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DISSERTATION

Special sets of real numbers and variants of the Borel Conjecture

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Kurzfassung

In meiner Dissertation beschäftige ich mich mit einem Teilgebiet der Mengenlehre: ich untersuche Fragen über “spezielle” Teilmengen der reellen Zahlen, die durch Konzepte aus der Analysis, Maßtheorie und Topologie motiviert sind. Ihre Lösung erfordert oft mengentheoretische Methoden wie zum Beispiel die sogenannte “Erzwingungsmethode” (engl.: “forcing”), da die meisten dieser Fragen nicht durch die gewöhnlichen Axiome der Mengenlehre (ZFC) “entscheidbar” sind.

Insbesondere untersuche ich “kleine” (oder: “spezielle”) Mengen, typischerweise Elemente gewisser Sigma-Ideale auf den reellen Zahlen, wie beispielsweise die aus der Maßtheorie stammenden Lebesgueschen Nullmengen, oder die noch viel kleineren “starken Nullmengen” (engl.: “strong measure zero sets”).

ZFC erlaubt keine Folgerungen der Art “alle Mengen von kleiner/großer Kardinalität liegen innerhalb/außerhalb eines gegebenen Ideals”. Zum Beispiel kann die sogenannte “Borel-Vermutung” (engl.: “Borel Conjecture”; benannt nach Émile Borel; es ist die Aussage, daß alle starken Nullmengen höchstens abzählbar sind) aus den ZFC-Axiomen weder bewiesen noch widerlegt werden.

Galvin, Mycielski und Solovay bewiesen einen auf diesem Gebiet zentralen Satz, indem sie eine Verbindung (mittels Translationen) zwischen starken Nullmengen und sogenannten (aus der Topologie stammenden) mageren Mengen zeigten. Dieser Satz ermöglicht es, den zum Begriff der starken Nullmenge “dualen” Begriff der “stark mageren Menge” (engl.: “strongly meager set”) – und damit auch die “duale Borel-Vermutung” (engl.: “dual Borel Conjecture”) – einzuführen.

In Kapitel 1 wird eine kurze Übersicht über alle für die Dissertation relevanten Konzepte gegeben.

Kapitel 2 ist eine gemeinsame Arbeit mit meinem Dissertationsbetreuer Martin Goldstern, Jakob Kellner und Saharon Shelah: wir zeigen, daß es ein

Modell von ZFC gibt, in dem sowohl die Borel-Vermutung als auch die duale Borel-Vermutung gilt (mit anderen Worten: in dem es weder überabzählbare starke Nullmengen noch überabzählbare stark magere Mengen gibt).

In Kapitel 3 wird eine Verstärkung dieses Resultats gezeigt, in dem der Begriff der stark mageren Mengen durch den (schwächeren) Begriff der "sehr mageren Mengen" (engl.: "very meager sets") ersetzt wird.

In Kapitel 4 wird gezeigt, daß sowohl die Borel-Vermutung als auch die duale Borel-Vermutung mit einer definierbaren Wohlordnung der reellen Zahlen verträglich ist.

In Kapitel 5 wende ich mich wieder dem oben erwähnten Satz von Galvin-Mycielski-Solovay zu und zeige, daß man ihn auf verschiedene allgemeinere Strukturen ausdehnen kann.

In Kapitel 6 definiere ich eine neue Klasse von "kleinen Mengen" (und die zugehörige Variante der Borel-Vermutung) und entwickle Methoden zur Untersuchung derselben (unter Annahme der Kontinuumshypothese).

In Kapitel 7 beschreibe ich kurz, auf welche Weise die Konzepte aus Kapitel 6 weiter verallgemeinert werden können.

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Preface

In my thesis, I investigate questions about subsets of the real line. While these questions are motivated by concepts from analysis, measure theory, and topology, their resolution often needs set-theoretic methods – such as the *method of forcing* –, as these questions (the prototypical such question is Hilbert's first problem about the *continuum hypothesis*) may be not resolvable by ZFC (the usual axioms of set theory).

In particular, I investigate *small (special) sets*, i.e., elements of certain natural σ-ideals on the real numbers (or collections only closed under taking subsets), such as the measure zero sets and the much smaller *strong measure zero sets*.

ZFC typically does not allow conclusions of the form "all sets of small/large cardinality are inside/outside of a certain collection". For instance, the *Borel Conjecture* – the statement that all strong measure zero sets are at most countable – can neither be proved nor refuted from the ZFC axioms.

Annotated contents

Chapter 1 Introduction

I give some historical background and review several concepts and results that are relevant to my thesis, such as the notion of strong measure zero, the Galvin-Mycielski-Solovay characterization of strong measure zero sets via translations of meager sets, the notion of strongly meager, Borel Conjecture, dual Borel Conjecture, etc.; furthermore, I give a very informal overview of my joint paper [GKSW] (i.e., Chapter 2).

Chapter 2 Borel Conjecture and dual Borel Conjecture

This chapter contains my joint paper with Martin Goldstern, Jakob Kellner, and Saharon Shelah: We show that it is consistent that the Borel Conjecture and the dual Borel Conjecture hold simultaneously.

Chapter 3 A strengthening of the dual Borel Conjecture

We show that a strengthening of the dual Borel Conjecture holds in our model of BC+dBC: there are no uncountable very meager sets there (the very meager sets always form a σ-ideal containing all strongly meager sets). This is joint work with Saharon Shelah.

Chapter 4 A projective well-order of the reals and BC/dBC

Using methods from [FF10], we show how to modify Laver's proof of Con(BC) to get a model of "BC + there exists a projective well-order of the reals". Similarly, we show the analogous result for dBC, using methods from [FFZ11]. This is joint work with Sy D. Friedman.

Chapter 5 Galvin-Mycielski-Solovay theorem revisited

I give versions of the Galvin-Mycielski-Solovay theorem for more general settings: I provide a version for the generalized Cantor space 2^κ for weakly compact κ, as well as a version for separable locally compact groups. On the other hand, I show that the Galvin-Mycielski-Solovay characterization consistently fails for the Baer-Specker group $\mathbb{Z}^\omega$.

Chapter 6 Sacks dense ideals and Marczewski Borel Conjecture

Let MBC (Marczewski Borel Conjecture) be the assertion that there are no uncountable s_0-shiftable sets (those sets that can be translated away from each set in the Marczewski ideal s_0). So MBC is the analogue to BC (dBC) with meager (measure zero) replaced by s_0. To investigate whether MBC is consistent, I introduce the notion of "Sacks dense ideals" to explore the family of s_0-shiftable sets. Even though Con(MBC) remains unsettled, I present several results about Sacks dense ideals.

Chapter 7 $\mathbb{P}$ dense ideals for tree forcing notions

In Chapter 6, problems regarding the Marczewski ideal s_0 are considered, which is connected to Sacks forcing $\mathbb{S}$. In this chapter, I briefly discuss whether Sacks forcing can be replaced by other tree forcing notions (such as Silver forcing, Laver forcing, etc.) in the arguments of Chapter 6.

Contents

Chapter 1

Introduction

In Section 1.1 of this introductory chapter, we briefly comment on the history of set theory.

In Section 1.2, we review several concepts and results that are relevant to the thesis, such as the notion of strong measure zero, the Galvin-Mycielski-Solovay characterization of strong measure zero sets via translations of meager sets, the notion of strongly meager, Borel Conjecture, dual Borel Conjecture, etc.

In Section 1.3, we give a very informal overview of Chapter 2 (which is my joint paper [GKSW]).

1.1 Historical background

The topic of this PhD thesis belongs to the *set theory of the real numbers*. More specifically, various collections of "small sets" of real numbers are investigated: often so-called *ideals* (i.e., collections closed under taking subsets *and* unions), but sometimes collections that are *only* closed under taking subsets – the minimal requirement for a sensible "smallness notion". Why are we interested in small sets of real numbers?

Well, for a long time mathematicians have been studying properties of the real line. In the second half of the 19th century, Georg Cantor investigated sets of real numbers naturally appearing in the context of Fourier series. This eventually led him to the famous discovery that there are *infinite sets of different cardinalities* (meaning that there is no bijection between them). In particular, he proved that the set of real numbers is *uncountable*: there are more real numbers than natural numbers.

He conjectured that every infinite subset of the real line is either countable or has the "size of the continuum" (i.e., is equinumerous with the real line);

this statement is called the *Continuum Hypothesis* (CH); in other words, the Continuum Hypothesis says that the size of the continuum ($= 2^{\aleph_0}$) is the least uncountable cardinality ($= \aleph_1$). He was able to show "CH for closed sets" (i.e., each uncountable closed set is of size continuum), but the general question remained open.

David Hilbert put the question whether CH is true on top of his famous list of 23 open problems which he presented at the International Congress of Mathematicians in Paris in 1900.

Now the problem is "solved": Kurt Gödel was able to show that CH holds in his famous *constructible universe* L (the "smallest model of ZFC"); in 1963, Paul Cohen invented his groundbreaking *method of forcing* to obtain a model of ZFC in which CH fails. So CH is known to be *independent* from ZFC, i.e., neither provable nor refutable from ZFC (the standard axioms of set theory, also considered to be the fundamental axioms of all of mathematics).

This "solution" is not quite satisfactory though: it just means that these standard axioms of set theory are too weak to decide this question; it does not agree with our (perhaps naïve) intuition that such a statement should either be true or false.

So what can we do? All Borel sets actually "obey" the Continuum Hypothesis: provided that a Borel set (or even an analytic set, i.e., the continuous image of a Borel set) is uncountable, it contains a perfect set and hence is of size continuum. Therefore, one can sharpen his/her intuition about subsets of the real line by trying to understand more complicated or *pathological* sets, such as sets of intermediate cardinality (i.e., uncountable sets strictly smaller than the continuum) or non-Lebesgue-measurable sets. This leads us to exploring *special sets of real numbers* – the main theme of this thesis –, with the strong measure zero sets as a starting point.

1.2 Preliminaries

In this section, we review several concepts and results that are relevant to the thesis.

The standard reference for the respective area of set theory is the book "Set theory: On the structure of the real line" by Bartoszyński and Judah (see [BJ95]); in particular, [BJ95, Chapter 8] provides numerous results on strong measure zero sets and related concepts.

As an introduction to an even larger variety of *special sets of real numbers*, we recommend Miller's survey article "Special subsets of the real line" (see [Mil84]).

See also Jech's encyclopedic view of the current state of the art in set theory ([Jec03]). Kunen ([Kun80]) gives an introduction to forcing; an introduction to the iteration of *proper forcing* is Goldstern's "Tools for your forcing construction" ([Gol93]).

Forcing is a technique to generate new models of ZFC with prescribed properties. It was developed by Paul Cohen, who used it in his 1963 proof of the independence of the continuum hypothesis (and the axiom of choice). For details on forcing we refer to the references given above. Here, we only give a notational remark.

Traditionally, there are two (contradictory) notations for interpreting a partial order as a forcing notion. We use the "Boolean" or "downwards" notation: if $(\mathbb{P}, \leq)$ is a forcing partial order, $q \leq p$ means "q extends p", "q is stronger than p", or "q has more information than p".

To avoid confusion, we employ the *alphabet convention* (see [Gol98]):

> Whenever two conditions are comparable, the notation is chosen so that the variable used for the stronger condition comes "lexicographically" later.

Consequently, we write, e.g., $q \leq p$ (for q stronger than p), but try to avoid expressions such as $p \leq r$ (for p stronger than r).

2^ω — the reals

"Set theory of the reals" is concerned with (some version of) the real numbers: there are several versions, such as the classical real line $\mathbb{R}$, the unit interval $[0, 1]$, the Baire space ω^ω, and the Cantor space 2^ω. Each of them forms a so-called *Polish space* (i.e., a separable, completely metrizable topological space). For technical reasons, we will mainly work in the Cantor space.

The Cantor space 2^ω is a compact Hausdorff space, which is zero-dimensional, i.e., it has a clopen basis (and is therefore totally disconnected); for $s \in 2^{<\omega}$, let

$$[s] := \{x \in 2^\omega : x \supseteq s\};$$

then $([s])_{s \in 2^{<\omega}}$ forms such a clopen basis.

Working in the Cantor space allows to express many (e.g., topological) properties of sets in a purely combinatorial way.

From now on, we are referring to the elements of 2^ω as *the reals*. In other words, a real is just a subset of the natural numbers (by identifying it with its characteristic function).

Ideals on 2^ω

We consider various kinds of sets of reals, i.e., subsets of 2^ω. A natural question is how to "measure" (the size of) such a set.

One way is to "forget about" any structural properties of 2^ω, just talking about the cardinality of the set. From this point of view, one can, e.g., distinguish between countable and uncountable sets of reals. Note that the countable sets form a σ-ideal – actually the smallest σ-ideal containing all singletons.

Definition 1.1. A family $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ is an *ideal*[1] if it is closed under taking subsets and taking finite unions. It is called *σ-ideal* if it is even closed under taking countable unions.

If the continuum hypothesis (CH) holds, then all uncountable sets are of size continuum (i.e., of maximal size). Otherwise, there are "smaller" uncountable sets of reals; such sets are necessarily non-Borel, since any uncountable Borel set contains a perfect set and hence is of size continuum.

We now turn to *structural properties* of 2^ω to obtain various other "smallness notions". Two of the most prominent examples are the σ-ideal $\mathcal{M}$ of *meager sets* and the σ-ideal $\mathcal{N}$ of *null* (or *measure zero*) sets, connected to the notions of *category* and *measure*.

$\mathcal{M}$ — the σ-ideal of meager sets

The notion of *category* is based on the *topological structure* of 2^ω.

Definition 1.2. A set $F \subseteq 2^\omega$ is *nowhere dense* if for each $s \in 2^{<\omega}$ there exists a $t \in 2^{<\omega}$ such that $t \supseteq s$ and $[t] \cap F = \emptyset$ (i.e., each basic clopen contains a basic clopen disjoint from it).

Put differently, a set is nowhere dense if its (topological) closure has empty interior.

Note that every nowhere dense set is actually contained in a closed nowhere dense set.

The collection of nowhere dense sets forms an ideal, but not a σ-ideal.

Definition 1.3. A set $M \subseteq 2^\omega$ is *meager* ($M \in \mathcal{M}$) if there are countably many (closed) nowhere dense sets $(F_n)_{n<\omega}$ with $M \subseteq \bigcup_{n<\omega} F_n$.

[1]Whenever we talk about an ideal $\mathcal{I}$, we tacitly assume that it is a *proper ideal*, i.e., $2^\omega \notin \mathcal{I}$.

The meager sets are also called "sets of first category" (and the non-meager ones "sets of second category").

Note that the collection $\mathcal{M}$ of meager sets forms a σ-ideal which has a basis consisting of Borel sets (actually F_σ-sets). By the Baire category theorem, the whole space 2^ω is of second category, i.e., not meager.

$\mathcal{N}$ — the σ-ideal of (Lebesgue) measure zero sets

We can view 2^ω as a *probability space*, equipped with the standard product *measure*. The measure of a basic clopen set $[s]$ can be easily computed:

$$\mu([s]) = 2^{-|s|},$$

where $|s|$ denotes the length of s. This measure can be extended to all Lebesgue measurable sets (in particular, to all Borel sets).

We can derive another sort of "small sets", the (Lebesgue) measure zero sets (or *null* sets); more explicitly, we define:

Definition 1.4. A set $N \subseteq 2^\omega$ is *(Lebesgue) measure zero* ($N \in \mathcal{N}$) if for each $\varepsilon > 0$ there exists a countable sequence $([s_n])_{n<\omega}$ of basic clopen sets such that $\sum_{n<\omega} \mu([s_n]) < \varepsilon$ and $N \subseteq \bigcup_{n<\omega}[s_n]$.

Roughly speaking, a set is Lebesgue measure zero if it can be covered by countably many basic clopen sets of arbitrarily small total measure.

Note that the collection $\mathcal{N}$ of measure zero sets forms a σ-ideal which has a basis consisting of Borel sets (actually G_δ-sets).

Even though meager sets as well as measure zero sets are "small" in a certain sense (after all, both collections form a σ-ideal which does not contain the whole space 2^ω), these two notions are very different from each other (kind of "orthogonal"): 2^ω can actually be partitioned into a meager set and a measure zero set (sometimes called "*Marczewski partition*").

The notions of meager and measure zero are often called "dual" to each other: it is true that they share many properties and some proofs of statements about meager sets can be transformed into analogous proofs about measure zero sets or vice versa, but there are major differences as well; often, one can deal with meager sets more easily than with measure zero sets.

Note that there is always a perfect set (hence a set of size continuum) that is both meager and measure zero (a so-called Cantor set). Therefore (in ZFC) both the meager ideal $\mathcal{M}$ and the null ideal $\mathcal{N}$ contain sets of arbitrary cardinality (any cardinality less or equal the continuum).

$\mathcal{SN}$ — the σ-ideal of strong measure zero sets

In 1919, Borel introduced a strengthening of the notion of measure zero:

Definition 1.5 (Borel)**.** A set $X \subseteq 2^\omega$ is *strong measure zero* ($X \in \mathcal{SN}$) if for each sequence $(\varepsilon_n)_{n<\omega}$ of positive real numbers there is a sequence $([s_n])_{n<\omega}$ of basic clopen sets such that $\mu([s_n]) < \varepsilon_n$ for each $n < \omega$ and $X \subseteq \bigcup_{n<\omega}[s_n]$.

Roughly speaking, a set is strong measure zero if it can be covered by basic clopen sets of arbitrarily small prescribed measures.

An analogous definition applies to the classical real line $\mathbb{R}$: however, one uses *intervals* instead of *basic clopen* sets. Working in 2^ω has the advantage that Definition 1.5 can be expressed in a "purely combinatorial" way: a set $X \subseteq 2^\omega$ is strong measure zero if for each[2] sequence $(k_n)_{n<\omega}$ of natural numbers there is a sequence $([s_n])_{n<\omega}$ of basic clopen sets such that $|s_n| \geq k_n$ for each $n < \omega$ and $X \subseteq \bigcup_{n<\omega}[s_n]$.

Indeed, it is natural to generalize the notion of strong measure zero to arbitrary metric spaces, using small *balls* (or just sets of small *diameter*) instead of basic clopen sets or intervals; see Definition 1.6 below.

It follows directly from the definition that each strong measure zero set is measure zero (and it is also easy to see that they form a σ-ideal). Moreover, it can be shown that a perfect set cannot be strong measure zero; therefore the notions of measure zero and strong measure zero never coincide (recall that the Cantor set mentioned above is a perfect measure zero set).

Consequently, an uncountable Borel set cannot be strong measure zero; so the σ-ideal of strong measure zero sets is somewhat more complicated than the σ-ideal of measure zero sets or meager sets as it has no basis consisting of Borel sets (unless each strong measure zero set is countable, i.e., BC holds; see Definition 1.12 and Theorem 1.14).

Strong measure zero sets in metric spaces

For a *metric space* $(\mathcal{X}, d)$, and $\varepsilon > 0$, let

$$B(x, \varepsilon) = \{z \in \mathcal{X} : d(x, z) < \varepsilon\}$$

be the open ball around x with radius ε.

The notion of strong measure zero can be generalized to metric spaces as follows:

Definition 1.6. A set $X \subseteq \mathcal{X}$ is *strong measure zero with respect to d* ($X \in \mathcal{SN}(\mathcal{X}, d)$) if for each sequence of positive real numbers $(\varepsilon_n)_{n<\omega}$ there is a sequence $(x_n)_{n<\omega}$ of elements of $\mathcal{X}$ such that $X \subseteq \bigcup_{n<\omega} B(x_n, \varepsilon_n)$.

[2] Note that the "fast-growing" sequences are the only interesting ones. . .

It is well-known and easy to see that every strong measure zero metric space is separable.

There is a connection between different metric spaces concerning the existence of uncountable strong measure zero sets:

Theorem 1.7 (Carlson). *Assume*[3] *BC. Then every strong measure zero metric space is countable; in other words: for each metric space* $(\mathcal{X}, d)$*, and* $X \subseteq \mathcal{X}$*, we have*

$$X \in \mathcal{SN}(\mathcal{X}, d) \iff |X| = \aleph_0.$$

Proof. Under the Borel Conjecture, every strong measure zero metric space is countable, see [Car93, Theorem 3.2], or [BJ95, Theorem 8.1.8]. The reformulation follows from the fact that we can pass to the metric subspace $(\mathcal{X} \cap X, d)$ which is then strong measure zero itself. □

In general, the notion of strong measure zero may depend on the metric. For locally compact Polish spaces, however, it is independent of it:

Lemma 1.8. *Let* $\mathcal{X}$ *be a locally compact Polish space, and let* d_1 *and* d_2 *be any two compatible*[4] *metrics. Then for every* $X \subseteq \mathcal{X}$*, we have*

$$X \in \mathcal{SN}(\mathcal{X}, d_1) \iff X \in \mathcal{SN}(\mathcal{X}, d_2).$$

Proof. See, e.g., [Kys00, Stwierdzenie 5.2 on page 34] (it is in[5] Polish, however). □

For non-locally compact Polish spaces such as the Baire space ω^ω, this is not true any longer. The *Rothberger property* (which is a purely topological notion, and stronger than strong measure zero with respect to any fixed metric; see [BJ95, Definition 8.1.10]) is connected to the notion of strong measure zero as follows:

Theorem 1.9 (Fremlin-Miller). *Let* $\mathcal{X}$ *be a metrizable space, and let* $X \subseteq \mathcal{X}$*. Then* X *has the Rothberger property if and only if* $X \in \mathcal{SN}(\mathcal{X}, d)$ *for every compatible metric* d *(i.e., if* X *is strong measure zero with respect to any metric which gives* $\mathcal{X}$ *the same topology).*

Proof. See [FM88, Theorem 1], or [BJ95, Theorem 8.1.11]. □

For more information on these (and further similar) properties and their interconnections, have a look at [FM88].

[3] For the definition of BC (Borel Conjecture), see Definition 1.12.

[4] I.e., metrics generating the topology of the Polish space.

[5] Actually, not too uncommon for information concerning Polish spaces ;-)

Galvin-Mycielski-Solovay theorem

From now on, we also use the *algebraic structure* of 2^ω: for $x, y \in 2^\omega$, let $x+y$ denote the bitwise sum modulo 2, i.e., $(x+y)(n) = x(n)+y(n) \bmod 2$ for each $n \in \omega$.

Note that $(2^\omega, +)$ is an abelian group; moreover, $-x = x$ for each $x \in 2^\omega$, so there is no difference between addition and subtraction.

For $t \in 2^\omega$ and $Y \subseteq 2^\omega$, let $t+Y$ denote the set Y translated ("shifted") by t, i.e.,

$$t + Y = \{t + y : y \in Y\};$$

similarly, for $X, Y \subseteq 2^\omega$, let $X+Y$ be the set $\bigcup_{x \in X} x + Y$.

A collection $\mathcal{I}$ of subsets of 2^ω is *translation-invariant* if $t + Y \in \mathcal{I}$ whenever $Y \in \mathcal{I}$ (for any $t \in 2^\omega$). Note that the σ-ideals $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{SN}$ are translation-invariant.

The following important theorem gives an equivalent definition of strong measure zero sets (see [GMS73] for the "announcement" of the "unpublished" result, and [Mil84] or [BJ95, Theorem 8.1.16] for the proof); it also provides a general scheme for defining "smallness notions" (see Definition 1.11 and the very general Definition 1.16).

Theorem 1.10 (Galvin-Mycielski-Solovay; 1973)**.** *A set $X \subseteq 2^\omega$ is strong measure zero if and only if $X + M \neq 2^\omega$ for each meager set M.*

Note that $X+M \neq 2^\omega$ if and only if X can be "translated away" from M (meaning that there is a "*translation real*" $t \in 2^\omega$ such that $(X+t) \cap M = \emptyset$). So the Galvin-Mycielski-Solovay theorem actually says that a set is strong measure zero if and only if it can be translated away from each meager set.

One direction of the theorem is quite easy. If X can be translated away from each meager set, then it is strong measure zero: given a sequence of ε_n's, define an open dense set as the union of basic clopen sets with measures $< \varepsilon_n$; then its complement F is (closed) nowhere dense (hence in particular meager); since X can be translated away from F by assumption, X can be covered by basic clopen sets of appropriate measures. The other direction is more difficult and requires a compactness argument together with a certain tree construction.

In Chapter 5, more general versions of the Galvin-Mycielski-Solovay theorem (including detailed proofs) are presented; the version for 2^ω (i.e., Theorem 1.10) as well as the "classical" version for the real line $\mathbb{R}$ are special cases of the theorems given there.

$\mathcal{SM}$ — the collection of strongly meager sets

By replacing $\mathcal{M}$ by $\mathcal{N}$ in Theorem 1.10, Prikry defined the following notion "dual" to strong measure zero:

Definition 1.11 (Prikry)**.** A set $Y \subseteq 2^\omega$ is *strongly meager* ($Y \in \mathcal{SM}$) if $Y + N \neq 2^\omega$ for each measure zero set N.

In other words, a set is strongly meager if it can be translated away from each measure zero set.

Unlike the case of strong measure zero sets (according to Definition 1.5), it may not be obvious at first sight why a strongly meager set deserves its name (i.e., is meager); the proof, however, is easy: just consider the Marczewski partition (i.e., partition the reals into a meager and a measure zero part), and note that a strongly meager set can be translated away from the measure zero part of the partition; consequently, it is covered by a meager set, hence meager itself.[6]

Interestingly, the collection $\mathcal{SM}$ of strongly meager sets is not necessarily an ideal: assuming[7] the continuum hypothesis (CH), there are two strongly meager sets whose union is not strongly meager (for the very involved proof, see [BS01]).

Borel Conjecture (BC), dual Borel Conjecture (dBC)

Unlike the ideal $\mathcal{M}$ of meager sets or the ideal $\mathcal{N}$ of measure zero sets, there is no reason why the σ-ideal $\mathcal{SN}$ of strong measure zero sets or the collection $\mathcal{SM}$ of strongly meager sets should contain sets of size continuum (or at least uncountable sets) in general.

Note that every countable set of reals is both strong measure zero and strongly meager. For strong measure zero sets, this is completely obvious by the "elementary" Definition 1.5; but the following argument works for both (i.e., strongly meager sets according to Definition 1.11 and strong measure zero sets according to Theorem 1.10): suppose C is countable; then $C + N \neq 2^\omega$ for each measure zero set N, since $C + N = \bigcup_{t \in C} (t + N)$ is the countable union of measure zero sets and therefore measure zero.

[6]Of course, the same argument applies to the characterization of strong measure zero sets via the Galvin-Mycielski-Solovay theorem (but is – so to speak – not necessary because we can argue via the "elementary" definition of strong measure zero, i.e., Definition 1.5): each strong measure zero set is contained in a translate of the measure zero part of the Marczewski partition.

[7]This is also consistent with $\neg$CH, but not provable from ZFC (since it can happen that each strongly meager set is countable, i.e., dBC holds; see Definition 1.13 and Theorem 1.15).

Borel conjectured that the only strong measure zero sets are the countable sets of reals:

Definition 1.12. The *Borel Conjecture* (BC) is the statement that there is *no uncountable* strong measure zero set, in other words, $\mathcal{SN} = [2^\omega]^{\leq\aleph_0}$.

Under CH, the Borel Conjecture is false: this can be seen, e.g., by building a *Luzin set* (i.e., a set of size $\aleph_1 = 2^{\aleph_0}$ whose intersection with any meager set is countable); it is easy to show that a Luzin set is strong measure zero (witnessing the negation of BC).

Let us state the "dual version" of the Borel Conjecture:

Definition 1.13. The *dual Borel Conjecture* (dBC) is the statement that there is *no uncountable* strongly meager set, in other words, $\mathcal{SM} = [2^\omega]^{\leq\aleph_0}$.

Also dBC fails under CH: this time, the easiest[8] way to come up with an uncountable strongly meager set may be invoking[9] the general Lemma 1.17.

Actually, the failure of BC as well as the failure of dBC are also consistent with larger continuum. In fact, Martin's Axiom (MA) implies that there are uncountable strong measure zero and uncountable strongly meager sets.[10]

In 1976, Laver presented his famous "countable support" forcing method to show the consistency of the Borel Conjecture (see [BJ95, Theorem 8.3.4], or [Lav76] for the original paper):

Theorem 1.14 (Laver). *If* V *is a model of ZFC and* $\mathbb{P}_{\omega_2}$ *is a countable support iteration of Laver forcing of length* ω_2*, then in* $\mathrm{V}^{\mathbb{P}_{\omega_2}}$ *the Borel Conjecture holds.*

Proper forcing was introduced by Shelah around 1982; only then, the "modern version" of the proof (using general iteration theory for proper forcings) became available.

In 1993, Carlson published his result that the dual Borel Conjecture holds in the "Cohen model":

[8] We could also argue by inductively building a *Sierpiński set* (i.e., a set of size $\aleph_1 = 2^{\aleph_0}$ whose intersection with any measure zero set is countable); it can be shown quite easily that a Sierpiński set is the union of two strongly meager sets, yielding the existence of an uncountable strongly meager set under CH; indeed, every Sierpiński set is strongly meager itself (see [Paw96b]), but the proof of this fact is much harder.

[9] Of course, Lemma 1.17 can also be used to show the failure of BC under CH.

[10] The reason is the following: MA implies $\mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$ ($\mathrm{cov}(\mathcal{N}) = 2^{\aleph_0}$, resp.); but $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{SN})$ ($\mathrm{cov}(\mathcal{N}) \leq \mathrm{non}(\mathcal{SM})$, resp.), so BC (dBC, resp.) fails under MA + $\neg$CH.

Theorem 1.15 (Carlson)**.** *If* V *is a model of ZFC and* $\mathbb{P}_{\omega_2}$ *is a finite support iteration of Cohen forcing of length* ω_2*, then in* $\mathrm{V}^{\mathbb{P}_{\omega_2}}$ *the dual Borel Conjecture holds.*

One of the ideas of the proof is the following: it can be shown that a single Cohen forcing introduces a "generic" measure zero set N with the property that for each uncountable set $Y \subseteq 2^\omega$ that belongs to the ground model, $Y + N = 2^\omega$ holds (in the extension); in the end, this measure zero set N will be capable of witnessing that Y is not strongly meager.

$\mathcal{I}^*$ — the $\mathcal{I}$-shiftable sets

As mentioned above, the Galvin-Mycielski-Solovay characterization of strong measure zero sets (see Theorem 1.10) gives rise to a general scheme for defining "smallness notions".

Suppose[11] $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$.

Definition 1.16. A set $X \subseteq 2^\omega$ is $\mathcal{I}$*-shiftable* ($X \in \mathcal{I}^*$) if $X + Z \neq 2^\omega$ for each set $Z \in \mathcal{I}$.

In other words, a set belongs to $\mathcal{I}^*$ (i.e., is $\mathcal{I}$-shiftable) if it can be translated away from every set in $\mathcal{I}$.

By Galvin-Mycielski-Solovay (see Theorem 1.10), the strong measure zero sets are exactly the *meager-shiftable* sets, i.e.,

$$\mathcal{SN} = \mathcal{M}^*,$$

whereas the strongly meager sets are (by definition) exactly the *null-shiftable* sets, i.e.,

$$\mathcal{SM} = \mathcal{N}^*$$

(see Definition 1.11).

Obviously, the collection $\mathcal{I}^*$ is always closed under taking subsets (i.e., it is a "sensible smallness notion"). However, the collection $\mathcal{I}^*$ may fail to form an ideal (even if $\mathcal{I}$ itself is a σ-ideal): for instance, $\mathcal{SM} = \mathcal{N}^*$ fails to be an ideal under CH (see the discussion after Definition 1.11).

In the spirit of Definition 1.16, the (usual) Borel Conjecture could be called $\mathcal{M}$-BC (since it says that $\mathcal{M}^* = [2^\omega]^{\leq\aleph_0}$), whereas the respective name for the dual Borel Conjecture (i.e., $\mathcal{N}^* = [2^\omega]^{\leq\aleph_0}$) would be $\mathcal{N}$-BC.

In Chapter 6, we will investigate the collection s_0^*, i.e., the collection $\mathcal{I}^*$ for $\mathcal{I}$ being the σ-ideal s_0 of Marczewski null sets, having the s_0-BC in mind (i.e., the statement $s_0^* = [2^\omega]^{\leq\aleph_0}$; we also call it Marczewski Borel Conjecture).

[11] Typically, $\mathcal{I}$ will be an ideal on 2^ω; but actually any collection of sets of reals is fine.

Note that Definition 1.16 only uses the group operation of 2^ω; consequently, we can adopt the analogous definition for every group $(G,+)$.

Under certain assumptions, it is easy to construct uncountable sets in $\mathcal{I}^*$, as demonstrated by the following lemma (which we state without proof); in Chapter 5, we will use it to get an uncountable meager-shiftable set for the Baer-Specker group $\mathbb{Z}^\omega$ (see Lemma 5.58), whereas in Chapter 6 we will refer to it discussing why we *cannot* make use of it in the context of s_0-shiftable sets (see Remark 6.7).

Lemma 1.17. *Let $(G,+)$ be any abelian*[12] *group.*

Let $\mathcal{I} \subseteq \mathcal{P}(G)$ be a collection of subsets of G, and let κ be an infinite cardinal. Suppose that the following holds:

1. *$\mathcal{I}$ is translation-invariant, i.e.,*
$$\forall Z \in \mathcal{I}\ \forall g \in G\ \ (Z \in \mathcal{I} \iff Z + g \in \mathcal{I}),$$
2. *$\mathcal{I}$ is inverse-invariant, i.e.,*
$$\forall Z \in \mathcal{I}\ \ (Z \in \mathcal{I} \iff -Z \in \mathcal{I}),$$
3. *$\mathcal{I}$ contains any singleton, i.e.,*
$$\forall g \in G\ \ \{g\} \in \mathcal{I},$$
4. *$\operatorname{cof}(\mathcal{I}) \leq \kappa$, i.e.,*
$$\exists (B_\alpha)_{\alpha<\kappa} \subseteq \mathcal{I}\ \ \forall Z \in \mathcal{I}\ \ \exists \alpha < \kappa\ \ \ Z \subseteq B_\alpha,$$
5. *$\operatorname{cov}(\mathcal{I}) \geq \kappa$, i.e.,*
$$\forall \mathcal{C} \subseteq \mathcal{I}\ \ \ |\mathcal{C}| < \kappa \implies \bigcup \mathcal{C} \neq G.$$

Then there exists a set $X \subseteq G$ such that $|X| = \kappa$ and $X \in \mathcal{I}^*$ (i.e., $X + Z \neq G$ for each set $Z \in \mathcal{I}$).

Proof. One can build such a set X by a quite straightforward inductive construction. □

[12] Of course, there is an analogue for non-abelian groups (which we will not need anyway); however, one has to be cautious because there may be pairs of non-equivalent notions of "$\mathcal{I}$-shiftable" with respect to interchanging the order of the group operation; see also Remark 5.28.

Note that the lemma in particular shows that both BC and dBC fail under CH: both the σ-ideal $\mathcal{M}$ of meager sets and the σ-ideal $\mathcal{N}$ of measure zero sets have a basis consisting of Borel sets, hence CH yields $\operatorname{cov}(\mathcal{M}) = \operatorname{cof}(\mathcal{M}) = \aleph_1$ and $\operatorname{cov}(\mathcal{N}) = \operatorname{cof}(\mathcal{N}) = \aleph_1$; moreover, $\mathcal{M}$ and $\mathcal{N}$ are translation-invariant and contain all singletons, hence the lemma applies and yields an uncountable set in $\mathcal{M}^*$ (in $\mathcal{N}^*$, respectively).

$\mathcal{I}^{\circledast}$ and very meager sets

The concept of "very meager" sets $\mathcal{VM}$ was introduced in Marcin Kysiak's master thesis (see [Kys00, Definicja 5.4]; in Polish); for an English reference, see, e.g., his paper [KW04, Definition 2.4].

Let $\sigma\langle\mathcal{I}\rangle$ denote the σ-ideal generated by the sets in $\mathcal{I}$. Note that if $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{J}$ is a proper[13] σ-ideal, then $\sigma\langle\mathcal{I}\rangle$ is a proper σ-ideal and $\sigma\langle\mathcal{I}\rangle \subseteq \mathcal{J}$. Therefore (since $\mathcal{SM} \subseteq \mathcal{M}$) the collection $\sigma\langle\mathcal{SM}\rangle$ is a proper σ-ideal consisting only of meager[14] sets.

In general, the collection $\mathcal{SM}$ of strongly meager sets is not even an ideal; for the (very involved) proof, see [BS01] by Bartoszyński and Shelah. So in particular $\sigma\langle\mathcal{SM}\rangle \supsetneqq \mathcal{SM}$ is consistent (e.g., holds under CH). Note that also $\sigma\langle\mathcal{SM}\rangle = \mathcal{SM}$ is consistent (it holds in every model of dBC). (As opposed to this, $\mathcal{SN}$ is always a σ-ideal, i.e., $\sigma\langle\mathcal{SN}\rangle = \mathcal{SN}$.)

Now we describe the collection $\mathcal{VM}$ of *very meager* sets. We start with an explicit description of $\mathcal{SM}$ and $\sigma\langle\mathcal{SM}\rangle$, then we "switch quantifiers" to obtain the definition of $\mathcal{VM}$ (let "$\exists \bigcup_n Y_n = Y$" be an abbreviation for "there exists a partition of Y into countably many pieces $(Y_n)_{n\in\omega}$").

$$\begin{array}{lllllll} Y \in \mathcal{SM} & \Leftrightarrow & & \forall N \in \mathcal{N} & & Y + N \neq 2^\omega \\ Y \in \sigma\langle\mathcal{SM}\rangle & \Leftrightarrow & \exists \bigcup_n Y_n = Y & \forall N \in \mathcal{N} & \forall n \in \omega & Y_n + N \neq 2^\omega \end{array}$$

$$Y \in \mathcal{VM} \quad :\Leftrightarrow \quad \forall N \in \mathcal{N} \quad \exists \bigcup_n Y_n = Y \quad \forall n \in \omega \quad Y_n + N \neq 2^\omega$$

It is immediate that $\mathcal{SM} \subseteq \sigma\langle\mathcal{SM}\rangle \subseteq \mathcal{VM}$: for $Y \subseteq 2^\omega$, it is a (potentially) stronger requirement to belong to $\sigma\langle\mathcal{SM}\rangle$ than to $\mathcal{VM}$, since in case of $\sigma\langle\mathcal{SM}\rangle$ there has to be a "uniform" partition of Y, in case of $\mathcal{VM}$ the partition may depend on the set $N \in \mathcal{N}$.

The collection $\mathcal{VM}$ (and $\mathcal{SM}$) is derived from the ideal $\mathcal{N}$ of measure zero sets; this gives rise to the following general definition:

[13] We say $\mathcal{J}$ is *proper* if $2^\omega \notin \mathcal{J}$.

[14] Actually, even $\sigma\langle\mathcal{SM}\rangle \subseteq \mathcal{VM} \subseteq \mathcal{PM} \subseteq \mathcal{M}$ holds, where $\mathcal{PM}$ is the collection of *perfectly meager* sets (and $\mathcal{VM}$ is the collection of very meager sets).

Definition 1.18. Let $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$. Define

$$\mathcal{I}^{\circledast} := \{Y \subseteq 2^\omega : \forall Z \in \mathcal{I} \quad \exists \bigcup_n Y_n = Y \quad \forall n \in \omega \;\; Y_n + Z \neq 2^\omega\}.$$

As in the case of $\mathcal{I} = \mathcal{N}$, we have $\mathcal{I}^* \subseteq \sigma\langle\mathcal{I}^*\rangle \subseteq \mathcal{I}^{\circledast}$. Moreover, $\mathcal{I}^{\circledast}$ clearly is a σ-ideal, and sometimes it may be a more natural σ-ideal to consider than the σ-ideal $\sigma\langle\mathcal{I}^*\rangle$.

We give the following equivalent characterization of $\mathcal{I}^{\circledast}$:

Lemma 1.19. $\mathcal{I}^{\circledast} = \{Y \subseteq 2^\omega : \forall Z \in \mathcal{I} \quad \exists T \in [2^\omega]^{\leq \aleph_0} \;\; Y \subseteq T + (2^\omega \setminus Z)\}$.

Proof. The proof is straightforward. An easy computation shows that for any two sets $Y, Z \subseteq 2^\omega$, the following two assertions are equivalent:

1. $\exists \bigcup_n Y_n = Y \quad \forall n \in \omega \;\; Y_n + Z \neq 2^\omega$,
2. $\exists T \in [2^\omega]^{\leq \aleph_0} \;\; Y \subseteq T + (2^\omega \setminus Z)$.

This immediately yields the equality of the two sets. □

Using Definition 1.18 (or the "alternative definition" of Lemma 1.19), we can restate the definition of $\mathcal{VM}$ as follows:

Definition 1.20. A set $Y \subseteq 2^\omega$ is *very meager* ($Y \in \mathcal{VM}$) if it belongs to $\mathcal{N}^{\circledast}$.

Indeed, Kysiak's original definition of "very meager" is formulated as in Lemma 1.19 (see [Kys00, Definicja 5.4]): he says that a set Y is very meager (in Polish: "bardzo pierwszej kategorii") if for each measure one set Z', there is a *countable set* T such that Y is covered by $T + Z'$. (Note that Y is strongly meager if and only if for each measure one set Z', there is a *singleton* $\{t\}$ such that Y is covered by $\{t\} + Z'$.)

As mentioned above, $\mathcal{SM} \subseteq \sigma\langle\mathcal{SM}\rangle \subseteq \mathcal{VM}$, and $\mathcal{SM} \subsetneq \sigma\langle\mathcal{SM}\rangle$ is consistent (e.g., holds under CH); hence in particular $\mathcal{SM} \subsetneq \mathcal{VM}$ (in other words: $\mathcal{N}^* \subsetneq \mathcal{N}^{\circledast}$) is consistent. However, it seems to be open whether $\sigma\langle\mathcal{SM}\rangle \subsetneq \mathcal{VM}$ is consistent.

On the other hand, $[2^\omega]^{\leq \aleph_0} = \mathcal{SM} = \sigma\langle\mathcal{SM}\rangle = \mathcal{VM}$ holds true in Carlson's model of dBC (which can be shown by a modification of Carlson's original proof; see [Kys00, Twierdzenie 5.17]). Therefore also $\mathcal{SM} = \mathcal{VM}$ (in other words: $\mathcal{N}^* = \mathcal{N}^{\circledast}$) is consistent.

For the ideal $\mathcal{M}$ of meager sets, the respective situation is different; one can actually show in ZFC that $\mathcal{M}^* = \mathcal{M}^{\circledast}$:

Theorem 1.21. $\mathcal{SN} = \mathcal{M}^* = \sigma\langle\mathcal{M}^*\rangle = \mathcal{M}^{\circledast}$.

Proof. By[15] [Kys00, Stwierdzenie 5.6], we have[16] $\mathcal{SN} = \mathcal{M}^{\circledast}$.

Note that $\mathcal{M}^* \subseteq \sigma\langle\mathcal{M}^*\rangle \subseteq \mathcal{M}^{\circledast}$ (which is true for any $\mathcal{I}$); moreover, the Galvin-Mycielski-Solovay theorem says that $\mathcal{SN} = \mathcal{M}^*$ (see Theorem 1.10); therefore, all four collections coincide. $\square$

Note that the assertion $\mathcal{VM} = [2^\omega]^{\leq\aleph_0}$ (which is true in Carlson's model of dBC, see above) can be seen as a *strengthening of dBC*. Actually, this strengthening of the dual Borel Conjecture holds true in our model [GKSW] of Borel Conjecture and dual Borel Conjecture (Chapter 2) as well. The proof of this fact is the objective of Chapter 3: it explains how to adapt the argument (concerned with the dual Borel Conjecture, i.e., strongly meager sets) in Chapter 2 to show that there are not only no uncountable strongly meager sets in our final model of BC + dBC, but no uncountable *very meager* sets either.

1.3 An overview of the proof of Con(BC+dBC)

Recall that the *Borel Conjecture* (BC) is the statement that there are no uncountable strong measure zero (smz) sets, whereas the *dual Borel Conjecture* (dBC) is the statement that there are no uncountable strongly meager (sm) sets.

Chapter 2 contains my joint paper [GKSW] with Martin Goldstern, Jakob Kellner, and Saharon Shelah; it shows the following result:

Theorem 1.22. *It is consistent that the Borel Conjecture and the dual Borel Conjecture hold simultaneously, i.e., there is a model of ZFC such that both BC and dBC hold.*

I will give a very informal overview of the proof, explaining some of the major ideas involved, with the known proofs of Con(BC) (by Laver) and Con(dBC) (by Carlson) as a starting point.

Laver's proof of Con(BC)

In his 1976 paper [Lav76], Laver introduced the method of countable support iteration to get a model satisfying the Borel Conjecture:

Theorem 1.23 (Laver, 1976)**.** *If $\mathbb{P}_{\omega_2}$ is the countable support iteration of Laver forcing of length ω_2, then in a generic extension by $\mathbb{P}_{\omega_2}$ the Borel Conjecture holds.*

[15] It is even shown for arbitrary locally compact Polish groups there.

[16] Again, being in $\mathcal{M}^{\circledast}$ is formulated as in Lemma 1.19.

The key points of the proof[17] are:

- Each of the iterands $\underset{\sim}{Q}_\alpha$ is (standard) *Laver forcing* $\mathbb{L}$, which "kills" all old (uncountable) smz sets. Actually the following property of Laver forcing is used:

  $\mathbb{L}$ "kills" smz; more precisely: If $X \in V$ is an uncountable set of reals, then $\Vdash_{\mathbb{L}}$ "X is not smz", witnessed by the sequence $\varepsilon_n := 1/\underset{\sim}{\ell}_n$, where $(\underset{\sim}{\ell}_n)_{n\in\omega}$ is the Laver real added by $\mathbb{L}$.

- Once an uncountable X "has been killed", it "stays dead" (until the end of the iteration). For this, the so-called Laver property is used:

  1. Laver forcing $\mathbb{L}$ has the *Laver property*.
  2. The Laver property is preserved under (proper) *countable support* iterations.
  3. "X is not smz" remains true after a forcing satisfying the Laver property.

There is a theorem by Pawlikowski (see [Paw96a]) which says that a set X is not smz if and only if there is a closed null set F such that $X + F$ is *not null*. Since another theorem by Pawlikowski (see [Paw96c]) says that Laver forcing "*preserves random reals*" (an iterable property implying the preservation of "not null"), we can use the property "preserving random reals" instead of the Laver property in the above argument.

Carlson's proof of Con(dBC)

In his 1993 paper [Car93], Carlson showed that one can obtain a model of the dual Borel Conjecture by adding many Cohen reals:

Theorem 1.24 (Carlson, 1993). *If $\mathbb{P}_{\omega_2}$ is the finite support iteration of Cohen forcing of length ω_2, then in a generic extension by $\mathbb{P}_{\omega_2}$ the dual Borel Conjecture holds.*

The key points of the proof are:

- Each of the iterands $\underset{\sim}{Q}_\alpha$ is (single) *Cohen forcing* $\mathbb{C}$, which "kills" all old (uncountable) strongly meager sets:

[17] Actually, the "modern" proof proceeds like this; the original proof of Laver precedes concepts such as "properness" and "iteration theorems for proper forcings" (which were introduced by Shelah).

$\mathbb{C}$ "kills" strongly meager; more precisely: If $X \in V$ is an uncountable set of reals, then $\Vdash_{\mathbb{C}}$ "X is not sm", witnessed by the "generic G_δ null set" $\underset{\sim}{Z} := \bigcap_{n\in\omega} \bigcup_{m\geq n} \underset{\sim}{C}_m$ (i.e., $\Vdash_{\mathbb{C}} X + \underset{\sim}{Z} = 2^\omega$), where the sequence $(\underset{\sim}{C}_m)_{m\in\omega}$ consists of "Cohenly chosen" clopen sets with measures $\mu(\underset{\sim}{C}_m) \leq 2^{-m}$. (For the proof, a *combinatorial lemma of Erdős* is used, which is only concerned with finite sets.)

- Once an uncountable X "has been killed", it "stays dead" (until the end of the iteration). For this, the property precaliber $\aleph_1$ is used:
  1. Cohen forcing $\mathbb{C}$ is (countable, hence trivially) *precaliber* $\aleph_1$.
  2. Precaliber $\aleph_1$ is preserved under *finite support* iterations.
  3. "X is not sm" remains true after forcing with precaliber $\aleph_1$.

In our proof, we will actually replace "precaliber $\aleph_1$" by "σ*-centered*" which is a stronger property (and is preserved under finite support iterations of countable[18] length).

Obstacles in combining the proofs of Laver and Carlson

To get a model of ZFC satisfying both BC and dBC, one might attempt to combine the proofs by Laver and Carlson by just mixing the two sorts of iterands, i.e., alternately forcing with Laver and Cohen forcing. This would "kill" all smz and sm sets which belong to the respective intermediate model; however, there are severe problems with this approach.

The methods to preserve "X is not smz" and "X is not sm", respectively, do not fit to each other (cf. the items 1. and 2. in the respective lists): Cohen forcing $\mathbb{C}$ does not have the Laver property, whereas Laver forcing $\mathbb{L}$ is not precaliber $\aleph_1$ (in fact not even c.c.c.); moreover, countable support iteration is essential for the preservation of the Laver property, whereas finite support iteration is essential to preserve precaliber $\aleph_1$.

In case of Carlson's proof, it is even worse, since adding Cohen reals (which is not only because of the Cohen iterands, but also due to the finite support iteration which adds Cohen reals at limits) inevitably destroys the Borel Conjecture (actually yielding a strong failure of BC: adding many Cohen reals makes $\mathrm{cov}(\mathcal{M})$ large, so also $\mathrm{non}(\mathcal{SN})$ will be large, i.e., not only *there is*, but *all* sets of size $\aleph_1$ are smz).

[18]Indeed, it is even preserved under finite support iterations of length less than $(2^{\aleph_0})^+$.

The idea of the proof of Con(BC+dBC)

So how to reconcile the two strategies?

We have to come up with an iteration which works with respect to both tasks (i.e., killing both smz and sm, and preserving both "X not smz" and "X not sm"). This is accomplished by using a "generic" iteration (with "generic iterands" and "generic support", so to speak).

For this purpose, we start with a "preparatory" forcing $\mathbb{R}$ (which is σ-closed and $\aleph_2$-c.c.); any generic filter $G \subseteq \mathbb{R}$ will yield a c.c.c.(!) iteration $\bar{\mathbf{P}}$ of length ω_2, whose "generic iterands" are (alternately) forcings "killing smz" and forcings "killing sm". Additionally, both types of iterands are capable of preserving both "X not smz" and "X not sm" (either behavior is "prepared" within the preparatory forcing $\mathbb{R}$, hence both at the same time is possible); also, both preserving "X not smz" and preserving "X not sm" is preserved at limits (since both finite and countable support is "prepared" within the preparatory forcing $\mathbb{R}$). Finally, the forcing $\mathbb{R} * \mathbf{P}_{\omega_2}$ yields BC+dBC.

The iterands: Ultralaver forcing and Janus forcing

First of all, we define two (classes of) forcing notions: Ultralaver forcing and Janus forcing.

Ultralaver forcing Instead of Laver forcing (as in the proof of Con(BC)), we use "ultralaver" forcings $\mathbb{L}_{\bar{D}}$, which are "filtered" versions of the standard Laver forcing $\mathbb{L}$. An *ultralaver forcing* $\mathbb{L}_{\bar{D}}$ is based on a system $\bar{D} = (D_s)_{s\in\omega^{<\omega}}$ of ultrafilters on ω and consists of all Laver trees $p \subseteq \omega^{<\omega}$ with the property that for any node $s \in p$ (above the stem) the set of n with $s^\frown n \in p$ (is not only infinite but) belongs to the (non-principal) ultrafilter D_s.

Like Laver forcing, it "kills" smz sets. Moreover, it is clearly σ-centered (which was the incentive to replace Laver forcing by ultralaver forcings); this is towards preserving "X not sm". By choosing appropriate ultrafilters, it can also be made to "preserve random reals" (in place of the Laver property); this is towards preserving "X not smz".

Janus forcing "Janus" forcing has "two faces" (therefore the name). It will always be preceded by an ultralaver forcing (the definition is actually dependent on the ultralaver real which has been added by this ultralaver forcing).

First of all, a *Janus forcing* consists of a (countable) core which is (essentially) Cohen forcing; as in Carlson's proof, the core provides a null set $\underset{\sim}{Z} := \bigcap_{n\in\omega} \bigcup_{m\geq n} \underset{\sim}{C}_m$ which will kill any old sm set X.

Second, there are (countably or uncountably many) additional conditions which are "wrapped around" the core. On the one hand, a Janus forcing can be countable, and hence σ-centered; this is towards preserving "X not sm". On the other hand, a Janus forcing can be (equivalent to) random forcing; this is towards preserving "X not smz", since random forcing "preserves random reals" which is the iterable property in place of the Laver property.

In both cases, however, a Janus forcing has to satisfy a certain combinatorial property which makes sure that its core yields a null set $\underaccent{\sim}{Z}$ which indeed fulfills $X + \underaccent{\sim}{Z} = 2^\omega$ for old X (i.e., kills old sm sets X). The property essentially says that there is no condition outside the core which excludes too many potential clopen sets from being selected as one of the $\underaccent{\sim}{C}_m$; in other words, it is true that the sequence $(\underaccent{\sim}{C}_m)_{m\in\omega}$ of clopen sets is not completely "Cohenly chosen" (as in Carlson's original proof), but still the $\underaccent{\sim}{C}_m$'s are chosen in a sufficiently free way in order to make the arguments work. (In fact, a quite involved *combinatorial lemma from [BS10]* is used, thereby replacing the lemma of Erdős used in Carlson's proof of Con(dBC).)

Alternating (partial countable support) iterations

An *alternating (partial countable support) iteration* is a (proper) iteration $\bar{P} = (P_\alpha, \underaccent{\sim}{Q}_\alpha)_{\alpha<\omega_2}$ of length ω_2 with the following properties:

- For even α, the iterand $\underaccent{\sim}{Q}_\alpha$ is an ultralaver forcing.
- For odd α, the iterand $\underaccent{\sim}{Q}_\alpha$ is a Janus forcing.
- For limit δ, the forcing P_δ is a *partial countable support limit* of the $(P_\alpha)_{\alpha<\delta}$; this is a subset of the countable support limit containing the direct limit.

The concept "partial countable support iteration" gives the freedom to use either an iteration which is (close to) a finite support iteration or an iteration which is (close to) a countable support iteration. Using (almost) finite support iterations is towards preserving "X not sm", whereas using (almost) countable support iterations is towards preserving "X not smz".

The preparatory forcing $\mathbb{R}$ and M-complete embeddings

The *preparatory forcing* $\mathbb{R}$ consists of alternating iterations which approximate the desired (generic) alternating iteration $\bar{\mathbf{P}}$. More precisely, $\mathbb{R}$ consists of conditions $x = (M^x, \bar{P}^x)$, where M^x is a countable ord-transitive model of (some fragment of) ZFC, and $\bar{P}^x$ is an alternating iteration in M^x.

A (countable) *ord-transitive model* M is somewhere in between a transitive model and an elementary (sub)model (of some $H(\chi)$); it essentially equals a transitive model, but $M \cap \mathrm{Ord}$ is not a countable ordinal (as in the case of a real countable transitive model), but a non-transitive set of ordinals (as in the case of an elementary model). One gets an ord-transitive model when "ord-collapsing" an elementary model, i.e., collapsing everything except for ordinals (which are "treated as urelements"). The reason for using ord-transitive models instead of transitive ones is the following: Our conditions $x = (M^x, \bar{P}^x) \in \mathbb{R}$ are supposed to approximate a generic iteration of length ω_2, so our approximating iterations $\bar{P}^x$ have to allow for non-trivial iterands $\underset{\sim}{Q}_\alpha$ for coordinates α arbitrarily large in ω_2.

The *order on* $\mathbb{R}$ is defined as follows: a condition $y = (M^y, \bar{P}^y)$ is stronger than $x = (M^x, \bar{P}^x)$, i.e., $y \leq x$, if

- $M^x \in M^y$ (and M^y already "knows" that M^x is in fact countable),
- $\bar{P}^x$ is "canonically M^x-completely embeddable" into $\bar{P}^y$.

"*Canonically M^x-completely embeddable*" means the following: One can find a (canonical) "coherent" sequence $(i_\alpha)_{\alpha \leq \omega_2}$ of M^x-complete embeddings $i_\alpha : P^x_\alpha \to P^y_\alpha$. (An embedding $i : P \to Q$ is *M-complete* if each maximal antichain $A \subseteq P$ *which belongs to M* is mapped to a maximal antichain $i[A] \subseteq Q$.)

For $\bar{P}^x$ being canonically M^x-completely embedded into $\bar{P}^y$, it is in particular necessary that $\underset{\sim}{Q}^x_\alpha$ is (forced to be) an M^x-complete subforcing of $\underset{\sim}{Q}^y_\alpha$. In case of ultralaver forcing (i.e., even α), this will be true whenever the ultrafilter system $\bar{D}^y$ (defining $\underset{\sim}{Q}^y_\alpha = \mathbb{L}_{\bar{D}^y}$) "extends" the ultrafilter system $\bar{D}^x$, i.e., for each node $s \in \omega^{<\omega}$, $\underset{\sim}{D}^y_s \supseteq D^x_s$.

Almost finite support and almost countable support

At limit ordinals δ, we have the following situation (still assuming that $\bar{P}^x$ is canonically M^x-completely embedded into $\bar{P}^y$).

The forcing P^y_δ is a partial countable support limit of the $(P^y_\alpha)_{\alpha<\delta}$ such that the "canonically defined embedding" $i_\delta : P^x_\delta \to P^y_\delta$ (satisfies $i_\delta[P^x_\delta] \subseteq P^y_\delta$ and) is M^x-complete. Note that we would like to define P^y_δ as the finite (or countable) support limit of the $(P^y_\alpha)_{\alpha<\delta}$. However, this is not quite possible; instead, we will take the "almost finite support limit over x" and the "almost countable support limit over x", respectively, which will make sure that i_δ is an M^x-complete embedding.

More precisely, the *almost finite support limit over x* is the "minimal" partial countable support limit; it essentially equals the direct limit of $(P^y_\alpha)_{\alpha<\delta}$

together with the (countable) set $i_\delta[P^x_\delta]$. The *almost countable support limit over x* is kind of "maximal": it essentially consists of all those conditions in the countable support limit of $(P^y_\alpha)_{\alpha<\delta}$ which do not outright force that i_δ is not M^x-complete.

They are sufficiently close to the finite/countable support limit to make the respective arguments work: The almost finite/countable support limit will preserve σ-centeredness/"preserving random reals", which is towards preserving "X not sm"/"X not smz".

The generic alternating iteration $\bar{\mathbf{P}}$

Fix a generic filter $G \subseteq \mathbb{R}$. The filter G yields a directed system of alternating iterations together with (not quite) complete embeddings; we can derive a *"generic" alternating iteration* $\bar{\mathbf{P}}$ (with limit $\mathbf{P}_{\omega_2}$) from the limit of this directed system.

The generic iteration $\bar{\mathbf{P}}$ is approximated by all the $\bar{P}^x$ for which $x = (M^x, \bar{P}^x)$ belongs to the generic filter G. More precisely, each such $\bar{P}^x$ can be canonically M^x-completely embedded into $\bar{\mathbf{P}}$, via embeddings $i^x_\alpha : P^x_\alpha \to \mathbf{P}_\alpha$; in other words, whenever $H \subseteq \mathbf{P}_{\omega_2}$ is a generic filter over V[G], then the preimage of H under $i^x_{\omega_2}$ (which is a subset of $P^x_{\omega_2}$) is a generic filter over M^x.

The iterands $\tilde{\mathbf{Q}}_\alpha$ of the generic iteration $\bar{\mathbf{P}} = (\mathbf{P}_\alpha, \tilde{\mathbf{Q}}_\alpha)_{\alpha<\omega_2}$ are "generic" ultralaver forcings (in case of α even) and "generic" Janus forcings (in case of α odd). Surprisingly, $\bar{\mathbf{P}}$ is a c.c.c. iteration (which is essential for the proof of BC and dBC): The generic ultralaver forcings are σ-centered (hence c.c.c.), and the generic Janus forcings can also be shown to be c.c.c. (by constructing a model M which "catches" a countable part of the maximal antichain, and using M-completeness to preserve maximality); moreover, the c.c.c. is preserved at all limits (which can be shown by a similar argument).

The proof of BC and dBC

We can now prove that *both BC and dBC hold in the final model* $\mathbb{R} * \mathbf{P}_{\omega_2}$. This is due to the fact that the (generic) ultralaver/Janus forcings kill smz/sm sets, and the fact that the set of all $x = (M^x, \bar{P}^x) \in \mathbb{R}$ with either of the following two properties is dense in $\mathbb{R}$:

- $\bar{P}^x$ is build towards preserving "X not smz".
- $\bar{P}^x$ is build towards preserving "X not sm".

More precisely, *the proof of BC* is (roughly speaking) as follows:

Assume towards a contradiction that (in the final model) the uncountable set X is smz. We can assume that X is of size $\aleph_1$, and that X already appears at some stage $\alpha_0 < \omega_2$. Consider an ultralaver forcing $\underset{\sim}{\mathbf{Q}}_\alpha$ at any even stage $\alpha > \alpha_0$; it will kill X, by adding a witness F (a Borel code for a closed null set) such that $X + F$ is not null. Since X is smz in the final model, we know that $X + F$ will become null at some later stage, i.e., we can assume that (a Borel code for) a null set containing $X + F$ appears at some $\beta < \omega_2$ with $\beta > \alpha$.

Since $\bar{\mathbf{P}}$ is c.c.c., names for reals (such as Borel codes) are countable objects, so they can be "seen" by countable models. The whole situation can be "reflected down" to a condition x in the preparatory forcing. This condition x can be strengthened to a condition $y = (M^y, \bar{P}^y)$ which belongs to the first of the two dense sets mentioned above, i.e., $\bar{P}^y$ is build towards preserving "X not smz": The iterands are ultralaver forcings "preserving (certain) random reals", and Janus forcings which are equivalent to random forcing (hence "preserving random reals"), respectively; at limits we take almost countable support limits over x (which preserve "preserving random reals"). This shows that "$X + F$ not null" is preserved, which can be shown to lead to a contradiction (by using absoluteness of names for reals etc.).

The proof of dBC is similar:

We again get a condition x "capturing" the whole situation; we choose a stronger condition y which belongs to the second of the two dense sets, i.e., $\bar{P}^y$ is build towards preserving "X not sm": The iterands are ultralaver forcings which are σ-centered anyway, and Janus forcings which are countable (hence σ-centered), respectively; at limits we now take almost finite support limits over x (which preserve σ-centeredness). Again, this leads to a contradiction.

Chapter 2

Borel Conjecture and dual Borel Conjecture

This chapter contains my joint paper with Martin Goldstern, Jakob Kellner, and Saharon Shelah (see [GKSW]).

> We show that it is consistent that the Borel Conjecture and the dual Borel Conjecture hold simultaneously.

In Section 1.3, I have given a very informal overview of the proof.

Introduction

History

A set X of reals[1] is called "strong measure zero" (smz), if for all functions $f : \omega \to \omega$ there are intervals I_n of measure $\leq 1/f(n)$ covering X. Obviously, a smz set is a null set (i.e., has Lebesgue measure zero), and it is easy to see that the family of smz sets forms a σ-ideal and that perfect sets (and therefore uncountable Borel or analytic sets) are not smz.

At the beginning of the 20th century, Borel [Bor19, p. 123] conjectured:

> Every smz set is countable.

This statement is known as the "Borel Conjecture" (BC). In the 1970s it was proved that BC is *independent*, i.e., neither provable nor refutable.

[1] In this paper, we use 2^ω as the set of reals. ($\omega = \{0, 1, 2, \ldots\}$.) By well-known results both the definition and the theorem also work for the unit interval $[0, 1]$ or the torus $\mathbb{R}/\mathbb{Z}$. Occasionally we also write "x is a real" for "$x \in \omega^\omega$".

Let us very briefly comment on the notion of independence: A sentence φ is called independent of a set T of axioms, if neither φ nor $\neg\varphi$ follows from T. (As a trivial example, $(\forall x)(\forall y) x \cdot y = y \cdot x$ is independent from the group axioms.) The set theoretic (first order) axiom system ZFC (Zermelo Fraenkel with the axiom of choice) is considered to be the standard axiomatization of all of mathematics: A mathematical proof is generally accepted as valid iff it can be formalized in ZFC. Therefore we just say "φ is independent" if φ is independent of ZFC. Several mathematical statements are independent, the earliest and most prominent example is Hilbert's first problem, the Continuum Hypothesis (CH).

BC is independent as well: Sierpiński [Sie28] showed that CH implies ¬BC (and, since Gödel showed the consistency of CH, this gives us the consistency of ¬BC). Using the method of forcing, Laver [Lav76] showed that BC is consistent.

Galvin, Mycielski and Solovay [GMS73] proved the following conjecture of Prikry:

> $X \subseteq 2^\omega$ is smz if and only if every comeager (dense G_δ) set contains a translate of X.

Prikry also defined the following dual notion:

> $X \subseteq 2^\omega$ is called "strongly meager" (sm) if every set of Lebesgue measure 1 contains a translate of X.

The dual Borel Conjecture (dBC) states:

> Every sm set is countable.

Prikry noted that CH implies ¬dBC and conjectured dBC to be consistent (and therefore independent), which was later proved by Carlson [Car93].

Numerous additional results regarding BC and dBC have been proved: The consistency of variants of BC or of dBC, the consistency of BC or dBC together with certain assumptions on cardinal characteristics, etc. See [BJ95, Ch. 8] for several of these results. In this paper, we prove the consistency (and therefore independence) of BC+dBC (i.e., consistently BC and dBC hold simultaneously).

The problem

The obvious first attempt to force BC+dBC is to somehow combine Laver's and Carlson's constructions. However, there are strong obstacles:

Laver's construction is a countable support iteration of Laver forcing. The crucial points are:

- Adding a Laver real makes every old uncountable set X non-smz.
- And this set X remains non-smz after another forcing P, provided that P has the "Laver property".

So we can start with CH and use a countable support iteration of Laver forcing of length ω_2. In the final model, every set X of reals of size $\aleph_1$ already appeared at some stage $\alpha < \omega_2$ of the iteration; the next Laver real makes X non-smz, and the rest of the iteration (as it is a countable support iteration of proper forcings with the Laver property) has the Laver property, and therefore X is still non-smz in the final model.

Carlson's construction on the other hand adds ω_2 many Cohen reals in a finite support iteration (or equivalently: finite support product). The crucial points are:

- A Cohen real makes every old uncountable set X non-sm.
- And this set X remains non-sm after another forcing P, provided that P has precaliber $\aleph_1$.

So we can start with CH, and use more or less the same argument as above: Assume that X appears at $\alpha < \omega_2$. Then the next Cohen makes X non-sm. It is enough to show that X remains non-sm at all subsequent stages $\beta < \omega_2$. This is guaranteed by the fact that a finite support iteration of Cohen reals of length $< \omega_2$ has precaliber $\aleph_1$.

So it is unclear how to combine the two proofs: A Cohen real makes all old sets smz, and it is easy to see that whenever we add Cohen reals cofinally often in an iteration of length, say, ω_2, all sets of any intermediate extension will be smz, thus violating BC. So we have to avoid Cohen reals,[2] which also implies that we cannot use finite support limits in our iterations. So we have a problem even if we find a replacement for Cohen forcing in Carlson's proof that makes all old uncountable sets X non-sm and that does not add Cohen reals: Since we cannot use finite support, it seems hopeless to get precaliber $\aleph_1$, an essential requirement to keep X non-sm.

Note that it is the *proofs* of BC and dBC that are seemingly irreconcilable; this is not clear for the models. Of course Carlson's model, i.e., the Cohen model, cannot satisfy BC, but it is not clear whether maybe already the Laver model could satisfy dBC. (It is even still open whether a single Laver forcing makes every old uncountable set non-sm.) Actually, Bartoszyński and Shelah [BS03] proved that the Laver model does satisfy the following weaker variant of dBC (note that the continuum has size $\aleph_2$ in the Laver model):

[2] An iteration that forces dBC without adding Cohen reals was given in [BS10], using non-Cohen oracle-cc.

Every sm set has size less than the continuum.

In any case, it turns out that one *can* reconcile Laver's and Carlson's proof, by "mixing" them "generically", resulting in the following theorem:

Theorem. *If ZFC is consistent, then ZFC+BC+dBC is consistent.*

Prerequisites

To understand anything of this paper, the reader

- should have some experience with finite and countable support iteration, proper forcing, $\aleph_2$-cc, σ-closed, etc.,
- should know what a quotient forcing is,
- should have seen some preservation theorem for proper countable support iteration,
- should have seen some tree forcings (such as Laver forcing).

To understand everything, additionally the following is required:

- The "case A" preservation theorem from [She98], more specifically we build on the proof of [Gol93] (or [GK06]).
- In particular, some familiarity with the property "preservation of randoms" is recommended. We will use the fact that random and Laver forcing have this property.
- We make some claims about (a rather special case of) ord-transitive models in Section 2.3.A. The readers can either believe these claims, or check them themselves (by some rather straightforward proofs), or look up the proofs (of more general settings) in [She04] or [Kel12].

From the theory of strong measure zero and strongly meager, we only need the following two results (which are essential for our proofs of BC and dBC, respectively):

- Pawlikowski's result from [Paw96a] (which we quote as Theorem 2.2 below), and
- Theorem 8 of Bartoszyński and Shelah's [BS10] (which we quote as Lemma 2.55).

We do not need any other results of Bartoszyński and Shelah's paper [BS10]; in particular we do not use the notion of non-Cohen oracle-cc (introduced in [She06]); and the reader does not have to know the original proofs of Con(BC) and Con(dBC), by Laver and Carlson, respectively.

The third author claims that our construction is more or less the same as a non-Cohen oracle-cc construction, and that the extended version presented in [She10] is even closer to our preparatory forcing.

Notation and some basic facts on forcing, strongly meager (sm) and strong measure zero (smz) sets

We call a lemma "Fact" if we think that no proof is necessary — either because it is trivial, or because it is well known (even without a reference), or because we give an explicit reference to the literature.

Stronger conditions in forcing notions are smaller, i.e., $q \le p$ means that q is stronger than p.

Let $P \subseteq Q$ be forcing notions. (As usual, we abuse notation by not distinguishing between the underlying set and the quasiorder on it.)

- For $p_1, p_2 \in P$ we write $p_1 \perp_P p_2$ for "p_1 and p_2 are incompatible". Otherwise we write $p_1 \not\perp_P p_2$. (We may just write $\perp$ or $\not\perp$ if P is understood.)
- $q \le^* p$ (or: $q \le^*_P p$) means that q forces that p is in the generic filter, or equivalently that every $q' \le q$ is compatible with p. And $q =^* p$ means $q \le^* p \ \wedge \ p \le^* q$.
- P is *separative*, if $\le$ is the same as $\le^*$, or equivalently, if for all q, p with $q \not\ge p$ there is an $r \le p$ incompatible with q. Given any P, we can define its "separative quotient" Q by first replacing (in P) $\le$ by $\le^*$ and then identifying elements p, q whenever $p =^* q$. Then Q is separative and forcing equivalent to P.
- "*P is a subforcing of Q*" means that the relation $\le_P$ is the restriction of $\le_Q$ to P.
- "*P is an incompatibility-preserving subforcing of Q*" means that P is a subforcing of Q and that $p_1 \perp_P p_2$ iff $p_1 \perp_Q p_2$ for all $p_1, p_2 \in P$.

Let additionally M be a countable transitive[3] model (of a sufficiently large subset of ZFC) containing P.

[3]We will also use so-called ord-transitive models, as defined in Section 2.3.A.

- "P is an M-complete subforcing of Q" (or: $P \lessdot_M Q$) means that P is a subforcing of Q and: if $A \subseteq P$ is in M a maximal antichain, then it is a maximal antichain of Q as well. (Or equivalently: P is an incompatibility-preserving subforcing of Q and every predense subset of P in M is predense in Q.) Note that this means that every Q-generic filter G over V induces a P-generic filter over M, namely $G^M := G \cap P$ (i.e., every maximal antichain of P in M meets $G \cap P$ in exactly one point). In particular, we can interpret a P-name τ in M as a Q-name. More exactly, there is a Q-name τ' such that $\tau'[G] = \tau[G^M]$ for all Q-generic filters G. We will usually just identify τ and τ'.

- Analogously, if $P \in M$ and $i : P \to Q$ is a function, then i is called an *M-complete embedding* if it preserves $\leq$ (or at least $\leq^*$) and $\perp$ and moreover: If $A \in M$ is predense in P, then $i[A]$ is predense in Q.

There are several possible characterizations of sm ("strongly meager") and smz ("strong measure zero") sets; we will use the following as definitions:

A set X is not sm if there is a measure 1 set into which X cannot be translated; i.e., if there is a null set Z such that $(X + t) \cap Z \neq \emptyset$ for all reals t, or, in other words, $Z + X = 2^\omega$. To summarize:

$$X \text{ is } \textit{not} \text{ sm iff there is a Lebesgue null set } Z \text{ such that } Z + X = 2^\omega. \tag{2.1}$$

We will call such a Z a "witness" for the fact that X is not sm (or say that Z witnesses that X is not sm).

The following theorem of Pawlikowski [Paw96a] is central for our proof[4] that BC holds in our model:

Theorem 2.2. *$X \subseteq 2^\omega$ is smz iff $X + F$ is null for every closed null set F. Moreover, for every dense G_δ set H we can* construct *(in an absolute way) a closed null set F such that for every $X \subseteq 2^\omega$ with $X + F$ null there is $t \in 2^\omega$ with $t + X \subseteq H$.*

In particular, we get:

$$X \text{ is } \textit{not} \text{ smz iff there is a closed null set } F \text{ such that } X + F \text{ has positive outer Lebesgue measure.} \tag{2.3}$$

Again, we will say that the closed null set F "witnesses" that X is not smz (or call F a witness for this fact).

[4] We thank Tomek Bartoszyński for pointing out Pawlikowski's result to us, and for suggesting that it might be useful for our proof.

Annotated contents

An informal overview of the proof, including two illustrations, can be found at `http://arxiv.org/abs/1112.4424/`.

2.1 Ultralaver forcing

In this section, we define the family of *ultralaver forcings* $\mathbb{L}_{\bar{D}}$, variants of Laver forcing which depend on a system $\bar{D}$ of ultrafilters.

In the rest of the paper, we will use the following properties of $\mathbb{L}_{\bar{D}}$. (And we will use *only* these properties. So readers who are willing to take these properties for granted could skip to Section 2.2.)

1. $\mathbb{L}_{\bar{D}}$ is σ-centered, hence ccc.
   (This is Lemma 2.5.)

2. $\mathbb{L}_{\bar{D}}$ is separative.
   (This is Lemma 2.6.)

3. *Ultralaver kills smz:* There is a canonical $\mathbb{L}_{\bar{D}}$-name $\bar{\underaccent{\sim}{\ell}}$ for a fast growing real in ω^ω called the ultralaver real. From this real, we can define (in an absolute way) a closed null set F such that $X + F$ is positive for all uncountable X in V (and therefore F witnesses that X is not smz, according to Theorem 2.2).
(This is Corollary 2.24.)

4. Whenever X is uncountable, then $\mathbb{L}_{\bar{D}}$ forces that X is not "thin".
(This is Corollary 2.27.)

5. If $(M, \in)$ is a countable model of ZFC* and if $\mathbb{L}_{\bar{D}^M}$ is an ultralaver forcing in M, then for any ultrafilter system $\bar{D}$ extending $\bar{D}^M$, $\mathbb{L}_{\bar{D}^M}$ is an M-complete subforcing of the ultralaver forcing $\mathbb{L}_{\bar{D}}$.
(This is Lemma 2.8.)
Moreover, the real $\bar{\underaccent{\sim}{\ell}}$ of item (3) is so "canonical" that we get: If (in M) $\bar{\underaccent{\sim}{\ell}}^M$ is the $\mathbb{L}_{\bar{D}^M}$-name for the $\mathbb{L}_{\bar{D}^M}$-generic real, and if (in V) $\bar{\underaccent{\sim}{\ell}}$ is the $\mathbb{L}_{\bar{D}}$-name for the $\mathbb{L}_{\bar{D}}$-generic real, and if H is $\mathbb{L}_{\bar{D}}$-generic over V and thus $H^M := H \cap \mathbb{L}_{\bar{D}^M}$ is the induced $\mathbb{L}_{\bar{D}^M}$-generic filter over M, then $\bar{\underaccent{\sim}{\ell}}[H]$ is equal to $\bar{\underaccent{\sim}{\ell}}^M[H^M]$.
Since the closed null set F is constructed from $\bar{\underaccent{\sim}{\ell}}$ in an absolute way, the same holds for F, i.e., the Borel codes $F[H]$ and $F[H^M]$ are the same.

6. Moreover, given M and $\mathbb{L}_{\bar{D}^M}$ as above, and a random real r over M, we can choose $\bar{D}$ extending $\bar{D}^M$ such that $\mathbb{L}_{\bar{D}}$ forces that randomness of r is preserved (in a strong way that can be preserved in a countable support iteration).
(This is Lemma 2.33.)

2.1.A Definition of ultralaver

Notation. We use the following fairly standard notation:

A *tree* is a nonempty set $p \subseteq \omega^{<\omega}$ which is closed under initial segments and has no maximal elements.[5] The elements ("nodes") of a tree are partially ordered by $\subseteq$.

For each sequence $s \in \omega^{<\omega}$ we write $\mathrm{lh}(s)$ for the length of s.

For any tree $p \subseteq \omega^{<\omega}$ and any $s \in p$ we write $\mathrm{succ}_p(s)$ for one of the following two sets:

$$\{k \in \omega : s^\frown k \in p\} \quad \text{or} \quad \{t \in p : (\exists k \in \omega)\ t = s^\frown k\}$$

[5]Except for the proof of Lemma 2.8, where we also allow trees with maximal elements, and even empty trees.

and we rely on the context to help the reader decide which set we mean.

A *branch* of p is either of the following:

- A function $f : \omega \to \omega$ with $f{\restriction}n \in p$ for all $n \in \omega$.
- A maximal chain in the partial order $(p, \subseteq)$. (As our trees do not have maximal elements, each such chain C determines a branch $\bigcup C$ in the first sense, and conversely.)

We write $[p]$ for the set of all branches of p.

For any tree $p \subseteq \omega^{<\omega}$ and any $s \in p$ we write $p^{[s]}$ for the set $\{t \in p : t \supseteq s \text{ or } t \subseteq s\}$, and we write $[s]$ for either of the following sets:

$$\{t \in p : s \subseteq t\} \quad \text{or} \quad \{x \in [p] : s \subseteq x\}.$$

The stem of a tree p is the shortest $s \in p$ with $|\mathrm{succ}_p(s)| > 1$. (The trees we consider will never be branches, i.e., will always have finite stems.)

Definition 2.4.

- For trees q, p we write $q \leq p$ if $q \subseteq p$ ("q is stronger than p"), and we say that "*q is a pure extension of p*" ($q \leq_0 p$) if $q \leq p$ and $\mathrm{stem}(q) = \mathrm{stem}(p)$.
- A filter system $\bar{D}$ is a family $(D_s)_{s \in \omega^{<\omega}}$ of filters on ω. (All our filters will contain the Fréchet filter of cofinite sets.) We write D_s^+ for the collection of D_s-positive sets (i.e., sets whose complement is not in D_s).
- We define $\mathbb{L}_{\bar{D}}$ to be the set of all trees p such that $\mathrm{succ}_p(t) \in D_t^+$ for all $t \in p$ above the stem.
- The generic filter is determined by the generic branch $\bar{\ell} = (\ell_i)_{i \in \omega} \in \omega^\omega$, called the *generic real*: $\{\bar{\ell}\} = \bigcap_{p \in G}[p]$ or equivalently, $\bar{\ell} = \bigcup_{p \in G} \mathrm{stem}(p)$.
- An ultrafilter system is a filter system consisting of ultrafilters. (Since all our filters contain the Fréchet filter, we only consider nonprincipal ultrafilters.)
- An *ultralaver forcing* is a forcing $\mathbb{L}_{\bar{D}}$ defined from an ultrafilter system. The generic real for an ultralaver forcing is also called the *ultralaver real*.

Recall that a forcing notion $(P, \leq)$ is *σ-centered* if $P = \bigcup_n P_n$, where for all $n, k \in \omega$ and for all $p_1, \dots, p_k \in P_n$ there is $q \leq p_1, \dots, p_k$.

Lemma 2.5. *All ultralaver forcings $\mathbb{L}_{\bar{D}}$ are σ-centered (hence ccc).*

Proof. Every finite set of conditions sharing the same stem has a common lower bound. □

Lemma 2.6. $\mathbb{L}_{\bar{D}}$ *is separative.*[6]

Proof. If $q \not\geq p$, then there is $s \in p \setminus q$. Now $p^{[s]} \perp q$. □

If each D_s is the Fréchet filter, then $\mathbb{L}_{\bar{D}}$ is *Laver forcing* (often just written $\mathbb{L}$).

2.1.B M-complete embeddings

Note that for all ultrafilter systems $\bar{D}$ we have:

> Two conditions in $\mathbb{L}_{\bar{D}}$ are compatible if and only if their stems are comparable and moreover, the longer stem is an element of the condition with the shorter stem. (2.7)

Lemma 2.8. *Let M be countable.*[7] *In M, let $\mathbb{L}_{\bar{D}^M}$ be an ultralaver forcing. Let $\bar{D}$ be (in V) a filter system extending*[8] *$\bar{D}^M$. Then $\mathbb{L}_{\bar{D}^M}$ is an M-complete subforcing of $\mathbb{L}_{\bar{D}}$.*

Proof. For any tree[9] T, any filter system $\bar{E} = (E_s)_{s\in\omega^{<\omega}}$, and any $s_0 \in T$ we define a sequence $(T^\alpha_{\bar{E},s_0})_{\alpha\in\omega_1}$ of "derivatives" (where we may abbreviate $T^\alpha_{\bar{E},s_0}$ to T^α) as follows:

- $T^0 := T^{[s_0]}$.
- Given T^α, we let $T^{\alpha+1} := T^\alpha \setminus \bigcup\{[s] : s \in T^\alpha, s_0 \subseteq s, \mathrm{succ}_{T^\alpha}(s) \notin E^+_s\}$, where $[s] := \{t : s \subseteq t\}$.
- For limit ordinals $\delta > 0$ we let $T^\delta := \bigcap_{\alpha<\delta} T^\alpha$.

Then we have

(a) Each T^α is closed under initial segments. Also: $\alpha < \beta$ implies $T^\alpha \supseteq T^\beta$.

(b) There is an $\alpha_0 < \omega_1$ such that $T^{\alpha_0} = T^{\alpha_0+1} = T^\beta$ for all $\beta > \alpha_0$. We write T^∞ or $T^\infty_{\bar{E},s_0}$ for T^{α_0}.

---

[6] See page 36 for the definition.

[7] Here, we can assume that M is a countable transitive model of a sufficiently large finite subset ZFC* of ZFC. Later, we will also use ord-transitive models instead of transitive ones, which does not make any difference as far as properties of $\mathbb{L}_{\bar{D}}$ are concerned, as our arguments take place in transitive parts of such models.

[8] I.e., $D^M_s \subseteq D_s$ for all $s \in \omega^{<\omega}$.

[9] Here we also allow empty trees, and trees with maximal nodes.

(c) If $s_0 \in T^\infty_{\bar{E},s_0}$, then $T^\infty_{\bar{E},s_0} \in \mathbb{L}_{\bar{E}}$ with stem s_0.
Conversely, if $\mathrm{stem}(T) = s_0$, and $T \in \mathbb{L}_{\bar{E}}$, then $T^\infty = T$.

(d) If T contains a tree $q \in \mathbb{L}_{\bar{E}}$ with $\mathrm{stem}(q) = s_0$, then T^∞ contains $q^\infty = q$, so in particular $s_0 \in T^\infty$.

(e) Thus: T contains a condition in $\mathbb{L}_{\bar{E}}$ with stem s_0 iff $s_0 \in T^\infty_{\bar{E},s_0}$.

(f) The computation of T^∞ is absolute between any two models containing T and $\bar{E}$. (In particular, any transitive ZFC*-model containing T and $\bar{E}$ will also contain α_0.)

(g) Moreover: Let $T \in M$, $\bar{E} \in M$, and let $\bar{E}'$ be a filter system extending $\bar{E}$ such that for all s_0 and all $A \in \mathscr{P}(\omega) \cap M$ we have: $A \in (E_{s_0})^+$ iff $A \in (E'_{s_0})^+$. (In particular, this will be true for any $\bar{E}'$ extending $\bar{E}$, provided that each E_{s_0} is an M-ultrafilter.)
Then for each $\alpha \in M$ we have $T^\alpha_{\bar{E},s_0} = T^\alpha_{\bar{E}',s_0}$ (and hence $T^\alpha_{\bar{E}',s_0} \in M$). (Proved by induction on α.)

Now let $A = (p_i : i \in I) \in M$ be a maximal antichain in $\mathbb{L}_{\bar{D}^M}$, and assume (in V) that $q \in \mathbb{L}_{\bar{D}}$. Let $s_0 := \mathrm{stem}(q)$.

We will show that q is compatible with some p_i (in $\mathbb{L}_{\bar{D}}$). This is clear if there is some i with $s_0 \in p_i$ and $\mathrm{stem}(p_i) \subseteq s_0$, by (2.7). (In this case, $p_i \cap q$ is a condition in $\mathbb{L}_{\bar{D}}$ with stem s_0.)

So for the rest of the proof we assume that this is not the case, i.e.:

$$\text{There is no } i \text{ with } s_0 \in p_i \text{ and } \mathrm{stem}(p_i) \subseteq s_0. \tag{2.9}$$

Let $J := \{i \in I : s_0 \subseteq \mathrm{stem}(p_i)\}$. We claim that there is $j \in J$ with $\mathrm{stem}(p_j) \in q$ (which as above implies that q and p_j are compatible).

Assume towards a contradiction that this is not the case. Then q is contained in the following tree T:

$$T := (\omega^{<\omega})^{[s_0]} \setminus \bigcup_{j \in J} [\mathrm{stem}(p_j)]. \tag{2.10}$$

Note that $T \in M$. In V we have:

$$\text{The tree } T \text{ contains a condition } q \text{ with stem } s_0. \tag{2.11}$$

So by (e) (applied in V), followed by (g), and again by (e) (now in M) we get:

$$\text{The tree } T \text{ also contains a condition } p \in M \text{ with stem } s_0. \tag{2.12}$$

Now p has to be compatible with some p_i. The sequences $s_0 = \mathrm{stem}(p)$ and $\mathrm{stem}(p_i)$ have to be comparable, so by (2.7) there are two possibilities:

1. $\operatorname{stem}(p_i) \subseteq \operatorname{stem}(p) = s_0 \in p_i$. We have excluded this case in our assumption (2.9).

2. $s_0 = \operatorname{stem}(p) \subseteq \operatorname{stem}(p_i) \in p$. So $i \in J$. By construction of T (see (2.10)), we conclude $\operatorname{stem}(p_i) \notin T$, contradicting $\operatorname{stem}(p_i) \in p \subseteq T$ (see 2.12). $\square$

2.1.C Ultralaver kills strong measure zero

The following lemma appears already in [Bla88, Theorem 9]. We will give a proof below in Lemma 2.38.

Lemma 2.13. *If A is a finite set, $\underset{\sim}{\alpha}$ an $\mathbb{L}_{\bar{D}}$-name, $p \in \mathbb{L}_{\bar{D}}$, and $p \Vdash \underset{\sim}{\alpha} \in A$, then there is $\beta \in A$ and a pure extension $q \leq_0 p$ such that $q \Vdash \underset{\sim}{\alpha} = \beta$.*

Definition 2.14. Let $\bar{\ell}$ be an increasing sequence of natural numbers. We say that $X \subseteq 2^\omega$ is *smz with respect to $\bar{\ell}$*, if there exists a sequence $(I_k)_{k\in\omega}$ of basic intervals of 2^ω of measure $\leq 2^{-\ell_k}$ (i.e., each I_k is of the form $[s_k]$ for some $s_k \in 2^{\ell_k}$) such that $X \subseteq \bigcap_{m\in\omega} \bigcup_{k\geq m} I_k$.

Remark 2.15. It is well known and easy to see that the properties

- For all $\bar{\ell}$ there exists a sequence $(I_k)_{k\in\omega}$ of basic intervals of 2^ω of measure $\leq 2^{-\ell_k}$ such that $X \subseteq \bigcup_{k\in\omega} I_k$.
- For all $\bar{\ell}$ there exists a sequence $(I_k)_{k\in\omega}$ of basic intervals of 2^ω of measure $\leq 2^{-\ell_k}$ such that $X \subseteq \bigcap_{m\in\omega} \bigcup_{k\geq m} I_k$.

are equivalent. Hence, a set X is smz iff X is smz with respect to all $\bar{\ell} \in \omega^\omega$.

The following lemma is a variant of the corresponding lemma (and proof) for Laver forcing (see for example [Jec03, Lemma 28.20]): Ultralaver makes old uncountable sets non-smz.

Lemma 2.16. *Let $\bar{D}$ be a system of ultrafilters, and let $\underset{\sim}{\bar{\ell}}$ be the $\mathbb{L}_{\bar{D}}$-name for the ultralaver real. Then each uncountable set $X \in V$ is forced to be non-smz (witnessed by the ultralaver real $\underset{\sim}{\bar{\ell}}$).*

More precisely, the following holds:

$$\Vdash_{\mathbb{L}_{\bar{D}}} \forall X \in V \cap [2^\omega]^{\aleph_1} \;\; \forall (x_k)_{k\in\omega} \subseteq 2^\omega \;\; X \not\subseteq \bigcap_{m\in\omega} \bigcup_{k\geq m} [x_k \restriction \underset{\sim}{\ell}_k]. \tag{2.17}$$

We first give two technical lemmas:

Lemma 2.18. *Let $p \in \mathbb{L}_{\bar{D}}$ with stem $s \in \omega^{<\omega}$, and let $\underset{\sim}{x}$ be a $\mathbb{L}_{\bar{D}}$-name for a real in 2^ω. Then there exists a pure extension $q \leq_0 p$ and a real $\tau \in 2^\omega$ such that for every $n \in \omega$,*

$$\{i \in \mathrm{succ}_q(s) : \ q^{[s^\frown i]} \Vdash \underset{\sim}{x}\restriction n = \tau\restriction n\} \in D_s. \tag{2.19}$$

Proof. For each $i \in \mathrm{succ}_p(s)$, let $q_i \leq_0 p^{[s^\frown i]}$ be such that q_i decides $\underset{\sim}{x}\restriction i$, i.e., there is a t_i of length i such that $q_i \Vdash \underset{\sim}{x}\restriction i = t_i$ (this is possible by Lemma 2.13).

Now we define the real $\tau \in 2^\omega$ as the D_s-limit of the t_i's. In more detail: For each $n \in \omega$ there is a (unique) $\tau_n \in 2^n$ such that $\{i : \ t_i\restriction n = \tau_n\} \in D_s$; since D_s is a filter, there is a real $\tau \in 2^\omega$ with $\tau\restriction n = \tau_n$ for each n. Finally, let $q := \bigcup_i q_i$. $\square$

Lemma 2.20. *Let $p \in \mathbb{L}_{\bar{D}}$ with stem s, and let $(\underset{\sim}{x}_k)_{k\in\omega}$ be a sequence of $\mathbb{L}_{\bar{D}}$-names for reals in 2^ω. Then there exists a pure extension $q \leq_0 p$ and a family of reals $(\tau_\eta)_{\eta\in q,\, \eta\supseteq s} \subseteq 2^\omega$ such that for each $\eta \in q$ above s, and every $n \in \omega$,*

$$\{i \in \mathrm{succ}_q(\eta) : \ q^{[\eta^\frown i]} \Vdash \underset{\sim}{x}_{|\eta|}\restriction n = \tau_\eta\restriction n\} \in D_\eta. \tag{2.21}$$

Proof. We apply Lemma 2.18 to each node η in p above s (and to $\underset{\sim}{x}_{|\eta|}$) separately: We first get a $p_1 \leq_0 p$ and a $\tau_s \in 2^\omega$; for every immediate successor $\eta \in \mathrm{succ}_{p_1}(s)$, we get $q_\eta \leq_0 p_1^{[\eta]}$ and a $\tau_\eta \in 2^\omega$, and let $p_2 := \bigcup_\eta q_\eta$; in this way, we get a (fusion) sequence $(p, p_1, p_2, \ldots)$, and let $q := \bigcap_k p_k$. $\square$

Proof of Lemma 2.16. We want to prove (2.17). Assume towards a contradiction that X is an uncountable set in V, and that $(\underset{\sim}{x}_k)_{k\in\omega}$ is a sequence of names for reals in 2^ω and $p \in \mathbb{L}_{\bar{D}}$ such that

$$p \Vdash X \subseteq \bigcap_{m\in\omega} \bigcup_{k\geq m} [\underset{\sim}{x}_k\restriction \underset{\sim}{\ell}_k]. \tag{2.22}$$

Let $s \in \omega^{<\omega}$ be the stem of p.

By Lemma 2.20, we can fix a pure extension $q \leq_0 p$ and a family $(\tau_\eta)_{\eta\in q,\, \eta\supseteq s} \subseteq 2^\omega$ such that for each $\eta \in q$ above the stem s and every $n \in \omega$, condition (2.21) holds.

Since X is (in V and) uncountable, we can find a real $x^* \in X$ which is different from each real in the countable family $(\tau_\eta)_{\eta\in q,\, \eta\supseteq s}$; more specifically, we can pick a family of natural numbers $(n_\eta)_{\eta\in q,\, \eta\supseteq s}$ such that $x^*\restriction n_\eta \neq \tau_\eta\restriction n_\eta$ for any η.

We can now find $r \leq_0 q$ such that:

- For all $\eta \in r$ above s and all $i \in \mathrm{succ}_r(\eta)$ we have $i > n_\eta$.

- For all $\eta \in r$ above s and all $i \in \mathrm{succ}_r(\eta)$ we have $r^{[\eta^\frown i]} \Vdash \underset{\sim}{x}_{|\eta|}\restriction n_\eta = \tau_\eta\restriction n_\eta \neq x^*\restriction n_\eta$.

So for all $\eta \in r$ above s we have, writing k for $|\eta|$, that $r^{[\eta^\frown i]}$ forces $x^* \notin [\underset{\sim}{x}_k\restriction n_\eta] \supseteq [\underset{\sim}{x}_k\restriction \ell_k]$. We conclude that r forces $x^* \notin \bigcup_{k\geq|s|}[\underset{\sim}{x}_k\restriction \ell_k]$, contradicting (2.22). $\square$

Corollary 2.23. *Let $(t_k)_{k\in\omega}$ be a dense subset of 2^ω.*

Let $\bar{D}$ be a system of ultrafilters, and let $\underset{\sim}{\bar{\ell}}$ be the $\mathbb{L}_{\bar{D}}$-name for the ultralaver real. Then the set

$$\underset{\sim}{H} := \bigcap_{m\in\omega}\bigcup_{k\geq m}[t_k\restriction \underset{\sim}{\ell}_k]$$

is forced to be a comeager set with the property that $\underset{\sim}{H}$ does not contain any translate of any old uncountable set.

Pawlikowski's theorem 2.2 gives us:

Corollary 2.24. *There is a canonical name F for a closed null set such that $X + F$ is positive for all uncountable X in V.*

In particular, no uncountable ground model set is smz in the ultralaver extension.

2.1.D Thin sets and strong measure zero

For the notion of "(very) thin" set, we use an increasing function $B^*(k)$ (the function we use will be described in Corollary 2.56). We will assume that $\bar{\ell}^* = (\ell^*_k)_{k\in\omega}$ is an increasing sequence of natural numbers with $\ell^*_{k+1} \gg B^*(k)$. (We will later use a subsequence of the ultralaver real $\bar{\ell}$ as $\bar{\ell}^*$, see Lemma 2.26).

Definition 2.25. For $X \subseteq 2^\omega$ and $k \in \omega$ we write $X\restriction[\ell^*_k, \ell^*_{k+1})$ for the set $\{x\restriction[\ell^*_k, \ell^*_{k+1}) : x \in X\}$. We say that

- $X \subseteq 2^\omega$ is "*very thin with respect to $\bar{\ell}^*$ and B^**", if there are infinitely many k with $|X\restriction[\ell^*_k, \ell^*_{k+1})| \leq B^*(k)$.
- $X \subseteq 2^\omega$ is "*thin with respect to $\bar{\ell}^*$ and B^**", if X is the union of countably many very thin sets.

Note that the family of thin sets is a σ-ideal, while the family of very thin sets is not even an ideal. Also, every very thin set is covered by a closed very thin (in particular nowhere dense) set. In particular, every thin set is meager and the ideal of thin sets is a proper ideal.

Lemma 2.26. *Let B^* be an increasing function. Let $\bar{\ell}$ be an increasing sequence of natural numbers. We define a subsequence $\bar{\ell}^*$ of $\bar{\ell}$ in the following way: $\ell^*_k = \ell_{n_k}$ where $n_{k+1} - n_k = B^*(k) \cdot 2^{\ell^*_k}$.*
Then we get: If X is thin with respect to $\bar{\ell}^*$ and B^*, then X is smz with respect to $\bar{\ell}$.

Proof. Assume that $X = \bigcup_{i \in \omega} Y_i$, each Y_i very thin with respect to $\bar{\ell}^*$ and B^*. Let $(X_j)_{j \in \omega}$ be an enumeration of $\{Y_i : i \in \omega\}$ where each Y_i appears infinitely often. So $X \subseteq \bigcap_{m \in \omega} \bigcup_{j \geq m} X_j$.

By induction on $j \in \omega$, we find for all $j > 0$ some $k_j > k_{j-1}$ such that

$$|X_j \restriction [\ell^*_{k_j}, \ell^*_{k_j+1})| \leq B^*(k_j) \quad \text{hence} \quad |X_j \restriction [0, \ell^*_{k_j+1})| \leq B^*(k_j) \cdot 2^{\ell^*_{k_j}} = n_{k_j+1} - n_{k_j}.$$

So we can enumerate $X_j \restriction [0, \ell^*_{k_j+1})$ as $(s_i)_{n_{k_j} \leq i < n_{k_j+1}}$. Hence X_j is a subset of $\bigcup_{n_{k_j} \leq i < n_{k_j+1}} [s_i]$; and each s_i has length $\ell^*_{k_j+1} \geq \ell_i$, since $\ell^*_{k_j+1} = \ell_{n_{k_j+1}}$ and $i < n_{k_j+1}$. This implies

$$X \subseteq \bigcap_{m \in \omega} \bigcup_{j \geq m} X_j \subseteq \bigcap_{m \in \omega} \bigcup_{i \geq m} [s_i].$$

Hence X is smz with respect to $\bar{\ell}$. □

Lemma 2.16 and Lemma 2.26 yield:

Corollary 2.27. *Let B^* be an increasing function. Let $\bar{D}$ be a system of ultrafilters, and $\underset{\sim}{\bar{\ell}}$ the name for the ultralaver real. Let $\underset{\sim}{\bar{\ell}}^*$ be constructed from B^* and $\underset{\sim}{\bar{\ell}}$ as in Lemma 2.26.*
Then $\mathbb{L}_{\bar{D}}$ forces that for every uncountable $X \subseteq 2^\omega$:

- *X is not smz with respect to $\underset{\sim}{\bar{\ell}}$.*
- *X is not thin with respect to $\underset{\sim}{\bar{\ell}}^*$ and B^*.*

2.1.E Ultralaver forcing and preservation of Lebesgue positivity

It is well known that both Laver forcing and random forcing preserve Lebesgue positivity; in fact they satisfy a stronger property that is preserved under countable support iterations. (So in particular, a countable support iteration of Laver and random also preserves positivity.)

Ultralaver forcing $\mathbb{L}_{\bar{D}}$ will in general not preserve positivity. Indeed, if all ultrafilters D_s are equal to the same ultrafilter D^*, then the range $L := \{\ell_0, \ell_1, \ldots\} \subseteq \omega$ of the ultralaver real $\bar{\ell}$ will diagonalize D^*, so every ground model real $x \in 2^\omega$ (viewed as a subset of ω) will either almost contain L or be almost disjoint to L, which implies that the set $2^\omega \cap V$ of old reals is covered by a null set in the extension. However, later in this paper it will become clear that if we choose the ultrafilters D_s in a sufficiently generic way, then many old positive sets will stay positive. More specifically, in this section we will show (Lemma 2.33): If $\bar{D}^M$ is an ultrafilter system in a countable model M and r a random real over M, then we can find an extension $\bar{D}$ such that $\mathbb{L}_{\bar{D}}$ forces that r remains random over $M[H^M]$ (where H^M denotes the $\mathbb{L}_{\bar{D}}$-name for the restriction of the $\mathbb{L}_{\bar{D}}$-generic filter H to $\mathbb{L}_{\bar{D}^M} \cap M$). Additionally, some "side conditions" are met, which are necessary to preserve the property in forcing iterations.

In Section 2.3.D we will see how to use this property to preserve randoms in limits.

The setup we use for preservation of randomness is basically the notation of "Case A" preservation introduced in [She98, Ch.XVIII], see also [Gol93, GK06] or the textbook [BJ95, 6.1.B]:

Definition 2.28. We write CLOPEN for the collection of clopen sets on 2^ω. We say that the function $Z : \omega \to \text{CLOPEN}$ is a code for a null set, if the measure of $Z(n)$ is at most 2^{-n} for each $n \in \omega$.

For such a code Z, the set nullset(Z) coded by Z is

$$\text{nullset}(Z) := \bigcap_n \bigcup_{k \geq n} Z(k).$$

The set nullset(Z) obviously is a null set, and it is well known that every null set is contained in such a set nullset(Z).

Definition 2.29. For a real r and any code Z, we define $Z \sqsubset_n r$ by:

$$(\forall k \geq n)\ r \notin Z(k).$$

We write $Z \sqsubset r$ if $Z \sqsubset_n r$ holds for some n; i.e., if $r \notin \text{nullset}(Z)$.

For later reference, we record the following trivial fact:

$$p \Vdash \underset{\sim}{Z} \sqsubset r \text{ iff there is a name } \underset{\sim}{n} \text{ for an element of } \omega \text{ such that } p \Vdash \underset{\sim}{Z} \sqsubset_{\underset{\sim}{n}} r. \tag{2.30}$$

Let P be a forcing notion, and $\underset{\sim}{Z}$ a P-name of a code for a null set. An *interpretation* of $\underset{\sim}{Z}$ below p is some code Z^* such that there is a sequence $p = p_0 \geq p_1 \geq p_2 \geq \dots$ such that p_m forces $\underset{\sim}{Z}\restriction m = Z^*\restriction m$. Usually we demand (which allows a simpler proof of the preservation theorem at limit stages) that the sequence $(p_0, p_1, \dots)$ is inconsistent, i.e., p forces that there is an m such that $p_m \notin G$. Note that whenever P adds a new ω-sequence of ordinals, we can find such an interpretation for any $\underset{\sim}{Z}$.

If $\bar{\underset{\sim}{Z}} = (\underset{\sim}{Z}_1, \dots, \underset{\sim}{Z}_m)$ is a tuple of names of codes for null sets, then an interpretation of $\bar{\underset{\sim}{Z}}$ below p is some tuple $(Z_1^*, \dots, Z_m^*)$ such that there is a single sequence $p = p_0 \geq p_1 \geq p_2 \geq \dots$ interpreting each $\underset{\sim}{Z}_i$ as Z_i^*.

We now turn to preservation of Lebesgue positivity:

Definition 2.31.

1. A forcing notion P *preserves Borel outer measure*, if P forces $\mathrm{Leb}^*(A^V) = \mathrm{Leb}(A^{V[G_P]})$ for every code A for a Borel set. (Leb^* denotes the outer Lebesgue measure, and for a Borel code A and a set-theoretic universe V, A^V denotes the Borel set coded by A in V.)
2. P *strongly preserves randoms*, if the following holds: Let $N \prec H(\chi^*)$ be countable for a sufficiently large regular cardinal χ^*, let $P, p, \bar{\underset{\sim}{Z}} = (\underset{\sim}{Z}_1, \dots, \underset{\sim}{Z}_m) \in N$, let $p \in P$ and let r be random over N. Assume that in N, $\bar{Z}^*$ is an interpretation of $\bar{\underset{\sim}{Z}}$, and assume $Z_i^* \sqsubset_{k_i} r$ for each i. Then there is an N-generic $q \leq p$ forcing that r is still random over $N[G]$ and moreover, $\underset{\sim}{Z}_i \sqsubset_{k_i} r$ for each i. (In particular, P has to be proper.)
3. Assume that P is absolutely definable. P *strongly preserves randoms over countable models* if (2) holds for all countable (transitive[10]) models N of ZFC*.

It is easy to see that these properties are increasing in strength. (Of course (3)⇒(2) works only if ZFC* is satisfied in $H(\chi^*)$.)

In [KS05] it is shown that (1) implies (3), provided that P is nep ("non-elementary proper", i.e., nicely definable and proper with respect to countable models). In particular, every Suslin ccc forcing notion such as random forcing, and also many tree forcing notions including Laver forcing, are nep. However $\mathbb{L}_{\bar{D}}$ is not nicely definable in this sense, as its definition uses ultrafilters as parameters.

Lemma 2.32. *Both Laver forcing and random forcing strongly preserve randoms over countable models.*

Proof. For random forcing, this is easy and well known (see, e.g., [BJ95, 6.3.12]).

For Laver forcing: By the above, it is enough to show (1). This was done by Woodin (unpublished) and Judah-Shelah [JS90]. A nicer proof (including a variant of (2)) is given by Pawlikowski [Paw96c]. □

[10] Later we will introduce ord-transitive models, and it is easy to see that it does not make any difference whether we demand transitive or not; this can be seen using a transitive collapse.

Ultralaver will generally not preserve Lebesgue positivity, let alone randomness. However, we get the following "local" variant of strong preservation of randoms (which will be used in the preservation theorem 2.109). The rest of this section will be devoted to the proof of the following lemma.

Lemma 2.33. *Assume that M is a countable model, $\bar{D}^M$ an ultrafilter system in M and r a random real over M. Then there is (in V) an ultrafilter system $\bar{D}$ extending*[11] *$\bar{D}^M$, such that the following holds:*
If

- *$p \in \mathbb{L}_{\bar{D}^M}$,*
- *in M, $\underset{\sim}{\bar{Z}} = (\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_m)$ is a sequence of $\mathbb{L}_{\bar{D}^M}$-names for codes for null sets,*[12] *and $Z_1^*, \ldots, Z_m^*$ are interpretations under p, witnessed by a sequence $(p_n)_{n\in\omega}$ with strictly increasing*[13] *stems,*
- *$Z_i^* \sqsubset_{k_i} r$ for $i = 1, \ldots, m$,*

then *there is a $q \leq p$ in $\mathbb{L}_{\bar{D}}$ forcing that*

- *r is random over $M[G^M]$,*
- *$\underset{\sim}{Z}_i \sqsubset_{k_i} r$ for $i = 1, \ldots, m$.*

For the proof of this lemma, we will use the following concepts:

Definition 2.34. Let $p \subseteq \omega^{<\omega}$ be a tree. A "*front name below p*" is a function[14] $h : F \to \text{CLOPEN}$, where $F \subseteq p$ is a front (a set that meets every branch of p in a unique point). (For notational simplicity we also allow h to be defined on elements $\notin p$; this way, every front name below p is also a front name below q whenever $q \leq p$.)

If h is a front name and $\bar{D}$ is any filter system with $p \in \mathbb{L}_{\bar{D}}$, we define the corresponding $\mathbb{L}_{\bar{D}}$-name (in the sense of forcing) $\underset{\sim}{z}^h$ by

$$\underset{\sim}{z}^h := \{(\check{y}, p^{[s]}) : s \in F,\ y \in h(s)\}. \tag{2.35}$$

(This does not depend on the $\bar{D}$ we use, since we set $\check{y} := \{(\check{x}, \omega^{<\omega}) : x \in y\}$.)

Up to forced equality, the name $\underset{\sim}{z}^h$ is characterized by the fact that $p^{[s]}$ forces (in any $\mathbb{L}_{\bar{D}}$) that $\underset{\sim}{z}^h = h(s)$, for every s in the domain of h.

[11] This implies, by Lemma 2.8, that the $\mathbb{L}_{\bar{D}}$-generic filter G induces an $\mathbb{L}_{\bar{D}^M}$-generic filter over M, which we call G^M.

[12] Recall that $\text{nullset}(\underset{\sim}{Z}) = \bigcap_n \bigcup_{k\geq n} \underset{\sim}{Z}(k)$ is a null set in the extension.

[13] It is enough to assume that the lengths of the stems diverge to infinity; any thin enough subsequence will then have strictly increasing stems and will still interpret each $\underset{\sim}{Z}_i$ as Z_i^*.

[14] Instead of CLOPEN we may also consider other ranges of front names, such as the class of all ordinals, or the set ω.

Note that the same object h can be viewed as a front name below p with respect to different forcings $\mathbb{L}_{\bar{D}_1}$, $\mathbb{L}_{\bar{D}_2}$, as long as $p \in \mathbb{L}_{\bar{D}_1} \cap \mathbb{L}_{\bar{D}_2}$.

Definition 2.36. Let $p \subseteq \omega^{<\omega}$ be a tree. A "*continuous name below p*" is either of the following:

- An ω-sequence of front names below p.
- A $\subseteq$-increasing function $g : p \to \text{CLOPEN}^{<\omega}$ such that $\lim_{n\to\infty} \mathrm{lh}(g(c{\restriction}n)) = \infty$ for every branch $c \in [p]$.

For each n, the set of minimal elements in $\{s \in p : \mathrm{lh}(g(s)) > n\}$ is a front, so each continuous name in the second sense naturally defines a name in the first sense, and conversely. Being a continuous name below p does not involve the notion of $\Vdash$ nor does it depend on the filter system $\bar{D}$.

If g is a continuous name and $\bar{D}$ is any filter system, we can again define the corresponding $\mathbb{L}_{\bar{D}}$-name $\underset{\sim}{Z}^g$ (in the sense of forcing); we leave a formal definition of $\underset{\sim}{Z}^g$ to the reader and content ourselves with this characterization:

$$(\forall s \in p) : p^{[s]} \Vdash_{\mathbb{L}_{\bar{D}}} g(s) \subseteq \underset{\sim}{Z}^g. \tag{2.37}$$

Note that a continuous name below p naturally corresponds to a continuous function $F : [p] \to \text{CLOPEN}^{\omega}$, and $\underset{\sim}{Z}^g$ is forced (by p) to be the value of F at the generic real $\underset{\sim}{\bar{\ell}}$.

Lemma 2.38. $\mathbb{L}_{\bar{D}}$ *has the following "pure decision properties":*

1. *Whenever $\underset{\sim}{y}$ is a name for an element of* CLOPEN, *$p \in \mathbb{L}_{\bar{D}}$, then there is a pure extension $p_1 \leq_0 p$ such that $\underset{\sim}{y} = \underset{\sim}{z}^h$ (is forced) for a front name h below p_1.*
2. *Whenever $\underset{\sim}{Y}$ is a name for a sequence of elements of* CLOPEN, *$p \in \mathbb{L}_{\bar{D}}$, then there is a pure extension $q \leq_0 p$ such that $\underset{\sim}{Y} = \underset{\sim}{Z}^g$ (is forced) for some continuous name g below q.*
3. *(This is Lemma 2.13.) If A is a finite set, $\underset{\sim}{\alpha}$ a name, $p \in \mathbb{L}_{\bar{D}}$, and p forces $\underset{\sim}{\alpha} \in A$, then there is $\beta \in A$ and a pure extension $q \leq_0 p$ such that $q \Vdash \underset{\sim}{\alpha} = \beta$.*

Proof. Let $p \in \mathbb{L}_{\bar{D}}$, $s_0 := \mathrm{stem}(p)$, $\underset{\sim}{y}$ a name for an element of CLOPEN.

We call $t \in p$ a "good node in p" if $\underset{\sim}{y}$ is a front name below $p^{[t]}$ (more formally: forced to be equal to $\underset{\sim}{z}^h$ for a front name h). We can find $p_1 \leq_0 p$ such that for all $t \in p_1$ above s_0: If there is $q \leq_0 p_1^{[t]}$ such that t is good in q, then t is already good in p_1.

We claim that s_0 is now good (in p_1). Note that for any bad node s the set $\{\, t \in \mathrm{succ}_{p_1}(s) : \ t \text{ bad}\,\}$ is in D_s^+. Hence, if s_0 is bad, we can inductively construct $p_2 \leq_0 p_1$ such that all nodes of p_2 are bad nodes in p_1. Now let $q \leq p_2$ decide $\underaccent{\tilde}{y}$, $s := \mathrm{stem}(q)$. Then $q \leq_0 p_1^{[s]}$, so s is good in p_1, contradiction. This finishes the proof of (1).

To prove (2), we first construct p_1 as in (1) with respect to $\underaccent{\tilde}{y}_0$. This gives a front $F_1 \subseteq p_1$ deciding $\underaccent{\tilde}{y}_0$. Above each node in F_1 we now repeat the construction from (1) with respect to $\underaccent{\tilde}{y}_1$, yielding p_2, etc. Finally, $q := \bigcap_n p_n$.

To prove (3): Similar to (1), we can find $p_1 \leq_0 p$ such that for each $t \in p_1$: If there is a pure extension of $p_1^{[t]}$ deciding $\underaccent{\tilde}{\alpha}$, then $p_1^{[t]}$ decides $\underaccent{\tilde}{\alpha}$; in this case we again call t good. Since there are only finitely many possibilities for the value of $\underaccent{\tilde}{\alpha}$, any bad node t has D_t^+ many bad successors. So if the stem of p_1 is bad, we can again reach a contradiction as in (1). $\square$

Corollary 2.39. *Let $\bar{D}$ be a filter system, and let $G \subseteq \mathbb{L}_{\bar{D}}$ be generic. Then every $Y \in$ CLOPEN$^\omega$ in $V[G]$ is the evaluation of a continuous name $\underaccent{\tilde}{Z}^g$ by G.*

Proof. In V, fix a $p \in \mathbb{L}_{\bar{D}}$ and a name $\underaccent{\tilde}{Y}$ for an element of CLOPEN$^\omega$. We can find $q \leq_0 p$ and a continuous name g below q such that $q \Vdash \underaccent{\tilde}{Y} = \underaccent{\tilde}{Z}^g$. $\square$

We will need the following modification of the concept of "continuous names".

Definition 2.40. Let $p \subseteq \omega^{<\omega}$ be a tree, $b \in [p]$ a branch. An "*almost continuous name below p (with respect to b)*" is a $\subseteq$-increasing function $g : p \to$ CLOPEN$^{<\omega}$ such that $\lim_{n\to\infty} \mathrm{lh}(g(c{\restriction}n)) = \infty$ for every branch $c \in [p]$, except possibly for $c = b$.

Note that "except possibly for $c = b$" is the only difference between this definition and the definition of a continuous name.

Since for any $\bar{D}$ it is forced[15] that the generic real (for $\mathbb{L}_{\bar{D}}$) is not equal to the exceptional branch b, we again get a name $\underaccent{\tilde}{Z}^g$ of a function in CLOPEN$^\omega$ satisfying:

$$(\forall s \in p) : p^{[s]} \Vdash_{\mathbb{L}_{\bar{D}}} g(s) \subseteq \underaccent{\tilde}{Z}^g.$$

An almost continuous name naturally corresponds to a continuous function F from $[p] \setminus \{b\}$ into CLOPEN$^\omega$.

Note that being an almost continuous name is a very simple combinatorial property of g which does not depend on $\bar{D}$, nor does it involve the notion $\Vdash$. Thus, the same function g can be viewed as an almost continuous name for two different forcing notions $\mathbb{L}_{\bar{D}_1}$, $\mathbb{L}_{\bar{D}_2}$ simultaneously.

[15]This follows from our assumption that all our filters contain the Fréchet filter.

Lemma 2.41. *Let $\bar{D}$ be a system of filters (not necessarily ultrafilters).*

Assume that $\bar{p} = (p_n)_{n\in\omega}$ witnesses that Y^* is an interpretation of $\underaccent{\tilde}{Y}$, and that the lengths of the stems of the p_n are strictly increasing.[16] *Then there exists a sequence $\bar{q} = (q_n)_{n\in\omega}$ such that*

1. $q_0 \geq q_1 \geq \cdots$.
2. $q_n \leq p_n$ *for all* n.
3. $\bar{q}$ *also interprets* $\underaccent{\tilde}{Y}$ *as* Y^*. *(This follows from the previous two statements.)*
4. $\underaccent{\tilde}{Y}$ *is almost continuous below* q_0*, i.e., there is an almost continuous name* g *such that* q_0 *forces* $\underaccent{\tilde}{Y} = \underaccent{\tilde}{Z}^g$.
5. $\underaccent{\tilde}{Y}$ *is almost continuous below* q_n*, for all* n. *(This follows from the previous statement.)*

Proof. Let b be the branch described by the stems of the conditions p_n:

$$b := \{s : (\exists n)\, s \subseteq \mathrm{stem}(p_n)\}.$$

We now construct a condition q_0. For every $s \in b$ satisfying $\mathrm{stem}(p_n) \subseteq s \subsetneq \mathrm{stem}(p_{n+1})$ we set $\mathrm{succ}_{q_0}(s) = \mathrm{succ}_{p_n}(s)$, and for all $t \in \mathrm{succ}_{q_0}(s)$ except for the one in b we let $q_0^{[t]} \leq_0 p_n^{[t]}$ be such that $\underaccent{\tilde}{Y}$ is continuous below $q_0^{[t]}$. We can do this by Lemma 2.38(2).

Now we set

$$q_n := p_n \cap q_0 = q_0^{[\mathrm{stem}(p_n)]} \leq p_n.$$

This takes care of (1) and (2). Now we show (4): Any branch c of q_0 not equal to b must contain a node $s^\frown k \notin b$ with $s \in b$, so c is a branch in $q_0^{[s^\frown k]}$, below which $\underaccent{\tilde}{Y}$ was continuous. $\square$

The following lemmas and corollaries are the motivation for considering continuous and almost continuous names.

Lemma 2.42. *Let $\bar{D}$ be a system of filters (not necessarily ultrafilters). Let $p \in \mathbb{L}_{\bar{D}}$, let b be a branch, and let $g : p \to \mathrm{CLOPEN}^{<\omega}$ be an almost continuous name below p with respect to b; write $\underaccent{\tilde}{Z}^g$ for the associated $\mathbb{L}_{\bar{D}}$-name.*

Let $r \in 2^\omega$ be a real, $n_0 \in \omega$. Then the following are equivalent:

1. $p \Vdash_{\mathbb{L}_{\bar{D}}} r \notin \bigcup_{n\geq n_0} \underaccent{\tilde}{Z}^g(n)$*, i.e.,* $\underaccent{\tilde}{Z}^g \sqsubset_{n_0} r$.

[16] It is easy to see that for every $\mathbb{L}_{\bar{D}}$-name $\underaccent{\tilde}{Y}$ we can find such $\bar{p}$ and Y^*: First find $\bar{p}$ which interprets both $\underaccent{\tilde}{Y}$ and $\underaccent{\tilde}{\ell}$, and then thin out to get a strictly increasing sequence of stems.

2. *For all $n \geq n_0$ and for all $s \in p$ for which $g(s)$ has length $> n$ we have $r \notin g(s)(n)$.*

Note that (2) does not mention the notion $\Vdash$ and does not depend on $\bar{D}$.

Proof. $\neg(2) \Rightarrow \neg(1)$: Assume that there is $s \in p$ for which $g(s)$ equals $(C_0, \ldots, C_n, \ldots, C_k)$ and $r \in C_n$. Then $p^{[s]}$ forces that the generic sequence $\underset{\sim}{Z}^g = (\underset{\sim}{Z}(0), \underset{\sim}{Z}(1), \ldots)$ starts with $C_0, \ldots, C_n$, so $p^{[s]}$ forces $r \in \underset{\sim}{Z}^g(n)$.

$\neg(1) \Rightarrow \neg(2)$: Assume that p does not force $r \notin \bigcup_{n \geq n_0} \underset{\sim}{Z}^g(n)$. So there is a condition $q \leq p$ and some $n \geq n_0$ such that $q \Vdash r \in \underset{\sim}{Z}^g(n)$. By increasing the stem of q, if necessary, we may assume that $s := \mathrm{stem}(q)$ is not on b (the "exceptional" branch), and that $g(s)$ has already length $> n$. Let $C_n := g(s)(n)$ be the n-th entry of $g(s)$. So $p^{[s]}$ already forces $\underset{\sim}{Z}^g(n) = C_n$; now $q^{[s]} \leq p^{[s]}$, and $q^{[s]}$ forces the following statements: $r \in \underset{\sim}{Z}^g(n)$, $\underset{\sim}{Z}^g(n) = C_n$. Hence $r \in C_n$, so (2) fails. $\square$

Corollary 2.43. *Let $\bar{D}_1$ and $\bar{D}_2$ be systems of filters, and assume that p is in $\mathbb{L}_{\bar{D}_1} \cap \mathbb{L}_{\bar{D}_2}$. Let $g : p \to \text{CLOPEN}^{<\omega}$ be an almost continuous name of a sequence of clopen sets, and let $\underset{\sim}{Z}_1^g$ and $\underset{\sim}{Z}_2^g$ be the associated $\mathbb{L}_{\bar{D}_1}$-name and $\mathbb{L}_{\bar{D}_2}$-name, respectively.*

Then for any real r and $n \in \omega$ we have

$$p \Vdash_{\mathbb{L}_{\bar{D}_1}} \underset{\sim}{Z}_1^g \sqsubset_n r \quad \Leftrightarrow \quad p \Vdash_{\mathbb{L}_{\bar{D}_2}} \underset{\sim}{Z}_2^g \sqsubset_n r.$$

(We will use this corollary for the special case that $\mathbb{L}_{\bar{D}_1}$ is an ultralaver forcing, and $\mathbb{L}_{\bar{D}_2}$ is Laver forcing.)

Lemma 2.44. *Let $\bar{D}_1$ and $\bar{D}_2$ be systems of filters, and assume that p is in $\mathbb{L}_{\bar{D}_1} \cap \mathbb{L}_{\bar{D}_2}$. Let $g : p \to \text{CLOPEN}^{<\omega}$ be a continuous name of a sequence of clopen sets, let $F \subseteq p$ be a front and let $h : F \to \omega$ be a front name. Again we will write $\underset{\sim}{Z}_1^g, \underset{\sim}{Z}_2^g$ for the associated names of codes for null sets, and we will write $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ for the associated $\mathbb{L}_{\bar{D}_1}$- and $\mathbb{L}_{\bar{D}_2}$-names, respectively, of natural numbers.*

Then for any real r we have:

$$p \Vdash_{\mathbb{L}_{\bar{D}_1}} \underset{\sim}{Z}_1^g \sqsubset_{\underset{\sim}{n}_1} r \quad \Leftrightarrow \quad p \Vdash_{\mathbb{L}_{\bar{D}_2}} \underset{\sim}{Z}_2^g \sqsubset_{\underset{\sim}{n}_2} r.$$

Proof. Assume $p \Vdash_{\mathbb{L}_{\bar{D}_1}} \underset{\sim}{Z}_1^g \sqsubset_{\underset{\sim}{n}_1} r$. So for each $s \in F$ we have: $p^{[s]} \Vdash_{\mathbb{L}_{\bar{D}_1}} \underset{\sim}{Z}_1^g \sqsubset_{h(s)} r$. By Corollary 2.43, we also have $p^{[s]} \Vdash_{\mathbb{L}_{\bar{D}_2}} \underset{\sim}{Z}_2^g \sqsubset_{h(s)} r$. So also $p^{[s]} \Vdash_{\mathbb{L}_{\bar{D}_2}} \underset{\sim}{Z}_2^g \sqsubset_{\underset{\sim}{n}_2} r$ for each $s \in F$. Hence $p \Vdash_{\mathbb{L}_{\bar{D}_2}} \underset{\sim}{Z}_2^g \sqsubset_{\underset{\sim}{n}_2} r$. $\square$

Corollary 2.45. *Assume $q \in \mathbb{L}$ forces in Laver forcing that $\underset{\sim}{Z}^{g_k} \sqsubset r$ for $k = 1, 2, \ldots$, where each g_k is a continuous name of a code for a null set. Then there is a Laver condition $q' \leq_0 q$ such that for all filter systems $\bar{D}$ we have:*

If $q' \in \mathbb{L}_{\bar{D}}$, then q' forces (in ultralaver forcing $\mathbb{L}_{\bar{D}}$) that $\utilde{Z}^{g_k} \sqsubset r$ for all k.

Proof. By (2.30) we can find a sequence $(\utilde{n}_k)_{k=1}^{\infty}$ of $\mathbb{L}$-names such that $q \Vdash \utilde{Z}^{g_k} \sqsubset_{\utilde{n}_k} r$ for each k. By Lemma 2.38(2) we can find $q' \leq_0 q$ such that this sequence is continuous below q'. Since each $\utilde{n}_k$ is now a front name below q', we can apply the previous lemma. $\square$

Lemma 2.46. *Let M be a countable model, $r \in 2^\omega$, $\bar{D}^M \in M$ an ultrafilter system, $\bar{D}$ a filter system extending $\bar{D}^M$, $q \in \mathbb{L}_{\bar{D}}$. For any V-generic filter $G \subseteq \mathbb{L}_{\bar{D}}$ we write G^M for the (M-generic, by Lemma 2.8) filter on $\mathbb{L}_{\bar{D}^M}$.*

The following are equivalent:

1. *$q \Vdash_{\mathbb{L}_{\bar{D}}} r$ is random over $M[G^M]$.*
2. *For all names $\utilde{Z} \in M$ of codes for null sets: $q \Vdash_{\mathbb{L}_{\bar{D}}} \utilde{Z} \sqsubset r$.*
3. *For all continuous names $g \in M$: $q \Vdash_{\mathbb{L}_{\bar{D}}} \utilde{Z}^g \sqsubset r$.*

Proof. (1)$\Leftrightarrow$(2) holds because every null set is contained in a set of the form $\mathrm{nullset}(Z)$, for some code Z.

(2)$\Leftrightarrow$(3): Every code for a null set in $M[G^M]$ is equal to $\utilde{Z}^g[G^M]$, for some $g \in M$, by Corollary 2.39. $\square$

The following lemma may be folklore. Nevertheless, we prove it for the convenience of the reader.

Lemma 2.47. *Let r be random over a countable model M and $A \in M$. Then there is a countable model $M' \supseteq M$ such that A is countable in M', but r is still random over M'.*

Proof. We will need the following forcing notions, all defined in M:

$$\begin{tikzcd} M \arrow[r, "C"] \arrow[d, "B_1"'] & M^C \arrow[d, "\utilde{B}_2"] \\ M^{B_1} \arrow[r, "\utilde{P}=C*\utilde{B}_2/B_1"'] & M^{C*\utilde{B}_2} \end{tikzcd}$$

- Let C be the forcing that collapses the cardinality of A to ω with finite conditions.
- Let B_1 be random forcing (trees $T \subseteq 2^{<\omega}$ of positive measure).
- Let $\utilde{B}_2$ be the C-name of random forcing.

- Let $i : B_1 \to C * \underset{\sim}{B}_2$ be the natural complete embedding $T \mapsto (1_C, T)$.
- Let $\underset{\sim}{P}$ be a B_1-name for the forcing $C * \underset{\sim}{B}_2 / i[G_{B_1}]$, the quotient of $C * \underset{\sim}{B}_2$ by the complete subforcing $i[B_1]$.

The random real r is B_1-generic over M. In $M[r]$ we let $P := \underset{\sim}{P}[r]$. Now let $H \subseteq P$ be generic over $M[r]$. Then $r * H \subseteq B_1 * \underset{\sim}{P} \simeq C * \underset{\sim}{B}_2$ induces an M-generic filter $J \subseteq C$ and an $M[J]$-generic filter $K \subseteq \underset{\sim}{B}_2[J]$; it is easy to check that K interprets the $\underset{\sim}{B}_2$-name of the canonical random real as the given random real r.

Hence r is random over the countable model $M' := M[J]$, and A is countable in M'.

$$\begin{tikzcd} M \arrow[r, "J"] \arrow[d, "r"'] & M[J] \arrow[d, "K"] \\ M[r] \arrow[r, "H"'] & M[r][H] \end{tikzcd}$$

$\square$

Proof of Lemma 2.33. We will first describe a construction that deals with a single triple $(\bar{p}, \bar{\underset{\sim}{Z}}, \bar{Z}^*)$ (where $\bar{p}$ is a sequence of conditions with strictly increasing stems which interprets $\bar{\underset{\sim}{Z}}$ as $\bar{Z}^*$); this construction will yield a condition $q' = q'(\bar{p}, \bar{\underset{\sim}{Z}}, \bar{Z}^*)$. We will then show how to deal with all possible triples.

So let p be a condition, and let $\bar{p} = (p_k)_{k \in \omega}$ be a sequence interpreting $\bar{\underset{\sim}{Z}}$ as $\bar{Z}^*$, where the lengths of the stems of p_n are strictly increasing and $p_0 = p$. It is easy to see that it is enough to deal with a single null set, i.e., $m = 1$, and with $k_1 = 0$. We write $\underset{\sim}{Z}$ and Z^* instead of $\underset{\sim}{Z}_1$ and Z_1^*.

Using Lemma 2.41 we may (strengthening the conditions in our interpretation) assume (in M) that the sequence $(\underset{\sim}{Z}(k))_{k \in \omega}$ is almost continuous, witnessed by $g : p \to \text{CLOPEN}^{<\omega}$. By Lemma 2.47, we can find a model $M' \supseteq M$ such that $(2^\omega)^M$ is countable in M', but r is still random over M'.

We now work in M'. Note that g still defines an almost continuous name, which we again call $\underset{\sim}{Z}$.

Each filter in D_s^M is now countably generated; let A_s be a pseudo-intersection of D_s^M which additionally satisfies $A_s \subseteq \mathrm{succ}_p(s)$ for all $s \in p$ above the stem. Let D'_s be the Fréchet filter on A_s. Let $p' \in \mathbb{L}_{\bar{D}'}$ be the tree with the same stem as p which satisfies $\mathrm{succ}_{p'}(s) = A_s$ for all $s \in p'$ above the stem.

By Lemma 2.8, we know that $\mathbb{L}_{\bar{D}^M}$ is an M-complete subforcing of $\mathbb{L}_{\bar{D}'}$ (in M' as well as in V). We write G^M for the induced filter on $\mathbb{L}_{\bar{D}^M}$.

We now work in V. Note that below the condition p', the forcing $\mathbb{L}_{\bar{D}'}$ is just Laver forcing $\mathbb{L}$, and that $p' \leq_{\mathbb{L}} p$. Using Lemma 2.32 we can find a condition $q \leq p'$ (in Laver forcing $\mathbb{L}$) such that:

$$q \text{ is } M'\text{-generic.} \tag{2.48}$$

$$q \Vdash_{\mathbb{L}} r \text{ is random over } M'[G_{\mathbb{L}}] \text{ (hence also over } M[G^M]). \tag{2.49}$$

$$\text{Moreover, } q \Vdash_{\mathbb{L}} \underset{\sim}{Z} \sqsubset_0 r. \tag{2.50}$$

Enumerate all continuous $\mathbb{L}_{\bar{D}^M}$-names of codes for null sets from M as $\underset{\sim}{Z}^{g_1}, \underset{\sim}{Z}^{g_2}, \ldots$ Applying Corollary 2.45 yields a condition $q' \leq q$ such that for all filter systems $\bar{E}$ satisfying $q' \in \mathbb{L}_{\bar{E}}$, we have $q' \Vdash_{\mathbb{L}_{\bar{E}}} \underset{\sim}{Z}^{g_i} \sqsubset r$ for all i. Corollary 2.43 and Lemma 2.46 now imply:

$$\begin{array}{l}\text{For every filter system } \bar{E} \text{ satisfying } q' \in \mathbb{L}_{\bar{E}},\ q' \text{ forces in } \mathbb{L}_{\bar{E}} \\ \text{that } r \text{ is random over } M[G^M] \text{ and that } \underset{\sim}{Z} \sqsubset_0 r.\end{array} \tag{2.51}$$

By thinning out q' we may assume that

$$\text{For each } \nu \in \omega^\omega \cap M \text{ there is } k \text{ such that } \nu{\restriction}k \notin q'. \tag{2.52}$$

We have now described a construction of $q' = q'(\bar{p}, \underset{\sim}{Z}, Z^*)$.

Let $(\bar{p}^n, \underset{\sim}{Z}^n, Z^{*n})$ enumerate all triples $(\bar{p}, \underset{\sim}{Z}, Z^*) \in M$ where $\bar{p}$ interprets $\underset{\sim}{Z}$ as Z^* (and consists of conditions with strictly increasing stems). For each n write ν^n for $\bigcup_k \mathrm{stem}(p^n_k)$, the branch determined by the stems of the sequence $\bar{p}^n$. We now define by induction a sequence q^n of conditions:

- $q^0 := q'(\bar{p}^0, \underset{\sim}{Z}^0, Z^{*0})$.
- Given q^{n-1} and $(\bar{p}^n, \underset{\sim}{Z}^n, Z^{*n})$, we find k_0 such that $\nu^n{\restriction}k_0 \notin q^0 \cup \cdots \cup q^{n-1}$ (using (2.52)). Let k_1 be such that $\mathrm{stem}(p^n_{k_1})$ has length $> k_0$. We replace $\bar{p}^n$ by $\bar{p}' := (p^n_k)_{k \geq k_1}$. (Obviously, $\bar{p}'$ still interprets $\underset{\sim}{Z}^n$ as Z^{*n}.) Now let $q^n := q'(\bar{p}', \underset{\sim}{Z}^n, Z^{*n})$.

Note that the stem of q^n is at least as long as the stem of $p^n_{k_1}$, and is therefore not in $q^0 \cup \cdots \cup q^{n-1}$, so $\mathrm{stem}(q^i)$ and $\mathrm{stem}(q^j)$ are incompatible for all $i \neq j$. Therefore we can choose for each s an ultrafilter D_s extending D^M_s such that $\mathrm{stem}(q^i) \subseteq s$ implies $\mathrm{succ}_{q^i}(s) \in D_s$.

Note that all q^i are in $\mathbb{L}_{\bar{D}}$. Therefore, we can use (2.51). Also, $q^i \leq p^i_0$. $\square$

Below, in Lemma 2.109, we will prove a preservation theorem using the following "local" variant of "random preservation":

Definition 2.53. Fix a countable model M, a real $r \in 2^\omega$ and a forcing notion $Q^M \in M$. Let Q^M be an M-complete subforcing of Q. We say that "*Q locally preserves randomness of r over M*", if there is in M a sequence $(D_n^{Q^M})_{n\in\omega}$ of open dense subsets of Q^M such that the following holds:
Assume that

- M thinks that $\bar{p} := (p^n)_{n\in\omega}$ interprets $(\underset{\sim}{Z}_1, \dots, \underset{\sim}{Z}_m)$ as $(Z_1^*, \dots, Z_m^*)$ (so each $\underset{\sim}{Z}_i$ is a Q^M-name of a code for a null set and each Z_i^* is a code for a null set, both in M);
- moreover, each p^n is in $D_n^{Q^M}$ (we call such a sequence $(p^n)_{n\in\omega}$, or the according interpretation, "*quick*");
- r is random over M;
- $Z_i^* \sqsubset_{k_i} r$ for $i = 1, \dots, m$.

Then there is a $q \leq_Q p^0$ forcing that

- r is random over $M[G^M]$;
- $\underset{\sim}{Z}_i \sqsubset_{k_i} r$ for $i = 1, \dots, m$.

Note that this is trivially satisfied if r is not random over M.

For a variant of this definition, see Section 2.6.

Setting $D_n^{Q^M}$ to be the set of conditions with stem of length at least n, Lemma 2.33 gives us:

Corollary 2.54. *If Q^M is an ultralaver forcing in M and r a real, then there is an ultralaver forcing Q over*[17] *Q^M locally preserving randomness of r over M.*

2.2 Janus forcing

In this section, we define a family of forcing notions that has two faces (hence the name "*Janus forcing*"): Elements of this family may be countable (and therefore equivalent to Cohen), and they may also be essentially random.

In the rest of the paper, we will use the following properties of Janus forcing notions $\mathbb{J}$. (And we will use *only* these properties. So readers who are willing to take these properties for granted could skip to Section 2.3.)

Throughout the whole paper we fix a function $B^* : \omega \to \omega$ given by Corollary 2.56. The Janus forcings will depend on a real parameter $\bar{\ell}^* =$

[17] "Q over Q^M" just means that Q^M is an M-complete subforcing of Q.

$(\ell^*_m)_{m\in\omega} \in \omega^\omega$ which grows fast with respect to B^*. (In our application, $\bar\ell^*$ will be given by a subsequence of an ultralaver real.)

The sequence $\bar\ell^*$ and the function B^* together define a notion of a "thin set" (see Definition 2.25).

1. There is a canonical $\mathbb{J}$-name for a (code for a) null set $\underset{\sim}{Z}_\nabla$.
   Whenever $X \subseteq 2^\omega$ is not thin, and $\mathbb{J}$ is countable, then $\mathbb{J}$ forces that X is not strongly meager, witnessed[18] by nullset($\underset{\sim}{Z}_\nabla$) (the set we get when we evaluate the code $\underset{\sim}{Z}_\nabla$). Moreover, for any $\mathbb{J}$-name $\underset{\sim}{Q}$ of a σ-centered forcing, also $\mathbb{J} * \underset{\sim}{Q}$ forces that X is not strongly meager, again witnessed by nullset($\underset{\sim}{Z}_\nabla$).
   (This is Lemma 2.63; "thin" is defined in Definition 2.25.)

2. Let M be a countable transitive model and $\mathbb{J}^M$ a Janus forcing in M. Then $\mathbb{J}^M$ is a Janus forcing in V as well (and of course countable in V). (Also note that trivially the forcing $\mathbb{J}^M$ is an M-complete subforcing of itself.)
   (This is Fact 2.62.)

3. Whenever M is a countable transitive model and $\mathbb{J}^M$ is a Janus forcing in M, then there is a Janus forcing $\mathbb{J}$ such that

   - $\mathbb{J}^M$ is an M-complete subforcing of $\mathbb{J}$.
   - $\mathbb{J}$ is (in V) equivalent to random forcing (actually we just need that $\mathbb{J}$ preserves Lebesgue positivity in a strong and iterable way).

   (This is Lemma 2.70 and Lemma 2.74.)

4. Moreover, the name $\underset{\sim}{Z}_\nabla$ referred to in (1) is so "canonical" that it evaluates to the same code in the $\mathbb{J}$-generic extension over V as in the $\mathbb{J}^M$-generic extension over M.
   (This is Fact 2.61.)

2.2.A Definition of Janus

A Janus forcing $\mathbb{J}$ will consist of:[19]

- A countable "core" (or: backbone) ∇ which is defined in a combinatorial way from a parameter $\bar\ell^*$. (In our application, we will use a Janus

[18] in the sense of (2.1)

[19] We thank Andreas Blass and Jindřich Zapletal for their comments that led to an improved presentation of Janus forcing.

forcing immediately after an ultralaver forcing, and $\bar{\ell}^*$ will be a subsequence of the ultralaver real.) This core is of course equivalent to Cohen forcing.

- Some additional "stuffing" $\mathbb{J} \setminus \nabla$ (countable[20] or uncountable). We allow great freedom for this, we just require that the core ∇ is a "sufficiently" complete subforcing (in a specific combinatorial sense, see Definition 2.59(3)).

We will use the following combinatorial theorem from [BS10]:

Lemma 2.55 ([BS10, Theorem 8][21])**.** *For every $\varepsilon, \delta > 0$ there exists $N_{\varepsilon,\delta} \in \omega$ such that for all sufficiently large finite sets $I \subseteq \omega$ there is a family $\mathcal{A}_I$ with $|\mathcal{A}_I| \geq 2$ consisting of sets $A \subseteq 2^I$ with $\frac{|A|}{2^{|I|}} \leq \varepsilon$ such that if $X \subseteq 2^I$, $|X| \geq N_{\varepsilon,\delta}$ then*

$$\frac{|\{A \in \mathcal{A}_I : X + A = 2^I\}|}{|\mathcal{A}_I|} \geq 1 - \delta.$$

(Recall that $X + A := \{x + a : x \in X, a \in A\}$.)

Rephrasing and specializing to $\delta = \frac{1}{4}$ and $\varepsilon = \frac{1}{2^i}$ we get:

Corollary 2.56. *For every $i \in \omega$ there exists $B^*(i)$ such that for all finite sets I with $|I| \geq B^*(i)$ there is a nonempty family $\mathcal{A}_I$ with $|\mathcal{A}_I| \geq 2$ satisfying the following:*

- *$\mathcal{A}_I$ consists of sets $A \subseteq 2^I$ with $\frac{|A|}{2^{|I|}} \leq \frac{1}{2^i}$.*
- *For every $X \subseteq 2^I$ satisfying $|X| \geq B^*(i)$, the set $\{A \in \mathcal{A}_I : X+A = 2^I\}$ has at least $\frac{3}{4}|\mathcal{A}_I|$ elements.*

Assumption 2.57. We fix a sufficiently fast increasing sequence $\bar{\ell}^* = (\ell^*_i)_{i\in\omega}$ of natural numbers; more precisely, the sequence $\bar{\ell}^*$ will be a subsequence of an ultralaver real $\bar{\ell}$, defined as in Lemma 2.26 using the function B^* from Corollary 2.56. Note that in this case $\ell^*_{i+1} - \ell^*_i \geq B^*(i)$; so we can fix for each i a family $\mathcal{A}_i \subseteq \mathscr{P}(2^{L_i})$ on the interval $L_i := [\ell^*_i, \ell^*_{i+1})$ according to Corollary 2.56.

[20] Also the trivial case $\mathbb{J} = \nabla$ is allowed.

[21] The theorem in [BS10] actually says "for a sufficiently large I", but the proof shows that this should be read as "for *all* sufficiently large I". Also, the quoted theorem only claims that $\mathcal{A}_I$ will be nonempty, but for $\varepsilon \leq \frac{1}{2}$ and $|I| > N_{\varepsilon,\delta}$ it is easy to see that $\mathcal{A}_I$ cannot be a singleton $\{A\}$: The set $X := 2^I \setminus A$ has size $\geq 2^{|I|-1} \geq N_{\varepsilon,\delta}$ but satisfies $X + A \neq 2^I$, as the constant sequence $\bar{0}$ is not in $X + A$.

Definition 2.58. First we define the "*core*" $\nabla = \nabla_{\bar{\ell}^*}$ of our forcing:

$$\nabla = \bigcup_{i\in\omega} \prod_{j<i} \mathcal{A}_j.$$

In other words, $\sigma \in \nabla$ iff $\sigma = (A_0, \ldots, A_{i-1})$ for some $i \in \omega$, $A_0 \in \mathcal{A}_0$, $\ldots$, $A_{i-1} \in \mathcal{A}_{i-1}$. We will denote the number i by height(σ).

The forcing notion ∇ is ordered by reverse inclusion (i.e., end extension): $\tau \le \sigma$ if $\tau \supseteq \sigma$.

Definition 2.59. Let $\bar{\ell}^* = (\ell_i^*)_{i\in\omega}$ be as in the assumption above. We say that $\mathbb{J}$ is a *Janus forcing* based on $\bar{\ell}^*$ if:

1. $(\nabla, \supseteq)$ is an incompatibility-preserving subforcing of $\mathbb{J}$.
2. For each $i \in \omega$ the set $\{\sigma \in \nabla : \text{height}(\sigma) = i\}$ is predense in $\mathbb{J}$. So in particular, $\mathbb{J}$ adds a branch through ∇. The union of this branch is called $\underset{\sim}{C}^\nabla = (\underset{\sim}{C}_0^\nabla, \underset{\sim}{C}_1^\nabla, \underset{\sim}{C}_2^\nabla, \ldots)$, where $\underset{\sim}{C}_i^\nabla \subseteq 2^{L_i}$ with $\underset{\sim}{C}_i^\nabla \in \mathcal{A}_i$.
3. "Fatness":[22] For all $p \in \mathbb{J}$ and all real numbers $\varepsilon > 0$ there are arbitrarily large $i \in \omega$ such that there is a core condition $\sigma = (A_0, \ldots, A_{i-1}) \in \nabla$ (of length i) with

$$\frac{|\{A \in \mathcal{A}_i : \sigma^\frown A \not\perp_{\mathbb{J}} p\}|}{|\mathcal{A}_i|} \ge 1 - \varepsilon.$$

   (Recall that $p \not\perp_{\mathbb{J}} q$ means that p and q are compatible in $\mathbb{J}$.)
4. $\mathbb{J}$ is ccc.
5. $\mathbb{J}$ is separative.[23]
6. (To simplify some technicalities:) $\mathbb{J} \subseteq H(\aleph_1)$.

We now define $\underset{\sim}{Z}_\nabla$, which will be a canonical $\mathbb{J}$-name of (a code for) a null set. We will use the sequence $\underset{\sim}{C}^\nabla$ added by $\mathbb{J}$ (see Definition 2.59(2)).

Definition 2.60. Each $\underset{\sim}{C}_i^\nabla$ defines a clopen set $\underset{\sim}{Z}_i^\nabla = \{x \in 2^\omega : x \restriction L_i \in \underset{\sim}{C}_i^\nabla\}$ of measure at most $\frac{1}{2^i}$. The sequence $\underset{\sim}{Z}_\nabla = (\underset{\sim}{Z}_0^\nabla, \underset{\sim}{Z}_1^\nabla, \underset{\sim}{Z}_2^\nabla, \ldots)$ is (a name for) a code for the null set

$$\text{nullset}(\underset{\sim}{Z}_\nabla) = \bigcap_{n<\omega} \bigcup_{i\ge n} \underset{\sim}{Z}_i^\nabla.$$

[22]This is the crucial combinatorial property of Janus forcing. Actually, (3) implies (2).
[23]Separative is defined on page 36.

Since $\underset{\sim}{C}^\nabla$ is defined "canonically" (see in particular Definition 2.59(1),(2)), and $\underset{\sim}{Z}^\nabla$ is constructed in an absolute way from $\underset{\sim}{C}^\nabla$, we get:

Fact 2.61. If $\mathbb{J}$ is a Janus forcing, M a countable model and $\mathbb{J}^M$ a Janus forcing in M which is an M-complete subset of $\mathbb{J}$, if H is $\mathbb{J}$-generic over V and H^M the induced $\mathbb{J}^M$-generic filter over M, then $\underset{\sim}{C}^\nabla$ evaluates to the same real in $M[H^M]$ as in $V[H]$, and therefore $\underset{\sim}{Z}^\nabla$ evaluates to the same code (but of course not to the same set of reals).

For later reference, we record the following trivial fact:

Fact 2.62. Being a Janus forcing is absolute. In particular, if $V \subseteq W$ are set theoretical universes and $\mathbb{J}$ is a Janus forcing in V, then $\mathbb{J}$ is a Janus forcing in W. In particular, if M is a countable model in V and $\mathbb{J} \in M$ a Janus forcing in M, then $\mathbb{J}$ is also a Janus forcing in V.
Let $(M^n)_{n\in\omega}$ be an increasing sequence of countable models, and let $\mathbb{J}^n \in M^n$ be Janus forcings. Assume that $\mathbb{J}^n$ is M^n-complete in $\mathbb{J}^{n+1}$. Then $\bigcup_n \mathbb{J}^n$ is a Janus forcing, and an M^n-complete extension of $\mathbb{J}^n$ for all n.

2.2.B Janus and strongly meager

Carlson [Car93] showed that Cohen reals make every uncountable set X of the ground model not strongly meager in the extension (and that not being strongly meager is preserved in a subsequent forcing with precaliber $\aleph_1$). We show that a *countable* Janus forcing $\mathbb{J}$ does the same (for a subsequent forcing that is even σ-centered, not just precaliber $\aleph_1$). This sounds trivial, since any (nontrivial) countable forcing is equivalent to Cohen forcing anyway. However, we show (and will later use) that the canonical null set $\underset{\sim}{Z}_\nabla$ defined above witnesses that X is not strongly meager (and not just some null set that we get out of the isomorphism between $\mathbb{J}$ and Cohen forcing). The point is that while ∇ is not a complete subforcing of $\mathbb{J}$, the condition (3) of the Definition 2.59 guarantees that Carlson's argument still works, if we assume that X is non-thin (not just uncountable). This is enough for us, since by Corollary 2.27 ultralaver forcing makes any uncountable set non-thin.

Recall that we fixed the increasing sequence $\bar{\ell}^* = (\ell^*_i)_{i\in\omega}$ and B^*. In the following, whenever we say "(very) thin" we mean "(very) thin with respect to $\bar{\ell}^*$ and B^*" (see Definition 2.25).

Lemma 2.63. *If X is not thin, $\mathbb{J}$ is a countable Janus forcing based on $\bar{\ell}^*$, and $\underset{\sim}{R}$ is a $\mathbb{J}$-name for a σ-centered forcing notion, then $\mathbb{J} * \underset{\sim}{R}$ forces that X is not strongly meager witnessed by the null set $\underset{\sim}{Z}_\nabla$.*

Proof. Let $\underset{\sim}{c}$ be a $\mathbb{J}$-name for a function $\underset{\sim}{c} : \underset{\sim}{R} \to \omega$ witnessing that $\underset{\sim}{R}$ is σ-centered.

Recall that "$\underset{\sim}{Z}_\nabla$ witnesses that X is not strongly meager" means that $X + \underset{\sim}{Z}_\nabla = 2^\omega$. Assume towards a contradiction that $(p, r) \in \mathbb{J} * \underset{\sim}{R}$ forces that $X + \underset{\sim}{Z}_\nabla \neq 2^\omega$. Then we can fix a $(\mathbb{J} * \underset{\sim}{R})$-name $\underset{\sim}{\xi}$ such that $(p, r) \Vdash \underset{\sim}{\xi} \notin X + \underset{\sim}{Z}_\nabla$, i.e., $(p, r) \Vdash (\forall x \in X)\ \underset{\sim}{\xi} \notin x + \underset{\sim}{Z}_\nabla$. By definition of $\underset{\sim}{Z}_\nabla$, we get

$$(p, r) \Vdash (\forall x \in X)\, (\exists n \in \omega)\, (\forall i \geq n)\ \underset{\sim}{\xi}\restriction L_i \notin x\restriction L_i + \underset{\sim}{C}_i^\nabla.$$

For each $x \in X$ we can find $(p_x, r_x) \leq (p, r)$ and natural numbers $n_x \in \omega$ and $m_x \in \omega$ such that p_x forces that $\underset{\sim}{c}(r_x) = m_x$ and

$$(p_x, r_x) \Vdash (\forall i \geq n_x)\ \underset{\sim}{\xi}\restriction L_i \notin x\restriction L_i + \underset{\sim}{C}_i^\nabla.$$

So $X = \bigcup_{p \in \mathbb{J}, m \in \omega, n \in \omega} X_{p,m,n}$, where $X_{p,m,n}$ is the set of all x with $p_x = p$, $m_x = m$, $n_x = n$. (Note that $\mathbb{J}$ is countable, so the union is countable.) As X is not thin, there is some p^*, m^*, n^* such that $X^* := X_{p^*,m^*,n^*}$ is not very thin. So we get for all $x \in X^*$:

$$(p^*, r_x) \Vdash (\forall i \geq n^*)\ \underset{\sim}{\xi}\restriction L_i \notin x\restriction L_i + \underset{\sim}{C}_i^\nabla. \tag{2.64}$$

Since X^* is not very thin, there is some $i_0 \in \omega$ such that for all $i \geq i_0$

$$\text{the (finite) set } X^*\restriction L_i \text{ has more than } B^*(i) \text{ elements.} \tag{2.65}$$

Due to the fact that $\mathbb{J}$ is a Janus forcing (see Definition 2.59 (3)), there are arbitrarily large $i \in \omega$ such that there is a core condition $\sigma = (A_0, \ldots, A_{i-1}) \in \nabla$ with

$$\frac{|\{A \in \mathcal{A}_i : \sigma^\frown A \not\perp_{\mathbb{J}} p^*\}|}{|\mathcal{A}_i|} \geq \frac{2}{3}. \tag{2.66}$$

Fix such an i larger than both i_0 and n^*, and fix a condition σ satisfying (2.66).

We now consider the following two subsets of $\mathcal{A}_i$:

$$\{A \in \mathcal{A}_i : \sigma^\frown A \not\perp_{\mathbb{J}} p^*\} \text{ and } \{A \in \mathcal{A}_i : X^*\restriction L_i + A = 2^{L_i}\}. \tag{2.67}$$

By (2.66), the relative measure (in $\mathcal{A}_i$) of the left one is at least $\frac{2}{3}$; due to (2.65) and the definition of $\mathcal{A}_i$ according to Corollary 2.56, the relative measure of the right one is at least $\frac{3}{4}$; so the two sets in (2.67) are not disjoint, and we can pick an A belonging to both.

Clearly, $\sigma^\frown A$ forces (in $\mathbb{J}$) that $\underset{\sim}{C}_i^\nabla$ is equal to A. Fix $q \in \mathbb{J}$ witnessing $\sigma^\frown A \not\perp_{\mathbb{J}} p^*$. Then

$$q \Vdash_{\mathbb{J}} X^*\restriction L_i + \underset{\sim}{C}_i^\nabla = X^*\restriction L_i + A = 2^{L_i}. \tag{2.68}$$

Since p^* forces that for each $x \in X^*$ the color $\underset{\sim}{c}(r_x) = m^*$, we can find an r^* which is (forced by $q \leq p^*$ to be) a lower bound of the *finite* set $\{r_x : x \in X^{**}\}$, where $X^{**} \subseteq X^*$ is any finite set with $X^{**}\restriction L_i = X^*\restriction L_i$.

By (2.64),

$$(q, r^*) \Vdash \underset{\sim}{\xi}\restriction L_i \notin X^{**}\restriction L_i + \underset{\sim}{C}_i^\nabla = X^*\restriction L_i + \underset{\sim}{C}_i^\nabla,$$

contradicting (2.68). □

Recall that by Corollary 2.27, every uncountable set X in V will not be thin in the $\mathbb{L}_{\bar{D}}$-extension. Hence we get:

Corollary 2.69. *Let X be uncountable. If $\mathbb{L}_{\bar{D}}$ is any ultralaver forcing adding an ultralaver real $\bar{\ell}$, and $\bar{\ell}^*$ is defined from $\bar{\ell}$ as in Lemma 2.26, and if $\mathbb{J}$ is a countable Janus forcing based on $\bar{\ell}^*$, $\underset{\sim}{Q}$ is any σ-centered forcing, then $\mathbb{L}_{\bar{D}} * \underset{\sim}{\mathbb{J}} * \underset{\sim}{Q}$ forces that X is not strongly meager.*

2.2.C Janus forcing and preservation of Lebesgue positivity

We show that every Janus forcing in a countable model M can be extended to locally preserve a given random real over M. (We showed the same for ultralaver forcing in Section 2.1.E.)

We start by proving that every countable Janus forcing can be embedded into a Janus forcing which is equivalent to random forcing, preserving the maximality of countably many maximal antichains. (In the following lemma, the letter M is just a label to distinguish $\mathbb{J}^M$ from $\mathbb{J}$, and does not necessarily refer to a model.)

Lemma 2.70. *Let $\mathbb{J}^M$ be a countable Janus forcing (based on $\bar{\ell}^*$) and let $\{D_k : k \in \omega\}$ be a countable family of open dense subsets of $\mathbb{J}^M$. Then there is a Janus forcing $\mathbb{J}$ (based on the same $\bar{\ell}^*$) such that*

- *$\mathbb{J}^M$ is an incompatibility-preserving subforcing of $\mathbb{J}$.*
- *Each D_k is still predense in $\mathbb{J}$.*
- *$\mathbb{J}$ is forcing equivalent to random forcing.*

Proof. Without loss of generality assume $D_0 = \mathbb{J}^M$. Recall that $\nabla = \nabla^{\mathbb{J}^M}$ was defined in Definition 2.58. Note that for each j the set $\{\sigma \in \nabla : \text{height}(\sigma) = j\}$ is predense in $\mathbb{J}^M$, so the set

$$E_j := \{p \in \mathbb{J}^M : \exists \sigma \in \nabla : \text{height}(\sigma) = j,\ p \leq \sigma\} \tag{2.71}$$

is dense open in $\mathbb{J}^M$; hence without loss of generality each E_j appears in our list of D_k's.

Let $\{r^n : n \in \omega\}$ be an enumeration of $\mathbb{J}^M$.

We now fix n for a while (up to (2.73)). We will construct a finitely splitting tree $S^n \subseteq \omega^{<\omega}$ and a family $(\sigma^n_s, p^n_s, \tau^{*n}_s)_{s\in S^n}$ satisfying the following (suppressing the superscript n):

(a) $\sigma_s \in \nabla$, $\sigma_{\langle\rangle} = \langle\rangle$, $s \subseteq t$ implies $\sigma_s \subseteq \sigma_t$, and $s \perp_{S^n} t$ implies $\sigma_s \perp_\nabla \sigma_t$. (So in particular the set $\{\sigma_t : t \in \mathrm{succ}_{S^n}(s)\}$ is a (finite) antichain above σ_s in ∇.)

(b) $p_s \in \mathbb{J}^M$, $p_{\langle\rangle} = r^n$; if $s \subseteq t$ then $p_t \leq_{\mathbb{J}^M} p_s$ (hence $p_t \leq r^n$); $s \perp_{S^n} t$ implies $p_s \perp_{\mathbb{J}^M} p_t$.

(c) $p_s \leq_{\mathbb{J}^M} \sigma_s$.

(d) $\sigma_s \subseteq \tau^*_s \in \nabla$, and $\{\sigma_t : t \in \mathrm{succ}_{S^n}(s)\}$ is the set of all $\tau \in \mathrm{succ}_\nabla(\tau^*_s)$ which are compatible with p_s.

(e) The set $\{\sigma_t : t \in \mathrm{succ}_{S^n}(s)\}$ is a subset of $\mathrm{succ}_\nabla(\tau^*_s)$ of relative size at least $1 - \frac{1}{\mathrm{lh}(s)+10}$.

(f) Each $s \in S^n$ has at least 2 successors (in S^n).

(g) If $k = \mathrm{lh}(s)$, then $p_s \in D_k$ (and therefore also in all D_l for $l < k$).

Set $\sigma_{\langle\rangle} = \langle\rangle$ and $p_{\langle\rangle} = r^n$. Given s, σ_s and p_s, we construct $\mathrm{succ}_{S^n}(s)$ and $(\sigma_t, p_t)_{t\in\mathrm{succ}_{S^n}(s)}$: We apply fatness 2.59(3) to p_s with $\varepsilon = \frac{1}{\mathrm{lh}(s)+10}$. So we get some $\tau^*_s \in \nabla$ of height bigger than the height of σ_s such that the set B of elements of $\mathrm{succ}_\nabla(\tau^*_s)$ which are compatible with p_s has relative size at least $1 - \varepsilon$. Since $p_s \leq_{\mathbb{J}^M} \sigma_s$ we get that τ^*_s is compatible with (and therefore stronger than) σ_s. Enumerate B as $\{\tau_0, \ldots, \tau_{l-1}\}$. Set $\mathrm{succ}_{S^n}(s) = \{s^\frown i : i < l\}$ and $\sigma_{s^\frown i} = \tau_i$. For $t \in \mathrm{succ}_{S^n}(s)$, choose $p_t \in \mathbb{J}^M$ stronger than both σ_t and p_s (which is obviously possible since σ_t and p_s are compatible), and moreover $p_t \in D_{\mathrm{lh}(t)}$. This concludes the construction of the family $(\sigma^n_s, p^n_s, \tau^{*n}_s)_{s\in S^n}$.

So $(S^n, \subseteq)$ is a finitely splitting nonempty tree of height ω with no maximal nodes and no isolated branches. $[S^n]$ is the (compact) set of branches of S^n. The closed subsets of $[S^n]$ are exactly the sets of the form $[T]$, where $T \subseteq S^n$ is a subtree of S^n with no maximal nodes. $[S^n]$ carries a natural ("uniform") probability measure μ_n, which is characterized by

$$\mu_n((S^n)^{[t]}) = \frac{1}{|\mathrm{succ}_{S^n}(s)|} \cdot \mu_n((S^n)^{[s]})$$

for all $s \in S^n$ and all $t \in \mathrm{succ}_{S^n}(s)$. (We just write $\mu_n(T)$ instead of $\mu_n([T])$ to increase readability.)

We call $T \subseteq S^n$ positive if $\mu_n(T) > 0$, and we call T pruned if $\mu_n(T^{[s]}) > 0$ for all $s \in T$. (Clearly every positive tree T contains a pruned tree T' of the same measure, which can be obtained from T by removing all nodes s with $\mu_n(T^{[s]}) = 0$.)

Let $T \subseteq S^n$ be a positive pruned tree and $\varepsilon > 0$. Then on all but finitely many levels k there is an $s \in T$ such that

$$\mathrm{succ}_T(s) \subseteq \mathrm{succ}_{S^n}(s) \text{ has relative size } \geq 1 - \varepsilon. \tag{2.72}$$

(This follows from Lebesgue's density theorem, or can easily be seen directly: Set $C_m = \bigcup_{t \in T,\, \mathrm{lh}(t)=m} (S^n)^{[t]}$. Then C_m is a decreasing sequence of closed sets, each containing $[T]$. If the claim fails, then $\mu_n(C_{m+1})) \leq \mu_n(C_m) \cdot (1 - \varepsilon)$ infinitely often; so $\mu_n(T) \leq \mu_n(\bigcap_m C_m) = 0$.)

It is well known that the set of positive, pruned subtrees of S^n, ordered by inclusion, is forcing equivalent to random forcing (which can be defined as the set of positive, pruned subtrees of $2^{<\omega}$).

We have now constructed S^n for all n. Define

$$\mathbb{J} = \mathbb{J}^M \cup \bigcup_n \{\, (n, T) : T \subseteq S^n \text{ is a positive pruned tree} \,\} \tag{2.73}$$

with the following partial order:

- The order on $\mathbb{J}$ extends the order on $\mathbb{J}^M$.
- $(n', T') \leq (n, T)$ if $n = n'$ and $T' \subseteq T$.
- For $p \in \mathbb{J}^M$: $(n, T) \leq p$ if there is a k such that $p^n_t \leq p$ for all $t \in T$ of length k. (Note that this will then be true for all bigger k as well.)
- $p \leq (n, T)$ never holds (for $p \in \mathbb{J}^M$).

The lemma now easily follows from the following properties:

1. The order on $\mathbb{J}$ is transitive.
2. $\mathbb{J}^M$ is an incompatibility-preserving subforcing of $\mathbb{J}$.
   In particular, $\mathbb{J}$ satisfies item (1) of Definition 2.59 of Janus forcing.
3. For all k: the set $\{(n, T^{[t]}) : t \in T,\ \mathrm{lh}(t) = k\}$ is a (finite) predense antichain below (n, T).
4. $(n, T^{[t]})$ is stronger than p^n_t for each $t \in T$ (witnessed, e.g., by $k = \mathrm{lh}(t)$).
   Of course, $(n, T^{[t]})$ is stronger than (n, T) as well.

5. Since $p^n_t \in D_k$ for $k = \mathrm{lh}(t)$, this implies that each D_k is predense below each (n, S^n) and therefore in $\mathbb{J}$.
   Also, since each set E_j appeared in our list of open dense subsets (see (2.71)), the set $\{\sigma \in \nabla : \mathrm{height}(\sigma) = j\}$ is still predense in $\mathbb{J}$, i.e., item (2) of the Definition 2.59 of Janus forcing is satisfied.

6. The condition (n, S^n) is stronger than r^n, so $\{(n, S^n) : n \in \omega\}$ is predense in $\mathbb{J}$ and $\mathbb{J} \setminus \mathbb{J}^M$ is dense in $\mathbb{J}$.
   Below each (n, S^n), the forcing $\mathbb{J}$ is isomorphic to random forcing.
   Therefore, $\mathbb{J}$ itself is forcing equivalent to random forcing. (In fact, the complete Boolean algebra generated by $\mathbb{J}$ is isomorphic to the standard random algebra, Borel sets modulo null sets.) This proves in particular that $\mathbb{J}$ is ccc, i.e., satisfies property 2.59(4).

7. It is easy (but not even necessary) to check that $\mathbb{J}$ is separative, i.e., property 2.59(5). In any case, we could replace $\leq_{\mathbb{J}}$ by $\leq^*_{\mathbb{J}}$, thus making $\mathbb{J}$ separative without changing $\leq_{\mathbb{J}^M}$, since $\mathbb{J}^M$ was already separative.

8. Property 2.59(6), i.e., $\mathbb{J} \in H(\aleph_1)$, is obvious.

9. The remaining item of the definition of Janus forcing, fatness 2.59(3), is satisfied.
   I.e., given $(n, T) \in \mathbb{J}$ and $\varepsilon > 0$ there is an arbitrarily high $\tau^* \in \nabla$ such that the relative size of the set $\{\tau \in \mathrm{succ}_\nabla(\tau^*) : \tau \not\perp (n, T)\}$ is at least $1 - \varepsilon$. (We will show $\geq (1-\varepsilon)^2$ instead, to simplify the notation.)

We show (9): Given $(n, T) \in \mathbb{J}$ and $\varepsilon > 0$, we use (2.72) to get an arbitrarily high $s \in T$ such that $\mathrm{succ}_T(s)$ is of relative size $\geq 1 - \varepsilon$ in $\mathrm{succ}_{S^n}(s)$. We may choose s of length $> \frac{1}{\varepsilon}$. We claim that τ^*_s is as required:

- Let $B := \{\sigma_t : t \in \mathrm{succ}_{S^n}(s)\}$. Note that $B = \{\tau \in \mathrm{succ}_\nabla(\tau^*_s) : \tau \not\perp p_s\}$.
  B has relative size $\geq 1 - \frac{1}{\mathrm{lh}(s)} \geq 1 - \varepsilon$ in $\mathrm{succ}_\nabla(\tau^*_s)$ (according to property (e) of S^n).

- $C := \{\sigma_t : t \in \mathrm{succ}_T(s)\}$ is a subset of B of relative size $\geq 1 - \varepsilon$ according to our choice of s.

- So C is of relative size $(1-\varepsilon)^2$ in $\mathrm{succ}_\nabla(\tau^*_s)$.

- Each $\sigma_t \in C$ is compatible with (n, T), as $(n, T^{[t]}) \leq p_t \leq \sigma_t$ (see (4)). $\square$

So in particular if $\mathbb{J}^M$ is a Janus forcing in a countable model M, then we can extend it to a Janus forcing $\mathbb{J}$ which is in fact random forcing. Since random forcing strongly preserves randoms over countable models (see Lemma 2.32), it is not surprising that we get local preservation of randoms for Janus forcing, i.e., the analoga of Lemma 2.33 and Corollary 2.54. (Still, some additional argument is needed, since the fact that $\mathbb{J}$ (which is now random forcing) "strongly preserves randoms" just means that a random real r over M is preserved with respect to random forcing in M, not with respect to $\mathbb{J}^M$.)

Lemma 2.74. *If $\mathbb{J}^M$ is a Janus forcing in a countable model M and r a random real over M, then there is a Janus forcing $\mathbb{J}$ such that $\mathbb{J}^M$ is an M-complete subforcing of $\mathbb{J}$ and the following holds:*
If

- *$p \in \mathbb{J}^M$,*
- *in M, $\bar{\underset{\sim}{Z}} = (\underset{\sim}{Z}_1, \dots, \underset{\sim}{Z}_m)$ is a sequence of $\mathbb{J}^M$-names for codes for null sets, and $Z_1^*, \dots, Z_m^*$ are interpretations under p, witnessed by a sequence $(p_n)_{n\in\omega}$,*
- *$Z_i^* \sqsubset_{k_i} r$ for $i = 1, \dots, m$,*

***then** there is a $q \leq p$ in $\mathbb{J}$ forcing that*

- *r is random over $M[H^M]$,*
- *$\underset{\sim}{Z}_i \sqsubset_{k_i} r$ for $i = 1, \dots, m$.*

Remark 2.75. In the version for ultralaver forcings, i.e., Lemma 2.33, we had to assume that the stems of the witnessing sequence are strictly increasing. In the Janus version, we do not have any requirement of that kind.

Proof. Let $\mathcal{D}$ be the set of dense subsets of $\mathbb{J}^M$ in M. According to Lemma 2.47, we can first find some countable M' such that r is still random over M' and such that in M' both $\mathbb{J}^M$ and $\mathcal{D}$ are countable. According to Fact 2.62, $\mathbb{J}^M$ is a (countable) Janus forcing in M', so we can apply Lemma 2.70 to the set $\mathcal{D}$ to construct a Janus forcing $\mathbb{J}^{M'}$ which is equivalent to random forcing such that (from the point of V) $\mathbb{J}^M \lessdot_M \mathbb{J}^{M'}$. In V, let[24] $\mathbb{J}$ be random forcing. $\mathbb{J}^{M'}$ is an M'-complete subforcing of $\mathbb{J}$ and therefore $\mathbb{J}^M \lessdot_M \mathbb{J}$. Moreover, as was noted in Lemma 2.32, we even know that random forcing

[24] More precisely: Densely embed $\mathbb{J}^{M'}$ into $(\text{Borel/null})^{M'}$, the complete Boolean algebra associated with random forcing in M', and let $\mathbb{J} := (\text{Borel/null})^V$. Using the embedding, $\mathbb{J}^{M'}$ can now be viewed as an M'-complete subset of $\mathbb{J}$.

strongly preserves randoms over M' (see Definition 2.53). To show that $\mathbb{J}$ is indeed a Janus forcing, we have to check the fatness condition 2.59(3); this follows easily from Π^1_1-absoluteness (recall that incompatibility of random conditions is Borel).

So assume that (in M) the sequence $(p_n)_{n\in\omega}$ of $\mathbb{J}^M$-conditions interprets $\bar{\underaccent{\tilde}{Z}}$ as $\bar{Z}^*$. In M', $\mathbb{J}^M$-names can be reinterpreted as $\mathbb{J}^{M'}$-names, and the $\mathbb{J}^{M'}$-name $\bar{\underaccent{\tilde}{Z}}$ is interpreted as $\bar{Z}^*$ by the same sequence $(p_n)_{n\in\omega}$. Let $k_1, \dots, k_m$ be such that $Z^*_i \sqsubset_{k_i} r$ for $i = 1, \dots, m$. So by strong preservation of randoms, we can in V find some $q \leq p_0$ forcing that r is random over $M'[H^{M'}]$ (and therefore also over the subset $M[H^M]$), and that $\underaccent{\tilde}{Z}_i \sqsubset_{k_i} r$ (where $\underaccent{\tilde}{Z}_i$ can be evaluated in $M'[H^{M'}]$ or equivalently in $M[H^M]$). $\square$

So Janus forcing is locally preserving randoms (just as ultralaver forcing):

Corollary 2.76. *If Q^M is a Janus forcing in M and r a real, then there is a Janus forcing Q over Q^M (which is in fact equivalent to random forcing) locally preserving randomness of r over M.*

Proof. In this case, the notion of "quick" interpretations is trivial, i.e., $D_k^{Q^M} = Q^M$ for all k, and the claim follows from the previous lemma. $\square$

2.3 Almost finite and almost countable support iterations

A main tool to construct the forcing for BC+dBC will be "partial countable support iterations", more particularly "almost finite support" and "almost countable support" iterations. A partial countable support iteration is a forcing iteration $(P_\alpha, Q_\alpha)_{\alpha<\omega_2}$ such that for each limit ordinal δ the forcing notion P_δ is a subset of the countable support limit of $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ which satisfies some natural properties (see Definition 2.82).

Instead of transitive models, we will use ord-transitive models (which are transitive when ordinals are considered as urelements). Why do we do that? We want to "approximate" the generic iteration $\bar{\mathbf{P}}$ of length ω_2 with countable models; this can be done more naturally with ord-transitive models (since obviously countable transitive models only see countable ordinals). We call such an ord-transitive model a "candidate" (provided it satisfies some nice properties, see Definition 2.77). A basic point is that forcing extensions work naturally with candidates.

In the next few paragraphs (and also in Section 2.4), $x = (M^x, \bar{P}^x)$ will denote a pair such that M^x is a candidate and $\bar{P}^x$ is (in M^x) a partial countable support iteration; similarly we write, e.g., $y = (M^y, \bar{P}^y)$ or $x_n = (M^{x_n}, \bar{P}^{x_n})$.

We will need the following results to prove BC+dBC. (However, as opposed to the case of the ultralaver and Janus section, the reader will probably have to read this section to understand the construction in the next section, and not just the following list of properties.)

Given $x = (M^x, \bar{P}^x)$, we can construct by induction on α a partial countable support iteration $\bar{P} = (P_\alpha, Q_\alpha)_{\alpha<\omega_2}$ satisfying:

> There is a canonical M^x-complete embedding from $\bar{P}^x$ to $\bar{P}$.

In this construction, we can use at each stage β any desired Q_β, as long as P_β forces that Q^x_β is (evaluated as) an $M^x[H^x_\beta]$-complete subforcing of Q_β (where $H^x_\beta \subseteq P^x_\beta$ is the M^x-generic filter induced by the generic filter $H_\beta \subseteq P_\beta$). Moreover, we can demand either of the following two additional properties[25] of the limit of this iteration $\bar{P}$:

1. If all Q_β are forced to be σ-centered, and Q_β is trivial for all $\beta \notin M^x$, then P_{ω_2} is σ-centered.

2. If r is random over M^x, and all Q_β locally preserve randomness of r over $M^x[H^x_\beta]$ (see Definition 2.53), then also P_{ω_2} locally preserves the randomness of r.

Actually, we need the following variant: Assume that we already have P_{α_0} for some $\alpha_0 \in M^x$, and that $P^x_{\alpha_0}$ canonically embeds into P_{α_0}, and that the respective assumption on Q_β holds for all $\beta \geq \alpha_0$. Then we get that P_{α_0} forces that the quotient $P_{\omega_2}/P_{\alpha_0}$ satisfies the respective conclusion.

We also need:[26]

3. If instead of a single x we have a sequence x_n such that each P^{x_n} canonically (and M^{x_n}-completely) embeds into $P^{x_{n+1}}$, then we can find a partial countable support iteration $\bar{P}$ into which all P^{x_n} embed canonically (and we can again use any desired Q_β, assuming that $Q^{x_n}_\beta$ is an $M^{x_n}[H^{x_n}_\beta]$-complete subforcing of Q_β for all $n \in \omega$).

4. (A fact that is easy to prove but awkward to formulate.) If a Δ-system argument produces two x_1, x_2 as in Lemma 2.122(3), then we can find a partial countable support iteration $\bar{P}$ such that $\bar{P}^{x_i}$ canonically (and M^{x_i}-completely) embeds into $\bar{P}$ for $i = 1, 2$.

[25] The σ-centered version is central for the proof of dBC; the random preserving version for BC.

[26] This will give σ-closure and $\aleph_2$-cc for the preparatory forcing $\mathbb{R}$.

2.3.A Ord-transitive models

We will use "ord-transitive" models, as introduced in [She04] (see also the presentation in [Kel12]). We briefly summarize the basic definitions and properties (restricted to the rather simple case needed in this paper):

Definition 2.77. Fix a suitable finite subset ZFC* of ZFC (that is satisfied by $H(\chi^*)$ for sufficiently large regular χ^*).

1. A set M is called a *candidate*, if
   - M is countable,
   - $(M, \in)$ is a model of ZFC*,
   - M is ord-absolute: $M \models \alpha \in \mathrm{Ord}$ iff $\alpha \in \mathrm{Ord}$, for all $\alpha \in M$,
   - M is *ord-transitive*: if $x \in M \setminus \mathrm{Ord}$, then $x \subseteq M$,
   - $\omega + 1 \subseteq M$.
   - "α is a limit ordinal" and "$\alpha = \beta + 1$" are both absolute between M and V.
2. A candidate M is called *nice*, if "α has countable cofinality" and "the countable set A is cofinal in α" both are absolute between M and V. (So if $\alpha \in M$ has countable cofinality, then $\alpha \cap M$ is cofinal in α.) Moreover, we assume $\omega_1 \in M$ (which implies $\omega_1^M = \omega_1$) and $\omega_2 \in M$ (but we do not require $\omega_2^M = \omega_2$).
3. Let P^M be a forcing notion in a candidate M. (To simplify notation, we can assume without loss of generality that $P^M \cap \mathrm{Ord} = \emptyset$ (or at least $\subseteq \omega$) and that therefore $P^M \subseteq M$ and also $A \subseteq M$ whenever M thinks that A is a subset of P^M.) Recall that a subset H^M of P^M is M-generic (or: P^M-generic over M), if $|A \cap H^M| = 1$ for all maximal antichains A in M.
4. Let H^M be P^M-generic over M and $\underaccent{\tilde}{\tau}$ a P^M-name in M. We define the evaluation $\underaccent{\tilde}{\tau}[H^M]^M$ to be x if M thinks that $p \Vdash_{P^M} \underaccent{\tilde}{\tau} = \check{x}$ for some $p \in H^M$ and $x \in M$ (or equivalently just for $x \in M \cap \mathrm{Ord}$), and $\{\underaccent{\tilde}{\sigma}[H^M]^M : (\underaccent{\tilde}{\sigma}, p) \in \underaccent{\tilde}{\tau}, p \in H^M\}$ otherwise. Abusing notation we write $\underaccent{\tilde}{\tau}[H^M]$ instead of $\underaccent{\tilde}{\tau}[H^M]^M$, and we write $M[H^M]$ for $\{\underaccent{\tilde}{\tau}[H^M] : \underaccent{\tilde}{\tau}$ is a P^M-name in $M\}$.
5. For any set N (typically, an elementary submodel of some $H(\chi)$), the *ord-collapse* k (or k^N) is a recursively defined function with domain N: $k(x) = x$ if $x \in \mathrm{Ord}$, and $k(x) = \{k(y) : y \in x \cap N\}$ otherwise.

6. We define $\operatorname{ordclos}(\alpha) := \emptyset$ for all ordinals α. The ord-transitive closure of a non-ordinal x is defined inductively on the rank:

$$\begin{aligned}\operatorname{ordclos}(x) &= x \cup \bigcup\{\operatorname{ordclos}(y) : y \in x \setminus \mathrm{Ord}\}\\ &= x \cup \bigcup\{\operatorname{ordclos}(y) : y \in x\}.\end{aligned}$$

   So for $x \notin \mathrm{Ord}$, the set $\operatorname{ordclos}(x)$ is the smallest ord-transitive set containing x as a subset. HCON is the collection of all sets x such that the ord-transitive closure of x is countable. x is in HCON iff x is element of some candidate. In particular, all reals and all ordinals are HCON.

   We write HCON_α for the family of all sets x in HCON whose transitive closure only contains ordinals $< \alpha$.

The following facts can be found in [She04] or [Kel12] (they can be proven by rather straightforward, if tedious, inductions on the ranks of the according objects).

Fact 2.78. 1. The ord-collapse of a countable elementary submodel of $H(\chi^*)$ is a nice candidate.

2. Unions, intersections etc. are generally not absolute for candidates. For example, let $x \in M \setminus \mathrm{Ord}$. In M we can construct a set y such that $M \models y = \omega_1 \cup \{x\}$. Then y is not an ordinal and therefore a subset of M, and in particular y is countable and $y \neq \omega_1 \cup \{x\}$.

3. Let $j : M \to M'$ be the transitive collapse of a candidate M, and $f : \omega_1 \cap M' \to \mathrm{Ord}$ the inverse (restricted to the ordinals). Obviously M' is a countable transitive model of ZFC*; moreover M is characterized by the pair (M', f) (we call such a pair a "labeled transitive model"). Note that f satisfies $f(\alpha+1) = f(\alpha)+1$, $f(\alpha) = \alpha$ for $\alpha \in \omega \cup \{\omega\}$. $M \models (\alpha \text{ is a limit})$ iff $f(\alpha)$ is a limit. $M \models \operatorname{cf}(\alpha) = \omega$ iff $\operatorname{cf}(f(\alpha)) = \omega$, and in that case $f[\alpha]$ is cofinal in $f(\alpha)$. On the other hand, given a transitive countable model M' of ZFC* and an f as above, then we can construct a (unique) candidate M corresponding to (M', f).

4. All candidates M with $M \cap \mathrm{Ord} \subseteq \omega_1$ are hereditarily countable, so their number is at most $2^{\aleph_0}$. Similarly, the cardinality of HCON_α is at most continuum whenever $\alpha < \omega_2$.

5. If M is a candidate, and if H^M is P^M-generic over M, then $M[H^M]$ is a candidate as well and an end-extension of M such that $M \cap \mathrm{Ord} =$

$M[H^M] \cap \mathrm{Ord}$. If M is nice and (M thinks that) P^M is proper, then $M[H^M]$ is nice as well.

6. Forcing extensions commute with the transitive collapse j:

   If M corresponds to (M', f), then $H^M \subseteq P^M$ is P^M-generic over M iff $H' := j[H^M]$ is $P' := j(P^M)$-generic over M', and in that case $M[H^M]$ corresponds to $(M'[H'], f)$. In particular, the forcing extension $M[H^M]$ of M satisfies the forcing theorem (everything that is forced is true, and everything true is forced).

7. In case of elementary submodels, forcing extensions commute with ord-collapses:

   Let N be a countable elementary submodel of $H(\chi^*)$, $P \in N$, $k : N \to M$ the ord-collapse (so M is a candidate), and let H be P-generic over V. Then H is P-generic over N iff $H^M := k[H]$ is $P^M := k(P)$-generic over M; and in that case the ord-collapse of $N[H]$ is $M[H^M]$.

Assume that a nice candidate M thinks that $(\bar{P}^M, \bar{Q}^M)$ is a forcing iteration of length ω_2^V (we will usually write ω_2 for the length of the iteration, by this we will always mean ω_2^V and not the possibly different ω_2^M). In this section, we will construct an iteration $(\bar{P}, \bar{Q})$ in V, also of length ω_2, such that each P^M_α canonically and M-completely embeds into P_α for all $\alpha \in \omega_2 \cap M$. Once we know (by induction) that P^M_α M-completely embeds into P_α, we know that a P_α-generic filter H_α induces a P^M_α-generic (over M) filter which we call H^M_α. Then $M[H^M_\alpha]$ is a candidate, but nice only if P^M_α is proper. We will not need that $M[H^M_\alpha]$ is nice, actually we will only investigate sets of reals (or elements of $H(\aleph_1)$) in $M[H^M_\alpha]$, so it does not make any difference whether we use $M[H^M_\alpha]$ or its transitive collapse.

Remark 2.79. In the discussion so far we omitted some details regarding the theory ZFC* (that a candidate has to satisfy). The following "fine print" hopefully absolves us from any liability. (It is entirely irrelevant for the understanding of the paper.)

We have to guarantee that each $M[H^M_\alpha]$ that we consider satisfies enough of ZFC to make our arguments work (for example, the definitions and basic properties of ultralaver and Janus forcings should work). This turns out to be easy, since (as usual) we do not need the full power set axiom for these arguments (just the existence of, say, $\beth_5$). So it is enough that each $M[H^M_\alpha]$ satisfies some fixed finite subset of ZFC minus power set, which we call ZFC*.

Of course we can also find a bigger (still finite) set ZFC** that implies: $\beth_{10}$ exists, and each forcing extension of the universe with a forcing of size

$\leq \beth_4$ satisfies ZFC*. And it is provable (in ZFC) that each $H(\chi)$ satisfies ZFC** for sufficiently large regular χ.

We define "candidate" using the weaker theory ZFC*, and require that nice candidates satisfy the stronger theory ZFC**. This guarantees that all forcing extensions (by small forcings) of nice candidates will be candidates (in particular, satisfy enough of ZFC such that our arguments about Janus or ultralaver forcings work). Also, every ord-collapse of a countable elementary submodel N of $H(\chi)$ will be a nice candidate.

2.3.B Partial countable support iterations

We introduce the notion of "partial countable support limit": a subset of the countable support (CS) limit containing the union (i.e., the direct limit) and satisfying some natural requirements.

Let us first describe what we mean by "forcing iteration". They have to satisfy the following requirements:

- A "*topless forcing iteration*" $(P_\alpha, Q_\alpha)_{\alpha<\varepsilon}$ is a sequence of forcing notions P_α and P_α-names Q_α of quasiorders with a weakest element 1_{Q_α}. A "*topped iteration*" additionally has a final limit P_ε. Each P_α is a set of partial functions on α (as, e.g., in [Gol93]). More specifically, if $\alpha < \beta \leq \varepsilon$ and $p \in P_\beta$, then $p\restriction\alpha \in P_\alpha$. Also, $p\restriction\beta \Vdash_{P_\beta} p(\beta) \in Q_\beta$ for all $\beta \in \mathrm{dom}(p)$. The order on P_β will always be the "natural" one: $q \leq p$ iff $q\restriction\alpha$ forces (in P_α) that $q^{\mathrm{tot}}(\alpha) \leq p^{\mathrm{tot}}(\alpha)$ for all $\alpha < \beta$, where $r^{\mathrm{tot}}(\alpha) = r(\alpha)$ for all $\alpha \in \mathrm{dom}(r)$ and 1_{Q_α} otherwise. $P_{\alpha+1}$ consists of *all* p with $p\restriction\alpha \in P_\alpha$ and $p\restriction\alpha \Vdash p^{\mathrm{tot}}(\alpha) \in Q_\alpha$, so it is forcing equivalent to $P_\alpha * Q_\alpha$.

- $P_\alpha \subseteq P_\beta$ whenever $\alpha < \beta \leq \varepsilon$. (In particular, the empty condition is an element of each P_β.)

- For any $p \in P_\varepsilon$ and any $q \in P_\alpha$ ($\alpha < \varepsilon$) with $q \leq p\restriction\alpha$, the partial function $q \wedge p := q \cup p\restriction[\alpha, \varepsilon)$ is a condition in P_ε as well (so in particular, $p\restriction\alpha$ is a reduction of p, hence P_α is a complete subforcing of P_ε; and $q \wedge p$ is the weakest condition in P_ε stronger than both q and p).

- Abusing notation, we usually just write $\bar{P}$ for an iteration (be it topless or topped).

- We usually write H_β for the generic filter on P_β (which induces P_α-generic filters called H_α for $\alpha \leq \beta$). For topped iterations we call the filter on the final limit sometimes just H instead of H_ε.

We use the following notation for quotients of iterations:

- For $\alpha < \beta$, in the P_α-extension $V[H_\alpha]$, we let P_β/H_α be the set of all $p \in P_\beta$ with $p\restriction\alpha \in H_\alpha$ (ordered as in P_β). We may occasionally write P_β/P_α for the P_α-name of P_β/H_α.

- Since P_α is a complete subforcing of P_β, this is a quotient with the usual properties, in particular P_β is equivalent to $P_\alpha * (P_\beta/H_\alpha)$.

Remark 2.80. It is well known that quotients of proper countable support iterations are naturally equivalent to (names of) countable support iterations. In this paper, we can restrict our attention to proper forcings, but we do not really have countable support iterations. It turns out that it is not necessary to investigate whether our quotients can naturally be seen as iterations of any kind, so to avoid the subtle problems involved we will not consider the quotient as an iteration by itself.

Definition 2.81. Let $\bar{P}$ be a (topless) iteration of limit length ε. We define three limits of $\bar{P}$:

- The "*direct limit*" is the union of the P_α (for $\alpha < \varepsilon$). So this is the smallest possible limit of the iteration.

- The "*inverse limit*" consists of *all* partial functions p with domain $\subseteq \varepsilon$ such that $p\restriction\alpha \in P_\alpha$ for all $\alpha < \varepsilon$. This is the largest possible limit of the iteration.

- The "*full countable support limit* $P_\varepsilon^{\mathrm{CS}}$" of $\bar{P}$ is the inverse limit if $\mathrm{cf}(\varepsilon) = \omega$ and the direct limit otherwise.

We say that P_ε is a "*partial CS limit*", if P_ε is a subset of the full CS limit and the sequence $(P_\alpha)_{\alpha\leq\varepsilon}$ is a topped iteration. In particular, this means that P_ε contains the direct limit, and satisfies the following for each $\alpha < \varepsilon$: P_ε is closed under $p \mapsto p\restriction\alpha$, and whenever $p \in P_\varepsilon$, $q \in P_\alpha$, $q \leq p\restriction\alpha$, then also the partial function $q \wedge p$ is in P_ε.

So for a given topless $\bar{P}$ there is a well-defined inverse, direct and full CS limit. If $\mathrm{cf}(\varepsilon) > \omega$, then the direct and the full CS limit coincide. If $\mathrm{cf}(\varepsilon) = \omega$, then the direct limit and the full CS limit (=inverse limit) differ. Both of them are partial CS limits, but there are many more possibilities for partial CS limits. By definition, all of them will yield iterations.

Note that the name "CS limit" is slightly inappropriate, as the size of supports of conditions is not part of the definition. To give a more specific example: Consider a topped iteration $\bar{P}$ of length $\omega + \omega$ where P_ω is the direct limit and $P_{\omega+\omega}$ is the full CS limit. Let p be any element of the full CS limit of $\bar{P}\restriction\omega$ which is not in P_ω; then p is not in $P_{\omega+\omega}$ either. So not every countable subset of $\omega+\omega$ can appear as the support of a condition.

Definition 2.82. A forcing iteration $\bar{P}$ is called a "*partial CS iteration*", if

- every limit is a partial CS limit, and
- every Q_α is (forced to be) separative.[27]

The following fact can easily be proved by transfinite induction:

Fact 2.83. Let $\bar{P}$ be a partial CS iteration. Then for all α the forcing notion P_α is separative.

From now on, all iterations we consider will be partial CS iterations. In this paper, we will only be interested in proper partial CS iterations, but properness is not part of the definition of partial CS iteration. (The reader may safely assume that all iterations are proper.)

Note that separativity of the Q_α implies that all partial CS iterations satisfy the following (trivially equivalent) properties:

Fact 2.84. Let $\bar{P}$ be a topped partial CS iteration of length ε. Then:

1. Let H be P_ε-generic. Then $p\in H$ iff $p\restriction\alpha\in H_\alpha$ for all $\alpha<\varepsilon$.
2. For all $q,p\in P_\varepsilon$: If $q\restriction\alpha\leq^* p\restriction\alpha$ for each $\alpha<\varepsilon$, then $q\leq^* p$.
3. For all $q,p\in P_\varepsilon$: If $q\restriction\alpha\leq^* p\restriction\alpha$ for each $\alpha<\varepsilon$, then $q\not\perp p$.

We will be concerned with the following situation:

Assume that M is a nice candidate, $\bar{P}^M$ is (in M) a topped partial CS iteration of length ε (a limit ordinal in M), and $\bar{P}$ is (in V) a topless partial CS iteration of length $\varepsilon' := \sup(\varepsilon\cap M)$. (Recall that "$\mathrm{cf}(\varepsilon)=\omega$" is absolute between M and V, and that $\mathrm{cf}(\varepsilon)=\omega$ implies $\varepsilon'=\varepsilon$.) Moreover, assume that we already have a system of M-complete coherent[28] embeddings $i_\beta : P^M_\beta\to P_\beta$ for $\beta\in\varepsilon'\cap M=\varepsilon\cap M$. (Recall that any potential partial CS limit of $\bar{P}$ is a subforcing of the full CS limit $P^{\mathrm{CS}}_{\varepsilon'}$.) It is easy to see that there is only one possibility for an embedding $j : P^M_\varepsilon\to P^{\mathrm{CS}}_{\varepsilon'}$ (in fact, into any potential partial CS limit of $\bar{P}$) that extends the i_β's naturally:

[27] The reason for this requirement is briefly discussed in Section 2.6. Separativity, as well as the relations $\leq^*$ and $=^*$, are defined on page 36.

[28] I.e., they commute with the restriction maps: $i_\alpha(p\restriction\alpha)=i_\beta(p)\restriction\alpha$ for $\alpha<\beta$ and $p\in P^M_\beta$.

Definition 2.85. For a topped partial CS iteration $\bar{P}^M$ in M of length ε and a topless one $\bar{P}$ in V of length $\varepsilon' := \sup(\varepsilon \cap M)$ together with coherent embeddings i_β, we define $j : P^M_\varepsilon \to P^{\mathrm{CS}}_{\varepsilon'}$, the "*canonical extension*", in the obvious way: Given $p \in P^M_\varepsilon$, take the sequence of restrictions to M-ordinals, apply the functions i_β, and let $j(p)$ be the union of the resulting coherent sequence.

We do not claim that $j : P^M_\varepsilon \to P^{\mathrm{CS}}_{\varepsilon'}$ is M-complete.[29] In the following, we will construct partial CS limits $P_{\varepsilon'}$ such that $j : P^M_\varepsilon \to P_{\varepsilon'}$ is M-complete. (Obviously, one requirement for such a limit is that $j[P^M_\varepsilon] \subseteq P_{\varepsilon'}$.) We will actually define two versions: The almost FS ("almost finite support") and the almost CS ("almost countable support") limit.

Note that there is only one effect that the "top" of $\bar{P}^M$ (i.e., the forcing P^M_ε) has on the canonical extension j: It determines the domain of j. In particular it will generally depend on P^M_ε whether j is complete or not. Apart from that, the value of any given $j(p)$ does not depend on P^M_ε.

Instead of arbitrary systems of embeddings i_α, we will only be interested in "canonical" ones. We assume for notational convenience that Q^M_α is a subset of Q_α (this will naturally be the case in our application anyway).

Definition 2.86 (The canonical embedding)**.** Let $\bar{P}$ be a partial CS iteration in V and $\bar{P}^M$ a partial CS iteration in M, both topped and of length $\varepsilon \in M$. We construct by induction on $\alpha \in (\varepsilon + 1) \cap M$ the canonical M-complete embeddings $i_\alpha : P^M_\alpha \to P_\alpha$. More precisely: We try to construct them, but it is possible that the construction fails. If the construction succeeds, then we say that "$\bar{P}^M$ *(canonically) embeds into* $\bar{P}$", or "*the canonical embeddings work*", or just: "$\bar{P}$ *is over* $\bar{P}^M$", or "*over* P^M_ε".

- Let $\alpha = \beta + 1$. By induction hypothesis, i_β is M-complete, so a V-generic filter $H_\beta \subseteq P_\beta$ induces an M-generic filter $H^M_\beta := i^{-1}_\beta[H_\beta] \subseteq P^M_\beta$. We require that (in the H_β extension) the set $Q^M_\beta[H^M_\beta]$ is an $M[H^M_\beta]$-complete subforcing of $Q_\beta[H_\beta]$. In this case, we define i_α in the obvious way.

[29] For example, if $\varepsilon = \varepsilon' = \omega$ and if P^M_ω is the finite support limit of a nontrivial iteration, then $j : P^M_\omega \to P^{\mathrm{CS}}_\omega$ is not complete: For notational simplicity, assume that all Q^M_n are (forced to be) Boolean algebras. In M, let c_n be (a P^M_n-name for) a nontrivial element of Q^M_n (so $\neg c_n$, the Boolean complement, is also nontrivial). Let p_n be the P^M_n-condition $(c_0, \dots, c_{n-1})$, i.e., the truth value of "$c_m \in H(m)$ for all $m < n$". Let q_n be the P^M_{n+1}-condition $(c_0, \dots, c_{n-1}, \neg c_n)$, i.e., the truth value of "n is minimal with $c_n \notin H(n)$". In M, the set $A = \{q_n : n \in \omega\}$ is a maximal antichain in P^M_ω. Moreover, the sequence $(p_n)_{n \in \omega}$ is a decreasing coherent sequence, therefore $i_n(p_n)$ defines an element p_ω in P^{CS}_ω, which is clearly incompatible with all $j(q_n)$, hence $j[A]$ is not maximal.

- For α limit, let i_α be the canonical extension of the family $(i_\beta)_{\beta\in\alpha\cap M}$. We require that P_α contains the range of i_α, and that i_α is M-complete; otherwise the construction fails. (If $\alpha' := \sup(\alpha\cap M) < \alpha$, then i_α will actually be an M-complete map into $P_{\alpha'}$, assuming that the requirement is fulfilled.)

In this section we try to construct a partial CS iteration $\bar{P}$ (over a given $\bar{P}^M$) satisfying additional properties.

Remark 2.87. What is the role of $\varepsilon' := \sup(\varepsilon \cap M)$? When our inductive construction of $\bar{P}$ arrives at P_ε where $\varepsilon' < \varepsilon$, it would be too late[30] to take care of M-completeness of i_ε at this stage, even if all i_α work nicely for $\alpha \in \varepsilon \cap M$. Note that $\varepsilon' < \varepsilon$ implies that ε is uncountable in M, and that therefore $P^M_\varepsilon = \bigcup_{\alpha\in\varepsilon\cap M} P^M_\alpha$. So the natural extension j of the embeddings $(i_\alpha)_{\alpha\in\varepsilon\cap M}$ has range in $P_{\varepsilon'}$, which will be a complete subforcing of P_ε. So we have to ensure M-completeness already in the construction of $P_{\varepsilon'}$.

For now we just record:

Lemma 2.88. *Assume that we have topped iterations $\bar{P}^M$ (in M) of length ε and $\bar{P}$ (in V) of length $\varepsilon' := \sup(\varepsilon \cap M)$, and that for all $\alpha \in \varepsilon \cap M$ the canonical embedding $i_\alpha : P^M_\alpha \to P_\alpha$ works. Let $i_\varepsilon : P^M_\varepsilon \to P^{\mathrm{CS}}_{\varepsilon'}$ be the canonical extension.*

1. *If P^M_ε is (in M) a direct limit (which is always the case if ε has uncountable cofinality) then i_ε (might not work, but at least) has range in $P_{\varepsilon'}$ and preserves incompatibility.*
2. *If i_ε has a range contained in $P_{\varepsilon'}$ and maps predense sets $D \subseteq P^M_\varepsilon$ in M to predense sets $i_\varepsilon[D] \subseteq P_{\varepsilon'}$, then i_ε preserves incompatibility (and therefore works).*

Proof. (1) Since P^M_ε is a direct limit, the canonical extension i_ε has range in $\bigcup_{\alpha<\varepsilon'} P_\alpha$, which is subset of any partial CS limit $P_{\varepsilon'}$. Incompatibility in P^M_ε is the same as incompatibility in P^M_α for sufficiently large $\alpha \in \varepsilon \cap M$, so by assumption it is preserved by i_α and hence also by i_ε.

[30] For example: Let $\varepsilon = \omega_1$ and $\varepsilon' = \omega_1 \cap M$. Assume that $P^M_{\omega_1}$ is (in M) a (or: the unique) partial CS limit of a nontrivial iteration. Assume that we have a topless iteration $\bar{P}$ of length ε' in V such that the canonical embeddings work for all $\alpha \in \omega_1 \cap M$. If we set $P_{\varepsilon'}$ to be the full CS limit, then we cannot further extend it to any iteration of length ω_1 such that the canonical embedding i_{ω_1} works: Let p_α and q_α be as in footnote 29. In M, the set $A = \{q_\alpha : \alpha \in \omega_1\}$ is a maximal antichain, and the sequence $(p_\alpha)_{\alpha\in\omega_1}$ is a decreasing coherent sequence. But in V there is an element $p_{\varepsilon'} \in P^{\mathrm{CS}}_{\varepsilon'}$ with $p_{\varepsilon'}{\restriction}\alpha = j(p_\alpha)$ for all $\alpha \in \varepsilon \cap M$. This condition $p_{\varepsilon'}$ is clearly incompatible with all elements of $j[A] = \{j(q_\alpha) : \alpha \in \varepsilon \cap M\}$. Hence $j[A]$ is not maximal.

(2) Fix $p_1, p_2 \in P^M_\varepsilon$, and assume that their images are compatible in $P_{\varepsilon'}$; we have to show that they are compatible in P^M_ε. So fix a generic filter $H \subseteq P_{\varepsilon'}$ containing $i_\varepsilon(p_1)$ and $i_\varepsilon(p_2)$.

In M, we define the following set D:

$$D := \{q \in P^M_\varepsilon : (q \leq p_1 \wedge q \leq p_2) \text{ or } (\exists \alpha < \varepsilon : q{\restriction}\alpha \perp_{P^M_\alpha} p_1{\restriction}\alpha) \text{ or } \\ (\exists \alpha < \varepsilon : q{\restriction}\alpha \perp_{P^M_\alpha} p_2{\restriction}\alpha)\}.$$

Using Fact 2.84(3) it is easy to check that D is dense. Since i_ε preserves predensity, there is $q \in D$ such that $i_\varepsilon(q) \in H$. We claim that q is stronger than p_1 and p_2. Otherwise we would have without loss of generality $q{\restriction}\alpha \perp_{P^M_\alpha} p_1{\restriction}\alpha$ for some $\alpha < \varepsilon$. But the filter $H{\restriction}\alpha$ contains both $i_\alpha(q{\restriction}\alpha)$ and $i_\alpha(p_1{\restriction}\alpha)$, contradicting the assumption that i_α preserves incompatibility. □

2.3.C Almost finite support iterations

Recall Definition 2.85 (of the canonical extension) and the setup that was described there: We have to find a subset $P_{\varepsilon'}$ of $P^{\mathrm{CS}}_{\varepsilon'}$ such that the canonical extension $j : P^M_\varepsilon \to P_{\varepsilon'}$ is M-complete.

We now define the almost finite support limit. (The direct limit will in general not do, as it may not contain the range $j[P^M_\varepsilon]$. The almost finite support limit is the obvious modification of the direct limit, and it is the smallest partial CS limit $P_{\varepsilon'}$ such that $j[P^M_\varepsilon] \subseteq P_{\varepsilon'}$, and it indeed turns out to be M-complete as well.)

Definition 2.89. Let ε be a limit ordinal in M, and let $\varepsilon' := \sup(\varepsilon \cap M)$. Let $\bar{P}^M$ be a topped iteration in M of length ε, and let $\bar{P}$ be a topless iteration in V of length ε'. Assume that the canonical embeddings i_α work for all $\alpha \in \varepsilon \cap M = \varepsilon' \cap M$. Let i_ε be the canonical extension. We define the *almost finite support limit of $\bar{P}$ over $\bar{P}^M$* (or: almost FS limit) as the following subforcing $P_{\varepsilon'}$ of $P^{\mathrm{CS}}_{\varepsilon'}$:

$$P_{\varepsilon'} := \{\, q \wedge i_\varepsilon(p) \in P^{\mathrm{CS}}_{\varepsilon'} :\ p \in P^M_\varepsilon \text{ and } q \in P_\alpha \text{ for some } \alpha \in \varepsilon \cap M \\ \text{such that } q \leq_{P_\alpha} i_\alpha(p{\restriction}\alpha)\,\}.$$

Note that for $\mathrm{cf}(\varepsilon) > \omega$, the almost FS limit is equal to the direct limit, as each $p \in P^M_\varepsilon$ is in fact in P^M_α for some $\alpha \in \varepsilon \cap M$, so $i_\varepsilon(p) = i_\alpha(p) \in P_\alpha$.

Lemma 2.90. *Assume that $\bar{P}$ and $\bar{P}^M$ are as above and let $P_{\varepsilon'}$ be the almost FS limit. Then $\bar{P}^\frown P_{\varepsilon'}$ is a partial CS iteration, and i_ε works, i.e., i_ε is an M-complete embedding from P^M_ε to $P_{\varepsilon'}$. (As $P_{\varepsilon'}$ is a complete subforcing of P_ε, this also implies that i_ε is M-complete from P^M_ε to P_ε.)*

Proof. It is easy to see that $P_{\varepsilon'}$ is a partial CS limit and contains the range $i_\varepsilon[P^M_\varepsilon]$. We now show preservation of predensity; this implies M-completeness by Lemma 2.88.

Let $(p_j)_{j\in J} \in M$ be a maximal antichain in P^M_ε. (Since P^M_ε does not have to be ccc in M, J can have any cardinality in M.) Let $q\wedge i_\varepsilon(p)$ be a condition in $P_{\varepsilon'}$. (If $\varepsilon' < \varepsilon$, i.e., if $\mathrm{cf}(\varepsilon) > \omega$, then we can choose p to be the empty condition.) Fix $\alpha \in \varepsilon \cap M$ be such that $q \in P_\alpha$. Let H_α be P_α-generic and contain q, so $p{\restriction}\alpha$ is in H^M_α. Now in $M[H^M_\alpha]$ the set $\{p_j : j \in J, p_j \in P^M_\varepsilon/H^M_\alpha\}$ is predense in $P^M_\varepsilon/H^M_\alpha$ (since this is forced by the empty condition in P^M_α). In particular, p is compatible with some p_j, witnessed by $p' \leq p, p_j$ in $P^M_\varepsilon/H^M_\alpha$.

We can find $q' \leq_{P_\alpha} q$ deciding j and p'; since certainly $q' \leq^* i_\alpha(p'{\restriction}\alpha)$, we may assume even $\leq$ without loss of generality. Now $q' \wedge i_\varepsilon(p') \leq q \wedge i_\varepsilon(p)$ (since $q' \leq q$ and $p' \leq p$), and $q' \wedge i_\varepsilon(p') \leq i_\varepsilon(p_j)$ (since $p' \leq p_j$). $\square$

Definition and Claim 2.91. Let $\bar{P}^M$ be a topped partial CS iteration in M of length ε. We can construct by induction on $\beta \in \varepsilon + 1$ an *almost finite support iteration $\bar{P}$ over $\bar{P}^M$* (or: almost FS iteration) as follows:

1. As induction hypothesis we assume that the canonical embedding i_α works for all $\alpha \in \beta \cap M$. (So the notation $M[H^M_\alpha]$ makes sense.)

2. Let $\beta = \alpha + 1$. If $\alpha \in M$, then we can use any Q_α provided that (it is forced that) Q^M_α is an $M[H^M_\alpha]$-complete subforcing of Q_α. (If $\alpha \notin M$, then there is no restriction on Q_α.)

3. Let $\beta \in M$ and $\mathrm{cf}(\beta) = \omega$. Then P_β is the almost FS limit of $(P_\alpha, Q_\alpha)_{\alpha<\beta}$ over P^M_β.

4. Let $\beta \in M$ and $\mathrm{cf}(\beta) > \omega$. Then P_β is again the almost FS limit of $(P_\alpha, Q_\alpha)_{\alpha<\beta}$ over P^M_β (which also happens to be the direct limit).

5. For limit ordinals not in M, P_β is the direct limit.

So the claim includes that the resulting $\bar{P}$ is a (topped) partial CS iteration of length ε over $\bar{P}^M$ (i.e., the canonical embeddings i_α work for all $\alpha \in (\varepsilon + 1) \cap M$), where we only assume that the Q_α satisfy the obvious requirement given in (2). (Note that we can always find some suitable Q_α for $\alpha \in M$, for example we can just take Q^M_α itself.)

Proof. We have to show (by induction) that the resulting sequence $\bar{P}$ is a partial CS iteration, and that $\bar{P}^M$ embeds into $\bar{P}$. For successor cases, there is nothing to do. So assume that α is a limit. If P_α is a direct limit, it is trivially a partial CS limit; if P_α is an almost FS limit, then the easy part of Lemma 2.90 shows that it is a partial CS limit.

So it remains to show that for a limit $\alpha \in M$, the (naturally defined) embedding $i_\alpha : P^M_\alpha \to P_\alpha$ is M-complete. This was the main claim in Lemma 2.90. $\square$

The following lemma is natural and easy.

Lemma 2.92. *Assume that we construct an almost FS iteration $\bar{P}$ over $\bar{P}^M$ where each Q_α is (forced to be) ccc. Then P_ε is ccc (and in particular proper).*

Proof. We show that P_α is ccc by induction on $\alpha \leq \varepsilon$. For successors, we use that Q_α is ccc. For α of uncountable cofinality, we know that we took the direct limit coboundedly often (and all P_β are ccc for $\beta < \alpha$), so by a result of Solovay P_α is again ccc. For α a limit of countable cofinality not in M, just use that all P_β are ccc for $\beta < \alpha$, and the fact that P_α is the direct limit. This leaves the case that $\alpha \in M$ has countable cofinality, i.e., the P_α is the almost FS limit. Let $A \subseteq P_\alpha$ be uncountable. Each $a \in A$ has the form $q \wedge i_\alpha(p)$ for $p \in P^M_\alpha$ and $q \in \bigcup_{\gamma<\alpha} P_\gamma$. We can thin out the set A such that p are the same and all q are in the same P_γ. So there have to be compatible elements in A. $\square$

All almost FS iterations that we consider in this paper will satisfy the countable chain condition (and hence in particular be proper).

We will need a variant of this lemma for σ-centered forcing notions.

Lemma 2.93. *Assume that we construct an almost FS iteration $\bar{P}$ over $\bar{P}^M$ where only countably many Q_α are nontrivial (e.g., only those with $\alpha \in M$) and where each Q_α is (forced to be) σ-centered. Then P_ε is σ-centered as well.*

Proof. By induction: The direct limit of countably many σ-centered forcings is σ-centered, as is the almost FS limit of σ-centered forcings (to color $q \wedge i_\alpha(p)$, use p itself together with the color of q). $\square$

We will actually need two variants of the almost FS construction: Countably many models M^n; and starting the almost FS iteration with some α_0.

Firstly, we can construct an almost FS iteration not just over one iteration $\bar{P}^M$, but over an increasing chain of iterations. Analogously to Definition 2.89 and Lemma 2.90, we can show:

Lemma 2.94. *For each $n \in \omega$, let M^n be a nice candidate, and let $\bar{P}^n$ be a topped partial CS iteration in M^n of length*[31] *$\varepsilon \in M^0$ of countable cofinality, such that $M^m \in M^n$ and M^n thinks that $\bar{P}^m$ canonically embeds into $\bar{P}^n$,*

[31] Or only: $\varepsilon \in M^{n_0}$ for some n_0.

for all $m < n$. Let $\bar{P}$ be a topless iteration of length ε into which all $\bar{P}^n$ canonically embed.

Then we can define the almost FS limit P_ε over $(\bar{P}^n)_{n\in\omega}$ as follows: Conditions in P_ε are of the form $q \wedge i^n_\varepsilon(p)$ where $n \in \omega$, $p \in P^n_\varepsilon$, and $q \in P_\alpha$ for some $\alpha \in M^n \cap \varepsilon$ with $q \leq i^n_\alpha(p\restriction\alpha)$. Then P_ε is a partial CS limit over each $\bar{P}^n$.

As before, we get the following corollary:

Corollary 2.95. *Given M^n and $\bar{P}^n$ as above, we can construct a topped partial CS iteration $\bar{P}$ such that each $\bar{P}^n$ embeds M^n-completely into it; we can choose Q_α as we wish (subject to the obvious restriction that each Q^n_α is an $M^n[H^n_\alpha]$-complete subforcing). If we always choose Q_α to be ccc, then $\bar{P}$ is ccc; this is the case if we set Q_α to be the union of the (countable) sets Q^n_α.*

Proof. We can define P_α by induction. If $\alpha \in \bigcup_{n\in\omega} M^n$ has countable cofinality, then we use the almost FS limit as in Lemma 2.94. Otherwise we use the direct limit. If $\alpha \in M^n$ has uncountable cofinality, then $\alpha' := \sup(\alpha\cap M)$ is an element of M^{n+1}. In our induction we have already considered α' and have defined $P_{\alpha'}$ by Lemma 2.94 (applied to the sequence $(\bar{P}^{n+1}, \bar{P}^{n+2}, \ldots)$). This is sufficient to show that $i^n_\alpha : P^n_\alpha \to P_{\alpha'} \lessdot P_\alpha$ is M^n-complete. $\square$

Secondly, we can start the almost FS iteration after some α_0 (i.e., $\bar{P}$ is already given up to α_0, and we can continue it as an almost FS iteration up to ε), and get the same properties that we previously showed for the almost FS iteration, but this time for the quotient $P_\varepsilon/P_{\alpha_0}$. In more detail:

Lemma 2.96. *Assume that $\bar{P}^M$ is in M a (topped) partial CS iteration of length ε, and that $\bar{P}$ is in V a topped partial CS iteration of length α_0 over $\bar{P}^M\restriction\alpha_0$ for some $\alpha_0 \in \varepsilon\cap M$. Then we can extend $\bar{P}$ to a (topped) partial CS iteration of length ε over $\bar{P}^M$, as in the almost FS iteration (i.e., using the almost FS limit at limit points $\beta > \alpha_0$ with $\beta \in M$ of countable cofinality; and the direct limit everywhere else). We can use any Q_α for $\alpha \geq \alpha_0$ (provided Q^M_α is an $M[H^M_\alpha]$-complete subforcing of Q_α). If all Q_α are ccc, then P_{α_0} forces that $P_\varepsilon/H_{\alpha_0}$ is ccc (in particular proper); if moreover all Q_α are σ-centered and only countably many are nontrivial, then P_{α_0} forces that $P_\varepsilon/H_{\alpha_0}$ is σ-centered.*

2.3.D Almost countable support iterations

"Almost countable support iterations $\bar{P}$" (over a given iteration $\bar{P}^M$ in a candidate M) will have the following two crucial properties: There is a canonical M-complete embedding of $\bar{P}^M$ into $\bar{P}$, and $\bar{P}$ preserves a given random real (similar to the usual countable support iterations).

Definition and Claim 2.97. Let $\bar{P}^M$ be a topped partial CS iteration in M of length ε. We can construct by induction on $\beta \in \varepsilon + 1$ the *almost countable support iteration* $\bar{P}$ *over* $\bar{P}^M$ (or: almost CS iteration):

1. As induction hypothesis, we assume that the canonical embedding i_α works for every $\alpha \in \beta \cap M$. We set[32]
$$\delta := \min(M \setminus \beta), \quad \delta' := \sup(\alpha + 1 : \alpha \in \delta \cap M). \tag{2.98}$$
Note that $\delta' \leq \beta \leq \delta$.

2. Let $\beta = \alpha + 1$. We can choose any desired forcing Q_α; if $\beta \in M$ we of course require that
$$Q_\alpha^M \text{ is an } M[H_\alpha^M]\text{-complete subforcing of } Q_\alpha. \tag{2.99}$$
This defines P_β.

3. Let $\mathrm{cf}(\beta) > \omega$. Then P_β is the direct limit.

4. Let $\mathrm{cf}(\beta) = \omega$ and assume that $\beta \in M$ (so $M \cap \beta$ is cofinal in β and $\delta' = \beta = \delta$). We define $P_\beta = P_\delta$ as the union of the following two sets:
   - The almost FS limit of $(P_\alpha, Q_\alpha)_{\alpha<\delta}$, see Definition 2.89.
   - The set P_δ^{gen} of M-generic conditions $q \in P_\delta^{\mathrm{CS}}$, i.e., those which satisfy
   $$q \Vdash_{P_\delta^{\mathrm{CS}}} i_\delta^{-1}[H_{P_\delta^{\mathrm{CS}}}] \subseteq P_\delta^M \text{ is } M\text{-generic}.$$

5. Let $\mathrm{cf}(\beta) = \omega$ and assume that $\beta \notin M$ but $M \cap \beta$ is cofinal in β, so $\delta' = \beta < \delta$. We define $P_\beta = P_{\delta'}$ as the union of the following two sets:
   - The direct limit of $(P_\alpha, Q_\alpha)_{\alpha<\delta'}$.
   - The set $P_{\delta'}^{\mathrm{gen}}$ of M-generic conditions $q \in P_{\delta'}^{\mathrm{CS}}$, i.e., those which satisfy
   $$q \Vdash_{P_{\delta'}^{\mathrm{CS}}} i_\delta^{-1}[H_{P_{\delta'}^{\mathrm{CS}}}] \subseteq P_\delta^M \text{ is } M\text{-generic}.$$

   (Note that the M-generic conditions form an open subset of $P_\beta^{\mathrm{CS}} = P_{\delta'}^{\mathrm{CS}}$.)

6. Let $\mathrm{cf}(\beta) = \omega$ and $M \cap \beta$ not cofinal in β (so $\beta \notin M$). Then P_β is the full CS limit of $(P_\alpha, Q_\alpha)_{\alpha<\beta}$ (see Definition 2.81).

[32]So for successors $\beta \in M$, we have $\delta' = \beta = \delta$. For $\beta \in M$ limit, $\beta = \delta$ and δ' is as in Definition 2.85.

So the claim is that for every choice of Q_α (with the obvious restriction (2.99)), this construction always results in a partial CS iteration $\bar{P}$ over $\bar{P}^M$. The proof is a bit cumbersome; it is a variant of the usual proof that properness is preserved in countable support iterations (see e.g. [Gol93]).

We will use the following fact in M (for the iteration $\bar{P}^M$):

> Let $\bar{P}$ be a topped iteration of length ε. Let $\alpha_1 \le \alpha_2 \le \beta \le \varepsilon$. Let p_1 be a P_{α_1}-name for a condition in P_ε, and let D be an open dense set of P_β. Then there is a P_{α_2}-name p_2 for a condition in D such that the empty condition of P_{α_2} forces: $p_2 \le p_1{\restriction}\beta$ and: if p_1 is in $P_\varepsilon/H_{\alpha_2}$, then the condition p_2 is as well. (2.100)

(Proof: Work in the P_{α_2}-extension. We know that $p' := p_1 \restriction \beta$ is a P_β-condition. We now define p_2 as follows: If $p' \notin P_\beta/H_{\alpha_2}$ (which is equivalent to $p_1 \notin P_\epsilon/H_{\alpha_2}$), then we choose any $p_2 \le p'$ in D (which is dense in P_β). Otherwise (using that $D \cap P_\beta/H_{\alpha_2}$ is dense in P_β/H_{α_2}) we can choose $p_2 \le p'$ in $D \cap P_\beta/H_{\alpha_2}$.)

The following easy fact will also be useful:

> Let P be a subforcing of Q. We define $P{\restriction}p := \{r \in P : r \le p\}$. Assume that $p \in P$ and $P{\restriction}p = Q{\restriction}p$.
> Then for any P-name $\underset{\sim}{x}$ and any formula $\varphi(x)$ we have: $p \Vdash_P \varphi(\underset{\sim}{x})$ iff $p \Vdash_Q \varphi(\underset{\sim}{x})$. (2.101)

We now prove by induction on $\beta \le \varepsilon$ the following statement (which includes that the Definition and Claim 2.97 works up to β). Let δ, δ' be as in (2.98).

Lemma 2.102. *(a) The topped iteration $\bar{P}$ of length β is a partial CS iteration.*

(b) The canonical embedding $i_\delta : P^M_\delta \to P_{\delta'}$ works, hence also $i_\delta : P^M_\delta \to P_\delta$ works.

(c) Moreover, assume that

- *$\alpha \in M \cap \delta$,*
- *$\underset{\sim}{p} \in M$ is a P^M_α-name of a P^M_δ-condition,*
- *$q \in P_\alpha$ forces (in P_α) that $\underset{\sim}{p}{\restriction}\alpha[H^M_\alpha]$ is in H^M_α.*

Then there is a $q^+ \in P_{\delta'}$ (and therefore in P_β) extending q and forcing that $\underset{\sim}{p}[H^M_\alpha]$ is in H^M_δ.

Proof. First let us deal with the trivial cases. It is clear that we always get a partial CS iteration.

- Assume that $\beta = \beta_0 + 1 \in M$, i.e., $\delta = \delta' = \beta$. It is clear that i_β works. To get q^+, first extend q to some $q' \in P_{\beta_0}$ (by induction hypothesis), then define q^+ extending q' by $q^+(\beta_0) := \underset{\sim}{p}(\beta_0)$.
- If $\beta = \beta_0 + 1 \notin M$, there is nothing to do.
- Assume that $\mathrm{cf}(\beta) > \omega$ (whether $\beta \in M$ or not). Then $\delta' < \beta$. So $i_\delta : P^M_\delta \to P_{\delta'}$ works by induction, and similarly (c) follows from the inductive assumption. (Use the inductive assumption for $\beta = \delta'$; the δ that we got at that stage is the same as the current δ, and the q^+ we obtained at that stage will still satisfy all requirements at the current stage.)
- Assume that $\mathrm{cf}(\beta) = \omega$ and that $M \cap \beta$ is bounded in β. Then the proof is the same as in the previous case.

We are left with the cases corresponding to (4) and (5) of Definition 2.97: $\mathrm{cf}(\beta) = \omega$ and $M \cap \beta$ is cofinal in β. So either $\beta \in M$, then $\delta' = \beta = \delta$, or $\beta \notin M$, then $\delta' = \beta < \delta$ and $\mathrm{cf}(\delta) > \omega$.

We leave it to the reader to check that P_β is indeed a partial CS limit. The main point is to see that for all $p, q \in P_\beta$ the condition $q \wedge p$ is in P_β as well, provided $q \in P_\alpha$ and $q \leq p{\restriction}\alpha$ for some $\alpha < \beta$. If $p \in P^{\mathrm{gen}}_\beta$, then this follows because P^{gen}_β is open in P^{CS}_β; the other cases are immediate from the definition (by induction).

We now turn to claim (c). Assume $q \in P_\alpha$ and $\underset{\sim}{p} \in M$ are given, $\alpha \in M \cap \delta$.

Let $(D_n)_{n\in\omega}$ enumerate all dense sets of P^M_δ which lie in M, and let $(\alpha_n)_{n\in\omega}$ be a sequence of ordinals in M which is cofinal in β, where $\alpha_0 = \alpha$.

Using (2.100) in M, we can find a sequence $(\underset{\sim}{p}_n)_{n\in\omega}$ satisfying the following in M, for all $n > 0$:

- $\underset{\sim}{p}_0 = \underset{\sim}{p}$.
- $\underset{\sim}{p}_n \in M$ is a $P^M_{\alpha_n}$-name of a P^M_δ-condition in D_n.
- $\Vdash_{P^M_{\alpha_n}} \underset{\sim}{p}_n \leq_{P^M_\delta} \underset{\sim}{p}_{n-1}$.
- $\Vdash_{P^M_{\alpha_n}}$ If $\underset{\sim}{p}_{n-1}{\restriction}\alpha_n \in H^M_{\alpha_n}$, then $\underset{\sim}{p}_n{\restriction}\alpha_n \in H^M_{\alpha_n}$ as well.

Using the inductive assumption for the α_n's, we can now find a sequence $(q_n)_{n\in\omega}$ of conditions satisfying the following:

- $q_0 = q$, $q_n \in P_{\alpha_n}$.
- $q_n{\restriction}\alpha_{n-1} = q_{n-1}$.
- $q_n \Vdash_{P_{\alpha_n}} \underset{\sim}{p}_{n-1}{\restriction}\alpha_n \in H^M_{\alpha_n}$, so also $\underset{\sim}{p}_n{\restriction}\alpha_n \in H^M_{\alpha_n}$.

Let $q^+ \in P^{\mathrm{CS}}_\beta$ be the union of the q_n. Then for all n:

1. $q_n \Vdash_{P^{\mathrm{CS}}_\beta} \underset{\sim}{p}_n{\restriction}\alpha_n \in H^M_{\alpha_n}$, so also q^+ forces this.
(Using induction on n.)
2. For all n and all $m \geq n$: $q^+ \Vdash_{P^{\mathrm{CS}}_\beta} \underset{\sim}{p}_m{\restriction}\alpha_m \in H^M_{\alpha_m}$, so also $\underset{\sim}{p}_n{\restriction}\alpha_m \in H^M_{\alpha_m}$.
(As $\underset{\sim}{p}_m \leq \underset{\sim}{p}_n$.)
3. $q^+ \Vdash_{P^{\mathrm{CS}}_\beta} \underset{\sim}{p}_n \in H^M_\delta$.
(Recall that P^{CS}_β is separative, see Fact 2.83. So $i_\delta(\underset{\sim}{p}_n) \in H_\delta$ iff $i_{\alpha_n}(\underset{\sim}{p}{\restriction}\alpha_m) \in H_{\alpha_m}$ for all large m.)

As $q^+ \Vdash_{P^{\mathrm{CS}}_\beta} \underset{\sim}{p}_n \in D_n \cap H^M_\delta$, we conclude that $q^+ \in P^{\mathrm{gen}}_\beta$ (using Lemma 2.88, applied to P^{CS}_β). In particular, P^{gen}_β is dense in P_β: Let $q \wedge i_\delta(p)$ be an element of the almost FS limit; so $q \in P_\alpha$ for some $\alpha < \beta$. Now find a generic q^+ extending q and stronger than $i_\delta(p)$, then $q^+ \leq q \wedge i_\delta(p)$.

It remains to show that i_δ is M-complete. Let $A \in M$ be a maximal antichain of P^M_δ, and $p \in P_\beta$. Assume towards a contradiction that p forces in P_β that $i^{-1}_\delta[H_\beta]$ does not intersect A in exactly one point.

Since P^{gen}_β is dense in P_β, we can find some $q \leq p$ in P^{gen}_β. Let

$$P' := \{r \in P^{\mathrm{CS}}_\beta : r \leq q\} = \{r \in P_\beta : r \leq q\},$$

where the equality holds because P^{gen}_β is open in P^{CS}_β.

Let Γ be the canonical name for a P'-generic filter, i.e.: $\Gamma := \{(\check{r}, r) : r \in P'\}$. Let R be either P^{CS}_β or P_β. We write $\langle\Gamma\rangle_R$ for the filter generated by Γ in R, i.e., $\langle\Gamma\rangle_R := \{r \in R : (\exists r' \in \Gamma)\ r' \leq r\}$. So

$$q \Vdash_R H_R = \langle\Gamma\rangle_R. \tag{2.103}$$

We now see that the following hold:

– $q \Vdash_{P_\beta} i^{-1}_\delta[H_{P_\beta}]$ does not intersect A in exactly one point. (By assumption.)

– $q \Vdash_{P_\beta} i^{-1}_\delta[\langle\Gamma\rangle_{P_\beta}]$ does not intersect A in exactly one point. (By (2.103).)

– $q \Vdash_{P^{\mathrm{CS}}_\beta} i^{-1}_\delta[\langle\Gamma\rangle_{P_\beta}]$ does not intersect A in exactly one point. (By (2.101).)

– $q \Vdash_{P^{\mathrm{CS}}_\beta} i_\delta^{-1}[\langle\Gamma\rangle_{P^{\mathrm{CS}}_\beta}]$ does not intersect A in exactly one point. (Because i_δ maps A into $P_\beta \subseteq P^{\mathrm{CS}}_\beta$, so $A \cap i_\delta^{-1}[\langle Y\rangle_{P_\beta}] = A \cap i_\delta^{-1}[\langle Y\rangle_{P^{\mathrm{CS}}_\beta}]$ for all Y.)

– $q \Vdash_{P^{\mathrm{CS}}_\beta} i_\delta^{-1}[H_{P^{\mathrm{CS}}_\beta}]$ does not intersect A in exactly one point. (Again by (2.103).)

But this, according to the definition of P^{gen}_β, implies $q \notin P^{\mathrm{gen}}_\beta$, a contradiction. □

We can also show that the almost CS iteration of proper forcings Q_α is proper. (We do not really need this fact, as we could allow non-proper iterations in our preparatory forcing, see Section 2.6.A(4). In some sense, M-completeness replaces properness, so the proof of M-completeness was similar to the "usual" proof of properness.)

Lemma 2.104. *Assume that in Definition 2.97, every Q_α is (forced to be) proper. Then also each P_δ is proper.*

Proof. By induction on $\delta \le \varepsilon$ we prove that for all $\alpha < \delta$ the quotient P_δ/H_α is (forced to be) proper. We use the following facts about properness:

If P is proper and P forces that Q is proper, then $P * Q$ is proper. (2.105)

If $\bar{P}$ is an iteration of length ω and if each Q_n is forced to be proper, then the inverse limit P_ω is proper, as are all quotients P_ω/H_n. (2.106)

If $\bar{P}$ is an iteration of length δ with $\mathrm{cf}(\delta) > \omega$, and if all quotients P_β/H_α (for $\alpha < \beta < \delta$) are forced to be proper, then the direct limit P_δ is proper, as are all quotients P_δ/H_α. (2.107)

If δ is a successor, then our inductive claim easily follows from the inductive assumption together with (2.105).

Let δ be a limit of countable cofinality, say $\delta = \sup_n \delta_n$. Define an iteration $\bar{P}'$ of length ω with $Q'_n := P_{\delta_{n+1}}/H_{\delta_n}$. (Each Q'_n is proper, by inductive assumption.) There is a natural forcing equivalence between P^{CS}_δ and P'^{CS}_ω, the full CS limit of $\bar{P}'$.

Let $N \prec H(\chi^*)$ contain $\bar{P}, P_\delta, \bar{P}', M, \bar{P}^M$. Let $p \in P_\delta \cap N$. Without loss of generality $p \in P^{\mathrm{gen}}_\delta$. So below p we can identify P_δ with P^{CS}_δ and hence with P'^{CS}_ω; now apply (2.106).

The case of uncountable cofinality is similar, using (2.107) instead. □

Recall the definition of $\sqsubset_n$ and $\sqsubset$ from Definition 2.29, the notion of (quick) interpretation Z^* (of a name $\underset{\sim}{Z}$ of a code for a null set) and the definition of local preservation of randoms from Definition 2.53. Recall that we have seen in Corollaries 2.54 and 2.76:

Lemma 2.108.
- *If Q^M is an ultralaver forcing in M and r a real, then there is an ultralaver forcing Q over Q^M locally preserving randomness of r over M.*
- *If Q^M is a Janus forcing in M and r a real, then there is a Janus forcing Q over Q^M locally preserving randomness of r over M.*

We will prove the following preservation theorem:

Lemma 2.109. *Let $\bar{P}$ be an almost CS iteration (of length ε) over $\bar{P}^M$, r random over M, and $p \in P^M_\varepsilon$. Assume that each P_α forces that Q_α locally preserves randomness of r over $M[H^M_\alpha]$. Then there is some $q \leq p$ in P_ε forcing that r is random over $M[H^M_\varepsilon]$.*

What we will actually need is the following variant:

Lemma 2.110. *Assume that $\bar{P}^M$ is in M a topped partial CS iteration of length ε, and we already have some topped partial CS iteration $\bar{P}$ over $\bar{P}^M \restriction \alpha_0$ of length $\alpha_0 \in M \cap \varepsilon$. Let $\underset{\sim}{r}$ be a P_{α_0}-name of a random real over $M[H^M_{\alpha_0}]$. Assume that we extend $\bar{P}$ to length ε as an almost CS iteration[33] using forcings Q_α which locally preserve the randomness of $\underset{\sim}{r}$ over $M[H^M_\alpha]$, witnessed by a sequence $(D^{Q^M_\alpha}_k)_{k\in\omega}$. Let $p \in P^M_\varepsilon$. Then we can find a $q \leq p$ in P_ε forcing that $\underset{\sim}{r}$ is random over $M[H^M_\varepsilon]$.*

Actually, we will only prove the two previous lemmas under the following additional assumption (which is enough for our application, and saves some unpleasant work). This additional assumption is not really necessary; without it, we could use the method of [GK06] for the proof.

Assumption 2.111.
- For each $\alpha \in M \cap \varepsilon$, ($P^M_\alpha$ forces that) Q^M_α is either trivial[34] or adds a new ω-sequence of ordinals. Note that in the latter case we can assume without loss of generality that $\bigcap_{n\in\omega} D^{Q^M_\alpha}_n = \emptyset$ (and, of course, that the $D^{Q^M_\alpha}_n$ are decreasing).

[33] Of course our official definition of almost CS iteration assumes that we start the construction at 0, so we modify this definition in the obvious way.

[34] More specifically, $Q^M_\alpha = \{\emptyset\}$.

- Moreover, we assume that already in M there is a set $T \subseteq \varepsilon$ such that P^M_α forces: Q^M_α is trivial iff $\alpha \in T$. (So whether Q^M_α is trivial or not does not depend on the generic filter below α, it is already decided in the ground model.)

The result will follow as a special case of the following lemma, which we prove by induction on β. (Note that this is a refined version of the proof of Lemma 2.102 and similar to the proof of the preservation theorem in [Gol93, 5.13].)

Definition 2.112. Under the assumptions of Lemma 2.110 and Assumption 2.111, let $\underset{\sim}{Z}$ be a P_δ-name, $\alpha_0 \leq \alpha < \delta$, and let $\bar{p} = (p^k)_{k\in\omega}$ be a sequence of P_α-names of conditions in P_δ/H_α. Let Z^* be a P_α-name.

We say that $(\bar{p}, Z^*)$ is a *quick* interpretation of $\underset{\sim}{Z}$ if $\bar{p}$ interprets $\underset{\sim}{Z}$ as Z^* (i.e., P_α forces that p^k forces $\underset{\sim}{Z}\restriction k = Z^*\restriction k$ for all k), and moreover:

> Letting $\beta \geq \alpha$ be minimal with Q^M_β nontrivial (if such β exists): P_β forces that the sequence $(p^k(\beta))_{k\in\omega}$ is quick in Q^M_β, i.e., $p^k(\beta) \in D_k^{Q^M_\beta}$ for all k.

It is easy to see that:

$$\text{For every name } \underset{\sim}{Z} \text{ there is a quick interpretation } (\bar{p}, Z^*). \tag{2.113}$$

Lemma 2.114. *Under the same assumptions as above, let β, δ, δ' be as in* (2.98) *(so in particular we have $\delta' \leq \beta \leq \delta \leq \varepsilon$).*
Assume that

- $\alpha \in M \cap \delta$ $(= M \cap \beta)$ *and* $\alpha \geq \alpha_0$ *(so* $\alpha < \delta'$*),*
- $p \in M$ *is a* P^M_α*-name of a* P^M_δ*-condition,*
- $\underset{\sim}{Z} \in M$ *is a* P^M_δ*-name of a code for null set,*
- $Z^* \in M$ *is a* P^M_α*-name of a code for a null set,*
- P^M_α *forces:* $\bar{p} = (p^k)_{k\in\omega} \in M$ *is a quick sequence in* P^M_δ/H^M_α *interpreting* $\underset{\sim}{Z}$ *as* Z^* *(as in Definition 2.112),*
- P^M_α *forces: if* $p\restriction\alpha \in H^M_\alpha$*, then* $p^0 \leq p$*,*
- $q \in P_\alpha$ *forces* $p\restriction\alpha \in H^M_\alpha$*,*
- q *forces that* r *is random over* $M[H^M_\alpha]$*, so in particular there is (in* V*) a* P_α*-name* $\underset{\sim}{c}_0$ *below* q *for the minimal* c *with* $Z^* \sqsubset_c r$*.*

Then *there is a condition $q^+ \in P_{\delta'}$, extending q, and forcing the following:*

- $p \in H^M_\delta$,
- *r is random over $M[H^M_\delta]$,*
- $\underaccent{\tilde}{Z} \sqsubset_{c_0} r$.

We actually claim a slightly stronger version, where instead of Z^* and $\underaccent{\tilde}{Z}$ we have finitely many codes for null sets and names of codes for null sets, respectively. We will use this stronger claim as inductive assumption, but for notational simplicity we only prove the weaker version; it is easy to see that the weaker version implies the stronger version.

Proof. ***The nontrivial successor case:*** $\beta = \gamma + 1 \in M$.

If Q^M_γ is trivial, there is nothing to do.

Now let $\gamma_0 \geq \alpha$ be minimal with $Q^M_{\gamma_0}$ nontrivial. We will distinguish two cases: $\gamma = \gamma_0$ and $\gamma > \gamma_0$.

Consider first the case that $\gamma = \gamma_0$. Work in $V[H_\gamma]$ where $q \in H_\gamma$. Note that $M[H^M_\gamma] = M[H^M_\alpha]$. So r is random over $M[H^M_\gamma]$, and $(p^k(\gamma))_{k\in\omega}$ quickly interprets $\underaccent{\tilde}{Z}$ as Z^* in Q^M_γ. Now let $q^+ {\restriction} \gamma = q$, and use the fact that Q_γ locally preserves randomness to find $q^+(\gamma) \leq p^0(\gamma)$.

Next consider the case that Q^M_γ is nontrivial and $\gamma \geq \gamma_0 + 1$. Again work in $V[H_\gamma]$. Let k^* be maximal with $p^{k^*} {\restriction} \gamma \in H^M_\gamma$. (This k^* exists, since the sequence $(p^k)_{k\in\omega}$ was quick, so there is even a k with $p^k {\restriction} (\gamma_0 + 1) \notin H^M_{\gamma_0+1}$.) Consider $\underaccent{\tilde}{Z}$ as a Q^M_γ-name, and (using (2.113)) find a quick interpretation Z' of $\underaccent{\tilde}{Z}$ witnessed by a sequence starting with $p^{k^*}(\gamma)$. In $M[H^M_\alpha]$, Z' is now a P^M_γ / H^M_α-name. Clearly, the sequence $(p^k {\restriction} \gamma)_{k\in\omega}$ is a quick sequence interpreting Z' as Z^*. (Use the fact that $p^k {\restriction} \gamma$ forces $k^* \geq k$.)
Using the induction hypothesis, we can first extend q to a condition $q' \in P_\gamma$ and then (again by our assumption that Q_γ locally preserves randomness) to a condition $q^+ \in P_{\gamma+1}$.

The nontrivial limit case: $M \cap \beta$ unbounded in β, i.e., $\delta' = \beta$. (This deals with cases (4) and (5) in Definition 2.97. In case (4) we have $\beta \in M$, i.e., $\beta = \delta$; in case (5) we have $\beta \notin M$ and $\beta < \delta$.)

Let $\alpha = \delta_0 < \delta_1 < \cdots$ be a sequence of M-ordinals cofinal in $M \cap \delta' = M \cap \delta$. We may assume[35] that each $Q^M_{\delta_n}$ is nontrivial.

Let $(\underaccent{\tilde}{Z}_n)_{n\in\omega}$ be a list of all P^M_δ-names in M of codes for null sets (starting with our given null set $\underaccent{\tilde}{Z} = \underaccent{\tilde}{Z}_0$). Let $(E_n)_{n\in\omega}$ enumerate all open dense sets

[35]If from some γ on all Q^M_ζ are trivial, then $P^M_\delta = P^M_\gamma$, so by induction there is nothing to do. If Q^M_α itself is trivial, then we let $\delta_0 := \min\{\zeta : Q^M_\zeta \text{ nontrivial}\}$ instead.

of P^M_δ from M, without loss of generality[36] we can assume that:

$$E_n \text{ decides } \underset{\sim}{Z}_0{\restriction}n,\ \ldots,\ \underset{\sim}{Z}_n{\restriction}n. \tag{2.115}$$

We write p^k_0 for p^k, and $Z_{0,0}$ for Z^*; as mentioned above, $\underset{\sim}{Z} = \underset{\sim}{Z}_0$.

By induction on n we can now find a sequence $\bar{p}_n = (p^k_n)_{k\in\omega}$ and $P^M_{\delta_n}$-names $Z_{i,n}$ for $i \in \{0,\ldots,n\}$ satisfying the following:

1. $P^M_{\delta_n}$ forces that $p^0_n \leq p^k_{n-1}$ whenever $p^k_{n-1} \in P^M_\delta/H^M_{\delta_n}$.
2. $P^M_{\delta_n}$ forces that $p^0_n \in E_n$. (Clearly $E_n \cap P^M_\delta/H^M_{\delta_n}$ is a dense set.)
3. $\bar{p}_n \in M$ is a $P^M_{\delta_n}$-name for a quick sequence interpreting $(\underset{\sim}{Z}_0,\ldots,\underset{\sim}{Z}_n)$ as $(Z_{0,n},\ldots,Z_{n,n})$ (in $P^M_\delta/H^M_{\delta_n}$), so $Z_{i,n}$ is a $P^M_{\delta_n}$-name of a code for a null set, for $0 \leq i \leq n$.

Note that this implies that the sequence $(p^k_{n-1}{\restriction}\delta_n)$ is (forced to be) a quick sequence interpreting $(Z_{0,n},\ldots,Z_{n-1,n})$ as $(Z_{0,n-1},\ldots,Z_{n-1,n-1})$.

Using the induction hypothesis, we now define a sequence $(q_n)_{n\in\omega}$ of conditions $q_n \in P_{\delta_n}$ and a sequence $(c_n)_{n\in\omega}$ (where c_n is a P_{δ_n}-name) such that (for $n > 0$) q_n extends q_{n-1} and forces the following:

- $p^0_{n-1}{\restriction}\delta_n \in H^M_{\delta_n}$.
- Therefore, $p^0_n \leq p^0_{n-1}$.
- r is random over $M[H^M_{\delta_n}]$.
- Let c_n be the least c such that $Z_{n,n} \sqsubset_c r$.
- $Z_{i,n} \sqsubset_{c_i} r$ for $i = 0,\ldots,n-1$.

Now let $q = \bigcup_n q_n \in P^{\mathrm{CS}}_{\delta'}$. As in Lemma 2.102 it is easy to see that $q \in P^{\mathrm{gen}}_{\delta'} \subseteq P_{\delta'}$. Moreover, by (2.115) we get that q forces that $\underset{\sim}{Z}_i = \lim_n Z_{i,n}$. Since each set $C_{c,r} := \{x : x \sqsubset_c r\}$ is closed, this implies that q forces $\underset{\sim}{Z}_i \sqsubset_{c_i} r$, in particular $\underset{\sim}{Z} = \underset{\sim}{Z}_0 \sqsubset_{c_0} r$.

The trivial cases: In all other cases, $M \cap \beta$ is bounded in β, so we already dealt with everything at stage $\beta_0 := \sup(\beta \cap M)$. Note that δ'_0 and δ_0 used at stage β_0 are the same as the current δ' and δ. $\square$

[36]well, if we just enumerate a basis of the open sets instead of all of them...

2.4 The forcing construction

In this section we describe a σ-closed "preparatory" forcing notion $\mathbb{R}$; the generic filter will define a "generic" forcing iteration $\bar{\mathbf{P}}$, so elements of $\mathbb{R}$ will be approximations to such an iteration. In Section 2.5 we will show that the forcing $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces BC and dBC.

From now on, we assume CH in the ground model.

2.4.A Alternating iterations, canonical embeddings and the preparatory forcing $\mathbb{R}$

The preparatory forcing $\mathbb{R}$ will consist of pairs $(M, \bar{P})$, where M is a countable model and $\bar{P} \in M$ is an iteration of ultralaver and Janus forcings.

Definition 2.116. An *alternating iteration*[37] is a topped partial CS iteration $\bar{P}$ of length ω_2 satisfying the following:

- Each P_α is proper.[38]
- For α even, either both Q_α and $Q_{\alpha+1}$ are (forced by the empty condition to be) trivial,[39] or P_α forces that Q_α is an ultralaver forcing adding the generic real $\bar{\ell}_\alpha$, and $P_{\alpha+1}$ forces that $Q_{\alpha+1}$ is a Janus forcing based on $\bar{\ell}^*_\alpha$ (where $\bar{\ell}^*$ is defined from $\bar{\ell}$ as in Lemma 2.26).

We will call an even index an "ultralaver position" and an odd one a "Janus position".

As in any partial CS iteration, each P_δ for $\mathrm{cf}(\delta) > \omega$ (and in particular P_{ω_2}) is a direct limit.

Recall that in Definition 2.86 we have defined the notion "$\bar{P}^M$ canonically embeds into $\bar{P}$" for nice candidates M and iterations $\bar{P} \in V$ and $\bar{P}^M \in M$. Since our iterations now have length ω_2, this means that the canonical embedding works up to and including[40] ω_2.

In the following, we will use pairs $x = (M^x, \bar{P}^x)$ as conditions in a forcing, where $\bar{P}^x$ is an alternating iteration in the nice candidate M^x. We will adapt our notation accordingly: Instead of writing M, $\bar{P}^M$, P^M_α H^M_α (the induced filter), Q^M_α, etc., we will write M^x, $\bar{P}^x$, P^x_α, H^x_α, Q^x_α, etc. Instead of "$\bar{P}^x$

[37] See Section 2.6 for possible variants of this definition.

[38] This does not seem to be necessary, see Section 2.6, but it is easy to ensure and might be comforting to some of the readers and/or authors.

[39] For definiteness, let us agree that the trivial forcing is the singleton $\{\emptyset\}$.

[40] This is stronger than to require that the canonical embedding works for every $\alpha \in \omega_2 \cap M$, even though both P_{ω_2} and $P^M_{\omega_2}$ are just direct limits; see footnote 30.

canonically embeds into $\bar{P}$" we will say[41] "x canonically embeds into $\bar{P}$" or "$(M^x, \bar{P}^x)$ canonically embeds into $\bar{P}$" (which is a more exact notation anyway, since the test whether the embedding is M^x-complete uses both M^x and $\bar{P}^x$, not just $\bar{P}^x$).

The following rephrases Definition 2.86 of a canonical embedding in our new notation, taking into account that:

$$\mathbb{L}_{\bar{D}^x} \text{ is an } M^x\text{-complete subforcing of } \mathbb{L}_{\bar{D}} \quad \text{iff} \quad \bar{D} \text{ extends } \bar{D}^x$$

(see Lemma 2.8).

Fact 2.117. $x = (M^x, \bar{P}^x)$ canonically embeds into $\bar{P}$, if (inductively) for all $\beta \in \omega_2 \cap M^x \cup \{\omega_2\}$ the following holds:

- Let $\beta = \alpha + 1$ for α even (i.e., an ultralaver position). Then either Q^x_α is trivial (and Q_α can be trivial or not), or we require that (P_α forces that) the $V[H_\alpha]$-ultrafilter system $\bar{D}$ used for Q_α extends the $M^x[H^x_\alpha]$-ultrafilter system $\bar{D}^x$ used for Q^x_α.
- Let $\beta = \alpha + 1$ for α odd (i.e., a Janus position). Then either Q^x_α is trivial, or we require that (P_α forces that) the Janus forcing Q^x_α is an $M^x[H^x_\alpha]$-complete subforcing of the Janus forcing Q_α.
- Let β be a limit. Then the canonical extension $i_\beta : P^x_\beta \to P_\beta$ is M^x-complete. (The canonical extension was defined in Definition 2.85.)

Fix a sufficiently large regular cardinal χ^* (see Remark 2.79).

Definition 2.118. The "*preparatory forcing*" $\mathbb{R}$ consists of pairs

$$x = (M^x, \bar{P}^x)$$

such that $M^x \in H(\chi^*)$ is a nice candidate (containing ω_2), and $\bar{P}^x$ is in M^x an alternating iteration (in particular topped and of length ω_2).
We define y to be stronger than x (in symbols: $y \leq_{\mathbb{R}} x$), if the following holds: either $x = y$, or:

- $M^x \in M^y$ and M^x is countable in M^y.
- M^y thinks that $(M^x, \bar{P}^x)$ canonically embeds into $\bar{P}^y$.

[41] Note the linguistic asymmetry here: A symmetric and more verbose variant would say "$x = (M^x, \bar{P}^x)$ canonically embeds into $(V, \bar{P})$".

Note that this order on $\mathbb{R}$ is transitive.

We will sometimes write $i_{x,y}$ for the canonical embedding (in M^y) from $P^x_{\omega_2}$ to $P^y_{\omega_2}$.

There are several variants of this definition which result in equivalent forcing notions. We will briefly come back to this in Section 2.6.

The following is trivial by elementarity:

Fact 2.119. Assume that $\bar{P}$ is an alternating iteration (in V), that $x = (M^x, \bar{P}^x) \in \mathbb{R}$ canonically embeds into $\bar{P}$, and that $N \prec H(\chi^*)$ contains x and $\bar{P}$. Let $y = (M^y, \bar{P}^y)$ be the ord-collapse of $(N, \bar{P})$. Then $y \in \mathbb{R}$ and $y \leq x$.

This fact will be used, for example, to get from the following Lemma 2.120 to Corollary 2.121.

Lemma 2.120. *Given $x \in \mathbb{R}$, there is an alternating iteration $\bar{P}$ such that x canonically embeds into $\bar{P}$.*

Proof. For the proof, we use either of the partial CS constructions introduced in the previous chapter (i.e., an almost CS iteration or an almost FS iteration over $\bar{P}^x$). The only thing we have to check is that we can indeed choose Q_α that satisfy the definition of an alternating iteration (i.e., as ultralaver or Janus forcings) and such that Q^x_α is M^x-complete in Q_α.

In the ultralaver case we arbitrarily extend $\bar{D}^x$ to an ultrafilter system $\bar{D}$, which is justified by Lemma 2.8.

In the Janus case, we take $Q_\alpha := Q^x_\alpha$ (this works by Fact 2.62). Alternatively, we could extend Q^x_α to a random forcing (using Lemma 2.74). $\square$

Corollary 2.121. *Given $x \in \mathbb{R}$ and an HCON object $b \in H(\chi^*)$ (e.g., a real or an ordinal), there is a $y \leq x$ such that $b \in M^y$.*

What we will actually need are the following three variants:

Lemma 2.122.
1. *Given $x \in \mathbb{R}$ there is a σ-centered alternating iteration $\bar{P}$ above x.*
2. *Given a decreasing sequence $\bar{x} = (x_n)_{n\in\omega}$ in $\mathbb{R}$, there is an alternating iteration $\bar{P}$ such that each x_n embeds into $\bar{P}$. Moreover, we can assume that for all Janus positions β, the Janus [42] forcing Q_β is (forced to be) the union of the $Q^{x_n}_\beta$, and that for all limits α, the forcing P_α is the almost FS limit over $(x_n)_{n\in\omega}$ (as in Corollary 2.95).*

[42] If all $Q^{x_n}_\beta$ are trivial, then we may also set Q_β to be the trivial forcing, which is formally not a Janus forcing.

3. *Let $x, y \in \mathbb{R}$. Let j^x be the transitive collapse of M^x, and define j^y analogously. Assume that $j^x[M^x] = j^y[M^y]$, that $j^x(\bar{P}^x) = j^y(\bar{P}^y)$ and that there are $\alpha_0 \le \alpha_1 < \omega_2$ such that:*
   - *$M^x \cap \alpha_0 = M^y \cap \alpha_0$ (and thus $j^x \restriction \alpha_0 = j^y \restriction \alpha_0$).*
   - *$M^x \cap [\alpha_0, \omega_2) \subseteq [\alpha_0, \alpha_1)$.*
   - *$M^y \cap [\alpha_0, \omega_2) \subseteq [\alpha_1, \omega_2)$.*

   Then there is an alternating iteration $\bar{P}$ such that both x and y canonically embed into it.

Proof. For (1), use an almost FS iteration. We only use the coordinates in M^x, and use the (countable!) Janus forcings $Q_\alpha := Q^x_\alpha$ for all Janus positions $\alpha \in M^x$ (see Fact 2.62). Ultralaver forcings are σ-centered anyway, so P_ε will be σ-centered, by Lemma 2.93.

For (2), use the almost FS iteration over the sequence $(x_n)_{n\in\omega}$ as in Corollary 2.95, and at Janus positions α set Q_α to be the union of the $Q^{x_n}_\alpha$. (By Fact 2.62, $Q^{x_n}_\alpha$ is M^{x_n}-complete in Q_α, so Corollary 2.95 can be applied here.)

For (3), we again use an almost FS construction. This time we start with an almost FS construction over x up to α_1, and then continue with an almost FS construction over y. $\square$

As above, Fact 2.119 gives us the following consequences:

Corollary 2.123. 1. *$\mathbb{R}$ is σ-closed. Hence $\mathbb{R}$ does not add new HCON objects (and in particular: no new reals).*

2. *$\mathbb{R}$ forces that the generic filter $G \subseteq \mathbb{R}$ is σ-directed, i.e., for every countable subset B of G there is a $y \in G$ stronger than each element of B.*

3. *$\mathbb{R}$ forces CH. (Since we assume CH in V.)*

4. *Given a decreasing sequence $\bar{x} = (x_n)_{n\in\omega}$ in $\mathbb{R}$ and any HCON object $b \in H(\chi^*)$, there is a $y \in \mathbb{R}$ such that*
   - *$y \le x_n$ for all n,*
   - *M^y contains b and the sequence $\bar{x}$,*
   - *for all Janus positions β, M^y thinks that the Janus forcing Q^y_β is (forced to be) the union of the $Q^{x_n}_\beta$,*
   - *for all limits α, M^y thinks that P^y_α is the almost FS limit[43] over $(x_n)_{n\in\omega}$ (of $(P^y_\beta)_{\beta<\alpha}$).*

[43]constructed in Lemma 2.94

Proof. Item (4) directly follows from Lemma 2.122(2) and Fact 2.119.

Item (1) is a special case of (4), and (2) and (3) are trivial consequences of (1). $\square$

Another consequence of Lemma 2.122 is:

Lemma 2.124. *The forcing notion $\mathbb{R}$ is $\aleph_2$-cc.*

Proof. Recall that we assume that V (and hence $V[G]$) satisfies CH.

Assume towards a contradiction that $(x_i : i < \omega_2)$ is an antichain. Using CH we may without loss of generality assume that for each $i \in \omega_2$ the transitive collapse of $(M^{x_i}, \bar{P}^{x_i})$ is the same. Set $L_i := M^{x_i} \cap \omega_2$. Using the Δ-lemma we find some uncountable $I \subseteq \omega_2$ such that the L_i for $i \in I$ form a Δ-system with root L. Set $\alpha_0 = \sup(L) + 3$. Moreover, we may assume $\sup(L_i) < \min(L_j \setminus \alpha_0)$ for all $i < j$.

Now take any $i, j \in I$, set $x := x_i$ and $y := x_j$, and use Lemma 2.122(3). Finally, use Fact 2.119 to find $z \leq x_i, x_j$. $\square$

2.4.B The generic forcing $\mathbf{P}'$

Let G be $\mathbb{R}$-generic. Obviously G is a $\leq_{\mathbb{R}}$-directed system. Using the canonical embeddings, we can construct in $V[G]$ a direct limit $\mathbf{P}'_{\omega_2}$ of the directed system G: Formally, we set

$$\mathbf{P}'_{\omega_2} := \{(x,p) : x \in G \text{ and } p \in P^x_{\omega_2}\},$$

and we set $(y,q) \leq (x,p)$ if $y \leq_{\mathbb{R}} x$ and q is (in y) stronger than $i_{x,y}(p)$ (where $i_{x,y} : P^x_{\omega_2} \to P^y_{\omega_2}$ is the canonical embedding). Similarly, we define for each α

$$\mathbf{P}'_\alpha := \{(x,p) : x \in G,\ \alpha \in M^x \text{ and } p \in P^x_\alpha\}$$

with the same order.

To summarize:

Definition 2.125. For $\alpha \leq \omega_2$, the direct limit of the P^x_α with $x \in G$ is called $\mathbf{P}'_\alpha$.

Formally, elements of $\mathbf{P}'_{\omega_2}$ are defined as pairs (x,p). However, the x does not really contribute any information. In particular:

Fact 2.126. 1. Assume that (x, p^x) and (y, p^y) are in $\mathbf{P}'_{\omega_2}$, that $y \leq x$, and that the canonical embedding $i_{x,y}$ witnessing $y \leq x$ maps p^x to p^y. Then $(x, p^x) =^* (y, p^y)$.

2. (y,q) is in $\mathbf{P}'_{\omega_2}$ stronger than (x,p) iff for some (or equivalently: for any) $z \leq x, y$ in G the canonically embedded q is in $P^z_{\omega_2}$ stronger than the canonically embedded p. The same holds if "stronger than" is replaced by "compatible with" or by "incompatible with".

3. If $(x,p) \in \mathbf{P}'_\alpha$, and if y is such that $M^y = M^x$ and $\bar{P}^y \restriction \alpha = \bar{P}^x \restriction \alpha$, then $(y,p) =^* (x,p)$.

In the following, we will therefore often abuse notation and just write p instead of (x,p) for an element of $\mathbf{P}'_\alpha$.

We can define a natural restriction map from $\mathbf{P}'_{\omega_2}$ to $\mathbf{P}'_\alpha$, by mapping (x,p) to $(x, p\restriction\alpha)$. Note that by the fact above, we can assume without loss of generality that $\alpha \in M^x$. More exactly: There is a $y \leq x$ in G such that $\alpha \in M^y$ (according to Corollary 2.121). Then in $\mathbf{P}'_{\omega_2}$ we have $(x,p) =^* (y,p)$.

Fact 2.127. The following is forced by $\mathbb{R}$:

- $\mathbf{P}'_\beta$ is completely embedded into $\mathbf{P}'_\alpha$ for $\beta < \alpha \leq \omega_2$ (witnessed by the natural restriction map).
- If $x \in G$, then P^x_α is M^x-completely embedded into $\mathbf{P}'_\alpha$ for $\alpha \leq \omega_2$ (by the identity map $p \mapsto (x,p)$).
- If $\mathrm{cf}(\alpha) > \omega$, then $\mathbf{P}'_\alpha$ is the union of the $\mathbf{P}'_\beta$ for $\beta < \alpha$.
- By definition, $\mathbf{P}'_{\omega_2}$ is a subset of V.

G will always denote an $\mathbb{R}$-generic filter, while the $\mathbf{P}'_{\omega_2}$-generic filter over $V[G]$ will be denoted by H'_{ω_2} (and the induced $\mathbf{P}'_\alpha$-generic by H'_α). Recall that for each $x \in G$, the map $p \mapsto (x,p)$ is an M^x-complete embedding of $P^x_{\omega_2}$ into $\mathbf{P}'_{\omega_2}$ (and of P^x_α into $\mathbf{P}'_\alpha$). This way $H'_\alpha \subseteq \mathbf{P}'_\alpha$ induces an M^x-generic filter $H^x_\alpha \subseteq P^x_\alpha$.

So $x \in \mathbb{R}$ forces that $\mathbf{P}'_\alpha$ is approximated by P^x_α. In particular we get:

Lemma 2.128. *Assume that $x \in \mathbb{R}$, that $\alpha \leq \omega_2$ in M^x, that $p \in P^x_\alpha$, that $\varphi(t)$ is a first order formula of the language $\{\in\}$ with one free variable t and that $\dot{\tau}$ is a P^x_α-name in M^x. Then $M^x \models p \Vdash_{P^x_\alpha} \varphi(\dot{\tau})$ iff $x \Vdash_{\mathbb{R}} (x,p) \Vdash_{\mathbf{P}'_\alpha} M^x[H^x_\alpha] \models \varphi(\dot{\tau}[H^x_\alpha])$.*

Proof. "$\Rightarrow$" is clear. So assume that $\varphi(\dot{\tau})$ is not forced in M^x. Then some $q \leq_{P^x_\alpha} p$ forces the negation. Now x forces that $(x,q) \leq (x,p)$ in $\mathbf{P}'_\alpha$; but the conditions (x,p) and (x,q) force contradictory statements. $\square$

2.4.C The inductive proof of ccc

We will now prove by induction on α that $\mathbf{P}'_\alpha$ is (forced to be) ccc and (equivalent to) an alternating iteration. Once we know this, we can prove Lemma 2.143, which easily implies all the lemmas in this section. So in particular these lemmas will only be needed to prove ccc and not for anything else (and they will probably not aid the understanding of the construction).

In this section, we try to stick to the following notation: $\mathbb{R}$-names are denoted with a tilde underneath (e.g., $\underset{\sim}{\tau}$), while P^x_α-names or $\mathbf{P}'_\alpha$-names (for any $\alpha \leq \omega_2$) are denoted with a dot accent (e.g., $\dot{\tau}$). We use both accents when we deal with $\mathbb{R}$-names for $\mathbf{P}'_\alpha$-names (e.g., $\underset{\sim}{\dot{\tau}}$).

We first prove a few lemmas that are easy generalizations of the following straightforward observation:

Assume that $x \Vdash_{\mathbb{R}} (\underset{\sim}{z}, \underset{\sim}{p}) \in \mathbf{P}'_\alpha$. In particular, $x \Vdash \underset{\sim}{z} \in G$. We first strengthen x to some x_1 that decides $\underset{\sim}{z}$ and $\underset{\sim}{p}$ to be z^* and p^*. Then $x_1 \leq^* z^*$ (the order $\leq^*$ is defined on page 36), so we can further strengthen x_1 to some $y \leq z^*$. By definition, this means that z^* is canonically embedded into $\bar{P}^y$; so (by Fact 2.126) the $P^{z^*}_\alpha$-condition p^* can be interpreted as a P^y_α-condition as well. So we end up with some $y \leq x$ and a P^y_α-condition p^* such that $y \Vdash_{\mathbb{R}} (\underset{\sim}{z}, \underset{\sim}{p}) =^* (y, p^*)$.

Since $\mathbb{R}$ is σ-closed, we can immediately generalize this to countably many ($\mathbb{R}$-names for) $\mathbf{P}'_\alpha$-conditions:

Fact 2.129. Assume that $x \Vdash_{\mathbb{R}} \underset{\sim}{p}_n \in \mathbf{P}'_\alpha$ for all $n \in \omega$. Then there is a $y \leq x$ and there are $p^*_n \in P^y_\alpha$ such that $y \Vdash_{\mathbb{R}} \underset{\sim}{p}_n =^* p^*_n$ for all $n \in \omega$.

Recall that more formally we should write: $x \Vdash_{\mathbb{R}} (\underset{\sim}{z}_n, \underset{\sim}{p}_n) \in \mathbf{P}'_\alpha$; and $y \Vdash_{\mathbb{R}} (\underset{\sim}{z}_n, \underset{\sim}{p}_n) =^* (y, p^*_n)$.

We will need a variant of the previous fact:

Lemma 2.130. *Assume that $\mathbf{P}'_\beta$ is forced to be ccc, and assume that x forces (in $\mathbb{R}$) that $\underset{\sim}{\dot{r}}_n$ is a $\mathbf{P}'_\beta$-name for a real (or an HCON object) for every $n \in \omega$. Then there is a $y \leq x$ and there are P^y_β-names $\dot{r}^*_n$ in M^y such that $y \Vdash_{\mathbb{R}} (\Vdash_{\mathbf{P}'_\beta} \underset{\sim}{\dot{r}}_n = \dot{r}^*_n)$ for all n.*

(Of course, we mean: $\underset{\sim}{\dot{r}}_n$ is evaluated by $G * H'_\beta$, while $\dot{r}^*_n$ is evaluated by H^y_β.)

Proof. The proof is an obvious consequence of the previous fact, since names of reals in a ccc forcing can be viewed as a countable sequence of conditions.

In more detail: For notational simplicity assume all $\underset{\sim}{\dot{r}}_n$ are names for elements of 2^ω. Working in V, we can find for each $n, m \in \omega$ names for a maximal antichain $\underset{\sim}{A}_{n,m}$ and for a function $\underset{\sim}{f}_{n,m} : \underset{\sim}{A}_{n,m} \to 2$ such that x forces that ($\mathbf{P}'_\beta$ forces that) $\underaccent{\tilde}{\dot r}_n(m) = \underaccent{\tilde}{f}_{n,m}(a)$ for the unique $a \in \underaccent{\tilde}{A}_{n,m} \cap H'_\beta$. Since $\mathbf{P}'_\beta$ is ccc, each $\underaccent{\tilde}{A}_{n,m}$ is countable, and since $\mathbb{R}$ is σ-closed, it is forced that the sequence $\underaccent{\tilde}{\Xi} = (\underaccent{\tilde}{A}_{n,m}, \underaccent{\tilde}{f}_{n,m})_{n,m\in\omega}$ is in V.

In V, we strengthen x to x_1 to decide $\underaccent{\tilde}{\Xi}$ to be some Ξ^*. We can also assume that $\Xi^* \in M^{x_1}$ (see Corollary 2.121). Each $A^*_{n,m}$ consists of countably many a such that x_1 forces $a \in \mathbf{P}'_\beta$. Using Fact 2.129 iteratively (and again the fact that $\mathbb{R}$ is σ-closed) we get some $y \leq x_1$ such that each such a is actually an element of P^y_β. So in M^y, we can use $(A^*_{n,m}, f^*_{n,m})_{n,m\in\omega}$ to construct P^y_β-names $\dot r^*_n$ in the obvious way.

Now assume that $y \in G$ and that H'_β is $\mathbf{P}'_\beta$-generic over $V[G]$. Fix any $a \in A^*_{n,m} = \underaccent{\tilde}{A}_{n,m}$. Since $a \in P^y_\beta$, we get $a \in H^y_\beta$ iff $a \in H'_\beta$. So there is a unique element a of $A^*_{n,m} \cap H^y_\beta$, and $\dot r^*_n(m) = f^*_{n,m}(a) = \underaccent{\tilde}{f}_{n,m}(a) = \underaccent{\tilde}{\dot r}_n(m)$. $\square$

We will also need the following modification:

Lemma 2.131. *(Same assumptions as in the previous lemma.) In $V[G][H'_\beta]$, let $\mathbf{Q}_\beta$ be the union of $Q^z_\beta[H^z_\beta]$ for all $z \in G$. In V, assume that x forces that each $\underaccent{\tilde}{\dot r}_n$ is a name for an element of $\mathbf{Q}_\beta$. Then there is a $y \leq x$ and there is in M^y a sequence $(\dot r^*_n)_{n\in\omega}$ of P^y_β-names for elements of Q^y_β such that y forces $\underaccent{\tilde}{\dot r}_n = \dot r^*_n$ for all n.*

So the difference to the previous lemma is: We additionally assume that $\underaccent{\tilde}{\dot r}_n$ is in $\bigcup_{z\in G} Q^z_\beta$, and we additionally get that $\dot r^*_n$ is a name for an element of Q^y_β.

Proof. Assume $x \in G$ and work in $V[G]$. Fix n. $\mathbf{P}'_\beta$ forces that there is some $y_n \in G$ and some $P^{y_n}_\beta$-name $\tau_n \in M^{y_n}$ of an element of $Q^{y_n}_\beta$ such that $\underaccent{\tilde}{\dot r}_n$ (evaluated by H'_β) is the same as τ_n (evaluated by $H^{y_n}_\beta$). Since we assume that $\mathbf{P}'_\beta$ is ccc, we can find a countable set $Y_n \subseteq G$ of the possible y_n, i.e., the empty condition of $\mathbf{P}'_\beta$ forces $y_n \in Y_n$. (As $\mathbb{R}$ is σ-closed and $Y_n \subseteq \mathbb{R} \subseteq V$, we must have $Y_n \in V$.)

So in V, there is (for each n) an $\mathbb{R}$-name $\underaccent{\tilde}{Y}_n$ for this countable set. Since $\mathbb{R}$ is σ-closed, we can find some $z_0 \leq x$ deciding each $\underaccent{\tilde}{Y}_n$ to be some countable set $Y^*_n \subseteq \mathbb{R}$. In particular, for each $y \in Y^*_n$ we know that $z_0 \Vdash_{\mathbb{R}} y \in G$, i.e., $z_0 \leq^* y$; so using once again that $\mathbb{R}$ is σ-closed we can find some z stronger than z_0 and all the $y \in \bigcup_{n\in\omega} Y^*_n$. Let X contain all $\tau \in M^y$ such that for some $y \in \bigcup_{n\in\omega} Y^*_n$, τ is a P^y_β-name for a Q^y_β-element. Since $z \leq y$, each $\tau \in X$ is actually[44] a P^z_β-name for an element of Q^z_β.

So X is a set of P^z_β-names for Q^z_β-elements; we can assume that $X \in M^z$. Also, z forces that $\underaccent{\tilde}{\dot r}_n \in X$ for all n. Using Lemma 2.130, we can additionally

[44]Here we use two consequences of $z \leq y$: Every P^y_β-name in M^y can be canonically interpreted as a P^z_β-name in M^z, and Q^y_β is (forced to be) a subset of Q^z_β.

assume that there are names P^z_β-name $\dot{r}^*_n$ in M^z such that z forces that $\underaccent{\tilde}{r}_n = \dot{r}^*_n$ is forced for each n. By Lemma 2.128, we know that M^z thinks that P^z_β forces that $\dot{r}^*_n \in X$. Therefore $\dot{r}^*_n$ is a P^z_β-name for a Q^z_β-element. $\square$

We now prove by induction on α that $\mathbf{P}'_\alpha$ is equivalent to a ccc alternating iteration:

Lemma 2.132. *The following holds in $V[G]$ for $\alpha < \omega_2$:*

1. *$\mathbf{P}'_\alpha$ is equivalent to an alternating iteration. More formally: There is an iteration $(\mathbf{P}_\beta, \mathbf{Q}_\beta)_{\beta<\alpha}$ with limit $\mathbf{P}_\alpha$ that satisfies the definition of alternating iteration (up to α), and there is a naturally defined dense embedding $j_\alpha : \mathbf{P}'_\alpha \to \mathbf{P}_\alpha$, such that for $\beta < \alpha$ we have $j_\beta \subseteq j_\alpha$, and the embeddings commute with the restrictions.*[45] *Each $\mathbf{Q}_\alpha$ is the union of all Q^x_α with $x \in G$. For $x \in G$ with $\alpha \in M^x$, the function $i_{x,\alpha} : P^x_\alpha \to \mathbf{P}_\alpha$ that maps p to $j_\alpha(x,p)$ is the canonical M^x-complete embedding.*
2. *In particular, a $\mathbf{P}'_\alpha$-generic filter H'_α can be translated into a $\mathbf{P}_\alpha$-generic filter which we call H_α (and vice versa).*
3. *$\mathbf{P}_\alpha$ has a dense subset of size $\aleph_1$.*
4. *$\mathbf{P}_\alpha$ is ccc.*
5. *$\mathbf{P}_\alpha$ forces CH.*

Proof. $\alpha = 0$ is trivial (since $\mathbf{P}_0$ and $\mathbf{P}'_0$ both are trivial: $\mathbf{P}_0$ is a singleton, and $\mathbf{P}'_0$ consists of pairwise compatible elements).

So assume that all items hold for all $\beta < \alpha$.

Proof of (1).

Ultralaver successor case: Let $\alpha = \beta + 1$ with β an ultralaver position. Let H_β be $\mathbf{P}_\beta$-generic over $V[G]$. Work in $V[G][H_\beta]$. By induction, for every $x \in G$ the canonical embedding $i_{x,\beta}$ defines a P^x_β-generic filter over M^x called H^x_β.

Definition of $\mathbf{Q}_\beta$ (and thus of $\mathbf{P}_\alpha$): In $M^x[H^x_\beta]$, the forcing notion Q^x_β is defined as $\mathbb{L}_{\bar{D}^x}$ for some system of ultrafilters $\bar{D}^x$ in $M^x[H^x_\beta]$. Fix some $s \in \omega^{<\omega}$. If $y \leq x$ in G, then D^y_s extends D^x_s. Let D_s be the union of all D^x_s with $x \in G$. So D_s is a proper filter. It is even an ultrafilter: Let r be a $\mathbf{P}_\beta$-name for a real. Using Lemma 2.130, we know that there is some $y \in G$ and some P^y_β-name $\underaccent{\tilde}{r}^y \in M^y$ such that (in $V[G][H_\beta]$) we have $\underaccent{\tilde}{r}^y[H^y_\beta] = r$. So

[45] I.e., $j_\beta(x, p{\restriction}\beta) = j_\alpha(x, p{\restriction}\beta) = j_\alpha(x,p){\restriction}\beta$.

$r \in M^y[H^y_\beta]$, hence either r or its complement is in D^y_s and therefore in D_s. So all filters in the family $\bar{D} = (D_s)_{s\in\omega^{<\omega}}$ are ultrafilters.

Now work again in $V[G]$. We set $\mathbf{Q}_\beta$ to be the $\mathbf{P}_\beta$-name for $\mathbb{L}_{\bar{D}}$. (Note that $\mathbf{P}_\beta$ forces that $\mathbf{Q}_\beta$ literally is the union of the $Q^x_\beta[H^x_\beta]$ for $x \in G$, again by Lemma 2.130.)

Definition of j_α: Let (x, p) be in $\mathbf{P}'_\alpha$. If $p \in P^x_\beta$, then we set $j_\alpha(x, p) = j_\beta(x, p)$, i.e., j_α will extend j_β. If $p = (p\restriction\beta, p(\beta))$ is in P^x_α but not in P^x_β, we set $j_\alpha(x, p) = (r, s) \in \mathbf{P}_\beta * \mathbf{Q}_\beta$ where $r = j_\beta(x, p\restriction\beta)$ and s is the ($\mathbf{P}_\alpha$-name for) $p(\beta)$ as evaluated in $M^x[H^x_\beta]$. From $\mathbf{Q}_\beta = \bigcup_{x\in G} Q^x_\beta[H^x_\beta]$ we conclude that this embedding is dense.

The canonical embedding: By induction we know that $i_{x,\beta}$ which maps $p \in P^x_\beta$ to $j_\beta(x, p)$ is (the restriction to P^x_β of) the canonical embedding of x into $\mathbf{P}_{\omega_2}$. So we have to extend the canonical embedding to $i_{x,\alpha} : P^x_\alpha \to \mathbf{P}_\alpha$. By definition of "canonical embedding", $i_{x,\alpha}$ maps $p \in P^x_\alpha$ to the pair $(i_{x,\beta}(p\restriction\beta), p(\beta))$. This is the same as $j_\alpha(x, p)$. We already know that D^x_s is (forced to be) an $M^x[H^x_\beta]$-ultrafilter that is extended by D_s.

Janus successor case: This is similar, but simpler than the previous case: Here, $\mathbf{Q}_\beta$ is just defined as the union of all $Q^x_\beta[H^x_\beta]$ for $x \in G$. We will show below that this union satisfies the ccc; just as in Fact 2.62, it is then easy to see that this union is again a Janus forcing.

In particular, $\mathbf{Q}_\beta$ consists of hereditarily countable objects (since it is the union of Janus forcings, which by definition consist of hereditarily countable objects). So since $\mathbf{P}_\beta$ forces CH, $\mathbf{Q}_\beta$ is forced to have size $\aleph_1$. Also note that since all Janus forcings involved are separative, the union (which is a limit of an incompatibility-preserving directed system) is trivially separative as well.

Limit case: Let α be a limit ordinal.

Definition of $\mathbf{P}_\alpha$ and j_α: First we define $j_\alpha : \mathbf{P}'_\alpha \to \mathbf{P}^{\mathrm{CS}}_\alpha$: For each $(x, p) \in \mathbf{P}'_\alpha$, let $j_\alpha(x, p) \in \mathbf{P}^{\mathrm{CS}}_\alpha$ be the union of all $j_\beta(x, p\restriction\beta)$ (for $\beta \in \alpha\cap M^x$). (Note that $\beta_1 < \beta_2$ implies that $j_{\beta_1}(x, p\restriction\beta_1)$ is a restriction of $j_{\beta_2}(x, p\restriction\beta_2)$, so this union is indeed an element of $\mathbf{P}^{\mathrm{CS}}_\alpha$.)

$\mathbf{P}_\alpha$ is the set of all $q \wedge p$, where $p \in j_\alpha[\mathbf{P}'_\alpha]$, $q \in \mathbf{P}_\beta$ for some $\beta < \alpha$, and $q \le p\restriction\beta$.

It is easy to check that $\mathbf{P}_\alpha$ actually is a partial countable support limit, and that j_α is dense. We will show below that $\mathbf{P}_\alpha$ satisfies the ccc, so in particular it is proper.

The canonical embedding: To see that $i_{x,\alpha}$ is the (restriction of the) canonical embedding, we just have to check that $i_{x,\alpha}$ is M^x-complete. This is the case since $\mathbf{P}'_\alpha$ is the direct limit of all P^y_α for $y \in G$ (without loss of generality $y \le x$), and each $i_{x,y}$ is M^x-complete (see Fact 2.127).

Proof of (3).

Recall that we assume CH in the ground model.

The successor case, $\alpha = \beta + 1$, follows easily from (3)–(5) for $\mathbf{P}_\beta$ (since $\mathbf{P}_\beta$ forces that $\mathbf{Q}_\beta$ has size $2^{\aleph_0} = \aleph_1 = \aleph_1^V$).

If $\mathrm{cf}(\alpha) > \omega$, then $\mathbf{P}_\alpha = \bigcup_{\beta<\alpha} \mathbf{P}_\beta$, so the proof is easy.

So let $\mathrm{cf}(\alpha) = \omega$. The following straightforward argument works for any ccc partial CS iteration where all iterands $\mathbf{Q}_\beta$ are of size $\leq \aleph_1$.

For notational simplicity we assume $\Vdash_{\mathbf{P}_\beta} \mathbf{Q}_\beta \subseteq \omega_1$ for all $\beta < \alpha$ (this is justified by inductive assumption (5)). By induction, we can assume that for all $\beta < \alpha$ there is a dense $\mathbf{P}^*_\beta \subseteq \mathbf{P}_\beta$ of size $\aleph_1$ and that every $\mathbf{P}^*_\beta$ is ccc. For each $p \in \mathbf{P}_\alpha$ and all $\beta \in \mathrm{dom}(p)$ we can find a maximal antichain $A^p_\beta \subseteq \mathbf{P}^*_\beta$ such that each element $a \in A^p_\beta$ decides the value of $p(\beta)$, say $a \Vdash_{\mathbf{P}_\beta} p(\beta) = \gamma^p_\beta(a)$. Writing[46] $p \sim q$ if $p \leq q$ and $q \leq p$, the map $p \mapsto (A^p_\beta, \gamma^p_\beta)_{\beta\in\mathrm{dom}(p)}$ is 1-1 modulo $\sim$. Since each A^p_β is countable, there are only $\aleph_1$ many possible values, therefore there are only $\aleph_1$ many $\sim$-equivalence classes. Any set of representatives will be dense.

Alternatively, we can prove (3) directly for $\mathbf{P}'_\alpha$. I.e., we can find a $\leq^*$-dense subset $\mathbf{P}'' \subseteq \mathbf{P}'_\alpha$ of cardinality $\aleph_1$. Note that a condition $(x, p) \in \mathbf{P}'_\alpha$ essentially depends only on p (cf. Fact 2.126). More specifically, given (x, p) we can "transitively[47] collapse x above α", resulting in a $=^*$-equivalent condition (x', p'). Since $|\alpha| = \aleph_1$, there are only $\aleph_1^{\aleph_0} = 2^{\aleph_0}$ many such candidates x' and since each x' is countable and $p' \in x'$, there are only $2^{\aleph_0}$ many pairs (x', p').

Proof of (4).

Ultralaver successor case: Let $\alpha = \beta + 1$ with β an ultralaver position. We already know that $\mathbf{P}_\alpha = \mathbf{P}_\beta * \mathbf{Q}_\beta$ where $\mathbf{Q}_\beta$ is an ultralaver forcing, which in particular is ccc, so by induction $\mathbf{P}_\alpha$ is ccc.

Janus successor case: As above it suffices to show that $\mathbf{Q}_\beta$, the union of the Janus forcings $Q^x_\beta[H^x_\beta]$ for $x \in G$, is (forced to be) ccc.

Assume towards a contradiction that this is not the case, i.e., that we have an uncountable antichain in $\mathbf{Q}_\beta$. We already know that $\mathbf{Q}_\beta$ has size $\aleph_1$ and therefore the uncountable antichain has size $\aleph_1$. So, working in V, we

[46] Since $\leq$ is separative, $p \sim q$ iff $p =^* q$, but this fact is not used here.

[47] In more detail: We define a function $f : M^x \to V$ by induction as follows: If $\beta \in M^x \cap \alpha + 1$ or if $\beta = \omega_2$, then $f(\beta) = \beta$. Otherwise, if $\beta \in M^x \cap \mathrm{Ord}$, then $f(\beta)$ is the smallest ordinal above $f[\beta]$. If $a \in M^x \setminus \mathrm{Ord}$, then $f(a) = \{f(b) : b \in a \cap M^x\}$. It is easy to see that f is an isomorphism from M^x to $M^{x'} := f[M^x]$ and that $M^{x'}$ is a candidate. Moreover, the ordinals that occur in $M^{x'}$ are subsets of $\alpha + \omega_1$ together with the interval $[\omega_2, \omega_2 + \omega_1]$; i.e., there are $\aleph_1$ many ordinals that can possibly occur in $M^{x'}$, and therefore there are $2^{\aleph}_0$ many possible such candidates. Moreover, setting $p' := f(p)$, it is easy to check that $(x, p) =^* (x', p')$ (similarly to Fact 2.126).

assume towards a contradiction that

$$x_0 \Vdash_{\mathbb{R}} p_0 \Vdash_{\mathbf{P}_\beta} \{\dot{\underset{\sim}{a}}_i : i \in \omega_1\} \text{ is a maximal (uncountable) antichain in } \mathbf{Q}_\beta. \tag{2.133}$$

We construct by induction on $n \in \omega$ a decreasing sequence of conditions such that x_{n+1} satisfies the following:

(i) For all $i \in \omega_1 \cap M^{x_n}$ there is (in $M^{x_{n+1}}$) a $P_\beta^{x_{n+1}}$-name $\dot{a}_i^*$ for a $Q_\beta^{x_{n+1}}$-condition such that

$$x_{n+1} \Vdash_{\mathbb{R}} p_0 \Vdash_{\mathbf{P}_\beta} \dot{\underset{\sim}{a}}_i = \dot{a}_i^*.$$

Why can we get that? Just use Lemma 2.131.

(ii) If τ is in M^{x_n} a $P_\beta^{x_n}$-name for an element of $Q_\beta^{x_n}$, then there is $k^*(\tau) \in \omega_1$ such that

$$x_{n+1} \Vdash_{\mathbb{R}} p_0 \Vdash_{\mathbf{P}_\beta} (\exists i < k^*(\tau))\ \dot{\underset{\sim}{a}}_i \not\perp_{\mathbf{Q}_\beta} \tau.$$

Also, all these $k^*(\tau)$ are in $M^{x_{n+1}}$.
Why can we get that? First note that $x_n \Vdash p_0 \Vdash (\exists i \in \omega_1)\ \dot{\underset{\sim}{a}}_i \not\perp \tau$. Since $\mathbf{P}_\beta$ is ccc, x_n forces that there is some bound $\underset{\sim}{k}(\tau)$ for i. So it suffices that x_{n+1} determines $\underset{\sim}{k}(\tau)$ to be $k^*(\tau)$ (for all the countably many τ).

Set $\delta^* := \omega_1 \cap \bigcup_{n\in\omega} M^{x_n}$. By Corollary 2.123(4), there is some y such that

- $y \leq x_n$ for all $n \in \omega$,
- $(x_n)_{n\in\omega}$ and $(\dot{a}_i^*)_{i\in\delta^*}$ are in M^y,
- (M^y thinks that) P_β^y forces that Q_β^y is the union of $Q_\beta^{x_n}$, i.e., as a formula: $M^y \models P_\beta^y \Vdash Q_\beta^y = \bigcup_{n\in\omega} Q_\beta^{x_n}$.

Let G be $\mathbb{R}$-generic (over V) containing y, and let H_β be $\mathbf{P}_\beta$-generic (over $V[G]$) containing p_0.

Set $A^* := \{\dot{a}_i^*[H_\beta^y] : i < \delta^*\}$. Note that A^* is in $M^y[H_\beta^y]$. We claim

$$A^* \subseteq Q_\beta^y[H_\beta^y] \text{ is predense.} \tag{2.134}$$

Pick any $q_0 \in Q_\beta^y$. So there is some $n \in \omega$ and some τ which is in M^{x_n} a $P_\beta^{x_n}$-name of a $Q_\beta^{x_n}$-condition, such that $q_0 = \tau[H_\beta^{x_n}]$. By (ii) above, x_{n+1} and therefore y forces (in $\mathbb{R}$) that for some $i < k^*(\tau)$ (and therefore some $i < \delta^*$) the condition p_0 forces the following (in $\mathbf{P}_\beta$):

The conditions $\dot{\underaccent{\tilde}{a}}_i$ and τ are compatible in $\mathbf{Q}_\beta$. Also, $\dot{\underaccent{\tilde}{a}}_i = \dot{a}^*_i$ and τ both are in Q^y_β, and Q^y_β is an incompatibility-preserving subforcing of $\mathbf{Q}_\beta$. Therefore $M^y[H^y_\beta]$ thinks that $\dot{a}^*_i$ and τ are compatible.

This proves (2.134).

Since $Q^y_\beta[H^y_\beta]$ is $M^y[H^y_\beta]$-complete in $\mathbf{Q}_\beta[H_\beta]$, and since $A^* \in M^y[H^y_\beta]$, this implies (as $\dot{a}^*_i[H^y_\beta] = \dot{\underaccent{\tilde}{a}}_i[G * H_\beta]$ for all $i < \delta^*$) that $\{\dot{\underaccent{\tilde}{a}}_i[G * H_\beta] : i < \delta^*\}$ already is predense, a contradiction to (2.133).

Limit case: We work with $\mathbf{P}'_\alpha$, which by definition only contains HCON objects.

Assume towards a contradiction that $\mathbf{P}'_\alpha$ has an uncountable antichain. We already know that $\mathbf{P}'_\alpha$ has a dense subset of size $\aleph_1$ (modulo $=^*$), so the antichain has size $\aleph_1$.

Again, work in V. We assume towards a contradiction that

$$x_0 \Vdash_{\mathbb{R}} \{\underaccent{\tilde}{a}_i : i \in \omega_1\} \text{ is a maximal (uncountable) antichain in } \mathbf{P}'_\alpha. \tag{2.135}$$

So each $\underaccent{\tilde}{a}_i$ is an $\mathbb{R}$-name for an HCON object (x, p) in V.

To lighten the notation we will abbreviate elements $(x, p) \in \mathbf{P}'_\alpha$ by p; this is justified by Fact 2.126.

Fix any HCON object p and $\beta < \alpha$. We will now define the $(\mathbb{R} * \mathbf{P}'_\beta)$-names $\underaccent{\tilde}{\dot{\iota}}(\beta, p)$ and $\underaccent{\tilde}{\dot{r}}(\beta, p)$: Let G be $\mathbb{R}$-generic and containing x_0, and H'_β be $\mathbf{P}'_\beta$-generic. Let R be the quotient $\mathbf{P}'_\alpha / H'_\beta$. If p is not in R, set $\underaccent{\tilde}{\dot{\iota}}(\beta, p) = \underaccent{\tilde}{\dot{r}}(\beta, p) = 0$. Otherwise, let $\underaccent{\tilde}{\dot{\iota}}(\beta, p)$ be the minimal i such that $\underaccent{\tilde}{a}_i \in R$ and $\underaccent{\tilde}{a}_i$ and p are compatible (in R), and set $\underaccent{\tilde}{\dot{r}}(\beta, p) \in R$ to be a witness of this compatibility. Since $\mathbf{P}'_\beta$ is (forced to be) ccc, we can find (in $V[G]$) a countable set $\underaccent{\tilde}{X}^\iota(\beta, p) \subseteq \omega_1$ containing all possibilities for $\underaccent{\tilde}{\dot{\iota}}(\beta, p)$ and similarly $\underaccent{\tilde}{X}^r(\beta, p)$ consisting of HCON objects for $\underaccent{\tilde}{\dot{r}}(\beta, p)$.

To summarize: For every $\beta < \alpha$ and every HCON object p, we can define (in V) the $\mathbb{R}$-names $\underaccent{\tilde}{X}^\iota(\beta, p)$ and $\underaccent{\tilde}{X}^r(\beta, p)$ such that

$$x_0 \Vdash_{\mathbb{R}} \Vdash_{\mathbf{P}'_\beta} \Big(p \in \mathbf{P}'_\alpha / H'_\beta \rightarrow (\exists i \in \underaccent{\tilde}{X}^\iota(\beta, p))\, (\exists r \in \underaccent{\tilde}{X}^r(\beta, p))\, r \leq_{\mathbf{P}'_\alpha / H'_\beta} p, \underaccent{\tilde}{a}_i \Big). \tag{2.136}$$

Similarly to the Janus successor case, we define by induction on $n \in \omega$ a decreasing sequence of conditions such that x_{n+1} satisfies the following: For all $\beta \in \alpha \cap M^{x_n}$ and $p \in P^{x_n}_\alpha$, x_{n+1} decides $\underaccent{\tilde}{X}^\iota(\beta, p)$ and $\underaccent{\tilde}{X}^r(\beta, p)$ to be some $X^{\iota *}(\beta, p)$ and $X^{r*}(\beta, p)$. For all $i \in \omega_1 \cap M^{x_n}$, x_{n+1} decides $\underaccent{\tilde}{a}_i$ to be some $a^*_i \in P^{x_{n+1}}_\alpha$. Moreover, each such $X^{\iota *}$ and X^{r*} is in $M^{x_{n+1}}$, and every $r \in X^{r*}(\beta, p)$ is in $P^{x_{n+1}}_\alpha$. (For this, we just use Fact 2.129 and Lemma 2.130.)

Set $\delta^* := \omega_1 \cap \bigcup_{n\in\omega} M^{x_n}$, and set $A^* := \{a^*_i : i \in \delta^*\}$.

By Corollary 2.123(4), there is some y such that

$$y \leq x_n \text{ for all } n \in \omega, \tag{2.137}$$

$$\bar{x} := (x_n)_{n\in\omega} \text{ and } A^* \text{ are in } M^y, \tag{2.138}$$

$$(M^y \text{ thinks that) } P^y_\alpha \text{ is defined as the almost FS limit over } \bar{x}. \tag{2.139}$$

We claim that y forces

$$A^* \text{ is predense in } P^y_\alpha. \tag{2.140}$$

Since P^y_α is M^y-completely embedded into $\mathbf{P}'_\alpha$, and since $A^* \in M^y$ (and since $\underaccent{\tilde}{a}_i = a^*_i$ for all $i \in \delta^*$) we get that $\{\underaccent{\tilde}{a}_i : i \in \delta^*\}$ is predense, a contradiction to (2.135).

So it remains to show (2.140). Let G be $\mathbb{R}$-generic containing y. Let r be a condition in P^y_α; we will find $i < \delta^*$ such that r is compatible with a^*_i. Since P^y_α is the almost FS limit over $\bar{x}$, there is some $n \in \omega$ and $\beta \in \alpha \cap M^{x_n}$ such that r has the form $q \wedge p$ with p in $P^{x_n}_\alpha$, $q \in P^y_\beta$ and $q \leq p{\restriction}\beta$.

Now let H'_β be $\mathbf{P}'_\beta$-generic containing q. Work in $V[G][H'_\beta]$. Since $q \leq p{\restriction}\beta$, we get $p \in \mathbf{P}'_\alpha/H'_\beta$. Let ι^* be the evaluation by $G * H'_\beta$ of $\underaccent{\tilde}{i}(\beta, p)$, and let r^* be the evaluation of $\underaccent{\tilde}{\dot{r}}(\beta, p)$. Note that $\iota^* < \delta^*$ and $r^* \in P^y_\alpha$. So we know that $a^*_{\iota^*}$ and p are compatible in $\mathbf{P}'_\alpha/H'_\beta$ witnessed by r^*. Find $q' \in H'_\beta$ forcing $r^* \leq_{\mathbf{P}'_\alpha/H'_\beta} p, a^*_{\iota^*}$. We may find $q' \leq q$. Now $q' \wedge r^*$ witnesses that $q \wedge p$ and $a^*_{\iota^*}$ are compatible in P^y_α.

To summarize: The crucial point in proving the ccc is that "densely" we choose (a variant of) a finite support iteration, see (2.139). Still, it is a bit surprising that we get the ccc, since we can also argue that densely we use (a variant of) a countable support iteration. But this does not prevent the ccc, it only prevents the generic iteration from having direct limits in stages of countable cofinality.[48]

Proof of (5).

This follows from (3) and (4). $\square$

2.4.D The generic alternating iteration $\bar{\mathbf{P}}$

In Lemma 2.132 we have seen:

[48]Assume that x forces that $\mathbf{P}'_\alpha$ is the union of the $\mathbf{P}'_\beta$ for $\beta < \alpha$; then we can find a stronger y that uses an almost CS iteration over x. This almost CS iteration contains a condition p with unbounded support. (Take any condition in the generic part of the almost CS limit; if this condition has bounded domain, we can extend it to have unbounded domain, see Definition 2.97.) Now p will be in $\mathbf{P}'_\alpha$ and have unbounded domain.

Corollary 2.141. *Let G be $\mathbb{R}$-generic. Then we can construct*[49] *(in $V[G]$) an alternating iteration $\bar{\mathbf{P}}$ such that the following holds:*

- *$\bar{\mathbf{P}}$ is ccc.*
- *If $x \in G$, then x canonically embeds into $\bar{\mathbf{P}}$. (In particular, a $\mathbf{P}_{\omega_2}$-generic filter H_{ω_2} induces a $P^x_{\omega_2}$-generic filter over M^x, called $H^x_{\omega_2}$.)*
- *Each $\mathbf{Q}_\alpha$ is the union of all $Q^x_\alpha[H^x_\alpha]$ with $x \in G$.*
- *$\mathbf{P}_{\omega_2}$ is equivalent to the direct limit $\mathbf{P}'_{\omega_2}$ of G: There is a dense embedding $j : \mathbf{P}'_{\omega_2} \to \mathbf{P}_{\omega_2}$, and for each $x \in G$ the function $p \mapsto j(x,p)$ is the canonical embedding.*

Lemma 2.142. *Let $x \in \mathbb{R}$. Then $\mathbb{R}$ forces the following: $x \in G$ iff x canonically embeds into $\bar{\mathbf{P}}$.*

Proof. If $x \in G$, then we already know that x canonically embeds into $\bar{\mathbf{P}}$.

So assume (towards a contradiction) that y forces that x embeds, but $y \Vdash x \notin G$. Work in $V[G]$ where $y \in G$. Both x (by assumption) and $y \in G$ canonically embed into $\bar{\mathbf{P}}$. Let N be an elementary submodel of $H^{V[G]}(\chi^*)$ containing $x, y, \bar{\mathbf{P}}$; let $z = (M^z, \bar{P}^z)$ be the ord-collapse of $(N, \bar{\mathbf{P}})$. Then $z \in V$ (as $\mathbb{R}$ is σ-closed) and $z \in \mathbb{R}$, and (by elementarity) $z \leq x, y$. This shows that $x \not\perp_{\mathbb{R}} y$, i.e., y cannot force $x \notin G$, a contradiction. □

Using ccc, we can now prove a lemma that is in fact stronger than the lemmas in the previous Section 2.4.C:

Lemma 2.143. *The following is forced by $\mathbb{R}$: Let $N \prec H^{V[G]}(\chi^*)$ be countable, and let y be the ord-collapse of $(N, \bar{\mathbf{P}})$. Then $y \in G$. Moreover, if $x \in G \cap N$, then $y \leq x$.*

Proof. Work in $V[G]$ with $x \in G$. Pick an elementary submodel N containing x and $\bar{\mathbf{P}}$. Let y be the ord-collapse of $(N, \bar{\mathbf{P}})$ via a collapsing map k. As above, it is clear that $y \in \mathbb{R}$ and $y \leq x$. To show $y \in G$, it is (by the previous lemma) enough to show that y canonically embeds. We claim that k^{-1} is the canonical embedding of y into $\bar{\mathbf{P}}$. The crucial point is to show M^y-completeness. Let $B \in M^y$ be a maximal antichain of $P^y_{\omega_2}$, say $B = k(A)$ where $A \in N$ is a maximal antichain of $\mathbf{P}_{\omega_2}$. So (by ccc) A is countable, hence $A \subseteq N$. So not only $A = k^{-1}(B)$ but even $A = k^{-1}[B]$. Hence k^{-1} is an M^y-complete embedding. □

[49] in an "absolute way": Given G, we first define $\mathbf{P}'_{\omega_2}$ to be the direct limit of G, and then inductively construct the $\mathbf{P}_\alpha$'s from $\mathbf{P}'_{\omega_2}$.

Remark 2.144. We used the ccc of $\mathbf{P}_{\omega_2}$ to prove Lemma 2.143; this use was essential in the sense that we can in turn easily prove the ccc of $\mathbf{P}_{\omega_2}$ if we assume that Lemma 2.143 holds. In fact Lemma 2.143 easily implies all other lemmas in Section 2.4.C as well.

2.5 The proof of BC+dBC

We first[50] prove that no uncountable X in V will be smz or sm in the final extension $V[G * H]$. Then we show how to modify the argument to work for all uncountable sets in $V[G * H]$.

2.5.A BC+dBC for ground model sets.

Lemma 2.145. *Let $X \in V$ be an uncountable set of reals. Then $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that X is not smz.*

Proof.

1. Fix any even $\alpha < \omega_2$ (i.e., an ultralaver position) in our iteration. The ultralaver forcing $\mathbf{Q}_\alpha$ adds a (canonically defined code for a) closed null set $\dot{F}$ constructed from the ultralaver real $\bar{\ell}_\alpha$. (Recall Corollary 2.24.) In the following, when we consider various ultralaver forcings $\mathbf{Q}_\alpha$, Q_α, Q^x_α, we treat $\dot{F}$ not as an actual name, but rather as a definition which depends on the forcing used.

2. According to Theorem 2.2, it is enough to show that $X + \dot{F}$ is non-null in the $\mathbb{R} * \mathbf{P}_{\omega_2}$-extension, or equivalently, in every $\mathbb{R} * \mathbf{P}_\beta$-extension ($\alpha < \beta < \omega_2$). So assume towards a contradiction that there is a $\beta > \alpha$ and an $\mathbb{R} * \mathbf{P}_\beta$-name $\dot{\underset{\sim}{Z}}$ of a (code for a) Borel null set such that some $(x, p) \in \mathbb{R} * \mathbf{P}_{\omega_2}$ forces that $X + \dot{F} \subseteq \dot{\underset{\sim}{Z}}$.

3. Using the dense embedding $j_{\omega_2} : \mathbf{P}'_{\omega_2} \to \mathbf{P}_{\omega_2}$, we may replace (x, p) by a condition $(x, p') \in \mathbb{R} * \mathbf{P}'_{\omega_2}$. According to Fact 2.129 (recall that we now know that $\mathbf{P}_{\omega_2}$ satisfies ccc) and Lemma 2.130 we can assume that p' is already a P^x_β-condition p^x and that $\dot{\underset{\sim}{Z}}$ is (forced by x to be the same as) a P^x_β-name $\dot{Z}^x$ in M^x.

4. We construct (in V) an iteration $\bar{P}$ in the following way:

[50] Note that for this weak version, it would be enough to produce a generic iteration of length 2 only, i.e., $\mathbf{Q}_0 * \mathbf{Q}_1$, where $\mathbf{Q}_0$ is an ultralaver forcing and $\mathbf{Q}_1$ a corresponding Janus forcing.

(a) Up to α, we take an arbitrary alternating iteration into which x embeds. In particular, P_α will be proper and hence force that X is still uncountable.

(b) Let Q_α be any ultralaver forcing (over Q^x_α in case $\alpha \in M^x$). So according to Corollary 2.24, we know that Q_α forces that $X + \dot{F}$ is not null.

Therefore we can pick (in $V[H_{\alpha+1}]$) some $\dot{r}$ in $X + \dot{F}$ which is random over (the countable model) $M^x[H^x_{\alpha+1}]$, where $H^x_{\alpha+1}$ is induced by $H_{\alpha+1}$.

(c) In the rest of the construction, we preserve randomness of $\dot{r}$ over $M^x[H^x_\zeta]$ for each $\zeta \leq \omega_2$. We can do this using an almost CS iteration over x where at each Janus position we use a random version of Janus forcing and at each ultralaver position we use a suitable ultralaver forcing; this is possible by Lemma 2.108. By Lemma 2.110, this iteration will preserve the randomness of $\dot{r}$.

(d) So we get $\bar{P}$ over x (with canonical embedding i_x) and $q \leq_{P_{\omega_2}} i_x(p^x)$ such that $q{\restriction}\beta$ forces (in P_β) that $\dot{r}$ is random over $M^x[H^x_\beta]$, in particular that $\dot{r} \notin \dot{Z}^x$.

We now pick a countable $N \prec H(\chi^*)$ containing everything and ord-collapse $(N, \bar{P})$ to $y \leq x$. (See Fact 2.119.) Set $X^y := X \cap M^y$ (the image of X under the collapse). By elementarity, M^y thinks that (a)–(d) above holds for $\bar{P}^y$ and that X^y is uncountable. Note that $X^y \subseteq X$.

5. This gives a contradiction in the obvious way: Let G be $\mathbb{R}$-generic over V and contain y, and let H_β be $\mathbf{P}_\beta$-generic over $V[G]$ and contain $q{\restriction}\beta$. So $M^y[H^y_\beta]$ thinks that $r \notin \dot{Z}^x$ (which is absolute) and that $r = x + f$ for some $x \in X^y \subseteq X$ and $f \in F$ (actually even in F as evaluated in $M^y[H^y_{\alpha+1}]$). So in $V[G][H_\beta]$, r is the sum of an element of X and an element of F. So $(y, q) \leq (x, p')$ forces that $\dot{r} \in (X + \dot{F}) \setminus \dot{\underaccent{\sim}{Z}}$, a contradiction to (2). $\square$

Of course, we need this result not just for ground model sets X, but for $\mathbb{R}*\mathbf{P}_{\omega_2}$-names $\dot{\underaccent{\sim}{X}} = (\dot{\underaccent{\sim}{x}}_i : i \in \omega_1)$ of uncountable sets. It is easy to see that it is enough to deal with $\mathbb{R}*\mathbf{P}_\beta$-names for (all) $\beta < \omega_2$. So given $\dot{\underaccent{\sim}{X}}$, we can (in the proof) pick α such that $\dot{\underaccent{\sim}{X}}$ is actually an $\mathbb{R} * \mathbf{P}_\alpha$-name. We can try to repeat the same proof; however, the problem is the following: When constructing $\bar{P}$ in (4), it is not clear how to simultaneously make all the uncountably many names $(\dot{\underaccent{\sim}{x}}_i)$ into $\bar{P}$-names in a sufficiently "absolute" way. In other words: It is not clear how to end up with some M^y and $\dot{X}^y$ uncountable in M^y such that it is guaranteed that $\dot{X}^y$ (evaluated in $M^y[H^y_\alpha]$) will be a subset of $\dot{\underset{\sim}{X}}$ (evaluated in $V[G][H_\alpha]$). We will solve this problem in the next section by factoring $\mathbb{R}$.

Let us now give the proof of the corresponding weak version of dBC:

Lemma 2.146. *Let $X \in V$ be an uncountable set of reals. Then $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that X is not strongly meager.*

Proof. The proof is parallel to the previous one:

1. Fix any even $\alpha < \omega_2$ (i.e., an ultralaver position) in our iteration. The Janus forcing $\mathbf{Q}_{\alpha+1}$ adds a (canonically defined code for a) null set $\dot{Z}_\nabla$. (See Definition 2.60 and Fact 2.61.)

2. According to (2.1), it is enough to show that $X + \dot{Z}_\nabla = 2^\omega$ in the $\mathbb{R} * \mathbf{P}_{\omega_2}$-extension, or equivalently, in every $\mathbb{R} * \mathbf{P}_\beta$-extension ($\alpha < \beta < \omega_2$). (For every real r, the statement $r \in X + \dot{Z}_\nabla$, i.e., $(\exists x \in X)\ x + r \in \dot{Z}_\nabla$, is absolute.) So assume towards a contradiction that there is a $\beta > \alpha$ and an $\mathbb{R} * \mathbf{P}_\beta$-name $\dot{\underset{\sim}{r}}$ of a real such that some $(x, p) \in \mathbb{R} * \mathbf{P}_{\omega_2}$ forces that $\dot{\underset{\sim}{r}} \notin X + \dot{Z}_\nabla$.

3. Again, we can assume that $\dot{\underset{\sim}{r}}$ is a P^x_β-name $\dot{r}^x$ in M^x.

4. We construct (in V) an iteration $\bar{P}$ in the following way:

   (a) Up to α, we take an arbitrary alternating iteration into which x embeds. In particular, P_α again forces that X is still uncountable.

   (b1) Let Q_α be any ultralaver forcing (over Q^x_α). Then Q_α forces that X is not thin (see Corollary 2.27).

   (b2) Let $Q_{\alpha+1}$ be a countable Janus forcing. So $Q_{\alpha+1}$ forces $X + \dot{Z}_\nabla = 2^\omega$. (See Lemma 2.63.)

   (c) We continue the iteration in a σ-centered way. I.e., we use an almost FS iteration over x of ultralaver forcings and countable Janus forcings, using trivial Q_ζ for all $\zeta \notin M^x$; see Lemma 2.93.

   (d) So P_β still forces that $X + \dot{Z}_\nabla = 2^\omega$, and in particular that $\dot{r}^x \in X + \dot{Z}_\nabla$. (Again by Lemma 2.63.)

   Again, by collapsing some N as in the previous proof, we get $y \leq x$ and $X^y \subseteq X$.

5. This again gives the obvious contradiction: Let G be $\mathbb{R}$-generic over V and contain y, and let H_β be $\mathbf{P}_\beta$-generic over $V[G]$ and contain p. So $M^y[H^y_\beta]$ thinks that $r = x + z$ for some $x \in X^y \subseteq X$ and $z \in Z_\nabla$ (this time, $\dot{Z}_\nabla$ is evaluated in $M^y[H^y_\beta]$), contradicting (2). $\square$

2.5.B A factor lemma

We can restrict $\mathbb{R}$ to any $\alpha^* < \omega_2$ in the obvious way: Conditions are pairs $x = (M^x, \bar{P}^x)$ of nice candidates M^x (containing α^*) and alternating iterations $\bar{P}^x$, but now M^x thinks that $\bar{P}^x$ has length α^* (and not ω_2). We call this variant $\mathbb{R}{\restriction}\alpha^*$.

Note that all results of Section 2.4 about $\mathbb{R}$ are still true for $\mathbb{R}{\restriction}\alpha^*$. In particular, whenever $G \subseteq \mathbb{R}{\restriction}\alpha^*$ is generic, it will define a direct limit (which we call $\mathbf{P}'^*$), and an alternating iteration of length α^* (called $\bar{\mathbf{P}}^*$); again we will have that $x \in G$ iff x canonically embeds into $\bar{\mathbf{P}}^*$.

There is a natural projection map from $\mathbb{R}$ (more exactly: from the dense subset of those x which satisfy $\alpha^* \in M^x$) into $\mathbb{R}{\restriction}\alpha^*$, mapping $x = (M^x, \bar{P}^x)$ to $x{\restriction}\alpha^* := (M^x, \bar{P}^x{\restriction}\alpha^*)$. (It is obvious that this projection is dense and preserves $\leq$.)

There is also a natural embedding φ from $\mathbb{R}{\restriction}\alpha^*$ to $\mathbb{R}$: We can just continue an alternating iteration of length α^* by appending trivial forcings.

φ is complete: It preserves $\leq$ and $\perp$. (Assume that $z \leq \varphi(x), \varphi(y)$. Then $z{\restriction}\alpha^* \leq x, y$.) Also, the projection is a reduction: If $y \leq x{\restriction}\alpha^*$ in $\mathbb{R}{\restriction}\alpha^*$, then let M^z be a model containing both x and y. In M^z, we can first construct an alternating iteration of length α^* over y (using almost FS over y, or almost CS — this does not matter here). We then continue this iteration $\bar{P}^z$ using almost FS or almost CS over x. So x and y both embed into $\bar{P}^z$, hence $z = (M^z, \bar{P}^z) \leq x, y$.

So according to the general factor lemma of forcing theory, we know that $\mathbb{R}$ is forcing equivalent to $\mathbb{R}{\restriction}\alpha^* * (\mathbb{R}/\mathbb{R}{\restriction}\alpha^*)$, where $\mathbb{R}/\mathbb{R}{\restriction}\alpha^*$ is the quotient of $\mathbb{R}$ and $\mathbb{R}{\restriction}\alpha^*$, i.e., the ($\mathbb{R}{\restriction}\alpha^*$-name for the) set of $x \in \mathbb{R}$ which are compatible (in $\mathbb{R}$) with all $\varphi(y)$ for $y \in G{\restriction}\alpha^*$ (the generic filter for $\mathbb{R}{\restriction}\alpha^*$), or equivalently, the set of $x \in \mathbb{R}$ such that $x{\restriction}\alpha^* \in G{\restriction}\alpha^*$. So Lemma 2.142 (relativized to $\mathbb{R}{\restriction}\alpha^*$) implies:

$$\mathbb{R}/\mathbb{R}{\restriction}\alpha^* \text{ is the set of } x \in \mathbb{R} \text{ that canonically embed (up to } \alpha^*\text{) into } \mathbf{P}_{\alpha^*}. \tag{2.147}$$

Setup. *Fix some $\alpha^* < \omega_2$ of uncountable cofinality.*[51] *Let $G{\restriction}\alpha^*$ be $\mathbb{R}{\restriction}\alpha^*$-generic over V and work in $V^* := V[G{\restriction}\alpha^*]$. Set $\bar{\mathbf{P}}^* = (\mathbf{P}^*_\beta)_{\beta<\alpha^*}$, the generic alternating iteration added by $\mathbb{R}{\restriction}\alpha^*$. Let $\mathbb{R}^*$ be the quotient $\mathbb{R}/\mathbb{R}{\restriction}\alpha^*$.*

We claim that $\mathbb{R}^*$ satisfies (in V^*) all the properties that we proved in Section 2.4 for $\mathbb{R}$ (in V), with the obvious modifications. In particular:

(A)$_{\alpha^*}$ $\mathbb{R}^*$ is $\aleph_2$-cc, since it is the quotient of an $\aleph_2$-cc forcing.

[51] Probably the cofinality is completely irrelevant, but the picture is clearer this way.

(B)$_{\alpha^*}$ $\mathbb{R}^*$ does not add new reals (and more generally, no new HCON objects), since it is the quotient of a σ-closed forcing.[52]

(C)$_{\alpha^*}$ Let G^* be $\mathbb{R}^*$-generic over V^*. Then G^* is $\mathbb{R}$-generic over V, and therefore Corollary 2.141 holds for G^*. (Note that $\mathbf{P}'_{\omega_2}$ and then $\mathbf{P}_{\omega_2}$ is constructed from G^*.) Moreover, it is easy to see[53] that $\bar{\mathbf{P}}$ starts with $\bar{\mathbf{P}}^*$.

(D)$_{\alpha^*}$ In particular, we get a variant of Lemma 2.143: The following is forced by $\mathbb{R}^*$: Let $N \prec H^{V[G^*]}(\chi^*)$ be countable, and let y be the ord-collapse of $(N, \bar{\mathbf{P}})$. Then $y \in G^*$. Moreover: If $x \in G^* \cap N$, then $y \leq x$.

We can use the last item to prove the $\mathbb{R}^*$-version of Fact 2.129:

Corollary 2.148. *In V^*, the following holds:*

1. *Assume that $x \in \mathbb{R}^*$ forces that $p \in \mathbf{P}_{\omega_2}$. Then there is a $y \leq x$ and a $p^y \in P^y_{\omega_2}$ such that y forces $p^y =^* p$.*
2. *Assume that $x \in \mathbb{R}^*$ forces that $\dot{\underset{\sim}{r}}$ is a $\mathbf{P}_{\omega_2}$-name of a real. Then there is a $y \leq x$ and a $P^y_{\omega_2}$-name $\dot{r}^y$ such that y forces that $\dot{r}^y$ and $\dot{\underset{\sim}{r}}$ are equivalent as $\mathbf{P}_{\omega_2}$-names.*

Proof. We only prove (1), the proof of (2) is similar.

Let G^* contain x. In $V[G^*]$, pick an elementary submodel N containing $x, p, \bar{\mathbf{P}}$ and let $(M^z, \bar{P}^z, p^z)$ be the ord-collapse of $(N, \bar{\mathbf{P}}, p)$. Then $z \in G^*$. This whole situation is forced by some $y \leq z \leq x \in G^*$. So y and p^y is as required, where $p^y \in P^y_{\omega_2}$ is the canonical image of p^z. $\square$

We also get the following analogue of Fact 2.119:

$$\text{In } V^* \text{ we have: Let } x \in \mathbb{R}^*. \text{ Assume that } \bar{P} \text{ is an alternating iteration that extends } \bar{\mathbf{P}}\restriction\alpha^* \text{ and that } x = (M^x, \bar{P}^x) \in \mathbb{R} \text{ canonically embeds into } \bar{P}, \text{ and that } N \prec H(\chi^*) \text{ contains } x \text{ and } \bar{P}. \text{ Let } y = (M^y, \bar{P}^y) \text{ be the ord-collapse of } (N, \bar{P}). \text{ Then } y \in \mathbb{R}^* \text{ and } y \leq x. \tag{2.149}$$

[52]It is easy to see that $\mathbb{R}^*$ is even σ-closed, by "relativizing" the proof for $\mathbb{R}$, but we will not need this.

[53]For $\beta \leq \alpha^*$, let $\mathbf{P}'^*_\beta$ be the direct limit of $(G\restriction\alpha^*)\restriction\beta$ and $\mathbf{P}'_\beta$ the direct limit of $G^*\restriction\beta$. The function $k_\beta : \mathbf{P}'^*_\beta \to \mathbf{P}'_\beta$ that maps (x, p) to $(\varphi(x), p)$ preserves $\leq$ and $\perp$ and is surjective modulo $=^*$, see Fact 2.126(3). So it is clear that defining $\bar{\mathbf{P}}^*\restriction\beta$ by induction from $\mathbf{P}'^*_\beta$ yields the same result as defining $\bar{\mathbf{P}}\restriction\beta$ from $\mathbf{P}'_\beta$.

We now claim that $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces BC+dBC. We know that $\mathbb{R}$ is forcing equivalent to $\mathbb{R}\restriction\alpha^* * \mathbb{R}^*$. Obviously we have

$$\mathbb{R} * \mathbf{P}_{\omega_2} = \mathbb{R}\restriction\alpha^* * \mathbb{R}^* * \mathbf{P}_{\alpha^*} * \mathbf{P}_{\alpha^*,\omega_2}$$

(where $\mathbf{P}_{\alpha^*,\omega_2}$ is the quotient of $\mathbf{P}_{\omega_2}$ and $\mathbf{P}_{\alpha^*}$). Note that $\mathbf{P}_{\alpha^*}$ is already determined by $\mathbb{R}\restriction\alpha^*$, so $\mathbb{R}^* * \mathbf{P}_{\alpha^*}$ is (forced by $\mathbb{R}\restriction\alpha^*$ to be) a product $\mathbb{R}^* \times \mathbf{P}_{\alpha^*} = \mathbf{P}_{\alpha^*} \times \mathbb{R}^*$.

But note that this is not the same as $\mathbf{P}_{\alpha^*} * \mathbb{R}^*$, where we evaluate the definition of $\mathbb{R}^*$ in the $\mathbf{P}_{\alpha^*}$-extension of $V[G\restriction\alpha^*]$: We would get new candidates and therefore new conditions in $\mathbb{R}^*$ after forcing with $\mathbf{P}_{\alpha^*}$. In other words, we can *not* just argue as follows:

Wrong argument. $\mathbb{R} * \mathbf{P}_{\omega_2}$ is the same as $(\mathbb{R}\restriction\alpha^* * \mathbf{P}_{\alpha^*}) * (\mathbb{R}^* * \mathbf{P}_{\alpha^*,\omega_2})$; so given an $\mathbb{R} * \mathbf{P}_{\omega_2}$-name X of a set of reals of size $\aleph_1$, we can choose α^* large enough so that X is an $(\mathbb{R}\restriction\alpha^* * \mathbf{P}_{\alpha^*})$-name. Then, working in the $(\mathbb{R}\restriction\alpha^* * \mathbf{P}_{\alpha^*})$-extension, we just apply Lemmas 2.145 and 2.146.

So what do we do instead? Assume that $\dot{\underaccent{\tilde}{X}} = \{\dot{\underaccent{\tilde}{\xi}}_i : i \in \omega_1\}$ is an $\mathbb{R} * \mathbf{P}_{\omega_2}$-name for a set of reals of size $\aleph_1$. So there is a $\beta < \omega_2$ such that $\dot{\underaccent{\tilde}{X}}$ is added by $\mathbb{R} * \mathbf{P}_\beta$. In the $\mathbb{R}$-extension, $\mathbf{P}_\beta$ is ccc, therefore we can assume that each $\dot{\underaccent{\tilde}{\xi}}_i$ is a system of countably many countable antichains $\underaccent{\tilde}{A}_i^m$ of $\mathbf{P}_\beta$, together with functions $\underaccent{\tilde}{f}_i^m : \underaccent{\tilde}{A}_i^m \to \{0,1\}$. For the following argument, we prefer to work with the equivalent $\mathbf{P}'_\beta$ instead of $\mathbf{P}_\beta$. We can assume that each of the sequences $B_i := (\underaccent{\tilde}{A}_i^m, \underaccent{\tilde}{f}_i^m)_{m\in\omega}$ is an element of V (since $\mathbf{P}'_\beta$ is a subset of V and since $\mathbb{R}$ is σ-closed). So each B_i is decided by a maximal antichain Z_i of $\mathbb{R}$. Since $\mathbb{R}$ is $\aleph_2$-cc, these $\aleph_1$ many antichains all are contained in some $\mathbb{R}\restriction\alpha^*$ with $\alpha^* \geq \beta$.

So in the $\mathbb{R}\restriction\alpha^*$-extension V^* we have the following situation: Each ξ_i is a very “absolute[54]” $\mathbb{R}^* * \mathbf{P}_{\alpha^*}$-name (or equivalently, $\mathbb{R}^* \times \mathbf{P}_{\alpha^*}$-name), in fact they are already determined by antichains that are in $\mathbf{P}_{\alpha^*}$ and do not depend on $\mathbb{R}^*$. So we can interpret them as $\mathbf{P}_{\alpha^*}$-names.

Note that:

$$\text{The } \xi_i \text{ are forced (by } \mathbb{R}^* * \mathbf{P}_{\alpha^*}) \text{ to be pairwise different, and therefore already by } \mathbf{P}_{\alpha^*}. \tag{2.150}$$

Now we are finally ready to prove that $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that every uncountable X is neither smz nor sm. It is enough to show that for every name $\dot{\underaccent{\tilde}{X}}$ of an uncountable set of reals of size $\aleph_1$ the forcing $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that $\dot{\underaccent{\tilde}{X}}$ is neither smz nor sm. For the rest of the proof we fix such a name $\dot{\underaccent{\tilde}{X}}$, the

[54] or: “nice” in the sense of [Kun80, 5.11]

corresponding $\dot{\underset{\sim}{\xi}}_i$'s (for $i \in \omega_1$), and the appropriate α^* as above. From now on, we work in the $\mathbb{R}{\restriction}\alpha^*$-extension V^*.

So we have to show that $\mathbb{R}^* * \mathbf{P}_{\omega_2}$ forces that $\dot{\underset{\sim}{X}}$ is neither smz nor sm.

After all our preparations, we can now just repeat the proofs of BC (Lemma 2.145) and dBC (Lemma 2.146) of Section 2.5.A, with the following modifications. The modifications are the same for both proofs; for better readability we describe the results of the change only for the proof of dBC.

1. Change: Instead of an arbitrary ultralaver position $\alpha < \omega_2$, we obviously have to choose $\alpha \geq \alpha^*$.
   For the dBC: We choose $\alpha \geq \alpha^*$ an arbitrary ultralaver position. The Janus forcing $\mathbf{Q}_{\alpha+1}$ adds a (canonically defined code for a) null set $\dot{Z}_\nabla$.

2. Change: No change here. (Of course we now have an $\mathbb{R}^* * \mathbf{P}_{\alpha^*}$-name $\dot{\underset{\sim}{X}}$ instead of a ground model set.)
   For the dBC: It is enough to show that $\dot{\underset{\sim}{X}} + \dot{Z}_\nabla = 2^\omega$ in the $\mathbb{R}^* * \mathbf{P}_{\omega_2}$-extension of V^*, or equivalently, in every $\mathbb{R}^* * \mathbf{P}_\beta$-extension ($\alpha < \beta < \omega_2$). So assume towards a contradiction that there is a $\beta > \alpha$ and an $\mathbb{R}^* * \mathbf{P}_\beta$-name $\dot{\underset{\sim}{r}}$ of a real such that some $(x, p) \in \mathbb{R}^* * \mathbf{P}_{\omega_2}$ forces that $\dot{\underset{\sim}{r}} \notin \dot{\underset{\sim}{X}} + \dot{Z}_\nabla$.

3. Change: No change here. (But we use Corollary 2.148 instead of Lemma 2.130.)
   For dBC: Using Corollary 2.148(2), without loss of generality x forces $p^x =^* p$ and there is a P^x_β-name $\dot{r}^x$ in M^x such that $\dot{r}^x = \dot{\underset{\sim}{r}}$ is forced.

4. Change: The iteration obviously has to start with the $\mathbb{R}{\restriction}\alpha^*$-generic iteration $\bar{\mathbf{P}}^*$ (which is ccc), the rest is the same.
   For dBC: In V^* we construct an iteration $\bar{P}$ in the following way:

   (a1) Up to α^*, we use the iteration $\bar{\mathbf{P}}^*$ (which already lives in our current universe V^*). As explained above in the paragraph preceding (2.150), $\dot{\underset{\sim}{X}}$ can be interpreted as a $\mathbf{P}_{\alpha^*}$-name $\dot{X}$, and by (2.150), $\dot{X}$ is forced to be uncountable.

   (a2) We continue the iteration from α^* to α in a way that embeds x and such that P_α is proper. So P_α will force that $\dot{X}$ is still uncountable.

   (b1) Let Q_α be any ultralaver forcing (over Q^x_α). Then Q_α forces that $\dot{X}$ is not thin.

   (b2) Let $Q_{\alpha+1}$ be a countable Janus forcing. So $Q_{\alpha+1}$ forces $\dot{X} + \dot{Z}_\nabla = 2^\omega$.

(c) We continue the iteration in a σ-centered way. I.e., we use an almost FS iteration over x of ultralaver forcings and countable Janus forcings, using trivial Q_ζ for all $\zeta \notin M^x$.

(d) So P_β still forces that $\dot{X} + \dot{Z}_\nabla = 2^\omega$, and in particular that $\dot{r}^x \in \dot{X} + \dot{Z}_\nabla$.

We now pick (in V^*) a countable $N \prec H(\chi^*)$ containing everything and ord-collapse $(N, \bar{P})$ to $y \leq x$, by (2.149). The HCON object y is of course in V (and even in $\mathbb{R}$), but we can say more: Since the iteration $\bar{P}$ starts with the $(\mathbb{R}\restriction\alpha^*)$-generic iteration $\bar{\mathbf{P}}^*$, the condition y will be in the quotient forcing $\mathbb{R}^*$.
Set $\dot{X}^y := \dot{X} \cap M^y$ (which is the image of $\dot{X}$ under the collapse, since we view $\dot{X}$ as a set of HCON-names). By elementarity, M^y thinks that (a)–(d) above holds for $\bar{P}^y$ and that $\dot{X}^y$ is forced to be uncountable. Note that $\dot{X}^y \subseteq \dot{X}$ in the following sense: Whenever $G^* * H$ is $\mathbb{R}^* * \mathbf{P}_{\omega_2}$-generic over V^*, and $y \in G^*$, then the evaluation of $\dot{X}^y$ in $M^y[H^y]$ is a subset of the evaluation of $\dot{X}$ in $V^*[G^* * H]$.

5. Change: No change here.
For dBC: We get our desired contradiction as follows:
Let G^* be $\mathbb{R}^*$-generic over V^* and contain y. Let H_β be $\mathbf{P}_\beta$-generic over $V^*[G^*]$ and contain p. So $M^y[H^y_\beta]$ thinks that $r = x + z$ for some $x \in X^y \subseteq X$ and[55] $z \in Z_\nabla$, contradicting (2).

2.6 A word on variants of the definitions

The following is not needed for understanding the paper, we just briefly comment on alternative ways some notions could be defined.

2.6.A Regarding "*alternating iterations*"

We call the set of $\alpha \in \omega_2$ such that Q_α is (forced to be) nontrivial the "*true domain*" of $\bar{P}$ (we use this notation in this remark only). Obviously $\bar{P}$ is naturally isomorphic to an iteration whose length is the order type of its true domain. In Definitions 2.116 and 2.118, we could have imposed the following additional requirements. All these variants lead to equivalent forcing notions.

[55] Note that we get the same Borel code, whether we evaluate $\dot{Z}_\nabla$ in $M^y[H^y_\beta]$ or in $V^*[G^* * H_\beta]$. Accordingly, the actual Borel set of reals coded by Z_∇ in the smaller universe is a subset of the corresponding Borel set in the larger universe.

1. M^x is (an ord-collapse of) an *elementary* submodel of $H(\chi^*)$.
   This is equivalent, as conditions coming from elementary submodels are dense in our $\mathbb{R}$, by Fact 2.119.
   While this definition looks much simpler and therefore nicer (we could replace ord-transitive models by the better understood elementary models), it would not make things easier and just "hides" the point of the construction: For example, we use models M^x that are (an ord-collapse of) an elementary submodel of $H^{V'}(\chi^*)$ for some forcing extension V' of V.

2. Require that (M^x thinks that) the true domain of $\bar{P}^x$ is ω_2.
   This is equivalent for the same reason as (1) (and this requirement is compatible with (1)).
   This definition would allow to drop the "trivial" option from the definition. The whole proof would still work with minor modifications — in particular, because of the following fact: [56]

   The finite support iteration of σ-centered forcing notions of length $< (2^{\aleph_0})^+$ is again σ-centered. (2.151)

   We chose our version for two reasons: first, it seems more flexible, and second, we were initially not aware of (2.151).

3. Alternatively, require that (M^x thinks that) the true domain of $\bar{P}^x$ is countable.
   Again, equivalence can be seen as in (1), again (3) is compatible with (1) but obviously not with (2).
   This requirement would not make the definition easier, so there is no reason to adopt it. It would have the slight inconvenience that instead of using ord-collapses as in Fact 2.119, we would have to put another model on top to make the iteration countable. Also, it would have the (purely aesthetic) disadvantage that the generic iteration itself does not satisfy this requirement.

4. Also, we could have dropped the requirement that the iteration is proper. It is never directly used, and "densely" $\bar{P}$ is proper anyway. (E.g., in Lemma 2.145(4)(a), we would just construct $\bar{P}$ up to α to be proper or even ccc, so that X remains uncountable.)

[56] We are grateful to Stefan Geschke and Andreas Blass for pointing out this fact. The only references we are aware of are [Tal94, proof of Lemma 2] and [Bla11].

2.6.B Regarding "*almost CS iterations and separative iterands*"

Recall that in Definition 2.82 we required that each iterand Q_α in a partial CS iteration is separative. This implies the property (actually: the three equivalent properties) from Fact 2.84. Let us call this property "*suitability*" for now. Suitability is a property of the limit P_ε of $\bar{P}$. Suitability always holds for finite support iterations and for countable support iterations. However, if we do not assume that each Q_α is separative, then suitability may fail for partial CS iterations. We could drop the separativity assumption, and instead add suitability as an additional natural requirement to the definition of partial CS limit.

The disadvantage of this approach is that we would have to check in all constructions of partial CS iterations that suitability is indeed satisfied (which we found to be straightforward but rather cumbersome, in particular in the case of the almost CS iteration).

In contrast, the disadvantage of assuming that Q_α is separative is minimal and purely cosmetic: It is well known that every quasiorder Q can be made into a separative one which is forcing equivalent to the original Q (e.g., by just redefining the order to be $\leq^*_Q$).

2.6.C Regarding "*preservation of random and quick sequences*"

Recall Definition 2.53 of local preservation of random reals and Lemma 2.108.

In some respect the dense sets D_n are unnecessary. For ultralaver forcing $\mathbb{L}_{\bar{D}}$, the notion of a "quick" sequence refers to the sets D_n of conditions with stem of length at least n.

We could define a new partial order on $\mathbb{L}_{\bar{D}}$ as follows:

$$q \leq' p \;\Leftrightarrow\; (q = p) \text{ or } (q \leq p \text{ and the stem of } q \text{ is strictly longer than the stem of } p).$$

Then $(\mathbb{L}_{\bar{D}}, \leq)$ and $(\mathbb{L}_{\bar{D}}, \leq')$ are forcing equivalent, and any $\leq'$-interpretation of a new real will automatically be quick.

Note however that $(\mathbb{L}_{\bar{D}}, \leq')$ is now not separative any more. Therefore we chose not to take this approach, since losing separativity causes technical inconvenience, as described in Section 2.6.B.

Chapter 3

A strengthening of the dual Borel Conjecture

In Chapter 2 (i.e., [GKSW]), we proved Con(BC+dBC), i.e., we constructed a model of ZFC and showed that both BC and dBC hold in this model; in other words, the model satisfies

$$\mathcal{M}^* = \mathcal{SN} = [2^\omega]^{\leq\aleph_0} = \mathcal{SM} = \mathcal{N}^*$$

(recalling that $\mathcal{M}^* = \mathcal{SN}$ by Galvin-Mycielski-Solovay, and $\mathcal{SM} = \mathcal{N}^*$ by definition).

In this chapter, we strengthen[1] the result by showing that there is no uncountable *very meager* set in the model of Chapter 2. For the concept of "very meager" set, see Definition 1.20 on page 23 (and the discussion there).

This is joint work with Saharon Shelah.

Strengthening of dBC in our model of BC+dBC

Let us state the result once again:

Theorem 3.1. *In the model for Con(BC+dBC) of Chapter 2, we even have*

$$\mathcal{VM} = [2^\omega]^{\leq\aleph_0}.$$

[1]I thank Marcin Kysiak for asking me (at the Winterschool 2011 in Hejnice, Czech Republic) whether this strengthening of dBC holds true in our model [GKSW] of BC+dBC.

Note that $[2^\omega]^{\leq\aleph_0} = \mathcal{SM} = \sigma\langle\mathcal{SM}\rangle$ holds true in the model anyway (by dBC). Actually, Kysiak's inducement for the question was that he wondered whether it could be the "first model" with $\sigma\langle\mathcal{SM}\rangle \subsetneq \mathcal{VM}$. Theorem 3.1 shows that it is not.

Before we describe how to adapt the proof (of dBC) in Chapter 2 to obtain the above theorem, let us point out that the result is *not* – as one might think at first sight – an "asymmetric strengthening" of Con(BC+dBC), only being concerned with dBC. However, the respective "strengthening of BC" (i.e., $\mathcal{M}^{\circledast} = [2^\omega]^{\leq\aleph_0}$) is void anyway. More precisely:

Corollary 3.2. *In the model for Con(BC+dBC) of Chapter 2, we have*

$$\mathcal{M}^{\circledast} = \mathcal{M}^{*} = \mathcal{SN} = [2^\omega]^{\leq\aleph_0} = \mathcal{SM} = \mathcal{N}^{*} = \mathcal{N}^{\circledast}.$$

Proof. Since BC and dBC hold in the model, we know that

$$\mathcal{M}^{*} = \mathcal{SN} = [2^\omega]^{\leq\aleph_0} = \mathcal{SM} = \mathcal{N}^{*}$$

holds true (without using Theorem 3.1).

Now recall that $\mathcal{VM} = \mathcal{N}^{\circledast}$ holds by definition, so Theorem 3.1 indeed yields $\mathcal{N}^{\circledast} = [2^\omega]^{\leq\aleph_0}$. Moreover, Theorem 1.21 says that $\mathcal{SN} = \mathcal{M}^{\circledast}$ holds anyway (i.e., in ZFC). $\square$

3.1 Janus forcing kills very meager sets

Recall that (according to Definition 1.18) $X \in \mathcal{VM} = \mathcal{N}^{\circledast}$ if and only if

$$\forall Z \in \mathcal{N} \quad \exists \bigcup_l X_l = X \quad \forall l \in \omega \quad X_l + Z \neq 2^\omega$$

(where "$\exists \bigcup_l X_l = X$" is an abbreviation for "there exists a partition of X into countably many pieces $(X_l)_{l\in\omega}$").

Definition 3.3. Let $Z \subseteq 2^\omega$ be a null set (i.e., $Z \in \mathcal{N}$).

We say that X *is not very meager witnessed by* Z if the following holds: whenever $\bigcup_l X_l = X$ is a partition of X, there exists an $l \in \omega$ such that $X_l + Z = 2^\omega$.

It is obvious by definition that the following holds:

$X \notin \mathcal{VM} \iff \exists Z \in \mathcal{N}$ such that "X is not very meager witnessed by Z".

We now adapt Lemma 2.63 of Section 2.2.B to the setting of "killing very meager sets" (instead of "killing strongly meager sets").

The original lemma (i.e., Lemma 2.63) reads:

> If X is not thin, $\mathbb{J}$ is a countable Janus forcing based on $\bar{\ell}^*$, and $\underaccent{\tilde}{R}$ is a $\mathbb{J}$-name for a σ-centered forcing notion, then $\mathbb{J} * \underaccent{\tilde}{R}$ forces that X is not **strongly meager** witnessed by the null set $\underaccent{\tilde}{Z}_\nabla$.

Recall that we have a fixed increasing sequence $\bar{\ell}^* = (\ell^*_i)_{i\in\omega}$ and B^*, and that whenever we say "(very) thin" we mean "(very) thin with respect to $\bar{\ell}^*$ and B^*" (see Section 2.1.D, in particular Definition 2.25).

The adapted lemma reads as follows (and will be used in Section 3.2 to obtain the strengthening of dBC, i.e., $\mathcal{VM} = [2^\omega]^{\leq\aleph_0}$, in the final model):

Lemma 3.4. *If X is not thin, $\mathbb{J}$ is a countable Janus forcing based on $\bar{\ell}^*$, and $\underset{\sim}{R}$ is a $\mathbb{J}$-name for a σ-centered forcing notion, then $\mathbb{J} * \underset{\sim}{R}$ forces that X is not* **very meager** *witnessed by the null set $\underset{\sim}{Z}_\nabla$.*

Proof. Let $\underset{\sim}{c}$ be a $\mathbb{J}$-name for a function $\underset{\sim}{c} : \underset{\sim}{R} \to \omega$ witnessing that $\underset{\sim}{R}$ is σ-centered.

Assume towards a contradiction that $(p, r) \in \mathbb{J} * \underset{\sim}{R}$ forces the opposite. So we can fix $(\mathbb{J} * \underset{\sim}{R})$-names $(\underset{\sim}{\xi}_l)_{l<\omega}$ and "partition labels" $(\underset{\sim}{l}_x)_{x\in X}$ (i.e., the name $\underset{\sim}{l}_x$ tells us which part of the partition of X the element x belongs to) such that $(p, r) \Vdash (\forall x \in X)\ \underset{\sim}{\xi}_{\underset{\sim}{l}_x} \notin x + \underset{\sim}{Z}_\nabla$. By definition of $\underset{\sim}{Z}_\nabla$, we get

$$(p, r) \Vdash (\forall x \in X)\,(\exists n \in \omega)\,(\forall i \geq n)\ \underset{\sim}{\xi}_{\underset{\sim}{l}_x}{\restriction}L_i \notin x{\restriction}L_i + \underset{\sim}{C}^\nabla_i.$$

For each $x \in X$ we can find $(p_x, r_x) \leq (p, r)$ and natural numbers $n_x \in \omega$, $m_x \in \omega$ and $l_x \in \omega$ such that p_x forces that $\underset{\sim}{c}(r_x) = m_x$, and that

$$(p_x, r_x) \Vdash \underset{\sim}{l}_x = l_x$$

and

$$(p_x, r_x) \Vdash (\forall i \geq n_x)\ \underset{\sim}{\xi}_{l_x}{\restriction}L_i \notin x{\restriction}L_i + \underset{\sim}{C}^\nabla_i.$$

So $X = \bigcup_{p\in\mathbb{J}, m\in\omega, n\in\omega, l\in\omega} X_{p,m,n,l}$, where $X_{p,m,n,l}$ is the set of all x with $p_x = p$, $m_x = m$, $n_x = n$, $l_x = l$. (Note that $\mathbb{J}$ is countable, so the union is countable.) As X is not thin, there is some p^*, m^*, n^*, l^* such that $X^* := X_{p^*,m^*,n^*,l^*}$ is not very thin.

So[2] we get for all $x \in X^*$:

$$(p^*, r_x) \Vdash (\forall i \geq n^*)\ \underset{\sim}{\xi}_{l^*}{\restriction}L_i \notin x{\restriction}L_i + \underset{\sim}{C}^\nabla_i. \tag{3.1}$$

Since X^* is not very thin, there is some $i_0 \in \omega$ such that for all $i \geq i_0$

$$\text{the (finite) set } X^*{\restriction}L_i \text{ has more than } B^*(i) \text{ elements.} \tag{3.2}$$

Due to the fact that $\mathbb{J}$ is a Janus forcing (see Definition 2.59 (3)), there are arbitrarily large $i \in \omega$ such that there is a core condition $\sigma = (A_0, \ldots, A_{i-1}) \in \nabla$ with

$$\frac{|\{A \in \mathcal{A}_i : \sigma^\frown A \not\perp_{\mathbb{J}} p^*\}|}{|\mathcal{A}_i|} \geq \frac{2}{3}. \tag{3.3}$$

[2]From here on, the proof is literally the same as the original proof of Lemma 2.63 (with $\underset{\sim}{\xi}$ replaced by $\underset{\sim}{\xi}_{l^*}$).

Fix such an i larger than both i_0 and n^*, and fix a condition σ satisfying (3.3).

We now consider the following two subsets of $\mathcal{A}_i$:

$$\{A \in \mathcal{A}_i : \sigma^\frown A \not\perp_{\mathbb{J}} p^*\} \text{ and } \{A \in \mathcal{A}_i : X^*\restriction L_i + A = 2^{L_i}\}. \tag{3.4}$$

By (3.3), the relative measure (in $\mathcal{A}_i$) of the left one is at least $\frac{2}{3}$; due to (3.2) and the definition of $\mathcal{A}_i$ according to Corollary 2.56, the relative measure of the right one is at least $\frac{3}{4}$; so the two sets in (3.4) are not disjoint, and we can pick an A belonging to both.

Clearly, $\sigma^\frown A$ forces (in $\mathbb{J}$) that $\underset{\sim}{C}_i^\nabla$ is equal to A. Fix $q \in \mathbb{J}$ witnessing $\sigma^\frown A \not\perp_{\mathbb{J}} p^*$. Then

$$q \Vdash_{\mathbb{J}} X^*\restriction L_i + \underset{\sim}{C}_i^\nabla = X^*\restriction L_i + A = 2^{L_i}. \tag{3.5}$$

Since p^* forces that for each $x \in X^*$ the color $\underset{\sim}{c}(r_x) = m^*$, we can find an r^* which is (forced by $q \leq p^*$ to be) a lower bound of the *finite* set $\{r_x : x \in X^{**}\}$, where $X^{**} \subseteq X^*$ is any finite set with $X^{**}\restriction L_i = X^*\restriction L_i$.

By (3.1),

$$(q, r^*) \Vdash \underset{\sim}{\xi}_{l^*}\restriction L_i \notin X^{**}\restriction L_i + \underset{\sim}{C}_i^\nabla = X^*\restriction L_i + \underset{\sim}{C}_i^\nabla,$$

contradicting (3.5). $\square$

3.2 Strengthening of dBC in the final model

We begin with a reformulation of Definition 3.3:

Lemma 3.5. *Let $X \subseteq 2^\omega$, and let $Z \subseteq 2^\omega$ be a null set. Then the following are equivalent:*

1. *X is not very meager witnessed by Z (i.e., whenever $\bigcup_l X_l = X$ is a partition of X, there exists an $l \in \omega$ such that $X_l + Z = 2^\omega$),*
2. *for each countable set $T \subseteq 2^\omega$, we have $X \nsubseteq T + (2^\omega \setminus Z)$.*

Proof. An easy computation shows that the two assertions are equivalent (as already mentioned in Section 1.2 where the notion of "very meager" was introduced; compare with items (1) and (2) in the proof of Lemma 1.19). $\square$

Now we are prepared to present the adapted version of Lemma 2.146 of Section 2.5.A:

Lemma 3.6. *Let $X \in V$ be an uncountable set of reals. Then $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that X is not* **very meager**.

Note that (as the lemmas in Section 2.5.A) the lemma shows the strengthening of dBC *only* for sets in the *ground model* V.

However, the transition from the proof of Lemma 3.6 to the arguments required to show the general case (i.e., for arbitrary sets X) is not influenced by the replacement of "strongly meager" by "very meager", and therefore completely analogous to the transition from Lemma 2.146 to the general case in Section 2.5.B (using the "factor lemma").

So we do not repeat the arguments given there, i.e., Lemma 3.6 finishes the proof of Theorem 3.1.

Proof of Lemma 3.6. The proof is parallel to the one of Lemma 2.146 (and therefore also to the one of Lemma 2.145) of Section 2.5.A:

1. Fix any even $\alpha < \omega_2$ (i.e., an ultralaver position) in our iteration. The Janus forcing $\mathbf{Q}_{\alpha+1}$ adds a (canonically defined code for a) null set $\dot{Z}_\nabla$. (See Definition 2.60 and Fact 2.61.)

   In the following, when we consider various Janus forcings $\mathbf{Q}_{\alpha+1}$, $Q_{\alpha+1}$, $Q^x_{\alpha+1}$, we treat $\dot{Z}_\nabla$ not as an actual name, but rather as a definition which depends on the forcing used.

2. According to the definition of very meager (see also the comment after Definition 3.3), it is enough to show that "X is not very meager witnessed by $\dot{Z}_\nabla$" holds in the $\mathbb{R} * \mathbf{P}_{\omega_2}$-extension; by Lemma 3.5, this is equivalent to saying that $X \not\subseteq T + (2^\omega \setminus \dot{Z}_\nabla)$ holds for every countable set $T \subseteq 2^\omega$.

   Assume towards a contradiction that we have $X \subseteq T + (2^\omega \setminus \dot{Z}_\nabla)$ for some fixed countable $T \subseteq 2^\omega$ (in the $\mathbb{R} * \mathbf{P}_{\omega_2}$-extension). We can fix a β with $\alpha < \beta < \omega_2$ such that T already exists in the $\mathbb{R} * \mathbf{P}_\beta$-extension; note that $X \subseteq T + (2^\omega \setminus \dot{Z}_\nabla)$ holds there as well (by absoluteness). So we can fix a condition $(x, p) \in \mathbb{R} * \mathbf{P}_{\omega_2}$ and an $\mathbb{R} * \mathbf{P}_\beta$-name $\dot{\underset{\sim}{T}}$ of a countable set of reals such that

   $$(x, p) \Vdash X \subseteq \dot{\underset{\sim}{T}} + (2^\omega \setminus \dot{Z}_\nabla). \tag{3.6}$$

3. Using the dense embedding $j_{\omega_2} : \mathbf{P}'_{\omega_2} \to \mathbf{P}_{\omega_2}$, we may replace (x, p) by a condition $(x, p') \in \mathbb{R} * \mathbf{P}'_{\omega_2}$. According to Fact 2.129 (recall that we know that $\mathbf{P}_{\omega_2}$ satisfies ccc) and Lemma 2.130 (note that Lemma 2.130 allows for countably many reals, so it is no problem to apply it to our name $\dot{\underset{\sim}{T}}$ of a countable[3] set of reals) we can assume that p' is already a P^x_β-condition p^x and that $\dot{\underset{\sim}{T}}$ is (forced by x to be the same as) a P^x_β-name $\dot{T}^x$ in M^x.

[3] Actually, at this point it is crucial that we have argued via the equivalent formulation

4. We construct (in V) an iteration $\bar{P}$ in the following way:

   (a) Up to α, we take an arbitrary alternating iteration into which x embeds. In particular, P_α again forces that X is still uncountable.

   (b1) Let Q_α be any ultralaver forcing (over Q^x_α). Then Q_α forces that X is not thin (see Corollary 2.27).

   (b2) Let $Q_{\alpha+1}$ be a countable Janus forcing. So $Q_{\alpha+1}$ forces "X is not very meager witnessed by $\dot{Z}_\nabla$". Here we apply our adapted lemma, i.e., we use Lemma 3.4 instead of Lemma 2.63.

   (c) We continue the iteration in a σ-centered way. I.e., we use an almost FS iteration over x of ultralaver forcings and countable Janus forcings, using trivial Q_ζ for all $\zeta \notin M^x$; see Lemma 2.93.

   (d) So P_β still forces that "X is not very meager witnessed by $\dot{Z}_\nabla$", i.e., $X \nsubseteq T + (2^\omega \setminus \dot{Z}_\nabla)$ for each countable $T \subseteq 2^\omega$ (recall Lemma 3.5). This is again due to Lemma 3.4 (instead of Lemma 2.63).
   So in particular, it is forced that $X \nsubseteq \dot{T}^x + (2^\omega \setminus \dot{Z}_\nabla)$.

   As usual, we pick a countable $N \prec H(\chi^*)$ containing everything and ord-collapse $(N, \bar{P})$ to $y \leq x$. (See Fact 2.119.) Set $X^y := X \cap M^y$ (the image of X under the collapse). By elementarity, M^y thinks that (a)–(d) above holds for $\bar{P}^y$ and that X^y is uncountable. Note that $X^y \subseteq X$.

5. As always, this gives a contradiction: Let G be $\mathbb{R}$-generic over V and contain y, and let H_β be $\mathbf{P}_\beta$-generic over $V[G]$ and contain p; then $M^y[H^y_\beta]$ thinks that $X^y \nsubseteq T+(2^\omega \setminus Z_\nabla)$ (where T is $\dot{T}^x$ evaluated by H^y_β); so there is an $x \in X^y$ which is not in $T + (2^\omega \setminus Z_\nabla)$; but $x \in X^y \subseteq X$, and $\dot{\underaccent{\tilde}{T}}$ is forced to be the same as $\dot{T}^x$ (see (3)), contradicting (3.6). $\square$

of "X is not very meager witnessed by $\dot{Z}_\nabla$" given in Lemma 3.5: it is no problem to "capture" a *countable* set T by a condition of the preparatory forcing; if we would use the original formulation (i.e., "there exists a partition of $X\dots$"), we would run into troubles because it is not clear how to capture ω_1 many "partition labels" (telling to which part of the partition each of the reals of X belongs to).

Chapter 4

A projective well-order of the reals and BC/dBC

In this chapter, we show that the existence of a *projective well-order* of the reals is consistent with the Borel Conjecture (and the dual Borel Conjecture, respectively). Actually, the respective well-orders are Δ^1_3 *definable*.

To prove our results, we describe how to apply the techniques in [FF10] and [FFZ11]; the presentation is by far not self-contained, but heavily relies on these two papers.

In Section 4.1, we show that the existence of a Δ^1_3 definable well-order of the reals is *consistent with BC*, using the machinery of [FF10].

In Section 4.2, we show that the existence of a Δ^1_3 definable well-order of the reals is *consistent with dBC*, using the machinery of [FFZ11].

This is joint work with Sy D. Friedman.

Historical information

Quoting from the introduction of [FF10] (which is – according to the authors – the first work on projective well-orders and cardinal characteristics of the continuum), we give some historical information:

> If $V = L$ then there exists a Σ^1_2 well-ordering of the reals. Furthermore, by Mansfield's Theorem (see [Jec03, Theorem 25.39]) the existence of a Σ^1_2 well-ordering of the reals implies that every real is constructible. Using a finite support iteration of ccc posets, L. Harrington showed that the existence of a $\boldsymbol{\Delta^1_3}$ wellordering of the reals is consistent with the continuum being arbitrarily large (see [Har77, Theorem A]). S. D. Friedman showed that Martin's

Axiom (and not CH) is consistent with the existence of a Δ^1_3 definable wellordering of the reals (see [Fri00] and see [Har77] for the corresponding boldface result).

MA makes all cardinal characteristics of the continuum large, and is also incompatible with BC as well as dBC. So the results mentioned are clearly not sufficient to get a model of "BC (or dBC) + a projective well-order of the reals".

Question

With Chapter 2 in mind, it is natural to ask whether the respective techniques can be combined:

Question 4.1. Is the existence of a projective well-order of the reals consistent with[1] BC+dBC?

4.1 A projective well-order and BC

In this section, we describe how to combine Laver's proof of Con(BC) with the methods from the paper "Cardinal characteristics and projective wellorders" by Vera Fischer and Sy D. Friedman (see [FF10]) in order to obtain a model of ZFC satisfying "BC + there exists a projective well-order of the reals":

Theorem 4.2. *The existence of a Δ^1_3 definable well-order of the reals is consistent with the Borel Conjecture (and $2^{\aleph_0} = \aleph_2$).*

By a theorem of Judah, Shelah, and Woodin (see [JSW90]), there is a model of the Borel Conjecture with large continuum (i.e., $2^{\aleph_0} > \aleph_2$). The involved proof demonstrates that BC remains valid when *adding many random reals* to Laver's model of BC.

Since the known methods for getting projective well-orders do not seem to allow for random reals, it is unclear to me whether it is possible to obtain such a model with a projective well-order:

Question 4.3. Is the existence of a Δ^1_3 definable[2] well-order of the reals consistent with the Borel Conjecture and $2^{\aleph_0} \geq \aleph_3$?

[1] Or, thinking of the strengthening given in Chapter 3, even with $\mathcal{N}^{\circledast} = \mathcal{M}^{\circledast} = [2^\omega]^{\leq\aleph_0}$ (cf. Corollary 3.2)?

[2] ... or any other complexity in the projective hierarchy

Proof of Theorem 4.2. For the key points of Laver's proof of Con(BC), see Theorem 1.23 and the subsequent discussion.

To obtain a model of BC, it is sufficient to proceed as follows:

1. start with a *model of CH*,
2. perform a *countable support iteration* (of length ω_2),
3. make sure that (at least) **cofinally** many of the iterands "*kill old (uncountable) strong measure zero sets*" (e.g., Laver forcing does),
4. *avoid resurrection* of "sets that have been killed"; this is guaranteed provided the tail of the forcing iteration has the *Laver property*, which can be enforced as follows:
   (a) make sure that **all** the iterands have the *Laver property*,
   (b) use some type of forcing iteration that *preserves the Laver property* (e.g., a countable support iteration of proper forcings),
5. make sure that ω_1 *is preserved*,
6. make sure that ω_2 *is preserved*.

So let us check that the above can be arranged within the framework of [FF10] (which leads to a Δ^1_3 definable well-order of the reals).

The (template for the) forcing iteration used in [FF10] is defined as follows: according to [FF10, Section 5], $\mathbb{P}_{\omega_2}$ is a countable support iteration, with iterands $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$, where

I. $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for an **arbitrary** proper forcing notion (of cardinality at most $\aleph_1$),

II. $\dot{\mathbb{Q}}^1_\alpha$ is either (a name for) the trivial poset, or $\dot{\mathbb{Q}}^1_\alpha = \dot{\mathbb{K}}^0_\alpha * \dot{\mathbb{K}}^1_\alpha * \dot{\mathbb{K}}^2_\alpha$, where

   i. $\dot{\mathbb{K}}^0_\alpha$ is composed of "club shooting" forcings of the form $\mathcal{Q}(S)$ (with $S \subseteq \omega_1$ stationary, co-stationary),
   ii. $\dot{\mathbb{K}}^1_\alpha$ is (a name for) a "localization" forcing $\mathcal{L}(\phi_\alpha)$",
   iii. $\dot{\mathbb{K}}^2_\alpha$ is (a name for) a "coding" forcing $\mathcal{C}(Y_\alpha)$ (whose conditions are perfect trees, i.e., $\mathcal{C}(Y_\alpha)$ is similar to Sacks forcing).

We now perform an iteration according to this template, and let $\dot{\mathbb{Q}}^0_\alpha$ to be (a name for) Laver forcing for all α (note that this is allowed by the template since Laver forcing is proper; see (I)).

In this way, we obtain a model with a Δ^1_3 definable well-order of the reals. In order to confirm that BC holds in this final model, we have to check that items (1)–(6) above are satisfied:

1. our ground model is L, hence CH is satisfied;
2. by definition, $\mathbb{P}_{\omega_2}$ is a countable support iteration;
3. *cofinally many iterands kill* old (uncountable) strong measure zero sets, since we have chosen all our $\dot{\mathbb{Q}}^0_\alpha$'s to be Laver forcing;
4. the tails of the iteration have the *Laver property*:
   (a) all involved forcings have the *Laver property* (and are proper, or at least *S-proper*, for a fixed stationary set $S \subseteq \omega_1$ that belongs to the ground model):
      I. $\dot{\mathbb{Q}}^0_\alpha$ is always Laver forcing (which has the *Laver property*, and is *proper*);
      II. $\dot{\mathbb{Q}}^1_\alpha$ is either the trivial poset (hence has the *Laver property*, and is *proper*), or we have $\dot{\mathbb{Q}}^1_\alpha = \dot{\mathbb{K}}^0_\alpha * \dot{\mathbb{K}}^1_\alpha * \dot{\mathbb{K}}^2_\alpha$, where
         i. $\dot{\mathbb{K}}^0_\alpha$ doesn't add new reals by [FF10, Lemma 9], hence vacuously has the *Laver property*;
         moreover, it is *S-proper* (see [FF10, Section 4]);
         ii. $\dot{\mathbb{K}}^1_\alpha$ doesn't add new reals by [FF10, Lemma 4], hence again vacuously has the *Laver property*;
         moreover, it is *proper* by Lemma [FF10, Lemma 3];
         iii. also $\dot{\mathbb{K}}^2_\alpha$ has the *Laver property*, but [FF10, Lemma 8] is not quite sufficient for that:

            it only shows that $\dot{\mathbb{K}}^2_\alpha$ is ω^ω-bounding; but actually, it is implicit in the proof of [FF10, Lemma 8] that $\dot{\mathbb{K}}^2_\alpha$ has the Laver property (and hence the "Sacks property"); just directly use the finite sets d_k instead of their maxima; they have size 2^k, so the limit of the "fusion sequence" indeed forces that $\dot{f}$ is not just bounded but contained in a 2^k-slalom of the ground model (yielding the Laver property);

         moreover, $\dot{\mathbb{K}}^2_\alpha$ is *proper* by Lemma [FF10, Lemma 7];

(b) the *Laver property is preserved* under countable support iterations of *(S-)proper* forcings (analogous to [FF10, Lemma 18]);

5. ω_1 is preserved since the iteration is S-proper (by [FF10, Lemma 18]);

6. ω_2 is preserved since the iteration is $\aleph_2$-cc. $\square$

4.2 A projective well-order and dBC

In this section, we describe how to combine Carlson's proof of Con(dBC) with the methods from the paper "Projective wellorders and mad families with large continuum" by Vera Fischer, Sy D. Friedman and Lyubomyr Zdomskyy (see [FFZ11]) in order to obtain a model of ZFC satisfying "dBC + there exists a projective well-order of the reals":

Theorem 4.4. *The existence of a Δ^1_3 definable well-order of the reals is consistent with the dual Borel Conjecture (and both $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_0} = \aleph_3$).*

I do not know whether it is possible to get even larger[3] continuum:

Question 4.5. Is the existence of a Δ^1_3 definable[4] well-order of the reals consistent with the dual Borel Conjecture and $2^{\aleph_0} \geq \aleph_4$?

Proof of Theorem 4.4. We describe the version with $2^{\aleph_0} = \aleph_3$ (closely following the framework in [FFZ11], which is concerned with large continuum as well); the case $2^{\aleph_0} = \aleph_2$ is supposed to be similar (alternatively, one can easily derive it from the version with $2^{\aleph_0} = \aleph_3$, as demonstrated in Remark 4.6).

For the key points of Carlson's proof of Con(dBC), see Theorem 1.24 and the subsequent discussion; note that it is no problem to perform a finite support iteration of length more than ω_2 in Carlson's argument.

To obtain a model of dBC, it is sufficient to proceed as follows:

1. start with a *model of CH*,

2. perform a *finite support iteration* (of length $\geq \omega_2$),

3. make sure that (at least) **cofinally** many of the iterands "*kill old (uncountable) strongly meager sets*" (e.g., Cohen forcing does),

[3] It is no problem to get models of dBC with large continuum: just use a long finite support iteration of Cohen forcings.

[4] …or any other complexity in the projective hierarchy

4. *avoid resurrection* of "sets that have been killed"; this is guaranteed provided the tail of the forcing iteration is *precaliber* $\aleph_1$, which can be enforced as follows:

   (a) make sure that **all** the iterands have *precaliber* $\aleph_1$,

   (b) use some type of forcing iteration that *preserves precaliber* $\aleph_1$ (e.g., finite support iteration).

So let us check that the above can be arranged within the framework of [FFZ11] (which leads to a Δ^1_3 definable well-order of the reals).

The (involved) forcing machinery used in [FFZ11] consists of two parts (see [FFZ11, Section 2, Step 3]):

I. (a preparatory part) the poset $\mathbb{P}_0 := \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ (it is ω-distributive according to [FFZ11, Lemma 1]),

II. the finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma \colon \alpha \leq \omega_3, \gamma < \omega_3 \rangle$, starting with the above $\mathbb{P}_0$; and, for each $\alpha < \omega_3$, $\dot{\mathbb{Q}}_\alpha$ is a (name for a) σ-centered poset.

In this way, we obtain a model with a Δ^1_3 definable well-order of the reals, as shown in [FFZ11].

We claim that it is easy to arrange that dBC holds in this final model (alternatively, we can even argue that dBC holds true "automatically"); so let us go through items (1)–(4) above to make sure that dBC holds:

1. the *model of CH* that is our "ground model for Carlson's proof" is not the actual ground model $V = L$, but rather the forcing extension by the forcing $\mathbb{P}_0$ from (I): note that it is still a model of CH since the forcing $\mathbb{P}_0$ is ω-distributive (hence adds no reals);

2. after $\mathbb{P}_0$, the iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma \colon \alpha \leq \omega_3, \gamma < \omega_3 \rangle$ is a finite support iteration of length ω_3 (see (II));

3. it is no problem to explicitly add Cohen reals in between (along the way up to ω_3); then *cofinally many* of the iterands perform the task of *killing* old (uncountable) strongly meager sets;

   alternatively, one can argue as follows: as described in [FFZ11, Section 2, Step 3, Case 2], cofinally many of the $\dot{\mathbb{Q}}_\alpha$'s will be forcings similar to Hechler forcing, which add Cohen reals anyway (recall that the "Hechler real modulo 2" is a Cohen real); so we actually do not have to change the machinery in [FFZ11] at all to obtain a model of dBC;

4. the tails of the iteration are *precaliber* $\aleph_1$:

   (a) all the iterands $\dot{\mathbb{Q}}_\alpha$ are σ-centered (see (II)), hence in particular *precaliber* $\aleph_1$;

   (b) *precaliber* $\aleph_1$ *is preserved* under finite support iterations. $\square$

Remark 4.6. As mentioned above, the respective model for which $2^{\aleph_0} = \aleph_2$ can be obtained in a similar way.

However, we can also derive it directly from the $2^{\aleph_0} = \aleph_3$ case: just collapse the continuum (which is $\aleph_3$) to $\aleph_2$ by a $<\aleph_2$-closed forcing; neither new reals, nor new sets of reals of size $\aleph_1$ are added by the collapse; consequently, dBC remains true, and the "definition of the well-order" is not influenced by the collapse either.

Chapter 5

Galvin-Mycielski-Solovay theorem revisited

In this chapter, we revisit the Galvin-Mycielski-Solovay theorem (GMS), and give versions of the theorem for more general settings.

In Section 5.1, we consider the generalized Cantor space 2^κ and show that (a generalization of) the theorem holds for κ weakly compact (including $\kappa = \omega$, of course).

In Section 5.2, we mainly prove that (a suitable generalization of) the Galvin-Mycielski-Solovay theorem holds for every separable locally compact group.

In Section 5.3, we demonstrate that we definitely need certain assumptions to prove the theorem: we show that the Galvin-Mycielski-Solovay theorem (consistently) fails for the Baer-Specker group $\mathbb{Z}^\omega$ (see Theorem 5.53). This is one of the main results of the chapter and answers a question I was asked[1] by Marcin Kysiak.

5.1 GMS for 2^κ ($\kappa \geq \aleph_0$ weakly compact)

In this section, we consider the generalized Cantor space 2^κ and show that the respective generalization of the Galvin-Mycielski-Solovay theorem holds for weakly compact κ (Theorem 5.10). The proof comprises the well-known proof of the Galvin-Mycielski-Solovay theorem for the usual Cantor space 2^ω as a special[2] case.

[1]. . . at the Winterschool 2011 in Hejnice (Czech Republic)

[2]This is actually no surprise since "$\kappa = \omega$ is weakly compact" holds anyway (except for uncountability).

A systematic study of the (generalized) notion of "strong measure zero" in the generalized Cantor space 2^κ (and the generalized Baire space κ^κ) was started in Aapo Halko's thesis, see [Hal96]; more results[3] can be found[4] in [HS01].

Throughout the section, let $\kappa \geq \aleph_0$ be an infinite regular cardinal.

Topology on 2^κ, (closed) nowhere dense and meager sets

We consider the *generalized Cantor space* 2^κ, equipped with the topology generated by the collection $\{[s] \colon s \in 2^\kappa\}$ of *"basic clopens"*, where $[s]$ denotes the set of all "(generalized) reals" $f \in 2^\kappa$ extending s, i.e.,

$$[s] := \{f \in 2^\kappa \colon s \subseteq f\}.$$

So a set $O \subseteq 2^\kappa$ is *open* if it is the union of "basic clopen" sets (i.e., if there is a family $(s_i)_{i\in I}$ with $s_i \in 2^{<\kappa}$ for each $i < \kappa$ such that $O = \bigcup_{i\in I}[s_i]$), or, equivalently, if for each $f \in O$ there is an $i < \kappa$ such that $[f{\restriction}i] \subseteq O$.

As usual when dealing with the generalized Cantor space 2^κ (or the generalized Baire space κ^κ), e.g., in so-called generalized descriptive set theory, we assume[5] that there are "only few" basic clopen sets $[s]$ ($s \in 2^{<\kappa}$):

Assumption 5.1. We assume that $|2^{<\kappa}| = |\kappa^{<\kappa}| = \kappa$.

Remark 5.2. Note that Assumption 5.1 means that 2^κ has a basis consisting of basic clopens which is as small as possible (namely of size κ).

It is quite easy to see that $|2^{<\kappa}|$ also equals the smallest size of a dense subset of 2^κ. (Recall that a set $D \subseteq 2^\kappa$ is dense if for each $s \in 2^{<\kappa}$ there is an $f \in D \cap [s]$.) Consequently, $|2^{<\kappa}| = \kappa$ is also equivalent to the statement that there is a dense subset of 2^κ of size κ, in other words, "generalized separability".

So Assumption 5.1 makes sure that 2^κ behaves "analogous" to 2^ω regarding the size of a basis and separability. It clearly follows from κ being

[3]One of the main results of [HS01] is the theorem that the "generalized Borel Conjecture" (i.e., the statement that there are no "generalized" strong measure zero subsets of 2^κ of size κ^+) necessarily fails for successor cardinals $\kappa > \aleph_0$ (under Assumption 5.1).

In [GS13], one can find an alternative (quite different) approach to generalize the Borel Conjecture to higher cardinals κ.

[4]I thank Sy D. Friedman for suggesting to present results from this paper in his student seminar at the Kurt Gödel Research Center. This was the incentive for the investigations of the generalized Cantor space 2^κ presented in this section.

[5]See, e.g, the introduction of [FK], where they assume $|\kappa^{<\kappa}| = \kappa$ for their study of Borel equivalence relations on the generalized Baire space. Note that this assumption is the same as saying $|2^{<\kappa}| = \kappa$ (since $|2^{<\kappa}| = |\kappa^{<\kappa}|$ holds for all regular cardinals). The same assumption is used in [HS01].

inaccessible[6], so in particular from κ being weakly compact, which is the assumption of Theorem 5.10; in this sense, we have Assumption 5.1 for free anyway. Some lemmas, however, will deal with arbitrary regular cardinals κ, provided that $|2^{<\kappa}| = \kappa$ (e.g., Lemma 5.16). So we are going to mention Assumption 5.1 explicitly whenever it is used.

A set $C \subseteq 2^\kappa$ is *closed* if its complement $2^\kappa \setminus C$ is open; equivalently, if there is a tree $T \subseteq 2^{<\kappa}$ such that C is the set of branches through T, i.e., $C = [T]$, where

$$[T] := \{f \in 2^\kappa \colon \forall i < \kappa \, (f{\restriction}i \in T)\}.$$

It is easy to show that the family of closed sets is closed under unions of size strictly less than κ (and under arbitrary intersections), whereas the family of open sets is closed under intersections of size strictly less than κ (and under arbitrary unions).

A set C is *nowhere dense* if for each $s \in 2^{<\kappa}$ there exists a $t \in 2^{<\kappa}$ such that $t \supseteq s$ and $[t] \cap C = \emptyset$. It is easy to see that each nowhere dense set is contained in a closed nowhere dense set. Moreover, a set $C \subseteq 2^\kappa$ is closed nowhere dense if and only if $2^\kappa \setminus C$ is open dense.

It is easy to show that the union of *strictly less* than κ many (closed) nowhere dense sets is (closed) nowhere dense:

Lemma 5.3. *Let $(C_i)_{i<\alpha}$, $\alpha < \kappa$, be (closed) nowhere dense sets, then $\bigcup_{i<\alpha} C_i$ is (closed) nowhere dense.*

The union of κ many (closed) nowhere dense sets is typically not nowhere dense:

Definition 5.4. A set $M \subseteq 2^\kappa$ is *meager* if it is covered by κ many (closed) nowhere dense sets, i.e., if there are $(C_i)_{i<\kappa}$ with C_i (closed) nowhere dense (for each $i < \kappa$) such that $M \subseteq \bigcup_{i<\kappa} C_i$.

Note that in particular each (closed) nowhere dense set is meager. Moreover, it easily follows from Lemma 5.3 that each meager set can be written as (covered by) an increasing union of (closed) nowhere dense sets.

$\mathcal{SN}(2^\kappa)$ — the strong measure zero sets on 2^κ

The following definition is the natural generalization of the "elementary definition" of strong measure zero to the generalized Cantor space 2^κ (see [Hal96, Definition 4.1], or [HS01, Definition 2.2]):

[6] As well as from GCH, of course.

Definition 5.5. Let $\kappa \geq \aleph_0$ be a regular cardinal. A set $X \subseteq 2^\kappa$ is *strong measure zero* ($X \in \mathcal{SN}(2^\kappa)$) if for each (strictly increasing) sequence[7] $(\alpha_i)_{i<\kappa}$ with $\alpha_i < \kappa$ (for each $i < \kappa$) there is a sequence $(u_i)_{i<\kappa}$ with $u_i \in 2^{\alpha_i}$ (for each $i < \kappa$) such that

$$X \subseteq \bigcup_{i<\kappa} [u_i].$$

The following lemma can also be found in [Hal96, Proposition 7.4]:

Lemma 5.6. *A set $X \subseteq 2^\kappa$ is strong measure zero if and only if for each (strictly increasing) sequence $(\alpha_i)_{i<\kappa}$ with $\alpha_i < \kappa$ (for each $i < \kappa$) there is a sequence $(u_i)_{i<\kappa}$ with $u_i \in 2^{\alpha_i}$ (for each $i < \kappa$) such that*

$$X \subseteq \bigcap_{j\in\kappa} \bigcup_{i\geq j} [u_i].$$

Note that the lemma just says that the definition of strong measure zero set doesn't change when we require each element of the set to be in cofinally many of the $[u_i]$'s instead of only one.

Proof. Partition $\kappa = \bigcup_{l<\kappa} A_l$ into κ many sets A_l, each of size κ.

Suppose $X \subseteq 2^\kappa$ is strong measure zero, and fix $(\alpha_i)_{i<\kappa}$. For each $l < \kappa$, apply the definition of strong measure zero to the subsequence $(\alpha_i)_{i\in A_l}$ to get a sequence $(u_i)_{i\in A_l}$ with $u_i \in 2^{\alpha_i}$ (for each $i \in A_l$) such that $X \subseteq \bigcup_{i\in A_l} [u_i]$. So altogether we have a sequence $(u_i)_{i<\kappa}$ such that $X \subseteq \bigcap_{j\in\kappa} \bigcup_{i\geq j} [u_i]$ (due to the fact that for each $j \in \kappa$, there is an $l < \kappa$ such that the minimum of A_l is above j). $\square$

$\mathcal{M}^*(2^\kappa)$ — the meager-shiftable sets on 2^κ

Recall that (for $X, Y \subseteq 2^\kappa$ and $z \in 2^\kappa$) $X + Y := \{x + y : x \in X, y \in Y\}$, and $X + z := \{x + z : x \in X\}$, where, given two elements $x, y \in 2^\kappa$, its sum $x + y$ is the "bitwise sum modulo 2", i.e., $x + y$ is the "real" satisfying $(x + y)(i) = x(i) + y(i) \bmod 2$ for each $i < \kappa$.

Definition 5.7. A set $X \subseteq 2^\kappa$ is *meager-shiftable* ($X \in \mathcal{M}^*(2^\kappa)$) if for each meager set $M \subseteq 2^\kappa$ we have $X + M \neq 2^\kappa$.

Note that $X + M \neq 2^\kappa$ if and only if X can be "translated away" from M (i.e., there is a "*translation real*" $z \in 2^\kappa$ such that $(X + z) \cap M = \emptyset$).

[7] This sequence is the analogue of the "ε_n-sequence" in the usual definition of strong measure zero in $\mathbb{R}$ and other metric spaces (see Definition 1.6)...

Galvin-Mycielski-Solovay theorem for weakly compact κ

We are going to prove that the (generalized) Galvin-Mycielski-Solovay theorem holds for all weakly compact cardinals κ (and the same proof shows it for $\kappa = \omega$ as well).

Definition 5.8. A regular cardinal κ has the *tree property* if there is no κ-Aronszajn tree, i.e., if every tree of height κ with levels of size strictly less than κ has a cofinal branch.

Note that the cardinal $\kappa = \aleph_0$ has the tree property: this is just König's lemma.

One of the many (equivalent) definitions of "weak compactness" is the following (see [Jec03, Lemma 9.26]):

Definition 5.9. A cardinal κ is *weakly compact* if it is inaccessible and has the tree property.

Now we can state the main result of this section:

Theorem 5.10. *Let κ be weakly compact, or $\kappa = \omega$. Suppose that $X \subseteq 2^\kappa$. Then X is strong measure zero if and only if X is meager-shiftable (i.e., X can be translated away from each meager set):*

$$\mathcal{SN}(2^\kappa) = \mathcal{M}^*(2^\kappa).$$

The rest of the section is devoted to the proof of this theorem.

Tree property vs. "compactness property"

In our proof of Theorem 5.10 we will use the following property of κ:

Definition 5.11. A regular cardinal κ has the *compactness property*[8] if each cover of 2^κ by basic clopen[9] sets has a subcover of size strictly less than κ; more explicitly[10], for each $\{s_i : i < \kappa\}$ with $s_i \in 2^{<\kappa}$ satisfying $\bigcup_{i<\kappa}[s_i] = 2^\kappa$, there is an $\alpha < \kappa$ such that $\bigcup_{i<\alpha}[s_i] = 2^\kappa$.

We start with a trivial fact:

[8] This is somehow the assertion that 2^κ is "compact", but – so to speak – in the sense of generalized descriptive set theory, not with respect to the Tychonoff (i.e., usual product) topology (with respect to which 2^κ is always compact).

[9] Note that we could equivalently say "open set" instead of "basic clopen", since every open set is just a union of basic clopens.

[10] It is enough to consider collections of only κ many basic clopens since $2^{<\kappa}$ has only κ many elements, provided that we adopt Assumption 5.1.

Lemma 5.12. *The cardinal $\kappa = \aleph_0$ has the compactness property.*

Proof. This is just the "usual compactness" of 2^ω (or König's lemma, respectively). $\square$

For inaccessibles, the compactness property is equivalent to the tree property (actually, even more can be shown; see Corollary 5.15):

Lemma 5.13. *Suppose κ is inaccessible. Then κ is weakly compact if and only if it has the compactness property.*

Proof. Fix an inaccessible cardinal κ. We show that κ has the tree property if and only if it has the compactness property.

So assume[11] that κ does *not* have the compactness property, i.e., assume we have a collection $\{s_i : i < \kappa\}$ with $s_i \in 2^{<\kappa}$ such that $\bigcup_{i<\kappa}[s_i] = 2^\kappa$, but

$$\forall \alpha < \kappa \quad \bigcup_{i<\alpha}[s_i] \neq 2^\kappa. \tag{5.1}$$

Now define a tree $T \subseteq 2^{<\kappa}$ by removing all nodes $t \in 2^{<\kappa}$ from the full tree $2^{<\kappa}$ which are above an s_i for some $i < \kappa$. Clearly, T is a tree, its levels are of size strictly less than κ (since κ is inaccessible), and all levels of T are non-empty (i.e., T is of height κ): otherwise, there is a $\gamma < \kappa$ such that for each node $t \in 2^\gamma$ there is an i with $t \supseteq s_i$; since $|2^\gamma| < \kappa$, there are only $<\kappa$ many such i's (say, $\alpha < \kappa$ is a bound), yielding that every element of 2^κ belongs to some $[s_i]$ with $i < \alpha$, which is impossible by (5.1). Moreover, T does not have a cofinal branch: such a branch $f \in 2^\kappa$ would not belong to any of the $[s_i]$'s, which is impossible by our assumption that the $[s_i]$'s cover 2^κ. Therefore, T is a κ-Aronszajn tree, i.e., κ does *not* have the tree property.

The other direction is very similar: assume that κ does *not* have the tree property, i.e., we can fix an Aronszajn tree $T \subseteq 2^{<\kappa}$. Now define $\{s_i : i < \kappa\} \subseteq 2^{<\kappa}$ to be the collection of the nodes not in T (or the minimal ones among them). It easy to see that $\bigcup_{i<\kappa}[s_i] = 2^\kappa$ (since there is no cofinal branch through T); on the other hand, $\bigcup_{i<\alpha}[s_i] \neq 2^\kappa$ for each $\alpha < \kappa$ (otherwise, the levels of T would be empty from some point on). Therefore, κ does *not* have the compactness property. $\square$

Remark 5.14. In general, we cannot drop the inaccessibility assumption when proving the equivalence of the tree property and the compactness property. This is due to the fact (as we argue below) that a regular cardinal

[11] We only need this direction (i.e., the one from left to right) for our Theorem 5.10 above.

$\kappa > \aleph_0$ *cannot* have the compactness property if it is not inaccessible (meaning strongly inaccessible). As opposed to this, it is consistent (modulo large cardinals) that successor cardinals such as $\aleph_2$ *do have the tree property.*

Let $\kappa = \mu^+$ be a successor cardinal: 2^μ is (at least[12]) κ, therefore there are (at least) κ many nodes on the μth level of $2^{<\kappa}$; let $\{s_i : i < 2^\mu\} := 2^\mu \subseteq 2^{<\kappa}$ be an enumeration of these nodes; now the $[s_i]$'s form a (disjoint) cover of 2^κ, but obviously there is no "proper subcover" (due to the disjointness), let alone of size less than κ; so κ does not have the compactness property.

For a regular limit cardinal κ being not inaccessible (i.e., a weakly inaccessible which is not strongly inaccessible), the argument is the same: since κ is not a strong limit, 2^μ will be (at least) κ (for sufficiently large $\mu < \kappa$); for any such μ, the above argument again shows that κ does not have the compactness property.

Corollary 5.15. *A regular cardinal $\kappa \geq \aleph_0$ has the compactness property if and only if κ is weakly compact or $\kappa = \aleph_0$.*

Proof. This follows from Lemma 5.12, Lemma 5.13, and Remark 5.14. □

The easy direction (for arbitrary $\kappa \geq \aleph_0$)

We first prove the "easy direction" of Theorem 5.10; actually, we prove it for arbitrary regular κ; however, we additionally have to invoke Assumption 5.1 (which follows anyway from the assumption in Theorem 5.10, namely that κ is weakly compact, or $\kappa = \omega$):

Lemma 5.16. *Let $\kappa \geq \aleph_0$ be regular, and assume that $|2^{<\kappa}| = \kappa$. Suppose that $X \subseteq 2^\kappa$. If X is meager-shiftable[13], then X is strong measure zero:*

$$\mathcal{SN}(2^\kappa) \supseteq \mathcal{M}^*(2^\kappa).$$

Proof. Let $X \in \mathcal{M}^*(2^\kappa)$. We have to show that $X \in \mathcal{SN}(2^\kappa)$. So fix a sequence $(\alpha_i)_{i<\kappa}$ with $\alpha_i < \kappa$ for each i.

[12] We can again adopt Assumption 5.1, i.e., $|2^{<\kappa}| = \kappa$, so the value will be *exactly* κ.

[13] Actually, it is sufficient to assume that X is "closed nowhere dense shiftable" (i.e., X can be translated away from each closed nowhere dense set); in other words, the proof of the lemma actually shows the following stronger result (note that trivially (closed nowhere dense)$^*(2^\kappa) \supseteq \mathcal{M}^*(2^\kappa)$):

$$\mathcal{SN}(2^\kappa) \supseteq (\text{closed nowhere dense})^*(2^\kappa).$$

We construct the following nowhere dense set $C \subseteq 2^\kappa$ (canonically corresponding to $(\alpha_i)_{i<\kappa}$): by the assumption that $|2^{<\kappa}| = \kappa$, we fix an enumeration $(s_i)_{i<\kappa}$ of $2^{<\kappa}$, and let $(t_i)_{i<\kappa}$ be any sequence of nodes in $2^{<\kappa}$ satisfying[14]

- $t_i \subseteq s_i$ or[15] $s_i \subseteq t_i$, and
- $|t_i| = \alpha_i$ (i.e., $t_i \in 2^{\alpha_i}$),

for each $i < \kappa$; now let $O := \bigcup_{i<\kappa}[t_i]$, and let $C := 2^\kappa \setminus O$; it is easy to see that O is open dense, so C is closed nowhere dense (in particular meager).

By our assumption that $X \in \mathcal{M}^*(2^\kappa)$, we know that $X + C \neq 2^\kappa$, in other words, we can fix a "translation real" $z \in 2^\kappa$ such that $X \cap (C+z) = \emptyset$, hence $X \subseteq O + z$. Note that $O + z = \bigcup_{i<\kappa}[t_i + z{\restriction}|t_i|]$, so letting $u_i := t_i + z{\restriction}|t_i|$, we have $|u_i| = \alpha_i$ for each i, and $X \subseteq \bigcup_{i<\kappa}[u_i]$, finishing the proof. $\square$

The difficult direction for weakly compact κ (and $\kappa = \omega$)

It remains to prove the "difficult" direction of Theorem 5.10. We now use the weak compactness of κ: we first prove a lemma making use of the fact that κ has the "compactness property" (see Definition 5.11 and Lemma 5.13); the lemma is quite similar to Lemma 8.1.17 in [BJ95]; the compactness property of κ replaces the compactness of $[0,1]$ used there.

Lemma 5.17. *Let κ be weakly compact (or $\kappa = \omega$).*

Suppose that $C \subseteq 2^\kappa$ is closed nowhere dense, and $s \in 2^{<\kappa}$. Then there exists a family $\mathcal{A} \subseteq 2^{<\kappa}$ with $|\mathcal{A}| < \kappa$ and $t \supseteq s$ for each $t \in \mathcal{A}$, and an $\alpha < \kappa$, such that the following holds: for every $u \in 2^{<\kappa}$ with $|u| \geq \alpha$ there exists a $t \in \mathcal{A}$ such that $([u] + [t]) \cap C = \emptyset$.

Proof. We first prove the following

Claim 5.18. For each $f \in 2^\kappa$ there exists a $u_f \in 2^{<\kappa}$ with $u_f \subseteq f$ (i.e., $f \in [u_f]$) and a $t_f \in 2^{<\kappa}$ with $t_f \supseteq s$ such that $([u_f] + [t_f]) \cap C = \emptyset$.

[14] Alternatively, we could think of it as follows: fix any dense set $(f_i)_{i<\kappa} \subseteq 2^\kappa$ of size κ, e.g., the "rationals"

$$\mathbb{Q}(2^\kappa) := \{f \in 2^\kappa : \exists\beta < \kappa\, \forall\gamma \geq \beta\; f(\gamma) = 0\},$$

and let $t_i := f_i{\restriction}\alpha_i$ for each $i < \kappa$.

[15] The second case $s_i \subseteq t_i$ is of course the "important" one, not the first one: it would not change Definition 5.5 if we would require the basic clopen sets to have "length" at least α_i (instead of exactly α_i).

Proof. Fix the $f \in 2^\kappa$ and note that $C + f$ is closed nowhere dense as well. So we can fix $t_f \in 2^{<\kappa}$ with $t_f \supseteq s$ such that $[t_f] \cap (C + f) = \emptyset$. Now let $u_f := f{\restriction}|t_f|$ (so $u_f \subseteq f$). It is straightforward to compute that $([u_f] + [t_f]) \cap C = \emptyset$, finishing the proof of the claim. $\square$

Fix (according to the claim) families $(u_f)_{f\in 2^\kappa}$ and $(t_f)_{f\in 2^\kappa}$. The family[16] $([u_f])_{f\in 2^\kappa}$ of basic clopens clearly covers the entire space 2^κ. Since κ is weakly compact (or $\kappa = \aleph_0$, resp.), it has the compactness property[17] by Lemma 5.13 (or Lemma 5.12, resp.). So the cover $([u_f])_{f\in 2^\kappa}$ has a small subcover, i.e., we can fix a $<\kappa$-sized set of indices $\{f_j : j < \gamma\}$ ($\gamma < \kappa$) such that

$$\bigcup_{j<\gamma} [u_{f_j}] = 2^\kappa. \tag{5.2}$$

Let $\mathcal{A} := \{t_{f_j} : j < \gamma\}$, and let[18] $\alpha < \kappa$ be such that $\alpha \geq |u_{f_j}|$ for all $j < \gamma$ (α is less than κ since κ is regular). The family $\mathcal{A} \subseteq 2^{<\kappa}$ has size $|\mathcal{A}| < \kappa$ and $t \supseteq s$ for each $t \in \mathcal{A}$, as required.

It remains to show that the conclusion of the lemma holds. Fix any $u \in 2^{<\kappa}$ with $|u| \geq \alpha$. We can easily pick a $j < \gamma$ such that $[u_{f_j}] \supseteq [u]$: pick any element $g \in [u]$, and let $j < \gamma$ such that $g \in [u_{f_j}]$ (this is possible by (5.2)); since $|u_{f_j}| \leq \alpha \leq |u|$, we have $u_{f_j} \subseteq u \subseteq g$, hence $[u_{f_j}] \supseteq [u]$. By choice of our families $(u_f)_{f\in 2^\kappa}$ and $(t_f)_{f\in 2^\kappa}$ (according to the claim), we have $([u_{f_j}] + [t_{f_j}]) \cap C = \emptyset$. Let $t := t_{f_j}$. So $t \in \mathcal{A}$, and (since $[u_{f_j}] \supseteq [u]$) $([u] + [t]) \cap C = \emptyset$, finishing the proof of the lemma. $\square$

Proof of Theorem 5.10. The "easy direction" has already been proved: since κ is weakly compact (or $\kappa = \omega$), we have (κ regular and) $|2^{<\kappa}| = \kappa$, hence Lemma 5.16 applies.

So suppose $X \subseteq 2^\kappa$ is strong measure zero, i.e., $X \in \mathcal{SN}(2^\kappa)$. We have to show that X is meager-shiftable ($X \in \mathcal{M}^*(2^\kappa)$): fix a meager set $M \subseteq 2^\kappa$; we will find a "translation real" $z \in 2^\kappa$ such that $(X + z) \cap M = \emptyset$.

First, let us fix an *increasing* family $(C_i)_{i<\kappa}$ of closed nowhere dense sets covering M, i.e., a family such that $i < j < \kappa$ implies $C_i \subseteq C_j$ and $M \subseteq \bigcup_{i<\kappa} C_i$ (this is possible by Lemma 5.3, and the comment after Definition 5.4, respectively); in particular, we have

$$M \subseteq \bigcup_{j<\kappa} \bigcap_{i\geq j} C_i. \tag{5.3}$$

[16] The family is indexed by the elements of 2^κ, but of course there are only $|2^{<\kappa}| = \kappa$ many basic clopens, so the family is actually of size κ.

[17] See Definition 5.11.

[18] Note that α is the "generalized" Lebesgue "number" of the covering…

Repeatedly using the above Lemma 5.17, we are going to build a tree $\mathcal{T} \subseteq \kappa^{<\kappa}$ of height κ. We prepare ourselves with the following

Claim 5.19. Let κ be inaccessible, and let $\mathcal{T} \subseteq \kappa^{<\kappa}$ be a $<\kappa$-branching tree (more explicitly: for each node $\tau \in \mathcal{T}$ there are strictly less than κ many successors, i.e., $|\mathrm{succ}_{\mathcal{T}}(\tau)| < \kappa$, where $\mathrm{succ}_{\mathcal{T}}(\tau) := \{\alpha < \kappa \colon \tau^\frown\langle\alpha\rangle \in \mathcal{T}\}$).

Then all levels of the tree have size strictly less than κ, i.e., for each $i < \kappa$, we have $|\mathcal{T}_i| < \kappa$, where $\mathcal{T}_i := \{\tau \in \mathcal{T} \colon |\tau| = i\}$.

Proof. Fix a $<\kappa$-branching tree $\mathcal{T} \subseteq \kappa^{<\kappa}$, and prove that $|\mathcal{T}_i| < \kappa$ by induction on $i < \kappa$.

For $i = j + 1$, we have $|\mathcal{T}_i| = \sum_{\tau\in\mathcal{T}_j} |\mathrm{succ}_{\mathcal{T}}(\tau)| \leq |\mathcal{T}_j| \cdot \sup_{\tau\in\mathcal{T}_j} |\mathrm{succ}_{\mathcal{T}}(\tau)|$; this is below κ since $|\mathcal{T}_j| < \kappa$ by induction hypothesis, and all the ($<\kappa$ many) values $|\mathrm{succ}_{\mathcal{T}}(\tau)|$ are below κ by the assumption that $\mathcal{T}$ is $<\kappa$-branching (hence its supremum is less than κ as well, due to the fact that κ is regular).

For $i < \kappa$ limit, we have $|\mathcal{T}_i| \leq \prod_{j<i} |\mathcal{T}_j| \leq (\sup_{j<i} |\mathcal{T}_j|)^{|i|}$; note that all the $|\mathcal{T}_j|$'s (for $j < i$) are below κ by induction hypothesis, hence (since κ is regular) its supremum is less than κ as well; so there is a $\mu < \kappa$ such that $(\sup_{j<i} |\mathcal{T}_j|)^{|i|} \leq \mu^\mu = 2^\mu$; since κ is strong limit, we have $2^\mu < \kappa$, finishing the proof of the claim. $\square$

We now build a $<\kappa$-branching tree $\mathcal{T} \subseteq \kappa^{<\kappa}$ together with families $\{t_\tau : \tau \in \mathcal{T}\} \subseteq 2^{<\kappa}$ and $\{\alpha_\tau : \tau \in \mathcal{T}\} \subseteq \kappa$ assigned to its nodes such that the following properties hold:

1. For each $\tau_0, \tau_1 \in \mathcal{T}$: whenever $\tau_0 \subseteq \tau_1$, then $t_{\tau_0} \subseteq t_{\tau_1}$.
2. For each $\tau \in \mathcal{T}$:

   for each $u \in 2^{<\kappa}$ with $|u| \geq \alpha_\tau$, there is an immediate successor $\tau^\frown\langle\xi\rangle \in \mathcal{T}$ of τ (i.e., $\xi \in \mathrm{succ}_{\mathcal{T}}(\tau)$) such that[19]

$$([u] + [t_{\tau^\frown\langle\xi\rangle}]) \cap C_{|\tau|} = \emptyset.$$

How can we build such a tree? Start with $\langle\rangle \in \mathcal{T}$, and let $t_{\langle\rangle}$ be any element of $2^{<\kappa}$, e.g., let[20] $t_{\langle\rangle} := \langle\rangle$.

Whenever we have constructed a node τ, we apply Lemma 5.17 to get (the successors of τ and) the t's assigned to the successors of τ, and the α assigned to τ itself. More precisely: we apply Lemma 5.17 to the set $C_{|\tau|}$

[19]Note that $|\tau|$ is the length of τ, i.e., at level i of the tree, we deal with the closed nowhere dense set C_i.

[20]Actually, the construction will yield a "translation real" z which belongs to the basic clopen $[t_{\langle\rangle}]$; since it doesn't matter "where" we start, we actually obtain that the set of possible translation reals z is dense in 2^κ.

(as the "C" in the lemma) and to t_τ (as the "s" in the lemma) to obtain a family $\mathcal{A} \subseteq 2^{<\kappa}$ of extensions of t_τ with $|\mathcal{A}| < \kappa$ and an $\alpha < \kappa$; we let $\alpha_\tau := \alpha$, and we let $\mathrm{succ}_{\mathcal{T}}(\tau) := |\mathcal{A}|$ (which is of size less than κ), and let $\mathcal{A} =: \{t_{\tau^\frown\langle\xi\rangle} : \xi \in \mathrm{succ}_{\mathcal{T}}(\tau)\}$; note that property (1) above remains true by induction (since the elements of $\mathcal{A}$ extend t_τ), and property (2) above (for the node τ dealt with here) is exactly the conclusion of Lemma 5.17.

At limits $i < \kappa$, we "take limits"; more precisely, we put all $\tau \in \kappa^i$ into $\mathcal{T}$ which satisfy $\tau \restriction j \in \mathcal{T}$ for each $j < i$, and (for all $\tau \in \mathcal{T}_i$) we define $t_\tau := \bigcup_{j<i} t_{\tau\restriction j} \in 2^{<\kappa}$ (note that, again, property (1) above remains true by induction); then we proceed as above (to get α_τ, the successors, ...).

Since $\mathcal{T}$ is a $<\kappa$-branching tree, and κ is (weakly compact, hence) inaccessible, we can apply Claim 5.19 above to obtain that all levels $\mathcal{T}_i$ of the tree $\mathcal{T}$ have size strictly less than κ:

$$\forall i < \kappa : \ |\mathcal{T}_i| < \kappa. \tag{5.4}$$

Therefore we can define α_i to be the supremum of all the α_τ's on level i (which is still below κ by (5.4)), i.e, we let (for each $i < \kappa$)

$$\alpha_i := \sup_{\tau \in \mathcal{T}_i} \alpha_\tau < \kappa.$$

Now[21] we use the fact that X was supposed to be strong measure zero, and apply Lemma 5.6 to the sequence $(\alpha_i)_{i<\kappa}$ to get a sequence $(u_i)_{i<\kappa}$ with $u_i \in 2^{\alpha_i}$ (i.e., $|u_i| = \alpha_i$) for each $i < \kappa$ such that

$$X \subseteq \bigcap_{j\in\kappa} \bigcup_{i \geq j} [u_i]. \tag{5.5}$$

By induction, we build a branch $b \in [\mathcal{T}]$ through our tree $\mathcal{T}$ such that for each $i < \kappa$

$$([u_i] + [t_{b\restriction(i+1)}]) \cap C_i = \emptyset. \tag{5.6}$$

To do so, we use property (2) of the tree $\mathcal{T}$ (see page 139) at successor steps; more precisely, we apply property (2) to $b\restriction i \in \mathcal{T}_i$ and u_i (note that $|u_i| = \alpha_i \geq \alpha_{b\restriction i}$) to obtain $b\restriction(i+1) \in \mathcal{T}_{i+1}$ such that (5.6) holds; at limit steps i, we just let $b\restriction i := \bigcup_{j<i} b\restriction j$ which belongs to $\mathcal{T}$ (by construction of $\mathcal{T}$).

By property (1) of the tree $\mathcal{T}$, we know that $(t_{b\restriction i})_{i<\kappa}$ is increasing (in other words, the $[t_{b\restriction i}]$'s form a decreasing family of basic clopens), so we can pick a $z \in \bigcap_{i<\kappa}[t_{b\restriction i}]$ (just take any[22] $z \in 2^\kappa$ with $z \supseteq t_{b\restriction i}$ for each $i < \kappa$).

[21] Note that we didn't look at X so far in the proof. In fact, we actually prove here that from each meager set M we can "compute" a sequence $(\alpha_i)_{i<\kappa}$ such that each set X which is "strong measure zero with respect to $(\alpha_i)_{i<\kappa}$" (i.e., there exists a sequence $(u_i)_{i<\kappa}$ with $u_i \in 2^{\alpha_i}$ such that (5.5) holds for X) can be translated away from M.

[22] Or, typically, the unique one.

Then $(X + z) \cap M = \emptyset$: otherwise, we can fix a $y \in (X + z) \cap M$; since $y \in M$, we can fix (by (5.3)) a $j < \kappa$ such that for any $i \geq j$, we have $y \in C_i$; since $y \in X + z$ (hence $y + z \in X$), we can fix (by (5.5)) an $i \geq j$ such that $y + z \in [u_i]$; let $x := y + z$; note that $x \in [u_i]$ and $z \in [t_{b\restriction(i+1)}]$, and $x + z = y$ belongs to C_i, which contradicts (5.6), finishing the proof of Theorem 5.10. $\square$

Questions

I wonder whether Theorem 5.10 is optimal, i.e., whether the generalized Galvin-Mycielski-Solovay theorem for 2^κ *only* holds for κ's that are weakly compact.

Question 5.20. Let $\kappa > \aleph_0$ be a regular uncountable cardinal that is not weakly compact (i.e., either an uncountable successor cardinal or an inaccessible without the tree property).

Can we show that $\mathcal{SN}(2^\kappa) \neq \mathcal{M}^*(2^\kappa)$ (is consistent)?

Since some sort of "compactness" seems to be an essential ingredient of all Galvin-Mycielski-Solovay like theorems/proofs (see also Theorem 5.38), I believe that the answer is yes. Note that there is a counterexample to Galvin-Mycielski-Solovay for the Baer-Specker group $\mathbb{Z}^\omega$ (see Theorem 5.53). I actually tried to adapt the idea of the proof to settle the above question for 2^{ω_1}, but it didn't work in a straightforward way; yet I think it should be possible to show that the Galvin-Mycielski-Solovay theorem (for, e.g., 2^{ω_1}) fails (at least consistently, e.g., under some combinatorial principle such as $\diamondsuit$, etc.).

Remark 5.21. The seventh chapter of Halko's thesis [Hal96] is concerned with the question whether there is some Galvin-Mycielski-Solovay type characterization for strong measure zero sets of the generalized Cantor space 2^{ω_1}. He does not give an analogue of Theorem 5.10 for $\kappa = \omega_1$; however, he introduces the notion of "stationary strong measure zero" and shows – under the assumption of $\diamondsuit^*$ – that each stationary strong measure zero set is closed nowhere dense shiftable[23] (see [Hal96, Theorem 7.8]).

A set X is *stationary*[24] *strong measure zero* (for the case $\kappa = \omega_1$, the definition was given in [Hal96, Definition 7.5]), if for each (strictly increasing)

[23]See also footnote 13 on page 136 for "closed nowhere dense shiftable".

[24]Interestingly enough, I came up with the very same notion independently some time ago, in a quite different context; instead of "stationary strong measure zero", I named it "club strong measure zero".

sequence $(\alpha_i)_{i<\kappa}$ with $\alpha_i < \kappa$ (for each $i < \kappa$) there is a sequence $(u_i)_{i<\kappa}$ with $u_i \in 2^{\alpha_i}$ (for each $i < \kappa$) such that for each club $C \subseteq \kappa$,

$$X \subseteq \bigcup_{i \in C} [u_i];$$

in other words, the u_i's are required to be chosen such that each element of the set X is in *stationarily* many of the $[u_i]$'s (instead of only one, or equivalently, cofinally many, see Lemma 5.6).

At the very end of the chapter about stationary strong measure zero sets, Halko conjectures that the stationary strong measure zero sets and the strong measure zero sets coincide (for $\kappa = \omega_1$, see [Hal96, Conjecture 7.9]); this would imply that (under $\diamondsuit^*$) being strong measure zero and being closed nowhere dense shiftable is the same. Therefore his conjecture "conflicts" with my conjecture that the answer to Question 5.20 is "yes" (since I think that this positive answer would even separate (closed nowhere dense)$^*(2^{\omega_1})$ from $\mathcal{SN}(2^{\omega_1})$, as it is the case for $\mathbb{Z}^\omega$; see Theorem 5.53 and footnote 54 on page 164).

5.2 GMS for separable locally compact groups

In this section, we mainly prove that (a suitable generalization of) the Galvin-Mycielski-Solovay theorem holds for every separable locally compact group (Theorem 5.46). On the way there, we also prove slightly more general results for the "difficult direction" of the theorem (see, e.g., Corollary 5.42).

I actually proved the version for compact groups (i.e., Theorem 5.38, or, rather, Corollary 5.39) (by generalizing the usual Galvin-Mycielski-Solovay theorem for $\mathbb{R}$ using Lebesgue's covering lemma for groups) before I learned[25] that Marcin Kysiak had already done the same for locally compact Polish groups (see [Kys00]). Nevertheless, I decided to include my version of the proof (and its generalization to certain locally compact groups which comprise the locally compact Polish groups Kysiak gave his proof for), for several reasons: first of all, my proof is perhaps slightly more general (see also item (5) on page 159), second, I did it without using any metrics (which doesn't increase the difficulty of the proof), using "Rothberger bounded" instead of "strong measure zero for metric spaces" (see Definition 5.23 and Remark 5.24), and third, Marcin Kysiak's presentation is in Polish, so this may be the first "English version" of the Galvin-Mycielski-Solovay theorem for locally compact Polish groups (Corollary 5.48).

[25] I thank Piotr Zakrzewski for pointing this out to me.

Topological groups $(G, +)$

A *topological group* $(G, +)$ is a group together with a topology such that both the group operation $+ : G \times G \to G$ (where $G \times G$ is equipped with the product topology) and the inverse function $- : G \to G$ are continuous. Since we use the additive notation for groups, we denote the identity (i.e., the neutral element) by 0.

We assume that all the groups we consider satisfy the separation axiom[26] T_3 (i.e., they are Hausdorff and regular).

Since G is in particular a topological space, we can talk about *open*, *closed*, *compact*, *dense*, *open dense*, and *(closed) nowhere dense* subsets of G. As usual, we say that a set $M \subseteq G$ is *meager*, if it is covered by countably many closed nowhere dense sets. Clearly, the collection of meager sets forms a σ-ideal.

By definition, the group structure of G "respects" its topological structure, hence all the (topological) notions mentioned above are invariant[27] under translations (both from the left and the from the right) and under taking inverses. In particular, a set $M \subseteq G$ is meager if and only if (for any $y \in G$) the translated set $y + M$ (or $M + y$) is meager, if and only if its inverse $(-M)$ is meager.

Let $\mathcal{U}(0)$ denote (a basis of) the system of (open) *neighborhoods* of the identity 0. Recall that for each $y \in G$, the collection $(y + U)_{U \in \mathcal{U}(0)}$ (as well as $(U + y)_{U \in \mathcal{U}(0)}$) is a system of neighborhoods of y. In particular, given an open set $O \subseteq G$ with $x \in O$, we can find a neighborhood $U \in \mathcal{U}(0)$ such that $x + U \subseteq O$ (or $U + x \subseteq O$). Moreover, for each $U \in \mathcal{U}(0)$, there is a $V \in \mathcal{U}(0)$ with $V + V \subseteq U$, as well as $(-V) \subseteq U$.

A topological group $(G, +)$ is *separable* if it has a countable dense subset. It is *compact* if every open cover has a finite subcover. It is *locally compact* if there is an open neighborhood $W \in \mathcal{U}(0)$ of the identity[28] with compact (topological) closure $\overline{W}$ (or, equivalently, if there is a neighborhood basis consisting of compact sets).

It is *abelian* (or: commutative) if for each $x, y \in G$, we have $x + y = y + x$.

[26] A topological group is T_0 if and only if it is $T_{3\frac{1}{2}}$, by a well-known theorem of Kakutani.

[27] Actually, for each $y \in G$, the function sending $x \mapsto y + x$ as well as the function sending $x \mapsto x + y$ is a homeomorphism from G to itself; the same is true for the inverse function.

[28] Or, equivalently, of every element of the group.

Lebesgue covering lemma for topological groups

We will need the following generalization of the well-known Lebesgue covering lemma (or: "Lebesgue number lemma") to topological groups:

Lemma 5.22. *Let $(G, +)$ be a topological group, and let $K \subseteq G$ be a compact subset.*

Let $\mathcal{O}$ be an open cover of K (i.e., $\mathcal{O}$ is a family of open[29] *sets with $\bigcup \mathcal{O} \supseteq K$). Then there exists a neighborhood*[30] *$U \in \mathcal{U}(0)$ of the identity of G such that for each $x \in K$ there is an $O \in \mathcal{O}$ with $x + U \subseteq O$.*

Roughly speaking, it says the following: whenever a compact set in a group is covered by open sets, then each sufficiently small subset of the compact set is contained in a single one of these open sets (where "sufficiently small" is measured in terms of the uniform structure on the group given by translates of the neighborhoods of the identity).

Proof. Let $\mathcal{O}$ be an open cover of K, i.e., for each $z \in K$, there is an $O \in \mathcal{O}$ with $z \in O$. Therefore we can fix a family $(V_z)_{z \in K} \subseteq \mathcal{U}(0)$ of neighborhoods of the identity such that for each $z \in K$ there is an $O \in \mathcal{O}$ with $z + V_z \subseteq O$.

Now note that for each $V \in \mathcal{U}(0)$ there is a $V' \in \mathcal{U}(0)$ with $V' + V' \subseteq V$. Consequently, we can fix $(V'_z)_{z \in K} \subseteq \mathcal{U}(0)$ such that for each $z \in K$ there is an $O \in \mathcal{O}$ with $z + V'_z + V'_z \subseteq O$.

Since $(z + V'_z)_{z \in K}$ is a cover of the compact set K, we can fix a *finite* set $\{z_i : i < n\} \subseteq K$ such that $(z_i + V'_{z_i})_{i<n}$ is still a cover of K. Define

$$U := \bigcap_{i<n} V'_{z_i} \in \mathcal{U}(0).$$

It remains to show that for each $x \in K$, there is an $O \in \mathcal{O}$ such that $x + U \subseteq O$. Fix $x \in K$. Since $(z_i + V'_{z_i})_{i<n}$ covers K, we can fix $i < n$ such that $x \in z_i + V'_{z_i}$. Therefore $x + U \subseteq z_i + V'_{z_i} + U \subseteq z_i + V'_{z_i} + V'_{z_i}$, so (by choice of the family $(V'_z)_{z \in K}$) there is an $O \in \mathcal{O}$ with $x + U \subseteq O$. □

Note that a completely analogous proof shows that (in Lemma 5.22) we can also find a neighborhood $U \in \mathcal{U}(0)$ such that for each $x \in K$ there is an $O \in \mathcal{O}$ with $U + x \subseteq O$ (i.e., $x + U$ is replaced by $U + x$; in non-abelian groups, this may make a difference).

[29] I.e., open in G.

[30] This is the "Lebesgue neighborhood U" for the cover $\mathcal{O}$, so to speak, analogous to the usual Lebesgue number in the context of metric spaces.

$\mathcal{SN}(G)$ — the smz (i.e., Rothberger bounded) sets

Let $(G, +)$ be a topological group. We now define the notion of being strong measure zero for subsets of G. Note that we slightly abuse notation here, since "strong measure zero" is normally reserved for metric spaces (see Definition 1.6), and the notion given here is "officially" called Rothberger bounded. We justify our (abuse of) notation in the remark after the definition.

Definition 5.23. A set $X \subseteq G$ is *strong measure zero*, or *Rothberger bounded* ($X \in \mathcal{SN}(G)$) if for every sequence of neighborhoods $(U_n)_{n<\omega} \subseteq \mathcal{U}(0)$, there exists a sequence $(x_n)_{n<\omega}$ of elements of G such that $X \subseteq \bigcup_{n<\omega}(x_n + U_n)$.

Remark 5.24. Let us explain in which way the notion of "strong measure zero" given above can be viewed as a proper generalization of the usual notion of strong measure zero for metric spaces (see Definition 1.6).

The Birkhoff-Kakutani theorem (see [Kec95, Theorem 9.1]) says that a topological group $(G, +)$ is metrizable if and only if it is Hausdorff[31] and first-countable (i.e., has a countable neighborhood basis of the identity). Moreover, every metrizable group $(G, +)$ admits a compatible[32] metric d which is left-invariant[33], i.e.,

$$\forall x, z_1, z_2 \in G \quad d(z_1, z_2) = d(x + z_1, x + z_2).$$

Let $(G, +)$ be a metrizable group, and let d be a left-invariant compatible metric. Then X is *Rothberger bounded* (i.e., $X \in \mathcal{SN}(G)$ according to the above definition) if and only if X is *strong measure zero with respect to d* (i.e., for each sequence $(\varepsilon_n)_{n<\omega}$ there is a sequence $(x_n)_{n<\omega}$ such that $X \subseteq \bigcup_{n<\omega} B(x_n, \varepsilon_n)$).

To see this, note that the open balls $B(0, \varepsilon) = \{z \in \mathcal{X} : d(0, z) < \varepsilon\}$ form a neighborhood basis of the identity. So prescribing a $U \in \mathcal{U}(0)$ amounts to the same as prescribing an $\varepsilon > 0$: for each $U \in \mathcal{U}(0)$ there is an $\varepsilon > 0$ such that $B(0, \varepsilon) \subseteq U$ (and vice versa). The left-invariance of d easily yields

$$x + B(0, \varepsilon) = B(x, \varepsilon)$$

for all $x \in G$. Therefore, a cover $X \subseteq \bigcup_{n<\omega} B(x_n, \varepsilon_n)$ in the definition of strong measure zero yields a cover $X \subseteq \bigcup_{n<\omega}(x_n + U_n)$ in the definition of Rothberger bounded (and vice versa), showing that the two notions are equivalent.

[31] We assume all our groups to be Hausdorff anyway.

[32] A metric is called compatible if it generates the topology of the space/group.

[33] Similarly, there exists a (maybe different) right-invariant metric.

In the context of the Galvin-Mycielski-Solovay theorem, we therefore consider it natural to call Rothberger bounded sets “strong measure zero”: first of all, it is most natural to look at the notion of being strong measure zero with respect to translation-invariant metrics since we are dealing with translations here, and second, the main scope of the Galvin-Mycielski-Solovay theorem are locally compact Polish groups (see Corollary 5.48), and for them, it turns out that the notion of being strong measure zero is independent of the metric anyway (see Lemma 1.8).

Note that every countable set is trivially strong measure zero (and subsets of strong measure zero sets are again strong measure zero). Moreover, the following holds:

Lemma 5.25. *The collection $\mathcal{SN}(G)$ is a left-translation-invariant σ-ideal. More precisely:*

1. *Let $(X_n)_{n<\omega} \subseteq \mathcal{SN}(G)$ be a countable sequence of strong measure zero sets. Then $\bigcup_{n<\omega} X_n \in \mathcal{SN}(G)$.*
2. *Let $z \in G$, and $X \in \mathcal{SN}(G)$. Then $z + X \in \mathcal{SN}(G)$.*

Proof. To prove (1), fix a sequence $(U_n)_{n\in\omega} \subseteq \mathcal{U}(0)$; partition $\omega = \bigcup_{l\in\omega} A_l$ into infinitely many infinite sets, and apply (for each $l \in \omega$) the definition of “being in $\mathcal{SN}(G)$” to $X_l \in \mathcal{SN}(G)$ and the sequence $(U_n)_{n\in A_l}$ to obtain a sequence $(x_n)_{n\in A_l} \subseteq G$ such that $X_l \subseteq \bigcup_{n\in A_l}(x_n + U_n)$; note that altogether we got a sequence $(x_n)_{n\in\omega} \subseteq G$ such that $\bigcup_{l\in\omega} X_l \subseteq \bigcup_{n\in\omega}(x_n + U_n)$.

To prove (2), fix a sequence $(U_n)_{n<\omega} \subseteq \mathcal{U}(0)$, and use the fact that $X \in \mathcal{SN}(G)$ to obtain a sequence $(x_n)_{n<\omega}$ such that $X \subseteq \bigcup_{n<\omega}(x_n + U_n)$; but then $z + X \subseteq \bigcup_{n<\omega}(z + x_n + U_n)$, i.e., the sequence $(z + x_n)_{n<\omega}$ witnesses that $z + X \in \mathcal{SN}(G)$. $\square$

Lemma 5.26. *Let $X \subseteq G$. Then $X \in \mathcal{SN}(G)$ if and only if for every sequence of neighborhoods $(U_n)_{n<\omega} \subseteq \mathcal{U}(0)$, there exists a sequence $(x_n)_{n<\omega}$ of elements of G such that $X \subseteq \bigcap_{m<\omega} \bigcup_{n\geq m}(x_n + U_n)$.*

Proof. Similar to the proof of Lemma 5.25 (1): just partition $\omega = \bigcup_{l\in\omega} A_l$ into infinitely many infinite sets, and (given $(U_n)_{n\in\omega} \subseteq \mathcal{U}(0)$) find (for each $l \in \omega$) witnesses $(x_n)_{n\in A_l}$ such that the $(x_n + U_n)_{n\in A_l}$ cover X; then $(x_n)_{n\in\omega}$ is as required. $\square$

$\mathcal{SN}(G)$ vs. ${}_{\leftrightharpoons}\mathcal{SN}(G)$ for non-abelian groups

If $(G, +)$ is an abelian group, it clearly doesn't matter whether we write $(x_n + U_n)$ or $(U_n + x_n)$ in the above definition of $\mathcal{SN}(G)$ (see Definition 5.23).

In general, however, it may yield a different collection of sets, which we call ${}_{\leftrightharpoons}\mathcal{SN}(G)$:

Definition 5.27. For a set $X \subseteq G$, we say that $X \in {}_{\leftrightharpoons}\mathcal{SN}(G)$ if for every sequence of neighborhoods $(U_n)_{n<\omega} \subseteq \mathcal{U}(0)$, there exists a sequence $(x_n)_{n<\omega}$ of elements of G such that $X \subseteq \bigcup_{n<\omega}(U_n + x_n)$.

We can prove analogous versions of Lemma 5.25 and Lemma 5.26 for the collection ${}_{\leftrightharpoons}\mathcal{SN}(G)$. In Lemma 5.25, for instance, we would obtain that ${}_{\leftrightharpoons}\mathcal{SN}(G)$ is a σ-ideal which is right-translation-invariant (i.e., $z \in G$ and $X \in {}_{\leftrightharpoons}\mathcal{SN}(G)$ implies $X + z \in {}_{\leftrightharpoons}\mathcal{SN}(G)$).

Remark 5.28. Whenever we have a definition involving the group operation $+$, we can give an *"interchanged version"* of the definition by just interchanging the two operands. In this way, we obtained ${}_{\leftrightharpoons}\mathcal{SN}(G)$ from $\mathcal{SN}(G)$ as its "interchanged version".

Similarly, all theorems involving such notions give rise to their "interchanged counterparts".

Under some circumstances, however, we can prove that the collections $\mathcal{SN}(G)$ and ${}_{\leftrightharpoons}\mathcal{SN}(G)$ are the same, even if the group is not abelian. For example, the following holds:

Lemma 5.29. *Let $(G, +)$ be a compact (or abelian) group. Then*

$$\mathcal{SN}(G) = {}_{\leftrightharpoons}\mathcal{SN}(G).$$

Proof. If G is abelian, then $\mathcal{SN}(G) = {}_{\leftrightharpoons}\mathcal{SN}(G)$ by definition.

So suppose that G is compact, and let $X \in \mathcal{SN}(G)$. We will show that $X \in {}_{\leftrightharpoons}\mathcal{SN}(G)$.

The Lebesgue covering lemma (Lemma 5.22) easily yields the following

Claim 5.30. For each (open) neighborhood $V \in \mathcal{U}(0)$ there exists a $U \in \mathcal{U}(0)$ such that[34] for each $x \in G$ there is a $z \in G$ with $x + U \subseteq V + z$.

Proof. Let $\mathcal{O} := (V + z)_{z \in G}$; then $\mathcal{O}$ is an open cover of $K := G$ (which is compact by assumption). So Lemma 5.22 implies that there exists a neighborhood $U \in \mathcal{U}(0)$ such that for each $x \in G$ there is an $O \in \mathcal{O}$ with $x + U \subseteq O$; in other words, for each $x \in G$ there is an $z \in G$ with $x + U \subseteq V + z$. $\square$

[34] Actually, one could refine the argument and even show that for each $x \in G$, we have $x + U \subseteq V + x$ (i.e., z can be chosen equal to x); but this is not necessary for the proof of the lemma.

To show that $X \in {}_{\leftrightharpoons}\mathcal{SN}(G)$ (see Definition 5.27), fix a sequence of neighborhoods $(V_n)_{n<\omega} \subseteq \mathcal{U}(0)$. By the claim, we can fix a sequence $(U_n)_{n<\omega} \subseteq \mathcal{U}(0)$ such that for each $n \in \omega$, we have

$$\forall x \in G \;\; \exists z \in G\colon \; x + U_n \subseteq V_n + z. \tag{5.7}$$

Now apply the fact that $X \in \mathcal{SN}(G)$ (see Definition 5.23) to find a sequence $(x_n)_{n<\omega} \subseteq G$ such that $X \subseteq \bigcup_{n<\omega}(x_n + U_n)$. By (5.7), we can fix a sequence $(z_n)_{n<\omega} \subseteq G$ such that for each $n \in \omega$, we have $x_n + U_n \subseteq V_n + z_n$. Therefore, $X \subseteq \bigcup_{n<\omega}(V_n + z_n)$, finishing the proof of $\mathcal{SN}(G) \subseteq {}_{\leftrightharpoons}\mathcal{SN}(G)$.

The other direction (i.e., ${}_{\leftrightharpoons}\mathcal{SN}(G) \subseteq \mathcal{SN}(G)$) is completely analogous, just use the "other version" of the Lebesgue covering lemma instead (see the remark after the proof of Lemma 5.22). $\square$

$\mathcal{M}^*(G)$ — the meager-shiftable sets

Recall that (for $X, Z \subseteq G$) $X + Z := \{x + z : x \in X, z \in Z\}$ denotes the "complex sum" of X and Z; for $y \in G$, let

$$X + y := \{x + y \colon x \in X\}$$

be the right-translate of X by y (and $y + X := \{y + x : x \in X\}$ the left-translate of X by y). Furthermore, let $(-X) := \{-x \colon x \in X\}$.

Definition 5.31. A set $X \subseteq G$ is *meager-shiftable* ($X \in \mathcal{M}^*(G)$) if for each meager set $M \subseteq G$ we have $M + X \neq G$.

Note that every countable set is meager-shiftable provided that the entire group is not meager: if $X \subseteq G$ is countable, then for each meager set M, we have $M + X$ meager (since the meager sets form a translation-invariant σ-ideal), hence $M + X \neq G$; moreover, subsets of meager-shiftable sets are clearly meager-shiftable as well. However, I think there is no reason to believe that the meager-shiftable sets form a σ-ideal in general (compare with the case of null-shiftable – i.e., strongly meager – sets, where CH even prevents them from being an ideal; see [BS01]); of course, they do form a σ-ideal, if we are in a situation where the Galvin-Mycielski-Solovay theorem holds (due to the fact that the strong measure zero sets form a σ-ideal).

A set X in $\mathcal{M}^*(G)$ is called meager-shiftable because it can be "translated away" from each meager set M.

However, in case of non-abelian groups, one may need to distinguish between left-translates and right-translates here. The following lemma explicates the equivalent versions of being in $\mathcal{M}^*(G)$.

Lemma 5.32. *For a set $X \subseteq G$, the following are equivalent:*

1. $X \in \mathcal{M}^*(G)$*, i.e.,* $\forall M \subseteq G$ *meager* $(M + X \neq G)$*.*
2. $\forall M \subseteq G$ *meager* $\exists y \in G$ *such that* $(X + y) \cap M = \emptyset$*.*
3. $\forall M \subseteq G$ *meager* $\exists y \in G$ *such that* $X \cap (M + y) = \emptyset$*.*

Proof. By easy computations, one can show that the following are equivalent:

i. $y \notin M + X$.

ii. $(X - y) \cap (-M) = \emptyset$.

iii. $X \cap ((-M) + y) = \emptyset$.

To show that, e.g., property (2) in the lemma implies property (1), fix an X satisfying property (2), and a meager M; now note that $(-M)$ is meager as well, and use property (2) for $(-M)$ to get a $y \in G$ such that $(X + y) \cap (-M) = \emptyset$; therefore (due to (ii) implies (i) for $-y$), we have $-y \notin M + X$, hence $M + X \neq G$.

So the point is (for any of the implications) that we can easily derive the equivalence of (1)–(3) from the equivalence of (i)–(iii) by noting that the family of meager sets is closed under taking inverses (i.e., M is meager if and only if $(-M)$ is meager), so there is no problem with passing from M to $(-M)$ since it is universally quantified; moreover, the y is existentially quantified, so no problem with changing the sign of y either; in this respect, the only thing we have to take care of is the fixed set X. $\square$

In general, there may be another version (the "interchanged" one) of "meager-shiftable":

Definition 5.33. For a set $X \subseteq G$, we say that $X \in {}_{\leftrightharpoons}\mathcal{M}^*(G)$ if for each meager set $M \subseteq G$ we have $X + M \neq G$.

Again, there are the respective equivalent versions of being in ${}_{\leftrightharpoons}\mathcal{M}^*(G)$.

Lemma 5.34. *For a set $X \subseteq G$, the following are equivalent:*

1. $X \in {}_{\leftrightharpoons}\mathcal{M}^*(G)$*, i.e.,* $\forall M \subseteq G$ *meager* $(X + M \neq G)$*.*
2. $\forall M \subseteq G$ *meager* $\exists y \in G$ *such that* $(y + X) \cap M = \emptyset$*.*
3. $\forall M \subseteq G$ *meager* $\exists y \in G$ *such that* $X \cap (y + M) = \emptyset$*.*

Proof. Completely analogous to the proof of Lemma 5.32. $\square$

Even though $\mathcal{M}^*(G)$ and ${}_{\leftrightharpoons}\mathcal{M}^*(G)$ may be different, there is the following easy connection between the two collections:

Lemma 5.35. *Let $X \subseteq G$. Then $X \in \mathcal{M}^*(G)$ if and only if $(-X) \in {}_{\leftrightharpoons}\mathcal{M}^*(G)$.*

Proof. Similar to the proof of Lemma 5.32: it is easy to compute that

$$y \notin M + X \iff -y \notin (-X) + (-M);$$

to finish the proof, we again use the fact that M is universally quantified in the definitions of $\mathcal{M}^*(G)$ and ${}_{\leftrightharpoons}\mathcal{M}^*(G)$, and that the family of meager sets is closed under taking inverses. $\square$

Easy direction of GMS, using separability

We now prove the "easy direction" of the (generalized) Galvin-Mycielski-Solovay theorem. The only assumption is separability:

Theorem 5.36. *Let $(G, +)$ be a separable group. Then*[35] ${}_{\leftrightharpoons}\mathcal{M}^*(G) \subseteq \mathcal{SN}(G)$.

Proof. Since G is separable, we can fix a countable set $\{z_n : n < \omega\} \subseteq G$ which is dense in G.

Let $X \in {}_{\leftrightharpoons}\mathcal{M}^*(G)$. Given a sequence $(U_n)_{n<\omega}$ of elements of $\mathcal{U}(0)$, we define $F := G \setminus \bigcup_{n<\omega}(z_n + U_n)$. Clearly, F is closed nowhere dense, hence in particular meager, so (see Lemma 5.34) there is a $y \in G$ such that $X \cap (y + F) = \emptyset$. In other words, $X \subseteq \bigcup_{n<\omega}((y + z_n) + U_n)$, which finishes the proof. $\square$

Of course, there is the respective "interchanged counterpart" of the above theorem, which is clearly true by interchanging everything (see also Remark 5.28). Yet we give the complete proof again this once.

Theorem 5.37. *Let $(G, +)$ be a separable group. Then $\mathcal{M}^*(G) \subseteq {}_{\leftrightharpoons}\mathcal{SN}(G)$.*

Proof. Again, let $\{z_n : n < \omega\} \subseteq G$ be a countable set dense in G.

Let $X \in \mathcal{M}^*(G)$. Given a sequence $(U_n)_{n<\omega}$ of elements of $\mathcal{U}(0)$, we define $F := G \setminus \bigcup_{n<\omega}(U_n + z_n)$. Clearly, F is closed nowhere dense, hence in particular meager, so (see Lemma 5.32) there is a $y \in G$ such that $X \cap (F + y) = \emptyset$. In other words, $X \subseteq \bigcup_{n<\omega}(U_n + (z_n + y))$, which finishes the proof. $\square$

[35] Actually, the proof will even show the stronger result that (not only every meager-shiftable set but also) every set which is "closed nowhere dense shiftable" (i.e., which can be translated away from each closed nowhere dense set) is in $\mathcal{SN}(G)$; in other words:

$${}_{\leftrightharpoons}(\text{closed nowhere dense})^*(G) \subseteq \mathcal{SN}(G).$$

However, it seems to be unclear whether one can show $\mathcal{M}^*(G) \subseteq \mathcal{SN}(G)$ (or ${}_{\leftrightharpoons}\mathcal{M}^*(G) \subseteq {}_{\leftrightharpoons}\mathcal{SN}(G)$) without any further assumptions.

Difficult direction of GMS, using compactness

We first prove[36] Theorem 5.38: it is the "core" of the proof, so to speak, and is sufficient to immediately yield a version of the Galvin-Mycielski-Solovay theorem for compact groups.

We can then use it to derive even more general versions for certain classes of locally compact groups.

Theorem 5.38. *Let $(G, +)$ be a locally compact group, and fix a witness, i.e., a neighborhood $W \in \mathcal{U}(0)$ of the identity with the property that its topological closure $\overline{W}$ is compact.*

Then for every $X \in \mathcal{SN}(G)$ with $X \subseteq W$, we have $X \in \mathcal{M}^*(G)$.

Before we prove the theorem, we give the "difficult direction of the Galvin-Mycielski-Solovay theorem" for compact groups as a corollary:

Corollary 5.39. *Let $(G, +)$ be a compact group. Then*

$$\mathcal{SN}(G) \subseteq \mathcal{M}^*(G).$$

Proof. Let $X \in \mathcal{SN}(G)$. Since G is compact, we just let $G =: W \in \mathcal{U}(0)$ be the "neighborhood" of the identity with compact closure $\overline{W} = W = G$. So $X \subseteq W$ is a void assumption, and Theorem 5.38 yields $X \in \mathcal{M}^*(G)$. $\square$

In particular, Corollary 5.39 applies to the Cantor space $(2^\omega, +)$. This is the second time we obtain the Galvin-Mycielski-Solovay theorem for 2^ω as a special case of a more general theorem: Theorem 5.10 in Section 5.1 also yields $\mathcal{SN}(2^\omega) = \mathcal{M}^*(2^\omega)$.

Proof of Theorem 5.38. First note that not only W itself, but also $W + W$ has compact closure (to see this, recall that $\overline{W} \times \overline{W} \subseteq G \times G$ is compact, so its image under the addition mapping $+ : G \times G \to G$ is compact as well, i.e., $+[\overline{W} \times \overline{W}] = \overline{W} + \overline{W} \subseteq G$ is compact; therefore $\overline{W + W}$ is compact since it is a closed subset of the compact set $\overline{W} + \overline{W}$). Let

$$K := \overline{W + W}$$

[36] This is the theorem (or actually rather Corollary 5.39) I proved independently of Marcin Kysiak (by generalizing the "usual" Galvin-Mycielski-Solovay theorem for $\mathbb{R}$ etc. using Lebesgue's covering lemma for groups) before I learned at the Winterschool 2013 in Hejnice (Czech Republic) that Kysiak had already done the same for locally compact Polish groups (see [Kys00]). I thank Piotr Zakrzewski for pointing this out to me.

denote this compact set (we will actually only use that K is any compact set with $W + W \subseteq K$).

We prepare ourselves with the following lemma:

Lemma 5.40. *Let $F \subseteq G$ be a closed nowhere dense set, and let $J \subseteq G$ be a closed set with non-empty interior. Then there exists a* finite *family $\mathcal{A}$ of closed subsets of J with non-empty interior, and a set $U \in \mathcal{U}(0)$ such that for every $x \in K$ there is a $J' \in \mathcal{A}$ with $((x + U) + J') \cap F = \emptyset$.*

Proof. We first prove the following

Claim 5.41. For any[37] $x \in K$, we can find an open set $O_x \ni x$ and a closed set $J_x \subseteq J$ with non-empty interior such that $(O_x + J_x) \cap F = \emptyset$.

Proof. To prove the claim, fix $x \in K$. Note that $(-x) + (G \setminus F)$ is an open dense set; since J has non-empty interior, we can pick $z \in G$ and $V \in \mathcal{U}(0)$ such that $V + z \subseteq J$ and $V + z \subseteq (-x) + (G \setminus F)$. Now choose $V' \in \mathcal{U}(0)$ such that $V' + V' \subseteq V$, and choose[38] an open $V'' \in \mathcal{U}(0)$ such that its topological closure $\overline{V''} \subseteq V'$. Define $O_x := x + V''$, and define $J_x := \overline{V''} + z$. Clearly, O_x is open and $x \in O_x$, and J_x is closed, has non-empty interior, and $J_x \subseteq V + z \subseteq J$. Finally, $O_x + J_x = (x + V'') + (\overline{V''} + z) \subseteq x + (V' + V') + z \subseteq x + V + z \subseteq G \setminus F$, which finishes the proof of the claim. □

The family $(O_x)_{x \in K}$ given by the claim covers the compact set K, so there is a finite set $\{x_j : j < n\} \subseteq K$ such that $\bigcup_{j<n} O_{x_j} = K$. Define $\mathcal{A} := \{J_{x_j} : j < n\}$. Now apply Lemma 5.22 to the cover $\mathcal{O} := \{O_{x_j} : j < n\}$ to obtain $U \in \mathcal{U}(0)$ such that for each $x \in K$ there is a $j < n$ with $x + U \subseteq O_{x_j}$.

Fix any $x \in K$: pick $j < n$ such that $x + U \subseteq O_{x_j}$; since $(O_{x_j} + J_{x_j}) \cap F = \emptyset$, also $((x + U) + J_{x_j}) \cap F = \emptyset$, which finishes the proof of the lemma. □

Suppose now that $X \in \mathcal{SN}(G)$ with $X \subseteq W$; we want to show that $X \in \mathcal{M}^*(G)$. So let $M \subseteq G$ be a meager set; we will find (see Lemma 5.32) a $y \in G$ such that $(X + y) \cap M = \emptyset$.

Let $\{F_n : n < \omega\}$ be an increasing family of closed nowhere dense sets covering M; in particular, we have

$$M \subseteq \bigcup_{m<\omega} \bigcap_{n \geq m} F_n. \tag{5.8}$$

Using Lemma 5.40, we inductively build a finitely branching tree $\mathcal{T} \subseteq \omega^{<\omega}$ together with a family $\{J_\tau : \tau \in \mathcal{T}\}$ of closed sets with non-empty interior and a family of neighborhoods $\{U_\tau : \tau \in \mathcal{T}\} \subseteq \mathcal{U}(0)$ such that the following holds:

[37] Actually, the claim itself is even true for any $x \in G$.

[38] Here we use the fact that G satisfies the separation axiom T_3.

0. $J_{\langle\rangle} = K$.[39]

1. For each $\tau_0, \tau_1 \in \mathcal{T}$: whenever $\tau_0 \subseteq \tau_1$, then $J_{\tau_0} \supseteq J_{\tau_1}$.

2. For each $\tau \in \mathcal{T}$:

   for every $x \in K$, there is an immediate successor $\tau^\frown\langle j\rangle \in \mathcal{T}$ of τ such that[40]
$$((x + U_\tau) + J_{\tau^\frown\langle j\rangle}) \cap F_{|\tau|} = \emptyset. \tag{5.9}$$

It is straightforward to construct such a tree and the corresponding families: just note that K is a compact (hence closed) set with non-empty interior, so we can start with $J_{\langle\rangle} := K$ and apply Lemma 5.40 to it (and F_0) to obtain $U_{\langle\rangle}$ and the finite set $\mathcal{A} =: \{J_{\langle j\rangle} : j < |\mathcal{A}|\}$ satisfying the required properties; then continue by induction, repeatedly applying Lemma 5.40.

Note that the tree $\mathcal{T}$ is finitely branching (hence each level is finite), so we can define for each $n < \omega$
$$U_n := (-W) \cap \bigcap_{\tau \in \mathcal{T} \cap \omega^n} U_\tau \in \mathcal{U}(0). \tag{5.10}$$

Since $X \in \mathcal{SN}(G)$, we can (see Lemma 5.26) fix a sequence $(x_n)_{n<\omega}$ of elements of X such that
$$X \subseteq \bigcap_{m<\omega} \bigcup_{n \geq m} (x_n + U_n). \tag{5.11}$$

Clearly, we can assume without loss of generality that $(x_n + U_n) \cap X \neq \emptyset$ for all $n \in \omega$: otherwise, some of the $(x_n + U_n)$ would not contribute to the union anyway (so we could just omit them, or change the respective x_n's to "artificially make them contribute"). Consequently, we can assume that each x_n belongs to our compact set K: recall that $U_n \subseteq (-W)$ and $X \subseteq W$; therefore, $(x_n + U_n) \cap X \subseteq (x_n + (-W)) \cap W \neq \emptyset$, hence $x_n \in W + W \subseteq K$.

By induction, we construct a branch b through $\mathcal{T}$ (i.e., $b \in [\mathcal{T}]$) such that for each $n < \omega$, we have
$$((x_n + U_n) + J_{b\restriction(n+1)}) \cap F_n = \emptyset; \tag{5.12}$$
to do so, we just apply (in step n) property (2) of the tree $\mathcal{T}$ for $\tau := b\restriction n$ to $x_n \in K$ to obtain $b\restriction(n+1)$ satisfying (5.9), which yields (5.12) due to $U_n \subseteq U_\tau$ (see (5.10)).

[39] Note that we can actually prove $(X + y) \cap M = \emptyset$ for "densely many" translation elements y, by letting $J_{\langle\rangle}$ be any small closed subset of K with non-empty interior instead of K itself (see also (5.13)).

[40] Note that $|\tau|$ is the length of τ, i.e., at level n of the tree, we deal with the closed nowhere dense set F_n.

Now note that the sequence $(J_{b\restriction n})_{n\in\omega}$ is a decreasing sequence of non-empty closed subsets of K (see properties (0) and (1) of the tree $\mathcal{T}$), hence (by compactness of K) also their intersection $\bigcap_{n\in\omega} J_{b\restriction n}$ is non-empty (otherwise $(G \setminus J_{b\restriction n})_{n<\omega}$ would be an open cover of K without a finite subcover). So we can pick any y (our "translation element") from there:

$$y \in \bigcap_{n\in\omega} J_{b\restriction n}. \tag{5.13}$$

Then $(X + y) \cap M = \emptyset$: otherwise, we can fix a $z \in (X + y) \cap M$; since $z \in M$, we can fix (by (5.8)) an $m \in \omega$ such that for any $n \geq m$, we have $z \in F_n$; since $z \in X + y$ (hence, $z - y \in X$), we can fix (by (5.11)) an $n \geq m$ such that $z - y \in x_n + U_n$; but $y \in J_{b\restriction(n+1)}$ (see (5.13)), and $(z-y)+y = z$ belongs to F_n, contradicting (5.12); so the proof of Theorem 5.38 is finished. □

Now we can derive a more general version:

Corollary 5.42. *Let $(G, +)$ be a locally compact group, and let $W \in \mathcal{U}(0)$ be a neighborhood with compact closure $\overline{W}$. Moreover, suppose*[41] *that there is a $C \subseteq G$ with $|C| \leq \aleph_0$ and $C + W = G$.*

Then $\mathcal{SN}(G) \subseteq \mathcal{M}^*(G)$.

Proof. We first give the idea of the proof: enlarge the given set $X \in \mathcal{SN}(G)$ (as well as the meager set M) to make it invariant[42] under translations by elements from C (this is no problem since both $\mathcal{SN}(G)$ and the meager sets form translation-invariant σ-ideals); then (by the assumption that $C + W$ covers the entire group) all the information about X can be found within W; so we can make advantage of the local compactness, i.e., we can apply Theorem 5.38 to finish the proof.

More precisely, we proceed as follows. Suppose that $X \in \mathcal{SN}(G)$; we first modify X to "push its information into W": let $X' := ((-C)+X) \cap W$. Note that also $X' \in \mathcal{SN}(G)$: since $(-C) + X = \bigcup_{c\in C}(-c + X)$, we have $(-C) + X \in \mathcal{SN}(G)$ by Lemma 5.25, hence its subset X' is in $\mathcal{SN}(G)$ as well.

Since $X' \in \mathcal{SN}(G)$ with $X' \subseteq W$, we can apply Theorem 5.38 to obtain that $X' \in \mathcal{M}^*(G)$.

We want to show that $X \in \mathcal{M}^*(G)$ (see Definition 5.31). So let $M \subseteq G$ be a meager set; we will show that $M + X \neq G$.

---

[41]We could call such a group G a "*CW group*".

[42]For the intuition, we can w.l.o.g. assume that C is a subgroup of G, so we can think of the enlarged sets X and M as C-periodic; for the proof, however, it is not necessary to assume that C is a group.

We now also modify M: let $M' := M + C$. Note that also M' is meager since the collection of meager sets is a translation-invariant σ-ideal.

Since $X' \in \mathcal{M}^*(G)$, we know that $M' + X' \neq G$. Fix $y \notin M' + X'$. To finish the proof, we show that $y \notin M + X$.

So assume towards a contradiction that $y = z + x$ for some $z \in M$ and $x \in X$. Since $C + W = G$, there are $c \in C$ and $w \in W$ with $x = c + w$, i.e., $-c + x = w$. Note that $-c + x \in X'$, and $z + c \in M'$, so

$$y = (z + c) + (-c + x) \in M' + X',$$

a contradiction. $\square$

Corollary 5.43. *Let $(G, +)$ be a locally compact group, and let $W \in \mathcal{U}(0)$ be a neighborhood with compact closure $\overline{W}$. Moreover, suppose*[43] *that there is a $C \subseteq G$ with $|C| \leq \aleph_0$ and $W + C = G$.*

Then ${}_{\leftrightharpoons}\mathcal{SN}(G) \subseteq {}_{\leftrightharpoons}\mathcal{M}^*(G)$.

Proof. Just note that Corollary 5.43 is the "interchanged[44] version" of Corollary 5.42 (see also Remark 5.28). $\square$

Even though (the difficult direction of) the Galvin-Mycielski-Solovay theorem for the classical real line $\mathbb{R}$ (see [GMS73]) will follow from Theorem 5.46 anyway (since $\mathbb{R}$ is separable and locally compact), we present it right now to illustrate Corollary 5.42:

Corollary 5.44. *For the classical real line $(\mathbb{R}, +)$, we have*[45]

$$\mathcal{SN}(\mathbb{R}) \subseteq \mathcal{M}^*(\mathbb{R}).$$

Proof. Let, for instance, $W := (-1, 1)$ be the open interval of length 2 centered at 0, and let $C := \mathbb{Z}$ be the integers. Then $\overline{W} = [-1, 1]$ is compact, $|C| = \aleph_0$, and $C + W = \mathbb{R}$. So Corollary 5.42 yields $\mathcal{SN}(\mathbb{R}) \subseteq \mathcal{M}^*(\mathbb{R})$. $\square$

GMS for separable locally compact groups

The following easy fact is well-known:

Lemma 5.45. *Let $W \in \mathcal{U}(0)$ be a neighborhood of the identity, and let $D \subseteq G$ be dense in G. Then $D + W = G$ (and $W + D = G$).*

[43] We could call such a group G a "*WC group*".

[44] Of course, not only the notions in the conclusion are the "interchanged" ones, but also in the assumption: $C + W = G$ is replaced by $W + C = G$.

[45] Of course, we even have ${}_{\leftrightharpoons}\mathcal{SN}(\mathbb{R}) = \mathcal{SN}(\mathbb{R}) = \mathcal{M}^*(\mathbb{R}) = {}_{\leftrightharpoons}\mathcal{M}^*(\mathbb{R})$: first of all, ${}_{\leftrightharpoons}\mathcal{SN}(\mathbb{R}) = \mathcal{SN}(\mathbb{R})$ and $\mathcal{M}^*(\mathbb{R}) = {}_{\leftrightharpoons}\mathcal{M}^*(\mathbb{R})$ trivially hold since $(\mathbb{R}, +)$ is commutative; second, $\mathbb{R}$ is separable, so also $\mathcal{SN}(\mathbb{R}) \supseteq \mathcal{M}^*(\mathbb{R})$ by Theorem 5.36.

Proof. Let $V \in \mathcal{U}(0)$ such that $(-V) \subseteq W$.

To show that $D + W = G$, let $x \in G$ be arbitrary. Since D is dense, we can fix $d \in D$ with $d \in x + V$. So, $x \in d + (-V) \subseteq d + W$, hence $x \in D + W$.

The proof that $W + D = G$ is analogous. □

We can now conclude the main theorem of Section 5.2. Note that we do not assume that $(G, +)$ is abelian.

Theorem 5.46. *Let $(G, +)$ be a separable, locally compact group. Then*

$$ {}_{\leftrightharpoons}\mathcal{SN}(G) = \mathcal{SN}(G) = \mathcal{M}^*(G) = {}_{\leftrightharpoons}\mathcal{M}^*(G). $$

Proof. Since G is locally compact, we can fix a neighborhood $W \in \mathcal{U}(0)$ such that its closure $\overline{W}$ is compact. Since G is separable, we can fix a dense set $C \subseteq G$ with $|C| \leq \aleph_0$.

By Lemma 5.45, we know that $C + W = G$; so we can use Corollary 5.42 to conclude

$$ \mathcal{SN}(G) \subseteq \mathcal{M}^*(G). $$

Again due to the separability of G, we can use Theorem 5.37 to obtain

$$ \mathcal{M}^*(G) \subseteq {}_{\leftrightharpoons}\mathcal{SN}(G). $$

Analogously, Corollary 5.43 yields (using $W + C = G$, again by Lemma 5.45) ${}_{\leftrightharpoons}\mathcal{SN}(G) \subseteq {}_{\leftrightharpoons}\mathcal{M}^*(G)$; lastly, Theorem 5.36 yields ${}_{\leftrightharpoons}\mathcal{M}^*(G) \subseteq \mathcal{SN}(G)$, finishing the proof of the theorem. □

Remark 5.47. In case of separable groups (e.g., $\mathbb{R}$, or other locally compact – but non-compact – Polish groups such as $\mathbb{R}^n$, etc.), it is typically an "overkill" to use a *dense* set C in the assumption of Corollary 5.42 to get $C + W = G$.

Indeed, a *discrete* set C may be enough for this purpose, as in the proof of Corollary 5.44, where $C = \mathbb{Z} \subseteq \mathbb{R}$. In that proof, we could have even chosen W to be the half-open[46] interval $[-\frac{1}{2}, \frac{1}{2})$ of length 1; this would turn our covering $\bigcup_{z \in \mathbb{Z}} z + W$ into a "tiling" of $\mathbb{R}$, i.e., each element $x \in \mathbb{R}$ is represented in a *unique* way as $x = z + w$ with $z \in \mathbb{Z}$ and $w \in W$.

In my opinion, this illustrates (even though we need separability for the easy direction of the Galvin-Mycielski-Solovay theorem anyway, and it is easy to get a countable set C with $C + W = G$ from separability[47] via Lemma 5.45) that always using a dense set C rather hides the point of the idea how to pass from the compact setting (i.e., Theorem 5.38) to, e.g., locally compact Polish groups.

[46] Note that W need not be an open neighborhood for the proofs of this section...

[47] For a situation without separability, see also item (6) on page 159.

GMS for Polish groups

Theorem 5.46 particularly yields the Galvin-Mycielski-Solovay theorem for locally compact Polish groups (which has already been proved in [Kys00, Twierdzenie 5.5 (Galvin-Mycielski-Solovay) on page 34]):

Corollary 5.48. *Let $(G, +)$ be a locally compact Polish group. Then the notions of being strong measure zero and being meager-shiftable coincide:*

$$_{\leftrightharpoons}\mathcal{SN}(G) = \mathcal{SN}(G) = \mathcal{M}^*(G) = {}_{\leftrightharpoons}\mathcal{M}^*(G).$$

Proof. Since Polish groups are separable, the assumptions of Theorem 5.46 are satisfied. □

Here it is no problem to just talk about "strong measure zero" and "meager-shiftable" without mentioning whether $\mathcal{SN}(G)$ or $_{\leftrightharpoons}\mathcal{SN}(G)$ (and $\mathcal{M}^*(G)$ or $_{\leftrightharpoons}\mathcal{M}^*(G)$, respectively) are meant because (in the context of Theorem 5.46) the respective two notions coincide anyway.

But we can say even more: for locally compact Polish groups, the notion of strong measure zero the way we use it in this section (i.e., being in $\mathcal{SN}(G)$ according to Definition 5.23, which is "officially" called *Rothberger bounded*, see also Remark 5.24) coincides with the usual notion of *strong measure zero in metric spaces* (see Definition 1.6), *regardless of the metric* being used (see Lemma 1.8).

Examples and remarks

We go through several topological groups in order to illustrate Theorem 5.46 (and Corollary 5.42, respectively), also mentioning some "borderline cases", and we show limitations of these theorems.

1. Typical examples of groups to which Theorem 5.46 can be applied include compact Polish groups such as the Cantor space $(2^\omega, +)$ with bitwise addition modulo 2, and the unit interval $[0, 1]$ with addition modulo 1 (in other words, the one-dimensional circle S^1 with rotation as the group operation), as well as locally compact (but not compact) Polish groups such as the classical real line $(\mathbb{R}, +)$ and the topological vector spaces[48] $(\mathbb{R}^n, +)$ etc. (all in the scope of Corollary 5.48).

[48] Also here (see Remark 5.47), it is not necessary to use separability for the difficult direction of the Galvin-Mycielski-Solovay theorem, but W being a unit (hyper)cube and C being the set $\mathbb{Z}^n$ of integer lattice points is fine, again forming a "tiling" of $\mathbb{R}^n$.

2. As a trivial (but weird) "instance" of Theorem 5.46, let us consider the finite cyclic group $(\mathbb{Z}_{17}, +)$ of integers with addition modulo 17 (necessarily[49] with the discrete topology): clearly, it is separable and compact (since it is finite), so Theorem 5.46 yields

$$\mathcal{SN}(\mathbb{Z}_{17}) = \mathcal{M}^*(\mathbb{Z}_{17}).$$

Let us also check this "by hand": obviously, every set is in $\mathcal{SN}(\mathbb{Z}_{17})$, i.e., $\mathcal{SN}(\mathbb{Z}_{17}) = \mathcal{P}(\mathbb{Z}_{17})$; on the other hand, every singleton $\{z\}$ (for $z \in \mathbb{Z}_{17}$) is open, hence *not* nowhere dense, and so only the empty set $\emptyset$ is meager; therefore *every* set $X \subseteq \mathbb{Z}_{17}$ can be translated away from each meager set, i.e., $\mathcal{M}^*(\mathbb{Z}_{17}) = \mathcal{P}(\mathbb{Z}_{17})$, and everything is fine.

3. Similarly, let us consider the infinite cyclic group $(\mathbb{Z}, +)$ of integers with the discrete topology: again, it is separable (just because it is countable); it is not compact, but locally compact ($\{0\} \in \mathcal{U}(0)$ is compact); so Theorem 5.46 again yields $\mathcal{SN}(\mathbb{Z}) = \mathcal{M}^*(\mathbb{Z})$. As in (2), only the empty set is meager, so once more we actually have

$$\mathcal{SN}(\mathbb{Z}) = \mathcal{M}^*(\mathbb{Z}) = \mathcal{P}(\mathbb{Z}).$$

4. Let us now consider the group $(\mathbb{Q}, +)$ of rational numbers with the usual topology (i.e., the topology inherited from $\mathbb{R}$). Again, each set is in $\mathcal{SN}(\mathbb{Q})$ (just because it is countable); however, each singleton $\{q\}$ (for $q \in \mathbb{Q}$) is clearly nowhere dense, hence the *entire group* $\mathbb{Q}$ *is meager*; therefore only the empty set $\emptyset$ can be "translated away" from *each* meager set, i.e.,

$$\mathcal{P}(\mathbb{Q}) = \mathcal{SN}(\mathbb{Q}) \neq \mathcal{M}^*(\mathbb{Q}) = \{\emptyset\}.$$

So the (difficult[50] direction of the) Galvin-Mycielski-Solovay theorem for $(\mathbb{Q}, +)$ fails: this is because $\mathbb{Q}$ is not locally compact.

However, $(\mathbb{Q}, +)$ is clearly σ-compact. This shows that the assumptions of Corollary 5.42 cannot be weakened to just requiring that a compact *set*[51] W with $C + W = G$ (for some countable C) exists.

[49] Recall that we assume T_3 for all our groups.

[50] The easy direction is satisfied; this is no surprise since $\mathbb{Q}$ is separable.

[51] The point where the proofs of Theorem 5.38 and Corollary 5.42 would break down seems to be the following: by $C + W = G$, we can "push all the information" of X into W; but to make the compactness argument of Theorem 5.38 work, we have to make sure that not just the relevant part of X itself belongs to a compact set, but all the centers (x_n) involved in the covering of X (see (5.11) on page 153 and the paragraph thereafter) belong to a compact set K; therefore we have to assume that not just W, but W plus some neighborhood is still compact.

5. We say that a topological group $(G, +)$ is an *insect* if there exists an uncountable set in $\mathcal{SN}(G)$ and $\mathcal{SN}(G) = \mathcal{M}^*(G)$.

   We claim that there are *non-metrizable insects.*

   For a cardinal $\kappa \geq \omega$, let $(2^\kappa_\times, +)$ denote the group 2^κ (with bitwise addition modulo 2) equipped with the usual product (i.e., Tychonoff) topology. (Note that for $\kappa > \omega$, the topology of $(2^\kappa_\times, +)$ is different from the topology of the group $(2^\kappa, +)$ considered in Section 5.1.) By Tychonoff's theorem, $(2^\kappa_\times, +)$ is compact for all κ. We claim that $(2^{\omega_1}_\times, +)$ is a non-metrizable insect (even in ZFC). First of all, it is clearly not first-countable, hence not metrizable. Moreover, it is well-known that the (Tychonoff) product of at most continuum many separable spaces is separable, so (since $\omega_1 \leq \mathfrak{c}$) $(2^{\omega_1}_\times, +)$ is separable; hence we can use Theorem 5.46 to conclude that $\mathcal{SN}(2^{\omega_1}_\times) = \mathcal{M}^*(2^{\omega_1}_\times)$. Lastly, the set

   $$\{f \in 2^{\omega_1} : \exists!\, i < \omega_1 \;\; f(i) = 1\}$$

   is of size $\aleph_1$ and in $\mathcal{SN}(2^{\omega_1}_\times)$: given a sequence of U_n's – we can view them as basic clopen neighborhoods of $\overline{0}$ with finite "supports" (where the 0's are fixed) – we let i^* be the supremum of all these supports; now we save one of the U_n's for later use, and cover all those (countably many) f's that have their 1 below i^* by the remaining countably many U_n's; all the ω_1 many remaining f's have only 0's up to i^*, so we can cover them with the single saved neighborhood.

   This shows that Theorem 5.46 is indeed a proper generalization of Corollary 5.48 (which is for locally compact Polish groups only): it provides non-trivial information about non-metrizable groups.

6. As in (5) above, let $(2^\kappa_\times, +)$ be equipped with the product topology. All these groups are compact. To get locally compact, non-compact groups, we can add one single $\mathbb{Z}$ component, i.e., consider the groups

   $$(\mathbb{Z} \times 2^\kappa_\times, +) \tag{5.14}$$

   with the product topology (where plus is the component-wise addition).

   In case of $\kappa = \omega$, we just have an instance of Corollary 5.48 for locally compact Polish groups.

   In case of $\omega < \kappa \leq \mathfrak{c}$, we have a separable locally compact group, so we *have* to use the more general Theorem 5.46 to get $\mathcal{SN}(\mathbb{Z} \times 2^\kappa_\times) = \mathcal{M}^*(\mathbb{Z} \times 2^\kappa_\times)$.

In case of $\kappa \geq \mathfrak{c}^+$, our group in (5.14) is not separable any more, so we may not expect to get the easy inclusion of the Galvin-Mycielski-Solovay theorem. The difficult one, however, still holds by Corollary 5.42: but this time we even *have* to use a countable C which is *not dense* to obtain $\mathcal{SN}(\mathbb{Z} \times 2^\kappa_\times) \subseteq \mathcal{M}^*(\mathbb{Z} \times 2^\kappa_\times)$ (just because there is no dense C available; see also Remark 5.47); e.g., let $C = \mathbb{Z} \times \{\overline{0}\}$ and $W = \{0\} \times 2^\kappa_\times$.

7. The Baer-Specker group $(\mathbb{Z}^\omega, +)$ (which we will investigate in Section 5.3) is a Polish group which is *not locally compact*. So none of the theorems of this Section 5.2 can be applied to obtain the difficult inclusion $\mathcal{SN}(\mathbb{Z}^\omega) \subseteq \mathcal{M}^*(\mathbb{Z}^\omega)$. Indeed, the main result of Section 5.3 (see Theorem 5.53) shows that the (difficult direction of the) Galvin-Mycielski-Solovay theorem (consistently) fails for $(\mathbb{Z}^\omega, +)$.

Questions

Regarding the pairs ("interchanged versions") of definitions for strong measure zero sets (see Definitions 5.23 and 5.27) and meager-shiftable sets (see Definitions 5.31 and 5.33), there is the following natural problem:

Question 5.49. Are there (Polish) groups $(G, +)$ for which

1. $\mathcal{SN}(G) \neq {}_{\leftrightharpoons}\mathcal{SN}(G)$, or
2. $\mathcal{M}^*(G) \neq {}_{\leftrightharpoons}\mathcal{M}^*(G)$.

Note that the only candidates are (Polish) groups that are neither abelian nor locally compact (see Theorem 5.46).

I believe that the group $(S_\infty, \circ)$ of permutations of ω could be a good candidate to find sets distinguishing the respective collections.

5.3 Failure of GMS for $\mathbb{Z}^\omega$

In this section, we investigate the Baer-Specker group $\mathbb{Z}^\omega$. We show that the Galvin-Mycielski-Solovay theorem (consistently) fails for $\mathbb{Z}^\omega$ (see Theorem 5.53). This answers a question I was asked by Marcin Kysiak during the Winterschool 2011 in Hejnice (Czech Republic). I would like to thank him for asking me this question, and for many interesting conversations there.

The Baer-Specker group $\mathbb{Z}^\omega$

The *Baer-Specker group* is the topological group $(\mathbb{Z}^\omega, +)$, where addition is defined component-wise, i.e., for $x, y \in \mathbb{Z}^\omega$, its sum $x + y$ is given by $(x + y)(n) := x(n) + y(n)$ for each $n < \omega$; furthermore, $\mathbb{Z}^\omega$ is equipped with the product topology; in other words, the topology is generated by the basic clopen sets $[s] := \{z \in \mathbb{Z}^\omega : z \supseteq s\}$ (for $s \in \mathbb{Z}^{<\omega}$).

Note that $(\mathbb{Z}^\omega, +)$ is an abelian Polish group (in particular, it is separable). However, $(\mathbb{Z}^\omega, +)$ is *not locally compact*: this is because a basic clopen set $[s]$ *can never be compact* since its open cover $([s^\frown n])_{n<\omega}$ (indeed, it is a partition of $[s]$) obviously has no finite subcover.

$\mathcal{SN}(\mathbb{Z}^\omega)$ — the strong measure zero sets

Let $\omega^{\omega\uparrow}$ denote the collection of strictly increasing functions in ω^ω.

Note that a set $X \subseteq \mathbb{Z}^\omega$ is *strong measure zero* (i.e., $X \in \mathcal{SN}(\mathbb{Z}^\omega)$ according to Definition 5.23) if and only if for every strictly increasing function $f \in \omega^{\omega\uparrow}$, there exists a sequence $(s_n)_{n<\omega}$ in $\mathbb{Z}^{<\omega}$ with $|s_n| \geq f(n)$ for all n such that $X \subseteq \bigcup_{n<\omega}[s_n]$.

Also note that whenever d is a (compatible) *translation-invariant*[52] *metric*, the above notion of strong measure zero coincides with the notion of strong measure zero with respect to d (see Definition 1.6); however, the notion of strong measure zero in the metric sense is not independent of the metric in this case (after all, Lemma 1.8 does not apply since $\mathbb{Z}^\omega$ is not locally compact).

$\mathcal{M}^*(\mathbb{Z}^\omega)$ — the meager-shiftable sets

A set $X \subseteq \mathbb{Z}^\omega$ is *meager-shiftable* (i.e., $X \in \mathcal{M}^*(\mathbb{Z}^\omega)$ according to Definition 5.31) if and only if for every meager set $M \subseteq \mathbb{Z}^\omega$, there is a $y \in \mathbb{Z}^\omega$ such that $(X + y) \cap M = \emptyset$ (see Lemma 5.32).

Easy relations between $\mathcal{SN}(\mathbb{Z}^\omega)$ and $\mathcal{M}^*(\mathbb{Z}^\omega)$

The following inclusions are easy:

Lemma 5.50. *In ZFC, we have* $[\mathbb{Z}^\omega]^{\leq\aleph_0} \subseteq \mathcal{M}^*(\mathbb{Z}^\omega) \subseteq \mathcal{SN}(\mathbb{Z}^\omega)$.

Proof. The first inclusion follows from the fact that the collection of meager sets on $\mathbb{Z}^\omega$ forms a *proper* translation-invariant σ-ideal (see also the paragraph after Definition 5.31 on page 148).

[52] Since $(\mathbb{Z}^\omega, +)$ is abelian, we need not talk about left- and right-invariant metrics.

The second inclusion is the "easy direction" of the Galvin-Mycielski-Solovay theorem: recall that $\mathbb{Z}^\omega$ is separable (and abelian), and apply Theorem 5.36. $\square$

We are going to show that the (difficult direction of the) Galvin-Mycielski-Solovay theorem "fails" for $\mathbb{Z}^\omega$, i.e., $\mathcal{SN}(\mathbb{Z}^\omega) \not\subseteq \mathcal{M}^*(\mathbb{Z}^\omega)$. However, we cannot expect to get it in ZFC. The reason is the following:

Lemma 5.51. *Suppose that the Borel Conjecture holds. Then*

$$[\mathbb{Z}^\omega]^{\leq\aleph_0} = \mathcal{M}^*(\mathbb{Z}^\omega) = \mathcal{SN}(\mathbb{Z}^\omega).$$

Proof. Since BC holds, Theorem 1.7 implies $\mathcal{SN}(\mathbb{Z}^\omega) = [\mathbb{Z}^\omega]^{\leq\aleph_0}$. Therefore all three collections are the same (see Lemma 5.50). $\square$

So the "Galvin-Mycielski-Solovay characterization for strong measure zero sets" consistently holds in $\mathbb{Z}^\omega$ (in a trivial way, though).

Remark 5.52. Alternatively, we can argue in an elementary way here. The assertions $\mathcal{SN}(2^\omega) = [2^\omega]^{\leq\aleph_0}$ (i.e., the "usual" BC) and the assertion $\mathcal{SN}(\mathbb{Z}^\omega) = [\mathbb{Z}^\omega]^{\leq\aleph_0}$ (i.e., "BC for $\mathbb{Z}^\omega$") are trivially equivalent, for the following reason: for the direction from left to right (this is the one we actually need for Lemma 5.51), note that each element of $\mathbb{Z}^\omega$ can be canonically[53] mapped to an element of 2^ω (with infinitely many 1's), hence each uncountable set in $\mathcal{SN}(\mathbb{Z}^\omega)$ can be "interpreted" as an uncountable set in 2^ω which is in $\mathcal{SN}(2^\omega)$ because the diameters of basic clopen sets only become smaller under this mapping; for the other direction, note that 2^ω can be (literally) viewed as a subset of $\mathbb{Z}^\omega$ (i.e., the mapping is the "identity"), so the argument is even simpler.

The main theorem: $\mathcal{SN}(\mathbb{Z}^\omega) \not\subseteq \mathcal{M}^*(\mathbb{Z}^\omega)$ (under CH)

We now prove the main theorem of this section: the difficult direction of the Galvin-Mycielski-Solovay theorem consistently fails. We present the theorem under the assumption of CH, but actually less is sufficient to make the arguments work (see Remark 5.57).

Theorem 5.53. *Assume CH. Then $\mathcal{SN}(\mathbb{Z}^\omega) \not\subseteq \mathcal{M}^*(\mathbb{Z}^\omega)$.*

This actually says (see Lemma 5.50) that the meager-shiftable sets form a proper subcollection of the strong measure zero sets, i.e., $\mathcal{M}^*(\mathbb{Z}^\omega) \subsetneqq \mathcal{SN}(\mathbb{Z}^\omega)$.

[53]Think of the homeomorphism between $2^\omega \setminus \mathbb{Q}$ and ω^ω via counting (for an element of $2^\omega \setminus \mathbb{Q}$) the numbers of 0's between two 1's.

Proof of Theorem 5.53. For $s \in \mathbb{Z}^{<\omega}$, let $[s]$ denote the set $\{t \in \mathbb{Z}^{<\omega} : t \supseteq s\}$ (or $\{x \in \mathbb{Z}^{\omega} : x \supseteq s\}$, depending on the context); for $s_0, s_1 \in \mathbb{Z}^{<\omega}$, let $s_0 {}^\frown s_1$ be their concatenation; let $0^{(k)} = \langle 0, \ldots, 0 \rangle$ be the element of $\mathbb{Z}^{<\omega}$ consisting of k zeros; and let $|s|$ denote the length of s. For a tree $F \subseteq \mathbb{Z}^{<\omega}$, let $[F] \subseteq \mathbb{Z}^{\omega}$ be the set of all branches through F.

Fix a function $\iota : \omega \to \mathbb{Z}^{<\omega}$ such that $\{n \in \omega : \iota(n) = s\}$ is infinite for each $s \in \mathbb{Z}^{<\omega}$. For $g \in \omega^{\omega\uparrow}$, let

$$F^g := \mathbb{Z}^{<\omega} \setminus \bigcup_{n \in \omega} [\iota(n) {}^\frown \langle 2\, g(n) \rangle {}^\frown 0^{(g(n))}].$$

Note that $[F^g] \subseteq \mathbb{Z}^{\omega}$ is a closed nowhere dense set.

Let $\mathbf{m} : \mathbb{Z}^{<\omega} \to \omega$ be defined by $\mathbf{m}(s) := \max(\{s(i) : i < |s|\} \cup \{1\})$. Now let

$$F^* := \mathbb{Z}^{<\omega} \setminus \bigcup_{s \in \mathbb{Z}^{<\omega}} [s {}^\frown 0^{(\mathbf{m}(s))}].$$

Again, note that $[F^*] \subseteq \mathbb{Z}^{\omega}$ is a closed nowhere dense set.

We need the following feature of F^*:

Lemma 5.54. *Let $s \in \mathbb{Z}^{<\omega}$, and let $k \in \omega \setminus \{0\}$. If s is in F^*, then each $t \in \mathbb{Z}^{<\omega}$ with $t \supseteq s {}^\frown \langle k \rangle$ and $|t| \leq |s| + k$ is in F^* as well.*

Proof. Let $s \in F^*$ and $k > 0$; suppose $t \supseteq s {}^\frown \langle k \rangle$ and $|t| \leq |s| + k$. Assume (towards contradiction) that t is not in F^*. Then there is an $s' \in \mathbb{Z}^{<\omega}$ such that $t \supseteq s' {}^\frown 0^{(\mathbf{m}(s'))}$; since $t \supseteq s {}^\frown \langle k \rangle$, either $s {}^\frown \langle k \rangle \subseteq s' {}^\frown 0^{(\mathbf{m}(s'))}$ or $s' {}^\frown 0^{(\mathbf{m}(s'))} \subseteq s {}^\frown \langle k \rangle$; in either case, we reach a contradiction: in the first case, $s {}^\frown \langle k \rangle \subseteq s'$ (recall $k \neq 0$), hence (by definition of $\mathbf{m}$) we have $\mathbf{m}(s') \geq k$ and therefore $|t| \geq |s'| + k \geq |s| + 1 + k$; in the second case, $s' {}^\frown 0^{(\mathbf{m}(s'))} \subseteq s$, hence $s \notin F^*$. $\square$

For $s \in \mathbb{Z}^{<\omega}$ and $y \in \mathbb{Z}^{\omega}$, we will abbreviate $s + y \restriction |s|$ by $s \boxplus y$. Note that $s \boxplus y$ is in $\mathbb{Z}^{<\omega}$ (and $|s \boxplus y| = |s|$), not in $\mathbb{Z}^{\omega}$.

Lemma 5.55. *Let $y \in \mathbb{Z}^{\omega}$, and let $g \in \omega^{\omega\uparrow}$. If $s \in \mathbb{Z}^{<\omega}$ satisfies $s \boxplus y \in F^*$, then we can find an extension $t \supsetneq s$ such that $t = \iota(n) {}^\frown \langle 2\, g(n) \rangle {}^\frown 0^{(g(n))}$ for some $n \in \omega$ and $t \boxplus y \in F^*$.*

Proof. Since $\{n \in \omega : \iota(n) = s\}$ is infinite for any s, and g is strictly increasing, we can choose $n \in \omega$ such that $\iota(n) = s$ and $g(n) < 2\, g(n) + y(|s|)$. (Note that $y(|s|) \in \mathbb{Z}$ can be negative, so this is not vacuously true.)

Define $t := \iota(n) {}^\frown \langle 2\, g(n) \rangle {}^\frown 0^{(g(n))}$. Since $\iota(n) = s$, we have $t \supsetneq s$. It remains to show that $t \boxplus y \in F^*$.

Let $k := 2\,g(n) + y(|s|)$. We have $t \boxplus y \supseteq (s \boxplus y)^\frown\langle k\rangle$. Note that $k > 0$ and $|t \boxplus y| = |t| = |s| + 1 + g(n) \leq |s| + k = |s \boxplus y| + k$. Since $s \boxplus y \in F^*$, the previous lemma yields $t \boxplus y \in F^*$. □

Lemma 5.56. *Let $y \in \mathbb{Z}^\omega$, and let $\{g_i : i \in \omega\} \subseteq \omega^{\omega\uparrow}$ be a countable set of strictly increasing functions. Then there is an $x \in \mathbb{Z}^\omega$ such that*

1. $x \notin \bigcup_{i\in\omega}[F^{g_i}]$,
2. $x + y \in [F^*]$.

Proof. By induction, we will construct a sequence $s_0 \subsetneq s_1 \subsetneq s_2 \cdots \in \mathbb{Z}^{<\omega}$. The required $x \in \mathbb{Z}^\omega$ will then be $\bigcup_{i\in\omega} s_i$.

We start with $s_0 := \langle\rangle \in \mathbb{Z}^{<\omega}$. Note that $\langle\rangle \in F^*$ (by definition of F^*), so $s_0 \boxplus y = \langle\rangle \in F^*$. Given s_i with $s_i \boxplus y \in F^*$, we can find (by Lemma 5.55) an extension $s_{i+1} \supsetneq s_i$ such that $s_{i+1} = \iota(n)^\frown\langle 2\,g_i(n)\rangle^\frown 0^{(g_i(n))}$ for some $n \in \omega$ and $s_{i+1} \boxplus y \in F^*$. Define $x := \bigcup_{i\in\omega} s_i$.

For $i \in \omega$, we have $x \supseteq s_{i+1} = \iota(n)^\frown\langle 2\,g_i(n)\rangle^\frown 0^{(g_i(n))}$ for some $n \in \omega$, hence $x \notin [F^{g_i}]$. And since $s_i \boxplus y \in F^*$ for each $i \in \omega$, we have $x + y \in [F^*]$, which finishes the proof of the lemma. □

To finish the proof of the theorem, we will construct a set $X \subseteq \mathbb{Z}^\omega$ which belongs to $\mathcal{SN}(\mathbb{Z}^\omega)$ but does not belong[54] to $\mathcal{M}^*(\mathbb{Z}^\omega)$.

Assuming CH, we can fix enumerations $(y_\alpha : \alpha < \omega_1)$ of $\mathbb{Z}^\omega$ and $(g_\alpha : \alpha < \omega_1)$ of $\omega^{\omega\uparrow}$. For each $\alpha < \omega_1$, we apply Lemma 5.56 to get an x_α such that $x_\alpha \notin \bigcup_{\beta<\alpha}[F^{g_\beta}]$ and $x_\alpha + y_\alpha \in [F^*]$. Define $X := \{x_\alpha : \alpha < \omega_1\}$. It remains to show that $X \in \mathcal{SN}(\mathbb{Z}^\omega)$ but $X \notin \mathcal{M}^*(\mathbb{Z}^\omega)$.

First note that $X \notin \mathcal{M}^*(\mathbb{Z}^\omega)$. This is witnessed by the set $[F^*] \subseteq \mathbb{Z}^\omega$ which is nowhere dense, hence in particular meager. It suffices to show that for each $y \in \mathbb{Z}^\omega$, we have $(X + y) \cap [F^*] \neq \emptyset$: fix $y \in \mathbb{Z}^\omega$; pick $\alpha < \omega_1$ such that $y_\alpha = y$; then $x_\alpha + y_\alpha \in (X + y) \cap [F^*]$.

To show that $X \in \mathcal{SN}(\mathbb{Z}^\omega)$, fix a function $f \in \omega^{\omega\uparrow}$. We will find a sequence $(s_n)_{n\in\omega}$ in $\mathbb{Z}^{<\omega}$ with $|s_n| \geq f(n)$ for each n such that $X \subseteq \bigcup_{n\in\omega}[s_n]$. We split f into two functions $g, h \in \omega^{\omega\uparrow}$, more precisely, let g and h be defined by $g(n) = f(2n)$ and $h(n) = f(2n+1)$. Fix $\alpha < \omega_1$ such that $g_\alpha = g$; note that for any $\beta > \alpha$, we have $x_\beta \notin [F^{g_\alpha}] = [F^g]$; in other words, $\{x_\beta : \alpha < \beta < \omega_1\} \subseteq \bigcup_{n\in\omega}[\iota(n)^\frown\langle 2\,g(n)\rangle^\frown 0^{(g(n))}]$. Let $s_{2n} := \iota(n)^\frown\langle 2\,g(n)\rangle^\frown 0^{(g(n))}$; then $|s_{2n}| \geq f(2n)$. Moreover, choose s_{2n+1} such that $|s_{2n+1}| \geq f(2n+1)$ and the

[54]Note that we actually show that X does not belong to (closed nowhere dense)$^*(\mathbb{Z}^\omega)$ either (which is a potentially stronger result) since we provide a witness (namely $[F^*]$) which is even closed nowhere dense (not just meager); see also footnote 35 on page 150.

I do not know, however, whether (closed nowhere dense)$^*(\mathbb{Z}^\omega)$ and $\mathcal{M}^*(\mathbb{Z}^\omega)$ coincide or not.

countable set $\{x_\beta : \beta < \alpha\}$ is covered by $\bigcup_{n\in\omega}[s_{2n+1}]$. Then $X \subseteq \bigcup_{n\in\omega}[s_n]$, and the proof of Theorem 5.53 is finished. $\square$

Remark 5.57. It is quite easy to see that it is not necessary to assume "full CH" in Theorem 5.53. In fact, $\mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$ (i.e., MA(countable)) is sufficient: in Lemma 5.56 (as well as within the final argument to show that $X \in \mathcal{SN}(\mathbb{Z}^\omega)$), one can use density arguments for Cohen forcing to replace "countable" by "less than continuum"; note that Lemma 5.55 tells us that the required sets in Cohen forcing are dense.

Under CH, the first inclusion of Lemma 5.50 is a proper inclusion as well:

Lemma 5.58. *Assume CH. Then $[\mathbb{Z}^\omega]^{\leq\aleph_0} \subsetneqq \mathcal{M}^*(\mathbb{Z}^\omega)$, i.e., there exists an uncountable set in $\mathcal{M}^*(\mathbb{Z}^\omega)$.*

Proof. This is a special case of Lemma 1.17: the meager sets on $\mathbb{Z}^\omega$ form a translation-invariant and inverse-invariant σ-ideal containing all singletons (and have a basis of F_σ sets), hence CH implies that

$$\aleph_1 = \mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = 2^{\aleph_0},$$

so the lemma applies and yields an $\aleph_1$ sized set in $\mathcal{M}^*(\mathbb{Z}^\omega)$. $\square$

Remark 5.59. As in Theorem 5.53, the weaker assumption[55] $\mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$ is sufficient: as in the proof above, it yields $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = 2^{\aleph_0}$, and hence the existence of a set of size continuum in $\mathcal{M}^*(\mathbb{Z}^\omega)$.

Corollary 5.60. *The following two statements are consistent with ZFC:*

1. $[\mathbb{Z}^\omega]^{\leq\aleph_0} = \mathcal{M}^*(\mathbb{Z}^\omega) = \mathcal{SN}(\mathbb{Z}^\omega)$,
2. $[\mathbb{Z}^\omega]^{\leq\aleph_0} \subsetneqq \mathcal{M}^*(\mathbb{Z}^\omega) \subsetneqq \mathcal{SN}(\mathbb{Z}^\omega)$.

Proof. The first statement holds under BC (by Lemma 5.51).

The second statement holds under CH: the first inclusion is proper by Lemma 5.58, and the second inclusion is proper by Theorem 5.53. $\square$

Questions

I do not know whether any of the (two) remaining options is consistent with ZFC:

Question 5.61. Is either of the following statements consistent with ZFC:

[55]... or here even $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$, yielding an uncountable set of size $\mathrm{cov}(\mathcal{M})$.

1. $[\mathbb{Z}^\omega]^{\leq\aleph_0} = \mathcal{M}^*(\mathbb{Z}^\omega) \subsetneq \mathcal{SN}(\mathbb{Z}^\omega)$,
2. $[\mathbb{Z}^\omega]^{\leq\aleph_0} \subsetneq \mathcal{M}^*(\mathbb{Z}^\omega) = \mathcal{SN}(\mathbb{Z}^\omega)$.

Note that the *first scenario* above could be called *weak BC for* $\mathbb{Z}^\omega$ (but without the actual BC): to obtain a such model one could try to iteratively kill all uncountable meager-shiftable sets while avoiding to get full BC, i.e., it is imaginable that one would need a kind of “gentle” Laver forcing.

A model for the *second scenario* would be a model in which the Galvin-Mycielski-Solovay characterization for $\mathbb{Z}^\omega$ holds in a non-trivial way, i.e., unlike in (1) of Corollary 5.60 (somehow “accidentally”, without the usual reason “local compactness”).

Using the terminology of item (5) on page 159, Question 5.61 (2) asks whether $\mathbb{Z}^\omega$ can be an insect in some model of ZFC.

I actually do not know the answer for any (non locally compact) Polish group:

Question 5.62. Is every Polish insect[56] locally compact?

In other words: Is local compactness of a Polish group $(G, +)$ the only imaginable reason for satisfying $\mathcal{M}^*(G) = \mathcal{SN}(G)$ in a non-trivial way?

More basically, I also tried to generalize Theorem 5.53 to the group $(\mathbb{R}^\omega, +)$, but unfortunately it didn’t work in a straightforward way. I conjecture that it can be done, though:

Question 5.63. Is their an analogue of Theorem 5.53 for other non locally compact Polish groups (such as $(\mathbb{R}^\omega, +)$ or the group $(S_\infty, \circ)$ of permutations of ω)?

Let us mention one more question (see also footnote 54 on page 164):

Question 5.64. Is it consistent[57] that the two collections

$$\text{(closed nowhere dense)}^*(G) \quad\text{and}\quad \mathcal{M}^*(G)$$

differ for any Polish group $(G, +)$?

[56]Let us say that a *butterfly* is a Polish insect that is not locally compact. So the existence of a model with a butterfly would answer the question to the negative.

[57]Note that BC must fail in this case; furthermore, the group must not be locally compact since the collection (closed nowhere dense)$^*(G)$ is in between $\mathcal{M}^*(G)$ and $\mathcal{SN}(G)$.

Chapter 6

Sacks dense ideals and Marczewski Borel Conjecture

In this chapter, I consider the Marczewski Borel Conjecture (MBC), a variant of the Borel Conjecture. Motivated by the question whether MBC is consistent, I introduce the notion of "Sacks dense ideal". Even though Con(MBC) remains unsettled, I present several results about Sacks dense ideals.

In Section 6.1, we recall the class s_0 of Marczewski null sets, consider the class s_0^* of s_0-shiftable sets, and introduce the Marczewski Borel Conjecture (the assertion that there are no uncountable s_0-shiftable sets).

In Section 6.2, we introduce the main concept of this chapter: the notion of "Sacks dense ideal". We prove – under CH – that any s_0-shiftable set belongs to all Sacks dense ideals.

In Section 6.3, we consider continuum many Sacks dense ideals $(\mathcal{J}_f)_{f \in \omega^\omega}$ in order to confine the class of s_0-shiftable sets (under CH); we derive that s_0-shiftable sets are "very small" (namely null-additive, in particular strong measure zero). Moreover, we show the existence of uncountable sets that belong to all $\mathcal{J}_f$.

In Section 6.4, we confine the s_0-shiftable sets even further by introducing another Sacks dense ideal, the "Vitali" Sacks dense ideal $\mathcal{E}_0$.

In Section 6.5, we explore the intersection of arbitrary families (of various sizes) of Sacks dense ideals. Among other results, we show that the intersection of $\aleph_1$ many Sacks dense ideals always contains uncountable sets.

In Section 6.6, we present one of the main results of this chapter (see Theorem 6.48) which yields an abundance of Sacks dense ideals (under CH).

In Section 6.7, we comment on the collection s_0^{**}.

I thank Thilo Weinert for suggesting to consider the Marczewski Borel Conjecture and the question whether it is consistent.

6.1 The Marczewski ideal s_0 and the Marczewski Borel Conjecture

In this section, we introduce a new variant of the Borel Conjecture: we replace the ideal $\mathcal{M}$ in the definition of BC (the ideal $\mathcal{N}$ in the definition of dBC, respectively) by the Marczewski ideal s_0. We obtain an assertion which we call the Marczewski Borel Conjecture (MBC).

s_0 — the Marczewski null sets

Recall that a (non-empty) set $P \subseteq 2^\omega$ is called *perfect* if it is closed and has no isolated points (in other words: if it is the set $[T]$ of branches through a perfect tree $T \subseteq 2^{<\omega}$).

Definition 6.1. A set $Z \subseteq 2^\omega$ is *Marczewski null* ($Z \in s_0$) if for each perfect set $P \subseteq 2^\omega$ there is a perfect subset $Q \subseteq P$ with $Q \cap Z = \emptyset$.

It is well-known that s_0 is a translation-invariant σ-ideal: Actually, the σ-closure can be shown by constructing a fusion-sequence of perfect sets (similar to the proof that Sacks forcing satisfies Axiom A, the Sacks property, etc.). Clearly, no perfect set (hence no uncountable Borel or analytic set) is in s_0.

Note that each Z of size less than the continuum belongs to s_0:

Lemma 6.2. *Let $Z \subseteq 2^\omega$ be such that $|Z| < 2^{\aleph_0}$. Then $Z \in s_0$.*

Proof. Fix a perfect set $P \subseteq 2^\omega$; we have to find a perfect subset $Q \subseteq P$ such that $Q \cap Z = \emptyset$.

Split P into "perfectly many" (hence $2^{\aleph_0}$ many) perfect sets $(P_\alpha)_{\alpha<2^{\aleph_0}}$. Then there is a $\beta < 2^{\aleph_0}$ such that the perfect set $Q := P_\beta \subseteq P$ is disjoint from Z. $\square$

Moreover, s_0 also contains "large" sets, i.e., sets of size continuum. This result (as well as several related results) can be found in Miller's survey article "Special Subset of the Real Line" (see [Mil84, Theorem 5.10]). We present[1] a proof using a maximal almost disjoint family ("mad family") of perfect sets (in forcing terminology: a maximal antichain in Sacks forcing):

Lemma 6.3. *There exists a set $Z \subseteq 2^\omega$ of size continuum with $Z \in s_0$.*

[1] I thank Thilo Weinert for coming up with this proof during the Young Set Theory Workshop 2009 in Barcelona.

Proof. A family $(P_\alpha : \alpha < 2^{\aleph_0})$ of perfect subsets of 2^ω is *almost disjoint* if for all $\alpha, \beta < 2^{\aleph_0}$, $\alpha \neq \beta$ implies $|P_\alpha \cap P_\beta| \leq \aleph_0$. (Note that for two perfect sets $P, Q \subseteq 2^\omega$, $P \cap Q$ is closed, hence either at most countable or of size continuum, therefore we have in general $|P \cap Q| \leq \aleph_0$ if and only if $|P \cap Q| < 2^{\aleph_0}$.)

Such a family is called *maximal* if for any perfect set $P \subseteq 2^\omega$, there is an $\alpha < 2^{\aleph_0}$ such that $|P \cap P_\alpha| > \aleph_0$.

Fix a maximal almost disjoint family $(P_\alpha : \alpha < 2^{\aleph_0})$ of perfect sets. To obtain such a family, just start with any family of continuum many disjoint perfect sets (e.g., partition 2^ω into "perfectly many" perfect sets), and extend this (almost disjoint) family to a maximal one (by Zorn's lemma).

Now we construct $Z = \{z_\alpha : \alpha < 2^{\aleph_0}\}$ of size continuum as follows: for any $\alpha < 2^{\aleph_0}$, we pick $z_\alpha \notin \bigcup_{\beta<\alpha} P_\beta \cup \{z_\beta : \beta < \alpha\}$; this is always possible, since $|P_\alpha \cap P_\beta| = \aleph_0$ for every $\beta < \alpha$, hence $|P_\alpha \cap \bigcup_{\beta<\alpha} P_\beta| < 2^{\aleph_0}$, i.e., $P_\alpha \setminus (\bigcup_{\beta<\alpha} P_\beta \cup \{z_\beta : \beta < \alpha\}) \neq \emptyset$ (so we can actually pick $z_\alpha \in P_\alpha$ if we wish). Clearly, $|Z| = 2^{\aleph_0}$.

It remains to show that $Z \in s_0$. Fix a perfect set $P \subseteq 2^\omega$; we have to find a perfect subset $Q \subseteq P$ such that $Q \cap Z = \emptyset$. By the maximality of our family $(P_\alpha : \alpha < 2^{\aleph_0})$, we can fix $\beta < 2^{\aleph_0}$ such that $|P \cap P_\beta| > \aleph_0$. Since $P \cap P_\beta$ is closed, there exists a perfect set $Q' \subseteq P \cap P_\beta$. By construction, we have $z_\alpha \notin P_\beta \supseteq Q'$ for any $\alpha > \beta$, i.e., $Q' \cap Z \subseteq \{z_\gamma : \gamma \leq \beta\}$. But $|\{z_\gamma : \gamma \leq \beta\}| < 2^{\aleph_0}$, so by Lemma 6.2, there exists a perfect set $Q \subseteq Q'$ such that $Q \cap \{z_\gamma : \gamma \leq \beta\} = \emptyset$, i.e., we have found our perfect set $Q \subseteq P$ with $Q \cap Z = \emptyset$. $\square$

s_0^* — the s_0-shiftable sets

Recall that (for $Y, Z \subseteq 2^\omega$ and $t \in 2^\omega$) $Y + Z := \{y + z : y \in Y, z \in Z\}$, and $Y + t := \{y + t : y \in Y\}$, where, given two elements $y, z \in 2^\omega$, its sum $y + z$ is the "bitwise sum modulo 2", i.e., $y + z$ is the real satisfying $(y + z)(n) = y(n) + z(n) \bmod 2$ for each $n < \omega$.

Note that $y + z = z + y$, and $-y = y$, so it is "very easy" to rearrange equations etc.

Definition 6.4. A set $Y \subseteq 2^\omega$ is *s_0-shiftable* ($Y \in s_0^*$) if for each set $Z \in s_0$ we have $Y + Z \neq 2^\omega$.

Note that $Y + Z \neq 2^\omega$ if and only if Y can be "translated away" from Z (i.e., there is a "translation real" $t \in 2^\omega$ such that $(Y + t) \cap Z = \emptyset$).

Since s_0 is a translation-invariant σ-ideal, it is easy to see that the collection s_0^* is translation-invariant, and contains all countable sets of reals, i.e.,

$$[2^\omega]^{\leq\aleph_0} \subseteq s_0^*. \tag{6.1}$$

However, I think there is no reason to believe that the collection s_0^* always forms a σ-ideal (compare with the case of null-shiftable – i.e., strongly meager – sets, where CH even prevents them from being an ideal; see [BS01]).

MBC — the Marczewski Borel Conjecture

Recall that the Borel Conjecture is the assertion that there are no uncountable strong measure zero sets; by the Galvin-Mycielski-Solovay theorem for 2^ω, we know that the meager-shiftable sets coincide with the strong measure zero sets; by definition, the strongly meager sets are the null-shiftable sets; the dual Borel Conjecture is the assertion that there are no uncountable strongly meager sets:

$$\begin{array}{ccccccc} \text{BC} & \Longleftrightarrow & \mathcal{M}^* & = & \mathcal{SN} & = & [2^\omega]^{\leq\aleph_0} \\ \\ \text{dBC} & \Longleftrightarrow & \mathcal{N}^* & = & \mathcal{SM} & = & [2^\omega]^{\leq\aleph_0} \end{array}$$

We introduce the respective variant of BC for the Marczewski ideal s_0:

Definition 6.5. The *Marczewski Borel Conjecture* (MBC) is the assertion that $s_0^* = [2^\omega]^{\leq\aleph_0}$.

In other words, MBC is the "s_0-BC" (as BC is the $\mathcal{M}$-BC, and dBC is the $\mathcal{N}$-BC).

What about the status of MBC (in models of ZFC)? In particular, I'm interested[2] in the following question:

Question 6.6. Is MBC consistent with ZFC?

Actually, it is not too difficult to see that the negation of MBC is consistent. Indeed, $\mathrm{cov}(s_0) > \aleph_1$ implies that all sets of size $\aleph_1$ are in s_0^* (hence MBC fails): the Marczewski ideal s_0 is translation-invariant, so given any Y with $|Y| = \aleph_1$ and any $Z \in s_0$, the sum $Y + Z = \bigcup_{y\in Y} y + Z$ is the union of only $\aleph_1$ many sets in s_0, hence $Y + Z \neq 2^\omega$.

But $\mathrm{cov}(s_0) = \aleph_2 = 2^{\aleph_0}$ holds true in the Sacks model (the model obtained by a countable support iteration of Sacks forcing $\mathbb{S}$ of length ω_2): intuitively

[2]I thank Thilo Weinert for asking me this question during the Young Set Theory Workshop 2009 in Barcelona, and for many fruitful conversations about this topic. Actually, my investigations presented in this chapter only originated because he wondered what is going to happen when $\mathcal{M}$ (or $\mathcal{N}$) in the definition of (d)BC is replaced by s_0.

speaking, this is because Sacks reals tend to avoid sets in s_0 since being disjoint from such sets is a dense property in Sacks forcing (by definition of s_0); since those "dense sets" are not really in the respective ground model, one has to refine the argument; for the details, see [JMS92, Theorem 1.2].

On the other hand, I do not know whether MBC is consistent. To investigate this question, I introduced the concept of "Sacks dense ideal" (see Definition 6.9), and established a connection between s_0^* and Sacks dense ideals (see Lemma 6.10). However, this connection only holds under CH. This was the incentive to study Sacks dense ideals – mainly in the context of CH. Even though the question whether MBC is consistent (with CH) remains unsettled, I consider Sacks dense ideals interesting for their own sake; so they will be the main focus of the chapter.

Remark 6.7. One may ask why MBC does not obviously fail under CH, i.e., why it is not straightforward to construct an uncountable set in s_0^* under CH. After all, it is rather easy to derive the failure of BC (or dBC, respectively) from CH: just perform a Luzin type construction of a strong measure zero set, i.e., use the fact that there is a Borel basis (hence a basis of size $\aleph_1$) of the σ-ideal of meager sets, etc.; alternatively, we can also use the general Lemma 1.17 to obtain uncountable sets in $\mathcal{M}^*$ or $\mathcal{N}^*$; applying Lemma 1.17 is no problem since CH implies that $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \aleph_1$ and $\mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \aleph_1$.

In contrast to the ideal $\mathcal{M}$ of meager sets and the ideal $\mathcal{N}$ of measure zero sets, the Marczewski ideal s_0 *does not have* a basis consisting of Borel sets (since any uncountable Borel set contains a perfect set which definitely does not belong to s_0). But even more is true: there is no basis of s_0 of size less or equal the continuum, i.e., $\mathrm{cof}(s_0) > 2^{\aleph_0}$; this was noted by[3] Fremlin; a slightly stronger result (namely $\mathrm{cf}(\mathrm{cof}(s_0)) > 2^{\aleph_0}$) is shown in [JMS92, Theorem 1.3].

So we always have (also under CH) $\mathrm{cov}(s_0) \leq 2^{\aleph_0} < \mathrm{cof}(s_0)$ which destroys the hope for an easy construction of an uncountable s_0-shiftable set with a method similar to the one in Lemma 1.17.

6.2 Sacks dense ideals

In this section, we introduce the main concept of this chapter: the notion of "Sacks dense ideal". We will investigate them in order to learn more about the collection s_0^* ("towards MBC", so to speak), but we are also interested in

[3] This is mentioned in the introduction of [JMS92] (right before Theorem 1.3 is listed).

them for their own sake. Note that many of the results require CH, in particular, Lemma 6.10 below, which establishes the main connection between s_0^* and Sacks dense ideals. Therefore we will often (but not always) restrict[4] our attention to the CH case.

σ-ideals dense in Sacks forcing and "Sacks dense ideals"

Let us first introduce a weaker notion:

Definition 6.8. A collection $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ is a *σ-ideal dense in Sacks forcing* if

1. $\mathcal{J}$ is a σ-ideal,
2. $\mathcal{J}$ contains all singletons (hence all countable sets),
3. $\mathcal{J}$ is "dense in Sacks forcing $\mathbb{S}$", i.e., each perfect set P contains a perfect subset $Q \subseteq P$ which belongs to $\mathcal{J}$.

However, we are mainly interested in the following stronger notion:

Definition 6.9. A collection $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ is a *Sacks dense ideal* if

1. $\mathcal{J}$ is a σ-ideal,

2b. $\mathcal{J}$ is *translation-invariant*, i.e.,

$$\forall Y \in \mathcal{J} \;\; \forall t \in 2^\omega \;\; (Y \in \mathcal{J} \iff Y + t \in \mathcal{J}),$$

3. $\mathcal{J}$ is "dense in Sacks forcing $\mathbb{S}$", i.e., each perfect set P contains a perfect subset $Q \subseteq P$ which belongs to $\mathcal{J}$.

Note that a Sacks dense ideal contains all singletons; therefore the Sacks dense ideals are exactly the σ-ideals dense in Sacks forcing (according to Definition 6.8) that are (in addition) translation-invariant.

To emphasize the difference to (not translation-invariant) σ-ideals dense in Sacks forcing, we may sometimes say "*translation-invariant* Sacks dense ideal" instead of just "Sacks dense ideal".

[4]Whenever a theorem needs CH, we will explicitly say so.

Connecting s_0^* with Sacks dense ideals (under CH)

The following lemma is central to the investigation of s_0^* and was the incentive for coming up with the notion of Sacks dense ideal in the first place:

Lemma 6.10. *Assume CH. Let $\mathcal{J}$ be any Sacks dense ideal. Then s_0^* is a subset of $\mathcal{J}$.*

Proof. Let $Y \notin \mathcal{J}$. We have to prove that $Y \notin s_0^*$; for that purpose, we will construct a set $Z \in s_0$ such that $Y + Z = 2^\omega$ (i.e., "Z witnesses $Y \notin s_0^*$").

As in Lemma 6.3, we fix a maximal almost disjoint family $(P_\alpha \colon \alpha < 2^{\aleph_0})$ of perfect sets, but this time "within our Sacks dense ideal $\mathcal{J}$", i.e., with the additional property that $P_\alpha \in \mathcal{J}$ for any $\alpha < 2^{\aleph_0}$. To obtain such a family, we again start with any family of continuum many disjoint perfect sets, then we replace each perfect set of this family with a perfect subset which belongs to the ideal $\mathcal{J}$ (note that this is possible since $\mathcal{J}$ is "dense in Sacks forcing"), and then we extend it to a maximal[5] almost disjoint family of perfect sets belonging to $\mathcal{J}$. (Note that such a family is automatically also maximal with respect to any perfect set, i.e., for any perfect set $P \subseteq 2^\omega$, there is an $\alpha < 2^{\aleph_0}$ with $|P \cap P_\alpha| > \aleph_0$. This corresponds to the easy "forcing fact" that every antichain which is maximal in a dense subforcing is also dense in the whole forcing.)

Moreover, let us fix any enumeration $(x_\alpha : \alpha < 2^{\aleph_0})$ of the reals, i.e., $2^\omega = \{x_\alpha \colon \alpha < 2^{\aleph_0}\}$.

We now inductively construct $Z = \{z_\alpha : \alpha < 2^{\aleph_0}\}$. For any $\alpha < 2^{\aleph_0}$, we pick $z_\alpha \in (Y + x_\alpha) \setminus \bigcup_{\beta<\alpha} P_\beta$. This is always possible, for the following reason: by assumption, $Y \notin \mathcal{J}$, so $Y + x_\alpha \notin \mathcal{J}$ holds as well (since $\mathcal{J}$ is translation-invariant); but all the P_β belong to $\mathcal{J}$, hence also $\bigcup_{\beta<\alpha} P_\beta$ is in $\mathcal{J}$ (since $\mathcal{J}$ is a σ-ideal[6]); therefore $(Y + x_\alpha) \setminus \bigcup_{\beta<\alpha} P_\beta \neq \emptyset$.

We claim that $Z \in s_0$ and $Y + Z = 2^\omega$. The latter is obvious by construction: for any $\alpha < 2^{\aleph_0}$, $x_\alpha \in Y + Z$, since z_α was chosen to be in $Y + x_\alpha$.

So it remains[7] to show that $Z \in s_0$. Fix a perfect set $P \subseteq 2^\omega$; we have to find a perfect subset $Q \subseteq P$ such that $Q \cap Z = \emptyset$. Since $\mathcal{J}$ is "dense in Sacks forcing", there is a perfect set $P' \subseteq P$ with $P' \in \mathcal{J}$. By the maximality of $(P_\alpha \colon \alpha < 2^{\aleph_0})$ within $\mathcal{J}$, we can fix $\beta < 2^{\aleph_0}$ such that $|P' \cap P_\beta| > \aleph_0$. Since $P' \cap P_\beta$ is closed, there exists a perfect set $P'' \subseteq P' \cap P_\beta$. By construction, we have $z_\alpha \notin P_\beta \supseteq P''$ for any $\alpha > \beta$, i.e., $P'' \cap Z \subseteq \{z_\gamma : \gamma \leq \beta\}$. But $|\{z_\gamma : \gamma \leq \beta\}| < 2^{\aleph_0}$, so by Lemma 6.2, there exists a perfect set $Q \subseteq P''$

[5] I.e., for any perfect set $P \in \mathcal{J}$, there is an $\alpha < 2^{\aleph_0}$ with $|P \cap P_\alpha| > \aleph_0$.

[6] For the argument to go through in ZFC, we would have to assume $\mathrm{add}(\mathcal{J}) = 2^{\aleph_0}$ here (instead of just σ-ideal – under CH, it is the same anyway); see also Remark 6.11.

[7] The rest of the argument is essentially the same as in Lemma 6.3.

such that $Q \cap \{z_\gamma : \gamma \leq \beta\} = \emptyset$, i.e., we have found our perfect set $Q \subseteq P$ with $Q \cap Z = \emptyset$. $\square$

In other words, the lemma says: A set in s_0^* belongs to the intersection of all Sacks dense ideals.

Remark 6.11. An analogue of Lemma 6.10 holds true in general (i.e., without assuming CH) provided that we require our Sacks dense ideal to have additivity continuum; in other words: ZFC proves that s_0^* is a subset of every Sacks dense ideal $\mathcal{J}$ satisfying $\operatorname{add}(\mathcal{J}) = 2^{\aleph_0}$ (see footnote 6 for the point in the proof where it is used).

However, this doesn't seem to help finding a model of MBC: if CH fails, a Sacks dense ideal with large additivity contains all $\aleph_1$ sized sets, and so does their intersection; with this approach, we therefore cannot hope for excluding all uncountable sets from being in s_0^*.

$\mathfrak{R}$ — the intersection of all Sacks dense ideals

Let $\mathfrak{R}$ denote[8] the intersection of all Sacks dense ideals (we also refer to the elements of $\mathfrak{R}$ as *completely Sacks dense* sets[9] of reals):

Definition 6.12. $\mathfrak{R} := \bigcap\{\mathcal{J} : \mathcal{J} \text{ is a Sacks dense ideal}\}$.

Note that clearly $\mathfrak{R}$ is a translation-invariant σ-ideal. With this notation, Lemma 6.10 says that $s_0^* \subseteq \mathfrak{R}$ (under CH).

$\mathfrak{R}$ and $s_0^{\circledast}$

Under CH, we have $s_0^* \subseteq \mathfrak{R}$. We do not know whether the reverse inclusion $\mathfrak{R} \subseteq s_0^*$ can be shown or not. However, it becomes true when replacing s_0^* by $s_0^{\circledast}$.

In Chapter 1, we defined (for every $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$) the collection $\mathcal{I}^{\circledast}$ (see Definition 1.18 on page 22 and the discussion there). Recall that the collection $s_0^{\circledast}$ is related to s_0^* in the same way as the collection $\mathcal{N}^{\circledast} = \mathcal{VM}$ of very meager sets is related to the collection $\mathcal{N}^* = \mathcal{SM}$ of strongly meager sets: a set Y is in $s_0^{\circledast}$, if for every set $Z \in s_0$ there exists a partition of Y into countably many pieces $(Y_n)_{n<\omega}$ such that $Y_n + Z \neq 2^\omega$ for each n.

[8] $\mathfrak{R}$ stands for "Raach", the place (near Vienna) where the Young Set Theory Workshop 2010 took place. I came up with the notion of Sacks dense ideal (and Lemma 6.10) during this conference.

[9] In a model of CH, let us say that a *squirrel* is an uncountable set in $\mathfrak{R}$, i.e., an uncountable completely Sacks dense set of reals; see also Question 6.46.

Lemma 6.13. $\mathfrak{R} \subseteq s_0^{\circledast}$.

Proof. Let $Y \in \mathfrak{R}$. We have to show that $Y \in s_0^{\circledast}$. More explicitly, we have to show that for every set $Z \in s_0$ there exists a partition of Y into countably many pieces $(Y_n)_{n<\omega}$ such that for all $n \in \omega$, we have $Y_n + Z \neq 2^\omega$.

So let's fix $Z \in s_0$. We define a related Sacks dense ideal $\mathcal{J}_Z$ as follows: for every perfect set $P \subseteq 2^\omega$, let us fix a perfect subset $Q(P) \subseteq P$ such that $Q(P) \cap Z = \emptyset$ (this is possible since $Z \in s_0$); define $\mathcal{J}_Z$ to be the σ-ideal generated by all translates of the sets $Q(P)$, i.e., let

$$\mathcal{J}_Z := \sigma\langle\{Q(P) + t \colon P \subseteq 2^\omega \text{ perfect, } t \in 2^\omega\}\rangle.$$

It is easy to see that $\mathcal{J}_Z$ is a Sacks dense ideal.

By assumption, $Y \in \mathfrak{R} = \bigcap\{\mathcal{J} : \mathcal{J} \text{ is a Sacks dense ideal}\}$, so in particular we have $Y \in \mathcal{J}_Z$, i.e., for some family $(P_n)_{n<\omega}$ of perfect sets and some family $(t_n)_{n<\omega}$ of "translation reals", we have $Y \subseteq \bigcup_{n<\omega}(Q(P_n) + t_n)$. To finish the proof, it is enough to show that $(Q(P_n) + t_n) + Z \neq 2^\omega$ (for any $n < \omega$), or, more generally, that $(Q(P) + t) + Z \neq 2^\omega$ (for any perfect set $P \subseteq 2^\omega$ and any real $t \in 2^\omega$). But this is obvious, since $t \notin (Q(P) + t) + Z$ is equivalent to $Q(P) \cap Z = \emptyset$ which is true by our choice of the $Q(P)$'s. $\square$

Remark 6.14. Note that we didn't need CH for the proof of Lemma 6.13. However, the CH issue is somewhat hidden, in the following sense. As discussed in Remark 6.11, in order to make $s_0^* \subseteq \mathfrak{R}$ hold true in general, we would have to adapt the definition of $\mathfrak{R}$: replace "σ-ideal" by "additivity continuum" in the definition of Sacks dense ideal. With respect to this adapted definition of $\mathfrak{R}$, Lemma 6.13 (i.e., $\mathfrak{R} \subseteq s_0^{\circledast}$) only stays true when we also adapt the definition of $s_0^{\circledast}$ accordingly (replace "there is a partition into countable many pieces" by "there is a partition into less than continuum many pieces").

Note that (6.1), Lemma 6.10, the fact that $\mathfrak{R}$ is a σ-ideal, and Lemma 6.13 together yield the following:

$$\text{CH} \quad \longrightarrow \quad [2^\omega]^{\leq\aleph_0} \subseteq s_0^* \subseteq \sigma\langle s_0^*\rangle \subseteq \mathfrak{R} \subseteq s_0^{\circledast}. \tag{6.2}$$

Remark 6.15. In my opinion, (6.2) makes the connection between s_0^* and $\mathfrak{R}$ (given by Lemma 6.10) even "tighter", and hence more interesting.

Let me explain in more detail what I actually mean. I do not know whether MBC(+CH) is consistent, but in any model of MBC+CH, we would obviously have $[2^\omega]^{\leq\aleph_0} = s_0^* = \sigma\langle s_0^*\rangle$, so either $[2^\omega]^{\leq\aleph_0} = \mathfrak{R}$ holds there as well, or we would have $[2^\omega]^{\leq\aleph_0} = \sigma\langle s_0^*\rangle \subsetneqq s_0^{\circledast}$. If the latter holds, we would have found the "remarkable example" of an ideal $\mathcal{I}$ (namely $\mathcal{I} = s_0$) with the property that $\sigma\langle\mathcal{I}^*\rangle \neq \mathcal{I}^{\circledast}$. It seems to be unknown, whether this situation is consistent for[10] $\mathcal{I} = \mathcal{N}$, i.e., whether it is consistent that the σ-ideal generated by the strongly meager sets differs from the collection of very meager sets (see Definition 1.20 on page 23 and the subsequent discussion). In the model of BC+dBC of our joint paper [GKSW] (see Chapter 2), this is not the case, i.e., $\sigma\langle\mathcal{N}^*\rangle = \mathcal{N}^{\circledast}$ holds there (see Theorem 3.1; actually, the whole Chapter 3 is devoted to the proof of it).

By the above, MBC (i.e., $s_0^* = [2^\omega]^{\leq\aleph_0}$) and $\mathfrak{R} = [2^\omega]^{\leq\aleph_0}$ are "almost equivalent" — in the sense that any counterexample would yield $\sigma\langle s_0^*\rangle \subsetneqq s_0^{\circledast}$, i.e., the above "remarkable case".

Finding Sacks dense ideals "towards MBC"

Recall that my original incentive for studying Sacks dense ideals (and the collection $\mathfrak{R}$) was the question whether the Marczewski Borel Conjecture MBC (i.e., the statement $[2^\omega]^{\leq\aleph_0} = s_0^*$) is consistent or not. Since Lemma 6.10 tells us how to confine s_0^* by Sacks dense ideals (but only under CH), the question ("towards MBC") is whether we can – under CH – find "many Sacks dense ideals" (at least consistently).

It is straightforward to check that the ideal $\mathcal{M}$ of meager sets as well as the ideal $\mathcal{N}$ of measure zero sets forms a Sacks dense ideal (and the σ-ideal $\mathcal{E}$ generated by the closed measure zero sets too), whereas for instance the ideal $\mathcal{SN}$ of strong measure zero sets does not. Nevertheless (as we will show in the next section) the strong measure zero sets can be "approximated from above" (by Sacks dense ideals), meaning that each set in the intersection $\mathfrak{R}$ of *all* Sacks dense ideals (and hence each set in s_0^*) is strong measure zero.

For now, let us just summarize what we have seen so far:

$$\mathrm{CH} \quad\longrightarrow\quad [2^\omega]^{\leq\aleph_0} \subseteq s_0^* \subseteq \mathfrak{R} \subseteq \mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}.$$

6.3 Confining s_0^* by Sacks dense ideals $(\mathcal{J}_f)_{f\in\omega^\omega}$

In this section, we investigate continuum many Sacks dense ideals $\mathcal{J}_f$, in order to confine the class s_0^* (under CH). Moreover, we show that sets of reals that belong to all of the $\mathcal{J}_f$'s (and hence – under CH – sets in s_0^*) are "very small" (namely null-additive, i.e., particularly strong measure zero). We also construct (under CH) uncountable sets which belong to all $\mathcal{J}_f$.

Let me remark that some of my theorems (in particular Lemma 6.27 and Theorem 6.28) are very reminiscent of the construction in the first section of

[10] It is not for $\mathcal{I} = \mathcal{M}$; see Theorem 1.21.

Tomek Bartoszyński's paper[11] "Remarks on small sets of reals" (see [Bar03]), even though I came up with the proofs completely independently.

Sacks dense ideal $\mathcal{J}_f$ generated by f-tiny sets

For $X \subseteq 2^\omega$ and $k \in \omega$, let $X{\restriction}k$ abbreviate $\{x{\restriction}k : x \in X\}$. Note that $X{\restriction}k \subseteq 2^k$.

Definition 6.16. Let[12] $f \in \omega^\omega$.

- We say that a set $X \subseteq 2^\omega$ is *f-tiny* if for almost all $k \in \omega$, we have $|X{\restriction}f(k)| \leq k$.
- Let $\mathcal{J}_f$ be the σ-ideal generated by the f-tiny sets:

$$\mathcal{J}_f := \sigma\langle\{X \subseteq 2^\omega : X \text{ is } f\text{-tiny}\}\rangle.$$

Remark 6.17. We could have defined f-tiny by demanding $|X{\restriction}f(k)| \leq k$ "for all[13] $k > 0$" instead of "for almost all k". This wouldn't make a big difference though, since the resulting σ-ideal $\mathcal{J}_f$ is the same for both versions.

Moreover, note that the (perfect kernel of the) closure of an f-tiny set X is again f-tiny, therefore

$$\mathcal{J}_f = \sigma\langle\{P \subseteq 2^\omega : P \text{ perfect}, P \text{ is } f\text{-tiny}\}\rangle;$$

in other words, we can think of $\mathcal{J}_f$ as generated by the *perfect* f-tiny sets only.

By definition, every $\mathcal{J}_f$ is a σ-ideal; even more holds:

Lemma 6.18. *Let $f \in \omega^\omega$. Then $\mathcal{J}_f$ is a Sacks dense ideal.*

Proof. To show that the σ-ideal $\mathcal{J}_f$ is translation-invariant it suffices to note that being f-tiny is a translation-invariant property (which is obvious).

It is also easy to see that $\mathcal{J}_f$ is dense in Sacks forcing. Fix a perfect set $P \subseteq 2^\omega$; we can think of it as a perfect tree, i.e., let $T \subseteq 2^{<\omega}$ be the (unique) perfect tree such that $P = [T]$ (where $[T]$ is the set of branches through T). Recall that $t \in T$ is a splitting node if both $t^\frown 0$ and $t^\frown 1$ are in T. We thin

[11] I thank Tomasz Weiss for pointing out this paper to me during the conference "Trends in set theory" in Warsaw 2012. I gave a talk there presenting part of the material of this chapter (including Theorem 6.48 which I had proved very shortly before the conference).

[12] We can w.l.o.g. assume that f is an increasing function.

[13] Note that restricting it to $k > 0$ is necessary because $|X{\restriction}f(0)| \leq 0$ only holds for the empty set $X = \emptyset$;-(

out the tree by removing[14] sufficiently many splitting nodes. In this way, it is not difficult to obtain a "sufficiently thin" perfect subtree $T' \subseteq T$ such that $Q := [T']$ is f-tiny; in particular, the perfect set $Q \subseteq P$ is in $\mathcal{J}_f$. $\square$

Definition 6.19. We say that X is *completely tiny* if X belongs to all $\mathcal{J}_f$'s, i.e.,

$$X \in \bigcap_{f \in \omega^\omega} \mathcal{J}_f.$$

Note that Lemma 6.18 and Lemma 6.10 imply that (under CH) every s_0-shiftable set is completely tiny:

$$\text{CH} \quad \longrightarrow \quad s_0^* \subseteq \bigcap_{f \in \omega^\omega} \mathcal{J}_f. \tag{6.3}$$

We will show that a completely tiny set is "quite small" (see Theorem 6.28). Let us first outline the idea of why a completely tiny set is strong measure zero[15] (which will follow from Theorem 6.28 anyway), i.e.,

$$\bigcap_{f \in \omega^\omega} \mathcal{J}_f \subseteq \mathcal{SN}. \tag{6.4}$$

Fix $X \in \bigcap_{f \in \omega^\omega} \mathcal{J}_f$, and let[16] $(k_n)_{n<\omega}$ be an increasing fast-growing sequence of natural numbers; we can "translate" this sequence to an even faster growing[17] function $g \in \omega^\omega$; since $X \in \mathcal{J}_g$, the set X can be covered by countably many (perfect) sets that are g-tiny; each of these sets can be covered by "few" basic clopen sets $[s_i]$ with $|s_i|$ "large"; so if g was chosen appropriately, X can be covered by $\bigcup_n [s_n]$ with $|s_n| \geq k_n$ for each $n < \omega$. This shows that X is strong measure zero.

Remark 6.20. It may be tempting to try to derive Con(MBC) from (6.3) and (6.4) by considering a model of BC. However, this doesn't work[18] since BC requires $2^{\aleph_0} > \aleph_1$, whereas (6.3) only holds under CH.

[14] "To remove a splitting node t" is supposed to mean "to keep either $t^\frown 0$ or $t^\frown 1$ within the tree, but to remove the other node (and each node extending it) from the tree".

[15] Similarly, one can easily show that each completely tiny set is perfectly meager (which also follows from Theorem 6.28).

[16] This is – so to speak – the "ε_n-sequence" of the "elementary definition" of strong measure zero; see Definition 1.5.

[17] E.g., choose g such that $g(1) \geq k_1$, $g(2) \geq k_{1+2}$, $g(3) \geq k_{1+2+3}$, etc.; such a g should work...

[18] An analogous remark applies to dBC and the fact that $s_0^* \subseteq \mathcal{SM}$ under CH (see Corollary 6.30).

Fast-increasing towers are completely tiny

We now show (under CH) the existence of uncountable sets (namely "fast-increasing towers") that are completely tiny, i.e., in $\bigcap_{f\in\omega^\omega}\mathcal{J}_f$; so let us recall the concept of a (fast-increasing) tower.

For each $x \in 2^\omega$, we identify x with the set $\{n : x(n) = 1\} \subseteq \omega$ (i.e., x is the characteristic function of this set). Let $\mathbb{Q} := \{z \in 2^\omega : \forall^\infty n\ (z(n) = 0)\}$ denote the "rationals" of 2^ω. If $x \in 2^\omega \setminus \mathbb{Q}$, i.e., x is (the characteristic function of) an infinite subset of ω, we let $\mathrm{enum}(x) \in \omega^\omega$ be its enumerating function, i.e., we let $\mathrm{enum}(x)$ be the unique strictly increasing function f such that its range $\{f(k) : k < \omega\}$ equals $\{n < \omega : x(n) = 1\}$.

For two functions $f, g \in \omega^\omega$, let $f \leq g$ denote "$f(n) \leq g(n)$ for all $n < \omega$", and let $f \leq^* g$ denote "$f(n) \leq g(n)$ for almost all $n < \omega$". As usual, we say that a subfamily of ω^ω is a *dominating family* if it is cofinal in ω^ω with respect to $\leq^*$; the *dominating number* $\mathfrak{d}$ is the smallest size of a dominating family.

We are going to use the same notation for reals $x, y \in 2^\omega \subseteq \omega^\omega$, i.e., for reals $x, y \in 2^\omega$, let $x \leq^* y$ denote "$x(n) \leq y(n)$ for almost all $n < \omega$". Note that $x \leq^* y$ if and only if $x \subseteq^* y$ when x and y are viewed as subsets of ω (where $\subseteq^*$ denotes almost inclusion, i.e., $x \subseteq^* y$ means $|x \setminus y| < \aleph_0$), and $x \leq y$ if and only if $x \subseteq y$.

Definition 6.21. A set $X \subseteq 2^\omega \setminus \mathbb{Q}$ is a *tower of length* $\gamma \leq 2^{\aleph_0}$ if

$$X = \{x_\alpha : \alpha < \gamma\},$$

and for each $\alpha \leq \beta < \gamma$, we have $x_\beta \leq^* x_\alpha$.

A tower $X = \{x_\alpha : \alpha < \gamma\} \subseteq 2^\omega \setminus \mathbb{Q}$ of length γ is a *fast-increasing tower* if the set $\{\mathrm{enum}(x_\alpha) : \alpha < \gamma\} \subseteq \omega^\omega$ of its enumerating functions forms a dominating family, i.e.,

$$\forall g \in \omega^\omega\ \exists \alpha < \gamma\ (g \leq^* \mathrm{enum}(x_\alpha)).$$

We now show that under certain circumstances concerning cardinal characteristics (in particular under CH), it is easy to construct a fast-increasing tower.

The *tower number* $\mathfrak{t}$ is the smallest γ such that there exists a tower $\{x_\alpha : \alpha < \gamma\}$ of length γ without a *pseudointersection*, i.e., without any $y \in 2^\omega \setminus \mathbb{Q}$ with $y \leq^* x_\alpha$ for each $\alpha < \gamma$. It is well-known that $\aleph_1 \leq \mathfrak{t} \leq \mathfrak{d} \leq 2^{\aleph_0}$.

Lemma 6.22. *Let $\mathfrak{t} = \mathfrak{d}$. Then there is a fast-increasing tower (of length $\mathfrak{d}$).*

Proof. Let $\{g_\alpha\colon \alpha < \mathfrak{d}\} \subseteq \omega^\omega$ be a dominating family, i.e., for each $g \in \omega^\omega$, there is an $\alpha < \mathfrak{d}$ such that $g \leq^* g_\alpha$. We will construct a fast-increasing tower $X = \{x_\alpha\colon \alpha < \mathfrak{d}\} \subseteq 2^\omega \setminus \mathbb{Q}$ of length $\mathfrak{d}$.

We construct the sequence $(x_\alpha)_{\alpha<\mathfrak{d}}$ by induction. We start with any $x_0 \in 2^\omega \setminus \mathbb{Q}$ such that $g_0 \leq \mathrm{enum}(x_0)$. Given x_α, we let $x_{\alpha+1} \in 2^\omega \setminus \mathbb{Q}$ be such that $x_{\alpha+1} \leq x_\alpha$ and $g_{\alpha+1} \leq \mathrm{enum}(x_{\alpha+1})$. At limits α, we use the fact that $\alpha < \mathfrak{t} = \mathfrak{d}$: given $\{x_\beta\colon \beta < \alpha\}$, we can therefore find a pseudointersection $y \in 2^\omega \setminus \mathbb{Q}$ such that $y \leq^* x_\beta$ for every $\beta < \alpha$; then, we again let $x_\alpha \in 2^\omega \setminus \mathbb{Q}$ be such that $x_\alpha \leq y$ and $g_\alpha \leq \mathrm{enum}(x_\alpha)$ (in particular, we have $x_\alpha \leq^* x_\beta$ for every $\beta < \alpha$).

So $X = \{x_\alpha\colon \alpha < \mathfrak{d}\}$ is a tower. It is also fast-increasing: by construction, $g_\alpha \leq \mathrm{enum}(x_\alpha)$ for every $\alpha < \mathfrak{d}$; now fix any $g \in \omega^\omega$; then there is $\alpha < \mathfrak{d}$ such that $g \leq^* g_\alpha$, hence $g \leq^* \mathrm{enum}(x_\alpha)$. □

In particular, CH implies the existence of a fast-increasing tower (of length $\omega_1 = 2^{\aleph_0}$).

Lemma 6.23. *Let $X \subseteq 2^\omega$ be a fast-increasing tower of length ω_1. Then X is completely tiny, i.e., $X \in \bigcap_{f\in\omega^\omega} \mathcal{J}_f$.*

Proof. Suppose that $X = \{x_\alpha\colon \alpha < \omega_1\}$ is a fast-increasing tower. We have to prove that X is completely tiny. So fix[19] an $f \in \omega^\omega$; we will show that $X \in \mathcal{J}_f$.

Let $g \in \omega^\omega$ be defined by $g(n) := f(2^{n+1})$ for each $n < \omega$. By the fact that $\{x_\alpha : \alpha < \omega_1\}$ is fast-increasing, we can fix an $\alpha < \omega_1$ such that $g \leq^* \mathrm{enum}(x_\alpha)$.

We first claim that the set $Y := \{y \in 2^\omega : y \leq x_\alpha\}$ (i.e., the "set of all subsets of x_α") is f-tiny. Let $\mathrm{enum}(x_\alpha) =: h \in \omega^\omega$. Since $g \leq^* h$, we can fix $N \in \omega$ such that $g(n) \leq h(n)$ for each $n \geq N$. We will show that for each $k \geq 2^N$, we have $|Y{\restriction}f(k)| \leq k$. So fix $k \geq 2^N$. Let $n < \omega$ be such that $k \in [2^n, 2^{n+1})$. Note that $n \geq N$, so $g(n) \leq h(n)$ by our assumption on N. Moreover, we have $f(k) < f(2^{n+1}) = g(n) \leq h(n)$. We have

$$|Y{\restriction}f(k)| = |\{s \in 2^{f(k)}\colon s \leq x_\alpha{\restriction}f(k)\}| = 2^{|x_\alpha \cap f(k)|},$$

where $x_\alpha \cap f(k)$ actually denotes the set $\{j < f(k) : x_\alpha(j) = 1\}$ (i.e., we view x_α as a subset of ω here). But $|x_\alpha \cap f(k)| \leq |x_\alpha \cap h(n)| = n$ (since $h = \mathrm{enum}(x_\alpha)$ enumerates x_α), therefore $|Y{\restriction}f(k)| \leq 2^n \leq k$, *finishing the proof of our claim* that Y is f-tiny.

To show that $X \in \mathcal{J}_f$, note that $\mathcal{J}_f$ is a σ-ideal containing all countable sets of reals (since every singleton trivially is f-tiny, hence in $\mathcal{J}_f$). Since $X = \{x_\beta\colon \beta < \alpha\} \cup \{x_\beta\colon \beta \geq \alpha\}$, it is enough to show that $\{x_\beta\colon \beta \geq \alpha\} \in \mathcal{J}_f$.

[19] We can assume w.l.o.g. that f is strictly increasing.

Recall that $\mathbb{Q} = \{z \in 2^\omega : \forall^\infty j \ (z(j) = 0)\}$ denotes the "rationals" of 2^ω. Since X is a tower, $x_\beta \leq^* x_\alpha$ for each $\beta \geq \alpha$; in other words: for each $\beta \geq \alpha$, there exists a $y \leq x_\alpha$ (i.e., $y \in Y$) and a $q \in \mathbb{Q}$ such that $x_\beta = q + y$, i.e.,

$$\{x_\beta : \beta \geq \alpha\} \subseteq \mathbb{Q} + Y = \bigcup_{q \in \mathbb{Q}} (q + Y) \in \mathcal{J}_f$$

(because Y is f-tiny, hence in $\mathcal{J}_f$, and $\mathcal{J}_f$ is a translation-invariant σ-ideal); so $\{x_\beta : \beta \geq \alpha\} \in \mathcal{J}_f$, finishing the proof. $\square$

So we have uncountable completely tiny sets under CH:

Corollary 6.24. *Under CH, there exists an uncountable set in $\bigcap_{f \in \omega^\omega} \mathcal{J}_f$.*

Proof. CH implies $\aleph_1 = \mathfrak{t} = \mathfrak{d} = 2^{\aleph_0}$, hence Lemma 6.22 yields the existence of a fast-increasing tower X of length ω_1. So by Lemma 6.23, X is in $\bigcap_{f \in \omega^\omega} \mathcal{J}_f$. Note that X is uncountable (since it is a dominating family). $\square$

Remark 6.25. We comment on another family of Sacks dense ideals (as an alternative to $(\mathcal{J}_f)_{f \in \omega^\omega}$).

Recall that we can think of a perfect set $P \subseteq 2^\omega$ as a perfect tree $T \subseteq 2^{<\omega}$ (with $P = [T]$). Further recall that $t \in T$ is a splitting node ($t \in \operatorname{split}(T)$) if both $t^\frown 0$ and $t^\frown 1$ are in T. Let us say that the splitting nodes of T are *monotonously enumerated by g* if $g : \omega \to \operatorname{split}(T)$ is bijective and $i < j$ implies $|g(i)| \leq |g(j)|$.

Let $f \in \omega^\omega$. Consider a tree T with the property that its splitting nodes are monotonously enumerated by some g with $|g(i)| \geq f(i)$ for (almost) every $i \in \omega$. Note that this more or less means that the perfect set $[T]$ is f-tiny.

We now define the following stronger notion: let us say that a perfect tree T (or the respective perfect set $P = [T]$) is *f-sparse* if its splitting nodes are monotonously enumerated by some g with $|g(i+1)| - |g(i)| > f(i)$ for every $i \in \omega$. Let $\mathcal{J}_f^{\text{sparse}}$ be the σ-ideal generated by the f-sparse perfect sets:

$$\mathcal{J}_f^{\text{sparse}} := \sigma\langle\{P \subseteq 2^\omega : P \text{ is } f\text{-sparse}\}\rangle.$$

As in Lemma 6.18 for $\mathcal{J}_f$, it is quite easy to check that $\mathcal{J}_f^{\text{sparse}}$ is a Sacks dense ideal.

Note that a function g with $|g(i+1)| - |g(i)| > f(i)$ (i.e., a witness for being f-sparse) in particular satisfies that the mapping $i \mapsto |g(i)|$ is injective, therefore all trees with more than one splitting node at the same height are excluded; especially, "uniform sets" such as the set $Y = \{y \in 2^\omega : y \leq x_\alpha\}$ in the proof of Lemma 6.23 can never be f-sparse (but f-tiny for appropriate f).

Therefore it is not clear how to find (under CH) an uncountable set in $\bigcap_{f\in\omega^\omega} \mathcal{J}_f^{\text{sparse}}$ (by a tower construction, as in Corollary 6.24). However, the much more general Theorem 6.41 below will indeed yield such an uncountable set (under CH).

Completely tiny sets are null-additive

We now prove that each completely tiny set (and hence – under CH – each s_0-shiftable set) is null-additive (hence strongly meager), and therefore (see Theorem 6.29) also meager-additive, so in particular strong measure zero.

Recall the notion of $\mathcal{I}$-additive for arbitrary $\mathcal{I}$:

Definition 6.26. Let $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$. Define

$$\mathcal{I}\text{-additive} := \{X \subseteq 2^\omega : X + Z \in \mathcal{I} \text{ for every set } Z \in \mathcal{I}\}.$$

Clearly, $\mathcal{I}$-additive $\subseteq \mathcal{I}^*$ (provided $2^\omega \notin \mathcal{I}$, which we always tacitly assume). In particular, each $\mathcal{N}$-additive (i.e., *null-additive*) set is strongly meager, and each $\mathcal{M}$-additive (i.e., *meager-additive*) set is strong measure zero. Moreover, whenever $\mathcal{I}$ is a σ-ideal (translation-invariant), the collection of $\mathcal{I}$-additive sets is a σ-ideal (translation-invariant) as well.

Lemma 6.27. *Given a measure zero set $N \in \mathcal{N}$, we can find a function $f \in \omega^\omega$ such that for each $X \subseteq 2^\omega$,*

$$X \textit{ is } f\textit{-tiny} \Rightarrow X + N \in \mathcal{N}.$$

Proof. Fix $N \in \mathcal{N}$. We find a function $f \in \omega$ as follows. It is easy to show (see, e.g., [Gol93, Fact 6.7(2)]) that there is a sequence of clopen sets $(C_n)_{n<\omega}$ with measure $\mu(C_n) \leq 2^{-n}$ for each $n < \omega$ such that $N \subseteq \bigcap_{m<\omega} \bigcup_{n\geq m} C_n$; now each C_n is the union of finitely many basic clopen sets: for each $n < \omega$, let $l_n < \omega$ and $s_{n,i} \in 2^{<\omega}$ (for each $i < l_n$) such that $C_n = \bigcup_{i<l_n} [s_{n,i}]$; without loss of generality, we can assume that for each $n < \omega$, all the $(s_{n,i})_{i<l_n}$ are (distinct and) of the same length, and let $f(n)$ be this common length; i.e., let $f \in \omega^\omega$ be such that $f(n) = |s_{n,0}|$ for each $n < \omega$.[20]

We claim that this f works: so fix a set X which is f-tiny; we have to show that $X + N \in \mathcal{N}$. Since X is f-tiny, we can fix an $N \in \omega$ such that $|X \restriction f(n)| \leq n$ for every $n \geq N$.

[20] Note that $\mu(C_n) \leq 2^{-n}$ implies $l_n \leq 2^{-n} \,.\, 2^{f(n)}$.

For the moment, fix $n \geq N$. Note that[21]

$$X + C_n = X + \bigcup_{i<l_n} [s_{n,i}] = X \restriction f(n) + \bigcup_{i<l_n} [s_{n,i}], \tag{6.5}$$

hence "adding X to C_n" increases the measure of C_n by (at most) a factor of $|X \restriction f(n)| \leq n$, i.e.,[22]

$$\mu(X + C_n) \leq |X \restriction f(n)| \,.\, \mu(C_n) \leq n \,.\, 2^{-n}.$$

Now for each $m < \omega$, $N \subseteq \bigcup_{n \geq m} C_n$, hence $X + N \subseteq \bigcup_{n \geq m}(X + C_n)$. Therefore, for each $m \geq N$,

$$\mu(X + N) \leq \sum_{n \geq m} \mu(X + C_n) \leq \sum_{n \geq m} n \,.\, 2^{-n} \quad \longrightarrow \quad 0 \quad (\text{for } m \to \infty),$$

so $\mu(X + N) = 0$, i.e., $X + N \in \mathcal{N}$. $\square$

Theorem 6.28. *Every completely tiny set (i.e., every set in $\bigcap_{f \in \omega^\omega} \mathcal{J}_f$) is $\mathcal{N}$-additive.*

Proof. Suppose that X is completely tiny, i.e., $X \in \bigcap_{g \in \omega^\omega} \mathcal{J}_g$. We have to prove that X is $\mathcal{N}$-additive. So fix a null set $N \in \mathcal{N}$; we will show that $X + N \in \mathcal{N}$.

By Lemma 6.27, we can find a function $f \in \omega^\omega$ such that $Y + N \in \mathcal{N}$ whenever $Y \subseteq 2^\omega$ is f-tiny.

By assumption, $X \in \mathcal{J}_f$ (the σ-ideal generated by the f-tiny sets), i.e., there are f-tiny sets $(X_k)_{k<\omega}$ such that $X \subseteq \bigcup_{k<\omega} X_k$. So $X_k + N \in \mathcal{N}$ for each $k < \omega$, hence

$$X + N \subseteq \bigcup_{k<\omega} (X_k + N) \in \mathcal{N}$$

(using the fact that $\mathcal{N}$ is a σ-ideal), i.e., $X + N \in \mathcal{N}$. $\square$

To derive that the set is small with respect to the other "notions of smallness" mentioned above, we use the following theorem of Shelah:

[21] Actually, $X \restriction f(n)$ is not a set of reals, but of elements of $2^{f(n)}$ (of size at most n); so what we really mean with the right-most expression of (6.5) is the following: since the question whether a real belongs to $\bigcup_{i<l_n}[s_{n,i}]$ or not only depends on its restriction to $f(n)$, it is sufficient to take any set of (at most n) "representative" reals for $X \restriction f(n)$ (its restrictions to $f(n)$ belong to $X \restriction f(n)$), yielding the same sum as X would yield.

[22] More formally: $z \in X + C_n \;\leftrightarrow\; z \restriction f(n) \in X \restriction f(n) + \{s_{n,i} : i < l_n\} \subseteq 2^{f(n)}$; but $|X \restriction f(n) + \{s_{n,i} : i < l_n\}| \leq |X \restriction f(n)| \,.\, |\{s_{n,i} : i < l_n\}| \leq n \,.\, l_n \leq n \,.\, 2^{-n} \,.\, 2^{f(n)}$, hence $\mu(X + C_n) \leq n \,.\, 2^{-n}$.

Theorem 6.29 (Shelah)**.** *Every $\mathcal{N}$-additive set is $\mathcal{M}$-additive.*

Proof. See, e.g., [BJ95, Theorem 2.7.20], or the original paper [She95]. $\square$

Let us summarize our results so far on "how small" s_0-shiftable sets are:

Corollary 6.30. *Assume CH. Then every s_0-shiftable set is $\mathcal{N}$-additive (and $\mathcal{M}$-additive), so in particular both strong measure zero and strongly*[23] *meager:*

$$CH \quad \longrightarrow \quad s_0^* \subseteq \mathcal{SN} \cap \mathcal{SM}.$$

Proof. By Lemma 6.10, Lemma 6.18, and Theorem 6.28,

$$s_0^* \subseteq \mathfrak{R} = \bigcap \{\mathcal{J} : \mathcal{J} \text{ is a Sacks dense ideal}\} \subseteq \bigcap_{f \in \omega^\omega} \mathcal{J}_f \subseteq \mathcal{N}\text{-additive}.$$

Since $\mathcal{N}$-additive $\subseteq \mathcal{N}^* = \mathcal{SM}$, each set in s_0^* is strongly meager.

By Theorem 6.29, we have $\mathcal{N}$-additive $\subseteq \mathcal{M}$-additive $\subseteq \mathcal{M}^* = \mathcal{SN}$, so each set in s_0^* is strong measure zero. $\square$

6.4 Confining s_0^* even more: the Vitali Sacks dense ideal $\mathcal{E}_0$

In this section, we introduce another Sacks dense ideal which we name $\mathcal{E}_0$ (derived from the "Vitali equivalence relation" on 2^ω); we will prove (under CH) that being in the intersection of the Sacks dense ideals $(\mathcal{J}_f)_{f \in \omega^\omega}$ is not enough for being in the intersection $\mathfrak{R}$ of *all* Sacks dense ideals, i.e., $\mathcal{E}_0$ "contributes" to this intersection in a non-trivial way (see Corollary 6.35). In other words,

$$\text{CH} \quad \longrightarrow \quad \bigcap_{f \in \omega^\omega} \mathcal{J}_f \supsetneqq \mathcal{E}_0 \cap \bigcap_{f \in \omega^\omega} \mathcal{J}_f \supseteq \mathfrak{R} = \bigcap \{\mathcal{J} : \mathcal{J} \text{ is a Sacks dense ideal}\}.$$

Later (see Corollary 6.49) we will show (again under CH) that the second inclusion is a proper inclusion as well.

[23] Hence also very meager (i.e., in $\mathcal{VM}$; see Definition 1.20), and perfectly meager. In fact, the following holds (where $\mathcal{PM}$ denotes the collection of perfectly meager sets):

$$\mathcal{N}\text{-additive} \subseteq \mathcal{N}^* = \mathcal{SM} \subseteq \sigma\langle\mathcal{SM}\rangle \subseteq \mathcal{VM} = \mathcal{N}^{\circledast} \subseteq \mathcal{PM} \subseteq \mathcal{M}.$$

Let us recall the *Vitali equivalence relation* E_0 on the real numbers: let $x, y \in 2^\omega$; then x is E_0-equivalent to y if x differs from y at only finitely many places, i.e.,

$$x \, E_0 \, y \quad :\Longleftrightarrow \quad \exists s \in 2^{<\omega} : \; x + s^\frown 00^{\cdots} = y \quad \Longleftrightarrow \quad x + y \in \mathbb{Q}.$$

A perfect set $P \subseteq 2^\omega$ is called a *perfect partial selector for* E_0 if $\neg(x \, E_0 \, y)$ for any two distinct $x, y \in P$.

We consider the σ-ideal generated by these perfect[24] partial selectors:

Definition 6.31. We define

$$\mathcal{E}_0 := \sigma\langle\{P \subseteq 2^\omega \colon P \text{ perfect partial selector for } E_0\}\rangle.$$

By definition, the collection $\mathcal{E}_0$ is a σ-ideal; even more holds:

Lemma 6.32. *$\mathcal{E}_0$ is a Sacks dense ideal.*

Proof. To show that the σ-ideal $\mathcal{E}_0$ is translation-invariant it suffices to show that being a perfect partial selector for E_0 is a translation-invariant property (which is obvious).

It only remains to show that $\mathcal{E}_0$ is dense in Sacks forcing. Fix a perfect set $P \subseteq 2^\omega$ (as in Lemma 6.18, we can think of it as a perfect tree $T \subseteq 2^{<\omega}$). We thin out this perfect tree T in such a way that we obtain a perfect subtree T' with the property that each two distinct branches through T' differ at infinitely many places (in other words, $[T']$ is a perfect partial selector for E_0).

More precisely, we proceed as follows. (For the notion of nth splitting node, fusion sequence etc., see Definition 6.37.) At step n of the inductive process of thinning out the tree (i.e., of building the fusion sequence), we keep the nth splitting nodes of the tree constructed so far, and thin out the tree by removing[25] splitting nodes along the way up the 2^{n+1} branches above the nth splitting nodes (and do not allow new splitting nodes for the time being) to make sure that each pair from these 2^{n+1} branches differs at (at least) one place somewhere above the nth splitting nodes. To accomplish this task, use the fact that the tree is perfect (hence above each node we

[24]When defining $\mathcal{J}_f$ (see Definition 6.16 and Remark 6.17), it does not matter whether perfect or arbitrary f-tiny sets are used. Here, the difference is essential: replacing "perfect partial selector" by "arbitrary (partial) selector" completely destroys the definition, since the whole space can be covered by countably many "Vitali sets" (actually $2^\omega = \mathbb{Q} + X$ whenever X is any "full selector for E_0").

[25]As in footnote 14, "to remove a splitting node t" is supposed to mean "to keep either $t^\frown 0$ or $t^\frown 1$ within the tree, but to remove the other node (and each node extending it) from the tree".

can find a splitting node), and note that – for a pair of branches – whenever there is a splitting node at some place in (at least) one of these branches, we can remove this splitting node in such a way that the values of the two branches become different (at the next place). In this way, we fulfill our task (of making the branches differ at some place) for all pairs, only then we allow the next – the $(n+1)$th – splitting nodes.

The limit of the fusion sequence will be a tree $T' \subseteq T$ with the property that each two branches through T' differ at infinitely many places. Therefore the set $Q := [T']$ is a perfect partial selector for E_0; in particular, the perfect set $Q \subseteq P$ is in $\mathcal{E}_0$. $\square$

We now construct a fast-increasing tower not in $\mathcal{E}_0$ by "diagonalizing against" all perfect partial selectors for E_0.

Theorem 6.33. *Assume CH. Then we can construct a fast-increasing tower $X = \{x_\alpha \colon \alpha < \omega_1\}$ such that $X \notin \mathcal{E}_0$.*

Proof. We start with the following lemma:

Lemma 6.34. *Let $\{P_m : m < \omega\}$ be a countable family of perfect partial selectors for E_0, and let $x \in 2^\omega \setminus \mathbb{Q}$. Then we can find a $y \in 2^\omega \setminus \mathbb{Q}$ such that $y \leq x$ and $y \notin \bigcup_{m<\omega} P_m$.*

Proof. Let $t_{-1} = \langle\rangle \in 2^{<\omega}$. We will construct a sequence $(t_m)_{m<\omega}$ such that the following holds for each $m < \omega$:

1. $t_m \in 2^{<\omega}$, $t_m \supseteq t_{m-1}$,
2. $t_m \leq x {\restriction} |t_m|$,
3. $[t_m] \cap P_m = \emptyset$,
4. $\exists j \in \mathrm{dom}(t_m) \setminus \mathrm{dom}(t_{m-1})$ with $t_m(j) = 1$.

Note that properties (1) and (2) above actually say that the sequence $(t_m)_{m<\omega}$ is a branch through the "uniform" perfect tree

$$T := \{t \in 2^{<\omega} \colon t \leq x {\restriction} |t|\}$$

(whose body $[T]$ is the set $\{z \in 2^\omega \colon z \leq x\}$). We define $y := \bigcup_{m<\omega} t_m$. Then $y \in [T]$, i.e., $y \leq x$. The only purpose of property (4) is to ensure that the real y takes value 1 infinitely often, i.e., we have $y \in 2^\omega \setminus \mathbb{Q}$. Clearly, $y \supseteq t_m$ for each $m < \omega$, hence property (3) implies $y \notin \bigcup_{m<\omega} P_m$.

It remains to construct the sequence $(t_m)_{m<\omega}$ satisfying the above properties. We proceed by induction on $m < \omega$. Fix m, and suppose that we have already got t_{m-1}. First pick a node $t' \in T$ extending t_{m-1} such that both $t'^\frown 0 \in T$ and $t'^\frown 1 \in T$ (i.e., t' is a splitting node of T above t_{m-1}), and such that t' takes value 1 at least once above the domain of t_{m-1} (this will yield property (4)). Let $z_0 := t'^\frown 0^\frown 000\cdots$ and $z_1 := t'^\frown 1^\frown 000\cdots$, and note that both z_0 and z_1 belong to $\mathbb{Q} \cap [T]$. In particular, $z_0 + z_1 \in \mathbb{Q}$. Since P_m is a (perfect) partial selector for E_0, at most one of the two reals z_0 and z_1 belongs to P_m, so we can pick $i \in 2$ such that $z_i \notin P_m$. But the complement of the perfect set P_m is open, so for sufficiently large $k < \omega$, we have $[z_i \restriction k] \cap P_m = \emptyset$, in other words, we can pick $t_m \in T$ (extending $t' \supseteq t_{m-1}$) satisfying property (3) (and the other properties anyway). $\square$

We will construct a fast-increasing tower $X = \{x_\alpha \colon \alpha < \omega_1\} \subseteq 2^\omega \setminus \mathbb{Q}$ of length ω_1 in a similar way as in Lemma 6.22 (note that CH implies $\mathfrak{t} = \mathfrak{d} = \omega_1$); we will use Lemma 6.34 to make sure that $X \notin \mathcal{E}_0$.

Let $\{P_\alpha : \alpha < \omega_1\}$ be an enumeration of *all* perfect partial selectors for E_0 (using CH). We construct the sequence $(x_\alpha)_{\alpha<\omega_1}$ by induction. For each $\alpha < \omega_1$, we first obtain an $x_\alpha^{\mathrm{old}} \in 2^\omega \setminus \mathbb{Q}$ satisfying

$$\forall \beta < \alpha : x_\alpha^{\mathrm{old}} \leq^* x_\beta \text{ and } g_\alpha \leq \mathrm{enum}(x_\alpha^{\mathrm{old}}) \tag{6.6}$$

as in Lemma 6.22 (see there for "what is g_α?"). Then we apply Lemma 6.34 to the countable family $\{P_\beta \colon \beta < \alpha\}$ of perfect partial selectors for E_0 and $x_\alpha^{\mathrm{old}} \in 2^\omega \setminus \mathbb{Q}$ to obtain an $x_\alpha \in 2^\omega \setminus \mathbb{Q}$ such that $x_\alpha \leq x_\alpha^{\mathrm{old}}$ and $x_\alpha \notin \bigcup_{\beta<\alpha} P_\beta$. Note that x_α (replacing x_α^{old}) still satisfies (6.6) above, therefore the proof that $X = \{x_\alpha \colon \alpha < \omega_1\}$ is a fast-increasing tower is exactly the same as in Lemma 6.22.

So it only remains to show[26] that $X \notin \mathcal{E}_0$. Assume towards a contradiction that $X \in \mathcal{E}_0$. Then there is a countable family $\{P_n \colon n < \omega\}$ of perfect partial selectors for E_0 such that $X \subseteq \bigcup_{n<\omega} P_n$. Since $(P_\alpha)_{\alpha<\omega_1}$ lists *all* perfect partial selectors for E_0, we can fix $\alpha < \omega_1$ such that $X \subseteq \bigcup_{n<\omega} P_n \subseteq \bigcup_{\beta<\alpha} P_\beta$. By construction, $x_\alpha \notin \bigcup_{\beta<\alpha} P_\beta$ (but $x_\alpha \in X$), a contradiction, and the proof of Theorem 6.33 is finished. $\square$

Corollary 6.35. *Assume CH. Then there is an (uncountable) set that is completely tiny (i.e., in $\bigcap_{f\in\omega^\omega} \mathcal{J}_f$) but not in $\mathcal{E}_0$.*

Proof. By Theorem 6.33, we get a fast-increasing tower X such that $X \notin \mathcal{E}_0$; by Lemma 6.23, X is completely tiny. $\square$

[26]The proof actually even shows that X is "hereditarily not in $\mathcal{E}_0$", i.e., for each $Y \subseteq X$ of size $\aleph_1$, we still have $Y \notin \mathcal{E}_0$.

Note that this actually shows[27] that being completely tiny (or $\mathcal{N}$-additive, respectively) is not sufficient for being in $\mathfrak{R}$ (the intersection of all Sacks dense ideals), let alone for being s_0-shiftable; more formally:

$$\mathrm{CH} \quad \longrightarrow \quad \mathcal{N}\text{-additive} \supseteq \bigcap_{f\in\omega^\omega} \mathcal{J}_f \supsetneqq \mathcal{E}_0 \cap \bigcap_{f\in\omega^\omega} \mathcal{J}_f \supseteq \mathfrak{R} \supseteq s_0^*.$$

Remark 6.36. It was suggested to me to look at the concept of "γ-*set*" (see, e.g., [GN82] or [GM84]), and to investigate whether a γ-set could be a good candidate for an uncountable set necessarily being in $\mathfrak{R}$ (i.e., the intersection of *all* Sacks dense ideals).

However, it turns out that this is not the case: since γ-sets are easily derived from towers, (the proof of) Corollary 6.35 also shows that there is a γ-set that is not in $\mathcal{E}_0$. Therefore not every γ-set is in $\mathfrak{R}$, witnessed by the Sacks dense ideal $\mathcal{E}_0$, so to speak. Alternatively[28], this can be shown as follows: by a theorem of Bartoszyński and Recław (see [BR96]), there is a γ-set that is not strongly meager, but every set in $\bigcap_{f\in\omega^\omega} \mathcal{J}_f$ (hence every set in $\mathfrak{R}$) is $\mathcal{N}$-additive by Theorem 6.28 (hence strongly meager).

6.5 Intersecting σ-ideals dense in Sacks forcing

In this section, we explore the result of intersecting countably many, $\aleph_1$ many, or all σ-ideals dense in Sacks forcing (or translation-invariant Sacks dense ideals, respectively).

Intersecting countably many

We first show that the intersection of countably many Sacks dense ideals results in a Sacks dense ideal; in other words, the class of Sacks dense ideals is closed under countable intersections.

Let's recall the notion of a fusion sequence of perfect trees (i.e., a fusion sequence for Sacks forcing $\mathbb{S}$):

Definition 6.37. Let $T \subseteq 2^{<\omega}$ be a perfect tree. A node $t \in T$ is a *splitting node* of T ($t \in \mathrm{split}(T)$) if both $t^\frown 0$ and $t^\frown 1$ belong to T. A node $t \in T$ is

[27] Indeed, aiming at this conclusion was my incentive for coming up with the Sacks dense ideal $\mathcal{E}_0$.

[28] My first approach to prove that a γ-set need not be in $\mathfrak{R}$ was the one via $\mathcal{E}_0$ (it was actually before I realized that every set in $\bigcap_{f\in\omega^\omega} \mathcal{J}_f$ is strongly meager).

called *nth splitting node* if

$$t \in \mathrm{split}(T) \quad \text{and} \quad |\{s \in T \colon s \subsetneqq t,\, s \in \mathrm{split}(T)\}| = n.$$

Note that for each $n \in \omega$, the number of nth splitting nodes of T is 2^n.

Let $S, T \subseteq 2^{<\omega}$ be perfect trees, and let $n \in \omega$. We say that $T \leq_n S$ if $T \subseteq S$, and T and S have the same nth splitting nodes (and hence also the same mth splitting nodes for $m < n$); more formally: $T \leq_n S$ if $T \subseteq S$, and "$t \in S$ is nth splitting node of S" implies "$t \in \mathrm{split}(T)$".

A sequence $T^0 \geq_0 T^1 \geq_1 T^2 \geq_2 \dots$ is called *fusion sequence*. Note that $T' := \bigcap_{n<\omega} T^n$ is again a perfect tree (the *limit* of the fusion sequence) with the property that $T' \leq_n T^n$ for each $n \in \omega$.

Given a perfect set $P \subseteq 2^\omega$, there is a (unique) perfect tree $T \subseteq 2^{<\omega}$ such that $P = [T]$ (and vice versa), where $[T]$ (the *body* of T) is the set of all branches through T, i.e.,

$$[T] := \{x \in 2^\omega \colon \forall n < \omega\ (x{\restriction}n \in T)\}.$$

Using this correspondence, we can define $Q \leq_n P$ for perfect sets $P, Q \subseteq 2^\omega$. Given a fusion sequence of perfect sets $P^0 \geq_0 P^1 \geq_1 P^2 \geq_2 \dots$, its limit $Q := \bigcap_{n<\omega} P^n$ is again perfect (with $Q \leq_n P^n$ for each n).

For a perfect tree T and a node $t \in T$, let $T^{[t]}$ denote the collection of those nodes in T that are comparable with t, i.e., let

$$T^{[t]} := \{s \in T \colon s \subseteq t \vee t \subseteq s\}.$$

Recall Definitions 6.8 and 6.9 for the notions of "σ-ideal dense in Sacks forcing" and "Sacks dense ideal" (which is its translation-invariant version), respectively.

Lemma 6.38. *Let $\mathcal{J}$ be a σ-ideal*[29] *dense in Sacks forcing. Then for each perfect set $P \subseteq 2^\omega$ and each $n \in \omega$ there exists a perfect set Q such that $Q \leq_n P$ and $Q \in \mathcal{J}$.*

Proof. Given P and n, we let $T \subseteq 2^{<\omega}$ be the perfect tree with $P = [T]$. Let $\{t_k \colon k < 2^n\}$ be an enumeration of the nth splitting nodes of T.

For each $k < 2^n$, consider the tree $T^{[t_k]}$ and apply the fact that $\mathcal{J}$ is dense in Sacks forcing to pick a perfect tree $T_k \subseteq T^{[t_k]}$ such that $[T_k] \in \mathcal{J}$. Let $T' := \bigcup_{k<2^n} T_k$, and let $Q := [T']$. Then Q is perfect with $Q \leq_n P$, and $Q \in \mathcal{J}$ (since $\mathcal{J}$ is an ideal and $Q = \bigcup_{k<2^n} [T_k]$). ☐

[29]Actually, being an ideal (and dense in Sacks forcing) would be enough for the proof of this lemma.

The class of σ-ideals dense in Sacks forcing is closed under countable intersections:

Theorem 6.39. *Let $\{\mathcal{J}_n : n < \omega\}$ be a countable family of σ-ideals dense in Sacks forcing. Then also $\bigcap_{n<\omega} \mathcal{J}_n$ is a σ-ideal dense in Sacks forcing. In particular, there exists a perfect (hence uncountable) set in $\bigcap_{n<\omega} \mathcal{J}_n$.*

Proof. Being a σ-ideal and containing all singletons is clearly preserved under arbitrary intersections (not only countable ones).

It remains to show that $\bigcap_{n<\omega} \mathcal{J}_n$ is dense in Sacks forcing. Fix a perfect set $P \subseteq 2^\omega$. We have to find a perfect $Q \subseteq P$ such that $Q \in \bigcap_{n<\omega} \mathcal{J}_n$. By induction on $n < \omega$, we construct (repeatedly using Lemma 6.38) a fusion sequence $P \geq_0 P^0 \geq_1 P^1 \geq_2 P^2 \geq_3 \dots$ such that for each $n < \omega$, we have $P^n \in \mathcal{J}_n$. Let $Q := \bigcap_{n<\omega} P^n$ be its limit. Then $Q \subseteq P$ is perfect, and for each $n < \omega$, we have $Q \subseteq P^n \in \mathcal{J}_n$, hence $Q \in \mathcal{J}_n$ (since $\mathcal{J}_n$ is closed under subsets); consequently, $Q \in \bigcap_{n<\omega} \mathcal{J}_n$. $\square$

It follows that also the class of (translation-invariant) Sacks dense ideals is closed under countable intersections:

Corollary 6.40. *Let $\{\mathcal{J}_n : n < \omega\}$ be a countable family of Sacks dense ideals. Then also $\bigcap_{n<\omega} \mathcal{J}_n$ is a Sacks dense ideal. In particular, there exists a perfect (hence uncountable) set in $\bigcap_{n<\omega} \mathcal{J}_n$.*

Proof. Since each of the $\mathcal{J}_n$'s is a σ-ideal dense in Sacks forcing, also $\bigcap_{n<\omega} \mathcal{J}_n$ is a σ-ideal dense in Sacks forcing (by Theorem 6.39). But $\bigcap_{n<\omega} \mathcal{J}_n$ is even a Sacks dense ideal (i.e., additionally translation-invariant), since translation-invariance is preserved under (arbitrary) intersections. $\square$

Intersecting $\aleph_1$: Todorčević's Aronszajn construction

We now show that the intersection of $\aleph_1$ many Sacks dense ideals (even though not being a Sacks dense ideal any longer) always contains an uncountable set of reals. Actually, translation-invariance is not relevant here, i.e., the theorem is valid for arbitrary σ-ideals dense in Sacks forcing.

The idea of the proof comes from[30] Todorčević's famous proof that $\diamondsuit$ implies the existence of a hereditary γ-set (see [GM84, Theorem 4]), where he argues using an "*Aronszajn tree of perfect sets*".

Note that we do not assume CH. However, we always consider only $\aleph_1$ many ideals (regardless whether CH holds or not), *not* continuum many.

[30] I thank Stevo Todorčević for fruitful conversations during the Hajnal birthday conference in Budapest 2011. When I told him about Sacks dense ideals (remarkably during a total lunar eclipse), he suggested to me that I "should look at" this proof of him.

Under CH, of course, the theorem applies to continuum many, for instance, to the family $(\mathcal{J}_f)_{f\in\omega^\omega}$ (see Definition 6.16), or to the family $(\mathcal{J}_f^{\text{sparse}})_{f\in\omega^\omega}$ (see Remark 6.25). In this (much more general) way, it reproves Corollary 6.24.

Theorem 6.41. *Let $\{\mathcal{J}_\alpha\colon \alpha<\omega_1\}$ be any $\aleph_1$ sized family of σ-ideals dense in Sacks forcing. Then there exists a set X in $\bigcap_{\alpha<\omega_1}\mathcal{J}_\alpha$ of size $\aleph_1$.*

In particular, the lemma is applicable to a family of Sacks dense ideals (but doesn't use their translation-invariance).

Proof. For notational convenience, we "renumber" the given $\mathcal{J}_\alpha$'s with successor ordinals only, i.e., we assume without loss of generality that $\{\mathcal{J}_{\beta+1}:\beta<\omega_1\}$ is a full list of our σ-ideals dense in Sacks forcing.

We will construct an "Aronszajn[31] tree $\mathcal{T}$ of perfect sets" (of height ω_1). More specifically, we will construct a tree $\mathcal{T}\subseteq\omega^{<\omega_1}$, together with perfect sets $(R_\eta)_{\eta\in\mathcal{T}}$ assigned to the nodes of $\mathcal{T}$, satisfying the following properties:

1. $\forall\alpha<\omega_1\ (|\mathcal{T}_\alpha|\le\aleph_0)$ (where
$$\mathcal{T}_\alpha:=\{\eta\in\mathcal{T}\colon |\eta|=\alpha\}$$
is the α'th level of the tree $\mathcal{T}$),
2. $\forall\eta\in\mathcal{T}\ (R_\eta\subseteq 2^\omega$ is a perfect set),
3. $\forall\eta,\xi\in\mathcal{T}\ (\eta\subseteq\xi\rightarrow R_\eta\supseteq R_\xi)$
4. $\forall\eta\in\mathcal{T}\ \forall n\in\omega\ (\eta^\frown n\in\mathcal{T}\wedge R_\eta\ge_n R_{\eta^\frown n})$
5. $\forall\eta\in\mathcal{T}\ \forall n\in\omega\ \forall\alpha>|\eta|\ \exists\xi\in\mathcal{T}(|\xi|=\alpha\wedge\eta\subseteq\xi\wedge R_\eta\ge_n R_\xi)$
6. $\forall\beta<\omega_1\ \forall\eta\in\mathcal{T}_{\beta+1}\ (R_\eta\in\mathcal{J}_{\beta+1})$

Provided that we have such a tree $\mathcal{T}$ and perfect sets $(R_\eta)_{\eta\in\mathcal{T}}$, we inductively construct a set $X=\{x_\alpha:\alpha<\omega_1\}$ as follows: for each $\alpha<\omega_1$, we choose any (or, e.g., the "left-most") $\eta\in\mathcal{T}_\alpha$, and pick (using $|R_\eta|>\aleph_0$)
$$x_\alpha\in R_\eta\setminus\{x_\beta\colon\beta<\alpha\}.$$
Clearly, $|X|=\aleph_1$.

[31] Typically, the tree will be an Aronszajn tree (since there are no strictly decreasing sequences of perfect sets/trees of length ω_1), even though it won't be used anywhere in the proof.

We have to prove that $X \in \bigcap_{\beta<\omega_1} \mathcal{J}_{\beta+1}$. Fix $\alpha := \beta+1 < \omega_1$. To show that $X \in \mathcal{J}_\alpha$, recall that $\mathcal{J}_\alpha$ is a σ-ideal containing[32] all singletons (hence containing all countable sets of reals). Since $X = \{x_\gamma \colon \gamma < \alpha\} \cup \{x_\gamma \colon \gamma \geq \alpha\}$, it is enough to show that $\{x_\gamma : \gamma \geq \alpha\} \in \mathcal{J}_\alpha$. Using property (1) and property (6) of the above list, and the fact that $\mathcal{J}_\alpha$ is a σ-ideal, we get

$$\bigcup\{R_\eta \colon \eta \in \mathcal{T}_\alpha\} \in \mathcal{J}_\alpha.$$

We claim that $\{x_\gamma : \gamma \geq \alpha\} \subseteq \bigcup\{R_\eta : \eta \in \mathcal{T}_\alpha\}$: fix $\gamma \geq \alpha$; then there is $\xi \in \mathcal{T}_\gamma$ such that $x_\gamma \in R_\xi$; we can find $\eta \in \mathcal{T}_\alpha$ with $\eta \subseteq \xi$ (namely $\eta := \xi \restriction \alpha$); by property (3) above, $R_\xi \subseteq R_\eta$, so we have $x_\gamma \in \bigcup\{R_\eta \colon \eta \in \mathcal{T}_\alpha\}$, finishing the proof that $X \in \mathcal{J}_\alpha$.

It remains to show how to construct the tree $\mathcal{T}$ and the sets $(R_\eta)_{\eta\in\mathcal{T}}$ with the desired properties. We proceed by induction on the levels of the tree.

($\alpha = 0$) Put the empty sequence $\langle\rangle$ into $\mathcal{T}$, and let $R_{\langle\rangle}$ be any perfect set (e.g., let $R_{\langle\rangle} := 2^\omega$).

(Successor step $\alpha = \beta + 1$) For each $\eta \in \mathcal{T}_\beta$ and each $n < \omega$, we put $\eta^\frown n$ into $\mathcal{T}$, and (using Lemma 6.38) let $R_{\eta^\frown n}$ be a perfect set in $\mathcal{J}_\alpha$ with $R_{\eta^\frown n} \leq_n R_\eta$.

(Limit step α) For each $\eta \in \mathcal{T}\restriction\alpha$ (the tree constructed so far) and each $n < \omega$, we will construct a $\xi \supseteq \eta$ with $|\xi| = \alpha$ (and put it into $\mathcal{T}$) and a perfect set $R_\xi \leq_n R_\eta$. So fix $\eta =: \eta_0 \in \mathcal{T}\restriction\alpha$ and $n < \omega$. Choose an ω-sequence $|\eta| =: \beta_0 < \beta_1 < \beta_2 < \ldots$ cofinal in α, and pick (by induction on $i < \omega$) nodes $\eta_{i+1} \in \mathcal{T}\restriction\alpha$ with $|\eta_{i+1}| = \beta_{i+1}$ such that $R_{\eta_{i+1}} \leq_{n+i} R_{\eta_i}$ (this is possible by inductive assumption, i.e., by property (5) for $\mathcal{T}\restriction\alpha$). Now put $\xi := \bigcup_{i<\omega} \eta_i$ into $\mathcal{T}$, and let $R_\xi := \bigcap_{i<\omega} R_{\eta_i}$ be the limit of the fusion sequence $R_{\eta_0} \geq_n R_{\eta_1} \geq_{n+1} R_{\eta_2} \geq_{n+2} \ldots$; observe that $|\xi| = \alpha$ and $R_\xi \leq_n R_\eta$ (i.e., property (5) holds up to α as well). $\square$

Remark 6.42. In [PR95], Pawlikowski and Recław introduced a *Cichoń diagram for classes of small sets* (as an analogue of the classical Cichoń diagram for the cardinal invariants). These "classes of small sets" are closely related to classes of small sets we are dealing with in this chapter, such as null-additive, meager-additive, strongly meager, and strong measure zero sets.

When discussing my proof of Theorem 6.41 with other people, I learned about Jörg Brendle's excellent paper[33] "Generic constructions of small sets of

[32] See Definition 6.8.

[33] I thank Lyubomyr Zdomskyy for suggesting to study this paper.

reals" (see [Bre96]). In this paper, Brendle shows (under CH) that there are no relations between the classes introduced in [PR95] except for those given by the Cichoń diagram. To prove this, he constructs certain sets by fixing an increasing elementary sequence of countable models $\{M_\alpha : \alpha < \omega_1\}$ of length ω_1 (with $\bigcup_{\alpha<\omega_1} M_\alpha \supseteq 2^\omega$), and then picking reals $\{x_\alpha \colon \alpha < \omega_1\}$ such that each x_α is generic over all models M_β for $\beta \leq \alpha$ for an appropriately chosen forcing[34] notion.

It may be not apparent at first sight, but actually there is a close connection between the Aronszajn construction of Theorem 6.41 and one of Brendle's constructions, namely the one using Sacks forcing $\mathbb{S}$. This particular construction yields a set $X := \{x_\alpha \colon \alpha < \omega_1\}$ (with the x_α Sacks generic[35] over the previous models) which belongs to the class $\mathrm{Add}(\mathcal{N})$ at the bottom-left of the "Cichoń diagram for classes of small sets" (i.e., the smallest of the classes).

Such a set X will automatically belong to each σ-ideal dense in Sacks forcing that is "seen by (one of) the models"; roughly speaking, this is because of the following: consider a model M_α and a set D dense in Sacks forcing (and seen by the model, i.e., $D \in M$), then being a real Sacks generic over M_α basically implies being in one of the Sacks conditions from the dense set D that belong to M_α; since M_α is countable, $\{x_\delta \colon \delta > \alpha\} \subseteq \bigcup(D \cap M_\alpha)$ belongs to the σ-ideal generated by D. Therefore X is in every σ-ideal dense in Sacks forcing that is "seen by some model".

Moreover, all sets in $\mathrm{Add}(\mathcal{N})$ are in particular null-additive, so X will be null-additive. This is no surprise, since each of the Sacks dense ideals $\mathcal{J}_f$ (see Definition 6.16) is "easily definable" from the respective real $f \in \omega^\omega$, hence will be seen by some model, therefore X will be in $\bigcap_{f\in\omega^\omega} \mathcal{J}_f$ and hence null-additive (by Theorem 6.28). Also, $\mathcal{E}_0$ (see Definition 6.31) is "easily definable", so X will be in $\mathcal{E}_0$ as well.

Intersecting all

On the other hand, intersecting *all*[36] σ-ideals dense in Sacks forcing "excludes" every uncountable set of reals.

[34] As he also mentions: for Cohen (random, respectively) forcing, this is nothing else than a modern way of representing the construction of a Luzin (Sierpiński, respectively) set; recall that a Luzin (Sierpiński, resp.) set is a set that has countable intersection with every meager (null, resp.) set.

[35] Actually, generic conditions are obtained using fusion sequences, as in the proof that Axiom A implies α-proper for every $\alpha < \omega_1$.

[36] Here it is crucial to allow also the ones that are not translation-invariant...

The key to the proof is the following lemma which comes up with "new" σ-ideals dense in Sacks forcing:

Lemma 6.43. *Let $Z \in s_0$ be uncountable. Then there exists a σ-ideal $\mathcal{J}_Z$ dense in Sacks forcing such that $Z \notin \mathcal{J}_Z$.*

Proof. Fix $Z \in s_0$. We define $\mathcal{J}_Z$ as follows: for every perfect set $P \subseteq 2^\omega$, fix a perfect subset $Q(P) \subseteq P$ such that $Q(P) \cap Z = \emptyset$ (this is possible since $Z \in s_0$); define $\mathcal{J}_Z$ to be the σ-ideal generated by all the $Q(P)$'s (and containing all countable sets), i.e., let

$$\mathcal{J}_Z := \sigma\langle \{Q(P) \colon P \subseteq 2^\omega \text{ perfect}\} \cup \{\{x\} \colon x \in 2^\omega\}\rangle.$$

Clearly, $\mathcal{J}_Z$ is a σ-ideal dense in Sacks forcing (see Definition 6.8).

Now assume towards a contradiction that $Z \in \mathcal{J}_Z$. Then there are perfect sets $(P_n)_{n<\omega}$ and reals $(x_n)_{n<\omega}$ such that

$$Z \subseteq \bigcup_{n<\omega} Q(P_n) \cup \{x_n \colon n < \omega\}.$$

But all our $Q(P)$'s were chosen to be disjoint from Z, so Z is actually a subset of $\{x_n \colon n < \omega\}$, contradicting our assumption that Z is uncountable. □

Marczewski[37] has proven that sufficiently small sets (such as strong measure zero sets, or perfectly meager sets) are in the Marczewski ideal s_0:

Theorem 6.44 (Marczewski)**.** $\mathcal{SN} \subseteq s_0$.

Proof. The result follows from Theorem 5.1 and Theorem 5.9 of Miller's survey article [Mil84] "Special Subset of the Real Line". □

We can now "compute" the intersection of all σ-ideals dense in Sacks forcing:

Theorem 6.45. *There is no uncountable set that belongs to* every *σ-ideal dense in Sacks forcing:*

$$[2^\omega]^{\leq \aleph_0} = \bigcap \{\mathcal{J} : \mathcal{J} \textit{ is a } \sigma\textit{-ideal dense in Sacks forcing}\}.$$

Proof. ("$\subseteq$") A countable set of reals clearly belongs to every σ-ideal dense in Sacks forcing (see Definition 6.8).

("$\supseteq$") Fix a set X that belongs to every σ-ideal dense in Sacks forcing. We have to show that X is countable.

[37] His name was Szpilrajn back then.

Recall that – for each $f \in \omega^\omega$ – the ideal $\mathcal{J}_f$ (see Definition 6.16) is a Sacks dense ideal (see Lemma 6.18), so in particular it is a σ-ideal dense in Sacks forcing. Therefore X belongs to $\bigcap_{f\in\omega^\omega} \mathcal{J}_f$. Now recall that $\bigcap_{f\in\omega^\omega} \mathcal{J}_f \subseteq \mathcal{SN}$: see either (6.4) on page 179 and the subsequent discussion for a (sketch of an) elementary proof, or argue with Theorem 6.28 and Theorem 6.29 (as in the proof of Corollary 6.30). So X is strong measure zero, hence (by Theorem 6.44) X is in s_0.

Assume towards a contradiction that X is uncountable. Since X is in s_0, Lemma 6.43 yields a σ-ideal $\mathcal{J}'$ dense in Sacks forcing such that $X \notin \mathcal{J}'$, but X belongs to every σ-ideal dense in Sacks forcing, a contradiction. □

However, the situation becomes unclear when we restrict ourselves to (translation-invariant) Sacks dense ideals.

This is the most important[38] open question of this chapter:

Question 6.46. Assume CH. Does the collection

$$\mathfrak{R} = \bigcap \{\mathcal{J} : \mathcal{J} \text{ is a Sacks dense ideal}\}$$

contain any uncountable sets[39] of reals (at least consistently)??

Remark 6.47. If we would consider σ-ideals $\mathcal{J}$ dense in Sacks forcing that are only invariant under translations by reals from $\mathbb{Q}$ (or any other fixed countable set) instead of Sacks dense ideals (which are invariant under translations by any real) in the question above, then the answer would be the same as the one for (not translation-invariant) σ-ideals dense in Sacks forcing (given by Theorem 6.45): only the countable sets of reals are in the intersection of all "$\mathbb{Q}$-*invariant*" $\mathcal{J}$'s. The reason is similar: given a set X that belongs to every $\mathbb{Q}$-invariant $\mathcal{J}$, we can argue as in the proof of Theorem 6.45 to show that $X \in s_0$, and would then derive another $\mathbb{Q}$-invariant $\mathcal{J}'$ from X with $X \notin \mathcal{J}'$ (somewhat as in Lemma 6.43 or Lemma 6.54, making use of the fact that s_0 is a translation-invariant σ-ideal).

[38] Indeed, of all open problems of my thesis, this is the one I would appreciate to know the answer the most.

Actually, either way of an answer would be interesting in some sense: if there were no uncountable sets in $\mathfrak{R}$ (under CH), this would immediately yield MBC under CH (hence solving the original problem whether MBC is consistent, which was the incentive for coming up with Sacks dense ideals and $\mathfrak{R}$); on the other hand, an uncountable set in $\mathfrak{R}$ would show that $\mathfrak{R}$ is a (non-trivial!) new class of sets of reals.

Of course, most interesting would be if both options were consistent with CH.

[39] In other words (according to the naming of footnote 9 on page 175): Is it consistent that there is a squirrel?

Note that Question 6.46 would have a positive answer if the total number of Sacks dense ideals were only $\aleph_1$ (see Theorem 6.41 which particularly applies to Sacks dense ideals). We do not know any quick argument though that there are more than $\aleph_1$ many Sacks dense ideals (under CH). However, it can be shown quite indirectly: Theorem 6.48 (which is the central result of Section 6.6) produces $\aleph_2$ many Sacks dense ideals under CH (see Corollary 6.50) in a rather "collateral way".

As opposed to this, combining Theorem 6.41 and Theorem 6.45 immediately demonstrates the existence of at least $\aleph_2$ many σ-ideals dense in Sacks forcing (in ZFC actually, without assuming CH).

6.6 More and more Sacks dense ideals

In this section, we "improve" Theorem 6.41 and obtain the following theorem which shows that there is an abundance of (translation-invariant!) Sacks dense ideals (under CH).

Theorem 6.48. *Assume CH. Let $\{\mathcal{J}_\alpha : \alpha < \omega_1\}$ be any $\aleph_1$ sized family of*[40] *Sacks dense ideals. Then there exist a Sacks dense ideal $\mathcal{J}'$ and a set X in $\bigcap_{\alpha<\omega_1} \mathcal{J}_\alpha$ (of size $\aleph_1$) such that $X \notin \mathcal{J}'$.*

The whole section is devoted to the proof of Theorem 6.48. Let us discuss some easy consequences first.

Corollary 6.49. *Assume CH. Then the intersection of* all *Sacks dense ideals is strictly smaller than the intersection of* any $\aleph_1$ sized family *of Sacks dense ideals.*

In other words, for any family $\{\mathcal{J}_\alpha : \alpha < \omega_1\}$ of Sacks dense ideals, we have

$$\bigcap_{\alpha<\omega_1} \mathcal{J}_\alpha \supsetneq \mathfrak{R}.$$

Proof. For the given family $\{\mathcal{J}_\alpha : \alpha < \omega_1\}$ of Sacks dense ideals, Theorem 6.48 gives us a set X which belongs to $\bigcap_{\alpha<\omega_1} \mathcal{J}_\alpha$ but (not to the Sacks dense ideal $\mathcal{J}'$, hence) not to $\mathfrak{R}$. $\square$

In particular, we have many (translation-invariant) Sacks dense ideals under CH:

[40] Again, it is sufficient to assume that each $\mathcal{J}_\alpha$ is a σ-ideal dense in Sacks forcing; i.e., as in Theorem 6.41, translation-invariance is not relevant here. However, the resulting Sacks dense ideal $\mathcal{J}'$ in the conclusion is indeed translation-invariant: this is actually the whole point of the theorem.

Corollary 6.50. *Assume CH. Then there are at least $\aleph_2$ many (i.e., more than continuum many) Sacks dense ideals.*

Proof. If there were only $\aleph_1$ many, we would have $\mathfrak{R} = \bigcap_{\alpha<\omega_1} \mathcal{J}_\alpha$, contradicting Corollary 6.49. $\square$

However, it is unclear to me (even under CH) whether there are $2^{\aleph_1}$ many Sacks dense ideals (in case of $2^{\aleph_1} > \aleph_2$), or whether there are $2^{(2^{\aleph_0})}$ many Sacks dense ideals (in general). (Compare with Theorem 7.12 about "Silver dense ideals".)

Remark 6.51. Note that Corollary 6.50 may be a weaker statement than Corollary 6.49: it is imaginable that there are "many different" Sacks dense ideals, but "only few really contribute" to their intersection $\mathfrak{R}$; in other words, there may be a – so to speak – "small basis towards $\mathfrak{R}$" for the class of all Sacks dense ideals.

Before we begin working towards the proof of Theorem 6.48, let us summarize the situation (under CH) for the collection s_0^* (the s_0-shiftable sets) which had been the starting point of our investigations:

$$\text{CH} \quad \longrightarrow \quad \mathcal{M} \cap \mathcal{N} \supsetneqq \mathcal{E} \supsetneqq \bigcap_{f \in \omega^\omega} \mathcal{J}_f \supsetneqq \mathcal{E}_0 \cap \bigcap_{f \in \omega^\omega} \mathcal{J}_f \supsetneqq \mathfrak{R} \supseteq s_0^* \supseteq [2^\omega]^{\leq \aleph_0}.$$

I do not know whether the right-most two inclusions are (consistently) proper.

The translatively Marczewski null sets s_0^{trans}

We are aiming at an analogue of Lemma 6.43 capable of producing "new" (translation-invariant!) Sacks dense ideals. Recall that Lemma 6.43 used s_0 in its hypothesis. Therefore we will introduce a "translative" variant of s_0.

First of all, observe the following: in the definition of s_0, it actually doesn't make any difference whether we require the "existing perfect subset" to be disjoint, or just to have countable intersection:

Lemma 6.52. *A set $X \subseteq 2^\omega$ is in s_0 if and only if for each perfect set $P \subseteq 2^\omega$ there is a perfect subset $Q \subseteq P$ with $|Q \cap X| \leq \aleph_0$.*

Proof. Each set in s_0 trivially satisfies the characterization on the right side.

Conversely, suppose X satisfies this characterization, and fix a perfect set $P \subseteq 2^\omega$; then there is a perfect subset $Q' \subseteq P$ such that $|Q' \cap X| \leq \aleph_0$. The rest of the argument is actually Lemma 6.2: split Q' into "perfectly many" (hence uncountably many) perfect sets $(Q_\alpha)_{\alpha<2^{\aleph_0}}$; then there is a $\beta < 2^{\aleph_0}$ such that the perfect set $Q := Q_\beta \subseteq Q' \subseteq P$ is disjoint from X. $\square$

In the following definition, we require the same for all *translates* of the perfect subset (yielding a strengthening of being Marczewski null):

Definition 6.53. A set $X \subseteq 2^\omega$ is *translatively Marczewski null* ($X \in s_0^{\text{trans}}$) if for each perfect set $P \subseteq 2^\omega$ there is a perfect subset $Q \subseteq P$ such that for each real $t \in 2^\omega$, we have $|(Q + t) \cap X| \leq \aleph_0$.

It is quite easy to show that $s_0^{\text{trans}} \subseteq s_0$ is a translation-invariant σ-ideal (for proving "σ-ideal" again use a fusion sequence).

This is the analogue of Lemma 6.43 "producing" (translation-invariant!) Sacks dense ideals, as promised above:

Lemma 6.54. *Let $Z \in s_0^{\text{trans}}$ be uncountable. Then there exists a Sacks dense ideal $\mathcal{J}_Z$ such that $Z \notin \mathcal{J}_Z$.*

Proof. Fix $Z \in s_0^{\text{trans}}$. We define a Sacks dense ideal $\mathcal{J}_Z$ as follows: for every perfect set $P \subseteq 2^\omega$, fix a perfect subset $Q(P) \subseteq P$ such that for each real $t \in 2^\omega$, we have $|(Q(P)+t) \cap Z| \leq \aleph_0$ (this is possible since $Z \in s_0^{\text{trans}}$); define $\mathcal{J}_Z$ to be the σ-ideal generated by all translates of the sets $Q(P)$, i.e., let

$$\mathcal{J}_Z := \sigma\langle\{Q(P) + t \colon P \subseteq 2^\omega \text{ perfect}, t \in 2^\omega\}\rangle.$$

It is easy to see that $\mathcal{J}_Z$ is a Sacks dense ideal (in particular, it is translation-invariant by definition).

Now assume towards a contradiction that $Z \in \mathcal{J}_Z$. Then there are perfect sets $(P_n)_{n<\omega}$ and reals $(t_n)_{n<\omega}$ such that

$$Z \subseteq \bigcup_{n<\omega} (Q(P_n) + t_n).$$

But all our $Q(P)$'s were chosen in such a way that $Q(P) + t$ has only countable intersection with Z (for any $t \in 2^\omega$), so Z is actually countable, a contradiction. □

A technical strengthening of s_0^{trans}

To construct sets in s_0^{trans}, we first take a closer look at the heights of splitting nodes of perfect trees:

Definition 6.55. Let $T \subseteq 2^{<\omega}$ be a perfect tree. We say that $n \in \omega$ is a *splitting level* of T ($n \in \text{splitlev}(T)$) if there is a splitting node s of T which is of length n. In other words,

$$\text{splitlev}(T) = \{|s| \colon s \in \text{split}(T)\} \subseteq \omega.$$

Given a perfect set $P \subseteq 2^\omega$, there is a (unique) perfect tree $T \subseteq 2^{<\omega}$ such that $P = [T]$ (and vice versa). Using this correspondence, we can define $\mathrm{splitlev}(P)$ to be $\mathrm{splitlev}(T)$.

It is quite obvious by definition that a natural number n belongs to $\mathrm{splitlev}(P)$ if and only if there are two reals in P with the property that n is the first place where they differ; more formally:

$$n \in \mathrm{splitlev}(P) \iff \exists x_0, x_1 \in P \ \ (x_0{\restriction}n = x_1{\restriction}n \ \wedge \ x_0(n) \neq x_1(n)). \quad (6.7)$$

Lemma 6.56. *The following hold true:*

1. *Let $P \subseteq 2^\omega$ be perfect and $t \in 2^\omega$. Then* $\mathrm{splitlev}(P) = \mathrm{splitlev}(P + t)$.
2. *Let $P_0, P_1 \subseteq 2^\omega$ be perfect sets with* $\mathrm{splitlev}(P_0) \cap \mathrm{splitlev}(P_1) =^* \emptyset$. *Then*[41] $|P_0 \cap P_1| < \aleph_0$.
3. *Suppose that $P_0, P_1 \subseteq 2^\omega$ are perfect sets, and $n_0, n_1 \in \omega$. Then there exist perfect sets Q_0 and Q_1 such that $Q_0 \leq_{n_0} P_0$ and $Q_1 \leq_{n_1} P_1$, and* $\mathrm{splitlev}(Q_0) \cap \mathrm{splitlev}(Q_1) =^* \emptyset$.
4. *Let $P, Q \subseteq 2^\omega$ be perfect sets with $Q \subseteq P$. Then* $\mathrm{splitlev}(Q) \subseteq \mathrm{splitlev}(P)$.

Proof. (1) Just note that for two reals $x_0, x_1 \in 2^\omega$, the first place where x_0 and x_1 differ is the same as the first place where $x_0 + t$ and $x_1 + t$ do. So (6.7) implies $\mathrm{splitlev}(P) = \mathrm{splitlev}(P + t)$.

(2) Since $\mathrm{splitlev}(P_0) \cap \mathrm{splitlev}(P_1) =^* \emptyset$, we can fix an $n^* \in \omega$ such that $\mathrm{splitlev}(P_0) \cap \mathrm{splitlev}(P_1) \subseteq n^*$. We claim that for any $s \in 2^{n^*}$, there is at most one real $x \in P_0 \cap P_1$ with $x \supseteq s$. This suffices because it yields $|P_0 \cap P_1| \leq |2^{n^*}| < \aleph_0$.

To prove our claim, fix $s \in 2^{n^*}$, and assume towards a contradiction that there are two distinct reals $x \neq y$ in $P_0 \cap P_1$ with $x, y \supseteq s$. Let $n \in \omega$ be minimal with $x(n) \neq y(n)$. Since x and y belong to both P_0 and P_1, the number n belongs to both $\mathrm{splitlev}(P_0)$ and $\mathrm{splitlev}(P_1)$ (see (6.7)). But $x, y \supseteq s \in 2^{n^*}$, so $n \geq n^*$, contradicting $\mathrm{splitlev}(P_0) \cap \mathrm{splitlev}(P_1) \subseteq n^*$.

(3) As in Lemma 6.18 and Lemma 6.32, we can think of the given perfect sets $P_0, P_1 \subseteq 2^\omega$ as perfect trees $T_0, T_1 \subseteq 2^{<\omega}$.

Let $n := \max(n_0, n_1)$. (For the notion of nth splitting node, etc., see Definition 6.37.) We thin out both trees (while keeping their nth splitting nodes) in such a way that we get perfect subtrees T_0' and T_1' with almost

[41] Actually, obtaining $|P_0 \cap P_1| \leq \aleph_0$ would be sufficient for our application.

disjoint sets splitlev(T'_0) and splitlev(T'_1) (actually, they are going to be really disjoint above the highest of the involved nth splitting nodes).

More precisely, we proceed as follows. We keep all the nth splitting nodes (of both trees), and remove[42] – alternately – splitting nodes from T_0 and T_1: first, we remove (in any way) every splitting node of T_0 (above the nth splitting nodes) up to the height at which all of the succeeding (i.e., $(n+1)$th) splitting nodes of T_1 have already appeared; then we turn to T_1, keep all these $(n+1)$th splitting nodes, and remove sufficiently many splitting nodes from T_1, until we reach the height of all the succeeding splitting nodes of the (already thinned out) tree T_0; we keep them, before we continue thinning out, etc.

We keep going like this; it is easy to see that we end up with two perfect trees $T'_0 \subseteq T_0$ and $T'_1 \subseteq T_1$ such that $T'_0 \leq_{n_0} T_0$ and $T'_1 \leq_{n_1} T_1$ and $\mathrm{splitlev}(T'_0) \cap \mathrm{splitlev}(T'_1) =^* \emptyset$.

(4) Obvious (again, we can use (6.7)). $\square$

We now introduce the collection s_0^{split} (which will be included in s_0^{trans}) since it is more transparent to aim for sets in s_0^{split} than in s_0^{trans}.

Definition 6.57. A set $X \subseteq 2^\omega$ is in s_0^{split} if for each perfect set P, there is a perfect subset $Q \subseteq P$ and perfect sets $(R_n)_{n<\omega}$ such that

$$|X \setminus \bigcup_{n<\omega} R_n| \leq \aleph_0 \ \wedge \ \forall n \in \omega \ \ (\mathrm{splitlev}(Q) \cap \mathrm{splitlev}(R_n) =^* \emptyset).$$

Remark 6.58. Note that replacing $|X \setminus \bigcup_{n<\omega} R_n| \leq \aleph_0$ by $X \subseteq \bigcup_{n<\omega} R_n$ actually wouldn't change the above definition of s_0^{split}: given the perfect subset Q there, and any single point $x \in 2^\omega$, it is easy to construct a perfect set R'_x containing x and satisfying $\mathrm{splitlev}(Q) \cap \mathrm{splitlev}(R'_x) =^* \emptyset$ (using the fact that splitlev(Q) cannot be co-finite); so once we have countably many perfect sets $(R_n)_{n<\omega}$ covering all but countably many points of X, we can cover these countably many points $(x_n)_{n<\omega}$ by countably many additional perfect sets $(R'_{x_n})_{n<\omega}$ with the desired property.

Nevertheless, we decided to give the definition of s_0^{split} the way we did since then it is more natural to argue that the set X in Lemma 6.60 is in s_0^{split} (and the proof of Lemma 6.59 doesn't change).

Being in s_0^{split} is indeed stronger than being in s_0^{trans}:

Lemma 6.59. $s_0^{\mathrm{split}} \subseteq s_0^{\mathrm{trans}}$.

[42] As in footnote 14, "to remove a splitting node t" is supposed to mean "to keep either $t^\frown 0$ or $t^\frown 1$ within the tree, but to remove the other node (and each node extending it) from the tree".

Proof. Suppose that X is in s_0^{split}. To show that X is in s_0^{trans}, let $P \subseteq 2^\omega$ be perfect; by definition of s_0^{split}, we can fix a perfect subset $Q \subseteq P$ such that the following holds: there are perfect sets $(R_n)_{n<\omega}$ such that $|X \setminus \bigcup_{n<\omega} R_n| \leq \aleph_0$ and for each $n \in \omega$, $\text{splitlev}(Q)$ and $\text{splitlev}(R_n)$ are almost disjoint. To finish the proof, we derive that for each real $t \in 2^\omega$, we have $|(Q+t) \cap X| \leq \aleph_0$.

Fix $t \in 2^\omega$. By Lemma 6.56 (1), we have $\text{splitlev}(Q) = \text{splitlev}(Q+t)$, hence $\text{splitlev}(Q+t) \cap \text{splitlev}(R_n) =^* \emptyset$ for each $n \in \omega$. So Lemma 6.56 (2) implies that $|(Q+t) \cap R_n| < \aleph_0$ for each $n \in \omega$, and this easily yields $|(Q+t) \cap X| \leq \aleph_0$. □

Refining the Aronszajn tree construction

Finally, we are prepared to prove our improved version of Theorem 6.41 which in turn will easily yield Theorem 6.48.

Lemma 6.60. *Assume CH. Let $\{\mathcal{J}_\alpha : \alpha < \omega_1\}$ be any $\aleph_1$ sized family of σ-ideals dense in Sacks forcing. Then there exists a set X in $\bigcap_{\alpha<\omega_1} \mathcal{J}_\alpha$ of size $\aleph_1$ that belongs to s_0^{split} (hence, by Lemma 6.59, to s_0^{trans}).*

Proof. We will modify the proof of Theorem 6.41 to obtain a set X which is additionally in s_0^{split}.

Using[43] CH, we can enumerate all perfect sets using only (successor) ordinals less than ω_1 as indices; i.e., let $\{P_{\beta+1} : \beta < \omega_1\}$ be a list of all perfect subsets of 2^ω.

As in Theorem 6.41, we construct a tree $\mathcal{T}$ and perfect sets $(R_\eta)_{\eta\in\mathcal{T}}$ satisfying the properties (1) to (6) demanded there. Simultaneously, we construct perfect sets $\{Q_{\beta+1} : \beta < \omega_1\}$ such that the following two additional properties hold:

7. $\forall \beta < \omega_1 \ (P_{\beta+1} \supseteq Q_{\beta+1})$
8. $\forall \beta < \omega_1 \ \forall \eta \in \mathcal{T}_{\beta+1} \ (\text{splitlev}(Q_{\beta+1}) \cap \text{splitlev}(R_\eta) =^* \emptyset)$

Provided that we have a tree $\mathcal{T}$, perfect sets $(R_\eta)_{\eta\in\mathcal{T}}$, and perfect sets $(Q_{\beta+1})_{\beta<\omega_1}$ satisfying all our properties (1) to (8), we inductively construct $X = \{x_\alpha : \alpha < \omega_1\}$ exactly the same way as in Theorem 6.41 (i.e., for each $\alpha < \omega_1$, we choose any $\eta \in \mathcal{T}_\alpha$, and we pick $x_\alpha \in R_\eta \setminus \{x_\beta : \beta < \alpha\}$). Clearly, $|X| = \aleph_1$, and again, we have $X \in \bigcap_{\beta<\omega_1} \mathcal{J}_{\beta+1}$ (the proof is exactly the same as in Theorem 6.41).

The only additional thing to prove is that X belongs to s_0^{split}. Suppose $P \subseteq 2^\omega$ is perfect; fix $\beta < \omega_1$ such that $P = P_{\beta+1}$, and let $Q := Q_{\beta+1}$;

[43] This is the only place in the proof where CH is used.

by property (7), the perfect set Q is a subset of P. Now note that $\{R_\eta : \eta \in \mathcal{T}_{\beta+1}\}$ is a countable collection of perfect sets (see property (1)), and $\{x_\gamma : \gamma \geq \beta+1\} \subseteq \bigcup\{R_\eta : \eta \in \mathcal{T}_{\beta+1}\}$ (the details are given in the proof of Theorem 6.41), i.e., $|X \setminus \{R_\eta : \eta \in \mathcal{T}_{\beta+1}\}| \leq \aleph_0$. Moreover, for any $\eta \in \mathcal{T}_{\beta+1}$, we have $\mathrm{splitlev}(Q) \cap \mathrm{splitlev}(R_\eta) =^* \emptyset$ (see property (8)), finishing the proof that $X \in s_0^{\mathrm{split}}$.

It remains to show how to construct the tree $\mathcal{T}$, the sets $(R_\eta)_{\eta\in\mathcal{T}}$, and the sets $(Q_{\beta+1})_{\beta<\omega_1}$ with the 8 desired properties. We proceed pretty much the same as in Theorem 6.41; at successor steps, however, we do additional work to construct the perfect set $Q_{\beta+1} \subseteq P_{\beta+1}$ and to fulfill property (8).

($\alpha = 0$) As in Theorem 6.41.

(Successor step $\alpha = \beta + 1$) For each $\eta \in \mathcal{T}_\beta$ and each $n < \omega$, let $R'_{\eta^\frown n}$ be a perfect set in $\mathcal{J}_\alpha$ with $R'_{\eta^\frown n} \leq_n R_\eta$ (using Lemma 6.38).

Now we go through all the (countably many) pairs $(\eta, n) \in \mathcal{T}_\beta \times \omega$ again and (using Lemma 6.56 (3)) construct perfect sets $R_{\eta^\frown n} \leq_n R'_{\eta^\frown n}$ and a fusion sequence $P_{\beta+1} =: P^{-1} \geq_0 P^0 \geq_1 P^1 \geq_2 \ldots$ (with limit $Q_{\beta+1}$) with the property that for each pair (η, n), the set $\mathrm{splitlev}(R_{\eta^\frown n})$ is almost disjoint from $\mathrm{splitlev}(P^k)$ for some $k < \omega$ (and hence from $\mathrm{splitlev}(Q_{\beta+1})$).

In more detail: Fix a (one-to-one) enumeration of $\mathcal{T}_\beta \times \omega$, i.e., let $\iota : \omega \to \mathcal{T}_\beta \times \omega$ be bijective. We proceed by induction on $k < \omega$. Consider the pair $(\eta, n) := \iota(k)$, look at the perfect sets P^{k-1} and $R'_{\eta^\frown n}$, and apply Lemma 6.56 (3) to obtain perfect sets P^k and $R_{\eta^\frown n}$ such that $P^k \leq_k P^{k-1}$ and $R_{\eta^\frown n} \leq_n R'_{\eta^\frown n}$, and $\mathrm{splitlev}(P^k) \cap \mathrm{splitlev}(R_{\eta^\frown n}) =^* \emptyset$. Finally, let $Q_{\beta+1} := \bigcap_{k<\omega} P^k$ be the limit of the fusion sequence $P_{\beta+1} = P^{-1} \geq_0 P^0 \geq_1 P^1 \geq_2 \ldots$; since $Q_{\beta+1}$ is a subset of every P^k, we have (see Lemma 6.56 (4)) $\mathrm{splitlev}(Q_{\beta+1}) \cap \mathrm{splitlev}(R_{\eta^\frown n}) =^* \emptyset$ for each (η, n), thereby fulfilling property (8).

Note that $\leq_n$ is transitive, so $R_{\eta^\frown n} \leq_n R_\eta$ holds, fulfilling property (4) (of Theorem 6.41). Moreover, $\mathcal{J}_\alpha = \mathcal{J}_{\beta+1}$ is an ideal and $R_{\eta^\frown n} \subseteq R'_{\eta^\frown n} \in \mathcal{J}_\alpha$, so $R_{\eta^\frown n} \in \mathcal{J}_\alpha$, fulfilling property (6).

(Limit step α) As in Theorem 6.41. □

Proof of Theorem 6.48. Let $\{\mathcal{J}_\alpha : \alpha < \omega_1\}$ be the given $\aleph_1$ sized family of Sacks dense ideals. Note that the $\mathcal{J}_\alpha$'s are in particular σ-ideals dense in Sacks forcing, so we can apply Lemma 6.60 to obtain a set $X \in \bigcap_{\alpha<\omega_1} \mathcal{J}_\alpha$ of size $\aleph_1$ that is in s_0^{split}. By Lemma 6.59, X is in s_0^{trans}, so (since X is uncountable) Lemma 6.54 implies the existence of a (translation-invariant) Sacks dense ideal $\mathcal{J}'$ such that $X \notin \mathcal{J}'$, finishing the proof of the theorem. □

6.7 A little corollary about s_0^{**}

In this section, we briefly comment on the collection s_0^{**}.

Recall that, for any $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$,

$$\mathcal{I}^* = \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{I}\}$$

(the collection of $\mathcal{I}$-shiftable sets). We can apply this *star operation* twice, yielding the collection $(\mathcal{I}^*)^* =: \mathcal{I}^{**}$ of $\mathcal{I}^*$-shiftable sets.

It is easy to check that the mapping $\mathcal{I} \mapsto \mathcal{I}^{**}$ is a *closure operation*, i.e., $\mathcal{I} \subseteq \mathcal{I}^{**}$ holds true for any collection $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$, and applying it twice doesn't change the collection any further.

Under CH, it can be shown that both the ideal $\mathcal{M}$ of meager sets and the ideal $\mathcal{N}$ of measure zero sets are "closed" under this closure operation, i.e.,

$$\text{CH} \quad \longrightarrow \quad \mathcal{M} = \mathcal{M}^{**} \;\wedge\; \mathcal{N} = \mathcal{N}^{**}. \tag{6.8}$$

Also, the collection $\mathcal{C} = [2^\omega]^{\leq\aleph_0}$ of countable sets of reals is closed in this sense (even without assuming CH), i.e., ZFC proves that $\mathcal{C}^{**} = \mathcal{C}$ (see [Sol03] for the original proof, or [PS08] for a simpler proof of this fact).

The situation for the Marczewski ideal s_0 is different:

Corollary 6.61. *Assume CH. Then* $s_0 \subsetneqq s_0^{**}$.

Proof. By[44] Corollary 6.30, we have[45] $s_0^* \subseteq \mathcal{SN} = \mathcal{M}^*$. Now note that $\mathcal{I}_1 \subseteq \mathcal{I}_2$ implies $\mathcal{I}_1^* \supseteq \mathcal{I}_2^*$ (for any two $\mathcal{I}_1, \mathcal{I}_2$), hence $s_0^* \subseteq \mathcal{M}^*$ yields $s_0^{**} \supseteq \mathcal{M}^{**}$. But $\mathcal{M}^{**} \supseteq \mathcal{M}$ (since[46] "$**$" is a closure operation); consequently, s_0^{**} contains perfect sets (since $\mathcal{M}$ does), whereas s_0 does not. $\square$

[44] Or already by (6.3) and (6.4), with easier arguments (see the paragraph after (6.4)).

[45] Alternatively, we could also start with $s_0^* \subseteq \mathcal{SM} = \mathcal{N}^*$ and argue in the dual way.

[46] Indeed, $\mathcal{M}^{**} = \mathcal{M}$ (see (6.8)), but we do not need that.

Chapter 7

$\mathbb{P}$ dense ideals for tree forcing notions

In this final chapter, we briefly discuss whether Sacks forcing $\mathbb{S}$ can be replaced by other tree forcing notions (such as Silver forcing $\mathbb{V}$, Laver forcing $\mathbb{L}$, etc.) in the concepts of Chapter 6.

p_0 — the $\mathbb{P}$-null sets

Recall that Definition 6.1 of the notion of Marczewski null sets is based on perfect sets; in other words, it is connected to the *Sacks forcing notion* $\mathbb{S}$ (which is the family of perfect trees of $2^{<\omega}$ ordered by inclusion).

Sacks forcing $\mathbb{S}$ belongs to the class of tree forcing notions (see, e.g., Giorgio Laguzzi's thesis [Lag12, Definition 14]):

Definition 7.1. A forcing $(\mathbb{P}, \leq)$ is a *tree forcing* (or *arboreal forcing*) *on* 2^ω (*on* ω^ω, resp.) if every element $p \in \mathbb{P}$ is a perfect tree of $2^{<\omega}$ (or $\omega^{<\omega}$, resp.), for every node $s \in p$, also[1] $p^{[s]} \in \mathbb{P}$, and $\mathbb{P}$ is ordered by inclusion.

Note that $q \leq p$ if and only if $[q] \subseteq [p]$, for any two conditions $q, p \in \mathbb{P}$ (where $[p] = \{x \in \omega^\omega : \forall n < \omega\ (x \restriction n \in p)\}$ denotes the body of p).

Clearly, Sacks forcing $\mathbb{S}$ is a tree forcing on 2^ω, whereas *Laver forcing* $\mathbb{L}$ and *Miller forcing* $\mathbb{M}$ are tree forcings on ω^ω.

Moreover, Cohen forcing $\mathbb{C}$, random forcing $\mathbb{B}$, and *Silver forcing* $\mathbb{V}$ can be represented as tree forcing notions as well.

Let us briefly explain the case of Silver forcing $\mathbb{V}$ (for more, we refer to [Lag12], in particular to the discussion after Definition 14).

[1] where $p^{[s]} = \{t \in p : t \leq s \vee s \leq t\}$

Usually, Silver forcing is defined as the set of partial functions from ω to 2 with co-infinite domain (ordered as Cohen forcing). For each such function g, we can also consider the corresponding *Silver tree* p_g, i.e., the tree with body

$$[p_g] = \{x \in 2^\omega : \ \forall n \in \mathrm{dom}(g) \ \ (x(n) = g(n))\};$$

in other words, p_g is a uniform subtree of $2^{<\omega}$ with the property that whenever $n \notin \mathrm{dom}(g)$, every node of length n is a splitting node, otherwise the tree "uniformly copies" g. In this way, Silver forcing is a tree forcing on 2^ω.

Let us restate the definition of Marczewski null:

> A set $Z \subseteq 2^\omega$ is *Marczewski null* ($Z \in s_0$) if for each $p \in \mathbb{S}$ there is a stronger condition $q \leq p$ with $[q] \cap Z = \emptyset$.

In the above definition, one can replace $\mathbb{S}$ by other tree forcings $\mathbb{P}$ (see, e.g., [Lag12, Definition 15], or [Kho12, Definition 2.1.9] for a definition presented in the context of Zapletal's "idealized forcing" framework [Zap08]):

Definition 7.2. A set $Z \subseteq 2^\omega$ (or $Z \subseteq \omega^\omega$, resp.) is $\mathbb{P}$-*null* ($Z \in p_0$) if for each $p \in \mathbb{P}$ there is a stronger condition $q \leq p$ with $[q] \cap Z = \emptyset$.

It is easy to see that the $\mathbb{C}$-null sets are exactly the nowhere dense sets, whereas the $\mathbb{B}$-null sets are the Lebesgue measure zero sets.

For Sacks forcing $\mathbb{S}$, we obtain the Marczewski null sets s_0 considered in Chapter 6; Silver forcing $\mathbb{V}$ yields the collection $v_0 \subseteq \mathcal{P}(2^\omega)$, whereas Laver forcing $\mathbb{L}$ and Miller forcing $\mathbb{M}$ yield the collections $l_0 \subseteq \mathcal{P}(\omega^\omega)$ and $m_0 \subseteq \mathcal{P}(\omega^\omega)$.

These collections have been extensively studied; e.g., in Brendle's paper [Bre95] "Strolling through paradise".

Note that the $\mathbb{P}$-null sets do not necessarily form a σ-ideal (e.g., in case of $\mathbb{P} = \mathbb{C}$). However, tree forcings with certain "fusion properties" (such as[2] Sacks forcing $\mathbb{S}$, Silver forcing $\mathbb{V}$, Laver forcing $\mathbb{L}$, Miller forcing $\mathbb{M}$, etc.) always yield σ-ideals.

Assumption 7.3. Here, we are only interested in tree forcings $\mathbb{P}$ with the property that the respective collection p_0 of $\mathbb{P}$-null sets forms a σ-ideal.

[2]This also applies to *Mathias forcing*, i.e., the "Mathias null sets" (better known under the name *Ramsey null sets*) form a σ-ideal, but it is not clear to me how to represent Mathias forcing in order to make it "translation-invariant" in some sense (see Assumption 7.4), so I believe that it does not fit into the framework of Chapter 6.

p_0^* — the p_0-shiftable sets

Several tree forcings $\mathbb{P}$ (such as Sacks forcing $\mathbb{S}$ and Silver forcing $\mathbb{V}$) are "translation-invariant" families of trees; in other words, a tree p is in $\mathbb{P}$ if and only if all of its translates "$p+t$" (for $t \in 2^\omega$) are in $\mathbb{P}$.

Since the Baire space ω^ω is not equipped with a group operation, there is no natural notion of translation available for tree forcings on ω^ω.

However, we can identify ω^ω with the Baer-Specker group $\mathbb{Z}^\omega$ (see also Section 5.3) by just using (componentwise) any bijection between ω and the group $(\mathbb{Z}, +)$ of integers.

In this way, we can view tree forcings on ω^ω such as Laver forcing $\mathbb{L}$ and Miller forcing $\mathbb{M}$ as tree forcings on $\mathbb{Z}^\omega$. Note that both $\mathbb{L}$ and $\mathbb{M}$ are translation-invariant (and "inverse-invariant") in this sense.

Assumption 7.4. From now on, we only consider *translation-invariant* (and inverse-invariant) tree forcing notions $\mathbb{P}$ (on 2^ω, or on $\mathbb{Z}^\omega$).

It easily follows that the corresponding notion of $\mathbb{P}$-null (i.e., the collection p_0) is translation-invariant as well (for instance, the notion of Marczewski null we dealt with in Chapter 6).

Remark 7.5. Recall that the Cantor space 2^ω and the Baire space ω^ω are "almost homeomorphic" (i.e., with only countably many "exceptional points"). So one can transfer subsets of 2^ω to ω^ω (and vice versa). This is done in Brendle's [Bre95] in order to be able to compare the collections s_0, v_0, l_0, and m_0 (among others) within one single space.

However, the translation-invariance of a tree forcing notion $\mathbb{P}$ as well as the translation-invariance of the respective collection p_0 is lost when it is transferred to the "wrong" space: for instance, one can view $\mathbb{L}$ as a family of trees of $2^{<\omega}$ (and l_0 as a collection of subsets of 2^ω), but then $\mathbb{L}$ is not translation-invariant any more.

Therefore I believe that it is sensible to stick to $\mathbb{Z}^\omega$ in case of Laver and Miller forcing.

For a tree forcing notion $\mathbb{P}$ satisfying the above assumptions, it makes perfect sense to consider the collection of p_0-shiftable sets:

Definition 7.6. A set $Y \subseteq 2^\omega$ ($Y \subseteq \mathbb{Z}^\omega$, resp.) is *$p_0$-shiftable* ($Y \in p_0^*$) if for each set $Z \in p_0$ we have $Y + Z \neq 2^\omega$ ($Y + Z \neq \mathbb{Z}^\omega$, resp.).

Special instances are, e.g., the collection $s_0^* \subseteq \mathcal{P}(2^\omega)$ considered in Chapter 6, as well as the collections $v_0^* \subseteq \mathcal{P}(2^\omega)$, $l_0^* \subseteq \mathcal{P}(\mathbb{Z}^\omega)$, and $m_0^* \subseteq \mathcal{P}(\mathbb{Z}^\omega)$.

Note that the two assumptions above together particularly imply that all countable subsets of 2^ω (of $\mathbb{Z}^\omega$, resp.) are p_0-shiftable.

So it is natural to consider the respective "Borel Conjecture":

Definition 7.7. The $\mathbb{P}$-BC is the statement that there are no uncountable p_0-shiftable sets, i.e., $p_0^* = [2^\omega]^{\leq\aleph_0}$ (or $p_0^* = [\mathbb{Z}^\omega]^{\leq\aleph_0}$, resp.).

Note that the $\mathbb{C}$-BC is actually[3] nothing else than the "usual" Borel Conjecture BC (by Galvin-Mycielski-Solovay), whereas the $\mathbb{B}$-BC (for random forcing $\mathbb{B}$) is the dBC (by definition).

Furthermore, the $\mathbb{S}$-BC (for Sacks forcing $\mathbb{S}$) is the Marczewski Borel Conjecture (MBC) discussed in Chapter 6.

Assuming certain properties for the tree forcing $\mathbb{P}$ (in particular, Assumption 7.9 below), one can show that $\mathrm{cof}(p_0) > 2^{\aleph_0}$ holds (at least under CH); see also Remark 6.7. So, again, Lemma 1.17 *cannot* be applied to show the failure of $\mathbb{P}$-BC under CH (in particular, this applies to $\mathbb{S}$, $\mathbb{V}$, $\mathbb{L}$, and $\mathbb{M}$).

$\mathbb{P}$ dense ideals

We now generalize the notion of "Sacks dense ideal" (see Definition 6.9); again, we either talk about collections of subsets of 2^ω or collections of subsets of $\mathbb{Z}^\omega$:

Definition 7.8. A collection $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ (or $\mathcal{J} \subseteq \mathcal{P}(\mathbb{Z}^\omega)$, respectively) is a $\mathbb{P}$ *dense ideal* if

1. $\mathcal{J}$ is a σ-ideal,
2. $\mathcal{J}$ is *translation-invariant*, i.e.,

$$\forall Y \in \mathcal{J} \;\; \forall t \in 2^\omega \;\; (Y \in \mathcal{J} \iff Y + t \in \mathcal{J}),$$

2'. $\mathcal{J}$ is *inverse-invariant*, i.e.,

$$\forall Y \in \mathcal{J} \;\; (Y \in \mathcal{J} \iff -Y \in \mathcal{J}),$$

3. $\mathcal{J}$ is "dense in $\mathbb{P}$", i.e.,

$$\forall p \in \mathbb{P} \;\; \exists q \leq p \;\; [q] \in \mathcal{J}.$$

Note that $-x = x$ for all $x \in 2^\omega$, so inverse-invariance is a void requirement for tree forcings on 2^ω (in particular, the notion of $\mathbb{S}$ dense ideal coincides with the notion of Sacks dense ideal as defined in Chapter 6).

To make the arguments of Chapter 6 (for connecting p_0^* and $\mathbb{P}$ dense ideals) work, we have to exclude ccc forcing notions (such as Cohen forcing or random forcing):

[3] Cohen forcing $\mathbb{C}$ does not satisfy Assumption 7.3, nevertheless we could adopt Definition 7.7 also for Cohen forcing.

Assumption 7.9. From now on, we assume that the tree forcing $\mathbb{P}$ has the property that there is an antichain of size continuum below every condition in $\mathbb{P}$.

For tree forcings $\mathbb{P}$ satisfying all the above assumptions, the parallel of Lemma 6.10 holds true:

Lemma 7.10. *Assume CH. Let $\mathcal{J}$ be any $\mathbb{P}$ dense ideal. Then p_0^* is a subset of $\mathcal{J}$.*

As in Chapter 6, let $\mathfrak{R}(\mathbb{P})$ denote the intersection of all $\mathbb{P}$ dense ideals (the elements of $\mathfrak{R}(\mathbb{P})$ are called *completely* $\mathbb{P}$ *dense* sets of reals):

Definition 7.11. $\mathfrak{R}(\mathbb{P}) := \bigcap\{\mathcal{J} : \mathcal{J}$ is a $\mathbb{P}$ dense ideal$\}$.

The collection $\mathfrak{R}(\mathbb{P})$ is a translation-invariant (as well as inverse-invariant) σ-ideal, and $p_0^* \subseteq \mathfrak{R}(\mathbb{P})$ under CH (by Lemma 7.10). In particular, $\mathfrak{R}(\mathbb{S})$ is equal to $\mathfrak{R}$ (of Definition 6.12).

Note that not only Sacks forcing, but also Silver forcing, Laver forcing, and Miller forcing satisfy all our above assumptions, so

$$CH \quad \longrightarrow \quad v_0^* \subseteq \mathfrak{R}(\mathbb{V}) \tag{7.1}$$

as well as $l_0^* \subseteq \mathfrak{R}(\mathbb{L})$, $m_0^* \subseteq \mathfrak{R}(\mathbb{M})$ and (as already known) $s_0^* \subseteq \mathfrak{R}$.

Again, we do not know whether $\mathfrak{R}(\mathbb{V}) = [2^\omega]^{\leq\aleph_0}$ is consistent with CH (cf. Question 6.46); so also Con($\mathbb{V}$-BC) remains unsettled.

However, it is easy to see that $\mathcal{J}_f$ (see Definition 6.16) is a *Silver dense ideal*[4] for each $f \in \omega^\omega$, so (see Theorem 6.28, etc.)

$$\mathfrak{R}(\mathbb{V}) \subseteq \bigcap_{f\in\omega^\omega} \mathcal{J}_f \subseteq \mathcal{N}\text{-additive} \subseteq \mathcal{SN} \cap \mathcal{SM};$$

in particular, CH implies (see (7.1)) that v_0^* only contains "very small" sets of reals.

Aronszajn tree constructions

We can also adopt the arguments of Chapter 6 for constructing Aronszajn trees of (perfect) sets to prove analogues of Theorem 6.41 and Theorem 6.48 (and its corollaries), provided the tree forcing $\mathbb{P}$ allows for fusion relations $\leq_n$ with certain nice properties. In particular, this is the case for Silver forcing $\mathbb{V}$ (as well as for $\mathbb{L}$ and $\mathbb{M}$).

[4] As opposed to this, $\mathcal{E}_0$ (see Definition 6.31) *does not* belong to the class of Silver dense ideals.

For instance, the intersection of any $\aleph_1$ many Silver dense ideals[5] always contains an uncountable set.

Moreover, CH implies that the intersection of *all* Silver dense ideals is strictly smaller than the intersection of *any* $\aleph_1$ *many* Silver dense ideals (cf. Corollary 6.49); in other words,

$$\bigcap_{\alpha<\omega_1} \mathcal{J}_\alpha \supsetneqq \mathfrak{R}(\mathbb{V}) \tag{7.2}$$

for any family $\{\mathcal{J}_\alpha \colon \alpha < \omega_1\}$ of Silver dense ideals.

In particular, there are (at least) $\aleph_2$ many Silver dense ideals under CH.

$2^{\mathfrak{c}}$ many Silver dense ideals

I conclude the thesis with the following theorem, the parallel of which for Sacks[6] dense ideals I was not able to prove.

Theorem 7.12. *There are $2^{\mathfrak{c}}$ many Silver dense ideals.*

Proof (Sketch). Fix a mad family $(A_i)_{i<\mathfrak{c}}$ of size continuum (of infinite subsets of ω). For each $i < \mathfrak{c}$, partition

$$A_i =: B_i^{00} \,\dot\cup\, B_i^{01} \,\dot\cup\, B_i^{10} \,\dot\cup\, B_i^{11}$$

into 4 infinite sets.

For each function $F\colon \mathfrak{c} \to 2$, we define $\mathcal{J}_F \subseteq \mathcal{P}(2^\omega)$ – which is going to be a Silver dense ideal – as follows. For each $i < \mathfrak{c}$, partition A_i into two sets C_i^0 and C_i^1, depending on $F(i)$: if $F(i) = 0$, let

$$C_i^0 := B_i^{00} \cup B_i^{01} \text{ and } C_i^1 := B_i^{10} \cup B_i^{11};$$

if $F(i) = 1$, let

$$C_i^0 := B_i^{00} \cup B_i^{10} \text{ and } C_i^1 := B_i^{01} \cup B_i^{11}.$$

Let $\mathcal{J}_F$ be the σ-ideal generated by the bodies $[p]$ of those Silver trees $p \in \mathbb{V}$ that satisfy

$$\exists i < \mathfrak{c}\ \exists j \in 2\ \operatorname{splitlev}(p) \subseteq C_i^j$$

[5] As in Theorem 6.41, one actually doesn't need translation-invariance here.

[6] Note that Theorem 7.12 holds in ZFC, whereas Theorem 6.48 (and hence Corollary 6.50, which says that there are $\aleph_2$ many Sacks dense ideals) only works under CH; so, even under CH, the existence of only $\mathfrak{c}^+$ many Sacks dense ideals is shown. On the other hand, it is not clear whether all of the $2^{\mathfrak{c}}$ many Silver dense ideals given by Theorem 7.12 really "contribute" (cf. Remark 6.51) to the intersection $\mathfrak{R}(\mathbb{V})$ of all Silver dense ideals; so under CH, the instance of (the parallel of) Corollary 6.49 for Silver forcing $\mathbb{V}$ (see (7.2)) may give information that is not given by Theorem 7.12.

(where $\operatorname{splitlev}(p)$ is the set of all n such that some (each) node of length n is a splitting node).

It can be shown that $\mathcal{J}_{F_1} \neq \mathcal{J}_{F_2}$ for any two functions $F_1 \neq F_2$ (hence there are $2^{\mathfrak{c}}$ many).

Finally, for each $F : \mathfrak{c} \to 2$, the collection $\mathcal{J}_F$ is a Silver dense ideal: clearly, $\mathcal{J}_F$ is a translation-invariant σ-ideal, and the maximality of $(A_i)_{i<\mathfrak{c}}$ is responsible for $\mathcal{J}_F$ being dense in Silver forcing. $\square$

Bibliography

[Bar03] Tomek Bartoszynski. Remarks on small sets of reals. *Proc. Amer. Math. Soc.*, 131(2):625–630 (electronic), 2003.

[BJ95] Tomek Bartoszyński and Haim Judah. *Set theory*. A K Peters Ltd., Wellesley, MA, 1995. On the structure of the real line.

[Bla88] Andreas Blass. Selective ultrafilters and homogeneity. *Ann. Pure Appl. Logic*, 38(3):215–255, 1988.

[Bla11] Andreas Blass (mathoverflow.net/users/6794). Finite support iterations of σ-centered forcing notions. MathOverflow, 2011. `http://mathoverflow.net/questions/84129` (version: 2011-12-23).

[Bor19] E. Borel. Sur la classification des ensembles de mesure nulle. *Bull. Soc. Math. France*, 47:97–125, 1919.

[BR96] Tomek Bartoszyński and Ireneusz Recław. Not every γ-set is strongly meager. In *Set theory (Boise, ID, 1992–1994)*, volume 192 of *Contemp. Math.*, pages 25–29. Amer. Math. Soc., Providence, RI, 1996.

[Bre95] Jörg Brendle. Strolling through paradise. *Fund. Math.*, 148(1):1–25, 1995.

[Bre96] Jörg Brendle. Generic constructions of small sets of reals. *Topology Appl.*, 71(2):125–147, 1996.

[BS01] Tomek Bartoszynski and Saharon Shelah. Strongly meager sets do not form an ideal. *Journal of Mathematical Logic*, 1:1–34, 2001. math.LO/9805148.

[BS03] Tomek Bartoszynski and Saharon Shelah. Strongly meager sets of size continuum. *Arch. Math. Logic*, 42(8):769–779, 2003.

[BS10] Tomek Bartoszynski and Saharon Shelah. Dual Borel conjecture and Cohen reals. *J. Symbolic Logic*, 75(4):1293–1310, 2010.

[Car93] Timothy J. Carlson. Strong measure zero and strongly meager sets. *Proc. Amer. Math. Soc.*, 118(2):577–586, 1993.

[FF10] Vera Fischer and Sy David Friedman. Cardinal characteristics and projective wellorders. *Ann. Pure Appl. Logic*, 161(7):916–922, 2010.

[FFZ11] Vera Fischer, Sy David Friedman, and Lyubomyr Zdomskyy. Projective wellorders and mad families with large continuum. *Ann. Pure Appl. Logic*, 162(11):853–862, 2011.

[FK] Sy D. Friedman and Vadim Kulikov. Failures of the Silver dichotomy in the generalised Baire space. *Preprint.*

[FM88] David H. Fremlin and Arnold W. Miller. On some properties of Hurewicz, Menger, and Rothberger. *Fund. Math.*, 129(1):17–33, 1988.

[Fri00] Sy D. Friedman. *Fine structure and class forcing*, volume 3 of *de Gruyter Series in Logic and its Applications.* Walter de Gruyter & Co., Berlin, 2000.

[GK06] Martin Goldstern and Jakob Kellner. New reals: can live with them, can live without them. *MLQ Math. Log. Q.*, 52(2):115–124, 2006.

[GKSW] Martin Goldstern, Jakob Kellner, Saharon Shelah, and Wolfgang Wohofsky. Borel Conjecture and dual Borel Conjecture. *To appear in: Trans. Amer. Math. Soc.* `http://arxiv.org/abs/1105.0823`.

[GM84] Fred Galvin and Arnold W. Miller. γ-sets and other singular sets of real numbers. *Topology Appl.*, 17(2):145–155, 1984.

[GMS73] Fred Galvin, Jan Mycielski, and Robert M. Solovay. Strong measure zero sets. *Notices of the AMS*, pages A–280, 1973.

[GN82] J. Gerlits and Zs. Nagy. Some properties of $C(X)$. I. *Topology Appl.*, 14(2):151–161, 1982.

[Gol93] Martin Goldstern. Tools for Your Forcing Construction. In Haim Judah, editor, *Set Theory of The Reals*, volume 6 of *Israel Mathematical Conference Proceedings*, pages 305–360. American Mathematical Society, 1993.

[Gol98] Martin Goldstern. A taste of proper forcing. In Carlos Augusto Di Prisco, editor, *Set theory: techniques and applications.*, pages 71–82, Dordrecht, 1998. Kluwer Academic Publishers. Proceedings of the conferences, Curacao, Netherlands Antilles, June 26–30, 1995 and Barcelona, Spain, June 10–14, 1996.

[GS13] Fred Galvin and Marion Scheepers. Borel's conjecture in topological groups. *J. Symbolic Logic*, 78(1):168–184, 2013.

[Hal96] Aapo Halko. Negligible subsets of the generalized Baire space $\omega_1^{\omega_1}$. *PhD thesis*, 1996.

[Har77] Leo Harrington. Long projective wellorderings. *Ann. Math. Logic*, 12(1):1–24, 1977.

[HS01] Aapo Halko and Saharon Shelah. On strong measure zero subsets of ${}^\kappa 2$. *Fundamenta Mathematicae*, 170:219–229, 2001. math.LO/9710218.

[Jec03] Thomas Jech. *Set theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[JMS92] Haim Judah, Arnold W. Miller, and Saharon Shelah. Sacks forcing, Laver forcing, and Martin's axiom. *Arch. Math. Logic*, 31(3):145–161, 1992.

[JS90] Haim Judah and Saharon Shelah. The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing). *J. Symbolic Logic*, 55(3):909–927, 1990.

[JSW90] Haim Judah, Saharon Shelah, and W. H. Woodin. The Borel conjecture. *Ann. Pure Appl. Logic*, 50(3):255–269, 1990.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kel12] Jakob Kellner. Non-elementary proper forcing. *To appear in: Rend. Semin. Mat. Univ. Politec. Torino*, 2012. `http://arxiv.org/abs/0910.2132`.

[Kho12] Yurii Khomskii. Regularity properties and definability in the real number continuum: Idealized forcing, polarized partitions, Hausdorff gaps and mad families in the projective hierarchy. *PhD thesis. University of Amsterdam, ILLC Dissertation Series DS-2012-04*, 2012.

[KS05] Jakob Kellner and Saharon Shelah. Preserving preservation. *J. Symbolic Logic*, 70(3):914–945, 2005.

[Kun80] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1980. An introduction to independence proofs.

[KW04] Marcin Kysiak and Tomasz Weiss. Small subsets of the reals and tree forcing notions. *Proc. Amer. Math. Soc.*, 132(1):251–259 (electronic), 2004.

[Kys00] Marcin Kysiak. On Erdős-Sierpiński duality between Lebesgue measure and Baire category (in Polish). *Master's thesis*, 2000.

[Lag12] Giorgio Laguzzi. Arboreal forcing notions and regularity properties of the real line. *PhD thesis. Universität Wien*, 2012.

[Lav76] Richard Laver. On the consistency of Borel's conjecture. *Acta Math.*, 137(3-4):151–169, 1976.

[Mil84] Arnold W. Miller. Special subsets of the real line. In *Handbook of set-theoretic topology*, pages 201–233. North-Holland, Amsterdam, 1984.

[Paw96a] Janusz Pawlikowski. A characterization of strong measure zero sets. *Israel J. Math.*, 93:171–183, 1996.

[Paw96b] Janusz Pawlikowski. Every Sierpiński set is strongly meager. *Arch. Math. Logic*, 35(5-6):281–285, 1996.

[Paw96c] Janusz Pawlikowski. Laver's forcing and outer measure. In *Set theory (Boise, ID, 1992–1994)*, volume 192 of *Contemp. Math.*, pages 71–76. Amer. Math. Soc., Providence, RI, 1996.

[PR95] Janusz Pawlikowski and Ireneusz Recław. Parametrized Cichoń's diagram and small sets. *Fund. Math.*, 147(2):135–155, 1995.

[PS08] Janusz Pawlikowski and Marcin Sabok. Two stars. *Arch. Math. Logic*, 47(7-8):673–676, 2008.

[She95] Saharon Shelah. Every null additive set of reals is meager additive. *Israel Journal of Mathematics*, 89:357–376, 1995. math.LO/9406228.

[She98] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[She04] S. Shelah. Properness without elementaricity. *J. Appl. Anal.*, 10(2):169–289, 2004.

[She06] S. Shelah. Non-Cohen oracle C.C.C. *J. Appl. Anal.*, 12(1):1–17, 2006.

[She10] Saharon Shelah. Large continuum, oracles. *Cent. Eur. J. Math.*, 8(2):213–234, 2010.

[Sie28] W. Sierpiński. Sur un ensemble non dénombrable, dont toute image continue est de mesure nulle. *Fund. Math.*, 11:302–304, 1928.

[Sol03] Sławomir Solecki. Translation invariant ideals. *Israel J. Math.*, 135:93–110, 2003.

[Tal94] Franklin D. Tall. σ-centred forcing and reflection of (sub)metrizability. *Proc. Amer. Math. Soc.*, 121(1):299–306, 1994.

[Zap08] Jindřich Zapletal. *Forcing idealized*, volume 174 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2008.

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