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Variational criteria for stiffness maxima and minima of proportionally loaded structures as solutions of an inverse problem

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ABSTRACT

Stiffness is a key term of structural mechanics. The same applies to the mechanical properties *stiffening* and *softening* of structures subjected to proportional loading. In the course of the loading process, originally stiffening (softening) structures may become softening (stiffening) structures. This occurs at the unknown load level at which the stiffness of the structure concerned attains a maximum (minimum) value. Extreme values of the stiffness of proportionally loaded structures are turning points of their mechanical behavior. Therefore, it is astonishing that analytical criteria for stiffness maxima and minima of such structures do not exist. In recent publications it has been claimed that points of inflection of eigenvalue functions of a special linear eigenvalue problem in the framework of the Finite Element Method (FEM) mark extreme values of the stiffness of proportionally loaded structures. The task of this work is to present the scientific foundation of this assertion in the form of criteria for stiffness maxima and minima based on variational calculus, termed variational criteria. This amounts to the solution of an inverse problem, with the mentioned numerical results as the observed effect and the sought variational criteria as the unknown cause. In general, analytical solutions for extreme values of the stiffness of proportionally loaded structures are inaccessible. Nevertheless, knowledge of their scientific basis is not only a fundamental scientific value in its own right, but also enhances the understanding of the intricacies of FE analysis for numerical determination of the load level at stiffness maxima and minima. The main motive for this work is to fill a significant scientific void of the literature in the area of the mechanics of engineering structures.

1. Introduction

Stiffness is an important property of engineering structures. A well-known category of the term *stiffness* are the cross-sectional stiffnesses of beams in the form of their axial, bending, shear, and torsional stiffness. Linguistically related to this term are the adjectives *stiffening* and *softening* in the context of proportionally loaded structures. They are lacking a generally accepted mathematical definition. An exception are elastic springs, representing single degree-of-freedom (d-o-f) systems. *E.g.*, the stiffness of a tension spring is defined as $k(u) = dP/du$, where u denotes the elongation due to a proportionally increasing tensile force P. A positive sign of $dk/du = d^2P/du^2$ refers to a stiffening spring and a negative sign to a softening spring.

For proportionally loaded structures the situation is more complicated. An originally stiffening (softening) structure may *e.g.* become a softening (stiffening) structure. This occurs at the unknown load level at which the stiffness of the structure attains a maximum (minimum) value. For an elastic spring this would mean that $dk/du = d^2P/du^2 = 0$. The individual d-o-f of multi d-o-f structures, however, do not behave alike. In particular, points of inflection on different load–displacement diagrams occur at different load levels, noting that there are also d-o-f without such points. This may explain why analytical criteria for extreme values of the stiffness of proportionally loaded structures do not exist. Nevertheless, lack of such criteria is astonishing, because the transition from a stiffening (softening) to a softening (stiffening) structure at its maximum (minimum) stiffness represents, in a figurative sense, a turning point of the mechanical behavior of the structure concerned.

With this in mind, recent assertions that points of inflection of fundamental eigenvalue functions of a special linear eigenvalue problem, generated by means of the Finite Element Method (FEM), mark extreme values of the stiffness of proportionally loaded structures [1] are worthy of attention. More precisely, it has been claimed that the condition

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for the transition from a softening to a stiffening structure, *i.e.* for a minimum value of the stiffness, is the one of a point of inflection of the real part of a partially complex fundamental eigenvalue function of this eigenvalue problem [2]. While the existence of such a transition was known beforehand, the necessity of a linear eigenvalue problem with the ability to produce complex eigenvalues in the prebuckling domain in order to determine the load level of this transition was nevertheless initially surprising. The following quest for a condition for the transition from a stiffening to a softening structure resulted in the assertion that the sought condition is the one of a point of inflection of the real fundamental eigenvalue function of the same eigenvalue problem [1]. Fig. 1 shows what is meant by *transition from a stiffening to a softening structure*. The illustration contains a $\chi_1 - \lambda$ diagram, where $\chi_1(\lambda)$ is the fundamental eigenvalue function of the aforementioned special linear eigenvalue problem, with λ as a dimensionless load parameter [2]. It refers to the prebuckling domain of a structure with loss of stability at point S, where $\chi_1 = 0$. The analytical background of this condition is the positive semidefiniteness of the functional $\delta^2 \Pi$. It represents the second variation of the potential energy Π of a system with infinitely many d-o-f, subjected to proportionally increasing, conservative forces. $\delta^2 \Pi$ is based on arbitrary variations of the state variables in the expression for Π at the considered load level, see *e.g.* [3]. It has been claimed in [1] that $d^2\chi_1/d\lambda^2 = 0$, see point I in Fig. 1, is the condition for the aforementioned transition. It is formally analogous to the previously mentioned condition for an extreme value of the stiffness of an elastic spring. This prompts the question for the analytical background of this assertion.

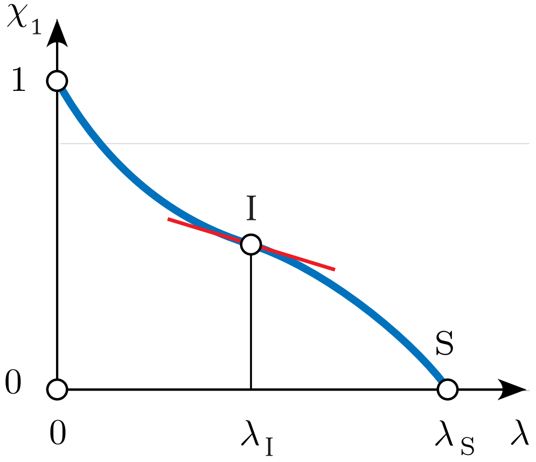


Fig. 1. $\chi_1 - \lambda$ diagram in the prebuckling domain of a proportionally loaded structure with loss of stability at point S and with the **asserted** transition from a stiffening to a softening structure at point I.

As mentioned previously, a transition from a softening to a stiffening structure was found to occur in a subregion of the prebuckling domain, where χ_1 is a complex eigenvalue [2]. Irrespective of their plausibility, the numerical conditions for the mentioned transitions, *i.e.* for extreme values of the stiffness of proportionally loaded structures, are still lacking a scientific basis.

The task of this work is to present this scientific foundation in the form of criteria based on variational calculus, termed variational criteria. This is done by drawing conclusions from the mentioned results from FE analysis, *i.e.* from the investigation of systems with a finite number of d-o-f. The availability of these numerical results has been the motivation for the search of their analytical basis, which is the topic of this work. The recourse to these results is imperative for the solution of the given inverse problem. *In the sense of such a problem these numerical results are the observed effect, whereas the sought variational criteria are the unknown cause.* It is practically impossible to obtain analytical solutions for the load level of extreme values of the stiffness of proportionally loaded structures. Nevertheless, knowledge of the basis for such inaccessible solutions is not only a fundamental scientific value in its own right, but also enhances the understanding of the intricacies of FE analysis to determine numerical approximations of the load level of stiffness maxima and minima.

Inverse analysis is an important element of mathematical quantifications in the sciences, not least in structural mechanics. More recent examples from the large body of literature in this scientific area are papers on inverse analysis in fracture mechanics [4], for pressure load identification [5], and for damage detection in beam and truss structures [6].

To the best knowledge of the authors, literature by other researchers or research groups on criteria for extreme values of the stiffness of proportionally loaded structures does not seem to exist. This explains the one-sidedness and sparseness of the literature cited in this work. Since the present paper is restricted to the prebuckling region, see point I in Fig. 1, as well as to structures that are not buckling at all, literature on buckling and postbuckling, both analytical and numerical, is not relevant to this work. Closing a significant scientific void of the literature in the area of the mechanics of engineering structures is the main motive for this publication.

The organization of the paper orientates itself by its character as the solution of an inverse problem. Section 2 deals with the determination of variational criteria for extreme values of the stiffness of proportionally loaded structures. The mechanical basis for these criteria is a hypothetical extension of the well-known functional appearing in the definition of stable equilibrium. Section 2.5 contains a summary of features of the extended functional, termed *Intermediate résumé*. The differences between the criteria for stiffness maxima and the ones for stiffness minima are highlighted. Section 3 begins with an investigation of correlations of particular properties of the previously mentioned linear eigenvalue problem, in the framework of the FEM, with key features of the said functional. The existence of such correlations serves as the basis for the subsequent verification of the previously mentioned claims that points of inflection on eigenvalue curves, obtained from this eigenvalue problem, signal extreme values of the stiffness of proportionally loaded structures. Section 3 ends with remarks about proportionally loaded structures without an extreme value of the stiffness in the prebuckling region. They represent counterparts of structures with such extreme values. Section 4 contains the conclusions drawn from the present investigation.

2. Determination of variational criteria for extreme values of the stiffness of proportionally loaded structures

2.1. Preliminaries

The following paragraph of this section is state-of-the-art. Its purpose is to provide the basis for the transition to new knowledge.

The considered structures represent systems with infinitely many d-o-f. They are assumed to be loaded by proportionally increasing, conservative forces. The change of the potential energy Π due to admissible variations of the state variables at a certain load level can formally be written as, see *e.g.* [7],

$$\Delta\Pi = \delta\Pi + \frac{1}{2!}\delta^2\Pi + \frac{1}{3!}\delta^3\Pi + \frac{1}{4!}\delta^4\Pi + \cdots, \tag{1}$$

where $\delta\Pi$ ($(1/2!)\delta^2\Pi$, $(1/3!)\delta^3\Pi$, $(1/4!)\delta^4\Pi$...) denotes the term in the expression for $\Delta\Pi$ that is linear (quadratic, cubic, quartic...) in these variations. Dirichlet's criteria for stable equilibrium, to which this work is restricted, are given as, see *e.g.* [3,7],

$$\delta\Pi = 0,\ \delta^2\Pi > 0,\ \text{for } 0 < \lambda < \lambda_S, \tag{2}$$

where λ_S denotes the value of λ at incipient buckling, noting, however, that a stability limit need not exist. Hence, for $0 < \lambda < \lambda_S$, $\delta^2\Pi$ is a positive-definite functional. For $\lambda = \lambda_S$,

$$\delta^2\Pi \geq 0, \tag{3}$$

indicating that there is at least one special variation of the state variables, for which

$$\delta^2 \Pi = 0. \tag{4}$$

Thus, for $\lambda = \lambda_S$, $\delta^2 \Pi$ is a positive-semidefinite functional. This leads to Trefftz's criterion for incipient buckling, given as, see *e.g.* [7],

$$\delta(\delta^2 \Pi) = 0, \tag{5}$$

herein referred to as a variational criterion. It is a necessary condition for $\bar{\delta}^2 \Pi = 0$ as the minimum value of $\delta^2 \Pi$ at the stability limit. ($\bar{\delta}^2 \Pi$ denotes the second variation of Π due to those special variations of the state variables that have led to (4).)

As mentioned previously, this work deals with extreme values of the stiffness of proportionally loaded structures that occur at stable equilibrium. This prompts the question for the relevance of Trefftz's criterion to the present publication. It will be shown in Sections 2.3 and 2.4 that the mathematical form of this criterion, which is a condition for the minimum value of a positive-semidefinite functional, *i.e.* for zero, may be viewed as the basis for the considerably more complicated variational criteria for extreme values of the stiffness of conservative systems with infinitely many d-o-f. The FEM-based counterpart of Trefftz's criterion is the positive semidefiniteness and singularity of the tangent stiffness matrix at incipient buckling. *Conversely, there must be variational criteria for extreme values of the stiffness of proportionally loaded structures with infinitely many d-o-f that are the scientific basis of FEM-related conditions for stiffness maxima and minima.* Otherwise the previously mentioned points of inflection on special FEM-based eigenvalue curves would be lacking physical significance.

2.2. Requirements for determination of variational criteria for stiffness maxima and minima of proportionally loaded structures

The basic requirement for determination of such criteria is the existence of a functional, with remarkable physical properties, as part of the expression for $\Delta \Pi$, see (1). For at least one variation of the state variables it must be a minimum (maximum) value, equal to zero, at the load level of maximum (minimum) stiffness. Recalling that for stable equilibrium of structures, to which this work is restricted, $\delta^2 \Pi$ is a positive-definite functional, see (2), it needs a supplementary functional that is negative in the interval $0 < \lambda < \lambda_S$ to obtain, in sum with $(1/2!)\delta^2 \Pi > 0$, a functional that is able to satisfy the aforementioned conditions for stiffness maxima and minima. *Nota bene*, this does not mean that this supplementary functional cannot be positive or zero, keeping in mind that the described situation is irrelevant to problems without extreme values of the stiffness in the prebuckling region. The supplementary functional is hypothetically assumed as $(1/3!)\delta^3 \Pi$. Basically, $\delta^3 \Pi$ is an indefinite functional. However, its sign in the prebuckling region is assumed to have a physical background. There are indications that it depends on the character of the stiffness change of proportionally loaded structures. For a *superlinear* stiffness change, $\delta^3 \Pi > 0$, whereas for a *sublinear* stiffness change, $\delta^3 \Pi < 0$. Hence, for each one of these two kinds of nonlinear stiffness changes in the prebuckling region, $\delta^3 \Pi$ is a definite functional. Thus, $\delta^3 \Pi = 0$ is the condition for a *linear* stiffness change. Consistent with this situation is the ability of the linear eigenvalue problem, in the framework of the FEM, used in [1,2], to distinguish between a *superlinear*, *sublinear*, and *linear* stiffness change in the prebuckling region. In the sense of inverse analysis, plausible numerical results for the load levels of stiffness maxima and minima, reported in these two papers, represent the observed effect, whereas the postulated constraint on the sign of $\delta^3 \Pi$ for problems with extreme values of the stiffness is part of the unknown cause. The hypothetical assumption for the sign of $\delta^3 \Pi$ will be considered as verified if characteristic features of the variational criteria for extreme values of the stiffness of proportionally loaded structures, to be derived in this work, correlate with fundamental properties of the linear eigenvalue problem, in the framework of the FEM, which is the basis for the derivation of the numerical conditions for such extreme values. A further rationale behind the assumption of $\delta^3 \Pi < 0$, for $0 < \lambda < \lambda_S$, for structures with an extreme value of the stiffness, is the possibility of a change of the sign of $(1/2!)\delta^2 \Pi + (1/3!)\delta^3 \Pi$ for all variations of the state variables. At this change, (1) disintegrates into

$$\delta \Pi = 0, \quad \frac{1}{2!}\delta^2 \Pi + \frac{1}{3!}\delta^3 \Pi = 0 \tag{6}$$

and

$$\Delta \Pi = \frac{1}{4!}\delta^4 \Pi + \cdots. \tag{7}$$

This occurs *e.g.* at the lower and the upper bound of a subregion of $0 < \lambda < \lambda_S$, inside which the functional $(1/2!)\delta^2 \Pi + (1/3!)\delta^3 \Pi$ is non-positive. The mechanical reason for this subregion is the existence of a minimum value of the stiffness inside of it, as will be shown in Section 2.4. As follows from (2) and (6),

$$\frac{1}{3!}\delta^3 \Pi = -\frac{1}{2!}\delta^2 \Pi < 0. \tag{8}$$

At the onset of loading, $\Pi = 0$. Hence, for $\lambda = 0$, $\Delta \Pi = 0$, implying

$$\delta^2 \Pi = 0, \ \delta^3 \Pi = 0 \quad \Rightarrow \quad \frac{1}{2!}\delta^2 \Pi + \frac{1}{3!}\delta^3 \Pi = 0. \tag{9}$$

At the stability limit, the mathematical character of the functional $(1/2!)\delta^2 \Pi + (1/3!)\delta^3 \Pi$ must be the same as the one of the functional $\delta^2 \Pi$. Hence, for $\lambda = \lambda_S$,

$$\delta^2 \Pi \geq 0 \quad \Rightarrow \quad \frac{1}{2!}\delta^2 \Pi + \frac{1}{3!}\delta^3 \Pi \geq 0 \quad \Rightarrow \quad \delta^3 \Pi \leq 0, \tag{10}$$

which is consistent with the assumption of $\delta^3 \Pi < 0$, for $0 < \lambda < \lambda_S$, for problems with stiffness maxima and minima. The relationships (10) imply the existence of variations of the state variables at the stability limit that lead to

$$\bar{\delta}^2 \Pi = \bar{\delta}^3 \Pi = 0. \tag{11}$$

As will be elucidated in Section 3.1, (9) and (10) have a decisive influence on the form of the linear eigenvalue problem, in the framework of the FEM, that serves as the tool for the numerical computation of extreme values of the stiffness of proportionally loaded structures. This is the rationale behind the presentation of (10) and (11), irrespective of the fact that this work is restricted to the prebuckling domain. In account with the character of an inverse problem, the assumed problem-dependent properties of $\delta^3 \Pi$ are elements of the unknown cause, in the form of the assumed functional $(1/2!)\delta^2 \Pi + (1/3!)\delta^3 \Pi$, of the observed effect, in the form of the said eigenvalue problem. This functional is the basis for determination of the sought variational criteria. The eigenvalue problem is the means for derivation of the corresponding numerical conditions for stiffness maxima and minima.

2.3. Criteria for stiffness maxima of proportionally loaded structures

Before reaching the load level $\lambda = \lambda_I$, at which the stiffness of a proportionally loaded structure becomes a maximum value, and thereafter, so long as the structure does not buckle, $(1/2!)\delta^2 \Pi + (1/3!)\delta^3 \Pi$ is assumed to be a positive quantity, *i.e.*

$$\frac{1}{2!}\delta^2 \Pi + \frac{1}{3!}\delta^3 \Pi > 0, \qquad 0 < \lambda < \lambda_I, \quad \lambda_I < \lambda < \lambda_S. \tag{12}$$

For $\lambda = \lambda_I$, analogous to Dirichlet's criterion for the start of buckling, see (3),

$$\frac{1}{2!}\delta^2 \Pi + \frac{1}{3!}\delta^3 \Pi \geq 0, \qquad \delta^2 \Pi > 0, \qquad \delta^3 \Pi < 0, \tag{13}$$

indicating that there is at least one special variation of the state variables that leads to

$$\frac{1}{2!}\delta^2 \Pi + \frac{1}{3!}\delta^3 \Pi = 0. \tag{14}$$

As follows from (12) and (13), for variations of the state variables leading to (14), the functional $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi$ must satisfy the condition

$$\frac{d}{d\lambda}\left(\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi\right) = 0. \tag{15}$$

It indicates that $(1/2!)\bar{\delta}^2\varPi + (1/3!)\bar{\delta}^3\varPi = 0$, based on such variations, is an extreme value of the respective functional. Because of (12), it is a minimum. Hence, the task is to find a stationary value of the functional $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi$, subjected to the subsidiary condition (15). By means of the Lagrange multiplier method this functional is expanded to

$$\left(\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi\right) + \varLambda\frac{d}{d\lambda}\left(\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi\right) \tag{16}$$

where $\varLambda$ denotes a Lagrange multiplier. The stationarity condition for the expanded functional reads

$$\delta\left[\left(\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi\right) + \varLambda\frac{d}{d\lambda}\left(\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi\right)\right] = 0. \tag{17}$$

It results in

$$\boxed{\begin{aligned} &\delta\left(\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi\right) + \varLambda\delta\left(\frac{d}{d\lambda}\left(\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi\right)\right) = 0, \\ &\frac{d}{d\lambda}\left(\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi\right) = 0 \end{aligned}} \tag{18}$$

as the sought variational criteria for a maximum value of the stiffness of proportionally loaded structures.

The difference between these criteria and Trefftz's criterion for loss of stability is threefold. Firstly, in contrast to Trefftz's criterion, the criteria for a maximum value of the stiffness of proportionally loaded structures also contain $\delta^3\varPi$. Secondly, contrary to Trefftz's criterion, the stationarity condition for the expanded functional, see (17), results in two vanishing expressions, see (18), the second of which is a subsidiary condition. While Trefftz's condition refers to the transition of $\delta^2\varPi$ from a positive-definite functional in the prebuckling region to an indefinite functional afterwards, via a positive-semidefinite functional at incipient buckling, $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi > 0$ has held before this extreme value was reached and holds again thereafter, so long as the structure does not buckle. Thirdly, contrary to Trefftz's criterion, the first criterion for maximum stiffness contains a term with a Lagrange multiplier. Its presence in the expression for the expanded functional, see (16), has enabled the derivation of the two conditions that represent the variational criteria for maximum stiffness, see (18). As will be shown in Section 3, the existence of the mentioned subsidiary condition and, furthermore, the vanishing of $\delta^2\varPi$ and $\delta^3\varPi$ at $\lambda = 0$ and the positive (negative) semidefiniteness of $\delta^2\varPi$ ($\delta^3\varPi$) at $\lambda = \lambda_S$ determine the form of the linear eigenvalue problem in the framework of the FEM that serves as the mathematical tool for numerical computation of the load levels at extreme values of the stiffness of proportionally loaded structures. The significance of the presented criteria for maximum stiffness rests on their sheer existence and on their mathematical form.

2.4. *Criteria for stiffness minima of proportionally loaded structures*

Immediately before reaching the load level $\lambda = \lambda_I$, at which the stiffness of a proportionally loaded structure becomes a minimum value, and afterwards, $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi$ is assumed to be a negative quantity, *i.e.*

$$\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi < 0, \quad \lambda_L < \lambda < \lambda_I, \quad \lambda_I < \lambda < \lambda_U. \tag{19}$$

For $\lambda = \lambda_I$,

$$\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi \le 0, \qquad \delta^2\varPi > 0, \qquad \delta^3\varPi < 0, \tag{20}$$

indicating that there is at least one special variation of the state variables that leads to

$$\frac{1}{2!}\delta^2\varPi + \frac{1}{3!}\delta^3\varPi = 0. \tag{21}$$

As follows from (19) and (20), for such a variation the functional $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi$ satisfies the condition (15). However, it now refers to a maximum value of this functional.

At $\lambda = \lambda_L$, the transition from solely positive values to solely negative values of the functional occurs. It is characterized by the relationships (6) and (7). The reason for the upper bound λ_U of the subinterval $[\lambda_I, \lambda_U]$ is that the subregion of solely negative values of the considered functional around $\lambda = \lambda_I$ is kind of a mechanical *cordon sanitaire*, inside which loss of stability is impossible. However, precluding the possibility of loss of stability at a higher load level just because of the existence of a minimum value of the stiffness would be an unphysical restriction. Hence, there is a load level, $\lambda = \lambda_U$, at which the temporarily solely negative values of the functional $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi$ become again solely positive values, *i.e.* at which again (6) and (7) hold.

By means of the Lagrange multiplier method it can be shown that the relations (18) are also the variational criteria for a minimum value of the stiffness of proportionally loaded structures. Thus, these relations are necessary and sufficient conditions for extreme values of the stiffness. However, for a minimum value of the stiffness, contrary to a maximum value, the stationarity condition for the expanded functional involves $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi < 0$, before this extreme value was reached and thereafter, while the subsidiary condition, $d/d\lambda((1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi) = 0$, refers to zero as the maximum value of the respective functional. The temporary "escape" of $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi$ into a negative quantity, see (20), reflects the tendency of the mechanical behavior of the respective structure to postpone the possibility of loss of stability to a load level at which the structure is stiffening. In passing, it is noted that buckling is not restricted to softening structures.

2.5. *Intermediate résumé*

- The functional $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi$, which is assumed to be the basis for the sought variational criteria for stiffness maxima and minima, is an element of the expression for $\Delta\varPi$, denoting the change of the potential energy due to admissible variations of the state variables at the considered load level, see (1).
- It is assumed to be a positive-semidefinite quantity at the load level of maximum stiffness and a negative-semidefinite quantity at the one of minimum stiffness. These assumptions are basic requirements for the presented variational criteria for extreme values of the stiffness of proportionally loaded structures, see (18).
- The rationale behind these assumptions is the hypothesis that $\delta^3\varPi < 0$, for $0 < \lambda_L < \lambda_S$, for structures which experience a *sublinear* stiffness change in the course of proportional loading.
- Before reaching the load level at maximum stiffness of a structure and afterwards, so long as the structure does not buckle, the functional is a positive quantity.
- This explains the necessity of a supplement to the functional $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi$ in the form of a product of the Lagrange multiplier $\varLambda$ and the term $d/d\lambda\,((1/2!)\,\delta^2\varPi + (1/3!)\,\delta^3\varPi)$, see (16), which is required to be zero.
- Before reaching the load level of minimum stiffness of a structure and afterwards $(1/2!)\delta^2\varPi + (1/3!)\delta^3\varPi$ is a negative quantity.

- This explains the necessity of a supplement to the functional $(1/2!)\delta^2\Pi + (1/3!)\delta^3\Pi$ in the form of a product of a Lagrange multiplier Λ and the term $d/d\lambda((1/2!)\delta^2\Pi + (1/3!)\delta^3\Pi)$, see (6), which is required to be zero.
- A minimum value of the stiffness occurs in a subregion of the prebuckling domain, $[\lambda_L, \lambda_U]$, where $0 < \lambda_L < \lambda_I$ and $\lambda_I < \lambda_U < \lambda_S$. At $\lambda = \lambda_L$ and $\lambda = \lambda_U$, the relationships (6) and (7) hold.
- At the onset of loading, $\Pi = 0$, resulting in $\delta\Pi = 0$, $\delta^2\Pi = 0$, $\delta^3\Pi = 0, \ldots.$.
- At the stability limit, $\delta^2\Pi$ ($\delta^3\Pi$) is a positive- (negative-)semidefinite functional.

In Section 3 it will be shown that the conditions for maximum and minimum stiffness of proportionally loaded structures, in the framework of the FEM, see [1], are consistent with the presented variational criteria for extreme values of the stiffness.

3. Translation of the variational criteria for extreme values of the stiffness of proportionally loaded structures into conditions in the framework of the FEM

3.1. Correlations of properties of an FEM-based linear eigenvalue problem with characteristic features of the underlying functional

The complexity of the presented variational criteria for extreme values of the stiffness of proportionally loaded structures renders analytical determination of the load levels at stiffness maxima and minima unfeasible. Hence, these load levels must be computed numerically. The method of choice is the FEM. It is based on the reduction of systems with infinitely many d-o-f to systems with finite numbers of d-o-f. If the said criteria are the sought solutions of an inverse problem, they must define, together with the underlying functional, the kind of the linear eigenvalue problem in the framework of the FEM that serves as the tool for derivation of numerical conditions for extreme values of the stiffness. This requires correlations of particular properties of this eigenvalue problem with characteristic features of the underlying functional.

Albeit for another purpose, Malendowski, one of the authors of [8], has used a novel linear eigenvalue problem that has later turned out to be the sought mathematical tool. Formally, this linear eigenvalue problem can be written as follows [1,2]:

$$\left[\begin{array}{c:c} \mathbf{K}-\chi\mathbf{K}_0 & \mathbf{G}^T-\chi\mathbf{G}_0^T \\ \hdashline \mathbf{G}-\chi\mathbf{G}_0 & \mathbf{0} \end{array}\right] \cdot \left\{\begin{array}{c} \mathbf{r} \\ \hdashline \mathbf{t} \end{array}\right\} = \left\{\begin{array}{c} \mathbf{0} \\ \hdashline \mathbf{0} \end{array}\right\}. \tag{22}$$

In (22), $\mathbf{K}$ denotes the tangent stiffness matrix, whereas $\mathbf{K}_0 = \mathbf{K}(\lambda = 0)$ stands for the small-displacement stiffness matrix. $\mathbf{G}$ and $\mathbf{G}_0 = \mathbf{G}(\lambda = 0)$ are ingredients of the subsidiary conditions

$$(\mathbf{G} - \chi\mathbf{G}_0) \cdot \mathbf{r} = \mathbf{0} \tag{23}$$

for the components of the subvector $\mathbf{r}$ of the eigenvector $\lfloor \mathbf{r}^T, \mathbf{t}^T \rfloor$ of the linear eigenvalue problem (22). χ denotes the eigenvalue, and $\mathbf{t}$ stands for the second subvector of the eigenvector. Mathematically, $\mathbf{t}$ is interpreted as a vector of Lagrange multipliers. The superscript "T" marks the transpose of the respective submatrix. (22) is the mathematical formulation of a linear eigenvalue problem, established with the help of hybrid beam elements, available in the library of the commercial computer program Abaqus [9]. These elements are extensions of displacement elements. The motivation for these extensions was to avoid numerical problems with the displacement elements in case of nearly incompressible material [9]. However, this aspect is not relevant to the present work. The linear eigenvalue problem (22), with the subsidary conditions (23), allows determination of the load level at which the stiffness of a proportionally loaded structure becomes an extreme value.

The two coefficient matrices of (22),

$$\left[\begin{array}{c:c} \mathbf{K} & \mathbf{G}^T \\ \hdashline \mathbf{G} & \mathbf{0} \end{array}\right] \quad \wedge \quad \left[\begin{array}{c:c} \mathbf{K}_0 & \mathbf{G}_0^T \\ \hdashline \mathbf{G}_0 & \mathbf{0} \end{array}\right], \tag{24}$$

are assemblages of extensions of element tangent stiffness matrices.They are indefinite matrices. The indefiniteness of both coefficient matrices of a linear eigenvalue problem is a necessary condition for complex eigenvalues, see *e.g.* [10]. A mechanical reason for their occurrence, at least as far as the fundamental eigenvalue is concerned, is the existence of a minimum value of the stiffness of a proportionally loaded structure [1,2]. Conversely, a maximum value of the stiffness requires a real fundamental eigenvalue [1]. The linear eigenvalue problem (22) is a mathematical tool that enables to account for both, maximum and minimum values of the stiffness. The scientific background of this ability are the variational criteria for stiffness maxima of proportionally loaded structures, see (18). It was shown that these criteria also hold for stiffness minima. The different sign combinations of the functional $(1/2!)\delta^2\Pi + (1/3!)\delta^3\Pi$ in (13) and (20) are distinguishing features of stiffness maxima and minima. Their correlation with real eigenvalues and real parts of conjugate complex eigenvalues, respectively, as corresponding distinguishing features in the framework of the FEM, is an element of the solution of the inverse problem to which this work is devoted.

The eigenvalue problem (22) is an unconventional novel form of eigenvalue problems used in the early days of nonlinear structural stability analyses by the FEM as the mathematical tool for so-called *accompanying linear eigenvalue analysis*, see *e.g.* [11]. The purpose of such analyses was to circumvent numerical problems in the vicinity of the stability limit. With the exception of the sought zero position of the fundamental eigenvalue function, $\chi(\lambda_S) \equiv \chi_1(\lambda_S) = 0$, at the start of buckling, this function was mechanically insignificant. In the given case, however, it is able to identify also the load levels at extreme values of the stiffness in the prebuckling region.

Specialization of (22) for $\lambda = 0$ gives [2]

$$\chi_i = 1, \qquad i = \{1, 2, \ldots, n\}, \tag{25}$$

representing an n-fold eigenvalue, where n denotes the number of d-o-f. The background of (25) is $\Pi(\lambda = 0) = 0$, resulting in $\delta\Pi = 0$, $\delta^2\Pi = 0$, $\delta^3\Pi = 0, \ldots$, at the onset of loading.

The second coefficient matrix of the linear eigenvalue problem (22) is equal to the matrix obtained from specialization of the first coefficient matrix for $\lambda = 0$. Thus, the coefficients of the characteristic equation of

$$\left|\begin{array}{c:c} \mathbf{K}-\chi\mathbf{K}_0 & \mathbf{G}^T-\chi\mathbf{G}_0^T \\ \hdashline \mathbf{G}-\chi\mathbf{G}_0 & \mathbf{0} \end{array}\right| = 0 \tag{26}$$

can be rendered dimensionless [2]. This is a necessary condition for an eigenvalue function, suitable for determination of the load level of an extreme value of the stiffness of a proportionally loaded structure. Specialization of (26) for a stability limit, *i.e.* for $\chi = 0$, gives

$$\left|\begin{array}{c:c} \mathbf{K} & \mathbf{G}^T \\ \hdashline \mathbf{G} & \mathbf{0} \end{array}\right| = 0. \tag{27}$$

The vanishing of the determinant of the tangent stiffness matrix, *i.e.*,

$$|\mathbf{K}| = 0, \tag{28}$$

is a necessary and sufficient condition for (27) and, thus, for $\chi = 0$. In other words, the extension of displacement elements to hybrid elements has obviously no influence on the stability limit.

3.2. Intermediate summary

- The form of the linear eigenvalue problem (22), characterized by constraint conditions for the components of the subvector $\mathbf{r}$ of the eigenvector $\lfloor \mathbf{r}^T, \mathbf{t}^T \rfloor$, correlates with the one of the variational conditions (18).
- The character of the fundamental eigenvalue function $\chi_1(\lambda)$ as a real quantity at the load level of maximum stiffness and a complex quantity at the one of minimum stiffness correlates with the form of the functional $(1/2!)\delta^2\Pi + (1/3!)\delta^3\Pi$ as a positive-semidefinite and a negative-semidefinite quantity, respectively.

- The n-fold eigenvalue at the onset of loading, equal to 1, which results in the vanishing of the matrix in (22), correlates with the vanishing of $\delta^2\Pi$ and $\delta^3\Pi$ as elements of the underlying functional $(1/2!)\delta^2\Pi+(1/3!)\delta^3\Pi$.
- The positive semidefiniteness and singularity of the tangent stiffness matrix $\mathbf{K}$ at the stability limit, which is a necessary and sufficient condition for the singularity of the first one of the two indefinite coefficient matrices of the linear eigenvalue problem (22), see (24), correlates with the positive semidefiniteness of $\delta^2\Pi$ and, depending on the kind of the nonlinear stiffness change in the prebuckling region, with the positive or negative semidefiniteness of $\delta^3\Pi$ at incipient buckling.

The delineated correlations of particular properties of the linear eigenvalue problem (22) with characteristic features of the functional for determination of variational criteria for extreme values of the stiffness of proportionally loaded structures are evidence of the functional's role as the scientific basis for an eigenvalue problem as a means for derivation of FEM-based conditions for stiffness maxima and minima. As will be shown in Section 3.3, this derivation requires computation of the second derivative of (22) with respect to λ.

3.3. *Derivation of numerical conditions for stiffness maxima of proportionally loaded structures*

Derivation of (22) with respect to λ gives

$$\left[\begin{array}{c:c} \dot{\mathbf{K}}-\dot{\chi}\mathbf{K}_0 & \dot{\mathbf{G}}^T-\dot{\chi}\mathbf{G}_0^T \\ \hdashline \dot{\mathbf{G}}-\dot{\chi}\mathbf{G}_0 & \mathbf{0} \end{array}\right]\cdot\left\{\begin{array}{c} \mathbf{r} \\ \hdashline \mathbf{t} \end{array}\right\}+$$

$$\left[\begin{array}{c:c} \mathbf{K}-\chi\mathbf{K}_0 & \mathbf{G}^T-\chi\mathbf{G}_0^T \\ \hdashline \mathbf{G}-\chi\mathbf{G}_0 & \mathbf{0} \end{array}\right]\cdot\left\{\begin{array}{c} \dot{\mathbf{r}} \\ \hdashline \dot{\mathbf{t}} \end{array}\right\}=\left\{\begin{array}{c} \mathbf{0} \\ \hdashline \mathbf{0} \end{array}\right\}, \tag{29}$$

where $\dot{}\ := d/d\lambda$. Premultiplication of (29) by $\lfloor\mathbf{r}^T,\mathbf{t}^T\rfloor$ and consideration of (22) yields

$$\lfloor\mathbf{r}^T,\mathbf{t}^T\rfloor\cdot\left[\begin{array}{c:c} \dot{\mathbf{K}}-\dot{\chi}\mathbf{K}_0 & \dot{\mathbf{G}}^T-\dot{\chi}\mathbf{G}_0^T \\ \hdashline \dot{\mathbf{G}}-\dot{\chi}\mathbf{G}_0 & \mathbf{0} \end{array}\right]\cdot\left\{\begin{array}{c} \mathbf{r} \\ \hdashline \mathbf{t} \end{array}\right\}=0. \tag{30}$$

Solving (30) for $\dot{\chi}$ results in

$$\dot{\chi}=\frac{\mathbf{r}^T\cdot\dot{\mathbf{K}}\cdot\mathbf{r}+2\,\mathbf{r}^T\cdot\dot{\mathbf{G}}^T\cdot\mathbf{t}}{\mathbf{r}^T\cdot\mathbf{K}_0\cdot\mathbf{r}+2\,\mathbf{r}^T\cdot\mathbf{G}_0^T\cdot\mathbf{t}}. \tag{31}$$

Derivation of (29) with respect to λ gives

$$\left[\begin{array}{c:c} \ddot{\mathbf{K}}-\ddot{\chi}\mathbf{K}_0 & \ddot{\mathbf{G}}^T-\ddot{\chi}\mathbf{G}_0^T \\ \hdashline \ddot{\mathbf{G}}-\ddot{\chi}\mathbf{G}_0 & \mathbf{0} \end{array}\right]\cdot\left\{\begin{array}{c} \mathbf{r} \\ \hdashline \mathbf{t} \end{array}\right\}+$$

$$2\left[\begin{array}{c:c} \dot{\mathbf{K}}-\dot{\chi}\mathbf{K}_0 & \dot{\mathbf{G}}^T-\dot{\chi}\mathbf{G}_0^T \\ \hdashline \dot{\mathbf{G}}-\dot{\chi}\mathbf{G}_0 & \mathbf{0} \end{array}\right]\cdot\left\{\begin{array}{c} \dot{\mathbf{r}} \\ \hdashline \dot{\mathbf{t}} \end{array}\right\}+$$

$$\left[\begin{array}{c:c} \mathbf{K}-\chi\mathbf{K}_0 & \mathbf{G}^T-\chi\mathbf{G}_0^T \\ \hdashline \mathbf{G}-\chi\mathbf{G}_0 & \mathbf{0} \end{array}\right]\cdot\left\{\begin{array}{c} \ddot{\mathbf{r}} \\ \hdashline \ddot{\mathbf{t}} \end{array}\right\}=\left\{\begin{array}{c} \mathbf{0} \\ \hdashline \mathbf{0} \end{array}\right\}. \tag{32}$$

Premultiplication of (32) by $\lfloor\mathbf{r}^T,\mathbf{t}^T\rfloor$ and consideration of (22) yields

$$\lfloor\mathbf{r}^T,\mathbf{t}^T\rfloor\cdot\left[\begin{array}{c:c} \ddot{\mathbf{K}}-\ddot{\chi}\mathbf{K}_0 & \ddot{\mathbf{G}}^T-\ddot{\chi}\mathbf{G}_0^T \\ \hdashline \ddot{\mathbf{G}}-\ddot{\chi}\mathbf{G}_0 & \mathbf{0} \end{array}\right]\cdot\left\{\begin{array}{c} \mathbf{r} \\ \hdashline \mathbf{t} \end{array}\right\}+$$

$$2\lfloor\mathbf{r}^T,\mathbf{t}^T\rfloor\cdot\left[\begin{array}{c:c} \dot{\mathbf{K}}-\dot{\chi}\mathbf{K}_0 & \dot{\mathbf{G}}^T-\dot{\chi}\mathbf{G}_0^T \\ \hdashline \dot{\mathbf{G}}-\dot{\chi}\mathbf{G}_0 & \mathbf{0} \end{array}\right]\cdot\left\{\begin{array}{c} \dot{\mathbf{r}} \\ \hdashline \dot{\mathbf{t}} \end{array}\right\}=0. \tag{33}$$

Making use of (29) in (33) gives

$$\lfloor\mathbf{r}^T,\mathbf{t}^T\rfloor\cdot\left[\begin{array}{c:c} \ddot{\mathbf{K}}-\ddot{\chi}\mathbf{K}_0 & \ddot{\mathbf{G}}^T-\ddot{\chi}\mathbf{G}_0^T \\ \hdashline \ddot{\mathbf{G}}-\ddot{\chi}\mathbf{G}_0 & \mathbf{0} \end{array}\right]\cdot\left\{\begin{array}{c} \mathbf{r} \\ \hdashline \mathbf{t} \end{array}\right\}-$$

$$2\lfloor\dot{\mathbf{r}}^T,\dot{\mathbf{t}}^T\rfloor\cdot\left[\begin{array}{c:c} \mathbf{K}-\chi\mathbf{K}_0 & \mathbf{G}^T-\chi\mathbf{G}_0^T \\ \hdashline \mathbf{G}-\chi\mathbf{G}_0 & \mathbf{0} \end{array}\right]\cdot\left\{\begin{array}{c} \dot{\mathbf{r}} \\ \hdashline \dot{\mathbf{t}} \end{array}\right\}=0. \tag{34}$$

To obtain an expression for $\ddot{\chi}$ by replacing $\dot{\mathbf{K}}$ and $\dot{\mathbf{G}}^T$ in the expression for $\dot{\chi}$, see (31), by $\ddot{\mathbf{K}}$ and $\ddot{\mathbf{G}}^T$, respectively, (34) must disintegrate into two vanishing quadratic forms. The rationale behind this disintegration, representing a constraint on (34), are the subsidiary conditions for $\mathbf{r}$, see (23), in the linear eigenvalue problem (22) as a consequence of the use of special hybrid elements. Mathematically, this disintegration is possible, because both matrices in (34) are indefinite matrices. The second matrix is, moreover, singular, with $\lfloor\mathbf{r}^T,\mathbf{t}^T\rfloor$ as the eigenvector. The mentioned disintegration of (34) results in

$$\chi=\frac{\dot{\mathbf{r}}^T\cdot\mathbf{K}\cdot\dot{\mathbf{r}}+2\,\dot{\mathbf{r}}^T\cdot\mathbf{G}^T\cdot\dot{\mathbf{t}}}{\dot{\mathbf{r}}^T\cdot\mathbf{K}_0\cdot\dot{\mathbf{r}}+2\,\dot{\mathbf{r}}^T\cdot\mathbf{G}_0^T\cdot\dot{\mathbf{t}}} \tag{35}$$

and

$$\ddot{\chi}=\frac{\mathbf{r}^T\cdot\ddot{\mathbf{K}}\cdot\mathbf{r}+2\,\mathbf{r}^T\cdot\ddot{\mathbf{G}}^T\cdot\mathbf{t}}{\mathbf{r}^T\cdot\mathbf{K}_0\cdot\mathbf{r}+2\,\mathbf{r}^T\cdot\mathbf{G}_0^T\cdot\mathbf{t}}. \tag{36}$$

The formal similarity of (35) and (36) hints at a physical relevance of $\ddot{\chi}=0$ analogous to the one of $\chi=0$.
At the stability limit,

$$\chi\equiv\chi_1=0, \tag{37}$$

where χ_1 denotes the fundamental eigenvalue. Substitution of (37) into (35) gives

$$\dot{\mathbf{r}}^T\cdot\mathbf{K}\cdot\dot{\mathbf{r}}+2\,\dot{\mathbf{r}}^T\cdot\mathbf{G}^T\cdot\dot{\mathbf{t}}=0, \tag{38}$$

which underscores the explicit independence of loss of stability of $\mathbf{K}_0$ and $\mathbf{G}_0$. It explains the mechanical rationale behind the disintegration of (34) into (35) and (36). At the stability limit, $\mathbf{K}$ is a positive-semidefinite matrix which is singular, with $\mathbf{r}$ as the eigenvector. Since $\dot{\mathbf{r}}$ is not an eigenvector of $\mathbf{K}$,

$$\dot{\mathbf{r}}^T\cdot\mathbf{K}\cdot\dot{\mathbf{r}}>0, \tag{39}$$

implying

$$2\dot{\mathbf{r}}^T\cdot\mathbf{G}^T\cdot\dot{\mathbf{t}}<0. \tag{40}$$

Nota bene, the load level $\lambda=\lambda_S$ at loss of stability could alternatively be obtained from accompanying linear eigenvalue analysis based on the linear eigenvalue problem

$$\left[\hat{\mathbf{K}}-\hat{\chi}\,\hat{\mathbf{K}}_0\right]\cdot\hat{\mathbf{r}}=\mathbf{0}. \tag{41}$$

It would take the place of the linear eigenvalue problem (22) if displacement elements were used instead of hybrid elements. In (41), $\hat{\mathbf{K}}$ denotes the tangent stiffness matrix, $\hat{\mathbf{K}}_0=\hat{\mathbf{K}}(\lambda=0)$ stands for the small-displacement stiffness matrix, and $\hat{\chi}$ and $\hat{\mathbf{r}}$ denote the eigenvalue and the eigenvector, respectively. In contrast to (22), (41) does not contain constraint conditions for the components of the eigenvector. This is consistent with Trefftz's variational criterion for loss of stability, see (5), insofar as this criterion does not contain a variational subsidiary condition.

Analogous to the situation at the stability limit, for a maximum value of the stiffness,

$$\boxed{\ddot{\chi}=0}\ , \tag{42}$$

[1]. Substitution of (42) into (36) gives

$$\boxed{\mathbf{r}^T\cdot\ddot{\mathbf{K}}\cdot\mathbf{r}+2\mathbf{r}^T\cdot\ddot{\mathbf{G}}^T\cdot\mathbf{t}=0}\ , \tag{43}$$

which is a new result, formally analogous to (38). (42) and (43) are the FEM-based counterparts of the variational criteria for stiffness maxima

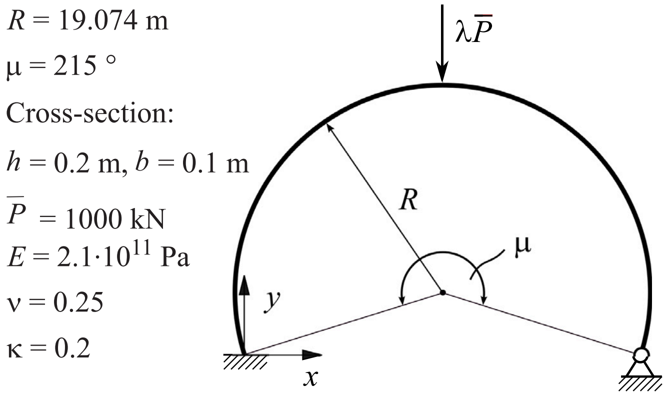


Fig. 2. Segment of a circular arch, clamped at one end and hinged at the other one, subjected to a vertical point load at its apex [1].

of proportionally loaded structures, see (18). (43) is independent of $\mathbf{K}_0$ and $\mathbf{G}_0$, which again explains the mechanical rationale behind the disintegration of (34) into (35) and (36). This is consistent with the independence of the said variational criteria of mechanical quantities referring to the start of loading.

If displacement elements were used instead of hybrid elements,

$$\hat{\mathbf{r}}^T \cdot \left[\ddot{\hat{\mathbf{K}}} - \ddot{\hat{\chi}}\hat{\mathbf{K}}_0\right] \cdot \hat{\mathbf{r}} - 2\,\dot{\hat{\mathbf{r}}}^T \cdot \left[\dot{\hat{\mathbf{K}}} - \dot{\hat{\chi}}\hat{\mathbf{K}}_0\right] \cdot \dot{\hat{\mathbf{r}}} = 0 \tag{44}$$

would take the place of (34). In [1] it was shown that for the displacement elements, serving as the basis for their extension to the hybrid elements used in this work, $\ddot{\hat{\chi}} = 0$ implies

$$\dot{\hat{\mathbf{r}}}^T \cdot \hat{\mathbf{K}} \cdot \dot{\hat{\mathbf{r}}} = 0, \tag{45}$$

which, however, is restricted to $\lambda > \lambda_S$, where $\hat{\mathbf{K}}$ is an indefinite matrix. This restriction is in conflict with the possibility of extreme values of the stiffness of proportionally loaded structures in the prebuckling region. *Hence, displacement elements are unsuitable for determination of stiffness maxima of such structures.*

3.4. Verification of $\ddot{\chi} \equiv \ddot{\chi}_1 = 0$ as a numerical condition for stiffness maxima of proportionally loaded structures

$\ddot{\chi}_1 = 0$, resulting from accompanying linear eigenvalue analysis by the FEM, was found to correlate with the presented variational criteria for a maximum value of the stiffness of proportionally loaded structures, see (18). This correlation confirms a plausible, albeit so far theoretically unproved numerical result, obtained for a segment of a circular arch, clamped at one end and hinged at the other one, subjected to a vertical point load at its apex, see Fig. 2, taken from [1]. h and b denote the height and the width of the rectangular cross-section; E, ν, and κ stand for the modulus of elasticity, Poisson's ratio, and the shear coefficient, respectively.

Fig. 3, adapted from an illustration in [1], shows the $\chi_1 - \lambda/\lambda_S$ diagram obtained with Abaqus hybrid elements B32H [9]. Point S denotes the zero position of χ_1 at the end of this curve. It indicates loss of stability of the structure by snap-through. Point I denotes the point of inflection on this curve, indicating the normalized load level at which the stiffness of the arch becomes a maximum value. The non-negative quantity $(1/2!)\delta^2\Pi + (1/3!)\delta^3\Pi$ at both points, albeit of a different kind, provides evidence of their mechanical significance. At point S, variations of the state variables resulting in $\bar{\delta}^2\Pi = 0$ and $\bar{\delta}^3\Pi = 0$ are responsible for the vanishing of this functional. The existence of $\bar{\delta}^2\Pi = 0$ is the basis of the positive semidefiniteness of the functional $\delta^2\Pi$ at the stability limit, resulting in Trefftz's criterion, see (11). It entails $\bar{\delta}^3\Pi = 0$ at the stability limit. At point I, however, it is $(1/3!)\bar{\delta}^3\Pi = -(1/2!)\bar{\delta}^2\Pi < 0$ that results in the vanishing of the expanded functional. Moreover, the special variations $\bar{\delta}^2\Pi$ and $\bar{\delta}^3\Pi$ must satisfy the subsidiary condition $d/d\lambda((1/2!)\delta^2\Pi + (1/3!)\delta^3\Pi) = 0$, see (18). At this point,

$$\ddot{\chi}_1 = 0, \quad \dddot{\chi}_1 < 0. \tag{46}$$

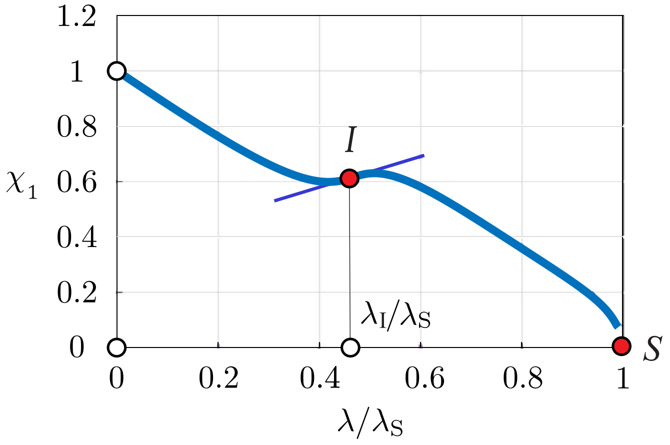


Fig. 3. Segment of a circular arch, clamped at one end and hinged at the other one, subjected to a vertical point load at its apex: $\chi_1 - \lambda/\lambda_S$ diagram obtained with Abaqus hybrid elements B32H, adapted from Fig. 7 in [1].

In contrast to the negative sign of $\dddot{\chi}_1$, the sign of $\dot{\chi}_1$ is indeterminate. Depending on the input parameters, it may be positive, as in Fig. 3, negative as in Fig. 1, or, by chance, zero. Hence, contrary to the point of inflection, the two extreme values of $\chi_1(\lambda)$ in Fig. 3 are mechanically insignificant. The reason for this is the absence of a variational criterion as their scientific basis. Since the sign of $\dot{\chi}_1$ at point I has no mechanical significance, it is not a distinguishing feature of stiffness maxima and minima. It is the character of $\chi_1(\lambda = \lambda_I)$ as a real quantity that marks $\ddot{\chi}_1(\lambda = \lambda_I) = 0$ as the condition for a stiffness maximum.

3.5. Derivation of numerical conditions for stiffness minima of proportionally loaded structures

A minimum value of the stiffness of a proportionally loaded structure is characterized by the negative semidefiniteness of the functional $(1/2!)\delta^2\Pi + (1/3!)\delta^3\Pi$. Before and after the corresponding load level, the functional is a negative quantity. This situation is restricted to a subregion of the prebuckling domain, see the subinterval $\lfloor\lambda_L, \lambda_U\rfloor$ in Fig. 5(b).

At $\lambda = \lambda_L, \chi \equiv \chi_1 = \chi_j > 0$, and

$$\lim_{\epsilon\to 0} \dot{\chi}_1(\lambda_L - \epsilon) = -\lim_{\epsilon\to 0} \dot{\chi}_j(\lambda_L - \epsilon) = \infty, \tag{47}$$

representing left-sided limits of $\dot{\chi}_1$ and $\dot{\chi}_j$. Substitution of (47), with $\ddot{\chi}_1 \equiv \ddot{\chi}$, into (31) results in

$$\mathbf{r}^T \cdot \mathbf{K}_0 \cdot \mathbf{r} + 2\mathbf{r}^T \cdot \mathbf{G}_0^T \cdot \mathbf{t} = 0 \tag{48}$$

as the corresponding solution at the "eigenvector level". At $\lambda = \lambda_U$, $\lambda_U > \lambda_I$, $\chi \equiv \chi_1 = \chi_j < 0$. Now, $\dot{\chi}_1 = -\dot{\chi}_j = \infty$ are the right-sided limits of $\dot{\chi}_1(\lambda)$ and $\dot{\chi}_j(\lambda)$. The scientific basis of the upper limit of the interval $[\lambda_L, \lambda_U]$ is the return of the temporarily negative quantity $(1/2!)\delta^2\Pi + (1/3!)\delta^3\Pi$ to a positive quantity, without which loss of stability at a higher load level would be impossible. Inside the interval $[\lambda_L, \lambda_U]$, the real part of the conjugate complex eigenvalues χ_1 and $\chi_j = \overline{\chi}_1$ replaces the real eigenvalue $\chi_1(\lambda)$ before and after this interval as the quantity that is relevant to the present research.

Both the right-sided limits of $\dot{\chi}_1(\lambda)$ and $\dot{\chi}_j(\lambda)$ at $\lambda = \lambda_L$ and the left-sided limits at $\lambda = \lambda_U$ are characterized by

$$d\chi_1 = d\chi_j = 0 \qquad \wedge \qquad d\lambda = 0, \tag{49}$$

see Fig. 5. Hence, D_L and D_U in this figure are singular points.

To obtain the conditions for a minimum value of the stiffness of proportionally loaded structures, (36) is rewritten as

$$
\begin{aligned}
&((\Re(\chi))^{\cdot\cdot} + i(\Im(\chi))^{\cdot\cdot}) \cdot [(\Re(\mathbf{r})^T + i\Im(\mathbf{r})^T) \cdot \mathbf{K}_0 \cdot (\Re(\mathbf{r}) + i\Im(\mathbf{r}))\\
&+ 2(\Re(\mathbf{r})^T + i\Im(\mathbf{r})^T) \cdot \mathbf{G}_0^T \cdot (\Re(\mathbf{t}) + i\Im(\mathbf{t}))] =\\
&(\Re(\mathbf{r})^T + i\Im(\mathbf{r})^T) \cdot \ddot{\mathbf{K}} \cdot (\Re(\mathbf{r}) + i\Im(\mathbf{r})) + 2(\Re(\mathbf{r})^T + i\Im(\mathbf{r})^T) \cdot \ddot{\mathbf{G}}^T \cdot (\Re(\mathbf{t}) + i\Im(\mathbf{t})),
\end{aligned}
\tag{50}
$$

where $\Re(\,)$ $(\Im(\,))$ is the symbol for the real (imaginary) part of the scalar or vector inside the parentheses. The real part of (50) is obtained as

$$
\begin{aligned}
&(\Re(\chi))^{\cdot\cdot} \cdot [\Re(\mathbf{r})^T \cdot \mathbf{K}_0 \cdot \Re(\mathbf{r}) - \Im(\mathbf{r})^T \cdot \mathbf{K}_0 \cdot \Im(\mathbf{r}) + 2\Re(\mathbf{r})^T \cdot \mathbf{G}_0^T \cdot \Re(\mathbf{t})\\
&- 2\Im(\mathbf{r})^T \cdot \mathbf{G}_0^T \cdot \Im(\mathbf{t})]\\
&- (\Im(\chi))^{\cdot\cdot} \cdot [2\Re(\mathbf{r})^T \cdot \mathbf{K}_0 \cdot \Im(\mathbf{r}) + 2\Re(\mathbf{r})^T \cdot \mathbf{G}_0^T \cdot \Im(\mathbf{t}) + 2\Im(\mathbf{r})^T \cdot \mathbf{G}_0^T \cdot \Re(\mathbf{t})] =\\
&\Re(\mathbf{r})^T \cdot \ddot{\mathbf{K}} \cdot \Re(\mathbf{r}) - \Im(\mathbf{r})^T \cdot \ddot{\mathbf{K}} \cdot \Im(\mathbf{r}) + 2\Re(\mathbf{r})^T \cdot \ddot{\mathbf{G}}^T \cdot \Re(\mathbf{t}) - 2\Im(\mathbf{r})^T \cdot \ddot{\mathbf{G}}^T \cdot \Im(\mathbf{t}).
\end{aligned}
\tag{51}
$$

At the load level of minimum stiffness, (51) disintegrates into

$$
\boxed{(\Re(\chi))^{\cdot\cdot} = 0}\ ,
\tag{52}
$$

which was previously reported in [2], and

$$
\boxed{\Re(\mathbf{r})^T \cdot \ddot{\mathbf{K}} \cdot \Re(\mathbf{r}) + 2\Re(\mathbf{r})^T \cdot \ddot{\mathbf{G}}^T \cdot \Re(\mathbf{t}) = 0}
\tag{53}
$$

and

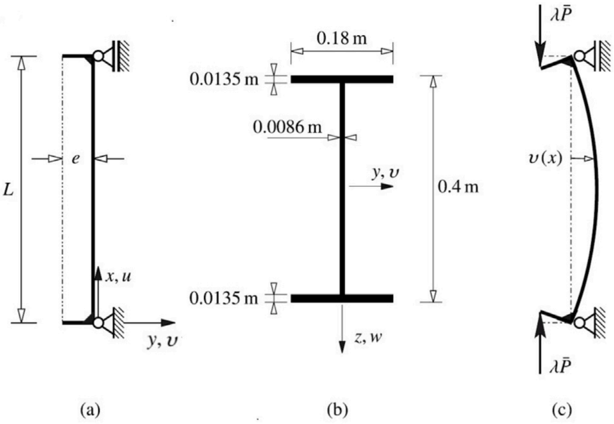
$$
\boxed{
\begin{aligned}
&(\Im(\chi))^{\cdot\cdot} =\\
&\frac{1}{2} \cdot \frac{\Im(\mathbf{r})^T \cdot \ddot{\mathbf{K}} \cdot \Im(\mathbf{r}) + 2\,\Im(\mathbf{r})^T \cdot \ddot{\mathbf{G}}^T \cdot \Im(\mathbf{t})}{\Re(\mathbf{r})^T \cdot \mathbf{K}_0 \cdot \Im(\mathbf{r}) + \Re(\mathbf{r})^T \cdot \mathbf{G}_0^T \cdot \Im(\mathbf{t}) + \Im(\mathbf{r})^T \cdot \mathbf{G}_0^T \cdot \Re(\mathbf{t})}
\end{aligned}
}\ ,
\tag{54}
$$

which are new results. (52)–(54) are the FEM-based counterparts of the variational criteria for stiffness minima of proportionally loaded structures, see (18). (For $\Im(\chi) = 0$, $\Re(\chi) = \chi$, $\Re(\mathbf{r}) = \mathbf{r}$ and $\Re(\mathbf{t}) = \mathbf{t}$. In this case, (52) and (53) become the conditions for a maximum value of the stiffness, see (42) and (43).)

3.6. *Verification of* $(\Re(\chi))^{\cdot\cdot} = 0$ *as a numerical condition for stiffness minima of proportionally loaded structures*

$(\Re(\chi_1))^{\cdot\cdot} = 0$, resulting from accompanying linear eigenvalue analysis by the FEM, was found to correlate with the presented variational criteria for a minimum value of the stiffness of proportionally loaded structures. These criteria were shown to be formally equal to the ones for a maximum value, see (18). This correlation confirms a plausible, albeit so far theoretically unproved numerical result, obtained for a bar, subjected to an eccentric compressive force, see Fig. 4, taken from [2]. The length of the simply supported IPE 400 bar, L, is equal to 5 m. Its eccentricity, e, is equal to 0.0404 m. The reference load $\bar{P}$ is chosen as the Euler load of a bar subjected to centric compression. The area of the cross-section, A, the modulus of elasticity, E, and Poisson's ratio, ν, are given as $8.0678 \cdot 10^{-3}$ m^2, $210 \cdot 10^9$ Pa, and 0.3, respectively.

Fig. 5(a), adapted from an illustration in [2], shows isometric plots of $\chi_1(\lambda) = \Re(\chi_1(\lambda)) + i\Im(\chi_1(\lambda))$ and $\chi_j(\lambda) = \overline{\chi}_1(\lambda) = \Re(\chi_1(\lambda)) + i\Im(\chi_1(\lambda))$ obtained with Abaqus hybrid elements B32OSH. The complex region of the eigenvalue functions $\chi_1(\lambda)$ and $\chi_j(\lambda)$ enables changes of the sign of their real parts from positive to negative, without χ_1 and χ_j intersecting the λ – axis. This proves that the bar does not buckle. Fig. 5(b), adapted from the same illustration, shows projections of these plots onto the $i\Im(\chi_1) = i\Im(\chi_j) = 0$ plane. $I_1 = I_j = I$ in Fig. 5(b) denotes the point of inflection on the $\Re(\chi_1) - \lambda = \Re(\chi_j) - \lambda$ diagram, replacing the $\chi_1 - \lambda$ and the $\chi_j - \lambda$ diagram inside the interval of conjugate complex eigenvalues $\chi_1(\lambda)$ and $\chi_j(\lambda)$. This point indicates the load level at which the stiffness of the bar becomes a minimum value. The presented variational criteria for stiffness maxima, see (18), which were shown to hold also for stiffness minima, provide evidence of the mechanical significance of point I. At points D_L and D_U, $\chi_1 = \chi_j$, elucidating the situation at the beginning and the end, respectively, of the interval $[\lambda_L, \lambda_U]$, inside which $\chi_1(\lambda)$ and $\chi_j(\lambda)$ are conjugate complex variables. At point $I_1 = I_j = I$,



(a) (b) (c)

Fig. 4. Bar, subjected to an eccentric compressive force: (a) undeformed structure, (b) cross-section, (c) deformed structure [2].

$$
(\Re(\chi_1))^{\cdot} < 0, \quad (\Re(\chi_1))^{\cdot\cdot} = 0, \quad (\Re(\chi_1))^{\cdot\cdot\cdot} > 0.
\tag{55}
$$

The sign of $\Re(\chi_1)^{\cdot\cdot\cdot}$ at point $I_1 = I_j = I$ in Fig. 5(b) is opposed to the one of $\dddot{\chi}_1$ at point I in Fig. 3. Initially, the sign of the curvature of $\chi_1(\lambda)$ in Fig. 5(b) is the same as the one in Fig. 3. Hence, a point of inflection with $\dddot{\chi}_1(\lambda) > 0$ instead of $(\Re(\chi_1))^{\cdot\cdot\cdot} > 0$ would have to be preceded by a point of inflection with $\dddot{\chi}_1(\lambda) < 0$. This means that a minimum value of the stiffness of proportionally loaded structures would have to be preceded by a maximum value, which, however, would be an unreasonable restriction for stiffness minima. The existence of a subregion of $0 < \lambda < \lambda_S$, inside which χ_1 and χ_j are conjugate complex eigenvalues, is a way out of the described problem. The discontinuous transition from $\ddot{\chi}_1 > 0$ before $\lambda = \lambda_L$ to $(\Re(\chi_1))^{\cdot\cdot} = (\Re(\chi_j))^{\cdot\cdot} < 0$ afterwards, see Fig. 5(b), paves the way to $(\Re(\chi_1))^{\cdot\cdot\cdot} > 0$ at $I_1 = I_j = I$.

3.7. *Proportionally loaded structures without an extreme value of the stiffness in the prebuckling region*

The variational conditions for extreme values of the stiffness of proportionally loaded structures in the prebuckling region, see (13) and (20), respectively, require $\delta^3\Pi < 0$. Thus, it is plausible that $\delta^3\Pi \geq 0$ is a characteristic feature of structures without such an extreme value in this region. Accordingly, $\chi_1 - \lambda$ diagrams obtained from accompanying linear eigenvalue analysis of such structures by means of the linear eigenvalue problem (22) in the framework of the FEM do not contain points of inflection in the prebuckling region. This is confirmed by the $\chi_1 - \lambda$ diagram obtained from accompanying linear eigenvalue analysis of a cantilever IPE 400 beam subjected to a point load at its free end. Fig. 6, taken from [2], refers to this analysis. Fig. 6(a) shows the structure. It also contains the geometric and the material data needed for the analysis. The blue curve in Fig. 6(b) shows the $\chi_1 - P$ diagram obtained with Abaqus hybrid elements

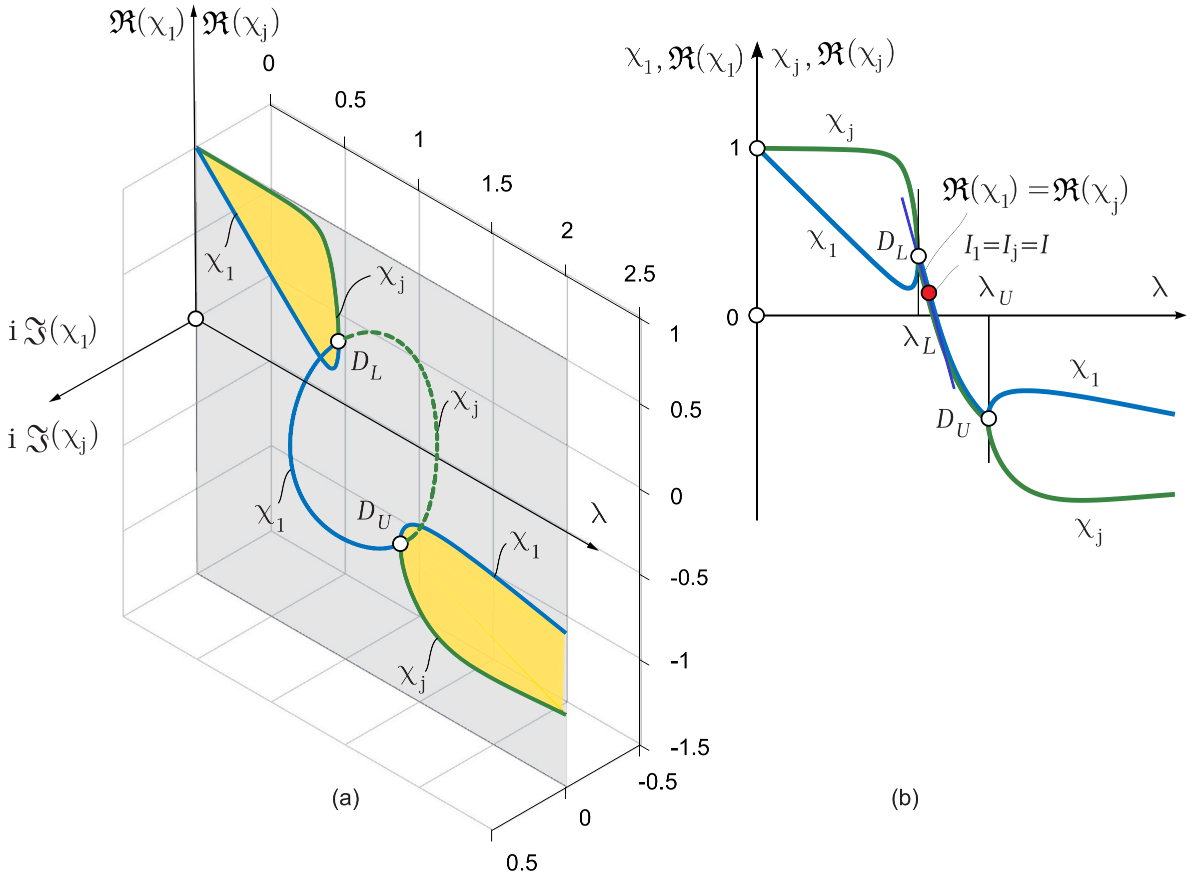


Fig. 5. Bar subjected to an eccentric compressive force: (a) isometric plots of $\chi_1(\lambda) = \Re(\chi_1(\lambda)) + i\Im(\chi_1(\lambda))$ and $\chi_j(\lambda) = \overline{\chi}_1(\lambda) = \Re(\chi_1(\lambda)) - i\Im(\chi_1(\lambda))$, obtained with Abaqus hybrid elements B32OSH, (b) projections of these plots onto the $i\Im(\chi_1) = i\Im(\chi_j) = 0$ plane, adapted from Fig. 6 in [2].

B32OSH [9]. P_S denotes the load level at the start of flexural–torsional buckling of the beam. The curvature of this curve is negative. This indicates a *superlinear* decrease of the stiffness of the structure in the prebuckling region. (For the limiting case *pure bending*, $\chi_1(\lambda)$ would be a straight line with a negative slope, see the chord $\overline{1\,S}$ in Fig. 6, indicating a *linear* stiffness decrease.) In constrast to it, the curvature of the $\chi_1 - P$ diagram in Fig. 3 and Fig. 5 is positive for $0 \leq \lambda < \lambda_I$ and $0 \leq \lambda < \lambda_L$, respectively. This indicates a *sublinear* increase and decrease, respectively, of the stiffness of the structure concerned before the load level of maximum stiffness at point I in Fig. 3 and the one of minimum stiffness at point I in Fig. 5, respectively, is reached. Consequently, $\delta^3 \Pi = 0$ is the varitional condition for the special case of a linear stiffness change. It correlates with a linear eigenvalue function, resulting from degeneration of the first one of the two coefficient matrices (24) of the linear eigenvalue problem (22) to a matrix with linear cofficients. The red curve in Fig. 6(b) shows the $\widehat{\chi}_1 - P$ diagram, obtained from accompanying linear eigenvalue analysis by means of the linear eigenvalue problem (41), with Abaqus displacement elements B32OS [9]. The hybrid finite elements B32OSH are extensions of the displacement elements B32OS [9]. The curvature of the $\widehat{\chi}_1 - P$ diagram is positive, as is the case with the curvatures of the $\widehat{\chi}_1 - P$ diagrams of the two previous examples, shown in [1,2]. None of these two diagrams contains a point of inflection. Thus, standard displacement finite elements are unsuitable for identification of the load level at an extreme value of the stiffness. On the other hand, as follows from Fig. 6, hybrid finite elements are not needed if the goal of accompanying linear eigenvalue analysis is restricted to determination of the buckling load of a proportionally loaded structure. For this purpose, however, accompanying linear eigenvalue analysis is an outdated method.

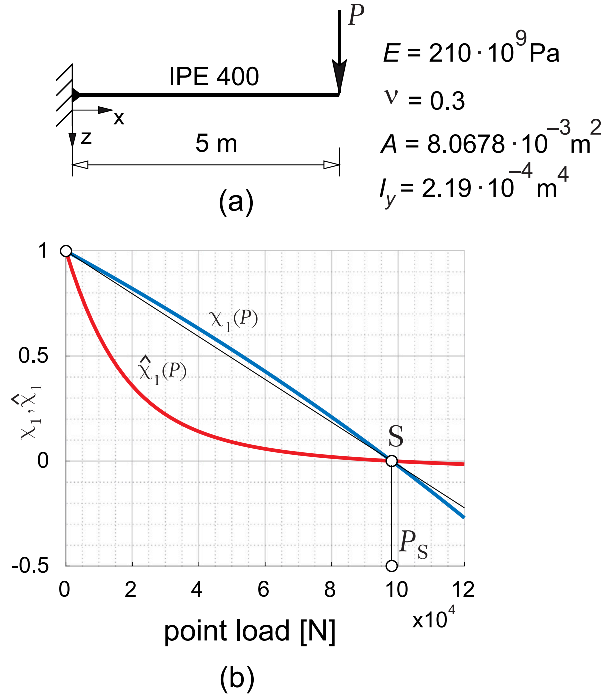


Fig. 6. Cantilever beam subjected to a point load at its free end: (a) structure, (b) $\chi_1 - P$ diagram and $\widehat{\chi}_1 - P$ diagram obtained with Abaqus hybrid elements B32OSH and Abaqus displacement elements B32OS, respectively, adapted from Fig. 3 in [2].

4. Conclusions

The presented variational criteria for extreme values of the stiffness of proportionally loaded structures were shown to be the scientific basis of previously reported numerical results for stiffness maxima and minima, obtained by the FEM. The existence of subsidiary conditions as elements of these criteria explains the necessity of a special linear eigenvalue problem as the tool for accompanying linear eigenvalue analysis by this method. Its mathematical formulation contains constraint conditions for the components of one of the two subvectors of the underlying linear eigenvalue problem. The positive semidefiniteness of the underlying functional at the load level of a stiffness maximum correlates with a point of inflection on a real eigenvalue curve, obtained from this eigenvalue problem. In contrast to it, the negative semidefiniteness of this functional at the load level of minimum stiffness correlates with a point of inflection on the real part of conjugate complex eigenvalue curves. They are restricted to a section between two sections with two different real eigenvalue curves. This explains the requirement of a linear eigenvalue problem with two indefinite coefficient matrices, representing a necessary condition for complex eigenvalues. At the lower and the upper bound of the load interval, inside which the stiffness becomes a minimum value, the underlying functional vanishes for all admissible variations of the state variables. Inside this interval the real part of two conjugate complex eigenvalues is the quantity that has been relevant to the present work. The existence of such a subregion, inside which buckling is impossible, reflects the tendency of the mechanical behavior of the respective structure to postpone the possibility of loss of its stability to a higher load level. In other words, the transition of an originally softening to a stiffening structure cannot once and for all preclude the possibility of buckling. After all, buckling is not restricted to softening structures.

The similarity of the criteria for maximum and minimum values of the stiffness of proportionally loaded structures manifests itself, on the one hand, in the form of the underlying functional and, on the other hand, in points of inflection on special eigenvalue curves. The difference between these criteria is reflected, on the one hand, in the sign of this functional and, on the other hand, in the nature of the point of inflection as regards its location either on a real eigenvalue curve or on the real part of partially conjugate complex eigenvalue curves as solution of a special FEM-based linear eigenvalue problem. The significance of the presented variational criteria for extreme values of the stiffness of proportionally loaded structures is based on their sheer existence as *conditiones sine qua non* for conditions for stiffness maxima and minima in the framework of the FEM and on their mathematical form. They determine the kind of the novel non-standard linear eigenvalue problem that serves as the tool for the numerical computation of these extreme values. *In the sense of the solution of an inverse problem the presented variational criteria were shown to be the originally unknown cause of a plausible effect in the form of points of inflection on eigenvalue curves obtained from a special linear eigenvalue problem.*

These criteria are based on a physically motivated hypothetical assumption for the sign of the third variation of the potential energy for problems with extreme values of the stiffness. Since analytical solutions of such problems are inaccessible, verification of this assumption had to be restricted to checking the correlations of characteristic features of the derived variational criteria for extreme values of the stiffness of proportionally loaded structures with properties of the novel FEM-based linear eigenvalue problem, representing the starting point for the derivation of the numerical conditions for such extreme values.

Extreme values of the stiffness of proportionally loaded structures are important elements of their mechanical answer to the loading to which they are subjected. They provide valuable information about the mechanical behavior to be expected after their occurrence. In the authors' opinion, the absence of analytical criteria for stiffness maxima and minima has been a significant scientific void of the literature in the area of structural mechanics. The purpose of this paper was to fill this void.

List of symbols

Nomenclature		
Variable	Base	Explanation
b	L	Width of the rectangular cross-section of a segment of a circular arch, shown in Fig. 1
e	L	Eccentricity of the normal force, illustrated in Fig. 3(a)
h	L	Height of the rectangular cross-section of a segment of a circular arch, shown in Fig. 1
i	–	Imaginary unit, introduced in the description of Fig. 4
n	–	Number of degrees-of-freedom in the FE simulations, introduced in (25)
$\mathbf{r}$	various	Subvector of an eigenvector of the linear eigenvalue problem (22)
$\hat{\mathbf{r}}$	various	Eigenvector of the linear eigenvalue problem (41)
$\mathbf{t}$	various	Subvector of an eigenvector of the linear eigenvalue problem (22)
D_L	–	Point on the eigenvalue curves $\chi_1(\lambda)$ and $\chi_j(\lambda)$, characterized by $\chi_1(\lambda_L) = \chi_j(\lambda_L)$, shown first in Fig. 4
D_U	–	Point on the eigenvalue curves $\chi_1(\lambda)$ and $\chi_j(\lambda)$, characterized by $\chi_1(\lambda_U) = \chi_j(\lambda_U)$, shown first in Fig. 4
E	M L^{-1} T^{-2}	Modulus of elasticity
$\mathbf{G}$	various	Submatrix of one of the two coefficient matrices of the linear eigenvalue problem (22)
$\mathbf{G}_0$	various	$\mathbf{G}(\lambda = 0)$, submatrix of the other coefficient matrix of the linear eigenvalue problem (22)
I	–	Point of inflection of $\chi_1(\lambda)$, shown first in Fig. 2
I_1	–	Point of inflection of $\Re(\chi_1(\lambda))$, shown in Fig. 4
I_j	–	Point of inflection of $\Re(\chi_j(\lambda))$, shown in Fig. 4
$\mathbf{K}$	various	Tangent stiffness matrix, representing a submatrix of one of the two coefficient matrices of the linear eigenvalue problem (22)
$\mathbf{K}_0$	various	$\mathbf{K}(\lambda = 0)$, submatrix of the other coefficient matrix of the linear eigenvalue problem (22)
$\hat{\mathbf{K}}$	various	Tangent stiffness matrix obtained with displacement elements
$\hat{K}_0$	various	$\hat{K}(\lambda = 0)$, coefficient of the linear eigenvalue problem (41)
L	L	Length of a bar, introduced in Fig. 3
$\overline{P}$	M L T^{-2}	Reference load, introduced first in Fig. 2
R	L	Radius of a segment of a circular arch, shown in Fig. 2
S	–	Subscript standing for “stability limit”
T	–	Superscript standing for “transpose”
U	M L T^{-2}	Strain energy
U_M	M L T^{-2}	Contribution of bending, shear, and torsion to U
u	L	Tangential displacement, introduced in Fig. 4(a)
v	L	Normal displacement, introduced in Figs. 4(a), (b), and (c)

w	L	Displacement component normal to u and v, shown in Fig. 4(b)
x	L	Component of the co-ordinate system, shown in Figs. 1 and 4(a)
y	L	Component of the co-ordinate system, shown in Figs. 1 and 4(a)
z	L	Component of the co-ordinate system, shown in Fig. 4(b)
$\Delta\Pi$	M L^2 T^{-2}	Change (Δ) of the potential energy (Π) due to admissible variations of the state variables at a certain load level, introduced in (1)
Λ	–	Lagrange multiplier, introduced in (16)
$\delta\Pi$	M L^2 T^{-2}	Term in the expression for $\Delta\Pi$ that is linear in admissible variations of the state variables, introduced in (1)
$\delta^2\Pi, \delta^3\Pi, \delta^4\Pi$	M L^2 T^{-2}	Ingredient of the term in the expression for $\Delta\Pi$ that is quadratic, cubic, and quartic, respectively, in admissible variations of the state variables, introduced in (1)
κ	–	Shear coefficient, introduced in Fig. 2
λ	–	Proportionality factor of reference forces, shown in Figs. 1, 2, 3,4(c), and 5
λ_L	–	Lower bound of the complex region of χ_1 and $\chi_j = \overline{\chi}_1$, introduced in (19)
λ_S	–	Value of λ at the stability limit S, introduced in (2)
λ_U	–	Upper bound of the complex region of χ_1 and $\chi_j = \overline{\chi}_1$, introduced in (19)
μ	–	Read angle of a segment of a circular arch, shown in Fig. 2
ν	–	Poisson's ratio
χ	–	Eigenvalue of the linear eigenvalue problem (22)
χ_1	–	Fundamental eigenvalue of the linear eigenvalue problem (22), introduced in (37)
$\overline{\chi}_1$	–	$\overline{\chi}_1 = \chi_j$, fundamental conjugate complex eigenvalue in the complex region of χ_1 and χ_j; χ_j is shown first in Fig. 5
$\hat{\chi}$	–	Eigenvalue of the linear eigenvalue problem (41)
$\Re(\mathbf{r})$	various	Real part of the subvector **r** of the eigenvector, introduced in (50)
$\Re(\mathbf{t})$	various	Real part of the subvector **t** of the eigenvector, introduced in (50)
$\Re(\chi_1)$	–	Real part of the complex eigenvalue χ_1 and of the conjugate complex eigenvalue $\chi_j = \overline{\chi}_1$, shown in Fig. 5
$\Im(\mathbf{r})$	various	Imaginary part of the subvector **r** of the eigenvector, introduced in (50)
$\Im(\mathbf{t})$	various	Imaginary part of the subvector **t** of the eigenvector, introduced in (50)
$\Im(\chi_1)$	–	Imaginary part of the complex eigenvalue χ_1, shown in Fig. 5
$\dot{}$	–	Symbol for $d/d\lambda$, introduced in (29)
$\mid\ \mid$	–	Symbol for a determinant

CRediT authorship contribution statement

Herbert A. Mang: Conception, theory, writing of original draft. **Mehdi Aminbaghai:** Writing of further drafts, reviewing and editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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