Diese Dissertation haben begutachtet:

....

DISSERTATION

Stable blow up dynamics for the radial wave equation with focusing power type nonlinearities

ausgeführt zum Zwecke der Erlangung des akademischen Grades
einer Doktorin der technischen Wissenschaften unter der Leitung von

Univ.-Prof. Dr.rer.nat. Ansgar Jüngel

E101
Institut für Analysis und Scientific Computing

eingereicht an der Technischen Universität Wien
Fakultät für Mathematik und Geoinformation

von

Mag.rer.nat. Birgit Schörkhuber

Matrikelnummer 0103043

Wien, am

Kurzfassung

In vielen Bereichen von Naturwissenschaft und Technik spielt die Modellierung natürlicher Phänomene durch zeitabhängige partielle Differentialgleichungen (PDEs) eine wesentliche Rolle, beschreiben diese doch die räumliche und zeitliche Änderung kontinuierlicher Größen im Rahmen physikalischer Theorien. Um allgemeine Aussagen über das Verhalten von Lösungen eines gegebenen Modells zu treffen, werden meist ausgeklügelte numerische und analytische Techniken benötigt, da nur in den wenigsten Fällen die Lösungen explizit bekannt sind.

Für zeitabhängige Gleichungen wird oft das sogenannte *Anfangswertproblem* betrachtet, d.h., die Frage nach der Existenz und Eindeutigkeit von Lösungen für gegebene Anfangsbedingungen zur Zeit $t = 0$. In vielen Fällen kann für gewisse Klassen von Anfangsdaten die Existenz von Lösungen zumindest für endliche Zeitintervalle $[0, T)$ garantiert werden. Die Frage, ob solche lokalen Lösungen auch global, also für *alle* Zeiten $t > 0$, fortgesetzt werden können, muss gerade für nichtlineare PDEs oft negativ beantwortet werden. Der Grund hierfür ist, dass Nichtlinearitäten selbstverstärkende Prozesse modellieren, die zum *Blow-up* in endlicher Zeit, also zur „Explosion“ von Lösungen für $t \to T$, $T < \infty$, führen können. In solchen Szenarien divergieren typischerweise entweder die Amplitude der Lösung selbst, oder gewisse Ableitungen, was die Ausbildung von Unstetigkeiten zur Folge hat. Beispiele für Blow-up in endlicher Zeit finden sich in Modellen der nichtlinearen Optik, in der Modellierung von chemischen Reaktion und Verbrennungsprozessen sowie in vielen Modellen der theoretischen Physik, um nur einige Bereiche zu nennen.

Erlaubt eine Gleichung Blow-up in endlicher Zeit, so interessiert man sich insbesondere für das Verhalten von *generischen* Blow-up Lösungen, also solchen, die nicht auf eine sehr spezielle Wahl der Anfangsbedingungen zurückzuführen sind. Diesbezüglich können numerische Experimente oft aufschlussreiche Hinweise liefern, z.B. in Bezug auf die Geschwindigkeit, mit der sich Singularitäten ausbilden, oder das Profil der Lösungen im Limes $t \to T$. Für viele verschiedene Gleichungen (die gewisse Skalierungseigenschaften aufweisen) beobachtet man in diesem Zusammenhang die Konvergenz von generischen Blow-up Lösungen gegen ein (modellabhängiges) sebstähnliches, d.h. skalierungsinvariantes Profil.

Die vorliegende Arbeit beschäftigt sich mit Blow-up von Lösungen nichtlinearer Wellengleichungen in $\mathbb{R}^{3+1}$ der Form

$$\partial_t^2 \psi - \Delta \psi = |\psi|^{p-1} \psi, \quad p > 1.$$

Solche Modelle finden sich in der theoretischen Physik beispielsweise im Rahmen skalarer Feldtheorien für masselose Teilchen. In der Mathematik begründet sich das Interesse vor allem durch die Tatsache, dass es sich um die einfachsten semilinearen Wellengleichungen handelt, die abhängig vom Exponenten p dennoch ein sehr komplexes dynamisches Verhalten zulassen.
Seit den 70er Jahren ist bekannt, dass für alle $p > 1$ Lösungen existieren, die in endlicher Zeit singulär werden. Eine ausführliche Zusammenfassung der wichtigsten Resultate in diesem Zusammenhang findet sich in Kapitel 2. Für $p = 3$, sowie für ungerade Exponenten $p \geq 7$, wurde die Existenz einer abzählbaren Familie von selbstähnlichen radialsymmetrischen Blow-up Lösungen bewiesen, die im Ursprung $r = 0$ divergieren. In numerischen Experimenten für $p \in \{3, 5, 7\}$ wurde die Konvergenz generischer Blow-up Lösungen gegen den *Grundzustand* ψ^T beobachtet, der als einzige selbstähnliche Lösung explizit bekannt ist. Analytisch wurde diese Konvergenz für $1 < p \leq 3$ im Falle radialer Lösungen gezeigt, die außerhalb des Ursprungs divergieren. Im allgemeinen Fall ist es schwierig zu beweisen, dass sich das generische Verhalten von Blow-up Lösungen durch ψ^T charakterisieren lässt, vor allem aufgrund der Nichteindeutigkeit des Grundzustandes in der Klasse selbstähnlicher Lösungen. Die vorliegende Arbeit befasst sich daher mit einem in diesem Zusammenhang notwendigen Kriterium und untersucht die nichtlineare Stabilität der selbstähnlichen Grundzustandslösung ψ^T unter kleinen Störungen. Das Hauptresultat wird in Kapitel 3 formuliert und lautet qualitativ wie folgt.

Theorem. *Für festes $p > 1$ existiert in einer geeignet gewählten Topologie eine offene Umgebung von radialen Anfangsdaten, sodass die zugehörigen Lösungen für $t \to T$ gegen ψ^T mit geeignetem T konvergieren, wobei diese Konvergenz im Rückwärtslichtkegel des Blow-up Punktes $(T, 0)$ betrachtet wird.*

Der Beweis dieses Resultats findet sich in Kapitel 4 und basiert auf der Formulierung der nichtlinearen Gleichung für kleine Störungen als System erster Ordnung im Rückwärtslichtkegel des Blow-up Punktes $(T, 0)$. Das System wird in angepassten Koordinaten (τ, ρ) studiert, wobei der Grenzwert $t \to T$ dem Limes $\tau \to \infty$ entspricht. Das linearisierte Problem wird als Operatorgleichung in einem geeignet definierten Hilbertraum formuliert. Im Falle $1 < p \leq 3$ entspricht die Norm einer lokalen Energie-Norm, für $3 < p < \infty$ wird ein zusätzlicher Ableitungsgrad verlangt. Der betrachtete Differentialoperator ist weder selbstadjungiert noch symmetrisch, weshalb zur Untersuchung der linearisierten Gleichung Methoden aus der Theorie stark stetiger Halbgruppen herangezogen werden. Anwendung finden neben Operatortheorie auch Resultate aus der Theorie gewöhnlicher Differentialgleichungen zur Untersuchung des Spektrums des Generators der Halbgruppe. Die Invarianz der Wellengleichung unter Zeittranslationen manifestiert sich hier in der Existenz einer instabilen Mode. Auf einem geeignet definierten Unterraum jedoch klingen Lösungen der linearisierten Gleichung exponentiell schnell ab. Die Behandlung der Nichtlinearität erfolgt perturbativ unter Zuhilfenahme der Duhamel-Formel. Um das instabile Verhalten der Symmetriemode auszugleichen, werden in einem ersten Schritt die

Anfangsdaten um einen entsprechenden Korrekturfaktor modifiziert. Die Existenz von exponentiell abklingenden Lösungen für solche Anfangsdaten folgt durch Anwendung des Banach'schen Fixpunktsatzes. In einem weiteren Schritt wird gezeigt, dass der Korrekturterm für eine geeignete Wahl der Blow-up Zeit T in Abhängigkeit der Anfangsdaten verschwindet und so das Orginalproblem gelöst wird. In physkalischen Koordinaten (t, r) beweist dies, dass kleine Störungen um den Grundzustand gegen ψ^T mit geeignetem T konvergieren.
Mögliche Verbesserungen und Verallgemeinerungen des obigen Resultats, sowie die Anwendung der vorgestellten Techniken auf andere Gleichungen werden in Kapitel 5 diskutiert. Erste konkrete Ideen diesbezüglich finden sich in Kapitel 6 für den *Yang-Mills Heat Flow*, einer partiellen Differentialgleichung, die in der Differentialgeometrie eine wichtige Rolle spielt. In dem betrachteten geometrischen Setting reduziert sich das Modell auf eine semilineare Wärmeleitungsgleichung, für die eine selbstähnliche Blow-up Lösung explizit bekannt ist.

Contents

Kurzfassung **iii**

1 Introduction **1**
1.1 Overview . . . 1
1.2 The main results . . . 3
1.3 Outline of the thesis . . . 5
1.4 Notation . . . 6

2 Three-dimensional wave equations with focusing power type nonlinearities **7**
2.1 Basic concepts . . . 8
2.2 Symmetries, scaling and criticality . . . 10
2.3 Well-posedness . . . 11
2.3.1 Local well-posedness . . . 11
2.3.2 Global well-posedness . . . 12
2.4 Known results for singularity formation . . . 13
2.4.1 The existence of self-similar blow up solutions . . . 13
2.4.2 Time evolution for large initial data . . . 15
2.4.3 Blow up rate and bounds in the subcritical case . . . 16
2.4.4 The blow up profile . . . 18
2.4.5 The energy critical equation . . . 18
2.4.6 Known results in the supercritical case . . . 21

3 Stable blow up dynamics **23**
3.1 Formulation of the main results . . . 23
3.1.1 (Sub)conformal nonlinearities $1 < p \leq 3$ . . . 24
3.1.2 Superconformal nonlinearities $3 < p < \infty$ . . . 26
3.2 Outline of the proof . . . 27
3.2.1 First order formulation in similarity coordinates . . . 27
3.2.2 Abstract formulation . . . 28
3.2.3 Well-posedness of the linearized equation . . . 29
3.2.4 Estimates for the linear time evolution . . . 30
3.2.5 The nonlinear problem . . . 31

4 Proof of the Main Results **35**
4.1 Proof of Theorem A 36
4.2 Proof of Theorem B 58

5 Discussion of possible extensions and generalizations **77**
5.1 Refinement of the results for the wave equation 77
5.2 Generalization of the method to Banach spaces 78
5.3 Semilinear heat equations 79

6 Further results and outlook **83**
6.1 Stability analysis for the Yang-Mills heat flow 83
6.1.1 Self-adjoint formulation - The eigenvalue problem 85
6.1.2 A non-self-adjoint setting 89
6.2 Summary and Conclusion 92

Appendix A Results from semigroup theory **93**
A.1 Semigroups and generators 93
A.2 Spectral bounds and growth estimates 96

Bibliography **99**

Acknowledgements

Curriculum vitae

Chapter 1

Introduction

1.1 Overview

Many models in theoretical and applied physics are based on the mathematical description of real world phenomena in terms of partial differential equations (PDEs). However, only in very special cases explicit solutions of such equations are known. In general, one has to rely on sophisticated numerical and analytic techniques to investigate the behavior of solutions of the model under consideration.
For time–dependent PDEs one is particularly interested in the *initial value problem*, i.e., the existence of solutions for initial data prescribed at $t = 0$. The initial value problem is said to be *locally well-posed* for certain classes of initial data, if existence and uniqueness of solutions can be established for all $t \in [0, T)$, where $T > 0$. Furthermore, it is required that the solution depends continuously on the data. Naturally, the question arises if all local solutions can be extended globally in time such that $T = \infty$ is possible. If this is the case, the initial value problem is said to be *globally well-posed*. For nonlinear PDEs, global well-posedness is a delicate issue and does not hold in general. The reason for this is that self-interactions may counteract certain smoothing mechanisms such as dissipation or dispersion, leading to *blow up* of solutions in finite time $T < \infty$. Typically, in such a scenario either the amplitude of the solution itself blows up or certain derivatives diverge resulting in a loss of regularity. Nonlinear PDEs admitting finite time blow up solutions can be found in various fields of physics ranging from hydrodynamics, combustion theory or non-linear optics, cf. [35], to models of theoretical physics. One of the most exciting examples is certainly the formation of black holes in the context of general relativity, which can be described by a system of nonlinear PDEs known as the Einstein equations.

If for a given equation break down of solutions is known to occur, one is of course interested in conditions on the data which may allow to predict the formation of singularities, as well as on the details of this process. A particular focus is on the behavior of *generic* blow up solutions, i.e., those which do not correspond to finetuned initial data. Here, numerical experiments can shed some light on possible generic features, such as the blow up rate or the profile of singular solutions for $t \to T$.

In the case that the model under consideration is invariant under a certain scaling of space and time, it is natural to investigate the existence of *self-similar solutions*, which are scale invariant by definition. Such solutions can provide explicit examples for finite time blow up and often play an important role in the dynamics of a system. For many models it is observed in numerical experiments that solutions corresponding to generic large initial data approach a certain universal self-similar profile as $t \to T$. Examples include semilinear wave and heat equations, higher-order semilinear parabolic problems as well as the Schrödinger equation for certain nonlinearities or more complicated coupled systems arising in chemotaxis, see for example [12], [14], [13] and the references therein.

Mathematically, it is a highly nontrivial problem to prove that the generic blow up behavior of a system can be described in terms of a certain self-similar solution ψ^T. In particular, if the equation under consideration admits other (possibly even infinitely many) self-similar profiles. Here, a necessary condition is the stability of the self-similar solution ψ^T under small perturbations.

In this thesis we address this question for semilinear wave equations with nonlinearities of focusing power type and investigate the nonlinear stability of a certain self-similar blow up solution, also known as the *ODE blow up solution*.

Stable blow up dynamics for semilinear wave equations

The basic ideas of the approach that is pursued in this thesis were developed by R. Donninger in [22] as well as by R. Donninger, the author of this thesis and P.C. Aichelburg in [28]. There, we investigated co-rotational wave maps from Minkowski space to the three-sphere, also known as the $SU(2)$-sigma model of particle physics. In this setting, the wave maps equation reduces to

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0 \tag{1.1}$$

for a radial function $\psi(t,r)$. For this model, the self-similar ground state solution is given by

$$\psi^T_{WM}(t,r) = 2\arctan(\tfrac{r}{T-t}).$$

Note that for $t \to T^-$ the gradient of ψ^T_{WM} blows up while the solution itself stays bounded.
Recently, similar techniques as in [28], [22], [26], [27] were also applied in [23] to investigate the equation

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{3}{r^2}\psi(\psi+1)(\psi+2) = 0 \tag{1.2}$$

for radial functions $\psi(t, r)$ where $r = |x|$, $x \in \mathbb{R}^5$. This model arises in the context of gauge theories in theoretical physics and will be referred to as the Yang-Mills (wave) equation. Here, the ground state solution is given by

$$\psi^T_{YMW}(t, r) = f(\tfrac{r}{T-t}) - 1, \quad f(\rho) = \tfrac{1-\rho^2}{1+\frac{3}{5}\rho^2}. \tag{1.3}$$

Note that the stability results in [22], [28] and [23] are conditional in the sense that they rely on a spectral assumption, which is strongly corroborated by numerical studies. However, a completely rigorous proof is still open.

1.2 The main results

In this thesis stable blow up dynamics for semilinear wave equations of the form

$$\partial_t^2 \psi - \Delta \psi = |\psi|^{p-1}\psi, \tag{1.4}$$

for $p \in \mathbb{R}$, $p > 1$, are investigated, where we restrict ourselves to $\mathbb{R}^{3+1}$. Such equations arise for example in certain field theories of theoretical physics describing massless scalar particles. However, our main motivation here to study this model is the fact that it provides one of the simplest nonlinear wave equations exhibiting a rich phenomenology. As such, it can be viewed as a toy model for more complicated problems, for which no suitable analytic techniques are available at the present time.

The sign of the nonlinearity in Eq. (1.4) corresponds to the *focusing case*, since the self-interaction counteracts the dispersive effects of the Laplacian. Moreover, the conserved energy associated to Eq. (1.4) is not positive definite and it is well-known that initial data with negative energy lead to solutions that blow up in finite time, see [68]. Eq. (1.4) is invariant under a certain scaling transformation. Furthermore, it is known that it admits a countable family of self-similar blow up solutions for $p = 3$, see [11], and $p \geq 7$ for p an odd integer, see [10]. The self-similar ground state or *ODE blow up solution* is given by

$$\psi^T(t, x) = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} (T - t)^{-\frac{2}{p-1}}.$$

The scaling behavior of the conserved energy gives rise to the classification into energy subcritical nonlinearities $1 < p < 5$ and energy supercritical nonlinearities $p > 5$. For $p = 5$ the equation is energy critical.

Semilinear wave equations have been the subject of extensive mathematical research in the past decades. The high number of publications that appeared in recent years, investigating also Eq. (1.4) in various space dimensions, indicate that this is still a very active field. Known results in regard to singularity formation for Eq. (1.4) in three space dimensions are discussed in detail in Chapter 2.

At this point, we only mention a few contributions. For $1 < p \leq 3$ the blow up rate in a local energy norm was established in [78], [79], [80]. Recently, upper bounds for the blow up rate of various quantities were obtained in [56] and [47] for $3 < p < 5$. In regard to the blow up profile there is numerical evidence, cf. [9], that ψ^T describes the behavior of generic blow up solutions for $p \in \{3, 5, 7\}$. On a rigorous level this has been investigated in [83] for $1 < p \leq 3$ for the case of radial solutions that blow up outside the origin.

The critical equation $p = 5$ is of particular interest. On the one hand, the ground state is unique in the class of self-similar solutions in this case. On the other hand, a static solution is known explicitly, which gives rise to another different blow up mechanism. Again, the generic behavior is supposed to be described by ψ^T.

For $p > 5$ much less is known. Although some interesting results have been obtained recently, see Chapter 2, the investigation of energy supercritical equations is still in its infancy. Note that Eq. (1.1) and Eq. (1.2) are of energy supercritical type as well. The fact that our approach can be applied to supercritical problems is possibly one of its greatest strengths.

In this thesis we rigorously prove the following results.

Theorem 1.2.1 (Main result, qualitative formulation). *For any $1 < p \leq 3$ there exists an open set of radial initial data in the energy topology such that the corresponding solution of Eq. (1.4) converges to ψ^T in the backward lightcone of the blow up point $(T, 0)$ as $t \to T^-$ where T is a suitable blow up time, depending on the data. For $3 < p < \infty$ the same holds true in a suitable topology stronger than the energy. In this sense, the blow up described by ψ^T is stable.*

The proof of the above theorem is completely independent of any previous well-posedness results for the nonlinear wave equation. Furthermore, it involves the construction of solutions corresponding to large initial data for energy supercritical wave equations, which may be regarded as an interesting result in its own right.

Our approach is functional analytic and based on the formulation of the nonlinear equation for small perturbations as a first order system in coordinates adapted to self-similarity. This system is then studied in the backward lightcone of the blow up point $(T, 0)$. One of the key ingredients is the choice of an appropriate function space such that the problem is studied in its most natural setting. Here we distinguish between $1 < p \leq 3$ and $3 < p < \infty$. The linearized problem is investigated by means of semigroup methods, cf. Appendix A. Moreover, operator theory and ODE methods are used to study the spectrum of the corresponding linear operator, which is neither selfadjoint nor symmetric. For $1 < p \leq 3$ we use partial results from [21] for the linearized equation which are further refined. The nonlinearity is then considered as a perturbation of the linearized problem. The full nonlinear problem is studied in integral form using the Duhamel formula. Here, difficulties arise due to the presence of an unstable symmetry mode of the linear operator, which we account for by a kind of modulation theory with respect to the blow up time T. In regard to this, we use a different strategy as in [22] avoiding the use of the implicit function theorem.

1.3 Outline of the thesis

In Chapter 2 we review known results for the focusing wave equation with nonlinearities of power type in $\mathbb{R}^{3+1}$. First, we recall some basic facts from the linear theory and discuss important properties and concepts for the nonlinear equation. Furthermore, we summarize results on local and global well-posedness of the initial value problem. Since the focus of this work is on the blow up dynamics, we discuss the most important results concerning singularity formation, where we concentrate particularly on recent developments in this field.

In Chapter 3 we define appropriate function spaces and present the quantitative formulation of Theorem 1.2.1. Furthermore, we give an outline of the proof and also discuss differences to [22], [28].

Chapter 4 contains the detailed proofs of the stability results that were stated in Chapter 3. The case $1 < p \leq 3$ is treated in Section 4.1. The corresponding proof for $3 < p < \infty$ can be found in Section 4.2.

In Chapter 5, possible refinements of our results for the wave equation are discussed. Furthermore, a slight improvement of the method is proposed, avoiding the use of the Hilbert space structure of the underlying function space. In view of possible future applications of our techniques we discuss known results concerning blow up via self-similar solutions for semilinear heat equations. We consider the heat equation with power type nonlinearities, the heat flow for harmonic maps as well as the Yang-Mills heat flow in the setting of a trivial principal $SO(n)-$bundle over $\mathbb{R}^n$ and $SO(n)-$equivariant connections. These examples provide the natural analogues of the above discussed wave equations.

In Chapter 6, the application of our approach to the Yang-Mills heat flow in five space dimensions is considered and some preliminary results are presented. We formulate the nonlinear equation for small perturbations in self-similar variables and discuss some aspects of the problem in different functional analytic settings. Since a rigorous stability theory requires some information on the spectrum of the operator corresponding to the linearized equation, we investigate this in a self-adjoint formulation. Although this setting is not suitable to study the full nonlinear problem, the obtained results can possibly be helpful also in the non-self-adjoint formulation, which is proposed at the end of Chapter 6. Some final remarks conclude the thesis.

For the reader's convenience, basic results from the theory of strongly continuous one-parameter semigroups are summarized in Appendix A.

1.4 Notation

In the sequel, $a \lesssim b$ means $a \leq cb$ for an absolute constant $c > 0$ and we also write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. Furthermore, for two positive functions f, g defined on a neighborhood of $T \in \mathbb{R} \cup \{\pm\infty\}$ we write $f \ll g$ if and only if $\lim_{t\to T} f(t)/g(t) = 0$.

Partial derivatives are denoted by $\psi_t(t,x) = \partial_t\psi(t,x)$. In the equations we sometimes omit the arguments of a function for the sake of readability. Furthermore, we use $\psi(t)$ and $\psi(t,\cdot)$ interchangeably.

The conventions concerning the notation of operators on function spaces are slightly different in Section 4.1 and Section 4.2. These sections contain the proofs of the main results for $1 < p \leq 3$ and $p > 3$, respectively. In both sections operators are denoted by capital letters. In Section 4.2, operators acting between product spaces are printed in bold. This is not the case in Section 4.1. In the outline of the proof given in Chapter 3, we adopt the notation from Section 4.1.

For a closed linear operator L we write $\sigma(L)$ and $\sigma_p(L)$ for the spectrum and point spectrum, respectively. Furthermore, we set $R_L(\lambda) := (\lambda - L)^{-1}$ for $\lambda \in \mathbb{C} \setminus \sigma(L)$. The Fréchet derivative of a map f is denoted by Df and we also use the notation $D_y f(x,y)$ for the partial Fréchet derivative with respect to the second variable.

In this thesis, $L^p(\Omega)$, $1 \leq p \leq \infty$, $\Omega \subseteq \mathbb{R}^n$ denote Lebesgue spaces defined as usual. For $s \in \mathbb{R}$, the L^2-based Sobolev spaces are denoted by $H^s(\Omega)$ in the inhomogenous case and by $\dot{H}^s(\Omega)$ in the homogeneous case. For the precise definitions, we refer to standard textbooks, such as [94]. By $L^p([0,T), L^q(\Omega))$ we denote the space of L^q-valued functions defined on $[0,T)$, for which the space-time norm $\|\cdot\|_{L^p_t L^q_x}$ is finite, i.e.,

$$\|\psi\|_{L^p_t L^q_x} = \left(\int_0^T \|\psi(t,\cdot)\|^p_{L^q(\Omega)}\right)^{1/p} < \infty.$$

As usual $C^k([0,T), X)$ is the space of k-times continuously differentiable functions, which are defined on $[0,T)$ and attain values in X, where X denotes some Banach space.

Declaration of authorship

Some parts of Chapter 3 and almost the entire Chapter 4 were published in [26], [27]. Needless to say that this work is based on many discussions and continuous exchange of ideas with the co-author Roland Donninger. However, those parts of the publications that appear in the present document were worked out by myself.

Chapter 2

Three-dimensional wave equations with focusing power type nonlinearities

In this chapter we discuss important properties and concepts for the nonlinear wave equation

$$\partial_t^2 \psi - \Delta \psi = |\psi|^{p-1} \psi, \tag{NLW}$$

for $p > 1$, where $\psi : I \times \mathbb{R}^3 \to \mathbb{R}$ and I is an interval. We recall some well-known results regarding local and global well-posedness of the Cauchy-problem for the (NLW) with initial data

$$\psi(0, \cdot) = \psi_0, \quad \partial_t \psi(0, \cdot) = \psi_1$$

in suitable function spaces. In order to put our main results into a broader context, we give an overview on what is known concerning the formation of singularities, where we focus in particular on developments in recent years.

The investigation of nonlinear wave equations is a highly active field of research and this summary is not intended to be exhaustive. Moreover, a comprehensive discussion covering arbitrary space dimensions and/or the defocusing case, which is the above equation with the sign of the nonlinearity reversed, is beyond the scope of this work. However, most of the results presented below were established in a more general setting and we refer to the cited literature as well as to monographs such as [91], [90], [92], [1] or [94] for detailed expositions.

In the following we restrict the discussion to the wave equation in three space dimensions. In this chapter we therefore omit the domain $\mathbb{R}^3$ in the notation and abbreviate for example $L^2 := L^2(\mathbb{R}^3)$.

2.1 Basic concepts

Many concepts and techniques for nonlinear equations rely on a well developed theory for the linear problem. Therefore, we briefly review some basic facts for the linear wave equation

$$\partial_t^2\psi - \Delta\psi = h. \tag{2.1}$$

One of the most important features of the wave equation is the fact that for $h \equiv 0$ the energy

$$E_0(\psi(t,\cdot),\partial_t\psi(t,\cdot)) = \tfrac{1}{2}\|(\psi(t,\cdot),\partial_t\psi(t,\cdot))\|^2_{\dot{H}^1\times L^2} \tag{2.2}$$

is a conserved quantity, cf. the proof of Lemma 2.2.1. The function space $\dot{H}^1 \times L^2$ will be referred to as the *energy space* in the sequel. For initial data

$$\psi(0,\cdot) = \psi_0, \quad \partial_t\psi(0,\cdot) = \psi_1$$

there are various explicit representations of the solution of Eq. (2.1). Using for example the (forward) fundamental solution together with Duhamel's principle yields

$$\psi(t) = \partial_t R(t) * \psi_0 + R(t) * \psi_1 + \int_0^t R(t-s) * h(s)ds, \tag{2.3}$$

where

$$R(t,x) = \frac{1}{4\pi t}\delta(t-|x|), \tag{2.4}$$

in three space dimensions, cf. for example [90]. For initial data $(\psi_0,\psi_1) \in C^2 \times C^1$ and $h \in C^1$, Eq. (2.3) defines a classical solution, cf. Section 4.1 in [90]. The fact that the fundamental solution is supported only in a bounded spacetime region naturally leads to the concept of lightcones and finite speed of propagation. We define the *backward light cone* with vertex (T,x_0) by

$$\mathcal{C}_T(x_0) := \{(t,x) : t \in (0,T), |x-x_0| \leq T-t\}.$$

If the vertex is located at $x_0 = 0$ we abbreviate the notation and set $\mathcal{C}_T := \mathcal{C}_T(0)$. If ψ is a smooth solution of the *free* wave equation, i.e., $h \equiv 0$ in Eq. (2.1), with initial data supported in the ball $|x| \leq a$ then finite speed of propagation implies that $\psi \equiv 0$ for $|x| > |t| + a$. Note that this holds for any space dimension n. Moreover, if n is odd and $n \geq 3$, then the fundamental solution is supported only at the boundary of a forward lightcone, which implies that waves propagate exactly at the speed of light (where we set $c = 1$). This is also known as the (strong) Huygens' principle. In the homogeneous case for $n = 3$ it implies that the solution at some point (t_0,x_0) only depends on the values of the data at the intersection of the backward lightcone

$\overline{\mathcal{C}_{t_0}(x_0)}$ with the initial hypersurface $\{t=0\}\times\mathbb{R}^3$. As a consequence, if ψ is a smooth solution with data supported in the ball $|x|\le a$ then $\psi\equiv 0$ not only in $|x|>|t|+a$ but also in the double cone $|x|<|t|-a$ for $|t|>a$.

The easiest way to obtain a solution of Eq. (2.1) is via Fourier transform. Using the notation $\hat{\psi}(t)=\mathcal{F}\psi(t)$ yields

$$\hat{\psi}(t)(\xi)=\cos(t|\xi|)\hat{\psi}_0(\xi)+\tfrac{\sin(t|\xi|)}{|\xi|}\hat{\psi}_1(\xi)+\int_0^t \tfrac{\sin((t-s)|\xi|)}{|\xi|}\hat{h}(s,\xi)ds. \tag{2.5}$$

Comparing this equation with Eq. (2.3) one can read off the representation

$$R(t)=\mathcal{F}^{-1}\left(\tfrac{\sin(t|\xi|)}{|\xi|}\right). \tag{2.6}$$

Eq. (2.5) is often written as

$$\begin{aligned}\psi(t)=&\cos(t\sqrt{-\Delta})\psi_0+\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\psi_1\\&+\int_0^t\frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}h(s,\cdot)ds\end{aligned} \tag{2.7}$$

The above representation formulas make sense for very large classes of initial data and thus provide a certain concept of weak solutions. In particular, for $(\psi_0,\psi_1)\in\dot{H}^1\times L^2$ and $h\in L^1(\mathbb{R},L^2)$, Eq. (2.7) yields a solution $\psi\in C(\mathbb{R},\dot{H}^1)$, $\partial_t\psi\in C(\mathbb{R},L^2)$, which is also referred to as a *strong energy class solution*, see again [90], Section 4.2. While the energy is conserved for solutions of the free wave equation, one has the following standard estimate in the inhomogeneous case,

$$\|(\psi(t,\cdot),\partial_t\psi(t,\cdot))\|_{\dot{H}^1\times L^2}\lesssim\|(\psi_0,\psi_1)\|_{\dot{H}^1\times L^2}+\int_0^t\|h(\xi,\cdot)\|_{L^2}d\xi, \tag{2.8}$$

for all $t\ge 0$ given that h is sufficiently integrable. More general, for $(\psi_0,\psi_1)\in H^s\times H^{s-1}$, $s\ge 1$ and $h\in L^1([0,T],H^{s-1})$ solutions of Eq. (2.1) satisfy

$$\|\psi(t,\cdot)\|_{H^s}+\|\partial_t\psi(t,\cdot)\|_{H^{s-1}}\lesssim(1+t)\left(\|\psi_0\|_{H^s}+\|\psi_1\|_{H^{s-1}}+\int_0^t\|h(\xi,\cdot)\|_{H^{s-1}}d\xi\right),$$

for each $0\le t\le T$. A detailed proof is given for example in [67] and relies on Eq. (2.5), Eq. (2.6), the properties of the Fourier transform and Eq. (2.8). Note that the L^2-norm of solutions grows linearly in time. The dispersive character of the wave equation is obvious in the following estimate, which holds for solutions of the free wave equation in n space dimensions and sufficiently regular initial data, cf. [90], Section 4.3,

$$\|\psi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}\lesssim t^{-\frac{n-1}{2}}.$$

For more (weighted) energy-type inequalities and estimates exploiting the geometric structure of the wave equation, we refer to the standard literature on the subject, e.g. [2] or [90]. Particularly important for the analysis of the (NLW) are *Strichartz estimates* for the wave equation providing bounds on space-time integral norms. The theory goes back to [51], [44], [69], [53], see also [94] for a detailed discussion and further references.

2.2 Symmetries, scaling and criticality

The existence of a conserved energy for the free wave equation is a consequence of the invariance of the equation under time translation. This symmetry is still present in the nonlinear case implicating the following conservation law.

Lemma 2.2.1. *Let $\psi \in C_0^\infty$ be a smooth solution of the* (NLW)*. Then the total energy*

$$E(\psi(t,\cdot),\partial_t\psi(t,\cdot)) := \tfrac{1}{2}\|(\psi(t,\cdot),\partial_t\psi(t,\cdot))\|^2_{\dot{H}^1\times L^2} - \tfrac{1}{p+1}\|\psi(t,\cdot)\|^{p+1}_{L^{p+1}} \tag{2.9}$$

is conserved.

Proof. Multiplication of the (NLW) by ψ_t yields

$$\partial_t\left(\tfrac{1}{2}|\nabla\psi|^2 + \tfrac{1}{2}|\psi_t|^2 - \tfrac{1}{p+1}|\psi|^{p+1}\right) - \nabla\cdot(\nabla\psi\psi_t) = 0$$

Integration over $\mathbb{R}^3$ implies that $\partial_t E(\psi(t,\cdot),\partial_t\psi(t,\cdot)) = 0$. □

The nonlinear wave equation is invariant under the full Poincaré group including translations in space and time, spatial rotations and Lorentz transformations. Moreover, the equation is invariant under dilation

$$\psi(t,x) \mapsto \psi_\lambda(t,x) := \lambda^{-\frac{2}{p-1}}\psi(t/\lambda, x/\lambda)$$

for a scaling parameter $\lambda > 0$. The corresponding scale-invariant spaces are the homogeneous Sobolev spaces

$$\dot{H}^{s_c}\times\dot{H}^{s_c-1}, \quad s_c := \tfrac{3}{2} - \tfrac{2}{p-1}, \tag{2.10}$$

where s_c is usually referred to as the *critical regularity.* Note that for $p = 5$ the energy space $\dot{H}^1\times L^2$ is critical. The total energy scales as

$$E(\psi_\lambda(t,\cdot),\partial_t\psi_\lambda(t,\cdot)) = \lambda^{\frac{p-5}{p-1}}E(\psi(\tfrac{t}{\lambda},\cdot),\partial_1\psi(\tfrac{t}{\lambda},\cdot)) \tag{2.11}$$

giving rise to the notion of *energy criticality.* The nonlinear wave equation is called *energy subcritical* for $1 < p < 5$, *critical* for $p = 5$ and *supercritical* for $p > 5$.

For $p = 3$ the (NLW) is conformally invariant, i.e., if ψ is a classical solution defined in a forward lightcone $|x| < t$ for $t > 0$ then so is

$$\tilde{\psi}(t,x) := \frac{1}{(t^2 - |x|^2)}\psi\left(\frac{t}{t^2-|x|^2}, \frac{x}{t^2-|x|^2}\right),$$

see for example [94] or Section 7 in [56] for a discussion. In the sequel we distinguish between *(sub)conformal* nonlinearities for $1 < p \leq 3$ and *superconfomal* nonlinearities for $p > 3$.

An important property that carries over from the linear case is finite speed of propagation. For the critical equation this is elaborated for example in [54], Remark 2.12. The argument is based on the integral formulation of the nonlinear wave equation

$$\begin{aligned} \psi(t,\cdot) = &\cos(t\sqrt{-\Delta})\psi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\psi_1 \\ &+ \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}|\psi(s,\cdot)|^{p-1}\psi(s,\cdot)ds, \end{aligned} \tag{2.12}$$

the method of successive approximation and the above described properties of the free equation. For smooth solutions we refer to Prop. 3.3 in [94], or to [37]. Consequently, for two solutions of the (NLW) defined on $[0, T) \times \mathbb{R}^3$, for which the corresponding initial data agree on $|x| \leq a$ for some $a > 0$, finite speed of propagation implies that the solutions agree in the (truncated) backward lightcone $\mathcal{C}_a \cap ([0, T) \times \mathbb{R}^3)$.

2.3 Well-posedness

2.3.1 Local well-posedness

In regard to local well-posedness we restrict ourselves to existence and uniqueness of solutions in the formulation of the results. However, continuous dependence on the initial data can be established as well and we refer to the cited literature for the details.

Lemma 2.3.1 (see [91])**.** *Let $p > 1$ and set $f(u) := |u|^{p-1}u$. Let $(\psi_0, \psi_1) \in C_0^{k+1} \times C_0^k$ be compactly supported initial data. If $f \in C^k(\mathbb{R})$, then there exists a (maximal) $T > 0$ such that the* (NLW) *has a unique solution $\psi \in C^k([0, T) \times \mathbb{R}^3)$. If $T < \infty$ then*

$$\lim_{t \to T} \|\psi(t,\cdot)\|_{L^\infty} = \infty.$$

Note that $f \in C^1(\mathbb{R})$ for all $p > 1$. In the case that the solution exists only for finite time, it stays regular up to the maximal time of existence, whereas the amplitude blows up. Non-classical solutions of the wave equation have to be interpreted in a suitable weak sense, i.e., via the Duhamel formula (2.12) or as distributional solutions as defined for example in [91], p. 10.

For energy subcritical nonlinearities, local existence of energy class solutions satisfying the (NLW) in the Duhamel sense can be established using the Banach fixed point theorem. The required bounds on the nonlinearity follow from energy estimates and Sobolev embedding in the (sub)conformal case $p \leq 3$, whereas Strichartz estimates are required for $3 < p < 5$.

Theorem 2.3.2. *Let* $(\psi_0, \psi_1) \in \dot{H}^1 \times L^2$ *and* $3 < p < 5$. *Then there exists a* $T > 0$ *sufficiently small and a unique solution* ψ *of the* (NLW) *where*

$$\psi \in C([0,T), \dot{H}^1) \cap C^1([0,T), L^2) \cap L^{\frac{2p}{p-3}}([0,T); L^{2p}).$$

A detailed proof can be found for example in [67]. In view of the above result, it is natural to ask about the minimal regularity required for local well-posedness. This was investigated by Lindblad and Sogge in [69]. It was shown that for $p \geq 3$ the (NLW) is ill-posed below the critical regularity, whereas for $2 < p < 3$ even slightly more regularity is required. We only state the corresponding result for $p \geq 3$ in three space dimensions, see also [91] for a more general formulation. The following theorem also provides the local existence theory for $p = 5$ in the energy space. For supercritical nonlinearities, a higher Sobolev space is required.

Theorem 2.3.3 (Lindblad-Sogge, [69])**.** *For* $p \geq 3$ *set* $s_c := \frac{3}{2} - \frac{2}{p-1}$. *Let*

$$(\psi_0, \psi_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c - 1}.$$

Then there exists a (maximal) $T > 0$ *and a unique solution* ψ *of the* (NLW) *with*

$$\psi \in C([0,T), \dot{H}^{s_c}) \cap C^1([0,T), \dot{H}^{s_c-1}) \cap L^{2p-2}([0,T) \times \mathbb{R}^3).$$

Moreover, there is an $\varepsilon > 0$ *depending on p such that if*

$$\|\psi_0\|_{\dot{H}^{s_c}} + \|\psi_1\|_{\dot{H}^{s_c-1}} < \varepsilon,$$

the corresponding solution is global in time. If $T < \infty$ *then* $\psi \notin L^{2p-2}([0,T] \times \mathbb{R}^3)$, *i.e.,*

$$\|\psi\|_{L^{2p-2}([0,T]\times\mathbb{R}^3)} = \infty.$$

2.3.2 Global well-posedness

The theorem by Lindblad and Sogge implies the existence of global solutions for sufficiently small initial data for nonlinearities with exponents $p \geq 3$. However, for smaller values of p the issue is more delicate. Note that there is a classical result by John [49] for the equation

$$\partial_t^2 \psi - \Delta \psi = |\psi|^p$$

and $1 < p < 1 + \sqrt{2}$ implying that blow up always occurs even for small, compactly supported smooth initial data, see also [37] for a proof and [91] for a broader discussion. In general, global well-posedness for arbitrary data does not hold for the focusing wave equation as will be discussed in the following section.

For the defocusing wave equation and $1 < p < 5$, global existence and regularity was shown by Jörgens [50] in 1961. For initial data in $\dot{H}^1 \times L^2$, global well-posedness in the energy space can also be established using the local theory, energy conservation and the fact that the full energy is positive definite in the defocusing case, providing a uniform bound for the energy norm, see for example [67]. For $p = 5$ global well-posedness and scattering to zero was established in [93], [45], [88], [89] and [5], where the (well-defined) notion of *scattering* refers to the fact that the solution behaves linearly for large times as the effects of the nonlinearity become negligible. For supercritical nonlinearities in the defocusing case, the question of large data global existence is still open.

2.4 Known results for singularity formation

For the focusing nonlinear wave equation, finite time blow up of solutions is known to occur for all powers $p > 1$. One of the first results in this direction was obtained by Levine [68] in 1974, based on the fact that the energy associated to the (NLW) given by Eq. (2.9) is not positive definite. By considering two time derivatives of $\|\psi(t,\cdot)\|^2_{L^2}$, it was proved that if

$$E(\psi_0, \psi_1) < 0,$$

the corresponding solution cannot exit globally in time, see also [37] for a discussion. A more recent result in this spirit was obtained for (sub)critical nonlinearities and initial data in $(\psi_0, \psi_1) \in (\dot{H}^1 \times L^2) \cap (\dot{H}^{s_c} \times \dot{H}^{s_c - 1})$, see Theorem 3.1 in [56] for a precise formulation.

2.4.1 The existence of self-similar blow up solutions

In the following, a solution of the wave equation is called *self-similar* if it is invariant under the natural scaling of the equation.

The most obvious evidence for the failure of global well-posedness is the existence of an explicit solution which blows up in finite time $T < \infty$. For the (NLW) such a solution can be obtained by neglecting the Laplacian and solving the ordinary differential equation

$$\partial_t^2 \psi = |\psi|^{p-1} \psi. \tag{2.13}$$

This yields the *ODE blow up solution*

$$\psi^T(t,x) = \kappa_p^{\frac{1}{p-1}}(T-t)^{-\frac{2}{p-1}}, \quad \kappa_p = \frac{2(p+1)}{(p-1)^2}. \tag{2.14}$$

The solution is constant in space and thus it has infinite energy. Nevertheless, using smooth cut off functions and invoking finite speed of propagation one can construct smooth compactly supported initial data such that the corresponding solution equals ψ^T inside a backward lightcone and blows up in the sense that

$$\lim_{t\to T} \|\psi(t,\cdot)\|_\infty = \infty.$$

In particular, the ODE blow up solution is self-similar, since it is of the form

$$\psi(t,x) = (T-t)^{-\frac{2}{p-1}} f\left(\tfrac{x}{T-t}\right)$$

with constant profile

$$f_0 = \kappa_p^{\frac{1}{p-1}} \in \mathbb{R}.$$

For the focusing nonlinear wave equation in one space dimension it was proved by Merle and Zaag in [81] that ψ^T is in fact the unique nontrivial self-similar solution up to symmetries of the equation. This, however, is not the case in higher dimensions. In [9] Bizoń, Chmaj and Tabor investigated the (NLW) for radial initial data, $p \geq 3$ and p an odd integer. Corresponding solutions satisfy the radial equation

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \tfrac{2}{r}\psi_r - |\psi|^{p-1}\psi = 0, \\ &\psi(0,\cdot) = \psi_0, \psi_t(0,\cdot) = \psi_1 \end{aligned} \tag{2.15}$$

where $\psi(t,r) := \psi(t,|x|)$ Inserting the ansatz

$$\psi(t,r) = (T-t)^{-\frac{2}{p-1}} f(\tfrac{r}{T-t}) \tag{2.16}$$

into (2.15) and introducing a new radial coordinate

$$\rho = \frac{r}{T-t}$$

yields a nonlinear ODE for self-similar profiles. For $p = 3$ and $p = 7$, a countable family of profiles $\{f_n\}$, $n \in \mathbb{N}_0$, was obtained numerically in [9] using shooting methods. Here, f_0 corresponds to the constant profile of ψ^T, which is therefore also referred to as the *self-similar ground state solution.* By finite speed of propagation no regularity or fall-off conditions have to be imposed on the solutions of the ODE for $\rho > 1$. In fact, for $p = 3$ all solutions obtained in [9] have singularities outside the backward lightcone $\mathcal{C}_T$ and can be labeled according to their number of zeros, whereas for $p = 7$ all profiles decrease monotonically having no zeros at all. Based on the numerical evidence, the existence of a countable family of self-similar solutions was rigorously proved by Bizoń, Maison and Wasserman [10] for $p \geq 7$ with p an odd integer and by Bizoń, Breitenlohner, Maison and Wasserman for $p = 3$, see [11].

In the critical case, the ground state is the unique self-similar solution, see for example [9] for a discussion. Another remarkable fact is the existence of a static solution of the (NLW) for $p = 5$ given by

$$W(x) = \left(1 + \tfrac{|x|^2}{3}\right)^{-\frac{1}{2}}, \tag{2.17}$$

which is referred to as *the ground state soliton* and will be discussed in more detail below.

2.4.2 Time evolution for large initial data

The time evolution for solutions of the radial wave equation (2.15) was studied numerically by Bizoń, Chmaj and Tabor for $p \in \{3, 5, 7\}$. The authors investigated the asymptotic blow up profile as well as the threshold for singularity formation by considering one-parameter families of initial data, as for example Gaussian profiles with variable amplitudes a. Global existence for small data suggests the existence of a critical value a^* such that data with $a < a^*$ lead to global in time solutions, whereas for $a > a^*$ one has finite time blow up. This was also observed in the numerical simulations. Moreover, for data near the threshold a^*, the solution approaches an intermediate attractor called *the critical solution* before it either blows up or disperses. In the following, we qualitatively summarize some of the outcomes of [9], cf. also the references therein, which suggest that the *generic* blow up behavior of solutions can be described in terms of the self-similar ground state solution.

Numerical observation (Bizoń-Chmaj-Tabor [9]). For the radial wave equation (2.15) with $p \in \{3, 5, 7\}$ the following observations were made.

- For solutions that blow up at some point (T, r_0) for $T > 0$ and $r_0 \geq 0$ it is observed that
$$\lim_{t \to T} (T-t)^{\frac{2}{p-1}} \psi(t, r_0) = f_0 = \left[\tfrac{2(p+1)}{(p-1)^2}\right]^{\frac{1}{p-1}}.$$
- For very large amplitudes the solutions blow up outside the origin at some point $r_0 > 0$. Furthermore, $r_0 \to 0$ as $a \to a_0$ for some amplitude $a_0 > a^*$.
- For $a^* < a < a_0$ the spatial pattern of the developing singularity can formally be described in terms of eigenmodes corresponding to the linearization around ψ^T in the backward lightcone of the blow up point $(T, 0)$.
- For $p = 7$ the critical solution is given by
$$(T-t)^{-\frac{1}{3}} f_1(\rho).$$
For $p = 5$ the critical solution can be identified with the static solution (2.17).

From an analytic point of view a lot of progress has been made in recent years concerning the description of the details in the process of singularity formation. This includes the results presented in Chapter 3 and Chapter 4, which prove nonlinear stability of ψ^T under small perturbations.

In the following, we discuss some of the most important contributions in this field, where we focus in particular on recent works. However, this summary is not intended to be exhaustive and we refer to the references in the cited literature for further results.

2.4.3 Blow up rate and bounds in the subcritical case

In the analysis of singular solutions for the (NLW) it is common to define the *blow up surface* of a solution as the graph of a function $x \mapsto T(x)$ such that the solution cannot be extended beyond its *domain of maximal (forward) extension*,

$$\{(t,x) : x \in \mathbb{R}^3, 0 \leq t < T(x)\}.$$

In this context, *the blow up time* is defined as $T := \min_x T(x)$. A point $x_0 \in \mathbb{R}^3$ is called *non-characteristic* if the domain of maximal extension contains a cone with vertex $(T(x_0), x_0)$ and slope smaller than one. For the precise definitions we refer to the literature cited below.

For the focusing (NLW) in one space dimension, a rather complete picture of singularity formation was obtained by Merle and Zaag in [81], [82], [85], [84], see also Côte and Zaag for a recent result [19]. In higher dimensions, the blow up rate was established by Merle and Zaag in [78], [79], [80] for (sub)conformal nonlinearities, i.e., $p \leq 3$ in three space dimensions. Their findings will be discussed in more detail below.

Bounds on the blow up rate in the full subcritical regime $1 < p < 5$ were proved by Killip, Stovall and Visan [56] in the more general framework of the nonlinear Klein-Gordon equation. Among other things, it was shown that for $3 \leq p < 5$, blow up of solutions of the (NLW) is always accompanied with divergence of the critical Sobolev norm $\dot{H}^{s_c} \times \dot{H}^{s_c - 1}$, $s_c = \frac{3}{2} - \frac{2}{p-1}$, (for lower values of p this is still true under some additional assumptions).

An explicit calculation shows that in any backward lightcone with vertex (T, x_0) the $H^1 \times L^2$- norm of the ground state solution $\psi^T(t, \cdot)$ blows up in the energy subcritical case, since

$$\begin{aligned}
&\|(\psi^T(t,\cdot), \partial_t \psi^T(t,\cdot))\|^2_{H^1(B_{T-t}(x_0)) \times L^2(B_{T-t}(x_0))} \\
&\qquad = \|\psi^T(t,\cdot)\|^2_{L^2(B_{T-t}(x_0))} + \|\partial_t \psi^T(t,\cdot)\|^2_{L^2(B_{T-t}(x_0))} \\
&\qquad = c_1 (T-t)^{-\frac{7-3p}{p-1}} + c_2 (T-t)^{-\frac{5-p}{p-1}}
\end{aligned} \tag{2.18}$$

for some constants $c_1, c_2 > 0$.

For arbitrary blow up solutions of the (NLW) lower bounds in lightcones follow from scaling and the local well-posedness theory in $H^1 \times L^2$. This is elaborated for example in [56], Cor. 2.7, as well as in [78], [79]. We qualitatively state the corresponding results for the three dimensional wave equation.

Lemma 2.4.1. *For $1 < p < 5$, let ψ be a solution of the* (NLW) *that blows up at (T, x_0) for $T > 0$ and $x_0 \in \mathbb{R}^3$. Then*

$$\begin{aligned} 1 \lesssim (T-t)^{\frac{7-3p}{p-1}} &\int_{|x-x_0|\leq T-t} |\psi(t,x)|^2 dx \\ &+ (T-t)^{\frac{5-p}{p-1}} \int_{|x-x_0|\leq T-t} \left(|\nabla\psi(t,x)|^2 + |\psi_t(t,x)|^2\right) dx \end{aligned} \tag{2.19}$$

for all $0 < t < T$ sufficiently close to T.

In the (sub)conformal case, the self-similar rate is also an upper bound, as was demonstrated by Merle and Zaag in [78], [79], see also Antonini and Merle [3]. Inspired by previous works on the nonlinear heat equation solutions of the wave equation with blow up time T were studied in adapted coordinates

$$y = \frac{x - x_0}{T - t}, \quad \tau = -\log(T - t) \tag{2.20}$$

for each $x_0 \in \mathbb{R}^3$. Note that self-similar solutions correspond to stationary solutions of the wave equation formulated in self-similar variables (τ, y).

Theorem 2.4.2 (Merle-Zaag, [78], [79])**.** *Let $1 < p \leq 3$ and let ψ be a solution of the* (NLW) *that blows up at time $T > 0$. For $x_0 \in \mathbb{R}^3$ define*

$$\omega_{x_0}(y,\tau) := e^{-\frac{2}{p-1}\tau}\psi(ye^{-\tau} + x_0, T - e^{-\tau}).$$

Then

$$\sup_{x_0\in\mathbb{R}^3} \|\omega_{x_0}(\tau)\|_{H^1(B_1(0))} + \|\partial_\tau \omega_{x_0}(\tau)\|_{L^2(B_1(0))} \simeq 1 \tag{2.21}$$

for all $\tau \geq -\log T + 1$.

The proof of the upper bound essentially relies on the existence of a Lyapunov functional, which was introduced in [3]. In physical coordinates, the expression (2.21) yields

$$\begin{aligned} (T-t)^{\frac{7-3p}{p-1}} &\int_{|x-x_0|\leq T-t} |\psi(t,x)|^2 dx \\ &+ (T-t)^{\frac{5-p}{p-1}} \int_{|x-x_0|\leq T-t} \left(|\nabla\psi(t,x)|^2 + |\psi_t(t,x)|^2\right) dx \lesssim 1. \end{aligned} \tag{2.22}$$

The results of [78], [79] were further extended in [80], where an equivalent upper bound was established for arbitrary points on the blow up surface. Moreover, for non-chracteristic points also a lower bound of the form (2.19) is available which yields the blow up rate in the non-degenerate case.
For $3 < p < 5$ upper bounds in global norms as well as in light cones were established by Killip, Stovall and Visan in [56]. Furthermore, it was proved that the results of Merle and Zaag remain valid for the Klein-Gordon equation with (sub)conformal nonlinearities. Some of the estimates in [56] were recently improved by Hamza and Zaag [47], who considered the problem again in similarity coordinates.

2.4.4 The blow up profile

The above results on the blow up rate give no information on the *blow up profile* for subcritical nonlinearities. In the case that the blow up does not occur at the origin, Merle and Zaag [83] could extend their analysis for the one-dimensional wave equation to the radial wave equation (2.15) with (sub)conformal nonlinearities $1 < p \leq 3$. It was shown that solutions converge to a Lorentz-transform of the ground state solution in backward lightcones of non characteristic points *outside the origin.* For solutions that blow up at the origin in the radial case, the existence of infinitely many self-similar blow up solutions implies that there cannot exist a universal profile that is approached by *any* singular solution of Eq. (2.15). However, as mentioned above, it is suggested that the self-similar ground state solution describes the asymptotic blow up profile of *generic* solutions.

2.4.5 The energy critical equation

For $p = 5$ the energy space is critical and local well-posedness in $\dot{H}^1 \times L^2$ follows from Theorem 2.3.3, which also implies that in case of finite time blow up the solution diverges in $L^8([0,T) \times \mathbb{R}^3)$, i.e.,

$$\|\psi\|_{L^8([0,T)\times\mathbb{R}^3)} = \infty. \tag{2.23}$$

However, contrary to the subcritical case, the solution does not necessarily diverge in the critical norm, as will be explained below. First, we consider again the ODE blow up solution

$$\psi^T(t,x) = (\tfrac{3}{4})^{\frac{1}{4}}(T-t)^{-\frac{1}{2}}, \tag{2.24}$$

which, up to symmetries, is the unique self-similar solution of the (NLW) for $p = 5$. With smooth cut off functions one can construct initial data such that the corresponding solution blows up at $t = T$ at least on a ball $|x| < T$, cf. for example [66]. Furthermore, the energy norm of this solution becomes infinite, i.e.,

$$\lim_{t\to T} \|(\psi(t,\cdot), \partial_t\psi(t,\cdot))\|_{\dot{H}^1\times L^2} \to \infty. \tag{2.25}$$

Singular solutions of the energy critical wave equation for which (2.25) holds are referred to as *type I*, whereas solutions which blow up according to (2.23) but stay bounded in the energy norm are called *type II*.

Tightly connected to the rich dynamics of the critical wave equation is the existence of the static solution W defined in Eq. (2.17), which is also called the ground state soliton. Note that the solution scales as

$$W_\lambda = \lambda^{\frac{1}{2}} W(\lambda x), \quad \text{for } \lambda \in \mathbb{R}.$$

The ground state soliton is in $\dot{H}^1(\mathbb{R}^3)$ and thus provides a global energy class solution that does not scatter to zero in future time. In regard to singularity formation, the negative energy blow up criterion does not apply for data close to the static solution since the energy of W_λ is positive. However, the soliton itself provides a threshold for the dynamical behavior of solutions, as was shown by Kenig and Merle in [54] for data with energies below $E(W,0)$. A similar characterization of solutions with $E(\psi_0,\psi_1) = E(W,0)$ was given by Duyckaerts and Merle and we refer to [34] for the result.

Theorem 2.4.3 (Kenig-Merle [54])**.** *Let $p = 5$ and let $\psi \in \dot{H}^1 \times L^2$ be a solution of the* (NLW) *for initial data (ψ_0,ψ_1) with energy*

$$E(\psi_0,\psi_1) < E(W,0).$$

- *If $\|\nabla\psi_0\|_{L^2} < \|\nabla W\|_{L^2}$ then the solution exists for all $t \in \mathbb{R}$ and scatters to zero.*
- *If $\|\nabla\psi_0\|_{L^2} > \|\nabla W\|_{L^2}$ then the solution blows up in finite time in both temporal directions.*

In regard to the existence of type II blow up solutions explicit radial examples were first constructed in the seminal work of Krieger, Schlag and Tataru. We only state a qualitative version and refer to Theorem 1.1 in [65] and [64], respectively, for a precise formulation.

Theorem 2.4.4 (Krieger-Schlag-Tataru [65], Krieger-Schlag [64])**.** *Consider the* (NLW) *for $p = 5$ in the radial case and let $\nu > 0$ be given. There exists an energy-class solution that blows up at $r = |x| = 0$ as $t \to T$, $T > 0$ and which is of the form*

$$\psi(t,r) = \lambda(t)^{1/2} W(\lambda(t)r) + \eta(t,r), \ \text{for } r \leq T - t \tag{2.26}$$

where

$$\lambda(t) = (T-t)^{-1-\nu}$$

and the error term η is small in a suitable sense.

One of the remarkable features of these solutions is that the blow up rate can be prescribed arbitrarily. More exotic type II blow up solutions with rates $\lambda(t)$ oscillating between pure power-laws were established recently by Donninger, Krieger, Huang and Schlag [24]. The spectrum of possible dynamics was further enriched by the results of Donninger and Krieger [25] who proved the existence of solutions blowing up at infinity, again with a continuum of rates, and of vanishing solutions that do not scatter.
In [31], [30] Duyckaerts, Kenig and Merle showed that type II blow up solutions with energies slightly above the energy of the soliton can always be decomposed according to (2.26). This was further investigated in [30] and finally the authors were able to prove a full characterization of possible dynamics in the radial case.

Theorem 2.4.5 (Duyckaerts-Kenig-Merle [32]). *Let $p = 5$ and let ψ be a radial solution of the* (NLW) *with maximal (forward) time of existence T. Then one of the following scenarios is realized.*

- *Type I blow up: $T < \infty$ and $\lim_{t\to T} \|(\psi(t), \partial_t\psi(t))\|_{\dot{H}^1\times L^2} = \infty$.*
- *Type II blow up: $T < \infty$ and there exist $(v_0, v_1) \in \dot{H}^1 \times L^2$, a number $N \in \mathbb{N}$, positive functions $\lambda_j(t)$*

$$\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_N(t) \ll T - t \text{ as } t \to T$$

  defined close to T and signs $\iota_j \in \{\pm 1\}$ such that

$$\lim_{t\to T} \left\| (\psi(t), \partial_t\psi(t)) - \left(v_0 + \sum_{j=1}^{N} \frac{\iota_j}{\lambda_j^{1/2}(t)} W\left(\tfrac{x}{\lambda_j(t)}\right), v_1 \right) \right\|_{\dot{H}^1\times L^2} = 0.$$

- *Global solution: $T = +\infty$ and there exists a solution v_L of the linear wave equation, a number $N \in \mathbb{N}_0$, positive functions $\lambda_j(t)$*

$$\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_N(t) \ll t \text{ as } t \to +\infty$$

  defined for large t and signs $\iota_j \in \{\pm 1\}$ such that

$$\lim_{t\to+\infty} \left\| (\psi(t), \partial_t\psi(t)) - \left(v_L(t) + \sum_{j=1}^{N} \frac{\iota_j}{\lambda_j^{1/2}(t)} W\left(\tfrac{x}{\lambda_j(t)}\right), \partial_t v_L(t) \right) \right\|_{\dot{H}^1\times L^2} = 0.$$

A different line of investigation was pursued by Krieger and Schlag, partially motivated by the numerical observations discussed above. In [63] perturbations around W_λ were studied in a topology strictly stronger than the energy topology. Note that the static solution is linearly unstable. A Lipschitz-manifold of co-dimension one passing through the ground state was constructed and its role as a local threshold between blow up and scattering to zero was established by Krieger, Schlag and

Nakanishi in [61]. In [59], [60] the authors started to investigate the dynamics for solutions with energies slightly above $E(W, 0)$ in the topology of the energy space. The authors prove that the behavior of solutions which do not stay close to the ground state can be classified into four cases: blow up in finite time or scattering to zero, both in forward or backward time direction. Moreover, it is shown that initial data for each case constitute a non-empty open set in the energy space. This work was completed recently in [62] with the construction of a smooth co-dimension one center-stable manifold incorporating all the solutions scattering to the ground state solitons, or staying close to. Since a precise formulation of the main results requires some notation, we refer the reader to Theorem 1.1 in [62]. In comparison with the results of [32], cf. Theorem 2.4.5, the manifold in [62] contains all global solutions with $N = 1$ as well as some of the type II blow up solutions. In this context also a contribution by Beceanu [6] should be mentioned, which is in a similar spirit.

These results confirm on a rigorous level the dynamical picture suggested by the numerical experiments described above. Despite the surprising features of type II blow up solutions, the above results also suggest that the *generic* behavior of finite time blow up solutions for the critical wave equation is of type I. In [66] Krieger and Wong proved the development of type I singularities for a certain concept of weak evolution for data near and above W. Krieger and Nahas studied stability of the type II blow up solutions as constructed in [65], [64] in the energy space. We qualitatively state their result and refer to Theorem 1.1 in [58] for the precise formulation.

Theorem 2.4.6 (Krieger-Nahas, [58]). *Let ψ be one of the type II blow up solutions constructed in [65], [64], with an error term small in a suitable sense. There exist open sets $\mathcal{U}$, $\mathcal{V}$ in the energy topology such that*

$$(\psi(0, \cdot), \partial_t \psi(0, \cdot)) \in \overline{\mathcal{U}} \cap \overline{\mathcal{V}}.$$

All data in $\mathcal{U}$ lead to solutions that exist globally and scatter to zero in forward time, whereas all data in $\mathcal{V}$ lead to solutions that blow up in finite time.

It has yet to be investigated if any open subset of $\mathcal{V}$ contains data leading to type I blow up.

2.4.6 Known results in the supercritical case

For the nonlinear wave equation in the energy supercritical regime $p > 5$ much less is known, although there has been important progress in recent times. In [55] Kenig and Merle considered the defocusing radial equation in three space dimensions and showed that any solution which is bounded in the critical Sobolev norm is globally defined and scatters. The symmetry assumption was then dropped by Killip and Visan in [57]. Similar results for the defocusing equation in higher space dimensions were obtained in [17], [16], [15].

Recently, the results of [55] were also established for the focusing wave equation in the radial case by Duyckaerts, Kenig and Merle in [33]. In particular, it was shown that any blow up solution diverges in the critical Sobolev norm.

Chapter 3

Stable blow up dynamics

The analytic and numerical results presented in the previous chapter suggest that the behavior of generic blow up solutions of the (NLW) can be described in terms of

$$\psi^T(t,x) = \kappa_p^{\frac{1}{p-1}}(T-t)^{-\frac{2}{p-1}}, \quad \kappa_p = \frac{2(p+1)}{(p-1)^2},$$

which we introduced as the *ODE blow up solution* or *self-similar ground state solution*, respectively.

In this chapter we present our main results, which prove nonlinear stability of ψ^T in the radial case for subcritical, critical and supercritical nonlinearities in suitable topologies. In particular, we construct open sets of radial initial data such that the corresponding solutions blow up at the origin in finite time and converge to ψ^T in the backward lightcone of the blow up point $(T,0)$.

3.1 Formulation of the main results

We consider the initial value problem for the nonlinear wave equation

$$\partial_t^2\psi - \Delta\psi = |\psi|^{p-1}\psi \tag{3.1}$$

where $\psi : I \times \mathbb{R}^3 \to \mathbb{R}$ and $p > 1$. We restrict ourselves to radial initial data

$$\psi(0,\cdot) = f, \quad \psi_t(0,\cdot) = g,$$

and study the Cauchy-problem in the backward lightcone

$$\mathcal{C}_T := \{(t,r) : t \in (0,T), r \in [0, T-t]\}.$$

Due to finite speed of propagation the evolution is completely independent of the behavior outside $\mathcal{C}_T$ and can be considered as a dynamical system of its own. Note that our analysis does not rely on any previous well-posedness result for the wave equation.

Since we are interested in the stability of ψ^T under small perturbations, we insert the ansatz

$$\psi = \psi^T + \varphi$$

into Eq. (3.1) to obtain the Cauchy-problem

$$\begin{cases} \varphi_{tt} - \varphi_{rr} - \frac{2}{r}\varphi_r - p(\psi^T)^{p-1}\varphi - N_T(\varphi) = 0 \text{ in } \mathcal{C}_T \\ \left.\begin{array}{l} \varphi(0,r) = f(r) - \psi^T(0,r) \\ \varphi_t(0,r) = g(r) - \psi^T_t(0,r) \end{array}\right\} \text{ for } r \in [0,T] \end{cases} \tag{3.2}$$

for the perturbation φ, where (f, g) are the free initial data of the original problem and

$$N_T(\varphi) = |\psi^T + \varphi|^{p-1}(\psi^T + \varphi) - |\psi^T|^{p-1}\psi^T - p|\psi^T|^{p-1}\varphi$$

is the nonlinear remainder. Regularity at the center yields the boundary condition for the derivative

$$\varphi_r(t, 0) = 0.$$

Note that there is no boundary condition for the perturbation itself at the origin.

3.1.1 (Sub)conformal nonlinearities $1 < p \leq 3$

The most natural setting for the nonlinear wave with subcritical nonlinearities is certainly the energy space. Note that the conserved energy associated to the *free radial equation*

$$\varphi_{tt} - \varphi_{rr} - \frac{2}{r}\varphi_r = 0, \tag{3.3}$$

is given by

$$\int_0^\infty r^2 \left(\varphi_t(t,r)^2 + \varphi_r(t,r)^2\right) dr. \tag{3.4}$$

Since we restrict the problem to a bounded space-time region we need a quantity that defines a *local energy norm* in the backward lightcone $\mathcal{C}_T$. However, the restriction of (3.4) to the interval $r \in (0, T - t)$ does not define a norm (only a semi-norm) due to the lack of a boundary condition for φ at $r = 0$. By integration by parts (and assuming sufficient decay at infinity) it can be easily seen that (3.4) is equivalent to

$$\mathcal{E}_\infty := \int_0^\infty \left(r^2\varphi_t(t,r)^2 + [r\varphi_r(t,r) + \varphi(t,r)]^2\right) dr. \tag{3.5}$$

Another way to obtain (3.5) is to define $\tilde{\varphi} := r\varphi$, such that Eq. (3.3) transforms to the $1 + 1$ wave equation

$$\tilde{\varphi}_{tt} - \tilde{\varphi}_{rr} = 0 \tag{3.6}$$

with conserved energy

$$\int_0^\infty \left(\tilde{\varphi}_t(t,r)^2 + \tilde{\varphi}_r(t,r)^2\right) dr. \tag{3.7}$$

Writing this expression in terms of the original field yields (3.5). The important observation is that, although the expressions (3.4) and (3.5) are equivalent on a global level, the latter has much nicer properties locally and can be used to define the required local energy norm.

Definition. For $R > 0$ we set $\tilde{\mathcal{E}}(R) := C^1[0,R] \times C[0,R]$ and define

$$\|(f,g)\|^2_{\mathcal{E}(R)} := \int_0^R |rf'(r) + f(r)|^2 dr + \int_0^R r^2 |g(r)|^2 dr. \tag{3.8}$$

Note that $\|\cdot\|_{\mathcal{E}(R)}$ defines a norm on $\tilde{\mathcal{E}}(R)$. We denote by $\mathcal{E}(R)$ the completion of $\tilde{\mathcal{E}}(R)$ with respect to $\|\cdot\|_{\mathcal{E}(R)}$ and refer to

$$(\mathcal{E}(R), \|\cdot\|_{\mathcal{E}(R)})$$

as the *local energy space.*

We consider (3.8) in the backward lightcone of the blow up point $(T,0)$, i.e., for $R = T - t$ and insert the fundamental self-similar solution to obtain

$$\|(\psi^T(t,\cdot), \psi^T_t(t,\cdot))\|_{\mathcal{E}(T-t)} = C_p (T-t)^{-\frac{5-p}{2(p-1)}} \tag{3.9}$$

where $C_p > 0$ denotes a p-dependent constant. Evidently, as $t \to T-$, this quantity blows up in the energy subcritical case. In the following theorem, ψ^1 denotes the ODE blow up solution ψ^T for $T = 1$.

Theorem A (**Main result for (sub)conformal nonlinearities [26]**). *Fix* $1 < p \leq 3$ *and* $\varepsilon > 0$. *Let* (f,g) *be radial initial data with*

$$\|(f,g) - (\psi^1(0,\cdot), \psi^1_t(0,\cdot))\|_{\mathcal{E}(\frac{3}{2})}$$

sufficiently small. Then there exists a $T > 0$ *close to* 1 *such that the Cauchy problem*

$$\begin{cases} \partial_t^2 \psi - \Delta\psi = |\psi|^{p-1}\psi \\ (\psi(0,\cdot), \psi_t(0,\cdot)) = (f,g) \end{cases} \tag{3.10}$$

has a unique radial solution $\psi : \mathcal{C}_T \to \mathbb{R}$ *which satisfies*

$$(T-t)^{\frac{5-p}{2(p-1)}} \|(\psi(t,\cdot), \psi_t(t,\cdot)) - (\psi^T(t,\cdot), \psi^T_t(t,\cdot))\|_{\mathcal{E}(T-t)} \leq C_\varepsilon (T-t)^{|\omega_p| - \varepsilon}$$

for all $t \in [0,T)$ *where*

$$\omega_p := \max\left\{-1, \tfrac{1}{2} - \tfrac{2}{p-1}\right\}$$

and $C_\varepsilon > 0$ *is a constant which depends on* ε.

By a solution we mean that ψ satisfies (3.10) in a suitable weak (Duhamel) sense, cf. below. Since ψ^T blows up in the local energy norm, the blow up rate of the perturbation has to be normalized with the rate (3.9) in order to obtain convergence towards ψ^T. The main steps in the proof will be discussed below. First, we state the respective result for superconformal powers and in particular, for energy supercritical nonlinearities.

3.1.2 Superconformal nonlinearities $3 < p < \infty$

The blow up rate of ψ^T given in Eq. (3.9) suggests that the above defined local energy norm is the appropriate setting for all energy subcritical nonlinearities $1 < p < 5$. However, similar to the local well-posedness theory for the nonlinear wave equation discussed in the previous chapter, additional estimates (such as Strichartz-estimates) would be necessary in order to control the nonlinearity for $3 < p \leq 5$.

For $p > 5$ we need a higher Sobolev space. This is already obvious from Theorem 2.3.3, which shows that $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$, $s_c = \frac{3}{2} - \frac{2}{p-1}$, is required for local well-posedness in the supercritical case. At this instant we want to avoid fractional derivatives for technical reasons. Therefore, we consider the above problem in a norm providing one additional degree of differentiability which allows us to estimate the nonlinearity for all $p > 3$ without further requirements. In order to obtain a suitable local norm, we differentiate Eq. (3.6) with respect to r, which shows that $\tilde{\varphi}_r$ again satisfies the one-dimensional wave equation. This immediately suggests to choose

$$\int_0^R [\tilde{\varphi}_{rt}(t,r)^2 + \tilde{\varphi}_{rr}(t,r)^2] dr.$$

However, $\tilde{\varphi}_r$ does not satisfy an appropriate boundary condition at the origin. Therefore, we simply add the energy term (3.7). This motivates the following definition.

Definition. For $R > 0$ we set $\tilde{\mathcal{E}}^h(R) := C^2[0,R] \times C^1[0,R]$ and define

$$\begin{aligned}\|(f,g)\|^2_{\mathcal{E}^h(R)} := &\int_0^R |rf'(r) + f(r)|^2 dr + \int_0^R |rf''(r) + 2f'(r)|^2 dr \\ &+ \int_0^R r^2 |g(r)|^2 dr + \int_0^R |rg'(r) + g(r)|^2 dr.\end{aligned}$$

We denote by $\mathcal{E}^h(R)$ the completion of $\tilde{\mathcal{E}}^h(R)$ with respect to $\|\cdot\|_{\mathcal{E}^h(R)}$ and refer to

$$(\mathcal{E}^h(R), \|\cdot\|_{\mathcal{E}^h(R)})$$

as the *local higher energy space*.

Inserting ψ^T shows that the fundamental self-similar solution blows up in the backward lightcone $\mathcal{C}_T$ with respect to the local higher energy norm, i.e.,

$$\begin{aligned}
&\|(\psi^T(t,\cdot),\psi^T_t(t,\cdot))\|^2_{\mathcal{E}^h(T-t)} \\
&= \int_0^{T-t} |\psi^T(t,r)|^2 dr + \int_0^{T-t} |r^2\psi^T_t(t,r)|^2 dr + \int_0^{T-t} |\psi^T_t(t,r)|^2 dr \\
&\simeq (T-t)^{\frac{p-5}{p-1}} + (T-t)^{-\frac{p+3}{p-1}} \simeq (T-t)^{-\frac{p+3}{p-1}}
\end{aligned} \tag{3.11}$$

for $t \in [0,T)$. With these preliminaries we are ready to formulate the main result for $p > 3$. As above, ψ^1 denotes the blow up solution ψ^T for fixed $T = 1$.

Theorem B (Main result for superconformal nonlinearities, [27]). *Fix $p \in \mathbb{R}$, $p > 3$. Choose $\varepsilon > 0$ such that*

$$\mu_p := \tfrac{2}{p-1} - \varepsilon > 0$$

and let (f,g) be radial initial data with

$$\|(f,g) - (\psi^1(0,\cdot),\psi^1_t(0,\cdot))\|_{\mathcal{E}^h(\frac{3}{2})}$$

sufficiently small. Then there exists a $T \in (\frac{1}{2},\frac{3}{2})$ such that the Cauchy problem

$$\begin{cases} \partial_t^2\psi - \Delta\psi = |\psi|^{p-1}\psi \\ (\psi(0,\cdot),\psi_t(0,\cdot)) = (f,g) \end{cases}$$

has a unique radial solution $\psi : \mathcal{C}_T \to \mathbb{R}$ which satisfies

$$(T-t)^{\frac{p+3}{2(p-1)}} \|(\psi(t,\cdot),\psi_t(t,\cdot)) - (\psi^T(t,\cdot),\psi^T_t(t,\cdot))\|_{\mathcal{E}^h(T-t)} \leq C_\varepsilon (T-t)^{\mu_p}$$

for all $t \in [0,T)$ and a constant $C_\varepsilon > 0$.

3.2 Outline of the proof

Since the main steps in the proofs of Theorem A and Theorem B are quite similar, we discuss both cases simultaneously. For $1 < p \leq 3$ we use some results from [21] for the linearized equation. These are partially reproved and further refined, see Lemma 4.1.7.

3.2.1 First order formulation in similarity coordinates

We formulate Eq. (3.2) as a first order system in similarity coordinates by introducing new variables

$$\varphi_1 = (T-t)^{\frac{2}{p-1}}(r\varphi)_t, \quad \varphi_2 = (T-t)^{\frac{2}{p-1}}(r\varphi)_r. \tag{3.12}$$

We study the problem in similarity coordinates (τ, ρ) given by

$$\rho = \frac{r}{T-t}, \quad \tau = -\log(T-t).$$

In these coordinates the backward lightcone $\mathcal{C}_T$ transforms to the infinite cylinder

$$\mathcal{Z}_T := \{(\tau, \rho) : \tau > -\log T, \rho \in [0,1]\},$$

such that the behavior of the perturbation for $t \to T$ is equivalent to the asymptotic behavior for $\tau \to \infty$. Setting $\phi_j(\tau, \rho) := \varphi_j(T - e^{-\tau}, e^{-\tau}\rho)$ for $j = 1, 2$ yields

$$\begin{aligned} \partial_\tau \phi_1 &= -\rho \partial_\rho \phi_1 + \partial_\rho \phi_2 - \tfrac{2}{p-1}\phi_1 \\ &\quad + p\kappa_p \int_0^\rho \phi_2(\tau, s)ds + \rho N(\rho^{-1} \int_0^\rho \phi_2(\tau, s)ds) \\ \partial_\tau \phi_2 &= -\rho \partial_\rho \phi_2 + \partial_\rho \phi_1 - \tfrac{2}{p-1}\phi_2 \end{aligned} \tag{3.13}$$

in $\mathcal{Z}_T$ with appropriately transformed initial data at $\tau = -\log T$. Note that the original field can be reconstructed by

$$\psi(t, r) = \psi^T(t, r) + (T-t)^{-\frac{2}{p-1}} r^{-1} \int_0^r \phi_2(-\log(T-t), \tfrac{r'}{T-t})dr'.$$

3.2.2 Abstract formulation

We formulate Eq. (3.13) as an abstract Cauchy-problem in a Hilbert space corresponding to the local (higher) energy space. In the new variables the function space simplifies to

$$\mathcal{H}^{p \le 3} := L^2(0,1)^2$$

in the (sub)conformal case $1 < p \le 3$. For $p > 3$ the higher energy norm can be written in terms of the standard norm on $H^1(0,1)^2$ and we set

$$\mathcal{H}^{p>3} := \{u \in H^1(0,1) : u(0) = 0\} \times H^1(0,1).$$

We introduce a function $\Phi : [-\log T, \infty) \to \mathcal{H}$ and rewrite the evolution equation for the perturbation as

$$\tfrac{d}{d\tau}\Phi(\tau) = \boldsymbol{L}\Phi(\tau) + \boldsymbol{N}(\Phi(\tau)), \quad \tau > -\log T \tag{3.14}$$

with corresponding initial data $\Phi(-\log T) = \boldsymbol{u}_0$. The operator $\boldsymbol{L}$ represents the linearized part of the above equation and $\boldsymbol{N}$ corresponds to the vector valued nonlinear remainder.

3.2.3 Well-posedness of the linearized equation

We first consider the linearized problem for an operator $\boldsymbol{L} = \boldsymbol{L}_0 + \boldsymbol{L}'$, where $(\boldsymbol{L}_0, \mathcal{D}(\boldsymbol{L}_0))$ is an appropriately defined unbounded linear operator and $\boldsymbol{L}'$ is a compact operator on both spaces $\mathcal{H}^{p\leq 3}$ and $\mathcal{H}^{p>3}$. Note that $\boldsymbol{L}_0$ is neither self-adjoint nor symmetric.

We use the framework of strongly continuous one-parameter semigroups to show well-posedness of the linearized equation and to obtain a growth estimate for linear perturbations, see Appendix A for a brief summary of standard results. By applying the Lumer-Phillips theorem, see A.1.2, we show that the free operator $(\boldsymbol{L}_0, \mathcal{D}(\boldsymbol{L}_0))$ generates a semigroup $\mathcal{S}_0 = (\mathbf{S}_0(\tau))_{\tau\geq 0}$ which satisfies an estimate of the form

$$\|\mathbf{S}_0(\tau)\| \leq M e^{\mu_0(\mathcal{S}_0)\tau}, \quad \forall \tau \geq 0, \tag{3.15}$$

for a constant $M \geq 1$ and some $\mu_0(\mathcal{S}_0) \in \mathbb{R}$. In $\mathcal{H}^{p\leq 3}$ we obtain

$$\mu_0(\mathcal{S}_0) = \frac{1}{2} - \frac{2}{p-1} \tag{3.16}$$

which is negative for $1 < p < 5$. Unfortunately, in $\mathcal{H}^{p>3}$ the L^2-part in the standard H^1-norm prevents us from proving a better bound than (3.16), which is too weak for $p \geq 5$. Note that this is different to the wave maps problem [28]. There, the corresponding higher energy norm for the wave maps equation could be defined as a homogeneous norm due to a different boundary condition. As a consequence, the required bounds for the free semigroup could be obtained in a straightforward manner. Here, the situation is more delicate. One of the main ingredients in the stability analysis is the definition of an equivalent norm on $\mathcal{H}^{p>3}$ by

$$\|\boldsymbol{u}\|_{\mathbf{1}}^2 := |u_1(1) + u_2(1)|^2 + \|u_1'\|_{L^2}^2 + \|u_2'\|_{L^2}^2. \tag{3.17}$$

This allows us to prove that

$$\mu_0(\mathcal{S}_0) = -\frac{2}{p-1}. \tag{3.18}$$

in $(\mathcal{H}^{p>3}, \|\cdot\|_{\mathbf{1}})$. By equivalence this yields a negative growth bound for the free semigroup with respect to the standard norm on $\mathcal{H}^{p>3}$ for all $p > 3$.

For the full operator we apply the bounded perturbation theorem, see A.1.3, and conclude that $(\boldsymbol{L}, \mathcal{D}(\boldsymbol{L}))$ generates a C_0-semigroup $\mathcal{S} = (\mathbf{S}(\tau))_{\tau\geq 0}$ with

$$\|\mathbf{S}(\tau)\| \leq M e^{(\mu_0(\mathcal{S}_0) + M\|\boldsymbol{L}'\|)\tau}, \quad \forall \tau \geq 0. \tag{3.19}$$

This also immediately implies well-posedness of the linearized problem on $\mathcal{H}^{p\leq 3}$ and $\mathcal{H}^{p>3}$ for initial data in $\mathcal{D}(\boldsymbol{L})$, cf. Appendix A.1.

3.2.4 Estimates for the linear time evolution

In order to improve the weak growth estimate in Eq. (3.19) we use the spectral properties the generator. The spectrum of $\boldsymbol{L}_0$ is contained in a left half plane

$$\sigma(\boldsymbol{L}_0) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq \mu_0(\mathcal{S}_0)\},$$

see A.1.1. Compactness of $\boldsymbol{L}'$ implies the following important characterization: If $\lambda \in \mathbb{C}$ is in the spectrum of $\boldsymbol{L}$ but not in the spectrum of $\boldsymbol{L}_0$ then λ is necessarily an eigenvalue of finite algebraic multiplicity. This allows us to reduce the spectral problem for the generator $\boldsymbol{L}$ to the investigation of the eigenvalue equation given that $\mathrm{Re}\lambda > \mu_0(\mathcal{S}_0)$. The corresponding second order ODE can be solved explicitly in terms of hypergeometric functions. In particular, our result is completely rigorous and does not rely on any numerical input. Note that this is not the case for the wave maps equation [28], [22], where the main result is conditional and relies on a spectral assumption.

We obtain the following characterization of the spectrum

$$\sigma(\boldsymbol{L}) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq \omega_p\} \cup \{1\}, \tag{3.20}$$

where

$$\omega_p = \begin{cases} \max\left\{-1, \frac{1}{2} - \frac{2}{p-1}\right\} & \text{for } 1 < p \leq 3 \\ -\frac{2}{p-1} & \text{for } p > 3. \end{cases} \tag{3.21}$$

Moreover, $\lambda_{\boldsymbol{g}} = 1$ is an eigenvalue with corresponding eigenfunction $\boldsymbol{g}_s$, which is referred to as the *symmetry mode*. Note that the existence of a positive eigenvalue is a consequence of the time translation invariance of the original nonlinear equation and the fact that ψ^T is a one-parameter family of solutions for $T > 0$. In the nonlinear theory we will account for this instability by adjusting the blow up time accordingly.

Since the symmetry eigenvalue $\lambda_{\boldsymbol{g}}$ is isolated, a spectral projection can be defined by

$$\boldsymbol{P} = \frac{1}{2\pi i} \int_\gamma \mathbf{R}_{\boldsymbol{L}}(\lambda) d\lambda,$$

see [52], p. 178, which decomposes the Hilbert space into $\mathcal{H} = \mathcal{N} \oplus \mathcal{M}$ where

$$\mathcal{N} = \ker\boldsymbol{P} = \mathrm{rg}(1 - \boldsymbol{P}), \quad \mathcal{M} := \mathrm{rg}\boldsymbol{P}.$$

The projection commutes with the resolvent and hence with the generator, see e.g. [52] p. 173. This implies that $\boldsymbol{L}$ is decomposed into parts $\boldsymbol{L}_{\mathcal{N}}$ and $\boldsymbol{L}_{\mathcal{M}}$ on the respective subspaces, where

$$\sigma(\boldsymbol{L}_{\mathcal{N}}) = \sigma(\boldsymbol{L}) \setminus \{1\}, \quad \sigma(\boldsymbol{L}_{\mathcal{M}}) = \{1\}.$$

A priori we only know that $\ker(\lambda - \boldsymbol{L}) \subseteq \mathrm{rg}\boldsymbol{P}$. Hence, we prove that $\mathrm{rg}\boldsymbol{P} = \langle \boldsymbol{g}_s \rangle$ using an abstract argument together with ODE methods. The semigroup commutes with the projection and thus, the stable and unstable subspaces are invariant under $\mathcal{S} = (\mathbf{S}(\tau))_{\tau \geq 0}$. By Lemma A.1.5

$$\mathcal{S}_{\mathcal{N}} = (\mathbf{S}|_{\mathcal{N}}(\tau))_{\tau \geq 0}$$

, is the subspace semigroup generated by $\boldsymbol{L}_{\mathcal{N}}$ on the subspace $\mathcal{N}$.
In general, the spectral bound of the generator and the growth bound of the semigroup defined in A.2.1 do not coincide. Therefore, we prove uniform bounds on the resolvent $\mathbf{R}_{\boldsymbol{L}}(\lambda)|_{\mathcal{N}}$ in the half-plane

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \omega_p + \varepsilon\}$$

for small $\varepsilon > 0$, where ω_p is defined in (3.21). This allows the application of the Gearhart-Prüss theorem A.2.1 to translate the negative spectral bound of the generator $\boldsymbol{L}$ on the stable subspace $\ker\boldsymbol{P}$ into a negative growth bound for the subspace semigroup.

3.2.5 The nonlinear problem

While in the linear perturbation theory the blow up time was considered to be fixed, we now consider T as a variable in order to control the instability arising from the symmetry mode. Therefore, we make the dependence of the equation on the blow up time more explicit. First, we define a 'universal' solution using the invariance of the equation under time translations and set

$$\Psi : [0, \infty) \to \mathcal{H}, \quad \Psi(\tau) := \Phi(\tau - \log T).$$

The initial data in Eq. (3.14) are of the form

$$(f, g) - (\psi^T(0, \cdot), \psi_t^T(0, \cdot))$$

modulo transformations of the original equation Eq. (3.2). We rewrite the data by introducing a function $\boldsymbol{v}$ which corresponds to the data relative to the ground state solution with blow up time $T = 1$,

$$\boldsymbol{v} \approx (f, g) - (\psi^1(0, \cdot), \psi_t^1(0, \cdot)).$$

In order to obtain a well-posed initial value problem, the initial data (f, g) have to be defined on the spatial interval $[0, T]$. Since we do not know the blow up time in advance, we restrict T to the interval $\mathcal{I} := (\frac{1}{2}, \frac{3}{2})$, which is no limitation since our argument is perturbative around $T = 1$ anyway. Hence, we rigorously define the initial data as a function $\boldsymbol{U}(\boldsymbol{v}, T)$ on a space $\mathfrak{H} \times \mathcal{I}$ where

$$\boldsymbol{U}(\boldsymbol{v}, T) \approx \boldsymbol{v} + (\psi^1(0, \cdot), \psi_t^1(0, \cdot)) - (\psi^T(0, \cdot), \psi_t^T(0, \cdot))$$

The space $\mathfrak{H}$ corresponds to $\mathcal{E}(\frac{3}{2})$ or $\mathcal{E}^h(\frac{3}{2})$, respectively. Note that

$$D_T \boldsymbol{U}(\mathbf{0}, T)|_{T=1} = c\boldsymbol{g}_s,$$

for a constant $c \in \mathbb{R}$, hence the symmetry mode is tangent to the curve $T \mapsto U(\mathbf{0}, T)$ at $T = 1$.

We study mild solutions of the equation

$$\begin{cases} \frac{d}{d\tau}\Psi(\tau) = \boldsymbol{L}\Psi(\tau) + \boldsymbol{N}(\Psi(\tau)) \text{ for } \tau > 0 \\ \Psi(0) = \boldsymbol{U}(\boldsymbol{v}, T) \end{cases}$$

such that a solution of Eq. (3.14) for a particular $T > 0$ can be obtained by setting $\Phi(\tau) := \Psi(\tau + \log T)$. Mild solutions satisfy the integral equation

$$\Psi(\tau) = \mathbf{S}(\tau)\boldsymbol{U}(\boldsymbol{v}, T) + \int_0^\tau \mathbf{S}(\tau - \tau')\boldsymbol{N}(\Psi(\tau'))d\tau' \quad \text{for} \quad \tau \geq 0 \tag{3.22}$$

which we consider on the space

$$\mathcal{X} := \left\{ \Psi \in C([0, \infty), \mathcal{H}) : \sup_{\tau > 0} e^{(|\omega_p| - \varepsilon)\tau} \|\Psi(\tau)\| < \infty \right\}.$$

For the nonlinearity we prove a local Lipschitz-estimate, i.e., for $\boldsymbol{u}, \boldsymbol{v}$ small

$$\|\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v})\| \lesssim (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)\|\boldsymbol{u} - \boldsymbol{v}\|.$$

In order to control the instability caused by $\boldsymbol{g}_s$ we first modify Eq. (3.22) by adding a correction term

$$\Psi(\tau) = \mathbf{S}(\tau)\boldsymbol{U}(\boldsymbol{v}, T) + \int_0^\tau \mathbf{S}(\tau - \tau')\boldsymbol{N}(\Psi(\tau'))d\tau' - e^\tau \boldsymbol{F}(\boldsymbol{v}, T)$$

where

$$\boldsymbol{F}(\boldsymbol{v}, T) = \boldsymbol{P}\left(\boldsymbol{U}(\boldsymbol{v}, T) + \int_0^\infty e^{-\tau'}\mathbf{N}(\Psi(\tau'))d\tau'\right).$$

Here, the projection $\boldsymbol{P}$ is the spectral projection defined above, hence the equation is corrected by an element of the unstable subspace. By applying the Banach fixed point theorem we prove the existence of a unique solution of the modified equation in $\mathcal{X}$ for small initial data $\boldsymbol{v}$ and T close to one.

In order to remove the correction and to obtain a solution of Eq. (3.22), it is important to note that $\boldsymbol{U}(\mathbf{0}, 1) = \mathbf{0}$ implies $\boldsymbol{F}(\mathbf{0}, 1) = \mathbf{0}$, which is due to the fact that the blow up time does not change if $(f, g) = (\psi^1(0, \cdot), \psi^1_t(0, \cdot))$ and $\varphi \equiv 0$. The idea is to extend this to a small neighborhood. Here, we use an improved argument which is different from [22], where one additional degree of differentiability was necessary due to an application of the implicit function theorem. We prove that for every small $\boldsymbol{v}$ there exists a T close to 1 such that

$$\boldsymbol{F}(\mathbf{v}, T) = \mathbf{0},$$

and thus obtain a solution of Eq. (3.22), which decays to zero according to

$$\|\Psi(\tau)\| \lesssim e^{-(|\omega_p|-\varepsilon)\tau}.$$

The formulation of this result in original variables and coordinates concludes the proof.

Chapter 4

Proof of the Main Results

The material presented in this Chapter was published in [26], [27], cf. Section 1.4.

As a starting point we consider the time evolution for the perturbation given by Eq. (3.2) in Chapter 3. Setting

$$\varphi_1 = (T-t)^{\frac{2}{p-1}}(r\varphi)_t, \quad \varphi_2 = (T-t)^{\frac{2}{p-1}}(r\varphi)_r, \tag{4.1}$$

transforms Eq. (3.2) into the first order system

$$\left\{\begin{array}{l} \left.\begin{array}{l} \partial_t\varphi_1 = \partial_r\varphi_2 - \frac{2}{p-1}(T-t)^{-1}\varphi_1 \\ \qquad\quad + p\kappa_p(T-t)^{-2}\int\varphi_2 + r(T-t)^{-2}N(r^{-1}\int\varphi_2) \\ \partial_t\varphi_2 = \partial_r\varphi_1 - \frac{2}{p-1}(T-t)^{-1}\varphi_2 \end{array}\right\} \text{ in } \mathcal{C}_T \\ \left.\begin{array}{l} \varphi_1(0,r) = T^{\frac{2}{p-1}} r\left[g(r) - \psi_t^T(0,r)\right] \\ \varphi_2(0,r) = T^{\frac{2}{p-1}}[rf'(r) + f(r) - \psi^T(0,r)] \end{array}\right\} \text{ for } r \in [0,T] \end{array}\right.$$

Note that

$$N_T\left((T-t)^{-\frac{2}{p-1}}r^{-1}\int\varphi_2\right) = (T-t)^{-\frac{2p}{p-1}}N(r^{-1}\int\varphi_2)$$

where $\int\varphi_2$ is shorthand for $\int_0^r\varphi_2(t,r')dr'$ and

$$N(x) = |\kappa_p^{\frac{1}{p-1}} + x|^{p-1}(\kappa_p^{\frac{1}{p-1}} + x) - \kappa_p^{\frac{p}{p-1}} - p\kappa_p x. \tag{4.2}$$

We formulate the problem in similarity coordinates (τ, ρ) given by

$$\rho = \frac{r}{T-t}, \quad \tau = -\log(T-t).$$

Derivatives transform according to $\partial_t = e^\tau(\partial_\tau + \rho\partial_\rho)$, $\partial_r = e^\tau\partial_\rho$.

Setting $\phi_j(\tau,\rho) := \varphi_j(T - e^{-\tau}, e^{-\tau}\rho)$ for $j = 1,2$ yields

$$\left\{\begin{array}{l} \left.\begin{array}{l} \partial_\tau\phi_1 = -\rho\partial_\rho\phi_1 + \partial_\rho\phi_2 - \frac{2}{p-1}\phi_1 \\ \qquad + p\kappa_p \int_0^\rho \phi_2(\tau,s)ds + \rho N\left(\rho^{-1}\int_0^\rho \phi_2(\tau,s)ds\right) \\ \partial_\tau\phi_2 = -\rho\partial_\rho\phi_2 + \partial_\rho\phi_1 - \frac{2}{p-1}\phi_2 \end{array}\right\} \text{ in } \mathcal{Z}_T \\ \left.\begin{array}{l} \phi_1(-\log T,\rho) = \rho(T^{\frac{p+1}{p-1}}g(T\rho) - \frac{2}{p-1}\kappa_p^{\frac{1}{p-1}}) \\ \phi_2(-\log T,\rho) = T^{\frac{2}{p-1}}(T\rho f'(T\rho) + f(T\rho)) - \kappa_p^{\frac{1}{p-1}} \end{array}\right\} \text{ for } \rho\in[0,1] \end{array}\right.$$

where $\mathcal{Z}_T := \{(\tau,\rho) : \tau > -\log T, \rho\in[0,1]\}$ and

$$N(x) = |\kappa_p^{\frac{1}{p-1}} + x|^{p-1}(\kappa_p^{\frac{1}{p-1}} + x) - \kappa_p^{\frac{p}{p-1}} - p\kappa_p x.$$

It is important to note that the original field can be reconstructed by

$$\begin{aligned} \psi(t,r) &= \psi^T(t,r) + (T-t)^{-\frac{2}{p-1}} r^{-1}\int_0^r \phi_2(-\log(T-t), \tfrac{r'}{T-t})dr', \\ \psi_t(t,r) &= \psi_t^T(t,r) + (T-t)^{-\frac{2}{p-1}} r^{-1}\phi_1(-\log(T-t), \tfrac{r}{T-t}). \end{aligned} \tag{4.3}$$

4.1 Proof of Theorem A

Throughout this section, we assume p to be a fixed real number with

$$1 < p \leq 3.$$

Note that most of the expressions and constants we are going to define below will depend on p. However, for the sake of readability, we do not indicate this dependence explicitly.

Linear Perturbation Theory

In the new variables and coordinates the local energy norm can be written as

$$\begin{aligned} &\|(\varphi(t,\cdot),\varphi_t(t,\cdot))\|^2_{\mathcal{E}(T-t)} \\ &= (T-t)^{\frac{p-5}{p-1}}\left(\int_0^1 |\phi_2(-\log(T-t),\rho)|^2 d\rho + \int_0^1 |\phi_1(-\log(T-t),\rho)|^2 d\rho\right). \end{aligned}$$

Therefore, we define the Hilbert space $\mathcal{H}$ as a product space,

$$\mathcal{H} := L^2(0,1)\times L^2(0,1), \quad \|\mathbf{u}\|^2_{\mathcal{H}} = \|u_1\|^2_{L^2} + \|u_2\|^2_{L^2},$$

for $\mathbf{u} = (u_1,u_2)^T \in \mathcal{H}$.

Note that some parts of the subsequent analysis for the linearized equation are also contained in [21]. However, in order to present a consistent picture, we summarize and reprove known results and supplement them by some new aspects, which will be important for the nonlinear theory, see for example Lemma 4.1.7.

Well-posedness of the linearized equation

We define operators $(\tilde{L}_0, \mathcal{D}(\tilde{L}_0))$ and $L' \in \mathcal{B}(\mathcal{H})$ by

$$\mathcal{D}(\tilde{L}_0) := \{\mathbf{u} \in C^1[0,1] \times C^1[0,1] : u_1(0) = 0\},$$

$$\tilde{L}_0 \mathbf{u}(\rho) := \begin{pmatrix} u_2'(\rho) - \rho u_1'(\rho) - \frac{2}{p-1} u_1(\rho) \\ u_1'(\rho) - \rho u_2'(\rho) - \frac{2}{p-1} u_2(\rho) \end{pmatrix}$$

and

$$L' \mathbf{u}(\rho) := \begin{pmatrix} p\kappa_p \int_0^\rho u_2(s) ds \\ 0 \end{pmatrix}$$

where $\mathbf{u} = (u_1, u_2)^T$. It is easy to see that $L' : \mathcal{H} \to \mathcal{H}$ is a compact operator.

Lemma 4.1.1. *The operator $\tilde{L}_0$ is closable and its closure L_0 generates a strongly continuous one–parameter semigroup $S_0 : [0, \infty) \to \mathcal{B}(\mathcal{H})$ satisfying*

$$\|S_0(\tau)\| \leq e^{\tilde{\omega}_p \tau}$$

for all $\tau \geq 0$ and $\tilde{\omega}_p := \frac{1}{2} - \frac{2}{p-1}$.

Proof. The claim is a consequence of the Lumer-Phillips Theorem (see [36], p. 83, Theorem 3.15). Indeed, a simple integration by parts yields the estimate

$$\mathrm{Re}(\tilde{L}_0 \mathbf{u} | \mathbf{u}) \leq \left(\tfrac{1}{2} - \tfrac{2}{p-1}\right) \|\mathbf{u}\|^2$$

and $\frac{1}{2} - \frac{2}{p-1} < 0$. Furthermore, for $\lambda := 1 - \frac{2}{p-1} > \tilde{\omega}_p$ the range of $\lambda - \tilde{L}_0$ is dense in $\mathcal{H}$. This follows from the very same calculation as in the proof of Lemma 2 in [21]. Since $\tilde{L}_0$ is densely defined, the Lumer-Phillips Theorem applies. ☐

Corollary 4.1.2. *The spectrum of L_0 is contained in a left half plane,*

$$\sigma(L_0) \subset \{\lambda \in \mathbb{C} : Re\lambda \leq \tilde{\omega}_p\},$$

with $\tilde{\omega}_p = \frac{1}{2} - \frac{2}{p-1}$ and the resolvent of L_0 satisfies

$$\|R_{L_0}(\lambda)\| \leq \frac{1}{Re\lambda - \tilde{\omega}_p}$$

for all $\lambda \in \mathbb{C}$ with $Re\lambda > \tilde{\omega}_p$.

Proof. The structure of the spectrum as well as the resolvent estimate follow by standard results of semigroup theory (see [36], p. 55, Theorem 1.10). ☐

The next corollary is a consequence of the Bounded Perturbation Theorem (see [36], p. 158).

Corollary 4.1.3 (well-posedness of the linearized equation)**.** *The operator* $L := L_0 + L'$, $\mathcal{D}(L) := \mathcal{D}(L_0)$ *generates a strongly continuous one–parameter semigroup* $S : [0,\infty) \to \mathcal{B}(\mathcal{H})$ *satisfying*

$$\|S(\tau)\| \leq e^{(\tilde{\omega}_p + p\kappa_p)\tau}$$

for all $\tau \geq 0$ *and* $\tilde{\omega}_p = \frac{1}{2} - \frac{2}{p-1}$. *In particular, the Cauchy problem*

$$\begin{cases} \frac{d}{d\tau}\Phi(\tau) = L\Phi(\tau) \text{ for } \tau > -\log T \\ \Phi(-\log T) = \mathbf{u} \end{cases}$$

has a unique solution given by $\Phi(\tau) = S(\tau + \log T)\mathbf{u}$ *for* $\mathbf{u} \in \mathcal{D}(L_0)$ *and all* $\tau \geq -\log T$.

Properties of the generator

In order to be able to describe the spectrum of L we characterize $\mathcal{D}(L)$ more explicitly.

Lemma 4.1.4. *Let* $\mathbf{u} \in \mathcal{D}(L)$. *Then* $\mathbf{u} \in C[0,1) \times C[0,1)$ *and* $u_1(0) = 0$. *Furthermore, for* $\mathbf{f} \in \mathcal{H}$ *the equation* $(\lambda - L)\mathbf{u} = \mathbf{f}$ *implies*

$$u_1(\rho) = \rho u_2(\rho) + (\lambda + \tfrac{2}{p-1} - 1)\int_0^\rho u_2(s)ds - \int_0^\rho f_2(s)ds$$

and

$$\begin{aligned} -(1-\rho^2)u''(\rho) &+ 2\left(\lambda + \tfrac{2}{p-1}\right)\rho u'(\rho) \\ &+ \left(\left(\lambda + \tfrac{2}{p-1}\right)\left(\lambda + \tfrac{2}{p-1} - 1\right) - p\kappa_p\right)u(\rho) \\ &= f_1(\rho) + \rho f_2(\rho) + (\lambda + \tfrac{2}{p-1})\int_0^\rho f_2(s)ds \end{aligned} \tag{4.4}$$

in a weak sense, where $u \in H^2_{\mathrm{loc}}(0,1) \cap C[0,1] \cap C^1[0,1)$ *is defined by* $u(\rho) := \int_0^\rho u_2(s)ds$.

Proof. Let $\mathbf{u} \in \mathcal{D}(L) = \mathcal{D}(L_0)$. By definition there exists a sequence $(\mathbf{u}_j) \subset \mathcal{D}(\tilde{L}_0) \subset C^1[0,1] \times C^1[0,1]$ such that $\mathbf{u}_j \to \mathbf{u}$ and $\tilde{L}_0\mathbf{u}_j \to L_0\mathbf{u}$ in $\mathcal{H}$. By combining the expressions for the individual components in an appropriate way we infer that $(1-\rho^2)u'_{1j}$ and $(1-\rho^2)u'_{2j}$ are convergent sequences in $L^2(0,1)$. Thus $u_1, u_2 \in H^1(0,1-\varepsilon) \hookrightarrow C[0,1-\varepsilon]$ for any $\varepsilon \in (0,1)$. This guarantees the boundary condition $u_1(0) = 0$.
Let $\mathbf{f} \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. Then $(\lambda - L)\mathbf{u} = \mathbf{f}$ implies

$$\begin{aligned} (\lambda + \tfrac{2}{p-1})u_1(\rho) + \rho u_1'(\rho) - u_2'(\rho) - p\kappa_p \int_0^\rho u_2(s)ds &= f_1(\rho) \\ (\lambda + \tfrac{2}{p-1})u_2(\rho) + \rho u_2'(\rho) - u_1'(\rho) &= f_2(\rho) \end{aligned}$$

in a weak sense. Thanks to the boundary condition we obtain from the second equation that

$$u_1(\rho) = \rho u_2(\rho) + (\lambda + \tfrac{2}{p-1} - 1)\int_0^\rho u_2(s)ds - \int_0^\rho f_2(s)ds.$$

Inserting this into the first equation yields

$$\begin{aligned}-(1-\rho^2)u_2'(\rho) + 2\left(\lambda + \tfrac{2}{p-1}\right)\rho u_2(\rho) + &\\ \left(\left(\lambda + \tfrac{2}{p-1}\right)\left(\lambda + \tfrac{2}{p-1} - 1\right) - p\kappa_p\right)&\int_0^\rho u_2(s)ds \\ = f_1(\rho) + \rho f_2(\rho) + \left(\lambda + \tfrac{2}{p-1}\right)&\int_0^\rho f_2(s)ds.\end{aligned}$$

We set $u(\rho) := \int_0^\rho u_2(s)ds$ and obtain Eq. (4.4). Finally, $u_2 \in L^2(0,1)$ implies $u \in H^1(0,1) \hookrightarrow C[0,1]$ and $\mathbf{u} \in \mathcal{D}(L)$ yields $u \in H^2_{\text{loc}}(0,1) \cap C^1[0,1)$. □

In order to improve the rough growth estimate given in Corollary 4.1.3, we analyse the spectrum of the generator. The next two Lemmas characterize the spectral properties of the generator L sufficiently accurate.

Lemma 4.1.5. *For the spectrum $\sigma(L)$ of the generator L we have*

$$\sigma(L) \subset \{\lambda \in \mathbb{C} : Re\lambda \leq \max\{\tilde{\omega}_p, -1\}\} \cup \{1\}$$

where $\tilde{\omega}_p = \frac{1}{2} - \frac{2}{p-1}$.

Proof. Set $M := \{\lambda \in \mathbb{C} : Re\lambda \leq \max\{\tilde{\omega}_p, -1\}\} \cup \{1\}$. Let $\lambda \in \sigma(L)$. If $Re\lambda \leq \tilde{\omega}_p$ then $\lambda \in M$. So let us assume that $Re\lambda > \tilde{\omega}_p$. Then, by Corollary 4.1.2, $\lambda \in \sigma(L) \setminus \sigma(L_0)$ and the identity

$$\lambda - L = [1 - L'R_{L_0}(\lambda)](\lambda - L_0)$$

together with the spectral theorem for compact operators imply that $\lambda \in \sigma_p(L)$. Thus, there exists a nontrivial $\mathbf{u} \in \mathcal{D}(L)$ such that $(\lambda - L)\mathbf{u} = \mathbf{0}$. By Lemma 4.1.4 this implies the existence of a weak solution u of Eq. (4.4) with right hand side equal to zero. Recall that $u \in H^1(0,1)$ and $u(0) = 0$. We transform Eq. (4.4) by substituting $\rho \mapsto z := \rho^2$ to obtain the hypergeometric equation (recall that $\kappa_p = \frac{2(p+1)}{(p-1)^2}$)

$$z(1-z)v''(z) + [c - (a+b+1)z]v'(z) - abv(z) = 0 \tag{4.5}$$

where $v(z) := u(\sqrt{z})$ and the parameters are given by

$$a = \tfrac{1}{2}(\lambda - 2), \quad b = \tfrac{1}{2}\left(\lambda + \tfrac{p+3}{p-1}\right), \quad c = \tfrac{1}{2}$$

For $\lambda \neq 1 - \frac{2}{p-1}$ a fundamental system around $z = 1$ is given by $\{v_1, \tilde{v}_1\}$,

$$\begin{aligned} v_1(z) &= {}_2F_1(a, b; a+b+1-c; 1-z), \\ \tilde{v}_1(z) &= (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c+1-a-b; 1-z) \end{aligned}$$

where ${}_2F_1$ is the standard hypergeometric function, see e.g., [86]. The exponent $c - a - b = 1 - \frac{2}{p-1} - \lambda$ vanishes for $\lambda = 1 - \frac{2}{p-1}$ and in this case one solution is still given by v_1 and the second one diverges logarithmically for $z \to 1$. Since we assume $Re\lambda > \frac{1}{2} - \frac{2}{p-1}$, v must be a multiple of v_1 for u to be in $H^1(0, 1)$. Around $z = 0$ there is a fundamental system given by $\{v_0, \tilde{v}_0\}$, where

$$\begin{aligned} v_0(z) &= z^{1-c} {}_2F_1(a+1-c, b+1-c; 2-c; z), \\ \tilde{v}_0(z) &= {}_2F_1(a, b; c; z). \end{aligned}$$

Thus, there exist constants c_1, c_2 such that $v_1 = c_1\tilde{v}_0 + c_2 v_0$. In order to satisfy the boundary condition $v(0) = 0$, the coefficient c_1, which can be given in terms of the Gamma function [86],

$$c_1 = \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)},$$

must vanish. Consequently, $c_1 = 0$ if and only if $a + 1 - c$ or $b + 1 - c$ is a pole, which yields $\frac{1}{2}(\lambda - 1) = -k$ or $\frac{\lambda}{2} + \frac{p+1}{p-1} = -k$ for a $k \in \mathbb{N}_0$. This implies that λ is real and

$$\lambda \in \left\{\omega \in \mathbb{R} : \omega > \tilde{\omega}_p \wedge \left(\omega = 1 - 2k \vee \omega = -2k - \tfrac{2p+2}{p-1}\right), k = 0, 1, \dots\right\}.$$

Since $1 < p \leq 3$ we have $\tilde{\omega}_p \leq -\frac{1}{2}$. If $\tilde{\omega}_p \geq -1$ then the only possibility is $\lambda = 1$. If $\tilde{\omega}_p < -1$ then either $\lambda = 1$ or $\lambda \leq -1$. In any case we conclude that $\lambda \in M$. □

Lemma 4.1.6. *The eigenvalue $1 \in \sigma_p(L)$ has geometric multiplicity equal to one. The associated geometric eigenspace is spanned by*

$$\mathbf{g}(\rho) := \begin{pmatrix} \frac{p+1}{p-1}\rho \\ 1 \end{pmatrix}. \tag{4.6}$$

In the following $\mathbf{g}$ *will be referred to as the symmetry mode.*

Proof. Note that $\mathbf{g} \in \mathcal{D}(L)$ and a straightforward calculation yields $(1 - L)\mathbf{g} = 0$. In particular by Lemma 4.1.4 and the definition of κ_p we infer that

$$g_1(\rho) = \rho g_2(\rho) + \tfrac{2}{p-1} \int_0^\rho g_2(s)ds \tag{4.7}$$

and

$$-(1-\rho^2)g''(\rho) + \tfrac{2(p+1)}{p-1}\rho g'(\rho) - \tfrac{2(p+1)}{p-1} g(\rho) = 0 \tag{4.8}$$

for $g(\rho) := \int_0^\rho g_2(s)ds = \rho$. Suppose there is another eigenfunction $\tilde{\mathbf{g}}$ for $\lambda = 1$. Then this corresponds to another (weak) solution $\tilde{g}(\rho) := \int_0^\rho \tilde{g}_2(s)ds$ of Eq. (4.8). A fundamental system of Eq. (4.8) is given by $\{h_0, h_1\}$, where $h_0(\rho) = \rho$ and

$$h_1(\rho) = (1-\rho^2)^{-\frac{2}{p-1}}\tilde{h}_1(\rho)$$

with $\tilde{h}_1(\rho) = {}_2F_1(1, \frac{1}{2} - \frac{p+1}{p-1}; \frac{1}{2}; \rho^2)$ and $\tilde{h}_1(1) \neq 0$ for $1 < p \leq 3$. However, by Lemma 4.1.4, $\tilde{g} \in C[0,1]$ and thus it must be a multiple of $h_0 = g$. Therefore, there exists a constant $c \in \mathbb{C}$ such that

$$\int_0^\rho \tilde{g}_2(s)ds = c\int_0^\rho g_2(s)ds$$

and we infer that $\tilde{g}_2 = cg_2$. Eq. (4.7) implies $\tilde{g}_1 = cg_1$ and we conclude that $\tilde{\mathbf{g}} = c\mathbf{g}$. $\square$

Spectral projection

The aim of this section is to remove the symmetry eigenvalue $\lambda = 1$ via a Riesz projection and to obtain a growth estimate for the solution of the linearized equation on the stable subspace. We define a operator

$$P = \frac{1}{2\pi i}\int_\gamma R_L(\xi)d\xi, \tag{4.9}$$

where γ is a circle that lies entirely in $\rho(L)$ and encloses the eigenvalue 1 in such a way that no other spectral points of L lie inside γ, see for example [52], p. 178. The operator P is a projection on $\mathcal{M} = P\mathcal{H} = \text{rg}P$ along $\mathcal{N} = (1-P)\mathcal{H} = \ker P$, which commutes with the resolvent $PR_L(\lambda) = R_L(\lambda)P$ for $\lambda \in \rho(L)$. According [52] Theorem 6.5, p. 173, this implies that P commutes with L in the sense that $PL \subset LP$, which means that whenever $\mathbf{u} \in \mathcal{D}(L)$ then $P\mathbf{u} \in \mathcal{D}(L)$ and $PL\mathbf{u} = LP\mathbf{u}$. As a consequence, the operator L is decomposed according to the decomposition of the Hilbert space $\mathcal{H} = \mathcal{N} \oplus \mathcal{M}$ into parts living on $\mathcal{M}$ and $\mathcal{N}$, respectively, see [52], p. 172. In particular, the operator on the stable subspace is denoted by $(L_\mathcal{N}, \mathcal{D}(L_\mathcal{N}))$ where

$$L_\mathcal{N}\mathbf{u} := L\mathbf{u}, \quad \mathcal{D}(L_\mathcal{N}) = \mathcal{D}(L) \cap \mathcal{N},$$

$L_\mathcal{M}$ is then defined analogously. Since $\mathcal{N}$ and $\mathcal{M}$ are closed subspaces $L_\mathcal{N}$ and $L_\mathcal{M}$ can be regarded as linear operators on the Hilbert spaces $\mathcal{N}$ and $\mathcal{M}$, respectively. For the spectrum, Theorem 6.17 in [52], p. 178, yields

$$\sigma(L_\mathcal{M}) = \{1\}, \quad \sigma(L_\mathcal{N}) = \sigma(L)\backslash\{1\}.$$

Since the projection P commutes with the semigroup the subspace $\mathcal{N}$ is invariant under $S(\tau)$. In particular, $S_\mathcal{N}(\tau) := S|_\mathcal{N}(\tau)$ is the subspace semigroup generated by $L_\mathcal{N}$, see also Section 4.5 in [28] for a detailed discussion. In regard to the dimension of the unstable subspace, we only know that $\ker(1-L) \subset \text{rg}P$. Therefore, the following result is crucial.

Lemma 4.1.7. *The unstable subspace $\mathcal{M}$ is spanned by the symmetry mode, i.e., $P\mathcal{H} = \langle \mathbf{g} \rangle$ and the algebraic multiplicity of $1 \in \sigma_p(L)$ is one.*

Proof. The case $\dim \mathcal{M} = \infty$ can be ruled out by an abstract argument: if $\dim \mathcal{M} = \infty$ then, by [52], p. 239, Theorem 5.28, 1 would belong to the essential spectrum of L which is stable under compact perturbations (see [52] p. 244, Theorem 5.35). However, $1 \notin \sigma(L_0)$ and this yields a contradiction. We conclude that $L_{\mathcal{M}}$ is in fact a finite-dimensional operator.
Since 1 is an eigenvalue of $L_{\mathcal{M}}$ and, according to Lemma 4.1.6, the corresponding geometric eigenspace is spanned by $\mathbf{g}$, we obtain $\mathbf{g} \in \mathcal{M}$ and thus, $\langle \mathbf{g} \rangle \subset \mathcal{M}$. It remains to prove the reverse inclusion. Note that $(1 - L_{\mathcal{M}})$ is nilpotent since 0 is the only eigenvalue, i.e., there exists an $m \in \mathbb{N}$ such that

$$(1 - L_{\mathcal{M}})^m \mathbf{u} = 0$$

for arbitrary $\mathbf{u} \in \mathcal{M}$. If $m = 1$ then $\mathcal{M} \subset \ker(1 - L_{\mathcal{M}}) = \langle \mathbf{g} \rangle$ and we are done. Suppose that $m \geq 2$. Then there exists a nontrivial $\mathbf{v} \in \operatorname{rg}(1 - L_{\mathcal{M}})$ such that $(1 - L_{\mathcal{M}})\mathbf{v} = 0$, i.e., $\mathbf{v} \in \ker(1 - L_{\mathcal{M}})$ and $\mathbf{v}$ must therefore be a multiple of the symmetry mode. This shows that there exists a $\mathbf{u} \in \mathcal{D}(L_{\mathcal{M}})$ with $(1 - L_{\mathcal{M}})\mathbf{u} = c\mathbf{g}$. We will show that this leads to a contradiction.
We set $c = 1$ without loss of generality. Suppose there exists a function $\mathbf{u}$ in $\mathcal{D}(L)$ such that $(1 - L)\mathbf{u} = \mathbf{g}$. Then, by Lemma 4.1.4,

$$-(1 - \rho^2)u''(\rho) + \tfrac{2(p+1)}{p-1}\rho u'(\rho) - \tfrac{2(p+1)}{p-1}u(\rho) = g(\rho) \tag{4.10}$$

for $u(\rho) := \int_0^\rho u_2(s)ds$ and

$$g(\rho) := g_1(\rho) + \rho g_2(\rho) + \tfrac{p+1}{p-1}\int_0^\rho g_2(s)ds = \tfrac{3p+1}{p-1}\rho.$$

For the homogeneous equation we have the fundamental system $\{h_0, h_1\}$ introduced in the proof of Lemma 4.1.6, where

$$h_0(\rho) = \rho, \quad h_1(\rho) = (1 - \rho^2)^{-\frac{2}{p-1}}\tilde{h}_1(\rho)$$

with $\tilde{h}_1$ continuous on $[0, 1]$ and $\tilde{h}_1(0) \neq 0$. The Wronskian is given by

$$W(h_0, h_1) = -(1 - \rho^2)^{-\frac{p+1}{p-1}}$$

and thus, a solution of the inhomogeneous equation must be of the form

$$\begin{aligned} u(\rho) = c_0 h_0(\rho) + c_1 h_1(\rho) - h_0(\rho) \int_{\rho_0}^\rho h_1(s)g(s)(1 - s^2)^{\frac{2}{p-1}}ds \\ + h_1(\rho) \int_{\rho_1}^\rho h_0(s)g(s)(1 - s^2)^{\frac{2}{p-1}}ds \end{aligned}$$

for some constants $c_0, c_1 \in \mathbb{C}$ and $\rho_0, \rho_1 \in [0,1]$. The boundary condition at $\rho = 0$ implies

$$c_1 = -\int_{\rho_1}^{0} h_0(s)g(s)(1-s^2)^{\frac{2}{p-1}}ds$$

and inserting the definitions of h_0, h_1 and g yields

$$u(\rho) = c_0\rho - \tfrac{3p+1}{p-1}\rho\int_{\rho_0}^{\rho} s\tilde{h}_1(s)ds + \tfrac{3p+1}{p-1}(1-\rho^2)^{-\frac{2}{p-1}}\tilde{h}_1(\rho)\int_0^{\rho} s^2(1-s^2)^{\frac{2}{p-1}}ds.$$

Since u belongs to $C[0,1]$ (Lemma 4.1.4), we must have

$$\int_0^1 s^2(1-s^2)^{\frac{2}{p-1}}ds = 0.$$

However, this is impossible since the integrand is strictly positive for all $s \in (0,1)$. $\square$

In order to improve the growth estimate in Lemma 4.1.3 we apply a well-known theorem by Gearhart, Prüss and Greiner. To this end we need the following result, which states that the resolvent is uniformly bounded in some right half plane. In the following we set

$$H_a := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq a\}$$

for $a \in \mathbb{R}$.

Lemma 4.1.8. *For any $\varepsilon > 0$ there exist constants $c_1, c_2 > 0$ such that*

$$\|R_L(\lambda)\| \leq c_1$$

for all $\lambda \in H_{\tilde{\omega}_p+\varepsilon}$ with $|\lambda| \geq c_2$.

Proof. Fix $\varepsilon > 0$ and let $\lambda \in H_{\tilde{\omega}_p+\varepsilon}$ where $\lambda \notin \{1 - \frac{2}{p-1}, 1\}$. We use the identity

$$R_L(\lambda) = R_{L_0}(\lambda)[1 - L'R_{L_0}(\lambda)]^{-1}$$

to obtain uniform bounds on the resolvent for $|\lambda|$ large. By definition of L' we have

$$L'R_{L_0}(\lambda)\mathbf{f} = \begin{pmatrix} p\kappa_p K[R_{L_0}(\lambda)\mathbf{f}]_2 \\ 0 \end{pmatrix}$$

where $K : L^2(0,1) \to L^2(0,1)$ is defined by $Ku(\rho) = \int_0^{\rho} u(s)ds$. For $\mathbf{f} \in \mathcal{H}$ consider the equation $(\lambda - L_0)\mathbf{u} = \mathbf{f}$. Its solution is given by $\mathbf{u} = R_{L_0}(\lambda)\mathbf{f}$. Lemma 4.1.4 yields

$$[R_{L_0}(\lambda)\mathbf{f}]_1(\rho) = (\lambda - 1 + \tfrac{2}{p-1})K[R_{L_0}(\lambda)\mathbf{f}]_2(\rho) + \rho[R_{L_0}(\lambda)\mathbf{f}]_2(\rho) - Kf_2(\rho).$$

The estimate in Lemma 4.1.2 implies

$$\|[R_{L_0}(\lambda)\mathbf{f}]_j\|_{L^2(0,1)} \leq \|R_{L_0}(\lambda)\mathbf{f}\| \leq \frac{\|\mathbf{f}\|}{|\mathrm{Re}\lambda - \tilde{\omega}_p|}$$

for $j = 1, 2$ and we obtain

$$\|K[R_{L_0}(\lambda)\mathbf{f}]_2\|_{L^2(0,1)} \lesssim \frac{\|\mathbf{f}\|}{|\lambda - 1 + \frac{2}{p-1}|}.$$

Thus, for $|\lambda|$ sufficiently large, the Neumann series

$$[1 - L'R_{L_0}(\lambda)]^{-1} = \sum_{k=0}^{\infty}[L'R_{L_0}(\lambda)]^k$$

converges and the claim follows. □

We conclude the linear perturbation theory with an estimate of the linear evolution on the stable subspace.

Proposition 4.1.9. *Let P be the spectral projection defined in Eq.* (4.9) *and set*

$$\omega_p := \max\left\{-1, \tfrac{1}{2} - \tfrac{2}{p-1}\right\}.$$

Then, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that the semigroup $S(\tau)$ given in Corollary 4.1.3 satisfies

$$\|S(\tau)(1-P)\mathbf{f}\| \leq C_\varepsilon e^{(-|\omega_p|+\varepsilon)\tau}\|(1-P)\mathbf{f}\| \tag{4.11}$$

for all $\tau \geq 0$ and $\mathbf{f} \in \mathcal{H}$. Furthermore, $S(\tau)P\mathbf{f} = e^\tau P\mathbf{f}$.

Proof. The operator $L_{\mathcal{N}}$ is the generator of the subspace semigroup $S|_{\mathcal{N}}(\tau)$ and its resolvent is given by $R_L(\lambda)|_{\mathcal{N}}$. The first estimate follows from the uniform boundedness of the resolvent in $H_{\omega_p+\varepsilon}$ (Lemma 4.1.8) and the theorem by Gearhart, Prüss and Greiner see Theorem A.2.1 in Appendix A.2. The second assertion follows from $P\mathcal{H} = \langle \mathbf{g} \rangle$ and the fact that $\mathbf{g}$ is an eigenfunction of the linear operator L with eigenvalue 1. □

Nonlinear perturbation theory

Now we turn to the full nonlinear problem. The following two lemmas will be used frequently.

Lemma 4.1.10. *If $u \in L^2(0,1)$ then $\tilde{u}$, defined by $\tilde{u}(\rho) := \frac{1}{\sqrt{\rho}}\int_0^\rho u(s)ds$, belongs to $L^\infty(0,1)$ and satisfies*

$$\|\tilde{u}\|_{L^\infty(0,1)} \leq \|u\|_{L^2(0,1)}.$$

Proof. First note that $\rho \mapsto \int_0^\rho u(s)ds$ is a continuous function on $[0,1]$ for $u \in L^2(0,1)$. Using the Cauchy-Schwarz inequality we esimate

$$|\tilde{u}(\rho)| = \left| \frac{1}{\sqrt{\rho}} \int\limits_0^\rho u(s)ds \right| \leq \|u\|_{L^2(0,1)}$$

for $\rho \in (0,1]$. Taking the essential supremum yields the claim. □

We will also use Hardy's inequality in the following form.

Lemma 4.1.11. *For $u \in L^2(0,1)$ we have*

$$\int_0^1 \frac{1}{\rho^2} \left| \int_0^\rho u(s)ds \right|^2 d\rho \lesssim \int_0^1 |u(\rho)|^2 d\rho.$$

Estimates for the nonlinearity

From now on we restrict ourselves to real–valued functions. We introduce a function $n : \mathbb{R} \times [0,1] \to \mathbb{R}$ defined by

$$n(x,\rho) := \rho \Big(|\kappa_p^{\frac{1}{p-1}} + x|^{p-1} (\kappa_p^{\frac{1}{p-1}} + x) - p\kappa_p x - \kappa_p^{\frac{p}{p-1}} \Big),$$

cf. Eq. (4.2). It is easy to see that

$$|n(x,\rho)| \lesssim \begin{cases} \rho |x|^2 & |x| < 1 \\ \rho |x|^p & |x| \geq 1. \end{cases}$$

A convenient way to write this is $|n(x,\rho)| \lesssim \rho |x|^2 \langle x \rangle^{p-2}$ with the "japanese bracket" $\langle x \rangle := \sqrt{1+|x|^2}$. In the following we denote by B_1 and $\mathcal{B}_1$ the open unit balls in $L^2(0,1)$ and $\mathcal{H}$, respectively. To (formally) define the nonlinearity we introduce an operator $A : L^2(0,1) \to L^2(0,1)$,

$$Au(\rho) := \frac{1}{\rho} \int_0^\rho u(s)ds.$$

An application of Hardy's inequality shows that A is bounded. We set

$$N(u)(\rho) := n(Au(\rho), \rho).$$

Lemma 4.1.12. *The operator N maps $L^2(0,1)$ into $L^2(0,1)$. Furthermore, there exist constants $c_1, c_2 > 0$ such that for $u, v \in B_1$*

$$\|N(u)\|_{L^2} \leq c_1 \|u\|_{L^2}^2$$

and

$$\|N(u) - N(v)\|_{L^2} \leq c_2 (\|u\|_{L^2} + \|v\|_{L^2}) \|u - v\|_{L^2}.$$

Proof. Note that for $1 < p \leq 3$ the function n defined as above is at least once continuously differentiable with respect to x and we have the bound

$$|\partial_1 n(x,\rho)| \lesssim \rho |x| \langle x \rangle^{p-2}$$

for all $x \in \mathbb{R}$ and $\rho \in [0,1]$ which, in particular, implies $\partial_1 n(0,\rho) = 0$ and hence, $N(0) = 0$.
By the fundamental theorem of calculus we infer that for $x, y \in \mathbb{R}$

$$\begin{aligned}
|n(x,\rho) - n(y,\rho)| &\leq |x-y| \int_0^1 |\partial_1 n(y + h(x-y), \rho)| dh \\
&\lesssim \rho |x-y| \int_0^1 |y + h(x-y)| \langle y + h(x-y) \rangle^{p-2} dh \\
&\lesssim \rho |x-y| \begin{cases} |x| + |y| & p \in (1,2] \\ |x|\langle x \rangle^{p-2} + |y| \langle y \rangle^{p-2} & p \in (2,3] \end{cases} .
\end{aligned}$$

Now we prove the estimate for the nonlinear operator N. The following argument works only for $1 < p \leq 3$, since for higher exponents the singular factors at $\rho = 0$ can no longer be controlled. For $u, v \in L^2(0,1)$ we write $\tilde{u}(\rho) := \int_0^\rho u(s) ds$ and $\tilde{v}(\rho) := \int_0^\rho v(s) ds$. We distinguish two cases.
If $p \in (1,2]$ we readily estimate

$$\begin{aligned}
\|N(u) - N(v)\|_{L^2}^2 &= \int_0^1 |n(Au(\rho), \rho) - n(Av(\rho), \rho)|^2 d\rho \\
&\lesssim \int_0^1 \rho^2 |Au(\rho) - Av(\rho)|^2 (|Au(\rho)|^2 + |Av(\rho)|^2) d\rho \\
&\lesssim (\|\tilde{u}\|_{L^\infty}^2 + \|\tilde{v}\|_{L^\infty}^2) \int_0^1 |Au(\rho) + Av(\rho)|^2 d\rho \\
&\lesssim (\|u\|_{L^2}^2 + \|v\|_{L^2}^2) \|u - v\|_{L^2}^2
\end{aligned}$$

by Lemma 4.1.10 and Hardy's inequality. On the other hand, if $p \in (2,3]$, we proceed similarly and obtain

$$\begin{aligned}
&\|N(u) - N(v)\|_{L^2}^2 \lesssim \int_0^1 \rho^2 |Au(\rho) - Av(\rho)|^2 \\
&\quad \times (|Au(\rho)|^2 \langle Au(\rho) \rangle^{2(p-2)} + |Av(\rho)|^2 \langle Av(\rho) \rangle^{2(p-2)}) d\rho \\
&\lesssim \int_0^1 \rho^{3-p} |Au(\rho) - Av(\rho)|^2 \\
&\quad \times \left(|\rho^{-\frac{1}{2}} \tilde{u}(\rho)|^2 \langle \rho^{-\frac{1}{2}} \tilde{u}(\rho) \rangle^{2(p-2)} + |\rho^{-\frac{1}{2}} \tilde{v}(\rho)|^2 \langle \rho^{-\frac{1}{2}} \tilde{v}(\rho) \rangle^{2(p-2)} \right) d\rho \\
&\lesssim \left(\|u\|_{L^2}^2 \langle \|u\|_{L^2} \rangle^{2(p-2)} + \|v\|_{L^2}^2 \langle \|v\|_{L^2} \rangle^{2(p-2)} \right) \|u - v\|_{L^2}^2
\end{aligned}$$

again by Lemma 4.1.10 and Hardy's inequality. Since $N(0) = 0$ we immediately conclude the boundedness of N on $L^2(0,1)$.

In particular, we have $\|N(u)\|_{L^2} \lesssim \|u\|_{L^2}^2$ for $u \in B_1$. For $u, v \in B_1$ the above estimates yield

$$\|N(u) - N(v)\|_{L^2} \lesssim (\|u\|_{L^2} + \|v\|_{L^2})\|u - v\|_{L^2}$$

as claimed. $\square$

Finally for $\mathbf{u} = (u_1, u_2)^T \in \mathcal{H}$ we define the vector valued nonlinearity by

$$\mathbf{N}(\mathbf{u}) := \begin{pmatrix} N(u_2) \\ 0 \end{pmatrix}.$$

Lemma 4.1.13. *The nonlinearity* $\mathbf{N}$ *maps* $\mathcal{H}$ *into* $\mathcal{H}$, $\mathbf{N}(\mathbf{0}) = \mathbf{0}$ *and there exist constants* $c_1, c_2 > 0$ *such that for* $\mathbf{u}$,$\mathbf{v} \in \mathcal{B}_1$

$$\|\mathbf{N}(\mathbf{u})\| \leq c_1 \|\mathbf{u}\|^2$$

and

$$\|\mathbf{N}(\mathbf{u}) - \mathbf{N}(\mathbf{v})\| \leq c_2(\|\mathbf{u}\| + \|\mathbf{v}\|)\|\mathbf{u} - \mathbf{v}\|.$$

Furthermore, $\mathbf{N}$ *is Fréchet differentiable at* $\mathbf{0}$ *and* $D\mathbf{N}(\mathbf{0}) = \mathbf{0}$.

Proof. For $\mathbf{u}, \mathbf{v} \in \mathcal{B}_1$ we apply the result of Lemma 4.1.12 to obtain

$$\begin{aligned}\|\mathbf{N}(\mathbf{u}) - \mathbf{N}(\mathbf{v})\|^2 &= \|N(u_2) - N(v_2)\|_{L^2}^2 \lesssim (\|u_2\|_{L^2}^2 + \|v_2\|_{L^2}^2)\|u_2 - v_2\|_{L^2}^2 \\ &\lesssim (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)\|\mathbf{u} - \mathbf{v}\|^2.\end{aligned}$$

This implies

$$\|\mathbf{N}(\mathbf{u}) - \mathbf{N}(\mathbf{v})\| \lesssim (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)^{\frac{1}{2}}\|\mathbf{u} - \mathbf{v}\| \lesssim (\|\mathbf{u}\| + \|\mathbf{v}\|)\|\mathbf{u} - \mathbf{v}\|.$$

We have $\mathbf{N}(\mathbf{0}) = \mathbf{0}$ which implies $\|\mathbf{N}(\mathbf{v})\| \lesssim \|\mathbf{v}\|^2$. In particular, there exists a constant c independent of $\mathbf{v}$ such that

$$\frac{\|\mathbf{N}(\mathbf{v})\|}{\|\mathbf{v}\|} \leq c\|\mathbf{v}\|.$$

Since the left hand side vanishes in the limit $\mathbf{v} \to \mathbf{0}$, we infer that $\mathbf{N}$ is Fréchet differentiable at zero with $D\mathbf{N}(\mathbf{0})$=$\mathbf{0}$. $\square$

Abstract formulation of the nonlinear equation

We turn to the full nonlinear problem and write Eq. (3.13) as an ordinary differential equation on $\mathcal{H}$. With the nonlinearity defined as above it reads

$$\begin{cases} \frac{d}{d\tau}\Phi(\tau) = L\Phi(\tau) + \mathbf{N}(\Phi(\tau)) \text{ for } \tau > -\log T \\ \Phi(-\log T) = \mathbf{u} \end{cases} \tag{4.12}$$

for $\Phi : [-\log T, \infty) \to \mathcal{H}$ and initial data $\mathbf{u} \in \mathcal{H}$. We rewrite the above system as an integral equation,

$$\Phi(\tau) = S(\tau + \log T)\mathbf{u} + \int_{-\log T}^{\tau} S(\tau - \tau')\mathbf{N}(\Phi(\tau'))d\tau' \quad \text{for} \quad \tau \geq -\log T,$$

i.e., we are looking for mild solutions of Eq. (4.12). In order to remove the dependence of the equation on the blow up time T we introduce a new variable $\Psi : [0, \infty) \to \mathcal{H}$ defined by

$$\Psi(\tau) := \Phi(\tau - \log T)$$

such that the above integral equation is now equivalent to

$$\Psi(\tau) = S(\tau)\mathbf{u} + \int_0^{\tau} S(\tau - \tau')\mathbf{N}(\Psi(\tau'))d\tau' \quad \text{for} \quad \tau \geq 0. \tag{4.13}$$

We study this equation on a Banach space $\mathcal{X}$ defined as

$$\mathcal{X} := \left\{ \Psi \in C([0, \infty), \mathcal{H}) : \sup_{\tau > 0} e^{\mu_p \tau} \|\Psi(\tau)\| < \infty \right\}$$

with norm

$$\|\Psi\|_{\mathcal{X}} := \sup_{\tau > 0} e^{\mu_p \tau} \|\Psi(\tau)\|$$

where

$$\mu_p := |\omega_p| - \varepsilon = \min\{1, \tfrac{2}{p-1} - \tfrac{1}{2}\} - \varepsilon,$$

cf. Proposition 4.1.9, where $\varepsilon > 0$ is arbitrary but fixed and without loss of generality we assume ε so small that $\mu_p > 0$. In the following, estimate (4.11) will be used frequently, hence most of the constants will depend on ε. However, for notational convenience we will only indicate this dependence in the proof of the main result.

Global existence for corrected (small) initial data

We follow the strategy of [22]. First we study the following equation,

$$\begin{aligned}\Psi(\tau) = S(\tau)(1 - P)\mathbf{u} - \int_0^{\infty} e^{\tau - \tau'} P\mathbf{N}(\Psi(\tau'))d\tau' \\ + \int_0^{\tau} S(\tau - \tau')\mathbf{N}(\Psi(\tau'))d\tau'.\end{aligned} \tag{4.14}$$

This is the original equation modified by a correction term in order to suppress the instability coming from the symmetry mode. We use a fixed point argument to show existence of solutions of Eq. (4.14). In a further step we account for the time translation symmetry of the problem and show that the correction can be annihilated by adjusting the blow up time T (which is now encoded in the initial data) such that we end up with a solution of Eq. (4.13). For $\delta > 0$ we define $\mathcal{X}_\delta \subset \mathcal{X}$ by

$$\mathcal{X}_\delta := \{\Psi \in \mathcal{X} : \|\Psi\|_{\mathcal{X}} \leq \delta\}.$$

Lemma 4.1.14. *For $0 < \delta < 1$ there exist constants $c_1, c_2 > 0$ such that*

$$\|\mathbf{N}(\Psi(\tau))\| \leq c_1 \delta^2 e^{-2\mu_p \tau}$$

and

$$\|\mathbf{N}(\Psi(\tau)) - \mathbf{N}(\Phi(\tau))\| \leq c_2 \delta e^{-\mu_p \tau} \|\Psi(\tau) - \Phi(\tau)\|$$

for $\Phi, \Psi \in \mathcal{X}_\delta$ and $\tau > 0$.

Proof. Let $\Psi \in \mathcal{X}_\delta$. Then $\|\Psi(\tau)\| \leq \delta e^{-\mu_p \tau} < 1$ for all $\tau > 0$ and $\delta < 1$. Lemma 4.1.13 implies that there exists a constant $c_1 > 0$ such that

$$\|\mathbf{N}(\Psi(\tau))\| \leq c_1 \|\Psi(\tau)\|^2 \leq c_1 \delta^2 e^{-2\mu_p \tau}.$$

Let $\Phi \in \mathcal{X}_\delta$. Then there exists a constant $c_2 > 0$ such that

$$\begin{aligned}\|\mathbf{N}(\Psi(\tau)) - \mathbf{N}(\Phi(\tau))\| &\leq \frac{c_2}{2}(\|\Psi(\tau)\| + \|\Phi(\tau)\|)\|\Psi(\tau) - \Phi(\tau)\| \\ &\leq c_2 \delta e^{-\mu_p \tau} \|\Psi(\tau) - \Phi(\tau)\|\end{aligned}$$

which implies the second estimate. □

We abbreviate the right hand side of Eq. (4.14) by defining the operator

$$\begin{aligned}\mathbf{K}(\Psi, \mathbf{u})(\tau) :=& S(\tau)(1 - P)\mathbf{u} - \int_0^\infty e^{\tau - \tau'} P\mathbf{N}(\Psi(\tau'))d\tau' \\ &+ \int_0^\tau S(\tau - \tau')\mathbf{N}(\Psi(\tau'))d\tau'.\end{aligned} \tag{4.15}$$

Lemma 4.1.15. *For $\delta > 0$ sufficiently small and fixed $\mathbf{u} \in \mathcal{H}$, with $\|\mathbf{u}\| \leq \delta^2$, the operator $\mathbf{K}$ maps $\mathcal{X}_\delta$ into itself and is contracting, in particular*

$$\|\mathbf{K}(\Phi, \mathbf{u}) - \mathbf{K}(\Psi, \mathbf{u})\|_{\mathcal{X}} \leq \frac{1}{2}\|\Phi - \Psi\|_{\mathcal{X}}$$

for $\Phi, \Psi \in \mathcal{X}_\delta$.

Proof. For fixed $(\Psi, \mathbf{u})$ with $\Psi \in \mathcal{X}_\delta$ and $\mathbf{u} \in \mathcal{H}$ the integrals occuring in the operator $\mathbf{K}$ can be viewed as Riemann integrals over continuous functions, which exist since $\|P\mathbf{N}(\Psi(\tau))\| \lesssim 1$ by Lemma 4.1.14. To see that $\mathbf{K}(\Psi, \mathbf{u}) \in \mathcal{X}_\delta$ for δ small enough we decompose the operator according to

$$\mathbf{K}(\Psi, \mathbf{u}) = P\mathbf{K}(\Psi, \mathbf{u}) + (1 - P)\mathbf{K}(\Psi, \mathbf{u}).$$

We apply the results of Proposition 4.1.9 and Lemma 4.1.14. Let $\|\mathbf{u}\| \leq \delta^2$. Then for $\tau \geq 0$ we obtain

$$\begin{aligned}&\|P\mathbf{K}(\Psi, \mathbf{u})(\tau)\| = \left\| - \int_0^\infty e^{\tau - \tau'} P\mathbf{N}(\Psi(\tau'))d\tau' + \int_0^\tau S(\tau - \tau')P\mathbf{N}(\Psi(\tau'))d\tau' \right\| \\ &\leq \int_\tau^\infty e^{\tau - \tau'} \|P\mathbf{N}(\Psi(\tau'))\| d\tau' \leq c_1 \delta^2 \int_\tau^\infty e^{\tau - \tau'(1 + 2\mu_p)} d\tau' \lesssim \delta^2 e^{-2\mu_p \tau},\end{aligned}$$

and

$$\begin{aligned}\|(1-P)\mathbf{K}(\Psi,\mathbf{u})(\tau)\| &\le \|S(\tau)(1-P)\mathbf{u}\| + \int_0^\tau \|S(\tau-\tau')(1-P)\mathbf{N}(\Psi(\tau'))\|\,d\tau' \\ &\lesssim e^{-\mu_p\tau}\|\mathbf{u}\| + \int_0^\tau e^{-\mu_p(\tau-\tau')}\|\mathbf{N}(\Psi(\tau'))\|d\tau' \\ &\lesssim \delta^2 e^{-\mu_p\tau} + \delta^2\int_0^\tau e^{-\mu_p(\tau+\tau')}d\tau' \lesssim \delta^2 e^{-\mu_p\tau}.\end{aligned}$$

We infer that there exist constants $c_1, c_2 > 0$ such that

$$\|P\mathbf{K}(\Psi,\mathbf{u})(\tau)\| \le c_1\delta^2 e^{-\mu_p\tau},$$

and

$$\|(1-P)\mathbf{K}(\Psi,\mathbf{u})(\tau)\| \le c_2\delta^2 e^{-\mu_p\tau}.$$

Thus for $\delta \le \min\{1, \frac{1}{2c_1}, \frac{1}{2c_2}\}$ we obtain

$$\begin{aligned}\|\mathbf{K}(\Psi,\mathbf{u})(\tau)\| &\le \|P\mathbf{K}(\Psi,\mathbf{u})(\tau)\| + \|(1-P)\mathbf{K}(\Psi,\mathbf{u})(\tau)\| \\ &\le \tfrac{\delta}{2}e^{-\mu_p\tau} + \tfrac{\delta}{2}e^{-\mu_p\tau} \le \delta e^{-\mu_p\tau}.\end{aligned}$$

Continuity of $\mathbf{K}(\Psi,\mathbf{u})(\tau)$ as a function of τ follows essentially from strong continuity of the semigroup (cf. Lemma 3.10 in [22]). It is left to show that $\mathbf{K}$ is contracting. Let $\Psi, \Phi \in \mathcal{X}_\delta$. Then

$$\begin{aligned}\|P\mathbf{K}(\Phi,\mathbf{u})(\tau) - P\mathbf{K}(\Psi,\mathbf{u})(\tau)\| &\le \int_\tau^\infty e^{\tau-\tau'}\|P\mathbf{N}(\Phi(\tau')) - P\mathbf{N}(\Psi(\tau'))\|d\tau' \\ \lesssim \delta\int_\tau^\infty e^{\tau-\tau'(1+\mu_p)}\|\Phi(\tau')-\Psi(\tau')\|d\tau' & \\ \lesssim \delta\sup_{\sigma>\tau} e^{\mu_p\sigma}\|\Phi(\sigma)-\Psi(\sigma)\|\int_\tau^\infty e^{\tau-\tau'(1+2\mu_p)}d\tau' &\lesssim \delta e^{-2\mu_p\tau}\|\Phi-\Psi\|_{\mathcal{X}}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\|(1-P)\mathbf{K}(\Phi,\mathbf{u})(\tau) - (1-P)\mathbf{K}(\Psi,\mathbf{u})(\tau)\| \\ &\le \int_0^\tau \|S(\tau-\tau')(1-P)(\mathbf{N}(\Phi(\tau')) - \mathbf{N}(\Psi(\tau')))\|d\tau' \\ &\lesssim \int_0^\tau e^{-\mu_p(\tau-\tau')}\|\mathbf{N}(\Phi(\tau')) - \mathbf{N}(\Psi(\tau'))\|d\tau' \lesssim \delta\int_0^\tau e^{-\mu_p\tau}\|\Phi(\tau')-\Psi(\tau')\|d\tau' \\ &\lesssim \delta\sup_{\sigma\in(0,\tau)} e^{\mu_p\sigma}\|\Phi(\sigma)-\Psi(\sigma)\|\int_0^\tau e^{-\mu_p(\tau+\tau')}d\tau' \lesssim \delta e^{-\mu_p\tau}\|\Phi-\Psi\|_{\mathcal{X}}.\end{aligned}$$

This shows that for δ sufficiently small,

$$\sup_{\tau>0} e^{\mu_p\tau}\|P\mathbf{K}(\Phi,\mathbf{u})(\tau) - P\mathbf{K}(\Psi,\mathbf{u})(\tau)\| \le \frac{1}{4}\|\Phi-\Psi\|_{\mathcal{X}},$$

and

$$\sup_{\tau>0} e^{\mu_p \tau} \|(1-P)\mathbf{K}(\Phi, \mathbf{u})(\tau) - (1-P)\mathbf{K}(\Psi, \mathbf{u})(\tau)\| \leq \frac{1}{4}\|\Phi - \Psi\|_{\mathcal{X}},$$

which implies the claim. □

Theorem 4.1.16. *For $\mathbf{u} \in \mathcal{B}_1 \subset \mathcal{H}$ sufficiently small, there exists a unique solution $\Psi(\cdot;\mathbf{u}) \in \mathcal{X}$ of*

$$\Psi(\cdot;\mathbf{u}) = \mathbf{K}(\Psi(\cdot;\mathbf{u}), \mathbf{u}). \tag{4.16}$$

Moreover, the map $\mathbf{\Psi} : \mathcal{U} \subset \mathcal{B}_1 \to \mathcal{X}$ defined by $\mathbf{\Psi}(\mathbf{u}) = \Psi(\cdot;\mathbf{u})$ is continuous and Fréchet differentiable at $\mathbf{u} = \mathbf{0}$ where $\mathcal{U}$ denotes a sufficiently small open neighborhood of zero in $\mathcal{H}$.

Proof. Lemma 4.1.15 and the fact that $\mathcal{X}_\delta$ is a closed subset yield a unique fixed point of Eq. (4.16) in $\mathcal{X}_\delta$. That this is indeed the unique solution in the whole space $\mathcal{X}$ follows by standard arguments (see also the proof of Theorem 4.1.20). Note that for $\mathbf{u}, \mathbf{v} \in \mathcal{U}$ we have $\mathbf{\Psi}(\mathbf{u}), \mathbf{\Psi}(\mathbf{v}) \in \mathcal{X}_\delta$ and

$$\begin{aligned}
&\|\mathbf{\Psi}(\mathbf{u}) - \mathbf{\Psi}(\mathbf{v})\|_{\mathcal{X}} = \|\mathbf{K}(\mathbf{\Psi}(\mathbf{u}), \mathbf{u}) - \mathbf{K}(\mathbf{\Psi}(\mathbf{v}), \mathbf{v})\|_{\mathcal{X}} \\
&\leq \|\mathbf{K}(\mathbf{\Psi}(\mathbf{u}), \mathbf{u}) - \mathbf{K}(\mathbf{\Psi}(\mathbf{v}), \mathbf{u})\|_{\mathcal{X}} + \|\mathbf{K}(\mathbf{\Psi}(\mathbf{v}), \mathbf{u}) - \mathbf{K}(\mathbf{\Psi}(\mathbf{v}), \mathbf{v})\|_{\mathcal{X}}.
\end{aligned}$$

By Lemma 4.1.14,

$$\|\mathbf{K}(\mathbf{\Psi}(\mathbf{u}), \mathbf{u}) - \mathbf{K}(\mathbf{\Psi}(\mathbf{v}), \mathbf{u})\|_{\mathcal{X}} \leq \frac{1}{2}\|\mathbf{\Psi}(\mathbf{u}) - \mathbf{\Psi}(\mathbf{v})\|_{\mathcal{X}}.$$

Inserting the definition of $\mathbf{K}$ yields

$$\|\mathbf{K}(\mathbf{\Psi}(\mathbf{v}), \mathbf{u})(\tau) - \mathbf{K}(\mathbf{\Psi}(\mathbf{v}), \mathbf{v})(\tau)\| = \|S(\tau)(1-P)(\mathbf{u} - \mathbf{v})\| \leq e^{-\mu_p \tau}\|\mathbf{u} - \mathbf{v}\|$$

and we conclude that

$$\|\mathbf{\Psi}(\mathbf{u}) - \mathbf{\Psi}(\mathbf{v})\|_{\mathcal{X}} \lesssim \|\mathbf{u} - \mathbf{v}\|, \tag{4.17}$$

which implies continuity. We claim that the solution map $\mathbf{\Psi}$ is Fréchet differentiable at $\mathbf{u} = \mathbf{0}$. We define an auxiliary operator $\tilde{D}\mathbf{\Psi}(\mathbf{0}) : \mathcal{H} \to \mathcal{X}$ by $[\tilde{D}\mathbf{\Psi}(\mathbf{0})\mathbf{v}](\tau) := S(\tau)(1-P)\mathbf{v}$ for $\mathbf{v} \in \mathcal{H}$. It is obvious that this defines a bounded linear operator from $\mathcal{H}$ into $\mathcal{X}$. We show that it is indeed the Fréchet derivative, i.e.,

$$\lim_{\mathbf{v} \to \mathbf{0}} \frac{1}{\|\mathbf{v}\|}\|\mathbf{\Psi}(\mathbf{v}) - \mathbf{\Psi}(\mathbf{0}) - \tilde{D}\mathbf{\Psi}(\mathbf{0})\mathbf{v}\|_{\mathcal{X}} = 0.$$

Recall that $\mathbf{N}(\mathbf{0}) = \mathbf{0}$, hence $\mathbf{\Psi}(\mathbf{0}) = \mathbf{0}$ is a solution of Eq (4.16) for $\mathbf{u} = \mathbf{0}$. We assume that $\mathbf{v} \in \mathcal{U}$, such that $\mathbf{\Psi}(\mathbf{v}) = \mathbf{K}(\mathbf{\Psi}(\mathbf{v}), \mathbf{v})$.
Inserting the definition of $\mathbf{K}$ we compute

$$\begin{aligned}
&\mathbf{\Psi}(\mathbf{v})(\tau) - S(\tau)(1-P)\mathbf{v} = \\
&= \int_0^\tau S(\tau - \tau')\mathbf{N}(\mathbf{\Psi}(\mathbf{v})(\tau'))d\tau' - \int_0^\infty e^{\tau - \tau'} P\mathbf{N}(\mathbf{\Psi}(\mathbf{v})(\tau'))d\tau' =: G(\mathbf{\Psi}(\mathbf{v}))(\tau).
\end{aligned}$$

Again we write

$$G(\mathbf{\Psi}(\mathbf{v}))(\tau) = P[G(\mathbf{\Psi}(\mathbf{v}))(\tau)] + (1-P)[G(\mathbf{\Psi}(\mathbf{v}))(\tau)].$$

Estimate (4.17) and calculations similar to those in the proof of Lemma 4.1.15 yield

$$\|P[G(\mathbf{\Psi}(\mathbf{v}))(\tau)]\| \leq \int_\tau^\infty e^{\tau-\tau'}\|P\mathbf{N}(\mathbf{\Psi}(\mathbf{v})(\tau'))\|d\tau'$$
$$\leq \int_\tau^\infty e^{\tau-\tau'}\|\mathbf{\Psi}(\mathbf{v})(\tau')\|^2 d\tau' \lesssim \|\mathbf{v}\|^2 \int_\tau^\infty e^{\tau-\tau'(1+2\mu_p)}d\tau' \lesssim \|\mathbf{v}\|^2 e^{-2\mu_p\tau}.$$

Similarly,

$$\|(1-P)[G(\mathbf{\Psi}(\mathbf{v}))(\tau)]\| \leq \int_0^\tau \|S(\tau-\tau')(1-P)\mathbf{N}(\mathbf{\Psi}(\mathbf{v})(\tau'))\|d\tau'$$
$$\lesssim \int_0^\tau e^{-\mu_p(\tau-\tau')}\|\mathbf{N}(\mathbf{\Psi}(\mathbf{v})(\tau'))\|d\tau' \lesssim \|\mathbf{v}\|^2 e^{-\mu_p\tau}.$$

We infer that $\|G(\mathbf{\Psi}(\mathbf{v}))\|_{\mathcal{X}} \lesssim \|\mathbf{v}\|^2$ and thus $\lim_{\mathbf{v}\to\mathbf{0}} \frac{1}{\|\mathbf{v}\|}\|G(\mathbf{\Psi}(\mathbf{v}))\|_{\mathcal{X}} = 0$, which implies the claim. $\square$

Global existence for arbitrary (small) initial data

The aim of this section is to use the existence result of Theorem 4.1.16 to obtain a solution of the original wave equation for arbitrary initial data (close to ψ^T). Up to now we implicitly assumed the blow up time T to be fixed. However, arbitrary perturbations of the initial data will change the blow up time and we account for this fact by allowing T to vary. Recall that the initial data we want to prescribe are of the form

$$\Psi(0)(\rho) = \begin{pmatrix} \rho T^{\frac{p+1}{p-1}} g(T\rho) - \frac{2\rho}{p-1}\kappa_p^{\frac{1}{p-1}} \\ T^{\frac{2}{p-1}}\left(T\rho f'(T\rho) + f(T\rho)\right) - \kappa_p^{\frac{1}{p-1}} \end{pmatrix}, \tag{4.18}$$

see Eq. (3.13). We separate the dependence on T and the free data (f,g) by introducing

$$\mathbf{v}(\rho) := \begin{pmatrix} \rho g(\rho) - \frac{2\rho}{p-1}\kappa_p^{\frac{1}{p-1}} \\ \rho f'(\rho) + f(\rho) - \kappa_p^{\frac{1}{p-1}} \end{pmatrix}, \quad \kappa(\rho) := \kappa_p^{\frac{1}{p-1}} \begin{pmatrix} \frac{2\rho}{p-1} \\ 1 \end{pmatrix}, \tag{4.19}$$

which are the initial data relative to the fundamental self–similar solution for $T=1$. We rewrite the right hand side of (4.18) and define

$$\mathbf{U}(\mathbf{v},T)(\rho) := T^{\frac{2}{p-1}}[\mathbf{v}(T\rho) + \kappa(T\rho)] - \kappa(\rho).$$

The data have to be prescribed on the interval $[0,T]$ and we are confronted with the problem that we do not know T in advance. As in [22] the argument will be perturbative around $T = 1$ and therefore it suffices to restrict T to the interval $I = (\frac{1}{2}, \frac{3}{2})$. In the following we set

$$\tilde{\mathcal{H}} := L^2(0, \tfrac{3}{2}) \times L^2(0, \tfrac{3}{2}).$$

Lemma 4.1.17. *The function* $\mathbf{U} : \tilde{\mathcal{H}} \times I \to \mathcal{H}$ *is continuous and* $\mathbf{U}(\mathbf{0}, 1) = \mathbf{0}$. *Furthermore* $\mathbf{U}(\mathbf{0}, \cdot) : I \to \mathcal{H}$ *is Fréchet differentiable and*

$$[D_T \mathbf{U}(\mathbf{0}, T)|_{T=1} \lambda](\rho) = \tfrac{2\lambda}{p-1} \kappa_p^{\frac{1}{p-1}} \mathbf{g}(\rho),$$

where $\lambda \in \mathbb{R}$ *and* $\mathbf{g}$ *denotes the symmetry mode, cf. Eq.* (4.6)*.*

Proof. The proof of continuity is similar to the proof of Lemma 3.14 in [22]. We define $J : L^2(0, \frac{3}{2}) \times I \to L^2(0,1)$ by $J(v,T)(\rho) := v(T\rho)$. For fixed T the map $J(\cdot, T) : L^2(0, \frac{3}{2}) \to L^2(0,1)$ is Lipschitz-continuous since

$$\begin{aligned} \|J(v,T) - J(\tilde{v},T)\|^2_{L^2(0,1)} &= \int_0^1 |v(T\rho) - \tilde{v}(T\rho)|^2 d\rho \\ &= \frac{1}{T} \int_0^T |v(\rho) - \tilde{v}(\rho)|^2 d\rho \leq 2\|v - \tilde{v}\|^2_{L^2(0,\frac{3}{2})} \end{aligned}$$

and the continuity is uniform with respect to T. It is therefore sufficient to show that for fixed $v \in L^2(0, \frac{3}{2})$ the function $J(v, \cdot) : I \to L^2(0,1)$ is continuous. This can be seen by noting that for all $v, \tilde{v} \in L^2(0, \frac{3}{2})$ and $T, \tilde{T} \in I$

$$\begin{aligned} \|J(v,T) - J(v,\tilde{T})\|_{L^2(0,1)} &\leq \|J(v,T) - J(\tilde{v},T)\|_{L^2(0,1)} \\ &\quad + \|J(\tilde{v},T) - J(\tilde{v},\tilde{T})\|_{L^2(0,1)} + \|J(\tilde{v},\tilde{T}) - J(v,\tilde{T})\|_{L^2(0,1)} \\ &\lesssim \|v - \tilde{v}\|_{L^2(0,\frac{3}{2})} + \|J(\tilde{v},T) - J(\tilde{v},\tilde{T})\|_{L^2(0,1)}. \end{aligned}$$

Thus, for any given $\epsilon > 0$ we can find a $\tilde{v} \in C[0, \frac{3}{2}]$ such that

$$\|J(v,T) - J(v,\tilde{T})\|_{L^2(0,1)} < \frac{\epsilon}{2} + c \left(\int_0^1 |\tilde{v}(T\rho) - \tilde{v}(\tilde{T}\rho)|^2 d\rho \right)^{\frac{1}{2}} \tag{4.20}$$

for some constant $c > 0$ since $C[0, \frac{3}{2}]$ is dense in $L^2(0, \frac{3}{2})$. By the continuity of $\tilde{v}$, the integral vanishes in the limit $T \to \tilde{T}$. The above results imply continuity of $J : L^2(0, \frac{3}{2}) \times I \to L^2(0,1)$ and thus,

$$\mathbf{U}(\mathbf{v}, T) = \begin{pmatrix} T^{\frac{2}{p-1}} (J(v_1, T) + J(\kappa_1, T)) - \kappa_1 \\ T^{\frac{2}{p-1}} (J(v_2, T) + J(\kappa_2, T)) - \kappa_2 \end{pmatrix}$$

is continuous for $\mathbf{v} = (v_1, v_2)^T \in \tilde{\mathcal{H}}$ and $\kappa = (\kappa_1, \kappa_2)^T$ as defined in Eq. (4.19).

To show differentiability we set $\mathbf{v} = \mathbf{0}$ and consider $\mathbf{U}(\mathbf{0}, \cdot) : I \to \mathcal{H}$, which is given by

$$\mathbf{U}(\mathbf{0}, T)(\rho) = T^{\frac{2}{p-1}}\kappa(T\rho) - \kappa(\rho) = \kappa_p^{\frac{1}{p-1}} \begin{pmatrix} \frac{2\rho}{p-1}\left(T^{\frac{p+1}{p-1}} - 1\right) \\ T^{\frac{2}{p-1}} - 1 \end{pmatrix}$$

The map is obviously differentiable for all $T \in I$. Recalling the definition of the symmetry mode in Lemma 4.1.5 we obtain

$$D_T \mathbf{U}(\mathbf{0}, T)|_{T=1}\lambda](\rho) = \frac{2\lambda}{p-1}\kappa_p^{\frac{1}{p-1}} \begin{pmatrix} \frac{(p+1)}{p-1}\rho \\ 1 \end{pmatrix} = \frac{2\lambda}{p-1}\kappa_p^{\frac{1}{p-1}}\mathbf{g}(\rho)$$

for $\lambda \in \mathbb{R}$. □

With these technical results at hand we now turn to the original problem. In the previous section we showed existence of solutions for the modified integral equation (4.14) with initial data $\mathbf{u} \in \mathcal{U}$, where $\mathcal{U}$ denotes a sufficiently small neighborhood of $\mathbf{0} \in \mathcal{H}$. We rewrite the initial data in terms of T and $\mathbf{v}$ as defined in Eq. (4.19). Inserting in the definition yields $\mathbf{U}(\mathbf{0}, 1) = \mathbf{0}$. By continuity $\mathbf{U}(\mathbf{v}, T) \in \mathcal{U}$ provided that $(\mathbf{v}, T) \in \mathcal{V} \times \tilde{I}$ where $\mathcal{V}$ and $\tilde{I}$ are sufficiently small neighborhoods of $\mathbf{0} \in \tilde{\mathcal{H}}$ and $1 \in I$, respectively. By Theorem 4.1.16 there exists a solution $\mathbf{U}(\mathbf{v}, T) \mapsto \mathbf{\Psi}(\mathbf{U}(\mathbf{v}, T)) \in \mathcal{X}$. Recall that Eq. (4.14) is Eq. (4.13) modified by an exponential factor times the function $\mathbf{F} : \mathcal{V} \times \tilde{I} \to \langle \mathbf{g} \rangle$ defined by

$$\mathbf{F}(\mathbf{v}, T) := P\left(\mathbf{U}(\mathbf{v}, T) + \int_0^\infty e^{-\tau'}\mathbf{N}(\mathbf{\Psi}(\mathbf{U}(\mathbf{v}, T))(\tau'))d\tau'\right).$$

Evaluation yields $\mathbf{F}(\mathbf{0}, 1) = \mathbf{0}$, i.e., for $\mathbf{v} = \mathbf{0}$ and $T = 1$ the correction vanishes and $\mathbf{\Psi}(\mathbf{U}(\mathbf{0}, 1)) = \mathbf{0}$ is also a solution of Eq. (4.13). In the following we show that for every small $\mathbf{v}$ there exists a T close to one, such that this still holds true. We need the next lemma as a prerequisite.

Lemma 4.1.18. $\mathbf{F} : \mathcal{V} \times \tilde{I} \subset \tilde{H} \times I \to \langle \mathbf{g} \rangle$ *is continuous. Moreover* $\mathbf{F}(\mathbf{0}, \cdot) : \tilde{I} \to \langle \mathbf{g} \rangle$ *is Fréchet differentiable at* $T = 1$ *and*

$$D_T \mathbf{F}(\mathbf{0}, T)|_{T=1}\lambda = \tfrac{2\lambda}{p-1}\kappa_p^{\frac{1}{p-1}}\mathbf{g}$$

for $\lambda \in \mathbb{R}$.

Proof. To rewrite $\mathbf{F}$ in a more abstract way we introduce the integral operator $\mathbf{B} : \mathcal{X} \to \mathcal{H}, \Psi \mapsto \int_0^\infty e^{-\tau'}\Psi(\tau')d\tau'$, which is linear and bounded since

$$\|\mathbf{B}\Psi\| \leq \int_0^\infty e^{-\tau'}\|\Psi(\tau')\|d\tau' \leq \sup_{\tau'>0}\|\Psi(\tau')\| \leq \|\Psi\|_{\mathcal{X}}.$$

We define $\tilde{\mathbf{N}} : \mathcal{X} \to \mathcal{X}$ by $\tilde{\mathbf{N}}(\Psi)(\tau) := \mathbf{N}(\Psi(\tau))$. We claim that $\tilde{\mathbf{N}}$ is Fréchet differentiable at $\mathbf{0} \in \mathcal{X}$ and the Fréchet derivative at zero is given by $D\tilde{\mathbf{N}}(\mathbf{0})\Psi = \mathbf{0}$ for $\Psi \in \mathcal{X}$. This follow from $\tilde{\mathbf{N}}(\mathbf{0}) = \mathbf{0}$ and

$$\|\tilde{\mathbf{N}}(\Psi)\|_{\mathcal{X}} = \sup_{\tau>0} e^{\mu_p \tau} \|\mathbf{N}(\Psi(\tau))\| \lesssim \sup_{\tau>0} e^{\mu_p \tau} \|\Psi(\tau)\|^2 \lesssim \|\Psi\|_{\mathcal{X}}^2 \quad \text{for} \quad \Psi \in \mathcal{X}_\delta.$$

Thus

$$\frac{\|\tilde{\mathbf{N}}(\Psi)\|_{\mathcal{X}}}{\|\Psi\|_{\mathcal{X}}} \lesssim \|\Psi\|_{\mathcal{X}}$$

with a constant independent of Ψ, which implies the claim. Now

$$\mathbf{F}(\mathbf{v}, T) = P\left[\mathbf{U}(\mathbf{v}, T) + \mathbf{B}\tilde{\mathbf{N}}(\mathbf{\Psi}(\mathbf{U}(\mathbf{v}, T)))\right].$$

By Lemma 4.1.17 and the continuity of $\tilde{\mathbf{N}}$ and $\mathbf{\Psi}$, respectively, we see that $\mathbf{F}$ is continuous. To show differentiability we set $\mathbf{v} = \mathbf{0}$ and obtain

$$\mathbf{F}(\mathbf{0}, T) = P\left[\mathbf{U}(\mathbf{0}, T) + \mathbf{B}\tilde{\mathbf{N}}(\mathbf{\Psi}(\mathbf{U}(\mathbf{0}, T)))\right].$$

The right hand side is differentiable at $T = 1$ by Theorem 4.1.16, Lemma 4.1.17 and the above considerations. We conclude that

$$\begin{aligned} D_T\mathbf{F}(\mathbf{0}, T)\big|_{T=1}\lambda &= PD_T\mathbf{U}(\mathbf{0}, T)\big|_{T=1}\lambda \\ &\quad + P\mathbf{B}D\tilde{\mathbf{N}}(\mathbf{0})D\mathbf{\Psi}(\mathbf{0})D_T\mathbf{U}(\mathbf{0}, T)\big|_{T=1}\lambda \\ &= PD_T\mathbf{U}(\mathbf{0}, T)\big|_{T=1}\lambda = \tfrac{2\lambda}{p-1}\kappa_p^{\frac{1}{p-1}}\mathbf{g}. \end{aligned}$$

$\square$

Lemma 4.1.19. *Let $\tilde{\mathcal{V}} \subset \tilde{H}$ be a sufficiently small neighborhood of $\mathbf{0}$. For every $\mathbf{v} \in \tilde{\mathcal{V}}$ there exists a $T \in \tilde{I} \subset (\frac{1}{2}, \frac{3}{2})$, such that $\mathbf{F}(\mathbf{v}, T) = \mathbf{0}$.*

Proof. The range of $\mathbf{F}$ is contained in $\langle \mathbf{g} \rangle$, which is a one-dimensional vector space. Thus, there exists an isomorphism $i : \langle \mathbf{g} \rangle \to \mathbb{R}$ such that $i(c\mathbf{g}) = c$ for $c \in \mathbb{R}$. We set $f := i \circ \mathbf{F}$, where $f : \mathcal{V} \times \tilde{I} \to \mathbb{R}$ is continuous and $\mathbf{F}(\mathbf{0}, 1) = \mathbf{0}$ implies $f(\mathbf{0}, 1) = 0$. Lemma 4.1.18 shows that $f(\mathbf{0}, \cdot) : \tilde{I} \to \mathbb{R}$ is differentiable at $T = 1$ and $D_T f(\mathbf{0}, T)\big|_{T=1} \neq 0$. Consequently, there exist values $T_1, T_2 \in \tilde{I}$ such that $f(\mathbf{0}, T_1) > 0$ and $f(\mathbf{0}, T_2) < 0$. Continuity of f with respect to the first variable implies that there exists an open neighborhood $\tilde{\mathcal{V}} \subset \mathcal{V}$ such that $f(\mathbf{v}, T_1) > 0$ and $f(\mathbf{v}, T_2) < 0$ for $\mathbf{v} \in \tilde{\mathcal{V}}$. For $\mathbf{v} \in \tilde{\mathcal{V}}$ consider $f(\mathbf{v}, \cdot) : \tilde{I} \to \mathbb{R}$. By continuity of $f(\mathbf{v}, T)$ with respect to T and the intermediate value theorem we conclude that there exists a $T^* \in (T_1, T_2)$ such that

$$f(\mathbf{v}, T^*) = 0.$$

$\square$

This yields the next result.

Theorem 4.1.20. *Let $\mathbf{v} \in \tilde{H}$ be sufficiently small. Then there exists a T close to 1 such that*

$$\Psi(\tau) = S(\tau)\mathbf{U}(\mathbf{v}, T) + \int_0^\tau S(\tau - \tau')\mathbf{N}(\Psi(\tau'))d\tau', \quad \tau \geq 0 \tag{4.21}$$

has a continuous solution $\Psi : [0, \infty) \to \mathcal{H}$ satisfying

$$\|\Psi(\tau)\| \leq \delta e^{-\mu_p \tau}$$

for all $\tau \geq 0$ and some $\delta \in (0, 1)$. Moreover, this solution is unique in $C([0, \infty), \mathcal{H})$.

Proof. The existence of a solution $\Psi \in \mathcal{X}_\delta$ follows from the above considerations. Let $\Phi \in C([0, \infty), \mathcal{H})$ be another solution satisfying the same equation. We assume that $\Psi \neq \Phi$. By continuity, there exists an $\varepsilon \in (0, \frac{1-\delta}{2})$ and a $\tau_0 > 0$ such that

$$\varepsilon < \|\Psi(\tau_0) - \Phi(\tau_0)\|$$

and

$$\|\Psi(\tau) - \Phi(\tau)\| < 2\varepsilon, \quad \tau \in [0, \tau_0],$$

which yields $\|\Phi(\tau)\| < 1$. For $\tau \in [0, \tau_0]$ we obtain

$$\begin{aligned}\|\Psi(\tau) - \Phi(\tau)\| &\leq c \int_0^\tau e^{\tau - \tau'} \|\mathbf{N}(\Psi(\tau')) - \mathbf{N}(\Phi(\tau'))\| d\tau' \\ &\leq C(\tau_0)(e^\tau - 1) \sup_{\tau' \in [0, \tau]} \|\Psi(\tau') - \Phi(\tau')\|\end{aligned}$$

by applying Lemma 4.1.13. We infer that there exists a $\tau_1 \in (0, \tau_0]$ such that

$$\sup_{\tau \in [0, \tau_1]} \|\Psi(\tau) - \Phi(\tau)\| \leq \frac{1}{2} \sup_{\tau \in [0, \tau_1]} \|\Psi(\tau) - \Phi(\tau)\|$$

which implies $\Psi(\tau) = \Phi(\tau)$ for all $\tau \in [0, \tau_1]$. Iterating this argument yields $\Psi(\tau) = \Phi(\tau)$ for $\tau \in [0, \tau_0]$, which contradicts $\|\Psi(\tau_0) - \Phi(\tau_0)\| > \varepsilon$. ☐

Proposition 4.1.21. *(Global existence for arbitrary, small initial data) Let $\varepsilon > 0$ be small enough such that $\mu_p = |\omega_p| - \varepsilon > 0$. Let $\mathbf{v} \in L^2(0, \frac{3}{2}) \times L^2(0, \frac{3}{2})$ be sufficiently small. Then there exists a T close to 1 such that*

$$\Phi(\tau) = S(\tau + \log T)\mathbf{U}(\mathbf{v}, T) + \int_{-\log T}^\tau S(\tau - \tau')\mathbf{N}(\Phi(\tau'))d\tau' \tag{4.22}$$

has a continuous solution $\Phi : [-\log T, \infty) \to \mathcal{H}$ satisfying

$$\|\Phi(\tau)\| \leq C_\varepsilon e^{-\mu_p \tau}$$

for all $\tau \geq -\log T$ and a constant $C_\varepsilon > 0$ depending on ε. Moreover, this solution is unique in $C([-\log T, \infty), \mathcal{H})$. Thus, Φ is the unique global mild solution of Eq. (4.12) with initial data $\Phi(-\log T) = \mathbf{U}(\mathbf{v}, T)$.

Proof of Theorem A

Proof. We translate the result of Proposition 4.1.21 back to the original coordinates (t,r). Let (f,g) satisfy the assumption of Theorem A. For the fundamental self–similar solution with $T=1$ we have

$$\psi^1(0,r)=\kappa_p^{\frac{1}{p-1}},\quad \psi^1_t(0,r)=\tfrac{2}{p-1}\kappa_p^{\frac{1}{p-1}}.$$

We define

$$v_1(\rho):=\rho g(\rho)-\tfrac{2\rho}{p-1}\kappa_p^{\frac{1}{p-1}},\quad v_2(\rho):=f(\rho)+\rho f'(\rho)-\kappa_p^{\frac{1}{p-1}},$$

such that $\mathbf{v}=(v_1,v_2)^T\in L^2(0,\frac{3}{2})\times L^2(0,\frac{3}{2})$ and

$$\|\mathbf{v}\|_{\tilde{\mathcal{H}}}=\|(f,g)-(\psi^1(0,\cdot),\psi^1_t(0,\cdot))\|_{\mathcal{E}(\frac{3}{2})}$$

We may assume $\mathbf{v}$ small enough to satisfy the assumptions of Proposition 4.1.21 and we infer that there exists a unique global mild solution $\Phi\in C([-\log T,\infty),\mathcal{H})$ of Eq. (4.12) for T close to 1 with initial data $\Phi(-\log T)=\mathbf{U}(\mathbf{v},T)$ and

$$\|\Phi(\tau)\|\le C_\varepsilon e^{-(|\omega_p|-\varepsilon)\tau}$$

for all $\tau\ge-\log T$. By definition

$$\Phi(\tau)(\rho)=(\phi_1(\tau,\rho),\phi_2(\tau,\rho))^T$$

is a solution of Eq. (3.13) and Eq. (4.3) yields

$$\psi(t,r)=\psi^T(t,r)+(T-t)^{-\frac{2}{p-1}}r^{-1}\int_0^r\phi_2(-\log(T-t),\tfrac{r'}{T-t})dr'$$

and

$$\psi_t(t,r)=\psi^T_t(t,r)+(T-t)^{-\frac{2}{p-1}}r^{-1}\phi_1(-\log(T-t),\tfrac{r}{T-t}).$$

For $\varphi=\psi-\psi^T$ we obtain

$$\begin{aligned}
\|(\varphi(t,\cdot),\varphi_t(t,\cdot))\|^2_{\mathcal{E}(T-t)}&=(T-t)^{-\frac{4}{p-1}}\int_0^{T-t}|\phi_2(-\log(T-t),\tfrac{r}{T-t})|^2dr\\
&+(T-t)^{-\frac{4}{p-1}}\int_0^{T-t}|\phi_1(-\log(T-t),\tfrac{r}{T-t})|^2dr\\
&=(T-t)^{\frac{p-5}{p-1}}\left(\int_0^1|\phi_2(-\log(T-t),\rho)|^2d\rho+\int_0^1|\phi_1(-\log(T-t),\rho)|^2d\rho\right)\\
&=(T-t)^{\frac{p-5}{p-1}}\|\Phi(-\log(T-t))\|^2\le C_\varepsilon^2(T-t)^{\frac{p-5}{p-1}+2(|\omega_p|-\varepsilon)}.
\end{aligned}$$

Thus,

$$\|(\psi(t,\cdot),\psi_t(t,\cdot))-(\psi^T(t,\cdot),\psi^T_t(t,\cdot))\|_{\mathcal{E}(T-t)}\le C_\varepsilon(T-t)^{\frac{p-5}{2(p-1)}+|\omega_p|-\varepsilon}.$$

$\square$

4.2 Proof of Theorem B

Throughout this section we assume p to be a fixed real number with

$$p > 3.$$

Linear perturbation theory

We address the linearized problem in the higher energy space. In the new variables the higher energy norm corresponds to the standard norm on $H^1(0,1)^2$ which is

$$\|\boldsymbol{u}\|^2 := \|u_1\|_{H^1}^2 + \|u_2\|_{H^1}^2,$$

where

$$\|u\|_{H^1}^2 = \int_0^1 |u(\rho)|^2 d\rho + \int_0^1 |u'(\rho)|^2 d\rho.$$

We set

$$\mathcal{H} = \{u \in H^1(0,1) : u(0) = 0\} \times H^1(0,1).$$

Now consider the sesquilinear form

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{1}} := (u_1(1) + u_2(1))\overline{(v_1(1) + v_2(1))} + \int_0^1 u_1'(\rho)\overline{v_1'(\rho)} d\rho + \int_0^1 u_2'(\rho)\overline{v_2'(\rho)} d\rho$$

and the associated quantity

$$\|\boldsymbol{u}\|_{\mathbf{1}}^2 := (\boldsymbol{u}, \boldsymbol{u})_{\mathbf{1}} = |u_1(1) + u_2(1)|^2 + \|u_1'\|_{L^2}^2 + \|u_2'\|_{L^2}^2.$$

Lemma 4.2.1. *The quantity $\|\cdot\|_{\mathbf{1}}$ defines a norm on $\mathcal{H}$ which is equivalent to $\|\cdot\|$.*

Proof. The map $\|\cdot\|_{\mathbf{1}}$ indeed defines a norm on $\mathcal{H}$ since $\|\boldsymbol{u}\|_{\mathbf{1}} = 0$ implies $u_1 = c_1$, $u_2 = c_2$ for constants c_1, c_2 as well as $u_1(1) = -u_2(1)$. The boundary condition $u_1(0) = 0$ shows that $c_1 = 0$ and thus, $c_2 = 0$.
Next, we prove equivalence of the norms. Using the fact that $\|u_j\|_{L^\infty} \lesssim \|u_j\|_{H^1}$ for $j = 1, 2$ we immediately obtain

$$\|\boldsymbol{u}\|_{\mathbf{1}}^2 \lesssim |u_1(1)|^2 + |u_2(1)|^2 + \|u_1'\|_{L^2}^2 + \|u_2'\|_{L^2}^2 \lesssim \|u_1\|_{H^1}^2 + \|u_2\|_{H^1}^2 \lesssim \|\boldsymbol{u}\|^2.$$

In order to prove the reverse inequality we require estimates for the L^2-norms of the individual components. By using the fundamental theorem of calculus for absolutely continuous functions, the boundary condition for u_1, and the Cauchy-Schwarz inequality, we obtain

$$|u_1(\rho)| \leq \int_0^\rho |u_1'(s)| ds \leq \|u_1'\|_{L^2}.$$

Squaring and integrating yields $\|u_1\|_{L^2} \leq \|u_1'\|_{L^2}$. To derive a similar estimate for $\|u_2\|_{L^2}$ we use the identity

$$\int_\rho^1 u_j'(s)ds = u_j(1) - u_j(\rho)$$

for $j = 1, 2$ to infer that

$$\begin{aligned}|u_1(\rho) + u_2(\rho)| &\leq |u_1(1) + u_2(1)| + \int_\rho^1 |u_1'(s)|ds + \int_\rho^1 |u_2'(s)|ds \\ &\leq |u_1(1) + u_2(1)| + \|u_1'\|_{L^2} + \|u_2'\|_{L^2}\end{aligned}$$

by Cauchy-Schwarz. Hence,

$$\begin{aligned}|u_2(\rho)| = |u_2(\rho) + u_1(\rho) - u_1(\rho)| &\leq |u_2(\rho) + u_1(\rho)| + |u_1(\rho)| \\ &\leq |u_1(1) + u_2(1)| + 2\|u_1'\|_{L^2} + \|u_2'\|_{L^2}\end{aligned}$$

where we used the above estimate for u_1. Squaring and integrating yields

$$\|u_2\|_{L^2}^2 \lesssim |u_1(1) + u_2(1)|^2 + \|u_1'\|_{L^2}^2 + \|u_2'\|_{L^2}^2 \lesssim \|\boldsymbol{u}\|_{\boldsymbol{1}}^2.$$

We conclude that

$$\|\boldsymbol{u}\|^2 = \|u_1\|_{L^2}^2 + \|u_1'\|_{L^2}^2 + \|u_2\|_{L^2}^2 + \|u_2'\|_{L^2}^2 \lesssim \|\boldsymbol{u}\|_{\boldsymbol{1}}^2.$$

□

Operator formulation – well-posedness of the linearized problem

In correspondence with the right-hand side of the linearization of Eq. (3.13) we define the operators $(\tilde{\boldsymbol{L}}_0, \mathcal{D}(\tilde{\boldsymbol{L}}_0))$ and $\boldsymbol{L}' \in \mathcal{B}(\mathcal{H})$ by

$$\tilde{\boldsymbol{L}}_0 \boldsymbol{u}(\rho) := \begin{pmatrix} u_2'(\rho) - \rho u_1'(\rho) \\ u_1'(\rho) - \rho u_2'(\rho) \end{pmatrix} - \tfrac{2}{p-1}\boldsymbol{u}(\rho)$$

where $\mathcal{D}(\tilde{\boldsymbol{L}}_0) := \{\boldsymbol{u} \in C^2[0,1] \times C^2[0,1] : u_1(0) = 0, u_2'(0) = 0\}$ and

$$\boldsymbol{L}'\boldsymbol{u}(\rho) := \begin{pmatrix} p\kappa_p \int_0^\rho u_2(s)ds \\ 0 \end{pmatrix}.$$

It follows by inspection that $\tilde{\boldsymbol{L}}_0$ has range in $\mathcal{H}$. It is also immediate that $\tilde{\boldsymbol{L}}_0$ is densely defined in $\mathcal{H}$. Furthermore, by exploiting the compactness of the embedding $H^1(0,1) \hookrightarrow L^2(0,1)$ it is easy to see that $\boldsymbol{L}'$ is a compact operator.

Lemma 4.2.2. *The operator $(\tilde{\boldsymbol{L}}_0, \mathcal{D}(\tilde{\boldsymbol{L}}_0))$ is closable and we denote its closure by $(\boldsymbol{L}_0, \mathcal{D}(\boldsymbol{L}_0))$. Consequently,*

$$\boldsymbol{L} := \boldsymbol{L}_0 + \boldsymbol{L}', \quad \mathcal{D}(\boldsymbol{L}) = \mathcal{D}(\boldsymbol{L}_0)$$

is a well-defined closed linear operator and $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L})$ implies that $u_j \in C[0,1] \cap C^1[0,1)$ for $j = 1,2$ with the boundary conditions $u_1(0) = u_2'(0) = 0$.
Furthermore, $\boldsymbol{L}$ is the generator of a strongly continuous one-parameter semigroup $\mathbf{S} : [0,\infty) \to \mathcal{B}(\mathcal{H})$ which satisfies

$$\|\mathbf{S}(\tau)\boldsymbol{u}\| \leq M e^{\omega\tau} \|\boldsymbol{u}\| \tag{4.23}$$

for all $\tau \geq 0$, a constant $M \geq 1$, and a p-dependent exponent $\omega > 0$.

Proof. We consider the Hilbert space $\mathcal{H}$ equipped with the norm $\|\cdot\|_{\mathbf{1}}$. First, we show that $\tilde{\boldsymbol{L}}_0$ is a closable operator and its closure is the generator of a C_0-semigroup. The next estimate is crucial for our approach. By definition of $(\cdot|\cdot)_{\mathbf{1}}$ we have

$$\begin{aligned}\mathrm{Re}(\tilde{\boldsymbol{L}}_0\boldsymbol{u}|\boldsymbol{u})_{\mathbf{1}} &= \mathrm{Re}\int_0^1 [u_2''(\rho) - \rho u_1''(\rho) - u_1'(\rho)]\overline{u_1'(\rho)}d\rho \\ &\quad + \mathrm{Re}\int_0^1 [u_1''(\rho) - \rho u_2''(\rho) - u_2'(\rho)]\overline{u_2'(\rho)}d\rho - \tfrac{2}{p-1}\|\boldsymbol{u}\|_{\mathbf{1}}^2.\end{aligned}$$

Since $\mathrm{Re}(u'\overline{u}) = \frac{1}{2}(|u|^2)'$, an integration by parts yields

$$\mathrm{Re}(\tilde{\boldsymbol{L}}_0\boldsymbol{u}|\boldsymbol{u})_{\mathbf{1}} \leq -\tfrac{2}{p-1}\|\boldsymbol{u}\|_{\mathbf{1}}^2.$$

Next, we show that $\mathrm{rg}(\lambda - \tilde{\boldsymbol{L}}_0)$ is dense for $\lambda := 1 - \frac{2}{p-1} > -\frac{2}{p-1}$. For arbitrary $\boldsymbol{f} = (f_1, f_2) \in \{(u_1,u_2) \in C^\infty[0,1]^2 : u_1(0) = 0\}$ (which is dense in $\mathcal{H}$) we set

$$F(\rho) := f_1(\rho) + \rho f_2(\rho) + \int_0^\rho f_2(s)ds$$

and define

$$u_1(\rho) := \rho u_2(\rho) - \int_0^\rho f_2(s)ds, \quad u_2(\rho) := \frac{1}{1-\rho^2}\int_\rho^1 F(s)ds.$$

By Taylor's theorem it is immediate that $u_j \in C^2[0,1]$ for $j = 1,2$ and we have $u_1(0) = 0$ as well as $u_2'(0) = -F(0) = 0$ which implies $\boldsymbol{u} = (u_1,u_2) \in \mathcal{D}(\tilde{\boldsymbol{L}}_0)$. A direct calculation shows that $(\lambda - \tilde{\boldsymbol{L}}_0)\boldsymbol{u} = \boldsymbol{f}$. Consequently, the Lumer-Phillips Theorem (see [36], p. 83, Theorem 3.15) shows that $(\tilde{\boldsymbol{L}}_0, \mathcal{D}(\tilde{\boldsymbol{L}}_0))$ is closable and its closure $(\boldsymbol{L}_0, \mathcal{D}(\boldsymbol{L}_0))$ generates a strongly continuous one-parameter semigroup $\mathbf{S}_0 : [0,\infty) \to \mathcal{B}(\mathcal{H})$ which satisfies

$$\|\mathbf{S}_0(\tau)\boldsymbol{u}\|_{\mathbf{1}} \leq e^{-\frac{2}{p-1}\tau}\|\boldsymbol{u}\|_{\mathbf{1}}$$

for $\boldsymbol{u} \in \mathcal{H}$. Equivalence of the norms $\|\cdot\|_{\mathbf{1}}$ and $\|\cdot\|$ on $\mathcal{H}$, which was shown in Lemma 4.2.1, implies the existence of a constant $M \geq 1$ such that

$$\|\mathbf{S}_0(\tau)\boldsymbol{u}\| \leq M e^{-\frac{2}{p-1}\tau}\|\boldsymbol{u}\|. \tag{4.24}$$

Next, we add the perturbation $\boldsymbol{L}' \in \mathcal{B}(\mathcal{H})$ and set $\boldsymbol{L} := \boldsymbol{L}_0 + \boldsymbol{L}'$. Boundedness of $\boldsymbol{L}'$ implies that $\mathcal{D}(\boldsymbol{L}) = \mathcal{D}(\boldsymbol{L}_0)$. The Bounded Perturbation Theorem (see [36], p. 158) shows that $\boldsymbol{L}$ is the generator of a strongly continuous one-parameter semigroup $\mathbf{S} : [0,\infty) \to \mathcal{B}(\mathcal{H})$ satisfying

$$\|\mathbf{S}(\tau)\boldsymbol{u}\| \leq M e^{(M\|\boldsymbol{L}'\| - \frac{2}{p-1})\tau}\|\boldsymbol{u}\|.$$

Finally, to characterize the generator in more detail assume that $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}) = \mathcal{D}(\boldsymbol{L}_0)$. The fact that $u_j \in C[0,1]$ for $j = 1,2$ and $u_1(0) = 0$ follows immediately by Sobolev embedding since $\boldsymbol{u} \in \mathcal{H}$. By definition of the closure there exists a sequence $(\boldsymbol{u}_k) \subset \mathcal{D}(\tilde{\boldsymbol{L}}_0) \subset C^2[0,1] \times C^2[0,1]$ such that $\boldsymbol{u}_k \to \boldsymbol{u}$ and $\boldsymbol{L}_0\boldsymbol{u}_k \to \boldsymbol{L}_0\boldsymbol{u}$ in $\mathcal{H}$. Sobolev embedding implies uniform convergence of the individual components and a suitable combination of the respective expressions shows that $(1-\cdot^2)u_{1,k}' \to (1-\cdot^2)u_1'$ and $(1-\cdot^2)u_{2,k}' \to (1-\cdot^2)u_2'$ uniformly. We infer that for $j=1,2$, $u_{j,k}' \to u_j'$ in $L^\infty(a,b)$ for any $(a,b] \subset (0,1)$ which shows that $u_j \in C^1[0,1)$. As a consequence, $u_{2,k}'(\rho) \to u_2'(\rho)$ pointwise for $\rho \in [0,1)$ which yields the boundary condition $u_2'(0) = 0$. $\square$

Corollary 4.2.3. *The Cauchy problem*

$$\begin{cases} \frac{d}{d\tau}\Psi(\tau) = \boldsymbol{L}\Psi(\tau) \text{ for } \tau > 0 \\ \Psi(0) = \boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}) \end{cases}$$

has a unique solution $\Psi \in C^1([0,\infty),\mathcal{H})$ which is given by $\Psi(\tau) = \mathbf{S}(\tau)\boldsymbol{u}$ for all $\tau \geq 0$.

Spectral analysis of the generator

The growth estimate for the semigroup $\mathbf{S}$ obtained in Lemma 4.2.2 by abstract results is not optimal. In order to refine (4.23) we investigate the spectral properties of the generator.

Lemma 4.2.4. *We have $\sigma(\boldsymbol{L}) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -\frac{2}{p-1}\} \cup \{1\}$. The spectral point $\lambda_{\boldsymbol{g}} = 1$ is an eigenvalue and the associated one-dimensional geometric eigenspace is spanned by the symmetry mode*

$$\boldsymbol{g}(\rho) := \begin{pmatrix} \frac{p+1}{p-1}\rho \\ 1 \end{pmatrix}. \tag{4.25}$$

Proof. We set $\mathcal{S} := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -\frac{2}{p-1}\} \cup \{1\}$. Let $\lambda \in \sigma(\boldsymbol{L})$. If $\mathrm{Re}\lambda \leq -\frac{2}{p-1}$ then $\lambda \in \mathcal{S}$ trivially, hence assume that $\mathrm{Re}\lambda > -\frac{2}{p-1}$. We show that under this assumption $\lambda \in \sigma_p(\boldsymbol{L})$ and $\lambda = 1$.

From (4.24) and standard results from semigroup theory (see [36], p. 55, Theorem 1.10) we infer that

$$\sigma(\boldsymbol{L}_0) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -\tfrac{2}{p-1}\}.$$

In particular, the above assumption on λ implies that $\lambda \notin \sigma(\boldsymbol{L}_0)$. We use the identity

$$\lambda - \boldsymbol{L} = [1 - \boldsymbol{L}'\mathbf{R}_{\boldsymbol{L}_0}(\lambda)](\lambda - \boldsymbol{L}_0)$$

which shows that $1 \in \sigma(\boldsymbol{L}'\mathbf{R}_{\boldsymbol{L}_0}(\lambda))$, hence 1 is an eigenvalue of the compact operator $\boldsymbol{L}'\mathbf{R}_{\boldsymbol{L}_0}(\lambda)$. Let $\boldsymbol{f} \in \mathcal{H}$ denote the corresponding eigenvector. Setting $\boldsymbol{u} := \mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}$ yields $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0) = \mathcal{D}(\boldsymbol{L})$, $\boldsymbol{u} \neq \mathbf{0}$, as well as $(\lambda - \boldsymbol{L})\boldsymbol{u} = \mathbf{0}$, and we conclude that $\lambda \in \sigma_p(\boldsymbol{L})$.

The eigenvalue equation $(\lambda - \boldsymbol{L})\boldsymbol{u} = \mathbf{0}$ implies that (see Section 4.1)

$$u_1(\rho) = \rho u_2(\rho) + (\lambda + \tfrac{3-p}{p-1}) \int_0^\rho u_2(s)ds, \tag{4.26}$$

as well as

$$\begin{aligned}(1-\rho^2)u''(\rho) &- \left(2\lambda + \tfrac{4}{p-1}\right)\rho u'(\rho) \\ &- \left[\left(\lambda + \tfrac{2}{p-1}\right)\left(\lambda + \tfrac{3-p}{p-1}\right) - p\kappa_p\right] u(\rho) = 0 \end{aligned}\tag{4.27}$$

where $u(\rho) := \int_0^\rho u_2(s)ds$. Since $u_2 \in H^1(0,1)$ for $\boldsymbol{u} \in \mathcal{H}$ we have $u \in H^2(0,1)$. Furthermore, $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L})$ yields $u \in C^2[0,1)$ and the boundary conditions $u(0) = u''(0) = 0$, see Lemma 4.2.2. We substitute $\rho \mapsto z := \rho^2$ to obtain the hypergeometric differential equation

$$z(1-z)v''(z) + [c - (a+b+1)z]v'(z) - abv(z) = 0 \tag{4.28}$$

where $v(z) := u(\sqrt{z})$ and the parameters are given by

$$a = \tfrac{1}{2}(\lambda - 2), \quad b = \tfrac{1}{2}(\lambda + \tfrac{p+3}{p-1}), \quad c = \tfrac{1}{2}.$$

At $z = 1$ the exponents of the indicial equation are $\{0, c-a-b\}$, where $c - a - b = \frac{p-3}{p-1} - \lambda$. The assumption $\mathrm{Re}\lambda > -\frac{2}{p-1}$ implies $\mathrm{Re}(c-a-b) < 1$ and thus, by Frobenius' method it follows that there exist two linearly independent solution v_1 and $\tilde{v}_1$ of Eq. (4.28) with the asymptotic behavior

$$v_1(z) \sim 1, \quad \tilde{v}_1(z) \sim (1-z)^{c-a-b} \text{ as } z \to 1-,$$

at least if $c-a-b \neq 0$. In the degenerate case $c-a-b = 0$ we have $\tilde{v}_1(z) \sim \log(1-z)$ as $z \to 1-$. In fact, v_1 is given explicitly by

$$v_1(z) = {}_2F_1(a, b; a+b+1-c; 1-z)$$

where ${}_2F_1$ denotes the standard hypergeometric function, see e.g. [86]. The assumption $\text{Re}\lambda > -\frac{2}{p-1}$ implies that $v = \alpha v_1$ for some constant $\alpha \in \mathbb{C}$ because otherwise the corresponding $u(\rho) = v(\rho^2)$ would not belong to $H^2(0,1)$. Another fundamental system $\{v_0, \tilde{v}_0\}$ of Eq. (4.28) is given by

$$\begin{aligned}\tilde{v}_0(z) &= {}_2F_1(a,b;c;z),\\ v_0(z) &= z^{1/2}{}_2F_1(a+1-c,b+1-c;2-c;z),\end{aligned}$$

see [86], and there must exist constants c_0, $c_1 \in \mathbb{C}$ such that

$$v_1 = c_0\tilde{v}_0 + c_1 v_0.$$

The connection coefficients c_0 and c_1 are known explicitly in terms of the Γ-function, see [86]. The condition $u(0) = 0$ implies that $v(0) = v_1(0) = 0$ and thus,

$$c_0 = \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}$$

must vanish. This can only be the case when at least one of the Gamma functions in the denominator has a pole, which is equivalent to

$$\tfrac{1}{2}(\lambda - 1) = -k \quad \text{or} \quad \tfrac{\lambda}{2} + \tfrac{p+1}{p-1} = -k \quad \text{for} \quad k \in \mathbb{N}_0.$$

The latter condition can be rewritten as $\lambda = -2k - \frac{2p}{p-1} - \frac{2}{p-1}$ which implies that $\lambda < -\frac{2}{p-1}$ but this is excluded by assumption. The first condition is satisfied if $\lambda = 1 - 2k \in \{1, -1, -3, \cdots\}$ and since $-\frac{2}{p-1} \in (-1, 0)$, we see that $\lambda = 1$ is the only possibility. We denote this particular eigenvalue by $\lambda_{\boldsymbol{g}}$. For $\lambda = \lambda_{\boldsymbol{g}} = 1$ we have $v_1(z) = c_1\sqrt{z}$ and $u(\rho) = \alpha\rho$ for some $\alpha \in \mathbb{C}$. In particular, u satisfies the boundary conditions $u(0) = u''(0) = 0$. Finally, from Eq. (4.26) we obtain $u_1(\rho) = \alpha\frac{p+1}{p-1}\rho$, $u_2(\rho) = \alpha$ which shows that the geometric eigenspace associated to $\lambda_{\boldsymbol{g}}$ is spanned by $\boldsymbol{g}$ as claimed. □

Resolvent bounds

Lemma 4.2.5. *Fix $\varepsilon > 0$. Then there exist constants $c_1, c_2 > 0$ such that*

$$\|\mathbf{R}_{\boldsymbol{L}}(\lambda)\| \leq c_2$$

for all $\lambda \in \mathbb{C}$ with $\text{Re}\lambda \geq -\frac{2}{p-1} + \varepsilon$ and $|\lambda| \geq c_1$.

Proof. In view of the identity

$$\mathbf{R}_{\boldsymbol{L}}(\lambda) = \mathbf{R}_{\boldsymbol{L}_0}(\lambda)[1 - \boldsymbol{L}'\mathbf{R}_{\boldsymbol{L}_0}(\lambda)]^{-1}$$

it suffices to prove smallness of $\|\boldsymbol{L}'\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\|$. First note that semigroup theory yields (see [36], p. 55, Theorem 1.10)

$$\|[\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_j\|_{H^1} \leq \|\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}\| \leq \frac{M\|\boldsymbol{f}\|}{\mathrm{Re}\lambda + \frac{2}{p-1}} \tag{4.29}$$

for $j = 1, 2$, $\boldsymbol{f} \in \mathcal{H}$, and M is the constant from Lemma 4.2.2. Suppose we have $\|\boldsymbol{L}'\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\| \leq c < 1$ for $c > 0$ and $|\lambda| \geq c_1$ large enough. Then this implies

$$\|\mathbf{R}_{\boldsymbol{L}}(\lambda)\| \leq \|\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\|(1 - \|\boldsymbol{L}'\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\|)^{-1} \leq c_2$$

where $c_2 \to \infty$ as $\varepsilon \to 0+$. Note that

$$\boldsymbol{L}'\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f} = \begin{pmatrix} p\kappa_p V[\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_2 \\ 0 \end{pmatrix}$$

where $V : H^1(0,1) \to \{u \in H^1(0,1) : u(0) = 0\}$ is a bounded operator defined by $Vu(\rho) := \int_0^\rho u(s)ds$. For all $\boldsymbol{f} \in \mathcal{H}$ we have $(\lambda - \boldsymbol{L}_0)\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f} = \boldsymbol{f}$ which implies

$$[\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1(\rho) = (\lambda - \tfrac{p-3}{p-1})V[\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_2(\rho) + \rho[\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_2(\rho) - Vf_2(\rho)$$

and this yields the estimate

$$|\lambda - \tfrac{p-3}{p-1}|\|V[\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_2\|_{H^1} \lesssim \|[\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1\|_{H^1} + \|[\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_2\|_{H^1} + \|f_2\|_{H^1}.$$

Using (4.29) we obtain

$$\|\boldsymbol{L}'\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}\| = p\kappa_p\|V[\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_2\|_{H^1} \lesssim \frac{\|\boldsymbol{f}\|}{|\lambda - \frac{p-3}{p-1}|}$$

such that $\|\boldsymbol{L}'\mathbf{R}_{\boldsymbol{L}_0}(\lambda)\| \leq \frac{1}{2}$ for all $|\lambda|$ sufficiently large. $\square$

A growth estimate for the linearized evolution

Lemma 4.2.6. *Let $\varepsilon > 0$ be fixed and so small that*

$$\mu_p := \tfrac{2}{p-1} - \varepsilon > 0.$$

Then there exists a projection $\boldsymbol{P} \in \mathcal{B}(\mathcal{H})$ onto $\langle \boldsymbol{g} \rangle$ which commutes with the semigroup $\mathbf{S}(\tau)$ for all $\tau \geq 0$ and

$$\begin{aligned} \|\mathbf{S}(\tau)(1 - \boldsymbol{P})\boldsymbol{f}\| &\leq C_\varepsilon e^{-\mu_p\tau}\|(1 - \boldsymbol{P})\boldsymbol{f}\| \\ \mathbf{S}(\tau)\boldsymbol{P}\boldsymbol{f} &= e^\tau \boldsymbol{P}\boldsymbol{f} \end{aligned}$$

for all $\tau \geq 0$, $\boldsymbol{f} \in \mathcal{H}$, and a constant $C_\varepsilon > 0$.

Proof. Let γ be a (positively oriented) circle around $\lambda_{\boldsymbol{g}}$ with radius $r_\gamma = \frac{1}{2}$. By Lemma 4.2.4, γ belongs to the resolvent set of $\boldsymbol{L}$ and no spectral points lie inside of γ except for $\lambda_{\boldsymbol{g}}$. According to [52], p. 178, Theorem 6.5, a spectral projection $\boldsymbol{P} \in \mathcal{B}(\mathcal{H})$ is defined by

$$\boldsymbol{P} = \frac{1}{2\pi i} \int_\gamma \mathbf{R}_{\boldsymbol{L}}(\lambda) d\lambda,$$

where $\boldsymbol{P}$ commutes with $\boldsymbol{L}$ in the sense that $\boldsymbol{PL} \subset \boldsymbol{LP}$. Furthermore, $\boldsymbol{P}$ commutes with the resolvent of $\boldsymbol{L}$, see [52] p. 173, Theorem 6.5. This implies that $\boldsymbol{P}$ commutes with the linear time evolution, i.e., $\boldsymbol{P}\mathbf{S}(\tau) = \mathbf{S}(\tau)\boldsymbol{P}$ for $\tau \geq 0$, where $\mathbf{S} : [0, \infty) \to \mathcal{B}(\mathcal{H})$ is the semigroup generated by $\boldsymbol{L}$.
Most important is that $\boldsymbol{L}$ is decomposed according to the decomposition of the Hilbert space $\mathcal{H} = \ker\boldsymbol{P} \oplus \mathrm{rg}\boldsymbol{P}$ into parts $\boldsymbol{L}|_{\ker\boldsymbol{P}}$ and $\boldsymbol{L}|_{\mathrm{rg}\boldsymbol{P}}$, where $\mathcal{D}(\boldsymbol{L}|_{\ker\boldsymbol{P}}) = \mathcal{D}(\boldsymbol{L}) \cap \ker \boldsymbol{P}$ and $\boldsymbol{L}|_{\ker\boldsymbol{P}}\boldsymbol{u} = \boldsymbol{Lu}$ for $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}|_{\ker\boldsymbol{P}})$ (an analogous definition holds for $\boldsymbol{L}|_{\mathrm{rg}\boldsymbol{P}}$). Moreover,

$$\sigma(\boldsymbol{L}|_{\ker\boldsymbol{P}}) = \sigma(\boldsymbol{L}) \setminus \{1\}, \quad \sigma(\boldsymbol{L}|_{\mathrm{rg}\boldsymbol{P}}) = \{1\}.$$

Since $\boldsymbol{L}$ is not self-adjoint, we only know a priori that $\ker(\lambda_{\boldsymbol{g}} - \boldsymbol{L}) = \langle \boldsymbol{g} \rangle \subseteq \mathrm{rg}\boldsymbol{P}$ and it remains to show that $\mathrm{rg}\boldsymbol{P} = \langle \boldsymbol{g} \rangle$. This is equivalent to the fact that the algebraic multiplicity of $\lambda_{\boldsymbol{g}}$ is equal to one and for this we refer to Section 4.1, where the proof of Lemma 4.1.7 can be copied verbatim.
Having this, it is easy to see that $\mathbf{S}(\tau)\boldsymbol{Pf} = e^\tau \boldsymbol{Pf}$ for $\boldsymbol{f} \in \mathcal{H}$. In order to obtain an estimate on the stable subspace, we use the structure of the spectrum of $\boldsymbol{L}|_{\ker\boldsymbol{P}}$ and Lemma 4.2.5 which imply that for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geq -\frac{2}{p-1} + \varepsilon$, the restriction of the resolvent $\mathbf{R}_{\boldsymbol{L}}(\lambda)$ to $\ker\boldsymbol{P}$ (which equals the resolvent of $\boldsymbol{L}|_{\ker\boldsymbol{P}}$) exists and is uniformly bounded. Since $\ker\boldsymbol{P}$ is a Hilbert space, we can apply the theorem by Gearhart, Prüss, and Greiner (see e.g. [36], p. 302, Theorem 1.11 or [48]) to obtain the claimed estimate. □

Nonlinear Perturbation Theory

The aim of this section is to prove the existence of solutions of the full nonlinear equation (3.13) which retain the exponential decay of the linearized problem on the stable subspace, cf. Lemma 4.2.6.
Note that the exponential growth of the semigroup on the unstable subspace $\boldsymbol{P}\mathcal{H}$ has its origin in the time translation symmetry of the original equation, In fact, we are perturbing around a one-parameter family of solutions and it is clear that a generic perturbation around ψ^{T^*} for a particular fixed value T^* will change the blow up time. Therefore, we expect the solution to converge to ψ^T where in general $T^* \neq T$.
Without loss of generality we set $T^* = 1$ and study perturbations around ψ^1. The blow up time T will be considered as a variable that will be fixed later on in the proof.

Note that from now on we restrict ourselves to real-valued functions.

Estimates for the nonlinearity

We formally set
$$Ku(\rho) := \frac{1}{\rho}\int_0^\rho u(s)ds.$$

Lemma 4.2.7. *Let $u \in H^1(0,1)$. Then $Ku \in L^\infty(0,1)$ and there exists a $c > 0$ such that $\|Ku\|_\infty \leq c\|u\|_{H^1}$.*

Proof. By the continuous embedding $H^1(0,1) \hookrightarrow L^\infty(0,1)$ we see that $u \in L^\infty(0,1)$. In particular,
$$|Ku(\rho)| \leq \frac{1}{\rho}\int_0^\rho |u(s)|ds \leq \|u\|_\infty \lesssim \|u\|_{H^1}$$
for all $\rho \in [0,1]$. □

In order to define the nonlinearity we introduce the auxiliary function $N : \mathbb{R} \times [0,1] \to \mathbb{R}$ given by
$$N(x,\rho) := \rho\left[|\kappa_p^{\frac{1}{p-1}} + x|^{p-1}(\kappa_p^{\frac{1}{p-1}} + x) - p\kappa_p x - \kappa_p^{\frac{p}{p-1}}\right], \tag{4.30}$$
cf. Eq. (4.2). Since $p > 3$, the function N is at least twice continuously differentiable on $\mathbb{R}$ with respect to the first variable. Furthermore, for any fixed $\rho \in [0,1]$, we have $N(x,\rho) = O(x^2)$ as $x \to 0$. Thus, it is easy to see that with $\langle x\rangle := \sqrt{1+|x|^2}$,
$$\begin{aligned} |N(x,\rho)| &\lesssim \rho|x|^2\langle x\rangle^{p-2} & \qquad |\partial_1 N(x,\rho)| &\lesssim \rho|x|\langle x\rangle^{p-2} \\ |\partial_1^2 N(x,\rho)| &\lesssim \rho\langle x\rangle^{p-2} & \qquad |\partial_2 N(x,\rho)| &\lesssim |x|^2\langle x\rangle^{p-2} \end{aligned} \tag{4.31}$$
for all $x \in \mathbb{R}$ and $\rho \in [0,1]$.
We formally define a vector-valued nonlinearity by
$$\boldsymbol{N}(\boldsymbol{u})(\rho) := \begin{pmatrix} N(Ku_2(\rho),\rho) \\ 0 \end{pmatrix}.$$

With this definition, Eq. (3.13) can be (formally) written as an ordinary differential equation for a function $\Phi : [-\log T, \infty) \to \mathcal{H}$ given by
$$\tfrac{d}{d\tau}\Phi(\tau) = \boldsymbol{L}\Phi(\tau) + \boldsymbol{N}(\Phi(\tau)), \quad \tau > -\log T \tag{4.32}$$
with initial data
$$\Phi(-\log T)(\rho) = \begin{pmatrix} \rho T^{\frac{p+1}{p-1}} g(T\rho) - \frac{2\rho}{p-1}\kappa_p^{\frac{1}{p-1}} \\ T^{\frac{2}{p-1}}\left(T\rho f'(T\rho) + f(T\rho)\right) - \kappa_p^{\frac{1}{p-1}} \end{pmatrix}. \tag{4.33}$$

In the following we denote by $\mathcal{B}_1$ the open unit ball in $(\mathcal{H}, \|\cdot\|)$.

Lemma 4.2.8. *The operator $\boldsymbol{N}$ maps $\mathcal{H}$ to $\mathcal{H}$ and there exists a constant $c > 0$ such that*

$$\|\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v})\| \leq c(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)\|\boldsymbol{u} - \boldsymbol{v}\|$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{B}_1$. Furthermore, $\boldsymbol{N}(\mathbf{0}) = \mathbf{0}$ and $\boldsymbol{N}$ is Fréchet differentiable at $\mathbf{0}$ with $D\mathbf{N}(\mathbf{0}) = \mathbf{0}$.

Proof. First, we derive some estimates for the real-valued function N defined in (4.30). We use the fundamental theorem of calculus and (4.31) to obtain

$$\begin{aligned} |\partial_1 N(x,\rho) - \partial_1 N(y,\rho)| &\leq |x-y| \int_0^1 |\partial_1^2 N(y + h(x-y), \rho)| dh \\ &\lesssim \rho |x-y| \int_0^1 \langle y + h(x-y) \rangle^{p-2} dh \\ &\lesssim \rho |x-y| [\langle x \rangle^{p-2} + \langle y \rangle^{p-2}] \end{aligned} \tag{4.34}$$

for $x, y \in \mathbb{R}$. Similarly,

$$\begin{aligned} |N(x,\rho) - N(y,\rho)| &\leq |x-y| \int_0^1 |\partial_1 N(y + h(x-y), \rho)| dh \\ &\lesssim \rho |x-y| [|x| \langle x \rangle^{p-2} + |y| \langle y \rangle^{p-2}]. \end{aligned} \tag{4.35}$$

Note that $\|\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v})\| = \|[\boldsymbol{N}(\boldsymbol{u})]_1 - [\boldsymbol{N}(\boldsymbol{v})]_1\|_{H^1}$ and we obtain

$$\begin{aligned} \|[\boldsymbol{N}(\boldsymbol{u})]_1 - [\boldsymbol{N}(\boldsymbol{v})]_1\|_{H^1}^2 = &\int_0^1 |N(Ku_2(\rho), \rho) - N(Kv_2(\rho), \rho)|^2 d\rho \\ &+ \int_0^1 \left| \tfrac{d}{d\rho} [N(Ku_2(\rho), \rho) - N(Kv_2(\rho), \rho)] \right|^2 d\rho. \end{aligned}$$

With Lemma 4.2.7 and (4.35) we get

$$\begin{aligned} I_0 := &\int_0^1 |N(Ku_2(\rho), \rho) - N(Kv_2(\rho), \rho)|^2 d\rho \\ \lesssim &\int_0^1 \rho^2 |Ku_2(\rho) - Kv_2(\rho)|^2 \Big[|Ku_2(\rho)|^2 \langle Ku_2(\rho) \rangle^{2(p-2)} \\ &+ |Kv_2(\rho)|^2 \langle Kv_2(\rho) \rangle^{2(p-2)} \Big] d\rho \\ \lesssim &\left[\|Ku_2\|_\infty^2 \langle \|Ku_2\|_\infty \rangle^{2(p-2)} + \|Kv_2\|_\infty^2 \langle \|Kv_2\|_\infty \rangle^{2(p-2)} \right] \\ &\times \|K(u_2 - v_2)\|_\infty^2 \\ \lesssim &\left[\|u_2\|_{H^1}^2 \langle \|u_2\|_{H^1} \rangle^{2(p-2)} + \|v_2\|_{H^1}^2 \langle \|v_2\|_{H^1} \rangle^{2(p-2)} \right] \|u_2 - v_2\|_{H^1}^2. \end{aligned}$$

For the second term we obtain

$$
\begin{aligned}
&\int_0^1 \left|\tfrac{d}{d\rho}[N(Ku_2(\rho),\rho) - N(Kv_2(\rho),\rho)]\right|^2 d\rho \\
&\quad\lesssim \int_0^1 |\partial_1 N(Ku_2(\rho),\rho)(Ku_2)'(\rho) - \partial_1 N(Kv_2(\rho),\rho)(Kv_2)'(\rho)|^2 \, d\rho \\
&\qquad + \int_0^1 |\partial_2 N(Ku_2(\rho),\rho) - \partial_2 N(Kv_2(\rho),\rho)|^2 d\rho \\
&\quad\lesssim I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &:= \int_0^1 |(Ku_2)'(\rho)|^2 |\partial_1 N(Ku_2(\rho),\rho) - \partial_1 N(Kv_2(\rho),\rho)|^2 d\rho \\
I_2 &:= \int_0^1 |\partial_1 N(Kv_2(\rho),\rho)|^2 |(Ku_2)'(\rho) - (Kv_2)'(\rho)|^2 d\rho \\
I_3 &:= \int_0^1 |\partial_2 N(Ku_2(\rho),\rho) - \partial_2 N(Kv_2(\rho),\rho)|^2 d\rho.
\end{aligned}
$$

Estimate (4.34) and Lemma 4.2.7 yield

$$
\begin{aligned}
I_1 &\lesssim \int_0^1 |u_2(\rho) - Ku_2(\rho)|^2 \, |Ku_2(\rho) - Kv_2(\rho)|^2 \\
&\qquad \times [\langle Ku_2(\rho)\rangle^{2(p-2)} + \langle Kv_2(\rho)\rangle^{2(p-2)}] d\rho \\
&\lesssim \left[\langle \|Ku_2\|_\infty\rangle^{2(p-2)} + \langle \|Kv_2\|_\infty\rangle^{2(p-2)}\right] \|u_2\|_{H^1}^2 \|K(u_2 - v_2)\|_\infty^2 \\
&\lesssim \left[\langle \|u_2\|_{H^1}\rangle^{2(p-2)} + \langle \|v_2\|_{H^1}\rangle^{2(p-2)}\right] \|u_2\|_{H^1}^2 \|u_2 - v_2\|_{H^1}^2.
\end{aligned}
$$

With estimate (4.31) we obtain

$$
\begin{aligned}
I_2 &\lesssim \int_0^1 |Kv_2(\rho)|^2 \langle Kv_2(\rho)\rangle^{2(p-2)} \left[|u_2(\rho) - v_2(\rho)|^2 + |K(u_2 - v_2)(\rho)|^2\right] d\rho \\
&\lesssim \|Kv_2\|_\infty^2 \langle \|Kv_2\|_\infty\rangle^{2(p-2)} [\|u_2 - v_2\|_{L^2}^2 + \|K(u_2 - v_2)\|_\infty^2] \\
&\lesssim \|v_2\|_{H^1}^2 \langle \|v_2\|_{H^1}\rangle^{2(p-2)} \|u_2 - v_2\|_{H^1}^2.
\end{aligned}
$$

Since $\partial_2 N(x,\rho) = \rho^{-1} N(x,\rho)$, the third term can be estimated using (4.35)

$$
\begin{aligned}
I_3 &= \int_0^1 \rho^{-2} |N(Ku_2(\rho),\rho) - N(Kv_2(\rho),\rho)|^2 d\rho \\
&\lesssim \left[\|u_2\|_{H^1}^2 \langle \|u_2\|_{H^1}\rangle^{2(p-2)} + \|v_2\|_{H^1}^2 \langle \|v_2\|_{H^1}\rangle^{2(p-2)}\right] \|u_2 - v_2\|_{H^1}^2.
\end{aligned}
$$

Summing up yields

$$
\begin{aligned}
I_0 + I_1 + I_2 + I_3 \lesssim &\left[\|u_2\|_{H^1}^2 \langle \|u_2\|_{H^1}\rangle^{2(p-2)} + \|v_2\|_{H^1}^2 \langle \|v_2\|_{H^1}\rangle^{2(p-2)}\right. \\
&\left. + \|u_2\|_{H^1}^2 \langle \|v_2\|_{H^1}\rangle^{2(p-2)}\right] \|u_2 - v_2\|_{H^1}^2.
\end{aligned}
$$

In particular, for $\boldsymbol{u} \in \mathcal{B}_1$ we have $\|u_2\|_{H_1} \leq 1$ and thus $\langle \|u_2\|_{H^1} \rangle \lesssim 1$. This yields

$$\begin{aligned}\|[\boldsymbol{N}(\boldsymbol{u})]_1 - [\boldsymbol{N}(\boldsymbol{v})]_1\|_{H^1}^2 &\lesssim \left(\|u_2\|_{H^1}^2 + \|v_2\|_{H^1}^2\right) \|u_2 - v_2\|_{H^1}^2 \\ &\lesssim \left(\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2\right) \|\boldsymbol{u} - \boldsymbol{v}\|^2\end{aligned}$$

for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{B}_1$ and we conclude that

$$\|\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v})\| \lesssim (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|) \|\boldsymbol{u} - \boldsymbol{v}\|.$$

The fact that $N(0, \rho) = 0$ for all $\rho \in [0, 1]$ yields $\boldsymbol{N}(\boldsymbol{0}) = \boldsymbol{0}$ such that the above estimate implies $\|\boldsymbol{N}(\boldsymbol{u})\| \lesssim \|\boldsymbol{u}\|^2$. In particular,

$$\frac{\|\boldsymbol{N}(\boldsymbol{u})\|}{\|\boldsymbol{u}\|} \to 0$$

for $\boldsymbol{u} \to \boldsymbol{0}$ which proves that $\boldsymbol{N}$ is differentiable at zero with $D\boldsymbol{N}(\boldsymbol{0}) = \boldsymbol{0}$. □

Abstract formulation of the nonlinear problem

Next, we rewrite the initial data Eq. (4.33) by setting

$$\boldsymbol{U}(\boldsymbol{v}, T)(\rho) := T^{\frac{2}{p-1}}[\boldsymbol{v}(T\rho) + \boldsymbol{\kappa}(T\rho)] - \boldsymbol{\kappa}(\rho)$$

where

$$\boldsymbol{v}(\rho) := \begin{pmatrix} \rho g(\rho) \\ \rho f'(\rho) + f(\rho) \end{pmatrix} - \boldsymbol{\kappa}(\rho), \quad \boldsymbol{\kappa}(\rho) := \kappa_p^{\frac{1}{p-1}} \begin{pmatrix} \frac{2\rho}{p-1} \\ 1 \end{pmatrix}.$$

Eq. (4.33) is equivalent to $\Phi(-\log T) = \boldsymbol{U}(\boldsymbol{v}, T)$. The point is that $\boldsymbol{v}$ denotes the data relative to ψ^1 such that we have clearly separated the functional dependence of the initial data on the free functions (f, g) (or $\boldsymbol{v}$, respectively) and the blow up time T. In the following we are interested in mild solutions of the equation

$$\begin{cases} \frac{d}{d\tau}\Psi(\tau) = \boldsymbol{L}\Psi(\tau) + \boldsymbol{N}(\Psi(\tau)) \text{ for } \tau > 0 \\ \Psi(0) = \boldsymbol{U}(\boldsymbol{v}, T) \end{cases} \tag{4.36}$$

such that a solution of Eq. (4.32) for a particular $T > 0$ can be obtained by setting $\Phi(\tau) := \Psi(\tau + \log T)$. As discussed in Section 3.2.5 we restrict T to the interval $\mathcal{I} := (\frac{1}{2}, \frac{3}{2})$, which is no limitation since our argument is perturbative around $T = 1$. We define the initial data as a function of the free data $\boldsymbol{v}$ and the blow up time T on $\mathfrak{H} \times \mathcal{I}$ where

$$\mathfrak{H} := \{u \in H^1(0, \tfrac{3}{2}) : u(0) = 0\} \times H^1(0, \tfrac{3}{2}).$$

Lemma 4.2.9. *The function $\boldsymbol{U} : \mathfrak{H} \times \mathcal{I} \to \mathcal{H}$ is continuous and $\boldsymbol{U}(\boldsymbol{0}, 1) = \boldsymbol{0}$. Furthermore $\boldsymbol{U}(\boldsymbol{0}, \cdot) : \mathcal{I} \to \mathcal{H}$ is Fréchet differentiable and*

$$[D_T \boldsymbol{U}(\boldsymbol{0}, T)|_{T=1} \lambda](\rho) = \tfrac{2\lambda}{p-1} \kappa_p^{\frac{1}{p-1}} \boldsymbol{g}(\rho),$$

for $\lambda \in \mathbb{R}$ where $\boldsymbol{g}$ denotes the symmetry mode.

Proof. We first consider the function $M : H^1(0,\frac{3}{2}) \times \mathcal{I} \to H^1(0,1)$ defined by $M(v,T)(\rho) := v(T\rho)$ and show that it is continuous. The estimate

$$\begin{aligned}
&\|M(v,T) - M(\tilde{v},T)\|^2_{H^1(0,1)} \\
&= \int_0^1 |v(T\rho) - \tilde{v}(T\rho)|^2 d\rho + T^2 \int_0^1 |v'(T\rho) - \tilde{v}'(T\rho)|^2 d\rho \\
&= \frac{1}{T}\int_0^T |v(\rho) - \tilde{v}(\rho)|^2 d\rho + T \int_0^T |v'(\rho) - \tilde{v}'(\rho)|^2 d\rho \\
&\leq 2\|v - \tilde{v}\|^2_{H^1(0,\frac{3}{2})}
\end{aligned}$$

implies that $M(v,T)$ is continuous with respect to v, uniformly in $T \in \mathcal{I}$. Hence, it is sufficient to prove continuity with respect to T. For any $v, \tilde{v} \in H^1(0,\frac{3}{2})$ and $T, \tilde{T} \in \mathcal{I}$ we have

$$\begin{aligned}
\|M(v,T) - M(v,\tilde{T})\|_{H^1(0,1)} \leq\ & \|M(v,T) - M(\tilde{v},T)\|_{H^1(0,1)} \\
&+ \|M(\tilde{v},T) - M(\tilde{v},\tilde{T})\|_{H^1(0,1)} \\
&+ \|M(\tilde{v},\tilde{T}) - M(v,\tilde{T})\|_{H^1(0,1)} \\
\lesssim\ & \|v - \tilde{v}\|_{H^1(0,\frac{3}{2})} + \|M(\tilde{v},T) - M(\tilde{v},\tilde{T})\|_{H^1(0,1)}
\end{aligned}$$

We use the density of $C^1[0,\frac{3}{2}]$ in $H^1(0,\frac{3}{2})$ to infer that for any given $\varepsilon > 0$ there exists a $\tilde{v} \in C^1[0,\frac{3}{2}]$ such that

$$\begin{aligned}
\|M(v,T) - M(v,\tilde{T})\|^2_{H^1(0,1)} <& \frac{\varepsilon^2}{2} + C\int_0^1 |\tilde{v}(T\rho) - \tilde{v}(\tilde{T}\rho)|^2 d\rho \\
&+ C\int_0^1 |T\tilde{v}'(T\rho) - \tilde{T}\tilde{v}'(\tilde{T}\rho)|^2 d\rho
\end{aligned}$$

and the integral terms tend to zero in the limit $\tilde{T} \to T$ by continuity of $\tilde{v}$ and $\tilde{v}'$. This implies the claimed continuity of M on $H^1(0,\frac{3}{2}) \times \mathcal{I}$. Thus, for $\boldsymbol{v} = (v_1, v_2) \in \mathfrak{H}$, $T \in \mathcal{I}$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ as defined above, the function $\boldsymbol{U}$ can be written as

$$\boldsymbol{U}(\boldsymbol{v}, T) = \begin{pmatrix} T^{\frac{2}{p-1}}[M(v_1,T) + M(\kappa_1,T)] - \kappa_1 \\ T^{\frac{2}{p-1}}[M(v_2,T) + M(\kappa_2,T)] - \kappa_2 \end{pmatrix}.$$

The properties of M imply that $[\boldsymbol{U}(\boldsymbol{v},T)]_j \in H^1(0,1)$ for $j = 1,2$. Furthermore, we have $[\boldsymbol{U}(\boldsymbol{v},T)]_1(0) = 0$ and $\boldsymbol{U}$ depends continuously on $(\boldsymbol{v}, T)$.
Evaluation yields

$$\boldsymbol{U}(\boldsymbol{0},T)(\rho) = \kappa_p^{\frac{1}{p-1}} \begin{pmatrix} \frac{2\rho}{p-1}\left[T^{\frac{p+1}{p-1}} - 1\right] \\ T^{\frac{2}{p-1}} - 1 \end{pmatrix}$$

and obviously, $\boldsymbol{U}(\boldsymbol{0},\cdot) : \mathcal{I} \to \mathcal{H}$ is differentiable for all $T \in I$. In particular, we have

$$[D_T \boldsymbol{U}(\boldsymbol{0},T)|_{T=1}\lambda](\rho) = \frac{2\lambda}{p-1}\kappa_p^{\frac{1}{p-1}} \begin{pmatrix} \frac{p+1}{p-1}\rho \\ 1 \end{pmatrix}$$

which concludes the proof. □

Since we interested in mild solutions of (4.36), we use Duhamel's formula to obtain

$$\Psi(\tau) = \mathbf{S}(\tau)\boldsymbol{U}(\boldsymbol{v}, T) + \int_0^\tau \mathbf{S}(\tau - \tau')\boldsymbol{N}(\Psi(\tau'))d\tau' \quad \text{for} \quad \tau \geq 0. \tag{4.37}$$

In the following, (4.37) will be studied in the function space $\mathcal{X}$ given by

$$\mathcal{X} := \left\{ \Psi \in C([0, \infty), \mathcal{H}) : \sup_{\tau > 0} e^{\mu_p \tau} \|\Psi(\tau)\| < \infty \right\}.$$

where the exponent μ_p was defined in Lemma 4.2.6.

Remark. We note that in the above formulation of the problem the particular properties of the underlying function space $\mathcal{H}$ are hidden in the abstract setting. From now on, the proofs mainly rely on the estimates for the nonlinearity (Lemma 4.2.8) and the semigroup on the stable and unstable subspaces (Lemma 4.2.6). Therefore, most of the subsequent analysis can be copied from Section 4.1. Hence, we will only sketch the proofs of the following results and refer the reader to Section 4.1 for the details of the calculations.

Global existence for corrected (small) initial data

The main problem which has to be addressed first is the exponential growth of the semigroup on the unstable subspace. As in the previous section we introduce a correction term and consider the fixed point problem

$$\Psi = \mathbf{K}(\Psi, \boldsymbol{U}(\boldsymbol{v}, T)) \tag{4.38}$$

where

$$\begin{aligned} \mathbf{K}(\Psi, \boldsymbol{u})(\tau) := &\mathbf{S}(\tau)(1 - \boldsymbol{P})\boldsymbol{u} - \int_0^\infty e^{\tau - \tau'} \boldsymbol{P}\boldsymbol{N}(\Psi(\tau'))d\tau' \\ &+ \int_0^\tau \mathbf{S}(\tau - \tau')\boldsymbol{N}(\Psi(\tau'))d\tau'. \end{aligned} \tag{4.39}$$

Note that $\Psi = \mathbf{K}(\Psi, \boldsymbol{U}(\boldsymbol{v}, T))$ corresponds to the original equation (4.37) for initial data modified by

$$-\boldsymbol{P}\left[\boldsymbol{U}(\boldsymbol{v}, T) + \int_0^\infty e^{-\tau'} \boldsymbol{N}(\Psi(\tau'))d\tau'\right],$$

an element of the unstable subspace $\boldsymbol{P}\mathcal{H}$ depending on the solution itself. As we will see, this correction forces decay of the solution. In the following we restrict ourselves to a closed ball $\mathcal{X}_\delta \subset \mathcal{X}$ defined by

$$\mathcal{X}_\delta := \{\Psi \in \mathcal{X} : \|\Psi\|_{\mathcal{X}} \leq \delta\}$$

for $\delta > 0$. Recall that $\boldsymbol{U}(\boldsymbol{0}, 1) = \boldsymbol{0}$, such that by continuity $\|\boldsymbol{U}(\boldsymbol{v}, T)\|$ is small for $\boldsymbol{v}$ small and T close to 1.

Theorem 4.2.10. *Let $\mathcal{U} \subset \mathcal{H}$ be a sufficiently small neighborhood of $\mathbf{0}$. Then, for any $\boldsymbol{u} \in \mathcal{U}$, there exists a unique $\Psi_{\boldsymbol{u}} \in \mathcal{X}$ which satisfies*

$$\Psi_{\boldsymbol{u}} = \mathbf{K}(\Psi_{\boldsymbol{u}}, \boldsymbol{u}).$$

Furthermore, the map $\boldsymbol{\Psi} : \mathcal{U} \to \mathcal{X}$ defined by $\boldsymbol{\Psi}(\boldsymbol{u}) := \Psi_{\boldsymbol{u}}$ is Fréchet differentiable at $\boldsymbol{u} = \mathbf{0}$. In particular, $\boldsymbol{\Psi}(\mathbf{U}(\boldsymbol{v}, T))$ exists provided $\boldsymbol{v} \in \mathfrak{H}$ is sufficiently small and T is sufficiently close to 1.

Proof. In the following we refer the reader to Section 4.1 for the details of the calculations.
Using the results of Lemma 4.2.8 one immediately obtains

$$\begin{aligned} \|\boldsymbol{N}(\Psi(\tau))\| &\lesssim \delta^2 e^{-2\mu_p \tau}, \\ \|\boldsymbol{N}(\Psi(\tau)) - \boldsymbol{N}(\Phi(\tau))\| &\lesssim \delta e^{-\mu_p \tau} \|\Psi(\tau) - \Phi(\tau)\| \end{aligned} \tag{4.40}$$

for $0 < \delta < 1$, $\Phi, \Psi \in \mathcal{X}_\delta$ and all $\tau \geq 0$.
Note, that the integrals in (4.39) exist as Riemann integrals over continuous functions for $(\Psi, \boldsymbol{u}) \in \mathcal{X}_\delta \times \mathcal{H}$. We decompose $\mathbf{K}$ according to

$$\mathbf{K}(\Psi, \boldsymbol{u})(\tau) = \boldsymbol{P}\mathbf{K}(\Psi, \boldsymbol{u})(\tau) + (1 - \boldsymbol{P})\mathbf{K}(\Psi, \boldsymbol{u})(\tau)$$

and show that $\mathbf{K}(\Psi, \boldsymbol{u}) \in \mathcal{X}_\delta$ for $\Psi \in \mathcal{X}_\delta$, $\|\boldsymbol{u}\| \leq \delta^2$, and δ sufficiently small. Using the estimates for the semigroup $\mathbf{S}$ on $\boldsymbol{P}\mathcal{H}$ and $(1 - \boldsymbol{P})\mathcal{H}$, cf. Lemma 4.2.6, together with (4.40) it is easy to see that for $\|\boldsymbol{u}\| \leq \delta^2$ and $\tau \geq 0$ we have

$$\begin{aligned} \|\boldsymbol{P}\mathbf{K}(\Psi, \boldsymbol{u})(\tau)\| &\lesssim \delta^2 e^{-2\mu_p \tau}, \\ \|(1 - \boldsymbol{P})\mathbf{K}(\Psi, \boldsymbol{u})(\tau)\| &\lesssim \delta^2 e^{-\mu_p \tau} \end{aligned}$$

which implies $\|\mathbf{K}(\Psi, \boldsymbol{u})(\tau)\| \leq \delta e^{-\mu_p \tau}$ provided $\delta > 0$ is sufficiently small. Continuity of $\mathbf{K}(\Psi, \boldsymbol{u})$ with respect to τ follows essentially from strong continuity of the semigroup $\mathbf{S}$ and we conclude that $\mathbf{K}(\Psi, \boldsymbol{u}) \in \mathcal{X}_\delta$.
To see that $\mathbf{K}(\cdot, \boldsymbol{u})$ is contracting we again use Lemma 4.2.6 and (4.40) to infer that

$$\begin{aligned} \|\boldsymbol{P}[\mathbf{K}(\Phi, \boldsymbol{u})(\tau) - \mathbf{K}(\Psi, \boldsymbol{u})(\tau)]\| &\lesssim \delta e^{-2\mu_p \tau} \|\Phi - \Psi\|_{\mathcal{X}} \\ \|(1 - \boldsymbol{P})[\mathbf{K}(\Phi, \boldsymbol{u})(\tau) - \mathbf{K}(\Psi, \boldsymbol{u})(\tau)]\| &\lesssim \delta e^{-\mu_p \tau} \|\Phi - \Psi\|_{\mathcal{X}} \end{aligned}$$

for $\Psi, \Phi \in \mathcal{X}_\delta$ and $\tau \geq 0$. In particular, for δ sufficiently small, we obtain

$$\|\mathbf{K}(\Phi, \boldsymbol{u}) - \mathbf{K}(\Psi, \boldsymbol{u})\|_{\mathcal{X}} \leq \tfrac{1}{2} \|\Phi - \Psi\|_{\mathcal{X}}.$$

We apply the Banach fixed point theorem to infer that for any $\boldsymbol{u} \in \mathcal{U}$, the equation

$$\Psi = \mathbf{K}(\Psi, \boldsymbol{u})$$

has a unique solution $\Psi_{\boldsymbol{u}}$ in the closed subset $\mathcal{X}_\delta$ provided $\mathcal{U} \subset \mathcal{H}$ is a sufficiently small neighborhood around $\mathbf{0}$. Furthermore, standard arguments imply that this is in fact the unique solution in the whole space $\mathcal{X}$.
The Banach fixed point theorem implies that the solution depends continuously on the initial data, i.e., the map $\mathbf{\Psi} : \mathcal{U} \to \mathcal{X}$ is continuous. In particular, for $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in \mathcal{U}$ and the corresponding solutions $\mathbf{\Psi}(\boldsymbol{u}), \mathbf{\Psi}(\tilde{\boldsymbol{u}}) \in \mathcal{X}_\delta$ it is easy to see that

$$\|\mathbf{\Psi}(\boldsymbol{u}) - \mathbf{\Psi}(\tilde{\boldsymbol{u}})\|_{\mathcal{X}} \lesssim \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|, \tag{4.41}$$

cf. the proof of Theorem 4.1.16 in Section 4.1.
In order to prove differentiability of $\mathbf{\Psi}(\boldsymbol{u})$ at $\boldsymbol{u} = \mathbf{0}$ we define

$$[\tilde{D}\mathbf{\Psi}(\mathbf{0})\boldsymbol{u}](\tau) := \mathbf{S}(\tau)(1 - \boldsymbol{P})\boldsymbol{u}$$

and note that $\tilde{D}\mathbf{\Psi}(\mathbf{0}) : \mathcal{H} \to \mathcal{X}$ is linear and bounded. We claim that $\tilde{D}\mathbf{\Psi}(\mathbf{0})$ is the Fréchet derivative of $\mathbf{\Psi}$ at $\mathbf{0}$. To prove this, we have to show that (recall that $\mathbf{\Psi}(\mathbf{0}) = \mathbf{0}$)

$$\lim_{\tilde{\boldsymbol{u}} \to \mathbf{0}} \frac{1}{\|\tilde{\boldsymbol{u}}\|} \|\mathbf{\Psi}(\tilde{\boldsymbol{u}}) - \tilde{D}\mathbf{\Psi}(\mathbf{0})\tilde{\boldsymbol{u}}\|_{\mathcal{X}} = 0.$$

For small $\tilde{\boldsymbol{u}}$ we have $\mathbf{\Psi}(\tilde{\boldsymbol{u}}) = \mathbf{K}(\mathbf{\Psi}(\tilde{\boldsymbol{u}}), \tilde{\boldsymbol{u}})$ and by definition we infer

$$\begin{aligned}\mathbf{\Psi}(\tilde{\boldsymbol{u}}) - \tilde{D}\mathbf{\Psi}(\mathbf{0})\tilde{\boldsymbol{u}} = &\int_0^\tau \mathbf{S}(\tau - \tau')\boldsymbol{N}(\mathbf{\Psi}(\tilde{\boldsymbol{u}})(\tau'))d\tau' \\ &- \int_0^\infty e^{\tau-\tau'}\boldsymbol{P}\boldsymbol{N}(\mathbf{\Psi}(\tilde{\boldsymbol{u}})(\tau'))d\tau' =: \mathbf{G}(\tilde{\boldsymbol{u}})(\tau)\end{aligned}$$

By using the decomposition

$$\mathbf{G}(\tilde{\boldsymbol{u}})(\tau) = \boldsymbol{P}[\mathbf{G}(\tilde{\boldsymbol{u}})(\tau)] + (1 - \boldsymbol{P})[\mathbf{G}(\tilde{\boldsymbol{u}})(\tau)],$$

the estimates for the nonlinearity and the semigroup, as well as (4.41) we obtain

$$\|\mathbf{G}(\tilde{\boldsymbol{u}})\|_{\mathcal{X}} \lesssim \|\tilde{\boldsymbol{u}}\|^2$$

which implies the claim.
Finally, since $\mathbf{U} : \mathfrak{H} \times \mathcal{I} \to \mathcal{H}$ is continuous and $\mathbf{U}(\mathbf{0}, 1) = \mathbf{0}$ (Lemma 4.2.9), it follows that $\mathbf{U}(\boldsymbol{v}, T) \in \mathcal{U}$ for all $\boldsymbol{v}$ sufficiently small and T sufficiently close to 1. $\square$

Global existence for arbitrary (small) initial data

We use the results of the previous section to obtain a global solution of the integral equation (4.37). In the following let $\mathfrak{U} \subset \mathfrak{H}$ be a sufficiently small open neighborhood of $\mathbf{0}$ and let $\mathcal{J} \subset \mathcal{I}$ be a sufficiently small open neigborhood of 1. For $(\boldsymbol{v}, T) \in \mathfrak{U} \times \mathcal{J}$, Theorem 4.2.10 yields the existence of a global solution $\boldsymbol{\Psi}(\boldsymbol{U}(\boldsymbol{v},T)) \in \mathcal{X}$ of the modified equation, which can be written as

$$\begin{aligned}\boldsymbol{\Psi}(\boldsymbol{U}(\boldsymbol{v},T))(\tau) =& \mathbf{S}(\tau)\boldsymbol{U}(\boldsymbol{v},T) + \int_0^\tau \mathbf{S}(\tau-\tau')\boldsymbol{N}(\boldsymbol{\Psi}(\boldsymbol{U}(\boldsymbol{v},T))(\tau'))d\tau' \\ &- e^\tau \boldsymbol{F}(\boldsymbol{v},T) \end{aligned} \tag{4.42}$$

for $\tau \geq 0$ where

$$\boldsymbol{F}(\boldsymbol{v},T) := \boldsymbol{P}\left[\boldsymbol{U}(\boldsymbol{v},T) + \int_0^\infty e^{-\tau'}\boldsymbol{N}(\boldsymbol{\Psi}(\boldsymbol{U}(\boldsymbol{v},T))(\tau'))d\tau'\right].$$

Note that for $\boldsymbol{v} = \mathbf{0}$ and $T = 1$ we have $\boldsymbol{U}(\mathbf{0},1) = \mathbf{0}$ and thus, $\boldsymbol{F}(\mathbf{0},1) = \mathbf{0}$. Hence, (4.42) reduces to (4.37) and $\boldsymbol{\Psi}(\boldsymbol{U}(\mathbf{0},1)) = \mathbf{0}$ solves the original equation. In the following, we extend this to a neighborhood of $(\mathbf{0},1)$.

Lemma 4.2.11. *The function* $\boldsymbol{F} : \mathfrak{U}\times\mathcal{J} \subset \mathfrak{H}\times\mathcal{I} \to \langle \boldsymbol{g}\rangle$ *is continuous. Furthermore,* $\boldsymbol{F}(\mathbf{0},\cdot) : \mathcal{J} \to \langle \boldsymbol{g}\rangle$ *is Fréchet differentiable at* 1 *and*

$$D_T\boldsymbol{F}(\mathbf{0},T)|_{T=1}\lambda = \tfrac{2\lambda}{p-1}\kappa_p^{\frac{1}{p-1}}\boldsymbol{g}$$

for all $\lambda \in \mathbb{R}$*. As a consequence, for every* $\boldsymbol{v} \in \mathfrak{U}$ *there exists a* $T \in \mathcal{J}$ *such that* $\boldsymbol{F}(\boldsymbol{v},T) = \mathbf{0}$*.*

Proof. To show continuity we rewrite the correction in a more abstract way by introducing operators $\mathbf{B} : \mathcal{X} \to \mathcal{H}$ and $\mathbf{N} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathbf{B}\Psi := \int_0^\infty e^{-\tau}\Psi(\tau)d\tau, \quad \mathbf{N}(\Psi)(\tau) := \boldsymbol{N}(\Psi(\tau)).$$

One can easily check that $\mathbf{B}$ is linear and bounded. Furthermore, the properties of the operator $\boldsymbol{N}$ described in Lemma 4.2.8 imply the $\mathbf{N}$ is continuous, differentiable at $\mathbf{0} \in \mathcal{X}$, and

$$D\mathbf{N}(\mathbf{0})\Psi = \mathbf{0} \quad \text{for} \quad \Psi \in \mathcal{X}, \tag{4.43}$$

see also Section 4.1, proof of Lemma 4.1.18. Thus, $\boldsymbol{F}$ can be written as a composition of continuous operators

$$\boldsymbol{F}(\boldsymbol{v},T) = \boldsymbol{P}\left[\boldsymbol{U}(\boldsymbol{v},T) + \mathbf{B}\mathbf{N}(\boldsymbol{\Psi}(\boldsymbol{U}(\boldsymbol{v},T)))\right].$$

For $\boldsymbol{v} = \mathbf{0}$ fixed the right-hand side is differentiable with respect to T at $T = 1$, see Lemma 4.2.9, (4.43) and Theorem 4.2.10, and we obtain

$$\begin{aligned} D_T\boldsymbol{F}(\mathbf{0},T)|_{T=1}\lambda &= \boldsymbol{P}D_T\boldsymbol{U}(\mathbf{0},T)|_{T=1}\lambda \\ &\quad + \boldsymbol{P}\mathbf{B}D\mathbf{N}(\mathbf{0})D\boldsymbol{\Psi}(\mathbf{0})D_T\mathbf{U}(\mathbf{0},T)|_{T=1}\lambda \\ &= \boldsymbol{P}D_T\boldsymbol{U}(\mathbf{0},T)|_{T=1}\lambda = \tfrac{2\lambda}{p-1}\kappa_p^{\frac{1}{p-1}}\boldsymbol{g}. \end{aligned}$$

Now we prove the second claim using the fact that the range of $\boldsymbol{F}$ is contained in the one-dimensional vector space $\langle \boldsymbol{g} \rangle$. Let $I : \langle \boldsymbol{g} \rangle \to \mathbb{R}$ be the isomorphism given by $I(c\boldsymbol{g}) = c$ for $c \in \mathbb{R}$. We define a real-valued, continuous function $f : \mathfrak{U} \times \mathcal{J} \to \mathbb{R}$ by $f = I \circ \boldsymbol{F}$. In particular, $f(\mathbf{0}, \cdot) : \mathcal{J} \to \mathbb{R}$ is continuous, differentiable at 1, and $D_T f(\mathbf{0}, T)|_{T=1} \neq \mathbf{0}$. Consequently, there exist $T^+, T^- \in \mathcal{J}$ such that $f(\mathbf{0}, T^-) < 0$ and $f(\mathbf{0}, T^+) > 0$. Since f is continuous in the first argument, we have $f(\boldsymbol{v}, T^+) > 0$ and $f(\boldsymbol{v}, T^-) < 0$ for all $\boldsymbol{v} \in \tilde{\mathfrak{U}} \subset \mathfrak{U}$ provided $\tilde{\mathfrak{U}}$ is sufficiently small. Consequently, by the intermediate value theorem we conclude that there exists a T^* (depending on $\boldsymbol{v}$) such that $f(\boldsymbol{v}, T^*) = I(\boldsymbol{F}(\boldsymbol{v}, T^*)) = 0$ implying that $\boldsymbol{F}(\boldsymbol{v}, T^*) = \mathbf{0}$. ☐

Theorem 4.2.12. *Let $\boldsymbol{v} \in \mathfrak{H}$ be sufficiently small. Then there exists a T close to 1 such that*

$$\Psi(\tau) = \mathbf{S}(\tau)\mathbf{U}(\boldsymbol{v}, T) + \int_0^\tau \mathbf{S}(\tau - \tau')\boldsymbol{N}(\Psi(\tau'))d\tau'$$

has a continuous solution $\Psi : [0, \infty) \to \mathcal{H}$ satisfying

$$\|\Psi(\tau)\| \leq \delta e^{-\mu_p \tau}$$

for all $\tau \geq 0$ and some $\delta \in (0, 1)$. Moreover, the solution is unique in $C([0, \infty), \mathcal{H})$.

Proof. The existence of a unique solution in $\mathcal{X}_\delta$ is a direct consequence of Theorem 4.2.10 and Lemma 4.2.11. The stated decay estimate follows from the definition of the space $\mathcal{X}_\delta$. For the uniqueness of the solution in the space $C([0, \infty), \mathcal{H})$ we refer the reader to the proof of Theorem 4.1.20 in Section 4.1. ☐

Proof of Theorem B

Proof. Choose $\varepsilon > 0$ such that $\mu_p = \frac{2}{p-1} - \varepsilon > 0$ and let the initial data (f, g) satisfy the assumptions of Theorem B. We set

$$v_1(r) := rg(r) - \tfrac{2r}{p-1}\kappa_p^{\frac{1}{p-1}}, \quad v_2(r) := f(r) + rf'(r) - \kappa_p^{\frac{1}{p-1}}.$$

By definition of the respective function spaces it is easy to see that

$$\|\boldsymbol{v}\|_{\mathfrak{H}} = \|(f, g) - (\psi^1(0, \cdot), \psi^1_t(0, \cdot))\|_{\mathcal{E}^h(\frac{3}{2})}$$

where $\boldsymbol{v} = (v_1, v_2)$. Hence, the smallness condition in Theorem B implies that $\boldsymbol{v}$ is so small that Theorem 4.2.12 applies. We infer that for a certain value $T > 0$ close to one (depending on $\boldsymbol{v}$) we obtain a unique mild solution $\Psi \in C([0, \infty), \mathcal{H})$ of (4.36). Setting $\Phi(\tau) := \Psi(\tau + \log T)$ yields a unique mild solution $\Phi \in C((-\log T, \infty], \mathcal{H})$ of

$$\begin{cases} \frac{d}{d\tau}\Phi(\tau) = \boldsymbol{L}\Psi(\tau) + \boldsymbol{N}(\Psi(\tau)) \text{ for } \tau > -\log T \\ \Phi(-\log T) = \boldsymbol{U}(\boldsymbol{v}, T) \end{cases}$$

satisfying

$$\|\Phi(\tau)\| \leq C_\varepsilon e^{-\mu_p \tau}$$

for all $\tau \geq -\log T$ and a constant $C_\varepsilon > 0$. By definition,

$$\Phi(\tau)(\rho) = (\phi_1(\tau,\rho), \phi_2(\tau,\rho))$$

is a solution of the original system (3.13). Using the identity (4.3) we infer

$$\begin{aligned}\|(\psi(t,\cdot),&\psi_t(t,\cdot)) - (\psi^T(t,\cdot),\psi_t^T(t,\cdot))\|^2_{\mathcal{E}^h(T-t)} = \\ &(T-t)^{-\frac{4}{p-1}} \int_0^{T-t} \Big(|\phi_1(-\log(T-t), \tfrac{r}{T-t})|^2 \\ &+ |\phi_2(-\log(T-t), \tfrac{r}{T-t})|^2 \Big) dr \\ &+ (T-t)^{-\frac{2(p+1)}{p-1}} \int_0^{T-t} \Big(|\partial_2\phi_1(-\log(T-t), \tfrac{r}{T-t})|^2 \\ &+ |\partial_2\phi_2(-\log(T-t), \tfrac{r}{T-t})|^2 \Big) dr\end{aligned}$$

and thus,

$$\begin{aligned}\|(\psi(t,\cdot),\psi_t(t,\cdot)) &- (\psi^T(t,\cdot),\psi_t^T(t,\cdot))\|^2_{\mathcal{E}^h(T-t)} = \\ &= (T-t)^{\frac{p-5}{p-1}} \int_0^1 \Big(|\phi_1(-\log(T-t),\rho)|^2 + |\phi_2(-\log(T-t),\rho)|^2 \Big) d\rho \\ &\quad + (T-t)^{-\frac{p+3}{p-1}} \int_0^1 \Big(|\partial_\rho\phi_1(-\log(T-t),\rho)|^2 \\ &\quad + |\partial_\rho\phi_2(-\log(T-t),\rho)|^2 \Big) d\rho \\ &\leq (T-t)^{-\frac{p+3}{p-1}} \|\Phi(-\log(T-t))\|^2 \leq C_\varepsilon (T-t)^{-\frac{p+3}{p-1}+\frac{4}{p-1}-2\varepsilon}\end{aligned}$$

which implies the claimed estimate. □

Chapter 5

Discussion of possible extensions and generalizations

5.1 Refinement of the results for the wave equation

For $1 < p \leq 3$ we proved nonlinear stability of the self-similar ground state solution ψ^T of the (NLW) in a topology corresponding to a local energy space. For $p > 3$, this was established in a topology which is stronger than the energy. However, as mentioned in Chapter 3, the blow up of ψ^T can be detected in the local energy norm for $1 < p < 5$. This strongly suggests that the result for the (sub)conformal case can be extended to hold in the full energy subcritical range. The reason for the limitation to (sub)conformal powers is the fact that the required estimates for the nonlinearity cannot be established for higher values of p without further regularity assumptions. In view of the local well-posedness theory for the nonlinear wave equation for $3 < p < 5$ in the energy space, where additional regularity is required in form of Strichartz-estimates, it would be interesting to investigate the existence of similar estimates for the linear equation in self-similar coordinates.

Certainly, this could also be important to improve the result for $p = 5$. However, the fact that we do not obtain exponential decay for the free semigroup in the local energy space, cf. Chapter 3, complicates matters. Nevertheless, it is certainly a challenging future project to prove stability of the ODE blow up solution for the critical wave equation in the energy topology.

Another obvious generalization is to remove the symmetry assumption. This will yield additional symmetry modes for the linearized problem. However, the ground state solution ψ^T is invariant under spatial translations, rotations and scaling. Hence, only Lorentz-transformations will play a role, apart from the time translation invariance, which is of course still present. For the conformal wave equation this is demonstrated in a recent work by Donninger and Zenginoğlu, where the hyperboloidal initial value problem for the (NLW) with $p = 3$ is considered without any symmetry assumptions. The authors show the existence of a co-dimension 4

Lipschitz manifold of initial data that lead to global solutions in forward time which do not scatter to free waves, see [29] for a precise formulation of the problem and the results. The proof of the main result partially relies on the generalization of the techniques described in Chapter 3 to non-radial perturbations. A detailed discussion of the main ideas from [29] would go beyond the scope of this work. Nevertheless, we want to point out that this is certainly a new and exciting development which may also open up future fields of application.

5.2 Generalization of the method to Banach spaces

For further applications it might be useful to generalize the presented approach to arbitrary Banach spaces, i.e., to remove the requirement of a Hilbert space structure. Concerning the linear stability analysis the latter condition is in fact crucial for the application of the theorem by Gearhart, Prüss, Hwang and Greiner, see Lemma A.2.1, which does not hold in arbitrary Banach spaces. We therefore discuss a different strategy to prove that the growth bound of the subspace semigroup $\mathcal{S}_{\mathcal{N}}$ is equal to the spectral bound of its generator $\boldsymbol{L}_{\mathcal{N}}$. This argument is actually quite simple and relies on the compactness of the perturbation. It can be formulated using the notion of the essential growth bound of a strongly continuous semigroup as defined in Section A.2. In the following discussion we restrict ourselves to $p > 3$.

Recall from Chapter 3 that the linearized operator is given by $\boldsymbol{L} = \boldsymbol{L}_0 + \boldsymbol{L}'$. For the semigroup $\mathcal{S}_0 = (\mathbf{S}_0(\tau))_{\tau \geq 0}$ on $\mathcal{H}^{p>3}$ generated by $\boldsymbol{L}_0$ we obtain

$$\omega_0(\mathcal{S}_0) \leq \mu_0(\mathcal{S}_0) = -\tfrac{2}{p-1}$$

where $\omega_0(\mathcal{S}_0)$ denotes the growth bound of the free semigroup. For the essential growth bound this implies that

$$\omega_{\text{ess}}(\mathcal{S}_0) \leq \omega_0(\mathcal{S}_0) \leq -\tfrac{2}{p-1}.$$

Adding the compact perturbation yields the semigroup $\mathcal{S} = (\mathbf{S}(\tau))_{\tau \geq 0}$ generated by $\boldsymbol{L}$. Now, the two semigroups can be related by the variation of parameters formula,

$$\mathbf{S}(\tau) = \mathbf{S}_0(\tau) + \int_0^\tau \mathbf{S}_0(\tau - s)\boldsymbol{L}'\mathbf{S}(s)ds \tag{5.1}$$

where compactness of the perturbation implies compactness of the integral term, see [36], p. 258, Prop. 2.12. In view of the stability of the essential spectrum under compact perturbations we obtain

$$r_{\text{ess}}(\mathbf{S}(\tau)) = r_{\text{ess}}(\mathbf{S}_0(\tau))$$

such that

$$\omega_{\text{ess}}(\mathcal{S}) = \omega_{\text{ess}}(\mathcal{S}_0) \leq \omega_0(\mathcal{S}_0) \leq -\tfrac{2}{p-1}.$$

Recall that the spectral projection $\boldsymbol{P}$ as defined in the proof of Lemma 4.2.6 is of rank one. Hence, the operator $\mathbf{S}(\tau)\boldsymbol{P}$ is compact for all $\tau \geq 0$ such that

$$r_{\text{ess}}(\mathbf{S}(\tau)) = r_{\text{ess}}(\mathbf{S}(\tau)(1-\boldsymbol{P})) = r_{\text{ess}}(\mathbf{S}_{\mathcal{N}}(\tau))$$

and

$$\omega_{\text{ess}}(\mathcal{S}_{\mathcal{N}}) = \omega_{\text{ess}}(\mathcal{S}).$$

Having this, we are interested in the growth bound $\omega_0(\mathcal{S}_{\mathcal{N}})$. Recall that on the stable subspace the spectral bound of the generator satisfies

$$s(\boldsymbol{L}_{\mathcal{N}}) \leq -\tfrac{2}{p-1}.$$

Furthermore, we know that

$$s(\boldsymbol{L}_{\mathcal{N}}) \leq \omega_0(\mathcal{S}_{\mathcal{N}}), \quad \omega_{\text{ess}}(\mathcal{S}_{\mathcal{N}}) \leq \omega_0(\mathcal{S}_{\mathcal{N}}).$$

Along the lines of [36], p. 258, we argue as follows. Assume that $\omega_{\text{ess}}(\mathcal{S}_{\mathcal{N}}) < \omega_0(\mathcal{S}_{\mathcal{N}})$. Then there is an eigenvalue μ of $\mathbf{S}_{\mathcal{N}}(\tau)$ satisfying

$$|\mu| = r(\mathbf{S}_{\mathcal{N}}(\tau)) = e^{\omega_0(\mathcal{S}_{\mathcal{N}})\tau}.$$

By the spectral mapping theorem, [36], p. 277, Theorem 3.7,

$$\sigma_p(\mathbf{S}_{\mathcal{N}}(\tau)) \setminus \{0\} = e^{\tau\sigma_p(\boldsymbol{L}_{\mathcal{N}})}$$

which shows that there exists an eigenvalue λ of $\boldsymbol{L}_{\mathcal{N}}$ with $\text{Re}\lambda = \omega_0(\mathcal{S}_{\mathcal{N}})$ such that $s(\boldsymbol{L}_{\mathcal{N}}) = \omega_0(\mathcal{S}_{\mathcal{N}})$. We conclude that

$$\omega_0(\mathcal{S}_{\mathcal{N}}) = \max\{s(\boldsymbol{L}_{\mathcal{N}}), \omega_{\text{ess}}(\mathcal{S}_{\mathcal{N}})\} \leq -\tfrac{2}{p-1}.$$

Note that all other arguments that were used in the linear perturbation theory can be generalized to arbitrary Banach spaces. In particular, the Lumer-Phillips theorem, which is given in Lemma A.1.2 for Hilbert spaces, can be formulated in a much more general setting, see [36], p. 83, Theorem 3.15.

5.3 Semilinear heat equations

In view of possible future applications of the presented techniques, we discuss known results regarding the formation of singularities via self-similar solutions for semilinear heat equations. In the stability analysis for hyperbolic problems such as the (NLW), the wave maps equation (1.1) and the Yang-Mills equation (1.2), the Cauchy-problem could be restricted to the backward lightcone of the blow up point. Note that this is no longer the case for parabolic problems.

The heat equation with focusing power nonlinearity

As an analogue of the (NLW) we consider the semilinear heat equation

$$\psi_t - \Delta\psi = |\psi|^{p-1}\psi. \tag{NLH}$$

Similarly, there is an ODE blow up solution which is given by

$$\psi^T_{NLH} = \beta^\beta (T-t)^{-\frac{1}{p-1}}, \quad \beta = \tfrac{1}{p-1}. \tag{5.2}$$

This model has been the subject of extensive research throughout the past decades. Typically, the (NLH) is considered for $x \in \mathbb{R}^n$ or on a bounded domain $\Omega \subset \mathbb{R}^n$ with appropriate boundary conditions. Again we restrict the discussions to $n = 3$. As opposed to the (NLW) the solution defined in Eq. (5.2) is the *unique* self-similar solution in the case $1 < p \leq 5$, see [41]. In a series of papers, [41], [42], [43], Giga and Kohn considered the (NLH) in similarity coordinates

$$\xi = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t).$$

They showed that a solution which blows up at $(T, 0)$ converges uniformly to ψ^T_{NLH} in any backward space-time parabola. A perturbation analysis around ψ^T_{NLH} in self-similar variables was already discussed by Fillipas and Kohn in [39], partially on a formal level, see also the references therein. The corresponding linearized operator is self-adjoint on a weighted Lebesgue-space with exponentially decaying weight function. The eigenvalue problem can be solved in terms of Hermite polynomials. However, the considerations in [39] imply that due to the presence of neutral modes, generic perturbations approach the ground state with a rate which is algebraic rather than exponential (in similarity coordinates). Moreover, the authors discuss complications in the formulation of the problem as an abstract ODE. These are connected to difficulties in regard to the definition of an appropriate function space. Certain aspects of this problem will also play a role for the Yang-Mills heat flow considered in Chapter 5, see also below.

Further details of singularity formation in the subcritical case were investigated by Merle and Zaag, see for example [75], [76], [77] and the references therein. For supercritical nonlinearities $p > 5$, the self-similar solution given in Eq. (5.2) is no longer unique, which severely complicates the analysis. In the radial case, convergence to ψ^T_{NLH} was established by Matos in [74], [73] for solutions that blow up outside the origin. The supercritical equation was further studied by Matano and Merle in a series of papers [70], [71], [72]. In the last-mentioned work the authors investigate the time evolution for one-parameter families of initial data. It is proved that for all but finitely many values of the parameter above a certain threshold, the corresponding solution blows up in a self-similar manner and converges to the constant profile β^β.

The harmonic maps heat flow

Another equation for which blow up via self-similar solutions can be observed is the *heat flow for co-rotational harmonic maps* from $\mathbb{R}^n$ to the $n-$sphere in dimensions $3 \leq n \leq 6$. The corresponding equation is given by

$$\psi_t = \psi_{rr} + \frac{n-1}{r}\psi_r - \frac{n-1}{2r^2}\sin(2\psi) \tag{5.3}$$

where $\psi(t,r) := \psi(t,|x|)$ for $x \in \mathbb{R}^n$, see for example [8] for a derivation. As far as scaling is concerned, we note that if ψ is a solution then so is

$$\psi_\lambda(t,r) := \psi(t/\lambda^2, r/\lambda).$$

The existence of a countable family of self-similar blow up solutions was rigorously established by Fan [38]. The self-similar profiles were also constructed numerically in [8]. In [8] Biernat and Bizoń performed numerical experiments which show profile convergence of generic blow up solutions to the self-similar ground state solution. As for the models described above, it is strongly suggested that this describes the generic behavior of singular solutions. Here, a stability analysis is severely complicated by the fact that the ground state is not known in closed form.

The Yang-Mills heat flow

We discuss another model which is related to Eq. (1.2) presented in Chapter 1. Note that Eq. (1.2) is a special case of a Yang-Mills equation for a certain geometrical setting. Yang-Mills theories play a major role in particle physics as gauge theories with non-abelian symmetry groups. The central object is the Yang-Mills functional which is usually considered on Minkowski space. Critical points of this functional satisfy a PDE of hyperbolic type, referred to as the Yang-Mills equation describing the time evolution of gauge fields. In mathematics, gauge theories can be formulated in the context of fiber bundles and differential geometry, where gauge fields are represented by connections on principal $G-$bundles. A deeper discussion of this correspondence would exceed the scope of this work. However, it should be mentioned that Yang-Mills theory led to many important developments in modern differential geometry. To study critical points of the Yang-Mills functional considered over Riemannian manifolds it was suggested to investigate the corresponding gradient flow, see for example [4] or [20]. The resulting PDE is also known as Yang-Mills heat flow, considered here for connections on the trivial bundle $\mathbb{R}^n \times SO(n)$ for $5 \leq n \leq 9$. For a derivation of the model we refer for example to [46] and the references therein. In the $SO(n)-$equivariant setting it reduces to a scalar equation for a radial real valued function $\psi(r,t)$, $r = |x|$, $x \in \mathbb{R}^n$, which reads

$$\psi_t = \psi_{rr} + \frac{n-3}{r}\psi_r - \frac{n-2}{r^2}F(\psi), \tag{5.4}$$

see [46], where

$$F(\psi) = \psi(\psi - 1)(\psi - 2).$$

Finite time blow up of solutions for Eq. (5.4) was established by Grotowski in [46] and it was shown by Gastel [40] that self-similar blow up solutions exist. In [96] Weinkove gave an explicit example of such a solution which reads

$$\psi^T_{\mathrm{YM}}(t, r) = f_{\mathrm{YM}}(\tfrac{r}{\sqrt{T-t}}), \quad f_{YM}(\xi) = \frac{\xi^2}{d_n \xi^2 + c_n} \tag{5.5}$$

with coefficients

$$d_n = \frac{\sqrt{n-2}}{2\sqrt{2}}, \quad c_n = \frac{1}{2}\left(6n - 12 - (n+2)\sqrt{2n-4}\right). \tag{5.6}$$

Note that there is a Lyapunov functional for Eq. (5.4) given by

$$F_\psi = \frac{1}{2}\int_0^\infty r^2 \left(\psi_r^2 + \frac{3\psi^2(\psi-2)^2}{2r^2}\right) dr.$$

To the best knowledge of the author, no numerical experiments have been performed so far in regard to singularity formation for Eq. (5.4). However, a comparison with the above models suggests that the solution given in Eq. (5.5) plays some role in the dynamics of blow up solutions. It was pointed out by Biernat and Bizoń in [8] that the shooting methods used by Fan in [38] could also be applied to Eq. (5.4) to establish the existence of a countable family of self-similar solutions. Due to similarities of (5.5) with the corresponding solution of the Yang-Mills equation, ψ^T_{YM} will be referred to as the ground state solution of Eq. (5.4). In the next Chapter we will discuss the possible application of our approach in order to prove nonlinear stability of this solution under small perturbations.

Chapter 6

Further results and outlook

In the following we present some preliminary results and ideas to prove nonlinear stability of the self-similar ground state solution for the supercritical Yang-Mills heat flow defined in Eq. (5.4).

6.1 Stability analysis for the Yang-Mills heat flow

We restrict ourselves to $n = 5$ for simplicity and study the initial value problem

$$\begin{cases} \psi_t - \psi_{rr} - \frac{2}{r}\psi_r + \frac{3}{r^2}F(\psi) = 0 \text{ for } (t,r) \in (0,T) \times (0,\infty), \\ \psi(0,\cdot) = \psi_0 \text{ for } r \in [0,\infty). \end{cases} \tag{6.1}$$

Recall that

$$F(\psi) = \psi(\psi - 1)(\psi - 2).$$

Regularity at the center requires that $\psi_r(t,0) = 0$ as well as $\psi(t,0) \in \{0,1,2\}$. Since $\psi^T_{\mathrm{YM}}(t,0) = 0$ we restrict ourselves to solutions with $\psi(t,0) = 0$.
We investigate small perturbations and insert the ansatz $\psi = \psi^T_{\mathrm{YM}} + \varphi$ which yields

$$\begin{cases} \varphi_t - \varphi_{rr} - \frac{2}{r}\varphi_r + \frac{3}{r^2}F'(\psi^T_{\mathrm{YM}})\varphi + \frac{3}{r^2}N^T(\varphi) = 0 \text{ on } (0,T) \times (0,\infty), \\ \varphi(0,r) = \psi_0(r) - \psi^T_{\mathrm{YM}}(0,r) \text{ for } r \in [0,\infty), \\ \varphi(t,0) = \varphi_r(t,0) = 0. \end{cases} \tag{6.2}$$

The nonlinearity is given by

$$N^T(\varphi) = F(\psi^T_{\mathrm{YM}} + \varphi) - F(\psi^T_{\mathrm{YM}}) - F'(\psi^T_{\mathrm{YM}})\varphi = \varphi^3 + 3(\psi^T_{\mathrm{YM}} - 1)\varphi^2.$$

We study the problem in similarity coordinates

$$\xi = \frac{r}{\sqrt{T - t}}, \quad \tau = -\log(T - t),$$

where $\xi \in [0, \infty)$ and $\tau \in [-\log T, \infty)$. The derivatives transform according to

$$\partial_t = e^{\tau}(\partial_\tau + \tfrac{1}{2}\xi\partial_\xi), \quad \partial_r = e^{\frac{\tau}{2}}\partial_\xi.$$

Note that in physical coordinates the lines $\xi = \text{const}$ correspond to backward space-time parabolae with vertex $(T, 0)$. In self-similar coordinates Eq. (6.2) reads

$$\phi_\tau - \phi_{\xi\xi} - \frac{2}{\xi}\phi_\xi + \frac{\xi}{2}\phi_\xi + \frac{3F'(f_{\mathrm{YM}})}{\xi^2}\phi + \frac{3}{\xi^2}N(\phi, \cdot) = 0 \tag{6.3}$$

for $\phi(\tau, \xi) := \varphi(T - e^{-\tau}, \xi e^{-\tau/2})$ with boundary conditions

$$\phi(\tau, 0) = \phi_\xi(\tau, 0) = 0 \quad \text{for all } \tau > -\log T$$

and appropriately transformed initial data. In the above equation

$$N(x, \xi) = x^3 + 3x^2(f_{\mathrm{YM}}(\xi) - 1), \quad f_{YM}(\xi) = \frac{\xi^2}{d_5\xi^2 + c_5}$$

where the constants d_5, c_5 are defined in Eq. (5.6). Note that

$$3F'(f_{\mathrm{YM}}(\xi)) = 6 - \xi^2 V(\xi)$$

for a bounded, smooth potential

$$V(\xi) := \frac{72(36 - 14\sqrt{6} + \sqrt{6}\xi^2 - 2\xi^2)}{(36 - 14\sqrt{6} + \sqrt{6}\xi^2)^2} \tag{6.4}$$

with $V(\xi) = O(\xi^{-2})$ for $\xi \to \infty$.

Abstract operator formulation

The aim is to rewrite Eq. (6.3) as an abstract operator equation of the form

$$\begin{aligned} \Phi(\tau) &= A\Phi(\tau) + N_{YM}(\Phi(\tau)) \\ \Phi(-\log T) &= \Phi_0 \end{aligned} \tag{6.5}$$

for a function $\Phi : [-\log T, \infty) \to X$, where X is some function space that has yet to be defined. We define a formal differential expression

$$au(\xi) = u''(\xi) + \frac{2}{\xi}u'(\xi) - \frac{\xi}{2}u'(\xi) - \frac{6}{\xi^2}u(\xi) + V(\xi)u(\xi) \tag{6.6}$$

Note that the free operator (without potential V) corresponds to a three dimensional radial Schrödinger operator for $\ell = 2$ and a harmonic oscillator potential.

For the linearized equation the most natural setting is certainly the exponentially weighted space $L^2_\mu(0,\infty)$ for a weight function

$$\mu(\xi) := \xi^2 e^{-\frac{\xi^2}{4}},$$

since

$$au(\xi) = \frac{1}{\mu(\xi)}\left[\frac{d}{d\xi}(\mu(\xi)u'(\xi)) + q(\xi)u(\xi)\right]$$

where $q(\xi) := \mu(\xi)(V(\xi) - 6\xi^{-2})$. This is a singular Sturm-Liouville problem and it can be checked that both endpoints $\xi = 0$ and $\xi = \infty$ are of the limit-point type. Thus, the operator A defined as the maximal operator is self-adjoint, cf. for example [95].

In view of the full nonlinear equation, Sobolev spaces equipped with exponentially decaying weight functions are problematic, since we cannot prove the required estimates for the nonlinearity. We therefore propose a different setting, that will be described in Section 6.1.2. First, however, we discuss another aspect of the problem.

One of the fundamental ingredients in the stability analysis is certainly the solution of the spectral problem for the linearized equation. While for the (NLW) solutions of the eigenvalue equation could be given in terms of hypergeometric functions, the corresponding eigenvalue problem for the Yang-Mills heat flow cannot be solved explicitly. In order to gain a better understanding for the equation, we first consider the linearized problem in the self-adjoint setting.

6.1.1 Self-adjoint formulation - The eigenvalue problem

We study the properties of the (maximally defined) linear operator $(A, \mathcal{D}(A))$ where $Au := au$ and

$$\mathcal{D}(A) = \{u \in L^2_\mu(0,\infty) : u, \mu u' \in AC_{loc}(0,\infty), au \in L^2_\mu(0,\infty)\}. \tag{6.7}$$

Since $(A, \mathcal{D}(A))$ is self-adjoint on $L^2_\mu(0,\infty)$, cf. for example [95], the spectrum is real. Furthermore, it is easy to check that $\lambda = 1$ is an eigenvalue with corresponding eigenfunction

$$g(\xi) = \frac{c_5\xi^2}{(d_5\xi^2 + c_5)^2}, \tag{6.8}$$

which can be identified as the symmetry mode. The aim is to show that this is the only non-negative eigenvalue of the operator $(A, \mathcal{D}(A))$. We transform the differential expression defining A to normal form which yields

$$-au(\xi) = \tfrac{1}{\xi} e^{\frac{\xi^2}{8}} h\left(\xi e^{\frac{-\xi^2}{8}} u(\xi)\right)$$

where

$$hv(\xi) := -v''(\xi) + V^{\text{eff}}(\xi)v(\xi)$$

for an effective potential

$$V^{\text{eff}}(\xi) := \frac{\xi^2}{16} + \frac{6}{\xi^2} - V(\xi) - \frac{3}{4}. \tag{6.9}$$

Setting

$$bu(\xi) := u'(\xi) + \alpha(\xi)u(\xi), \quad b^+u(\xi) := -u'(\xi) + \alpha(\xi)u(\xi) \tag{6.10}$$

with

$$\alpha(\xi) := \frac{3}{\xi} - \frac{\xi}{4} - \frac{4\xi}{6\sqrt{6} - 14 + \xi^2}$$

we infer that $h = bb^+ - 1$. One can check that $\ker(b^+) = \operatorname{span}\{v_g\}$, where

$$v_g(\xi) = \xi e^{-\frac{\xi^2}{8}} g(\xi).$$

The supersymmetric partner

The factorization of the expression h turns out to be extremely useful for the solution of the eigenvalue problem. We define the formal differential expression

$$h_s v(\xi) := b^+b - 1 = -v''(\xi) + V_s^{\text{eff}}(\xi)v(\xi)$$

for $\xi \in (0, \infty)$ where

$$V_s^{\text{eff}}(\xi) = \frac{\xi^2}{16} + \frac{12}{\xi^2} + \frac{3}{4} + V_s(\xi)$$

and

$$V_s(\xi) := \frac{-\left(\xi^2 + 24\sqrt{6} - 44\right)\xi^2 + 384\sqrt{6} - 956}{\left(\xi^2 + 6\sqrt{6} - 14\right)^2}.$$

The potential V_s is smooth and bounded. In the following, we define the *supersymmetric partner* of the operator H. Such constructions have a long tradition in quantum mechanics and we refer for example to [18] for a nice overview.

Lemma 6.1.1. *Let $H_s v := h_s v$ and set*

$$\mathcal{D}(H_s) := \{v \in L^2(0,\infty) : v, v' \in AC_{loc}(0,\infty), h_s v \in L^2(0,\infty)\}.$$

Then the operator $(H_s, \mathcal{D}(H_s))$ is self-adjoint on $L^2(0,\infty)$ and bounded below. In particular,

$$\sigma(H_s) \subseteq [\omega_s, \infty),$$

where $\omega_s > 0$.

Proof. Since h_s is in the limit point case for both endpoints of the interval, see for example [95], the (maximal) operator $(H_s, \mathcal{D}(H_s))$, $H_s u := h_s u$,

$$\mathcal{D}(H_s) := \{u \in L^2(0,\infty) : u, u' \in AC_{loc}(0,\infty), h_s u \in L^2(0,\infty)\}$$

is self-adjoint on $L^2(0,\infty)$. Furthermore, the maximal operator is the closure of $(\tilde{H}_s, \mathcal{D}(\tilde{H}_s))$, where $\tilde{H}_s u := h_s u$ and

$$\mathcal{D}(\tilde{H}_s) = \{u \in \mathcal{D}(H_s) : u \text{ has compact support }\}.$$

In order to prove that H_s is bounded below it therefore suffices to prove that

$$(\tilde{H}_s u, u)_{L^2} \geq \omega_s \|u\|_{L^2}$$

for all $u \in \mathcal{D}(\tilde{H}_s)$. The Cauchy-Schwarz inequality and the fact that $u \in \mathcal{D}(\tilde{H}_s)$ vanishes at the origin yield

$$\int_0^\gamma |u(\xi)|^2 d\xi \leq \gamma^2 \int_0^\gamma |u'(\xi)|^2 d\xi \tag{6.11}$$

for all $\gamma > 0$. In the following we set $\gamma = \frac{5}{2}$. The effective supersymmetric potential attains its global minimum $V^{\text{eff}}_{s,min}$ in the interval $(0,\gamma)$, in fact

$$-\frac{3}{25} < V^{\text{eff}}_{s,min} < 0, \quad \frac{1}{\gamma^2} + V^{\text{eff}}_{s,min} > \frac{1}{25} > V^{\text{eff}}_s(\gamma) > 0.$$

For $\xi \geq \gamma$ the potential V^{eff}_S is strictly positive and monotonically increasing.
For all $u \in \mathcal{D}(\tilde{H}_s)$ we readily estimate using partial integration and Eq. (6.11),

$$\begin{aligned}
(\tilde{H}_s u, u)_{L^2} &= -\int_0^\infty u''(\xi)\overline{u(\xi)} d\xi + \int_0^\infty V_s(\xi)|u(\xi)|^2 d\xi \\
&= \int_0^\infty |u'(\xi)|^2 d\xi + \int_0^\infty V^{\text{eff}}_s(\xi)|u(\xi)|^2 d\xi = \int_0^\gamma |u'(\xi)|^2 d\xi \\
&+ \int_\gamma^\infty |u'(\xi)|^2 d\xi + \int_0^\gamma V^{\text{eff}}_s(\xi)|u(\xi)|^2 d\xi + \int_\gamma^\infty V^{\text{eff}}_s(\xi)|u(\xi)|^2 d\xi \\
&\geq \int_0^\gamma |u'(\xi)|^2 d\xi + V^{\text{eff}}_{s,min} \int_0^\gamma |u(\xi)|^2 d\xi + V^{\text{eff}}_s(\gamma) \int_\gamma^\infty |u(\xi)|^2 d\xi \\
&\geq \left(\frac{1}{\gamma^2} + V^{\text{eff}}_{s,min}\right) \int_0^\gamma |u(\xi)|^2 d\xi + V^{\text{eff}}_s(\gamma) \int_\gamma^\infty |u(\xi)|^2 d\xi \\
&\geq \frac{1}{25} \int_\gamma^\infty |u(\xi)|^2 + V^{\text{eff}}_s(\gamma) \int_\gamma^\infty |u(\xi)|^2 d\xi \\
&\geq V^{\text{eff}}_s(\gamma) \int_0^\infty |u(\xi)|^2 d\xi = \omega_s \|u\|^2_{L^2}
\end{aligned}$$

where we set $\omega_s := V^{\text{eff}}_s(\frac{5}{2}) \approx 0.017$. This estimate can now be extended to hold for all $u \in \mathcal{D}(H_s)$ by approximation. Standard results, see for example [52], p. 278, imply that the spectrum is bounded below by ω_s. □

Lemma 6.1.2. *Let $\omega_s > 0$ denote the constant determined in Lemma 6.1.1. The operator $(A, \mathcal{D}(A))$ has no eigenvalues in $(-\omega_s, \infty)$ except for $\lambda = 1$.*

Proof. We argue by contradiction. Suppose that $\lambda > -\omega_s$, $\lambda \neq 1$ is an eigenvalue with corresponding eigenfunction u_λ, i.e., $u_\lambda \in \mathcal{D}(A)$ is a solution of the eigenvalue equation

$$(\lambda - a)u_\lambda = 0.$$

Since the coefficients of this second order differential equation are smooth in $(0, \infty)$, ODE theory yields $u_\lambda \in C^\infty(0, \infty)$. The endpoint $\xi = 0$ is a regular singular point and the Frobenius method implies the existence of two linearly independent solutions $\{u^0, u^1\}$ near the origin with asymptotic behavior

$$u^0 \sim \xi^2, \quad u^1 \sim \xi^{-3}.$$

The fact that $u_\lambda \in L^2_\mu(0, \infty)$ implies that $u_\lambda \sim \xi^2$. The endpoint $\xi = \infty$ is a singular point and the Frobenius method does not apply. However, the behavior of solutions for large argument will be discussed below.
We set $v_\lambda(\xi) := \xi e^{-\frac{\xi^2}{8}} u(\xi)$. Then $v_\lambda \in L^2(0, \infty)$ and it satisfies the equation

$$(\hat{\lambda} - h)v_\lambda = 0$$

where $\hat{\lambda} = -\lambda$. By assumption $b^+ v_\lambda \neq 0$ such that

$$\hat{v}_\lambda := b^+ v_\lambda$$

is a nontrivial solution of the equation

$$(\hat{\lambda} - h_s)\hat{v}_\lambda = 0.$$

Thus, $\hat{v}_\lambda$ is an eigenfunction of H_s given that $\hat{v}_\lambda \in \mathcal{D}(H_s)$.
Since v_λ is smooth in $(0, \infty)$ it remains to show that $\hat{v}_\lambda, h_s \hat{v}_\lambda \in L^2(0, \infty)$. Note that since $\hat{v}_\lambda$ satisfies the eigenvalue equation it suffices to prove that $\hat{v}_\lambda \in L^2(0, \infty)$, which depends on the behavior of the function at the boundaries of the interval. At the origin, $v_\lambda \sim \xi^3$ implies that $\hat{v}_\lambda \in L^2(0, R)$ for each $R > 0$.
To investigate the behavior of $\hat{v}_\lambda$ at infinity we consider the equation

$$(\hat{\lambda} - h)v = 0.$$

Since $V = O(\xi^{-2})$ for $\xi \to \infty$ the potential can be neglected for large arguments and the dominant part of the equation is given by

$$v''(\xi) + \left(\hat{\lambda} + \frac{3}{4} - \frac{\xi^2}{16}\right) v(\xi) = 0. \tag{6.12}$$

By rescaling, the equation can be transformed to the Weber equation

$$y''(x) + \left(\nu + \frac{1}{2} - \frac{x^2}{4}\right) y(x) = 0$$

where $x = \xi/\sqrt{2}$, $y(x) := v(\sqrt{2}x)$ and $\nu = 2\hat{\lambda} + 1$. Solutions of the Weber equation can be given in terms of parabolic cylinder functions. According to [7], p. 133, there are two linearly independent solutions $\{D_\nu, D_{-\nu-1}\}$ where $D_{-\nu-1}$ grows exponentially and

$$D_\nu(x) \sim x^\nu e^{-\frac{x^2}{4}} \quad \text{for } x \to \infty.$$

Adding the potential V does not change the asymptotic exponential behavior and solutions of Eq. (6.12) can be described in terms of exponentially growing/decaying functions for $\xi \to \infty$. We infer that v_λ decays exponentially fast, since otherwise it would contradict the fact that $v_\lambda \in L^2(0,\infty)$. This implies in particular that $b^+ v_\lambda \in L^2(0,\infty)$. Hence, $\hat{\lambda}$ is an eigenvalue of H_s with corresponding eigenfunction $\hat{v}_\lambda$. However, since we assumed that $\lambda > -\omega_s$ this implies that $\hat{\lambda} < \omega_s$ which contradicts Lemma 6.1.1. □

6.1.2 A non-self-adjoint setting

As mentioned above, Sobolev spaces with exponentially decaying weight functions are not useful to study the nonlinear problem. In order to find a suitable functional analytic set up we pursue the same strategy as for previous problems. There are some basic requirements, which a suitable normed space has to satisfy. First, the norm should be derived from a quantity which is naturally associated to the free equation

$$\varphi_t - \varphi_{rr} - \frac{2}{r}\varphi_r + \frac{6}{r^2}\varphi = 0 \tag{6.13}$$

with boundary conditions $\varphi(t,0) = \varphi_r(t,0) = 0$, since the nonlinear term in Eq. (6.3) is considered as a perturbation. Furthermore, the norm should be strong enough to detect the blow up, i.e., $\|\psi^T_{\mathrm{YM}}(t)\| \to \infty$ as $t \to T$. Finally, a certain degree of regularity is required to estimate the nonlinearity. In order to meet the last requirement we consider derivatives of Eq. (6.13). To avoid a pure third order equation for φ_r we first transform Eq. (6.13) by setting $\hat{\varphi} = r^k \varphi$. The choice $k = -2$ yields

$$\hat{\varphi}_t - \hat{\varphi}_{rr} - \frac{6}{r}\hat{\varphi}_r = 0,$$

which corresponds to a radial heat equation for $n = 7$. However, in this case the boundary condition at the origin is lost and we cannot define a homogeneous norm depending for example only on the derivative $\hat{\varphi}_r$. This however often turned out to be useful for the analysis. Therefore, we choose $k = 3$ to obtain

$$\hat{\varphi}_t - \hat{\varphi}_{rr} + \frac{4}{r}\hat{\varphi}_r = 0. \tag{6.14}$$

We set $D_r := \frac{1}{r}\partial_r$ and apply D_r^2 to Eq. (6.14). Note that

$$D_r^2\left(\hat{\varphi}_t - \hat{\varphi}_{rr} + \frac{4}{r}\hat{\varphi}_r\right) = \partial_t(D_r^2\hat{\varphi}) - \partial_r^2(D_r^2\hat{\varphi}),$$

which implies that $\tilde{\varphi} := D_r^2(r^3\varphi)$ satisfies the one-dimensional heat equation. Hence we suggest to define a norm based on the quantity

$$\int_0^\infty |\tilde{\varphi}_r(t,r)|^2 dr.$$

This can be considered as a kind of higher energy which is dissipative for solutions of the free equation (6.13). In order to handle the nonlinearity, a rough estimate implies that two additional derivatives are required. We therefore suggest to consider the stability of the self-similar ground state solution for the Yang-Mills heat flow in $(\mathcal{E}_{\mathrm{YM}}, \|\cdot\|_{\mathcal{E}_{\mathrm{YM}}})$, which is the closure of

$$\widetilde{\mathcal{E}_{\mathrm{YM}}} := \{f \in C^4[0,\infty) : f(0) = f'(0) = 0, \|f\|_{\mathcal{E}_{\mathrm{YM}}} < \infty\},$$

equipped with the norm

$$\begin{aligned}\|f\|^2_{\mathcal{E}_{\mathrm{YM}}} := &\int_0^\infty \left|\mathscr{D}(r^3 f(r))\right|^2 dr + \int_0^\infty \left|\tfrac{d}{dr}[\mathscr{D}(r^3 f(r))]\right|^2 dr \\ &+ \int_0^\infty \left|\tfrac{d^2}{dr^2}[\mathscr{D}(r^3 f(r))]\right|^2 dr,\end{aligned}$$

where

$$\mathscr{D}f(r) := \frac{1}{r}\frac{d}{dr}\left(\frac{1}{r}f'(r)\right).$$

Note that

$$\begin{aligned}&\|\psi^T_{\mathrm{YM}}(t,\cdot)\|^2_{\mathcal{E}} = (T-t)^{-1/2}\int_0^\infty |\xi f''_{YM}(\xi) + 5f'_{YM}(\xi) + \tfrac{3}{\xi}f_{YM}(\xi)|^2 d\xi \\ &+ (T-t)^{-3/2}\int_0^\infty |\xi f^{(3)}_{YM}(\xi) + 6f''_{YM}(\xi) + \tfrac{3}{\xi^2}(\xi f'_{YM}(\xi) - 3f_{YM}(\xi))|^2 d\xi \\ &+ (T-t)^{-5/2}\int_0^\infty |\xi f^{(4)}_{YM}(\xi) + 7f^{(3)}_{YM}(\xi) + \tfrac{3}{\xi^3}(\xi^2 f''_{YM}(\xi) - 2\xi f'_{YM}(\xi) + 2f_{YM}(\xi))|^2 d\xi \\ &\simeq (T-t)^{-5/2}.\end{aligned}$$

We consider the linearized equation in similarity coordinates and set $\hat{\phi} := \xi^3\phi$ which yields

$$\hat{\phi}_\tau - \hat{\phi}_{\xi\xi} + \frac{4}{\xi}\hat{\phi}_\xi + \frac{\xi}{2}\hat{\phi}_\xi - \frac{3}{2}\hat{\phi} - V\phi = 0. \tag{6.15}$$

We suggest to formulate this equation as an abstract problem in a suitably defined Hilbert space $\mathcal{H}$ with inner product

$$\begin{aligned}(u|v)_{\mathcal{H}} := &\int_0^\infty \mathscr{D}u(\xi)\overline{\mathscr{D}v(\xi)}d\xi + \int_0^\infty (\mathscr{D}u)'(\xi)\overline{(\mathscr{D}v)'(\xi)}d\xi \\ &+ \int_0^\infty (\mathscr{D}u)''(\xi)\overline{(\mathscr{D}v)''(\xi)}d\xi.\end{aligned} \tag{6.16}$$

Formally, we define the free operator A_0 as

$$A_0u(\xi) := u''(\xi) - \left(\frac{4}{\xi} + \frac{\xi}{2}\right)u'(\xi) + \frac{3}{2}u(\xi)$$

and consider the potential as a perturbation $\hat{A}$, where $\hat{A}u(\xi) = V(\xi)u(\xi)$. Recall that the potential is a smooth, bounded function and $V(\xi) = O(\xi^{-2})$ for $\xi \to \infty$, which implies that $\hat{A}$ is bounded. Moreover, in view of the decay of the potential at infinity we are confident to prove that $\hat{A}$ is compact relative to A_0. To show that the free operator generates a strongly continuous semigroup, a dissipative estimate for A_0 is required. The following identities simplify the calculations

$$\begin{aligned}
\mathscr{D}A_0u &= \mathcal{A}_0\mathscr{D}u, & \mathcal{A}_0v(\xi) &:= v''(\xi) - \frac{\xi}{2}v'(\xi) - \frac{1}{2}v(\xi),\\
(\mathscr{D}A_0u)' &= \mathcal{A}_1[(\mathscr{D}u)'], & \mathcal{A}_1v(\xi) &:= v''(\xi) - \frac{\xi}{2}v'(\xi) - v(\xi),\\
(\mathscr{D}A_0u)'' &= \mathcal{A}_2[(\mathscr{D}u)''], & \mathcal{A}_2v(\xi) &:= v''(\xi) - \frac{\xi}{2}v'(\xi) - \frac{3}{2}v(\xi).
\end{aligned}$$

Assuming that the boundary terms vanish in the following computation we formally obtain

$$\begin{aligned}
\mathrm{Re}(A_0u|u)_{\mathcal{H}} = \mathrm{Re}&\left(\int_0^\infty \mathcal{A}_0\mathscr{D}u(\xi)\overline{\mathscr{D}u(\xi)}d\xi + \int_0^\infty \mathcal{A}_1(\mathscr{D}u)'(\xi)\overline{(\mathscr{D}u)'(\xi)}d\xi\right.\\
&\left.+\int_0^\infty \mathcal{A}_1(\mathscr{D}u)''(\xi)\overline{(\mathscr{D}u)''(\xi)}d\xi\right) = \sum_{k=0}^{2}\left\{\mathrm{Re}\left(\int_0^\infty \mathscr{D}u^{(k+2)}(\xi)\overline{\mathscr{D}u^{(k)}(\xi)}d\xi\right)\right.\\
&\left.-\frac{1}{4}\int_0^\infty \xi\tfrac{d}{d\xi}|\mathscr{D}u^{(k)}(\xi)|^2d\xi - \frac{(k+1)}{2}\int_0^\infty |\mathscr{D}u^{(k)}(\xi)|^2d\xi\right\} =\\
&\sum_{k=0}^{2}\left\{\mathrm{Re}\left(\lim_{\xi\to\infty}\mathscr{D}u^{(k+1)}(\xi)\overline{\mathscr{D}u^{(k)}(\xi)} - \mathscr{D}u^{(k+1)}(0)\overline{\mathscr{D}u^{(k)}(0)}\right) - \|\mathscr{D}u^{(k+1)}\|_{L^2}^2\right.\\
&\left.-\frac{1}{4}\lim_{\xi\to\infty}\xi|\mathscr{D}u^{(k)}(\xi)|^2 + \frac{1}{4}\|\mathscr{D}u^{(k)}\|_{L^2}^2 - \frac{(k+1)}{2}\|\mathscr{D}u^{(k)}\|_{L^2}^2\right\} \le -\frac{1}{4}\|u\|_{\mathcal{H}}^2.
\end{aligned}$$

This estimate may be used to establish the existence of a C_0-semigroup $(S_0(\tau))_{\tau\geq 0}$ generated by the closed operator A_0, which satisfies

$$\|S_0(\tau)\| \le e^{-\frac{1}{4}\tau}.$$

This would allow to infer well-posedness of the linearized equation in $\mathcal{H}$ by application of the bounded perturbation theorem. In order to improve the growth estimate for the full problem we use again the spectral properties of the generator $A := A_0 + \hat{A}$. For $\lambda \in \sigma(A)$ and $\mathrm{Re}\lambda > -\frac{1}{4}$ only eigenvalues with finite algebraic multiplicity have to be considered, since relative compact perturbations leave the essential spectrum invariant. In regard to the eigenvalue problem we expect that the results that were obtained in Section 6.1.1 in the self-adjoint setting can be helpful to prove an equivalent statement in $\mathcal{H}$.

Having this, the time evolution can be restricted to a stable subspace using a spectral projection. It remains to show that a negative spectral bound for the generator restricted to the subspace implies exponential decay of solutions. The abstract argument for compact perturbations that was presented in Section 5.2 cannot be generalized to relative compact perturbation in a straightforward manner and the Gearhart-Prüss theorem has to be applied instead. For wave equations it was surprisingly straightforward to prove the required bounds on the resolvent, which seems to be connected to the first order formulation and the structure of the equations. Here, this seems to be a more delicate issue and a different approach, using for example ODE methods, is required. Regarding the nonlinear theory it should be possible to prove local Lipschitz estimates for the nonlinearity. The remaining parts of the proof could then be performed as described in Chapter 3.

6.2 Summary and Conclusion

In this thesis, blow up for wave equations with focusing power type nonlinearities was investigated. It was shown that the self-similar ground state (ODE blow up) solution ψ^T is stable under small radial perturbations. For nonlinearities with exponents $1 < p \leq 3$ this was established in a local energy space, whereas for $p > 3$ one additional degree of regularity was required. These results strongly support the conjecture that the generic blow behavior of solutions of the (NLW) can be described in terms of the ODE blow up.

Possible future refinements of the obtained results include an improvement of the topology for $3 < p < 5$, i.e., a proof of nonlinear stability of ψ^T in the energy topology in the full energy subcritical regime. This can be established possibly even in the energy critical case $p = 5$.

Blow up via self-similar solutions is not restricted to nonlinear wave equations but can be observed for a variety of different models. Some examples of nonlinear heat equations were given in Chapter 5. Further generalizations of the presented techniques could therefore be used to address similar problems for other types of nonlinear PDEs. This expectation is also supported by the preliminary results and ideas that were presented above for the Yang-Mills heat flow.

Appendix A

Results from semigroup theory

We collect some important facts and definitions from the theory of strongly continuous one-parameter semigroups. Some of the results below are applied in the proofs presented in Chapter 4. The following material can be found in [36], if no other references are given.

A.1 Semigroups and generators

In the sequel, X is assumed to be a Banach space.

Definition A.1.1. A family $\mathcal{S} = (S(t))_{t\geq 0}$ of bounded linear operators on X is called *strongly continuous one-parameter semigroup* or *C_0-semigroup* if for all $t, s \geq 0$,

$$S(t+s) = S(t)S(s), \tag{A.1}$$

$$S(0) = I \tag{A.2}$$

and if for every $x \in X$ the orbit map

$$\xi_x : t \mapsto \xi_x(t) := S(t)x$$

is continuous from $\mathbb{R}^+$ into X.

Right from the definition the following important property can be obtained.

Proposition A.1.2. For every C_0-semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that for all $\tau \geq 0$,

$$\|S(\tau)\| \leq Me^{\omega\tau}. \tag{A.3}$$

Definition A.1.3. The generator $A : \mathcal{D}(A) \subseteq X \to X$ of a C_0-semigroup $(S(t))_{t\geq 0}$ on X is the linear operator operator

$$Ax := \lim_{h\to 0} \frac{1}{h}(S(h)x - x)$$

defined for every x in its domain

$$D(A) := \{x \in X : \lim_{h\to 0} \frac{1}{h}(S(h)x - x) \text{ exists } \}.$$

The generator is a closed, densely defined linear operator that determines the semi-group uniquely. Moreover, the following relations hold.

Lemma A.1.4. Let $(A, \mathcal{D}(A))$ be the generator of a C_0-semigroup $(S(t))_{t\geq 0}$. If $x \in \mathcal{D}(A)$, then $S(t)x \in D(A)$ and for all $t \geq 0$,

$$\frac{d}{dt}S(t)x = S(t)Ax = AS(t)x.$$

The next result relates the semigroup to the resolvent of its generator, revealing some information on the spectrum.

Theorem A.1.1. Let the operator $(A, \mathcal{D}(A))$ be the generator of a C_0-semigroup $(S(t))_{t\geq 0}$. Let $\omega \in \mathbb{R}$ and $M \geq 1$ be constants such that

$$\|S(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0.$$

i) If $\lambda \in \mathbb{C}$ such that the integral

$$R_A(\lambda)x = \int_0^\infty e^{-\lambda t} S(t)x dt$$

exists for all $x \in X$, then λ is in $\rho(A) := \mathbb{C} \setminus \sigma(A)$ and $R_A(\lambda)$ is the resolvent of the generator A.

ii) If $\mathrm{Re}\lambda > \omega$, then $\lambda \in \rho(A)$ and the resolvent is given by the above integral expression.

iii) For all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega$

$$\|R_A(\lambda)\| \leq \frac{M}{\mathrm{Re}\lambda - \omega}.$$

For the following theorem, which is very important for our analysis, we also refer to [87], Theorem 12.22. A more general version for Banach spaces is given in [36], Theorem II.3.15.

Theorem A.1.2 (Lumer-Phillips, 1961)**.** Let $\mathcal{H}$ be a Hilbert space and let $(A, \mathcal{D}(A))$ be a densely defined linear operator on $\mathcal{H}$. Assume that there exists a constant $\omega \in \mathbb{R}$ such that

$$\mathrm{Re}(Ax, x) \leq \omega\|x\|^2,$$

for all $x \in \mathcal{D}(A)$. Then the following statements are equivalent:

i) The closure of A is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ satisfying

$$\|S(t)\| \leq e^{\omega t}, \quad \forall t \geq 0.$$

ii) $\mathrm{rg}(\lambda - A)$ is dense in $\mathcal{H}$ for some $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega$.

The next important theorem can be found in [36], Theorem III.1.3.

Theorem A.1.3 (Bounded perturbation theorem)**.** Let $(A_0, \mathcal{D}(A_0))$ be the generator of a strongly continuous semigroup $(S_0(t))_{t\geq 0}$ on X satisfying

$$\|S_0(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0$$

for some $\omega \in \mathbb{R}$, $M \geq 1$. If A' is a bounded operator on X then $A = A_0 + A'$ with $\mathcal{D}(A) = \mathcal{D}(A_0)$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ which satisfies

$$\|S(t)\| \leq Me^{(\omega + M\|A'\|)t}, \quad \forall t \geq 0.$$

We also need the following result, which can be derived from [36], Examples I.5.12 and II.2.3.

Lemma A.1.5 (Subspace semigroups)**.** Let $(A, \mathcal{D}(A))$ be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X. Let $\mathcal{N}$ be a closed subspace $\mathcal{N} \subset X$. If $\mathcal{N}$ is invariant under the semigroup, i.e.,

$$S(t)\mathcal{N} \subset \mathcal{N}$$

for all $t \geq 0$, then the restrictions

$$S_{\mathcal{N}}(t) := S|_{\mathcal{N}}(t)$$

form a strongly continuous semigroup $(S_{\mathcal{N}}(t))_{t\geq 0}$ called the *subspace semigroup* on the Banach space $\mathcal{N}$. The generator of $(S_{\mathcal{N}}(t))_{t\geq 0}$ is $(A_{\mathcal{N}}, \mathcal{D}(A_{\mathcal{N}}))$ where

$$A_{\mathcal{N}}f = Af, \quad \mathcal{D}(A_{\mathcal{N}}) = \mathcal{D}(A) \cap \mathcal{N}.$$

Well-posedness for evolution equations

Concerning well-posedness of abstract initial-value problems of the form

$$\begin{cases} \frac{d}{dt}\Psi(t) = A\Psi(t) & \text{for } t \geq 0, \\ \Psi(0) = x \end{cases} \tag{A.4}$$

for functions Ψ with values in X, $A : \mathcal{D}(A) \subset X \to X$ a closed linear operator and $x \in X$ the initial value, the following result can be stated, see [36], Prop. II.6.2.

Proposition A.1.6. Let $(A, \mathcal{D}(A))$ be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X. Then, for every $x \in \mathcal{D}(A)$, the function

$$\Psi : t \mapsto \Psi(t) := S(t)x$$

is the unique classical solution of the above abstract initial value problem, i.e., Ψ is continuously differentiable with respect to X, $\Psi(t) \in \mathcal{D}(A)$ for all $t \geq 0$, and Eq. (A.4) holds.

Moreover, including continuous dependence on the data into the notion of well-posedness, the following important observation can be made, see [36], Cor. II.6.9.

Corollary A.1.7. For a closed operator $A : \mathcal{D}(A) \subset X \to X$, the associated abstract initial-value problem is well-posed if and only if A generates a strongly continuous semigroup on X.

A.2 Spectral bounds and growth estimates

In the following, we define the *spectral radius* of a bounded operator S on X by

$$r(S) := \sup\{|\lambda| : \lambda \in \sigma(S)\}.$$

Definition A.2.1. For a C_0-semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ the growth bound $\omega_0(\mathcal{S})$ is defined as

$$\omega_0(\mathcal{S}) := \inf\{\omega \in \mathbb{R} : \exists M_\omega \geq 1 \text{ such that } \|S(t)\| \leq M_\omega e^{\omega t}, \forall t \geq 0\}.$$

It can be shown, cf. [36] Prop. IV.2.2 that

$$\omega_0(\mathcal{S}) = \frac{1}{t_0} \log r(S(t_0)), \tag{A.5}$$

for each $t_0 > 0$. In particular, the spectral radius of the operator $S(t)$ is given by

$$r(S(t)) = e^{\omega_0 t}, \quad \forall t \geq 0.$$

Similarly, one can characterize the *essential spectral radius* by

$$r_{\text{ess}}(S) := \inf\{r > 0 : \lambda \in \sigma(S), |\lambda| > r \text{ is a pole of finite algebraic multiplicity}\},$$

which naturally induces the notion of the *essential growth bound* defined by

$$\omega_{\text{ess}}(\mathcal{S}) = \frac{1}{t_0} \log r_{\text{ess}}(S(t_0)) < \frac{1}{t_0} \log r(S(t_0)) = \omega_0(\mathcal{S}). \tag{A.6}$$

Finally, the *spectral bound* of an operator $(A, \mathcal{D}(A))$ on X is defined as

$$s(A) := \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\}.$$

A consequence of the above presented relations between semigroup, generator and resolvent is the following corollary, cf. [36], Cor. II.1.13.

Corollary A.2.2. In general one has

$$-\infty \leq s(A) \leq \omega_0(\mathcal{S}) < +\infty.$$

The following important result was obtained by various people in different formulations and is often referred to as the Gearhart-Prüss Theorem, see for example V.1.11 and V.1.13 in [36]. Here we use a formulation of the theorem as given in [48].

Theorem A.2.1 (Gearhart-Prüss-Hwang-Greiner)**.** Let $\mathcal{H}$ be a Hilbert space and let $(A, \mathcal{D}(A))$ be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $\mathcal{H}$. Assume that $\|R_A(\lambda)\|$ is uniformly bounded in the half-plane

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \omega\}.$$

Then there exists a constant $M > 0$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0. \tag{A.7}$$

Conversely, if (A.7) holds for some $\omega \in \mathbb{R}$, then for every $\alpha > \omega$, $\|R_A(\lambda)\|$ is uniformly bounded in the half-plane $\mathrm{Re}\lambda \geq \alpha$.

Bibliography

[1] Serge Alinhac. *Blowup for Nonlinear Hyperbolic Equations.* Number 17 in Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser Boston, 1 edition, 1995.

[2] Serge Alinhac. *Hyperbolic Partial Differential Equations.* Springer New York Berlin Heidelberg, 2009.

[3] Christophe Antonini and Frank Merle. Optimal bounds on positive blow-up solutions for a semilinear wave equation. *International Mathematics Research Notices*, (21):1141–1167, 2001.

[4] Michael Atiyah and Raoul Bott. The Yang-Mills equations over Riemann surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 308(1505):523–615, 1983.

[5] Hajer Bahouri and Jalal Shatah. Decay estimates for the critical semilinear wave equation. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 15(6):783 – 789, 1998.

[6] Marius Beceanu. A centre-stable manifold for the energy-critical wave equation in $\mathbb{R}^3$ in the symmetric setting. *ArXiv e-prints*, December 2012.

[7] Carl M. Bender and Steven A. Orszag. *Advanced mathematical methods for scientists and engineers.* McGraw-Hill Book Company, 1978.

[8] Pawel Biernat and Piotr Bizoń. Shrinkers, expanders, and the unique continuation beyond generic blowup in the heat flow for harmonic maps between spheres. *Nonlinearity*, 24(8):2211–2228, 2011.

[9] Piotr Bizoń, Tadeusz Chmaj, and Zbisław Tabor. On blowup for semilinear wave equations with a focusing nonlinearity. *Nonlinearity*, 17(6):2187–2201, 2004.

[10] Piotr Bizoń, Dieter Maison, and Arthur Wasserman. Self-similar solutions of semilinear wave equations with a focusing nonlinearity. *Nonlinearity*, 20(9):2061–2074, 2007.

[11] Poitr Bizoń, Peter Breitenlohner, Dieter Maison, and Arthur Wasserman. Self-similar solutions of the cubic wave equation. *Nonlinearity*, 23(2):225–236, 2010.

[12] Chris J. Budd, Shaohua Chen, and Robert D. Russell. New self-similar solutions of the nonlinear Schrödinger equation with moving mesh computations. *J. Comput. Phys*, 152:756–789, 1999.

[13] Chris J. Budd, Shaohua Chen, and Robert D. Russell. Self-similar blow-up for a chemotaxis system in higher dimensional domains. *RIMS Kôkyûroku Bessatsu*, B15:87–99, 2009.

[14] Chris J. Budd, JF Williams, and Victor Galaktionov. Self-similar blow-up in higher-order semilinear parabolic equations. *SIAM Journal on Applied Mathematics*, 64(5):1775–1809, 2004.

[15] Aynur Bulut. The defocusing energy-supercritical cubic nonlinear wave equation in dimension five. *Preprint arXiv:1112.0629v1*, 2011.

[16] Aynur Bulut. The radial defocusing energy-supercritical cubic nonlinear wave equation in $\mathbb{R}^{1+5}$. *Preprint arXiv:1104.2002*, 2011.

[17] Aynur Bulut. Global well-posedness and scattering for the defocusing energy-supercritical cubic nonlinear wave equation. *Journal of Functional Analysis*, 263(6):1609 – 1660, 2012.

[18] David J. Fernández C. Supersymmetric Quantum Mechanics. In M. Garcia Rocha, R. Lopez Fernandez, L. F. Rojas Ochoa, and G. Torres Vega, editors, *American Institute of Physics Conference Series*, volume 1287 of *American Institute of Physics Conference Series*, pages 3–36, October 2010.

[19] Raphaël Côte and Hatem Zaag. Construction of a multisoliton blowup solution to the semilinear wave equation in one space dimension. *Communications on Pure and Applied Mathematics*, 66(10):1541–1581, 2013.

[20] Simon K. Donaldson. Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. *Proceedings of the London Mathematical Society*, s3-50(1):1–26, 1985.

[21] Roland Donninger. The radial wave operator in similarity coordinates. *J. Math. Phys.*, 51(2):023527, 10, 2010.

[22] Roland Donninger. On stable self-similar blowup for equivariant wave maps. *Commun. Pure Appl. Math.*, 64(8):1095–1147, 2011.

[23] Roland Donninger. Stable self-similar blowup in energy supercritical Yang-Mills theory. *Preprint arXiv:1202.1389*, 2012.

[24] Roland Donninger, Min Huang, Joachim Krieger, and Willhelm Schlag. Exotic blowup solutions for the u^5 focusing wave equation in $\mathbb{R}^3$. *Preprint arXiv:1212.4718*, 2012.

[25] Roland Donninger and Joachim Krieger. Nonscattering solutions and blowup at infinity for the critical wave equation. *Preprint arXiv:1201.3258*, 2012.

[26] Roland Donninger and Birgit Schörkhuber. Stable self-similar blow up for energy subcritical wave equations. *Dyn. Partial Differ. Equ.*, 9(1):63–87, 2012.

[27] Roland Donninger and Birgit Schörkhuber. Stable blow up dynamics for energy supercritical wave equations. *accepted for publication in Transactions of the American Mathematical Society*, 2013.

[28] Roland Donninger, Birgit Schörkhuber, and Peter C. Aichelburg. On stable self-similar blow up for equivariant wave maps: The linearized problem. *Annales Henri Poincaré*, 13(1):103–144, 2012.

[29] Roland Donninger and Anil Zenginoğlu. Nondispersive decay for the cubic wave equation. *ArXiv e-prints*, April 2013.

[30] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. Universality of the blow-up profile for small type II blow-up solutions of energy-critical wave equation: the non-radial case. *Preprint arXiv:1003.0625*, 2010.

[31] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. Universality of blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation. *J. Eur. Math. Soc. (JEMS)*, 13(3):533–599, 2011.

[32] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. Classification of radial solutions of the focusing, energy-critical wave equation. *Preprint arXiv:1204.0031*, 2012.

[33] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. Scattering for radial, bounded solutions of focusing supercritical wave equations. *ArXiv e-prints*, August 2012.

[34] Thomas Duyckaerts and Frank Merle. Dynamics of threshold solutions for energy-critical wave equation. *International Mathematics Research Papers*, 2008.

[35] Jens Eggers and Marco A Fontelos. The role of self-similarity in singularities of partial differential equations. *Nonlinearity*, 22(1):R1, 2009.

[36] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T.

Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[37] Lawrence C. Evans. *Partial Differential Equations*. Graduate Studies in Mathematics. American Mathematical Society, second edition, 2010.

[38] Huijun Fan. Existence of the self-similar solutions in the heat flow of harmonic maps. *Science in China. Series A. Mathematics*, 42(2):113–132, 1999.

[39] Stathis Filippas and Robert V. Kohn. Refined asymptotics for the blowup of $u_t - \delta u = u^p$. *Communications on Pure and Applied Mathematics*, 45(7):821–869, 1992.

[40] Andreas Gastel. Singularities of first kind in the harmonic map and Yang-Mills heat flows. *Mathematische Zeitschrift*, 242(1):47–62, 2002.

[41] Yoshikazu Giga and Robert V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Communications on Pure and Applied Mathematics*, 38(3):297–319, 1985.

[42] Yoshikazu Giga and Robert V. Kohn. Characterizing blowup using similarity variables. *Indiana Univ. Math. J.*, 36:1–40, 1987.

[43] Yoshikazu Giga and Robert V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Communications on Pure and Applied Mathematics*, 42(6):845–884, 1989.

[44] J. Ginibre and G. Velo. Generalized Strichartz inequalities for the wave equation. *Journal of Functional Analysis*, 133(1):50 – 68, 1995.

[45] Manoussos G. Grillakis. Regularity and asymptotic behavior of the wave equation with a critical nonlinearity. *Annals of Mathematics*, 132(3):pp. 485–509, 1990.

[46] Joseph F. Grotowski. Finite time blow-up for the Yang-Mills heat flow in higher dimensions. *Mathematische Zeitschrift*, 237(2):321–333, 2001.

[47] Mohamed-Ali Hamza and Hatem Zaag. Blow-up results for semilinear wave equations in the super-conformal case. *Preprint arXiv:1301.0473*, 2013.

[48] Bernd Hellffer and Johannes Sjöstrand. From resolvent bounds to semigroup bounds. *Preprint arXiv:1001.4171*, 2010.

[49] Fritz John. Blow-up of solutions of nonlinear wave equations in three space dimensions. *Manuscripta Mathematica*, 28(1-3):235–268, 1979.

[50] Konrad Jörgens. Das Anfangswertproblem im Grossen für eine Klasse nichtlinearer Wellengleichungen. *Mathematische Zeitschrift*, 77(1):295–308, 1961.

[51] L.V. Kapitanskiĭ. Some generalizations of the Strichartz-Brenner inequality. *Leningr. Math. J.*, 1(3):693–726, 1990.

[52] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[53] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math*, 120:955–980.

[54] Carlos E. Kenig and Frank Merle. Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation. *Acta Math.*, 201(2):147–212, 2008.

[55] Carlos E. Kenig and Frank Merle. Radial solutions to energy supercritical wave equations in odd dimensions. *Discrete Contin. Dyn. Syst.*, 31(4):1365–1381, 2011.

[56] Rowan Killip, Betsy Stovall, and Monica Visan. Blowup behaviour for the nonlinear Klein-Gordon equation. *Preprint arXiv:1203.4886*, 2012.

[57] Rowan Killip and Monica Visan. The defocusing energy-supercritical non-linear wave equation in three space dimensions. *Trans. Amer. Math. Soc.*, 363(7):3893–3934, 2011.

[58] Joachim Krieger and Joules Nahas. Instability of type II blow up for the quintic nonlinear wave equation on $\mathbb{R}^{3+1}$. *Preprint arXiv:1212.4628*, 2012.

[59] Joachim Krieger, Kenji Nakanishi, and Wilhelm Schlag. Global dynamics away from the ground state for the energy-critical nonlinear wave equation. *Preprint arXiv:1010.3799*, 2010.

[60] Joachim Krieger, Kenji Nakanishi, and Wilhelm Schlag. Global dynamics of the nonradial energy-critical wave equation above the ground state energy. *Preprint arXiv:1112.5663*, 2011.

[61] Joachim Krieger, Kenji Nakanishi, and Wilhelm Schlag. Threshold phenomenon for the quintic wave equation in three dimensions. *Preprint arXiv:1209.0347*, 2012.

[62] Joachim Krieger, Kenji Nakanishi, and Wilhelm Schlag. Center-stable manifold of the ground state in the energy space for the critical wave equation. *Preprint arXiv:1303.2400*, 2013.

[63] Joachim Krieger and Wilhelm Schlag. On the focusing critical semi-linear wave equation. *Amer. J. Math.*, 129(3):843–913, 2007.

[64] Joachim Krieger and Wilhelm Schlag. Full range of blow up exponents for the quintic wave equation in three dimensions. *Preprint arXiv:1212.3795*, 2012.

[65] Joachim Krieger, Wilhelm Schlag, and Daniel Tataru. Slow blow-up solutions for the $H^1(\mathbb{R}^3)$ critical focusing semilinear wave equation. *Duke Math. J.*, 147(1):1–53, 2009.

[66] Joachim Krieger and Willie Wong. A note on type I blow up formation for the critical NLW. *Preprint arXiv:1301.4378*, 2013.

[67] Andrew Lawrie. Nonlinear wave equations. *Expository Notes, available on http://math.berkeley.edu/ alawrie/.*

[68] Howard A. Levine. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. *Trans. Amer. Math. Soc.*, 192:1–21, 1974.

[69] Hans Lindblad and Christopher D. Sogge. On existence and scattering with minimal regularity for semilinear wave equations. *Journal of Functional Analysis*, 130(2):357 – 426, 1995.

[70] Hiroshi Matano and Frank Merle. On nonexistence of type II blowup for a supercritical nonlinear heat equation. *Communications on Pure and Applied Mathematics*, 57(11):1494–1541, 2004.

[71] Hiroshi Matano and Frank Merle. Classification of type I and type II behaviors for a supercritical nonlinear heat equation. *Journal of Functional Analysis*, 256(4):992 – 1064, 2009.

[72] Hiroshi Matano and Frank Merle. Threshold and generic type I behaviors for a supercritical nonlinear heat equation. *Journal of Functional Analysis*, 261(3):716 – 748, 2011.

[73] Júlia Matos. Convergence of blow-up solutions of nonlinear heat equations in the supercritical case. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 129:1197 – 1227, 1999.

[74] Júlia Matos. Unfocused blow up solutions of semilinear parabolic equations. *Discrete Contin. Dyn. Syst.*, 5(4):905 – 928, 1999.

[75] Frank Merle and Hatem Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997.

[76] Frank Merle and Hatem Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Communications on Pure and Applied Mathematics*, 51(2):139–196, 1998.

[77] Frank Merle and Hatem Zaag. O.D.E. type behavior of blow-up solutions of nonlinear heat equations. *Discrete and Continuous Dynamical Systems - Series A*, 8(2):435–450, 2002.

[78] Frank Merle and Hatem Zaag. Determination of the blow-up rate for the semilinear wave equation. *Amer. J. Math.*, 125(5):1147–1164, 2003.

[79] Frank Merle and Hatem Zaag. Determination of the blow-up rate for a critical semilinear wave equation. *Math. Ann.*, 331(2):395–416, 2005.

[80] Frank Merle and Hatem Zaag. On growth rate near the blowup surface for semilinear wave equations. *Int. Math. Res. Not.*, (19):1127–1155, 2005.

[81] Frank Merle and Hatem Zaag. Existence and universality of the blow-up profile for the semilinear wave equation in one space dimension. *J. Funct. Anal.*, 253(1):43–121, 2007.

[82] Frank Merle and Hatem Zaag. Openness of the set of non-characteristic points and regularity of the blow-up curve for the 1 D semilinear wave equation. *Comm. Math. Phys.*, 282(1):55–86, 2008.

[83] Frank Merle and Hatem Zaag. Blow-up behavior outside the origin for a semilinear wave equation in the radial case. *Bull. Sci. Math.*, 135(4):353–373, 2011.

[84] Frank Merle and Hatem Zaag. Existence and classification of characteristic points at blow-up for a semilinear wave equation in one space dimension. *American Journal of Mathematics*, 134(3):581–648, 2012.

[85] Frank Merle and Hatem Zaag. Isolatedness of characteristic points at blow-up for a semilinear wave equation in one space dimension. *Duke Math. J.*, 161(15):2837–2908, 2012.

[86] Frank Olver, Daniel Lozier, Ronald Boisvert, and Charles Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce National Institute of Standards and Technology, 2010.

[87] Michael Renardy and Robert C. Rogers. *An Introduction to Partial Differential Equations*. Number 13 in Texts in Applied Mathematics. Springer New York Berlin Heidelberg, 2004.

[88] Jalal Shatah and Michael Struwe. Regularity results for nonlinear wave equations. *Ann. of Math. (2)*, 138(3):503–518, 1993.

[89] Jalal Shatah and Michael Struwe. Well-posedness in the energy space for semilinear wave equations with critical growth. *International Mathematics Research Notices*, (7):303ff., approx. 7 pp. (electronic), 1994.

[90] Jalal Shatah and Michael Struwe. *Geometric wave equations.* Number 2 in Courant Lect. Notes. American Mathematical Society, Courant Institute of Mathematical Sciences at New York University, 2000.

[91] Christopher D. Sogge. *Lectures on non-linear wave equations.* International Press, Boston, MA, second edition, 2008.

[92] Walter A. Strauss. *Nonlinear Wave Equations.* CMBS Regional Conference Series in Mathematics. American Mathematical Society, 1989.

[93] Michael Struwe. Globally regular solutions to the u^5 Klein-Gordon equation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 15(3):495–513 (1989), 1988.

[94] Terence Tao. *Nonlinear Dispersive Equations: Local and Global Analysis.* Number 106 in CBMS Regional Conference Series in Mathematics. American Mathematical Society, second edition, 2006.

[95] Joachim Weidmann. *Spectral Theory of Ordinary Differential Operators*, volume 1258 of *Lecture Notes in Mathematics.* Springer Berlin Heidelberg, Boston, MA, 1987.

[96] Ben Weinkove. Singularity formation in the Yang-Mills flow. *Calculus of Variations and Partial Differential Equations*, 19(2):211–220, 2004.

Acknowledgements

First and foremost I want to thank my advisor Professor Ansgar Jüngel for his professional support, his guidance throughout the time of my PhD and his contributions to our joint projects. He was extremely supportive and encouraged me to pursue my personal research interests, always ready to offer sound advice.

My gratitude goes to Dr. Roland Donninger, whose enthusiasm for his work inspired me for so many years. On many occasions, I was impressed by his deep mathematical insight and his ability to reduce a complicated problems to the essential. Among all the other important things I have learned from him, he convinced me to acknowledge the value of a good toy model.

I would also like to thank Professor Joachim Krieger from the École Polytechnique Fédérale de Lausanne and Professor Wilhelm Schlag from the University of Chicago for showing interest in my work and for giving me the opportunity to present our results on various occasions.

Furthermore, I want to thank Professor Thomas Meurer from the University of Kiel for the contributions to our joint research on boundary control problems. Although these results are not included in this thesis, I regard our joint work as a substantial part of my PhD studies.

I want to express my gratitude to the Vienna University of Technology as well as the Austrian Science Fund (FWF) for financing the projects I was working on.

I would like to acknowledge all current and previous members of our working group as well as the colleagues at the Institute for Analysis and Scientific Computing, who were always helpful and supportive. I want to thank Bertram, Carola, Sabine and Maja for their hospitality, as well as Mario, Ines, Lara and Stefan for the very pleasant working atmosphere in our office. Furthermore, I want to thank Dominik for his friendship and the many enlightening discussions we had. Finally, I thank Jan-Frederik for appreciating my cakes and for his company throughout the time of my PhD.

This would not have been possible without the constant and unconditional support from my parents Burgi and Gerald, who were always great role models and never stopped me from asking questions, making me the person I am today.

Props go out to my beloved brothers Chrilli and Andi, who I deeply admire for their intelligence, their creativity and their huge talent to make life feel easy. A thousand thanks to Susi, Julia, Harre and my friends, old and new, for their patience, their support and all the good times we had. I also want to thank Emma and Herbert for their encouragement during the past years.

These days, I am probably the only scientist in the world, who can report on the discovery of a superpartner - thank you Wolfi, for being mine :)

Curriculum Vitae

Personal information

Name	Birgit Maria Schörkhuber
Address	Schönbrunnerstraße 26/6, 1050 Vienna, Austria
Date of Birth	December 1, 1981, Steyr, Austria
Parents	Gerald and Notburga Schörkhuber
Brothers	Christian and Andreas Schörkhuber

Educational background

since 03/2010	PhD Studies in Mathematics
	Vienna University of Technology, Austria
Feb 23, 2010	Diploma exam in Physics, with distinction
	Degree: Mag.rer.nat.
	Major: Theoretical Physics
	Advisor: Univ.-Prof.i.R. Peter Christian Aichelburg
02/2006 – 07/2006	Universidad Autónoma de Madrid, Spain
10/2003 – 02/2010	Physics, University of Vienna, Austria
Apr 25, 2005	Bachelor in Astronomy, with distinction
	Degree: Bakk.rer.nat.
10/2001 –04/2005	Astronomy, University of Vienna, Austria
11/2000– 05/2001	Development aid volunteer in Jacobina, Brasil
June 21, 2000	Matura, with distinction
	Bundesgymnasium Steyr Werndlpark

Occupation

04/2013–current	Project assistant, Vienna University of Technology
03/2010–03/2013	University assistant, Vienna University of Technology
10/2008–10/2009	Vienna University Children's Office
	Project manager, science communication.
10/2008–06/2009	Contract work, University of Vienna,
	Research study in didactics of physics
03/2007–07/2008	Tutor, University of Vienna, *Project e-Learn Physik*
12/2006–09/2008	Austrian Educational Competence Centre Physics

Teaching

2010 - 2012	Exercise classes, Vienna University of Technology
	Analysis 1/2/3 für Technische Mathematik, 2011/2012/2010
	Höherer Analysis für Lehramt Mathematik, 2010/2012
	Analysis 2 für Lehramt Mathematik, 2011
2009	Exercise class, University of Vienna, *Scientific Computing*,

Publications

1. R. Donninger and B. Schörkhuber. *Stable blow up dynamics for energy supercritical wave equations.* Transactions of the American Mathematical Society, accepted for publication.

2. B. Schörkhuber, T. Meurer, A. Jüngel. *Flatness of semilinear parabolic PDEs - A generalized Chauchy-Kowalevski approach.* IEEE Transactions on Automatic Control, Vol. 58, No. 9, p. 2277-2291, 2013.

3. B. Schörkhuber, T. Meurer, A. Jüngel. *Flatness-based trajectory planning for semilinear parabolic PDEs.* 51st IEEE Conference on Decision and Control, p. 3538 - 3543, 2012.

4. R. Donninger and B. Schörkhuber. *Stable self-similar blow up for energy subcritical wave equations.* Dynamics of PDE, Vol. 9, No. 1, p. 63-87, 2012.

5. R. Donninger, B. Schörkhuber, P.C. Aichelburg. *On stable self-similar blow up for equivariant wave maps - The linearized problem.* Annales Henri Poincaré, Vol. 13, No. 1, p. 103-144, 2012.

Talks

1. *Flatness-based trajectory planning for semilinear parabolic PDEs*, Dec 11, 2012, 51st IEEE Conference on Decision and Control, Maui, USA

2. *Stable self–similar blow up for the semilinear wave equation*, July 02, 2012, Conference *Nonlinear Hamiltonian PDEs*, Ascona, Switzerland

3. *On stable self–similar blow up for semilinear wave equations*, April 20, 2012, École Polytechnique Fédérale de Lausanne, Switzerland

4. *On linear stability of self-similar blow up for co-rotational wave maps*, January 24, 2011, Erwin Schrödinger International Institute for Mathematical Physics (ESI), Follow-up workshop to the 2010 ESI programme on "Quantitative Studies of Nonlinear Wave Phenomena", Vienna, Austria

Workshops, schools and conferences

1. 51st IEEE Conference on Decision and Control, Dec. 10-13, 2012, Maui, USA

2. Conference *Nonlinear Hamiltonian PDEs*, July 2012, Ascona, Switzerland

3. 4th Elgersburg School 2012, Topics: *Hamiltonian Control Systems, Pseudospectra and Nonnormal Dynamical Systems*, March 2012, Elgersburg, Germany

4. Workshop *Nonlinear Dispersive Equations*, June 2011, ETH Zurich, Switzerland

5. Follow-up workshop to the 2010 ESI programme on *Quantitative Studies of Nonlinear Wave Phenomena*, Jan 2011, ESI, Vienna, Austria

6. Symposium *Analysis and Control of Infinite-Dimensional Systems in the Engineering Sciences*, Nov 2010, Max Planck Institute, Magdeburg, Germany

7. 5th Summer School on *Methods and Models of Kinetic Theory*, June 2010, Porto Ercole, Italy