})| > s, x_n < 0\}$ and the Hessian matrix $\mathrm{D}^2 w_s(x)$ has $(n-2)$ eigenvalues equal to $1/|(x_1, \dots, x_{n-1})|$ and two eigenvalues equal to zero on this set. Thus,

$$[\mathrm{D}^2 w_s(x)]_j = \begin{cases} \binom{n-2}{j} \frac{1}{|(x_1,\dots,x_{n-1})|^j} & \text{if } j \in \{1, \dots, n-2\}, \\ 0 & \text{if } j = n-1, \end{cases}$$

for every $x \in A_3$. Therefore, by (2.3.3) we trivially have

$$\int_{A_3} \zeta(|x|) x_n \, \mathrm{d}\Phi_{n-1}(w_s; x) = 0.$$

Furthermore, for $j \in \{1, \dots, n-2\}$ we use polar coordinates to obtain

$$\begin{aligned} \int_{A_3} \zeta(|x|) x_n \, \mathrm{d}\Phi_j(w_s; x) &= \binom{n-2}{j} \int_{A_3} \frac{\zeta(|x|) x_n}{|(x_1, \dots, x_{n-1})|^j} \, \mathrm{d}x \\ &= -\binom{n-2}{j} \int_{\{y \in \mathbb{R}^{n-1} : |y| > s\}} \frac{1}{|y|^j} \int_0^\infty \zeta(\sqrt{|y|^2 + x_n^2}) x_n \, \mathrm{d}x_n \, \mathrm{d}y \\ &= -(n-1)\kappa_{n-1} \binom{n-2}{j} \int_s^\infty r^{n-2-j} \int_0^\infty \zeta(\sqrt{r^2 + x_n^2}) x_n \, \mathrm{d}x_n \, \mathrm{d}r \\ &= -(n-1)\kappa_{n-1} \binom{n-2}{j} \int_s^\infty r^{n-2-j} \int_r^\infty t\zeta(t) \, \mathrm{d}t \, \mathrm{d}r \\ &= \kappa_{n-1} \binom{n-1}{j} \left(s^{n-1-j} \int_s^\infty t\zeta(t) \, \mathrm{d}t - \int_s^\infty r^{n-j} \zeta(r) \, \mathrm{d}r \right), \end{aligned}$$

where we have implicitly used that $n - 1 \geq 2$, together with the substitution $t = \sqrt{r^2 + x_n^2}$ and integration by parts.

- Let $A_4 = \{x \in \mathbb{R}^n : |x| = s, x_n > 0\} = s\mathbb{S}^{n-1} \cap \{x \in \mathbb{R}^n : x_n > 0\}$. Since Hessian measures are locally determined (see (2.3.4)), it follows that $\Phi_j(w_s; \cdot)$ coincides with $\Phi_j(v_s; \cdot)$ on $\{x \in \mathbb{R}^n : x_n > 0\}$ and, in particular, on A_4. Observe that v_s is a radially symmetric function and therefore, $\Phi_j(v_s; \cdot)$ is invariant under rotations, which implies that $\Phi_j(v_s; \cdot)$, restricted to $s\mathbb{S}^{n-1}$, must be the Haar measure on this set. This means that there exists a constant $c_{n,j,s}$ such that

$$\int_{s\mathbb{S}^{n-1}} b(z) \, \mathrm{d}\Phi_j(v_s; z) = c_{n,j,s} \int_{s\mathbb{S}^{n-1}} b(z) \, \mathrm{d}\mathcal{H}^{n-1}(z)$$

for every Borel measurable function b on $s\mathbb{S}^{n-1}$. By Lemma 2.6.1 we have

$$\kappa_n \binom{n}{j} s^{n-j} = \Phi_j(v_s; s\mathbb{S}^{n-1}) = c_{n,j,s} \, \mathcal{H}^{n-1}(s\mathbb{S}^{n-1}) = c_{n,j,s} \, n\kappa_n s^{n-1},$$

and thus, $c_{n,j,s} = \frac{1}{n}\binom{n}{j}\frac{1}{s^{j-1}}$. Summing up, we obtain

$$\begin{aligned} \int_{A_4} \zeta(|x|) x_n \, \mathrm{d}\Phi_j(w_s; x) &= \int_{A_4} \zeta(|x|) x_n \, \mathrm{d}\Phi_j(v_s; x) \\ &= \frac{1}{n}\binom{n}{j} \frac{1}{s^{j-1}} \int_{s\mathbb{S}^{n-1} \cap \{x \in \mathbb{R}^n : x_n > 0\}} \zeta(|z|) z_n \, \mathrm{d}\mathcal{H}^{n-1}(z) \\ &= \frac{1}{n}\binom{n}{j} s^{n-j+1} \zeta(s) \int_{\{z \in \mathbb{S}^{n-1} : z_n > 0\}} z_n \, \mathrm{d}\mathcal{H}^{n-1}(z) \\ &= \frac{\kappa_{n-1}}{n}\binom{n}{j} s^{n-j+1} \zeta(s) \end{aligned}$$

where we have used (2.6.4).

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- Lastly, let $A_5 = \{x \in \mathbb{R}^n : |(x_1, \dots, x_{n-1})| = s, x_n < 0\}$. Since

$$w_s(x_1, \dots, x_n) = v_s|_{\mathbb{R}^{n-1}}(x_1, \dots, x_{n-1})$$

for every $(x_1, \dots, x_n) \in \{x \in \mathbb{R}^n : x_n < 0\}$, it follows from Lemma 2.3.2 and the fact that Hessian measures are locally determined that

$$\mathrm{d}\Phi_j(w_s; (x_1, \dots, x_n)) = \mathrm{d}\Phi_j^{(n-1)}(v_s|_{\mathbb{R}^{n-1}}; (x_1, \dots, x_{n-1}))\,\mathrm{d}x_n$$

on A_5. By Lemma 2.6.1 we have $\Phi_j^{(n-1)}(v_s|_{\mathbb{R}^{n-1}}; s\mathbb{S}^{n-2}) = \kappa_{n-1}\binom{n-1}{j} s^{n-1-j}$, and thus

$$\begin{aligned}\int_{A_5} \zeta(|x|)x_n\,\mathrm{d}\Phi_j(w_s; x) &= -\int_0^\infty \int_{s\mathbb{S}^{n-2}} \zeta(\sqrt{|y|^2 + x_n^2})x_n\,\mathrm{d}\Phi_j^{(n-1)}(v_s|_{\mathbb{R}^{n-1}}; y)\,\mathrm{d}x_n \\ &= -\kappa_{n-1}\binom{n-1}{j} s^{n-1-j} \int_0^\infty \zeta(\sqrt{s^2 + x_n^2})x_n\,\mathrm{d}x_n \\ &= -\kappa_{n-1}\binom{n-1}{j} s^{n-1-j} \int_s^\infty t\zeta(t)\,\mathrm{d}t.\end{aligned}$$

Since $\mathbb{R}^n$ is the disjoint union of the sets $A_0, \dots, A_n$, we finally obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} \zeta(|x|)x_n\,\mathrm{d}\Phi_j(w_s; x) \\ &\quad= \kappa_{n-1}\binom{n-1}{j}\int_s^\infty r^{n-j}\zeta(r)\,\mathrm{d}r + \kappa_{n-1}\binom{n-1}{j}\left(s^{n-1-j}\int_s^\infty t\zeta(t)\,\mathrm{d}t - \int_s^\infty r^{n-j}\zeta(r)\,\mathrm{d}r\right) \\ &\qquad+ \frac{\kappa_{n-1}}{n}\binom{n}{j}s^{n-j+1}\zeta(s) - \kappa_{n-1}\binom{n-1}{j}s^{n-1-j}\int_s^\infty t\zeta(t)\,\mathrm{d}t \\ &\quad= \frac{\kappa_{n-1}}{n}\binom{n}{j}s^{n-j+1}\zeta(s),\end{aligned}$$

when $j \in \{1, \dots, n-2\}$. Similarly, for $j = n-1$,

$$\begin{aligned}\int_{\mathbb{R}^n} \zeta(|x|)x_n\,\mathrm{d}\Phi_{n-1}(w_s; x) &= \kappa_{n-1}\binom{n-1}{n-1}\int_s^\infty r^{n-(n-1)}\zeta(r)\,\mathrm{d}r + \frac{\kappa_{n-1}}{n}\binom{n}{n-1}s^{n-(n-1)+1}\zeta(s) \\ &\quad- \kappa_{n-1}\binom{n-1}{n-1}s^{n-1-(n-1)}\int_s^\infty t\zeta(t)\,\mathrm{d}t \\ &= \kappa_{n-1}s^2\zeta(s),\end{aligned}$$

which gives the desired result. □

To conclude this section, we show that the statement of Proposition 2.6.2 also holds true for $j = n$, including the case $n = 1$. For this, we first observe that

$$w_s + s = h_{D^n} \vee s$$

for every $s \geq 0$, where we write $D^n = B^n \cap \{x \in \mathbb{R}^n : x_n \geq 0\}$ and where we also use this as a definition of w_s when $n = 1$ (with $D^1 = [0, 1]$). It now follows from [6, Lemma 2] and (2.3.12) that the convex conjugate $(w_s + s)^* \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is the largest lower semicontinuous, convex function that is pointwise bounded from above by $\mathbf{I}_{D^n} \wedge (\mathbf{I}_{\{o\}} - s)$, or equivalently

$$w_s^*(x) = \mathbf{I}_{D^n}(x) + s|x| \tag{2.6.5}$$

for every $x \in \mathbb{R}^n$ and $s \geq 0$.

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Lemma 2.6.3. *If $\zeta \in T_n^n$, then*

$$\mathrm{t}^*_{n,\zeta}(w_s) = \frac{\kappa_{n-1}}{n} s\zeta(s)\, e_n$$

for every $s \geq 0$.

Proof. Let $\zeta \in T_n^n$ be given. As in the proof of Proposition 2.6.2, it follows from the symmetries of w_s and the $\mathrm{O}(n)$ equivariance of $\mathrm{t}^*_{n,\zeta}$, that $\mathrm{t}^*_{n,\zeta}(w_s)$ is parallel to e_n. By (2.3.7) and (2.6.5) we now have

$$\begin{aligned}
\big(\mathrm{t}^*_{n,\zeta}(w_s)\big)_n = \big(\mathrm{t}_{n,\zeta}(w_s^*)\big)_n &= \int_{\mathbb{R}^n} \zeta(|y|) y_n \,\mathrm{dMA}^*(w_s^*; y) \\
&= \int_{D^n} \zeta(|\nabla w_s^*(x)|) \frac{\partial w_s^*}{\partial x_n}(x)\,\mathrm{d}x \\
&= \int_{D^n} \zeta(s) \frac{s x_n}{|x|}\,\mathrm{d}x \\
&= s\zeta(s) \int_{\{z \in \mathbb{S}^{n-1} : z_n \geq 0\}} \int_0^1 z_n r^{n-1}\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^{n-1}(z) \\
&= \frac{1}{n} s\zeta(s) \int_{\{z \in \mathbb{S}^{n-1} : z_n \geq 0\}} z_n\,\mathrm{d}\mathcal{H}^{n-1}(z) \\
&= \frac{\kappa_{n-1}}{n} s\zeta(s)
\end{aligned}$$

for every $s \geq 0$, where we have used (2.6.4) in the last step. □

Remark 2.6.4. Proposition 2.6.2 and Lemma 2.6.3 together with the definition of the classes T_j^n show that $t^*_{j,\zeta}(w_0) = o$ for every $j \in \{1, \dots, n\}$ and $\zeta \in T_j^n$. This follows independently from the fact that $w_0 = h_{D^n}$ and that $K \mapsto t^*_{j,\zeta}(h_K)$ defines a continuous, translation invariant, rotation equivariant valuation on $\mathcal{K}^n$, which by Proposition 2.1.1 must be trivial.

2.6.2 The Abel Transform

To mimic Klain's approach to Hadwiger's theorem [47], we need to understand how the functional Minkowski vectors $\mathrm{t}_{j,\zeta}$ behave on functions with lower dimensional domains. For this we consider the *Abel transform* of a function $\zeta \in C_b((0,\infty))$, which is given by

$$\mathcal{A}\zeta(s) = \int_{-\infty}^{\infty} \zeta(\sqrt{s^2 + t^2})\,\mathrm{d}t$$

for $s > 0$. Applying the Abel transform k times, $k \in \mathbb{N}$, gives

$$\mathcal{A}^k \zeta(s) = \int_{\mathbb{R}^k} \zeta(\sqrt{|x|^2 + s^2})\,\mathrm{d}x \tag{2.6.6}$$

for $s > 0$ and we remark that twofold application of the Abel transform generates a differentiable function (cf. [25, Lemma 2.4]). For $\xi \in C_b^1((0,\infty))$, it is easy to verify that the inverse Abel transform is given by

$$\mathcal{A}^{-1}\xi(s) = -\frac{1}{\pi} \int_s^{\infty} \frac{\xi'(t)}{\sqrt{t^2 - s^2}}\,\mathrm{d}t \tag{2.6.7}$$

for $s > 0$. See, for example, [15, Chapter 13]. For the following result, we write $C_c^\infty([0,\infty))$ for the space of continuous functions with compact support on $[0,\infty)$ functions that are infinitely differentiable on $(0,\infty)$ and have one-sided derivatives of all orders at 0^+.

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Lemma 2.6.5. *Let $\tilde{\psi} \in C_c^\infty([0,\infty))$. If $\xi \in C_c^\infty([0,\infty))$ is such that*

$$\xi(s) = \tilde{\psi}(s^2) \tag{2.6.8}$$

for $s \geq 0$, then there exists $\psi \in C_c^\infty([0,\infty))$ such that

$$\mathcal{A}^{-1}\,\xi(s) = \psi(s^2)$$

for $s \geq 0$.

Proof. Let $\tilde{\psi}$ and ξ be as in (2.6.8). By (2.6.7) we have

$$\mathcal{A}^{-1}\,\xi(s) = -\frac{2}{\pi}\int_s^\infty \frac{\tilde{\psi}'(t^2)t}{\sqrt{t^2-s^2}}\,\mathrm{d}t = -\frac{2}{\pi}\int_0^\infty \tilde{\psi}'(s^2+r^2)\,\mathrm{d}r$$

for $s > 0$, where we have used the substitution $r = \sqrt{t^2-s^2}$. Since $\tilde{\psi}$ has compact support, this expression and all its derivatives continuously extend to $s = 0$, completing the proof. □

Lemma 2.6.6. *Let $k \in \{1,\dots,n-1\}$, $j \in \{1,\dots,k\}$ and $\zeta \in C_c([0,\infty))$. If $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is such that $\mathrm{dom}(u) \subseteq \mathbb{R}^k$, then*

$$\int_{\mathbb{R}^n} \zeta(|y|)y_i\,\mathrm{d}\Psi_j(u;y) = \int_{\mathbb{R}^k} (\mathcal{A}^{n-k}\,\zeta)(|z|)z_i\,\mathrm{d}\Psi_j^{(k)}(u|_{\mathbb{R}^k};z)$$

for every $i \in \{1,\dots,k\}$ and

$$\int_{\mathbb{R}^n} \zeta(|y|)y_i\,\mathrm{d}\Psi_j(u;y) = 0$$

for every $i \in \{k+1,\dots,n\}$.

Proof. If $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is such that $\mathrm{dom}(u) \subseteq \mathbb{R}^k$, then there exists $w \in \mathrm{Conv}(\mathbb{R}^k;\mathbb{R})$ such that

$$u^*(x_1,\dots,x_n) = w(x_1,\dots,x_k)$$

for every $(x_1,\dots,x_n) \in \mathbb{R}^n$. Thus, by (2.3.5), Lemma 2.3.2 together with Fubini's theorem, and (2.6.6), we now have

$$\begin{aligned}
\int_{\mathbb{R}^n} \zeta(|y|)y_i\,\mathrm{d}\Psi_j(u;y) &= \int_{\mathbb{R}^n} \zeta(|x|)x_i\,\mathrm{d}\Phi_j(u^*;x)\\
&= \int_{\mathbb{R}^k}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \zeta(\sqrt{|z|^2+|(x_{k+1},\dots,x_n)|^2})z_i\,\mathrm{d}x_{k+1}\cdots\mathrm{d}x_n\,\mathrm{d}\Phi_j^{(k)}(w;z)\\
&= \int_{\mathbb{R}^k} (\mathcal{A}^{n-k}\,\zeta)(|z|)z_i\,\mathrm{d}\Phi_j^{(k)}(w;z)\\
&= \int_{\mathbb{R}^k} (\mathcal{A}^{n-k}\,\zeta)(|z|)z_i\,\mathrm{d}\Psi_j^{(k)}(u|_{\mathbb{R}^k};z)
\end{aligned}$$

for every $i \in \{1,\dots,k\}$, where we have used that $w^* = u|_{\mathbb{R}^k}$ when considering convex conjugation with respect to the ambient space $\mathbb{R}^k$. Similarly, for $i \in \{k+1,\dots,n\}$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \zeta(|y|)y_i\,\mathrm{d}\Psi_j(u;y) &= \int_{\mathbb{R}^k}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \zeta(\sqrt{|z|^2+|(x_{k+1},\dots,x_n)|^2})x_i\,\mathrm{d}x_{k+1}\cdots\mathrm{d}x_n\,\mathrm{d}\Phi_j^{(k)}(w;z)\\
&= 0,
\end{aligned}$$

where we have used that $x_i \mapsto \zeta(\sqrt{|z|^2+|(x_{k+1},\dots,x_n)|^2})x_i$ is an odd function on $\mathbb{R}$. □

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2.6.3 Classification of Smooth Valuations

Our proof of Theorem C is based on the extended Klain approach developed in [23], which in turn is based on Klain's strategy of reproving Hadwiger's theorem in [47]. See also [16]. Based on a classification of simple valuations, Klain used an induction argument that involves extending valuations defined on lower-dimensional convex bodies to valuations defined on general convex bodies. As the results of Section 2.6.2 show, such an extension is not possible in general for continuous functional Minkowski vectors since the relevant densities may not be in the image of the Abel transform (see [29] for a related problem). Thus, in order to prove Theorem C, we will first establish a classification result under additional smoothness assumptions. Here, a continuous, dually epi-translation invariant valuation $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is called *smooth* when each of the coordinates of

$$K \mapsto \mathrm{z}(h_K(\cdot,-1))$$

defines a smooth valuation on $\mathcal{K}^{n+1}$ (see [52, Proposition 6.4]). A continuous, translation invariant valuation $\mathrm{Z}\colon \mathcal{K}^{n+1} \to \mathbb{R}$ is in turn *smooth* when $\vartheta \mapsto \mathrm{Z}\circ\vartheta$ is an infinitely differentiable map from $\mathrm{GL}(n)$ into the Banach space of continuous, translation invariant valuations on $\mathcal{K}^{n+1}$ (we refer to [79, Section 6.5] for an overview). For our purposes, the following simple result will be sufficient. Cf. [82, Theorem 4.1] or [23, (2.1)].

Lemma 2.6.7. *Let $j \in \{1,\dots,n\}$ and let $\varphi\colon \mathbb{S}^n \to \mathbb{R}$ be continuous. The valuation*

$$K \mapsto \int_{\mathbb{S}^n} \varphi(z)\,\mathrm{d}S_j(K,z), \quad K \in \mathcal{K}^{n+1},$$

is smooth, if and only if φ is smooth.

For the announced classification result for smooth valuations, we first need to show that the relevant valuations are smooth. For this, we need the following consequence of [86, Theorem 2].

Lemma 2.6.8. *Let $i \in \{1,\dots,n\}$ and $\alpha \in T_n^n$. The map*

$$x \mapsto \alpha(|x|)x_i$$

is of class C^∞ on $\mathbb{R}^n$, if and only if there exists $\psi \in C_c^\infty([0,\infty))$ such that $\alpha(s) = \psi(s^2)$ for every $s > 0$.

Proposition 2.6.9. *Let $j \in \{1,\dots,n\}$ and $\alpha \in T_n^n$. The valuation*

$$v \mapsto \int_{\mathbb{R}^n} \alpha(|x|)x\,\mathrm{d}\Phi_j(v;x), \quad v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}), \tag{2.6.9}$$

is smooth, if and only if there exists $\psi \in C_c^\infty([0,\infty))$ such that $\alpha(s) = \psi(s^2)$ for every $s > 0$.

Proof. Throughout the proof, we consider the ith coordinate of (2.6.9) with some arbitrary, fixed $i \in \{1,\dots,n\}$. If $v(x) = h_K(x,-1)$, $x \in \mathbb{R}^n$, with some $K \in \mathcal{K}^{n+1}$, then (2.4.8) and Theorem 2.5.10 show that

$$\int_{\mathbb{R}^n} \alpha(|x|)x_i\,\mathrm{d}\Phi_j(v;x) = \big(\mathrm{t}^*_{j,\alpha}(v)\big)_i = \binom{n}{j}\int_{\mathbb{S}^n_-} (\mathcal{T}^{n-j}\,\alpha)(|\operatorname{gno}(z)|)z_i\,\mathrm{d}S_j(K,z).$$

Thus, by Lemma 2.6.7, the ith coordinate of (2.6.9) is smooth, if and only if

$$\varphi(z) = (\mathcal{T}^{n-j}\,\alpha)(|\operatorname{gno}(z)|)z_i, \quad z \in \mathbb{S}^n_-,$$

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is smooth, where we remark that we can trivially extend φ from $\mathbb{S}^n_-$ to $\mathbb{S}^n$ with $\varphi(z) = 0$ on $\mathbb{S}^n \setminus \mathbb{S}^n_-$ since α has bounded support. As the gnomonic projection and its inverse are smooth, smoothness of φ is equivalent to smoothness of

$$x \mapsto \varphi(\mathrm{gno}^{-1}(x)) = (\mathcal{T}^{n-j}\,\alpha)(|x|)\sqrt{1+|x|^2}\,x_i$$

on $\mathbb{R}^n$. By Lemma 2.6.8 together with the properties of $\mathcal{T}$ and its inverse, this is equivalent to $\alpha(s) = \psi(s^2)$, $s > 0$, for some $\psi \in C_c^\infty([0,\infty))$. □

Remark 2.6.10. We remark that the statement of Proposition 2.6.9 has previously been claimed in the more general [25, Lemma 4.6]. However, the proof of the latter is erroneous, and it, therefore, cannot be used for the situation considered here. Cf. Remark 2.5.11.

Corresponding to the definition of smooth valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, we call a continuous, epi-translation invariant valuation $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ *smooth* when

$$v \mapsto \mathrm{z}(v^*), \quad v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}),$$

defines a smooth valuation. We, therefore, immediately obtain the following dual version of Proposition 2.6.9.

Proposition 2.6.11. *Let $j \in \{1,\dots,n\}$ and $\alpha \in T^n_n$. The valuation*

$$u \mapsto \int_{\mathbb{R}^n} \alpha(|y|)y\,\mathrm{d}\Psi_j(u;y), \quad u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n),$$

is smooth, if and only if there exists $\psi \in C_c^\infty([0,\infty))$ such that $\alpha(s) = \psi(s^2)$ for every $s > 0$.

We now treat the one-dimensional case of the main result of this section.

Proposition 2.6.12. *A map $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}) \to \mathbb{R}$ is a smooth, epi-translation invariant, reflection equivariant valuation, if and only if there exists $\psi \in C_c^\infty([0,\infty))$ such that*

$$\mathrm{z}(u) = \int_{\mathbb{R}} \psi(|y|^2)y\,\mathrm{d}\Psi_1(u;y) \tag{2.6.10}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R})$.

Proof. It follows from Proposition 2.3.6 that (2.6.10) defines a continuous, epi-translation invariant, reflection equivariant valuation. Furthermore, by Proposition 2.6.11, the valuation is also smooth.

Conversely, if z is a smooth, epi-translation invariant, reflection equivariant valuation, then it follows from Theorem 2.3.8 that z can be decomposed in epi-homogeneous terms with degrees 0 and 1. Theorem 2.3.9 and Theorem 2.3.10, together with Proposition 2.6.11, now show that z must be as in (2.6.10). □

The following result is a reformulation of [22, Lemma 8.3], where it was obtained as a direct consequence of the functional Hadwiger theorem together with the properties of the transform $\mathcal{R}$. In fact, Proposition 2.6.13 can be regarded as equivalent to the functional Hadwiger theorem, and we note that a new proof of this statement was recently presented in [54, Section 5]. Here, we use the family of functions $u_s \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, $s \geq 0$, defined as $u_s(x) = v_s^*(x) = \mathbf{I}_{B^n} + s|x|$ for $x \in \mathbb{R}^n$ (cf. (2.6.3)).

Proposition 2.6.13. *Let $n \geq 2$ and $j \in \{0,\dots,n\}$. If $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous, epi-translation and rotation invariant valuation that is epi-homogeneous of degree j such that $\mathrm{Z}(u_s) = 0$ for every $s \geq 0$, then $\mathrm{Z} \equiv 0$.*

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Throughout the remainder of this section we embed $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$ via the identification $\mathbb{R}^{n-1} = e_n^\perp$ and consider $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$ as the set of all $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ such that $\mathrm{dom}(u) \subseteq \mathbb{R}^{n-1}$.

Lemma 2.6.14. *Let $n \geq 3$. If $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is a continuous, epi-translation invariant, rotation equivariant valuation, then $\mathrm{z}(u) \in \mathbb{R}^{n-1}$ for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$. For $n = 2$, the statement holds if we replace rotation equivariance with $\mathrm{O}(2)$ equivariance.*

Proof. Let $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ be given and define $\tilde{\mathrm{Z}}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1}) \to \mathbb{R}$ as the nth coordinate of the restriction of z to $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$. That is,

$$\tilde{\mathrm{Z}}(u) = (\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u))_n$$

for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$. We need to show that $\tilde{\mathrm{Z}}$ vanishes identically.

We start with the case $n \geq 3$. By Theorem 2.3.8, applied coordinate-wise, we may assume without loss of generality that z is epi-homogeneous of degree $j \in \{0, \dots, n\}$. If $j = n$, then Theorem 2.3.10 immediately shows that z vanishes on functions with lower dimensional domains and thus $\tilde{\mathrm{Z}} \equiv 0$. For $j < n$, we observe that the properties of z imply that also $\tilde{\mathrm{Z}}$ is a continuous, epi-translation invariant valuation that is epi-homogeneous of degree j. Now for $\vartheta \in \mathrm{SO}(n-1)$ let $\Theta \in \mathrm{SO}(n)$ be given by

$$\Theta = \begin{pmatrix} \vartheta & 0 \\ 0 & 1 \end{pmatrix}.$$

The rotation equivariance of z shows that

$$\tilde{\mathrm{Z}}(u \circ \vartheta^{-1}) = (\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u \circ \Theta^{-1}))_n = (\Theta\,\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u))_n = (\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u))_n = \tilde{\mathrm{Z}}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$ and, therefore, $\tilde{\mathrm{Z}}$ is also rotation invariant. Next, for $s \geq 0$ let $\tilde{u}_s \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$ be given by $\tilde{u}_s(x) = \mathbf{I}_{B^{n-1}}(x) + s|x|$. For $\eta = \mathrm{diag}(1, \dots, 1, -1, -1) \in \mathrm{SO}(n)$ we have

$$\begin{aligned}
\tilde{\mathrm{Z}}(\tilde{u}_s) &= (\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u_s))_n \\
&= (\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u_s \circ \eta^{-1}))_n \\
&= (\eta\,\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u_s))_n \\
&= -(\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u_s))_n \\
&= -\tilde{\mathrm{Z}}(u_s)
\end{aligned}$$

for every $s \geq 0$. Proposition 2.6.13, applied with respect to an ambient space of dimension $n - 1 \geq 2$, now shows that $\tilde{\mathrm{Z}}(u) = 0$ for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$.

Lastly, when $n = 2$ and z is $\mathrm{O}(2)$ equivariant, we use $\Theta = \mathrm{diag}(1, -1) \in \mathrm{O}(2)$ to obtain

$$\tilde{\mathrm{Z}}(u) = (\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R})}(u \circ \Theta^{-1}))_2 = (\Theta\,\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R})}(u))_2 = -(\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R})}(u))_2 = -\tilde{\mathrm{Z}}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R})$, which shows that $\tilde{\mathrm{Z}} \equiv 0$. □

Theorem 2.6.15. *For $n \geq 3$, a map $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is a smooth, epi-translation invariant, rotation equivariant valuation, if and only if there exist functions $\psi_1, \dots, \psi_n \in C_c^\infty([0, \infty))$ such that*

$$\mathrm{z}(u) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} \psi_j(|y|^2) y \, \mathrm{d}\Psi_j(u; y) \tag{2.6.11}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. For $n \leq 2$, the same representation holds if we replace rotation equivariance with $\mathrm{O}(n)$ equivariance.

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Proof. If $\psi_1, \dots, \psi_n \in C_c^\infty([0,\infty))$, then it follows from Proposition 2.3.6 together with Proposition 2.6.11 that (2.6.11) is a smooth, epi-translation invariant, $\mathrm{O}(n)$ equivariant valuation.

Conversely, let $\mathrm{z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ be a smooth, epi-translation invariant, rotation equivariant (or $\mathrm{O}(n)$ equivariant for $n \leq 2$) valuation. We use induction on the dimension n to show that z is as in (2.6.11). For the base case $n = 1$, the statement follows from Proposition 2.6.12. Now let $n \geq 2$ and assume that the statement is true for the case $n-1$. We denote by $\tilde{\mathrm{z}}$ the restriction of z to $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$ and note that Lemma 2.6.14 shows that $\tilde{\mathrm{z}}$ takes values in $\mathbb{R}^{n-1}$. Clearly, $\tilde{\mathrm{z}}$ is a smooth, epi-translation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$. Now for $\vartheta \in \mathrm{O}(n-1)$ let $\Theta \in \mathrm{SO}(n)$ be given by

$$\Theta = \begin{pmatrix} \vartheta & 0 \\ 0 & \det(\vartheta) \end{pmatrix}.$$

The rotation equivariance of z shows that

$$\tilde{\mathrm{z}}(u \circ \vartheta^{-1}) = \mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u \circ \Theta^{-1}) = \Theta\,\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u) = \vartheta \tilde{\mathrm{z}}(u),$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$, where we have used that the nth coordinate of $\mathrm{z}\,|_{\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})}(u)$ vanishes. Thus, $\tilde{\mathrm{z}}$ is also $\mathrm{O}(n-1)$ equivariant (and in particular $\mathrm{SO}(n-1)$ equivariant for $n-1 \geq 3$) and it follows from the induction assumption that there exist functions $\tilde{\psi}_1, \dots, \tilde{\psi}_{n-1} \in C_c^\infty([0,\infty))$ such that

$$\tilde{\mathrm{z}}(u) = \sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} \tilde{\psi}_j(|y|^2) y \, \mathrm{d}\Psi_j^{(n-1)}(u; y)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^{n-1})$. By Lemma 2.6.5 and Lemma 2.6.6 there exist functions $\psi_j \in C_c^\infty([0,\infty))$, $j \in \{1, \dots, n-1\}$, such that

$$\mathrm{z}(u) = \sum_{j=1}^{n-1} \int_{\mathbb{R}^n} \psi_j(|y|^2) y \, \mathrm{d}\Psi_j^{(n)}(u; y) \tag{2.6.12}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ such that $\mathrm{dom}(u) \subseteq \mathbb{R}^{n-1}$. Now let $\bar{\mathrm{z}}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ be given by

$$\bar{\mathrm{z}}(u) = \mathrm{z}(u) - \sum_{j=1}^{n-1} \int_{\mathbb{R}^n} \psi_j(|y|^2) y \, \mathrm{d}\Psi_j^{(n)}(u; y) \tag{2.6.13}$$

for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. It follows from the assumptions on z together with the first part of the proof that $\bar{\mathrm{z}}$ is a smooth, epi-translation invariant, rotation equivariant (or $\mathrm{O}(2)$ equivariant when $n = 2$) valuation. Moreover, if $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is such that $\mathrm{dom}(u) \subseteq E$ for some $(n-1)$-dimensional linear subspace $E \subset \mathbb{R}^n$, then we can find a rotation $\vartheta_E \in \mathrm{SO}(n)$ such that $\mathrm{dom}(u \circ \vartheta_E^{-1}) \subseteq \mathbb{R}^{n-1}$. The rotation equivariance of $\bar{\mathrm{z}}$ and (2.6.12) now show that

$$\bar{\mathrm{z}}(u) = \vartheta_E^{-1}\, \bar{\mathrm{z}}(u \circ \vartheta_E^{-1}) = o.$$

Together with the epi-translation invariance of $\bar{\mathrm{z}}$, this implies that $\bar{\mathrm{z}}$ is simple. Thus, it follows from Theorem 2.3.11, and Proposition 2.6.11 that there exists $\psi_n \in C_c^\infty([0,\infty))$ such that

$$\bar{\mathrm{z}}(u) = \int_{\mathbb{R}^n} \psi_n(|y|^2) y \, \mathrm{d}\Psi_n^{(n)}(u; y)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Together with (2.6.13), this gives the desired representation (2.6.11). □

We close this section with the equivalent formulation of the last result for smooth valuations on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$.

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Theorem 2.6.16. *For $n \geq 3$, a map $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is a smooth, dually epi-translation invariant, rotation equivariant valuation, if and only if there exist functions $\psi_1, \dots, \psi_n \in C_c^\infty([0,\infty))$ such that*

$$\mathrm{z}(v) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} \psi_j(|x|^2) x \,\mathrm{d}\Phi_j(v;x)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. For $n \leq 2$, the same representation holds if we replace rotation equivariance with $\mathrm{O}(n)$ equivariance.

2.6.4 Approximation and Support

The following result, which shows that continuous valuations can be approximated by smooth valuations, is a consequence of [52, Theorem 1].

Lemma 2.6.17. *If $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is a continuous, dually epi-translation invariant valuation that is homogeneous of degree j, then there exists a sequence of smooth, dually epi-translation invariant valuations $\mathrm{z}_k\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$, $k \in \mathbb{N}$, that converges to z and such that each of the z_k is homogeneous of degree j.*

Here, the relevant spaces of valuations are equipped with the topology of uniform convergence on compact subsets, for which we have the following description [50, Proposition 2.4].

Lemma 2.6.18. *A set $V \subset \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is relatively compact, if and only if for every compact subset $A \subset \mathbb{R}^n$ there exists a constant c_A such that*

$$\sup_{x\in A} |v(x)| \leq c_A$$

for every $v \in V$.

A version of Lemma 2.6.17 is known in which the approximating sequence for rotation invariant z can also be selected as rotation invariant [52, Proposition 6.6]. For our purposes, however, we need a variant for rotation equivariant valuations.

Lemma 2.6.19. *Let z and z_k, $k \in \mathbb{N}$, be as in Lemma 2.6.17. If z is rotation equivariant, then also the valuations z_k, $k \in \mathbb{N}$, can be chosen as rotation equivariant. Similarly, if z_k is $\mathrm{O}(n)$ equivariant, then also z_k, $k \in \mathbb{N}$, can be chosen to have this property.*

Proof. Let a continuous, dually epi-translation invariant, rotation equivariant valuation z on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ be given that is homogeneous of degree j, and let z_k, $k \in \mathbb{N}$, be an approximating sequence of smooth valuations as in Lemma 2.6.17. Define $\bar{\mathrm{z}}_k\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ as

$$\bar{\mathrm{z}}_k(v) = \int_{\mathrm{SO}(n)} \vartheta\, \mathrm{z}_k(v \circ \vartheta)\,\mathrm{d}\vartheta$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $k \in \mathbb{N}$, where we integrate with respect to the Haar probability measure on $\mathrm{SO}(n)$. Clearly, each $\bar{\mathrm{z}}_k$ is a smooth, epi-translation invariant valuation, where the proof of smoothness is analogous to [52, Proposition 6.6]. Moreover,

$$\bar{\mathrm{z}}_k(v \circ \eta^{-1}) = \int_{\mathrm{SO}(n)} \vartheta\, \mathrm{z}_k(v \circ \eta^{-1} \circ \vartheta)\,\mathrm{d}\vartheta = \int_{\mathrm{SO}(n)} \eta\,\varphi\, \mathrm{z}_k(v \circ \varphi)\,\mathrm{d}\varphi = \eta\, \bar{\mathrm{z}}_k(v)$$

for every $\eta \in \mathrm{SO}(n)$ and $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where we used $\varphi = \eta^{-1}\vartheta$. Thus, the valuations $\bar{\mathrm{z}}_k$, $k \in \mathbb{N}$, are also rotation equivariant.

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It remains to show that $\bar{\mathrm{z}}_k$ converges to z as $k \to \infty$. For this, let V be a compact subset of $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and let $\varepsilon > 0$ be arbitrary. It easily follows from Lemma 2.6.18 that also the set $\bar{V} = \{v \circ \vartheta : v \in V, \vartheta \in \mathrm{SO}(n)\}$ is compact. Thus, by the properties of the sequence z_k, there exists $k_0 \in \mathbb{N}$ such that

$$|\,\mathrm{z}_k(v) - \mathrm{z}(v)| < \varepsilon$$

for every $v \in \bar{V}$ and $k \geq k_0$. Using that $\int_{\mathrm{SO}(n)} \vartheta\, \mathrm{z}(v \circ \vartheta)\,\mathrm{d}\vartheta = \mathrm{z}(v)$ as z is rotation equivariant, we now have

$$\begin{aligned}
|\bar{\mathrm{z}}_k(v) - \mathrm{z}(v)| &= \left| \int_{\mathrm{SO}(n)} \vartheta\, \mathrm{z}_k(v \circ \vartheta)\,\mathrm{d}\vartheta - \int_{\mathrm{SO}(n)} \vartheta\, \mathrm{z}(v \circ \vartheta)\,\mathrm{d}\vartheta \right| \\
&\leq \int_{\mathrm{SO}(n)} |\vartheta\,(\mathrm{z}_k(v \circ \vartheta) - \mathrm{z}(v \circ \vartheta))|\,\mathrm{d}\vartheta \\
&= \int_{\mathrm{SO}(n)} |\mathrm{z}_k(v \circ \vartheta) - \mathrm{z}(v \circ \vartheta)|\,\mathrm{d}\vartheta \\
&< \varepsilon
\end{aligned}$$

for every $v \in V$ and $k \geq k_0$, which shows that $\bar{\mathrm{z}}_k$ converges uniformly to z on compact subsets.

Lastly, we remark that the proof for $\mathrm{O}(n)$ equivariant valuations is analogous to the above. □

We continue with the *support* $\mathrm{supp}(\mathrm{z})$ of a continuous, dually epi-translation invariant valuation $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$, for which we have the following description due to [50, Proposition 6.3] and where we remark that each such valuation has compact support [50, Corollary 6.2].

Lemma 2.6.20. *Let $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ be a continuous, dually epi-translation invariant valuation. The support of z is minimal (with respect to inclusion) among all closed sets $A \subset \mathbb{R}^n$ with the following property: if $v, w \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ are such that $v = w$ on an open neighborhood of A, then $\mathrm{z}(v) = \mathrm{z}(w)$.*

We give a simple proof of the following result, which can be independently derived from [51, Corollary 2.11].

Lemma 2.6.21. *Let $j \in \{1,\dots,n\}$ and $\zeta \in T_j^n$. If $R = \max\{|x| : x \in \mathrm{supp}(\mathrm{t}^*_{j,\zeta})\}$, then $\mathrm{supp}(\zeta) \subseteq [0,R]$.*

Proof. Assume that there exists $\bar{s} > R$ such that $\zeta(\bar{s}) \neq 0$. By definition, the function $w_{\bar{s}}$ coincides with the constant zero function $\mathbf{0} \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ on a neighborhood of RB^n. However, $\mathrm{t}^*_{j,\zeta}(\mathbf{0}) = o$, while Proposition 2.6.2 and the choice of $\bar{s}$ show that $\mathrm{t}^*_{j,\zeta}(w_{\bar{s}}) \neq o$. By Lemma 2.6.20, this is a contradiction. □

Our interest in the support of the valuations $\mathrm{t}^*_{j,\zeta}$ stems from the following result, which was shown in [50, Proposition 1.2].

Proposition 2.6.22. *If $\mathrm{z}_k\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$, $k \in \mathbb{N}$, is a sequence of continuous, dually epi-translation invariant valuations that converges to a continuous, dually epi-translation invariant valuation $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$, then there exists a compact set $A \subset \mathbb{R}^n$ such that the supports of z and z_k, $k \in \mathbb{N}$, are contained in A.*

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2.6.5 Proof of Theorem C

For given $\zeta_1 \in T_1^n, \dots, \zeta_n \in T_n^n$ it follows from Theorem A that $\mathrm{z} = \sum_{j=1}^n \mathrm{t}^*_{j,\zeta_j}$ defines a continuous, dually epi-translation invariant, rotation equivariant valuation. In particular, this valuation is also $\mathrm{O}(n)$ equivariant.

Now let a continuous, dually epi-translation invariant, rotation equivariant (or $\mathrm{O}(n)$ equivariant when $n \leq 2$) valuation $\mathrm{z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ be given. Considering Theorem 2.3.7 we may assume that z is homogeneous of degree $j \in \{0,1,\dots,n\}$. By Theorem 2.3.9 and Theorem 2.3.10 together with duality it is enough to consider $j \in \{1,\dots,n-1\}$. Using Lemma 2.6.17, we can find a sequence of smooth, dually epi-translation invariant, rotation equivariant (or $\mathrm{O}(n)$ equivariant) valuations $\mathrm{z}_k\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$, $k \in \mathbb{N}$, each homogeneous of degree j, that converge to z. Theorem 2.6.16 shows that for each $k \in \mathbb{N}$ there exists $\zeta_k \in C_c^\infty([0,\infty))$ (of the form $\zeta_k(t) = \psi_k(t^2)$) such that

$$\mathrm{z}_k(v) = \int_{\mathbb{R}^n} \zeta_k(|x|)x \,\mathrm{d}\Phi_j(v;x) = \mathrm{t}^*_{j,\zeta_k}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where the second equation follows from (2.4.8). Since trivially $\zeta_k \in T_j^n$, it follows from Proposition 2.6.2 that

$$\big(\mathrm{z}_k(w_s)\big)_n = \binom{n}{j}\frac{\kappa_{n-1}}{n}\beta_k(s)$$

for every $s \geq 0$ and $k \in \mathbb{N}$, where $\beta_k \in C_c([0,\infty))$ is such that $\beta_k(s) = s^{n-j+1}\zeta_k(s)$ for $s > 0$ and $\beta_k(0) = 0$. In particular, $\big(\mathrm{z}_k(w_0)\big)_n = 0$. Let now $\beta\colon [0,\infty) \to \mathbb{R}$ be such that

$$\big(\mathrm{z}(w_s)\big)_n = \binom{n}{j}\frac{\kappa_{n-1}}{n}\beta(s)$$

for $s \geq 0$. For any compact set $C \subset [0,\infty)$ it follows from Lemma 2.6.18 that the sequence z_k converges uniformly to z on $\{w_s : s \in C\}$ as $k \to \infty$. This implies that the functions $\beta_k(s)$ converge uniformly to β on any such set C, which implies that $\beta \in C([0,\infty))$ and that $\beta(0) = 0$. We remark that these properties can also be derived from the properties of z. Furthermore, Lemma 2.6.21 and Proposition 2.6.22 show that there exists $R > 0$ such that $\mathrm{supp}(\zeta_k), \mathrm{supp}(\beta_k) \subseteq [0,R]$ for every $k \in \mathbb{N}$. Thus, we conclude that also $\beta \in C_c([0,\infty))$ and furthermore, that β_k converges uniformly to β on $[0,\infty)$.

We now set $\zeta(s) = \beta(s)/s^{n-j+1}$ for $s > 0$. The properties of β and the definition of the class T_j^n imply that $\zeta \in T_j^n$. We want to show that $\mathrm{z} = \mathrm{t}^*_{j,\zeta}$. By Theorem B this is equivalent to

$$\mathrm{z}(v) = \int_{\mathbb{R}^n} \xi(x)\,\mathrm{dMA}_j(v;x) \tag{2.6.14}$$

for every $v \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, where $\xi(x) = \binom{n}{j}(\mathcal{R}^{n-j}\zeta)(|x|)x$ for $x \in \mathbb{R}^n \setminus \{o\}$ and $\xi(o) = o$. In particular, Lemma 2.3.4 and Lemma 2.4.2 show that $\xi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$. Similarly, for $k \in \mathbb{N}$ let $\xi_k \in C_c(\mathbb{R}^n;\mathbb{R}^n)$ be given by $\xi_k(x) = \binom{n}{j}(\mathcal{R}^{n-j}\zeta_k)(|x|)x$ for $x \in \mathbb{R}^n \setminus \{o\}$ and $\xi_k(o) = o$. Since β_k converges uniformly to β on $[0,\infty)$, it follows from

$$\begin{aligned}
|\xi_k(x) - \xi(x)| &= \binom{n}{j}\left|(\beta_k(|x|) - \beta(|x|))\frac{x}{|x|} + (n-j)\left(\int_{|x|}^\infty \frac{\beta_k(t) - \beta(t)}{t^2}\,\mathrm{d}t\right)x\right| \\
&\leq \binom{n}{j}\left(\|\beta_k - \beta\|_\infty + (n-j)\|\beta_k - \beta\|_\infty \int_{|x|}^\infty \frac{1}{t^2}\,\mathrm{d}t\;|x|\right) \\
&= (1+n-j)\binom{n}{j}\|\beta_k - \beta\|_\infty
\end{aligned}$$

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for $x \in \mathbb{R}^n \setminus \{o\}$ and $k \in \mathbb{N}$, that also ξ_k converges uniformly to ξ on $\mathbb{R}^n$ as $k \to \infty$. Moreover, $\operatorname{supp}(\xi_k), \operatorname{supp}(\xi) \subseteq RB^n$ for every $k \in \mathbb{N}$. Thus, using Theorem B together with the dominated convergence theorem and the fact that $\mathrm{MA}_j(v; RB^n) < \infty$, we obtain

$$\mathrm{z}(v) = \lim_{k\to\infty} \mathrm{t}^*_{j,\zeta_k}(v) = \lim_{k\to\infty} \int_{\mathbb{R}^n} \xi_k(x)\,\mathrm{dMA}_j(v;x) = \int_{\mathbb{R}^n} \xi(x)\,\mathrm{dMA}_j(v;x)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, which shows (2.6.14).

Lastly, if $\mathrm{z} = \sum_{j=1}^n \mathrm{t}^*_{j,\zeta_j}$ with $\zeta_j \in T^n_j$ for $j \in \{1,\dots,n\}$, then by homogeneity together with Proposition 2.6.2 and Lemma 2.6.3 we have

$$\mathrm{z}(\lambda\, w_s) = \frac{\kappa_{n-1}}{n} \sum_{j=1}^{n} \binom{n}{j} \lambda^j s^{n-j+1} \zeta_j(s) e_n$$

for every $s, \lambda \geq 0$, which uniquely determines the densities ζ_j. □

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3 Additive Kinematic Formulas for Functional Minkowski Vectors

3.1 Introduction

Let $\mathcal{K}^n$ denote the set of *convex bodies* in $\mathbb{R}^n$, that is, the set of non-empty, compact, and convex subsets of $\mathbb{R}^n$ where $n \in \mathbb{N}$. One of the most relevant functionals defined on $\mathcal{K}^n$ is the volume of a convex body K, meaning its n-dimensional Lebesgue measure that we denote by $V_n(K)$.

Let B^n denote the n-dimensional Euclidean unit ball in $\mathbb{R}^n$. For $\lambda > 0$, we have the following Steiner formula

$$V_n(K + \lambda B^n) = \sum_{j=0}^{n} \lambda^{n-j} \kappa_{n-j} V_j(K)$$

where κ_k is the k-dimensional volume of the unit ball in $\mathbb{R}^k$ and

$$K + L = \{x + y;\ x \in K \text{ and } y \in L\}$$

denotes the Minkowski sum of K and L. The functionals V_j are called the intrinsic volumes. An important formula involving the intrinsic volumes in integral geometry is given in the following theorem, which is known as the additive kinematic formula.

Theorem 3.1.1 ([46], Theorem 5.13)**.** *Let $0 \leq j \leq n$ and $K, L \in \mathcal{K}^n$, then*

$$\int_{\mathrm{SO}(n)} V_j(K + \vartheta L)\, \mathrm{d}\vartheta = \sum_{j=0}^{k} \frac{\binom{2n-j}{n-j} \kappa_{n-k} \kappa_{n+k-j}}{\binom{2n-j}{n-k} \kappa_n \kappa_{n-j}} V_k(K) V_{j-k}(L),$$

where we denote by $\mathrm{SO}(n)$ the group of special orthogonal transformations on $\mathbb{R}^n$, and $\mathrm{d}\vartheta$ denotes integration with respect to the Haar probability measure on $\mathrm{SO}(n)$.

An elegant proof of this result relies on Hadwiger's characterization theorem, which is given below, through the notion of a *valuation*. We say that a map $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is a valuation if the following holds for every $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$,

$$\mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) = \mathrm{Z}(K) + \mathrm{Z}(L).$$

Theorem 3.1.2 ([38], Satz IV)**.** *A map $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is a continuous, translation and rotation invariant valuation if and only if there exists $c_0, \ldots, c_n \in \mathbb{R}$ such that*

$$\mathrm{Z}(K) = \sum_{j=0}^{n} c_j V_j(K).$$

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Here, the continuity of operators on $\mathcal{K}^n$ is understood to be with respect to the Hausdorff metric; see [79, Section 1.8]. The map Z is said to be translation invariant if $\mathrm{Z}(K+x)=\mathrm{Z}(K)$ for all $K\in\mathcal{K}^n$ and for $x\in\mathbb{R}^n$ and is said to be rotation invariant if $\mathrm{Z}(\vartheta K)=\mathrm{Z}(K)$ for all $K\in\mathcal{K}^n$ and all $\vartheta\in\mathrm{SO}(n)$.

We point out that other additive kinematic formulas were derived for more general valuations in [11, 43, 44] and for different groups that act on the space of convex bodies in [10, 16]. For general references on integral geometry, see [49, 80].

This article aims to establish an additive kinematic formula for the functional Minkowski vectors introduced in [69] and characterized in [70]. Let

$$\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})=\{v:\mathbb{R}^n\to\mathbb{R}:v\text{ is convex}\}$$

denote the space of (finite-valued) convex functions on $\mathbb{R}^n$. We equip this space and its subspaces with the topology of *epi-convergence* and note that in this setting, this topology coincides with the topology of pointwise convergence. We observe that the support function of a convex body $K\in\mathcal{K}^n$, denoted by h_K, is an element of $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, and it is given by

$$h_K(x)=\max_{y\in K}\langle x,y\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^n$. We remark that $h_{B^n}(x)=|x|$ where $|x|$ denotes the Euclidean norm of $x\in\mathbb{R}^n$.

Let $C_b((0,\infty))$ denote the space of continuous functions with bounded support on $(0,\infty)$. The functional Minkowski vectors are operators of the form

$$v\to\int_{\mathbb{R}^n}\alpha(|x|)x\,\mathrm{dMA}(v[j],h_{B^n}[n-j];x)$$

where $v\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, and $\alpha\in C_b((0,\infty))$ is such that $\lim_{s\to0^+}\alpha(s)s=0$. Here, $\mathrm{MA}(v_1,\dots,v_n;\cdot)$ denotes the *mixed Monge–Ampère measure* of the functions $v_1,\dots,v_n\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, and the notation $v[j]$ means that the function v is repeated j times, while $h_{B^n}[n-j]$ means that h_{B^n} is repeated $n-j$ times. Under C^2 assumptions on the entries, the mixed Monge–Ampère measure is absolutely continuous with respect to the Lebesgue measure and

$$\mathrm{dMA}(v_1,\dots,v_n;x)=\det(\mathrm{D}^2v_1(x),\dots,\mathrm{D}^2v_n(x))\,\mathrm{d}x$$

where $\det:(\mathbb{R}^{n\times n})\to\mathbb{R}$ denotes the mixed discriminant. We use the abbreviated notation,

$$\mathrm{MA}_j(v;\cdot)=\mathrm{MA}_j(v[j],h_{B^n}[n-j];\cdot).\tag{3.1.1}$$

The functional Minkowski vectors vanish identically on support functions of convex bodies, that is, if $K\in\mathcal{K}^n$, then for $j\in\{0,\dots,n-1\}$,

$$\int_{\mathbb{R}^n}\alpha(|x|)x\,\mathrm{dMA}(h_K[j],h_{B^n}[n-j];x)\simeq\int_{\mathbb{S}^{n-1}}z\,\mathrm{d}S_j(K,z)=0\tag{3.1.2}$$

where the last equality is the Minkowski relation and where $\simeq$ denotes equality up to constant factors, depending on n, j, and α, and $S_j(K,\cdot)$ are the area measures; see [79, (5.30) and (5.52)], and [70, (4.3)].

For $\xi\in C_b((0,\infty))$ and $s>0$, we define

$$\mathcal{R}\,\xi(s)=s\xi(s)+\int_s^\infty\xi(r)\,\mathrm{d}r$$

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and we have $\mathcal{R}\,\xi \in C_b((0,\infty))$. For $m \geq 0$, we have (Cf. [27, Lemma 3.6])

$$\mathcal{R}^m\,\xi(s) = s^m\xi(s) + m\int_s^\infty r^{m-1}\xi(r)\,\mathrm{d}r, \tag{3.1.3}$$

where

$$\mathcal{R}^m\,\xi := \underbrace{\mathcal{R}\circ\cdots\circ\mathcal{R}}_{m}\xi.$$

Suppose that $\gamma : (0,\infty)\times(0,\infty) \to \mathbb{R}$, then we denote by the transform $\mathcal{R}_i$ the integral transform $\mathcal{R}$ on the i-th entry, where $i \in \{1,2\}$, that is,

$$\mathcal{R}_1\,\gamma(s,t) = s\gamma(s,t) + \int_s^\infty \gamma(r,t)\,\mathrm{d}r$$

for every $t > 0$. Similarly, we define $\mathcal{R}_2$.

We are now ready to state the main theorem of this article.

Theorem 3.1.3. *If $0 \leq j \leq n$ and $\alpha \in C_b((0,\infty))$ is such that $\lim_{s\to 0^+}\alpha(s)s = 0$, then*

$$\begin{aligned}&\kappa_n\int_{\mathrm{SO}(n)}\int_{\mathbb{R}^n}\alpha(|y|)y\,\mathrm{dMA}_j(u+(v\circ\vartheta^{-1});y)\,\mathrm{d}\vartheta\\ &=\sum_{k=1}^{j}\binom{j}{k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathcal{R}_1^{n-k}\left(\mathcal{R}^{-n+k}\,\alpha\right)(\max\{|x|,|y|\})\,x\,\mathrm{dMA}_{j-k}(v;y)\,\mathrm{dMA}_k(u;x)\end{aligned} \tag{3.1.4}$$

for every $u, v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

We bring the reader's attention to the fact that D. Hug, F. Mussnig, and J. Ulivelli [43] established an additive kinematic formula for functional intrinsic volumes on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ which serves as a functional analog to Theorem 3.1.1. It is given in the following theorem.

Theorem 3.1.4 ([43], Theorem 1.4)**.** *If $0 \leq j \leq n$ and $\alpha : [0,\infty) \to [0,\infty)$ is a measurable function, then*

$$\begin{aligned}&\kappa_n\int_{\mathrm{SO}(n)}\int_{\mathbb{R}^n}\alpha(|y|)\,\mathrm{dMA}_j(u+(v\circ\vartheta^{-1}),y)\,\mathrm{d}\vartheta\\ &=\sum_{i=0}^{j}\binom{j}{i}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\alpha(\max\{|x|,|y|\})\,\mathrm{dMA}_{j-i}(v,y)\,\mathrm{dMA}_i(u,x)\end{aligned} \tag{3.1.5}$$

for every $v, v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

Although both formulas (3.1.4) and (3.1.5) establish additive kinematic formulas on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, they differ in two important aspects. First, formula (3.1.4) deals with vector-valued functionals, while formula (3.1.5) deals with scalar-valued functionals. Secondly, one can observe that the expression in (3.1.5) is symmetric in the functions u and v, while formula (3.1.4) is rotation invariant in v but rotation equivariant in u; see Section 3.3.

If we choose $u = h_K$ for $K \in \mathcal{K}^n$ in Theorem 3.1.3, then (3.1.2) shows that (3.1.4) vanishes. However, if the support function of K is substituted for v, we obtain the following result.

Corollary 3.1.5. *If $K \in \mathcal{K}^n$, and $\alpha \in C_b((0,\infty))$ is such that $\lim_{s\to 0^+}\alpha(s)s = 0$, then*

$$\begin{aligned}&\kappa_n\int_{\mathrm{SO}(n)}\int_{\mathbb{R}^n}\alpha(|y|)y\,\mathrm{dMA}_j(u+(h_K\circ\vartheta^{-1});y)\,\mathrm{d}\vartheta\\ &\qquad=\sum_{k=1}^{j}\frac{\binom{j}{k}}{\binom{n}{j-k}}\kappa_{n-j+k}V_{j-k}(K)\int_{\mathbb{R}^n}\alpha(|x|)x\,\mathrm{dMA}_k(u;x)\end{aligned}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

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3.2 Preliminaries

We work in n-dimensional Euclidean space $\mathbb{R}^n$, with $n \geq 1$, endowed with the Euclidean norm $|\cdot|$ and the usual scalar product $\langle \cdot, \cdot \rangle$. We write $e_1, \dots, e_n$ for the canonical basis vectors of $\mathbb{R}^n$ and use the coordinates, $x = (x_1, \dots, x_n)$, for $x \in \mathbb{R}^n$ with respect to this basis.

Let $B^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the Euclidean unit ball and $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. We denote by κ_n the n-dimensional volume of B^n and by ω_n the $(n-1)$-dimensional Hausdorff measure, that we denote by $\mathcal{H}^{n-1}$, of $\mathbb{S}^{n-1}$. In this section, we will collect some results on (finite-valued) convex functions. We refer to [75, 76] for general references.

Convex functions

Let $\partial v(x)$ denote the subdifferential of the function $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ at x given by

$$\partial v(x) = \{y \in \mathbb{R}^n : v(z) \geq v(x) + \langle y, z - x \rangle \ \forall z \in \mathbb{R}^n\}.$$

This set is closed and convex for any $x \in \mathbb{R}^n$, and the function v is differentiable if and only if the set $\partial v(x)$ contains one element, namely the gradient of v at x, that is $\nabla v(x)$. In addition, we have the following result that can be found in [75, Theorem 23.8].

Theorem 3.2.1. *If u and v are elements of $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, then for all $x \in \mathbb{R}^n$,*

$$\partial(u + v)(x) = \partial u(x) + \partial v(x)$$

Since the topology of epi-convergence on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ coincides with the topology of pointwise convergence, the following result is easy to see.

Lemma 3.2.2 ([76], Theorem 7.46)**.** *Let $u_j, v_j \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ for $j \in \mathbb{N}$. If u_j epi-converges to $u \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ and v_j epi-converges to $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, then $u_j + v_j$ epi-converges to $u + v$.*

Another result that will be needed is the following.

Lemma 3.2.3 ([27], Lemma 3.3)**.** *The map*

$$(\vartheta, v) \to v \circ \vartheta^{-1}$$

is jointly continuous on $\mathrm{SO}(n) \times \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$.

Hessian measures

To each $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, we assign a family of nonnegative, locally finite Borel measures that we denote by $\Phi_j^n(v, \cdot)$, where $j \in \{0, \dots, n\}$, through the relation

$$\mathcal{H}^n(\{x + sy : x \in B, y \in \partial v(x)\}) = \sum_{j=0}^{n} s^j \Phi_j^n(v, B)$$

for every Borel set $B \subset \mathbb{R}^n$ and every $s \geq 0$. They are called the *Hessian measures* associated with v. Furthermore, if v is C^2 on some open set $A \subset \mathbb{R}^n$, then $\Phi_j^n(v, \cdot)$ is absolutely continuous with respect to the Lebesgue measure and

$$\mathrm{d}\Phi_j^n(v, x) = [\mathrm{D}^2 v(x)]_j \, \mathrm{d}x \tag{3.2.1}$$

on the open set A for all $0 \leq j \leq n$. Here, $\mathrm{D}^2 v$ denotes the Hessian matrix of v at $x \in \mathbb{R}^n$, and we write $[A]_j$ for the jth elementary symmetric function of the eigenvalues of the symmetric matrix

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$A \in \mathbb{R}^{n\times n}$. We refer the reader to [19, 22, 84] for more results and details on Hessian measures. By (3.2.1) and the weak continuity of Hessian measures with respect to epi-convergence, these measures are locally determined, that is, if $u, v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ are such that $u = v$ on some open set $A \subset \mathbb{R}^n$, then

$$\Phi_j^n(u, A \cap B) = \Phi_j^n(v, A \cap B) \tag{3.2.2}$$

for every relatively compact Borel set $B \subset \mathbb{R}^n$.

The following lemma provides a decomposition of Hessian measures of functions depending on less than n variables.

Lemma 3.2.4 ([25], Lemma 2.9). *Let $k \in \{1,\dots,n-1\}$, $j \in \{0,\dots,k\}$, and $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. If there exists $w \in \mathrm{Conv}(\mathbb{R}^k;\mathbb{R})$ such that*

$$v(x_1,\dots,x_n) = w(x_1,\dots,x_k)$$

for every $(x_1,\dots,x_n) \in \mathbb{R}^n$, then

$$\mathrm{d}\Phi_j^n(v;(x_1,\dots,x_n)) = \mathrm{d}\Phi_j^{(k)}(w;(x_1,\dots,x_k))\,\mathrm{d}x_{k+1}\cdots\mathrm{d}x_n.$$

We say that a map $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is a valuation if

$$\mathrm{Z}(v \vee w) + \mathrm{Z}(v \wedge w) = \mathrm{Z}(v) + \mathrm{Z}(w)$$

for all $v, w \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ such that their pointwise maximum $v \vee w$ and their pointwise minimum $v \wedge w$ are elements of $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. In the following theorem, A. Colesanti, M. Ludwig, and F. Mussnig established a full characterization of valuations with specific properties, which represents a functional version of Hadwiger's Theorem 3.1.2.

Let

$$D_j^n = \{\xi \in C_b((0,\infty)) \,;\, \lim_{r\to 0^+} r^{n-j}\xi(r) = 0 \text{ and } \lim_{r\to 0^+}\int_r^\infty p^{n-j-1}\xi(p)\,\mathrm{d}p \text{ exists and is finite}\}$$

for $0 \le j \le n-1$ and $D_n^n = \{\xi \in C_b((0,\infty)) \,;\, \lim_{r\to 0^+}\xi(r) \text{ exists and is finite}\}$.

Theorem 3.2.5 ([26], Theorem 1.5). *A functional $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is a continuous, dually epi-translation invariant, rotation invariant valuation if and only if there exist functions $\xi_0 \in D_0^n, \dots, \xi_n \in D_n^n$ such that*

$$\mathrm{Z}(v) = \sum_{i=0}^n \mathrm{V}_{i,\xi_i}^*(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where

$$\mathrm{V}_{i,\xi_i}^*(v) = \int_{\mathbb{R}^n} \xi_i(|x|)\,\mathrm{d}\Phi_i^n(v;x) \tag{3.2.3}$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$.

Here, Z is called *dually epi-translation invariant* if $\mathrm{Z}(v+f) = \mathrm{Z}(v)$ for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and every affine map $f : \mathbb{R}^n \to \mathbb{R}$. It is said to be *rotation invariant* if $\mathrm{Z}(v\circ\vartheta^{-1}) = \mathrm{Z}(v)$ for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and every $\vartheta \in \mathrm{SO}(n)$. Lastly, the map Z is said to be homogeneous of degree j if $\mathrm{Z}(\lambda v) = \lambda^j \mathrm{Z}(v)$ for all $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\lambda > 0$. Theorem 3.2.5 was first established in [26] and was also obtained by J. Knoerr in [55].

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Let

$$\mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R}) = \{v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) : v \text{ is of class } C^2 \text{ around the origin}\},$$

then the representation given in (3.2.3) also holds for $v \in \mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$. That is,

$$\mathrm{V}^*_{i,\xi_i}(v) = \int_{\mathbb{R}^n} \xi_i(|x|)\,\mathrm{d}\Phi^n_i(v;x) \tag{3.2.4}$$

for every $v \in \mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$ where $\xi_i \in D^n_i$ for each $i \in \{0,\dots,n\}$. See [26, Lemma 3.23].

Very recently, together with F. Mussnig, we introduced a new family of vector-valued valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and characterized them. These valuations are called *functional Minkowski vectors*. They arise from applying a Steiner formula to a functional moment vector defined on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. See [57, 69].

For $0 \le j \le n$, let

$$T^n_j = \{\xi \in C_b((0,\infty)) \,;\, \lim_{r\to 0^+} r^{n-j+1}\xi(r) = 0\}.$$

Theorem 3.2.6 ([70], Theorem C)**.** *A functional* $\mathrm{t} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ *is a continuous, dually epi-translation invariant, rotation equivariant valuation if and only if there exist functions* $\xi_1 \in T^n_1,\dots,\xi_n \in T^n_n$ *such that*

$$\mathrm{t}(v) = \sum_{k=1}^{n} \mathrm{t}^*_{k,\xi_k}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$*, where*

$$\mathrm{t}^*_{k,\xi_k}(v) = \int_{\mathbb{R}^n} \xi_k(|x|)x\,\mathrm{d}\Phi^n_k(v;x) \tag{3.2.5}$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$.

Here t is said to be rotation equivariant if $\mathrm{t}(v\circ\vartheta^{-1}) = \vartheta\,\mathrm{t}(v)$ for all $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\vartheta \in \mathrm{SO}(n)$.

One can also extend the representation given in (3.2.5) to $v \in \mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$; see [70, Lemma 4.11]. This means that,

$$\mathrm{t}^*_{k,\xi_k}(v) = \int_{\mathbb{R}^n} \xi_k(|x|)x\,\mathrm{d}\Phi^n_k(v;x) \tag{3.2.6}$$

for every $v \in \mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$ where $\xi_k \in T^n_k$ for every $k \in \{1,\dots,n\}$.

For $x \in \mathbb{R}^n$, we denote by $|x|_{n-1} := \sqrt{x_1^2+\cdots+x_{n-1}^2}$. For $t,s \ge 0$, we set

$$v_t(x) = \begin{cases} 0 & \text{if } |x| \le t \\ |x| - t & \text{if } |x| \ge t, \end{cases}$$

and

$$w_s(x) = \begin{cases} 0 & \text{if } |x| \le s \text{ and } x_n \ge 0 \\ 0 & \text{if } |x|_{n-1} \le s \text{ and } x_n < 0 \\ |x| - s & \text{if } |x| > s \text{ and } x_n \ge 0 \\ |x|_{n-1} - s & \text{if } |x|_{n-1} > s \text{ and } x_n < 0, \end{cases}$$

for every $x \in \mathbb{R}^n$. We note that v_t and w_s are elements of $\mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$ for every $t,s>0$. These two functions are key in establishing Theorem 3.2.5 and Theorem 3.2.6, respectively. We present some results that are essential to proving Theorem 3.1.3.

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Lemma 3.2.7 ([26], Lemma 2.15). *Let $1 \leq j \leq n-1$ and $\xi \in D_j^n$. Then, for every $t > 0$,*

$$\mathrm{V}^*_{j,\xi}(v_t) = \kappa_n \binom{n}{j} \left[t^{n-j} \xi(t) + (n-j) \int_t^{+\infty} r^{n-j-1} \xi(r)\, \mathrm{d}r \right]$$

Lemma 3.2.8 ([70], Proposition 6.2). *If $1 \leq j \leq n$ and $\xi \in T_j^n$, then for every $s > 0$,*

$$\mathrm{t}^*_{j,\xi}(w_s) = \frac{1}{n} \binom{n}{j} \kappa_{n-1} s^{n-j+1} \xi(s)\, e_n.$$

Mixed Monge–Ampère measures

We define the *Monge–Ampère measure* of $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, a Radon measure on $\mathbb{R}^n$, as

$$\mathrm{MA}(v, A) = V_n \left(\bigcup_{x \in A} \partial v(x) \right)$$

for every Borel set $A \subset \mathbb{R}^n$. The *mixed Monge–Ampère measure* which is associated to an n-tuple of functions in $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, is defined as

$$\mathrm{MA}(\lambda_1 v_1 + \cdots + \lambda_m v_m; \cdot) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} \mathrm{MA}(v_{i_1}, \ldots, v_{i_n}; \cdot) \tag{3.2.7}$$

where $m \in \mathbb{N}$, $v_1, \ldots, v_m \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\lambda_1, \ldots, \lambda_m \geq 0$. Equation (3.2.7) uniquely determines the mixed Monge–Ampère measure if we additionally assume that it is symmetric in its entries; see [84].

The following proposition gives a description of the measures MA_0 defined in (3.1.1).

Proposition 3.2.9 ([24], Theorem 4.5). *Let $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. We have*

$$\mathrm{MA}_0(v, B) = \kappa_n \delta_0(B)$$

for every Borel set $B \subset \mathbb{R}^n$, where δ_0 is the Dirac measure at the origin.

We point out that the functionals defined in Theorem 3.2.5 and Theorem 3.2.6 can be expressed in terms of the MA_j measures as follows.

Theorem 3.2.10 ([24], Theorem 1.6). *If $0 \leq i \leq n$ and $\xi_i \in D_i^n$, then*

$$\mathrm{V}^*_{i,\xi_i}(v) = \int_{\mathbb{R}^n} \alpha_i(|x|)\, \mathrm{dMA}_i(v, x) \tag{3.2.8}$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where $\alpha_i \in D_n^n$ and is given by

$$\alpha_i(p) = \binom{n}{i} \mathcal{R}^{n-i}\, \xi_i(p),$$

for every $p > 0$, where the integral transform $\mathcal{R}$ is defined in (3.1.3).

Theorem 3.2.11 ([70], Theorem B). *Let $1 \leq k \leq n$ and $\xi_k \in T_k^n$, then*

$$\mathrm{t}^*_{k,\xi_k}(v) = \int_{\mathbb{R}^n} \alpha_k(|x|) x\, \mathrm{dMA}_k(v, x) \tag{3.2.9}$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where $\alpha_k \in T_n^n$ and is given by

$$\alpha_k(p) = \binom{n}{k} \mathcal{R}^{n-k}\, \xi_k(p)$$

for every $p > 0$.

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The relevant properties of the operators appearing in Theorem 3.2.10 and Theorem 3.2.11 are consequences of the following statements, which can be found in [24, Proposition 5.3].

Lemma 3.2.12. *Let $\psi \in C_c(\mathbb{R}^n)$ and $0 \leq j \leq n$. If $v_1, \dots, v_{n-j} \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, then*

$$v \to \int_{\mathbb{R}^n} \psi(x)\, \mathrm{dMA}(v[j], v_1, \dots, v_{n-j} : x)$$

defines a continuous, dually epi-translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ that is homogeneous of degree j.

The following lemma on the integral transform $\mathcal{R}$ will be useful in the proof of Theorem 3.1.3.

Lemma 3.2.13 ([70], Lemma 4.2)**.** *For $0 \leq k \leq n$ and $0 \leq l \leq n-k$, the map $\mathcal{R}^l : T_k^n \to T_k^{n-l}$ is a bijection.*

3.3 Proof of Theorem 3.1.3

Let $1 \leq j \leq n$, and $\xi \in T_j^n$. Define

$$\mathrm{z}_{j,\xi}(u,v) = \int_{\mathrm{SO}(n)} \mathrm{t}^*_{j,\xi}(u + (v \circ \vartheta^{-1}))\, \mathrm{d}\vartheta \tag{3.3.1}$$

for every $u, v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ where the functional $\mathrm{t}^*_{j,\xi}$ is defined in (3.2.9).

Lemma 3.3.1. *Let $1 \leq j \leq n$ and $\xi \in T_j^n$. For a fixed $\bar{v} \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, the map $\mathrm{z}_{j,\xi}(\cdot,\bar{v}) : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}^n$ is a continuous, dually epi-translation invariant and rotation equivariant valuation.*

Proof. Let $1 \leq j \leq n$, and $\xi \in T_j^n$. For a fixed $\bar{v} \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, we have

$$\mathrm{z}_{j,\xi}(u,\bar{v}) = \int_{\mathrm{SO}(n)} \mathrm{t}^*_{j,\xi}(u + (\bar{v} \circ \vartheta^{-1}))\, \mathrm{d}\vartheta \tag{3.3.2}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

1. We begin by showing that the map

$$(\vartheta, u) \to \mathrm{t}^*_{j,\xi}(u + (\bar{v} \circ \vartheta^{-1})) \tag{3.3.3}$$

is jointly continuous on $\mathrm{SO}(n) \times \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Since the functional $\mathrm{t}^*_{j,\xi}$ is rotation equivariant,

$$\begin{aligned} \mathrm{t}^*_{j,\xi}(u + (\bar{v} \circ \vartheta^{-1})) &= \mathrm{t}^*_{j,\xi}(((u \circ \vartheta) + \bar{v}) \circ \vartheta^{-1}) \\ &= \vartheta\, \mathrm{t}^*_{j,\xi}((u \circ \vartheta) + \bar{v})) \end{aligned}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\vartheta \in \mathrm{SO}(n)$. Thus, by Lemma 3.2.2, Lemma 3.2.3, and Lemma 3.2.12, the map (3.3.3) is jointly continuous on $\mathrm{SO}(n) \times \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

Now, let $u_i \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ with $i \in \mathbb{N}$ be such that u_i epi-converges to some $\bar{u} \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, meaning that $\{u_i : i \in \mathbb{N}\} \cup \{\bar{u}\}$ is sequentially compact. And since $\mathrm{SO}(n)$ is compact and (3.3.3) is jointly continuous, it follows that the supremum

$$\sup\left\{ \left| \mathrm{t}^*_{j,\xi}(u_i + (\bar{v} \circ \vartheta^{-1})) \right| : i \in \mathbb{N}, \vartheta \in \mathrm{SO}(n) \right\}$$

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is finite. Thus, by applying the dominated convergence theorem, we obtain

$$\lim_{i\to\infty}\int_{\mathrm{SO}(n)} \mathrm{t}^*_{j,\xi}(u_i+(\bar{v}\circ\vartheta^{-1}))\,\mathrm{d}\vartheta = \int_{\mathrm{SO}(n)} \mathrm{t}^*_{j,\xi}(\bar{u}+(\bar{v}\circ\vartheta^{-1}))\,\mathrm{d}\vartheta$$

which shows that (3.3.2) is continuous.

2. Let $\phi\in\mathrm{SO}(n)$. We have

$$\begin{aligned}\mathrm{z}_{j,\xi}(u\circ\phi^{-1},\bar{v}) &= \int_{\mathrm{SO}(n)} \mathrm{t}^*_{j,\xi}((u\circ\phi^{-1})+(\bar{v}\circ\vartheta^{-1}))\,\mathrm{d}\vartheta\\ &= \int_{\mathrm{SO}(n)} \mathrm{t}^*_{j,\xi}(u+(\bar{v}\circ\vartheta^{-1}\phi))\circ\phi^{-1})\,\mathrm{d}\vartheta\\ &= \phi\int_{\mathrm{SO}(n)} \mathrm{t}^*_{j,\xi}(u+(\bar{v}\circ\psi^{-1}))\,\mathrm{d}\psi\\ &= \phi\,\mathrm{z}_{j,\xi}(u,\bar{v})\end{aligned}$$

for all $u\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where we have used the rotation equivariance of the functional $\mathrm{t}^*_{j,\xi}$ and the Haar measure's left invariance on $\mathrm{SO}(n)$. This shows that (3.3.2) is rotation equivariant.

3. Let $y\in\mathbb{R}^n$. We have,

$$((u+l_y)+(\bar{v}\circ\vartheta^{-1}))(x) = ((u+(\bar{v}\circ\vartheta^{-1}))+l_y)(x)$$

for all $u\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ where $l_y(x)=\langle x,y\rangle$ for all $x\in\mathbb{R}^n$. Since the functional $\mathrm{t}^*_{j,\xi}$ is dually epi-translation invariant, the map in (3.3.2) is dually epi-translation invariant.

4. If $u_1,u_2\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, then $(u_1\wedge u_2)+\bar{v}=(u_1+\bar{v})\wedge(u_2+\bar{v})$ and similarly $(u_1\vee u_2)+\bar{v}=(u_1+\bar{v})\vee(u_2+\bar{v})$. Since, $\mathrm{t}^*_{j,\xi}$ is a valuation, the map in (3.3.2) is also a valuation.

Thus, for a fixed $\bar{v}\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, the map $u\mapsto\mathrm{z}_{j,\xi}(u,\bar{v})$ is a continuous, dually epi-translation invariant and rotation equivariant valuation for every $u\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. □

Lemma 3.3.2. *For any $1\le j\le n$, a fixed $\bar{u}\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\xi\in T^n_j$, the map $\mathrm{z}_{j,\xi}(\bar{u},\cdot):\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})\to\mathbb{R}^n$ is a continuous, dually epi-translation invariant and rotation invariant valuation.*

Proof. Let $1\le j\le n$, $\xi\in T^n_j$, and fix $\bar{u}\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Then,

$$\mathrm{z}_{j,\xi}(\bar{u},v) = \int_{\mathrm{SO}(n)} \mathrm{t}^*_{j,\xi}(\bar{u}+(v\circ\vartheta^{-1}))\,\mathrm{d}\vartheta \tag{3.3.4}$$

for every $v\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

1. Similarly to Lemma 3.3.1, one can show that the map

$$(\vartheta,v)\to\int_{\mathbb{R}^n} \mathrm{t}^*_{j,\xi}(\bar{u}+(v\circ\vartheta^{-1}))\,\mathrm{d}\vartheta \tag{3.3.5}$$

is jointly continuous on $\mathrm{SO}(n)\times\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Let $v_i\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ be a sequence that epi-converges to $\bar{v}\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, and hence $\{v_i\ :i\in\mathbb{N}\}\cup\{\bar{v}\}$ is sequentially compact. Since $\mathrm{SO}(n)$ is compact and (3.3.5) is jointly continuous, it follows that

$$\sup\{\left|\mathrm{t}^*_{j,\xi}(\bar{u}+(v_i\circ\vartheta^{-1}))\right| : i\in\mathbb{N},\vartheta\in\mathrm{SO}(n)\}$$

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is finite, and hence, by applying the dominated convergence, we get

$$\lim_{i\to\infty}\int_{\mathrm{SO}(n)}\mathrm{t}^*_{j,\xi}(\bar{u}+(v_i\circ\vartheta^{-1}))\,\mathrm{d}\vartheta=\int_{\mathrm{SO}(n)}\mathrm{t}^*_{j,\xi}(\bar{u}+(\bar{v}\circ\vartheta^{-1}))\,\mathrm{d}\vartheta$$

which shows that (3.3.4) is continuous.

2. Since

$$\int_{\mathrm{SO}(n)}\mathrm{t}^*_{j,\xi}(\bar{u}+v\circ\vartheta^{-1})=\int_{\mathrm{SO}(n)}\vartheta\,\mathrm{t}^*_{j,\xi}((\bar{u}\circ\vartheta)+v)\,\mathrm{d}\vartheta, \tag{3.3.6}$$

then for every $\phi\in\mathrm{SO}(n)$, we have

$$\begin{aligned}\mathrm{z}_{j,\xi}(\bar{u},v\circ\phi^{-1})&=\int_{\mathrm{SO}(n)}\vartheta\,\mathrm{t}^*_{j,\xi}((\bar{u}\circ\vartheta)+(v\circ\phi^{-1}))\,\mathrm{d}\vartheta\\&=\int_{\mathrm{SO}(n)}\vartheta\,\mathrm{t}^*_{j,\xi}(((\bar{u}\circ\vartheta\phi)+v)\circ\phi^{-1})\,\mathrm{d}\vartheta\\&=\int_{\mathrm{SO}(n)}\vartheta\phi\,\mathrm{t}^*_{j,\xi}((\bar{u}\circ\vartheta\phi)+v)\,\mathrm{d}\vartheta\\&=\int_{\mathrm{SO}(n)}\psi\,\mathrm{t}^*_{j,\xi}((\bar{u}\circ\psi)+v)\,\mathrm{d}\psi\\&=\mathrm{z}_{j,\xi}(\bar{u},v)\end{aligned}$$

for all $v\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. We have used the rotation equivariance of the functional $\mathrm{t}^*_{j,\xi}$ in the third equality, and the left-invariance of the Haar measure on $\mathrm{SO}(n)$ in the fourth equality. Hence, the functional (3.3.4) is rotation invariant.

3. Let $y\in\mathbb{R}^n$. Since

$$(\bar{u}\circ\vartheta))(x)+(v+l_y)(x)=(((\bar{u}\circ\vartheta)+\bar{v})+l_y)(x)$$

for all $x\in\mathbb{R}^n$, then, by (3.3.6) and since $\mathrm{t}^*_{j,\xi}$ is dually epi-translation invariant, the functional given by (3.3.4) is dually epi-translation invariant.

4. If $v_1,v_2\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, then $\bar{u}+(v_1\wedge v_2)=(\bar{u}+v_1)\wedge(\bar{u}+v_2)$ and similarly $\bar{u}+(v_1\vee v_2)=(\bar{u}+v_1)\vee(\bar{u}+v_2)$. Hence, since $\mathrm{t}^*_{j,\xi}$ is a valuation, the map in (3.3.4) is a valuation.

Thus, for a fixed $\bar{u}\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, the map $v\mapsto\mathrm{z}_{j,\xi}(\bar{u},v)$ is a continuous, dually epi-translation invariant and rotation invariant valuation for every $v\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. □

To prove Theorem 3.1.3, we will use some tedious calculations (Proposition 3.A.3), which can be found in the Appendix 3.A.

Proof of Theorem 3.1.3. If $j=0$, the statement follows immediately. Thus, we assume that $1\leq j\leq n$ in the following. For $u,v\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, by Theorem 3.2.11, we have

$$\begin{aligned}\mathrm{z}_{j,\xi}(u,v)&=\int_{\mathrm{SO}(n)}\mathrm{t}^*_{j,\xi}(u+(v\circ\vartheta^{-1}))\,\mathrm{d}\vartheta\\&=\int_{\mathrm{SO}(n)}\int_{\mathbb{R}^n}\alpha(|x|)x\,\mathrm{dMA}_j(u+(v\circ\vartheta^{-1}))\,\mathrm{d}\vartheta.\end{aligned} \tag{3.3.7}$$

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where

$$\alpha(r) = \binom{n}{j} \mathcal{R}^{n-j}\, \xi(r)$$

for every $r > 0$ and we remark that $\alpha \in T_n^n$. For a fixed $\bar{v} \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, and according to Lemma 3.3.1, the map $u \mapsto \mathrm{z}_{j,\xi}(u,\bar{v})$ is a continuous, translation invariant, and rotation equivariant valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. By Theorem 3.2.6, and Theorem 3.2.11, there exists $\xi^j_{k,\bar{v}} \in T^n_k$ where $1 \le k \le n$, such that

$$\mathrm{z}_{j,\xi}(u,\bar{v}) = \sum_{k=1}^{n} \int_{\mathbb{R}^n} \alpha^j_{k,\bar{v}}(|x|)x \,\mathrm{dMA}_k(u;x) \tag{3.3.8}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where

$$\alpha^j_{k,\bar{v}}(p) = \binom{n}{k} \mathcal{R}^{n-k}\, \xi^j_{k,\bar{v}}(p),$$

for every $p > 0$. We recall that,

$$w_s(x) = \begin{cases} 0 & \text{if } |x| \le s \text{ and } x_n \ge 0 \\ 0 & \text{if } |x|_{n-1} \le s \text{ and } x_n < 0 \\ |x| - s & \text{if } |x| > s \text{ and } x_n \ge 0 \\ |x|_{n-1} - s & \text{if } |x|_{n-1} > s \text{ and } x_n < 0 \end{cases}$$

where $x \in \mathbb{R}^n$, is a function in $\mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$. By Lemma 3.2.8 and by the homogeneity of the map $z_{j,\xi}$, we have that

$$\mathrm{z}_{j,\xi}(\mu w_s,\bar{v}) = \frac{1}{n}\kappa_{n-1} \sum_{k=1}^{n} \mu^k \binom{n}{k} s^{n-k+1} \xi^j_{k,\bar{v}}(s)\; e_n$$

for every $s > 0$ and every $\mu > 0$.

Let us now fix $\bar{u} \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. By Lemma 3.3.2 and for a fixed $\bar{s} > 0$, each of the maps

$$v \mapsto \xi^j_{k,v}(\bar{s}), \quad \text{where } 1 \le k \le n$$

is a continuous, translation invariant, and rotation invariant valuation. By Theorem 3.2.5 and Theorem 3.2.10, there exists $\xi^j_{0,i}, \dots, \xi^j_{k,i} \in D^n_i$, where $0 \le i \le n$, and $1 \le k \le n$, such that

$$\xi^j_{k,v}(\bar{s}) = \sum_{i=0}^{n} \int_{\mathbb{R}^n} \alpha^j_{k,i}(\bar{s},|y|) \,\mathrm{dMA}_i(v;y)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where

$$\alpha^j_{k,i}(\bar{s},q) = \binom{n}{i} \mathcal{R}_2^{n-i}\, \xi^j_{k,i}(\bar{s},q),$$

for every $p > 0$. Thus, we have that

$$\mathrm{z}_{j,\xi}(u,v) = \sum_{i=0}^{n}\sum_{k=1}^{n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \alpha^j_{k,i}(|x|,|y|)x \,\mathrm{dMA}_i(v;y) \,\mathrm{dMA}_k(u;x)$$

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for every $u, v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Since (3.3.1) is homogeneous of degree j in the argument $u + (v \circ \vartheta^{-1})$, it follows that

$$\mathrm{z}_{j,\xi}(u,v) = \sum_{k=1}^{j} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \alpha^j_{k,j-k}(|x|,|y|)x \,\mathrm{dMA}_{j-k}(v;y)\,\mathrm{dMA}_k(u;x) \tag{3.3.9}$$

for every $u, v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, where

$$\alpha^j_{k,j-k}(p,q) = \binom{n}{k}\binom{n}{j-k}\mathcal{R}_1^{n-k}\left(\mathcal{R}_2^{n-j+k}\left(\xi^j_{k,j-k}\right)\right)(p,q) \tag{3.3.10}$$

for every $p, q > 0$. For simplicity, we will denote $\alpha^j_{k,j-k}$ by $\alpha_{k,j-k}$.

It remains to show the relationship between $\alpha_{k,j-k}$ and α. On the one hand, by evaluating the functional (3.3.9) at $u = \mu w_s$ and $v = \lambda v_t$ for $\lambda, \mu > 0$ and $s, t > 0$, we have by Proposition 3.2.9, together with Lemma 3.2.7, Lemma 3.2.8, and Lemma 3.2.13,

$$\mathrm{z}_{j,\xi}(\mu w_s, \lambda v_t) = \frac{1}{n}\kappa_{n-1}\kappa_n\left[\mu^j s^{n-j+1}\,\mathcal{R}_1^{-n+j}\,\alpha_{j,0}(s,0) + \sum_{k=1}^{j-1}\mu^k\lambda^{j-k}s^{n-k+1}\,\mathcal{R}_1^{-n+k}\,\alpha_{k,j-k}(s,t)\right]e_n, \tag{3.3.11}$$

for every $1 \le j \le n$. On the other hand, by Proposition 3.A.3, we have

$$\mathrm{z}_{j,\xi}(\mu w_s, \lambda v_t) = \frac{1}{n}\kappa_{n-1}\left[\mu^j s^{n-j+1}\,\mathcal{R}^{-n+j}\,\alpha(s) + \sum_{k=1}^{j-1}\binom{j}{k}\lambda^{j-k}\mu^k s^{n-k+1}\,\mathcal{R}^{-n+k}\,\alpha\left(\max\{s,t\}\right)\right]e_n \tag{3.3.12}$$

for every $1 \le j \le n$ and every $s, t > 0$. Since $\lambda, \mu > 0$ are arbitrary, then by comparing the coefficients of (3.3.11) with (3.3.12), we get

$$\kappa_n\,\mathcal{R}_1^{-n+j}\,\alpha_{j,0}(s,0) = \mathcal{R}^{-n+j}\,\alpha(s)$$

for $k = j$ such that $1 \le j \le n$, and

$$\kappa_n\,\mathcal{R}_1^{-n+k}\left(\alpha_{k,j-k}\right)(s,t) = \binom{j}{k}\mathcal{R}^{-n+k}\left(\alpha\right)\left(\max\{s,t\}\right)$$

for every $1 \le k \le j-1$ and every $2 \le j \le n$. By Lemma 3.2.13, we have that

$$\alpha_{j,0}(s,0) = \frac{1}{\kappa_n}\alpha(s)$$

or $k = j$ such that $1 \le j \le n$, and

$$\alpha_{k,j-k}(s,t) = \frac{1}{\kappa_n}\binom{j}{k}\mathcal{R}_1^{n-k}\left(\mathcal{R}^{-n+k}\left(\alpha\right)\right)\left(\max\{s,t\}\right)$$

for every $1 \le k \le j-1$, where $2 \le j \le n$. By observing that

$$\alpha_{j,0}(s,0) = \frac{1}{\kappa_n}\mathcal{R}_1^{n-j}\left(\mathcal{R}^{-n+j}\left(\alpha\right)\right)\left(\max\{s,0\}\right)$$

for $k = j$ where $1 \le j \le n$, the result follows. □

In order to obtain the result in Corollary 3.1.5, we need the following lemma.

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Lemma 3.3.3 ([42], Lemma 4.6)**.** *If $K \in \mathcal{K}^n$, then*

$$\binom{n}{j} \mathrm{dMA}_j(h_K[j], h_{B^n}[n-j], B) = \kappa_{n-j} V_j(K)\, \delta_0(B)$$

for every Borel set $B \subset \mathbb{R}^n$ and $0 \leq j \leq n$.

Proof of Corollary 3.1.5. Since $\alpha(\max\{|x|, 0\}) = \alpha(|x|)$ for all $x \in \mathbb{R}^n$, the proof follows from Theorem 3.1.3, and by applying Lemma 3.3.3. □

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3.A Appendix to Chapter 3

As the proof of Proposition 3.A.3 is long and technical, we decided to put it in the appendix.

We recall that for $t, s \geq 0$,

$$v_t(x) = \begin{cases} 0 & \text{if } |x| \leq t \\ |x| - t & \text{if } |x| \geq t \end{cases}$$

and

$$w_s(x) = \begin{cases} 0 & \text{if } |x| \leq s \text{ and } x_n \geq 0 \\ 0 & \text{if } |x|_{n-1} \leq s \text{ and } x_n < 0 \\ |x| - s & \text{if } |x| > s \text{ and } x_n \geq 0 \\ |x|_{n-1} - s & \text{if } |x|_{n-1} > s \text{ and } x_n < 0 \end{cases}$$

for every $x \in \mathbb{R}^n$ where $|x|_{n-1} = \sqrt{x_1^2 + \cdots + x_{n-1}^2}$. Then, the following lemma is easy to see.

Lemma 3.A.1. *For $\mu, \lambda \geq 0$, we have the following:*

1. *If $s \leq t$, then*

$$(\mu\, w_s + \lambda\, v_t)(x) = \begin{cases} 0 & \text{if } |x| \leq s \text{ and } x_n \geq 0 \\ \mu(|x| - s) & \text{if } s < |x| \leq t \text{ and } x_n \geq 0 \\ \mu(|x| - s) + \lambda(|x| - t) & \text{if } |x| > t \text{ and } x_n \geq 0 \\ 0 & \text{if } |x|_{n-1} \leq s,\ |x| \leq t \text{ and } x_n < 0 \\ \lambda(|x| - t) & \text{if } |x|_{n-1} \leq s,\ |x| > t \text{ and } x_n < 0 \\ \mu(|x|_{n-1} - s) & \text{if } |x|_{n-1} > s,\ |x| \leq t \text{ and } x_n < 0 \\ \mu(|x|_{n-1} - s) + \lambda(|x| - t) & \text{if } |x|_{n-1} > s,\ |x| > t \text{ and } x_n < 0. \end{cases}$$

2. *If $s > t$, then*

$$(\mu\, w_s + \lambda\, v_t)(x) = \begin{cases} 0 & \text{if } |x| \leq t \text{ and } x_n \geq 0 \\ \lambda(|x| - t) & \text{if } t < |x| \leq s \text{ and } x_n \geq 0 \\ \mu(|x| - s) + \lambda(|x| - t) & \text{if } |x| > s \text{ and } x_n \geq 0 \\ 0 & \text{if } |x| \leq t \text{ and } x_n < 0 \\ \lambda(|x| - t) & \text{if } |x|_{n-1} \leq s,\ |x| > t \text{ and } x_n < 0 \\ \mu(|x|_{n-1} - s) + \lambda(|x| - t) & \text{if } |x|_{n-1} > s \text{ and } x_n < 0. \end{cases}$$

To prove Proposition 3.A.3, an essential tool will be the *disintegration theorem*.

Theorem 3.A.2 ([64], Theorem 17.1)**.** *Let μ be a finite Borel measure concentrated on a Borel set $E \subset \mathbb{R}^n$, and let $\pi : E \to \mathbb{R}$ be the projection map into the last coordinate. Then the following property holds: there exists a Borel set $\Sigma \subset \mathbb{R}$ of full $(\pi_{\#}\mu)$-measure such that for every $\sigma \in \Sigma$, there exists $\mu_\sigma \in \mathcal{P}(\mathbb{R}^n)$ such that μ_σ is concentrated on $\pi^{-1}(\sigma)$, and*

$$\int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu = \int_{\Sigma} \mathrm{d}(P_{\#}\mu)(\sigma) \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu_\sigma,$$

for every bounded Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}$, where $(\pi_{\#}\mu)$ denotes the push-forward of the measure μ through the map π, that is,

$$(\pi_{\#}\mu)(B) = \mu(\pi^{-1}(B)),$$

for all $B \subset \mathbb{R}$ Borel set.

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Evaluating the map given by (3.3.1) at μw_s and λv_t where $s,t>0$, and $\mu,\lambda>0$, yields the following result.

Proposition 3.A.3. *If $1\le j\le n$ and $\xi\in T^n_j$, then*

$$\mathrm{z}_{j,\xi}(\mu w_s,\lambda v_t)=\frac{1}{n}\kappa_{n-1}\left[\mu^j s^{n-j+1}\,\mathcal{R}^{-n+j}\,\alpha(s)+\sum_{k=1}^{j-1}\binom{j}{k}\lambda^{j-k}\mu^k s^{n-k+1}\,\mathcal{R}^{-n+k}\,\alpha\left(\max\{s,t\}\right)\right]e_n$$

for every $s,t>0$ and every $\mu,\lambda>0$, where

$$\alpha(r)=\binom{n}{j}\mathcal{R}^{n-j}\,\xi(r)$$

for every $r>0$.

Proof. Let $1\le j\le n$, $\xi\in T^n_j$, and $\mu,\lambda>0$. Since the function $\mu w_s+\lambda v_t\in \mathrm{Conv}_{(o)}(\mathbb{R}^n;\mathbb{R})$, it follows by (3.2.6) and by the rotation invariance of the function v_t, that

$$\begin{aligned}\mathrm{z}_{j,\xi}(\mu w_s,\lambda v_t)&=\int_{\mathrm{SO}(n)}\mathrm{t}^*_{j,\xi}(\mu w_s+(\lambda v_t\circ\vartheta^{-1}))\,\mathrm{d}\vartheta\\&=\int_{\mathbb{R}^n}\xi(|x|)x\,\mathrm{d}\Phi^n_j(\mu w_s+\lambda v_t;x)\end{aligned}\tag{3.A.1}$$

for all $s,t>0$. By Lemma 3.A.1, we will need to distinguish between the cases when $s\le t$ and $s>t$ in the last integral of (3.A.1).

Let us first point out that for every $\vartheta\in\mathrm{SO}(n)$ such that $\vartheta e_n=e_n$, it follows from the definition of w_s and of v_t and by Lemma 3.3.1 and Lemma 3.3.2 that

$$\mathrm{z}_{j,\xi}(\mu w_s,\lambda v_t)=\mathrm{z}_{j,\xi}(\mu w_s\circ\vartheta^{-1},\mu u_t\circ\vartheta^{-1})=\vartheta\,\mathrm{z}_{j,\xi}(\mu w_s,\lambda v_t),$$

for every $s,t>0$. This implies that $\mathrm{t}(\mu w_s,\lambda v_t)$ must be parallel to e_n, and therefore we only need to compute the integral at the last coordinate x_n.

We recall that for $j\in\{0,\dots,n\}$, the measures $\Phi^n_j(v,\cdot)$ associated with v are given by the jth elementary symmetric function of the Hessian of v when the function v is C^2 and are given by (3.2.1).

1. Let $s\le t$. We will split the domain $\mathbb{R}^n$ into several parts.

   (a) The function $\mu w_s+\lambda v_t$ is of class C^2 on the set $A_1=\{x\in\mathbb{R}^n: s<|x|\le t, x_n\ge 0\}$, and the Hessian matrix of $\mu w_s+\lambda v_t$ at a point x in this set has $(n-1)$-eigenvalues equal to $\mu/|x|$ and the last eigenvalue is equal to zero. Hence,

   $$[\mathrm{D}^2(\mu w_s+\lambda v_t)(x)]_j=\begin{cases}\binom{n-1}{j}\frac{\mu^j}{|x|^j} & \text{if } 0\le j\le n-1\\ 0 & \text{if } j=n\end{cases}$$

   which implies, for $0\le j\le n-1$, that

   $$\begin{aligned}\int_{A_1}\xi(|x|)x_n[\mathrm{D}^2(\mu w_s+\lambda v_t)(x)]_j\,\mathrm{d}x&=\mu^j\binom{n-1}{j}\int_s^t r^{n-j}\xi(r)\,\mathrm{d}r\int_{\mathbb{S}^{n-1}\cap y_n\ge 0}y_n\,\mathrm{d}y\,e_n\\&=\mu^j\binom{n-1}{j}\frac{\omega_{n-1}}{n-1}\int_s^t r^{n-j}\xi(r)\,\mathrm{d}r.\end{aligned}$$

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(b) The function $\mu w_s + \lambda v_t$ is of class C^2 on $A_2 = \{x \in \mathbb{R}^n : |x| > t, x_n \geq 0\}$, and its Hessian matrix at a point x in this set has $(n-1)$-eigenvalues equal to $(\mu+\lambda)/|x|$ and the last eigenvalue is equal to zero. Hence,

$$[\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j = \begin{cases} \binom{n-1}{j} \frac{(\mu+\lambda)^j}{|x|^j} & \text{if } 0 \leq j \leq n-1 \\ 0 & \text{if } j = n \end{cases}$$

which implies, for $0 \leq j \leq n-1$, that

$$\begin{aligned} &\int_{A_2} \xi(|x|) x_n [\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j \, \mathrm{d}x \\ &\qquad = (\mu+\lambda)^j \binom{n-1}{j} \int_t^\infty r^{n-j} \xi(r) \, \mathrm{d}r \int_{\mathbb{S}^{n-1} \cap y_n \geq 0} y_n \, \mathrm{d}y \\ &\qquad = (\mu+\lambda)^j \binom{n-1}{j} \frac{\omega_{n-1}}{n-1} \int_t^\infty r^{n-j} \xi(r) \, \mathrm{d}r. \end{aligned}$$

(c) The function $\mu w_s + \lambda v_t$ is of class C^2 on $A_3 = \{x \in \mathbb{R}^n : |x|_{n-1} \leq s, |x| > t, x_n < 0\}$ and its Hessian matrix at a point x in this set has $(n-1)$-eigenvalues equal to $\lambda/|x|$ and the last eigenvalue is equal to zero. Hence,

$$[\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j = \begin{cases} \binom{n-1}{j} \frac{\lambda^j}{|x|^j} & \text{if } 0 \leq j \leq n-1 \\ 0 & \text{if } j = n \end{cases}$$

which implies, for $0 \leq j \leq n-1$, that

$$\begin{aligned} &\int_{A_3} \xi(|x|) x_n [\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j \, \mathrm{d}x \\ &\quad = \lambda^j \binom{n-1}{j} \int_0^{+\infty} r^{n-j} \xi(r) \int_{\mathbb{S}^{n-1} \cap e_n^\perp} \int_{-1}^1 \tau (1-\tau^2)^{\frac{n-3}{2}} \mathbb{1}_{A_3} \, \mathrm{d}\tau \, \mathrm{d}y \, \mathrm{d}r \\ &\quad = \lambda^j \binom{n-1}{j} \omega_{n-1} \int_t^{+\infty} r^{n-j} \xi(r) \int_{-1}^{-\sqrt{1-\frac{s^2}{r^2}}} \tau (1-\tau^2)^{\frac{n-3}{2}} \, \mathrm{d}\tau \, \mathrm{d}r \\ &\quad = -\lambda^j \binom{n-1}{j} \frac{\omega_{n-1}}{n-1} s^{n-1} \int_t^{+\infty} r^{1-j} \xi(r) \, \mathrm{d}r. \end{aligned}$$

(d) The function $\mu w_s + \lambda v_t$ is of class C^2 on $A_4 = \{x \in \mathbb{R}^n : |x|_{n-1} > s, |x| \leq t, x_n < 0\}$ and the eigenvalues of its Hessian at a point x in this set are given by

$$[\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j = \begin{cases} \binom{n-2}{j} \frac{\mu^j}{|x|_{n-1}^j} & \text{if } 0 \leq j \leq n-2 \\ 0 & \text{if } j \in \{n-1, n\} \end{cases}$$

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which implies, for $0 \le j \le n-2$, that

$$\begin{aligned}
&\int_{A_4} \xi(|x|)x_n[\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j \,\mathrm{d}x \\
&\qquad = \mu^j \binom{n-2}{j} \int_0^{+\infty} r^{n-j}\xi(r) \int_{\mathbb{S}^{n-1}\cap e_n^\perp} \int_{-1}^{1} \tau(1-\tau^2)^{\frac{n-j-3}{2}} \mathbb{1}_{A_4} \,\mathrm{d}\tau \,\mathrm{d}y \,\mathrm{d}r \\
&\qquad = \mu^j \binom{n-2}{j} \omega_{n-1} \int_s^t r^{n-j}\xi(r) \int_{-\sqrt{1-\frac{s^2}{r^2}}}^{0} \tau(1-\tau^2)^{\frac{n-j-3}{2}} \,\mathrm{d}\tau \,\mathrm{d}r \\
&\qquad = -\mu^j \binom{n-2}{j} \frac{\omega_{n-1}}{n-j-1} \left[\int_s^t r^{n-j}\xi(r)\,\mathrm{d}r - s^{n-j-1} \int_s^t r\xi(r)\,\mathrm{d}r \right] \\
&\qquad = -\mu^j \binom{n-1}{j} \frac{\omega_{n-1}}{n-1} \left[\int_s^t r^{n-j}\xi(r)\,\mathrm{d}r - s^{n-j-1} \int_s^t r\xi(r)\,\mathrm{d}r \right].
\end{aligned}$$

(e) The function $\mu w_s + \lambda v_t$ is of class C^2 on $A_5 = \{x \in \mathbb{R}^n : |x|_{n-1} > s, |x| > t, x_n > 0\}$ and the eigenvalues of its Hessian at a point x in this set are given by

$$[\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j = \begin{cases} \binom{n-2}{j} \left(\frac{\lambda}{|x|} + \frac{\mu}{|x|_{n-1}}\right)^j & \\ + & \text{if } 0 \le j \le n-2 \\ \binom{n-2}{j-1} \left(\frac{\lambda}{|x|} + \frac{\mu}{|x|_{n-1}}\right)^{j-1} \frac{\lambda}{|x|} & \\ \left(\frac{\lambda}{|x|} + \frac{\mu}{|x|_{n-1}}\right)^{n-2} \frac{\lambda}{|x|} & \text{if } j = n-1 \\ 0 & \text{if } j = n \end{cases}$$

this implies, for $0 \le j \le n-2$, that

$$\begin{aligned}
&\int_{A_5} \xi(|x|)x_n[\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j \,\mathrm{d}x = \\
&\omega_{n-1} \left[\binom{n-2}{j} \sum_{k=0}^{j} \binom{j}{k} \mu^k \lambda^{j-k} \int_t^{+\infty} r^{n-j}\xi(r) \int_{-\sqrt{1-\frac{s^2}{r^2}}}^{0} \tau(1-\tau^2)^{\frac{n-k-3}{2}} \,\mathrm{d}\tau \,\mathrm{d}r + \right. \\
&\left. \binom{n-2}{j-1} \sum_{k=0}^{j-1} \binom{j-1}{k} \mu^k \lambda^{j-k} \int_t^{+\infty} r^{n-j}\xi(r) \int_{-\sqrt{1-\frac{s^2}{r^2}}}^{0} \tau(1-\tau^2)^{\frac{n-k-3}{2}} \,\mathrm{d}\tau \,\mathrm{d}r \right] \\
&= -\omega_{n-1} \times \\
&\left[\binom{n-2}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{\mu^k \lambda^{j-k}}{n-k-1} \left(\int_t^{+\infty} r^{n-j}\xi(r)\,\mathrm{d}r - s^{n-k-1} \int_t^{+\infty} r^{k-j+1}\xi(r)\,\mathrm{d}r \right) + \right. \\
&\left. \binom{n-2}{j-1} \sum_{k=0}^{j-1} \binom{j-1}{k} \frac{\mu^k \lambda^{j-k}}{n-k-1} \left(\int_t^{+\infty} r^{n-j}\xi(r)\,\mathrm{d}r - s^{n-k-1} \int_t^{+\infty} r^{k-j+1}\xi(r)\,\mathrm{d}r \right) \right] \\
&= -\frac{\omega_{n-1}}{n-1} \binom{n-1}{j} \times \\
&\left[\sum_{k=0}^{j} \binom{j}{k} \mu^k \lambda^{j-k} \left(\int_t^{+\infty} r^{n-j}\xi(r)\,\mathrm{d}r - s^{n-k-1} \int_t^{+\infty} r^{k-j+1}\xi(r)\,\mathrm{d}r \right) \right],
\end{aligned}$$

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and

$$\begin{aligned}
&\int_{A_5} \xi(|x|)x_n[\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_{n-1}\,\mathrm{d}x \\
&= \omega_{n-1}\sum_{k=0}^{n-2}\binom{n-2}{k}\mu^k\lambda^{n-k-1}\int_t^{+\infty} r\xi(r)\int_{-\sqrt{1-\frac{s^2}{r^2}}}^{0}\tau(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau\,\mathrm{d}r \\
&= -\omega_{n-1}\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{\mu^k\lambda^{n-k-1}}{n-k-1}\left(\int_t^{+\infty} r\xi(r)\,\mathrm{d}r - s^{n-k-1}\int_t^{+\infty} r^{k-n+2}\xi(r)\,\mathrm{d}r\right) \\
&= -\frac{\omega_{n-1}}{n-1}\sum_{k=0}^{n-2}\binom{n-1}{k}\mu^k\lambda^{n-k-1}\left(\int_t^{+\infty} r\xi(r)\,\mathrm{d}r - s^{n-k-1}\int_t^{+\infty} r^{k-n+2}\xi(r)\,\mathrm{d}r\right),
\end{aligned}$$

for $j = n-1$.

(f) For the first set of singular points $B_1 = \{x \in \mathbb{R}^n : |x| = s,\ x_n \geq 0\}$, since Hessian measures are locally determined, by (3.2.2), it follows that $\Phi_j^n(\mu w_s + \lambda v_t, \cdot)|_{B_1} = \Phi_j^n(\mu v_s, \cdot)|_{B_1}$ and since v_s is radially symmetric, this implies that $\Phi_j^n(v_s, \cdot)|_{B_1}$ is rotation invariant. Hence, it is equivalent to the Haar measure up to a constant $C_{n,j,s}$ on B_1; this means that

$$\int_{B_1}\gamma(x)\,\mathrm{d}\Phi_j^n(v_s, x) = C_{n,j,s}\int_{B_1}\gamma(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

for every $\gamma \in C_b(\mathbb{R}^n)$, which implies that

$$\begin{aligned}
\Phi_j^n(v_s, B_1) &= C_{n,j,s}\int_{B_1}\mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \frac{\omega_n}{2}C_{n,j,s}\,s^{n-1}
\end{aligned}$$

and by a similar proof to Lemma 3.2.7, we have that

$$\Phi_j^n(v_s, B_1) = \frac{\kappa_n}{2}\binom{n}{j}s^{n-j}$$

for every $1 \leq j \leq n$. Hence,

$$C_{n,j,s} = \frac{1}{n}\binom{n}{j}s^{1-j}$$

for every $1 \leq j \leq n$ and,

$$\begin{aligned}
\int_{B_1}\xi(|x|)x_n\,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) &= \mu^j\int_{B_1}\xi(|x|)x_n\,\mathrm{d}\Phi_j^n(v_s, x) \\
&= \frac{1}{n}\mu^j\binom{n}{j}s^{1-j}\int_{B_1}\xi(|x|)x_n\,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \frac{1}{n}\mu^j\binom{n}{j}s^{n-j+1}\xi(s)\int_{\mathbb{S}^{n-1}\cap\{x_n\geq 0\}}x_n\,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \frac{1}{n}\mu^j\binom{n}{j}\frac{\omega_{n-1}}{n-1}s^{n-j+1}\xi(s).
\end{aligned}$$

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(g) For the second set of singular points $B_2 = \{x \in \mathbb{R}^n : |x| = t, x_n \geq 0\}$, since $\mu w_s + \lambda v_t$ is radially symmetric on this set, we have that $\Phi_j^n(\mu w_s + \lambda v_t, \cdot)|_{B_2}$ is rotation invariant. This implies that it is equivalent to a constant $C_{n,j,t}$ to the Haar measure. Hence,

$$\int_{B_2} \gamma(x) \, \mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) = C_{n,j,t} \int_{B_2} \gamma(x) \, \mathrm{d}\mathcal{H}^{n-1}(x)$$

for every $\gamma \in C_b(\mathbb{R}^n)$. It follows that,

$$\begin{aligned} \Phi_j^n(\mu w_s + \lambda v_t, B_2) &= C_{n,j,t} \int_{B_2} \mathrm{d}\mathcal{H}^{n-1}(x) \\ &= \frac{\omega_n}{2} C_{n,j,t} \, t^{n-1} \end{aligned}$$

and on the other hand, by Lemma 3.2.1,

$$\partial(\mu w_s + \lambda v_t)(x) = \left\{ \alpha \frac{x}{|x|} : \alpha \in [\mu, \mu + \lambda] \right\}$$

for every $x \in B_2$, which implies that

$$P_p(\mu w_s + \lambda v_t, B_2 \times \mathbb{R}^n) = \left\{ (t + \alpha)x \mid \alpha \in [\mu p, (\mu + \lambda)p], x \in \mathbb{S}^{n-1}, \ x_n \geq 0 \right\}.$$

Thus, for every $1 \leq j \leq n$,

$$\Phi_j^n(\mu w_s + \lambda v_t, B_2) = \binom{n}{j} \frac{\kappa_n}{2} t^{n-j} \sum_{k=0}^{j-1} \binom{j}{k} \mu^k \lambda^{j-k}.$$

It follows that,

$$C_{n,j,t} = \frac{1}{n} \binom{n}{j} t^{1-j} \sum_{k=0}^{j-1} \binom{j}{k} \mu^k \lambda^{j-k}$$

for every $1 \leq j \leq n$. Thus,

$$\begin{aligned} &\int_{B_2} \xi(|x|) x_n \, \mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) \\ &\quad = \frac{1}{n} \binom{n}{j} t^{1-j} \sum_{k=0}^{j-1} \binom{j}{k} \mu^k \lambda^{j-k} \int_{B_2} \xi(|x|) x_n \, \mathrm{d}\mathcal{H}^{n-1}(x) \\ &\quad = \frac{1}{n} \binom{n}{j} t^{n-j+1} \xi(t) \sum_{k=0}^{j-1} \binom{j}{k} \mu^k \lambda^{j-k} \int_{\mathbb{S}^{n-1} \cap \{x_n \geq 0\}} x_n \, \mathrm{d}\mathcal{H}^{n-1}(x) \\ &\quad = \frac{1}{n} \frac{\omega_{n-1}}{n-1} \binom{n}{j} t^{n-j+1} \xi(t) \sum_{k=0}^{j-1} \binom{j}{k} \mu^k \lambda^{j-k}. \end{aligned}$$

(h) For the third set of singular points $B_3 = \{x \in \mathbb{R}^n : |x|_{n-1} < s, |x| = t, x_n < 0\}$, since Hessian measures are locally determined, by (3.2.2), we have

$$\Phi_j^n(\mu w_s + \lambda v_t, \cdot)|_{B_3} = \Phi_j^n(\lambda v_t, \cdot)|_{B_3}$$

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and since v_t is radially symmetric, this implies that $\Phi_j^n(\lambda v_t, \cdot)|_{B_3}$ is equivalent to the Haar measure up to a constant $K_{n,j,t}$. Hence,

$$\int_{B_3} \gamma(x)\,\mathrm{d}\Phi_j^n(\lambda v_t, x) = K_{n,j,t} \int_{B_3} \gamma(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

for every $\gamma \in C_b(\mathbb{R}^n)$. We obtain that,

$$\begin{aligned}\Phi_j^n(\lambda v_t, B_3) &= K_{n,j,t} \int_{B_3} \mathrm{d}\mathcal{H}^{n-1}(x) \\ &= \omega_n C_{n,j,s}\, t^{n-1} \int_{-1}^{-\sqrt{1-\frac{s^2}{t^2}}} (1-\tau^2)^{\frac{n-3}{2}}\,\mathrm{d}\tau.\end{aligned}$$

On the other hand, using a similar proof of Lemma 3.2.7, for every $1 \le j \le n$, one can show that

$$\Phi_j^n(v_t, B_3) = \kappa_n \binom{n}{j} \lambda^j t^{n-j} \int_{-1}^{-\sqrt{1-\frac{s^2}{t^2}}} (1-\tau^2)^{\frac{n-3}{2}}\,\mathrm{d}\tau$$

It follows that,

$$K_{n,j,t} = \frac{1}{n}\binom{n}{j}\lambda^j t^{1-j}.$$

Hence,

$$\begin{aligned}&\int_{B_3} \xi(|x|)x_n\,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) \\ &\quad= \frac{1}{n}\binom{n}{j}\lambda^j t^{1-j} \int_{B_3} \xi(|x|)x_n\,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &\quad= \frac{1}{n}\binom{n}{j}\lambda^j t^{n-j+1}\xi(t) \int_{\mathbb{S}^{n-1}\cap|x|_{n-1}<\frac{s}{t}\cap x_n<0} x_n\,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &\quad= \frac{1}{n}\binom{n}{j}\lambda^j t^{n-j+1}\xi(t) \int_{\mathbb{S}^{n-1}\cap e_n^\perp} \int_{-1}^{-\sqrt{1-\frac{s^2}{t^2}}} \tau(1-\tau^2)^{\frac{n-3}{2}}\,\mathrm{d}\tau\,\mathrm{d}\omega \\ &\quad= -\frac{1}{n}\frac{\omega_{n-1}}{n-1}\binom{n}{j}\lambda^j t^{2-j} s^{n-1}\xi(t).\end{aligned}$$

(i) For the fourth set of singular points $B_4 = \{x \in \mathbb{R}^n : |x|_{n-1} = s, |x| \le t, x_n < 0\}$, since

$$(\mu w_s + \lambda v_t)(x_1, \dots, x_n) = \mu p_s(x_1, \dots, x_{n-1})$$

for every $(x_1, \dots, x_n) \in \{x \in \mathbb{R}^n : |x| = t, x_n < 0\}$ where $p_s(x) = \max\{0, |x|_{n-1} - s\}$, it follows from Lemma 3.2.4 and by (3.2.2), for every $1 \le j \le n-1$, that

$$\int_{B_4} \xi(|x|)x_n\,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) = \mu^j \int_{B_4} \xi(|x|)x_n\,\mathrm{d}\Phi_j^{n-1}(p_s, (x_1, \dots, x_{n-1}))\,\mathrm{d}x_n.$$

By Lemma 3.2.7,

$$\Phi_j^{n-1}(p_s, s\mathbb{S}^{n-1}) = \frac{\omega_{n-1}}{n-1}\binom{n-1}{j} s^{n-j-1}$$

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and hence for every $1 \leq j \leq n-1$,

$$\begin{aligned}\int_{B_4} &\xi(|x|)x_n \,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) \\ &= \mu^j \int_{B_4} \xi(|x|)x_n \,\mathrm{d}\Phi_j^{n-1}(p_s, (x, \cdots, x_{n-1}))\,\mathrm{d}x_n \\ &= \mu^j \binom{n-1}{j} \frac{\omega_{n-1}}{n-1} s^{n-j-1} \int_{\sqrt{s^2+x_n^2}<t \cap x_n<0} \xi\left(\sqrt{s^2+x_n^2}\right) x_n \,\mathrm{d}x_n \\ &= -\mu^j \binom{n-1}{j} \frac{\omega_{n-1}}{n-1} s^{n-j-1} \int_s^t r\xi(r)\,\mathrm{d}r.\end{aligned}$$

(j) For the fifth set of singular points given by $B_5 = \{x \in \mathbb{R}^n : |x|_{n-1} = s, |x| > t, x_n < 0\}$, we remark that $x_n = -\sqrt{|x|^2 - s^2}$. By Lemma 3.2.1, we have

$$\partial(\mu w_s + \lambda v_t)(x) = \partial(\mu w_s) + \partial(\lambda v_t) = \left\{\left(\alpha \frac{x_1}{|x|_{n-1}}, \ldots, \alpha \frac{x_{n-1}}{|x|_{n-1}}, 0\right) : \alpha \in [0,\mu]\right\} + \lambda \frac{x}{|x|}$$

for every $x \in B_5$, and this implies that

$$\begin{aligned}&P_p(\mu w_s + \lambda v_t, B_5) = \\ &\left\{x + p\alpha \frac{(x - x_n e_n)}{|x|_{n-1}} + p\lambda \frac{x}{|x|} : |x| > t, |x|_{n-1} = s, x_n = -\sqrt{|x|^2 - s^2}, \alpha \in [0,\mu]\right\} = \\ &\left\{(r + p\lambda)(\tau e_n + \sqrt{1-\tau^2} v) + p\alpha v : v \in \mathbb{S}^{n-1} \cap e_n^{\perp}, r > t, \tau = -\frac{1}{r}\sqrt{r^2 - s^2}, \alpha \in [0,\mu]\right\}\end{aligned}$$

and thus

$$\begin{aligned}&P_p(\mu w_s + \lambda v_t, B_5) = \\ &\left\{-\left(1 + p\frac{\lambda}{r}\right)\sqrt{r^2 - s^2} e_n + \left(s + p\left(\frac{\lambda s}{r} + \alpha\right)\right) v : v \in \mathbb{S}^{n-1} \cap e_n^{\perp}, r > t, \alpha \in [0,\mu]\right\}\end{aligned}$$

for all $p \geq 0$. Hence, the n-dimensional Hausdorff measure of B_5 is given by

$$\begin{aligned}&\omega_{n-1} \int_t^{+\infty} \int_0^{\mu} \left[p\lambda \frac{s^2}{r^2\sqrt{r^2-s^2}} + \frac{r}{\sqrt{r^2-s^2}}\right] p\left(s + p\left(\frac{\lambda s}{r} + \alpha\right)\right)^{n-2} \mathrm{d}\alpha\,\mathrm{d}r = \frac{\omega_{n-1}}{n-1} \times \\ &\int_t^{+\infty} \left[p\lambda \frac{s^2}{r^2\sqrt{r^2-s^2}} + \frac{r}{\sqrt{r^2-s^2}}\right] \left[\left(s + p\left(\frac{\lambda s}{r} + \mu\right)\right)^{n-1} - \left(s + p\frac{\lambda s}{r}\right)^{n-1}\right] \mathrm{d}r = \\ &\frac{\omega_{n-1}}{n-1} \sum_{j=1}^{n-1} p^j \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} \lambda^{j-k} \mu^k s^{n-k-1} \times \\ &\int_t^{+\infty} \left[p\lambda \frac{s^2}{r^2\sqrt{r^2-s^2}} + \frac{r}{\sqrt{r^2-s^2}}\right] \frac{1}{r^{j-k}}\,\mathrm{d}r = \\ &\frac{\omega_{n-1}}{n-1} \sum_{j=1}^{n-1} p^{j+1} \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} \lambda^{j-k+1} \mu^k s^{n-k+1} \int_t^{+\infty} \frac{1}{r^{j-k+2}\sqrt{r^2-s^2}}\,\mathrm{d}r + \\ &\frac{\omega_{n-1}}{n-1} \sum_{j=1}^{n-1} p^j \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} \lambda^{j-k} \mu^k s^{n-k-1} \int_t^{+\infty} \frac{1}{r^{j-k-1}\sqrt{r^2-s^2}}\,\mathrm{d}r.\end{aligned}$$

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Thus, for $j = 1$,

$$\begin{aligned}\Phi_1^n(\mu w_s + \lambda v_t, B_5) &= \omega_{n-1} s^{n-2} \mu \int_t^{+\infty} \frac{r}{\sqrt{r^2 - s^2}} \,\mathrm{d}r \\ &= \omega_{n-1} s^{n-2} \mu \int_{-\infty}^{-\sqrt{t^2-s^2}} \mathrm{d}x_n\end{aligned}$$

and for $2 \leq j \leq n-1$,

$$\begin{aligned}&\Phi_j^n(\mu w_s + \lambda v_t, B_5) \\ &= \frac{\omega_{n-1}}{n-1} \left[\binom{n}{j} \frac{1}{n} \sum_{k=1}^{j-1} \binom{j}{k} (j-k) s^{n-k+1} \lambda^{j-k} \mu^k \int_t^{+\infty} \frac{1}{r^{j-k+1}\sqrt{r^2-s^2}} \,\mathrm{d}r \right. \\ &\left. + \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} s^{n-k-1} \lambda^{j-k} \mu^k \int_t^{+\infty} \frac{1}{r^{j-k-1}\sqrt{r^2-s^2}} \,\mathrm{d}r \right] \\ &= \frac{\omega_{n-1}}{n-1} \left[\binom{n}{j} \frac{1}{n} \sum_{k=1}^{j-1} \binom{j}{k} (j-k) s^{n-k+1} \lambda^{j-k} \mu^k \int_{-\infty}^{-\sqrt{t^2-s^2}} (x_n^2 + s^2)^{\frac{k-j-2}{2}} \,\mathrm{d}x_n \right. \\ &\left. + \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} s^{n-k-1} \lambda^{j-k} \mu^k \int_{-\infty}^{-\sqrt{t^2-s^2}} (x_n^2 + s^2)^{\frac{k-j}{2}} \,\mathrm{d}x_n \right].\end{aligned}$$

Lastly, for $j = n$,

$$\begin{aligned}\Phi_n^n(\mu w_s + \lambda v_t, B_5) &= \frac{\omega_{n-1}}{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} s^{n-k+1} \lambda^{n-k} \mu^k \int_t^{+\infty} \frac{1}{r^{n-k+1}\sqrt{r^2-s^2}} \,\mathrm{d}r \\ &= \frac{\omega_{n-1}}{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} s^{n-k+1} \lambda^{n-k} \mu^k \int_{-\infty}^{-\sqrt{t^2-s^2}} (x_n^2 + s^2)^{\frac{k-j-2}{2}} \,\mathrm{d}x_n.\end{aligned}$$

On the other hand, let $\pi : \mathbb{R}^n \to \mathbb{R}$ be the projection of every point in $x \in \mathbb{R}^n$ to its last coordinate $x_n \in \mathbb{R}$. Since $\mathbb{R}$ is locally compact and separable, the set $B_5 \subset \mathbb{R}^n$ is a Borel set and the measure $\Phi_j^n(\mu w_s + \lambda v_t, \cdot) \llcorner B_5$ is concentrated on B_5, then by Theorem 3.A.2, there exists a measure η such that it is concentrated on $\Sigma = \left(-\infty, -\sqrt{t^2 - s^2}\right) \subset \mathbb{R}$ and for every $x_n \in \Sigma$ there exists a finite Borel measure $\mu_{x_n}^* \in \mathcal{P}(\mathbb{R}^n)$ such that $\mu_{x_n}^*(\mathbb{R}^n) > 0$ and it is concentrated on $B_5 \cap \pi^{-1}(x_n)$, and

$$\int_{B_5} \varphi(x) \,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) = \int_\Sigma \mathrm{d}\eta(x_n) \int_{\mathbb{R}^n} \varphi(x) \,\mathrm{d}\mu_{x_n}^*(x)$$

for every bounded Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}$. Since

$$\eta = (\pi|_{B_5})_\#(\Phi_j^n(\mu w_s + \lambda v_t, \cdot) \llcorner B_5) = \pi_\#(\Phi_j^n(\mu w_s + \lambda v_t, \cdot) \llcorner B_5),$$

and $\mu_{x_n}^* = \mu_{x_n}^{B_5}$ where $\mu_{x_n}^{B_5}$ denotes the disintegration of $\Phi_j^n(\mu w_s + \lambda v_t, \cdot) \llcorner B_5$ with respect to $\pi|_{B_5}$ satisfies the required properties and since $B_5 \cap \pi^{-1}(x_n) = s\mathbb{S}^{n-2}$ for every $x_n \in \Sigma$, then

$$\begin{aligned}&\int_{B_5} \varphi(x) \,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) \\ &\quad = \int_\Sigma \mathrm{d}\pi_\#(\Phi_j^n(\mu w_s + \lambda v_t, \cdot) \llcorner B_5)(x_n) \int_{s\mathbb{S}^{n-2}} \varphi(x', x_n) \,\mathrm{d}\mu_{x_n}^A(x').\end{aligned}$$

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Since the function $\mu w_s + \lambda v_t$ is $\mathrm{SO}(n-1)$ invariant on $B_5 \cap \pi^{-1}(x_n)$ for every $x_n \in \Sigma$, then $\mu_{x_n}^{B_5}$ is $\mathrm{SO}(n-1)$ invariant, this implies that there exist a function $\tilde{K}_{n,j}$ depending on x_n such that

$$\int_{B_5} \varphi(x)\,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) = \int_{\Sigma'} \mathrm{d}\pi_{\#}(\Phi_j^n(\mu w_s + \lambda v_t, \cdot)\llcorner B_5)(x_n) \int_{s\mathbb{S}^{n-2}} \tilde{K}_{n,j}(x_n)\varphi(x', x_n)\,\mathrm{d}\mathcal{H}^{n-2}(x'), \quad (3.3.14)$$

and since for every generating Borel set of $\mathbb{R}$, which is given by $(-\infty, c)$ for some $c \in \mathbb{R}$, we have

$$(\pi_{\#}(\Phi_j^n(\mu w_s + \lambda v_t, \cdot)\llcorner B_5))(Y) = \Phi_j^n(\mu w_s + \lambda v_t, B_5 \cap \pi^{-1}(Y)).$$

Then, by computing the parallel set of the function $\mu w_s + \lambda v_t$ at $B_5 \cap \pi^{-1}(Y)$, we have that for $j = 1$,

$$\begin{aligned}\Phi_1^n(\mu w_s + \lambda v_t, B_5 \cap \pi^{-1}(Y)) =& \omega_{n-1} s^{n-2} \mu \int_t^{+\infty} \mathbb{1}_Y\left(-\sqrt{r^2 - s^2}\right) \frac{r}{\sqrt{r^2 - s^2}}\,\mathrm{d}r \\ &= \omega_{n-1} s^{n-2} \mu \int_{-\infty}^{-\sqrt{t^2-s^2}} \mathbb{1}_Y(x_n)\,\mathrm{d}x_n\end{aligned}$$

while for $2 \le j \le n-1$,

$$\begin{aligned}&\Phi_j^n(\mu w_s + \lambda v_t, B_5 \cap \pi^{-1}(Y)) = \frac{\omega_{n-1}}{n-1} \times \\ &\left[\binom{n}{j}\frac{1}{n}\sum_{k=1}^{j-1}\binom{j}{k}(j-k)s^{n-k+1}\lambda^{j-k}\mu^k \int_{-\infty}^{-\sqrt{t^2-s^2}} \mathbb{1}_Y(x_n)(x_n^2 + s^2)^{\frac{k-j-2}{2}}\,\mathrm{d}x_n + \right. \\ &\left.\binom{n-1}{j}\sum_{k=1}^{j}\binom{j}{k}s^{n-k-1}\lambda^{j-k}\mu^k \int_{-\infty}^{-\sqrt{t^2-s^2}} \mathbb{1}_Y(x_n)(x_n^2 + s^2)^{\frac{k-j}{2}}\,\mathrm{d}x_n\right]\end{aligned}$$

and

$$\begin{aligned}&\Phi_n^n(\mu w_s + \lambda v_t, B_5 \cap \pi^{-1}(Y)) = \\ &\frac{\omega_{n-1}}{n-1}\sum_{k=1}^{n-1}\binom{n-1}{k}s^{n-k+1}\lambda^{n-k}\mu^k \int_{-\infty}^{-\sqrt{t^2-s^2}} \mathbb{1}_Y(x_n)(x_n^2 + s^2)^{\frac{k-n-2}{2}}\,\mathrm{d}x_n\end{aligned}$$

when $j = n$. This implies that the measure $\pi_{\#}(\Phi_j^n(\mu w_s + \lambda v_t, \cdot)\llcorner B_5)$ is absolutely continuous with respect to the Lebesgue measure. After substituting for $\varphi = 1$ in (3.3.14) and comparing with the total measure of $\Phi_j^n(\mu w_s + \lambda v_t, \cdot)\llcorner B_5$, we get that $K_{n,j}(x_n) = \frac{s^{2-n}}{\omega_{n-1}}$ for every $x_n \in \mathbb{R}$. Thus, for $j = 1$,

$$\begin{aligned}\int_{B_5} \xi(|x|)x_n\,\mathrm{d}\Phi_1^n(\mu w_s + \lambda v_t, x) &= \omega_{n-1}\mu s^{n-2} \int_\Sigma \int_{s\mathbb{S}^{n-2}} \xi(|x|)x_n\,\mathrm{d}\mathcal{H}^{n-2}(x')\,\mathrm{d}x_n \\ &= \omega_{n-1}\mu s^{n-2} \int_{-\infty}^{-\sqrt{t^2-s^2}} \xi\left(\sqrt{s^2 + x_n^2}\right) x_n\,\mathrm{d}x_n \\ &= -\omega_{n-1}\mu s^{n-2} \int_t^{+\infty} r\xi(r)\,\mathrm{d}r\end{aligned}$$

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and for every $1 \leq j \leq n-1$,

$$\begin{aligned}
&\int_{B_5} \xi(|x|) x_n \, \mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) \\
&= -\frac{\omega_{n-1}}{n-1} \left[\binom{n}{j} \frac{1}{n} \sum_{k=1}^{j-1} \binom{j}{k} (j-k) \lambda^{j-k} \mu^k s^{n-k+1} \int_t^{+\infty} r^{k-j-1} \xi(r) \, \mathrm{d}r \right. \\
&\quad \left. + \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} \lambda^{j-k} \mu^k s^{n-k-1} \int_t^{+\infty} r^{k-j+1} \xi(r) \, \mathrm{d}r \right]
\end{aligned}$$

whereas for $j = n$, we have,

$$\begin{aligned}
&\int_{B_5} \xi(|x|) x_n \, \mathrm{d}\Phi_n^n(\mu w_s + \lambda v_t, x) \\
&\qquad = -\frac{\omega_{n-1}}{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} \lambda^{j-k} \mu^k s^{n-k+1} \int_t^{+\infty} r^{k-n-1} \xi(r) \, \mathrm{d}r.
\end{aligned}$$

(k) For the sixth set of singular points given by $B_6 = \{x \in \mathbb{R}^n : |x| = t, |x|_{n-1} > s, x_n < 0\}$, we have by Lemma 3.2.1,

$$\begin{aligned}
\partial(\mu\, w_s + \lambda\, v_t)(x) &= \partial(\mu\, w_s)(x) + \partial(\lambda\, v_t)(x) \\
&= \left\{ \left(\mu \frac{x_1}{|x|_{n-1}}, \ldots, \mu \frac{x_{n-1}}{|x|_{n-1}}, 0 \right) \right\} + \left\{ \alpha \frac{x}{|x|} : \alpha \in [0, \lambda] \right\}
\end{aligned}$$

for every $x \in B_6$, and thus

$$\begin{aligned}
&P_p(\mu w_s + \lambda v_t, B_6) \\
&= \left\{ x + p\mu \frac{(x - x_n e_n)}{|x|_{n-1}} + p\alpha \frac{x}{|x|} : |x| = t, |x|_{n-1} > s, x_n < 0, \alpha \in [0, \lambda] \right\} \\
&= \left\{ (t + p\alpha)(\tau e_n + \sqrt{1-\tau^2} v) + p\mu v : v \in \mathbb{S}^{n-1} \cap e_n^{\perp}, \tau \in [-\sqrt{1 - \frac{s^2}{t^2}}, 0], \alpha \in [0, \lambda] \right\}
\end{aligned}$$

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for all $p \geq 0$. Hence, the n-dimensional Hausdorff measure of $_6$ is given by

$$\omega_{n-1}\int_0^\lambda\int_{-\sqrt{1-\frac{s^2}{r^2}}}^0 p(t+p\alpha)^{n-1}\left(1+\frac{p\mu}{(t+p\alpha)\sqrt{1-\tau^2}}\right)^{n-2}(1-\tau^2)^{\frac{n-3}{2}}\,\mathrm{d}\tau\,\mathrm{d}\alpha =$$

$$\omega_{n-1}\int_0^\lambda\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0 p(t+p\alpha)\left(t+p\left(\alpha+\frac{\mu}{\sqrt{1-\tau^2}}\right)\right)^{n-2}(1-\tau^2)^{\frac{n-3}{2}}\,\mathrm{d}\tau\,\mathrm{d}\alpha =$$

$$\omega_{n-1}\sum_{j=0}^{n-2}\binom{n-2}{j}\int_0^\lambda\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0 p(t+p\alpha)t^{n-j-2}p^j\left(\alpha+\frac{\mu}{\sqrt{1-\tau^2}}\right)^j(1-\tau^2)^{\frac{n-3}{2}}\,\mathrm{d}\tau\,\mathrm{d}\alpha$$

$$=\omega_{n-1}\sum_{j=0}^{n-2}\binom{n-2}{j}p^{j+1}t^{n-j-1}\sum_{k=0}^{j}\binom{j}{k}\mu^k\int_0^\lambda\alpha^{j-k}\,\mathrm{d}r\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau+$$

$$\omega_{n-1}\sum_{j=0}^{n-2}\binom{n-2}{j}p^{j+2}t^{n-j-2}\sum_{k=0}^{j}\binom{j}{k}\mu^k\int_0^\lambda\alpha^{j-k+1}\,\mathrm{d}r\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau =$$

$$\omega_{n-1}\sum_{j=0}^{n-2}\binom{n-2}{j}p^{j+1}t^{n-j-1}\sum_{k=0}^{j}\binom{j}{k}\frac{1}{j-k+1}\mu^k\lambda^{j-k+1}\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau+$$

$$\omega_{n-1}\sum_{j=0}^{n-2}\binom{n-2}{j}p^{j+2}t^{n-j-2}\sum_{k=0}^{j}\binom{j}{k}\frac{1}{j-k+2}\mu^k\lambda^{j-k+2}\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau.$$

Thus, for $j=1$,

$$\Phi_1^n(\mu w_s+\lambda v_t,B_6)=\omega_{n-1}\lambda\,t^{n-1}\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0(1-\tau^2)^{\frac{n-3}{2}}\,\mathrm{d}\tau$$

and

$$\Phi_j^n(\mu w_s+\lambda v_t,B_6)=\frac{\omega_{n-1}}{n-1}t^{n-j}\left[\binom{n-1}{j}\sum_{k=0}^{j-1}\binom{j}{k}\mu^k\lambda^{j-k}\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau\right.$$
$$\left.+\frac{1}{n}\binom{n}{j}\sum_{k=0}^{j-2}\binom{j}{k}(j-k-1)\mu^k\lambda^{j-k}\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau\right]$$
$$=\frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j}t^{n-j}\sum_{k=0}^{j-1}\binom{j}{k}(n-k-1)\mu^k\lambda^{j-k}\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau$$

for every $2\leq j\leq n-1$, whereas for $j=n$, we have,

$$\Phi_n^n(\mu w_s+\lambda v_t,B_6)=\frac{\omega_{n-1}}{n-1}\frac{1}{n}\sum_{k=0}^{n-2}\binom{n}{k}(n-k-1)\mu^k\lambda^{n-k}\int_{-\sqrt{1-\frac{s^2}{t^2}}}^0(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau$$

On the other hand, since B_6 is Borel set, then similarly to the argument above and by Theorem 3.A.2, there exists a measure $\eta=\pi_\#(\Phi_j^n(\mu w_s+\lambda v_t,\cdot)\llcorner B_6)$ such that it is concentrated on $\Sigma'=\left(-\sqrt{t^2-s^2},0\right)\subset\mathbb{R}$, and for every $x_n\in\Sigma'$ there exists a finite Borel measure $\mu^*_{x_n}=\mu^{B_6}_{x_n}$, where $\mu^{B_6}_{x_n}$ denotes the disintegration of $\Phi_j^n(\mu w_s+\lambda v_t,\cdot)\llcorner B_6$ with respect to $\pi|_{B_6}$ such that $\mu^{B_6}_{x_n}(\mathbb{R}^n)>0$ and it is concentrated on $B_6\cap$

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$\pi^{-1}(x_n) = \sqrt{t^2 - x_n^2}\mathbb{S}^{n-2}$, and

$$\int_{B_6} \varphi(x)\,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) = \int_{\Sigma'} \mathrm{d}\pi_{\#}(\Phi_j^n(\mu w_s + \lambda v_t, \cdot)\llcorner B_6)(x_n) \int_{\sqrt{t^2-x_n^2}\mathbb{S}^{n-2}} \varphi(x', x_n)\,\mathrm{d}\mu_{x_n}^{B_6}(x')$$

for every bounded Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}$. Since $\mu_{x_n}^{B_6}$ is concentrated on the set $\sqrt{t^2 - x_n^2}\,\mathbb{S}^{n-2}$ and the function $\mu w_s + \lambda v_t$ is $\mathrm{SO}(n-1)$ invariant on $B_6 \cap \pi^{-1}(x_n)$ for every $x_n \in \Sigma'$, then $\mu_{x_n}^{B_6}$ is $\mathrm{SO}(n-1)$ invariant; this implies that there exist a function $L_{n,j}$ depending on x_n such that

$$\int_{B_6} \varphi(x)\,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) = \int_{\Sigma'} \mathrm{d}\pi_{\#}(\Phi_j^n(\mu w_s + \lambda v_t, \cdot)\llcorner B_6)(x_n) \int_{\sqrt{t^2-x_n^2}\,\mathbb{S}^{n-2}} L_{n,j}(x_n)\varphi(x', x_n)\,\mathrm{d}\mathcal{H}^{n-2}(x'), \tag{3.3.15}$$

and since for every generating Borel set of $\mathbb{R}$, which is given by $(-\infty, c)$ for some $c \in \mathbb{R}$, we have

$$(\pi_{\#}(\Phi_j^n(\mu w_s + \lambda v_t, \cdot)\llcorner B_6))(Y) = \Phi_j^n(\mu w_s + \lambda v_t, B_6 \cap \pi^{-1}(Y)),$$

then by computing the parallel set of the function $\mu w_s + \lambda v_t$ at $B_6 \cap \pi^{-1}(Y)$, it follows that

$$\begin{aligned}\Phi_1^n(\mu w_s + \lambda v_t, B_6 \cap \pi^{-1}(Y)) &= \omega_{n-1}\lambda\, t^{n-1} \int_{-\sqrt{1-\frac{s^2}{t^2}}}^{0} \mathbb{1}_Y(t\tau)(1-\tau^2)^{\frac{n-3}{2}}\,\mathrm{d}\tau \\ &= \omega_{n-1}\lambda\, t \int_{-\sqrt{t^2-s^2}}^{0} \mathbb{1}_Y(x_n)(t^2 - x_n^2)^{\frac{n-3}{2}}\,\mathrm{d}x_n\end{aligned}$$

and for every $2 \le j \le n-1$, the measure $\Phi_j^n(\mu w_s + \lambda v_t, \cdot)$ at B_6 is given by

$$\begin{aligned}&\frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j} t^{n-j} \sum_{k=0}^{j-1}\binom{j}{k}(n-k-1)\mu^k\lambda^{j-k} \int_{-\sqrt{1-\frac{s^2}{t^2}}}^{0} \mathbb{1}_Y(t\tau)(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau = \\ &\frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j} t^{2-j} \sum_{k=0}^{j-1}\binom{j}{k}(n-k-1)\mu^k\lambda^{j-k} \int_{-\sqrt{t^2-s^2}}^{0} \mathbb{1}_Y(x_n)(t^2 - x_n^2)^{\frac{n-k-3}{2}}\,\mathrm{d}x_n\end{aligned}$$

whereas for $j = n$, we have that

$$\begin{aligned}&\Phi_n^n(\mu w_s + \lambda v_t, B_6) \\ &= \frac{\omega_{n-1}}{n-1}\frac{1}{n}\sum_{k=0}^{n-2}\binom{n}{k}(n-k-1)\mu^k\lambda^{n-k} \int_{-\sqrt{1-\frac{s^2}{t^2}}}^{0} \mathbb{1}_Y(t\tau)(1-\tau^2)^{\frac{n-k-3}{2}}\,\mathrm{d}\tau \\ &= \frac{\omega_{n-1}}{n-1}\frac{1}{n}\sum_{k=0}^{n-2}\binom{n}{k}(n-k-1)t^{2-n}\mu^k\lambda^{n-k} \int_{-\sqrt{t^2-s^2}}^{0} \mathbb{1}_Y(x_n)(t^2 - x_n^2)^{\frac{n-k-3}{2}}\,\mathrm{d}x_n\end{aligned}$$

This implies that the measure $\pi_{\#}(\Phi_j^n(\mu w_s + \lambda v_t, \cdot)\llcorner B_6)$ is absolutely continuous with respect to the Lebesgue measure. After substituting for $\varphi = 1$ in (3.3.15) and comparing with the total measure of $\Phi_j^n(\mu w_s + \lambda v_t, \cdot)$ at B_6, we get that $L_{n,j}(x_n) = \frac{(t^2-x_n^2)^{\frac{2-n}{2}}}{\omega_{n-1}}$

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for every $x_n \in \mathbb{R}$. Thus for $j = 1$, we have

$$\begin{aligned}
&\int_{B_6} \xi(|x|)x_n \,\mathrm{d}\Phi_1^n(\mu w_s + \lambda v_t, x) \\
&\qquad = \omega_{n-1}\lambda t \int_{\Sigma'} (t^2 - x_n^2)^{\frac{n-3}{2}} x_n L_{n,1}(x_n) \int_{\sqrt{t^2-x_n^2}\mathbb{S}^{n-1}} \xi(|x|)\,\mathrm{d}\mathcal{H}^{n-2}(x')\,\mathrm{d}x_n \\
&\qquad = \omega_{n-1}\lambda t^n \xi(t) \int_{-\sqrt{1-\frac{s^2}{t^2}}}^{0} (1-u^2)^{\frac{n-2}{2}} u \,\mathrm{d}u \\
&\qquad = \frac{\omega_{n-1}}{n-1}\lambda\, \xi(t)\left[-t^n + ts^{n-1}\right]
\end{aligned}$$

and for every $2 \le j \le n-1$, we have

$$\begin{aligned}
&\int_{B_6} \xi(|x|)x_n \,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) \\
&\quad = \frac{1}{n}\binom{n}{j}\frac{\omega_{n-1}}{n-1} t^{n-j+1}\xi(t) \sum_{k=0}^{j-1}\binom{j}{k}(n-k-1)\mu^k\lambda^{j-k}\int_{-\sqrt{1-\frac{s^2}{t^2}}}^{0} u(1-u^2)^{\frac{n-k-3}{2}}\,\mathrm{d}u \\
&\quad = \frac{\omega_{n-1}}{n-1} t^{n-j+1}\xi(t)\frac{1}{n}\binom{n}{j}\sum_{k=0}^{j-1}\binom{j}{k}\mu^k\lambda^{j-k}\left(-1+\frac{s^{n-k-1}}{t^{n-k-1}}\right),
\end{aligned}$$

and for $j = n$, we have

$$\begin{aligned}
&\int_{B_6} \xi(|x|)x_n \,\mathrm{d}\Phi_n^n(\mu w_s + \lambda v_t, x) \\
&\qquad = \frac{\omega_{n-1}}{n-1}\frac{1}{n}\sum_{k=0}^{n-2}\binom{n}{k}(n-k-1)\mu^k\lambda^{n-k}\xi(t)t\int_{-\sqrt{1-\frac{s^2}{t^2}}}^{0} u(1-u^2)^{\frac{n-k-3}{2}}\,\mathrm{d}u \\
&\qquad = -\frac{\omega_{n-1}}{n-1}\frac{1}{n}\sum_{k=0}^{n-2}\binom{n}{k}\mu^k\lambda^{n-k}\left[t - s^{n-k-1}t^{k+2-n}\right]\xi(t).
\end{aligned}$$

(l) For the last set of singular points $B_7 = \{x \in \mathbb{R}^n : |x|_{n-1} = s, |x| = t, x_n < 0\}$, we have by Lemma 3.2.1 that

$$\partial(\mu\, w_s + \lambda\, v_t)(x) = \left\{\alpha\frac{x}{|x|} : \alpha \in [0,\lambda]\right\} + \left\{(\beta\frac{x_1}{|x|_{n-1}}, \ldots, \beta\frac{x_{n-1}}{|x|_{n-1}}, 0) : \beta \in [0,\mu]\right\}$$

for every $x \in B_7$. This implies that

$$\begin{aligned}
&P_p(\mu w_s + \lambda v_t, B_7) = \\
&\left\{x + p\beta\frac{(x - x_n e_n)}{|x|_{n-1}} + p\alpha\frac{x}{|x|} : |x| = t, |x|_{n-1} = s, x_n < 0, \alpha \in [0,\lambda], \beta \in [0,\mu]\right\} = \\
&\left\{(t + p\alpha)(\tau e_n + \sqrt{1-\tau^2}v) + p\beta v : v \in \mathbb{S}^{n-1} \cap e_n^{\perp}, \tau = -\sqrt{1-\frac{s^2}{t^2}}, \alpha \in [0,\lambda], \beta \in [0,\mu]\right\}
\end{aligned}$$

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and thus, the n-dimensional Hausdorff measure of B_7 is given by

$$\omega_{n-1}\int_0^\lambda\int_0^\mu p^2\sqrt{1-\frac{s^2}{t^2}}\left(s+p\left(\frac{\alpha s}{t}+\beta\right)\right)^{n-2}\,\mathrm{d}\beta\,\mathrm{d}\alpha=$$
$$\omega_{n-1}\sum_{j=0}^{n-2}\binom{n-2}{j}\int_0^\lambda\int_0^\mu p^{j+2}\sqrt{1-\frac{s^2}{t^2}}s^{n-j-2}\left(\frac{\alpha s}{t}+\beta\right)^{j}\,\mathrm{d}\beta\,\mathrm{d}\alpha=$$
$$\omega_{n-1}\sum_{j=0}^{n-2}p^{j+2}\binom{n-2}{j}\sqrt{t^2-s^2}\sum_{k=0}^{j}\binom{j}{k}\frac{s^{n-k-2}}{t^{j-k+1}}\int_0^\lambda\alpha^{j-k}\,\mathrm{d}\alpha\int_0^\mu\beta^k\,\mathrm{d}\beta=$$
$$\omega_{n-1}\sum_{j=0}^{n-2}p^{j+2}\binom{n-2}{j}\sqrt{t^2-s^2}\sum_{k=0}^{j}\binom{j}{k}\frac{s^{n-k-2}}{t^{j-k+1}}\frac{\lambda^{j-k+1}}{j-k+1}\frac{\mu^{k+1}}{k+1}$$

This implies that

$$\Phi_j^n(\mu w_s+\lambda v_t,B_7)=\frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j}\sqrt{t^2-s^2}\sum_{k=0}^{j-2}\binom{j}{k+1}\frac{s^{n-k+2}}{t^{j-k-1}}\lambda^{j-k-1}\mu^{k+1}$$

for every $2\le j\le n$. Again, using the projection function to the last coordinate and since B_7 is a Borel set, then by Theorem 3.A.2, the measure $\pi_\#(\Phi_j^n(\mu w_s+\lambda v_t,\cdot)\llcorner B_7)$ is concentrated on $\Sigma'''=\left\{-\sqrt{t^2-s^2}\right\}\subset\mathbb{R}$ and for $x_n=-\sqrt{t^2-s^2}$ there exists a finite Borel measure $\mu^*_{x_n}$ with $\mu^*_{x_n}=\mu^{B_7}_{x_n}$ where $\mu^{B_7}_{x_n}$ denotes the disintegration of $\Phi_j^n(\mu w_s+\lambda v_t,\cdot)\llcorner B_7$ with respect to $\pi|_{B_7}$ such that $\mu^{B_7}_{x_n}(\mathbb{R}^n)>0$ and it is concentrated on $B_7\cap\pi^{-1}(-\sqrt{t^2-s^2})$, and

$$\int_{B_7}\varphi(x)\,\mathrm{d}\Phi_j^n(\mu w_s+\lambda v_t,x)$$
$$=\int_{\Sigma'''}\mathrm{d}\pi_\#(\Phi_j^n(\mu w_s+\lambda v_t,\cdot)\llcorner B_7)(x_n)\int_{s\mathbb{S}^{n-2}}\varphi(x',x_n)\,\mathrm{d}\mu^{B_7}_{x_n}(x')$$

Since $\mu^{B_7}_{x_n}$ is concentrated on $s\mathbb{S}^{n-2}$, the function $\mu w_s+\lambda v_t$ is $\mathrm{SO}(n-1)$ invariant on $B_7\cap\pi^{-1}(x_n)$, then $\mu^{B_7}_{x_n}$ is $\mathrm{SO}(n-1)$ invariant, this implies that there exist a constant $M_{n,j}$ such that

$$\int_{B_7}\varphi(x)\,\mathrm{d}\Phi_j^n(\mu w_s+\lambda v_t,x)=$$
$$\int_{\Sigma'''}\mathrm{d}\pi_\#(\Phi_j^n(\mu w_s+\lambda v_t,\cdot)\llcorner B_7)(x_n)\int_{s\mathbb{S}^{n-2}}M_{n,j}\,\varphi(x',x_n)\,\mathrm{d}\mathcal{H}^{n-2}(x')\qquad(3.3.16)$$

and since for every generating Borel set of $\mathbb{R}$, which is given by $(-\infty,c)$ for some $c\in\mathbb{R}$, we have

$$(\pi_\#(\Phi_j^n(\mu w_s+\lambda v_t,\cdot)\llcorner B_7))(Y)=\Phi_j^n(\mu w_s+\lambda v_t,B_7\cap\pi^{-1}(Y)),$$

then by computing the parallel set of the function $\mu w_s+\lambda v_t$ at $B_7\cap\pi^{-1}(Y)$, we have that for every $2\le j\le n-2$,

$$\Phi_j^n(\mu w_s+\lambda v_t,B_7\cap\pi^{-1}(Y))$$
$$=\frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j}\sqrt{t^2-s^2}\sum_{k=0}^{j-2}\binom{j}{k+1}\frac{s^{n-k+2}}{t^{j-k-1}}\lambda^{j-k-1}\mu^{k+1}\delta_{-\sqrt{t^2-s^2}}.$$

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This implies that the measure $\pi_{\#}(\Phi_j^n(\mu w_s+\lambda v_t,\cdot)\llcorner B_7)$ proportional to the Dirac delta measure concentrated at the point $x_n = -\sqrt{t^2-s^2}$. After substituting for $\varphi = 1$ in (3.3.16) and comparing with the total measure of $\Phi_j^n(\mu w_s+\lambda v_t,\cdot)$ at B_7, we get that $M_{n,j} = \frac{s^{2-n}}{\omega_{n-1}}$. Hence,

$$\begin{aligned}\int_{B_7} &\xi(|x|)x_n \,\mathrm{d}\Phi_j^n(\mu w_s+\lambda v_t,x)\\ &= -\frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j}\xi(t)\sum_{k=0}^{j-2}\binom{j}{k+1}\lambda^{j-k-1}\mu^{k+1}\left[\frac{s^{n-k-2}}{t^{j-k-3}}-\frac{s^{n-k}}{t^{j-k-1}}\right]\\ &= -\frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j}\xi(t)\sum_{k=1}^{j-1}\binom{j}{k}\lambda^{j-k}\mu^{k}\left[\frac{s^{n-k-1}}{t^{j-k-2}}-\frac{s^{n-k+1}}{t^{j-k}}\right]\end{aligned}$$

for every $2\le j\le n$.

Since the sets $A_1,\dots,A_5$ and $B_1,\dots,B_7$ are disjoint and their union is $\mathbb{R}^n$, then for every $s,t\ge 0$ such that $s\le t$, we have

$$\int_{\mathbb{R}^n}\xi(|x|)x_n\Phi_1^n(\mu w_s+\lambda v_t,x) = \frac{\omega_{n-1}}{n-1}\mu s^n\xi(s)$$

for $j=1$ and for every $2\le j\le n$,

$$\begin{aligned}\int_{\mathbb{R}^n} &\xi(|x|)x_n \,\mathrm{d}\Phi_j^n(\mu w_s+\lambda v_t,x)\\ &= \frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j}\left[\mu^j s^{n-j+1}\xi(s)+\sum_{k=1}^{j-1}\binom{j}{k}\lambda^{j-k}\mu^k s^{n-k+1}\times\right.\\ &\quad\left.\left(t^{k-j}\xi(t)-(j-k)\int_t^\infty r^{k-j-1}\xi(r)\,\mathrm{d}r\right)\right]\\ &= \frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j}\left[\mu^j s^{n-j+1}\xi(s)+\sum_{k=1}^{j-1}\binom{j}{k}\lambda^{j-k}\mu^k s^{n-k+1}\,\mathcal{R}^{k-j}\,\xi(t)\right]\end{aligned}$$

where the last equality holds by the definition of the $\mathcal{R}$ transform.

2. For $t<s$, we divide the domain $\mathbb{R}^n$ into several parts.

   (a) The function $\mu w_s+\lambda v_t$ is of class C^2 on the set $C_1=\{x\in\mathbb{R}^n : t<|x|\le s, x_n\ge 0\}$, and its Hessian matrix at a point x in this set has $(n-1)$-eigenvalues equal to $\lambda/|x|$ and the last eigenvalue is equal to zero. Hence,

$$[\mathrm{D}^2(\mu w_s+\lambda v_t)(x)]_j = \begin{cases}\binom{n-1}{j}\frac{\lambda^j}{|x|^j} & \text{if } 0\le j\le n-1\\ 0 & \text{if } j=n\end{cases}$$

   and for $1\le j\le n-1$,

$$\begin{aligned}\int_{C_1}\xi(|x|)x_n[\mathrm{D}^2(\mu w_s+\lambda v_t)(x)]_j\,\mathrm{d}x &= \lambda^j\binom{n-1}{j}\int_t^s r^{n-j}\xi(r)\,\mathrm{d}r\int_{\mathbb{S}^{n-1}\cap y_n\ge 0}y_n\,\mathrm{d}y\\ &= \lambda^j\binom{n-1}{j}\frac{\omega_{n-1}}{n-1}\int_t^s r^{n-j}\xi(r)\,\mathrm{d}r.\end{aligned}$$

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(b) The function $\mu w_s + \lambda v_t$ is of class C^2 on the set $C_2 = \{x \in \mathbb{R}^n : |x| > s, x_n \geq 0\}$, and its Hessian matrix at a point x on C_2 has $(n-1)$-eigenvalues equal to $(\mu+\lambda)/|x|$ and the last eigenvalue is equal to zero. Hence,

$$[\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j = \begin{cases} \binom{n-1}{j} \frac{(\mu+\lambda)^j}{|x|^j} & \text{if } 0 \leq j \leq n-1 \\ 0 & \text{if } j = n \end{cases}$$

and for $1 \leq j \leq n-1$,

$$\begin{aligned} &\int_{C_2} \xi(|x|) x_n [\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j \, \mathrm{d}x \\ &\qquad = (\mu+\lambda)^j \binom{n-1}{j} \int_s^\infty r^{n-j} \xi(r) \, \mathrm{d}r \int_{\mathbb{S}^{n-1} \cap y_n \geq 0} y_n \, \mathrm{d}y \\ &\qquad = (\mu+\lambda)^j \binom{n-1}{j} \frac{\omega_{n-1}}{n-1} \int_s^\infty r^{n-j} \xi(r) \, \mathrm{d}r. \end{aligned}$$

(c) The function $\mu w_s + \lambda v_t$ is also of class C^2 on $C_3 = \{x \in \mathbb{R}^n : |x|_{n-1} \leq s, |x| > t, x_n < 0\}$, and its Hessian matrix at a point $x \in C_3$ has $(n-1)$-eigenvalues equal to $\lambda/|x|$ and the last eigenvalue is equal to zero. Hence,

$$[\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j = \begin{cases} \binom{n-1}{j} \frac{\lambda^j}{|x|^j} & \text{if } 0 \leq j \leq n-1 \\ 0 & \text{if } j = n \end{cases}$$

and for $1 \leq j \leq n-1$, if we denote by $C_3^1 = \{x \in \mathbb{R}^n : t < |x| \leq s, x_n < 0\}$ and $C_3^2 = \{x \in \mathbb{R}^n : |x| > s, |x|_{n-1} < s, x_n < 0\}$, then $C_3^1 \cup C_3^2 = C_3$ and

$$\begin{aligned} &\int_{C_3} \xi(|x|) x_n [\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j \, \mathrm{d}x \\ &\quad = \lambda^j \binom{n-1}{j} \left[\int_{C_3^1} \xi(|x|) \frac{x_n}{|x|^j} \, \mathrm{d}x + \int_{C_3^2} \xi(|x|) \frac{x_n}{|x|^j} \, \mathrm{d}x \right] \\ &\quad = \lambda^j \binom{n-1}{j} \left[\int_t^s r^{n-j} \xi(r) \int_{\mathbb{S}^{n-1} \cap e_n^\perp} \int_{-1}^0 \tau (1-\tau^2)^{\frac{n-3}{2}} \, \mathrm{d}\tau \, \mathrm{d}y \, \mathrm{d}r \right. \\ &\quad \left. + \int_s^{+\infty} r^{n-j} \xi(r) \int_{\mathbb{S}^{n-1} \cap e_n^\perp} \int_{-1}^{-\sqrt{1-\frac{s^2}{r^2}}} \tau (1-\tau^2)^{\frac{n-3}{2}} \, \mathrm{d}\tau \, \mathrm{d}y \, \mathrm{d}r \right] \\ &\quad = -\frac{\omega_{n-1}}{n-1} \lambda^j \binom{n-1}{j} \left[\int_t^s r^{n-j} \xi(r) \, \mathrm{d}r + s^{n-1} \int_s^{+\infty} r^{1-j} \xi(r) \, \mathrm{d}r \right]. \end{aligned}$$

(d) The function $\mu w_s + \lambda v_t$ is again of class C^2 on $C_4 = \{x \in \mathbb{R}^n : |x|_{n-1} > s, x_n < 0\}$, and the eigenvalues of its Hessian matrix at a point x in this set are given by

$$[\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j = \begin{cases} \binom{n-2}{j} \left(\frac{\lambda}{|x|} + \frac{\mu}{|x|_{n-1}} \right)^j & \\ + & \text{if } 1 \leq j \leq n-2 \\ \binom{n-2}{j-1} \left(\frac{\lambda}{|x|} + \frac{\mu}{|x|_{n-1}} \right)^{j-1} \frac{\lambda}{|x|} & \\ \left(\frac{\lambda}{|x|} + \frac{\mu}{|x|_{n-1}} \right)^{n-2} \frac{\lambda}{|x|} & \text{if } j = n-1 \\ 0 & \text{if } j = n \end{cases}$$

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This implies, for $1 \le j \le n-2$, that

$$\begin{aligned}
&\int_{C_4} \xi(|x|) x_n [\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j \,\mathrm{d}x \\
&= \omega_{n-1} \Bigg[\binom{n-2}{j} \sum_{k=0}^{j} \binom{j}{k} \mu^k \lambda^{j-k} \int_s^{+\infty} r^{n-j} \xi(r) \int_{-\sqrt{1-\frac{s^2}{r^2}}}^{0} \tau (1-\tau^2)^{\frac{n-k-3}{2}} \,\mathrm{d}\tau \,\mathrm{d}r \\
&+ \binom{n-2}{j-1} \sum_{k=0}^{j-1} \binom{j-1}{k} \mu^k \lambda^{j-k} \int_s^{+\infty} r^{n-j} \xi(r) \int_{-\sqrt{1-\frac{s^2}{r^2}}}^{0} \tau (1-\tau^2)^{\frac{n-k-3}{2}} \,\mathrm{d}\tau \,\mathrm{d}r \Bigg] \\
&= -\omega_{n-1} \times \\
&\Bigg[\binom{n-2}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{\mu^k \lambda^{j-k}}{n-k-1} \left(\int_s^{+\infty} r^{n-j} \xi(r) \,\mathrm{d}r - s^{n-k-1} \int_s^{+\infty} r^{k-j+1} \xi(r) \,\mathrm{d}r \right) + \\
&\binom{n-2}{j-1} \sum_{k=0}^{j-1} \binom{j-1}{k} \frac{\mu^k \lambda^{j-k}}{n-k-1} \left(\int_s^{+\infty} r^{n-j} \xi(r) \,\mathrm{d}r - s^{n-k-1} \int_s^{+\infty} r^{k-j+1} \xi(r) \,\mathrm{d}r \right) \Bigg] \\
&= -\frac{\omega_{n-1}}{n-1} \binom{n-1}{j} \times \\
&\left[\sum_{k=0}^{j} \binom{j}{k} \mu^k \lambda^{j-k} \left(\int_s^{+\infty} r^{n-j} \xi(r) \,\mathrm{d}r - s^{n-k-1} \int_s^{+\infty} r^{k-j+1} \xi(r) \,\mathrm{d}r \right) \right],
\end{aligned}$$

and for $j = n-1$, we have

$$\begin{aligned}
&\int_{C_4} \xi(|x|) x_n [\mathrm{D}^2(\mu w_s + \lambda v_t)(x)]_j \,\mathrm{d}x \\
&= \omega_{n-1} \sum_{k=0}^{n-2} \binom{n-2}{k} \mu^k \lambda^{n-k-1} \int_s^{+\infty} r \xi(r) \int_{-\sqrt{1-\frac{s^2}{r^2}}}^{0} \tau (1-\tau^2)^{\frac{n-k-3}{2}} \,\mathrm{d}\tau \,\mathrm{d}r \\
&= -\omega_{n-1} \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\mu^k \lambda^{n-k-1}}{n-k-1} \left(\int_s^{+\infty} r \xi(r) \,\mathrm{d}r - s^{n-k-1} \int_s^{+\infty} r^{k-n+2} \xi(r) \,\mathrm{d}r \right) \\
&= -\frac{\omega_{n-1}}{n-1} \sum_{k=0}^{n-2} \binom{n-1}{k} \mu^k \lambda^{n-k-1} \left(\int_s^{+\infty} r \xi(r) \,\mathrm{d}r - s^{n-k-1} \int_s^{+\infty} r^{k-n+2} \xi(r) \,\mathrm{d}r \right).
\end{aligned}$$

(e) For the first set of singular points $D_1 = \{x \in \mathbb{R}^n : |x| = t, x_n \ge 0\}$, by (3.2.2), we have $\Phi_j^n(\mu w_s + \lambda v_t, \cdot)|_{D_1} = \Phi_j^n(\lambda v_t, \cdot)|_{D_1}$ and since v_t is radially symmetric, we have that $\Phi_j^n(v_t, \cdot)|_{D_1}$ is rotation invariant; this implies that it is equivalent to the Haar measure up to a constant $C_{n,j,t}$. Hence,

$$\int_{D_1} \gamma(x) \,\mathrm{d}\Phi_j^n(v_t, x) = C_{n,j,t} \int_{D_1} \gamma(x) \,\mathrm{d}\mathcal{H}^{n-1}(x)$$

for every $\gamma \in C_b(\mathbb{R}^n)$. It follows that,

$$\begin{aligned}
\Phi_j^n(v_t, D_1) &= C_{n,j,t} \int_{D_1} \mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \frac{\omega_{n-1}}{2} C_{n,j,t} t^{n-1},
\end{aligned}$$

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and by a similar proof of Lemma 3.2.7, we have

$$\Phi_j^n(v_t, D_1) = \frac{\kappa_n}{2}\binom{n}{j}t^{n-j}$$

and hence

$$C_{n,j,t} = \frac{1}{n}\binom{n}{j}t^{1-j}.$$

This implies that

$$\begin{aligned}\int_{D_1} \xi(|x|)x_n \,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) &= \int_{D_1} \xi(|x|)x_n \,\mathrm{d}\Phi_j^n(\lambda v_t, x)\\ &= \lambda^j \frac{1}{n}\binom{n}{j}t^{1-j}\int_{D_1}\xi(|x|)x_n \,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &= \lambda^j \frac{1}{n}\binom{n}{j}t^{n-j+1}\xi(t)\int_{\mathbb{S}^{n-1}\cap\{x_n\geq 0\}} x_n \,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &= \lambda^j \frac{1}{n}\binom{n}{j}\frac{\omega_{n-1}}{n-1}t^{n-j+1}\xi(t).\end{aligned}$$

(f) For the second set of singular points $D_2 = \{x \in \mathbb{R}^n : |x| = s, x_n \geq 0\}$, since $\mu w_s + \lambda v_t$ is radially symmetric on this set, we get that $\Phi_j^n(\mu w_s + \lambda v_t, \cdot)|_{D_2}$ is rotation invariant, this implies that it is equivalent to the Haar measure up to a constant $C_{n,j,s}$. Hence,

$$\int_{D_2}\gamma(x)\,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) = C_{n,j,s}\int_{D_2}\gamma(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

for every $\gamma \in C_b(\mathbb{R}^n)$. While,

$$\begin{aligned}\Phi_j^n(\mu w_s + \lambda v_t, D_2) &= C_{n,j,s}\int_{D_2}\mathrm{d}\mathcal{H}^{n-1}(x)\\ &= \frac{\omega_n}{2}C_{n,j,s}s^{n-1}.\end{aligned}$$

And on the other hand, since

$$P_p(\mu w_s + \lambda v_t, D_2 \times \mathbb{R}^n) = \Big\{(s+\alpha)x : \alpha \in [\lambda p, (\mu+\lambda)p], x \in \mathbb{S}^{n-1}, x_n \geq 0\Big\},$$

we have

$$\Phi_j^n(\mu w_s + \lambda v_t, D_2) = \binom{n}{j}\frac{\kappa_n}{2}s^{n-j}\sum_{k=1}^{j}\binom{j}{k}\lambda^{j-k}\mu^k$$

which implies that

$$C_{n,j,s} = \frac{1}{n}\binom{n}{j}s^{1-j}\sum_{k=1}^{j}\binom{j}{k}\lambda^{j-k}\mu^k,$$

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and thus

$$\begin{aligned}
&\int_{D_2} \xi(|x|)x_n \,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) \\
&\quad = \frac{1}{n}\binom{n}{j} s^{1-j} \sum_{k=1}^{j} \binom{j}{k} \lambda^{j-k}\mu^k \int_{D_2} \xi(|x|)x_n \,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&\quad = \frac{1}{n}\binom{n}{j} s^{n-j+1}\xi(s) \sum_{k=1}^{j} \binom{j}{k} \lambda^{j-k}\mu^k \int_{\mathbb{S}^{n-1}\cap\{x_n\geq 0\}} x_n \,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&\quad = \frac{1}{n}\frac{\omega_{n-1}}{n-1} s^{n-j+1}\xi(s) \sum_{k=1}^{j} \binom{j}{k} \lambda^{j-k}\mu^k.
\end{aligned}$$

(g) For the third set of singular points $D_3 = \{x \in \mathbb{R}^n : |x| = t, x_n < 0\}$, by (3.2.2), we have that $\Phi_j^n(\mu w_s + \lambda v_t, \cdot)|_{D_3} = \Phi_j^n(\lambda v_t, \cdot)|_{D_3} = \lambda^j \Phi_j^n(v_t, \cdot)|_{D_3}$ and since v_t is radially symmetric, this implies that $\Phi_j^n(v_t, \cdot)$ is rotation invariant on this set. Hence, it is equivalent to the Haar measure up to a constant $K_{n,j,t}$, and,

$$\int_{D_3} \gamma(x) \,\mathrm{d}\Phi_j^n(v_t, x) = K_{n,j,t} \int_{D_3} \gamma(x) \,\mathrm{d}\mathcal{H}^{n-1}(x)$$

for every $\gamma \in C_b(\mathbb{R}^n)$. It follows that

$$\begin{aligned}
\Phi_j^n(v_t, D_3) &= K_{n,j,t} \int_{D_3} \mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \frac{\omega_n}{2} C_{n,j,s} t^{n-1}
\end{aligned}$$

and by Lemma 3.2.7, we have

$$\Phi_j^n(v_t, D_3) = \frac{\kappa_n}{2}\binom{n}{j} t^{n-j}$$

and hence

$$K_{n,j,t} = \frac{1}{n}\binom{n}{j} t^{1-j}$$

which implies that

$$\begin{aligned}
\int_{D_3} \xi(|x|)x_n \,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) &= \frac{1}{n}\binom{n}{j} \lambda^j t^{1-j} \int_{D_3} \xi(|x|)x_n \,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \frac{1}{n}\binom{n}{j} \lambda^j t^{n-j+1}\xi(t) \int_{\mathbb{S}^{n-1}\cap\{x_n<0\}} x_n \,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&= -\frac{1}{n}\frac{\omega_{n-1}}{n-1}\binom{n}{j} \lambda^j t^{n-j+1}\xi(t).
\end{aligned}$$

(h) For the fourth set of singular points $D_4 = \{x \in \mathbb{R}^n : |x|_{n-1} = s, x_n < 0\}$, by Lemma 3.2.1,

$$\begin{aligned}
\partial(\mu w_s + \lambda v_t)(x) &= \partial(\mu w_s) + \partial(\lambda v_t) \\
&= \left\{\left(\alpha\frac{x_1}{|x|_{n-1}}, \ldots, \alpha\frac{x_{n-1}}{|x|_{n-1}}, 0\right) : \alpha \in [0, \mu]\right\} + \lambda\frac{x}{|x|}
\end{aligned}$$

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for every $x \in D_4$ and this implies that

$$
\begin{aligned}
&P_p(\mu w_s + \lambda v_t, D_4) = \\
&\left\{x + p\alpha \frac{(x - x_n e_n)}{|x|_{n-1}} + p\lambda \frac{x}{|x|} : |x|_{n-1} = s, x_n = -\sqrt{|x|^2 - s^2}, \alpha \in [0, \mu]\right\} = \\
&\left\{(r + p\lambda)(\tau e_n + \sqrt{1 - \tau^2} v) + p\alpha v : v \in \mathbb{S}^{n-1} \cap e_n^{\perp}, r > s, \tau = -\frac{1}{r}\sqrt{r^2 - s^2}, \alpha \in [0, \mu]\right\} \\
&= \left\{-\left(1 + p\frac{\lambda}{r}\right)\sqrt{r^2 - s^2} e_n + \left(s + p\left(\frac{\lambda s}{r} + \alpha\right)\right) v : v \in \mathbb{S}^{n-1} \cap e_n^{\perp}, r > s, \alpha \in [0, \mu]\right\}
\end{aligned}
$$

for all $p \geq 0$. Hence, the n-dimensional Hausdorff measure of D_4 is given by

$$
\begin{aligned}
&\omega_{n-1} \int_s^{+\infty} \int_0^{\mu} \left[p\lambda \frac{s^2}{r^2\sqrt{r^2 - s^2}} + \frac{r}{\sqrt{r^2 - s^2}}\right] p\left(s + p\left(\frac{\lambda s}{r} + \alpha\right)\right)^{n-2} \mathrm{d}\alpha \, \mathrm{d}r \\
&= \frac{\omega_{n-1}}{n-1} \int_s^{+\infty} \left[p\lambda \frac{s^2}{r^2\sqrt{r^2 - s^2}} + \frac{r}{\sqrt{r^2 - s^2}}\right] \times \\
&\left[\left(s + p\left(\frac{\lambda s}{r} + \mu\right)\right)^{n-1} - \left(s + p\frac{\lambda s}{r}\right)^{n-1}\right] \mathrm{d}r \\
&= \frac{\omega_{n-1}}{n-1} \sum_{j=1}^{n-1} p^j \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} s^{n-k-1} \int_s^{+\infty} \left[p\lambda \frac{s^2}{r^2\sqrt{r^2 - s^2}} + \frac{r}{\sqrt{r^2 - s^2}}\right] \frac{\lambda^{j-k}}{r^{j-k}} \mu^k \, \mathrm{d}r \\
&= \frac{\omega_{n-1}}{n-1} \sum_{j=1}^{n-1} p^{j+1} \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} s^{n-k+1} \lambda^{j-k+1} \mu^k \int_s^{+\infty} \frac{1}{r^{j-k+2}\sqrt{r^2 - s^2}} \mathrm{d}r \\
&\quad + \frac{\omega_{n-1}}{n-1} \sum_{j=1}^{n-1} p^j \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} s^{n-k-1} \lambda^{j-k} \mu^k \int_s^{+\infty} \frac{1}{r^{j-k-1}\sqrt{r^2 - s^2}} \mathrm{d}r.
\end{aligned}
$$

Thus, for $j = 1$ we have

$$
\begin{aligned}
\Phi_1^n(\mu w_s + \lambda v_t, D_4) &= \omega_{n-1} s^{n-2} \mu \int_s^{+\infty} \frac{r}{\sqrt{r^2 - s^2}} \mathrm{d}r \\
&= \omega_{n-1} s^{n-2} \mu \int_{-\infty}^{0} \mathrm{d}x_n
\end{aligned}
$$

and

$$
\begin{aligned}
&\Phi_j^n(\mu w_s + \lambda v_t, D_4) \\
&= \frac{\omega_{n-1}}{n-1} \left[\binom{n}{j} \frac{1}{n} \sum_{k=1}^{j-1} \binom{j}{k} (j-k) s^{n-k+1} \lambda^{j-k} \mu^k \int_s^{+\infty} \frac{1}{r^{j-k+1}\sqrt{r^2 - s^2}} \mathrm{d}r \right. \\
&\quad \left. + \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} s^{n-k-1} \lambda^{j-k} \mu^k \int_s^{+\infty} \frac{1}{r^{j-k-1}\sqrt{r^2 - s^2}} \mathrm{d}r\right] \\
&= \frac{\omega_{n-1}}{n-1} \left[\binom{n}{j} \frac{1}{n} \sum_{k=1}^{j-1} \binom{j}{k} (j-k) s^{n-k+1} \lambda^{j-k} \mu^k \int_{-\infty}^{0} (x_n^2 + s^2)^{\frac{k-j-2}{2}} \mathrm{d}x_n \right. \\
&\quad \left. + \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} s^{n-k-1} \lambda^{j-k} \mu^k \int_{-\infty}^{0} (x_n^2 + s^2)^{\frac{k-j}{2}} \mathrm{d}x_n\right]
\end{aligned}
$$

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for every $2 \leq j \leq n-1$, and for $j = n$

$$\Phi_n^n(\mu w_s + \lambda v_t, D_4) = \frac{\omega_{n-1}}{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} s^{n-k+1} \lambda^{n-k} \mu^k \int_s^{+\infty} \frac{1}{r^{n-k+1}\sqrt{r^2-s^2}}\, \mathrm{d}r$$
$$= \frac{\omega_{n-1}}{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} s^{n-k+1} \lambda^{n-k} \mu^k \int_{-\infty}^{0} (x_n^2 + s^2)^{\frac{k-j-2}{2}}\, \mathrm{d}x_n$$

On the other hand, since D_4 is a Borel set, following the same reasoning as in the cases of the sets B_5 and B_6, then by Theorem 3.A.2 and by (3.2.2), there exists a function $N_{n,j}$ depending on x_n such that

$$\int_{D_4} \varphi(x)\, \mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) =$$
$$\int_{\Sigma''''} \mathrm{d}\pi_{\#}(\Phi_j^n(\mu w_s + \lambda v_t, \cdot) \llcorner D_4)(x_n) \int_{s\mathbb{S}^{n-2}} N_{n,j}(x_n) \varphi(x', x_n)\, \mathrm{d}\mathcal{H}^{n-2}(x') \quad (3.3.17)$$

for every bounded Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}$, and since for every generating Borel set of $\mathbb{R}$ which is given by $(-\infty, c)$ for some $c \in \mathbb{R}$, we have

$$(\pi_{\#}(\Phi_j^n(\mu w_s + \lambda v_t, \cdot) \llcorner D_4))(Y) = \Phi_j^n(\mu w_s + \lambda v_t, D_4 \cap \pi^{-1}(Y)),$$

then by computing the parallel set of the function $\mu w_s + \lambda v_t$ at $D_4 \cap \pi^{-1}(Y)$, we retrieve the following

$$\Phi_1^n(\mu w_s + \lambda v_t, D_4 \cap \pi^{-1}(Y)) = \omega_{n-1} s^{n-2} \mu \int_s^{+\infty} \mathbb{1}_Y\left(-\sqrt{r^2-s^2}\right) \frac{r}{\sqrt{r^2-s^2}}\, \mathrm{d}r$$
$$= \omega_{n-1} s^{n-2} \mu \int_{-\infty}^{0} \mathbb{1}_Y(x_n)\, \mathrm{d}x_n$$

for $j = 1$, and

$$\Phi_j^n(\mu w_s + \lambda v_t, D_4 \cap \pi^{-1}(Y)) = \frac{\omega_{n-1}}{n-1} \times$$
$$\left[\binom{n}{j} \frac{1}{n} \sum_{k=1}^{j-1} \binom{j}{k} (j-k) s^{n-k+1} \lambda^{j-k} \mu^k \int_{-\infty}^{0} \mathbb{1}_Y(x_n)(x_n^2 + s^2)^{\frac{k-j-2}{2}}\, \mathrm{d}x_n + \right.$$
$$\left. \binom{n-1}{j} \sum_{k=1}^{j} \binom{j}{k} s^{n-k-1} \lambda^{j-k} \mu^k \int_{-\infty}^{0} \mathbb{1}_Y(x_n)(x_n^2 + s^2)^{\frac{k-j}{2}}\, \mathrm{d}x_n \right]$$

for every $2 \leq j \leq n-1$ and

$$\Phi_n^n(\mu w_s + \lambda v_t, D_4 \cap \pi^{-1}(Y)) =$$
$$\frac{\omega_{n-1}}{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} s^{n-k+1} \lambda^{n-k} \mu^k \int_{-\infty}^{0} \mathbb{1}_Y(x_n)(x_n^2 + s^2)^{\frac{k-n-2}{2}}\, \mathrm{d}x_n$$

when $j = n$. This implies that the measure $\pi_{\#}(\Phi_j^n(\mu w_s + \lambda v_t, \cdot) \llcorner D_4)$ is absolutely continuous with respect to the Lebesgue measure. After substituting for $\varphi = 1$ in (3.3.17) and comparing with the total measure of $\Phi_j^n(\mu w_s + \lambda v_t, \cdot) \llcorner D_4$, we get that

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$N_{n,j}(x_n) = \frac{s^{2-n}}{\omega_{n-1}}$ for every $x_n \in \mathbb{R}$. Thus, for $j = 1$

$$\begin{aligned}
&\int_{D_4} \xi(|x|)x_n \,\mathrm{d}\Phi_1^n(\mu w_s + \lambda v_t, x) \\
&\qquad = \omega_{n-1}s^{n-2}\mu \int_{-\infty}^{0} \int_{s\mathbb{S}^{n-1}} \xi(|x|)N_{n,1}(x_n)x_n \,\mathrm{d}\mathcal{H}^{n-2}(x') \,\mathrm{d}\mathcal{H}^1(x_n) \\
&\qquad = \omega_{n-1}s^{n-2}\mu \int_{-\infty}^{0} \xi\left(\sqrt{s^2 + x_n^2}\right) x_n \,\mathrm{d}\mathcal{H}^1(x_n) \\
&\qquad = -\omega_{n-1}s^{n-2}\mu \int_{s}^{+\infty} r\xi(r) \,\mathrm{d}r,
\end{aligned}$$

and for every $1 \le j \le n-1$

$$\begin{aligned}
&\int_{D_4} \xi(|x|)x_n \,\mathrm{d}\Phi_j^n(\mu w_s + \lambda v_t, x) \\
&\qquad = -\frac{\omega_{n-1}}{n-1}\left[\binom{n-1}{j}\sum_{k=1}^{j}\binom{j}{k}\lambda^{j-k}\mu^k s^{n-k-1}\int_s^{+\infty} r^{-j+k+1}\xi(r)\,\mathrm{d}r\right. \\
&\qquad \left. + \binom{n}{j}\frac{1}{n}\sum_{k=1}^{j-1}\binom{j}{k}(j-k)\lambda^{j-k}\mu^k s^{n-k+1}\int_s^{+\infty} r^{-j+k-1}\xi(r)\,\mathrm{d}r\right]
\end{aligned}$$

and for $j = n$, we have

$$\begin{aligned}
&\int_{D_4} \xi(|x|)x_n \,\mathrm{d}\Phi_n^n(\mu w_s + \lambda v_t, x) \\
&\qquad = -\frac{\omega_{n-1}}{n-1}\sum_{k=1}^{n-1}\binom{n-1}{k}\lambda^{j-k}\mu^k s^{n-k+1}\int_s^{+\infty} r^{k-n}\xi(r)\,\mathrm{d}r \\
&\qquad = -\frac{\omega_{n-1}}{n(n-1)}\sum_{k=1}^{n-1}\binom{n}{k}(n-k)\lambda^{j-k}\mu^k s^{n-k+1}\int_s^{+\infty} r^{k-n}\xi(r)\,\mathrm{d}r.
\end{aligned}$$

Since the sets $C_1, \dots, C_4$ and $D_1, \dots, D_4$ are disjoint and their union is $\mathbb{R}^n$, then for every $s, t \ge 0$ such that $t < s$ we have

$$\int_{\mathbb{R}^n} \xi(|x|)x_n\Phi_1^n(\mu w_s + \lambda v_t, x) = \frac{\omega_{n-1}}{n-1}\mu s^n \xi(s)$$

when $j = 1$ and for every $2 \le j \le n$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \xi(|x|)x_n\Phi_j^n(\mu w_s + \lambda v_t, x) \\
&\qquad = \frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j}\left[\mu^j s^{n-j+1}\xi(s) + \sum_{k=1}^{j-1}\binom{j}{k}\lambda^{j-k}\mu^k s^{n-k+1}\times\right. \\
&\qquad \left.\left(s^{k-j}\xi(s) - (j-k)\int_s^{\infty} r^{k-j-1}\xi(r)\,\mathrm{d}r\right)\right] \\
&\qquad = \frac{\omega_{n-1}}{n-1}\frac{1}{n}\binom{n}{j}\left[\mu^j s^{n-j+1}\xi(s) + \sum_{k=1}^{j-1}\binom{j}{k}\lambda^{j-k}\mu^k s^{n-k+1}\,\mathcal{R}^{k-j}\,\xi(s)\right]
\end{aligned}$$

where the last equality holds by the definition of the $\mathcal{R}$ transform.

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From the two cases above, and since $\alpha = \binom{n}{j} \mathcal{R}^{n-j} \xi \in T_n^n$, then by Lemma 3.2.13 and (3.1.3), we have that

$$z_{j,\xi}(\mu w_s, \lambda v_t) = \kappa_{n-1} \frac{1}{n} \left[\mu^j s^{n-j+1} \mathcal{R}^{-n+j} \alpha(s) + \sum_{k=1}^{j-1} \binom{j}{k} \lambda^{j-k} \mu^k s^{n-k+1} \mathcal{R}^{-n+k} \alpha\left(\max\{s,t\}\right) \right]$$

for every $s, t > 0$ and every $\mu, \lambda > 0$. □

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