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Algebraic Structures on Translation Invariant Valuations

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Preface

Let V be an n-dimensional vector space and $\mathcal{K}(V)$ the set of compact convex sets (convex bodies) in V. A valuation on V is a map $\phi : \mathcal{K}(V) \to \mathbb{R}$ that satisfies

$$\phi(A \cup B) + \phi(A \cap B) = \phi(A) + \phi(B),$$

whenever $A, B, A \cup B \in \mathcal{K}(V)$. The space of translation-invariant continuous valuations on V is denoted by Val. A valuation $\phi \in Val$ is called even if $\phi(-K) = \phi(K)$ and odd if $\phi(-K) = -\phi(K)$. We say ϕ is k-homogeneous if $\phi(\lambda K) = \lambda^k \phi(K), \ \lambda \geq 0$. In this thesis we study several algebraic operations defined on a certain dense subspace of Val - the subspace of smooth valuations Val^∞. Furthermore, an exposition of the relation between kinematic formulas and these algebraic operations is given. In particular we will prove the Fundamental Theorem of Algebraic Integral Geometry. We mainly follow the papers [4, 16] where these operations were introduced for the first time.

In Chapter 1 we collect tools and basic notions from different areas of mathematics. More precisely, we start with a very quick review of fundamental results from the theory of convex bodies. A good reference for this subject is [28]. In the short subsequent section we introduce the notion of Grassmannians and angle functions on them. We then continue by gathering some important tools from infinite-dimensional representation theory in the next section, in particular, we define the notion of smooth vectors. We have to assume some knowledge from finite-dimensional representation theory. Also the reader might want to confer a more thorough introduction to infinite-dimensional representation theory as in [22, 32, 34]. The last section of this background chapter deals with differential forms on the sphere bundle $V \times S(V)$, where V is an n-dimensional vector space. The purpose of this section is twofold. On the one hand, we use it to fix notation and recall the definition of contact manifolds and the Hodge $*$-operator. On the other hand, we introduce the spaces Ω_k^V of smooth, translation-invariant $(n-1)$-forms of bi-degree $(k, n-k-1)$. These spaces are closely related to smooth valuations as we will discuss in the next chapter.

Chapter 2 deals with the central notion of this thesis - translation-invariant valuations. In the first section we define this notion and consider several important examples. Moreover, we present the classical theorems of Hadwiger and McMullen. A very important consequence of the latter is that the space of translation-invariant valuations Val decomposes by degree of homogeneity. The next section starts with a discussion of Alesker's Irreducibility Theorem which states that the decomposition of Val by parity and degree of homogeneity is a decomposition into irreducible subspaces with respect to the natural $GL(V)$-action. This deep theorem is extremely powerful. For example it implies that functionals of the form $K \mapsto vol_n(K + A), \ \ K, A \in \mathcal{K}(V)$, denoted by $\mu_A(K)$, are dense in the space of translation-invariant valuations. After that we introduce smooth valuations, which are defined as the smooth vectors of Val. In the following section, we review the connection between smooth valuations and the differential forms from the spaces Ω_k^V defined in the previous chapter. It turns out that every smooth valuation can be represented by integrating some form over the so called normal cycle of the evaluated body. Lastly, we turn to even smooth valuations and ways to represent them via Klain functions and Crofton measures. This then leads to the definition of the Alesker-Fourier transform $\mathbb{F}$ for even valuations in the last section.

In Chapter 3 we define the multiplication on Val^∞. We follow [4], where it was considered for the first time. Initially this product is defined on valuations of the form $K \mapsto vol_n(K + A),\ \ K, A \in \mathcal{K}(V)$ by

$$vol_n(K + A) \cdot vol_n(K + B) = vol_{2n}(\Delta(K) + A \times B), \tag{1}$$

where $\Delta(K)$ is the diagonal embedding of K in $V \times V$. At this point we already know that these valuations form a dense subspace. However, the multiplication does not admit a continuous extension to all of Val. In Section 3.2 we prove, using tools from representation theory, that the product defined by (1) is continuous with respect to the topology on Val^∞ and thus extends to all of Val^∞. We then move towards establishing the Poincaré duality. Given two valuations $\phi, \psi \in Val^\infty$, we consider the n-homogeneous component of their product. This determines a pairing

$$\langle \cdot, \cdot \rangle : Val^\infty \times Val^\infty \to Val_n \cong \mathbb{R}.$$

The main theorem of Section 3.4 now states that this pairing is non-degenerate. At the end of this chapter we consider the product of even smooth valuations and its description in terms of Crofton measures.

In Chapter 4 we discuss another algebraic operation - the convolution. It was originally defined as the operation $\mathbb{F}^{-1}(\mathbb{F}\phi \cdot \mathbb{F}\psi)$ on even smooth valuations. We follow [16], where this operation was described in geometric terms and extended to Val^∞. In the first section we prove that

$$\mathbb{F}^{-1}(\mathbb{F}\mu_A \cdot \mathbb{F}\mu_B) = \mu_{A+B},$$

whenever the μ_A, μ_B, μ_{A+B} are even smooth valuations. In Section 4.2 we extend, using the representation of a smooth valuation via a differential form, the convolution on even smooth valuations to a continuous operation $* : Val^\infty \times Val^\infty \to Val^\infty$ satisfying

$$\mu_A * \mu_B = \mu_{A+B}, \quad A, B \in \mathcal{K}(V).$$

In Section 4.3 we briefly mention that the Alesker-Fourier transform has an extension to Val^∞ relating product and convolution.

In Chapter 5 we study kinematic formulas and their relation to the previously defined operations. The foundation for these considerations is the fact that any space of valuations Val^G invariant under a closed subgroup G of $SO(V)$ acting transitively on the sphere, is finite dimensional. We prove this in Section 1. An easy argument then shows that given a basis $\phi_1, \dots \phi_m$ of Val^G, we obtain formulas of the form

$$\int_V \int_G \phi_j(K \cap (x + gL))dgdx = \sum_{k,l=1}^{m} c^i_{k,l}\phi_k(K)\phi_l(L).$$

These are called kinematic formulas. They determine the kinematic operator $k_G : Val^\infty \to Val^\infty \otimes Val^\infty$. The content of the fundamental theorem that we will prove in Section 5.3 is that this kinematic operator is in fact dual to the product on Val^G.

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Contents

1 General background

In this first chapter, we recall basic facts from different areas of mathematics. By V we denote a real vector space of dimension n with a fixed scalar product $\langle \cdot, \cdot \rangle$. We write $B_r(x)$ for the Euclidean ball with radius r and center x in V.

1.1 Convex bodies

In this section, we review basic definitions, concepts and results regarding convex bodies. For more details cf. [28].

Definition 1. *A non-empty set $K \subseteq V$ is called a convex body whenever it is convex and compact. The set of all convex bodies is denoted by $\mathcal{K}(V)$. The subset of strictly convex bodies with smooth boundary is denoted by $\mathcal{K}^\infty(V)$.*

The Hausdorff metric on $\mathcal{K}(V)$ is defined by

$$d(A, B) = \inf\{\epsilon > 0 : A \subseteq B + B_\epsilon(0), B \subseteq A + B_\epsilon(0)\}. \tag{2}$$

Endowed with this metric, $\mathcal{K}(V)$ is a complete locally compact metric space.

A well known and important property of convex sets is that there exists a unique nearest point projection.

Lemma 2. *Let $K \in \mathcal{K}(V)$ and $x \in V$. There is a unique point $y \in K$ satisfying $\|y - x\| = d(K, x)$. Consequently, there exists a unique map $p_K : V \to K$ called the nearest point projection determined by*

$$\|p_K(x) - x\| = d(K, x).$$

Moreover, p_K is a contraction, that is,

$$\|p_K(x_1) - p_K(x_2)\| \leq \|x_1 - x_2\|, \quad x_1, x_2 \in V.$$

The existence and uniqueness of the nearest point projection is closely related to the notion of the support function of a convex set.

Definition 3. *Let $K \in \mathcal{K}(V)$. For $y \in V$, the support function of K is defined by*

$$h_K(y) = \max_{x \in K} \langle x, y \rangle.$$

Remark 4. The map $K \mapsto h_K$ is injective. Thus, every convex body is uniquely determined by its support function.

It turns out, that the map $K \mapsto h_K$ is also Minkowski-linear.

Lemma 5. *Let $K_1, K_2 \in \mathcal{K}(V)$ and $\lambda \geq 0$. Then $h_{K_1} + h_{K_2} = h_{K_1+K_2}$ and $\lambda h_{K_1} = h_{\lambda K_1}$.*

A function f on the space of continuous real valued functions on V, that is $f \in C(V)$, is called sublinear, if

$$f(x + y) \leq f(x) + f(y), \quad x, y \in V.$$

Support functions are precisely the sublinear and one-homogeneous functions on V.

Lemma 6. *A function $f \in C(V)$ is a support function of a convex body if and only if it is sublinear and one-homogeneous.*

Consider a strictly convex body with smooth boundary $K \in \mathcal{K}^\infty(V)$. Then for every direction $u \in S(V)$, there exists a unique boundary point of K with outer unit normal equal to u. It is easy to see, that this point is

$$\arg\max_{x \in K} \langle x, u \rangle.$$

The next lemma expresses this fact in terms of the gradient of the support function.

Lemma 7. *Let $K \in \mathcal{K}^\infty(V)$. Then $h_K(y) = \langle \nabla h_K(y), y \rangle$.*

Since every support function $h : V \to \mathbb{R}$ is one-homogeneous, we may identify it with its restriction to the sphere $h : S(V) \to \mathbb{R}$. The next result states, that the identification of convex bodies with their respective support functions is homeomorphic.

Lemma 8. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}(V)$. Then $K_n \to K$ in the Hausdorff metric if and only if $h_{K_n} \to h_K$ in $C(S(V))$ endowed with the maximum norm.*

Recall that there are canonic semi-norms $\|\cdot\|_m,\ m \leq k$ on the spaces $C^k(S(V))$ of k-times differentiable real valued functions on the sphere (cf. Section 1.4).

Theorem 9. *If $f \in C^2(S(V))$, then there exists a convex body $K \in \mathcal{K}(V)$ and $s > 0$ such that,*

$$f(u) = h_K(u) - sh(B_1(0), u), \qquad u \in S(V),$$

with $\|h_K\|_2 \leq c \cdot \|f\|_2$. In particular, every $f \in C^2(S(V))$ can be written as the difference of two support functions.

Remark 10. If $f \in C^n(S(V))$, then s and K can be chosen such that

$$\|h_K\|_n \leq c\|f\|_n.$$

As a corollary we obtain:

Corollary 11. *Any Minkowski-linear map Φ, defined on the set of support functions mapping in a topological vector space E, uniquely induces a linear map $\Phi : C^2(S(V)) \to E$. Moreover, if Φ is continuous on the space of support functions, then the induced map on $C^2(SV(V))$ is also continuous.*

Proof. Let $f \in C^2(S(V))$. Theorem 9 guarantees the existence of support functions h_1, h_2 with $f = h_1 - h_2$. We define

$$\Phi(f) = \Phi(h_1) - \Phi(h_2).$$

We have to check, that this is well-defined. Let $h_1 - h_2 = h_1' - h_2'$ with support functions h_1', h_2'. Then $\Phi(h_1 + h_2') = \Phi(h_1' + h_2)$ and hence

$$\Phi(h_1) - \Phi(h_2) = \Phi(h_1') - \Phi(h_2').$$

Clearly, the so defined map is linear. The statement concerning continuity immediately follows from the possibility to choose the support functions such that their norms are bounded by a multiple of $\|f\|_2$ (cf. Theorem 9).

□

The next notion is fundamental in the theory of convex bodies.

Definition 12. *The mixed volume of* $A_1, \dots, A_n \in \mathcal{K}(V)$ *is defined by*

$$V(A_1, \dots, A_n) = \frac{1}{n!} \sum_{m=1}^{n} (-1)^{n+m} \sum_{1 \le i_1 < \dots < i_m \le n} vol_n(A_{i_1} + \dots + A_{i_m}).$$

In the following we denote by $K[m] := (K, \dots, K)$ where K is taken m times.

Lemma 13. *Let* $K, L, A_1, \cdots, A_n \in \mathcal{K}(V)$. *Then*

1. $V : \mathcal{K}(V)^n \to \mathbb{R}$ *is continuous and positive.*
2. *For all* $\alpha, \beta \ge 0$,
$$V(\alpha K + \beta L, A_2, \cdots, A_n) = \alpha V(K, A_2, \dots, A_n) + \beta V(L, A_2, \dots, A_n).$$
3. $V(K[n]) = vol_n(K)$.
4. *Let* W *be another vector space with a fixed scalar product and* $f : V \to W$ *an isomorphism. Then*
$$V(f(A_1), \dots, f(A_n)) = |\det f| \cdot V(A_1, \dots, A_n).$$

The next theorem explains the name "mixed volume".

Theorem 14. *The mixed volumes are symmetric in their arguments and invariant under independend translations of their arguments. If* $A_1, \dots, A_k \in \mathcal{K}(V)$ *and* $\lambda_1, \dots, \lambda_k \ge 0$, *then*

$$vol_n(\lambda_1 A_1 + \dots + \lambda_k A_k) = \sum_{i_1, \dots, i_n = 1}^{k} \lambda_{i_1} \cdots \lambda_{i_n} V(A_{i_1}, \dots, A_{i_n}).$$

If we apply the last theorem for $\lambda_1 = 1$, then we obtain the following useful formula:

$$vol_n \left(K + \sum_{j=0}^{k} \lambda_j A_j \right) = \sum_{m=0}^{n} \binom{n}{m} \sum_{i_1, \dots, i_m = 1}^{k} \lambda_{i_1} \cdots \lambda_{i_m} V(K[n-m], A_{i_1}, \dots, A_{i_m}). \tag{3}$$

1.2 Grassmannians and angle functions

For the definitions in this section we refer to [11, 29]. We recall the definition of the i-th Grassmannian.

Definition 15. *The* i*-th Grassmannian* Gr_i *on* V *is the set of all* i*-dimensional subspaces of* V. *As usual, the sets of all continuous and smooth functions* $f : Gr_i \to \mathbb{C}$, *will be denoted by* $C(Gr_i)$ *and* $C^\infty(Gr_i)$, *respectively.*

Since Gr_i can also be viewed as the homogeneous space $O(n)/O(i) \times O(n-i)$, there exists a unique $O(n)$-invariant measure ν_i on Gr_i such that $\nu_i(Gr_i) = 1$. This is the canonical measure on Gr_i. Therefore, we set

$$\int_{Gr_i} f(E) dE := \int_{Gr_i} f(E) d\nu_i(E).$$

It is possible, to define the sine and the cosine of two subspaces.

Definition 16. *For $k \leq l$, the cosine $|\cos|: Gr_k \times Gr_l \to [0, \infty)$ is defined by*

$$|\cos(E, F)| = vol_k(M|F),$$

where $M \in E$ such that $vol_k(M) = 1$. (The definition is independent of the choice of M.) If $k > l$ then $|\cos(E, F)| := |\cos(E^\perp, F^\perp)|$. Furthermore, we define the map $|\sin|: Gr_k \times Gr_l \to [0, \infty)$ by

$$|\sin(E, F)| := |\cos(E, F^\perp)|.$$

The next easy lemma collects some basic properties of angle functions on Grassmannians.

Lemma 17. *Let E, F, G be arbitrary subspaces of V. Then*

1. $|\cos(E, F)| = |\cos(F, E)| = |\cos(E^\perp, F^\perp)|$.
2. $|\sin(E, F)| = |\sin(F, E)| = |\sin(E^\perp, F^\perp)|$.
3. *Let $G \perp F$. Then* $|\cos(E + G, F)| = |\cos(E, F)|$.
4. *Let $dim E = k,\ dim F = l$. If $k + l \leq n$, then $vol_{k+l}(A + B) = |\sin(E, F)| vol_k(A) vol_l(B)$ for all convex bodies $A \subseteq E, B \subseteq F$.*

Definition 18. *The cosine transform $C_i : C(Gr_i) \to C(Gr_i)$ is defined by*

$$(C_i f)(F) := \int_{Gr_i} |\cos(E, F)| f(E) dE.$$

The following lemma describes the projection of a convex body K onto the Minkowski-sum of two subspaces in terms of the union of the projections on each of those spaces.

Lemma 19. *Let $E \in Gr_k,\ \ F \in Gr_l$ such that $E \cap F = \{0\}$. Then,*

$$\begin{aligned} K|(E + F) &= \{w + u : u \in E,\ \ w \in F,\ \ \exists v \in (E + F)^\perp,\ \ w + u + v \in K\} \\ &= \bigcup_{w \in F} w + [((w + (E + F)^\perp + E) \cap K)|E]. \end{aligned}$$

In particular,

$$vol_{k+l}(K|(E + F)) = \int_F vol_k(((w + (E + F)^\perp + E) \cap K)|E)\ \ dw.$$

Proof. The equality between the sets is obvious. The equation for the volumes follows from the fact, that the union

$$\bigcup_{w \in F} w + [((w + (E + F)^\perp + E) \cap K)|E]$$

is disjoint. Indeed, let

$$x \in \Big(w + [((w + (E + F)^\perp + E) \cap K)|E]\Big) \cap \Big(w' + [((w + (E + F)^\perp + E) \cap K)|E]\Big).$$

Then $w + u = x = w' + u',\ \ u, u' \in E$, hence $w - w' \in E$ and we obtain $w = w'$. $\square$

1.3 Group actions and smooth vectors

We review a few facts from representation theory. In particular, we recall the concept of smooth vectors. For more details, we refer the reader to [22, 32, 34]. Let us begin with the definition of a Fréchet space.

Definition 20. *A Fréchet space is a topological vector space F satisfying the following additional properties:*

- *F is Hausdorff.*
- *The topology is induced by a countable family of semi norms $\|\cdot\|_k$, $k \in \mathbb{N}$. This means that a neighborhood basis of $v \in F$ is given by the sets $\{w \in F : \|v - w\|_k < \epsilon,\ k \leq m\}$, $\epsilon > 0, m \in \mathbb{N}$.*
- *F is complete with respect to the family of semi norms.*

Remark 21. We can always choose a monotone family of semi norms, i.e., a family such that $\|\cdot\|_k \leq \|\cdot\|_l$ for $k \leq l$. Indeed, given an arbitrary family of semi norms $\|\cdot\|'_k$, we can consider the equivalent family $\|\cdot\|_k := \sum_{i=1}^{k} \|\cdot\|'_i$. Let E, F be Fréchet spaces with monotone families of seminorms $\|\cdot\|_k$ and $f : E \to F$. We see, that f is continuous at a point $v_0 \in E$ if and only if for every $l \in \mathbb{N}$, there exists a $k \in \mathbb{N}$, such that

$$f : (E, \|\cdot\|_k) \to (F, \|\cdot\|_l)$$

is continuous at v_0.

We will now consider several important examples of Fréchet spaces. A sequence of compact sets $\Omega_N \subseteq X$ of a topological space X is called exhausting, if for every $x \in X$, there is k such that $x \in \Omega_k$.

1. Consider the space of continuous functions $C(\mathbb{R}^n)$. This space is a Fréchet space with the family of seminorms given by $\|f\|_{\Omega_k} = \sup_{x \in \Omega_k} \|f(x)\|$, where $\Omega_k \subseteq G$ is an exhausting sequence of compact sets.
2. More generally, let M be a smooth manifold and F a Fréchet space with a topology induced by a family of seminorms $\|\cdot\|_l$. Then there exists a Fréchet topology on $C(M, F)$ given by the family of seminorms $\|f\|_{\Omega_k, l} = \sup_{x \in \Omega_k} \|f(x)\|_l$, where again $\Omega_k \subseteq M$ is an exhausting sequence of compact sets. The existence of this is guranteed, since every smooth manifold is locally compact and second countable.
3. The space $C^\infty(\mathbb{R}^n)$ is also a Fréchet space. The seminorms are given by

   $$\|f\|_{\Omega_k, l} = \sup \left\{ \left| \frac{\partial^l}{\partial x_{j_1} \cdots \partial x_{j_l}} f(x) \right| : x \in \Omega_k,\ \ 0 \leq j_i \leq n \right\}$$

   with an exhausting sequence $\Omega_k \subseteq \mathbb{R}^n$ and $l \in \mathbb{N}_0$.

Definition 22. *Let G be a Lie group and F a Fréchet space. A homomorphism $\pi : G \to Aut(F)$ of G into the automorphisms on F, is called a representation or action of G on F. If the map $(g, v) \mapsto \pi(g)v$ of $G \times F$ to F is continuous, the representation is called continuous.*

Let (π, G) be a continuous representation of G on a Fréchet space F. We consider the map $\Phi_\pi : F \to C(G, F)$ given by

$$\Phi_\pi(v)(g) := \pi(g)v.$$

Obviously, this map is injective and linear. There is a useful description of the image $\Phi_\pi(F)$:

Lemma 23. *Let (π, G) be a continuous representation of G on a Fréchet space F. Then*

$$\Phi_\pi(F) = \{f \in C(G, F) : f(gh) = \pi(g)f(h), \;\; g, h \in G\}. \tag{4}$$

Proof. Let $v \in F$. Clearly,

$$\Phi_\pi(v)(gh) = \pi(gh)v = \pi(g)\pi(h)v = \pi(g)\Phi_\pi(v)(h).$$

Suppose on the other hand that $f \in C(G, F)$ satisfies the equation $f(gh) = \pi(g)f(h)$. Then

$$f(g) = f(ge) = \pi(g)f(e) = \Phi_\pi(f(e))(g).$$

□

The next lemma shows, that the map Φ_π is indeed a homeomorphic isomorphism.

Lemma 24. *Let (π, G) be a continuous representation of G on a Fréchet space F. Then the map $\Phi_\pi : F \to \Phi_\pi(F) \subseteq C(G, F)$ is a homeomorphic isomorphism. Moreover, $\Phi_\pi(F)$ is closed in $C(G, F)$.*

Proof. Let us start by showing that $\Phi(F)$ is closed in $C(G, F)$. Indeed, this follows from the previous lemma. Since the evaluation maps $\iota_g : C(G, F) \to F$, $f \mapsto f(g)$ are continuous, the defining equation in (4) is stable under limits and thus we have $\overline{\Phi_\pi(F)} = \Phi_\pi(F)$.

It remains to show, that the map is bi-continuous. Let $\|\cdot\|_k$ be a semi norm on F and $\Omega \subseteq G$ compact. Since $\|\pi(g)\|_k$ attains a maximum M on Ω we obtain

$$\|\Phi_\pi(v)\|_{k,\Omega} = \sup_{g \in \Omega} \|\pi(g)v\|_k \leq M\|v\|_k.$$

Hence, Φ_π is continuous. On the other hand, let $\Omega \subseteq G$ be compact and contain the neutral element e. Then we have

$$\|v\|_k = \|\pi(e)v\|_k \leq \|\Phi_\pi(v)\|_{k,\Omega}$$

and we see, that Φ_π^{-1} is also continuous.

□

For maps between a manifold M and a Fréchet space F there exists a notion of differentiability.

Definition 25. *Let F be a Fréchet space. A map $f : \mathbb{R}^n \to F$ is called differentiable at $x_0 \in \mathbb{R}^n$, if there is a linear map $Df : \mathbb{R}^n \to F$ such that*

$$\lim_{x_0 \to x} \frac{f(x) - f(x_0) - (Df)(x - x_0)}{\|x - x_0\|} = 0.$$

There is a characterization of smoothness using the concept of smooth curves (cf. [25]).

Remark 26. A map $f : M \to F$ from a manifold M into a Fréchet space F is smooth, if and only if f-images of smooth curves are smooth curves. A curve $\gamma : \mathbb{R} \to F$ is smooth if and only if the maps $l \circ \gamma : \mathbb{R} \to \mathbb{R}$ are smooth for every l in the topological dual F^*.

Using this concept of differentiability, we obtain another class of Fréchet spaces. Let F be a Fréchet space with a countable family of seminorms $\|\cdot\|_l$ inducing the topology. Consider the space $C^\infty(C,F)$ where $C \subseteq \mathbb{R}^n$ is a compact subset. An obvious way to define seminorms $\|\cdot\|_{k,l}$ on $C^\infty(C,F)$ is given by

$$\|f\|_{k,l} = \sup\left\{\left\|\frac{\partial^k}{\partial x_{j_1}\cdots\partial x_{j_k}} f(x)\right\|_l : x \in C,\ \ 0 \leq j_i \leq n,\ \ l \in \mathbb{N}\right\}.$$

More generally, take the space $C^\infty(M,F)$ where M is a smooth manifold. Now, choose a countable family of compact charts (C_i, φ) which cover M. We may define a countable family of semi norms on $C^\infty(M,F)$ by

$$\|f\|_{C_i,k,l} := \|f \circ \varphi\|_{k,l}.$$

In particular, we will be interested in those cases where the manifold is a Lie group G. In this setting, there is another coordinate-free way to describe the Fréchet topology. Let $\mathfrak{g}$ be the Lie algebra of G, let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ and for every $X \in \mathfrak{g}$ let $\tilde{X}$ denote the left-invariant vector field on G associated with X. Choosing an exhausting sequence Ω_i of G, the Fréchet topology on $C^\infty(G,F)$ is induced by the family of seminorms

$$\|f\|_{\Omega_i,k,l} := \sup\left\{\left\|\tilde{X_{j_1}} \circ \cdots \circ \tilde{X_{j_k}} f(g)\right\|_l : g \in \Omega_k,\ \ 1 \leq j_i \leq l,\ \ l \in \mathbb{N}\right\}.$$

We have seen that we can identify vectors on F with continuous functions from G to F. It thus makes sense to examine those vectors $v \in E$ such that $\Phi_\pi(v)$ is smooth.

Definition 27. *Let (π, G) be a continuous representation of G on a Fréchet space F. We call a vector $v \in F$ smooth if $\Phi_\pi(v) \in C^\infty(G,F)$. The subspace of all smooth vectors in F is denoted by F^∞. A representation on a Fréchet space F is called smooth if $F = F^\infty$.*

It is easy to see that $\Phi_\pi(F^\infty)$ is a closed subspace of $C^\infty(G,F)$. Indeed, as the topology on $C^\infty(G,F)$ is finer then that on $C(G,F)$ this immediately follows from Lemma 24. Therefore, the relative topology of $\Phi_\pi(F^\infty)$ in $C^\infty(G,F)$ induces a topology on F^∞. It is called the Garding topology. Henceforth, we will consider F^∞ endowed with the Garding topology unless otherwise stated. We may describe this topology in the following way (cf. [13]):

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}$ act on F^∞ by

$$\pi(X)v = \left.\frac{d}{dt}\right|_{t=0} \pi(exp(tX))v = \tilde{X}\pi(e)v, \quad X \in \mathfrak{g}.$$

We now choose a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$. Then the Garding topology is induced by the countable family of seminorms

$$p_{I,l}(v) = \|\pi(X_{i_1}) \circ \cdots \circ \pi(X_{i_m})v\|_l,\ \ l \in \mathbb{N}, \tag{5}$$

where the semi norms $\|\cdot\|_l$ induce the topology on F and $I = (i_1, \ldots, i_m)$ with $i_k \in \{1, \ldots, n\}$ (it suffices to consider monotone multi indices). Note, that these semi norms only consider the norm of derivatives at e. The reason - roughly speaking - that in this particular setting we nevertheless obtain an equivalent family of semi norms is that by equation (4) the value of $f \in \Phi_\pi(F)$ at any point g is just a linear transformation of the value at e.

For the next lemma we refer to the references given in the beginning of this section.

Lemma 28. *Let (π, G) be a continuous representation of G on a Fréchet space F. Then the following statments hold:*

1. *The space of smooth vectors F^∞ is G-invariant and dense in F.*
2. *(π, G) is a continuous representation on F^∞. Moreover, it is smooth, i.e., $(F^\infty)^\infty = F^\infty$.*

Continuous G-module homomorphisms induce continuous homomorphisms between the respective G-modules of smooth vectors.

Lemma 29. *Let π and ν be continuous representations of a Lie group G on Fréchet spaces E and F. Let $T : E \to F$ be a continuous and G-equivariant linear map. Then $T(E^\infty) \subseteq F^\infty$ and the induced map $T : E^\infty \to F^\infty$ is continuous.*

Proof. We will show, that smooth vectors are mapped to smooth vectors. To this end, fix $v \in E^\infty$. Then for every smooth curve $c : \mathbb{R} \to G$ and $l \in F^*$, the mapping

$$t \mapsto l \circ (\nu(c(t))Tv) = (l \circ T)(\pi(c(t))v), \qquad t \in \mathbb{R},$$

is smooth because v is a smooth vector and $l \circ T \in E^*$. Hence, $Tv \in F^\infty$. In order to show continuity we observe that $p_{I,l}(Tv) \leq \|T\|_{k,l} \cdot p_{I,l}(v)$ where $\|T\|_{k,l} := \sup\{\|Tv\|_l : \|v\|_k \leq 1\}$. The claim now follows from Remark 21 and the fact that T is linear.

□

Let (π, E) and (π', F) be representations of G on the Fréchet spaces E and F. Then there is an induced action of G on $E \times F$ given by

$$\rho(g)(v, v') = (\pi(g)v, \pi'(g)v'), \quad v \in E, \ v' \in F, \ g \in G.$$

It is obvious, that ρ is a continuous (smooth) representation if and only if π and π' are continuous (smooth) representations. In particular $(E \times F)^\infty = E^\infty \times F^\infty$.

Remark 30. A closer look at the argument in Lemma 29 shows that the lemma more generally holds for continuous and equivariant multilinear maps.

1.4 Differential forms on the sphere bundle

For this section we assume some familiarity with integration of differential forms on rectifiable sets. For an introduction confer [26, 36].

We denote the set of all (continuous) m-forms on a manifold M by $\Lambda^m(M)$. Moreover, we define $\Lambda(M) := \bigoplus_{m=0}^{\dim M} \Lambda^m(M)$. Let $I = (i_1, \ldots, i_m), \ i_k \in \{1, \ldots, n\}$ be a monotone multi index, i.e. $k \mapsto i_k$ is strictly monotone increasing. Let $(U; x_1, \ldots, x_n)$ be a coordinate chart on M. We set

$$dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_m}.$$

By $T_pM, \ p \in M$ we denote the tangent space to M at the point p. Recall that $\{dx_I|_p : |I| = m\}$ is a basis for $(T_p^*M)^m$ for every $p \in U$. Therefore, every $\omega \in \Lambda^m M$ has a coordinate representation of the form

$$\omega = \sum_{|I|=m} \omega_I(x) dx_I, \quad \omega_I \in C(V, \mathbb{R}).$$

A differential of the form $\omega_I dx_I$ is called simple.

We are particularily interested in the tangent bundle $TV = V \times V$ and the sphere bundle $SV = V \times S(V)$. There are canonical smooth structures on those spaces (for SV cf. [14]). For simplicity, in the following let X be either $S(V)$ or V. We also choose an orthonormal basis $x_1, \dots, x_n$ of V. Then for the remainder of this section let V be oriented by the orientation form

$$dvol_n = dx_1 \wedge \cdots \wedge dx_n.$$

Furthermore, put $p_1 : V \times X \to V$ and $p_2 : V \times X \to X$ for the projections onto the first and second component, respectively. Henceforth, we will not distinguish between differential forms $\omega_1 \in \Lambda V$ and $\omega_2 \in \Lambda X$ and their pullbacks $p_1^*\omega_1 \in \Lambda(V \times X)$ and $p_2^*\omega_2 \in \Lambda(V \times X)$. For the tangent space in a point $(x, y) \in V \times X$, we have the decomposition

$$T_{(x,y)}(V \times X) = T_x V \oplus T_y X.$$

Therefore, there also is a decomposition

$$\Lambda^m(V \times X) = \sum_{k=0}^{m} (\Lambda^k V \wedge \Lambda^{m-k} X).$$

We also need the definition of the Hodge $*$-operator:

Definition 31. *Let I be a monotone multi index and set $I^c := \{1, \dots n\} \setminus I$. The Hodge $*$-operator $* : \Lambda^k V \to \Lambda^{n-k} V$ is determined by*

$$*dx_I = \sigma_{I,I^c} dx_{I^c}$$

where σ_{I,I^c} is the sign of the permutation mapping (I, I^c) to $\{1, \dots, n\}$.

Remark 32. It is easy to see, that on k-forms $*^{-1} = (-1)^{k(n-k)} *$.

Let M be a manifold. Recall, that a contact element at a point $p \in M$ is a hyperplane in the tangent space $T_p M$. Clearly, every contact element is determined by the kernel of a 1-form α, that is unique up to multiplication with a non zero scalar.

Definition 33. *A contact structure on a manifold M of odd dimension $2m + 1$ is a field of contact elements on M that is locally determined by smooth 1-forms α satisfying*

$$\alpha \wedge (d\alpha)^n \neq 0.$$

There is a canonical contact structure on the $(2n - 1)$-dimensional manifold SV (cf [14]). It is determined by the contact form

$$\alpha = \sum_{i=1}^{n} y_i dx_i, \quad y_1^2 + \cdots + y_n^2 = 1,$$

where $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are coordinates on V. The contact element in a point (x, u) is easily seen to be $u^\perp \oplus T_u S(V) \subseteq T_x V \oplus T_u S(V)$.

Definition 34. *Let $\omega \in V \times X$.*

1. *ω is called translation-invariant if it is translation-invariant with respect to the first component, i.e., $\tau_t^* \omega = \omega$ for every $t \in V$, where $\tau_t(x, y) = (x + t, y)$, $y \in X$.*

2. *We say ω is of bi-degree (k, l) if $\omega \in \Lambda^k(V) \otimes \Lambda^l(X)$.*

3. *A form $\omega \in \Lambda(SV)$ is said to be vertical if $\alpha \wedge \omega = 0$.*

4. *$\omega \in \Lambda(TV)$ is called homogeneous of degree k if it is k-homogeneous in the second component, i.e. $h_\lambda^*\omega = \lambda^k\omega$ for every $\lambda \geq 0$, where $h_\lambda(x,y) = (x, \lambda y)$.*

5. *The space of smooth, translation-invariant $(n-1)$-forms of bi-degree $(k, n-k-1)$ on SV, such that $d\omega$ is vertical is denoted by Ω_k^V. We also set*

$$\Omega^V = \bigoplus_{k=0}^{n-1} \Omega_k^V.$$

6. *The space of translation-invariant, $(n-k)$-homogeneous closed differential forms of bi-degree $(k, n-k)$ on TV whose restriction to SV is vertical is denoted $\Omega_{B,k}^V$. Moreover, we set*

$$\Omega_B^V := \left(\bigoplus_{k=0}^{n-1} \Omega_{B,k}^V\right) \oplus span\{\pi_V^* dvol_n\}.$$

Remark 35. A few remarks are in order:

1. If $\omega \in \Lambda V$ is translation-invariant, then $\omega = \sum_{0 \leq |I| \leq n} c_I dx_I$ for some constants $c_I \in \mathbb{R}$. In particular, $d\omega = 0$.

2. At any point $p \in SV$ there are smooth dual-coordinates $\alpha, \beta_1, \dots, \beta_{n-1}$. Hence, $\omega \in \Lambda(SV)$ is vertical if and only if there is an $\omega' \in \Lambda(SV)$ such that $\omega = \alpha \wedge \omega'$. We call $\omega \in \Lambda(SV)$ tangential if it is of the form

$$\omega = \sum_{0 \leq |I| \leq n-1} \omega_I(x)\beta_I, \quad \omega_I \in C(SV).$$

3. It is easy to see that $dx_i \in \Lambda V$ is one-homogeneous. In particular, every form

$$\omega = f(x) dx_I \in \Lambda V,$$

where $|I| = k$ and $f \in C(V)$ is an l-homogeneous function, is $k+l$ homogeneous. Therefore, for simple forms in cartesian coordinates on V, the degree of homogeneity is given by the sum of the degree of homogeneity of the coordinate function and the degree of the form.

4. Let us now consider homogeneity in terms of polar coordinates $(r, \theta_1, \dots, \theta_{n-1})$ on V. Clearly, dr is one-homogeneous and $d\theta_i$ is 0-homogeneous. Moreover, every l-homogeneous function in polar coordinates is of the form

$$r^l \cdot g(\theta_1, \dots, \theta_{n-1}).$$

Hence, every k-homogeneous form on V is of the form

$$d(r^k) \wedge \sum_{0 \leq |I| \leq n-1} \omega_I'(\theta_1, \dots, \theta_{n-1}) d\theta_I + r^k \sum_{0 \leq |I| \leq n-1} \omega_I''(\theta_1, \dots, \theta_{n-1}) d\theta_I.$$

Let $\tilde{r}(x,y) := |y|,\ \ x, y \in V$ and $p : V \times V \setminus V \times \{0\} \to V \times S(V),\ \ (x,y) \mapsto (x, \frac{y}{\tilde{r}(x,y)})$.

Definition 36. *Given $\omega \in \Omega_k^V$, define*

$$\omega_B := \begin{cases} d(\tilde{r}^{n-k} p^* \omega) & \text{on} \quad TV \setminus V \times \{0\} \\ 0 & \text{on} \quad V \times \{0\} \end{cases}$$

Remark 37. Since $p^*\beta$ is 0-homogeneous for every $\beta \in \Lambda(SV)$, it is easy to see that $\omega_B \in \Omega_{B,k}^V$.

Lemma 38. *The linear map $\Phi : \Omega_k^V \to \Omega_{B,k}^V, \omega \mapsto \omega_B$ is surjective and we always have $\omega_B|_{SV} = d\omega$.*

Proof. Let $\theta \in \Omega_{B,k}^V$. Since θ is $(n-k)$-homogeneous, by Remark 35.4, there exist forms β, γ on SV of bi-degrees $(k, n-k-1)$ and $(k, n-k)$, respectively, such that

$$\theta = d(\tilde{r}^{n-k}) \wedge p^*\beta + \tilde{r}^{n-k} p^*\gamma.$$

As θ is closed,

$$0 = d\theta = -d(\tilde{r}^{n-k}) \wedge p^* d\beta + d(\tilde{r}^{n-k}) \wedge p^*\gamma$$

and thus $d\beta = \gamma$. Consequently, we obtain

$$\theta = d(\tilde{r}^{n-k} p^*\beta).$$

Clearly, $d\beta = \theta|_{SV}$. Hence, $d\beta$ is vertical so $\beta \in \Omega_k^V$ with $\theta = \beta_B$. $\square$

2 Valuations on convex bodies

2.1 Valuations

In this section we recall the notion of valuation. Apart from basic definitions we will state some well known results from the theory of valuations. For a more thorough introduction, we refer the reader to [27].

Definition 39. *A valuation is a real-valued map ϕ on $\mathcal{K}(V)$ such that*

$$\phi(A) + \phi(B) = \phi(A \cup B) + \phi(A \cap B) \tag{6}$$

whenever $A, B, A \cup B \in \mathcal{K}(V)$. A valuation ϕ is called

- *continuous if it is a continuous map with respect to the Hausdorff metric;*
- *translation-invariant if $\phi(K + v) = \phi(K)$ for all $K \in \mathcal{K}(V)$ and $v \in V$;*
- *of degree i if $\phi(\lambda K) = \lambda^i \phi(K)$ for all $K \in \mathcal{K}(V)$;*
- *even if $\phi(-K) = \phi(K)$;*
- *odd if $\phi(K) = -\phi(-K)$.*

The set of all continuous translation invariant valuations is denoted by $Val(V)$. For the subset of valuations of degree i we write $Val_i(V)$ and for the subset of even and odd ones we write $Val^+(V)$ and $Val^-(V)$, respectively. Finally, $Val_i^{\pm}(V) := Val_i(V) \cap Val^{\pm}(V)$.

Remark 40. If there is no possibility for confusion we will somtimes write Val instead of $Val(V)$.

Example 41. We consider a few examples.

1. The restriction of a measure on V to convex bodies is obviously a valuation. In particular, the ordinary volume $vol_n : \mathcal{K}(V) \to \mathbb{R}$ is an even valuation of degree n.
2. Let $A \in \mathcal{K}(V)$. Then $\mu_A(K) := vol_n(K + A)$ is a valuation. Clearly, it is even if and only if A is antipodally symmetric, that is, there is $x \in V$ such that $-A = x + A$.
3. The Euler characteristic χ defined by $\chi(K) = 1$ for all $K \in \mathcal{K}(V)$ is a valuation of degree 0.

In connection with the so-called Steiner formula we obtain some more important examples of valuations. Let $K \in \mathcal{K}(V)$, then Steiner's formula states that

$$\mu_{B_r(0)}(K) = vol_n(K + B_r(0)) = \sum_{i=0}^{n} r^{n-i} \omega_{n-i} V_i(K). \tag{7}$$

The coefficients

$$V_i(K) = \frac{\binom{n}{i}}{\omega_{n-i}} \cdot V(K[i], B_1(0)[n-i])$$

are called intrinsic volumes. The i-th intrinsic volume V_i is a valuation of degree i. These valuations are very natural and important geometric functionals on convex bodies. For example, V_n is the volume of the body and V_{n-1} is $\frac{1}{2}$ of the Hausdorff measure of ∂K. Note that the intrinsic volumes are $SO(V)$- and translation-invariant, i.e. $\phi(t + gK) = \phi(K)$ for all $K \in \mathcal{K}(V)$, $t \in \mathbb{R}$ and $g \in SO(V)$. Indeed, the intrinsic volumes form a basis of the space of $SO(V)$- and translation-invariant valuations (cf. [19, 27]).

Theorem 42 (Hadwiger)**.** *The space of $SO(V)$- and translation-invariant valuations on V is spanned by the intrinsic volumes $V_0, V_1, \dots, V_n$.*

Remark 43. It is worthwhile to remark, that $SO(V)$-invariant valuations are even. Indeed, it suffices to show that this is true for the intrinsic volumes. However, this is immediate from the definition of mixed volumes and the fact that $B_1(0) = -B_1(0)$.

The Steiner formula is a consequence of the decomposition of a valuation μ_A into its k-homogeneous parts, stated in Theorem 14.

Lemma 44. *Let $A \in \mathcal{K}(V)$. Then*

$$V_{A_1,\dots,A_k} := K \mapsto V(K[n-k], A_1, \dots, A_k) \in Val_k.$$

Moreover,

$$\mu_A(K) = vol_n(K + A) = \sum_{k=0}^{n} \binom{n}{k} V_{A[k]}(K). \tag{8}$$

Proof. Identity (8) follows from (3) with $m = 1, \lambda_1 = 1$.

In order to show that $V_{A_1,\dots,A_k} \in Val_k$, we apply the differential operator $\frac{\partial^k}{\partial\lambda_1\cdots\partial\lambda_k}\Big|_{\lambda_j=0}$ to both sides of equation (3),

$$\frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\Bigg|_{\lambda_j=0} vol_n\left(K + \sum_{j=0}^{k} \lambda_j A_j\right) = k!\binom{n}{k} V(K[n-k], A_1, \dots, A_k). \tag{9}$$

Since

$$\frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\Bigg|_{\lambda_j=0} vol_n\left(K + \sum_{j=0}^{k} \lambda_j A_j\right) = \frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\Bigg|_{\lambda_j=0} \mu_{\sum_{j=0}^{k} \lambda_j A_j}(K)$$

is a valuation, the right-hand side of (9) is also a valuation. □

Definition 45. *Let*

$$MVol := span\{\mu_A : A \in \mathcal{K}(V)\}$$

and let $MVol_k := MVol \cap Val_k$.

Remark 46. By the definition of mixed volumes, every mixed volume can be written as a linear combination of μ_{A_i}. Thus, we have

$$MVol = span\{V(K[n-k], A_1, \dots, A_k) : k = 0, \dots, n, \;\; A_1, \dots, A_n \in \mathcal{K}(V)\}.$$

There is a generalization of Theorem 14 for arbitrary valuations (cf. [24]).

Theorem 47 (McMullen)**.** *Let $\phi \in Val$. If $A_1, \dots, A_k \in \mathcal{K}(V)$ and $\lambda_1, \dots, \lambda_k \geq 0$, then*

$$\phi(\lambda_1 A_1 + \cdots + \lambda_k A_k)$$

is a polynomial in $\lambda_1, \dots, \lambda_k$ of degree at most n. The coefficient of $\lambda_1^{d_1} \cdots \lambda_k^{d_k}$ is a translation-invariant and continuous valuation in A_j of degree d_j for all $j = 1, \dots, k$.

If we apply this theorem for $k = 1$, then we obtain:

Corollary 48 (McMullen's decomposition)**.**

$$Val = \bigoplus_{i=0}^{n} Val_i^+ \oplus Val_i^-.$$

Proof. From an application of Theorem 47 for $k = 1$ we obtain $Val = \bigoplus_{i=0}^{n} Val_i$. Writing

$$\phi(K) = \frac{\phi(K) + \phi(-K)}{2} + \frac{\phi(K) - \phi(-K)}{2}$$

for $\phi \in Val_i$ completes the proof. □

By McMullen's decomposition, every $\phi \in Val$ can be decomposed into its odd and even components of degree k,

$$\phi = \sum_{i=0}^{n} \phi_i = \sum_{i=0}^{n} (\phi_i^+ + \phi_i^-).$$

We define a norm on Val by

$$\|\phi\| := \sup_{K \subseteq B_1(0)} \phi(K).$$

Corollary 49. *Endowed with the norm defined above, Val becomes a Banach space. Moreover, for every $0 \leq k \leq n$, Val_k^+ and Val_k^- are closed subspaces.*

Proof. Let $(\phi^{(i)})_{i \in \mathbb{N}}$ be a Cauchy sequence in Val. Since, the $\phi^{(i)}$ are linear combinations of at most n-homogeneous maps, the $\phi^{(i)}$ uniformly converge to a map ϕ on every set $\{K \in \mathcal{K}(V) : K \subseteq B_r(0)\}$. Therefore, the pointwise limit $\phi(K) := \lim_{i \to \infty} \phi^{(i)}(K)$ exists. Clearly, it is a translation invariant valuation. Moreover, since the convergence is uniform on bounded sets, it is also continuous. □

Definition 50. *Let $W \subseteq V$ be a subspace of V. Furthermore, let $\iota_W : \mathcal{K}(W) \to \mathcal{K}(V)$ be the imbedding given by $K \mapsto K$. Then the restriction map $r_W : Val(V) \to Val(W)$ is given by*

$$[r_W(\phi)](K) = \phi(\iota_W(K)), \quad \phi \in Val(V), K \in \mathcal{K}(W).$$

2.2 The Irreducibility Theorem and smooth valuations

There is a natural way for a group G to act on the space of valuations.

Definition 51. *Let G be a group that acts on the vector space V by linear transformations. Then the action of G on $Val(V)$ is given by*

$$[g \cdot \phi](K) = \phi(g^{-1}K), \quad K \in \mathcal{K}(V).$$

A valuation ϕ is called G-invariant if $[g \cdot \phi](K) = \phi(K)$ for all $K \in \mathcal{K}(V), g \in G$. A subset $W \subseteq Val(V)$ is called invariant if $g \cdot W = \{g \cdot \phi : \phi \in W\} \subseteq W$.

The following theorem of Alesker is fundamental in the theory of valuations (cf. [2]). It states, that McMullen's decomposition is a decomposition into irreducible spaces with respect to the action of $GL(V)$.

Theorem 52 (Alesker's Irreducibility Theorem)**.** *The representation of $GL(V)$ on $Val_k^+(V)$ and $Val_k^-(V)$ is irreducible, i.e. there exists no non-trivial $GL(V)$-invariant closed subspace.*

This theorem has many important consequences. One of them is the positive answer to a conjecture of McMullen:

Corollary 53. *Let $\epsilon = +, -$ and $0 \leq k \leq n$. Then the subspace $MVol_k^\epsilon$ is dense in Val_k^ϵ. In particular, $MVol$ is dense in Val.*

Proof. Clearly, $MVol_k$ is $GL(V)$-invariant. Furthermore, the intersections $MVol_k \cap Val_k^+$ and $MVol_k \cap Val_k^-$ are obviously non-trivial for all $0 \leq k \leq n$. By the Irreducibility Theorem they have to be dense. The second assertion immediately follows from Lemma 44 and McMullen's decomposition, Corollary 48. □

Remark 54. By the previous corollary and Lemma 13, we obtain that Val_n is one-dimensional and spanned by vol_n.

We next recall the definition of smooth valuations.

Definition 55. *A valuation $\phi \in Val$ is called smooth if it is a smooth vector of the natural action of $GL(V)$ on Val. The subspace of smooth valuations is denoted Val^∞.*

It is not particularly surprising that Val^∞ carries more structure than Val. Most importantly, the algebraic operations that we introduce in the later chapters will only be defined on the subspace of smooth valuations. It is worth mentioning that many of the most important examples of valuations are smooth. The following theorem shows that in the context of valuations $GL(V)$-smoothness is the same as $SO(V)$-smoothness (cf. [33]).

Theorem 56. *A valuation $\phi \in Val$ is smooth if and only if it is a smooth vector of the natural action of $SO(V)$ on Val. Furthermore, the $GL(V)$ and $SO(V)$ Garding topologies coincide.*

Most of the results we proved for the space Val also hold for Val^∞.

Corollary 57 (McMullen's decomposition)**.**

$$Val^\infty = \sum_{i=0}^{n} Val_i^{+,\infty} \oplus Val_i^{-,\infty}.$$

Proof. Let $\phi \in Val^\infty$. Theorem 47 yields

$$\phi(\lambda K) = \sum_{i=0}^{n} \lambda^i \phi_i(K)$$

with $\phi_i \in Val_i$. In order to see that $\phi_i \in Val^\infty$ we write down the equation above for $\lambda = 0, \dots, n$. This yields a system of linear equations represented by a Vandermonde matrix A with the distinct parameters $0, 1, \dots, n$. Since such a matrix is always invertible, we obtain

$$A^{-1} \begin{pmatrix} \phi(0K) \\ \phi(K) \\ \vdots \\ \phi(nK) \end{pmatrix} = \begin{pmatrix} \phi_0(K) \\ \phi_1(K) \\ \vdots \\ \phi_n(K) \end{pmatrix}.$$

Consequently, we have $\phi_i \in Val^\infty$ since $\phi(iK) \in Val^\infty$. Clearly, the even and odd part of a smooth valuation is also smooth. □

We set $MVol \cap Val^\infty := MVol^\infty$.

Corollary 58. *The subspace $MVol_k^\infty$ is dense in Val_k^∞. In particular, $MVol^\infty$ is dense in Val^∞.*

Proof. From Corollary 53 and Lemma 28 we obtain that $MVol^\infty$ is dense in Val. By Theorem 56 these spaces can be represented by respective subspaces of $C(SO(V), Val)$ and $C^\infty(SO(V), Val)$. Let $f \in C^\infty(SO(V), Val) \cong Val^\infty$. For every $m \in \mathbb{N}$, there is a sequence $\tilde{f}_i^m \in MVol^\infty \subseteq C^\infty(SO(V), Val)$ that converges to the derivative $\frac{\partial^{m^m}}{\partial x_1^m \cdots \partial x_m^m} f$ in the $\|\cdot\|_0$ norm (sup-norm). Since $SO(V)$ is compact, this implies that there is a sequence $f_i^m \in MVol^\infty \subseteq C^\infty(SO(V), Val)$, $f_i^n \to f$ (obtained by integration) that converges to f in the $\|\cdot\|_m$ norm. Hence, we obtain that the diagonal sequence f_m^m converges to f in $C^\infty(SO(V), Val)$.

□

Lemma 59. *If $A \in \mathcal{K}^\infty(V)$, then $\mu_A \in Val^\infty$.*

2.3 The normal cycle and smooth valuations

It turns out that smooth valuations can be represented by differential forms integrated over the normal cycle. We start by introducing the notion of normal cycle.

Definition 60. *Let $K \in \mathcal{K}(V)$ and $x \in K$. The tangent cone T_xK at the point x is the closure of the set*

$$\{y \in V : \exists \epsilon > 0 \quad x + \epsilon y \in K\}.$$

We define the normal cone at x as the set

$$Nor(K, x) = \{w \in V : \langle y, w \rangle \leq 0 \quad \forall y \in T_xK\}.$$

The normal cycle of a convex body K is given by

$$\overrightarrow{N}(K) = \{(x; u) : x \in \partial K, \ \ u \in Nor(K, x) \cap S(V)\} \subseteq SV.$$

Finally, we also need a slight modification of the normal cycle given by

$$\overrightarrow{N}_B(K) = \{(x; u) : x \in K, \ \ u \in Nor(K, x) \cap B_1(0)\} \subseteq TV.$$

In the next lemma the nearest point projection is used to construct a bi-Lipschitz map between the previously defined sets and certain rectifiable sets.

Lemma 61. *Let $K \in \mathcal{K}(V)$ and $K_1 = K + B_1(0)$. Then the maps*

- $P_K : \partial K_1 \to \overrightarrow{N}(K), \ \ x \mapsto (p_K(x); x - p_K(x))$
- $\tilde{P}_K : K_1 \to \overrightarrow{N}_B(K), \ \ x \mapsto (p_K(x); x - p_K(x))$

are bi-Lipschitz. In particular, $\overrightarrow{N}(K)$ and $\overrightarrow{N}_B(K)$ are rectifiable.

Proof. Since p_K is Lipschitz, P_K is also Lipschitz. Furthermore, it is easy to see that P_K is injective and surjective on ∂K_1. The inverse map is given by the Lipschitz map $(x; y) \mapsto x + y$. The case of $\tilde{P}_K$ works analogously. □

Remark 62. Lemma 61 directly implies that $\partial \overrightarrow{N}_B(K) = \overrightarrow{N}(K)$.

The next theorem shows that smooth valuations can be represented as differential forms integrated over the normal cycle of K (cf. Definition 60). By Lemma 61 those integrals are well-defined. The theorem stated here, is a special case of results from [5] and [12].

Theorem 63. *Let $0 \leq k \leq n-1$, $\omega \in \Omega^V_{k,tr}$. Then the map $\nu(\omega) : \mathcal{K}(V) \to \mathbb{R}$ given by*

$$\nu(\omega)(K) = \int_{\overrightarrow{N}(K)} \omega$$

is a smooth, translation-invariant valuation of degree k. The map

$$\nu : \Omega^V_k \to Val^\infty_k$$

is surjective with kernel

$$\ker \nu = \{\omega \in \Omega^V_k : \omega \text{ is exact}\}.$$

We also need a slightly different characterization in terms of the space Ω^V_B (cf. [16]).

Corollary 64. *Let ν be the map from the last theorem and $\omega \in \Omega^V_k$. Then*

$$\int_{\overrightarrow{N}_B(K)} \omega_B = \nu(\omega)(K).$$

In particular, the map $\nu_B : \Omega^V_{B,k} \to Val^\infty_k$ given by

$$\nu_B(\theta)(K) = \int_{\overrightarrow{N}_B(K)} \theta, \quad \theta \in \Omega^V_{B,k},$$

is surjective.

Proof. Clearly, $\omega_B = d(\tilde{r}^{n-k} p^* \omega)$ is the differential of a Lipschitz form if $\omega \in \Omega^V_k$. Therefore, we may apply Stokes theorem (cf. [15, 35]) to obtain

$$\int_{\overrightarrow{N}_B(K)} \omega_B = \int_{\partial \overrightarrow{N}_B(K))} \tilde{r}^{n-k} p^* \omega.$$

Since $\partial \overrightarrow{N}_B(K) = \overrightarrow{N}(K) \subseteq SV$, we see that this is equal to

$$\int_{\overrightarrow{N}(K)} \omega = \nu(\omega)(K).$$

□

Remark 65. Since

$$vol_n(K) = \int_{\overrightarrow{N}_B} p_1^*(dvol_n),$$

we have that

$$\nu_B : \Omega^V_B \to Val^\infty$$

is surjective. Moreover, it is easy to see that this map is continuous with respect to the C^∞ norm on Ω^V_B. Hence, the open mapping theorem implies that ν_B is open.

2.4 Even valuations and the Klain embedding

There are two very useful ways to describe a valuation $\phi \in Val_i^{+,\infty}$ in terms of functions on the Grassmannian Gr_i. These representations are related by the cosine transform C_i.

We start with an important theorem by Klain (cf. [20]). It characterizes the volume as the unique simple valuation $\phi \in Val^+$.

Definition 66. *A valuation $\phi \in Val$ is called simple if it vanishes on convex bodies with empty interior.*

Remark 67. A convex body $K \in \mathcal{K}(V)$ has empty interior if there is a hyperplane $H \in Gr_{n-1}$ such that $K \subseteq H$.

Theorem 68 (Klain)**.** *Let $\phi \in Val^+$ be simple. Then there is a $c \in \mathbb{R}$ such that $\phi = c \cdot vol_n$.*

This characterization allows us to connect even valuations of a given degree to functions on the Grassmannian. Let $\phi \in Val_i^+$ and $E \in Gr_i$. Then the restriction $\phi|_E$ is simple. Indeed, let $F \subseteq E$ be of minimal dimension such that $\phi|_F \neq 0$. It follows, that $\phi|_F$ is simple and hence a multiple of the volume on F. However, since ϕ is of degree i, this is possible only if $F = E$. Therefore, there is a constant $Kl_\phi(E)$ such that $\phi|_E = Kl_\phi(E) vol_i$. This leads to the following definition.

Definition 69. *Let $\phi \in Val_i^+$. Then $Kl_\phi : Gr_i \to \mathbb{C}$ is called the Klain function of ϕ. The map $Kl : Val_i^+ \to C(Gr_i)$, which maps a valuation to its Klain function, is called the Klain embedding.*

For the following result confer [1, 21].

Theorem 70. *The Klain embedding $Kl : Val_i^+ \to C(Gr_i)$ is a continuous and injective linear operator. In particular, a valuation $\phi \in Val_i^+$ is uniquely determined by its Klain function Kl_ϕ.*

Proof. It is obvious that Kl is linear. Let $\phi \in Val_i^+$ such that $Kl_\phi = 0$. Clearly, the restriction of $\phi|_E$ is a simple valuation for any $E \in Gr_{i+1}$. By Klain's theorem $\phi|_E = c \cdot vol_{i+1}$. However as ϕ is of degree i, we conclude that $\phi|_E = 0$. Proceeding by induction, we obtain $\phi = 0$. Continuity follows from

$$\|Kl\|_\phi \leq \frac{1}{\omega_i} \sup_{E \in Gr_i} |\phi(B \cap E)| \leq \frac{1}{\omega_i} \|\phi\|.$$

□

Example 71. The intrinsic volume V_i is characterized as the unique valuation $\phi \in Val_i^+$ such that $Kl_\phi = \omega_{n-i}(\binom{n}{i})^{-1}$. We therefore see that V_i does not depend on the ambient vector space. In other words, let $W \subseteq V$ be an m-dimensional subspace. Then

$$r_W(V_i^V) = V_i^W,$$

where V_i^V, V_i^W denote the intrinsic volumes on the respective spaces (V_i is set equal to 0 if i is greater than the dimension of the ambient space).

The following lemma can be found in [16].

Lemma 72. *Let $A \in \mathcal{K}^\infty(V)$ be centrally symmetric. Then the Klain function of the degree k component $\mu_{A,k}$ of μ_A is given by*

$$Kl_{\mu_{A,k}}(L) = V_{n-k}(p_{L^\perp} A), \quad L \in Gr_k.$$

Proof. For $L \in Gr_k(V)$, set $B_L(r) := B_r(0) \cap L$. Choose $C \in \mathbb{R}$ such, that $A \subseteq B_C(0)$. Then

$$B_L(r) + A \subseteq (B_L(r) + p_{L^\perp} A) \cup (\partial B_L(r) + B_C(0)). \tag{10}$$

In order to prove 10, let $b+a \in B_L(r)+A$ with $b \in B_L(r), a \in A$. If $b+p_L(a) \in B_L(r)$ it follows immediately that $b+a \in B_L(r) + p_{L^\perp} A$. On the other hand, if $b+p_L(a) \notin B_L(r)$, then there exists a $\lambda \leq 1$, such that $b+\lambda p_L(a) \in \partial B_L(r)$. Since $(1-\lambda)p_L(a) + p_{L^\perp}(a) \in B_C(r)$, we obtain $b+a \in \partial B_L(r) + B_c(0)$. We now compute

$$\mu_A(B_L(r)) = vol(B_L(r) + A) =$$

$$= vol(B_L(r) + p_{L^\perp} A) + O(vol(B_L(r) + B_C(0))) =$$
$$= V_k(B_L(r)) V_{n-k}(p_{L^\perp} A) + o(r^k).$$

Hence, it easily follows that

$$Kl_{\mu_{A,k}}(L) = \lim_{r \to \infty} \frac{\mu_A(B_L(r))}{V_k(B_L(r))} = V_{n-k}(p_{L^\perp} A).$$

□

One might ask the question which functions correspond to a valuation via the Klain embedding. It turns out that the range of the Klain embedding coincides with the range of the cosine transform, see Definition 18. The following theorem was obtained by Alesker from an application of the Casselman-Wallach theorem to the main results in [11].

Theorem 73. *Suppose that $1 \leq i \leq n-1$. The image of the Klain embedding on smooth valuations $Kl_i : Val_i^{+,\infty} \to C^\infty(Gr_i)$ coincides with the image of the cosine transform on smooth functions $C_i : C^\infty(Gr_i) \to C^\infty(Gr_i)$.*

For $1 \leq i \leq n-1$, we denote the image of all smooth functions on Gr_i under the cosine transform C_i by T_i^∞. Although C_i is not injective in general, its restriction to T_i^∞ always is, because C_i is selfadjoint. Moreover, Alesker deduced (cf. [3]) that

$$C_i(T_i^\infty) = T_i^\infty.$$

2.5 Crofton measures

Let us turn to the second way to describe even valuations. Consider a finite Borel measure μ on Gr_i and define

$$(Cr_i\mu)(K) = \int_{Gr_i} vol_i(K|E) d\mu(E). \tag{11}$$

Then $Cr_i\mu \in Val_i^+$. This motivates the next definition.

Definition 74. *A finite Borel measure μ on Gr_i is called a Crofton measure for the valuation $\phi \in Val_i^+$, if $Cr_i\mu = \phi$.*

The image of all Borel measures under the map Cr_i is obviously a $GL(V)$-invariant subspace of Val_i^+. Therefore, Aleskers Irreducibility Theorem, Theorem 52, yields that this image is dense in Val_i^+. It turns out, that all valuations in $Val_i^{+,\infty}$ can be represented by a Crofton measure. We restrict the map Cr_i, defined in (11) to smooth functions:

$$(Cr_i f)(K) = \int_{Gr_i} vol_i(K|E)f(E)dE, \quad f \in C^\infty(Gr_i).$$

The resulting valuation is obviously smooth. Now let $F \in Gr_i$. Then, for any convex body $K \subseteq F$ and $f \in C(Gr_i)$, by the definition of the cosine,

$$(Cr_i f)(K) = vol_i(K) \int_{Gr_i} |\cos(E,F)| f(E)dE.$$

Hence, we obtain,

$$Kl_{Cr_i f} = C_i f. \tag{12}$$

Theorem 73 implies:

Corollary 75. *For any valuation $\phi \in Val_i^{+,\infty}$, there exists a unique smooth measure $\mu \in T_i^\infty$ such that μ is a Crofton measure for ϕ.*

We will also need Crofton's formulas with respect to affine Grassmannians. Let $\overline{Gr_i}$ denote the i-th affine Grassmannian. Clearly, the product of the $O(n)$-invariant measure ν_i on Gr_i with the $(n-i)$-dimensional Lebesgue measure is a normalized $O(n)$- and translation-invariant measure on $\overline{Gr_i}$. We denote the orthogonal complement map by

$$\perp : Gr_i \to Gr_{n-i}, \quad 0 \le i \le n.$$

It is a bijection with $\perp_* \nu_i = \nu_{n-i}$.

Corollary 76. *Let $\phi \in Val_i^{+,\infty}$ and let m_ϕ be a Crofton measure for ϕ. Define $\overline{m_\phi} := \perp_*(m_\phi) \times vol_{n-i}$. Then*

$$\phi(K) = \int_{\overline{Gr_{n-i}}} \chi(K \cap E) d\overline{m_\phi}.$$

Proof.

$$\begin{aligned}\phi(K) &= \int_{Gr_i} vol_i(K|E)dm_\phi(E) = \int_{Gr_{n-i}} vol_i(K|E^\perp)dm_\phi(E^\perp) \\ &= \int_{Gr_{n-i}} \int_{E^\perp} \chi(K \cap (E+x))dx dm_\phi(E^\perp) = \int_{\overline{Gr_{n-i}}} \chi(K \cap E) d\overline{m_\phi}.\end{aligned}$$

□

2.6 The Alesker-Fourier transform for even valuations

For even valuations there exists a duality transform (cf. [3]). It is called the Alesker-Fourier transform as it shares properties with the classical Fourier transform with respect to the algebraic structures to be introduced in the later chapters.

Lemma 77. *Let $f \in C^\infty(Gr_i)$. Then*

$$(C_i f) \circ \perp = C_{n-i}(f \circ \perp).$$

Proof. Let $F \in Gr_{n-i}$. Then

$$\begin{aligned}[(C_i f) \circ \perp](F) &= \int_{Gr_i} |\cos(E, F^\perp)| f(E) d\nu_i(E) \\ &= \int_{Gr_{n-i}} |\cos(E^\perp, F^\perp)| (f \circ \perp)(E) d\nu_{n-i}(E) \\ &= \int_{Gr_{n-i}} |\cos(E, F)| (f \circ \perp)(E) d\nu_{n-i}(E) = [C_{n-i}(f \circ \perp)](F).\end{aligned}$$

□

For $f \in T_i^\infty$, by Theorem 73, there is a $g \in C^\infty(Gr_i)$ such that $f = C_i g$. The previous lemma now yields

$$f \circ \perp = (C_i g) \circ \perp = C_{n-i}(g \circ \perp) \in T_{n-i}^\infty. \tag{13}$$

We may therefore define the Alesker-Fourier transform for even and smooth valuations.

Definition 78. *Let $\phi \in Val_k^{+,\infty}$. Then the Alesker-Fourier transform*

$$\mathbb{F} : Val_k^{+,\infty} \to Val_{n-k}^{+,\infty}, \quad 0 \leq k \leq n$$

is defined by

$$Kl_{\mathbb{F}_\phi} = Kl_\phi \circ \perp.$$

Remark 79. Clearly, $\mathbb{F}$ is an involution on $Val_k^{+,\infty}$, i.e., $\mathbb{F} \circ \mathbb{F} = id$.

Lemma 80. *Let $\phi \in Val_k^{+,\infty}$ with Crofton measure $m_\phi = f_\phi \nu_k$. Then $\perp_* m_\phi$ is a Crofton measure for $\mathbb{F}\phi$.*

Proof. By definition, we have $\phi = Cr_k f_\phi$. Using (12) and (13) we obtain,

$$Kl_{\mathbb{F}Cr_k f_\phi} = Kl_{Cr_k f_\phi} \circ \perp = C_k f_\phi \circ \perp = C_{n-k}(f_\phi \circ \perp) = Kl_{Cr_{n-k}(f \circ \perp)}.$$

Hence, $\perp_* m_\phi$ is a Crofton measure for $\mathbb{F}\phi$. □

The next lemma shows how the Alesker-Fourier transform acts under restriction maps.

Lemma 81. *Let $W \subset V$ be an m-dimensional subspace, $\mathbb{F}_W : Val_*^+(W) \to Val_{m-*}^+(W)$ the Alesker-Fourier transform on W, $r_W : Val(V) \to Val(W)$ the restriction map and $A \in \mathcal{K}^\infty(V)$ centrally symmetric. Then*

$$r_W(\mathbb{F}\mu_A) = \mathbb{F}_W(\mu_{p_W A}). \tag{14}$$

Proof. It suffices to show, that the Klain functions of the degree k components are the same. Let us denote $p_W A := A'$. Take $0 \leq k \leq n$ and $L \in Gr_k(W)$. For the Klain function of the righthand side we obtain

$$Kl_{(\mathbb{F}_W(\mu_{A'}))_k}(L) = Kl_{\mathbb{F}_W(\mu_{A',m-k})}(L) = Kl_{\mu_{A',m-k}}(L^{\perp_W}).$$

On the other hand we have

$$Kl_{(r_W(\mathbb{F}\mu_A))_k}(L) = Kl_{\mathbb{F}\mu_{A,n-k}}(L) = Kl_{\mu_{A,n-k}}(L^{\perp_V}).$$

The previous lemma now yields

$$Kl_{\mu_{A',m-k}}(L^{\perp_W}) = Kl_{\mu_k}(p_L A) = Kl_{\mu_{A,n-k}}(L^{\perp_V}).$$

□

The Alesker-Fourier transform in general will be discussed at the end of Chapter 4.

3 Product of smooth translation invariant valuations

This chapter deals with the product on smooth valuations. It was introduced by Alesker in [4]. As before, let V be an n-dimensional real vector space with a fixed inner product $\langle \cdot, \cdot \rangle$.

3.1 The product on mixed volumes

In this section we define a product on $MVol$ and establish some of its basic properties.

Definition 82. *Let $A, B \in \mathcal{K}(V)$ and μ_A, μ_B be the respective valuations. Then the product of μ_A and μ_B is defined by*

$$\mu_A \cdot \mu_B(K) = vol_{2n}(\Delta(K) + A \times B), \tag{15}$$

where $\Delta(K) := \{(x, x) \in V \times V : \ x \in K\}$ is the diagonal embedding.

Lemma 83. *For $A, B \in \mathcal{K}(V)$, we have*

$$\mu_A \cdot \mu_B(K) = \int_V \mu_A(K \cap (y - B))dy. \tag{16}$$

Proof. We start by showing, that $(x, y) \in \Delta(K) + A \times B$ is equivalent to $x \in (K \cap (y - B)) + A$. If $(x, y) \in \Delta(K) + A \times B$, then there is a z such that $x = z + a, \ \ y = z + b$ for some $a \in A, \ \ b \in B$. Thus, $x = y - b + a \in (K \cap (y - B)) + A$. If, on the other hand, $x \in (K \cap (y - B)) + A$, then there are $a \in A, \ \ b \in B$ such that $x = y - b + a$ with $y - b \in K$. Hence, $(x, y) = (y - b + a, y - b + b) \in \Delta(K) + A \times B$. Using Fubini's theorem for the first equality, we obtain

$$\begin{aligned} \mu_A \cdot \mu_B(K) &= \int_V \int_V 1_{\Delta(K) + A \times B}(x, y)dxdy \\ &= \int_V \int_{(K \cap (y - B)) + A} 1dxdy = \int_V \mu_A(K \cap (y - B))dy. \end{aligned}$$

□

Using the last lemma, it is possible to linearly extend the product.

Corollary 84. *The product defined by (15) extends to a unique, associative, commutative and bilinear product on $MVol$ which is continuous in each argument, with the Euler characteristic χ as a unit. Moreover, this product is $GL(V)$-equivariant.*

Proof. The continuity in each argument easily follows from (16). The rest of the properties are trivial. By seperate continuity and the density of $MVol$ in Val, we obtain the following equation:

$$\phi \cdot \mu_A(K) = \int_V \phi(K \cap (y - A))dy, \quad \phi \in Val. \tag{17}$$

Putting $\phi = \chi$, we see that χ is a unit. In order to prove the equivariance property we compute

$$\begin{aligned} g \cdot (\mu_A \cdot \mu_B)(K) &= (\mu_A \cdot \mu_B)(g^{-1}K) = vol_{2n}(\Delta(g^{-1}K) + A \times B) \\ &= |\det(g^{-1})|^2 vol_{2n}(\Delta(K) + gA \times gB) = (g \cdot \mu_A) \cdot (g \cdot \mu_B)(K). \end{aligned}$$

□

For mixed volumes we derive a more explicit formula:

Lemma 85. *Let* $\phi = V_{A_1,\dots,A_k}$ *and* $\psi = V_{B_1,\dots,B_l}$. *Then,*

$$(\phi \cdot \psi)(K) = \frac{(2n)!}{n!(2n-k-l)!} \left[\frac{n!}{(n-k)!\ (n-l)!}\right]^{-1} \cdot V(\Delta(K)[2n-k-l], A_1 \times \{0\}, \dots, A_k \times \{0\}, \{0\} \times B_1, \dots, \{0\} \times B_l).$$

Proof. Using (9), bilinearity and the continuity in each argument we obtain

$$\left[k!\binom{n}{k} l!\binom{n}{l}\right] (\phi \cdot \psi)(K)$$
$$= \left.\frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\right|_{\lambda_j=0} \left.\frac{\partial^l}{\partial\mu_1 \cdots \partial\mu_l}\right|_{\mu_j=0} \left[\mu_{\sum_{i=0}^k \lambda_i A_{s,i}} \cdot \mu_{\sum_{i=0}^l \mu_i B_{t,i}}\right](K)$$

$$= \left.\frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\right|_{\lambda_j=0} \left.\frac{\partial^l}{\partial\mu_1 \cdots \partial\mu_l}\right|_{\mu_j=0} vol_{2n}\left(\Delta(K) + \sum_{i=0}^k \lambda_i (A_i \times \{0\}) + \sum_{i=0}^l \mu_i (\{0\} \times B_i)\right)$$

$$= \left[(k+l)!\binom{2n}{k+l}\right] V(\Delta(K)[2n-k-l], A_1 \times \{0\}, \dots, A_k \times \{0\}, \{0\} \times B_1, \dots, \{0\} \times B_l).$$

□

Remark 86. Lemmas 44 and 85 show that if $\phi \in MVol_k$ and $\psi \in MVol_l$, then $\phi \cdot \psi \in MVol_{k+l}$, where $MVol_m = \{0\}$ if $m > n$.

3.2 Continuity, the product on smooth valuations

We would like to extend the product defined in the last section for $MVol$ to all of Val. However, this is not possible in a continuous manner. Instead, we are now going to prove, that the product on $MVol$ induces a continuous product on $MVol^\infty$ and thus extends to all of Val^∞. In order to do that, we will represent mixed volumes by smooth function on powers of the sphere. We will need some facts concerning these function spaces. Consider the space $C^\infty(S(V)^k)$. By the Stone-Weierstrass theorem, it is easy to see that

$$span\{f_1(x_1) \cdots f_k(x_k) \in C^\infty(S(V)) : f_i \in C^\infty(S(V))\}$$

is dense in $C^\infty(S(V)^k)$. Consequently, for a function $F \in C^\infty(S(V)^k)$, we find functions $f_{i,j}$ such that

$$F(x_1, \dots, x_k) = \sum_{j=1}^\infty f_{1,j}(x_1) \cdots f_{k,j}(x_k).$$

Moreover, these functions may be chosen such that

$$\sum_{j=0}^\infty \|f_{1,j}\|_\infty \cdots \|f_{k,j}\|_\infty \le C\|F\|_m \tag{18}$$

with C and m depending only on k and n. For more details confer [23,30]. We need a theorem of L. Schwartz. The theorem presented here is a version of the theorem found in [31]. One may also find this result in [30].

Theorem 87 (L. Schwarz Kernel Theorem)**.** *Let $B : C^\infty(S(V)) \times C^\infty(S(V)) \to \mathbb{R}$ be a multilinear continuous map. Then there is a continuous linear operator $F : C^\infty(S(V)^2) \to \mathbb{R}$ such that*

$$B(f_1, f_2) = F(f_1 \cdot f_2).$$

Another result - from representation theory - that we will use is the Casselman-Wallach theorem (cf. [13]). Let (π, G, E) be a representation of a closed subgroup $G \subseteq GL(V)$ on a Fréchet space E and let $\|\cdot\|_l$, $l \in \mathbb{N}$ be a family of semi-norms inducing the topology on E. Recall that π is said to be of moderate growth if for each semi-norm $\|\cdot\|_i$ on E, there exists a semi-norm $\|\cdot\|_j$ on E and a $\lambda \in \mathbb{R}$ such that

$$\|\pi(g)v\|_i \leq |g|^\lambda \|v\|_j, \quad g \in G, v \in E,$$

where $|g| := \max\{\|g\|, \|g^{-1}\|\}$.

Theorem 88 (Casselman-Wallach)**.** *Let G be a real reductive group. Let (π, G, E) and (ρ, G, F) be smooth representations of moderate growth of G in Fréchet spaces E and F. Moreover, assume that F is an admissible G-module of finite length. Then any continuous and G-equivariant linear operator $\Phi : E \to F$ has closed image.*

Remark 89. For the notions of real reductive group and admissible module of finite length confer [22, 32, 34]. Note that $GL(V)$ is a real reductive group. It is known that the action of $GL(V)$ on Val^∞ makes this space into an admissible G-module of finite length. Furthermore, it is an easy consequence of McMullen's decomposition that this action is of moderate growth.

We define an action of $GL(V)$ on the space $C(S(V)^k)$ by

$$(g \cdot f)(\nu_1, \ldots, \nu_k) = \|g\nu_1\| \cdots \|g\nu_k\| f\left(\frac{g^{-1}\nu_1}{\|g^{-1}\nu_1\|}, \ldots, \frac{g^{-1}\nu_k}{\|g^{-1}\nu_k\|}\right). \tag{19}$$

It is easy to see that if h_K is the support function of $K \in \mathcal{K}(V)$, then

$$g \cdot h_K = h_{gK}, \quad g \in GL(V).$$

The next lemma is a slight modification of Lemma 1.27 in [33].

Lemma 90. *Let $\left[C(S(V)^k)\right]^\infty$ be the space of smooth vectors with respect to the $GL(V)$-action on $C(S(V)^k)$ defined in (19). Then*

$$\left[C(S(V)^k)\right]^\infty = C^\infty(S(V)^k)$$

as topological vector spaces. The induced representation on $C^\infty(S(V)^k)$ is of moderate growth.

Proof. Suppose that $f \in C^\infty(S(V)^k)$ and that $c : \mathbb{R} \to G$ is a smooth curve. Recall that by the Riesz representation theorem, every linear functional $l \in \left(C(S(V)^k)\right)^*$ is represented by a measure μ on $S(V)^k$. Hence, we see that

$$t \mapsto l(c(t) \cdot f) = \int_{S(V)^k} \|c(t)x_1\| \cdots \|c(t)x_k\| f\left(\frac{c(t)^{-1}x_1}{\|c(t)^{-1}x_1\|}, \ldots, \frac{c(t)^{-1}x_k}{\|c(t)^{-1}x_k\|}\right) d\mu(x_1, \ldots, x_k)$$

is smooth, that is f is a smooth vector. Conversely, let $f \in \left[C(S(V)^k)\right]^\infty$ be a smooth vector. Clearly, it is also a smooth vector with respect to the induced $O(n)$-action on $C^\infty(S(V)^k)$. Note that $S(V) = O(n)/O(n-1)$ as smooth manifolds. Denote by $\pi : O(n) \to O(n)/O(n-1) = S(V)$ the quotient map and let l be the functional which evaluates f at $\pi(id)$. Since f is a smooth vector the map

$$g \mapsto l(g \cdot f) = f(\pi(g^{-1})),$$

from $O(n)$ to $\mathbb{R}$ is smooth. Thus $f \circ \pi$ is smooth and therefore f as well. Using the description of the Garding topology via the semi norms (5), it is easily seen that

$$id : C^\infty(S(V)^k) \to \left[C(S(V)^k)\right]^\infty$$

is continuous. By the open mapping theorem, it is an isomorphism.

□

Remark 91. Lemma 90 shows that for every $j \in \mathbb{N}$, the space of smooth vectors in $C^j(S(V)^k)$ with respect to the action (19) is given by $C^\infty(S(V)^k)$. Indeed since the embeddings $C^\infty(S(V)^k) \hookrightarrow C^j(S(V)^k) \hookrightarrow C(S(V)^k)$ are continuous we have

$$C^\infty(S(V)^k) = \left[C^\infty(S(V)^k)\right]^\infty \subseteq \left[C^j(S(V)^k)\right]^\infty \subseteq \left[C(S(V)^k)\right]^\infty = C^\infty(S(V)^k).$$

Recall that Val_n is spanned by vol_n (cf. Remark 54).

Theorem 92. *For every $0 \le k \le n$, there is an open and continuous epimorphism*

$$\Theta_k : Val_n \otimes C^\infty(S(V)^k) \to Val^\infty_{n-k}$$

which is $GL(V)$-equivariant with respect to the $GL(V)$-action defined in (19).

Proof. Let $0 \le k \le n$. We define a continuous map $\Theta'_k : Val_n \otimes \mathcal{K}(V)^k \to Val_{n-k}$ by

$$\Theta'_k(c \cdot vol_n \otimes (A_1, \dots, A_k)) := c \cdot V_{A_1,\dots,A_k}.$$

By Lemma 13, Θ'_k is Minkowski-linear with respect to each A_j.

Since every compact set is determined by its support function, Θ'_k can be identified with the respective map on support functions:

$$\Theta'_k(c \cdot vol_n \otimes h_{A_1}, \dots, h_{A_k}) := c \cdot V_{A_1,\dots,A_k}.$$

This map is again Minkowski-linear and continuous (cf. Lemmas 5 and 8).

Using Equation (9), we see that it is also $GL(V)$-equivariant, as

$$\begin{aligned}
&(\Theta'_k(g \cdot (c \cdot vol_n) \otimes (g \cdot h_{A_1}, \dots, g \cdot h_{A_k})))(K) = \\
&= \left(k!\binom{n}{k}\right)^{-1} \frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\bigg|_{\lambda_j=0} c \cdot vol_n \left(g^{-1}\left(K + \sum_{j=0}^k \lambda_j (gA_j)\right)\right) \\
&= \left(k!\binom{n}{k}\right)^{-1} \frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\bigg|_{\lambda_j=0} c \cdot vol_n \left(g^{-1}K + \sum_{j=0}^k \lambda_j A_j\right) \\
&= g \cdot (\Theta'_k(c \cdot vol_n \otimes (h_{A_1}, \dots, h_{A_k}))(K).
\end{aligned}$$

By Corollary 11, Θ'_k extends to a continuous multilinear and $GL(V)$-equivariant map on $Val_n \otimes (C^2(S(V)))^k$. Applying Lemma 29 to this map and using Remark 91, we obtain a continuous $GL(V)$-equivariant map

$$\Theta'_k : Val_n \otimes C^\infty(S(V)^k) \to Val^\infty_{n-k}.$$

Having fixed a scalar product on V this map can be identified with a map defined on $C^\infty(S(V)^k)$. To this map we now apply the L. Schwartz kernel theorem $k-1$ times. This yields a continuous linear operator

$$\Theta_k : C^\infty(S(V)^k) \to Val^\infty_{n-k}.$$

This operator, in turn, may be identified with a map

$$\Theta_k : Val_n \otimes C^\infty(S(V)^k) \to Val^\infty_{n-k}$$

which is the desired operator. By construction we have

$$\Theta_k((c \cdot vol_n) \otimes (h_{A_1} \cdot \dots \cdot h_{A_k})) = c \cdot V_{A_1,\dots,A_k} \tag{20}$$

for all $A_1, \dots, A_k \in \mathcal{K}(V)$. It therefore easily follows that this map is $GL(V)$-equivariant. From Corollary 53 we deduce that Θ_k has dense image. Applying the Casselman-Wallach Theorem, Theorem 88, yields that Θ_k is indeed surjective. Now the open mapping theorem shows, that it is also open.

□

Remark 93. As we have already done in the proof of the theorem, we may identify Θ_k with a map

$$\Theta_k : C^\infty(S(V)^k) \to Val^\infty_k.$$

This of course is due to the fact that we have fixed a scalar product on V. In the original paper [4] by Alesker, Theorem 92 was stated invariant from any Euclidean structure on V. To this end we recall the notion of the oriented projectivized cotangent bundle defined by $\mathbb{P}_+(V^*) := (V^* \setminus \{0\})/\mathbb{R}_+$. That is, $\mathbb{P}_+(V^*) = \{[\xi]_+ : \xi \in V^*\}$ with $[\xi]_+ := \{c\xi : c > 0\}$. There is a line bundle $\pi : L \to \mathbb{P}_+(V^*)$ determined by

$$\pi^{-1}([\xi]_+) = ([\xi]_+, |(span(\xi))^*|),$$

where $|(span(\xi))^*|$ is the space of densities on $span(\xi)$, i.e. the space of functions $\mu : span(\xi) \to span(\xi)$ satisfying

$$\mu(ca\xi) = |c|\mu(a\xi), \quad a, c \in \mathbb{R}.$$

Support functions are then viewed as functions $h : \mathbb{P}_+(V^*) \to L$. The action (19) corresponds to the canonical action on functions, i.e., $g \cdot f(x) = f(g^{-1}x)$. (The actions on $\mathbb{P}_+(V^*)$ and $|(span(\xi))^*|$ are also defined in this way.)

Note that summing the Θ_k's yields a continuous and open epimorphism

$$\Theta = \sum_{k=0}^{n} \Theta_k : (\oplus_{k=0}^n C^\infty(S(V)^k, \mathbb{R})) \to Val^\infty.$$

The main result of this section is the following theorem.

Theorem 94. *Formula (15) uniquely determines an associative and commutative multiplication*

$$\cdot : Val^\infty \times Val^\infty \to Val^\infty.$$

The following properties hold.

- $\cdot : Val^\infty \times Val^\infty \to Val^\infty$ *is continuous.*
- *The Euler characteristic* χ *is a unit.*
- $Val_k^\infty \cdot Val_l^\infty \subseteq Val_{k+l}^\infty$*, i.e., if* $\phi \in Val_k$ *and* $\psi \in Val_l$*, then* $\phi \cdot \psi \in Val_{k+l}$ *where* $Val_m = \{0\}$ *if* $m > n$*.*
- *The product of valuations with given parity is only odd if their parity is different.*
- *The restriction map* $r_W : Val(V) \to Val(W)$ *commutes with the product, i.e.,* $r_W(\phi \cdot \psi) = (r_W\phi) \cdot (r_W\psi)$*.*
- *The product is* $GL(V)$*-equivariant, that is,*

$$g \cdot (\phi \cdot \psi) = (g \cdot \phi) \cdot (g \cdot \psi), \quad g \in GL(V).$$

Proof. The product is well defined on $MVol^\infty \times MVol^\infty$. Let us consider the following diagram:

$$\begin{array}{ccc}
\oplus_{k=0}^n (C^\infty(S(V)^k, \mathbb{R})) & \times & \oplus_{k=0}^n (C^\infty(S(V)^k, \mathbb{R})) \\
\Theta \big\downarrow & & \Theta \big\downarrow \\
Val^\infty & \times & Val^\infty \\
 & \cdot \big\downarrow & \\
 & Val &
\end{array}$$

As Θ is an open map, the continuity of the product is equivalent to that of the composition $\cdot \circ (\Theta \times \Theta)$. We will now prove that this is indeed the case. Therefore, take $F \in C^\infty(S(V)^k)$ and $G \in C^\infty(S(V)^{k'})$ with $\Theta(F), \Theta(G) \in MVol^\infty$. We choose functions $f_{l,j}, g_{l,j}$ representing F and G, respectively, that satisfy (18). Note that, by Theorem 9, the $f_{l,j}$ and $g_{l,j}$ can be written as differences of support functions with norms bounded by $c_1\|f_{l,j}\|_m$ and $c_1\|g_{l,j}\|_m$, respectively. For an arbitrary convex body $K \subseteq B_1(0)$, we have

$$\begin{aligned}
(\Theta(F) \cdot \Theta(G))(K) &= \sum_{s,t=1}^{\infty} [\Theta(f_{1,s} \cdots f_{k,s}) \cdot \Theta(g_{1,t} \cdots g_{k',t})](K) \\
&\leq \sum_{s,t=1}^{\infty} [\Theta(h_{1,s} \cdots h_{k,s}) \cdot \Theta(h'_{1,t} \cdots h'_{k',t})](K).
\end{aligned}$$

Let $A_{l,j}$ and $B_{l,j}$ be the convex bodies corresponding to $h_{l,j}$ and $h'_{l,j}$. We then obtain that the last term is equal to

$$
\begin{aligned}
&c_2 \sum_{s,t=1}^{\infty} V(\Delta(K)[2n-k-l], A_{1,s} \times \{0\}, \dots, A_{k,s} \times \{0\}, \{0\} \times B_{1,t}, \dots, \{0\} \times B_{k',t}) \\
&\leq c_1 c_2 \sum_{s,t=1}^{\infty} \prod_{l=1}^{k} \|f_{l,s}\|_0 \prod_{l'=1}^{k'} \|g_{l',t}\|_0 \cdot V(\Delta(K)[2n-k-k'], B_1(0)[k+k']) \\
&\leq c_3 \sum_{s,t=1}^{\infty} \prod_{m=1}^{k} \|f_{s,m}\|_0 \prod_{m'=1}^{k'} \|g_{t,m'}\|_0 \\
&\leq c_3 C \|F\|_m \|G\|_m.
\end{aligned}
$$

Hence, the product

$$\cdot : MVol^\infty \times MVol^\infty \to Val$$

is continuous. By Corollary 58, it extends to a continuous map

$$\cdot : Val^\infty \times Val^\infty \to Val.$$

Now Remark 30 proves the claim. The remaining properties are either straightforward to show or they follow from Corollary 84. □

3.3 A few examples

In this section we further examine the product of mixed volumes and calculate several concrete examples. The next two lemmas can be found in [4].

Lemma 95. *Let $V = V_1 \oplus V_2$ be an orthogonal decomposition of the vector space V with $\dim V_1 = m$. Let $M \in \mathcal{K}(V_1), A_1, \dots, A_{n-m} \in \mathcal{K}(V)$. Then*

$$V(M[m], A_1, \dots, A_{n-m}) = \binom{n}{m}^{-1} vol_m(M) \cdot V(p_{V_2} A_1, \dots, p_{V_2} A_{n-m}).$$

Proof. Let us denote the difference of the two sides by $\Phi(A_1, \dots, A_{n-m})$. Clearly, Φ is symmetric in $A_1, \dots, A_{n-m}$. Assume that $\Phi(A, \dots, A) = 0$ for all $A \in \mathcal{K}(V)$. Then a standard argument shows that $\Phi(A_{\sigma(1)}, \dots, A_{\sigma(n-m)}) = \operatorname{sgn}(\sigma)\Phi(A_1, \dots, A_{n-m})$, $\sigma \in S_{n-m}$ and thus, by the symmetry of Φ, $\Phi = 0$. It therefore suffices to prove the claim for $A_1, \dots, A_{n-m} = A$.

By (3), we have (cf. (9))

$$V(M[m], A[n-m]) = \left[m! \binom{n}{m}\right]^{-1} \frac{d^m}{d\epsilon}\Big|_{\epsilon=0} vol_n(A + \epsilon M).$$

An application of Lemma 19 to $vol_n(A + \epsilon M) = vol_n((A + \epsilon M)|(V_1 + V_2))$ yields

$$
\begin{aligned}
vol_n(A + \epsilon M) &= \int_{z \in V_2} vol_m((z + V_1) \cap (A + \epsilon M)) dz = \int_{z \in V_2} vol_m((A \cap (z + V_1)) + \epsilon M) dz \\
&= \int_{z \in \pi_{V_2} A} \epsilon^m vol_m(M) + O(\epsilon^{m-1}) dz = \epsilon^n vol_{n-m}(p_{V_2} A) vol_m(M) + O(\epsilon^{m-1}).
\end{aligned}
$$

This proves the lemma. □

We can use the last lemma to strengthen the result from Lemma 85 for $k + l = n$.

Lemma 96. *Let* $\phi(K) = V(K[i], A_1, \ldots, A_{n-i}), \psi(K) = V(K[n-i], B_1, \ldots, B_i)$ *with* $A_j, B_j \in \mathcal{K}(V)$. *Then*

$$(\phi \cdot \psi)(K) = \binom{n}{i}^{-1} V(A_1, \ldots, A_{n-i}, -B_1, \ldots, -B_i) \cdot vol(K).$$

Proof. From Lemma 85 we obtain

$$(\phi \cdot \psi)(K) = \binom{2n}{n}\binom{n}{i}^{-1} V(\Delta(K)[n], A_1 \times \{0\}, \ldots, A_{n-i} \times \{0\}, \{0\} \times B_1, \ldots, \{0\} \times B_i).$$

We have $V \times V = V_1 \oplus V_2$, where $V_1 = \{(x,x) : x \in V\} = \Delta(V)$ and $V_2 = \{(x,-x) : x \in V\} =: \Delta'(V)$. Note that

$$p_{V_2}((x,0)) = \frac{1}{2}(x,-x), \;\; p_{V_2}((0,x)) = \frac{1}{2}(-x,x).$$

The embeddings $\Delta : V \to V_1 \subseteq V \times V$ and $\Delta' : V \to V_2 \subseteq V \times V$ are obviously linear with $\det \Delta = \det \Delta' = 2^{-\frac{n}{2}}$. Hence, using the last lemma and Lemma 13 we obtain

$$\begin{aligned}(\phi \cdot \psi)(K) &= \binom{n}{i}^{-1} vol_n(\Delta(K)) \cdot V\left(\frac{\Delta'(A_1)}{2}, \ldots, \frac{\Delta'(A_{n-i})}{2}, \frac{-\Delta'(B_1)}{2}, \ldots, \frac{-\Delta'(B_i)}{2}\right) \\ &= \binom{n}{i}^{-1} V(A_1, \ldots, A_{n-i}, -B_1, \ldots, -B_i) \cdot vol_n(K).\end{aligned}$$

□

We can now determine the product of intrinsic volumes.

Example 97. Recall that the i-th intrinsic volume is given by $V_i(K) = \frac{\binom{n}{i}}{\omega_{n-i}} \cdot V(K[i], B[n-i])$. Moreover, let $\begin{bmatrix} n \\ i \end{bmatrix} := \frac{\omega_n \binom{n}{i}}{\omega_{n-i}\omega_i}$ be the flag coefficient.

- By the previous lemma,

$$\begin{aligned}(V_i \cdot V_{n-i})(K) &= \frac{\binom{n}{i}}{\omega_{n-i}\omega_i} V(B_1(0)[n-i], -B_1(0)[i]) \cdot vol_n(K) = \\ &= \begin{bmatrix} n \\ i \end{bmatrix} \cdot vol_n(K).\end{aligned}$$

- More generally, for $i + j \leq n$ we consider the product $V_i \cdot V_j$. Take any $i+j$-dimensional subspace $W \subseteq V$. By the previous lemma and Remark 71 we have

$$r_W(V_i \cdot V_j) = r_W(V_i) \cdot r_W(V_j) = V_i \cdot V_j = \begin{bmatrix} i+j \\ i \end{bmatrix} \cdot vol_{i+j}(\cdot).$$

Therefore, the Klain function of $V_i \cdot V_j$ is the constant function $\begin{bmatrix} i+j \\ i \end{bmatrix}$ and from Remark 71 we deduce

$$V_i \cdot V_j = \begin{bmatrix} i+j \\ i \end{bmatrix} \cdot V_{i+j}.$$

3.4 The Poincaré duality

In this section, we introduce the Poincaré duality (cf. [4]). Let us consider the topological dual Val^* of Val. Endowed with the operator norm, this space is again a Banach space. Furthermore, there is a natural $GL(V)$-action on it given by

$$g \cdot l(\phi) = l(g^{-1} \cdot \phi) \quad g \in GL(V), \;\; \phi \in Val.$$

We may therefore consider the subspace $(Val^*)^\infty$ of smooth vectors. As it will turn out, the multiplication induces an isomorphism between Val^∞ and $(Val^*)^\infty$ - the so called Poincaré duality. Let $\phi, \psi \in Val^\infty$ and let $\langle \phi, \psi \rangle \in Val_n$ denote the degree n component of the product $(\phi \cdot \psi)_n$. This defines a bilinear map

$$\langle \cdot, \cdot \rangle : Val^\infty \times Val^\infty \to Val_n.$$

Since $\langle \phi, \cdot \rangle \in Val^* \otimes Val_n$, we obtain a linear map

$$Val^\infty \to Val^* \otimes Val_n.$$

Since the product is continuous in each argument, this map is also continuous. Using Lemma 29, we ultimately obtain a continuous linear map

$$Val^\infty \to (Val^*)^\infty \otimes Val_n.$$

Definition 98. *The Poincaré duality*

$$p : Val^\infty \to (Val^*)^\infty \otimes Val_n$$

is defined by

$$[p(\phi)](\psi) := \langle \phi, \psi \rangle.$$

Let $g \in GL(V)$. Then we have

$$[p(g \cdot \phi)](\psi) = ((g \cdot \phi) \cdot \psi)_n = g \cdot (\phi \cdot (g^{-1} \cdot \psi))_n.$$

Thus p is $GL(V)$-equivariant.

Remark 99. Since we have fixed a scalar product on V, we may view p as a map

$$p : Val^\infty \to (Val^\infty)^*$$

which is than no longer $GL(V)$ but merely $O(n)$-invariant.

Before we can prove that p is indeed an isomorphism, we need one more lemma.

Lemma 100. *The multiplication*

$$\cdot : Val_1^{-,\infty} \times Val_1^{-,\infty} \to Val_2^{+,\infty}$$

has dense image.

Proof. Suppose that this multiplication is not trivial. By the equivariance of the multiplication, the image is $GL(V)$-invariant. The Irreducibility Theorem therefore implies that it is dense. Hence, it suffices to show that the multiplication is not trivial. We start by showing that we may assume $n = 2$. Let us fix a 2-dimensional subspace $W \subseteq V$. The restriction map $r_W : Val(V) \to Val(W)$ is non-zero and the image is $GL(W)$-invariant. Hence, by the Irreducibility Theorem, the image of this restriction is a dense subspace. Since the restriction commutes with the product, it is sufficient to show that the multiplication

$$Val(W)_1^{-,\infty} \times Val(W)_1^{-,\infty} \to Val_2(W)$$

is non-zero. By the previous lemma, we have for $A, B \in \mathcal{K}(W)$,

$$V(K, A) \cdot V(K, B) = c \cdot V(A, -B) \cdot vol_2(K),$$

where $c \neq 0$. For arbitrary $A, B \in \mathcal{K}(W)$, the valuations $\phi(K) := V(K, A) - V(K, -A)$, $\psi(K) := V(K, B) - V(K, -B)$ are odd. The product of these valuations is

$$(\phi \cdot \psi)(K) = 2c \cdot (V(A, B) - V(A, -B)) \cdot vol_2(K).$$

By the definition of mixed volumes, we have $V(A, B) - V(A, -B) = \frac{1}{2}[V(A+B) - V(A+(-B))]$. If we choose A and B so that $V(A + B) \neq V(A + (-B))$, then the above product does not vanish. A possible choice for instance would be $A = B = \{x \in \mathbb{R}^n : x_i \geq 0, \ \sum_{i=1}^n x_i \leq 1\}$. □

We now prove the main theorem of this section.

Theorem 101. *The map $p : Val^\infty \to (Val^*)^\infty \otimes Val_n$ is an isomorphism.*

Proof. We first observe that the multiplication

$$Val_i^{+,\infty} \times Val_{n-i}^{-,\infty} \to Val_n$$

is trivial, since the product is odd and vol_n is even. Therefore, the map p decomposes into the restrictions

$$p : Val_i^{\epsilon,\infty} \to ((Val_{n-i}^{\epsilon})^*)^\infty \otimes Val_n, \quad 0 \leq i \leq n, \ \ \epsilon = +, -.$$

Since p is $GL(V)$-equivariant, both kernel and image of these maps are also $GL(V)$-invariant. The kernel is a closed set, thus by the Irreducibility Theorem it has to be either trivial or the whole space. Assume that the corresponding products

$$Val_i^{\epsilon,\infty} \times Val_{n-i}^{\epsilon,\infty} \to Val_n$$

are non trivial. Then p is injective. Furthermore, by another application of the Irreducibility Theorem, we see that the image is a dense subspace. The Casselman-Wallach theorem then shows that the map is surjective. It thus remains to show, that the products are non trivial. In the even case this follows immediately from $(V_i \cdot V_{n-i})(K) = \begin{bmatrix} n \\ i \end{bmatrix} \cdot vol_n(K) \neq 0$, (c.f. Example 97).

Let us consider the odd case. We want to show, that the multiplication

$$Val_i^{-,\infty} \times Val_{n-i}^{-,\infty} \to Val_n$$

is non-zero. Clearly, it suffices to show that the composition

$$\begin{array}{ccc}
Val_1^{-,\infty} \times Val_{i-1}^{+,\infty} & \times & Val_1^{-,\infty} \times Val_{n-i+1}^{+,\infty} \\
\Big\downarrow \cdot & & \Big\downarrow \cdot \\
Val_i^{-,\infty} & \times & Val_{n-i}^{-,\infty} \\
& \Big\downarrow \cdot & \\
& Val_n &
\end{array}$$

is non trivial. Since the multiplication is commutative and associative, it is sufficient to show that the multiplication

$$Val_1^{-,\infty} \times Val_1^{-,\infty} \times Val_{i-1}^{+,\infty} \times Val_{n-i+1}^{+,\infty} \to Val_n$$

is non trivial. Consider the following diagram:

$$\begin{array}{ccc}
Val_1^{-,\infty} \times Val_1^{-,\infty} & \times & Val_{i-1}^{+,\infty} \times Val_{n-i+1}^{+,\infty} \\
\Big\downarrow \cdot & & \Big\downarrow \cdot \\
Val_2^{+,\infty} & \times & Val_{n-2}^{+,\infty} \\
& \Big\downarrow \cdot & \\
& Val_n &
\end{array}$$

The two multiplications on the first level have dense images. The left one has dense image by the previous lemma. For the right one we remark that $V_{i-1} \cdot V_{n-i+1} = C \cdot V_{n-2} \neq 0$ (cf. Example 97). Hence, the Irreducibility Theorem yields density. The multiplication on the second level is not trivial, as we already proved the statement for the even case. Therefore, the composition is non trivial.

□

3.5 Product of even valuations

We want to describe the multiplication of even valuations in terms of Crofton measures. It is easily seen how this works for Crofton measures with respect to affine Grassmannians (cf. [8]). We then use this to show how the multiplication works for "mean-projection Crofton measures".

Lemma 102. *Let $\phi \in Val_k^{+,\infty}$ with a Crofton measure m_ϕ and $\psi \in Val^\infty$. Then*

$$\phi \cdot \psi(K) = \int_{\overline{Gr_k}} \psi(K \cap E) d\overline{m_\phi(E)}.$$

Proof. Both sides of the equation are continuous and linear in ϕ. We may therefore assume that $\psi = \mu_A$ for $A \in \mathcal{K}^\infty(V)$. By (16), we have

$$\begin{aligned}\phi \cdot \psi(K) &= \int_V \phi(K \cap (x - A))dx = \int_V \int_{\overline{Gr_k}} \chi(K \cap (x - A) \cap E) d\overline{m_\phi} dx \\ &= \int_{\overline{Gr_k}} \int_V \chi(K \cap (x - A) \cap E) dx d\overline{m_\phi} = \int_{\overline{Gr_k}} \psi(K \cap E) d\overline{m_\phi}.\end{aligned}$$

□

Let $\phi \in Val_k^{+,\infty}$ and $\psi \in Val_l^{+,\infty}$ with respective Crofton measures $\overline{m_\phi}, \overline{m_\psi}$. By the previous lemma we have

$$\begin{aligned}\phi \cdot \psi(K) &= \int_{\overline{Gr_k}} \psi(K \cap E) d\overline{m_\phi(E)} \\ &= \int_{\overline{Gr_k}} \int_{\overline{Gr_l}} \chi(K \cap E \cap F) d\overline{m_\psi(F)} d\overline{m_\phi(E)}.\end{aligned}$$

We now have the means to prove the main theorem of this section.

Theorem 103. *Let $\phi \in Val_l^{+,\infty}, \psi \in Val_k^{+,\infty}$ and m_ϕ, m_ψ be their respective Crofton measures. Furthermore, let $\sigma : Gr_k \times Gr_l \setminus \Delta \to Gr_{k+l}$ denote the sum map $\sigma(E, F) = E + F$, where Δ is the null set of subspaces E, F, such that $dim(E + F) < k + l$. Then the pushforward $\sigma_*(|sin| m_\phi \times m_\psi) := m_{\phi \cdot \psi}$ is a Crofton measure for $\phi \cdot \psi$.*

Proof. As we have seen, the product can be written as

$$\begin{aligned}\phi \cdot \psi(K) &= \int_{\overline{Gr_k}} \int_{\overline{Gr_l}} \chi(K \cap E \cap F) d\overline{m_\psi(F)} d\overline{m_\phi(E)} \\ &= \int_{Gr_k} \int_{Gr_l} \int_E \int_F \chi(K \cap (E^\perp + p) \cap (F^\perp + q)) \; dq \; dp \; dm_\psi(F) \; dm_\phi(E) \\ &= \int_{Gr_k} \int_{Gr_l} \int_E vol_l((K \cap (E^\perp + p))|F) \; dp \; dm_\psi(F) \; dm_\phi(E).\end{aligned}$$

The assumption $\dim(E + F) = k + l$ yields $E^\perp = F' + (E + F)^\perp$ for an l-dimensional subspace F' with $E + F' = E + F$. We therefore have

$$\int_E vol_l((K \cap (E^\perp + p))|F) \; dp = \int_E vol_l((K \cap (F' + (E + F')^\perp + p))|F) \; dp.$$

By Lemma 17 and 19, we further obtain

$$\begin{aligned}&\int_E vol_l((K \cap (F' + (E + F')^\perp + p))|F) \; dp \\ &\quad = \int_E vol_l([(K \cap (F' + (E + F')^\perp + p))|F']|F) \; dp \\ &\quad = |\cos(F, F')| \cdot \int_E vol_l((K \cap (F' + (E + F')^\perp + p))|F') \; dp \\ &\quad = |\cos(F, E^\perp)| vol_{k+l}(K|E + F') = |\sin(F, E)| vol_{k+l}(K|E + F).\end{aligned}$$

So we conclude that

$$\phi \cdot \psi(K) = \int_{Gr_k} \int_{Gr_l} |\sin(E, F)| vol_{k+l}(K|E + F) dm_\psi(F) dm_\phi(E).$$

□

The next lemma is an easy consequence of the last theorem.

Lemma 104. *Let $\phi, \psi \in Val_k^{+,\infty}$. Then*

$$[p(\phi)](\mathbb{F}\psi) = \phi \cdot \mathbb{F}\psi = \int_{Gr_k}\int_{Gr_k} |\cos(E,F)| dm_\phi(E) dm_\psi(F) \cdot vol_n = \int_{Gr_k} Kl_\phi(F) dm_\psi(F) \cdot vol_n.$$

In particular,

$$\mathbb{F}^* \circ p = p \circ \mathbb{F}.$$

Proof. We have

$$\begin{aligned}
\phi \cdot \mathbb{F}\psi &= \int_{Gr_{n-k}}\int_{Gr_k} |\sin(E,F)| dm_\phi(E) d(\perp^* m_\psi)(F) \cdot vol_n \\
&= \int_{Gr_{n-k}}\int_{Gr_k} |\sin(E,F)| dm_\phi(E) dm_\psi(F^\perp) \cdot vol_n \\
&= \int_{Gr_k}\int_{Gr_k} |\sin(E,F^\perp)| dm_\phi(E) dm_\psi(F) \cdot vol_n \\
&= \int_{Gr_k}\int_{Gr_k} |\cos(E,F)| dm_\phi(E) dm_\psi(F) \cdot vol_n = \int_{Gr_k} Kl_\phi(F) dm_\psi(F) \cdot vol_n.
\end{aligned}$$

□

4 Convolution

As mentioned earlier, there exists another bilinear operator on the space of smooth valuations that relates to the multiplication via the Alesker-Fourier transform. It is obvious how to define this operation on even valuations (namely by $\phi * \psi = \mathbb{F}(\mathbb{F}\phi \cdot \mathbb{F}\psi)$). In the first section, we will describe the convolution on even valuations in geometric terms. In Section 2 the convolution will be extended to the entire space of smooth valutations using the characterization by differential forms. As usual, let V be a vector space of finite dimension n with a fixed scalar product $\langle \cdot, \cdot \rangle$. The presented results are taken from [16].

4.1 The convolution in the even case

Definition 105. *The convolution* $* : Val^{+,\infty} \times Val^{+,\infty} \to Val^{+,\infty}$ *is defined by*

$$\phi * \psi = \mathbb{F}(\mathbb{F}\phi \cdot \mathbb{F}\psi), \quad \phi, \psi \in Val^{+,\infty}.$$

Theorem 106. *If* $A, B \in \mathcal{K}^\infty(V)$ *are antipodally symmetric, i.e.* $\mu_A, \mu_B \in Val^{+,\infty}$*. Then*

$$\mathbb{F}(\mathbb{F}\mu_A \cdot \mathbb{F}\mu_B) = \mu_{A+B}. \tag{21}$$

Proof. Since $\mathbb{F} \circ \mathbb{F} = id$, it suffices to prove the equivalent equation

$$(\mathbb{F}\mu_A \cdot \mathbb{F}\mu_B) = \mathbb{F}\mu_{A+B}.$$

We will show that this formula holds for the degree n component and use that together with Corollary 81 to obtain the equality of the Klain functions of both sides. Let $\mu_A = \sum_{k=0}^n \mu_{A,k}$ and $\mu_B = \sum_{k=0}^n \mu_{B,k}$ be the decompositions by degree of homogeneity, and let $m_{B,k}$ be a smooth Crofton measure for $\mu_{B,k}$. By Lemma 104,

$$\mathbb{F}\mu_{A,n-k} \cdot \mathbb{F}\mu_{B,k} = \int_{Gr_k(V)} Kl_{\mathbb{F}\mu_{A,n-k}}(L) dm_{B,k}(L) \cdot vol_n = \int_{Gr_k(V)} Kl_{\mu_{A,n-k}}(L^\perp) dm_{B,k}(L) \cdot vol_n.$$

Applying Lemma 72 this can be written as

$$\int_{Gr_k(V)} V_k(p_L(A)) dm_{B,k}(L) \cdot vol_n = \mu_{B,k}(A) \cdot vol_n.$$

Hence, we obtain

$$(\mathbb{F}\mu_A \cdot \mathbb{F}\mu_B)_n = \sum_{k=0}^n \mu_{B,k}(A) \cdot vol_n = \mu_B(A) \cdot vol_n = vol_n(A+B) \cdot vol_n = (\mathbb{F}\mu_{A+B})_n. \tag{22}$$

Now let $W \subseteq V$ be an m-dimensional subspace. Using Corollary 81 and the fact that the product commutes with restrictions, we obtain

$$r_W(\mathbb{F}\mu_A \cdot \mathbb{F}\mu_B) = r_W(\mathbb{F}\mu_A) \cdot r_W(\mathbb{F}\mu_B) = \mathbb{F}_W(\mu_{p_W A}) \cdot \mathbb{F}_W(\mu_{p_W B}).$$

Thus, (22) and Corollary 81 yield

$$(r_W(\mathbb{F}\mu_A \cdot \mathbb{F}\mu_B))_m = (\mathbb{F}_W \mu_{p_W}(A+B))_m = (r_W(\mathbb{F}\mu_{A+B}))_m$$

for the degree m component of the restriction of the product. Since this holds for all $W \in Gr_m(V)$, the Klain functions of $(\mathbb{F}\mu_A \cdot \mathbb{F}\mu_B)_m$ and $(\mathbb{F}\mu_{A+B})_m$ coincide. It follows, that $(\mathbb{F}\mu_A \cdot \mathbb{F}\mu_B) = \mathbb{F}\mu_{A+B}$. □

4.2 Convolution in general

In order to extend the convolution to all of Val^∞, we will define a product on Ω_B^V which in turn induces a product on Val^∞ coinciding with the convolution for even valuations. First, we need the following operator.

Definition 107. *The operator $*_1 : \Lambda^k V \wedge \Lambda X \to \Lambda^{n-k} V \wedge \Lambda X$ is defined by*

$$*_1(\omega_1 \wedge \omega_2) = (-1)^{\binom{n-k}{2}} (*\omega_1) \wedge \omega_2$$

for $\omega_1 \in \Lambda^k V, \omega_2 \in \Lambda X$.

Lemma 108. *The operator $*_1$ has the following properties.*

1. *$*_1$ is an involution.*
2. *If ω is a translation invariant form of bi-degree (k,l), then $*_1\omega$ is a translation invariant form of bi-degree $(n-k,l)$.*
3. *If ω is vertical, then $*_1\omega$ is tangential. Conversely, if ω is tangential, then $*_1\omega$ is vertical.*
4. *Let $\omega_1 \in \Lambda^k V$ and $\omega_2 \in \Lambda^l X$. Then*

$$*_1(\omega_2 \wedge \omega_1) = (-1)^{nl} \omega_2 \wedge *_1\omega_1.$$

5. *On translation invariant forms we have $d \circ *_1 = (-1)^n *_1 \circ d$.*
6. *For all forms on SV, we have $*_1 \circ p^* = p^* \circ *_1$.*

Proof. The first two assertions are trivial. Claim 3 easily follows from Remark 35. Claim 4 is an easy computation:

$$\begin{aligned} *_1(\omega_2 \wedge \omega_1) &= (-1)^{kl} *_1 (\omega_1 \wedge \omega_2) = (-1)^{kl} *_1 \omega_1 \wedge \omega_2 \\ &= (-1)^{kl}(-1)^{l(n-k)} \omega_2 \wedge *_1\omega_1 = (-1)^{nl} \omega_2 \wedge *_1\omega_1. \end{aligned}$$

To prove Claim 5, let ω be a translation invariant form on $V \times X$. We may assume that $\omega = \omega_1 \wedge \omega_2,\ \omega_1 \in \Lambda^k V, \omega_2 \in \Lambda^l X$ since d and $*_1$ are linear operators. Since ω is translation invariant, we have $d\omega_1 = 0$ and also $d(*\omega_1) = 0$. Now

$$\begin{aligned} d(*_1(\omega_1 \wedge \omega_2)) &= (-1)^{\binom{n-k}{2}} d(*\omega_1 \wedge \omega_2) = (-1)^{\binom{n-k}{2}+n-k} (*\omega_1) \wedge d\omega_2 \\ &= (-1)^{\binom{n-k}{2}+n+k} (*\omega_1) \wedge d\omega_2 = (-1)^{n+k} *_1 (\omega_1 \wedge d\omega_2) \\ &= (-1)^n *_1 (d(\omega_1 \wedge \omega_2)). \end{aligned}$$

Claim 6: Again, because both operators are linear it suffices to prove the claim for $\omega = \omega_1 \wedge \omega_2,\ \omega_1 \in \Lambda^k V, \omega_2 \in \Lambda^l S(V)$. We have

$$\begin{aligned} *_1(p^*(\omega_1 \wedge \omega_2)) &= *_1(\omega_1 \wedge p^*\omega_2) = (-1)^{\binom{n-k}{2}} (*\omega_1) \wedge p^*\omega_2 \\ &= p^*((-1)^{\binom{n-k}{2}} (*\omega_1) \wedge \omega_2) = p^*(*_1(\omega_1 \wedge \omega_2)). \end{aligned}$$

□

We define the following operations on Ω^V and Ω^V_B, respectively.

Definition 109. *Let $\beta \in \Omega^V_k$, $\gamma \in \Omega^V_l$ and $\theta \in \Omega^V_{B,k}$, $\theta' \in \Omega^V_{B,l}$. Then*

- $\beta * \gamma := (2n-k-l)^{-1} *_1^{-1} \left((n-k) *_1 \beta \wedge *_1 d\gamma + (n-l) *_1 \gamma \wedge *_1 d\beta\right)$.
- $\theta *_B \theta' := *_1^{-1}(*_1\theta \wedge *_1\theta')$ *. Moreover,* $(p_1^* vol_V) *_B \theta := \theta$.

Recall that, by Theorem 63, there is a surjective map $\nu : \Omega^V_k \to Val_k^\infty$.

Theorem 110. *Let $\beta \in \Omega^V_k$ and $\gamma \in \Omega^V_l$. Then*

1. $\beta * \gamma \cong *_1^{-1}(*_1\beta \wedge *_1 d\gamma) \mod \ker \nu$.
2. $\beta * \gamma \in \Omega^V_{k+l-n}$.
3. $\beta_B *_B \gamma_B = (\beta * \gamma)_B$. *In particular,* $\theta *_B \theta' \in \Omega^V_{B,k+l-n}$ *for* $\theta \in \Omega^V_{B,k}$, $\theta' \in \Omega^V_{B,l}$.
4. $*$ *and* $*_B$ *extend to continuous, commutative and associative bilinear products of degree* $-n$ *on* Ω^V *and* Ω^V_B*, respectively.*
5. *If* $d\beta = 0$*, then* $d(\beta * \gamma) = 0$ *for all* $\beta, \gamma \in \Omega^V$.

Proof. In order to prove Claim 1, we use Lemma 108, to obtain

$$\beta * \gamma = *_1^{-1}(*_1\beta \wedge *_1 d\gamma) + d\left(\frac{n-l}{2n-k-l} *_1^{-1} (*_1\beta \wedge *_1\gamma)\right).$$

Claim 2: Clearly, $\beta * \gamma$ has the correct bi-degree and is translation invariant. From Claim 1, it is easy to see that

$$d(\beta * \gamma) = *_1^{-1}(*_1 d\beta \wedge *_1 d\gamma).$$

Now Lemma 108 immediately yields that $d(\beta * \gamma)$ is vertical.

Claim 3: For the following computation, we will use Lemma 108 several times. We start by computing the left-hand side:

$$\begin{aligned}
\beta_B *_B \gamma_B &= *_1^{-1} \left[*_1 d(\tilde{r}^{n-k} p^*\beta) \wedge *_1 d(\widetilde{r^{n-l}} p^*\gamma)\right] \\
&= *_1^{-1} \left[*_1(d(\tilde{r}^{n-k}) \wedge p^*\beta + \tilde{r}^{n-k} p^* d\beta) \wedge *_1(d(\widetilde{r^{n-l}}) \wedge p^*\gamma + \widetilde{r^{n-l}} p^* d\gamma)\right] \\
&= \underbrace{*_1^{-1}[*_1(d(\tilde{r}^{n-k}) \wedge p^*\beta) \wedge *_1(d(\widetilde{r^{n-l}}) \wedge p^*\gamma)]}_{:=A} + \underbrace{*_1^{-1}[*_1(d(\tilde{r}^{n-k}) \wedge p^*\beta) \wedge *_1(\widetilde{r^{n-l}} p^* d\gamma)]}_{:=B} \\
&\quad + \underbrace{*_1^{-1}[*_1(\tilde{r}^{n-k} p^* d\beta) \wedge *_1(d(\widetilde{r^{n-l}}) \wedge p^*\gamma)]}_{:=C} + \underbrace{*_1^{-1}[*_1(\tilde{r}^{n-k} p^* d\beta) \wedge *_1(\widetilde{r^{n-l}} p^* d\gamma)]}_{:=D}.
\end{aligned}$$

Note, that $A = 0$. Indeed, since $d(r^j) = j r^{j-1} dr \in \Lambda X$, we obtain

$$*_1(d(\tilde{r}^{n-k}) \wedge p^*\beta) \wedge *_1(d(\widetilde{r^{n-l}}) \wedge p^*\gamma) = (-1)^{2(n-k)-1)}(-1)^n(-1)^n d(\tilde{r}^{n-k}) \wedge d(\widetilde{r^{n-l}}) \wedge *_1 p^*\beta \wedge *_1 p^*\gamma = 0.$$

Now we compute the right-hand side:

$$\begin{aligned}
(\beta * \gamma)_B =& d(r^{2n-k-l} p^*(\beta * \gamma)) = d(r^{2n-k-l}) \wedge p^*(\beta * \gamma) + r^{2n-k-l} p^* d(\beta * \gamma) \\
&= \underbrace{d(r^{2n-k-l}) \wedge p^*[(2n-k-l)^{-1} *_1^{-1} ((n-k) *_1 \beta \wedge *_1 d\gamma + (n-l) *_1 \gamma \wedge *_1 d\beta)]}_{:=E} \\
&\quad + r^{2n-k-l} p^* *_1^{-1} [*_1 d\beta \wedge *_1 d\gamma] = E + D.
\end{aligned}$$

It remains to show, that $E = B + C$. To this end we compute

$$\begin{aligned}B + C =&(-1)^{n+n}(n-k)r^{2n-k-l-1}dr \wedge p^* *_1^{-1} [*_1\beta \wedge *_1 d\gamma] \\ &+ (-1)^{n+n+2(n-k)}(n-l)r^{2n-k-l-1}dr \wedge p^* *_1^{-1} [*_1 d\beta \wedge *_1\gamma] \\ =&r^{2n-k-l-1}dr \wedge p^* *_1^{-1} [(n-k) *_1 \beta \wedge *_1 d\gamma + (-1)^{2(n-k)(2(n-l)-1)}(n-l) *_1 d\beta \wedge *_1\gamma] \\ =&E.\end{aligned}$$

Claim 4: The products are obviously of degree $-n$ and continuous with respect to the C^∞-topology on the respective spaces of forms. By Claim 3, it suffices to prove commutativity and associativity for $*_B$. Commutativity immediately follows from the fact that $*_1\theta$ is of even degree for all $\theta \in \Omega_B^V$. For the associativity we compute,

$$(\theta *_B \theta') *_B \theta'' = *_1^{-1}(*_1\theta \wedge *_1\theta' \wedge *_1\theta'') = \theta *_B (\theta' *_B \theta'').$$

Claim 5 is immediate from Claim 1.

□

Remark 111. The last assertion shows, that $*_B$ induces a commutative and associative product bilinear product of degree $-n$ on Val^∞.

We will now construct certain differential forms in Ω_B^V that represent valuations μ_A. For $A \in \mathcal{K}^\infty(V)$, set

$$\eta_A(y) := \begin{cases} r\nabla h_A(y), & y \neq 0, \\ 0, & y = 0. \end{cases}$$

We define a Lipschitz map $G_A : TV \to V$ by

$$G_A(x,y) := x + \eta_A(y).$$

In view of Lemma 7, it is not surprising that

$$G_A(\overrightarrow{N}_B(K)) = K + A.$$

Indeed, since $A \in \mathcal{K}^\infty(V)$, we have

$$\overrightarrow{N}_B(K) = \{(x, \lambda n_x) : x \in \partial K, 0 \leq \lambda \leq 1\} \cup K,$$

where n_x is the outer unit normal at $x \in \partial K$. Also, by Lemma 7,

$$\langle y, \nabla h_A(y)\rangle = h_A(y).$$

This yields

$$\langle u, x + \eta(\lambda n_x)\rangle = \langle u, x\rangle + \lambda\langle u, \nabla h_A(n_x)\rangle \leq h_K(u) + \lambda h_A(u) \leq h_{K+A}(u).$$

As equality holds for $\lambda = 1$ and $x \in \partial K$ such that $u = n_x$, we conclude that $G_A(\overrightarrow{N}_B(K)) = K + A$.

Consequently,

$$\theta_A := G_A^*(dvol_n) = \bigwedge_{i=1}^n (dx_i + d(\eta_A)_i)$$

is the desired form representing μ_A. However, we have to check that it actually is in Ω_B^V.

Lemma 112. *Let $A \in \mathcal{K}^\infty(V)$. Then $\theta_A \in \Omega_B^V$ and*

$$\nu_B(\theta_A) = \mu_A.$$

Proof. For simplicity we set $\eta_A = \eta$. Clearly, θ_A is closed. Moreover, decomposing θ_A by bi-degree, it is easy to see, that every component is of the correct homogeneity. We now show, that $\theta_A|_{SV}$ is vertical. Note, that

$$d\eta_i = \sum_{j=1}^n \left[\frac{y_j}{|y|}\partial^i h_A(y) + |y|\partial^{ij} h_A(y)\right] dy_j.$$

Using Euler's rule for the differentiation of homogeneous functions, we obtain

$$\begin{aligned}\sum_{i=1}^n y_i d\eta_i\Big|_{SV} &= \sum_{j=1}^n \left[y_j \sum_{i=1}^n y_i \partial^i h_A(y) + \partial^j\left(\sum_{i=1}^n y_i\partial^i h_A(y)\right) - \partial^j h_A(y)\right] dy_j\Big|_{SV} \\ &= h_A(y)\left(\sum_{j=1}^n y_j dy_j\right)\Big|_{SV} = 0.\end{aligned}$$

Hence, we compute

$$\begin{aligned}\alpha \wedge \theta_A|_{SV} &= \left(\sum_{i=1}^n y_i dx_i\right) \wedge G_A^*(dvol_n)\Big|_{SV} \\ &= \left(\sum_{i=1}^n y_i(dx_i + d\eta_i)\right) \wedge G_A^*(dvol_n)\Big|_{SV} \\ &= \left(\sum_{i=1}^n y_i(G_A^* dx_i)\right) \wedge G_A^*(dvol_n)\Big|_{SV} \\ &= \sum_{i=1}^n y_i G_A^*(dx_i \wedge dvol_n)\Big|_{SV} = 0.\end{aligned}$$

By the considerations we made previous to this lemma, we obtain

$$\int_{\vec{N}_B(K)} \theta_A = \int_{G_A(\vec{N}_B(K))} dvol_n = \mu_A(K).$$

□

The next theorem shows, that the product on Val^∞ induced by $*_B$ coincides with the convolution for even valuations and hence extends the convolution to all of Val^∞

Theorem 113. *The convolution $* : Val^\infty \times Val^\infty \to Val^\infty$, given by*

$$\phi * \psi = \nu_B(\beta *_B \gamma)$$

for $\phi, \psi \in Val^\infty$ with $\beta, \gamma \in \Omega_B^V$ representing them, is well-defined. It is a bilinear, associative, commutative and continuous product of degree $-n$. Moreover, it satisfies

$$\mu_A * \mu_B = \mu_{A+B}, \quad A, B \in \mathcal{K}(V). \tag{23}$$

In particular, it extends the convolution of even smooth valuations.

Proof. From Theorem 110.5 it follows immediately that $*$ is well-defined. Clearly, $*$ is an associative, commutative bilinear product of degree $-n$. In order to show continuity let ϕ_j and ψ_j be sequences in Val^∞ converging to ϕ and ψ, respectively. Since ν_B is an open map (cf. Remark 65) we may now choose sequences $\beta_j, \gamma_j \in \Omega_B^V$ representing ϕ_j and γ_j converging to β, γ, respectively. By the continuity of the convolution on differential forms we have $\beta_j *_B \gamma_j \to \beta * \gamma$. Hence, by the continuity of ν_B, we obtain $\phi_j * \psi_j \to \phi * \psi$.

We will now prove (23). By the previous lemma, it suffices to show that

$$\theta_A *_B \theta_B = \theta_{A+B}.$$

First we note, that $\theta_A = \theta_A^{I,I^c} dx_I \wedge d\eta_{I^c}$, where $\theta_A^{I,I^c} = \sigma_{I,I^c}$ (where we are using Einstein's summation convention). Obviously,

$$(*_1\theta_A)^{I,I} = (-1)^{\binom{n-|I^c|}{2}} \sigma_{I^c,I} \theta_A^{I^c,I} = (-1)^{\binom{n-|I^c|}{2}}.$$

Consequently, we obtain

$$*_1\theta_A = \bigwedge_{i=1}^n (1 + \sigma_i dx_i \wedge d(\eta_A)_i).$$

Therefore,

$$\begin{aligned} *_1(\theta_A *_B \theta_B) &= *_1\theta_A \wedge *_1\theta_B \\ &= \bigwedge_{i=1}^n (1 + \sigma_i dx_i \wedge d(\eta_A)_i) \wedge (1 + \sigma_i dx_i \wedge d(\eta_B)_i) \\ &= \bigwedge_{i=1}^n (1 + \sigma_i dx_i \wedge (d(\eta_A + \eta_B)_i) = *_1\theta_{A+B}. \end{aligned}$$

□

Remark 114. By continuity, equation (23) extends to

$$\mu_A * \phi(K) = \phi(K + A), \quad \phi \in Val^\infty, \ A \in \mathcal{K}(V).$$

We will also give a characterization of the convolution in terms of mixed volumes.

Corollary 115. *Let $k + l \geq n$ and $A_1, \dots, A_{n-k}, B_1, \dots, B_{n-l} \in \mathcal{K}^\infty(V)$. Then*

$$V_{A_1,\dots,A_{n-k}} * V_{B_1,\dots,B_{n-l}} = \binom{k+l}{k}^{-1} \binom{k+l}{n} V_{A_1,\dots,A_{n-k},B_1,\dots,B_{n-l}}. \tag{24}$$

Proof. Using (9) and the continuity of the convolution, we obtain

$$\begin{aligned} V_{A_1,\dots,A_{n-k}} * V_{B_1,\dots,B_{n-l}} &= \frac{k!l!}{n!^2} \left.\frac{\partial^{n-k}}{\partial\lambda_1 \cdots \partial\lambda_{n-k}}\right|_{\lambda_j=0} \left.\frac{\partial^{n-l}}{\partial\mu_1 \cdots \partial\mu_{n-l}}\right|_{\mu_j=0} \mu_{\sum_{i=1}^{n-k} \lambda_i A_i + \sum_{j=1}^{n-l} \mu_j B_j} \\ &= \binom{k+l}{k}^{-1} \binom{k+l}{n} V_{A_1,\dots,A_{n-k},B_1,\dots,B_{n-l}}. \end{aligned}$$

□

4.3 The Alesker-Fourier transform

The transform defined in Chapter 2.6 has an extension to all of Val^∞. The following theorem was proven in [7].

Theorem 116. *There exists an isomorphism of topological vector spaces*

$$\mathbb{F}_V : Val^\infty(V) \to Val^\infty(V^*) \otimes Val_n(V)$$

which satisfies the following properties:

1. $\mathbb{F}_V$ *is* $GL(V)$*-equivariant.*

2. $\mathbb{F}_V$ *is an isomorphism of algebras when the domain is equipped with the product and the range with the convolution.*

3. *Consider the composition*

$$\epsilon_V := (\mathbb{F}_{V^*} \otimes id_{Val_n(V)}) \circ \mathbb{F}_V : Val^\infty \to Val^\infty.$$

   Then for all $\phi \in Val^\infty$ *and* $K \in \mathcal{K}(V)$,

$$(\epsilon_V \phi)(K) = \phi(-K).$$

5 The Fundamental Theorem of Algebraic Integral Geometry

In the last two chapters we have introduced several algebraic operations on smooth valuations. In this chapter we investigate their relation to kinematic formulas. It turns out, that roughly speaking the kinematic (additive kinematic) coproduct is the adjoint of the multiplication (convolution). This result is known as the Fundamental Theorem of Algebraic Integral Geometry. We follow [10, 18].

5.1 G-invariant valuations

Let $G \subseteq SO(V)$ be a closed subgroup. Since every such G is a Lie group, there exists a uniquely determined Haar measure on G, satisfying

$$\int_G dg = 1.$$

We set $\overline{G} := G \times V$. This space again is endowed with a Haar measure, namely the product of the Haar measure on G and the Lebesgue measure on V. By Val^G we will denote the subset of G-invariant valuations in Val. By Hadwigers Theorem, $Val^{SO(V)}$ is finite dimensional. A natural question is how "big" a subgroup G of $SO(V)$ has to be in order to have the property that Val^G is finite dimensional. The next theorem by Alesker (cf. [6]) answers this question.

Theorem 117. *Let $n \geq 2$. Then $\dim Val^G < \infty$ if G acts transitively on the unit sphere. If this is the case, then $Val^G \subseteq Val^\infty$.*

Proof. We start by showing, that $Val^G \cap Val^\infty$ is finite-dimensional. If $\psi \in (Val^G \cap Val^\infty)$, then there is a differential form $\omega \in \Omega^V$ such that

$$\psi(K) = c \cdot vol_n(K) + \int_{\overrightarrow{N}(K)} \omega.$$

Now consider a valuation $\phi \in (Val^G \cap Val^\infty)$. Then

$$\phi(K) = \int_G g \cdot \phi(K) dg = c \cdot vol_n(K) + \int_G \int_{\overrightarrow{N}(K)} \omega_g dg = c \cdot vol_n(K) + \int_{\overrightarrow{N}(K)} \int_G \omega_g dg.$$

Setting $\tilde{\omega} := \int_G \omega_g dg$ we see that

$$\phi(K) = c \cdot vol_n(K) + \int_{\overrightarrow{N}(K)} \tilde{\omega}.$$

Hence ϕ may be represented by a G-invariant differential form in Ω^V. However, any such form is uniquely determined by its value at a given point. We conclude, that $Val^G \cap Val^\infty$ is finite-dimensional. Now let ϕ be an arbitrary valuation in Val^G. As smooth vectors form a dense subspace, there is a sequence of smooth valuations $\phi_i \in Val^\infty$ which converges to ϕ. By the G-invariance of ϕ for every convex body $K \subseteq B_1(0)$, we obtain

$$\int_G g \cdot \phi_i(K) dg \to \int_G g \cdot \phi(K) dg = \phi(K)$$

so that there is also a sequence in $Val^G \cap Val^\infty$ that converges to ϕ. Consequently, we obtain

$$Val^G = \overline{Val^G \cap Val^\infty} = Val^G \cap Val^\infty,$$

where the last equality holds since $Val^G \cap Val^\infty$ is finite-dimensional. □

Remark 118. We also note that if Val^G is finite-dimensional, then G acts transitively on the sphere (cf. [17]).

There is a convenient way to obtain a basis of Val^G:

First note $MVol \cap Val^G = Val^G$. Indeed, $MVol \cap Val^G$ is dense in Val^G and since Val^G is finite dimensional it is the whole space. Let $b_1, \dots, b_m$ be an arbitrary basis of Val^G. Each b_i can be written as a finite linear combination of mixed volumes, say

$$b_i = \sum_{j=1}^{N_i} \mu_{A_j^i}.$$

Integrating over G yields

$$b_i(K) = \sum_{j=1}^{N_i} \int_G \mu_{gA_j^i} dg.$$

Hence, there exist convex bodies $A_1, \dots, A_m \in \mathcal{K}(V)$, such that

$$\mu_{A_i}^G := \int_G \mu_{g(-A_i)} dg$$

is a basis of Val^G.

We formulate this as a lemma:

Lemma 119. *Let G be a compact subgroup of $SO(V)$ which acts transitively on the sphere. Then there are convex bodies $A_1, \dots, A_m$ such that a basis of Val^G is given by the valuations*

$$\mu_{A_j}^G(K) = \int_G vol_n(K - gA_j) dg = \int_{\overline{G}} \chi(K \cap \overline{g}A_j) d\overline{g}, \quad j = 1, \dots, m.$$

Proof. We still need to prove the second equality. However, this follows from (17),

$$\begin{aligned} \int_{\overline{G}} \chi(K \cap \overline{g}A_j) d\overline{g} &= \int_G \int_V \chi(K \cap (x + gA_j)) dx dg \\ &= \int_G \chi \cdot \mu_{-A_j}(K) dg = \mu_{A_j}^G. \end{aligned}$$

□

5.2 Kimematic formulas

Let us consider a compact subgroup $G \subseteq SO(V)$ which acts transitively on the sphere. By the previous theorem there is a basis $\phi_1, \dots, \phi_m$ of Val^G. Let $1 \le i \le m$. We are interested in the expressions

$$\int_{\overline{G}} \phi_i(K \cap \overline{g}L) d\overline{g}, \quad K, L \in \mathcal{K}(V).$$

Note that

$$K \mapsto \int_{\overline{G}} \phi_i(K \cap \overline{g}L) d\overline{g}, \quad L \mapsto \int_{\bar{G}} \phi_i(K \cap \overline{g}L) d\overline{g}$$

are elements of Val^G. Therefore, we have

$$\int_{\overline{G}} \phi_i(K \cap \overline{g}L) d\overline{g} = \sum_{k=1}^{m} \lambda_k(L) \phi_k(K) = \sum_{k,l=0}^{m} c_{k,l}^i \phi_l(L) \phi_k(K). \tag{25}$$

These formulas are called kinematic formulas. In a similar way, we obtain additive kinematic formulas. These are of the form

$$\int_G \phi_i(K+gL)dg = \sum_{k,l=1}^{m} a^i_{k,l}\phi_l(L)\phi_k(K). \tag{26}$$

It is easy to see, that $c^i_{k,l} = c^i_{l,k}$ and $a^i_{k,l} = a^i_{l,k}$. Indeed this follows from

$$\int_{\overline{G}} \phi_i(K\cap \overline{g}L)d\overline{g} = \int_{\overline{G}} \phi_i(\overline{g}^{-1}K\cap L)d\overline{g} = \int_{\overline{G}} \phi_i(L\cap \overline{g}K)d\overline{g}$$

and the respective equation for the additive kinematic formulas.

Example 120. Let $G = SO(V)$. By explicitly calculating the right hand side of (25) for balls of different radii, we obtain

$$c^i_{k,l} = \begin{cases} \begin{bmatrix} n+i \\ i \end{bmatrix} \begin{bmatrix} n+i \\ k \end{bmatrix}^{-1}, & k+l = n+i, \\ 0, & k+l \neq n+i. \end{cases} .$$

Thus the kinematic formulas for $SO(V)$ are explicitly given by

$$\int_{\overline{SO(V)}} V_i(K\cap \overline{g}L)d\overline{g} = \sum_{k=0}^{n} \begin{bmatrix} n+i \\ i \end{bmatrix} \begin{bmatrix} n+i \\ k \end{bmatrix}^{-1} V_{n-k}(L)V_k(K).$$

In a similar manner we obtain

$$\int_{SO(V)} V_i(K+gL)dg = \sum_{k=0}^{n} \begin{bmatrix} 2n-i \\ n-i \end{bmatrix} \begin{bmatrix} 2n-i \\ n-k \end{bmatrix}^{-1} V_k(K)V_{n-k}(L).$$

The kinemtic formulas, can be encoded in the so called kinematic-operators:

Definition 121. *Let G be a closed subgroup of $SO(V)$ acting transitively on $S(V)$.*

1. *The kinematic operator $k_G : Val^G \to Val^G \otimes Val^G$ is defined by*

$$\phi_i \mapsto \sum_{k,l}^{m} c^i_{k,l}\phi_k \otimes \phi_l.$$

2. *The additive kinematic operator $a_G : Val^G \to Val^G \otimes Val^G$ is defined by*

$$\phi_i \mapsto \sum_{k,l}^{m} a^i_{k,l}\phi_k \otimes \phi_l.$$

Lemma 122. *The operators k_G and a_G are co-commutative, co-associative co-products on Val^G.*

Proof. Let $\iota : Val^G \otimes Val^G \to Val^G \otimes Val^G;\ (\phi,\psi) \mapsto (\psi,\phi)$. k_G is co-commutative if the following diagram commutes:

$$\begin{CD} Val^G @>k_G>> Val^G \otimes Val^G \\ @V{id}VV @VV{\iota}V \\ Val^G @>k_G>> Val^G \otimes Val^G. \end{CD}$$

Obviously, this just means that the coefficients satisfy $c^i_{k,l} = c^i_{l,k}$ which we already know. The co-associativity property is the commutativity of the following diagram:

$$\begin{CD} Val^G @>(k_G \otimes id) \circ k_G>> Val^G \otimes Val^G \otimes Val^G \\ @V{id}VV @VV{id \otimes id \otimes id}V \\ Val^G @>(id \otimes k_G) \circ k_G>> Val^G \otimes Val^G \otimes Val^G \end{CD}.$$

Let $K, L, M \in \mathcal{K}(V)$ and $\phi \in Val^G$. Then

$$(id \otimes k_G)(k_G(\phi))[K, L, M] = \int_{\overline{G}} \int_{\overline{G}} \phi(K \cap \bar{g}L \cap \bar{h}M) d\bar{g} d\bar{h} = (k_G \otimes id)(k_G(\phi))[K, L, M].$$

The case of the additive kinematic operator can be proved in a similar way.

□

Remark 123. It is easy to see that

$$\mu^G_{A_j}(K) = k_G(\chi)[K, A_j].$$

5.3 The Fundamental Theorem

In this final section, we relate the kinematic operators to the algebraic structures we introduced in the previous chapters. In particular, we will discuss the Fundamental Theorem of Algebraic Integral Geometry (FTAIG). First we need to consider the restrictions of those operations to a subspace Val^G, where G is a closed subgroup of $SO(V)$ which acts transitively on the unit sphere.

Let $\phi \in Val^G$. We have

$$(g \cdot p(\phi))[\psi] = \langle \phi, g^{-1} \cdot \psi \rangle = \langle g^{-1} \cdot \phi, g^{-1} \cdot \psi \rangle = g \cdot \langle \phi, \psi \rangle = p(\phi)[\psi].$$

Therefore, the Poincaré duality restricts to

$$p_G : Val^G \to Val^{G*}.$$

Since $\phi \cdot \psi,\ \phi * \psi \in Val^G$ for all $\phi,\ \psi \in Val^G$, the multiplication and convolution induce maps

$$m_G : Val^G \otimes Val^G \to Val^G,$$
$$c_G : Val^G \otimes Val^G \to Val^G,$$

respectively.

Recall, that $V \otimes V \cong Hom(V^*, V)$ for any vector space V. We may therefore view the kinematic operators evaluated at some $\phi \in Val^G$ as homomorphisms $k_G(\phi) : Val^{G*} \to Val^G$ and $a_G(\phi) : Val^{G*} \to Val^G$.

Remark 124. Using bilinearity and continuity of the product, we obtain

$$\phi \cdot \mu^G_{A_j}(K) = \int_{\overline{G}} \phi(K \cap \overline{g}A_j) d\overline{g}, \quad \phi \in Val^G.$$

Together with Remark 123 this immediately yields

$$k_G(\phi) = k_G(\phi \cdot \chi) = \phi \otimes \chi \cdot k_G(\chi).$$

Indeed the last equation holds for arbitrary $\psi \in Val^G$ instead of χ.

Lemma 125. *If $\phi, \psi \in Val^G$, then*

$$k_G(\phi \cdot \psi) = (\phi \otimes \chi) \cdot k_G(\psi).$$

Moreover, a similar formula holds for the additive kinematic operator and the convolution:

$$a_G(\phi * \psi) = (\psi \otimes vol_n) * a_G(\phi).$$

Proof. We may assume that $\phi = \mu^G_{A_j}$. Then

$$\begin{aligned} k_G(\mu^G_{A_j} \cdot \psi)[K, L] &= \int_{\overline{G}} \int_{\overline{G}} \psi(K \cap \overline{g}L \cap \overline{h}A_j) d\overline{h} d\overline{g} \\ &= \int_{\overline{G}} \int_{\overline{G}} \psi(K \cap \overline{g}L \cap \overline{h}A_j) d\overline{g} d\overline{h} = \int_{\overline{G}} k_G(\psi)[K \cap \overline{h}A_j, L] d\overline{h} \\ &= \left[k_G(\psi)[\cdot, L] \cdot \int_{\overline{G}} \chi[\cdot \cap \overline{h}A_j] d\overline{h} \right] (K) = (\psi \otimes \chi) \cdot k_G(\psi)[K, L]. \end{aligned}$$

In order to prove the second formula we compute

$$\begin{aligned} a_G(\phi * \psi)[K, L] &= \int_G \mu^G_{A_j} * \psi(K + gL) dg \\ &= \int_G \int_G \phi(K + gL - hA_j) dh dg = \int_G \int_G \phi(K + gL - hA_j) dg dh \\ &= \int_G a_G(\psi)[K - hA_j, L] dh = \left[a_G(\psi)[\cdot, L] * \int_G vol_n(\cdot - hA_j) dh \right] (K) \\ &= (\psi \otimes vol_n) * a_G(\phi)[K, L]. \end{aligned}$$

□

Lemma 126. *Let χ be the Euler characteristic. Then*

$$k_G(\chi) = p_G^{-1}.$$

Proof. Let $\tau_{A_j} \in Val^{G*}$ be given by $\tau_{A_j}(\phi) = \phi(A_j)$. By Remark 123,

$$k_G(\chi)(\tau_{A_j}) = \mu^G_{A_j}.$$

We immediately obtain that $k_G(\chi)$ is an isomorphism and that the τ_{A_j} are a basis of Val^{G*}. Also, for any $\phi \in Val^G$ and $K \in \mathcal{K}(V)$,

$$[\phi \cdot k_G(\chi)(\tau_{A_j})](K) = \phi \cdot \int_{\overline{G}} \chi(A_j \cap \overline{g}K) d\overline{g} = \int_{\overline{G}} \phi(A_j \cap \overline{g}K) d\overline{g}.$$

Let $K = B_r(0)$. Consider the sets $M_r := \{\overline{g} \in \overline{G} : A_j \subseteq \overline{g}B_r(0)\}$ and $M'_r := \{\overline{g} \in \overline{G} : A_j \cap \overline{g}B_r(0) \neq 0\} \cap M_r^c$. It is easy to see, that

$$\lim_{r\to\infty} \frac{\int_{M_r} d\overline{g}}{r^n} = vol_n(B_1(0)), \quad \lim_{r\to\infty} \frac{\int_{M'_r} d\overline{g}}{r^n} = 0.$$

Hence,

$$p_G(\phi)[k_G(\chi)(\tau_{A_j})] = (\phi \cdot k_G(\chi)(\tau_{A_j}))_n = \phi(A_j)vol_n = \tau_{A_j}(\phi)vol_n.$$

□

Theorem 127 (FTAIG)**.** *Let G be a closed subgroup of $SO(V)$ acting transitively on the sphere. Then $k_G = (p_G \otimes p_G)^{-1} \circ (m_G^* \circ p_G)$. This equation corresponds to the commuting diagram:*

$$\begin{array}{ccc} Val^G & \xrightarrow{k_G} & Val^G \otimes Val^G \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_G \otimes p_G} \\ Val^{G*} & \xrightarrow{m_G^*} & Val^{G*} \otimes Val^{G*}. \end{array}$$

Proof. Let $\phi, \psi \in Val^G$. We start by proving the formula $k_G = (p_G \otimes p_G)^{-1} \circ (m_G^* \circ p_G)$ evaluated at the Euler characteristic χ. Since $m_G^*(p_G(\chi))(\phi \otimes \psi) = \langle \phi, \psi \rangle$ this is equivalent to

$$(p_G(\phi) \otimes p_G(\psi))[k_G(\chi)] = k_G(\chi)[p_G(\phi) \otimes p_G(\psi)] = \langle \phi, \psi \rangle. \tag{27}$$

Let us choose a basis $e_1, \dots, e_m$ of Val^G and a corresponding dual basis $e_1^*, \dots, e_m^*$ of Va^{G*}. Calculating in coordinates yields

$$(M_{p_G}\phi)^T M_{k_G(\chi)}(M_{p_G}\psi) = (M_{p_G}\phi)^T M_{k_G(\chi)}(M_{k_G(\chi)}^{-1}\psi) = (M_{p_G}\phi)^T \psi = \langle \phi, \psi \rangle.$$

The general case now easily follows from Lemma 125. Indeed, for $\vartheta \in Val^G$, we obtain

$$k_G(\vartheta)[p_G(\phi) \otimes p_G(\psi)] = ((\phi \otimes \chi) \cdot k_G(\chi))[p_G(\phi) \otimes p_G(\psi)] = \langle \phi \cdot \phi, \psi \rangle.$$

□

Such a theorem also exists for additive kinematic formulas provided, $Val^G \subseteq Val^+$. Before we can prove it, we need one more lemma.

Lemma 128. *Let G be a closed subgroup of $SO(V)$ acting transitively on the sphere and such that $Val^G \subseteq Val^+$. Then*

$$(\mathbb{F} \otimes \mathbb{F})(k_G(\chi)) = k_G(\chi). \tag{28}$$

Proof. Lemma 104 and Theorem 127 yield

$$\begin{aligned}(\mathbb{F} \otimes \mathbb{F})(k_G(\chi)) &= (\mathbb{F} \otimes \mathbb{F}) \circ (p_G^{-1} \otimes p_G^{-1}) \circ m_G^*(vol_n^*) \\ &= (p_G^{-1} \otimes p_G^{-1}) \circ (\mathbb{F}^* \otimes \mathbb{F}^*) \circ m_G^*(vol_n^*).\end{aligned}$$

By Lemma 104 and the fact that $\mathbb{F}$ is an involution on even valuations, given any $\phi, \psi \in Val^G$, we have

$$\begin{aligned}((\mathbb{F}^* \otimes \mathbb{F}^*) \circ m_G^*(vol_n^*))[\phi \otimes \psi] &= vol_n^*[m_G \circ (\mathbb{F} \otimes \mathbb{F})(\phi \otimes \psi)] \\ &= vol_n^*[\mathbb{F}\phi \cdot \mathbb{F}\psi] = vol_n^*[\phi \cdot \psi] = (m_G^* \circ vol_n)[\phi \otimes \psi].\end{aligned}$$

Therefore, $(\mathbb{F}^* \otimes \mathbb{F}^*) \circ m_G^*(vol_n^*) = m_G^*(vol_n^*)$ and we obtain

$$(\mathbb{F} \otimes \mathbb{F})(k_G(\chi)) = (p_G^{-1} \otimes p_G^{-1}) \circ m_G^*(vol_n^*) = k_G(\chi).$$

□

Theorem 129 (FTAIG)**.** *Let G be a closed subgroup of $SO(V)$ acting transitively on the sphere and such that $Val^G \subseteq Val^+$. Then*

$$a_G = (\mathbb{F} \otimes \mathbb{F}) \circ k_G \circ \mathbb{F}. \tag{29}$$

Moreover, $a_G = (p_G \otimes p_G)^{-1} \circ c_G^* \circ p_G$. Hence, we obtain the following commuting diagramm:

$$\begin{array}{ccc} Val^{G*} & \xrightarrow{\quad m_G^* \quad} & Val^{G*} \otimes Va^{G*} \\ {\scriptstyle p_G}\uparrow & & \uparrow{\scriptstyle p_G \otimes p_G} \\ Val^G & \xrightarrow{\quad k_G \quad} & Val^G \otimes Val^G \\ {\scriptstyle \mathbb{F}}\uparrow & & \downarrow{\scriptstyle \mathbb{F} \otimes \mathbb{F}} \\ Val^G & \xrightarrow{\quad a_G \quad} & Val^G \otimes Val^G \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_G \otimes p_G} \\ Val^{G*} & \xrightarrow{\quad c_G^* \quad} & Val^{G*} \otimes Val^{G*} \end{array}$$

Proof. (28) yields

$$\begin{aligned} a_G(vol_n)(K, L) &= \int_G vol_n(K + GL) dg = \int_{\overline{G}} \chi(K \cap \overline{g}L) d\overline{g} \\ &= k_G(\chi)(K, L) = (\mathbb{F} \otimes \mathbb{F})(k_G(\chi))(K, L).\end{aligned}$$

That is, both sides of (29) give the same result when evaluated at vol_n. Let $\phi \in Val^G$. By Lemma 125, we have

$$\begin{aligned} a_G(\phi) &= a_G(\phi * vol_n) = (\phi \otimes vol_n) * a_G(vol_n) \\ &= (\mathbb{F} \otimes \mathbb{F})((\mathbb{F}\phi \otimes \chi) \cdot k_G(\chi)) = (\mathbb{F} \otimes \mathbb{F})(k_G(\mathbb{F}\phi)),\end{aligned}$$

as claimed. To show the relation between a_G and the convolution, note that by Theorem 106

$$c_G^* = (\mathbb{F}^* \otimes \mathbb{F}^*) \circ m_G^* \circ \mathbb{F}^*.$$

On the other hand Lemma 104 and what we have shown so far (the upper part of the diagramm) yields

$$\begin{aligned} a_G &= (\mathbb{F} \otimes \mathbb{F}) \circ (p_G^{-1} \otimes p_G^{-1}) \circ m_G^* \circ p_G \circ \mathbb{F} \\ &= (p_G^{-1} \otimes p_G^{-1}) \circ (\mathbb{F}^* \otimes \mathbb{F}^*) \circ m_G^* \circ \mathbb{F}^* \circ p_G \\ &= (p_G \otimes p_G)^{-1} \circ c_G^* \circ p_G. \end{aligned}$$

□

Remark 130. A detailed investigation of the spaces Val^G showed that indeed one always has $Val^G \subseteq Val^+$ (cf [9]).

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