Flexible rod partially sliding in an oscillating sleeve: ejection against gravity

Yury Vetyukov

TU Wien, Vienna, Austria

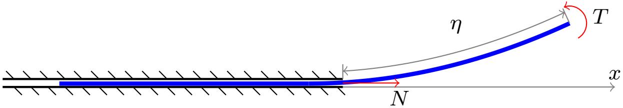
yury.vetyukov@tuwien.ac.at

INSTITUT FÜR MECHANIK UND MECHATRONIK
Mechanics & Mechatronics

TECHNISCHE UNIVERSITÄT WIEN
Vienna | Austria

Motivation
Configurational forces in beams

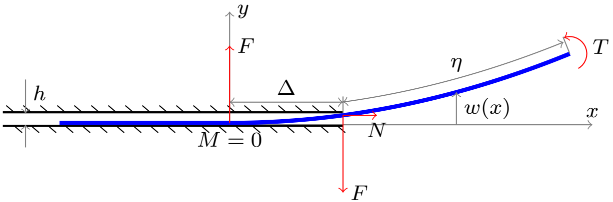
- Beam in a rigid sleeve with a moment at the tip: energy release rate



- N is a generalized force work conjugate to the length η:

$$\mathcal{E}_{\text{total}} = \mathcal{E}_{\text{elastic}} + \mathcal{E}_{\text{external}} = -\frac{T^2}{2a}\eta \quad \Rightarrow \quad N = \frac{T^2}{2a} \tag{1}$$

- Contact problem: finite sleeve thickness h, we integrate beam equation



- Axial contact force: $N = Fw'(\Delta) = T^2/(2a)$

Motivation
Dancing rod problem

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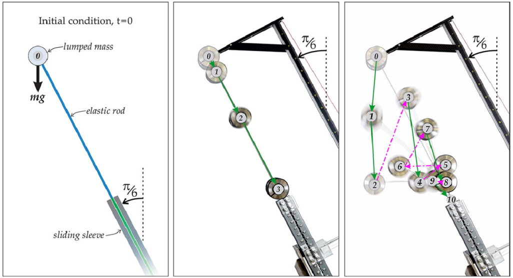
Configurational forces and nonlinear structural dynamics

C. Armanini, F. Dal Corso, D. Misseroni, D. Bigoni

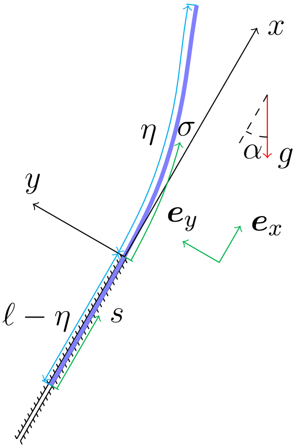


- Flexible rod, low friction, concentrated mass
- Fast bending oscillations with variable frequency
- Slow axial motion
- Configurational force prevents full injection
- Would it be possible to predict the maximum injection depth over one period?

Mathematical model

Flexible rod partially sliding in a fixed rigid sleeve

- **Previous work:** Vetyukov Yury, Alexander Humer, and Alois Steindl. "Nonlinear dynamics of a flexible rod partially sliding in a rigid sleeve under the action of gravity and configurational force." Journal of the Mechanics and Physics of Solids 193 (2024): 105854.
- Main features of the mathematical model
  - Inextensible unshearable rod, distributed mass
  - No friction $\Rightarrow$ conservative system
  - Material coordinate s, length ℓ
  - Lagrangian (material) kinematic description $\boldsymbol{x} = \boldsymbol{x}(s, t)$ is inefficient
  - Non-material mixed Eulerian-Lagrangian model
  - Length of the free part $\eta(t)$ is config. parameter
- Non-material f.e. model with mesh coordinate σ
- 2 d.o.f. model with one-mode approximation of $y(\sigma, t)$ accurate for small transverse vibrations
- Hamiltonian equations and adiabatic invariants allow for analytical solutions in closed form



Mathematical model

2 d.o.f. model with rotating sleeve: Kinematic description

- Rotation angle $\alpha(t)$ is given function of time
- Parametrization of the free part:

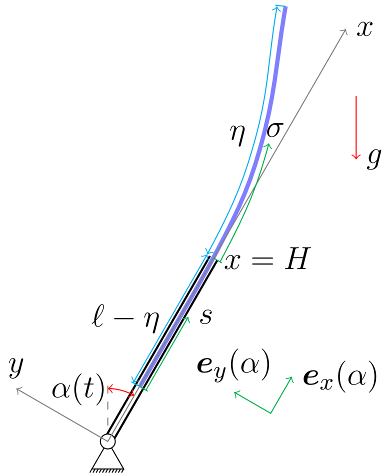
$$\boldsymbol{x} = x\boldsymbol{e}_x + y\boldsymbol{e}_y = \boldsymbol{x}(\sigma, t), \quad 0 \le \sigma \le 1 \quad (2)$$

- Mapping material coord. ⇔ mesh coord.:

$$s = l - \eta + \eta\sigma, \qquad \sigma = 1 - (l - s)/\eta \quad (3)$$

- Strain energy requires spatial derivatives:

$$\partial_s \boldsymbol{x} = \partial_\sigma \boldsymbol{x}\, \partial_s \sigma = \frac{1}{\eta}\partial_\sigma \boldsymbol{x} \quad (4)$$



Mathematical model

2 d.o.f. model with rotating sleeve: Kinematic description

- Material velocity of a particle:

$$\dot{\boldsymbol{x}} = \partial_t \boldsymbol{x}|_{s=\text{const}} = \partial_t \boldsymbol{x}|_{\sigma=\text{const}} + \dot{\sigma}\, \partial_\sigma \boldsymbol{x}$$

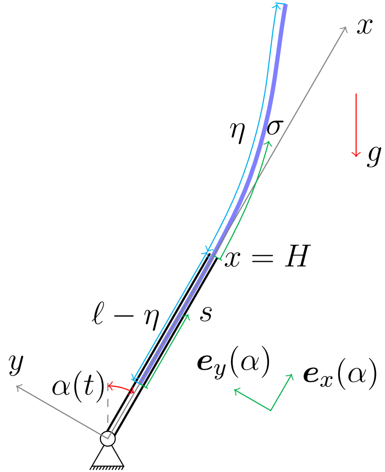
$$\dot{s} = 0 \quad \Rightarrow \quad \dot{\sigma} = \frac{(1-\sigma)\dot{\eta}}{\eta} \tag{5}$$

- Rotating frame: $\boldsymbol{e}_x(\alpha(t))$, $\boldsymbol{e}_y(\alpha(t))$
- Approximation with two generalized coords.:

$$\boldsymbol{x}(\sigma, t) = (H + \eta(t)\sigma)\boldsymbol{e}_x + \gamma(t) w(\sigma) \boldsymbol{e}_y \tag{6}$$

- Assumed small bending, first mode
- Velocity of a particle in the free segment:

$$\dot{\boldsymbol{x}} = \dot{\eta}\boldsymbol{e}_x + (\dot{\gamma} w + \dot{\sigma}\gamma\, \partial_\sigma w)\, \boldsymbol{e}_y + \dot{\alpha}\boldsymbol{e}_z \times \boldsymbol{x} \tag{7}$$



Mathematical model

2 d.o.f. model with rotating sleeve: Energies

- Strain energy of bending:

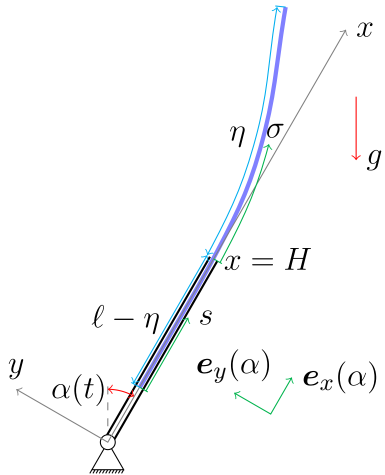
$$\mathcal{E}_{\text{elastic}} = \frac{1}{2}\int_0^1 EJ\kappa^2\eta \,\mathrm{d}\sigma = \frac{8a\gamma^2}{5\eta^3} \tag{8}$$

- Kinetic energy: free + in the sleeve

$$\begin{aligned}\mathcal{E}_{\text{kinetic}} &= \frac{1}{2}\int_0^1 \rho\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}\eta\,\mathrm{d}\sigma + \\ &+\frac{1}{2}\int_{H-(\ell-\eta)}^{H} \rho(\dot{\eta}^2 + (\dot{\alpha}x)^2)\,\mathrm{d}x\end{aligned} \tag{9}$$

- Potential in the field of gravity:

$$\mathcal{E}_{\text{external}} = \mathcal{E}_{\text{free}} + \mathcal{E}_{\text{sleeve}} \tag{10}$$



Mathematical model

2 d.o.f. model with rotating sleeve: Equations of motion

- Lagrange's equations of motion of 2nd kind for $q_1 = \eta$, $q_2 = \gamma$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_{\text{kinetic}}}{\partial \dot{q}_i}\right) - \frac{\partial E_{\text{kinetic}}}{\partial q_i} = -\frac{\partial(\mathcal{E}_{\text{elastic}} + \mathcal{E}_{\text{external}})}{\partial q_i} \tag{11}$$

- Eqs. of motion after processing with Wolfram Mathematica:

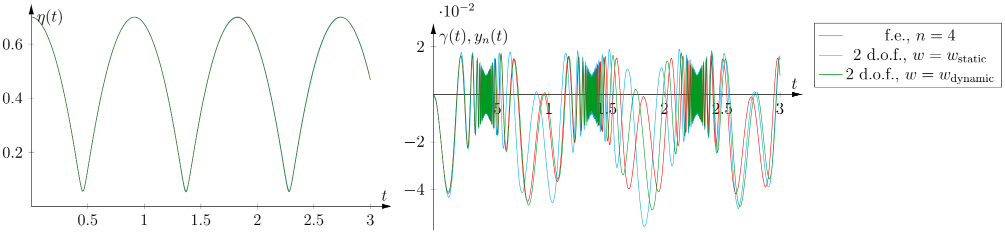
$$\begin{aligned}
&2\,\rho\,\gamma[t]\,\eta[t]^3\,\Big(80\,\gamma'[t]\,\eta'[t] + \\
&\quad \eta[t]\,\big(81\,g\,\mathrm{Sin}[\alpha[t]] + 9\,(9\,H - 5\,\eta[t])\,\big(-\mathrm{Sin}[\alpha[t]]\,\alpha'[t]^2 + \mathrm{Cos}[\alpha[t]]\,\alpha''[t]\big) + 26\,\gamma''[t]\big)\Big) + \\
&\tfrac{81}{2}\,\rho\,\eta[t]^4\,\Big(\mathrm{Cos}[\alpha[t]]\,\big(10\,g\,(-H + l) + \eta[t]\,\alpha'[t]\,(5\,(2\,H + \eta[t])\,\alpha'[t] - 8\,\gamma'[t])\big) + \\
&\quad 5\,\mathrm{Sin}[\alpha[t]]\,\eta[t]\,(2\,H + \eta[t])\,\alpha''[t] + 5\,l\,\big((-2\,H + l - 2\,\eta[t])\,\alpha'[t]^2 + 2\,\eta''[t]\big)\Big) + \gamma[t]^2\,\Big(-1944\,a + \\
&\quad 4\,\rho\,\eta[t]^2\,\big(-10\,\eta'[t]^2 + 13\,\eta[t]^2\,\big((-1 + \mathrm{Cos}[\alpha[t]])\,\alpha'[t]^2 + \mathrm{Sin}[\alpha[t]]\,\alpha''[t]\big) + 20\,\eta[t]\,\eta''[t]\big)\Big) == 0
\end{aligned}$$

$$\begin{aligned}
&\rho\,\eta[t]^3\,\Big(-9\,\mathrm{Sin}[\alpha[t]]\,\eta[t]\,\big(-18\,g + (18\,H + 13\,\eta[t])\,\alpha'[t]^2\big) + 9\,\mathrm{Cos}[\alpha[t]]\,\eta[t] \\
&\quad \big(36\,\alpha'[t]\,\eta'[t] + (18\,H + 13\,\eta[t])\,\alpha''[t]\big) + 104\,\big(\gamma'[t]\,\eta'[t] + \eta[t]\,\gamma''[t]\big)\Big) + 4\,\gamma[t]\,\Big(324\,a + \\
&\quad \rho\,\eta[t]^2\,\big(-20\,\eta'[t]^2 + 13\,\eta[t]\,\big(2\,\eta[t]\,\big((-1 + \mathrm{Cos}[\alpha[t]])\,\alpha'[t]^2 + \mathrm{Sin}[\alpha[t]]\,\alpha''[t]\big) + \eta''[t]\big)\big)\Big) == 0
\end{aligned}$$

Mathematical model

2 d.o.f. model with rotating sleeve: Time integration

- Time integration for $\alpha = \pi/4 = \mathrm{const}$, slenderness ratio $h/\ell = 2 \cdot 10^{-3}$



- Discrepancies because of higher vibration modes
- Amplitude and frequency changes are accurate
- $\eta(t)$ is a slow variable, $\gamma(t)$ is a fast one
- Analytical estimates for amplitude, frequency and maximum injection depth are available

Vibrating sleeve
Keeping height stationary: Analytics

- Aim: periodic excitation near vertical stabilizes vertical position

$$\alpha = \alpha_0 \sin \omega t \quad \Rightarrow \quad \eta \approx \eta_0 = \mathrm{const} \quad \text{but} \quad \alpha_0 = ? \quad \omega = ? \tag{12}$$

- Assuming the oscillations small, we substitute in the equations of motion

$$\alpha = \lambda \alpha_0 \sin \omega t, \quad \gamma = \lambda \gamma_0 \sin \omega t, \quad \eta = \eta_0. \tag{13}$$

We keep terms up to λ^2, average over a period and find algebraic relations for α_0, γ_0, η_0 and ω:

$$-972\, a\, \gamma0^2 + \frac{1}{4} \eta0^4 \rho \left(81\, g \left(5\, (H - l) \left(-4 + \alpha0^2\right) + 4\, \alpha0\, \gamma0\right) - \left(104\, \gamma0^2 + 162\, H\, \alpha0\, (5\, l\, \alpha0 + 2\, \gamma0) + 9\, \alpha0 \left(-45\, l^2\, \alpha0 + 90\, l\, \alpha0\, \eta0 + 52\, \gamma0\, \eta0\right)\right) \omega^2\right) == 0$$

$$1296\, a\, \gamma0 - \eta0^4 \rho \left(-162\, g\, \alpha0 + (162\, H\, \alpha0 + 104\, \gamma0 + 117\, \alpha0\, \eta0)\, \omega^2\right) == 0$$

- Solving for γ_0, we end up with an algebraic equation for η_0
- Just real roots with $0 < \eta_0 < 1$ are the physically meaningful

Vibrating sleeve

Keeping height stationary: Analytics

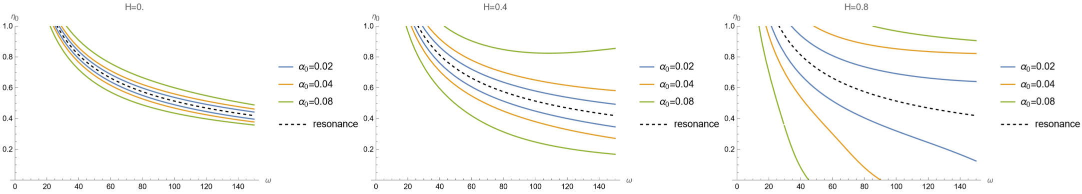
- Parameter set (SI units):

$$\ell = 1,\ h = 5 \cdot 10^{-3},\ E = 2.1 \cdot 10^{11},\ \rho_3 = 7800,\ g = 9.8,\ H = 0, 0.4, 0.8 \tag{14}$$

- Frequency of bending vibrations when $\alpha = 0$ and $\eta = \eta_0$:

$$13\eta_0^4 \rho \ddot{\gamma} + 162 a \gamma = 0 \quad \Rightarrow \quad \gamma = \gamma_0 \sin \omega_\gamma t, \quad \omega_\gamma = 26.4379/\eta_0^2 \tag{15}$$

- We solve the equation for η_0 varying ω, α_0 and H:

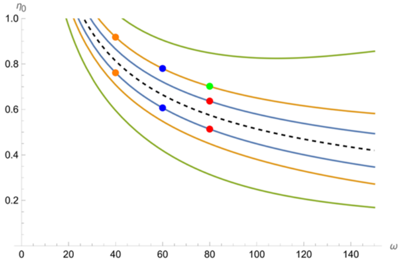


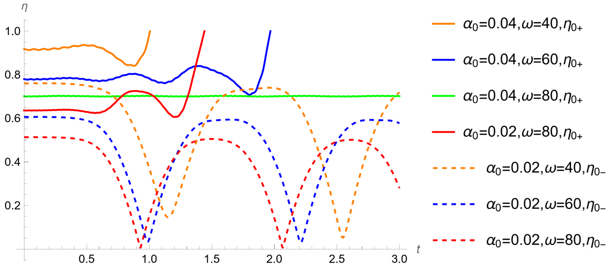
- Two solutions η_{0+} and η_{0-} exist at small α_0 close to resonance $\omega = \omega_\gamma$

Vibrating sleeve

Keeping height stationary: Numerical experiments

- We integrate over time for $H = 0.4$
- Analytical stationary solution with η_0 and γ_0 is used as initial condition
- Behavior depends on excitation amplitude, frequency and starting point





- "Stationary height" regime with $\eta_0 = \eta_{0+}$ may become stable for sufficiently high excitation frequencies and amplitudes

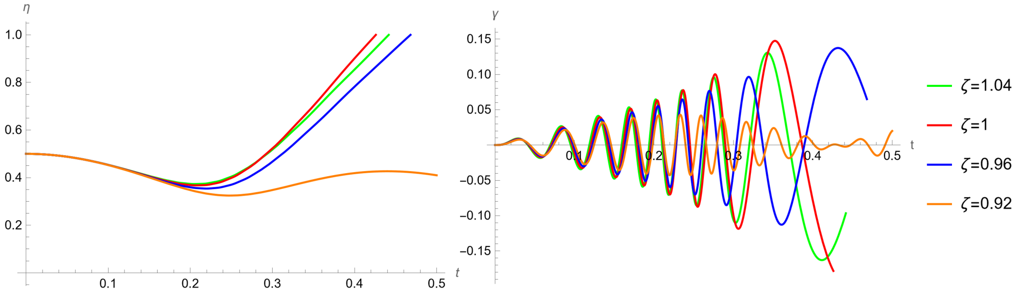
Vibrating sleeve

Quick ejection: Varying excitation frequency

- Excitation frequency is adjusted in accordance with $\eta(t)$:

$$\alpha(t) = \alpha_0 \sin \zeta\tau, \quad \dot{\tau} = \omega_\gamma(\eta) \tag{16}$$

  Here τ is the phase variable and ζ is detuning parameter
- We choose $H = 0$, $\alpha = 0.01$ and simulate for varying ζ:



- The fastest possible ejection is reached at continuous resonance $\zeta = 1$