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DIPLOMARBEIT

Pressure Robust Discretizations for Navier Stokes Equations: Divergence-free Reconstruction for Taylor-Hood Elements and High Order Hybrid Discontinuous Galerkin Methods

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Abstract

This thesis focuses on a well-known issue of discretization techniques for solving the incompressible Navier Stokes equations. Due to a weak treatment of the incompressibility constraint there are different disadvantages that appear, which can have a major impact on the convergence and physical behaviour of the solutions. First we approximate the equations with a well-known pair of elements and introduce an operator that creates a reconstruction into a proper space to fix the mentioned problems.
Afterwards we use an $H(\mathrm{div})$ conforming method that already handles the incompressibility constraint in a proper way. For a stable high order approximation an estimation for the saddlepoint structure of the Stokes equations is needed, known as the Ladyschenskaja-Babuška-Brezzi (LBB) condition. The independency of the estimation from the order of the polynomial degree is shown in this thesis. For that we introduce an H^2-stable extension that preserves polynomials.
All operators and schemes are implemented based on the finite element library Netgen/NGSolve and tested with proper examples.

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Contents

1 Introduction

Computational fluid dynamics, or as commonly abbreviated CFD, is a huge part in the field of numerical mathematics and engineering. For a long time, scientists and engineers have been trying to find a way to describe the motion of fluids and gases. Using the basic rules of physics, namely Newton's laws, one can derive different models for all kinds of fluids. This thesis considers the incompressible Navier Stokes equations which describe a flow of an incompressible fluid by the velocity u and the pressure p:

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \end{aligned}$$

with suitable initial and boundary conditions. Due to the nonlinear structure and the incompressibility constraint this set of partial differential equations can not always be solved in an explicit form, and a discretization technique has to be used. The method considered in this thesis is a mixed finite element approximation, therefore the approaching derivations are treated in a weak sense. Although this works for a large amount of problems it may happen that the resulting solutions are affected with a large error and show a non physical behaviour. The occurring problems can be distinguished when we look at the error estimation of the mixed problem:

$$\|u - u_h\|_V \preccurlyeq \inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q .$$

We observe that the error of the velocity not only depends on the best approximation of the velocity, but also on the best approximation of the pressure. A *pressure robust* scheme, as it is mentioned in [Lin13], overcomes this dependency and also results in proper physical solutions. In this thesis we introduce pressure robust methods with an optimal convergence order.

Outline of the thesis:

In the first two chapters we focus on the derivation of the Navier Stokes equations and post some known results and approximation properties. We are going to distinguish the main causes for the resulting lack that we mentioned above and try to understand how we can solve this problem.

In chapter three we introduce two versions of a reconstruction operator for the well-known Taylor-Hood element. This operator corrects the divergence of the solution by equilibrating the error in a proper space. We show that this reconstruction does not affect the convergence order of the method, thus an optimal error estimation is provided.

We finish this chapter with some numerical examples, including steady and unsteady flows and a two phase Stokes example.

In the fourth chapter we consider a discretization that was already introduced in a hybridised version of the method from Cockburn, Kanschat and Schötzau (see [CKS05]) by Christoph Lehrenfeld and Joachim Schöberl in [LS15], namely a high order hybrid discontinuous Galerkin ansatz. This mixed method fulfills the properties that lead to a proper physical description and an independent velocity error. Still, for a stable and optimal convergence it remains to show the independency of the LBB-constant β of the polynomial degree, so $\beta \neq \beta(k)$. For this proof we need an H^2-continuous extension that preserves polynomials. Under this assumption we show the k-robust LBB-condition and finish the chapter with a numerical example, namely an unsteady laminar flow around a cylinder.

The aim of the last chapter is to show the existence of such an H^2-continuous extension. We split the result in three theorems. In the first step we show the existence of an extension for fixed values of the tangential gradient on the boundary. The second theorem is used to *weakly* correct the values of normal derivative under certain assumptions of the input. Finally we close the statement by showing that the error due to the assumptions of the second step is *small* enough.

Implementation:

All numerical examples were implemented and tested in the finite element library Netgen/NGSolve, see [Sch97] and [J.14].

Notation:

In this thesis we stick to the following notation:

Ω	bounded subdomain of $\mathbb{R}^2$ or $\mathbb{R}^3$
$\mathcal{T}$	quasi uniform mesh on Ω
h	global element size of $\mathcal{T}$
$\hat{T}$	reference element
ω_V	vertex path
ω_T	element path
$\Pi^k(T)$	polynomials of order k on T
$\Pi^k(\mathcal{T})$	element wise polynomials of order k
$I_h^{\Pi^k}$	standard nodal interpolator
$I_h^{\mathrm{BDM}^k}$	BDM interpolator
$\Pi_h^{\mathcal{C}}$	Clement operator
$\Pi^{\mathcal{F}}$	Fortin operator
$\mathcal{P}_Q^{L^2}$	L^2 projection on a Hilbert space Q
$\mathcal{P}_{\mathbb{R}}^{L^2(\omega)}$	L^2 projection on constants on the domain ω
F	linear mapping from the element to $\hat{T}$
$\mathcal{P}$	Piola transformation
$\mathcal{C}$	covariant transformation
$\mathcal{R}_h^V$	reconstruction operator defined by a vertex equilibration
$\mathcal{R}_h^T$	reconstruction operator defined by an element equilibration

By $a \preccurlyeq b$ we mean that there exists a constant c independent of a, b, k, h such that $a \leq cb$

2 Navier Stokes equations

In this chapter we derive the incompressible Navier Stokes equations which are a system of non linear partial differential equations and describe the motion of a fluid in space and time. For the derivation we concern conservation properties derived from physics. We proceed as in [SS14], [SA08] and [Bra15].

2.1 Description model

A fluid can be described in two ways:

i. Using a **Lagrangian** model, one considers a fluid as an amount of particles and describes them with their trajectories and their streamlines. Using this description, one can determine the position in space for every particle at each point in time.

ii. Using an **Eulerian** model, one considers a fixed domain and describes the fluid by determining the velocity at each point in space for each point in time.

In fluid dynamics the Eulerian model is more common (see [SS14]) and is also used for this thesis, but we want to remind that both descriptions can always be transformed into the other one.

2.2 Physical quantities

In this thesis we assume a bounded fixed domain $\Omega \subset \mathbb{R}^d$ where $d = 2$ or 3, with $\partial\Omega = \Gamma_N \cup \Gamma_D$, where $\Gamma_D = \Gamma_{in} \cup \Gamma_{wall}$ is a Dirichlet boundary that describes either an inflow condition ($u(x,t) = u_{in}(x,t)$) or a no-slip condition on walls ($u(x,t) = 0$) and Γ_N the Neumann boundary where we assume a *do-nothing* outflow. For the time we define an interval $I_T = [0,T]$ with $T > 0$. The physical quantities that appear are

i. $\rho(x,t) \in C^1(\Omega \times I_T)$ ...Density of the fluid

ii. $u(x,t) \in [C^2(\Omega \times I_T)]^d$ ...Velocity of the fluid

iii. $p(x,t) \in C^1(\Omega \times I_T)$ ...Pressure of the fluid

iv. $f(x,t) \in [C^0(\Omega \times I_T)]^d$ ...Volume force that acts on the fluid

v. $\sigma(x,t) \in [C^1(\Omega \times I_T)]^{d\times d}$ ...Stress tensor on the surface

for all $x \in \Omega$ and $t \in I_T$.

2.3 Mass derivation

Due to the two models for a fluid we have to consider two different derivations. Let $b \in C^1$ be an arbitrary function then we define

i. $\frac{\partial b}{\partial t}$ ...The change of time one observes at a fixed point $x \in \Omega$

ii. $\frac{\mathrm{D}b}{\mathrm{D}t}$ ...The mass derivation which describes the change in time when one follows the particle.

Using those two descriptions we observe that in a fluid with the velocity u we get the equivalence (see [SA08])

$$\frac{\mathrm{D}b}{\mathrm{D}t} = \frac{\partial b}{\partial t} + (u \cdot \nabla)b. \tag{2.3.1}$$

2.4 Conservation of mass

Using the density ρ we can determine the mass of an arbitrary volume $V(t) \subset \Omega$ by

$$m := \int_{V(t)} \rho \,\mathrm{d}x \quad \forall t \in I_T.$$

The conservation of mass implies that

$$\frac{\mathrm{D}m}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t} \int_{V(t)} \rho \,\mathrm{d}x = 0.$$

As $V(t)$ is transported with velocity u in time, we have to be careful when we want to change the integral and the derivation. Using the mass derivation (2.3.1), the Reynolds transport theorem (see appendix, 7.1.1) and $V = V(0)$ we can show that

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_{V(t)} \rho \,\mathrm{d}x = \int_V \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \operatorname{div} u \,\mathrm{d}x = \int_V \frac{\partial \rho}{\partial t} + \operatorname{div} \rho u \,\mathrm{d}x = 0,$$

and thus, as V was arbitrary, we get the point wise equivalence

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0.$$

In this thesis we always assume a constant density in space and time, so we get the well-known incompressibility constraint for the velocity

$$\operatorname{div} u = 0. \tag{2.4.1}$$

2.5 Conservation of momentum

The first law of classical mechanics (Newton's first law) states that the rate of change of the momentum of a body is balanced by the forces applied on this body:

$$\frac{\mathrm{D}P}{\mathrm{D}t} = F.$$

Considering a fluid we can define the momentum of a body, thus a constant set of particles, by

$$P = \int_{V(t)} \rho u \, \mathrm{d}x.$$

Using the stress tensor σ on the boundary and the volume force f as acting forces on the fluid we get

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_{V(t)} \rho u \, \mathrm{d}x = \int_{\partial V(t)} \sigma \cdot n \, \mathrm{d}s + \int_{V(t)} \rho f \, \mathrm{d}x. \tag{2.5.1}$$

Similar to before, due to the the properties of the mass derivation and the transport theorem of Reynolds (7.1.1),

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_{V} \rho u \, \mathrm{d}x = \int_{V} \rho \frac{\mathrm{D}u}{\mathrm{D}t} \, \mathrm{d}x = \int_{V} \operatorname{div} \sigma + \rho f \, \mathrm{d}x, \tag{2.5.2}$$

and so, as V was arbitrary, we get the equations of the conservation of momentum

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = \operatorname{div} \sigma + \rho f.$$

2.6 Stokes fluid

To close the set of equations we have to derive a dependency of the stress tensor σ of the velocity u and the pressure p. For this thesis we consider a Newtonian and Stokes fluid which has the properties ([Bra15])

i. $\sigma = \sigma(\varepsilon(u))$ with $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$

ii. σ is homogeneous

iii. σ is isotropic

iv. If $\varepsilon(u) = 0$ then $\sigma_{ij} = -p\delta_{ij}$.

Using those properties we get

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = \mu \Delta u - \nabla p + \rho f,$$

or

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla \tilde{p} = f, \tag{2.6.1}$$

where μ is the dynamic viscosity, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity and $\tilde{p} = \frac{p}{\rho}$ is a scaled pressure.

Remark 1: From now on we always write p for the scaled pressure.

2.7 Navier Stokes equations

The incompressibility constraint (2.4.1) and the equations derived from the conservation of momentum (2.6.1) are called the *unsteady incompressible Navier Stokes equations*:

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p &= f && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= u_{in} && \text{on } \Gamma_{in} \\ u &= 0 && \text{on } \Gamma_{wall} \end{aligned} \tag{2.7.1}$$

2.8 Stokes equations

A characteristic number for fluid dynamics is the *Reynolds number* defined by

$$Re := \frac{UL}{\nu}, \tag{2.8.1}$$

where U and L are a characteristic velocity and length. This number can be used to measure the ratio between inertia and friction forces. In the case of a steady flow ($\partial u/\partial t = 0$) and a very small Reynolds number Re we can make an asymptotic expansion of the velocity to see that the convective term $(u \cdot \nabla)u$ in the Navier Stokes equations (2.7.1) vanishes ([Bra15]). We get the *steady Stokes equations*:

$$\begin{aligned} -\nu\Delta u + \nabla p &= f && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= u_{in} && \text{on } \Gamma_{in} \\ u &= 0 && \text{on } \Gamma_{wall} \end{aligned} \tag{2.8.2}$$

3 Discretization of the Stokes and the Navier Stokes problem

In this chapter we present discretization techniques for the steady Stokes problem (2.8.2) and introduce a time discretization for the unsteady Navier Stokes equations (2.7.1). We consider the weak formulation and analyze the problem to derive a condition that is necessary for a stable method. For the approximations we also study the properties of the error and identify a problem that arises using most of the standard methods. This leads us to the main aspects considered in chapter 4 and 5.

3.1 Weak formulation

3.1.1 Stokes problem

As the Stokes equations (2.8.2) can be seen as a set of partial differential equations for the velocity and the pressure we have two spaces for our solutions and test functions:

$$V := [H_0^1(\Omega)]^d \quad \text{and} \quad Q := L_0^2 = \{q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}x = 0\}. \tag{3.1.1}$$

We multiply the first equation of (2.8.2) with a test function $v \in V$ and the second equation with a test function $q \in Q$ and integrate over Ω to get

$$\begin{aligned} &\int_\Omega -\nu \Delta u v \, \mathrm{d}x + \int_\Omega \nabla p \; v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \\ &\int_\Omega \operatorname{div} u \; q \, \mathrm{d}x = 0. \end{aligned}$$

Next, assuming homogeneous boundary conditions $u = 0$ on $\partial\Omega$, we apply integration by part on the integrals of the first line

$$\int_\Omega -\nu \Delta u v \, \mathrm{d}x + \int_\Omega \nabla p \; v \, \mathrm{d}x = \int_\Omega \nu \nabla u : \nabla v \, \mathrm{d}x - \int_\Omega \operatorname{div} v \; p \, \mathrm{d}x,$$

and so by defining two bilinear forms

$$\begin{aligned} a(u, v) &:= \int_\Omega \nu \nabla u : \nabla v \, \mathrm{d}x \\ b(v, q) &:= \int_\Omega \operatorname{div} v \; q \, \mathrm{d}x, \end{aligned}$$

we get the variational formulation:

Problem 1: *Find $(u,p) \in V \times Q$ so that*

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v)_{L^2(\Omega)} && \forall v \in V \\ b(u,q) &= 0 && \forall q \in Q \end{aligned} \tag{3.1.2}$$

Remark 2: *To get a symmetric saddlepoint structure in the variational formulation the pressure p was scaled with a minus.*
Remark 3: *The space Q was chosen as for $v \in [H_0^1(\Omega)]^d$ and $q \in H^1(\Omega)$*

$$b(v,q) := \int_\Omega \operatorname{div} v \; q \; \mathrm{d}x = -\int_\Omega v \nabla q \; \mathrm{d}x,$$

so the bilinear form $b(v,q)$ does not change if we add a constant to q.

3.1.2 Convective term of the Navier Stokes equations

To discretize the Navier Stokes equations we also have to find a weak formulation including the term $(u \cdot \nabla u)$. For that we multiply the convective term with a test function $v \in V$ and integrate over Ω to get

$$c(u,u,v) := \int_\Omega (u \cdot \nabla) u \cdot v \; \mathrm{d}x.$$

By that we define the variational formulation

Problem 2: *Find $(u,p) \in V \times Q$ so that*

$$\begin{aligned} a(u,v) + b(v,p) + c(u,u,v) &= (f,v)_{L^2(\Omega)} && \forall v \in V \\ b(u,q) &= 0 && \forall q \in Q. \end{aligned} \tag{3.1.3}$$

A common approach to find a discretization of the Navier Stokes equations is to use this variational formulation as continuous basis, because the discretization properties of a convective term similar to $(b \cdot \nabla)u$, for an arbitrary but fixed function b, is well-known. A different approach is the following. We take a closer look on $c(u,u,v)$:

$$\begin{aligned} \int_\Omega (u \cdot \nabla) u \cdot v \; \mathrm{d}x &= \int_\Omega \sum_{i,j=1}^{d} u_i (\frac{\partial}{\partial x_i} u_j) v_j \; \mathrm{d}x \\ &= \underbrace{\int_\Omega \sum_{i,j=1}^{d} u_i (\frac{\partial}{\partial x_i} u_j) v_j \; \mathrm{d}x - \int_\Omega \sum_{i,j=1}^{d} u_i (\frac{\partial}{\partial x_j} u_i) v_j \; \mathrm{d}x}_{B} \\ &\quad + \underbrace{\int_\Omega \sum_{i,j=1}^{d} u_i (\frac{\partial}{\partial x_j} u_i) v_j \; \mathrm{d}x}_{A}. \end{aligned}$$

Using the product rule on one component

$$u_i(\frac{\partial}{\partial x_j}u_i)v_j = \frac{1}{2}\frac{\partial}{\partial x_j}(u_i)^2 v_j,$$

we can write A as

$$\int_\Omega \sum_{i,j=1}^{d} u_i(\frac{\partial}{\partial x_j}u_i)v_j \ \mathrm{d}x = \int_\Omega \sum_{i,j=1}^{d} \frac{1}{2}\frac{\partial}{\partial x_j}(u_i)^2 v_j \ \mathrm{d}x = \frac{1}{2}\int_\Omega \nabla(u^2)\cdot v \ \mathrm{d}x,$$

where $u^2 = u \cdot u$. Next we integrate by part, to get

$$\frac{1}{2}\int_\Omega \nabla(u^2)\cdot v \ \mathrm{d}x = \frac{1}{2}\int_\Omega (u^2)\operatorname{div} v \ \mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}(u^2)v\cdot n \ \mathrm{d}s.$$

For B we observe that in the case of $i = j$ the terms vanish, and so by using the outer product (for $d = 3$)

$$(u \otimes \nabla) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

and

$$\begin{aligned} u\otimes v - v\otimes u &= \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix} - \begin{pmatrix} v_1u_1 & v_1u_2 & v_1u_3 \\ v_2u_1 & v_2u_2 & v_2u_3 \\ v_3u_1 & v_3u_2 & v_3u_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & u_1v_2 - v_1u_2 & u_1v_3 - v_1u_3 \\ u_2v_1 - v_2u_1 & 0 & u_2v_3 - v_2u_3 \\ u_3v_1 - v_3u_1 & u_3v_2 - v_3u_2 & 0 \end{pmatrix}, \end{aligned}$$

we can write B also as

$$\int_\Omega \sum_{i,j=1}^{d} u_i(\frac{\partial}{\partial x_i}u_j)v_j - u_i(\frac{\partial}{\partial x_j}u_i)v_j \ \mathrm{d}x = \int_\Omega -(u\otimes\nabla) : [u\otimes v - v\otimes u] \ \mathrm{d}x,$$

or

$$\int_\Omega -(u\otimes\nabla) : [u\otimes v - v\otimes u] \ \mathrm{d}x = \int_\Omega (\nabla\times u)\cdot(u\times v) \ \mathrm{d}x.$$

Using the identity $a\cdot(b\times c) = c\cdot(a\times b) = (a\times b)\cdot c$ and $\nabla\times u = \operatorname{curl} u$ we get

$$\int_\Omega (\nabla\times u)\cdot(u\times v) \ \mathrm{d}x = \int_\Omega (\operatorname{curl} u\times u)\cdot v \ \mathrm{d}x,$$

and so

$$\int_\Omega (u\cdot\nabla)u\cdot v \ \mathrm{d}x = \int_\Omega (\operatorname{curl} u\times u)\cdot v \ \mathrm{d}x + \frac{1}{2}\int_\Omega (u^2)\operatorname{div} v \ \mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}(u^2)v\cdot n \ \mathrm{d}s.$$

Defining

$$c_{curl}(u,u,v) := \int_\Omega (\text{curl } u \times u) \cdot v \, \mathrm{d}x,$$

we get the variational formulation

Problem 3: *Find* $(u,p) \in V \times Q$ *so that*

$$\begin{aligned} a(u,v) + b(v, p + 1/2u^2) + c_{curl}(u,u,v) &= (f,v)_{L^2(\Omega)} \quad && \forall v \in V \\ b(u,q) &= 0 && \forall q \in Q. \end{aligned} \tag{3.1.4}$$

Remark 4: In this thesis we call (3.1.4) the curl formulation of the Navier Stokes equations.

Remark 5: Due to $b(v, p + 1/2u^2)$ *the pressure* p *is scaled with* $1/2u^2$*. We call this scaled pressure* $p_b := p + 1/2u^2$ *the Bernoulli pressure.*

Remark 6: As we defined the variational problem with homogeneous Dirichlet boundary conditions, the boundary integral $\int_{\partial\Omega}(u^2)v \cdot n$ *vanishes. When we want to solve a domain including Neumann boundaries, we have to include this integral on the right hand side of the variational formulation.*

3.2 Analysis of the saddle point problem

3.2.1 Abstract theory

A mixed variational formulation implies two Hilbert spaces V and Q, bilinear forms

$$\begin{aligned} a(u,v) &: V \times V \to \mathbb{R}, \\ b(u,q) &: V \times Q \to \mathbb{R} \end{aligned}$$

and continuous linear-forms

$$\begin{aligned} f(v) &: V \to \mathbb{R}, \\ g(q) &: Q \to \mathbb{R}. \end{aligned}$$

The problem is to find $u \in V$ and $p \in Q$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= f(v) \quad \forall v \in V \\ b(u,q) &= g(q) \quad \forall q \in Q. \end{aligned}$$

One can also add up the two lines and define the bilinear form $B(\cdot,\cdot) : (V \times Q) \times (V \times Q) \to \mathbb{R}$ by

$$B((u,p),(v,q)) = a(u,v) + b(u,q) + b(v,p), \tag{3.2.1}$$

to write the mixed method as a single variational problem:

$$\text{Find } (u,p) \in V \times Q : \quad B((u,p),(v,q)) = f(v) + g(q) \quad \forall (v,q) \in V \times Q, \tag{3.2.2}$$

see [Sch09].

3.2.2 LBB-condition and the Brezzi theorem

To guarantee a stable and unique solution for the mixed problem we introduce the theorem of Brezzi including the LBB-condition named after Olga Alexandrowna Ladyschenskaja, Ivo Babuška and Franco Brezzi.

Theorem 3.1 (Brezzi's theorem). *Assume that* $a(\cdot,\cdot)$ *and* $b(\cdot,\cdot)$ *are continuous bilinear forms*

$$\begin{aligned} a(u,v) &\leq \alpha_2 \|u\|_V \|v\|_V \quad \forall u,v \in V, \\ b(v,q) &\leq \beta_2 \|v\|_V \|q\|_Q \quad \forall v \in V, \forall q \in Q. \end{aligned}$$

Assume $a(.,.)$ *is coercive on the kernel, i.e.,*

$$a(u,u) \geq \alpha_1 \|u\|_V^2 \quad \forall u \in V_0$$

with $V_0 := \{v \in V : b(v,q) = 0 \ \forall q \in Q\}$*, and the LBB-condition is fulfilled*

$$\sup_{v \in V} \frac{b(v,q)}{\|v\|_V} \geq \beta_1 \|q\|_Q \quad \forall q \in Q. \tag{3.2.3}$$

Then the mixed problem (3.2.2) is uniquely solvable, and the solution fullfills the stability estimation

$$\|u\|_V + \|p\|_Q \leq c\{\|f\|_{V^*} + \|g\|_{Q^*}\},$$

with the constant c *depending on* $\alpha_1, \alpha_2, \beta_1, \beta_2$*.*

From the Brezzi theorem it also follows that the bilinear form $B(\cdot,\cdot)$ fulfills the inf-sup condition

$$\inf_{\substack{v,q \in V \times Q \\ v,q \neq 0}} \sup_{\substack{u,p \in V \times Q \\ u,p \neq 0}} \frac{B((u,p),(v,q))}{(\|v\|_V + \|q\|_Q)(\|u\|_V + \|p\|_Q)} \geq \beta.$$

See [Sch09].

3.2.3 Analysis of the Stokes problem

We now want to use the Brezzi theorem for the Stokes problem. For the analysis we use the norms

$$\begin{aligned} \|\cdot\|_V &:= \sqrt{\nu} \|\cdot\|_{H^1(\Omega)}, \\ \|\cdot\|_Q &:= \frac{1}{\sqrt{\nu}} \|\cdot\|_{L^2(\Omega)}. \end{aligned}$$

We already defined the bilinear forms for the weak formulation (3.1.2), so by using the Cauchy Schwarz inequality we see

$$a(u,v) = \int_\Omega \nu \nabla u \nabla v \ \mathrm{d}x \overset{\text{C.S.}}{\leq} \|u\|_V \|v\|_V ,$$
$$b(v,q) = \int_\Omega \operatorname{div} v \ q \ \mathrm{d}x \overset{\text{C.S.}}{\leq} \|\operatorname{div} v\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)} \leq \|v\|_V \|q\|_Q .$$

On the kernel $V_0 := \{v \in V : \int_\Omega \operatorname{div} v \ q \ \mathrm{d}x = 0 \ \forall q \in Q\}$ we observe coercivity by using the Poincare inequality (see appendix, theorem 7.18)

$$a(u,u) \geq \alpha_1 \|u\|_V^2 , \tag{3.2.4}$$

where α_1 depends on the shape of Ω. The LBB-condition was first shown by Nečas by proving an equivalent inequality for the dual operator of the divergence, namely the gradient. We observe that $\nabla : L^2(\Omega) \to [H^-1(\Omega)]^d$ has a closed range and Nečas ([Neč67]) showed that

$$\|q\|_{L^2(\Omega} \leq c(\Omega) \|\nabla q\|_{H^{-1}(\Omega)} \quad \forall q \in Q.$$

By that ([Bra00][154]) we get

$$\sup_{v \in V} \frac{\int_\Omega \operatorname{div} v \ q \ \mathrm{d}x}{\|v\|_V} \geq \beta_1 \|q\|_Q \quad \forall q \in Q. \tag{3.2.5}$$

Using Brezzi's theorem we find stable and unique solutions u and p of the steady Stokes problem.

3.3 Approximation of the Stokes problem

In this section we want to take a closer look on the approximation of the saddle point problem. Note that the discrete version of the LBB-condition can not be derived from the continuous one (3.2.5).

3.3.1 Basic results

We define the finite-dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$. The h refers to a quasi-uniform triangulation $\mathcal{T}$ (see appendix) which these approximation spaces are derived from. The discrete variational formulation is

Problem 4: *Find* $(u_h, p_h) \in V_h \times Q_h$ *so that*

$$\begin{aligned} a(u_h, v_h) + b(v_h, p_h) &= (f, v_h)_{L^2(\Omega)} && \forall v_h \in V_h \\ b(u_h, q_h) &= 0 && \forall q_h \in Q_h. \end{aligned} \tag{3.3.1}$$

The continuity results of the bilinear forms follow from the infinite dimensional case and the coercivity on $V_{h,0} := \{v_h \in V_h : \int_\Omega \operatorname{div} v_h \; q_h \; \mathrm{d}x = 0 \; \forall q_h \in Q_h\}$ again can be derived from the Poincare inequality. The discrete LBB-condition

$$\sup_{v_h \in V_h} \frac{\int_\Omega \operatorname{div} v_h \; q_h \; \mathrm{d}x}{\|v_h\|_V} \geq \beta_1 \|q_h\|_Q \quad \forall q_h \in Q_h \tag{3.3.2}$$

is the constraint that arises for the definitions of the approximation spaces. For example, it is not possible to use $V_h := [\Pi^2(\Omega)]^2$ and $Q_h := \Pi^2(\Omega)$ (for $d = 2$). In this thesis we use different couplings for V_h and Q_h. In section 3.4 we present two standard elements and in chapter 5 we use a hybrid discontinuous Galerkin ansatz introduced by Christoph Lehrenfeld and Joachim Schöberl ([LS15] and [Leh10]).

3.3.2 Error analysis

Due to an approximation we are always interested in the error $\|u - u_h\|_V$ and $\|p - p_h\|_Q$. We show different results similar to [BF91].

Remark 7: In this section we always refer on the constants that appear in Brezzi's theorem 3.1 considered for the discrete Stokes problem (3.3.1). As mentioned α_1 depends on Ω and $\alpha_2 = 1$, but also $\beta_2 = 1$ as the scaling with ν is hidden in $\|\cdot\|_Q$.

Proposition 1. *Let (u, p) be the solution of (3.1.2) and (u_h, p_h) be the solution of (3.3.1). Then we have*

$$\|u - u_h\|_V \leq \left(1 + \frac{\alpha_2}{\alpha_1}\right) \inf_{v_h \in V_{h,0}} \|u - v_h\|_V + \frac{\beta_2}{\alpha_1} \inf_{q_h \in Q_h} \|p - q_h\|_Q . \tag{3.3.3}$$

Proof. Let v_h be an arbitrary element of V_h. As $v_h - u_h \in V_{h,0}$ we have (coercivity)

$$\alpha_1 \|v_h - u_h\|_V^2 \leq a(v_h - u_h, v_h - u_h) = a(v_h - u, v_h - u_h) + a(u - u_h, v_h - u_h).$$

For the second term we observe

$$\begin{aligned} a(u - u_h, v_h - u_h) &= a(u, v_h - u_h) - a(u_h, v_h - u_h) \\ &= f(v_h - u_h) - b(v_h - u_h, p) - f(v_h - u_h) - b(v_h - u_h, p_h) \\ &= -b(v_h - u_h, p - p_h), \end{aligned}$$

and so

$$\begin{aligned} \alpha_1 \|v_h - u_h\|_V^2 &\leq a(v_h - u, v_h - u_h) - b(v_h - u_h, p - p_h) \\ &\leq \alpha_2 \|v_h - u\|_V \|v_h - u_h\|_V + \beta_2 \|v_h - u_h\|_V \|p - p_h\|_Q , \end{aligned}$$

and

$$\|v_h - u_h\|_V \leq \frac{\alpha_2}{\alpha_1} \|u - v_h\|_V + \frac{\beta_2}{\alpha_1} \|p - p_h\|_Q .$$

Using the triangle inequality

$$\|u - u_h\|_V \le \|u - v_h\|_V + \|v_h - u_h\|_V$$

we get the result. □

Until now we only have an estimation on the manifold V_{h_0} of discrete divergence-free test functions. For the error estimation on the full space V_h we show

Proposition 2. *Let (u,p) be the solution of (3.1.2) and (u_h, p_h) be the solution of (3.3.1). Assume that β_1 is the constant of the discrete LBB-condition (3.3.2). Then we have*

$$\inf_{v_h \in V_{h,0}} \|u - v_h\|_V \le \left(1 + \frac{\beta_2}{\beta_1}\right) \inf_{v_h \in V_h} \|u - v_h\|_V \, .$$

Proof. Let w_h be an arbitrary element of V_h. First we observe that the discrete LBB-condition is equivalent to the existence of a solution of the variational problem

Problem 5: *For all $g_h \in Q_h$ find $u_h \in V_h$ so that*

$$b(u_h, q_h) = (g_h, q_h)_{L^2(\Omega)} \quad \forall q_h \in Q_h \tag{3.3.4}$$

$$\|u_h\|_V \le \frac{1}{\beta_1} \|g_h\|_{Q*} \, . \tag{3.3.5}$$

We solve the equation

$$b(r_h, q_h) = (u - w_h, q_h) \quad \forall q_h \in Q_h, \tag{3.3.6}$$

and get

$$\|r_h\|_V \le \frac{1}{\beta_1} \|u - w_h\|_{Q*} = \frac{1}{\beta_1} \sup_{q_h \in Q_h} \frac{b(u - w_h, q_h)}{\|q_h\|_Q} \le \frac{\beta_2}{\beta_1} \|u - w_h\|_V \, .$$

Next we define $v_h := r_h + w_h$, and as

$$\begin{aligned} b(v_h, q_h) &= b(r_h, q_h) + b(w_h, q_h) \\ &= b(u, q_h) - b(w_h, q_h) + b(w_h, q_h) = 0, \end{aligned}$$

the test function v_h is an element of the kernel $V_{h,0}$. Finally we get

$$\|u - v_h\|_V = \|u - w_h - r_h\|_V \le \|u - w_h\|_V + \|r_h\|_V \le \left(1 + \frac{\beta_2}{\beta_1}\right) \|u - w_h\|_V \, .$$

□

So all together we get

Corollary 1. *Let (u,p) be the solution of (3.1.2) and (u_h,p_h) be the solution of (3.3.1). Assume that β_1 is the constant of the discrete LBB-condition, then*

$$\|u-u_h\|_V \leq \left(1+\frac{\alpha_2}{\alpha_1}\right)\left(1+\frac{\beta_2}{\beta_1}\right)\inf_{v_h\in V_h}\|u-v_h\|_V + \frac{\beta_2}{\alpha_1}\inf_{q_h\in Q_h}\|p-q_h\|_Q\,. \tag{3.3.7}$$

Two major aspects can be indicated looking at this error estimations. First of all, we observe that the error of the velocity approximation depends also on the solution of the pressure discretization, and secondly we notice that we can not guarantee a small error when the constant of the discrete LBB-condition tends to zero. Both problems can be eliminated by using proper spaces and a Fortin operator, respectively. Before we continue examining cases where we can improve the approximation error for the velocity we also bound the pressure error.

Proposition 3. *Let (u,p) be the solution of (3.1.2) and (u_h,p_h) be the solution of (3.3.1). Assume that β_1 is the constant of the discrete LBB-condition, then*

$$\|p-p_h\|_Q \leq (1+\frac{\beta_2}{\beta_1})\inf_{q_h\in Q_h}\|p-q_h\|_Q + \frac{\alpha_2}{\beta_1}\|u-u_h\|_V\,.$$

Proof. For $v=v_h$ and $q=q_h$ subtract equation (3.3.1) from (3.1.2) to get

$$a(u-u_h,v_h)+b(v_h,p-p_h)=0,$$

or

$$b(v_h,q_h-p_h)=-a(u-u_h,v_h)-b(v_h,p-q_h).$$

Using the discrete LBB-condition and the continuity of the bilinear forms we get

$$\begin{aligned}\|q_h-p_h\|_Q &\leq \frac{1}{\beta_1}\frac{b(v_h,q_h-p_h)}{\|v_h\|_V} = \frac{1}{\beta_1}\frac{-a(u-u_h,v_h)-b(v_h,p-q_h)}{\|v_h\|_V}\\ &\leq \frac{\alpha_2}{\beta_1}\|u-u_h\|_V + \frac{\beta_2}{\beta_1}\|p-q_h\|_Q\,.\end{aligned}$$

and thus

$$\|p-p_h\|_Q \leq (1+\frac{\beta_2}{\beta_1})\inf_{q_h\in Q_h}\|p-q_h\|_Q + \frac{\alpha_2}{\beta_1}\|u-u_h\|_V\,.$$

□

Discrete kernel subset property

Analysing the proof of proposition 1 we see that the error of the pressure arises in the continuity estimation of $b(v_h - u_h, p - p_h)$. We know that $v_h - u_h$ is an element of $V_{h,0}$, so the problem is that $b(v_h - u_h, p)$ does not have to be zero for every choice of V_h and Q_h. This is indeed the case when we have the *discrete kernel subset property*

$$V_{h,0} \subset V_0, \tag{3.3.8}$$

as in this case $b(v_h - u_h, p) = 0$.

Corollary 2. *Let (u, p) be the solution of (3.1.2) and (u_h, p_h) be the solution of (3.3.1), and assume that $V_{h,0} \subset V_h$. Then we have*

$$\|u - u_h\|_V \leq \left(1 + \frac{\alpha_2}{\alpha_1}\right) \inf_{v_h \in V_{h,0}} \|u - v_h\|_V \,. \tag{3.3.9}$$

Proof. Follows from proposition 1 and $b(v_h - u_h, p - p_h) = 0$. □

Fortin operator

As mentioned above the case $\beta_1 \to 0$ can cause a loss in precision and even a lack of convergence ([BF91][58]), so it is really important to study the behaviour of this constant. One approach is to use a Fortin operator $\Pi^{\mathcal{F}} : V \to V_h$ with the properties

$$\begin{aligned} &b(\Pi^{\mathcal{F}} u - u, q_h) = 0 \qquad \forall q_h \in Q_h \\ &\left\|\Pi^{\mathcal{F}} u\right\|_V \leq c \|u\|_V \,, \end{aligned} \tag{3.3.10}$$

with $c \neq c(h)$. If we have this properties we see that

$$\begin{aligned} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V} &\geq \sup_{v \in V} \frac{b(\Pi^{\mathcal{F}} v, q_h)}{\|\Pi^{\mathcal{F}} v\|_V} = \sup_{v \in V} \frac{b(v, q_h)}{\|\Pi^{\mathcal{F}} v\|_V} \\ &\geq \sup_{v \in V} \frac{b(v, q_h)}{c \|v\|_V} \geq \frac{\beta_1}{c} \|q\|_Q \,, \end{aligned}$$

and so the discrete LBB-condition follows from the continuous one. Using a Fortin operator also delivers an error estimation that is independent of the LBB constant β_1.

Corollary 3. *Let (u, p) be the solution of (3.1.2) and (u_h, p_h) be the solution of (3.3.1). Assume we have a Fortin operator $\Pi^{\mathcal{F}}$ with the properties (3.3.10), then we get*

$$\|u - u_h\|_V \leq \left(1 + \frac{\alpha_2}{\alpha_1}\right) \left\|u - \Pi^{\mathcal{F}} u\right\|_V + \frac{\beta_2}{\alpha_1} \inf_{q_h \in Q_h} \|p - q_h\|_Q \,.$$

Proof. Follows from proposition 1 and the properties (3.3.10) of the operator $\Pi^{\mathcal{F}}$. □

The construction of such a Fortin operator is normally not intuitively but is often constructed in the following way. First one constructs an operator $\Pi_1^{\mathcal{F}}$, which delivers a "best approximation" with a certain continuity. After that a second operator $\Pi_2^{\mathcal{F}}$ provides local corrections to preserve the properties (3.3.10), so we have:

Proposition 4. *Assume $\Pi_1^{\mathcal{F}} : V \to V_h$ and $\Pi_2^{\mathcal{F}} : V \to V_h$ such that*

$$\left\|\Pi_1^{\mathcal{F}} u\right\|_V \leq c_1 \left\|u\right\|_V,$$

and for all $u \in V$

$$\begin{aligned} b(\Pi_2^{\mathcal{F}} u - u, q_h) = 0 \quad \forall q_h \in Q_h, \\ \left\|\Pi_2^{\mathcal{F}}(u - \Pi_1^{\mathcal{F}} u)\right\|_V \leq c_2 \left\|u\right\|_V, \end{aligned}$$

then for $\Pi^{\mathcal{F}} u := \Pi_2^{\mathcal{F}}(u - \Pi_1^{\mathcal{F}} u) + \Pi_1^{\mathcal{F}} u$ properties (3.3.10) are fulfilled with $c = c_1 + c_2$.

Proof. We observe

$$\begin{aligned} b(\Pi^{\mathcal{F}} u, q_h) &= b(\Pi_2^{\mathcal{F}}(u - \Pi_1^{\mathcal{F}} u), q_h) + b(\Pi_1^{\mathcal{F}} u, q_h) \\ &= b(u - \Pi_1^{\mathcal{F}} u, q_h) + b(\Pi_1^{\mathcal{F}} u, q_h) \\ &= b(u, q_h), \end{aligned}$$

and

$$\left\|\Pi^{\mathcal{F}} u\right\|_V \leq \left\|\Pi_2^{\mathcal{F}}(u - \Pi_1^{\mathcal{F}} u)\right\|_V + \left\|\Pi_1^{\mathcal{F}} u\right\|_V \leq (c_1 + c_2) \left\|u\right\|_V.$$

□

3.3.3 Exact divergence-free

In section 3.3.1 we defined the weak formulation (3.3.1). Therefore, if one can solve the discrete problem, the velocity u_h only fulfills

$$\int_\Omega \operatorname{div} u_h \; q_h \; \mathrm{d}x = 0 \quad \forall q_h \in Q_h,$$

which we call *discrete divergence-free*. In this section we want to analyze the problem under an even sharper assumption for the approximation. Assume we have the property that the divergence of a velocity test function v_h is an element of the pressure space Q_h, thus

$$\operatorname{div} V_h \subset Q_h. \tag{3.3.11}$$

Then we have that from discrete divergence-free follows exact divergence-free, namely

$$\int_\Omega \operatorname{div} v_h \; q_h \; \mathrm{d}x = 0 \quad \forall q_h \in Q_h \quad \Rightarrow \quad \operatorname{div} v_h = 0, \tag{3.3.12}$$

and so also for the solution u_h

$$\operatorname{div} u_h = 0.$$

We mention three advantages if this property is fulfilled.

Remark 8: As $\operatorname{div} u \in L^2(\Omega)$ *a point evaluation is not legit, so* $\operatorname{div} v_h = 0$ *is meant almost everywhere in* Ω *.*

Remark 9: There are examples, which fulfill (3.3.11), e.g. the Scott Vogelius element (see [BF91]), but they are in general computationally expensive. In chapter 5 we use a recent approach that has the property (3.3.11) and is also cheap to compute. We also want to mention that one could use isogeometric methods from isogeometric analysis to fulfill this property.

Discrete kernel subset property

With (3.3.11) it is clear that we have $V_{0,h} \subset V_0$ as

$$b(u_h, q) = \int_\Omega \underbrace{\operatorname{div} u_h}_{=0} q = 0 \quad \forall q \in Q,$$

and so due to corollary 2 we get a better approximation of the velocity.

Energy losses

Consider the unsteady Navier Stokes equations (2.7.1) with density $\rho = 1$, $\nu = 1$, no volume forces f and homogeneous Dirichlet boundary conditions. Due to the friction of the particles the kinetic energy should decrease in time, so

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{L^2(\Omega)} \leq 0.$$

Using (2.7.1) and integration by part we observe

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\|u^2\right\|^2_{L^2(\Omega)} &= \int_\Omega u u' \, \mathrm{d}x = \int_\Omega u \left(\nu \Delta u - (u \cdot \nabla) u + \nabla p\right) \, \mathrm{d}x \\ &= \int_\Omega -\nu \nabla u : \nabla u - (u \cdot \nabla) u \; u - \operatorname{div} u \; p \, \mathrm{d}x.\end{aligned}$$

The convective term can then be written as

$$\begin{aligned}\int_\Omega (u \cdot \nabla) u \; u \, \mathrm{d}x &= \int_\Omega \sum_{i=1,2,3} u_i \frac{\partial u}{\partial x_i} u \, \mathrm{d}x = \int_\Omega \sum_{i=1,2,3} u_i \frac{1}{2} \frac{\partial u^2}{\partial x_i} u^2 \, \mathrm{d}x \\ &= \frac{1}{2} \int_\Omega u \cdot \nabla u^2 \, \mathrm{d}x = \frac{1}{2} \int_\Omega \operatorname{div} u \; u^2 \, \mathrm{d}x,\end{aligned}$$

and, using the in compressibility constraint $\operatorname{div} u = 0$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\|u^2\right\|^2_{L^2(\Omega)} = \int_\Omega -\nu \nabla u : \nabla u \, \mathrm{d}x \leq 0.$$

The question is if this is still valid for the approximation of u_h:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_h^2\right\|_{L^2(\Omega)}^2 = \underbrace{\int_\Omega -\nu\nabla u_h : \nabla u_h \,\mathrm{d}x}_{\leq 0} - \frac{1}{2}\int_\Omega \operatorname{div} u_h \, u_h^2 \,\mathrm{d}x - \underbrace{\int_\Omega \operatorname{div} u_h \, p_h \,\mathrm{d}x}_{=0}.$$

Due to $u_h^2 \notin Q_h$, the convective part does not vanish as u_h is only discrete divergence-free, but if (3.3.11) is fulfilled we also observe the proper physical behaviour

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_h^2\right\|_{L^2(\Omega)}^2 = \int_\Omega -\nu\nabla u_h : \nabla u_h \,\mathrm{d}x \leq 0.$$

Helmholtz decomposition

The last point we want to mention is an algebraic property that arises from exact divergence-free functions and the resulting impact on the Stokes and Navier Stokes equations (see [Lin13]). First we observe that an arbitrary irrotational field $\nabla\phi$ and a divergence-free field ω with homogeneous boundary values are orthogonal with respect to the L^2 scalar product as

$$\int_\Omega \nabla\phi \,\omega \,\mathrm{d}x = -\int_\Omega \phi \operatorname{div}\omega \,\mathrm{d}x + \int_{\partial\Omega} \phi\, \omega\cdot n \,\mathrm{d}s = 0. \tag{3.3.13}$$

Next we define the orthogonal complement of V_0 due to the scalarproduct induced by the bilinear form $a(\cdot,\cdot)$

$$V_0^\perp := \{v \in V : a(v,w) = 0 \quad \forall w \in V_0\}.$$

By that and the orthogonality (3.3.13) we observe that the weak formulation of the Stokes equations (3.1.2) splits into two parts. We get the variational formulation:

Problem 6: *Find $(u,p) \in V_0 \times Q$ so that*

$$\int_\Omega \nu\nabla u : \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall v \in V_0, \tag{3.3.14}$$

and

$$-\int_\Omega \nabla p v = \int_\Omega f v \quad \forall v \in V_0^\perp.$$

Assume we use a gradient field for the right hand side, namely $f_\nabla := \nabla\phi$, then we observe that

$$\int_\Omega \nu\nabla u : \nabla v \,\mathrm{d}x = \int_\Omega f_\nabla v \,\mathrm{d}x = 0 \quad \forall v \in V_0,$$

and

$$-\int_\Omega \nabla p v = \int_\Omega f_\nabla v \,\mathrm{d}x \quad \forall v \in V_0^\perp,$$

and thus the solution is $(u = 0, p = \phi)$, so the irrotational force was completely balanced by the pressure. When we want to approximate this we observe that in the case of $\operatorname{div} V_h \subset Q_h$ the solution is given by $(u_h = 0, p_h = \mathcal{P}^{L^2}_{Q_h}\phi)$, where $\mathcal{P}^{L^2}_{Q_h}$ is the L^2 projection on Q_h. This can be seen as for all $v_h \in V_h$ we have

$$\int_\Omega \underbrace{\operatorname{div} v_h}_{\in Q_h} \mathcal{P}^{L^2}_{Q_h}\phi \,\mathrm{d}x = \int_\Omega \operatorname{div} v_h \phi = \int_\Omega \nabla\phi v_h,$$

and so $(u_h = 0, p_h = \mathcal{P}^{L^2}_{Q_h}\phi)$ solves (3.1.2) with $f = \nabla\phi$. In the case of using discrete divergence-free velocity fields this may not be fulfilled and we get a nonphysical velocity field $u_h \neq (0, 0)$.

Analyzing the curl formulation of the Navier Stokes equations (3.1.4) we recall the identity $(u \cdot \nabla)u = (\nabla \times u) \times u + \frac{1}{2}\nabla(u^2)$, and the resulting pressure $p + \frac{1}{2}u^2$. Again using only discrete divergence-free test functions results in a bad approximation due to the appearing gradient term $\nabla(u^2)$. In chapter 4 we introduce a reconstruction operator to solve this problem and compare the results of discrete and exact divergence-free ansatz spaces in an unsteady Navier Stokes example.

3.3.4 Aubin Nitsche technique for the Stokes problem

In this section we present a standard Aubin Nitsche technique to show that the convergence rate in the L^2 norm of the velocity is one order higher than in the H^1 norm. For this we first define the dual problem of the Stokes problem using the bilinear form (3.2.1):

Problem 7: *Find $(w, \lambda) \in V \times Q$ so that*

$$B((v, q),(w, \lambda)) = (f, v)_{L^2(\Omega)} \quad \forall v \in V, \forall q \in Q. \tag{3.3.15}$$

For the Aubin Nitsche technique we have to assume a regularity property of the solution, which is presented in [KO76] on proper domains.

Theorem 3.2. *Assume that the solution of the discrete Stokes problem (3.3.1) fulfills*

$$\|u - u_h\|_{H^1(\Omega)} \preccurlyeq h^k \|u\|_{H^{k+1}(\Omega)} \quad \text{and} \quad \|p - p_h\|_{L^2(\Omega)} \preccurlyeq h^k \|p\|_{H^k(\Omega)},$$

and that the dual problem fulfills an H^2/H^1 regularity, namely

$$\|w\|_{H^2(\Omega} \preccurlyeq \|f\|_{L^2(\Omega)} \quad \text{and} \quad \|\lambda\|_{H^1(\Omega} \preccurlyeq \|f\|_{L^2(\Omega)},$$

then we have the L^2 estimation

$$\|u - u_h\|_{L^2(\Omega)} \preccurlyeq h^{k+1} \|u\|_{H^{k+1}(\Omega)} + h^{k+1} \|p\|_{H^k(\Omega)}.$$

Proof. We first solve the dual problem with the error $u - u_h$ as right hand side:

$$B((v,q),(w,\lambda)) = (u-u_h, v)_{L^2(\Omega)} \quad \forall v \in V, \forall q \in Q.$$

Next we choose the test functions $v := u - u_h$ and $q := p - p_h$, and get

$$(u-u_h, u-u_h)_{L^2(\Omega)} = B((u-u_h, p-p_h),(w,\lambda)).$$

Together with the Galerkin orthogonality of the Stokes problem

$$\int_\Omega \nabla(u-u_h)\nabla w_h + \int_\Omega \operatorname{div} w_h (p-p_h) = 0$$
$$\int_\Omega \operatorname{div}(u-u_h)\lambda_h = 0,$$

for an $w_h \in V_h$ and $\lambda_h \in Q_h$, and the standard interpolator $I_h^{\Pi^1}$ and $I_h^{\Pi^0}$ (see appendix), we get

$$\|u-u_h\|^2_{L^2(\Omega)} = B((u-u_h, p-p_h),(w - I_h^{\Pi^1} w, \lambda - I_h^{\Pi^0}\lambda)).$$

Using the continuity of the bilinear form B (see Brezzi's theorem for the Stokes problem 3.1) we first see

$$\begin{aligned} B((u-u_h,p-p_h),(w-I_h^{\Pi^1}w,\lambda-I_h^{\Pi^0}\lambda)) \preccurlyeq{}& \|u-u_h\|_{H^1(\Omega)} \left\|w - I_h^{\Pi^1} w\right\|_{H^1(\Omega)} \\ &+ \|u-u_h\|_{H^1(\Omega)} \left\|\lambda - I_h^{\Pi^0}\lambda\right\|_{L^2(\Omega)} \\ &+ \left\|w - I_h^{\Pi^1} w\right\|_{H^1(\Omega)} \|p-p_h\|_{L^2(\Omega)}, \end{aligned}$$

and so using the assumption of the discrete error and the error assumptions of $I_h^{\Pi^k}$ $(k = 0,1)$ we get

$$\begin{aligned} \|u-u_h\|^2_{L^2(\Omega)} \preccurlyeq{}& h^k \|u\|_{H^{k+1}(\Omega)} h \|w\|_{H^2(\Omega)} \\ &+ h^k \|u\|_{H^{k+1}(\Omega)} h \|\lambda\|_{H^1(\Omega)} \\ &+ h \|w\|_{H^2(\Omega)} h^k \|p\|_{H^{k-1}(\Omega)}. \end{aligned}$$

Together with the regularity estimates and after dividing one factor we have

$$\|u-u_h\|_{L^2(\Omega)} \preccurlyeq h^{k+1} \|u\|_{H^{k+1}(\Omega)} + h^{k+1} \|p\|_{H^k(\Omega)}.$$

□

3.4 Finite elements for the Stokes problem

We have seen in the last sections that the choice of the approximation spaces has to fulfill the discrete LBB-condition (3.3.2) to guarantee a stable and unique solution. In this chapter we introduce two different elements for the Stokes equations and show their approximation properties. Using the $P_2 - P_1$ Taylor-Hood element leads to a continuous pressure, but the test functions only preserve a discrete divergence-free property. How to eliminate this disadvantage is the subject of chapter 4. The $P_2 - P_0$ element leads to a discontinuous pressure and an element wise conservation of the divergence, but has a worse error convergence rate.

3.4.1 Taylor-Hood element

Trying to solve the mixed formulation (3.3.1) we could make the choice of using the same polynomial degree for the approximation of the velocity and the pressure. This was quite famous when people started to approximate the Stokes equations as the results looked satisfying. Soon it came up that in this case the stability of the solution highly depends on the mesh as the kernel of the gradient can become big for some triangulations ([BF91][210]). This unpredictable behaviour of equal order interpolation methods for the velocity and the pressure was the impetus to find proper elements. An approach was given by Taylor and Hood ([HT73]) by using an approximation for the pressure of one degree lower than the velocity. We give a 2D example by analyzing the $P_2 - P_1$ element. For a given triangulation $\mathcal{T}$ we choose the spaces:

$$V_h := [\Pi^2(\mathcal{T})]^2 \cap [C^0(\Omega)]^2 \quad \text{and} \quad Q_h := \Pi^1(\mathcal{T}) \cap C^0(\Omega).$$

Using a standard interpolation operator I_h^V and a Clement operator (see appendix) $\Pi_h^{\mathcal{C}}$ leads to

$$\left\|v - I_h^V v\right\|_{H^1(\Omega)} \leq h^2 |v|_{H^3(\Omega)} \quad \text{and} \quad \left\|q - \Pi_h^{\mathcal{C}} q\right\|_{L^2(\Omega)} \leq h^2 |p|_{H^2(\Omega)}.$$

Assuming enough regularity for the exact solutions u and p, we get the interpolation error of the solutions u_h and p_h of (3.3.1) for the Taylor-Hood element in the H^1 norm by

$$\|u - u_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \preccurlyeq h^2 |u|_{H^3(\Omega)} + h^2 |p|_{H^2(\Omega)}$$

and in the L^2 norm (see 3.2)

$$\|u - u_h\|_{L^2(\Omega)} \preccurlyeq h^3 |u|_{H^3(\Omega)} + h^3 |p|_{H^2(\Omega)},$$

thus an optimal convergence order of the velocity and pressure error. For the discrete LBB-condition we refer to [GR86], but state the needed assumption on the mesh $\mathcal{T}$:

$$\left\{ \begin{array}{l} \mathcal{T} \text{ has a set of interior nodes } \{V_r\}_{r=1}^R \text{ such that } \{\omega_{V_r}\}_{r=1}^R \text{ with} \\ \qquad\qquad \omega_{V_r} := \bigcup_{T : v_r \in T} T \\ \text{is a partition of } \Omega \end{array} \right\} \tag{3.4.1}$$

Although the fact that the Taylor-Hood elements are quite popular due to an easy implementation and analysis, they do not have the property $\operatorname{div} V_h \subset Q_h$, and so only provide discrete divergence-free solutions.

3.4.2 Discontinuous pressure

Instead of approximating the pressure with a continuous ansatz it is also possible to use a $P_2 - P_0$ element, namely an approximation with polynomials of order two for the velocity and a piecewise constant approximation for the pressure, so

$$V_h := [\Pi^2(\mathcal{T})]^2 \cap [C^0(\Omega)]^2 \quad \text{and} \quad Q_h := \Pi^0(\mathcal{T}).$$

By that the divergence-free condition can be written as

$$\int_T \operatorname{div} u_h \, \mathrm{d}x = \int_{\partial T} u_h \cdot n \, \mathrm{d}s = 0 \quad \forall T \in \mathcal{T},$$

which reads as a conservation of mass on each triangle. Using a standard interpolation operator I_h^V and the Clement operator (see appendix) $\Pi_h^{\mathcal{C}}$, we see that we lose one order of accuracy due to the poor approximation of the pressure

$$\left\| v - I_h^V v \right\|_{H^1(\Omega)} \leq h^2 |v|_{H^3(\Omega)} \quad \text{and} \quad \left\| q - \Pi_h^{\mathcal{C}} q \right\|_{L^2(\Omega)} \leq h |q|_{H^1(\Omega)}.$$

To show the LBB-condition for the $P_2 - P_0$ element, we construct a Fortin operator that fulfills (3.3.10) and proceed as in proposition 4. The idea of the construction of the second operator $\Pi_2^{\mathcal{F}}$ is that $b(u - \Pi_2^{\mathcal{F}} u, q_h) = 0$ now reads as

$$\int_T \operatorname{div} (u - \Pi_2^{\mathcal{F}} u) \, \mathrm{d}x = \int_{\partial T} \left(u - \Pi_2^{\mathcal{F}} u \right) \cdot n \, \mathrm{d}s = 0.$$

Lemma 3.3. *The choice*

$$V_h := [\Pi^2(\mathcal{T})]^2 \cap [C^0(\Omega)]^2 \quad \textit{and} \quad Q_h := \Pi^0(\mathcal{T})$$

fulfills the discrete LBB-condition (3.3.2).

Proof. Let $\Pi_1^{\mathcal{F}}$ be a Clement operator $\Pi_h^{\mathcal{C}}$. By that we get

$$\left\| \Pi_1^{\mathcal{F}} v \right\|_V \preccurlyeq \|v\|_V .$$
$$\left\| v - \Pi_1^{\mathcal{F}} v \right\|_Q \preccurlyeq h \|v\|_V .$$

Define $\Pi_2^{\mathcal{F}}$ on all triangles in the following way

$$(\Pi_2^{\mathcal{F}} v)(V_i) = 0 \quad i = 1,2,3$$
$$\int_{E_{ij}} \Pi_2^{\mathcal{F}} v \cdot n \, \mathrm{d}s = \int_{E_{ij}} v \cdot n \, \mathrm{d}s \quad i,j = 1,2,3 \quad i \neq j,$$

where V_i are the vertices of the triangle. Note that $\Pi_2^{\mathcal{F}} v \in V_h$, as the second condition can be fulfilled by a correct choice of $(\Pi_2^{\mathcal{F}} u)(V_{ij})$ where V_{ij} is the middle point of edge E_{ij}. Then by construction we get

$$\int_T \operatorname{div} (v - \Pi_2^{\mathcal{F}} v) q_h \, \mathrm{d}x = 0 \quad \forall q_h \in Q_h,$$

and using a scaling argument we get

$$\left\| \Pi_2^{\mathcal{F}} v \right\|_{V(T)} = \left\| \widehat{\Pi_2^{\mathcal{F}} v} \right\|_{V(\hat{T})} \preccurlyeq \|\hat{v}\|_{V(\hat{T})} \preccurlyeq \frac{1}{h} \|v\|_{Q(T)} + \|v\|_{V(T)} ,$$

where $\|\cdot\|_{V(T)}$ and $\|\cdot\|_{Q(T)}$ are with ν scaled $H^1(T)$ and $L^2(T)$ norms, respectively. Next we observe that

$$\left\|\Pi_2^{\mathcal{F}}(v - \Pi_1^{\mathcal{F}} v)\right\|_V^2 = \sum_{T\in\mathcal{T}} \left\|\Pi_2^{\mathcal{F}}(v - \Pi_1^{\mathcal{F}} v)\right\|_{V(T)}^2$$
$$\preccurlyeq \sum_{T\in\mathcal{T}} \frac{1}{h^2} \left\|(I - \Pi_1^{\mathcal{F}})v)\right\|_{Q(T)}^2 + \left\|(I - \Pi_1^{\mathcal{F}})v)\right\|_{V(T)}^2 \preccurlyeq \|v\|_V^2 .$$

Defining $\Pi^{\mathcal{F}} v := \Pi_1^{\mathcal{F}} v + \Pi_2^{\mathcal{F}}(v - \Pi_1^{\mathcal{F}} v)$ and using proposition 4 the lemma is shown. □

3.5 Time discretization

In this section we want to discuss the discretization of the unsteady Navier Stokes equations (2.7.1). For this we use an additive decomposition method, the IMEX scheme, which stands for an IMplicit EXplicit splitting method (see [ARS97]). The main idea is to handle the nonlinear convection term explicitly and use it as a force for the implicit scheme. The implicit part of the equation is the diffusion term and the incompressibility constraint. Due to that, we ensure that this constraint is fulfilled in each time step. For the ease, we use $f := 0$ in this section.

3.5.1 Basic definitions

For the time dependent discretization we use an approximation for the velocity u_h and the pressure p_h given by

$$u_h(x,t) := \sum_{i=1}^{N_u} u_i(t)\varphi_i(x) \quad \text{and} \quad p_h(x,t) := \sum_{i=1}^{N_p} p_i(t)\psi_i(x),$$

with $\underline{u}(t) := \{u_i(t)\}_{i=1}^{N_u}$, $\underline{p}(t) := \{p_i(t)\}_{i=1}^{N_p}$ and $\{\varphi_i\}_{i=1}^{N_u}$ as a basis for V_h and $\{\psi_i\}_{i=1}^{N_p}$ as a basis for Q_h. By that we define the matrices

$$\begin{aligned}
&M \in \mathbb{R}^{N_u\times N_u} && M_{i,j} := \int_\Omega \varphi_i \cdot \varphi_j \,\mathrm{d}x \quad \forall i,j = 1,\dots,N_u,\\
&A \in \mathbb{R}^{N_u\times N_u} && A_{i,j} := \int_\Omega \nu\nabla\varphi_i : \nabla\varphi_j \,\mathrm{d}x \quad \forall i,j = 1,\dots,N_u,\\
&D \in \mathbb{R}^{N_u\times N_p} && D_{i,j} := \int_\Omega \operatorname{div}\varphi_i \;\psi_j \,\mathrm{d}x \quad \forall i = 1,\dots,N_u \text{ and } \forall j = 1,\dots,N_p,
\end{aligned}$$

and for an arbitrary $\underline{v} := \{v_i\}_{i=1}^{N_u}$ and $\underline{w} := \{w_i\}_{i=1}^{N_u}$ the vector

$$C(\underline{v})\underline{w} \in \mathbb{R}^{N_u} \quad C_i := \int_\Omega (v_h \cdot \nabla w_h)\varphi_i \,\mathrm{d}x$$
$$\text{with} \quad v_h(x,t) := \sum_{i=1}^{N_u} v_i(t)\varphi_i(x) \quad \text{and} \quad w_h(x,t) := \sum_{i=1}^{N_u} w_i(t)\varphi_i(x).$$

Using those definitions and the finite element discretization for the spatial domain (see (3.1.2 and (3.1.3)) we have the problem:

γ	γ	0
1	$1-\gamma$	γ
	$1-\gamma$	γ

0	0	0	0
γ	γ	0	0
1	δ	$1-\delta$	0
	δ	$1-\delta$	0

Table 3.1: Butcher tableaus for 2 step, L-Stable IMEX scheme

Problem 8: *Find* $(\underline{u}, \underline{p}) \in \mathbb{R}^{N_u+N_p}$ *so that*

$$\begin{aligned} M\frac{\partial \underline{u}}{\partial t} + A\underline{u} + D\underline{p} + C(\underline{u})\underline{u} &= 0 && \text{in } I_T \\ D^T\underline{u} &= 0 && \text{in } I_T \\ \underline{u}(0) &= \underline{u}_0 \end{aligned} \tag{3.5.1}$$

3.5.2 First order IMEX

Assume that the time interval I_T is divided in equidistant steps

$$0 =: t_0 < t_i < \cdots < t_{N_t} := T,$$

with length $\Delta t = t_{i+1} - t_i$ for all $i = 1, \dots, N_t - 1$. We use the following approximation for the time derivation of $\underline{u}(t)$

$$\frac{\partial \underline{u}}{\partial t} := \frac{\underline{u}^{n+1} - \underline{u}^n}{\Delta t},$$

with $\underline{u}^n := \underline{u}(t_n)$, $\underline{p}^n := \underline{p}(t_n)$. Using an implicit Euler method for the stiffness A and the divergence constraint D and an explicit Euler method for the convection C leads to the system

$$\begin{aligned} (M + \Delta t A)\underline{u}^{n+1} + \Delta t D\underline{p}^{n+1} &= M\underline{u}^n - \Delta t C(\underline{u}^n)\underline{u}^n \\ D^T\underline{u}^{n+1} &= 0. \end{aligned}$$

Note that the convection term only appears on the right hand side of the system as the vector $C(\underline{u}^n)\underline{u}^n$ can be calculated for each new time step.

3.5.3 Second order IMEX

Instead of using first order Euler schemes for A, D and C we now use a diagonal Runge Kutta method. By that, the resulting IMEX scheme can be represented with the Butcher tableaus (3.1) with $\gamma := 1 - \sqrt{\frac{1}{2}}$ and $\delta := 1 - \frac{1}{2\gamma}$. The obtained systems are:

$$\begin{aligned} (M + \gamma\Delta t A)\underline{u}^1 + \gamma\Delta t D\underline{p}^1 &= M\underline{u}^n - \gamma\Delta t C(\underline{u}^n)\underline{u}^n \\ D^T\underline{u}^1 &= 0 \end{aligned}$$

and

$$\begin{aligned}(M+\gamma\Delta t A)\underline{u}^{n+1}+\gamma\Delta t D\underline{p}^{n+1} &= M\underline{u}^{n}-\Delta t(1-\gamma)A\underline{u}^{1}\\ &\quad -\Delta t\left((1-\delta)C(\underline{u}^{1})\underline{u}^{1}-\delta C(\underline{u}^{n})\underline{u}^{n}\right)\\ D^{T}\underline{u}^{n+1} &= 0.\end{aligned}$$

4 Reconstruction operator for the Taylor-Hood element

In the last chapter we introduced the $P_2 - P_1$ Taylor-Hood element using

$$V_h = [\Pi^2(\mathcal{T})]^2 \cap [C^0(\Omega)]^2 \quad \text{and} \quad Q_h = \Pi^1(\mathcal{T}) \cap C^0(\Omega)$$

as approximation spaces for the Stokes problem. The disadvantage of the element is that the velocity test functions preserve only a discrete divergence-free property

$$\int_\Omega \operatorname{div}(v_h)\ q_h \ \mathrm{d}x = 0 \quad \forall q_h \in Q_h.$$

In this chapter we introduce two versions of reconstruction operators $\mathcal{R}_h$ that cure this drawback. The first one deals with local problems on vertex patches and can handle the problem we observed due to the Helmholtz decomposition (3.3.3). The second one preserves also a proper approximation convergence and will fulfill enough regularity for an L^2 error estimation using an Aubin-Nitsche duality argument.

Remark 10: In this thesis we construct an operator for the $P_2 - P_1$ Taylor-Hood element, but we want to mention that the reconstruction is also possible for a bigger class of elements as for example the mini-element and high order Taylor-Hood elements. This is the topic of a paper in preparation with Joachim Schöberl, Alexander Linke and Christian Merdon.

4.1 Basic definitions

For the reconstruction we define the spaces

$$\Sigma_h := \mathrm{BDM}^2(\mathcal{T}) \subset H(\operatorname{div})(\Omega) \quad \text{and} \quad \widetilde{Q}_h := \Pi^1(\mathcal{T})/\mathbb{R} \subset L^2_0(\Omega). \tag{4.1.1}$$

Note that $\widetilde{Q}_h$ are piecewise polynomials of order one and do not have to be continuous over element edges. For the construction we use the L^2 projection onto constants defined by

$$\begin{aligned} \mathcal{P}_{\mathbb{R}}^{L^2(\omega)} :& L^2(\omega) \to \mathbb{R} \\ & u \mapsto \frac{1}{|\omega|} \int_\omega u \ \mathrm{d}x =: \overline{u}^\omega \end{aligned}$$

for all $\omega \subset \Omega$, and define the vertex patch $\omega_V \subset \Omega$ by

$$\omega_V := \bigcup_{T : V \in T} T$$

for all vertices $V \in \mathcal{T}$ and the element patch $\omega_T \subset \Omega$ by

$$\omega_T := \bigcup_{\tilde{T}:T\cap\tilde{T}\neq\emptyset} \tilde{T}$$

for all $T \in \mathcal{T}$. On this patches we furthermore define the local spaces

$$\Sigma_{h,0}(\omega_i) := \{\tau_h \in \Sigma_h : \tau_h \cdot n = 0 \text{ on } \partial\omega_i\} \quad \text{and} \quad \widetilde{Q}_h(\omega_i) = \Pi^1(\omega_i)/\mathbb{R},$$

with $i \in \{T, V\}$.

4.2 Vertex reconstruction

4.2.1 Construction of $\mathcal{R}_h^V$

The main idea for the reconstruction operator is to find a function $\sigma_h \in \Sigma_h$ that fulfills

$$\operatorname{div} \sigma_h = \operatorname{div} u_h$$

exactly and then define the vertex reconstruction by

$$\mathcal{R}_h^V(u_h) := u_h - \sigma_h.$$

To find σ_h we solve local problems on all vertex patches ω_V to get local solutions σ_V. By summing up the local corrections we then have σ_h.

Problem 9: *For a given $w_h \in \Pi^1(\omega_V)$, find $(\sigma_h^V, \lambda_h) \in \Sigma_{h,0}(\omega_V) \times \widetilde{Q}_h(\omega_V)/\mathbb{R}$ so that*

$$\int_{\omega_V} \sigma_h^V \cdot \tau_h \, \mathrm{d}x + \int_{\omega_V} \operatorname{div} \tau_h \lambda_h \, \mathrm{d}x = 0 \quad \forall \tau_h \in \Sigma_{h,0}(\omega_V) \tag{4.2.1}$$

$$\int_{\omega_V} \operatorname{div} \sigma_h^V \psi_h \, \mathrm{d}x = \int_{\omega_V} (\phi_V \operatorname{div} w_h)(\psi_h - \overline{\psi_h}^{\omega_V}) \, \mathrm{d}x \quad \forall \psi_h \in \widetilde{Q}_h(\omega_V)/\mathbb{R}$$

where ϕ_V is the hat function corresponding to the vertex V.

Remark 11: The right hand side of the second line of problem (4.2.1) is a valid linear form on the factorized space $\widetilde{Q}_h(\omega_V)/\mathbb{R}$ as it does not depend on the representative of one equivalence class. This is fulfilled as when we add a constant to ψ_h it is eliminated on the right hand side as we subtract the mean value, and on the left side due to $\sigma_h^V = 0$ on $\partial\omega_V$.

Lemma 4.1. *Equation (4.2.1) has a unique solution* (σ_h^V, λ_h) *satisfying*

i.

$$\operatorname{div} \sigma_h^V = \mathcal{P}^{L^2}_{\widetilde{Q}_h(\omega_V)} \Big((I - \mathcal{P}^{L^2(\omega_V)}_{\mathbb{R}}) \phi_V \operatorname{div} w_h \Big) \quad \text{on } \Omega, \tag{4.2.2}$$

where σ_h^V *is trivially extended by* 0 *onto* Ω. *In particular, if*

$$(\operatorname{div} w_h, q_h)_{L^2(\Omega)} = 0 \quad \forall q_h \in Q_h,$$

then $\operatorname{div} \sigma_h^V = \mathcal{P}^{L^2}_{\widetilde{Q}_h(\omega_V)} (\phi_V \operatorname{div} w_h)$.

ii. $\left\| \sigma_h^V \right\|_{L^2(\Omega)} \preccurlyeq h \left\| \operatorname{div} w_h \right\|_{L^2(\Omega)}$

Proof. First we show the existence and uniqueness of the saddlepoint problem (4.2.1). On the spaces $\Sigma_h(\omega_V)$ and $\widetilde{Q}_h(\omega_V)$ we choose the norms

$$\|\tau_h\|_{\Sigma_h(\omega_V)} := \|\tau_h\|_{L^2(\omega_V)} + h \left\| \operatorname{div} \tau_h \right\|_{L^2(\omega_V)},$$

and

$$\|\psi_h\|_{\widetilde{Q}_h(\omega_V)} := \frac{1}{h} \|\psi_h\|_{L^2(\omega_V)},$$

and get

$$\begin{aligned} a_\sigma(\sigma_h, \tau_h) &:= \int_{\omega_V} \sigma_h \cdot \tau_h \, \mathrm{d}x \overset{\text{C.S.}}{\leq} \|\sigma_h\|_{\Sigma_h(\omega_V)} \|\tau_h\|_{\Sigma_h(\omega_V)} \\ b_\sigma(\tau_h, \psi_h) &:= \int_{\omega_V} \operatorname{div} \tau_h \; \psi_h \, \mathrm{d}x \overset{\text{C.S.}}{\leq} \|\tau_h\|_{\Sigma_h(\omega_V)} \|\psi_h\|_{\widetilde{Q}_h(\omega_V)}. \end{aligned}$$

For the coercivity on the kernel we point out that due to $\operatorname{div} \Sigma_{h,0}(\omega_V) \subset \widetilde{Q}_h(\omega_V)/\mathbb{R}$ it follows that an element in the kernel fulfills an exact divergence-free property $\operatorname{div} \tau_h = 0$ and we get

$$a_\sigma(\tau_h, \tau_h) = \int_{\omega_V} \tau_h \cdot \tau_h = \|\tau_h\|^2_{L^2(\omega_V)} = \|\tau_h\|^2_{\widetilde{Q}_h(\omega_V)}.$$

It remains to show the discrete LBB-condition to use Brezzi's theorem. We first show the LBB-condition on the reference patch $\widehat{\omega_V}$ and then on ω_V. It should be mentioned that there exist different reference patches due to the number of elements that belong to a vertex, but for each triangulation $\mathcal{T}$ there exist a finite number of reference patches. We use the BDM interpolator $I_h^{\mathrm{BDM}^2}$ that provides (see appendix)

$$b_\sigma(I_h^{\mathrm{BDM}^2} \tau, \psi_h) = b_\sigma(\tau, \psi_h) \quad \forall \psi_h \in \widetilde{Q}_h(\widehat{\omega_V})$$

and

$$\left\| I_h^{\mathrm{BDM}^2} \tau \right\|_{H(\operatorname{div})(\widehat{\omega_V})} \preccurlyeq \|\tau\|_{H^1(\widehat{\omega_V})} \quad \forall \tau \in [H^1(\widehat{\omega_V})]^2.$$

As $[H^1(\widehat{\omega_V})]^2 \subset H(\operatorname{div})(\widehat{\omega_V})$ we get for all $\hat{\psi}_h \in \widetilde{Q}_h(\widehat{\omega_V})/\mathbb{R}$

$$\sup_{\hat{\tau}_h \in \Sigma_{h,0}(\widehat{\omega_V})} \frac{b_\sigma(\hat{\tau}_h, \hat{\psi}_h)}{\|\hat{\tau}_h\|_{H(\operatorname{div})(\widehat{\omega_V})}} \succcurlyeq \sup_{\hat{\tau} \in [H^1(\widehat{\omega_V})]^2} \frac{b_\sigma(I_h^{\mathrm{BDM}^2}\hat{\tau}, \hat{\psi}_h)}{\left\|I_h^{\mathrm{BDM}}\hat{\tau}\right\|_{H^1(\widehat{\omega_V})}} \succcurlyeq \sup_{\hat{\tau} \in [H^1(\widehat{\omega_V})]^2} \frac{b_\sigma(\hat{\tau}, \hat{\psi}_h)}{\|\hat{\tau}\|_{H^1(\widehat{\omega_V})}}$$

Next we use the continuous LBB-condition (3.2.5) to get

$$\sup_{\hat{\tau}_h \in \Sigma_{h,0}(\widehat{\omega_V})} \frac{b_\sigma(\hat{\tau}_h, \hat{\psi}_h)}{\|\hat{\tau}_h\|_{H(\operatorname{div})(\widehat{\omega_V})}} \succcurlyeq \left\|\hat{\psi}_h\right\|_{L^2(\widehat{\omega_V})}.$$

To show the condition on ω_V we choose for an arbitrary ψ_h the functions $\hat{\psi}_h = \psi_h$, and $\tau_h := \mathcal{P}(\hat{\tau}_h)$, where $\mathcal{P}$ is the Piola transformation (see appendix). By that we get

$$\begin{aligned}
\sup_{\tau_h \in \Sigma_{h,0}(\omega_V)} \frac{b_\sigma(\tau_h, \psi_h)}{\|\tau_h\|_{\Sigma_{h,0}}} &= \sup_{\tau_h \in \Sigma_{h,0}(\omega_V)} \frac{\int_{\omega_V} \operatorname{div} \tau_h \psi_h}{\|\tau_h\|_{L^2(\omega_V)} + h\,\|\operatorname{div} \tau_h\|_{L^2(\omega_V)}} \\
&= \sup_{\hat{\tau} \in \Sigma_{h,0}(\widehat{\omega_V})} \frac{\int_{\widehat{\omega_V}} \operatorname{div} \hat{\tau}_h \hat{\psi}_h}{\|\hat{\tau}_h\|_{L^2(\widehat{\omega_V})} + \|\operatorname{div} \hat{\tau}_h\|_{L^2(\widehat{\omega_V})}} \\
&\succcurlyeq \left\|\hat{\psi}_h\right\|_{L^2(\widehat{\omega_V})} = \frac{1}{h}\,\|\psi_h\|_{L^2(\omega_V)} = \|\psi_h\|_{\widetilde{Q}_h(\omega_V)}
\end{aligned}$$

Using Brezzi's theorem 3.1 the existence and uniqueness is proven. Next we observe that for $\psi_h \in \mathbb{R}$ we get $\mathcal{P}_{\mathbb{R}}^{L^2(\omega_V)}\psi_h = \psi_h$, and together with

$$\int_{\omega_V} \phi_V \operatorname{div} w_h (\psi_h - \overline{\psi_h}^{\omega_V})\ \mathrm{d}x = 0,$$

and

$$\int_{\omega_V} \operatorname{div} \sigma_h^V \psi_h\ \mathrm{d}x = \psi_h \int_{\partial\omega_V} \sigma_h^V \cdot n\ \mathrm{d}x = 0,$$

it follows that the second line of problem (4.2.1) is also fulfilled for constants and thus we get

$$\int_{\omega_V} \operatorname{div} \sigma_h^V \psi_h\ \mathrm{d}x = \int_{\omega_V} \phi_V \operatorname{div} w_h(\psi_h - \overline{\psi_h}^{\omega_V})\ \mathrm{d}x \quad \forall \psi_h \in \widetilde{Q}_h(\omega_V).$$

Using the locality of the L^2 projection on constants and an arbitrary $q \in L^2(\Omega)$ we see

$$\begin{aligned}
\int_\Omega \operatorname{div} \sigma_h^V\ q\ \mathrm{d}x = \int_{\omega_V} \operatorname{div} \sigma_h^V\ q\ \mathrm{d}x &= \int_{\omega_V} (\phi_V \operatorname{div} w_h)(I - \mathcal{P}_{\mathbb{R}}^{L^2(\omega_V)})\mathcal{P}^{L^2}_{\widetilde{Q}_h(\omega_V)}(q)\ \mathrm{d}x \\
&= \int_{\omega_V} \mathcal{P}^{L^2}_{\widetilde{Q}_h(\omega_V)} \Big((I - \mathcal{P}_{\mathbb{R}}^{L^2(\omega_V)})(\phi_V \operatorname{div} w_h)\Big)\ q\ \mathrm{d}x,
\end{aligned}$$

so the first equation is shown. If

$$(\operatorname{div} w_h, q_h)_{L^2(\Omega)} = 0 \quad \forall q_h \in Q_h,$$

we get

$$\mathcal{P}_{\mathbb{R}}^{L^2(\omega_V)}(\phi_V \operatorname{div} w_h) = \frac{1}{|\omega_V|}\int_{\omega_V} \phi_V \operatorname{div} w_h \,\mathrm{d}x = 0, \tag{4.2.3}$$

as $\phi_V \in Q_h$ and therefore $\operatorname{div}\sigma_h^V = \mathcal{P}^{L^2}_{\widetilde{Q}_h(\omega_V)}(\phi_V \operatorname{div} w_h)$. The estimation for the norm is given due to stability estimation provided by Brezzi's theorem, thus

$$\begin{aligned}\left\|\sigma_h^V\right\|_{L^2(\Omega)} = \left\|\sigma_h^V\right\|_{L^2(\omega_V)} &\preccurlyeq \left\|\mathcal{P}^{L^2}_{\widetilde{Q}_h(\omega_V)}(\phi_V \operatorname{div} w_h)\right\|_{\widetilde{Q}_h(\omega_V)'} \\ &\preccurlyeq h\left\|\operatorname{div} w_h\right\|_{L^2(\omega_V)} \preccurlyeq h\left\|\operatorname{div} w_h\right\|_{L^2(\Omega)}.\end{aligned}$$

□

Now we can construct the reconstruction operator. Note that $\sigma_h^V \in \Sigma_{h,0}(\omega_V)$, so it has a 0 normal trace and by that we can define

$$\sigma_h := \sum_{V\in\mathcal{T}} \sigma_h^V \in \Sigma_h.$$

Theorem 4.2. *The operator defined by*

$$\mathcal{R}_h^V(w_h) := w_h - \sigma_h,$$

fulfills

i. If $(\operatorname{div} w_h, q_h)_{L^2(\Omega)} = 0 \;\forall q_h \in Q_h$, *then* $(\operatorname{div}\mathcal{R}_h^V(w_h), \tilde{q}_h)_{L^2(\Omega)} = 0 \;\forall \tilde{q}_h \in \widetilde{Q}_h$

ii. $\left\|w_h - \mathcal{R}_h^V(w_h)\right\|_{L^2(\Omega)} \preccurlyeq h\left\|\operatorname{div} w_h\right\|_{L^2(\Omega)}$

Proof. We observe that

$$\begin{aligned}\operatorname{div}\mathcal{R}_h^V(w_h) &= \operatorname{div} w_h - \operatorname{div}\sigma_h = \operatorname{div} w_h - \sum_{V\in\mathcal{T}}\operatorname{div}\sigma_h^V \\ &= \operatorname{div} w_h - \sum_{V\in\mathcal{T}} \mathcal{P}^{L^2}_{\widetilde{Q}_h(\omega_V)}\Big((I - \mathcal{P}_{\mathbb{R}}^{L^2(\omega_V)})\phi_V \operatorname{div} w_h\Big) \\ &= \operatorname{div} w_h - \mathcal{P}^{L^2}_{\widetilde{Q}_h}\underbrace{\sum_{V\in\mathcal{T}}\phi_V}_{=1} \operatorname{div} w_h - \mathcal{P}^{L^2}_{\widetilde{Q}_h(\omega_V)}\sum_{V\in\mathcal{T}}\underbrace{\mathcal{P}_{\mathbb{R}}^{L^2(\omega_V)}\phi_V \operatorname{div} w_h}_{=0} \\ &= (I - \mathcal{P}^{L^2}_{\widetilde{Q}_h})\operatorname{div} w_h \perp_{L^2} \widetilde{Q}_h,\end{aligned}$$

and

$$\begin{aligned}\left\|w_h - \mathcal{R}_h^V(w_h)\right\|_{L^2(\Omega)} = \left\|\sigma_h\right\|_{L^2(\Omega)} &\preccurlyeq \sum_{V\in\mathcal{T}}\left\|\sigma_h^V\right\|_{L^2(\omega_V)} \\ &\preccurlyeq \sum_{V\in\mathcal{T}} h\left\|\operatorname{div} w_h\right\|_{L^2(\omega_V)} \preccurlyeq h\left\|\operatorname{div} w_h\right\|_{L^2(\Omega)}.\end{aligned}$$

□

4.2.2 Error analysis

The reconstruction operator is used for the right hand side. Now we have the problem

Problem 10: *Find $(u_h, p_h) \in V_h \times Q_h$ so that*

$$\begin{aligned} a(u_h, v_h) + b(v_h, p_h) &= (f, \mathcal{R}_h^V(v_h))_{L^2(\Omega)} && \forall v_h \in V_h \\ b(u_h, q_h) &= 0 && \forall q_h \in Q_h. \end{aligned} \tag{4.2.4}$$

Due to the Helmholtz decomposition, a discrete divergence-free velocity test function can result in a velocity $u_h \neq (0,0)$ when using a gradient field as force $f = \nabla\phi$ (see 3.3.3), but for the solution of the problem 4.2.4 we get

$$a(u_h, v_h) = (\nabla\phi, \mathcal{R}_h^V(v_h))_{L^2(\Omega)} = (\phi, \underbrace{\operatorname{div} \mathcal{R}_h^V(v_h)}_{=0})_{L^2(\Omega)} = 0 \quad \forall v_h \in V_{h,0}$$

and thus $u_h = (0,0)$. For the error analysis we use the first lemma of Strang that provides an estimation for the velocity error (see [BF91][108]) and the error estimation from section (1) to get

$$\|u - u_h\|_V \preccurlyeq \inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q + \sup_{w_h \in V_h} \frac{(f, w_h)_{L^2} - (f, \mathcal{R}_h^V(w_h))_{L^2}}{\|w_h\|_V}. \tag{4.2.5}$$

The last term can be estimated due to

$$\begin{aligned} (f, w_h)_{L^2(\Omega)} - (f, \mathcal{R}_h^V(w_h))_{L^2(\Omega)} &\leq \|f\|_{L^2(\Omega)} \left\|w_h - \mathcal{R}_h^V(w_h)\right\|_{L^2(\Omega)} \\ &\preccurlyeq \|f\|_{L^2(\Omega)} h \|w_h\|_{H^1(\Omega)}. \end{aligned}$$

And so, using the error analysis for the $P_2 - P_1$ Taylor-Hood element

$$\|u - u_h\|_V \preccurlyeq h^2 |u|_{H^3(\Omega)} + h^2 |p|_{H^2(\Omega)} + h \|f\|_{L^2(\Omega)}. \tag{4.2.6}$$

For the pressure we also use the Strang lemma to get

$$\|p - p_h\|_V \preccurlyeq \inf_{q_h \in Q_h} \|p - q_h\|_Q + \|u - u_h\|_V + \sup_{w_h \in V_h} \frac{(f, w_h)_{L^2} - (f, \mathcal{R}_h^V(w_h))_{L^2}}{\|w_h\|_V}.$$

Similar to before we get for the Taylor-Hood element

$$\|p - p_h\|_{L^2(\Omega)} \preccurlyeq h^2 |u|_{H^3(\Omega)} + h^2 |p|_{H^2(\Omega)} + h \|f\|_{L^2(\Omega)}. \tag{4.2.7}$$

Here we see the problem. Due to the dominant h term, the error has a worse convergence rate than we would expect ($\mathcal{O}(h^2)$). This is fixed in the next section.

4.3 Element reconstruction

4.3.1 Construction of $\mathcal{R}_h^T$

Analogue to $\mathcal{R}_h^V$ we solve local problems to find a global reconstruction operator. The difference is that we use element patches (see section 4.1) instead and a smoothing operator on the test functions of the mixed problem. We define

$$\mathcal{S} : \widetilde{Q}_h \to Q_h$$

with the properties

i. $\mathcal{S}|_{Q_h} = \text{id}$,

ii. if $\tilde{q}_h \in \mathbb{R} \Rightarrow \mathcal{S}(\tilde{q}_h) = \tilde{q}_h$,

iii. the smoothing operator is quasi local, namely if $\tilde{q}_h = 0$ on $\omega_T \Rightarrow \mathcal{S}(\tilde{q}_h) = 0$ on T.

Using the smoothing operator for the right hand side we define the problem on ω_T by

Problem 11: *For a given $w_h \in \Pi^1(\omega_T)$, find $(\sigma_h^T, \lambda_h) \in \Sigma_{h,0}(\omega_T) \times \widetilde{Q}_h(\omega_T)/\mathbb{R}$ so that*

$$\begin{aligned} \int_{\omega_T} \sigma_h^T \cdot \tau_h \, \mathrm{d}x + \int_{\omega_T} \operatorname{div} \tau_h \lambda_h \, \mathrm{d}x &= 0 \quad \forall \tau_h \in \Sigma_{h,0}(\omega_T) \\ \int_{\omega_T} \operatorname{div} \sigma_h^T \psi_h \, \mathrm{d}x &= \int_T \operatorname{div} w_h (\psi_h - \mathcal{S}(\psi_h)) \, \mathrm{d}x \quad \forall \psi_h \in \widetilde{Q}_h(\omega_T)/\mathbb{R}. \end{aligned} \tag{4.3.1}$$

Lemma 4.3. *Equation (4.3.1) has a unique solution (σ_h^T, λ_h) satisfying*

i. $(\operatorname{div} \sigma_h^T, \tilde{q}_h)_{L^2(\Omega)} = (\operatorname{div} w_h, \tilde{q}_h - \mathcal{S}\tilde{q}_h)_{L^2(T)} \quad \forall \tilde{q}_h \in \widetilde{Q}_h(\Omega)$

ii. $\|\sigma_h^T\|_{H^{-1}(\Omega)} \preccurlyeq h^2 \|\operatorname{div} u_h\|_{L^2(\Omega)}$

Proof. The existence and uniqueness is shown analogue as in lemma 4.1. Due to the second property of the smoothing operator $\mathcal{S}$ we have for a $\psi_h \in \mathbb{R}$

$$\int_T \operatorname{div} w_h (\psi_h - \mathcal{S}\psi_h) \, \mathrm{d}x = 0,$$

and due to the zero boundary values of σ_h^T also

$$\int_{\omega_T} \operatorname{div} \sigma_h^T \psi_h \, \mathrm{d}x = \psi_h \int_{\partial\omega_T} \sigma_h^T \cdot n \, \mathrm{d}x = 0.$$

By that the second line of (4.3.1) is also valid for constants and thus

$$\int_{\omega_T} \operatorname{div} \sigma_h^T \psi_h \, \mathrm{d}x = \int_T \operatorname{div} w_h (\psi_h - \mathcal{S}(\psi_h)) \, \mathrm{d}x \quad \forall \psi_h \in \widetilde{Q}_h(\omega_T).$$

Using the quasi locality we can finally expand the equation on $\widetilde{Q}_h$. To see this we choose an arbitrary $\tilde{q}_h \in \widetilde{Q}_h$ and split it into $\tilde{q}_h = \tilde{q}_h^1 + \tilde{q}_h^2$ with $\mathrm{supp}(\tilde{q}_h^1) = \Omega/\omega_T$ and $\mathrm{supp}(\tilde{q}_h^2) = \omega_T$. Then we get

$$\begin{aligned}\int_\Omega \operatorname{div} \sigma_h^T \tilde{q}_h \, \mathrm{d}x &= \int_{\Omega/\omega_T} \underbrace{\operatorname{div} \sigma_h^T}_{=0} \tilde{q}_h^1 \, \mathrm{d}x + \int_{\omega_T} \operatorname{div} \sigma_h^T \tilde{q}_h^2 \, \mathrm{d}x = \\ &= \int_T \operatorname{div} w_h(\tilde{q}_h^2 - \mathcal{S}(\tilde{q}_h^2)) \, \mathrm{d}x = \int_T \operatorname{div} w_h(\tilde{q}_h - \mathcal{S}(\tilde{q}_h)) \, \mathrm{d}x.\end{aligned}$$

Finally we show the estimation in the dual norm. For that we choose an arbitrary $g \in [\Pi^0(\omega_T)]^2$ and $\psi \in \Pi^1(\omega_T)$ such that $\nabla\psi = g$ and observe

$$\begin{aligned}\int_{\omega_T} \sigma_h^T \cdot g \, \mathrm{d}x &= \int_{\omega_T} \sigma_h^T \cdot \nabla\psi \, \mathrm{d}x = \int_{\partial\omega_T} \sigma_h^T \cdot n\psi \, \mathrm{d}s - \int_{\omega_T} \operatorname{div} \sigma_h^T \psi \, \mathrm{d}x \\ &= \int_T \operatorname{div} w_h \underbrace{(I - \mathcal{S})\psi}_{=0 \text{ on } T} \, \mathrm{d}x = 0,\end{aligned}$$

and so

$$\begin{aligned}\left\|\sigma_h^T\right\|_{H^{-1}(\Omega)} &= \sup_{w \in [H^1(\Omega)]^2} \frac{\int_\Omega \sigma_h^T \cdot w \, \mathrm{d}x}{\|w\|_{H^1(\Omega)}} = \sup_{w \in [H^1(\Omega)]^2} \frac{\int_{\omega_T} \sigma_h^T \cdot w \, \mathrm{d}x}{\|w\|_{H^1(\Omega)}} \\ &= \sup_{w \in [H^1(\Omega)]^2} \frac{\int_\Omega \sigma_h^T \cdot (I - [\mathcal{P}_{\mathbb{R}}^{L^2(\omega_T)}]^2) w \, \mathrm{d}x}{\|w\|_{H^1(\Omega)}} \preccurlyeq \sup_{w \in [H^1(\Omega)]^2} \frac{\left\|\sigma_h^T\right\|_{L^2(\omega_T)} h \|w\|_{H^1(\Omega)}}{\|w\|_{H^1(\Omega)}} \\ &= h \left\|\sigma_h^T\right\|_{L^2(\omega_T)}.\end{aligned}$$

Together with the estimation of Brezzi's theorem we have

$$\left\|\sigma_h^T\right\|_{L^2(\omega_T)} \preccurlyeq h \|\operatorname{div} w_h\|_{L^2(\omega_T)}.$$

□

Similar to before we can sum up all local reconstructions σ_h^T to define

$$\sigma_h := \sum_{T \in Tc} \sigma_h^T \in \Sigma_h.$$

Theorem 4.4. *Let* $\mathcal{R}_h^T w_h := w_h - \sigma_h$ *then*

i. *If* $(\operatorname{div} w_h, q_h)_{L^2(\Omega)} = 0 \; \forall q_h \in Q_h$, *then*

$$(\operatorname{div} \mathcal{R}_h^T(w_h), \tilde{q}_h)_{L^2(\Omega)} = 0 \; \forall \tilde{q}_h \in \widetilde{Q}_h.$$

ii. $\left\|w_h - \mathcal{R}_h^T(w_h)\right\|_{H^{-1}(\Omega)} \preccurlyeq h^2 \|\operatorname{div} w_h\|_{L^2(\Omega)}$

Proof. For an arbitrary w_h with $(\operatorname{div} w_h, q_h)_{L^2(\Omega)} = 0 \ \forall q_h \in Q_h$ and a $\tilde{q}_h \in \widetilde{Q}_h$ we have:

$$\begin{aligned}(\operatorname{div} \mathcal{R}_h^T(w_h), \tilde{q}_h)_{L^2(\Omega)} &= (\operatorname{div} w_h, \tilde{q}_h)_{L^2(\Omega)} - \sum_{T\in\mathcal{T}} (\operatorname{div} \sigma_h^T, \tilde{q}_h)_{L^2(\Omega)} \\ &= (\operatorname{div} w_h, \tilde{q}_h)_{L^2(\Omega)} - \sum_{T\in\mathcal{T}} (\operatorname{div} w_h, \tilde{q}_h - \mathcal{S}\tilde{q}_h)_{L^2(T)} \\ &= (\operatorname{div} w_h, \tilde{q}_h)_{L^2(\Omega)} - (\operatorname{div} w_h, \tilde{q}_h)_{L^2(\Omega)} + (\operatorname{div} w_h, \underbrace{\mathcal{S}\tilde{q}_h}_{\in Q_h})_{L^2(\Omega)}.\end{aligned}$$

For the estimation in the dual norm we use the properties of each σ_h^T shown in lemma 4.3, so for $w_h - \mathcal{R}_h^T w_h = \sigma_h$ we get

$$\begin{aligned}\|\sigma_h\|_{H^{-1}(\Omega)} &= \sup_{w\in[H^1(\Omega)]^2} \frac{\int_\Omega \sigma_h \cdot w \ \mathrm{d}x}{\|w\|_{H^1(\Omega)}} = \sup_{w\in[H^1(\Omega)]^2} \frac{\int_\Omega \sum_{T\in\mathcal{T}} \sigma_h^T \cdot w \ \mathrm{d}x}{\|w\|_{H^1(\Omega)}} \\ &= \sup_{w\in[H^1(\Omega)]^2} \frac{\sum_{T\in\mathcal{T}} \int_{\omega_T} \sigma_h^T \cdot w \ \mathrm{d}x}{\|w\|_{H^1(\Omega)}} = \sup_{w\in[H^1(\Omega)]^2} \frac{\sum_{T\in\mathcal{T}} \int_{\omega_T} \sigma_h^T \cdot (I - \mathcal{P}_{\mathbb{R}}^{L^2(\omega_T)}) w \ \mathrm{d}x}{\|w\|_{H^1(\Omega)}} \\ &\preccurlyeq \sup_{w\in[H^1(\Omega)]^2} \frac{\sum_{T\in\mathcal{T}} \left\|\sigma_h^T\right\|_{L^2(\omega_T)} h \|w\|_{H^1(\omega_T)}}{\|w\|_{H^1(\Omega)}} \preccurlyeq h \sum_{T\in\mathcal{T}} \left\|\sigma_h^T\right\|_{L^2(\omega_T)} \\ &\preccurlyeq h^2 \|\operatorname{div} w_h\|_{L^2(\Omega)}\end{aligned}$$

□

Remark 12: An example for the smoothing operator $\mathcal{S}$ from $\Pi^1(\mathcal{T}) \to \Pi^1(\mathcal{T}) \cap C^0(\Omega)$ is given by averaging the values of the discontinuous linear polynomials in each vertex of T, also called Oswald-interpolator see [Osw93].

4.3.2 Error analysis

As for the first operator $\mathcal{R}_h^V$ we observe the same properties for the element reconstruction $\mathcal{R}_h^T$ with respect to the problems due to the Helmholtz decomposition (see section 4.2.2). The main advantage can be seen in the error analysis. Again using Strang's lemma we get

$$\begin{aligned}\|u - u_h\|_V &\preccurlyeq \inf_{v_h\in V_h} \|u - v_h\|_V + \inf_{q_h\in Q_h} \|p - q_h\|_{V_h} + \sup_{w_h\in V_h} \frac{(f, w_h)_{L^2} - (f, \mathcal{R}_h^V(w_h))_{L^2}}{\|w_h\|_V} \\ \|p - p_h\|_V &\preccurlyeq \inf_{q_h\in Q_h} \|p - q_h\|_Q + \inf_{v_h\in V_h} \|u - u_h\|_V + \sup_{w_h\in V_h} \frac{(f, w_h)_{L^2} - (f, \mathcal{R}_h^V(w_h))_{L^2}}{\|w_h\|_V}.\end{aligned}$$

The last term can be estimated using the dual norms

$$\begin{aligned}(f, w_h - \mathcal{R}_h^T(w_h))_{L^2(\Omega)} &= \langle w_h - \mathcal{R}_h^T w_h, f\rangle_{H^{-1}\times H^1} \\ &\preccurlyeq \left\|w_h - \mathcal{R}_h^T(w_h)\right\|_{H^{-1}(\Omega)} \|f\|_{H^1(\Omega)} \\ &\preccurlyeq h^2 \|w_h\|_{H^1(\Omega)} \|f\|_{H^1(\Omega)}.\end{aligned}$$

Using the error analysis for the $P_2 - P_1$ Taylor-Hood element we get

$$\|u - u_h\|_V + \|p - p_h\|_{L^2(\Omega)} \preccurlyeq h^2 |u|_{H^3(\Omega)} + h^2 |p|_{H^2(\Omega)} + h^2 \|f\|_{H^1(\Omega)} .$$

We see that the error still has the convergence order $\mathcal{O}(h^2)$, so the reconstruction operator had no impact on this property. For the L^2 error we use an Aubin Nitsche technique as presented in 3.2 but due to the non conforming right hand side we can not proceed in the same way. As in section 3.2 we assume enough regularity of the dual problem and solve the problem with the error $u - u_h$ as right hand side to get

$$(u - u_h, u - u_h)_{L^2(\Omega)} = B((u - u_h, p - p_h), (w, \lambda)).$$

Similar to the proof of theorem 3.2 we use the Galerkin orthogonality to subtract $I_h^{\Pi^1} w$ and $I_h^{\Pi^0} \lambda$, but due to the different right hand sides another term appears and we get

$$\|u - u_h\|^2_{L^2(\Omega)} = B((u - u_h, p - p_h), (w - I_h^{\Pi^1} w, \lambda - I_h^{\Pi^0} \lambda)) - (f, I_h^{\Pi^1} w - \mathcal{R}_h^T (I_h^{\Pi^1} w)).$$

The first term is bounded similarly to theorem 3.2

$$\begin{aligned} B((u - u_h, p - p_h), (w - I_h^{\Pi^1} w, \lambda - I_h^{\Pi^0} \lambda)) &\preccurlyeq h^2 \|u\|_{H^3(\Omega)} h \|w\|_{H^2(\Omega)} \\ &+ h^2 \|u\|_{H^3(\Omega)} h \|\lambda\|_{H^1(\Omega)} \\ &+ h \|w\|_{H^2(\Omega)} h^2 \|p\|_{H^2(\Omega)} . \end{aligned}$$

For the second term we use

$$(f, I_h^{\Pi^1} w - \mathcal{R}_h^T (I_h^{\Pi^1} w))_{L^2(\Omega)} \leq \left\| I_h^{\Pi^1} w - \mathcal{R}_h^T (I_h^{\Pi^1} w) \right\|_{H^{-1}(\Omega)} \|f\|_{H^1(\Omega)} ,$$

and theorem 4.4

$$\left\| I_h^{\Pi^1} w - \mathcal{R}_h^T (I_h^{\Pi^1} w) \right\|_{H^{-1}(\Omega)} \preccurlyeq h^2 \left\| \operatorname{div} I_h^{\Pi^1} w \right\|_{L^2(\Omega)} .$$

As the solution of the dual problem fulfills $\operatorname{div} w = 0$, we can subtract this term to get

$$\begin{aligned} \left\| I_h^{\Pi^1} w - \mathcal{R}_h^T (I_h^{\Pi^1} w) \right\|_{H^{-1}(\Omega)} &\preccurlyeq h^2 \left\| \operatorname{div} I_h^{\Pi^1} w - \operatorname{div} w \right\|_{L^2(\Omega)} \\ &\preccurlyeq h^2 \left\| I_h^{\Pi^1} w - w \right\|_{H^1(\Omega)} \preccurlyeq h^3 \|w\|_{H^2(\Omega)} , \end{aligned}$$

and thus

$$\|u - u_h\|_{L^2(\Omega)} \preccurlyeq h^3 \|u\|_{H^3(\Omega)} + h^3 \|p\|_{H^2(\Omega)} + h^3 \|f\|_{H^1(\Omega)} .$$

4.4 Numerical examples

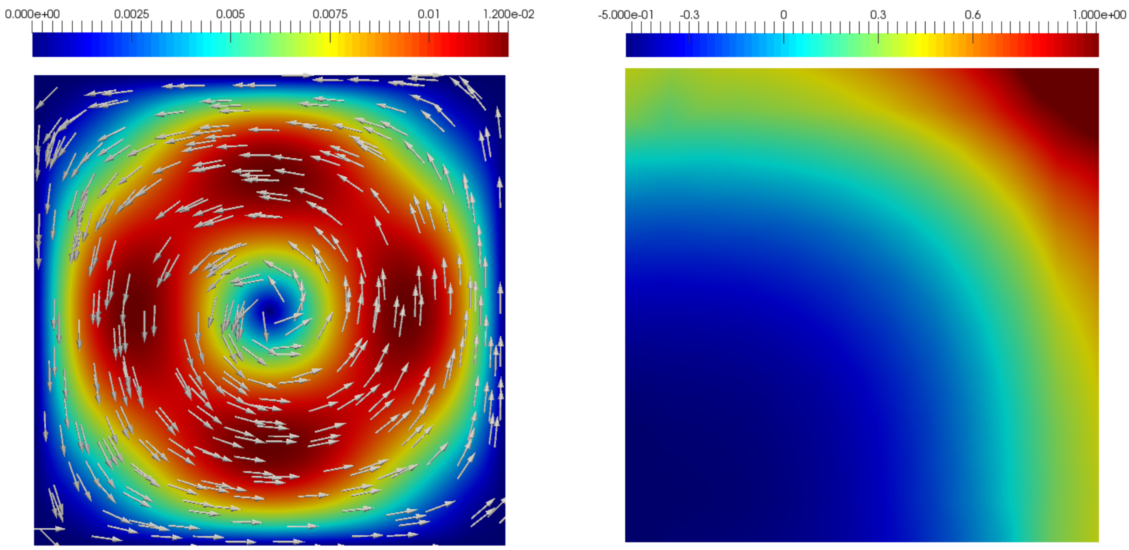
In this section we present four different numerical examples using the reconstruction operators $\mathcal{R}_h^V$ and $\mathcal{R}_h^T$. In the first example we compare the convergence orders of the standard Taylor-Hood element with and without the reconstruction for the right hand side. In the second example we solve a standard Stokes problem with non homogeneous boundary conditions that induce a flow and use the operators to correct the divergence of the solution. In the third problem we concern a Hagen-Poiseuille channel flow by solving the unsteady Navier Stokes equations in time. The convection term is implemented in a curl formulation (see equation 3.1.4) which results in a bad approximation due to the Helmholtz decomposition (see 3.3.3) when we use discrete divergence-free test functions. Finally we look at a two phase Stokes flow where the pressure is discontinuous over the surface of a bubble.

4.4.1 Convergence rates for a smooth solution

In the first example we want to examine the convergence rates of the pressure and the velocity error measured in the H^1 and the L^2 norm. The domain is the unit square $\Omega := (0,1) \times (0,1)$ and the exact velocity and pressure is given by

$$u := \operatorname{curl} \zeta \quad \text{with} \quad \zeta := x^2(x-1)^2 y^2 (y-1)^2$$
$$p = x^3 + y^3 - \frac{1}{2}.$$

In figure 4.1 we can see the exact solutions. For the error observation we use different viscosities $\nu = 1, 10^{-3}, 10^{-8}$ and either no reconstruction or $\mathcal{R}_h^V$ and $\mathcal{R}_h^T$ on the right hand side.



(a) Velocity field and absolute value $|u|$ (b) Pressure p

Figure 4.1: Exact solutions of the first example

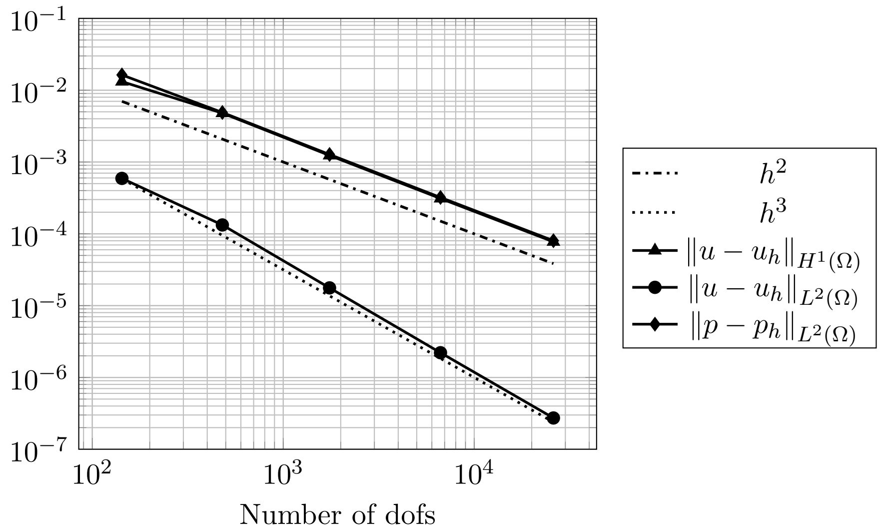


Figure 4.2: Convergence rates of the $P_2 - P_1$ Taylor-Hood element for $\nu = 1$

Classical Taylor -Hood

In section 3.4.1 we observed the classical convergence rates for the $P_2 - P_1$ Taylor-Hood element

$$\|u - u_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \preccurlyeq h^2|u|_{H^3(\Omega)} + h^2|p|_{H^2(\Omega)}$$

and

$$\|u - u_h\|_{L^2(\Omega)} \preccurlyeq h^3|u|_{H^3(\Omega)} + h^3|p|_{H^2(\Omega)}.$$

In figure 4.2 we observe the expected rates but we also distinguish a lack of accuracy that appears using a small viscosity, see figure 4.3. Although the convergence rate is still $\mathcal{O}(h^2)$ the error gets really high. This can be seen when we look at the error analysis in section 3.3.2 as the Q norm was scaled by $\frac{1}{\sqrt{\nu}}$ and the V norm with $\sqrt{\nu}$ and so

$$\|u - u_h\|_{H^1(\Omega)} \preccurlyeq \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} + \frac{1}{\nu} \inf_{q_h \in Q_h} \|p - q_h\|_Q .$$

Note, that the pressure is induced by a gradient field $(3x^2, 3y^2)$, so an error in the approximation of the pressure results in an even bigger error for the velocity.

Vertex reconstruction

Now we use the vertex reconstruction $\mathcal{R}_h^V$ for the right hand side and again analyze the error. Due to the properties of the operator we expect a lower convergence rate, namely

$$\|u - u_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \preccurlyeq h^2|u|_{H^3(\Omega)} + h^2|p|_{H^2(\Omega)} + h \|f\|_{L^2(\Omega)} ,$$

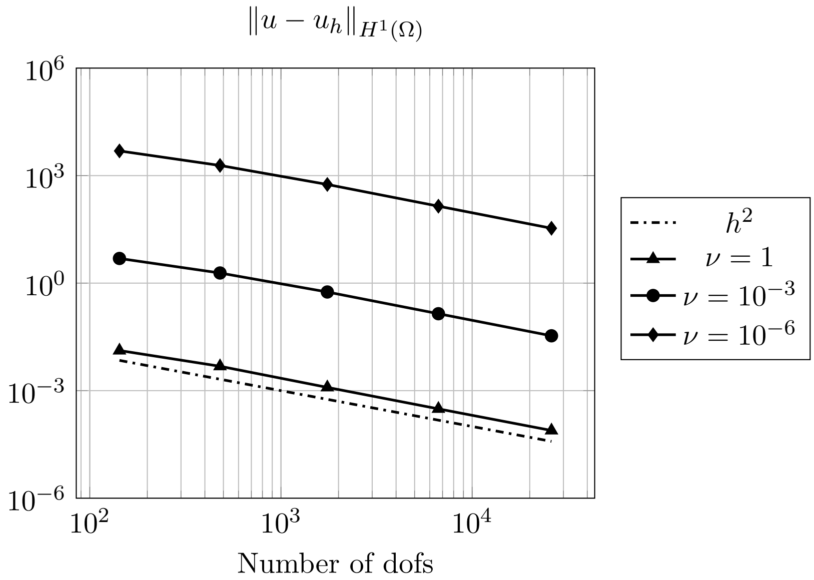


Figure 4.3: H^1 convergence rate for u of the $P_2 - P_1$ Taylor-Hood element for different viscosities

so only a linear convergence. In figure 4.4 we see that the error of the velocity still has an $\mathcal{O}(h^2)$ convergence, which is better than expected. The reason for this is not yet known, but still the convergence of the pressure error is less than compared to the Taylor-Hood element, namely $\mathcal{O}(h^{3/2})$. Anyway, the reconstruction helped in the dependency of ν as we can observe in figure 4.5, where we see no impact at all.

Element reconstruction

Finally we use the element reconstruction $\mathcal{R}_h^T$ for the right hand side. Again we observe the error and expect to see the same rates as for the Taylor-Hood element

$$\|u - u_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \preccurlyeq h^2 |u|_{H^3(\Omega)} + h^2 |p|_{H^2(\Omega)} + h^2 \|f\|_{H^1(\Omega)},$$

and

$$\|u - u_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \preccurlyeq h^3 |u|_{H^3(\Omega)} + h^3 |p|_{H^2(\Omega)} + h^3 \|f\|_{H^1(\Omega)},$$

but no impact of the viscosity. In figure 4.6 we see, that compared to the vertex reconstruction, the pressure has also an $\mathcal{O}(h^2)$ convergence rate and in figure 4.7 we can observe that changing the viscosity ν makes no difference.

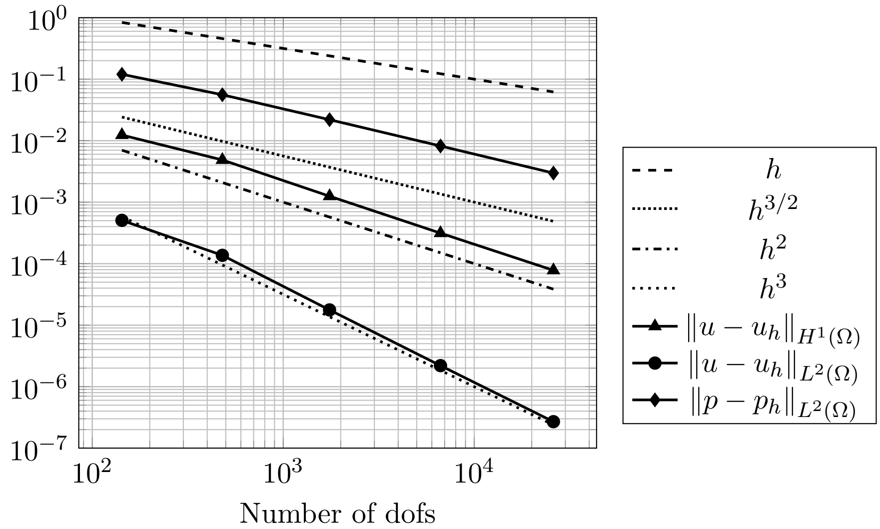


Figure 4.4: Convergence rates of the $P_2 - P_1$ Taylor-Hood element for $\nu = 1$ using $\mathcal{R}_h^V$ for the right hand side

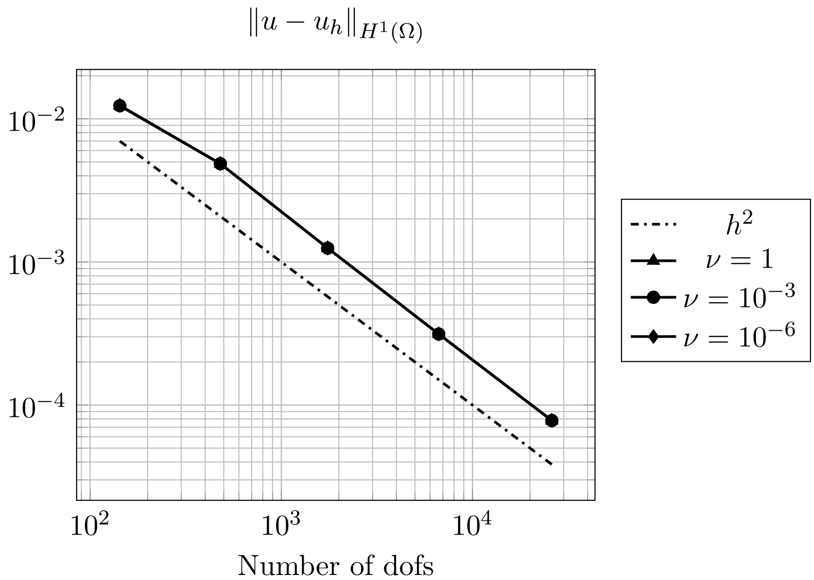


Figure 4.5: H^1 convergence rate for u of the $P_2 - P_1$ Taylor-Hood element for different viscosities using $\mathcal{R}_h^V$ for the right hand side

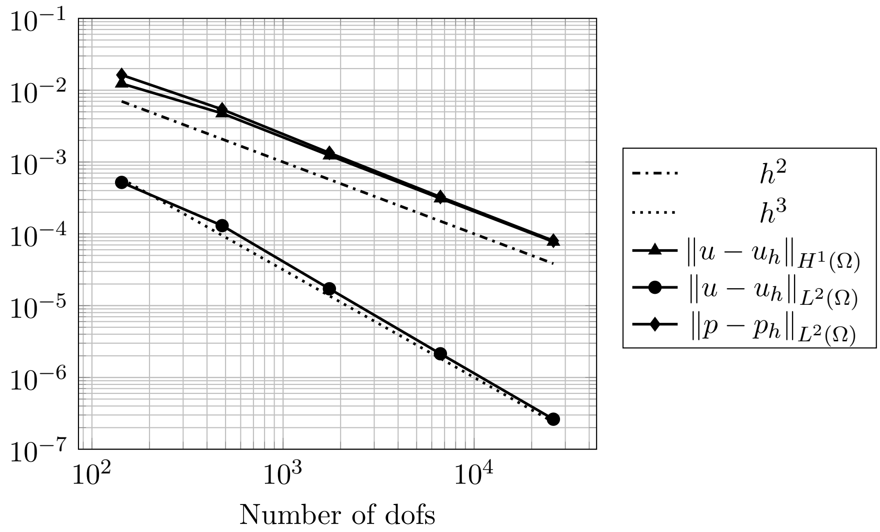


Figure 4.6: Convergence rates of the $P_2 - P_1$ Taylor-Hood element for $\nu = 1$ using $\mathcal{R}_h^T$ for the right hand side

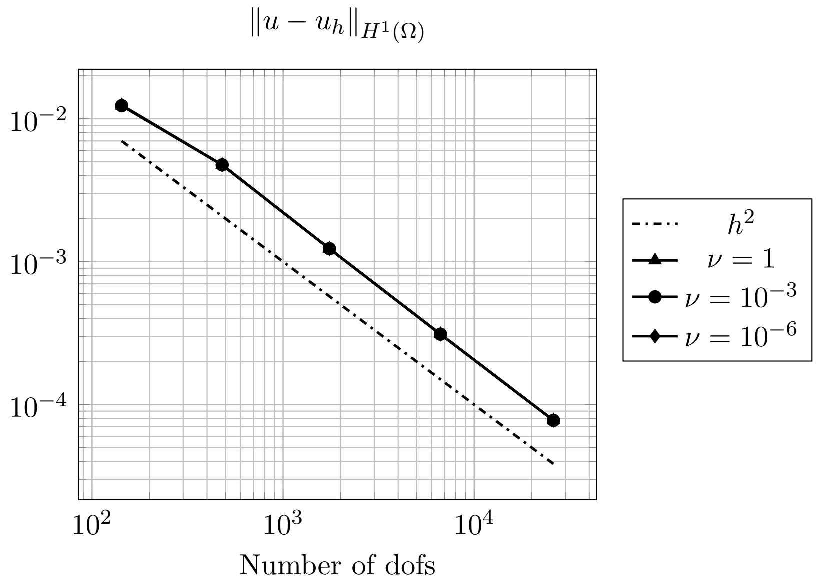


Figure 4.7: H^1 convergence rate for u of the $P_2 - P_1$ Taylor-Hood element for different viscosities using $\mathcal{R}_h^T$ for the right hand side

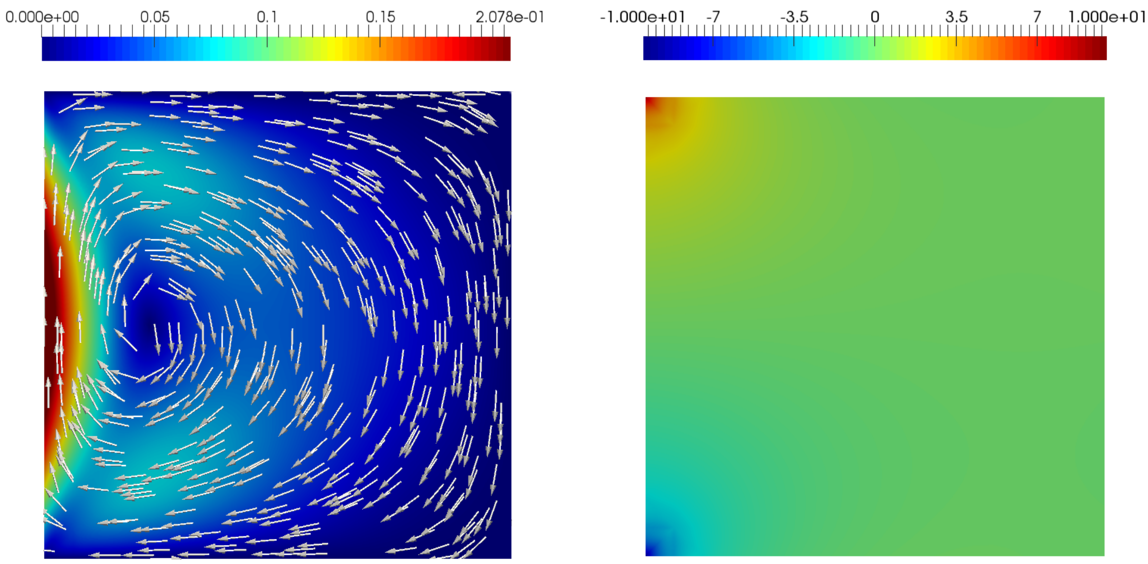
4.4.2 Stokes example with post processing

In this example we use the reconstruction operator $\mathcal{R}_h^T$ as a post processing tool to generate a divergence-free velocity $\mathcal{R}_h^T(u_h)$. The domain is the unit square $\Omega := (0, 1) \times$

(0, 1) and we induce a flow by non homogeneous boundary conditions

$$u = (0, y(1-y)) \quad \text{on} \quad \{0\} \times (0,1)$$
$$u = (0, 0) \quad \text{on} \quad \partial\Omega \setminus \{0\} \times (0,1).$$

The viscosity is chosen as $\nu = 1$. In figure 4.8 the induced flow field u_h and the pressure p_h is shown, and in figure 4.9 we can observe an $\mathcal{O}(h)$ convergence of the error $\|\text{div}\, u_h\|_{L^2(\Omega)}$ although the error is really high, so the incompressibility constraint is not approximated very well. When we use the reconstruction the divergence is reduced to a value $\mathcal{O}(1e^{-16})$ for each refinement level. Due to the error analysis of the reconstruction operator, we also expect a good convergence in the H^1 and the L^2 norm.



(a) Velocity field and absolute value $|u_h|$

(b) Pressure p_h

Figure 4.8: Approximated solutions of the second example

4.4.3 Hagen-Poiseuille channel flow using the curl formulation of the Navier Stokes equations

Until now we only focused on the steady Stokes equations. In this example we want to analyze a Hagen-Poiseuille channel flow approximated by the unsteady Navier Stokes equations. We consider the domain $\Omega = (0, 10) \times (0, 2)$ with Dirichlet and Neumann boundaries

$$\Gamma_N := \{10\} \times (0,2) \quad \text{and} \quad \Gamma_D := \partial\Omega \setminus \Gamma_N,$$

viscosity $\nu = 10^{-3}$ and together with

$$c_{curl}(u_h, u_h, v_h) = \int_\Omega (\text{curl } u_h \times u_h) \cdot v_h \quad \forall v_h \in V_h$$

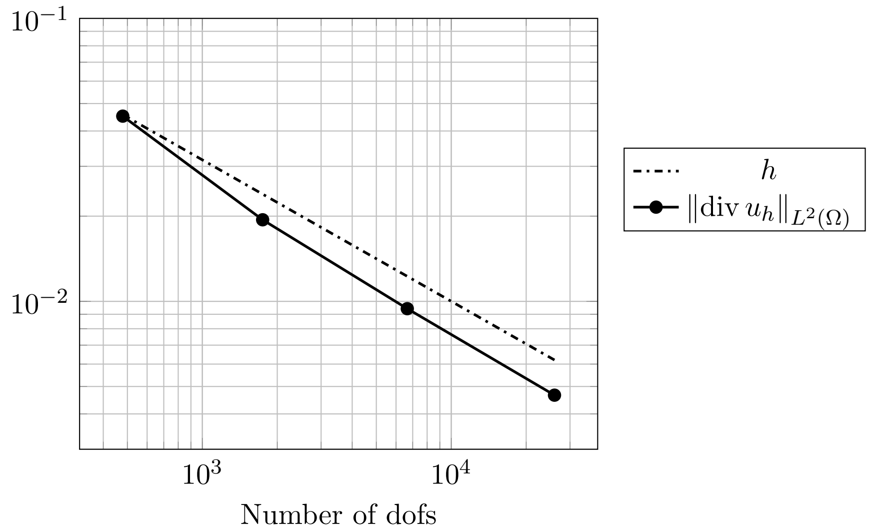


Figure 4.9: $\|\operatorname{div} u_h\|_{L^2(\Omega)}$ for the $P_2 - P_1$ Taylor-Hood element

we solve the unsteady Navier Stokes equations using a first order IMEX scheme introduced in section 3.5.2 with timestep $\Delta t = 0.001$ and a steady Stokes solution as startvalue $u(x, 0) = u_{stokes}$. Using the boundary conditions

$$u = (10y(2-y), 0) \quad \text{on} \quad \{0\} \times (0, 2)$$
$$u = (0, 0) \quad \text{on} \quad \Gamma_D \setminus (\{0\} \times (0, 2) \cup \Gamma_N),$$

the exact solution is given by:

$$u = (10y(2-y), 0) \quad \text{and} \quad p = 0.02x - 0.1 \tag{4.4.1}$$

This example is chosen so that the convective term of the Navier Stokes equations is equal to zero

$$(u \cdot \nabla)u = 0.$$

Using the identity

$$(u \cdot \nabla)u = \text{curl } u \times u + \frac{1}{2}\nabla u^2$$

we get

$$\text{curl } u \times u = -\frac{1}{2}\nabla u^2.$$

In this case the convective term is a gradient field and so due to the Helmholtz decomposition 3.3.3 we expect a bad approximation using discrete divergence-free test functions like the $P_2 - P_1$ Taylor-Hood element preserves. To solve this problem we use the reconstruction operator in the convective term, namely

$$c_{curl}(u_h, u_h, v_h) = \int_\Omega (\text{curl } u_h \times \mathcal{R}_h^T(u_h)) \cdot \mathcal{R}_h^T(v_h) \, \mathrm{d}x \quad \forall v_h \in V_h.$$

We use the reconstruction operator on the one hand for the test function v_h to get rid of the mentioned problem above and on the other hand on u_h to make the term antisymmetric. In figure 4.10 we compare the two approximations at $t = 0.5s$. We see that the reconstruction operator has a major impact on the exactness of the solution.

Remark 13: The curl formulation is chosen to present the properties of the extension operator. Although a standard convection formulation may result in a proper approximation for this example, there are cases where this is not valid anymore, so a curl formulation including the reconstruction can help.

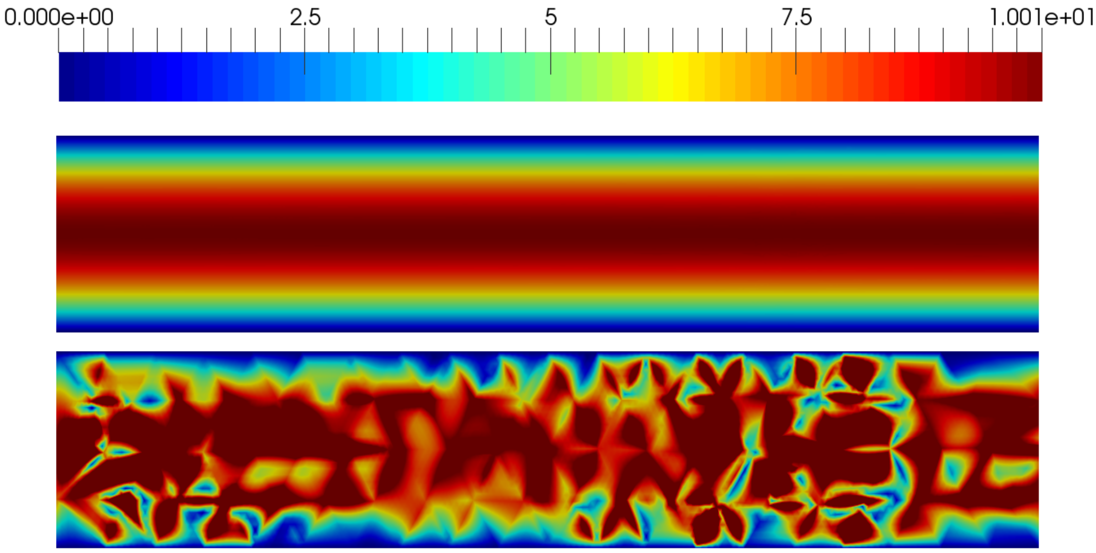


Figure 4.10: Absolute value $|u_h|$ of the approximation of the Hagen-Poiseuille flow at $t = 0.5s$. The upper picture is with the reconstruction $\mathcal{R}_h^T$ and the lower without

4.4.4 Two phase bubble Stokes example

The last example we give is a two phase Stokes flow. For this problem the exact solution is given by a discontinuous pressure that we approximate with continuous linear polynomials. Due to the coupling of the pressure and the velocity this bad approximation results in a non-physical behaviour of the velocity. We consider the domain $\Omega = \Omega_1 \cup \Omega_2$ with

$$\Omega_2 := \{(x, y) : x^2 + y^2 \leq 0.5\} \quad \text{and} \quad \Omega_1 := (-1, 1) \times (-1, 1) \setminus \Omega_2,$$

zero volume forces and introduce a force on the interface $\partial\Omega_2$ to solve:

$$\begin{aligned} -\nu\Delta u + \nabla p &= 0 \quad \text{in } \Omega_i, i = 1,2 \\ \operatorname{div} u &= 0 \quad \text{in } \Omega_i, i = 1,2 \\ [\![u]\!] &= 0 \quad \text{on } \partial\Omega_2 \\ [\![-\nu\nabla u \cdot n + p \cdot n]\!] &= f_s \quad \text{on } \partial\Omega_2 \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

On the interface we use the surface tension force given by

$$f_s = -\tau \int_{\partial\Omega_2} \kappa \cdot n \, \mathrm{d}s,$$

where τ is a constant and κ is the mean curvature. In the case of a circle it is given by

$$\kappa = \frac{1}{R},$$

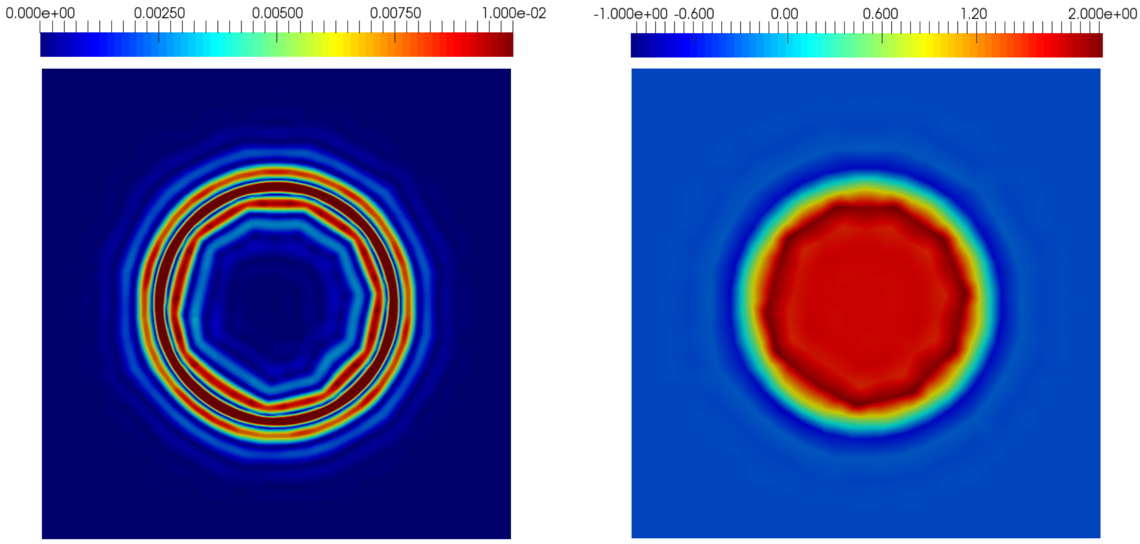
where R is the radius. For $\partial\Omega_2$ and $\tau = 1$ this results in

$$f_s = -2 \int_{\partial\Omega_2} n \, \mathrm{d}s.$$

With $K := \frac{\pi}{4}$ the exact velocity and pressure is given by

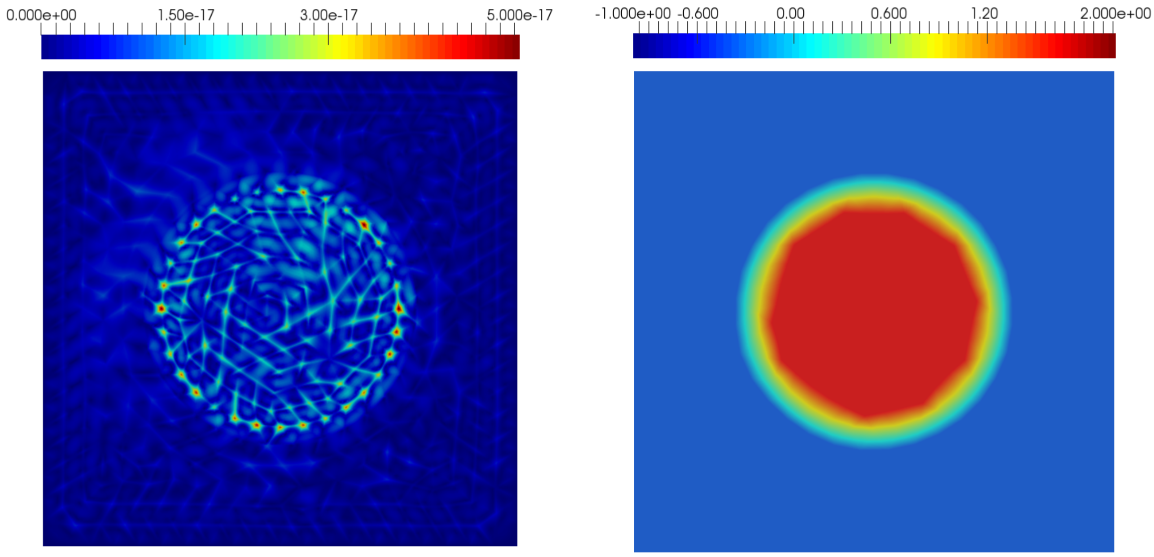
$$\begin{aligned} &u = (0,0) \text{ in } \Omega \\ &p = -0.5K \text{ in } \Omega_1 \\ &p = -0.5K + 2 \text{ in } \Omega_2. \end{aligned}$$

We see that the pressure is discontinuous across the boundary $\partial\Omega_2$ and the velocity is constant zero. Using the standard Taylor-Hood element results in a bad approximation of the pressure and due to that an influence on the velocity so $u_h \neq (0,0)$. In figure 4.11 we see the solution for $\nu = 1e^{-3}$ for the Taylor-Hood element and in figure 4.12 using the reconstruction on the surface tension. In figure 4.13 we also plotted the pressure of the two solutions along the line from $(-1,0)$ to $(1,0)$. One can see that the standard Taylor-Hood element produces oscillations in contrast to the reconstruction.



(a) Absolute value $|u_h|$ (b) Pressure p_h

Figure 4.11: Approximated solutions of the fourth example with the standard Taylor-Hood element



(a) Absolute value $|u_h|$ (b) Pressure p_h

Figure 4.12: Approximated solutions of the fourth example using the reconstruction operator on the surface tension

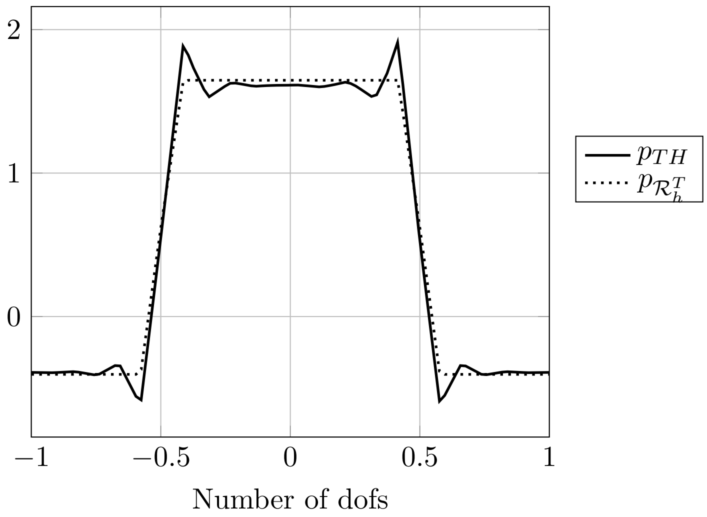


Figure 4.13: Pressure of the fourth example from $(-1, 0)$ to $(1, 0)$ for the Taylor-Hood element with and without using the reconstruction operator on the surface tension

In figure 4.14 we see the error of the standard Taylor-Hood element. The L^2 error of pressure and the H^1 error of the velocity seem to converge only with order $\mathcal{O}(h^{1/2})$ and the L^2 error of the velocity converges with a rate between $\mathcal{O}(h)$ and $\mathcal{O}(h^{3/2})$. Still, all errors are really high. In figure 4.15 we can see the effect of using the reconstruction operator $\mathcal{R}_h^T$ on the surface tension force. The L^2 and H^1 error of the velocity is reduced close to zero and the L^2 error seems to converge with order $\mathcal{O}(h^{1/2})$. The L^2 pressure error (see figure 4.16) also seems to converge with order $\mathcal{O}(h^{1/2})$.

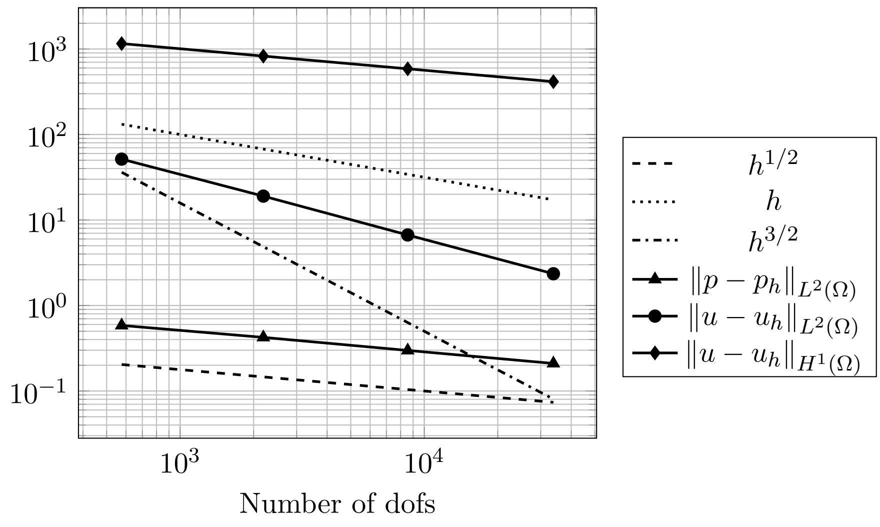


Figure 4.14: Error convergence of p_h and u_h for the two phase bubble flow without the reconstruction of the surface tension

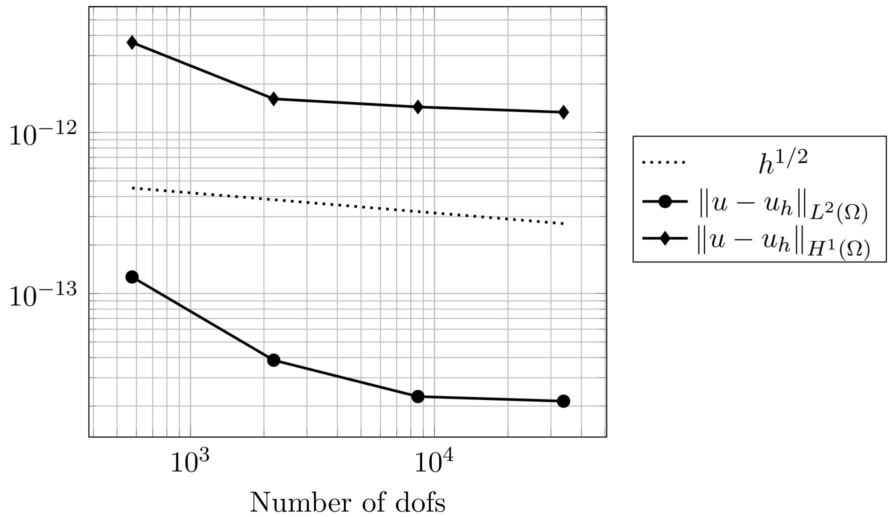


Figure 4.15: Error convergence of u_h for the two phase bubble flow with reconstruction of the surface tension

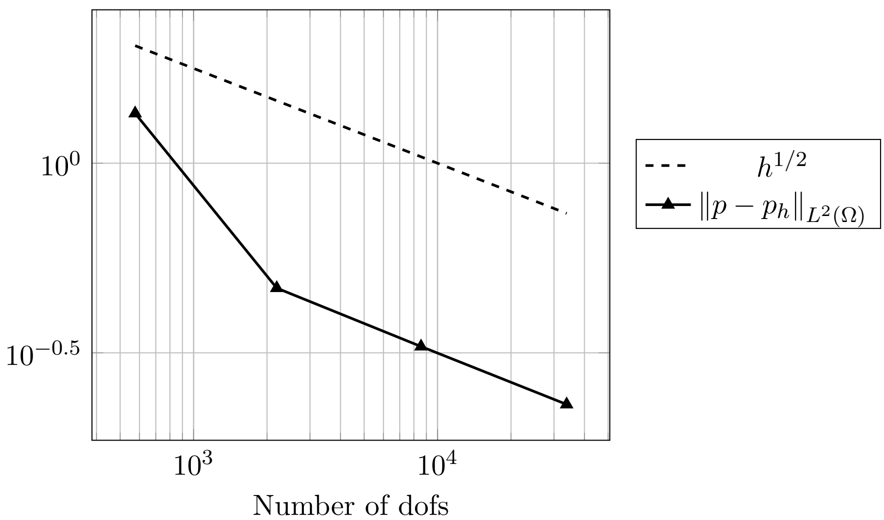


Figure 4.16: Error convergence of p_h for the two phase bubble flow with reconstruction of the surface tension

5 Hybrid discontinuous Galerkin method for the Navier Stokes equations

In this chapter we use a discretization method that was introduced by Joachim Schöberl and Christoph Lehrenfeld in [Leh10] and in [LS15]. In section 3.4 we already introduced a finite element pair with a discontinuous pressure that leads to an element wise divergence preserving velocity field but has a worse convergence order. The introduced method goes even further by using a hybridised discontinuous Galerkin approach for the velocity as well as a discontinuous pressure approximation. In the first section we present the method and show some existing results. We see that there already exists an h independent version of the discrete LBB-condition, so showing also the independence of the polynomial order k is the outcome of the second section. In the third section we introduce an implementation of the curl convection and finally show a numerical result in the last section.

5.1 Hybrid discontinuous Galerkin method for the Stokes equations

5.1.1 Hybrid discontinuous Galerkin method

We want to use a discontinuous Galerkin (DG) approach but want to construct a method that can generate exact divergence-free flow fields. This yields to an $H(\mathrm{div})(\Omega)$ conforming discretization, namely to normal continuous velocity components over edges of finite elements. Therefore we introduce the space

$$W_h := \{u_h^T \in [\Pi^k(\mathcal{T})]^d : [\![u_h \cdot n]\!]_E = 0 \; \forall E \in \mathcal{F}\},$$

where E is an element of the triangulation skeleton $\mathcal{F}$ and $[\![u_h \cdot n]\!]_E$ is the jump on the common edge E of two elements T_1 and T_2

$$[\![u_h \cdot n]\!]_E = \left(u_h|_{T_1} - u_h|_{T_2} \right) \cdot n_1.$$

For the pressure we use the space $Q_h := \Pi^{k-1}(\mathcal{T})$. By that we get the property $\mathrm{div}\,(W_H) = Q_h$, see ([LS15]). As functions of W_h do not belong to $H^1(\Omega)$ anymore we use the interior penalty method introduced in [Arn82] to weakly enforce continuity of the velocity flow field but use a hybridised version to avoid a full coupling between neighbours. For this we add additional unknowns on the skeleton and define the facet space as

$$F_h := \{u_h^F \in [\Pi^k(\mathcal{F})]^d : u_h^F \cdot n = 0\},$$

for the tangential trace of the velocity and introduce the velocity space as $V_h := W_h \times F_h$.

Remark 14: The space F_h was only introduced to implement a more efficient handling of linear systems, see [LS15], in fact it is essentially the same as the interior penalty DG formulation.

Remark 15: For the construction of W_h we used BDM^k *finite elements*

Remark 16: In this thesis we only consider the two dimensional case $d = 2$. By that we can define a tangential vector τ on each edge.

5.1.2 Approximation of the Stokes equations

Using the interior penalty method [Arn82] and the notation $u_h = (u_h^T, u_h^F)$, $v_h = (v_h^T, v_h^F)$ and $[\![v_h^\tau]\!] := [(v_h^T - v_h^F)\cdot\tau]\tau$ and $[\![u_h^\tau]\!] := [(u_h^T - u_h^F)\cdot\tau]\tau$ we define the bilinear forms for the Stokes problem as in [LS15] by

$$\begin{aligned} a_{HDG}(u_h, v_h) :=& \sum_{T\in\mathcal{T}} \int_T \nu\nabla u_h^T : \nabla v_h^T \; \mathrm{d}x - \int_{\partial T} \nu \frac{\partial u_h^T}{\partial n} [\![v_h^\tau]\!] \; \mathrm{d}s \\ &- \int_{\partial T} \nu \frac{\partial v_h^T}{\partial n} [\![u_h^\tau]\!] \; \mathrm{d}s + \int_{\partial T} \frac{\alpha}{h} [\![u_h^\tau]\!][\![v_h^\tau]\!] \; \mathrm{d}s, \\ b_{HDG}(u_h, q_h) :=& \sum_{T\in\mathcal{T}} \int_T \operatorname{div} u_h^T \; q_h \; \mathrm{d}x. \end{aligned}$$

Using the norm

$$\|v_h\|^2_{H^1_{HDG}(\Omega)} := \sum_{T\in\mathcal{T}} \|\nabla v_h\|^2_{L^2(T)} + \frac{k^2}{h} \|[\![v_h^\tau]\!]\|^2_{L^2(\partial T)}$$

we can show for sufficiently large α continuity of the bilinear forms $a_{HDG}(\cdot,\cdot)$ and $b_{HDG}(\cdot,\cdot)$ and coercivity on the kernel. This was shown for an h-version already in [Leh10] where an inverse trace inequality (see [WH03]) was needed

$$\|\nabla q_h \cdot n\|_{L^2(\partial T)} \preccurlyeq \frac{k^2}{h} \|\nabla q_h\|_{L^2(T)} \quad \forall q_h \in [\Pi^k(T)]^2.$$

We use this inequality also to see that $\|v_h\|^2_{H^1_{HDG}(\Omega)}$ is equivalent (on the discrete space W_h) to

$$\begin{aligned} \|v_h\|^2_{H^1_{HDG}(\Omega)} &\simeq \|v_h\|^2_{H^1_{HDG*}(\Omega)} \\ &:= \sum_{T\in\mathcal{T}} \|\nabla v_h\|^2_{L^2(T)} + \frac{k^2}{h} \|[\![v_h^\tau]\!]\|^2_{L^2(\partial T)} + \frac{h}{k^2} \left\| \frac{\partial v_h^T}{\partial n} \right\|^2_{L^2(\partial T)}. \end{aligned} \tag{5.1.1}$$

Also the discrete LBB-condition

$$\sup_{v_h\in V_h} \frac{\int_\Omega \operatorname{div} v_h \; q_h}{\|v_h\|_{H^1_{HDG}(\Omega)}} \geq \beta_h \|q_h\|_{L^2(\Omega)} \quad \forall q_h \in Q_h$$

with $\beta \neq \beta(h)$ was proven in [Leh10]. In the next section we present a proof that we also have $\beta \neq \beta(k)$, and so an independence of the polynomial degree k.

5.2 High order discrete LBB-condition

As mentioned in remark 14 the facet space F_h was only introduced for an efficient handling of linear systems. Due to that fact we change to a DG formulation and notation in this section. Under the assumption that the polynomial degree $k > 2$, we want to show the global discrete LBB-condition for $V_h = BDM_k(\mathcal{T})$ and $Q_h = \Pi^{k-1}(\mathcal{T})$, so

$$\sup_{v_h \in V_h} \frac{\int_\Omega \operatorname{div} v_h \; q_h \; \mathrm{d}x}{\|v_h\|_{H^1_{DG}(\Omega)}} \succcurlyeq \beta \|q_h\|_{L^2(\Omega)} \quad \forall q_h \in Q_h \tag{5.2.1}$$

with $\|v_h\|^2_{H^1_{DG}(\Omega)} = \sum_{T\in\mathcal{T}} \|\nabla v_h\|_{L^2(T)} + \sum_{E\in\mathcal{T}} \frac{k^2}{h} \|[\![v_h \cdot \tau]\!]\|^2_{L^2(E)}$ and $\beta \neq \beta(k)$. For this, we split the proof in two steps:

i. First we prove the local LBB-condition on the reference element $\hat{T}$ for test functions $q \in \Pi^{k-1}(\hat{T}) \cap L^2_0(\hat{T})$.

ii. We show the global LBB-condition using a transformation to the reference triangle.

For the first step we use an extension that is continuous with respect to the H^2 norm and preserves polynomials.

Theorem 5.1. *Assume a given function* $u \in [\Pi^k(\hat{T})]^2$ *where* $\hat{T}$ *is the reference Element and* $\int_{\partial\hat{T}} u \cdot n \; \mathrm{d}x = 0$. *Then there exists an operator* $\mathcal{E} : [\Pi^k(\hat{T})]^2 \to \Pi^{k+1}(\hat{T})$ *so that for* $\phi = \mathcal{E}(u)$ *we have*

$$\operatorname{curl} \phi \cdot n = u \cdot n \text{ on } \partial\hat{T} \tag{5.2.2}$$

$$\|(u - \operatorname{curl} \phi) \cdot \tau\|_{L^2(\partial\hat{T})} \preccurlyeq \frac{1}{k} \|u\|_{H^1(\hat{T})} \tag{5.2.3}$$

$$\|\phi\|_{H^2(\hat{T})} \preccurlyeq \|u\|_{H^1(\hat{T})} \tag{5.2.4}$$

The existence of such an extension is not trivial and is proven in the last chapter 6.

5.2.1 Local LBB-condition

Theorem 5.2. *Let* $\hat{T}$ *be the reference element with the vertices* $(0,0),(0,1),(1,0)$. *Then*

$$\sup_{v_h \in \mathrm{BDM}_0^k(\hat{T})} \frac{\int_{\hat{T}} \operatorname{div} v_h \; q_h \; \mathrm{d}x}{\|v_h\|_{H^1_{DG,0}(\hat{T})}} \geq \beta \|q_h\|_{L^2(\hat{T})} \quad \forall q_h \in \Pi^{k-1}(\hat{T}) \cap L^2_0(\hat{T}) \tag{5.2.5}$$

With $\mathrm{BDM}_0^k(\hat{T}) = \{v \in \mathrm{BDM}^k(\hat{T}) : v \cdot n = 0 \text{ on } \partial\hat{T}\}$ *and* $\|v_h\|^2_{H^1_{DG,0}(\hat{T})} := \|\nabla v_h\|^2_{L^2(\hat{T})} + \sum\limits_{E\subset\partial\hat{T}} k^2 \|v_h \cdot \tau\|^2_{L^2(E)}$, *and* $\beta \neq \beta(k)$.

Proof. Choose an arbitrary $p \in \Pi^{k-1}(\hat{T}) \cap L_0^2(\hat{T})$. We define a Poincare-Operator $\mathcal{Z}_a$ by

$$\begin{aligned}\mathcal{Z}_a :& \Pi^{k-1}(\hat{T}) \to [\Pi^k(\hat{T})]^2 \\ & p \mapsto \mathcal{Z}_a(p)(x) := (x-a)\int_0^1 tp(\gamma(t))\,\mathrm{d}t,\end{aligned}$$

with $a \in \hat{T}$ and $\gamma(t) := a + t(x-a)$, and define $\mathcal{Z}$ by

$$\mathcal{Z}(p)(x) := \int_{\hat{T}} \theta(a)\mathcal{Z}_a(p)(x)\,\mathrm{d}a,$$

where $\theta \in C_0^\infty(\hat{T})$ and $\int_{\hat{T}} \theta\,\mathrm{d}x = 1$. By that we get $\operatorname{div}\mathcal{Z}(p) = p$ and as the operator is also continuous with respect to the H^1 norm (see [CM08]) we have for $u_1 := \mathcal{Z}(p)$

$$\|u_1\|_{H^1(\hat{T})} \preccurlyeq \|p\|_{L^2(\hat{T})}. \tag{5.2.6}$$

In figure 5.1 we can see an example for $p = 12xy - 1$ and the corresponding velocity field $\mathcal{Z}_a$ for $a = (0.2, 0.2)$

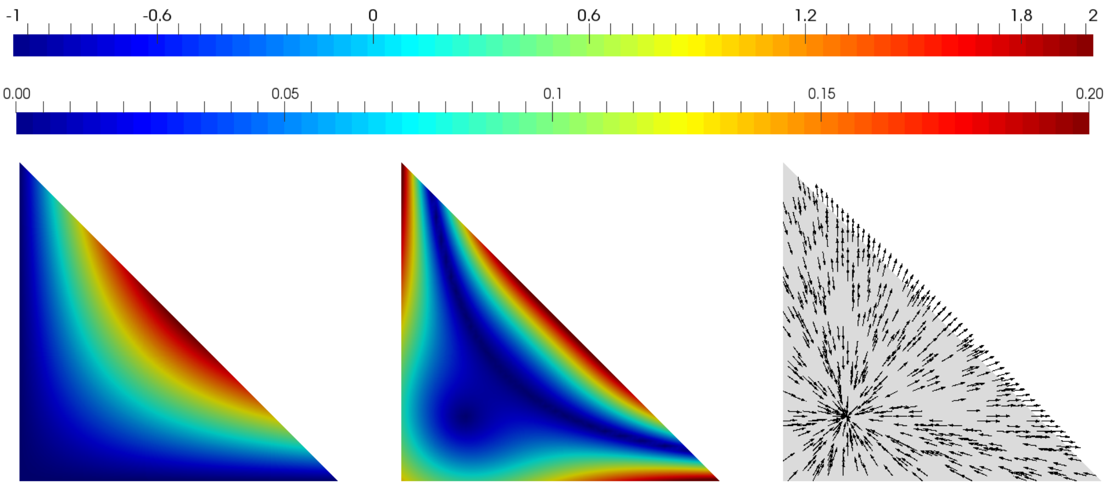


Figure 5.1: Pressure p, absolute value $|\mathcal{Z}_a|$ and the velocity field $\mathcal{Z}_a$. The upper scale is for the pressure, the lower for $|\mathcal{Z}_a|$

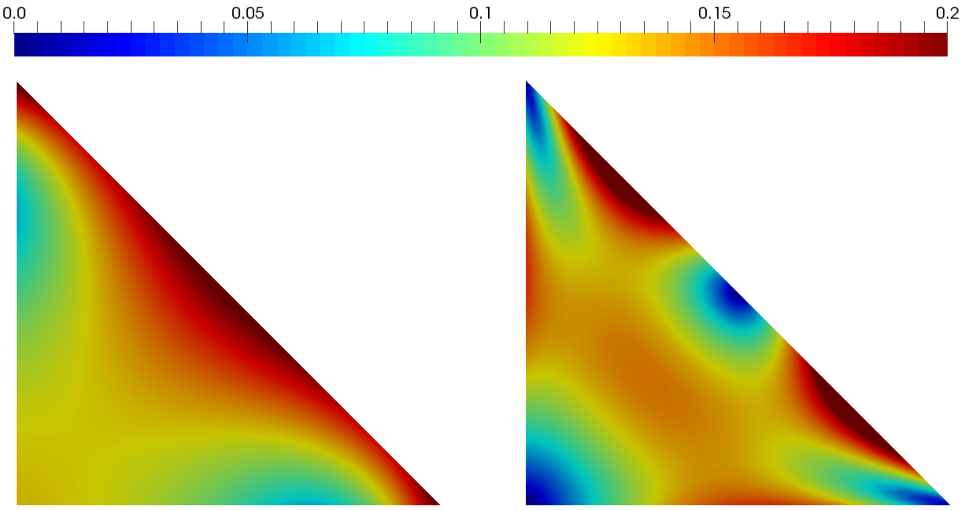
Next we use theorem 5.1 to find a function $\phi \in \Pi^{k+1}(\hat{T})$ with

$$\begin{aligned}&\operatorname{curl}\phi \cdot n = -u_1 \cdot n \quad \text{on} \quad \partial\hat{T} \\ &\|(u_1 + \operatorname{curl}\phi)\cdot\tau\|_{L^2(\partial\hat{T})} \preccurlyeq \frac{1}{k}\|u_1\|_{H^1(\hat{T})} \\ &\|\phi\|_{H^2(\hat{T})} \preccurlyeq \|u_1\|_{H^1(\hat{T})}.\end{aligned} \tag{5.2.7}$$

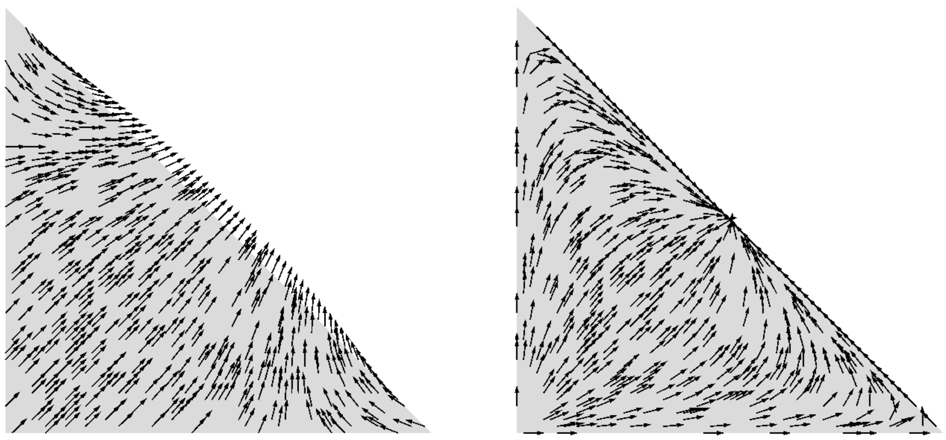
Let $u_2 := \operatorname{curl}\phi$ and $u := u_1 + u_2$, then

$$\begin{aligned}u\big|_{\partial\hat{T}} \cdot n = (u_1\big|_{\partial\hat{T}} + u_2\big|_{\partial\hat{T}}) \cdot n = (u_1\big|_{\partial\hat{T}} - u_1\big|_{\partial\hat{T}}) \cdot n = 0 \\ \operatorname{div} u = \operatorname{div} u_1 + \operatorname{div} u_2 = p + \operatorname{div}\operatorname{curl}\phi = p.\end{aligned}$$

In figure 5.2 we can see the vector field curl ϕ and the resulting velocity u for the example from above. Note that the normal trace of the result is equal to zero.



(a) Absolute value of $|\text{curl } \phi|$ and $|u|$



(b) Left we see the field of curl ϕ and right the resulting field u

Figure 5.2: Result after the correction $u = u_1 + \text{curl } \phi$

For the u we have the estimation

$$\begin{aligned}
\|u\|^2_{H^1_{DG,0}(\hat{T})} &\preccurlyeq \|u_1\|^2_{H^1_{DG,0}(\hat{T})} + \|u_2\|^2_{H^1_{DG,0}(\hat{T})} \\
&\preccurlyeq \|u_1\|^2_{H^1_{DG,0}(\hat{T})} + \underbrace{\|\nabla u_2\|^2_{L^2(\hat{T})}}_{\leq \|\phi\|^2_{H^2(\hat{T})}} + \sum_{E \subset \partial \hat{T}} k^2 \|u_2 \cdot \tau\|^2_{L^2(E)} \\
&\preccurlyeq \|u_1\|^2_{H^1(\hat{T})} + \sum_{E \subset \partial \hat{T}} k^2 \|u_1 \cdot \tau\|^2_{L^2(E)} + \sum_{E \subset \partial \hat{T}} k^2 \|u_2 \cdot \tau\|^2_{L^2(E)} .
\end{aligned}$$

Together with

$$\|(u_1 + u_2) \cdot \tau\|^2_{L^2(E)} \preccurlyeq \frac{1}{k^2} \|u_1\|^2_{H^1(\hat{T})}$$

and the triangle inequality we get

$$\|u\|^2_{H^1_{DG,0}(\hat{T})} \preccurlyeq \|u_1\|^2_{H^1(\hat{T})} \preccurlyeq \|p\|^2_{L^2(\hat{T})}. \tag{5.2.8}$$

With that we see

$$\begin{aligned}\sup_{v_h \in \mathrm{BDM}_0^k(\hat{T})} \frac{\int_\Omega \operatorname{div} v_h \; p \; \mathrm{d}x}{\|v_h\|_{H^1_{DG,0}(\hat{T})}} &\succcurlyeq \frac{\int_\Omega \overbrace{\operatorname{div} u}^{=p} \; p \; \mathrm{d}x}{\|u\|_{H^1_{DG,0}(\hat{T})}} = \frac{\|p\|^2_{L^2(\hat{T})}}{\|u\|_{H^1_{DG,0}(\hat{T})}} \\ &\succcurlyeq \frac{\|u\|_{H^1_{DG,0}(\hat{T})} \|p\|_{L^2}}{\|u\|_{H^1_{DG,0}(\hat{T})}} \succcurlyeq \|p\|_{L^2}. \end{aligned} \tag{5.2.9}$$

□

5.2.2 Global LBB-condition

Theorem 5.3. *Let $k > 2$, $V_h = BDM_k(\mathcal{T})$ and $Q_h = \Pi^{k-1}(\mathcal{T})$. Then we have*

$$\sup_{v_h \in V_h} \frac{\int_\Omega \operatorname{div} v_h \; q_h \; \mathrm{d}x}{\|v_h\|_{H^1_{DG}(\Omega)}} \geq \beta \|q_h\|_{L^2(\Omega)} \quad \forall q_h \in Q_h \tag{5.2.10}$$

with $\|v_h\|^2_{H^1_{DG}(\Omega)} = \sum_{T \in \mathcal{T}} \|\nabla v_h\|_{L^2(T)} + \sum_{E \in \mathcal{T}} \frac{k^2}{h} \|[\![v_h \cdot \tau]\!]\|^2_{L^2(E)}$ and $\beta \neq \beta(k)$.

Proof. To show the global LBB-condition we proceed in two steps. First we construct a low order function $u_{h,1}$ and then we define a correction $u_{h,2}$ using the local LBB-condition (5.2.5).
Step 1: Assume an arbitrary $p_h \in \Pi^{k-1}(\mathcal{T})$. First we construct a function $u_{h,1} \in \left([\Pi^2(\mathcal{T})]^2 \cap [C^0(\Omega)]^2\right) \subset [H^1(\Omega)]^2$ with

$$\overline{\operatorname{div} u_{h,1}}^{\mathcal{T}} = \overline{p_h}^{\mathcal{T}},$$

where

$$\overline{p_h}^{\mathcal{T}}\big|_T = \int_T p_h \; \mathrm{d}x \quad \forall T \in \mathcal{T}.$$

To find $u_{h,1}$ we solve a kind of Stokes problem on $[H^1(\Omega)]^2$ with a different right hand side for the divergence

Problem 12: *Find* $(u_S, p_S) \in ([H^1(\Omega)]^2 \times L_0^2)$ *so that*

$$\int_\Omega \nu \nabla u : \nabla v \;\mathrm{d}x + \int_\Omega \operatorname{div} p_S \; q \;\mathrm{d}x, = 0 \quad \forall v \in V$$
$$\int_\Omega \operatorname{div} u_S \; q \;\mathrm{d}x, = \int_\Omega \overline{p_h}^{\mathcal{T}} \; q \;\mathrm{d}x \quad \forall q \in Q$$

Due to the LBB-condition for the continuous Stokes problem (3.2.5) this problem has an unique solution and we have

$$\operatorname{div} u_S = \overline{p_h}^{\mathcal{T}} \quad \text{and} \quad \|u_S\|_{H^1(\Omega)} \preccurlyeq \left\|\overline{p_h}^{\mathcal{T}}\right\|_{L^2(\Omega)}.$$

Next we use the Fortin $\Pi^{\mathcal{F}}$ operator we introduced in section 3.4.2, to define

$$u_{h,1} := \Pi^{\mathcal{F}}(u_S) \quad \in [\Pi^2(\mathcal{T})]^2 \cap [C^0(\Omega)]^2$$

Using the properties of $\Pi^{\mathcal{F}}$ we have

$$\int_T \operatorname{div} u_{h,1} \;\mathrm{d}x = \int_T \operatorname{div} u_S \;\mathrm{d}x = \int_T \overline{p_h}^{\mathcal{T}} \;\mathrm{d}x \quad \forall T \in \mathcal{T}$$
$$\left\|u_h^1\right\|_{H^1(\Omega)} \preccurlyeq \|u_S\|_{H^1(\Omega)} \preccurlyeq \left\|\overline{p_h}^{\mathcal{T}}\right\|_{L^2(\Omega)}.$$

Step 2: Next we define

$$p_h^2 := p_h - \operatorname{div} u_h^1 \in \Pi_0^{k-1}(T) \cap L_0^2(T) \quad \forall T \in \mathcal{T}.$$

Now for all $T \in \mathcal{T}$ we look for a function u_h^T with

$$\operatorname{div} u_h^T = p_h^2\big|_T$$
$$u_h^T \cdot n = 0 \text{ on } \partial T$$
$$\left\|u_h^T\right\|_{H^1_{DG}(T)} \preccurlyeq \left\|p_h^2\right\|_{L^2(T)}.$$

We do that by solving a problem on the reference element. For each element we define a reference right hand side $\hat{p} := h^2 \; p_h^2\big|_T$. Due to the local LBB-condition (5.2.5) we find a function $\hat{u}$ with

$$\operatorname{div} \hat{u} = \hat{p}$$
$$\hat{u} \cdot n = 0 \text{ on } \partial \hat{T}$$
$$\|\hat{u}\|_{H^1_{DG,0}(\hat{T})} \preccurlyeq \|\hat{p}\|_{L^2(\hat{T})}.$$

Now we use the Piola transformation $\mathcal{P}$ (see appendix) to get the solution on T

$$u_h^T := \mathcal{P}(\hat{u}) = \frac{1}{\det(\mathcal{J})} \mathcal{J} \hat{u},$$

where $\mathcal{J}$ is the Jacobian matrix of the finite element mapping $F : \hat{T} \to T$. Due to the properties of $\mathcal{P}$ we get

$$\operatorname{div} u_h^T = \underbrace{\frac{1}{\det(\mathcal{J})}}_{=h^2} \operatorname{div} \hat{u} = \frac{1}{h^2}\hat{p} = p_h^2\big|_T$$

$$u_h^T \cdot n = 0 \text{ on } \partial\hat{T}$$

and as

$$\begin{aligned}\|\hat{u}\|_{H^1_{DG,0}(\hat{T})} &= \|\nabla\hat{u}\|^2_{L^2(\hat{T})} + \sum_{E\subset\partial\hat{T}} k^2 \|u_h \cdot \tau\|^2_{L^2(E)} \\ &= \underbrace{h^2 \left\|\nabla u_h^T\right\|^2_{L^2(T)} + h^2 \sum_{E\subset\partial T} \frac{k^2}{h} \left\|u_h^T \cdot \tau\right\|^2_{L^2(E)}}_{h^2\left\|u_h^T\right\|^2_{H^1_{DG}(T)}} \preccurlyeq \|\hat{p}\|^2_{L^2(\hat{T})} = h^2 \left\|p_h^2\right\|^2_{L^2(T)}\end{aligned}$$

we get

$$\left\|u_h^T\right\|_{H^1_{DG}(T)} \preccurlyeq \left\|p_h^2\right\|_{L^2(T)}.$$

Summing up the element wise solutions leads to $u_{h,2} := \sum_{T\in\mathcal{T}} u_h^T$ and we observe

$$\operatorname{div} u_{h,2} = p_h^2.$$

Using this correction $u_h := u_{h,1} + u_{h,2}$ we get

$$\operatorname{div}(u_h) = \operatorname{div}(u_{h,1}) + \operatorname{div}(u_{h,2}) = \operatorname{div}(u_{h,1}) + p_h - \operatorname{div}(u_{h,1}) = p_h,$$

and

$$\begin{aligned}\|u_h\|^2_{H^1_{DG}(\Omega)} &\le \underbrace{\|u_{h,1}\|^2_{H^1_{DG}(\Omega)}}_{=\left\|u_{h,1}\right\|^2_{H^1(\Omega)}} + \|u_{h,2}\|^2_{H^1_{DG}(\Omega)} \preccurlyeq \left\|\overline{p_h}^{\mathcal{T}}\right\|^2_{L^2(\Omega)} + \sum_{T\in\mathcal{T}} \|u_{h,2}\|^2_{H^1_{DG}(T)} \\ &\preccurlyeq \|p_h\|^2_{L^2(\Omega)} + \sum_{T\in\mathcal{T}} \left\|p_h^2\right\|^2_{L^2(T)} \preccurlyeq \|p_h\|^2_{L^2(\Omega)}.\end{aligned}$$

Finally we have

$$\begin{aligned}\sup_{v_h\in V_h} \frac{\int_\Omega \operatorname{div} v_h p_h \, \mathrm{d}x}{\|v_h\|_{H^1_{DG}(\Omega)}} &\succcurlyeq \frac{\int_\Omega \operatorname{div} u_h p_h \, \mathrm{d}x}{\|u_h\|_{H^1_{DG}(\Omega)}} = \frac{\int_\Omega (p_h)^2 \, \mathrm{d}x}{\|u_h\|_{H^1_{DG}(\Omega)}} \\ &\succcurlyeq \frac{\|u_h\|_{H^1_{DG}(\Omega)} \|p_h\|_{L^2(\Omega)}}{\|u_h\|_{H^1_{DG}(\Omega)}} \succcurlyeq \|p_h\|_{L^2(\Omega)}.\end{aligned}$$

□

5.2.3 Numerical estimations

In the previous section we showed that in the two dimensional case the discrete LBB constant is independent of the polynomial order k (on triangles). We also calculated the LBB constant numerically. The results are shown in table 5.1. As we can see the results are satisfying and also in the three dimensional case (for a tetrahedron) the independency seems to be fulfilled.

k	4	8	16	32
2D/trig	0.167	0.190	0.201	0.205
3D/tet	0.104	0.105	0.106	-

Table 5.1: Numerically calculated discrete LBB constant for the HDG method for 2 and 3 dimensions

5.3 Curl convection for the hybrid discontinuous Galerkin method

In this section we show an implementation of the curl convection $c_{curl}(u, u, v)$ for the HDG-method. For the time discretization we use the methods introduced in section 3.5, so the convection term only appears in an explicit form. The problem that appears is that for a function $u \in H(\mathrm{div})(\mathcal{T})$ the curl is only defined in a distributional sense. To see this choose an arbitrary function $\varphi \in C_0^\infty(\Omega)$ and the definition of the weak curl to formally get

$$\begin{aligned}\langle \mathrm{curl}\ u, \varphi \rangle &:= -\int_\Omega u \cdot \mathrm{curl}\ \varphi\ \mathrm{d}x = \sum_{T\in\mathcal{T}} \int_T u \cdot \mathrm{curl}\ \varphi\ \mathrm{d}x \\ &= \sum_{T\in\mathcal{T}} \int_T \mathrm{curl}\ u \cdot \varphi\ \mathrm{d}x + \sum_{E\in\mathcal{F}} \int_E [\![u \times n]\!] \cdot \varphi\ \mathrm{d}s.\end{aligned}$$

Using this for functions $w_h, u_h, v_h \in W_h$ and a standard DG upwind value $\hat{u}_h^T$ leads to

$$\begin{aligned}c_{curl}(w_h, u_h, v_v) = &\sum_{T\in\mathcal{T}} \int_T \mathrm{curl}\ w_h^T \cdot (u_h^T \times v_h^T)\ \mathrm{d}x \\ &+ \sum_{E\in\partial T} \int_E \left(\hat{w}_h^T \times n\right) \left(u_h^T v_h^F\right) \mathrm{d}s - \int_E \left(w_h^T \times n\right) \left(u_h^F v_h^T\right) \mathrm{d}s.\end{aligned}$$

It remains the implementation of the boundary term that leads to the Bernoulli pressure

$$\frac{1}{2} \int_{\partial\Omega} (u^2) v \cdot n\ \mathrm{d}s,$$

in the case of Neumann boundaries. For the HDG method this leads to

$$c_{bp}(u_h, v_h) := \sum_{E\in\Gamma_N} \int_E \frac{(u_h^T)^2 + (u_h^F)^2}{2} (v_h^T \cdot n)\ \mathrm{d}s.$$

Remark 17: *The choice of the upwind value appears only for the facet values v_h^F in the curl convection. This leads to a stable discretization although this is not natural for the tangential components.*

Remark 18: *We saw in section (4.4.3) that a reconstruction on exact divergence-free velocity functions improves the solution when using a curl formulation for the convective part of the Navier Stokes equations. We implemented the curl convection for the HDG method to show that due to the property* $\operatorname{div} V_h \subset Q_h$*, namely an exact divergence-free approximation, we can also use the curl formulation to get proper solutions. In [LS15] also the standard convection form was implemented.*

5.4 Approximation error

Due to the properties we are now able to show an optimal error convergence, namely an independent constant $c \neq c(k,h)$ in the error estimation. We start with the approximation error for the exact solutions (u,p) and the approximation u_h and the L^2 projection $\mathcal{P}_{Q_h}^{L^2} p$

$$\|u - u_h\|_{H^1_{HDG}(\Omega)} + \left\|\mathcal{P}_{Q_h}^{L^2} p - p_h\right\|_{L^2(\Omega)}.$$

Next we add and subtract a BDM interpolator to get

$$\begin{aligned}\|u - u_h\|_{H^1_{HDG}(\Omega)} + \left\|\mathcal{P}_{Q_h}^{L^2} p - p_h\right\|_{L^2(\Omega)} \leq{}& \left\|u - I_h^{\mathrm{BDM}^k} u\right\|_{H^1_{HDG}(\Omega)} \\ &+ \left\|I_h^{\mathrm{BDM}^k} u - u_h\right\|_{H^1_{HDG}(\Omega)} + \left\|\mathcal{P}_{Q_h}^{L^2} p - p_h\right\|_{L^2(\Omega)}.\end{aligned}$$

For the third and fourth term we use the inf-sup condition for the bilinear form B and the Galerkin orthogonality to get

$$\begin{aligned}\left\|I_h^{\mathrm{BDM}^k} u - u_h\right\|_{H^1_{HDG}(\Omega)} &+ \left\|\mathcal{P}_{Q_h}^{L^2} p - p_h\right\|_{L^2(\Omega)} \leq \\ &\frac{1}{\beta} \sup_{\substack{v_h, p_h \in V_h \times Q_h \\ v_h, q_h \neq 0}} \frac{B((I_h^{\mathrm{BDM}^k} u - u_h, \mathcal{P}_{Q_h}^{L^2} p - p_h), (v_h, q_h))}{\|v_h\|_{H^1_{HDG}(\Omega)} + \|q_h\|_{L^2(\Omega)}} \\ &= \frac{1}{\beta} \sup_{\substack{v_h, p_h \in V_h \times Q_h \\ v_h, q_h \neq 0}} \frac{B((I_h^{\mathrm{BDM}^k} u - u, \mathcal{P}_{Q_h}^{L^2} p - p), (v_h, q_h))}{\|v_h\|_{H^1_{HDG}(\Omega)} + \|q_h\|_{L^2(\Omega)}}.\end{aligned}$$

Next we use the definition of B and the commuting properties of the BDM interpolator and the L^2 projection, namely

$$b_{HDG}(I_h^{\mathrm{BDM}^k} u - u, q_h) = b_{HDG}(v_h, \mathcal{P}_{Q_h}^{L^2} p - p) = 0,$$

to get

$$\begin{aligned}\left\|I_h^{\mathrm{BDM}^k} u - u_h\right\|_{H^1_{HDG}(\Omega)} &+ \left\|\mathcal{P}_{Q_h}^{L^2} p - p_h\right\|_{L^2(\Omega)} \\ &\leq \frac{1}{\beta} \frac{a_{HDG}(I_h^{\mathrm{BDM}^k} u - u, v_h)}{\|v_h\|_{H^1_{HDG}(\Omega)}}.\end{aligned}$$

Due to the definition of a_{HDG} we can bound it with the $HDG*$ norm, and use the equivalence (5.1.1) to get

$$\begin{aligned}\frac{a_{HDG}(I_h^{\text{BDM}^k} u - u, v_h)}{\|v_h\|_{H^1_{HDG}(\Omega)}} &\preccurlyeq \frac{\left\| I_h^{\text{BDM}^k} u - u \right\|_{H^1_{HDG*}(\Omega)} \|v_h\|_{H^1_{HDG*}(\Omega)}}{\|v_h\|_{H^1_{HDG}(\Omega)}} \\ &\simeq \frac{\left\| I_h^{\text{BDM}^k} u - u \right\|_{H^1_{HDG*}(\Omega)} \|v_h\|_{H^1_{HDG}(\Omega)}}{\|v_h\|_{H^1_{HDG}(\Omega)}} \\ &\preccurlyeq \left\| I_h^{\text{BDM}^k} u - u \right\|_{H^1_{HDG*}(\Omega)},\end{aligned}$$

and so altogether

$$\|u - u_h\|_{H^1_{HDG}(\Omega)} + \left\| \mathcal{P}^{L^2}_{Q_h} p - p_h \right\|_{L^2(\Omega)} \leq c \left\| u - I_h^{\text{BDM}^k} u \right\|_{H^1_{HDG}(\Omega)},$$

with $c \neq c(h,k)$.

5.5 Numerical example

In the end we show an example using the curl formulation and the HDG-method to solve the Navier Stokes equations using a first order IMEX scheme 3.5.2. The example we chose is the well-known benchmark of a laminar flow around a cylinder, see [ST96]. We simulate the 2D example with the inflow condition

$$u(0,y) = \frac{6y(0.41-y)}{0.41^2},$$

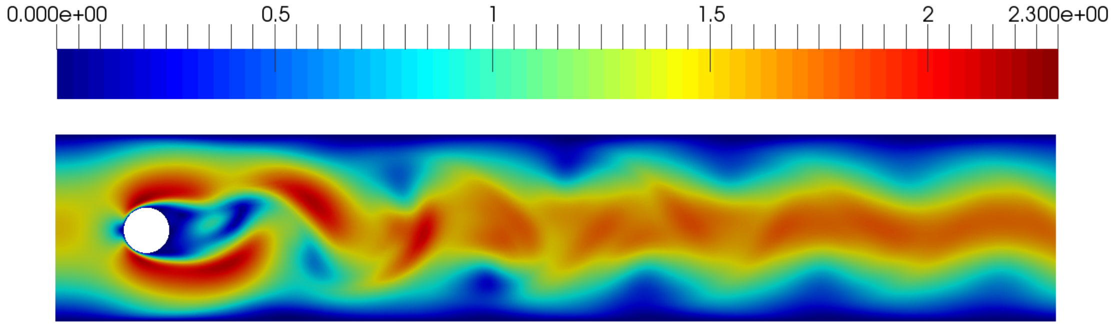
which yields to a Reynolds number $Re = 100$ and induces an unsteady flow. In figure 5.3 one can see the solution at $t = 3s$ and in figure 5.4 the drag and lift coefficient defined by

$$c_D = \frac{2F_D}{\rho \overline{u}^2 D} \quad \text{and} \quad c_L = \frac{2F_L}{\rho \overline{u}^2 D},$$

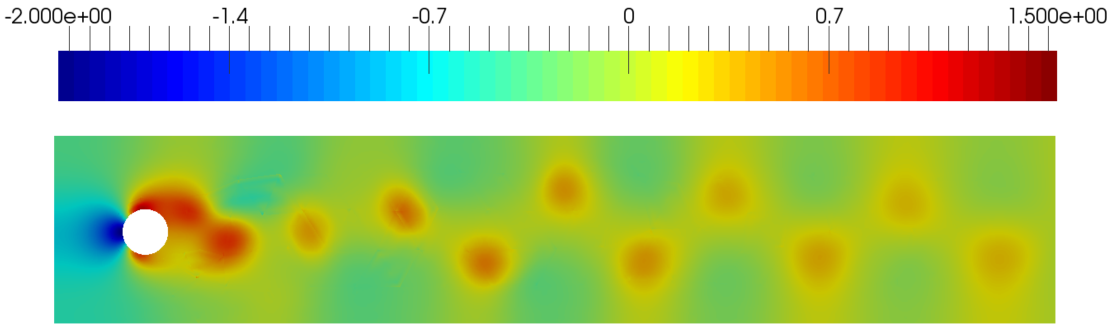
with

$$F_D = \int_{\Gamma_c} (\nu \frac{\partial u_t}{\partial n} n_y - p n_x)\, ds \quad \text{and} \quad F_L = -\int_{\Gamma_c} (\nu \frac{\partial u_t}{\partial n} n_x + p n_y)\, ds$$

where Γ_c is the boundary of the cylinder, are plotted. For the simulation we use a mesh with 315 elements and chose the polynomial degree $k = 3$.



(a) Velocity field and absolute value $|u_h|$



(b) Pressure p_h

Figure 5.3: Absolute value of the velocity $|u_h|$ and pressure p of a laminar flow around the cylinder

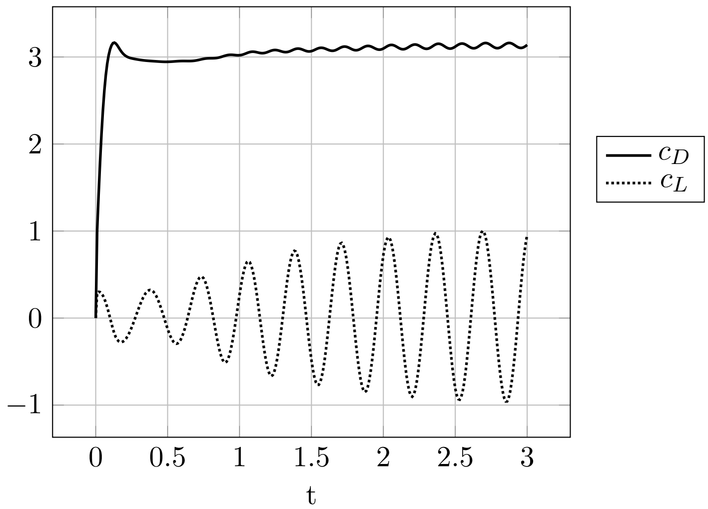


Figure 5.4: Drag and lift coefficient for a laminar flow around a cylinder

6 H^2-extension

In this section we prove the existence of an H^2-continuous operator $\mathcal{E}$. Note that $\mathcal{E}$ enforces exact tangential values of the gradient on $\partial\hat{T}$ although the properties in (5.2.7) demand an exact normal trace. This can be done by switching from the ∇ operator to the transposed ∇^T operator in 5.1. The proof is analogue, but the tangential and the normal values change their position.

Theorem 6.1. *Assume a given function $u \in [\Pi^k(\hat{T})]^2$ where $\hat{T}$ is the reference element and $\int_{\partial\hat{T}} u \cdot \tau \,\mathrm{d}x = 0$. Then there exists an operator $\mathcal{E} : [\Pi^k(\hat{T})]^2 \to \Pi^{k+1}(\hat{T})$ so that for $\phi = \mathcal{E}(u)$ we have*

$$\nabla\phi \cdot \tau = u \cdot \tau \text{ on } \partial\hat{T} \tag{6.0.1}$$

$$\|(u - \nabla\phi) \cdot n\|_{L^2(\partial\hat{T})} \preccurlyeq \frac{1}{k} \|u\|_{H^1(\hat{T})} \tag{6.0.2}$$

$$\|\phi\|_{H^2(\hat{T})} \preccurlyeq \|u\|_{H^1(\hat{T})} \,. \tag{6.0.3}$$

Due to the different demands of the extension $\mathcal{E}(u)$, we show three theorems with different kinds of properties and use them afterwards to prove theorem 6.1.

First we see that the tangential gradient of $\mathcal{E}(u)$ has the same values as the tangential trace of u. How to find a function that has this property is shown in the first theorem 6.3. We start with an extension that preserves a proper tangential gradient on the first edge. Afterwards we correct the error of the tangential gradients on the other two edges by defining two more extensions. The main challenge is the H^2-estimation where we use different techniques and the definition of the $H^{1/2}$ seminorm. The final result $\mathcal{E}^D(u)$ then fulfills

$$\nabla\mathcal{E}^D(u) \cdot \tau = u \cdot \tau \text{ on } \partial\hat{T}.$$

After the first step the normal gradient of $\mathcal{E}^D(u)$ does not coincide with the normal trace of u at all. The second theorem 6.4 is used to correct the error $u_c = \left(u - \nabla\mathcal{E}^D(u)\right) \cdot n$. This can be done under certain assumptions, so we split the error u_c in a *good* term u_c^g, where we can use theorem 6.4, and a *bad* term u_c^b. The constructed extension for the proof of the second theorem is also split in three parts correcting the values edge by edge. Again the H^2-estimation is a big challenge and uses similar techniques as in the proof of theorem 6.3.

That the splitting of u_c in two parts is a stable operation and how to handle u_c^b, is considered in the last theorem 6.5. We show that there exists a polynomial whose norm is bounded in a proper way by the polynomial order k. For that we use a Lagrangian function to find a minimum under certain conditions. Finally we show the proof of 6.1 in section 6.4.

Before we continue with the theorems 6.3, 6.4 and 6.5, we show a lemma which uses the technique of K-functionals.

Lemma 6.2. *Let $E_1 = (0,1)$ and assume a function $a(s) \in C^1(E_1)$ and $u \in H^{1/2}(E_1)$. For $(x,y) \in \hat{T}$ we define*

$$g(x,y) := \int_0^1 a(s)u(x+sy)ds.$$

Then we have

$$\|g\|_{H^1(\hat{T})} \preccurlyeq \|u\|_{H^{1/2}(E_1)}.$$

Proof. Due to $\|g\|^2_{H^1(T)} = \|g\|^2_{L^2(T)} + \|\nabla g\|^2_{L^2(T)}$ we show the proof in two steps. First we bound the L^2 norm and then the H^1 seminorm.
For an arbitrary y we define the horizontal line $L_y := \{(t,y) : t \in [0, 1-y]\}$. By that we get, using Cauchy Schwarzs inequality,

$$\begin{aligned} \|g\|^2_{L^2(L_y)} &= \int_0^{1-y} g(x,y)^2 \,\mathrm{d}x = \int_0^{1-y} \left(\int_0^1 a(s)u(x+sy)\,\mathrm{d}s\right)^2 \mathrm{d}x \\ &\overset{\text{C.S.}}{\preccurlyeq} \int_0^{1-y}\int_0^1 a(s)^2\,\mathrm{d}s \int_0^1 u(x+sy)^2\,\mathrm{d}s\,\mathrm{d}x \preccurlyeq \int_0^{1-y}\int_0^1 u(x+sy)^2\,\mathrm{d}s\,\mathrm{d}x. \end{aligned}$$

Using the substitution $t = x + sy$ we get

$$\int_0^{1-y}\int_0^1 u(x+sy)^2\,\mathrm{d}s\,\mathrm{d}x = \frac{1}{y}\int_0^{1-y}\int_x^{x+y} u(t)^2\,\mathrm{d}t\,\mathrm{d}x.$$

In the next step we use Fubini's theorem to first change the order of the integration variables and then increase the integrated area, namely

$$\int_0^{1-y}\int_x^{x+y} u(t)^2\,\mathrm{d}t\,\mathrm{d}x = \iint\limits_{\substack{0\le x\le 1-y\\ x\le t\le x+y}} u(t)^2\,\mathrm{d}(t,x) \overset{\text{Fub.}}{\preccurlyeq} \iint\limits_{\substack{0\le t\le 1\\ t-y\le x\le t}} u(t)^2\,\mathrm{d}(x,t),$$

and so

$$\|g\|^2_{L^2(L_y)} \preccurlyeq \frac{1}{y}\int_0^1 u(t)^2 \underbrace{\int_{t-y}^t \mathrm{d}x}_{=y}\,\mathrm{d}t = \int_0^1 u(t)^2\,\mathrm{d}t = \|u\|^2_{L^2(E_1)} \preccurlyeq \|u\|^2_{H^{1/2}(E_1)}.$$

By that we can bound the L^2 norm on $\hat{T}$

$$\|g\|^2_{L^2(\hat{T})} = \int_0^1 \|g\|^2_{L^2(L_y)}\,\mathrm{d}y \preccurlyeq \|u\|^2_{H^{1/2}(E_1)}.$$

To bound the H^1 seminorm we use the theory of interpolation spaces (see appendix 7.5). In the first step we make an estimation for the x derivation and then for the y derivation. Due to

$$\frac{\partial}{\partial x} g(x,y) = \int_0^1 u'(x+sy)a(s)\ \mathrm{d}s,$$

we get analogue as before

$$\left\| \frac{\partial}{\partial x} g(x,y) \right\|_{L^2(L_y)}^2 \preccurlyeq \|u'\|_{L^2(E_1)}^2 . \tag{6.0.4}$$

Next we observe that by using integration by part we can write the x derivation also as

$$\begin{aligned}
\frac{\partial}{\partial x} g(x,y) &= \int_0^1 \underbrace{u'(x+sy)}_{=\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}s}u(x+sy)}\ a(s)\ \mathrm{d}s \\
&= \underbrace{\frac{1}{y}\int_0^1 a'(s)u(x+sy)\ \mathrm{d}s}_{=:A} + \underbrace{\frac{1}{y}\left(a(1)u(x+y) - a(0)u(x)\right)}_{=:B} .
\end{aligned}$$

For the first term we proceed as before to get

$$\|A\|_{L^2(L_y)}^2 \preccurlyeq \frac{1}{y^2}\|u\|_{L^2(E_1)}^2 . \tag{6.0.5}$$

For the second term we get

$$\|B\|_{L^2(L_y)}^2 = \frac{1}{y^2}\int_0^{1-y} \left(a(1)u(x+y) - a(0)u(x)\right)^2\ \mathrm{d}x \preccurlyeq \frac{1}{y^2}\int_0^{1-y} u(x+y)^2 + u(x)^2\ \mathrm{d}x,$$

and together with the substitution $t = x+y$

$$\|B\|_{L^2(L_y)}^2 \preccurlyeq \frac{1}{y^2}\int_y^1 u(t)^2\ \mathrm{d}t + \frac{1}{y^2}\int_0^{1-y} u(x)^2\ \mathrm{d}x \preccurlyeq \frac{1}{y^2}\|u\|_{L^2(E_1)}^2 . \tag{6.0.6}$$

Using estimation (6.0.4), (6.0.5) and (6.0.6) and the definition of the K-functional we have

$$\left\| \frac{\partial}{\partial x} g(x,y) \right\|_{L^2(L_y)} \preccurlyeq \inf_{u=u_0+u_1} \sqrt{\frac{1}{y^2}\|u_0\|_{L^2(E_1)}^2 + \|u_1'\|_{L^2(E_1)}^2} \leq \frac{1}{y}K(y,u),$$

and so

$$\begin{aligned}
\left\| \frac{\partial}{\partial x} g(x,y) \right\|_{L^2(\hat{T})}^2 &= \int_0^1 \left\| \frac{\partial}{\partial x} g(x,y) \right\|_{L^2(L_y)}^2\ \mathrm{d}y \\
&= \int_0^1 \frac{1}{y^2}K(y,u)^2\ \mathrm{d}y \preccurlyeq \int_0^\infty \frac{1}{y^2}K(y,u)^2\ \mathrm{d}y = \|u\|_{H^{1/2}(E_1)}^2 .
\end{aligned}$$

For the y derivation we observe that

$$\frac{\partial}{\partial y} g(x,y) = \int_0^1 u'(x+sy)sa(s)\ \mathrm{d}s.$$

So by defining $\tilde{a}(s) = sa(s)$ we proceed similar to the x derivation to finish the proof. □

6.1 Tangential extension $\mathcal{E}^D$

Theorem 6.3. *Assume a given function $u \in [\Pi^k(\hat{T})]^2$ where $\hat{T}$ is the reference element and $\int_{\partial\hat{T}} u \cdot \tau \,\mathrm{d}s = 0$. Then there exists an operator $\mathcal{E}^D : [\Pi^k(\hat{T})]^2 \to \Pi^{k+1}(\hat{T})$ so that for $\phi_\tau = \mathcal{E}^D(u)$ we have*

$$\nabla\phi_\tau \cdot \tau = u \cdot \tau \text{ on } \partial\hat{T} \tag{6.1.1}$$

$$\|\phi_\tau\|_{H^2(\hat{T})} \preccurlyeq \|u\|_{H^1(\hat{T})}. \tag{6.1.2}$$

Proof. For the proof we split the extension $\mathcal{E}^D$ in three edge extensions $\mathcal{E}_1^{E_i}, \mathcal{E}_2^{E_{i,j}}$ and $\mathcal{E}_3^{E_i}$. The main idea is to first extend the values of the lower edge and correct the values on the other two edges afterwards. All following examples are visualized in a three dimensional perspective, see figure 6.1.

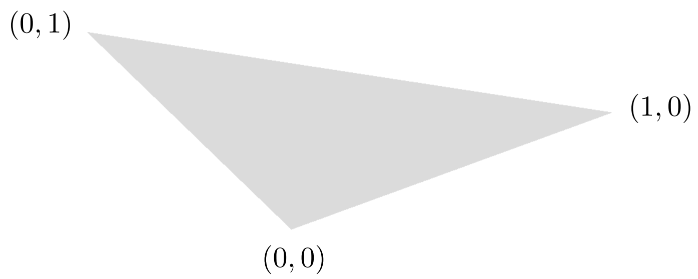


Figure 6.1: Three dimensional perspective of the reference element $\hat{T}$

Step 1: The first extension $\mathcal{E}_1^{E_i}$ has no further restrictions. For the ease we set $E_i = E_1$ and define τ_{E_1} as the tangential vector on E_1. For $u_\tau = u \cdot \tau_{E_1}$ we set

$$\psi(x) := \int_0^x u_\tau(s, 0) \,\mathrm{d}s + c,$$

where c can be an arbitrary constant and

$$\mathcal{E}_1^{E_1}(u)(x, y) = \phi_1(x, y) := \int_0^1 \psi(x + sy) \,\mathrm{d}s.$$

We observe that

$$\begin{aligned} \frac{\partial\phi_1}{\partial x} &= \int_0^1 \psi'(x + sy) \,\mathrm{d}s = \int_0^1 u_\tau(x + sy) \,\mathrm{d}s \\ \frac{\partial\phi_1}{\partial y} &= \int_0^1 \psi'(x + sy)s \,\mathrm{d}s = \int_0^1 u_\tau(x + sy)s \,\mathrm{d}s \end{aligned} \tag{6.1.3}$$

and so

$$\nabla\phi_1 \cdot \tau_1|_{E_1} = \left.\frac{\partial\phi_1}{\partial x}\right|_{E_1} = \int_0^1 u_\tau(x) \,\mathrm{d}s = u_\tau(x) = u \cdot \tau. \tag{6.1.4}$$

In figure 6.2 we can see the values and the partial derivation of the first extension ϕ_1 for $u_\tau = 6x(1-x) - 1$.

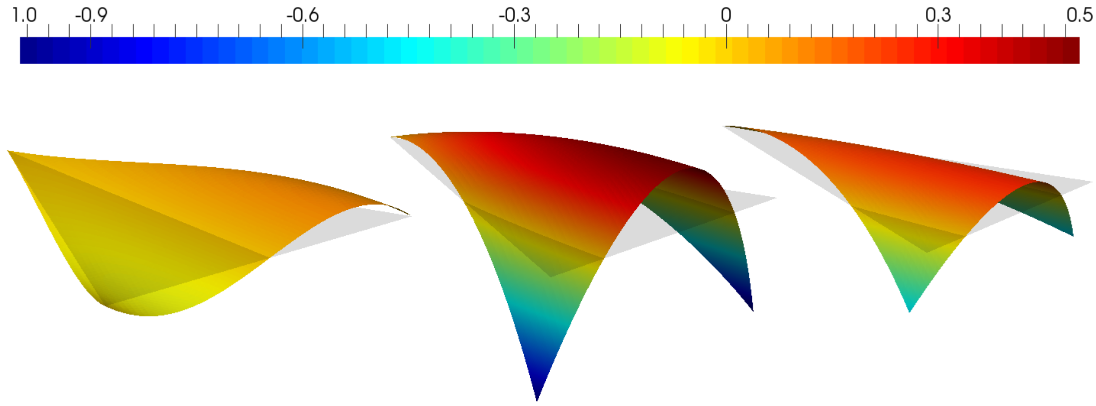


Figure 6.2: Extension ϕ_1, $\frac{\partial \phi_1}{\partial x}$ and $\frac{\partial \phi_1}{\partial y}$ for $u_\tau = 6x(1-x) - 1$

We now show

$$\|\phi_1\|^2_{H^2(\hat{T})} = \|\phi_1\|^2_{L^2(\hat{T})} + \|\nabla\phi_1\|^2_{H^1(\hat{T})} \preccurlyeq \|u\|^2_{H^1(\hat{T})}\,.$$

For the L^2 estimation we proceed as in the proof of lemma 6.2

$$\begin{aligned}
\|\phi_1\|^2_{L^2(L_y)} &= \int_0^{1-y} \Big(\int_0^1 \psi(x+sy)\ \mathrm{d}s\Big)^2\ \mathrm{d}x \overset{\text{C.S.}}{\preccurlyeq} \int_0^{1-y}\int_0^1 \psi(x+sy)^2\ \mathrm{d}s\ \mathrm{d}x \\
&\preccurlyeq \|\psi\|^2_{L^2(E_1)} = \int_0^1 \Big(\int_0^x u_\tau(s,0)\ \mathrm{d}s\Big)^2\ \mathrm{d}x \overset{\text{C.S.}}{\preccurlyeq} \int_0^1\int_0^x u_\tau(s,0)^2\ \mathrm{d}s\ \mathrm{d}x \\
&\preccurlyeq \|u_\tau\|^2_{L^2(E_1)} \preccurlyeq \|u_\tau\|^2_{H^{1/2}(E_1)} \preccurlyeq \|u\|^2_{H^1(\hat{T})}
\end{aligned}$$

and so

$$\|\phi_1\|^2_{L^2(\hat{T})} = \int_0^1 \|\phi_1\|^2_{L^2(L_y)}\ \mathrm{d}y \preccurlyeq \|u\|^2_{H^1(\hat{T})}\,.$$

To bound $\|\nabla\phi_1\|^2_{H^1(\hat{T})}$ we use lemma 6.2 for each partial derivative (6.1.3), and so we get

$$\|\nabla\phi_1\|^2_{H^1(\hat{T})} \preccurlyeq \|u_\tau\|^2_{H^{1/2}(E_1)} \preccurlyeq \|u\|^2_{H^1(\hat{T})}\,,$$

and

$$\|\phi_1\|_{H^2(\hat{T})} \preccurlyeq \|u\|^2_{H^1(\hat{T})}\,. \tag{6.1.5}$$

Step 2: For $\mathcal{E}_2^{E_{i,j}}$ we have the restriction that

$$\nabla\mathcal{E}_2^{E_{i,j}}(u)\cdot\tau_{E_j}\Big|_{E_j} = 0.$$

Again we set $E_i = E_1$ and $E_j = E_2$. For an arbitrary $\tilde{u}$ we define

$$u_{2,\tau} := \tilde{u} \cdot \tau_{E_1},$$

and

$$\psi_2(x) := \int_0^x u_{2,\tau}(s,0)\ \mathrm{d}s - \overline{\psi_2} \quad \text{with} \quad \overline{\psi_2} := \int_0^1 u_{2,\tau}(s,0)\ \mathrm{d}s.$$

With that we define the extension

$$\mathcal{E}_2^{E_{1,2}}(\tilde{u})(x,y) = \phi_2(x,y) := \int_0^1 \psi_2(x+sy)\ \mathrm{d}s - \frac{y}{1-x}\int_0^1 \psi_2(x+s(1-x))\ \mathrm{d}s,$$

or using integration by part for the second term

$$\begin{aligned}\frac{y}{1-x}\int_0^1 \psi_2(x+s(1-x))\ \mathrm{d}s &= -y\int_0^1 \psi_2'(x+s(1-x))s\ \mathrm{d}s\\ &= -y\int_0^1 u_{2,\tau}(x+s(1-x))s\ \mathrm{d}s,\end{aligned}$$

we can write the extension as

$$\phi_2(x,y) = \underbrace{\int_0^1 \psi_2(x+sy)\ \mathrm{d}s}_{=\mathcal{E}_1^{E_1}(\tilde{u})(x,y)} + \underbrace{y\int_0^1 u_{2,\tau}(x+s(1-x))s\ \mathrm{d}s}_{=:\phi_2^{corr}}.$$

On E_2 we have $y = 1-x$ and so

$$\phi_2|_{E_2} = 0,$$

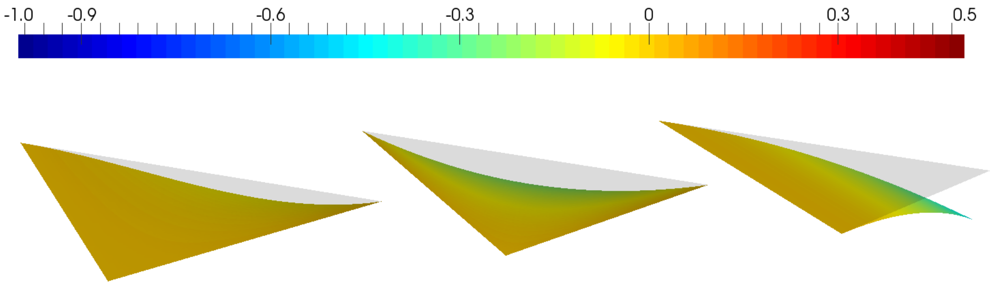
and due to a constant value on the edge E_2 also

$$\nabla\phi_2 \cdot \tau_{E_2}|_{E_2} = 0. \tag{6.1.6}$$

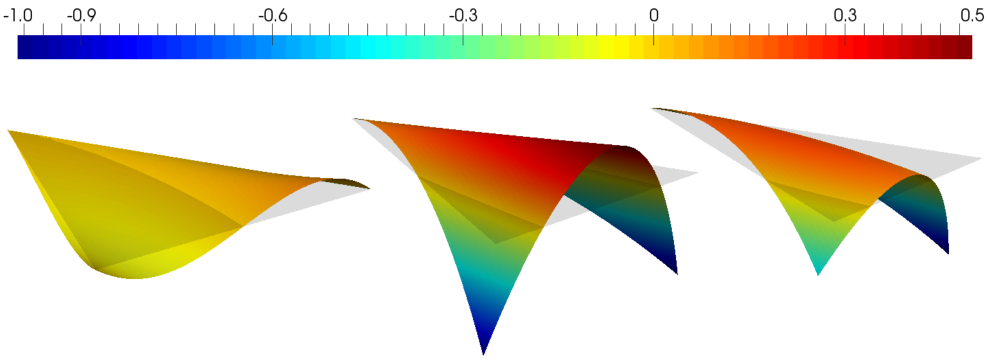
On E_1 we have $y = 0$ and so

$$\nabla\phi_2|_{E_1} \cdot \tau_{E_1} = \frac{\partial\psi_2(x)}{\partial x} = u_{2,\tau} = \tilde{u}\cdot\tau_{E_1}. \tag{6.1.7}$$

In figure 6.3 we can see the values and the partial derivations of the correction term ϕ_2^{corr} and the resulting extension ϕ_2 for the same example $u_{2,\tau} = 6x(1-x) - 1$. Note that the correction does not change the values and the derivation with respect to x on the edge E_1.



(a) Correction ϕ_2^{corr}, $\frac{\partial \phi_2^{corr}}{\partial x}$ and $\frac{\partial \phi_2^{corr}}{\partial y}$



(b) Extension ϕ_2, $\frac{\partial \phi_2}{\partial x}$ and $\frac{\partial \phi_2}{\partial y}$

Figure 6.3: Values and partial derivatives of the correction ϕ_2^{corr} and the extension ϕ_2 for $u_{2,\tau} = 6x(1-x) - 1$

It remains to show the H^2 estimation. For the first part $\mathcal{E}_1^{E_1}(\tilde{u})(x,y)$ we showed the estimation in step 1, so we have

$$\left\| \mathcal{E}_1^{E_1}(\tilde{u}) \right\|_{H^2(\hat{T})} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})} .$$

It remains the estimation for ϕ_2^{corr}. Again we split the norm in

$$\|\phi_2^{corr}\|_{H^2(\hat{T})}^2 = \|\phi_2^{corr}\|_{L^2(\hat{T})}^2 + \|\nabla \phi_2^{corr}\|_{L^2(\hat{T})}^2 + \left\|\nabla^2 \phi_2^{corr}\right\|_{L^2(\hat{T})}^2 .$$

For the first term we get

$$\begin{aligned}
\|\phi_2^{corr}\|^2_{L^2(\hat{T})} &= \int_0^1 \int_0^{1-x} (\phi_2^{corr})^2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \int_0^{1-x} \left(y \int_0^1 u_{2,\tau}(x + s(1-x)) s \, \mathrm{d}s \right)^2 \, \mathrm{d}y \, \mathrm{d}x \\
&\overset{\text{C.S.}}{\preccurlyeq} \int_0^1 \int_0^{1-x} \underbrace{\left(\frac{y}{1-x} \right)^2}_{\leq 1} \left(\int_x^1 u_{2,\tau}(t) \, \mathrm{d}t \right)^2 \, \mathrm{d}y \, \mathrm{d}x \\
&\preccurlyeq \int_0^1 \int_0^{1-x} \int_x^1 u_{2,\tau}(t)^2 \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}x \preccurlyeq \int_0^1 \int_0^{1-x} \int_0^1 u_{2,\tau}(t)^2 \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}x \\
&\preccurlyeq \|u_{2,\tau}\|_{L^2(E_1)} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})} .
\end{aligned}$$

Next we calculate all partial derivatives up to order 2 (always using integration by part for the $u'_{2,\tau}$ terms)

$$\begin{aligned}
\frac{\partial \phi_2^{corr}}{\partial x} &= y \int_0^1 \underbrace{u'_{2,\tau}(x + s(1-x))}_{= \frac{1}{1-x} \frac{\mathrm{d}}{\mathrm{d}s} u_{2,\tau}(x+s(1-x))} (1-s)s \, \mathrm{d}s = \\
&= \frac{y}{1-x} \left(\int_0^1 u_{2,\tau}(x + s(1-x))(2s-1) \, \mathrm{d}s + \underbrace{u_{2,\tau}(x + s(1-x))(1-s)s \Big|_0^1}_{=0} \right) \\
&= \frac{y}{1-x} \int_0^1 u_{2,\tau}(x + s(1-x))(2s-1) \, \mathrm{d}s,
\end{aligned}$$

$$\frac{\partial \phi_2^{corr}}{\partial y} = \int_0^1 u_{2,\tau}(x + s(1-x)) s \, \mathrm{d}s,$$

$$\begin{aligned}
\frac{\partial^2 \phi_2^{corr}}{\partial x \partial x} &= \frac{y}{(1-x)^2} \int_0^1 u_{2,\tau}(x + s(1-x))(2s-1) \, \mathrm{d}s \\
&+ \frac{y}{1-x} \int_0^1 u'_{2,\tau}(x + s(1-x))(2s-1)(1-s) \, \mathrm{d}s \\
&= \frac{y}{(1-x)^2} \int_0^1 u_{2,\tau}(x + s(1-x))(2s-1) \, \mathrm{d}s \\
&+ \frac{y}{(1-x)^2} \int_0^1 u_{2,\tau}(x + s(1-x))(4s-3) \, \mathrm{d}s + \frac{y}{(1-x)^2} u_{2,\tau}(x),
\end{aligned}$$

$$\frac{\partial^2 \phi_2^{corr}}{\partial x \partial y} = \frac{1}{1-x} \int_0^1 u_{2,\tau}(x + s(1-x))(2s-1) \, \mathrm{d}s,$$

and finally

$$\frac{\partial^2 \phi_2^{corr}}{\partial y \partial y} = 0.$$

We start with the mixed second order derivative. First note that

$$\int_0^1 2s - 1 \; \mathrm{d}s = 0,$$

by that we can substract the meanvalue

$$\overline{u_{2,\tau}}^{(x,1)} := \frac{1}{1-x} \int_x^1 u_{2,\tau}(s) \; \mathrm{d}s,$$

to get

$$\begin{aligned} \frac{\partial^2 \phi_2^{corr}}{\partial x \partial y} &= \frac{1}{1-x} \int_0^1 u_{2,\tau}(x + s(1-x))(2s-1) \; \mathrm{d}s \\ &= \frac{1}{1-x} \int_0^1 \left(u_{2,\tau}(x + s(1-x)) - \overline{u_{2,\tau}}^{(x,1)} \right) (2s-1) \; \mathrm{d}s. \end{aligned}$$

And so using the Cauchy-Schwarz inequality

$$\begin{aligned} \left\| \frac{\partial^2 \phi_2^{corr}}{\partial x \partial y} \right\|_{L^2(\hat{T})}^2 &= \int_0^1 \int_0^{1-x} \left(\frac{\partial^2 \phi_2^{corr}}{\partial x \partial y} \right)^2 \; \mathrm{d}y \; \mathrm{d}x \\ &= \int_0^1 \left(\frac{\partial^2 \phi_2^{corr}}{\partial x \partial y} \right)^2 \int_0^{1-x} \; \mathrm{d}y \; \mathrm{d}x \\ &\overset{\text{C.S.}}{\preccurlyeq} \int_0^1 \frac{1}{1-x} \int_0^1 \left(u_{2,\tau}(x + s(1-x)) - \overline{u_{2,\tau}}^{(x,1)} \right)^2 \; \mathrm{d}s \; \mathrm{d}x \\ &= \int_0^1 \frac{1}{(1-x)^2} \int_x^1 \left(u_{2,\tau}(t) - \overline{u_{2,\tau}}^{(x,1)} \right)^2 \; \mathrm{d}t \; \mathrm{d}x. \end{aligned}$$

The inner integral can also be written as

$$\int_x^1 \left(u_{2,\tau}(t) - \overline{u_{2,\tau}}^{(x,1)} \right)^2 \; \mathrm{d}t = \frac{1}{2(1-x)} \int_x^1 \int_x^1 (u_{2,\tau}(t) - u_{2,\tau}(s))^2 \; \mathrm{d}t \; \mathrm{d}s,$$

and so using Fubini's theorem

$$\begin{aligned} \left\| \frac{\partial^2 \phi_2^{corr}}{\partial x \partial y} \right\|_{L^2(\hat{T})}^2 &\preccurlyeq \int_0^1 \frac{1}{(1-x)^3} \int_x^1 \int_x^1 (u_{2,\tau}(t) - u_{2,\tau}(s))^2 \; \mathrm{d}t \; \mathrm{d}s \; \mathrm{d}x \\ &\overset{\text{Fub.}}{\preccurlyeq} \iiint\limits_{\substack{x \le s \\ x \le t \\ 0 \le s,t \le 1}} \frac{(u_{2,\tau}(t) - u_{2,\tau}(s))^2}{(1-x)^3} \; \mathrm{d}(s,t,x). \\ &\preccurlyeq \int_0^1 \int_0^1 \int_0^{\min(s,t)} \frac{1}{(1-x)^3} \; \mathrm{d}x \, (u_{2,\tau}(t) - u_{2,\tau}(s))^2 \; \mathrm{d}t \; \mathrm{d}s. \end{aligned}$$

For the inner integral and w.l.o.g. assuming that $s < t$ we get

$$\int_0^{\min(s,t)} \frac{1}{(1-x)^3}\,\mathrm{d}x = \left(\frac{1}{1-\min(s,t)}\right)^2 - 1 \le \left(\frac{1}{1-s}\right)^2 \le \left(\frac{1}{t-s}\right)^2,$$

and so

$$\left\|\frac{\partial^2 \phi_2^{corr}}{\partial x \partial y}\right\|^2_{L^2(\hat{T})} \preccurlyeq \int_0^1 \int_0^1 \frac{(u_{2,\tau}(t) - u_{2,\tau}(s))^2}{(t-s)^2}\,\mathrm{d}t\,\mathrm{d}s.$$

Using the definition of the Sobolev Slobodeckij norm (see appendix) we get

$$\left\|\frac{\partial^2 \phi_2^{corr}}{\partial x \partial y}\right\|^2_{L^2(\hat{T})} \preccurlyeq \int_0^1 \int_0^1 \frac{(u_{2,\tau}(t) - u_{2,\tau}(s))^2}{(t-s)^2}\,\mathrm{d}t\,\mathrm{d}s = |u_{2,\tau}|^2_{H^{1/2}(E_1)} \le \|\tilde{u}\|^2_{H^1(\hat{T})}.$$

Next we look at the second derivative of $\frac{\partial^2 \phi_2^{corr}}{\partial x \partial x}$. Note that we have $\frac{y}{1-x} \le 1$ and so we get

$$\begin{aligned}\frac{\partial^2 \phi_2^{corr}}{\partial x \partial x} \preccurlyeq\ & \frac{1}{1-x}\int_0^1 u_{2,\tau}(x+s(1-x))(2s-1)\,\mathrm{d}s \\ & + \frac{1}{1-x}\int_0^1 u_{2,\tau}(x+s(1-x))(4s-3)\,\mathrm{d}s + \frac{1}{1-x}u_{2,\tau}(x),\end{aligned}$$

or by adding and subtracting an integral

$$\begin{aligned}\frac{\partial^2 \phi_2^{corr}}{\partial x \partial x} \preccurlyeq\ & \frac{1}{1-x}\int_0^1 u_{2,\tau}(x+s(1-x))(2s-1)\,\mathrm{d}s \\ & + \frac{1}{1-x}\int_0^1 u_{2,\tau}(x+s(1-x))(4s-2)\,\mathrm{d}s \\ & + \underbrace{\frac{1}{1-x}\int_0^1 u_{2,\tau}(x) - u_{2,\tau}(x+s(1-x))\,\mathrm{d}s}_{=:A}.\end{aligned}$$

The first and the second term can be estimated in the same way as $\frac{\partial^2 \phi_2^{corr}}{\partial x \partial y}$. We focus on the third term A. Again using the Cauchy Schwarz inequality and Fubini's theorem the L^2 norm is bounded by

$$\begin{aligned}\|A\|^2_{L^2(\hat{T})} &= \int_0^1 \int_0^{1-x} A^2\,\mathrm{d}y\,\mathrm{d}x \overset{\text{C.S.}}{\preccurlyeq} \int_0^1 \frac{1}{1-x}\int_0^1 (u_{2,\tau}(x) - u_{2,\tau}(x+s(1-x))^2\,\mathrm{d}s\,\mathrm{d}x \\ &= \int_0^1 \frac{1}{(1-x)^2}\int_x^1 (u_{2,\tau}(x) - u_{2,\tau}(t))^2\,\mathrm{d}t\,\mathrm{d}x = \iint\limits_{t\ge x} \frac{(u_{2,\tau}(x) - u_{2,\tau}(t))^2}{(1-x)^2}\,\mathrm{d}(x,t) \\ &\overset{\text{Fub.}}{\preccurlyeq} \iint\limits_{t\ge x} \frac{(u_{2,\tau}(x) - u_{2,\tau}(t))^2}{(x-t)^2}\,\mathrm{d}(x,t) \preccurlyeq \int_0^1\int_0^1 \frac{(u_{2,\tau}(x) - u_{2,\tau}(t))^2}{(x-t)^2}\,\mathrm{d}x\,\mathrm{d}t \\ &= |u_{2,\tau}|^2_{H^{1/2}(E_1)} \preccurlyeq \|\tilde{u}\|^2_{H^1(\hat{T})},\end{aligned}$$

and by that the estimation for $\frac{\partial^2 \phi_2^{corr}}{\partial x \partial x}$ is given. It remains to bound the first order derivatives. The estimation for x derivation is similar to $\frac{\partial^2 \phi_2^{corr}}{\partial x \partial y}$ because $y \leq 1$, and for the y derivation we observe as before

$$\begin{aligned}\left\|\frac{\partial \phi_2^{corr}}{\partial y}\right\|_{L^2(\hat{T})}^2 &= \int_0^1 \int_0^{1-x} \left(\int_0^1 u_{2,\tau}(x+s(1-x)s \,\mathrm{d}s\right)^2 \,\mathrm{d}y \,\mathrm{d}x \\ &\leq \int_0^1 (1-x) \int_0^1 u_{2,\tau}(x+s(1-x)^2 \,\mathrm{d}s \,\mathrm{d}x = \int_0^1 \int_x^1 u_{2,\tau}(t)^2 \,\mathrm{d}t \,\mathrm{d}x \\ &\preccurlyeq \|u_{2,\tau}\|_{L^2(E_1)} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})} .\end{aligned}$$

All together we have

$$\|\phi_2\|_{H^2(\hat{T})} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})}^2 . \tag{6.1.8}$$

Step 3: For $\mathcal{E}_3^{E_i}$ we have the restriction that

$$\nabla \mathcal{E}_3^{E_i}(u) \cdot \tau_{E_j}\Big|_{E_j} = 0 \quad \text{for} \quad j \neq i.$$

Again we set $E_i = E_1$ and $j = 2, 3$. For an arbitrary $\tilde{u}$ with

$$\int_{\partial \hat{T}} \tilde{u} \cdot \tau \,\mathrm{d}s = 0 \quad \text{and} \quad \tilde{u}|_{E_2} \cdot \tau_{E_2} = \tilde{u}|_{E_3} \cdot \tau_{E_3} = 0 \tag{6.1.9}$$

we set

$$u_{3,\tau} := \tilde{u} \cdot \tau_{E_1},$$

and

$$\psi_3(x) := \int_0^x u_{3,\tau}(s,0) \,\mathrm{d}s.$$

By that we define the last extension as

$$\begin{aligned}\mathcal{E}_3^{E_1}(\tilde{u})(x,y) = \phi_3(x,y) := \int_0^1 \psi_3(x+sy) \,\mathrm{d}s - \underbrace{\frac{y}{1-x}\int_0^1 \psi_3(x+s(1-x)) \,\mathrm{d}s}_{=:\phi_3^{corr,1}} \\ - \underbrace{\frac{y}{x+y}\int_0^1 \psi_3(s(x+y)) \,\mathrm{d}s}_{=:\phi_3^{corr,2}} + \underbrace{y\int_0^1 \psi_3(s) \,\mathrm{d}s}_{=:\phi_3^{corr,3}} .\end{aligned}$$

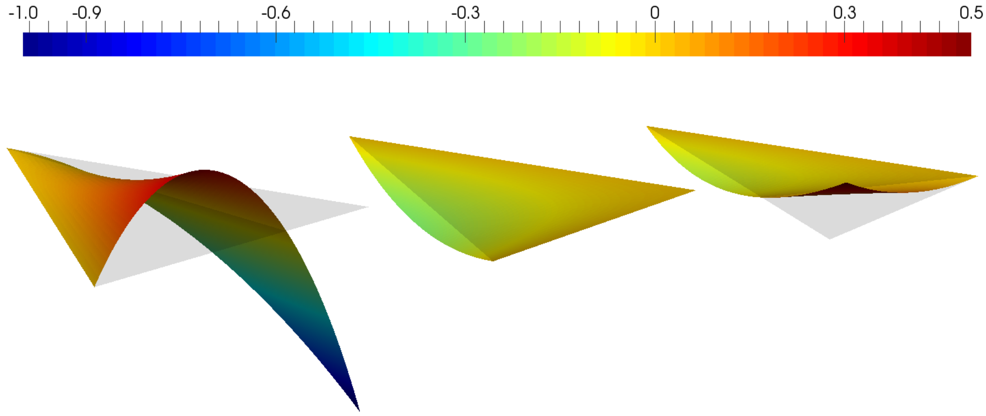
Similar to the second extension we observe that on E_2 due to $y = 1-x$ and on E_3 due to $x = 0$ the extension is constant, and so

$$\nabla \phi_3|_{E_2} \cdot \tau_{E_2} = \nabla \phi_3|_{E_3} \cdot \tau_{E_3} = 0, \tag{6.1.10}$$

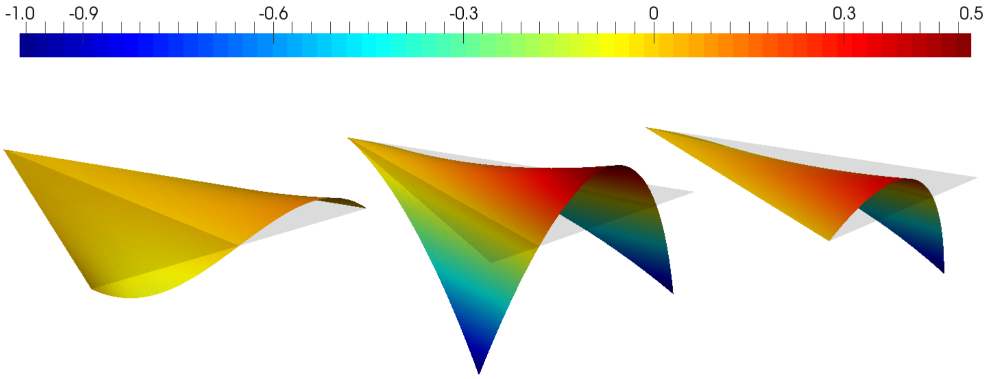
and

$$\nabla \phi_3|_{E_1} \cdot \tau_{E_1} = \psi_3' = u_{3,\tau} = \tilde{u} \cdot \tau_{E_1}. \tag{6.1.11}$$

In figure 6.4 we can see the values and the partial derivations of the correction $\phi_3^{corr,2}$ and the resulting extension ϕ_3 for the same example $u_{2,\tau} = 6x(1-x) - 1$. Note that on the edge E_3 the partial derivation with respect to y of the correction $\frac{\partial \phi_3^{corr,2}}{\partial y}$ is the negative value of $\frac{\partial \phi_2}{\partial y}$ in figure 6.3. This leads to $\nabla \phi_3|_{E_3} \cdot \tau_{E_3} = 0$.



(a) Correction $\phi_3^{corr,2}$, $\frac{\partial \phi_3^{corr,2}}{\partial x}$ and $\frac{\partial \phi_3^{corr,2}}{\partial y}$



(b) Extension ϕ_3, $\frac{\partial \phi_3}{\partial x}$ and $\frac{\partial \phi_3}{\partial y}$

Figure 6.4: Values and partial derivatives of the correction $\phi_3^{corr,2}$ and the extension ϕ_3 for $u_{2,\tau} = 6x(1-x) - 1$

Again we have to show the H^2 estimation. The first term of ϕ_3 can be written as $\mathcal{E}_1^{E_1}$ where we already have the estimation. The second term is the same as for $\mathcal{E}_2^{E_1}$, so it remains to show the estimations for $\phi_3^{corr,2}$ and $\phi_3^{corr,3}$. Note that we have $\psi_3(0) = 0$, and by that we can write

$$\begin{aligned}\phi_3^{corr,2} &= -\frac{y}{x+y}\int_0^1 \psi_3(s(x+y))\ \mathrm{d}s \\ &= y\int_0^1 \psi_3'(s(x+y))(s-1)\ \mathrm{d}s - \frac{y}{x+y}\psi_3(s(x+y))(s-1)\Big|_0^1 \\ &= y\int_0^1 u_{3,\tau}(s(x+y))(s-1)\ \mathrm{d}s\end{aligned}$$

and due to (6.1.9) also $\psi_3(1) = 0$, so

$$\phi_3^{corr,3} = y\int_0^1 \psi_3(x)\ \mathrm{d}s = -y\int_0^1 \psi_3'(s)s\ \mathrm{d}s + \psi_3(s)s\Big|_0^1 = -y\int_0^1 u_{3,\tau}(s)s\ \mathrm{d}s.$$

With similar techniques to the proofs for the estimation of $\mathcal{E}_1^{E_1}$ and $\mathcal{E}_2^{E_1}$ we can bound

$$\left\|\phi_3^{corr,3}\right\|_{H^2(\hat{T})} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})}.$$

For $\phi_3^{corr,2}$ we make the estimations part by part. Together with $\frac{y}{x+y} \leq 1$ the L^2 estimation follows from

$$\begin{aligned}\left\|\phi_3^{corr,2}\right\|_{L^2(\hat{T})} &\overset{\text{C.S.}}{\preccurlyeq} \int_0^1\int_0^{1-x} y^2 \int_0^1 u_{3,\tau}(s(x+y))^2\ \mathrm{d}s\ \mathrm{d}y\ \mathrm{d}x \\ &\preccurlyeq \int_0^1\int_0^{1-x} \frac{y^2}{x+y}\int_0^{x+y} u_{3,\tau}(s(x+y))^2\ \mathrm{d}s\ \mathrm{d}y\ \mathrm{d}x \\ &\preccurlyeq \|u_{3,\tau}\|_{L^2(E_1)} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})}.\end{aligned}$$

Next we calculate all partial derivatives using integration by part on all integrals where $u_{3,\tau}'$ appears to get:

$$\frac{\partial\phi_3^{corr,2}}{\partial x} = \frac{y}{x+y}\int_0^1 u_{3,\tau}(s(x+y))(1-2s)\ \mathrm{d}s$$

$$\frac{\partial\phi_3^{corr,2}}{\partial y} = \int_0^1 u(s(x+y)(s-1)\ \mathrm{d}s + \frac{y}{x+y}\int_0^1 u_{3,\tau}(s(x+y))(1-2s)\ \mathrm{d}s$$

$$\begin{aligned}\frac{\partial^2\phi_3^{corr,2}}{\partial x\partial x} &= \frac{-y}{(x+y)^2}\int_0^1 u_{3,\tau}(s(x+y))(1-2s)\ \mathrm{d}s \\ &+ \frac{y}{(x+y)^2}\int_0^1 u_{3,\tau}(s(x+y))(4s-2)\ \mathrm{d}s \\ &+ \frac{y}{(x+y)^2}\int_0^1 u_{3,\tau}(s(x+y)) - u_{3,\tau}(x+y)\ \mathrm{d}s\end{aligned}$$

$$\frac{\partial^2\phi_3^{corr,2}}{\partial x\partial y} = \frac{x}{(x+y)^2}\int_0^1 u_{3,\tau}(s(x+y))(1-2s)\ \mathrm{d}s$$
$$+\frac{y}{(x+y)^2}\int_0^1 u_{3,\tau}(s(x+y))(4s-2)\ \mathrm{d}s$$
$$+\frac{y}{(x+y)^2}\int_0^1 u_{3,\tau}(s(x+y)) - u_{3,\tau}(x+y)\ \mathrm{d}s$$

$$\frac{\partial^2\phi_3^{corr,2}}{\partial y\partial y} = \frac{1}{x+y}\int_0^1 u_{3,\tau}(s(x+y))(1-2s)\ \mathrm{d}s$$
$$+\frac{x}{(x+y)^2}\int_0^1 u_{3,\tau}(s(x+y))(1-2s)\ \mathrm{d}s$$
$$+\frac{y}{(x+y)^2}\int_0^1 u_{3,\tau}(s(x+y))(4s-2)\ \mathrm{d}s$$
$$+\frac{y}{(x+y)^2}\int_0^1 u_{3,\tau}(s(x+y)) - u_{3,\tau}(x+y)\ \mathrm{d}s.$$

Again we use that $\frac{x}{x+y} \leq 1$ and $\frac{y}{x+y} \leq 1$ on $\hat{T}$. We start with the L^2 norm of the first order x-derivation and proceed similar to the estimation of $\frac{\partial^2\phi_2^{corr}}{\partial x\partial y}$.

$$\begin{aligned}
\left\|\frac{\partial\phi_3^{corr,2}}{\partial x}\right\|^2_{L^2(\hat{T})} &\overset{\text{C.S.}}{\preccurlyeq} \int_0^1\int_0^{1-x}\int_0^1 \left(u_{3,\tau}(s(x+y)) - \overline{u_{3,\tau}}^{(0,x+y)}\right)^2\ \mathrm{d}s\ \mathrm{d}y\ \mathrm{d}x\\
&= \int_0^1\int_0^{1-x}\frac{1}{x+y}\int_0^{x+y}\left(u_{3,\tau}(t) - \overline{u_{3,\tau}}^{(0,x+y)}\right)^2\ \mathrm{d}t\ \mathrm{d}y\ \mathrm{d}x\\
&\preccurlyeq \int_0^1\int_0^{1-x}\frac{1}{(x+y)^2}\int_0^{x+y}\int_0^{x+y}\left(u_{3,\tau}(t) - u_{3,\tau}(s)\right)^2\ \mathrm{d}t\ \mathrm{d}s\ \mathrm{d}y\ \mathrm{d}x\\
&= \int_0^1\int_0^{1-x}\int_0^{x+y}\int_0^{x+y}\frac{\left(u_{3,\tau}(t) - u_{3,\tau}(s)\right)^2}{(x+y)^2}\ \mathrm{d}t\ \mathrm{d}s\ \mathrm{d}y\ \mathrm{d}x\\
&= \int_0^1\int_0^{1-x}\int_0^{x+y}\int_0^{x+y}\frac{\left(u_{3,\tau}(t) - u_{3,\tau}(s)\right)^2}{(t-s)^2}\ \mathrm{d}t\ \mathrm{d}s\ \mathrm{d}y\ \mathrm{d}x\\
&\leq \int_0^1\int_0^{1-x}\underbrace{\int_0^1\int_0^1\frac{\left(u_{3,\tau}(t) - u_{3,\tau}(s)\right)^2}{(t-s)^2}\ \mathrm{d}t\ \mathrm{d}s}_{=|u_{3,\tau}|^2_{H^{1/2}(E_1)}}\ \mathrm{d}y\ \mathrm{d}x,
\end{aligned}$$

as $t-s \leq t \leq x+y$, and so

$$\left\|\frac{\partial\phi_3^{corr,2}}{\partial x}\right\|_{L^2(\hat{T})} \preccurlyeq |u_{3,\tau}|_{H^{1/2}(E_1)} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})}.$$

We continue with the y derivation. First observe that the second integral is the same as the x derivation, so we have

$$\left\|\frac{\partial\phi_3^{corr,2}}{\partial y}\right\|^2_{L^2(\hat{T})} \preccurlyeq \int_0^1\int_0^{1-x}\int_0^1 u_{3,\tau}(s(x+y))^2\ \mathrm{d}s\ \mathrm{d}y\ \mathrm{d}x + \left\|\frac{\partial\phi_3^{corr,2}}{\partial x}\right\|^2_{L^2(\hat{T})}.$$

Using Fubini's theorem and $\zeta = x + y$ we observe

$$\begin{aligned}\int_0^1 \int_0^{1-x} \int_0^1 u_{3,\tau}(s(x+y))^2 \,\mathrm{d}s \,\mathrm{d}y \,\mathrm{d}x &= \int_0^1 \int_0^{1-x} \frac{1}{x+y} \int_0^1 u_{3,\tau}(t)^2 \,\mathrm{d}t \,\mathrm{d}y \,\mathrm{d}x \\ &\overset{\text{Fub.}}{=} \int_0^1 \int_0^{\zeta} \frac{1}{\zeta} \int_0^{\zeta} u_{3,\tau}(t)^2 \,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}\zeta \\ &\preccurlyeq \|u_{3,\tau}\|_{L^2(E_1)},\end{aligned}$$

and so also

$$\left\| \frac{\partial \phi_3^{corr,2}}{\partial y} \right\|_{L^2(\hat{T})} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})}.$$

Finally we define two functions

$$\xi := \frac{1}{x+y} \int_0^1 u_{3,\tau}(s(x+y))(1-2s) \,\mathrm{d}s$$

and

$$\theta := \frac{1}{x+y} \int_0^1 u_{3,\tau}(s(x+y)) - u_{3,\tau}(x+y) \,\mathrm{d}s,$$

to bound the L^2 norm of the second order derivation with the L^2 norm of linear combinations of ξ and θ. For the estimation of ξ we proceed similarly for $\frac{\partial \phi_3^{corr,2}}{\partial x}$ but have to use Fubini's theorem due to the higher power of the $x + y$ term.

$$\begin{aligned}\|\xi\|^2_{L^2(\hat{T})} &\preccurlyeq \int_0^1 \int_0^{1-x} \int_0^{x+y} \int_0^{x+y} \frac{(u_{3,\tau}(t) - u_{3,\tau}(s))^2}{(x+y)^3} \,\mathrm{d}t \,\mathrm{d}s \,\mathrm{d}y \,\mathrm{d}x \\ &\preccurlyeq \int_0^1 \iiint\limits_{\substack{s-x\le y\\ t-x\le y\\ 0\le t,s\le 1}} \frac{(u_{3,\tau}(t) - u_{3,\tau}(s))^2}{(x+y)^3} \,\mathrm{d}(s,t,y) \,\mathrm{d}x \\ &\overset{\text{Fub.}}{\preccurlyeq} \int_0^1 \int_0^1 \int_0^1 \int_{\max(s-x,t-x)}^1 \frac{1}{(x+y)^3} \,\mathrm{d}y \,(u_{3,\tau}(t) - u_{3,\tau}(s))^2 \,\mathrm{d}t \,\mathrm{d}s \,\mathrm{d}x.\end{aligned}$$

Without loss of generality let $s < t$ to get

$$\int_{\max(s-x,t-x)}^1 \frac{1}{(x+y)^3} \,\mathrm{d}y = \frac{-2}{(x+y)^2}\Big|_{\max(s-x,t-x)}^1 \le \frac{2}{t^2} \le \frac{2}{(t-s)^2},$$

and so

$$\begin{aligned}\|\xi\|^2_{L^2(\hat{T})} &\preccurlyeq \int_0^1 \int_0^1 \int_0^1 \frac{(u_{3,\tau}(t) - u_{3,\tau}(s))^2}{(t-s)^2} \,\mathrm{d}t \,\mathrm{d}s \,\mathrm{d}x. \\ &\preccurlyeq |u_{3,\tau}|_{H^{1/2}(E_1)} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})}.\end{aligned}$$

For θ we get

$$\begin{aligned}
\|\theta\|^2_{L^2(\hat{T})} &\preccurlyeq \int_0^1 \int_0^{1-x} \frac{1}{x+y} \int_0^1 \left(u_{3,\tau}(s(x+y)) - u_{3,\tau}(x+y)\right)^2 \; \mathrm{d}t \; \mathrm{d}y \; \mathrm{d}x \\
&\preccurlyeq \int_0^1 \int_0^{1-x} \frac{1}{(x+y)^2} \int_0^{x+y} \left(u_{3,\tau}(t) - u_{3,\tau}(x+y)\right)^2 \; \mathrm{d}t \; \mathrm{d}y \; \mathrm{d}x \\
&\preccurlyeq \int_0^1 \int_0^{1-x} \int_0^{x+y} \frac{\left(u_{3,\tau}(t) - u_{3,\tau}(x+y)\right)^2}{(x+y)^2} \; \mathrm{d}t \; \mathrm{d}y \; \mathrm{d}x.
\end{aligned}$$

Together with $|t-(x+y)| \leq |t| + |x+y| \leq 2|x+y|$ for $t \leq x+y$ we get

$$\begin{aligned}
\|\theta\|^2_{L^2(\hat{T})} &\preccurlyeq \int_0^1 \int_0^{1-x} \int_0^{x+y} \frac{\left(u_{3,\tau}(t) - u_{3,\tau}(x+y)\right)^2}{(t-(x+y))^2} \; \mathrm{d}t \; \mathrm{d}y \; \mathrm{d}x \\
&\preccurlyeq |u_{3,\tau}|_{H^{1/2}(E_1)} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})} .
\end{aligned}$$

As

$$\begin{aligned}
\left\| \frac{\partial^2 \phi_3^{corr,2}}{\partial x \partial x} \right\|_{L^2(\hat{T})} &\leq \|\theta\|^2_{L^2(\hat{T})} + \|\xi\|^2_{L^2(\hat{T})} \\
\left\| \frac{\partial^2 \phi_3^{corr,2}}{\partial y \partial y} \right\|_{L^2(\hat{T})} &\leq \|\theta\|^2_{L^2(\hat{T})} + \|\xi\|^2_{L^2(\hat{T})} \\
\left\| \frac{\partial^2 \phi_3^{corr,2}}{\partial x \partial y} \right\|_{L^2(\hat{T})} &\leq \|\theta\|^2_{L^2(\hat{T})} + \|\xi\|^2_{L^2(\hat{T})} ,
\end{aligned}$$

we finally have

$$\|\phi_3\|_{H^2(\hat{T})} \preccurlyeq \|\tilde{u}\|_{H^1(\hat{T})} . \tag{6.1.12}$$

Step 4: Now we can construct $\mathcal{E}^D$. Assume a given u with

$$\int_{\partial \hat{T}} u \cdot \tau \; \mathrm{d}s = 0.$$

First we define the linear transformations F_2 and F_3 defined by the transformations of the vertices

$$F_2 : \begin{array}{ccc} (1,0) & \to & (1,0) \\ (0,0) & \to & (0,1) \\ (0,1) & \to & (0,0) \end{array} \quad \text{and} \quad F_3 : \begin{array}{ccc} (0,0) & \to & (0,0) \\ (0,1) & \to & (1,0) \\ (1,0) & \to & (0,1) \end{array}$$

and the corresponding covariant transformations $\mathcal{C}_2, \mathcal{C}_3$ (see appendix) to define proper extensions from the other edges, so

$$\mathcal{E}_2^{E_{2,1}}(u)(x,y) = \left(\mathcal{E}_2^{E_{1,2}}(\mathcal{C}_2 u)\right)(F_2(x,y)),$$

with

$$\left\|\mathcal{E}_2^{E_{2,1}}(v)\right\|_{H^2(\hat{T})} \preccurlyeq \left\|\mathcal{E}_2^{E_{1,2}}(v)\right\|_{H^2(\hat{T})},$$

and

$$\mathcal{E}_3^{E_3}(u)(x,y) = \left(\mathcal{E}_3^{E_1}(\mathcal{C}_3 u)\right)(F_3(x,y)),$$

with

$$\left\|\mathcal{E}_3^{E_3}(v)\right\|_{H^2(\hat{T})} \preccurlyeq \left\|\mathcal{E}_3^{E_1}(v)\right\|_{H^2(\hat{T})}.$$

By that we define

$$\begin{aligned}
\phi_1 &= \mathcal{E}_1^{E_1}(u)\\
\phi_2 &= \phi_1 + \mathcal{E}_2^{E_{2,1}}\left(u - \nabla\phi_1\right)\\
\phi_3 &= \phi_2 + \mathcal{E}_3^{E_3}\left(u - \nabla\phi_2\right),
\end{aligned}$$

and set $\mathcal{E}^D(u) = \phi_3$. Note that due to Green's theorem the surface integral over $\partial\hat{T}$ of the tangential component of $\nabla\phi_2$ is 0, and so also

$$\int_{\partial\hat{T}} u - \nabla\phi_2 \cdot \tau \; \mathrm{d}s = 0,$$

and as

$$u - \nabla\phi_2|_{E_1} \cdot \tau_{E_1} = u - \nabla\phi_2|_{E_2} \cdot \tau_{E_2} = 0$$

assumption 6.1.9 for $\mathcal{E}_3^{E_3}$ is fulfilled. Also we see that due to the construction of the extensions by integrals of u we get a function $\mathcal{E}^D(u) \in \Pi^{k+1}(\hat{T})$, and using the properties (6.1.6), (6.1.7) and (6.1.10), (6.1.11) we have

$$\begin{aligned}
\nabla\mathcal{E}^D(u)\big|_{E_1} \cdot \tau_{E_1} &= \nabla\phi_2|_{E_1} \cdot \tau_{E_1} + \underbrace{\nabla\mathcal{E}_3^{E_3}\left(u - \nabla\phi_2\right)\Big|_{E_1} \cdot \tau_{E_1}}_{=0}\\
&= \nabla\phi_1|_{E_1} \cdot \tau_{E_1} + \underbrace{\nabla\mathcal{E}_2^{E_{2,1}}\left(u - \nabla\phi_1\right)\Big|_{E_1} \cdot \tau_{E_1}}_{=0}\\
&= \nabla\phi_1|_{E_1} \cdot \tau_{E_1} = u \cdot \tau_{E_1}
\end{aligned}$$

$$\begin{aligned}
\nabla\mathcal{E}^D(u)\big|_{E_2} \cdot \tau_{E_2} &= \nabla\phi_2|_{E_2} \cdot \tau_{E_2} + \underbrace{\nabla\mathcal{E}_3^{E_3}\left(u - \nabla\phi_2\right)\Big|_{E_1} \cdot \tau_{E_1}}_{=0}\\
&= \nabla\phi_1|_{E_2} \cdot \tau_{E_2} + \nabla\mathcal{E}_2^{E_{2,1}}\left(u - \nabla\phi_1\right)\Big|_{E_2} \cdot \tau_{E_2}\\
&= \nabla\phi_1|_{E_2} \cdot \tau_{E_2} + (u - \nabla\phi_1)|_{E_2} \cdot \tau_{E_2} = u \cdot \tau_{E_2}
\end{aligned}$$

$$\nabla \mathcal{E}^D(u)\big|_{E_3} \cdot \tau_{E_3} = \nabla\phi_2|_{E_3} \cdot \tau_{E_3} + \nabla\mathcal{E}_3^{E_3}(u - \nabla\phi_2)\Big|_{E_3} \cdot \tau_{E_3}$$
$$= \nabla\phi_2|_{E_3} \cdot \tau_{E_3} + (u - \nabla\phi_2)|_{E_3} \cdot \tau_{E_3} = u \cdot \tau_{E_3}.$$

The H^2 estimation follows from (6.1.5), (6.1.8) and (6.1.12), so

$$\begin{aligned}
\left\|\mathcal{E}^D(u)\right\|_{H^2(\hat{T})} &\preccurlyeq \|\phi_2\|_{H^2(\hat{T})} + \left\|\mathcal{E}_3^{E_3}(u - \nabla\phi_2)\right\|_{H^2(\hat{T})} \\
&\preccurlyeq \|\phi_2\|_{H^2(\hat{T})} + \|u - \nabla\phi_2\|_{H^1(\hat{T})} \preccurlyeq \|\phi_2\|_{H^2(\hat{T})} + \|u\|_{H^1(\hat{T})} \\
&\preccurlyeq \|\phi_1\|_{H^2(\hat{T})} + \left\|\mathcal{E}_2^{E_{2,1}}(u - \nabla\phi_1)\right\|_{H^2(\hat{T})} + \|u\|_{H^1(\hat{T})} \\
&\preccurlyeq \|\phi_1\|_{H^2(\hat{T})} + \|u - \nabla\phi_1\|_{H^1(\hat{T})} + \|u\|_{H^1(\hat{T})} \\
&\preccurlyeq \|\phi_1\|_{H^2(\hat{T})} + \|u\|_{H^1(\hat{T})} \preccurlyeq \|u\|_{H^1(\hat{T})}
\end{aligned}$$

□

6.2 Normal extension $\mathcal{E}^N$

Theorem 6.4. *Assume a given function $u \in \Pi^k(E_1)$ where $E_1 = [0,1]$ is an edge of the reference element and has a zero of order two in the vertices. Then there exists an operator $\mathcal{E}^N : \Pi^k(E_1) \to \Pi^{k+1}(\hat{T})$ so that for $\phi_n = \mathcal{E}^N(u)$ we have*

$$\phi_n = 0 \text{ on } \partial\hat{T} \tag{6.2.1}$$
$$\nabla\phi_n \cdot n = u \text{ on } E_1 \tag{6.2.2}$$
$$\nabla\phi_n \cdot n = 0 \text{ on } \partial\hat{T} \setminus E_1 \tag{6.2.3}$$
$$\|\phi_n\|_{H^2(\hat{T})} \preccurlyeq \|u\|_{H^{1/2}_{00}(E_1)}. \tag{6.2.4}$$

Proof. The idea of the proof is similar to the proof of theorem 6.3. First we extend the values of the lower edge and correct the trace and the normal derivative afterwards. For the corrections we use cubic blending coefficients. Due to that it is not trivial that the corresponding extension still fulfills $\mathcal{E}^N(u) \in \Pi^{k+1}(\hat{T})$. Before we start with the proof we remind the reader of the definition of the $\|\cdot\|_{H^{1/2}_{00}(E_1)}$ norm

$$\begin{aligned}
\|u\|^2_{H^{1/2}_{00}(E_1)} &= |u|_{H^{1/2}(E_1)} + \|u\|_{L^2_*(E_1)} \\
&= \int_0^1\int_0^1 \frac{(u(x)-u(y))^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y + \int_0^1 \left(\frac{1}{1-x} + \frac{1}{x}\right) u^2(x)\,\mathrm{d}x.
\end{aligned}$$

All following examples are visualized in a three dimensional perspective, see figure 6.1.
Step 1: In the first step we define

$$\phi_1(x,y) := y\int_0^1 a_1(s)u(x+sy)\,\mathrm{d}s,$$

with $a_1(s) = 6s(1-s)$, and so

$$\int_0^1 a_1(s)\,\mathrm{d}s = 1 \quad \text{and} \quad \int_0^1 a_1'(s)s\,\mathrm{d}s = -1 \quad \text{and} \quad a(0) = a(1) = 0.$$

For the gradient we use integration by part to get

$$\frac{\partial \phi_1}{\partial x} = \int_0^1 a_1'(s)u(x-sy)\,\mathrm{d}s,$$

and

$$\frac{\partial \phi_1}{\partial y} = \int_0^1 a_1'(s)su(x+ys)\,\mathrm{d}s.$$

By that we observe

$$\phi_1|_{E_1} = 0$$

and

$$\nabla\phi_1 \cdot n_{E_1}|_{E_1} = -\frac{\partial \phi_1}{\partial y} = -u \underbrace{\int_0^1 a_1'(s)s\,\mathrm{d}s}_{=-1} = u(x).$$

In figure 6.5 we can observe the first extension ϕ_1 for the example $u = 10x^2(1-x)^2$. Due to the constant value $\phi_1 = 0$ on the lower edge E_1 the derivation with respect to x is also zero.

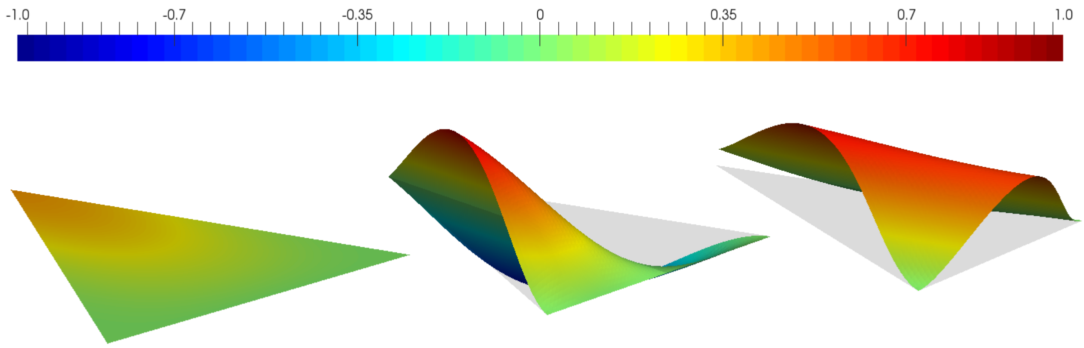


Figure 6.5: Values and partial derivatives of the extension ϕ_1 for $u = 10x^2(1-x)^2$

Using lemma 6.2 we observe that

$$\left\|\frac{\partial \phi_1}{\partial x}\right\|_{H^1(\hat{T})} \preccurlyeq \|u\|_{H^{1/2}(E_1)} \preccurlyeq \|u\|_{H^{1/2}_{00}(E_1)}$$
$$\left\|\frac{\partial \phi_1}{\partial y}\right\|_{H^1(\hat{T})} \preccurlyeq \|u\|_{H^{1/2}(E_1)} \preccurlyeq \|u\|_{H^{1/2}_{00}(E_1)}$$

and due to

$$\begin{aligned}
\|\phi_1\|^2_{L^2(\hat{T})} &= \int_0^1 \int_0^{1-x} y^2 (\int_0^1 a_1(s) u(x+sy)\, \mathrm{d}s)^2 \,\mathrm{d}y\, \mathrm{d}x \\
&\overset{\text{C.S.}}{\preccurlyeq} \int_0^1 \int_0^{1-x} y^2 \int_0^1 u(x+sy)^2 \,\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_0^1 \int_0^{1-x} \frac{y^2}{1-x} \int_x^1 u(t)^2 \,\mathrm{d}t\, \mathrm{d}y\, \mathrm{d}x \\
&\preccurlyeq \int_0^1 \int_0^{1-x} \int_0^1 u(x+sy)^2 \,\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}x \\
&\preccurlyeq \|u\|_{L^2(E_1)} \preccurlyeq \|u\|_{H^{1/2}_{00}(E_1)},
\end{aligned}$$

we have

$$\|\phi_1\|_{H^2(\hat{T})} \preccurlyeq \|u\|_{H^{1/2}_{00}(E_1)}.$$

Step 2: In the next step we correct the values on the second edge E_2. For that we use two cubic polynomial blending functions, namely

$$a(s) = 3s^2 - 2s^3 \quad \text{and} \quad b(s) = s^3 - s^2,$$

with the properties

$$a(0) = a'(0) = a'(1) = 0 \quad \text{and} \quad a(1) = 1 \tag{6.2.5}$$

$$b(0) = b'(0) = b(1) = 0 \quad \text{and} \quad b'(1) = 1. \tag{6.2.6}$$

and define

$$\phi_2(x,y) = \phi_1(x,y) - a\left(\frac{y}{1-x}\right)\phi_1(x,1-x) - b\left(\frac{y}{1-x}\right)(1-x)\frac{\partial \phi_1}{\partial y}(x,1-x).$$

The first polynomial a corrects the values, the second one b corrects the derivative on the edge E_2. Due to the zero values on the lower edge, the extension from the first step ϕ_1 does not change on E_1. In figure 6.6 we can see $a(s)$ and $b(s)$ on $(0,1)$.

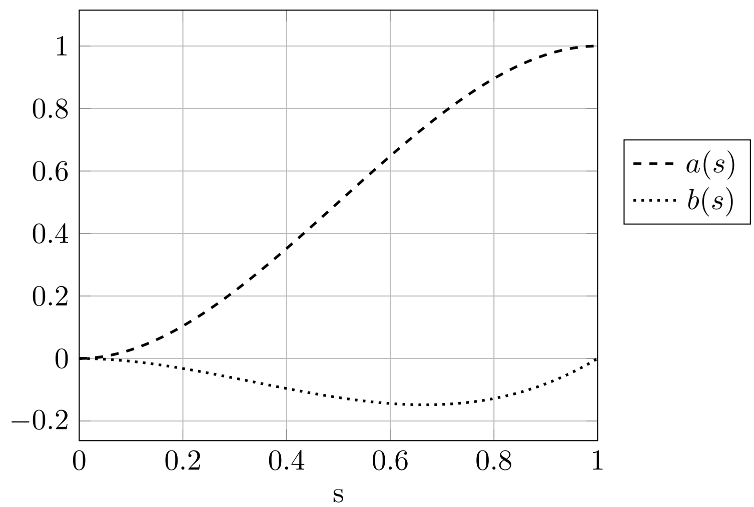


Figure 6.6: $a(s)$ and $b(s)$ on $(0,1)$

On the edges E_1 and E_2 we now have

$$\begin{aligned}
\phi_2|_{E_1} &= \phi_1|_{E_1} - a(0)\phi_1(x, 1-x) - b(0)(1-x)\frac{\partial \phi_1}{\partial y}(x, 1-x) = 0 \\
\phi_2|_{E_2} &= \phi_1|_{E_2} - a(1)\phi_1(x, 1-x) - b(1)(1-x)\frac{\partial \phi_1}{\partial y}(x, 1-x) \\
&= \phi_1(x, 1-x) - \phi_1(x, 1-x) = 0.
\end{aligned}$$

In figure 6.7 we can observe the extension ϕ_2 including the first correction for the example $u = 10x^2(1-x)^2$. Note that the values and partial derivatives on the second edge E_2 are corrected.

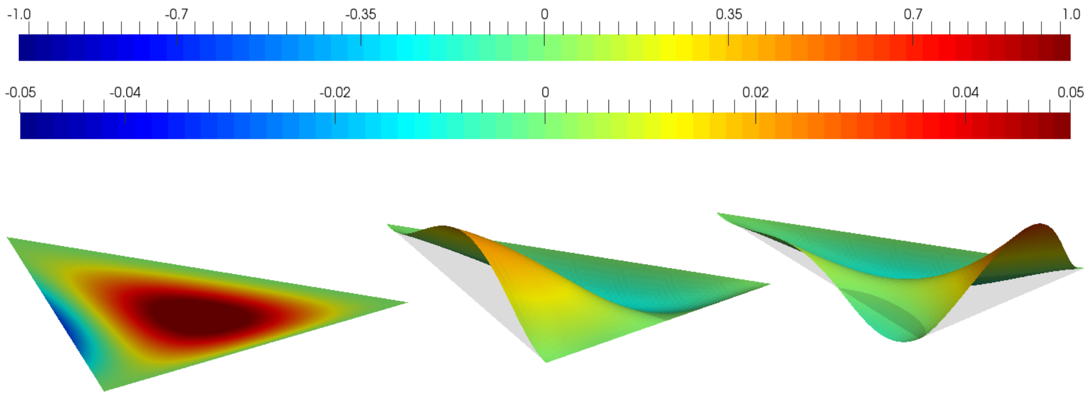


Figure 6.7: Values (lower scale) and partial derivatives (upper scale) of the extension ϕ_2 for $u = 10x^2(1-x)^2$

Next we calculate the derivations using the chain and the product rule

$$\begin{aligned}
\frac{\partial \phi_2}{\partial x}(x, y) = \frac{\partial \phi_1}{\partial x}(x, y) & \\
& - \frac{\partial}{\partial x}(\frac{y}{1-x})a'\left(\frac{y}{1-x}\right)\phi_1(x, 1-x) \\
& - a\left(\frac{y}{1-x}\right)\left(\frac{\partial \phi_1}{\partial x}(x, 1-x) - \frac{\partial \phi_2}{\partial y}(x, 1-x)\right) \\
& - \frac{\partial}{\partial x}\left(\frac{y}{1-x}\right)b'\left(\frac{y}{1-x}\right)\frac{\partial \phi_1}{\partial y}(x, 1-x) \\
& - b\left(\frac{y}{1-x}\right)\left(\frac{\partial^2 \phi_1}{\partial x \partial x}(x, 1-x) - \frac{\partial^2 \phi_1}{\partial x \partial y}(x, 1-x)\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \phi_2}{\partial y}(x, y) = \frac{\partial \phi_1}{\partial y}(x, y) & - \frac{\partial}{\partial y}(\frac{y}{1-x})a'\left(\frac{y}{1-x}\right)\phi_1(x, 1-x) \\
& - \frac{\partial}{\partial y}(\frac{y}{1-x})b'\left(\frac{y}{1-x}\right)(1-x)\frac{\partial \phi_1}{\partial y}(x, 1-x).
\end{aligned}$$

and observe

$$\begin{aligned}\left.\frac{\partial \phi_2}{\partial y}\right|_{E_1} &= \left.\frac{\partial \phi_1}{\partial y}\right|_{E_1} - \frac{a'(0)}{1-x}\phi_1(x,1-x) \\ &\quad - \frac{b'(0)}{1-x}(1-x)\frac{\partial \phi_1}{\partial y}(x,1-x) \\ &= \left.\frac{\partial \phi_1}{\partial y}\right|_{E_1} = u(x),\end{aligned}$$

so $\nabla\phi_2 \cdot n_{E_1}|_{E_1} = u(x)$, and

$$\begin{aligned}\left.\frac{\partial \phi_2}{\partial y}\right|_{E_2} &= \left.\frac{\partial \phi_1}{\partial y}\right|_{E_2} - \frac{a(1)}{1-x}\phi_1(x,1-x) \\ &\quad - \frac{b'(1)}{1-x}(1-x)\frac{\partial \phi_1}{\partial y}(x,1-x) \\ &= \frac{\partial \phi_1}{\partial y}(x,1-x) - \frac{\partial \phi_1}{\partial y}(x,1-x) = 0.\end{aligned}$$

Due to the constant value of $\phi_2 = 0$ on the edge E_2 we have

$$\nabla\phi_2 \cdot \tau_{E_2}|_{E_2} = 0 \Rightarrow -\left.\frac{\partial \phi_2}{\partial x}\right|_{E_2} = \left.\frac{\partial \phi_2}{\partial y}\right|_{E_2} = 0$$

and so $\nabla\phi_2 \cdot n_{E_2}|_{E_2} = 0$. It remains to show the H^2 estimation. For the first term of ϕ_2 we already showed the estimation in the first step. For the rest we split ϕ_2 into

$$\psi := a\left(\frac{y}{1-x}\right)\phi_1(x,1-x) \quad \text{and} \quad \xi := b\left(\frac{y}{1-x}\right)(1-x)\frac{\partial \phi_1}{\partial y}(x,1-x).$$

We first show the estimation for ψ

$$\psi(x,y) = \frac{y^2}{(1-x)^2}(3(1-x)-2y)\int_0^1 u(x+s(1-x))a_1(s)\ \mathrm{d}s.$$

For all estimations we use the Cauchy Schwarz inequality, Fubini's theorem and that $\frac{y}{1-x} \leq 1$ on $\hat{T}$. By that we get

$$\begin{aligned}\|\psi\|^2_{L^2(\hat{T})} &= \int_0^1\int_0^{1-x} \psi^2\ \mathrm{d}y\ \mathrm{d}x \\ &\overset{\text{C.S.}}{\preccurlyeq} \int_0^1\int_0^{1-x} (3(1-x)-2y)^2 \int_0^1 u(x+s(1-x))^2\ \mathrm{d}s\ \mathrm{d}y\ \mathrm{d}x \\ &= \int_0^1\int_0^{1-x} (3(1-x)-2y)^2 \frac{1}{1-x}\int_x^1 u(t)^2\ \mathrm{d}t\ \mathrm{d}y\ \mathrm{d}x \\ &\preccurlyeq \int_0^1\int_0^{1-x}\int_x^1 u(t)^2\ \mathrm{d}t\ \mathrm{d}y\ \mathrm{d}x \preccurlyeq \|u\|^2_{L^2(E_1)} \preccurlyeq \|u\|^2_{H^{1/2}_{00}(E_1)}.\end{aligned}$$

We continue with the first order derivatives.

$$\begin{aligned}
\frac{\partial\psi}{\partial y} &= 6\frac{y}{1-x}\left(1-\frac{y}{1-x}\right)\int_0^1 u(x+s(1-x))a_1(s)\ \mathrm{d}s\\
\frac{\partial\psi}{\partial x} &= 6\frac{y^2}{(1-x)^2}\left(1-\frac{y}{1-x}\right)\int_0^1 u(x+s(1-x))a_1(s)\ \mathrm{d}s\\
&+\frac{y^2}{(1-x)^2}\left(3-2\frac{y}{1-x}\right)\frac{\partial}{\partial x}\left(\phi_1(x,1-x)\right).\\
&= 6\frac{y^2}{(1-x)^2}\left(1-\frac{y}{1-x}\right)\int_0^1 u(x+s(1-x))a_1(s)\ \mathrm{d}s\\
&+\frac{y^2}{(1-x)^2}\left(3-2\frac{y}{1-x}\right)\int_0^1 (a_1'(s)-a_1'(s)s)u(x+s(1-x))\ \mathrm{d}s
\end{aligned}$$

For the y-derivation we have

$$\begin{aligned}
\left\|\frac{\partial\psi}{\partial y}\right\|^2_{L^2(\hat{T})} &\overset{\text{c.s.}}{\preccurlyeq} \int_0^1\int_0^{1-x}\int_0^1 u^2(x+s(1-x))\ \mathrm{d}s\ \mathrm{d}y\ \mathrm{d}x\\
&= \int_0^1\int_0^{1-x}\frac{1}{1-x}\int_x^1 u(t)^2\ \mathrm{d}t\ \mathrm{d}y\ \mathrm{d}x\\
&\preccurlyeq \int_0^1\int_0^{1-x}\int_x^1 \frac{u(t)^2}{1-t}\ \mathrm{d}t\ \mathrm{d}y\ \mathrm{d}x\\
&\preccurlyeq \|u\|^2_{L^2_*(E_1)} \preccurlyeq \|u\|^2_{H^{1/2}_{00}(E_1)}.
\end{aligned}$$

Due to the structure of the x-derivation we bound the norm similar to the y-derivation, to get

$$\left\|\frac{\partial\psi}{\partial x}\right\|^2_{L^2(\hat{T})} \preccurlyeq \|u\|^2_{H^{1/2}_{00}(E_1)}.$$

It remains the estimation for the second order derivatives.

$$\frac{\partial^2\psi}{\partial y\partial y} = 6\frac{1}{1-x}\left(1-\frac{2y}{1-x}\right)\int_0^1 u(x+s(1-x))a_1(s)\ \mathrm{d}s$$

$$\begin{aligned}
\frac{\partial^2\psi}{\partial y\partial x} &= \left(12\frac{y}{1-x}-18\frac{y^2}{(1-x)^2}\right)(1-x)\int_0^1 u(x+s(1-x))a_1(s)\ \mathrm{d}s\\
&+6\frac{y}{(1-x)^2}\left(1-\frac{y}{1-x}\right)\int_0^1 (a_1'(s)-a_1'(s)s)u(x+s(1-x))\ \mathrm{d}s
\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial x\partial x} &= \left(18\frac{y^2}{(1-x)^2} - 24\frac{y^3}{(1-x)^3}\right)(1-x)\int_0^1 u(x+s(1-x))a_1(s)\ \mathrm{d}s \\ &+ 6\frac{y^2}{(1-x)^3}\left(1-\frac{y}{1-x}\right)\int_0^1 (a_1'(s)s - a_1'(s))u(x+s(1-x))\ \mathrm{d}s \\ &+ \frac{y^2}{(1-x)^2}\left(3-2\frac{y}{1-x}\right)\frac{\partial}{\partial x}\left(\int_0^1 (a_1'(s) - a_1'(s)s)u(x+s(1-x))\ \mathrm{d}s\right) \\ &= \left(18\frac{y^2}{(1-x)^2} - 24\frac{y^3}{(1-x)^3}\right)(1-x)\int_0^1 u(x+s(1-x))a_1(s)\ \mathrm{d}s \\ &+ 6\frac{y^2}{(1-x)^3}\left(1-\frac{y}{1-x}\right)\int_0^1 (a_1'(s)s - a_1'(s))u(x+s(1-x))\ \mathrm{d}s \\ &+ \frac{y^2}{(1-x)^2}\left(3-2\frac{y}{1-x}\right)\frac{1}{1-x}\int_0^1 (36s^2-60s+18)u(x+s(1-x))\ \mathrm{d}s \\ &- \frac{y^2}{(1-x)^2}\left(3-2\frac{y}{1-x}\right)\frac{6}{1-x}\int_0^1 u(x) - u(x+s(1-x))\ \mathrm{d}s.\end{aligned}$$

We start with the second derivative with respect to y.

$$\begin{aligned}\left\|\frac{\partial^2\psi}{\partial y\partial y}\right\|^2_{L^2(\hat{T})} &\overset{\text{C.S.}}{\preccurlyeq} \int_0^1\int_0^{1-x}\frac{1}{(1-x)^2}\int_0^1 u(x+s(1-x))^2\ \mathrm{d}s\ \mathrm{d}y\ \mathrm{d}x \\ &= \int_0^1 \frac{1}{(1-x)^2}\int_x^1 u(t)^2\ \mathrm{d}t\ \mathrm{d}x \\ \iint\limits_{\substack{0\le x\le 1\\ x\le t}} \frac{u(t)^2}{(1-x)^2}\ \mathrm{d}(x,t) &\overset{\text{Fub.}}{=} \iint\limits_{\substack{0\le t\le 1\\ x\le t}} \frac{u(t)^2}{(1-x)^2}\ \mathrm{d}(x,t) \\ &= \int_0^1\int_0^t \frac{1}{(1-x)^2}\ \mathrm{d}x\ u(t)^2\ \mathrm{d}t \preccurlyeq \int_0^1 \frac{1}{1-t}u(t)^2\ \mathrm{d}t \preccurlyeq \|u\|^2_{H^{1/2}_{00}(E_1)}.\end{aligned}$$

The first term of the mixed derivation can be estimated as $\|\psi\|_{L^2(\hat{T})}$ and the second term as $\left\|\frac{\partial^2\psi}{\partial y\partial y}\right\|_{L^2(\hat{T})}$, so we also get

$$\left\|\frac{\partial^2\psi}{\partial y\partial x}\right\|^2_{L^2(\hat{T})} \preccurlyeq \|u\|^2_{H^{1/2}_{00}(E_1)}.$$

It remains the second order derivation of x. Again the first three terms can be estimated similar to before. It remains the last term

$$-\frac{y^2}{(1-x)^2}\left(3-2\frac{y}{1-x}\right)\frac{6}{1-x}\int_0^1 u(x) - u(x+s(1-x))\ \mathrm{d}s.$$

For this we refer to the estimation of $\left\|\frac{\partial^2\phi_2^{corr}}{\partial x\partial x}\right\|_{L^2(\hat{T})}$ in the proof of theorem 6.3 to bound the L^2 norm with $|u|_{H^{1/2}(E_1)}$. Altogether we have

$$\left\|\frac{\partial^2\psi}{\partial x\partial x}\right\|^2_{L^2(\hat{T})} \preccurlyeq \|u\|^2_{H^{1/2}_{00}(E_1)},$$

and so

$$\|\psi\|_{H^2(\hat{T})} \preccurlyeq \|u\|_{H_{00}^{1/2}(E_1)} .$$

The estimations for ξ can be done in the same way by using the introduced techniques, and thus we get

$$\|\psi\|_{H^2(\hat{T})} \preccurlyeq \|u\|_{H_{00}^{1/2}(E_1)} \quad \text{and} \quad \|\phi_2\|_{H^2(\hat{T})} \preccurlyeq \|u\|_{H_{00}^{1/2}(E_1)} .$$

Step 3: For the last step we define

$$\begin{aligned}
\mathcal{E}^N(u)(x,y) = \phi_3(x,y) := \phi_2(x,y) &- a\left(\frac{y}{x+y}\right)\phi_2(0,x+y) \\
&- b\left(\frac{y}{x+y}\right)(x+y)\frac{\partial \phi_2}{\partial x}(0,x+y) \\
&+ b\left(\frac{y}{x+y}\right)(x+y)\frac{\partial \phi_2}{\partial y}(0,x+y),
\end{aligned}$$

and have

$$\begin{aligned}
\phi_3|_{E_1} &= \phi_2|_{E_1} = 0 \\
\phi_3|_{E_2} &= \phi_2|_{E_2} - a(y)\underbrace{\phi_2(0,1)}_{=0} \\
&\quad - b(1-x)\underbrace{\frac{\partial \phi_2}{\partial x}(0,1)}_{=0} + b(1-x)\underbrace{\frac{\partial \phi_2}{\partial x}(0,1)}_{=0} = 0 \\
\phi_3|_{E_3} &= \phi_2|_{E_3} - a(1)\phi_2(0,y) = \\
&= \phi_2(0,y) - \phi_2(0,y) = 0.
\end{aligned}$$

In figure 6.8 we can observe the final extension ϕ_3 including the first and the second correction for the example $u = 10x^2(1-x)^2$.

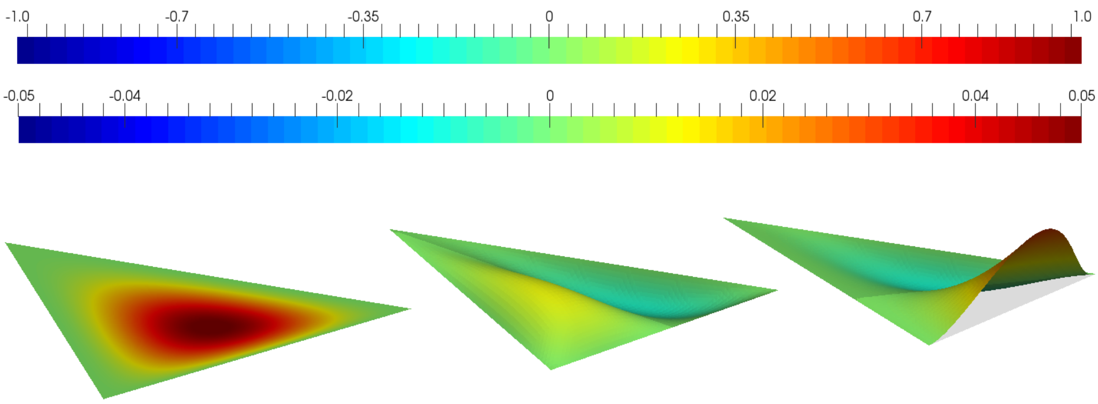


Figure 6.8: Values (lower scale) and partial derivatives (upper scale) of the resulting extension ϕ_3 for $u = 10x^2(1-x)^2$

The first order derivations are given by

$$\begin{aligned}\frac{\partial\phi_3}{\partial x}(x,y) &= \frac{\partial\phi_2}{\partial x}(x,y)\\ &- \frac{y}{(x+y)^2}a'\left(\frac{y}{x+y}\right)\phi_2(0,x+y)\\ &- a\left(\frac{y}{x+y}\right)\frac{\partial\phi_2}{\partial y}(0,x+y)\\ &- \frac{y}{(x+y)^2}b'\left(\frac{y}{x+y}\right)(x+y)\frac{\partial\phi_2}{\partial x}(0,x+y)\\ &- b\left(\frac{y}{x+y}\right)\left(\frac{\partial\phi_2}{\partial x}(0,x+y)+(x+y)\frac{\partial^2\phi_2}{\partial x\partial y}(0,x+y)\right)\\ &+ \frac{y}{(x+y)^2}b'\left(\frac{y}{x+y}\right)(x+y)\frac{\partial\phi_2}{\partial y}(0,x+y)\\ &+ b\left(\frac{y}{x+y}\right)\left(\frac{\partial\phi_2}{\partial y}(0,x+y)+(x+y)\frac{\partial^2\phi_2}{\partial y\partial y}(0,x+y)\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\phi_3}{\partial y}(x,y) &= \frac{\partial\phi_2}{\partial y}(x,y)\\ &- \frac{x}{(x+y)^2}a'\left(\frac{y}{x+y}\right)\phi_2(0,x+y)\\ &- a\left(\frac{y}{x+y}\right)\frac{\partial\phi_2}{\partial y}(0,x+y)\\ &- \frac{x}{(x+y)^2}b'\left(\frac{y}{x+y}\right)(x+y)\frac{\partial\phi_2}{\partial x}(0,x+y)\\ &- b\left(\frac{y}{x+y}\right)\left(\frac{\partial\phi_2}{\partial x}(0,x+y)+(x+y)\frac{\partial^2\phi_2}{\partial x\partial y}(0,x+y)\right)\\ &+ \frac{x}{(x+y)^2}b'\left(\frac{y}{x+y}\right)(x+y)\frac{\partial\phi_2}{\partial y}(0,x+y)\\ &+ b\left(\frac{y}{x+y}\right)\left(\frac{\partial\phi_2}{\partial y}(0,x+y)+(x+y)\frac{\partial^2\phi_2}{\partial y\partial y}(0,x+y)\right).\end{aligned}$$

On the edges we observe

$$\left.\frac{\partial\phi_3}{\partial y}\right|_{E_1} = \left.\frac{\partial\phi_2}{\partial y}\right|_{E_1} = u(x) \Rightarrow \nabla\phi_2\cdot n_{E_1}|_{E_1} = u(x),$$

and

$$\begin{aligned}
\left.\frac{\partial \phi_3}{\partial y}\right|_{E_2} &= \left.\frac{\partial \phi_2}{\partial y}\right|_{E_2} - x a'(1-x)\phi_2(0,1) - a(1-x)\frac{\partial \phi_2}{\partial y}(0,1) \\
&\quad - x b'(1-x)\frac{\partial \phi_2}{\partial x}(0,1) \\
&\quad - b(1-x)\left(\frac{\partial \phi_2}{\partial x}(0,1) + \frac{\partial^2 \phi_2}{\partial x \partial y}(0,1)\right) \\
&\quad + x b'(1-x)\frac{\partial \phi_2}{\partial y}(0,1) \\
&\quad + b(1-x)\left(\frac{\partial \phi_2}{\partial y}(0,1) + \frac{\partial^2 \phi_2}{\partial y \partial y}(0,1)\right) \\
&= \left.\frac{\partial \phi_2}{\partial y}\right|_{E_2} = 0,
\end{aligned}$$

so again similar to ϕ_2 it follows $\nabla\phi_2 \cdot n_{E_2}|_{E_2} = 0$. Finally on the last edge we have

$$\begin{aligned}
\left.\frac{\partial \phi_3}{\partial x}\right|_{E_3} &= \left.\frac{\partial \phi_2}{\partial x}\right|_{E_3} - \frac{1}{y} a'(1)\phi_2(0,y) - a(1)\frac{\partial \phi_2}{\partial y}(0,y) \\
&\quad - \frac{1}{y} b'(1) y \frac{\partial \phi_2}{\partial x}(0,y) - b(1)\left(\frac{\partial \phi_2}{\partial x}(0,y) + y\frac{\partial^2 \phi_2}{\partial x \partial y}(0,y)\right) \\
&\quad + \frac{1}{y} b'(1) y \frac{\partial \phi_2}{\partial y}(0,y) + b(1)\left(\frac{\partial \phi_2}{\partial y}(0,y) + y\frac{\partial^2 \phi_2}{\partial y \partial y}(0,y)\right) \\
&= \frac{\partial \phi_2}{\partial x}(0,y) - \frac{\partial \phi_2}{\partial y}(0,y) - \frac{\partial \phi_2}{\partial x}(0,y) + \frac{\partial \phi_2}{\partial y}(0,y) = 0,
\end{aligned}$$

and so also $\nabla\phi_2 \cdot n_{E_2}|_{E_3} = \left.\frac{\partial \phi_3}{\partial x}\right|_{E_3} = 0$. The H^2 estimation for ϕ_3 is analogue as the estimation for ϕ_2 so we have

$$\|\phi_3\|_{H^2(\hat{T})} \preccurlyeq \|u\|_{H^{1/2}_{00}(E_1)}.$$

It remains to show that the extension $\mathcal{E}^N(u)$ belongs to $\Pi^{k+1}(\hat{T})$. Due to the definition by integrals of u we raise the polynomial order by one, but by using the blending coefficients $a(y/(1-x))$, $b(y/(1-x))$, $a(y/(y+x))$ and $b(y/(y+x))$ the result may not be a polynomial anymore. To see that this is still fulfilled, we use that the given polynomial u has a zero of order two in the vertices, and so there exist polynomials $v, w \in \Pi^{k-2}(E)$ so that u can be written as

$$u(x) = (1-x)^2 v(x) \quad \text{and} \quad u(x) = x^2 w(x).$$

For ϕ_2 we observe

$$\begin{aligned}
\phi_2(x,y) &= \phi_1(x,y) - a\left(\frac{y}{1-x}\right)\phi_1(x,1-x) - b\left(\frac{y}{1-x}\right)(1-x)\frac{\partial\phi_1}{\partial y}(x,1-x) \\
&= y\int_0^1 a_1(s)u(x+sy)\ \mathrm{d}s \\
&\quad - \frac{3y^2(1-x)-2y^3}{(1-x)^3}(1-x)\int_0^1 a_1(s)u(x+s(1-x))\ \mathrm{d}s \\
&\quad - \frac{y^3-y^2(1-x)}{(1-x)^3}(1-x)\int_0^1 a_1'(s)su(x+(1-x)s)\ \mathrm{d}s.
\end{aligned}$$

And as

$$\begin{aligned}
u(x+s(1-x)) &= (1-x-s(1-x))^2\, v(x+s(1-x)) \\
&= ((1-x)(1-s))^2\, v(x+s(1-x)) \\
&= (1-x)^2(1-s)^2 v(x+s(1-x))
\end{aligned}$$

we have

$$\begin{aligned}
\phi_2(x,y) &= y\int_0^1 a_1(s)u(x+sy)\ \mathrm{d}s \\
&\quad - \frac{3y^2(1-x)-2y^3}{(1-x)^3}(1-x)^3\int_0^1 a_1(s)(1-s)^2 v(x+s(1-x))\ \mathrm{d}s \\
&\quad - \frac{y^3-y^2(1-x)}{(1-x)^3}(1-x)^3\int_0^1 a_1'(s)s(1-s)^2 v(x+(1-x)s)\ \mathrm{d}s,
\end{aligned}$$

and so $\phi_2 \in \Pi^{k+1}(\hat{T})$. Similar for ϕ_3 we have

$$\begin{aligned}
\phi_3(x,y) &= \phi_2(x,y) - a\left(\frac{y}{x+y}\right)\phi_2(0,x+y) \\
&\quad - b\left(\frac{y}{x+y}\right)(x+y)\frac{\partial\phi_2}{\partial x}(0,x+y) \\
&\quad + b\left(\frac{y}{x+y}\right)(x+y)\frac{\partial\phi_2}{\partial y}(0,x+y) \\
&= \phi_2(x,y) \\
&\quad - \frac{3y^2(x+y)-2y^3}{(x+y)^3}(x+y)\int_0^1 a_1(s)u(s(x+y))\ \mathrm{d}s \\
&\quad - \frac{y^3-y^2(x+y)}{(x+y)^3}(x+y)\int_0^1 a_1'(s)u(s(x+y))\ \mathrm{d}s \\
&\quad + \frac{y^3-y^2(x+y)}{(x+y)^3}(x+y)\int_0^1 a_1'(s)su(s(x+y))\ \mathrm{d}s,
\end{aligned}$$

and together with

$$u(s(x+y)) = s^2(x+y)^2 w(s(x+y))$$

we see

$$\begin{aligned}\phi_3(x,y) = \phi_2(x,y)\\ &- \frac{3y^2(x+y)-2y^3}{(x+y)^3}(x+y)^3\int_0^1 a_1(s)s^2 w(s(x+y))\,\mathrm{d}s\\ &- \frac{y^3-y^2(x+y)}{(x+y)^3}(x+y)^3\int_0^1 a_1'(s)s^2 w(s(x+y))\,\mathrm{d}s\\ &+ \frac{y^3-y^2(x+y)}{(x+y)^3}(x+y)^3\int_0^1 a_1'(s)s^3 w(s(x+y))\,\mathrm{d}s,\end{aligned}$$

and so $\mathcal{E}^N(u) \in \Pi^{k+1}(\hat{T})$. □

6.3 Correction theorem

Theorem 6.5. *Assume a given function $u \in \Pi^k(\partial\hat{T})$ and $u = 0$ on $\partial\hat{T} \setminus E_1$ where $E_1 = [0,1]$. Then we have*

i. $|u'(1)| \preccurlyeq k^2 \|u\|_{H_{00}^{1/2}(E_1)}$

ii. There exists a function $e_k \in \Pi^k(E)$ with $e_k'(1) = 1$ and $e_k'(0) = e_k(0) = e_k(1) = 0$ so that

$$\|e_k\|_{H_{00}^{1/2}(E_1)} \preccurlyeq \frac{1}{k^2} \tag{6.3.1}$$

$$\|e_k\|_{L^2(E_1)} \preccurlyeq \frac{1}{k^3}. \tag{6.3.2}$$

Proof. We first show statement *ii.* For the ease we show the estimation on $E = (-1,1)$, the theorem follows with a transformation to $E_1 = (0,1)$. We use a special basis for $\Pi^k(E)$, namely integrated Jacobi polynomials (see appendix and [BS06] and [AS65]). We define

$$\begin{aligned}\hat{P}_n^{(1,0)}(x) &:= -\int_x^1 P_{n-1}^{(1,0)}(s)\,\mathrm{d}s \quad 1 \le n \le k\\ \hat{P}_0^{(1,0)}(x) &:= 1,\end{aligned}$$

with the properties

$$\begin{aligned}\hat{P}_n^{(1,0)}(1) &= 0 \quad 1 \le n \le k\\ \hat{P}_n^{(1,0)\prime}(1) &= n \quad 0 \le n \le k\\ (2n+1)P_n^{(0,0)} &= (n+1)\hat{P}_n^{(1,0)} - n\hat{P}_{n-1}^{(1,0)}\\ P_m^{(1,0)} &= \frac{1}{m+1}\sum_{n=0}^{m}(2n+1)P_n^{(0,0)}.\end{aligned}$$

We also use integrated Legendre Polynomials

$$L_{n+1}(x) := -\int_x^1 P_n^{(0,0)}(s)\ \mathrm{d}s$$

with the representation (using $P_{-1}^{(0,0)} := -1$)

$$(2n+1)L_{n+1} = P_{n+1}^{(0,0)} - P_{n-1}^{(0,0)} \quad n \geq 1.$$

We are looking now for a function $\tilde{e}$ given by the argument of the minimum of the weighted H^1 seminorm

$$\tilde{e} := \arg \min_{\substack{v \in \Pi^k \\ v(1)=0 \\ v'(1)=1}} \int_{-1}^1 (1-x)v'(x)^2\ \mathrm{d}x = \arg \min_{\substack{v \in \Pi^k \\ v(1)=0 \\ v'(1)=1}} |v|^2_{H^1_\omega(E)},$$

with the weight $\omega = 1 - x$. For this we set

$$v(x) := \sum_{j=0}^k \alpha_j \hat{P}_j^{(1,0)}(x),$$

with $\alpha_0 = 0$. Note that due to the choice of our basis we have

$$\int_{-1}^1 (1-x)\hat{P}_n^{(1,0)\prime}(x)\hat{P}_m^{(1,0)\prime}(x)\ \mathrm{d}s = \int_{-1}^1 (1-x)P_{n-1}^{(1,0)}(x)P_{m-1}^{(1,0)}(x)\ \mathrm{d}s = \delta_{n,m}\frac{2}{n+1},$$

and so

$$\begin{aligned} |v|^2_{H^1_\omega(E)} &= \sum_{j=1}^k \alpha_j^2 |\hat{P}_j^{(1,0)}|^2_{H^1_\omega(E)} = \sum_{j=1}^k \alpha_j^2 \frac{2}{j+1} \\ v(1) &= 0 \\ v'(1) &= \sum_{j=1}^k \alpha_j j. \end{aligned}$$

We use the technique of Lagrangian multipliers to find the minimum. We define

$$L(\alpha_1, \ldots, \alpha_k, \lambda) = \sum_{j=1}^k \alpha_j^2 \frac{2}{j+1} + \lambda(\sum_{j=1}^k \alpha_j j - 1),$$

and get

$$\frac{\partial L}{\partial \alpha_j} = \frac{4}{1+j}\alpha_j + j\lambda \stackrel{!}{=} 0 \quad \Rightarrow \quad \alpha_j = -\frac{j(1+j)}{4}\lambda$$

and

$$\frac{\partial L}{\partial \lambda} = \sum_{j=1}^k \alpha_j j - 1 = \sum_{j=1}^k -\frac{j^2(1+j)}{4} - 1.$$

Solving this set of equations leads to

$$\lambda = \frac{-48}{k(k+1)(k+2)(3k+1)}$$
$$\alpha_j = \frac{-12j(1+j)}{k(k+1)(k+2)(3k+1)}.$$

Using this coefficients we have

$$\begin{aligned} |v|^2_{H^1_\omega(E)} &= \sum_{j=1}^k \alpha_j^2 \frac{2}{1+j} = \sum_{j=1}^k \frac{2}{1+j} \frac{12^2 j^2 (1+j)^2}{k^2(k+1)^2(k+2)^2(3k+1)^2} \\ &\preccurlyeq \sum_{j=1}^k \frac{j^3}{k^2(k+1)^2(k+2)^2(3k+1)^2} \preccurlyeq \sum_{j=1}^k \frac{j^3}{k^8} \leq \frac{k^3}{k^8} \sum_{j=1}^k 1 = \frac{1}{k^4}, \end{aligned}$$

and so

$$|v|_{H^1_\omega(E)} \preccurlyeq \frac{1}{k^2} \Rightarrow |\tilde{e}|_{H^1_\omega(E)} \preccurlyeq \frac{1}{k^2}.$$

Next we also bound the L^2 norm. Due to the properties of the chosen basis functions we can write v also as

$$\begin{aligned} v &= \sum_{j=1}^k -\alpha_j \int_x^1 P_{j-1}^{(1,0)}(s)\,\mathrm{d}s = \sum_{j=1}^k -\alpha_j \int_x^1 \frac{1}{j} \sum_{i=1}^{j-1} (2i+1) P_i^{(0,0)}(s)\,\mathrm{d}s \\ &= \sum_{j=1}^k -\alpha_j \frac{1}{j} \sum_{i=1}^{j-1} \underbrace{\int_x^1 (2i+1) P_i^{(0,0)}(s)\,\mathrm{d}s}_{=P_{i+1}^{(0,0)}(x) - P_{i-1}^{(0,0)}(x)} \\ &= \sum_{j=1}^k -\frac{\alpha_j}{j} \left(P_j^{(0,0)}(x) + P_{j-1}^{(0,0)}(x) - P_0^{(0,0)}(x) - P_{-1}^{(0,0)}(x) \right) \\ &= \sum_{j=1}^k -\frac{\alpha_j}{j} \left(P_j^{(0,0)}(x) + P_{j-1}^{(0,0)}(x) \right). \end{aligned}$$

Using this representation we get

$$\begin{aligned} \|v\|^2_{L^2(E)} &= \sum_{j=1}^k \alpha_j^2 \frac{1}{j^2} \left\| P_j^{(0,0)} + P_{j-1}^{(0,0)} \right\|^2_{L^2(E)} \\ &\preccurlyeq \sum_{j=1}^k \frac{j^4}{k^8} \frac{1}{j^2} \underbrace{\left\| P_j^{(0,0)} \right\|^2_{L^2(E)}}_{\frac{2}{2j+1}} \preccurlyeq \frac{j}{k^8} \sum_{j=1}^k 1 \preccurlyeq \frac{1}{k^6}, \end{aligned}$$

and so

$$\|v\|_{L^2(E)} \preccurlyeq \frac{1}{k^3} \Rightarrow \|\tilde{e}\|_{L^2(E)} \preccurlyeq \frac{1}{k^3}.$$

Due to [BM97][Page 253] we also have

$$\|\tilde{e}\|_{H^1(E)} \preccurlyeq \frac{1}{k}. \tag{6.3.3}$$

Next we use a transformation F from $E = (-1,1)$ to $E_1 = (0,1)$ and define $e_k(x) := \tilde{e}(F^{-1}(x))x^2$, and observe that $e_k(0) = e_k(1) = 0$ and also

$$e_k'(x) = 2\tilde{e}'(x)x^2 + \tilde{e}(x)2x \Rightarrow e_k'(0) = 0 \quad \text{and} \quad e_k'(1) = 1.$$

The norms can also be bounded by

$$\|e_k\|_{L^2(E_1)} \preccurlyeq \frac{1}{k^3}$$
$$\|e_k\|_{H^1(E_1)} \preccurlyeq \frac{1}{k}.$$

As e_k is zero in the vertices it also belongs to $H_0^1(E_1)$. Next we use the technique of interpolation spaces (see appendix) namely

$$H_{00}^{1/2}(E_1) = [L^2(E_1), H_0^1(E_1)] \quad \text{with} \quad \|u\|_{H_{00}^{1/2}(E_1)} \leq \sqrt{\|u\|_{L^2(E)}\,\|u\|_{H^1(E)}}$$

and so we have

$$\|e_k\|_{H_{00}^{1/2}(E_1)} \leq \sqrt{\frac{1}{k^4}} \leq \frac{1}{k^2}.$$

It remains statement i. For this we use an extension of u given by

$$\phi_1(x,y) := \int_0^1 a(s)u(x+sy)\ \mathrm{d}s \quad \text{with} \quad a(s) = 4 - 6s.$$

and so

$$u'(1) = \frac{\partial \phi_1}{\partial x}(1,0).$$

Note that we have

$$\int_0^1 a(s)\ \mathrm{d}s = 1 \quad \text{and} \quad \int_0^1 a(s)s\ \mathrm{d}s = 0,$$

so together with lemma 6.2 we have

$$\|\phi_1\|_{H^1(\hat{T})} \preccurlyeq \|u\|_{H^{1/2}(E_1)}\,.$$

Next we use an average operator to define

$$\overline{u}^y(x) := \frac{1}{1-x}\int_0^{1-x} \phi_1(x,s)\ \mathrm{d}s = \int_0^1 \phi_1(x,(1-x)s)\ \mathrm{d}s$$

with

$$|\overline{u}^y|_{H_\omega^1(E_1)} = \|\overline{u}^y\|_{H^1(\hat{T})} \preccurlyeq \|\phi_1\|_{H^1(\hat{T})} \preccurlyeq \|u\|_{H^{1/2}(E_1)}\,.$$

By the definition of $\tilde{e}$ as the minimum of the H^1_ω seminorm it also follows

$$|(\overline{u}^y)'(1)| \preccurlyeq k^2 |\overline{u}^y|_{H^1_\omega(E_1)} \preccurlyeq k^2 \|u\|_{H^{1/2}(E_1)} .$$

And so due to

$$(\overline{u}^y)'(x) = \int_0^1 \frac{\partial \phi_1}{\partial x}(x,(1-x)s) - s\frac{\partial \phi_1}{\partial y}(x,(1-x)s)\ \mathrm{d}s$$

we have

$$\begin{aligned}(\overline{u}^y)'(1) &= \frac{\partial \phi_1}{\partial x}(1,0)\int_0^1 a(s)\ \mathrm{d}s - \frac{1}{2}\frac{\partial \phi_1}{\partial y}(1,0)\int_0^1 a(s)s\ \mathrm{d}s \\ &= \frac{\partial \phi_1}{\partial x}(1,0) = u'(1)\end{aligned}$$

and so finally

$$|u'(1)| \preccurlyeq k^2 \|u\|_{H^{1/2}(E_1)} \preccurlyeq \|u\|_{H^{1/2}_{00}(E_1)}$$

□

6.4 Proof of the H^2-continuous extension $\mathcal{E}$

Proof of theorem 6.1. Using the extension of theorem 6.3 we set $\phi_\tau = \mathcal{E}^D(u)$ and define

$$u_c := u - \nabla \phi_\tau.$$

Due to the properties of $\mathcal{E}^D$ we see that

$$u_c \cdot \tau = u \cdot \tau - \nabla \phi_\tau \cdot \tau = 0 \quad \text{on} \quad \partial \hat{T}.$$

Now let n_i be the normal vector on the edge E_i of the reference triangle, and $u_i := u_c \cdot n_i$. In the first step we look on the lower edge $E_1 = [0,1]$. We want to use theorem 6.5 for u_1, but it is not zero on $\partial \hat{T} \setminus E_1$. Therefore we define

$$u_s^1 := u_c \cdot ((x,y) - V_2) \in \Pi^{k+1}(\hat{T}),$$

where $V_2 = (0,1)$ is the vertex opposite to E_1, and observe that

$$\begin{aligned} u_s^1\big|_{E_2} &= u_c \cdot (x - V_2)|_{E_2} = u_c \cdot (c\tau_2)|_{E_2} = 0 \\ u_s^1\big|_{E_3} &= u_c \cdot (x - V_2)|_{E_3} = u_c \cdot (c\tau_3)|_{E_3} = 0, \end{aligned}$$

and

$$u_s^1\big|_{E_1} = \begin{pmatrix} u_{s,x}^1 \\ u_{s,y}^1 \end{pmatrix} \cdot \begin{pmatrix} x-0 \\ y-1 \end{pmatrix}^T \overset{y=0 \text{ on } E_1}{=} -u_{c,y} = u_1,$$

so

$$u_s^1\big|_{\partial \hat{T}} \in \Pi^k(\partial \hat{T}).$$

Due to theorem 6.5 we can find a function $e_{k,1}$ and a function $e_{k,0}$ with

$$e'_{k,1}(1) = 1 \quad \text{and} \quad e'_{k,1}(0) = e_{k,1}(0) = e_{k,1}(1) = 0$$
$$e'_{k,0}(0) = 1 \quad \text{and} \quad e'_{k,0}(1) = e_{k,0}(0) = e_{k,0}(1) = 0,$$

where $e_{k,0}$ was defined by mirroring the edge in theorem 6.5. Using this functions we define

$$u_1^b := (u_s^1|_{E_1})'(1)e_{k,1} + (u_s^1|_{E_1})'(0)e_{k,0}$$
$$u_1^g := u_1|_{E_1} - u_1^b.$$

We observe that u_1^g has a zero of order two in the vertices and so we can use theorem 6.4 to define $\phi_n^1 := \mathcal{E}^N(u_1^g)$. As for u_1 we proceed analogue for the other edges E_2 and E_3 to get $\phi_n^2 := \mathcal{E}^N(u_2^g)$ and $\phi_n^3 := \mathcal{E}^N(u_3^g)$ and finally

$$\mathcal{E}(u) = \phi := \phi_\tau + \phi_n^1 + \phi_n^2 + \phi_n^3.$$

We observe that on $\partial\hat{T}$

$$\nabla\phi \cdot \tau = \underbrace{\nabla\phi_\tau \cdot \tau}_{=u\cdot\tau} + \nabla\phi_n^1 \cdot \tau + \nabla\phi_n^2 \cdot \tau + \nabla\phi_n^3 \cdot \tau,$$

and as $\phi_n^i = 0$ on $\partial\hat{T}$ the tangential gradient $\nabla\phi_n^i \cdot \tau = 0$, and we get property (6.0.1)

$$\nabla\phi \cdot \tau = u \cdot \tau.$$

Next we observe that

$$\|\phi\|^2_{H^2(\hat{T})} \preccurlyeq \|\phi_\tau\|^2_{H^2(\hat{T})} + \left\|\phi_n^1\right\|^2_{H^2(\hat{T})} + \left\|\phi_n^2\right\|^2_{H^2(\hat{T})} + \left\|\phi_n^3\right\|^2_{H^2(\hat{T})}.$$

For ϕ_n^1 we have (see theorem 6.4)

$$\begin{aligned}\left\|\phi_n^1\right\|_{H^2(\hat{T})} &\preccurlyeq \|u_1^g\|_{H^{1/2}_{00}(E)} = \|u_1 - u_1^b\|_{H^{1/2}_{00}(E)} \preccurlyeq \|u_1\|_{H^{1/2}_{00}(E)} + \|u_1^b\|_{H^{1/2}_{00}(E)} \\ &\preccurlyeq \|u_1\|_{H^{1/2}_{00}(E)} + |u_s^{1\prime}(1)|\|e_{k,1}\|_{H^{1/2}_{00}(E)} + |u_s^{1\prime}(0)|\|e_{k,0}\|_{H^{1/2}_{00}(E)},\end{aligned}$$

and as

$$|u_s^{1\prime}(1)| \preccurlyeq k^2\|u_s^1\|_{H^{1/2}_{00}(E)}$$
$$\|e_{k,1}\|_{H^{1/2}_{00}(E)} \preccurlyeq \frac{1}{k^2},$$

and the similar bounds for $e_{k,0}$ we get

$$\left\|\phi_n^1\right\|_{H^2(\hat{T})} \preccurlyeq \|u_1\|_{H^{1/2}_{00}(E)} + \|u_s^1\|_{H^{1/2}_{00}(E)}.$$

Note that on E_1 we have $u_1 = u_s^1$, and so as u_s^1 is 0 on $\partial\hat{T} \setminus E_1$ we can bound the $H^{1/2}_{00}(E_1)$ with the $H^1(\hat{T})$. To see this assume an arbitrary function v with $v = 0$ on E_3.

Then we can define the value of v on E_1 as an integral over L_y defined as the line from $(0,x)$ to $(x,0)$

$$v(x,0) = \int_0^x (1,-1)^{\mathrm{T}} \cdot \nabla u(s, x-y)\ \mathrm{d}s \overset{\text{C.S.}}{\preccurlyeq} \sqrt{x}\sqrt{\|\nabla v\|_{L^2(L_y)}},$$

and by that we can bound the weighted L^2 norm

$$\int_0^1 \frac{1}{x} v(x,0)^2 \preccurlyeq \|\nabla v\|^2_{L^2(\hat{T})}$$

so

$$\|v\|_{H^{1/2}_{00}(E_1)} \preccurlyeq \|\nabla v\|_{H^1(\hat{T})}.$$

The same can be done with the edge E_2. Using this for u_s^1 we get

$$\|\phi_n^1\|_{H^2(\hat{T})} \preccurlyeq \|u_s^1\|_{H^1(\hat{T})} \preccurlyeq \|u_s^1\|_{H^1(\hat{T})} \preccurlyeq \|u_c\|_{H^1(\hat{T})}.$$

Analogue we bound the H^2 norms for ϕ_n^2 and ϕ_n^3 and get

$$\|\phi\|^2_{H^2(\hat{T})} \preccurlyeq \underbrace{\|\phi_\tau\|^2_{H^2(\hat{T})}}_{\preccurlyeq \|u\|^2_{H^1(\hat{T})}} + \|u_c\|_{H^1(\hat{T})} = \|u\|^2_{H^1(\hat{T})} + \|u - \nabla\phi_\tau\|^2_{H^1(\hat{T})} \preccurlyeq \|u\|^2_{H^1(\hat{T})},$$

so the H^2 continuity (6.0.3) is shown. For the last inequality (6.0.3) we first observe that on the boundary $\partial\hat{T}$ we get

$$\begin{aligned}\nabla\phi \cdot n &= \nabla\phi_\tau \cdot n + \sum_i \nabla\phi_n^i \cdot n = \nabla\phi_\tau \cdot n + \sum_i u_i^g = \nabla\phi_\tau \cdot n + \underbrace{\sum_i u_i}_{u_c \cdot n} - \sum_i u_i^b \\ &= \nabla\phi_\tau \cdot n + u \cdot n - \nabla\phi_\tau \cdot n - \sum_i u_i^b = u \cdot n - \sum_i u_i^b,\end{aligned}$$

using $u_i^b\big|_{E_j} = 0$ for $i \neq j$, and so

$$\|(u - \nabla\phi)\cdot n\|_{L^2(\partial\hat{T})} \preccurlyeq \sum_i \left\|u_i^b\right\|_{L^2(E_i)}.$$

Taking a closer look on the first term u_1^b we see that

$$\begin{aligned}\left\|u_1^b\right\|_{L^2(E_1)} &\preccurlyeq |u_s^{1\prime}(1)|\, \|e_{k,1}\|_{L^2(E_1)} + |u_s^{1\prime}(0)|\, \|e_{k,0}\|_{L^2(E_1)} \preccurlyeq \frac{1}{k}\|u_s^1\|_{H^{1/2}_{00}(E)} \\ &\preccurlyeq \frac{1}{k}\|u_s^1\|_{H^1(\hat{T})} \preccurlyeq \frac{1}{k}\|u_c\|_{H^1(\hat{T})} = \frac{1}{k}\|u - \nabla\phi_\tau\|_{H^1(\hat{T})} \preccurlyeq \frac{1}{k}\|u\|_{H^1(\hat{T})},\end{aligned}$$

and so with the similar estimation for the other two terms we have

$$\|(u - \nabla\phi)\cdot n\|_{L^2(\partial\hat{T})} \preccurlyeq \sum_i \left\|u_i^b\right\|_{L^2(E_i)} \preccurlyeq \frac{1}{k}\|u\|_{H^1(\hat{T})}.$$

□

7 Appendix

In this chpater we show some basic definitions and results based on [Sch09],[BF91], [BL76], [SA08][32] and [Ste08]. We restrict the results to the two dimensional case in this thesis.

7.1 Transport theorem of Reynold

Theorem 7.1 (Reynold's transport theorem)**.** *Let $V(t) \subset \Omega$ be an arbitrary mass fixed volume in the fluid that is transported with the velocity u, and let $b \in C^1$ be an arbitrary function. With $V = V(0)$ we get*

$$\frac{D}{Dt}\int_V b \; \mathrm{d}x = \int_V \frac{\partial b}{\partial d} \; \mathrm{d}x + \int_{\partial V} b(u \cdot n) \; \mathrm{d}s. \tag{7.1.1}$$

7.2 Jacobi Polynomials

Definition 7.2. *Let $w = (1-x)^\alpha(1-x)^\beta$. We define the n^{th}-order Jacobi polynomials $P_n^{(\alpha,\beta)}$ by the Rodrigues' Formula as*

$$P_n^{(\alpha,\beta)}(x) := \frac{1}{(-2)^n n! w(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(w(x)(1-x^2)^n\right).$$

Theorem 7.3. *The Jacobi polynomials fulfill the orthogonality relation*

$$\int_{-1}^{1} w P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) \; \mathrm{d}x = \delta_{n,m} \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{(n+\alpha)!(n+\beta)!}{n!(n+\alpha+\beta)!},$$

and

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

Parameters can be shifted by

$$(2n+\alpha+\beta)P_n^{(\alpha-1,\beta)} = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x) - (n+\beta)P_{n-1}^{(\alpha,\beta)}(x).$$

7.3 Sobolev Slobotezki space

Definition 7.4. *We define the Sobolev-Slobodečki spaces for $s \in (0,1)$ and $k \in \mathbb{N}$ as*

$$W_p^{k+s}(\Omega) := \{u \in L^p(\Omega) : \|u\|_{W_p^{k+s}(\Omega)} < \infty\}$$

where

$$\|u\|^p_{W_p^{k+s}(\Omega)} := \|u\|^p_{W_p^k(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y.$$

7.4 Finite Elements

Ciarlet's definition of a finite element:

Definition 7.5 (Finite element)**.** *A finite element is a triple (T, V_T, Ψ_T), where*

i. T is a bounded set

ii. $V_T = \{\phi_T^1, \ldots, \phi_T^{N_T}\}$ is a function space on T of finite dimension N_T

iii. $\Psi_T = \{\psi_T^1, \ldots, \psi_T^{N_T}\}$ is a set of linearly independent functionals on V_T

Definition 7.6 (Triangulation)**.** *A regular triangulation $\mathcal{T} = \{T_1, \ldots, T_M\}$ of a domain Ω is the subdivision of a domain Ω in closed triangles T_i such that $\overline{\Omega} = \bigcup T_i$ and $T_i \cap T_j$ is either*

i. empty

ii. a common edge of T_i and T_j

iii. or $T_i = T_j$ in the case of $i = j$.

Definition 7.7 (Linear mapping). *We define the linear mapping of the reference triangle $\hat{T}$ to $T \in \mathcal{T}$ by*

$$F_T(\hat{x}) = a_T + B_T\hat{x},$$

with $a_T \in \mathbb{R}^2$ and $B_T \in \mathbb{R}^{2\times 2}$. For a shape regular triangulation, namely

$$|T| \succcurlyeq h_T^2 \quad \forall T \in \mathcal{T}$$

with $h_T = diam(T)$ we have

$$\|B_T\| \approx h_T$$
$$\left\|B_T^{-1}\right\| \approx h_T^{-1}.$$

We further call a triangulation quasi uniform if all elements are essentially of the same size, so there exists one global h such that

$$h_T \approx h \quad \forall T \in \mathcal{T}$$

Definition 7.8 (Standard nodal interpolator). *Let $v \in C^m(\overline{T})$. We define the standard nodal interpolator I_T as*

$$I_T v := \sum_{i=1}^{N_T} \psi_T^i(v)\phi_T^i \quad I_h^{\Pi^k} v = \sum_{T\in\mathcal{T}} I_T v.$$

Definition 7.9 (Clement operator). *We define $\Pi_h^{\mathcal{C}}$ as in [Clé75]. Assume a given function $u \in L^2(\Omega)$. Let ϕ_i be the basis of $\Pi^k(\mathcal{T})$, and let $S_i := supp(\phi_i)$. We define q_i as the L^2 best approximation of u on S_i, namely*

$$(u - q_i, v_i)_{L^2(S_i)} = 0 \quad \forall v_i \in \Pi^k(S_i).$$

We set

$$\Pi_h^{\mathcal{C}} = \sum_i \psi_i(q_i)\phi_i,$$

where ψ_i is the corresponding functional of ϕ_i. Then we have for $u \in H^m(\Omega)$

$$\left\|u - \Pi_h^{\mathcal{C}} u\right\|_{H^n(\Omega)} \preccurlyeq h^{m-n} \|u\|_{H^m(\Omega)}$$

for $m \geq 0, m \geq n, n \leq 1$ and $m \leq k+1$.

Lemma 7.10 (BDM interpolator)**.** *The BDM interpolator satisfies*

$$\int_T \operatorname{div}(I_h^{\mathrm{BDM}^k} u)\ q\ \mathrm{d}x = \int_T \operatorname{div} u\ q\ \mathrm{d}x \quad \forall q \in \Pi^{k-1}(T)$$

Lemma 7.11 (Bramble Hilbert lemma)**.** *Let U be some Hilbert space and $L : H^k(\Omega) \to U$ be a continuous linear operator such that $Lq = 0$ for polynomials $q \in \Pi^{k-1}(\Omega)$. Then we have*

$$\|Lv\|_U \leq |v|_{H^k(\Omega)}.$$

Lemma 7.12. *Let (T, V_T, Ψ_T) be a finite element such that the element space V_T contains polynomials up to order k. Then we have*

$$\|v - I_T v\|_{H^1(T)} \leq C|v|_{H^m(T)} \quad \forall v \in H^m(T) \tag{7.4.1}$$

for all $m \geq 1$ and $m \leq k+1$.

Theorem 7.13. *Assume that*

i. the solution of a problem i smooth: $u \in H^m(\Omega)$ for $m \geq 2$

ii. all element spaces V_T contain polynomials $\Pi^k(T)$ for $k \geq 1$

iii. the mesh is quasi uniform.

Then we have

$$h^{-1}\left\|u - I_h^{\Pi^k} u\right\|_{L^2(T)} + \left\|u - I_h^{\Pi^k} u\right\|_{H^1(T)} \preccurlyeq h^{\min\{m-1,k\}} \|u\|_{H^m(\Omega)}.$$

Definition 7.14 (Piola transformation)**.** *Let F_T be the linear mapping of T and $\hat{\sigma_\mathcal{P}} \in [L^2(\hat{T})]^2$ a given vector function. Then we define the Piola transformation as*

$$\sigma_\mathcal{P}(x) = \mathcal{P}\hat{\sigma_\mathcal{P}}(x) := \frac{1}{\det \mathcal{J}} \mathcal{J} \hat{\sigma_\mathcal{P}}(\hat{x}), \tag{7.4.2}$$

where $\mathcal{J}$ is the Jacobian of F_T.

Definition 7.15 (Covariant transformation)**.** *Let F_T be the linear mapping of T and $\hat{\sigma_{\mathcal{C}}} \in [L^2(\hat{T})]^2$ a given vector function. Then we define the covariant transformation as*

$$\sigma_{\mathcal{C}}(x) = \mathcal{C}\hat{\sigma_{\mathcal{C}}}(x) := \mathcal{J}^{-T}\hat{\sigma_{\mathcal{C}}}(\hat{x}), \tag{7.4.3}$$

where $\mathcal{J}$ is the Jacobian of F_T.

Lemma 7.16. *For $\hat{\sigma_{\mathcal{P}}} \in H(\mathrm{div})(\hat{T})$ we have*

$$\operatorname{div} \sigma_{\mathcal{P}} = \frac{1}{\det \mathcal{J}} \operatorname{div} \hat{\sigma_{\mathcal{P}}}. \tag{7.4.4}$$

Let $\hat{e}$ be an edge of the reference triangle $\hat{T}$ and $e = F(\hat{e})$ then

$$\int_e \sigma_{\mathcal{P}} \cdot n \; \mathrm{d}s = \int_{\hat{e}} \hat{\sigma_{\mathcal{P}}} \cdot n \; \mathrm{d}s.$$

Remark 19: *Due to the preservation of the normal flow we can construct $H(\mathrm{div})(\Omega)$ conforming approximations.*

Lemma 7.17. *For $\hat{\sigma_{\mathcal{C}}} \in H(\mathrm{curl})(\hat{T})$ we have*

$$\operatorname{curl} \sigma_{\mathcal{C}} = \frac{1}{\det \mathcal{J}} \operatorname{curl} \hat{\sigma_{\mathcal{C}}}. \tag{7.4.5}$$

Let $\hat{e}$ be an edge of the reference triangle $\hat{T}$ and $e = F(\hat{e})$ then

$$\int_e \sigma_{\mathcal{C}} \cdot \tau \; \mathrm{d}s = \int_{\hat{e}} \hat{\sigma_{\mathcal{C}}} \cdot \tau \; \mathrm{d}s.$$

Theorem 7.18 (Poincare inequality)**.** *Assume a bounded domain $\Omega \subset \mathbb{R}^2$ and let $\Gamma_D \subset \partial\Omega$ be of positive measure $|\Gamma_D|$. Let $V_D = \{v \in H^1(\Omega) : \mathrm{tr}_{\Gamma_D} v = 0\}$. Then*

$$\|v\|_{L^2(\Omega)} \preccurlyeq \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in V_D,$$

where tr *is the trace operator* $\mathrm{tr} : H^1(\Omega) \to L^2(\partial\Omega)$.

7.5 Interpolation spaces

Definition 7.19. *Let $V_1 \subset V_0$ be Banach spaces with a dense and continuous embedding and define the K-functional by*

$$\begin{aligned} K &: \mathbb{R}^+ \times V_0 \to \mathbb{R} \\ (t,u) &\mapsto K(t,u) := \inf_{\substack{u_0 \in V_0, u_1 \in V_1 \\ u = u_0 + u_1}} \sqrt{\|u_0\|_{V_0}^2 + t^2 \|u_1\|_{V_1}^2}. \end{aligned}$$

By that we define for $s \in (0,1)$ the interpolation norm as

$$\|u\|_s := \left(\int_0^\infty \frac{t^{-2s}}{t} K(t,u)^2 \, \mathrm{d}t \right)^{1/2},$$

and the interpolation space

$$V^s = [V_0, V_1] := \{u \in V_0 : \|u\|_s < \infty\},$$

with

$$\|u\|_{V_s} \leq \|u\|_{V_0}^{1-s} \|u\|_{V_1}^s .$$

Theorem 7.20. *Let $\Omega = (0,1)$. Then*

$$H^{1/2}(\Omega) = [L^2(\Omega), H^1(\Omega)],$$

and

$$\|u\|_{H^{1/2}(\Omega)} = \left(\int_0^\infty \frac{1}{t^2} K(t,u)^2 \, \mathrm{d}t \right)^{1/2}.$$

Theorem 7.21. *Let $\Omega = (0,1)$. Then*

$$H_{00}^{1/2}(\Omega) = [L^2(\Omega), H_0^1(\Omega)].$$

List of Figures

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