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TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DISSERTATION

A New Proof of Thoma's Theorem & Eight Cardinals in Cichoń's Diagram

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Ausgeführt zum Zwecke der Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften
unter der Leitung von

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To Patience
And certainly not my own.

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Kurzfassung der Dissertation

Die vorliegende Disseration besteht aus zwei unabhängigen Artikeln. Das aus [TT19] bestehende Kapitel 2 wurde in Kollaboration mit Asger Törnquist erarbeitet; Kapitel 3 folgt [KTT18] in Kollaboration mit Jakob Kellner and Anda Tanasie. Kapitel 1 präsentiert eine Einführung für die darauf folgenden beiden Kapitel. Kapitel 2 und 3 stellen für das Format einer Dissertation geänderte Fassungen der beiden Artikel dar.

Im Kontext der Darstellungstheorie von Gruppen durch unitäre Operatoren wird eine Gruppe vom *Typ I* genannt, falls alle Faktordarstellungen vom Typ I sind. In Folge eines gefeierten Theorems von James Glimm [Gli61b] sind die Gruppen vom Typ I genau jene, deren Klassifikationsproblem der irreduziblen unitären Darstellungen die Komplexität besitzen, die in der deskriptiven Mengenlehre heutzutage "konkret klassifizierbar" oder "glatt" genannt wird. Elmar Thoma bewies in [Tho64] die folgende überraschende Charakterisierung der abzählbaren diskreten Gruppen vom Typ I: Es sind genau solche, die eine abelsche Untergruppe mit endlichem Index besitzen. In Kapitel 2 präsentieren wir einen neuen, einfacheren Beweis des Theorems von Thoma, wobei wir darin nur relativ elementare Methoden anwenden.

Cichóns Diagramm beinhaltet Kardinalzahlcharakteristiken der reellen Zahlen, die entweder vom Ideal der Nullmengen (im Sinne von Maß) oder vom Ideal der mageren Mengen (im Sinne von Kategorie) abgeleitet sind. In Kapitel 3 zeigen wir, dass acht dieser Kardinalzahlen gleichzeitig in einem Modell von ZFC getrennt werden können. Genauer formuliert erhalten wir das folgende Resultat: Unter der Annahme von drei stark kompakten Kardinalzahlen is folgendes konsistent:

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

Aus der selben Annahme folgt auch die Konsistenz von

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

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Abstract

This thesis consists of two separate articles. The work in [TT19] is joint with Asger Törnquist, [KTT18] is joint with Jakob Kellner and Anda Tanasie. We provide two independent introductions in Chapter 1 of the thesis. Chapters 2 and 3 represent amended versions of the two articles that have been modified to suit the format of a thesis.

In the theory of unitary group representations, a group is called *type I* if all factor representations are of type I, and by a celebrated theorem of James Glimm [Gli61b], the type I groups are precisely those groups for which the irreducible unitary representations are what descriptive set theorists now call "concretely classifiable". Elmar Thoma [Tho64] proved the following surprising characterization of the countable discrete groups of type I: They are precisely those that contain a finite index abelian subgroup. In chapter 2 we give a new, simpler proof of Thoma's theorem, which relies only on relatively elementary methods.

Cichoń's diagram contains cardinal characterics based on the ideal of null sets (in the sense of measure) or the ideal of meagre sets (in the sense of category). In chapter 3 we show how to separate eight of the cardinals at the same time. More precisely, we obtain the following result: Assuming three strongly compact cardinals, it is consistent that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

Under the same assumption, it is consistent that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

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Acknowledgments

The author is currently a PhD student at the Institute of Discrete Mathematics and Geometry, TU Wien, and a DOC Fellow of the Austrian Academy of Sciences. The author furthermore gratefully acknowledges support by the Austrian Science Fund (FWF) grants no. P26737 and P30666.

I would like to thank Sven Raum for pointing out [Smi72] to us. We are indebted to Hiroshi Ando for helpful conversations and comments that influenced our presentation. I am furthermore grateful to Saharon Shelah for valuable discussions.

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To Anda, for talking about mathematics, and especially for being such a good friend to me.

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To my father, for challenging me in subtle and fruitful ways.

To my mother.

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1 Introduction

This thesis is comprised of two related subjects: set theory of the reals and descriptive set theory. Set theory of the reals concerns itself with properties of the real numbers, which can be shown to exhibit differing realizations in various models of ZFC set theory. Descriptive set theory, on the other hand, started out as a discipline trying to correlate synatictical properties of classes of real numbers with certain regularity properties they thus inherit. From a more modern standpoint, a lot of work has been put into figuring out the fine line between absolute properties of the reals and classes of real numbers where independence phenomena start to show.

Chapter 2 presents a new proof of a celebrated theorem of Thoma. Why do we concern ourselves with this? First of all, in the modern theory of Borel equivalence relations, which stems from descriptive set theory, applications to other areas of mathematics abound. One instance of these applications is exactly the classification problem of irreducible unitary representations (up to unitary equivalence) of discrete countable groups. Through a string of important results (see Section 1.1.2) and Thoma's result one strong dichotomy arose: if a discrete countable group possesses an abelian subgroup of finite index, then the associated classification problem is "smooth". If not, the associated classification problem is not classifiable by countable structures. We point out that our new proof is very elementary in nature, and only makes use of standard direct integral theory. We furthermore note that [Smi72] presents a different elementary proof. Our proof furthermore generalises naturally to *SIN* groups, which we outline at the end of Chapter 2.

Chapter 3 presents a proof that allows us to separate eight different cardinal characteristics in Cichoń's diagram simultaneously. Cichoń's diagram illustrates all ZFC relations between certain cardinal characteristics associated to the meagre ideal and to the null ideal. We note that this result has been superseded by the impressive [GKS19], which employs Boolean ultrapowers like our proof and manages to separate ten cardinals simultaneously, which is the maximum possible number for cardinals in Cichoń's diagram. Finally, with new methods and removing the large cardinal assumption, there is the recent [GKMS19], which definitively settles research in this direction.

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1.1 Prerequisites for Chapter 2

For all basic concepts of descriptive set theory we refer the reader to the standard treatise in [Kec95]. On the theory of von Neumann algebras, [Dix96] gives a thorough introduction, whereas [Bla06] presents a modern overview of the subject. A particularly accessible and complete presentation of direct integral decompositions in the context of von Neumann algebras and unitary group representations is [Nie80].

1.1.1 A brief History of Borel Equivalence Relations

The initial spark of the modern theory of Borel equivalence relations, set in the context of descriptive set theory, goes back to the seminal work of Friedman and Stanley in [FS89]. Soon after that, the major result of Harrington, Kechris and Louveau ([HKL90]) generalised a famous dichotomy result of Glimm and Effros, and set the stage for further enquiries into the nature of the quasi-order of Borel reducibility of Borel equivalence relations (compare with Silver's dichotomy for coanalytic equivalence relations). It bears mentioning that already [FS89] introduced a jump operator for Borel equivalence relations, which in can be used to yield an ω_1 tall tower of equivalence relations.

Already at the beginning, the relationship to model theory and countable structures via the actions of S_∞ was prominent, as it allows natural classification problems from other areas of mathematics to be used as benchmarks for investigating specific complexities. On a different level, the theory of countable Borel equivalence relations has proved to be a particularly useful field of reasearch, not least because of its immediate relationship to countable group actions (concerning the some structural properties, see e.g. [AK00] showing that the order of Borel sets modulo inclusion can be embedded into the reducibility hieararchy of the countable Borel equivalence relations).

More recently, Miller's new proof of the G_0-dichotomy has shown a new direction of research for the Borel reducibility hierarchy (see [Mil12] for an overview).

We end this section by listing a string of results pertaining to other areas of mathematics that are paradigmatic of the theory's scope and impact.

- Torsion-free abelian groups of rank $n \in \mathbb{N}$ [Tho03]: the complexity strictly increases with rank.
- Von Neumann factors [ST09]: the classification problem of isomorphism is not classifiable by countable structures for each type.

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1.1.2 Classifying Irreducible Unitary Representations

In recent years there has been a renewed interest in the classification problem for irreducible unitary representations. The current interest comes primarily from descriptive set theorists, who in the past 30 years have developed a vast theory, called *Borel reducibility theory*, for measuring the relative complexity of classification problems in mathematics. Turning these techniques toward the problem of classifying irreducible representations of groups (and C*-algebras) is, satisfyingly, a sort of homecoming: Modern Borel reducibility theory grew out of the smooth/non-smooth dichotomy that originated in the work of Mackey, Glimm and Effros, among others, exactly with the purpose of quantifying the phenomenon that for some groups and C*-algebras, their irreducible unitary representations are concretely classifiable (i.e., the unitary dual is "smooth"), while for other groups and C*-algebras, no reasonable classification seems to be possible (i.e., the unitary dual is not smooth, equivalently, is not countably separated). See [Gli61b, Gli61a, Eff65b, Eff81].

Glimm's celebrated "smooth dual if and only if type I" theorem (in ([Gli61b, Theorem 2]) is a wellspring: A string of results in descriptive set theory, [Hjo97, Far12, Tho15], have sharpened Glimm's result considerably, showing that when the unitary dual is not smooth, then the complexity of classifying the irreducible representations is very high. It is still an active research problem to determine the exact complexity of classifying the irreducible unitary representations for various groups (see [Tho15]).

In the important special case when we consider unitary representations of countable discrete groups, a cornerstone theorem is the following result due to Thoma:

Theorem 1.1 (Thoma 1964, [Tho64, Satz 6, p.133ff.])**.** *A countable discrete group is type I if and only if it is abelian-by-finite (i.e., contains a finite index abelian subgroup).*

1.2 Prerequisites for Chapter 3

1.2.1 Forcing and set theory of the reals

The standard introduction to forcing is [Kun83]. Even though not immediately used, the reader will benefit from having some familiarity with [BJ95] and the concepts therein contained. We assume the reader to be familiar with the main content of these books.

We briefly recall the origins of the method of forcing.

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The classical way to get a model of ¬CH uses the method of forcing, introduced by Cohen in [Coh63]. Very informally speaking: given a model V of ZFC, and a partial order P in V, we can find a so-called "generic object" G for P over V (which is not an element of V) and use this object to extend V to a bigger ZFC model $V[G]$. When a suitable partial order P is chosen, it is "forced" that $V[G]$ satisfies ¬CH. It turns out that many other mathematical statements φ can be forced by other partial orders, thus establishing the (relative) consistency of φ.

In this way, the independence of many classical mathematical problems has been established, such as the Souslin Hypothesis (solved by Solovay and Tennenbaum [ST71]), the Borel Conjecture (solved by Richard Laver [Lav76]) or Whitehead's Problem (solved by Saharon Shelah [She74]).

Over the years, forcing has been developed into a vast and sophisticated theory that continues to have new applications in many areas of mathematics (most commonly independence results, but sometimes also "direct" or "positive" results). Central to the theory of forcing is the notion of multiple forcing, in particular forcing iteration: after extending the universe V to some $V[G]$, we can again extend $V[G]$ to a bigger $V[G][H]$, and we can iterate this process transfinitely often. All the independence results mentioned in the previous paragraph use some form of multiple forcing.

1.2.2 A brief overview of cardinal characteristics

Recall the cardinal characteristics represented in Cichoń's diagram:

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

An arrow between $\mathfrak{x}$ and $\mathfrak{y}$ indicates that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$. Moreover,

$$\max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) = \mathrm{cof}(\mathcal{M}), \ \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) = \mathrm{add}(\mathcal{M}).$$

These are the only "simple" restrictions in the following sense: every assignment of $\aleph_1$ and $\aleph_2$ to the entries of Cichoń's diagram that honors these restrictions can be shown to be consistent. It is more challenging to get more than two simultaneously different values, for recent progress in this direction see, e.g., [Mej13, GMS16, FGKS17].

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Chapter 3 consists of two parts: In the first one, we present a finite support ccc iteration P^4 forcing that $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} = 2^{\aleph_0}$ (and actually something stronger, cf. Lemmas 3.18 and 3.20). This is nothing new: The forcing and all required properties were presented in [Mej13]. We recall all the facts that are required for our result, in a form convenient for our purposes.

In the second part, we investigate the (iterated) Boolean ultrapower P^7 of P^4. Assuming three strongly compact cardinals, this ultrapower (again a finite support ccc iteration) forces

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0},$$

i.e., we get the following values in the diagram (for some increasing cardinals λ_i):

$$\begin{tikzcd}
 & \lambda_2 \arrow[r] & \square \arrow[r] & \square \arrow[r] & \lambda_6 \arrow[r] & \lambda_7 \\
 & & \lambda_3 \arrow[r] & \lambda_4 & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[r] \arrow[uu] & \square \arrow[r] \arrow[u] & \square \arrow[r] \arrow[u] & \lambda_5 \arrow[uuu] &
\end{tikzcd}$$

It seems unlikely that the large cardinals assumption is actually needed, but we would expect a proof without it to be considerably more complicated.

The kind of Boolean ultrapower that we use was investigated in [Man71], and recently applied, e.g., in [MS16] and [RS] (where a Boolean ultrapower of a forcing notion is applied to cardinal characteristics of the reals). Recently Shelah developed a method of using Boolean ultrapowers to control characteristics in Cichoń's diagram. Chapter 3 is a relatively simple application of these methods.

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2 A new proof of Thoma's Theorem

2.1 Introduction

2.1.1 Scope

The purpose of this chapter is to present a new and more accessible proof of Theorem 1.1. The concrete and easy to understand statement of this theorem makes it broadly useful, not just for unitary representations of groups. For instance, it is often used in ergodic theory in combination with the so-called Gaussian construction to achieve a large family of non-conjugate measure-preserving ergodic actions of a given non-type I group, see e.g. [Hjo05, TÖ9, Ioa11, Eps07].

We thank Sven Raum for pointing out the work of M. Smith to us, which also gives an elementary proof of Thoma's theorem. However, we take a different approach than the proof presented in [Smi72]. Our proof naturally generalises to SIN-groups, we do not know if that is the case with Smith's proof. We clarify in Section 2.4 how to adapt the proof to this more general setting. Smith — more similarly to Thoma, but avoiding the use of direct integral theory — shows how to obtain an abelian subgroup of finite index in the case that the group is type I. Our approach shows the contraposition, i.e., we show how to embed arbitrarily large matrix algebras in the case that there is no abelian subgroup of finite index contained in the group.

Theorem 1.1 gives a dichotomy in the classification of irreducible unitary representations: A countable discrete group is either abelian-by-finite, and the classification of its irreducible unitary representations is smooth, or it is not abelian-by-finite, and the classification of its irreducible unitary representations is horribly difficult, as measured by Borel reducibility. It is interesting to note that Thomas [Tho15] has proved that for all *amenable* non-abelian-by-finite countable discrete groups, the classification of the irreducible unitary representations always form the same non-smooth Borel reducibility degree, independently of the group. Thomas has asked in [Tho15] if the complexity of classifying the irreducible unitary representations of *non-amenable* countable discrete groups is strictly higher, but this is not yet known.

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A theorem of this nature could be viewed as giving a second dichotomy for the classification of irreducible unitary representations, with Theorem 1.1 being the first dichotomy.

Comparison with Thoma's proof

Let us briefly comment on our proof in relation to Thoma's original proof. The difficult part of the theorem is to show "only if". Let us fix a countable discrete group G for this discussion.

Thoma's proof is based on the development of a "bespoke", or "custom made", direct integral decomposition theory for what he calls "traces" on the algebra $\mathfrak{A}(G)$ of all finitely supported complex valued functions on G. These traces in turn correspond to positive definite functions on G that are invariant under conjugacy ("class finite positive definite functions", in Thoma's terminology).

By contrast, our proof uses only the standard direct integral decomposition theory for von Neumann algebras into factors, and unitary representations into factor representations, and so our proof avoids building up a special, custom made direct integral decomposition theory altogether.

In our proof, as well as in Thoma's proof, a key role is played by the normal subgroup

$$G^{\mathrm{fc}} = \{g \in G : |[g]_G| < \infty\}$$

consisting of all the elements in G with finite conjugacy classes. Note: Throughout the chapter we will use the notation $[g]_G = \{hgh^{-1} : h \in G\}$.

The "easy" case is when $[G : G^{\mathrm{fc}}] = \infty$, i.e., G^{fc} has infinite index in G. In this case, our proof and Thoma's follow a rather similar route: We show in Section 2.2 that almost every factor representation in the factor decomposition of the regular representation is of type II_1. Our proof is essentially an elaboration on the well-known proof that the group von Neumann algebra $L(G)$ of an icc group is a II_1 factor. Thoma achieves the same conclusion in [Tho64, Satz 3], but he proves it by analyzing extensions of class finite positive definite functions from G^{fc} to all of G from the point of view of his custom made direct integral theory.

The "hard" case for us, as well as for Thoma, is the case when $[G : G^{\mathrm{fc}}] < \infty$, and it is also here that our proof differs most significantly from Thoma's. Our proof in this case is found in Section 2.3 below. Aiming to prove the contrapositve of the "only if" in Theorem 1.1, we assume that G is *not* abelian-by-finite, and proceed to construct inside of G^{fc} a sequence $(G_i)_{i\in\mathbb{N}}$ of pairwise commuting non-abelian

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subgroups. We show that the subgroups $G_1 \cdots G_N$ give rise to von Neumann subalgebras $S(G_1 \cdots G_N)$ of $L(G)$ with the property that at least half the factors in the factor decomposition of $S(G_1 \cdots G_N)$ will have dimension $> k$ when N is chosen sufficiently large. This in turn gives us that $L(G)$ contains a subalgebra which is of type II_1, and so $L(G)$ is not purely of type I, which implies the result.

Thoma takes quite a different path in the "hard" case. He assumes that G is type I, and assumes w.l.o.g. that $G = G^{\mathrm{fc}}$. If G is finitely generated, the assumption $G = G^{\mathrm{fc}}$ easily gives that G is abelian-by-finite. Therefore Thoma can assume that G is not finitely generated, and so there is a strictly increasing sequence of finitely generated subgroups $\mathfrak{K}_0 \subset \mathfrak{K}_1 \subset \cdots \subset G$, which exhausts G. He then considers an "extremal" class finite positive definite function α on G, which by his bespoke direct integral representation theory can be identified with a factor representation. He then analyses α in terms of its restrictions α_i to the subgroups $\mathfrak{K}_i$. The assumption that G is type I puts a bound on the growth of the dimensions of the factor representations that arise from the α_i. This bound, through several further arguments, allows Thoma to find a finite index abelian subgroup in G.

2.1.2 Preliminaries

In this section we review some of the basic decomposition theory of unitary representations and von Neumann algebras and some facts about group von Neumann algebras. A more detailed overview of these things can be found in [Bla06, III.5, III.3.3], and a fuller account may be found in [Dix96, Nie80, KR97]. We also fix notation that will be used in the rest of the paper.

Unitary representations and von Neumann algebras

Given a Hilbert space $\mathcal{H}$, we will denote by $\mathcal{B}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$, and by $\mathcal{U}(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$. A *unitary representation* of a countable discrete group G is simply a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$. To π we associate

$$\pi(G)' = \{T \in \mathcal{B}(H) : (\forall g \in G)\, T\pi(g) = \pi(g)T\}.$$

It is easy to show that $\pi(G)'$ is a *von Neumann algebra*, i.e., a $*$-subalgebra of $\mathcal{B}(H)$ which is closed in the weak operator topology (equivalently, in the ultraweak operator topology), and which contains the identity operator I on $\mathcal{H}$. The representation π is called a *factor representation* (or sometimes in older references, a *primary* rep-

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resentation) if $\pi(G)'$ is a factor[1] in the sense of von Neumann algebra theory, the definition of which is: a von Neumann algebra $M \subseteq \mathcal{B}(H)$ is a *factor* if the *center* of M,

$$Z(M) = \{x \in M : (\forall y \in M)\, xy = yx\},$$

consist of multiples of I, that is $Z(M) = \mathbb{C}I$. Von Neumann factors are naturally categorized in *types*, called I, II, and III, which break into further subtypes. The type of a factor representation π is the type of $\pi(G)'$. In this chapter we will only need to consider type I_n, for $n \in \mathbb{N}$, and II_1 factors, and we give a working definition of these at the end of this section.

Every von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$ admits a direct integral decomposition into factors. This means: There is a measure space (X, μ) and a measurable assignment $x \mapsto \mathcal{H}^x$ of Hilbert spaces, and a measurable assignment $x \mapsto F^x \subseteq \mathcal{B}(\mathcal{H}^x)$ of factors such that

$$M = \int F^x d\mu(x),$$

in the sense that, in a natural way, $\mathcal{H} = \int \mathcal{H}^x d\mu(x)$, and M is generated by the measurable functions $f : X \to \mathcal{B}(\mathcal{H}^x)$ where $f(x) \in F^x$ for μ-almost every x. In this picture, $Z(M)$ corresponds to the measurable functions $f : X \to \mathcal{B}(\mathcal{H}^x)$ where $f(x) = c^x I(x)$, where $c^x \in \mathbb{C}$ and $I(x)$ is the identity operator on $\mathcal{H}^x$.

We note that in the special case when $\mathcal{H}$ is separable (which will always be the case in this paper), the measure space (X, μ) in the decomposition is a standard measure space (in the sense of e.g. [Kec95]), and we could have replaced "measurable" with "Borel" above. See [Eff65a, Eff66, HW98, HW00] for a detailed development of the Borel theory of separably acting von Neumann algebras.

Turning our attention back to unitary representations, if we are given $\pi : G \to \mathcal{U}(\mathcal{H})$, the factor decomposition of $\pi(G)'$ gives rise to an *integral decomposition of the representation π into factor representations*. That is, for any unitary representation, we can find a measure space (X, μ) and a decomposition $\mathcal{H} = \int \mathcal{H}^x d\mu(x)$ of the Hilbert space, and a measurable assignment $x \mapsto \pi^x(g) \in \mathcal{U}(\mathcal{H}^x)$ for each $g \in G$, such that $g \mapsto \pi^x(g)$ is a factor representation on $\mathcal{H}^x$ (a.e).

[1] Unlike [Bla06], but similar to [Dix96, Nie80, HT12], we take the view that it is the structure of $\pi(G)'$ that is the most relevant to analyzing the representation π, and not $\pi(G)''$. This is because the structure of projections in $\pi(G)'$ is more relevant to analyzing π than the projections in $\pi(G)''$, since the projections in $\pi(G)'$ correspond to exactly invariant subspaces of π.

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The regular representation and the group von Neumann algebra

The left regular representation λ is the representation $\lambda : G \to l^2(G)$ given by $\lambda(g) = u_g$, where

$$(u_g(f))(h) = f(g^{-1}h).$$

The right regular representation $\rho : G \to l^2(G)$ is defined by $(\rho(g))(f)(h) = f(hg)$.

The (left) group von Neumann algebra $L(G)$ is the von Neumann algebra generated by $\lambda(G)$. We let $R(G)$ be the von Neumann algebra generated by $\rho(G)$. It turns out that $L(G)' = R(G)$ and $R(G)' = L(G)$ ([KR97, Theorem 6.7.2]). Since much of this chapter is be concerned with analyzing the factor decomposition of $L(G)$, it is worth noting that because $L(G) = \rho(G)'$, this actually corresponds to analyzing the factor decomposition of the *right* regular representation. Of course, we could have chosen to focus on $R(G)$ and the representation λ instead, with much the same result.

The group von Neumann algebra $L(G)$ is equipped with a natural *trace* $\tau : L(G) \to \mathbb{C}$, which is continuous w.r.t. the ultraweak operator topology (see [Bla06, p. 14]), and which is uniquely determined by requiring that

$$\tau(\sum_{g\in G} \alpha_g u_g) = \alpha_e$$

on all finite sums $\sum_{g\in G} \alpha_g u_g$, where $\alpha_g \in \mathbb{C}$. The trace is a linear functional, and satisfies $\tau(I) = 1$ and $\tau(ab) = \tau(ba)$ for all $a, b \in L(G)$. Moreover, the trace is *faithful*, meaning that $\tau(aa^*) = 0$ iff $a = 0$. In particular, if $p \in L(G)$ is a projection, i.e., $p = p^* = p^2$, then $\tau(p) = 0$ iff $p = 0$. The trace gives rise to an inner product $\langle a, b\rangle_\tau = \tau(ab^*)$ on $L(G)$, which in turn gives rise to a norm $\|a\|_\tau = \sqrt{\langle a, a\rangle_\tau}$. Note that unitaries in $L(G)$ are unit vectors in $\|\cdot\|_\tau$, and that the family $(u_g)_{g\in G}$ forms an orthonormal set in $(L(G), \|\cdot\|_\tau)$. Warning: $L(G)$ is in general not complete in this norm.

In our proofs, we will mostly be considering a fixed group G together with a subgroup $H < G$. In this setting, we will denote by $S(H)$ the von Neumann algebra generated by the unitaries u_h, where h ranges over H. The ambient group G will always be clear from context. Clearly, $S(H)$ is a von Neumann subalgebra of $L(G)$, and the restriction of the trace τ on $L(G)$ is a trace on $S(H)$.

We will often consider the factor decomposition of $L(G)$ or $S(H)$ for some $H < G$, e.g.

$$L(G) = \int_{(X,\mu)} F^x d\mu(x).$$

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Every central projection (i.e., projection in $Z(L(G))$) is of the form $p = \int_B I(x)d\mu(x)$ for some measurable $B \subseteq X$, and where $I(x)$ is the identity operator in F^x. So we can think of central projections as measurable subsets of X, and we let $\mathrm{supp}(p)$ denote the measurable set corresponding to the central projection p. We can then define a new measure on X by $\mu_\tau(B) = \tau(p)$, and since τ is faithful, this measure is absolutely equivalent to μ. For this reason, we could just as well have used the measure μ_τ coming from the trace in the factor decomposition of $L(G)$ (or $S(H)$), and this what we will always do from now on when decomposing $L(G)$ and $S(H)$.

When we use the measure coming from the trace in decomposing $L(G)$ or $S(H)$, then the trace itself decomposes nicely: We can find a measurable assignment $x \mapsto \tau^x$ of traces on each factor F^x in the decomposition so that

$$\tau(a) = \int \tau^x(a^x)d\mu_\tau(x).$$

Type I_n and II_1

We make the following practical working definition, which will suffice for the purposes of this paper. A von Neumann factor F is type I_n if it is isomorphic to the matrix algebra $M_n(\mathbb{C})$. A von Neumann factor F is type II_1 if it is infinite dimensional (as a vector space) and admits an ultraweakly continuous trace $\tau : F \to \mathbb{C}$ with $\tau(I) = 1$. See [Bla06, III.1].

Murray-von Neumann equivalence of projections

On one occasion in the proof we will make use of the following notion: In a von Neumann algebra M, two projections $p, q \in M$ are *Murray-von Neumann equivalent* if there is $u \in M$ (called a *partial unitary*) such that

$$u^*u = p \text{ and } uu^* = q\,.$$

Corners vs. subalgebras

Unlike C*-algebras, von Neumann algebras are always assumed to contain the identity operator (which is then the unit element of the algebra). In particular, a von Neumann subalgebra of another von Neumann algebra is always a *unital* subalgebra.

On a few occasions, we will consider the algebra $pMp = pM = Mp$, where M is a von Neumann algebra and $p \in Z(M)$ is a central projection. It is clear that this

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is a $*$-subalgebra, but it is not unital unless $p = I$. We will call such a subalgebra a *corner* of M.

Inclusions among abelian von Neumann algebras

Finally, we will on several occasions consider a situation where we are given abelian von Neumann subalgebras $A \subseteq B \subseteq L(G)$, where G is some countable discrete group.

In general, abelian von Neumann algebras are isomorphic to $L^\infty(X, \mu)$ acting by pointwise multiplication in $L^2(X, \mu)$, for some measure space (X, μ) (see [Bla06, p. 236]). If we identify A (resp. B) with $L^\infty(X_A, \mu_A)$ (resp. $L^\infty(X_B, \mu_B)$) in this way, we can always assume that the measure comes from the trace inherited from $L(G)$. Moreover, under this identification, the projections in A (resp. B), correspond to measurable subsets of X_A (resp. X_B), and vice versa. Since every projection in A is also a projection in B, this means that the measurable subsets of X_A can be identified with a sub-σ-algebra of the measurable subsets of X_B, and since the measures μ_A and μ_B are derived from the trace on $L(G)$, they agree on this sub-σ-algebra.

Another possible view to take is that the situation $A \subseteq B \subseteq L(G)$ gives rise to a measure-preserving surjection $\vartheta : X_B \to X_A$. However, we will work with the sub-σ-algebra view described above, as it is easier in our setting.

2.1.3 A note on the class of groups

We present the proof for the discrete countable case. The reader can (if so inclined) consider discrete countable groups, since the decomposition theory will work in an analogous way (see [Nie80]). We include a final section that provides an outline of how to adapt our proof for the case of separable *SIN* groups.

2.2 The "easy" case: $[G : G^{\mathrm{fc}}] = \infty$

In the case when G^{fc} has infinite index in G, Theorem 1.1 follows from:

Theorem 2.1 (Mautner)**.** *Let G be a countable discrete group, and suppose $[G : G^{\mathrm{fc}}] = \infty$. Then every factor in the factor decomposition of $L(G)$ is of type II_1.*

For completeness we include a simple proof of the result. We will furthermore utilize the ideas from the following two lemmata in the sequel.

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Lemma 2.2.
1) $Z(L(G)) \subseteq S(G^{\mathrm{fc}})$.
2) For any projection $p \in S(G^{\mathrm{fc}})$ and $h \in G \setminus G^{\mathrm{fc}}$ we have

$$\langle p, u_h \rangle_\tau = 0.$$

Proof.
1) We omit the proof, as this is *exactly* what the standard proof (see e.g. [KR97][Theorem 6.7.5]) of "$L(G)$ is a factor when G has infinite conjugacy classes (icc)" actually shows.

2) By definition, $p \in S(G^{\mathrm{fc}})$ can be approximated arbitrarily well in the ultraweak topology by finite sums $\sum_{g \in G^{\mathrm{fc}}} \alpha_g u_g$ for a sequence $\alpha_g \in \mathbb{C}$. So for $h \notin G^{\mathrm{fc}}$, the product $pu_{h^{-1}}$ is the ultraweak limit of finite sums of the form

$$su_{h^{-1}} = \sum_{g \in G^{\mathrm{fc}}} \alpha_g u_g u_{h^{-1}} = \sum_{g \in G^{\mathrm{fc}}} \alpha_g u_{gh^{-1}},$$

Since none of these sums contain u_e we get $\langle s, u_h \rangle_\tau = \tau(su_{h^{-1}}) = 0$, and so by continuity $\langle p, u_h \rangle_\tau = 0$. □

Lemma 2.3. *Let*

$$L(G) = \int_X F^x d\mu_\tau(x)$$

be the factor decomposition of $L(G)$ (w.r.t. the measure μ_τ arising from the trace), let τ^x be the trace on F^x (as in Section 2.1.2), and for each $g \in G$ decompose u_g as

$$u_g = \int u_g^x d\mu_\tau(x),$$

where u_g^x is a unitary in F^x. Let $g, h \in G$ with $g \notin hG^{\mathrm{fc}}$. Then

$$\langle u_g^x, u_h^x \rangle_{\tau^x} = 0$$

for almost all x.

Proof. Let

$$A = \{x \in X : \langle u_g^x, u_h^x \rangle_{\tau^x} \neq 0\}.$$

We will show that $\mu_\tau(A) = 0$.

Suppose instead $\mu_\tau(A) > 0$. Then for some $\delta > 0$ the set

$$A_\delta = \{x \in X : \mathrm{Re}(\langle u_g^x, u_h^x \rangle_{\tau^x}) > \delta\}$$

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has positive measure, and so there is $r_0 \in \mathbb{R}$ with $r_0 > \delta$ such that the set

$$B = \{x \in X : |\mathrm{Re}(\langle u_g^x, u_h^x \rangle_{\tau^x}) - r_0| < \frac{\delta}{2}\}$$

has positive μ_τ-measure. In $L(G)$, the set B corresponds to the projection

$$p = \int_B I(x) d\mu_\tau(x)$$

where $I(x)$ is the identity in F^x. As $p \in Z(L(G))$, by Lemma 2.2 we have $\langle p, u_{g^{-1}h} \rangle = 0$. However, the definition of B gives

$$\mathrm{Re}(\int_B \langle u_g^x, u_h^x \rangle_{\tau^x} d\mu_\tau(x)) = \int_B \mathrm{Re}(\langle u_g^x, u_h^x \rangle_{\tau^x}) d\mu_\tau(x) > \frac{\delta}{2}\mu_\tau(B),$$

and so since

$$\mathrm{Re}(\langle p, u_{g^{-1}h} \rangle_\tau) = \mathrm{Re}(\langle u_g p, u_h \rangle_\tau) = \mathrm{Re}(\int_B \langle u_g^x, u_h^x \rangle_{\tau^x} d\mu_\tau(x))$$

we get that $\langle p, u_{g^{-1}h} \rangle_\tau \neq 0$, which is a contradiction. □

Proof of Theorem 2.1. Let G be a countable discrete group, and suppose $[G : G^{\mathrm{fc}}] = \infty$. Let $(g_i)_{i \in \mathbb{N}}$ be a sequence in G such that $g_i G^{\mathrm{fc}}$ enumerates all left cosets injectively. Let F^x and u_g^x be as in the statement of the previous lemma. The previous lemma gives that the unitaries $u_{g_i}^x \in F^x$ are orthogonal w.r.t. $\langle \cdot, \cdot \rangle_{\tau_x}$ for almost all x. Therefore, almost every factor in the decomposition of $L(G)$ is infinite-dimensional, and thus of type II_1. □

2.3 The "hard" case: $[G : G^{\mathrm{fc}}] < \infty$

Throughout this section, G denotes a countable discrete group. In the case when G^{fc} has finite index in G, Theorem 1.1 follows immediately from (B) of the following theorem.

Theorem 2.4. *Let G be a countable discrete group.*

(A) *Suppose G contains a sequence of non-abelian pairwise commuting subgroups $(G_i)_{i \in \mathbb{N}}$ (i.e., each G_i is non-abelian, but for $i \neq j$ every element of G_i commutes with every element of G_j). Then $L(G)$ is not of type I.*

(B) *If $[G : G^{\mathrm{fc}}] < \infty$ and G is not abelian-by-finite, then G contains a sequence of non-abelian pairwise commuting subgroups $(G_i)_{i \in \mathbb{N}}$. Therefore, by* (A), *$L(G)$ is not type I, and so its factor decomposition contains a II_1 factor.*

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2.3.1 Proof of Part (A) of Theorem 2.4

The proof is developed in a sequence of lemmata.

Lemma 2.5. *Let $H < G$ be a subgroup, and let*

$$S(H) = \int F^x d\mu_\tau(x)$$

be the factor decomposition of $S(H)$. Suppose H is not abelian. Then the set

$$B = \{x : F^x \not\simeq \mathbb{C}\}$$

has $\mu_\tau(B) \geq \frac{1}{2}$.

Proof. Let $g, h \in H$ be non-commuting elements of $S(H)$. Since $gh \neq hg$, we have that u_{gh} and u_{hg} are orthonormal vectors w.r.t. $\langle \cdot, \cdot \rangle_\tau$, and so $\|u_{gh} - u_{hg}\|_\tau^2 = \|u_{gh}\|_\tau^2 + \|u_{hg}\|_\tau^2 = 2$. Thus

$$2 = \|u_{gh} - u_{hg}\|^2 = \int_{B^c} \|u^x_{gh} - u^x_{hg}\|^2 d\mu_\tau + \int_B \|u^x_{gh} - u^x_{hg}\|^2 d\mu_\tau \tag{2.6}$$

Since for $x \in B^c$ the factor F^x is commutative (being isomorphic to $\mathbb{C}$), we have $u^x_{gh} = u^x_g u^x_h = u^x_h u^x_g = u^x_{hg}$. Thus the first integral above is 0. On the other hand, since u^x_{gh} and u^x_{hg} are unit vectors in $\langle \cdot, \cdot \rangle_{\tau_x}$, we have $\|u^x_{gh} - u^x_{hg}\| \leq 2$, and so the second integral above can be at most $4\mu_\tau(B)$. Thus eq. (2.6) gives $2 \leq 4\mu_\tau(B)$, so $\mu_\tau(B) \geq \frac{1}{2}$ follows. □

Before the next proof, we remark that when $H_0, H_1 < G$ are commuting subgroups, then $H_0 H_1 = \{h_0 h_1 : h_0 \in H_0 \wedge h_1 \in H_1\}$ is a subgroup of G.

Lemma 1. *Suppose that $H_0, H_1 < G$ are commuting subgroups. Let $S(H_i) = \int F^x_i d\mu_i(x)$ and $S(H_0 H_1) = \int F^x d\mu(x)$ be the factor decompositions. Let $p_i \in S(H_i)$, $i \in \{0, 1\}$, be central projections. Then*

1. *$p_0 p_1$ is a central projection in $S(H_0 H_1)$.*
2. *If there are $n_0, n_1 \in \mathbb{N}$ such that F^x_i contains a unital copy of $M_{n_i}(\mathbb{C})$ for almost all $x \in \operatorname{supp}(p_i)$, then for almost all $x \in \operatorname{supp}(p_0 p_1)$ the factor F^x contains a copy of $M_{n_0 n_1}(\mathbb{C})$.*
3. *If in (2) we have $F^x_i \simeq M_{n_i}(\mathbb{C})$ for almost all $x \in \operatorname{supp}(p_i)$, then*

$$(\forall^\mu x \in \operatorname{supp}(p_0 p_1))\ F^x \simeq M_{n_0 n_1}(\mathbb{C}).$$

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Proof. (1) Since p_0 and p_1 commute, it is clear that p_0p_1 is a projection, and since each p_i commutes with all u_g when $g \in H_0 \cup H_1$, it follows that p_0p_1 is a central projection in $S(H_0H_1)$.

(2) We may assume that $p_0p_1 \neq 0$, otherwise there is nothing to show. By the Jankov-von Neumann uniformization Theorem [Kec95, Theorem 18.1], we can find measurable functions $x \mapsto e^{x,i}_{jk}$, where $x \in \operatorname{supp}(p_i)$, such that $(e^{x,i}_{jk})_{1\leq j,k\leq n_i}$ is a system of (non-zero) matrix units (see [Bla06, p. 159] or [RLL00, pp. 109–110]) in F^x_i. Clearly

$$e^i_{jk} = \int e^{x,i}_{jk} d\mu_i(x)$$

is then a set of $n_i \times n_i$ matrix units in $p_iS(H_i)$.

Claim. Let $p \in Z(S(H_0H_1))$ be a central projection with $0 \neq p \leq p_0p_1$. Then, for $1 \leq j,k,a \leq n_0$ and $1 \leq l,m,b \leq n_1$,

(a) $pe^0_{jk}e^1_{lm} \neq 0$;

(b) $(pe^0_{ja}e^1_{lb})(pe^0_{ak}e^1_{bm}) = pe^0_{jk}e^1_{lm}$;

(c) $(pe^0_{jk}e^1_{lm})^* = pe^0_{kj}e^1_{ml}$.

Thus

$$\{pe^0_{jk}e^1_{lm} : 1 \leq j,k \leq n_0 \wedge 1 \leq l,m \leq n_1\}$$

forms a system of $n_0n_1 \times n_0n_1$ matrix units.

Proof of Claim. (b) and (c) follow by direct calculation from $(e^0_{jk})_{1\leq j,k\leq n_0}$ and $(e^1_{lm})_{1\leq l,m\leq n_1}$ being matrix units. To see (1) it is enough to observe that

$$p_{km} = (pe^0_{jk}e^1_{lm})^*(pe^0_{jk}e^1_{lm}) = pe^0_{kk}e^1_{mm} \neq 0\,.$$

Since all the projections p_{km} are Murray-von Neumann equivalent by (b) and (c) above, it follows that if one of them is equal to 0 then all of them must be equal to 0. Since

$$\sum_{k,m} p_{km} = \sum_{k,m} pe^0_{kk}e^1_{mm} = p\,,$$

not all p_{km} can be zero. Claim. $\dashv$

The previous claim clearly gives that F^x contains a system of non-zero $n_0n_1 \times n_0n_1$ matrix units for almost every $x \in \operatorname{supp}(p_0p_1)$.

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(3) We will not need this part of the lemma. All the same, to see that $F^x \simeq M_{n_0 n_1}(\mathbb{C})$ under the additional assumptions of (3), setting $p = p_0 p_1$ we just need to show that the corner $pS(H_0 H_1)$ is generated by the set

$$\{qe^0_{jk}e^1_{lm} : 1 \leq j,k \leq n_0 \wedge 1 \leq l,m \leq n_1 \wedge q \in Z(S(H_0H_1)) \wedge q \leq p\}.$$

This follows since, by the definition of the e^i_{jk}, the corner $pS(H_i)$ is generated by

$$\{pe^i_{jk} : 0 \leq i \leq 1 \wedge 1 \leq j,k \leq n_i\} \cup \{q \in Z(S(H_i)) : q \leq p\},$$

and it is clear that $Z(S(H_i)) \subseteq Z(S(H_0H_1))$, and $pS(H_0H_1)$ is generated by $pS(H_0) \cup pS(H_1)$. □

The next two lemmata show how Lemma 2.5 and 1 together can be used to achieve dimension growth in the factor decomposition in the presence of a sequence of commuting non-abelian subgroups.

Lemma 2.7. *Let $\frac{1}{2} > \varepsilon > 0$, $k > 1$. Let G be a group which contains a sequence of commuting subgroups $(G_i)_{i\in\mathbb{N}}$, and suppose the factor decompositions $S(G_i) = \int_{X_i} F_i^x d\mu_i(x)$ satisfy that the sets*

$$B_i = \{x : \dim(F_i^x) \geq k\}$$

have $\mu_i(B_i) > \frac{1}{2} - \varepsilon$. Then there is $N \in \mathbb{N}$ such that the factor decomposition $S(G_0 \cdots G_N) = \int F^x d\mu(x)$ satisfies

$$\mu_\tau(\{x : \dim(F^x) \geq k^2\}) > \frac{1}{2} - \varepsilon.$$

Proof. Let $G_\infty = \langle \bigcup_{i\in\mathbb{N}} G_i \rangle$, i.e., G_∞ is the group generated by $\bigcup_{i\in\mathbb{N}} G_i$. Notice that the center of $S(G_i)$ and of $S(G_1 \cdots G_N)$ are contained in the center of $S(G_\infty)$, so we may consider all the sets B_i as belonging to the same probability space, namely the probability space corresponding to $Z(S(G_\infty))$, with the measure μ_τ derived from the trace. By the previous lemma, it holds that for any $N \in \mathbb{N}$ the factor decomposition $S(G_1 \cdots G_N) = \int F^x d\mu_\tau(x)$ satisfies

$$\{x : \dim(F^x) > k\} \supseteq \bigcup_{i\leq N} B_i.$$

As $N \mapsto \mu_\tau(\bigcup_{i<N} B_i)$ is bounded by 1, there must be some N such that $\mu_\tau(B_N \setminus \bigcup_{i<N} B_i) < \mu_\tau(B_N) - \frac{1}{2} + \varepsilon$. Then, by the previous lemma applied to the groups

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$G_1 \cdots G_{N-1}$ and G_N we have for the factor decomposition $S(G_1 \cdots G_N) = \int F^x d\mu_\tau(x)$ that

$$\{x : \dim(F^x) \geq k^2\} \supseteq B_N \cap \bigcup_{i<N} B_i,$$

and by the choice of N, the set on the right hand side has measure greater than $\mu_\tau(B_n) - (\mu_\tau(B_n) - \frac{1}{2} + \varepsilon) = \frac{1}{2} - \varepsilon$. □

Lemma 2.8. *Let $\frac{1}{2} > \varepsilon > 0$, $k \geq 1$. Let G be a group which contains a sequence of commuting non-abelian subgroups $(G_i)_{i\in\mathbb{N}}$. Then there is $N \in \mathbb{N}$ such that for the direct integral decomposition $S(G_1 \cdots G_N) = \int F^x d\mu(x)$ it holds that*

$$\mu(\{x : \dim(F^x) \geq 2^{2^{k-1}}\}) > \frac{1}{2} - \varepsilon.$$

Proof. By induction on k. The case $k = 1$ follows from Lemma 2.5. The inductive step follows from the previous lemma. □

Proof of Theorem 2.4 (A). Let G_∞ be as in the proof of the previous lemma, and let $S(G_\infty) = \int F^x d\mu_\tau(x)$ be the direct integral decomposition, with μ_τ the measure derived from the trace.

CLAIM. $\mu_\tau(\{x : \dim(F^x) = \infty\}) \geq \frac{1}{2}$, and so $S(G_\infty)$ is not type I.

Proof of Claim. Fix $\varepsilon > 0$. It suffices to show that $\mu_\tau(\{x : \dim(F^x) > k\}) > \frac{1}{2} - \varepsilon$ for all $k \in \mathbb{N}$.

Using the previous lemma, find $N \in \mathbb{N}$ such that for the direct integral decomposition $S(G_1 \cdots G_N) = \int E^x d\mu_\tau(x)$ we have

$$\mu_\tau(\{x : \dim(E^x) > k\}) > \frac{1}{2} - \varepsilon.$$

Let $G_{N+1,\infty} = \langle \bigcup_{i>N} G_i \rangle$. Since $G_\infty = (G_1 \cdots G_N) G_{N+1,\infty}$, it follows by Lemma 1 that

$$\mu_\tau(\{x : \dim(F^x) > k\}) > \frac{1}{2} - \varepsilon.$$

Claim. ⊣

As $L(G) \supseteq S(G_\infty)$, it follows that $L(G)$ is not type I. □

REMARK. The reader may note that the sequence of Lemmata 2.5–2.8 can be used to give an explicit embedding of the hyperfinite II_1 factor R into a corner of $L(G)$ under the assumptions of Theorem 2.4 (A).

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2.3.2 Proof of part (B) of Theorem 2.4

We continue to let G denote a countable discrete group. The following lemma gives the sequence of subgroups needed for Theorem 2.4 (B).

Lemma 2.9. *Suppose G^{fc} is not abelian-by-finite. Then there are sequences $(g_i)_{i\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ of elements of G^{fc}, and a sequence of subgroups $(G_i)_{i\in\omega}$ of G^{fc}, having the following properties:*

1. *g_i and h_i do not commute.*
2. *$G_i = \langle [g_i]_{G^{\mathrm{fc}}}, [h_i]_{G^{\mathrm{fc}}}\rangle$. In particular, G_i is non-abelian by (1).*
3. *G_i and G_j are commuting subgroups of G whenever $i \neq j$ (i.e., all elements of G_i commute with all elements of G_j whenever $i \neq j$).*

Proof. We will define the sequences $(g_i)_{i\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ recursively and see that (1), (2) and (3) hold.

First pick $g_1, h_1 \in G^{\mathrm{fc}}$ to be any two non-commuting elements of G^{fc}. Suppose then that $g_1, \dots, g_k$ and $h_1, \dots, h_k$ have been defined, and that (1), (2) and (3) are satisfied. Let

$$K = \bigcup \{[g_i]_{G^{\mathrm{fc}}}, [h_i]_{G^{\mathrm{fc}}} : 1 \leq i \leq k\},$$

and note that K is finite and invariant under conjugation in G^{fc}. For each $g \in G^{\mathrm{fc}}$, let $\gamma_g : G^{\mathrm{fc}} \to G^{\mathrm{fc}} : h \mapsto ghg^{-1}$. Note that $\gamma_g(K) = K$, and each γ_g is completely determined by its values on K. Thus $\psi : g \mapsto \gamma_g \restriction K$ is a homomorphism from G^{fc} to S_K, the group of permutations of K. Since K is finite, $\ker(\psi)$ has finite index in G^{fc}, and since $aba^{-1} = b$ for any $a \in \ker(\psi)$ and $b \in K$, it follows that the elements of $\ker(\psi)$ commute with all elements in the subgroups $G_1, \dots, G_k$. Since G^{fc} is not abelian-by-finite, $\ker(\psi)$ is not abelian, and so we can pick non-commuting elements $g_{k+1}, h_{k+1} \in \ker(\psi)$. Since $\ker(\psi)$ is normal in G^{fc} we have $[g_{k+1}]_{G^{\mathrm{fc}}}, [h_{k+1}]_{G^{\mathrm{fc}}} \subseteq \ker(\psi)$, and so $G_{k+1} = \langle [g_{k+1}]_{G^{\mathrm{fc}}}, [h_{k+1}]_{G^{\mathrm{fc}}}\rangle$ commutes with all the subgroups $G_1, \dots, G_k$. □

Proof of Theorem 2.4 (B). If $[G : G^{\mathrm{fc}}] < \infty$ and G is not abelian-by-finite, then G^{fc} is not abelian-by-finite. By the previous lemma, G^{fc}, and therefore G, contains a sequence of non-abelian pairwise commuting subgroups, as required. □

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2.4 Kaniuth's generalization to locally compact SIN groups

Recall that a topological group is called a *SIN group* if the identity has a neighborhood basis which is conjugacy invariant. Kaniuth [Kan71] used Thoma's method to prove the following generalization of Thoma's theorem to locally compact SIN groups:

Theorem 2.10 ([Kan71])**.** *Let G be a locally compact* SIN *group. Then the following are equivalent*

1. *G is type I*
2. *$L(G)$ is type I*
3. *G contains a co-compact abelian subgroup.*

We point out that by [Dix77, Proposition 13.10.5], $L(G)$ is a finite von Neumann algebra (i.e., the factor decomposition consists of type I_n and II_1 factors) if and only if it G is SIN.

Sketch of proof of Theorem 2.10 in the separable case. The difficult implication is (2) implies (3). We will give the proof assuming that G is separable, and therefore metrizable, and later discuss how the non-separable case follows.

Fix a compatible 2-sided invariant metric on G. Find $c > 0$ such that the closed d-ball $B = B(e, c)$ of radius c around the identity is compact, and let

$$\xi(x) = \begin{cases} d(e,x) - c & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

Notice that ξ is conjugation invariant. After possibly multiplying ξ (or d) with a constant, we can assume that $\|\xi\|_{L^2(G,\mu)} = 1$. Define $\tau : L(G) \to \mathbb{C}$ by $\tau(a) = \langle a\xi, \xi\rangle$. The conjugation invariance of ξ ensures that τ is a trace on G.

Let G^{cc} denote the subgroup of G generated by elements whose conjugacy class has compact closure.

THE "EASY" CASE: $[G : G^{\mathrm{cc}}] = \infty$. Lemma 2.2.(1) with G^{cc} replacing G^{fc}, is well-known. In the proof of Lemma 2.2(2), if we take $h \notin G \setminus G^{\mathrm{cc}}$, then for any finite sum $s = \sum_{g \in G^{\mathrm{cc}}} \alpha_g u_g$ approximating a central projection, we can find a conjugate $\tilde{h}$ of h such that $g\tilde{h}^{-1} \notin B$ for all g occurring in the sum s. Then $\tau(su_{h^{-1}}) = \tau(su_{\tilde{h}^{-1}}) = 0$, and the proof goes through as before.

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Lemma 2.3 goes through unchanged.

THE "HARD" CASE $[G : G^{cc}] < \infty$. We show how to adapt Lemma 2.9. From that, part (B) of Theorem 2.4 follows directly. Furthermore, (A) works exactly as for countable discrete groups, so no changes are needed.

For Lemma 2.9 the assumption is that G^{cc} is not abelian-by-finite, i.e. there does not exist a closed abelian co-compact subgroup. Suppose that $g_1, \dots, g_k$ and $h_1, \dots, h_k$ are defined such that properties (1),(2),(3) are satisfied. Construct K by taking the union of the closures of the conjugacy classes, i.e.

$$K = \bigcup \{\overline{[g_i]}_{G^{cc}}, \overline{[h_i]}_{G^{cc}} : 1 \leq i \leq k\},$$

which again makes K compact and invariant under each γ_g for $g \in G^{cc}$. Note that since the metric is 2-sided invariant, each γ_g is in fact an isometry of K. From the compactness of K it follows that the group of isometries of K is also compact, whence $\ker(\psi)$ is co-compact, which guarantees that we can continue the construction as described in Lemma 2.9.

□

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3 Eight Cardinals in Cichoń's Diagram

3.1 The initial forcing

3.1.1 Good iterations

The forcing P^4 we are about to define has many pleasant properties because it is "good", a notion first explored in [JS90] and [Bre91]. We now recall the basic facts of good iterations, and specify the instances of the relations we use.

Assumption 3.1. *We will consider binary relations* R *on* $X = \omega^\omega$ *(or on* $X = 2^\omega$*) that satisfy the following: There are relations* R^n *such that* $\mathrm{R} = \bigcup_{n\in\omega} \mathrm{R}^n$*, each* R^n *is a closed subset (and in fact absolutely defined) of* $X \times X$*, and for* $g \in X$ *and* $n \in \omega$*, the set* $\{f \in X : f \,\mathrm{R}^n\, g\}$ *is nowhere dense. Also, for all* $g \in X$ *there is some* $f \in X$ *with* $f \,\mathrm{R}\, g$*.*

We will actually use another space as well, the space $\mathcal{C}$ of strictly positive rational sequences $(q_n)_{n\in\omega}$ such that $\sum_{n\in\omega} q_n \leq 1$. It is easy to see that $\mathcal{C}$ is homeomorphic to ω^ω, when we equip the rationals with the discrete topology and use the product topology.

We use the following instances of relations R on X; it is easy to see that they all satisfy the assumption (in case of $X = \mathcal{C}$ we use the homeomorphism mentioned above):

Definition 3.2.

1. $X = \mathcal{C}$: $f \,\mathrm{R}_1\, g$ *if* $(\forall^* n \in \omega)\, f(n) \leq g(n)$.
   (We use "$\forall^* n \in \omega$" for "$(\exists n_0 \in \omega)(\forall n > n_0)$".)

2. $X = 2^\omega$: $f \,\mathrm{R}_2\, g$ *if* $(\forall^* n \in \omega)\, f \restriction I_n \neq g \restriction I_n$,
   where $(I_n)_{n\in\omega}$ *is the increasing interval partition of* ω *with* $|I_n| = 2^{n+1}$.

3. $X = \omega^\omega$: $f \,\mathrm{R}_3\, g$ *if* $(\forall^* n \in \omega)\, f(n) \leq g(n)$.

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We say "f is bounded by g" if $f \mathrm{R} g$; and, for $\mathcal{Y} \subseteq \omega^\omega$, "$f$ is bounded by $\mathcal{Y}$" if $(\exists y \in \mathcal{Y})\, f \mathrm{R} y$. We say "unbounded" for "not bounded". (I.e., f is unbounded by $\mathcal{Y}$ if $(\forall y \in \mathcal{Y}) \neg f \mathrm{R} y$.) We call $\mathcal{X}$ an R-unbounded family, if $\neg(\exists g)\,(\forall x \in \mathcal{X})\, x \mathrm{R} g$, and an R-dominating family if $(\forall f)\,(\exists x \in \mathcal{X})\, f \mathrm{R} x$. Let $\mathfrak{b}_i$ be the minimal size of an $\mathrm{R_i}$-unbounded family, and $\mathfrak{d}_i$ of an $\mathrm{R_i}$-dominating family.

We only need the following connection between $\mathrm{R_i}$ and the cardinal characteristics:

Lemma 3.3.

1. $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_1$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}_1$.
2. $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_2$ *and* $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}_2$.
3. $\mathfrak{b} = \mathfrak{b}_3$ *and* $\mathfrak{d} = \mathfrak{d}_3$.

Proof. (3) holds by definition. (1) can be found in [BJ95, 6.5.B]. To prove (2), note that for fixed $g \in 2^\omega$ the set $\{f \in 2^\omega : \neg g \mathrm{R}_2 f\}$ is a null set, call it N_g. Let $\mathcal{G}$ be an R_2-unbounded family. Then $\{N_g : g \in \mathcal{G}\}$ covers 2^ω: Fix $f \in 2^\omega$. As f does not bound $\mathcal{G}$, there is some $g \in \mathcal{G}$ unbounded by f, i.e., $f \in N_g$. Let X be a non-null set. Then X is R_2-dominating: For any $g \in 2^\omega$ there is some $x \in X \setminus N_g$, i.e., $g \mathrm{R}_2 x$. □

Definition 3.4. *[JS90] Let P be a ccc forcing, λ an uncountable regular cardinal, and* R *as above. P is* **(R, λ)-good***, if for each P-name $r \in \omega^\omega$ there is (in V) a nonempty set $\mathcal{Y} \subseteq \omega^\omega$ of size $<\lambda$ such that every f (in V) that is* R*-unbounded by $\mathcal{Y}$ is forced to be* R*-unbounded by r as well.*

Note that λ-good trivially implies μ-good if $\mu \geq \lambda$ are regular.

How do we get good forcings? Let us just quote the following results:

Lemma 3.5. *A FS iteration of Cohen forcing is good for any* (R, λ)*, and the composition of two* (R, λ)*-good forcings is* (R, λ)*-good.*
Assume that $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ is a FS ccc iteration. Then P_δ is (R, λ)*-good, if each Q_α is forced to satisfy the following:*

1. *For* $\mathrm{R} = \mathrm{R}_1$*: $|Q_\alpha| < \lambda$, or Q_α is σ-centered, or Q_α is a sub-Boolean-algebra of the random algebra.*
2. *For* $\mathrm{R} = \mathrm{R}_2$*: $|Q_\alpha| < \lambda$, or Q_α is σ-centered.*
3. *For* $\mathrm{R} = \mathrm{R}_3$*: $|Q_\alpha| < \lambda$.*

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Proof. (R, λ)-goodness is preserved by FS ccc iterations (in particular compositions), as proved in [JS90], cf. [BJ95, 6.4.11–12]. Also, ccc forcings of size $<\lambda$ are (R, λ)-good [BJ95, 6.4.7], which takes care of the case of Cohens and of $|Q_\alpha| < \lambda$. So it remains to show that (for $i = 1, 2$) the "large" iterands in the list are ($\mathrm{R_i}$, λ)-good. For R_1 this follows from [JS90] and [Kam89], cf. [BJ95, 6.5.17–18]. For R_2 this is proven in [Bre91]. □

Lemma 3.6. *Let $\lambda \leq \kappa \leq \mu$ be uncountable regular cardinals. After forcing with μ many Cohen reals $(c_\alpha)_{\alpha\in\mu}$, followed by an* (R, λ)*-good forcing, we get: For every real r in the final extension, the set $\{\beta \in \kappa : c_\beta$ is unbounded by $r\}$ is cobounded in κ. I.e., $(\exists\alpha \in \kappa)\,(\forall\beta \in \kappa \setminus \alpha)\,\neg c_\beta \,\mathrm{R}\, r$.*

(The Cohen real c_β can be interpreted both as Cohen generic element of 2^ω and as Cohen generic element of ω^ω; we use the interpretation suitable for the relation R.)

Proof. Work in the intermediate extension after κ many Cohen reals, let us call it V_κ. The remaining forcing (i.e., $\mu \setminus \kappa$ many Cohens composed with the good forcing) is good; so applying Definition 3.4 we get (in V_κ) a set $\mathcal{Y}$ of size $<\lambda$.

As the initial Cohen extension is ccc, and $\kappa \geq \lambda$ is regular, we get some $\alpha \in \kappa$ such that each element y of $\mathcal{Y}$ already exists in the extension by the first α many Cohens, call it V_α. The set of reals M_y bounded by y is meager (and absolute). Any c_β for $\beta \in \kappa \setminus \alpha$ is Cohen over V_α, and therefore not in M_y, i.e., not bounded by y. As this holds for all y, c_β is unbounded by $\mathcal{Y}$, and thus, according to the definition of good, unbounded by r as well. □

In the light of this result, let us revisit Lemma 3.3 with some new notation:

Definition 3.7. *For $i = 1, 2, 3$, $\lambda > \aleph_0$ regular, and P a ccc forcing notion, let $\circledcirc_i(P, \lambda)$ stand for: "There is a sequence $(x_\alpha)_{\alpha\in\lambda}$ of P-names such that for every P-name y we have $(\exists\alpha \in \lambda)\,(\forall\beta \in \lambda \setminus \alpha)\; P \Vdash \neg x_\beta \,\mathrm{R_i}\, y$."*

Lemma 3.8. *$\circledcirc_i(P, \lambda)$ implies $\mathfrak{b}_i \leq \lambda$ and $\mathfrak{d}_i \geq \lambda$. In particular:*

1. *$\circledcirc_1(P, \lambda)$ implies $P \Vdash (\,\mathrm{add}(\mathcal{N}) \leq \lambda \,\&\, \mathrm{cof}(\mathcal{N}) \geq \lambda\,)$.*
2. *$\circledcirc_2(P, \lambda)$ implies $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \leq \lambda \,\&\, \mathrm{non}(\mathcal{N}) \geq \lambda\,)$.*
3. *$\circledcirc_3(P, \lambda)$ implies $P \Vdash (\,\mathfrak{b} \leq \lambda \,\&\, \mathfrak{d} \geq \lambda\,)$.*

Proof. The set $\{x_\alpha : \alpha \in \lambda\}$ is certainly forced to be $\mathrm{R_i}$-unbounded; and given a set $Y = \{y_j : j < \theta\}$ of $\theta < \lambda$ many P-names, each has a bound α_j, so for any $\beta \in \lambda$ above all α_j we get $P \Vdash \neg x_\beta \,\mathrm{R_i}\, y_j$ for all j; i.e., Y cannot be dominating. □

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3.1.2 Ground model Borel functions, partial random forcing

The following lemma seems to be well known (but we are not aware of a good reference or an established notation):

Definition 3.9. *Let Q be a forcing notion, and let η be a Q-name for a real. We say that Q is "generically Borel determined (by η, via B)", if*

- *Q consists of reals,*
- *the Q-generic filter is determined by the real η, and moreover:*
- *$B \subseteq \mathbb{R}^2$ is a Borel relation such that for all $q \in Q$, $Q \Vdash (\, B(q, \eta) \leftrightarrow q \in G\,)$.*

We investigate iterations of such forcings:

Lemma 3.10. *Assume that $(P_\beta, Q_\beta)_{\beta<\alpha}$ is a FS ccc iteration such that each Q_β is generically Borel determined (in an absolute way already fixed in V). Then for each P_α-name r of a real, there is (in the ground model) a Borel function $F : \mathbb{R}^\omega \to \mathbb{R}$ and a sequence $(\alpha_i)_{i\in\omega}$ of ordinals in α such that P_α forces $r = F((\eta_{\alpha_i})_{i\in\omega})$.*

Proof. We prove by induction on $\gamma \leq \alpha$:

- For all $p \in P_\gamma$ there is a Borel relation $B^p \subseteq \mathbb{R}^\omega$ and a sequence $(\alpha_i^p)_{i\in\omega}$ of elements of γ such that $P_\gamma \Vdash B^p((\eta_{\alpha_i^p})_{i\in\omega}) \leftrightarrow p \in G_\gamma$.
- For each P_γ-name r of a real, there is a Borel function F^r and a sequence $(\alpha_i^r)_{i\in\omega}$ of elements of γ such that $P_\gamma \Vdash r = F^r((\eta_{\alpha_i^p})_{i\in\omega})$.

The second item follows from the first, as we can use the countable maximal antichains that decide $r(n) = m$.

If γ is a limit ordinal, then P_γ has no new elements, so there is nothing to do.

So assume $\gamma = \zeta + 1$. By our assumption, Q_ζ is generically Borel determined from η_ζ via a Borel relation B_ζ. Consider $(p, q) \in P_\zeta * Q_\zeta$. This is in G_γ iff $p \in G_\zeta$ (which, by induction, is Borel) and $q \in G(\zeta)$. As q is a real, it is forced that $q = B^q((\alpha_i^q)_{i\in\omega})$. Moreover, P_ζ forces that Q_ζ forces that $q \in G(\zeta)$ iff $B_\zeta(\eta_\zeta, q)$ iff $B_\zeta(\eta_\zeta, B^q((\alpha_i^q)_{i\in\omega}))$. □

Definition 3.11. *Given $(P_\beta, Q_\beta)_{\beta<\alpha}$ as above, and some $w \subseteq \alpha$, we define the P_α-name $\mathbb{R}^w$ to consist of all reals r such that in the ground model there are a Borel function F and a sequence $(\alpha_i)_{i\in\omega}$ of elements of w such that $r = F((\eta_{\alpha_i})_{i\in\omega})$.*

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The following is straightforward:

Facts 3.12.

- *Set (in V) $\mu = (|w| + 2)^{\aleph_0}$. Then it is forced that $\mathbb{R}^w$ has cardinality $\leq\mu$.*
- *If $w' \supseteq w$, then (it is forced that) $\mathbb{R}^{w'} \supseteq \mathbb{R}^w$.*
- *If w is the increasing union of $(w_\alpha)_{\alpha\in\gamma}$ with $\mathrm{cf}(\gamma) \geq \omega_1$, then (it is forced that) $\mathbb{R}^w = \bigcup_{\alpha\in\gamma} \mathbb{R}^{w_\alpha}$.*
- *For every P_α-name r of a real there is a countable w such that (it is forced that) $r \in \mathbb{R}^w$.*

Definition 3.13. *Let $\mathbb{B}$ be (the definition of) random forcing, i.e., positive pruned trees T, ordered by inclusion. Given $(P_\beta, Q_\beta)_{\beta<\alpha}$ as above, $w \subseteq \alpha$, we define the P_α-name $\mathbb{B}^w := \mathbb{B} \cap \mathbb{R}^w$ and call it "partial random forcing defined from w".*

Clearly $\mathbb{B}^w$ is a subforcing (not necessarily a complete one) of $\mathbb{B}$, and if p, q in $\mathbb{B}^w$ are incompatible in $\mathbb{B}^w$ then they are incompatible in random forcing. In particular $\mathbb{B}^w$ is ccc.

Note that $\mathbb{B}^w$ is forced to be generically Borel determined, in way already fixed in V: The generic real η is defined by $\{\eta\} = \bigcap\{[s] \in G : s \in 2^{<\omega}\}$, and the Borel relation by "$\eta \in [T]$".

Remark 3.14. *In this section, we have provided a very explicit notion of "partial random", using Borel functions. The use of Borel functions is not essential, we could use any other method of calculating reals from generic reals at certain restricted positions, provided this method satisfies Facts 3.12. One such alternative definition has been used in [GMS16]: We can define the sub-forcing $P_\alpha \restriction w$ of P_α in a natural way, and require that it is a complete subforcing (which is a closure property of w). Then we can take Q_α to be random forcing as evaluated in the $P_\alpha \restriction w$-extension.*

While this approach is basically equivalent (and may seem slightly more natural than the artificial use of Borel functions), it has the disadvantage that we have to take care of the closure property of w.

Definition 3.15. *Analogously to "partial random", we define the "partial Hechler" and "partial amoeba" forcings.*

These forcings are generically Borel determined as well.

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3.1.3 The inital forcing P^4

Assume that λ is regular uncountable and $\mu < \lambda$ implies $\mu^{\aleph_0} < \lambda$. Then $|w| < \lambda$ implies that the size of a partial forcing defined by w is $<\lambda$.

Definition 3.16. *Assume GCH and let $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ be regular cardinals. Set $\delta_4 = \lambda_4 + \lambda_4$. Partition $\delta_4 \setminus \lambda_4$ into unbounded sets S^1, S^2, and S^3. Fix for each $\alpha \in \delta_4 \setminus \lambda_4$ some $w_\alpha \subseteq \alpha$ such that each $\{w_\alpha : \alpha \in S^i\}$ is cofinal in $[\delta_4]^{<\lambda_i}$.*[1]

We now define $P^4 = (P_\alpha, Q_\alpha)_{\alpha\in\delta_4}$ to be the FS ccc iteration which first adds λ_4 many Cohen reals, and such that for each $\alpha \in \delta_4 \setminus \lambda_4$,

$$\text{if } \alpha \text{ is in} \left\{ \begin{matrix} S^1 \\ S^2 \\ S^3 \end{matrix} \right\}, \text{ then } Q_\alpha \text{ is the partial} \left\{ \begin{matrix} \textit{amoeba} \\ \textit{random} \\ \textit{Hechler} \end{matrix} \right\} \text{forcing defined from } w_\alpha.$$

The forcing results in $2^{\aleph_0} = \lambda_4$, which follows from the following easy and well-known fact:

Lemma 3.17. *Let $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ be a FS ccc iteration of length δ such that each Q_α is forced to consist of real numbers, and set $\lambda(\delta) := (2+\delta)^{\aleph_0}$. Then $P_\delta \Vdash 2^{\aleph_0} \leq \lambda(\delta)$.*

Proof. By induction on δ, we show that there is a dense subforcing of $D_\delta \subseteq P_\delta$ of size $\leq\lambda(\delta)$. Then the continuum has size at most $\lambda(\delta)$ (as each name of a real corresponds to a countable sequence of antichains, labeled with $0, 1$, in P_δ, without loss of generality in D_δ).

For $\delta + 1$, $D_\delta \subseteq P_\delta$ is dense and has size $\leq\lambda(\delta)$, and Q_δ is forced to have size $\leq\lambda(\delta)$. Without loss of generality we can identify Q_δ with a subset of $\lambda(\delta)$. Let $D_{\delta+1}$ consist of $(p, \check{\alpha}) \in P_{\delta+1}$ such that $p \in D_\delta$ forces $\alpha \in Q_\delta$.

For δ limit, the union of D_α is dense in $P_\delta = \bigcup_{\alpha\in\delta} P_\alpha$. □

According to Lemma 3.5 P^4 is $(\mathrm{R}_i, \lambda_i)$-good for $i = 1, 2, 3$, so Lemmas 3.6 and 3.8 gives us:

Lemma 3.18. $\circledcirc_i(P^4, \kappa)$ *holds for $i = 1, 2, 3$ and each regular cardinal κ in $[\lambda_i, \lambda_4]$.*

So in particular, P^4 forces $\mathrm{add}(\mathcal{N}) \leq \lambda_1$, $\mathrm{cov}(\mathcal{N}) \leq \lambda_2$, $\mathfrak{b} \leq \lambda_3$ *and* $\mathrm{cof}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \mathfrak{d} = 2^{\aleph_0}$.

Theorem 3.19. *[Mej13, Thm. 2] P^4 forces* $\mathrm{add}(\mathcal{N}) = \lambda_1$, $\mathrm{cov}(\mathcal{N}) = \lambda_2$, $\mathfrak{b} = \lambda_3$, *and* $\mathfrak{d} = \lambda_4 = 2^{\aleph_0}$.

[1] I.e., if $\alpha \in S^i$ then $|w_\alpha| < \lambda_i$, and for all $u \subseteq \delta_4$, $|u| < \lambda_i$ there is some $\alpha \in S^i$ with $w_\alpha \supseteq u$.

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Proof. It is easy to see that the partial amoebas take care of $\mathrm{add}(\mathcal{N}) \geq \lambda_1$: Let $(N_i)_{i\in\mu}$, $\aleph_1 \leq \mu < \lambda_1$ be a family of P^4-names of null sets. Each N_i is a Borel code, i.e., a real, and therefore Borel-computed from some countable set $w^i \subseteq \delta_4$. The union of the w^i is a set w^* of size $\leq\mu$ that already Borel-decides all N_i. There is some $\beta \in S^1$ such that $w_\beta \supseteq w^*$, so the partial amoeba forcing at β sees all the null sets N_i and therefore covers their union.

Analogously one proves $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathfrak{b} \geq \lambda_3$. □

We will reformulate the proof for $\mathrm{cov}(\mathcal{N})$ in a cumbersome manner that can be conveniently used later on:

Lemma 3.20. *Let $\boxplus_2(P, \lambda, \mu)$ stand for: "P is a ccc forcing notion, and there is a $<\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(r_s)_{s\in S}$ of P-names for reals such that for each P-name N of a null set $(\exists s \in S)\,(\forall t \succ s)\; P \Vdash r_t \notin N$."*

- $\boxplus_2(P, \lambda, \mu)$ *implies* $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \geq \lambda \;\&\; \mathrm{non}(\mathcal{N}) \leq \mu\,)$.
- $\boxplus_2(P^4, \lambda_2, \lambda_4)$ *holds.*

Proof. $\mathrm{cov}(\mathcal{N}) \geq \lambda$: Fix $<\lambda$ many P-names N_α of null sets. Each real has a "lower bound" $s_\alpha \in S$, i.e., $P \Vdash r_t \notin N_\alpha$ whenever $t \succ s_\alpha$. Let $t \succ s_\alpha$ for all α (this is possible as S is directed). So $P \Vdash r_t \notin N_\alpha$ for every α, i.e., the union doesn't cover the reals.

$\mathrm{non}(\mathcal{N}) \leq \mu$, as the set of all r_s is not null: For every name N of a null set there is some $s \in S$ such that $P \Vdash r_s \notin N$.

For P^4, we set $S = S^2$, $s \prec t$ if $w_s \subseteq w_t$, and we let r_s be the partial random real added at s. A P^4 name for a null set N depends (in a Borel way) on a countable index set $w^* \subseteq \delta_4$. Fix some $s \in S^2$ such that $w_s \supseteq w^*$, and pick any $t \succ s$. Then w_t contains all information to calculate the null set N, and therefore the partial random r_t over w_t will avoid N. □

3.2 The Boolean ultrapower of the forcing

3.2.1 Boolean ultrapowers

Boolean ultrapowers generalize regular ultrapowers by using arbitrary Boolean algebras instead of the power set algebra.

Assumption 3.21. *κ is strongly compact, B is a κ-distributive, κ^+-cc, atomless complete Boolean algebra.*

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Lemma 3.22. *[KT64] Every κ-complete filter on B can be extended to a κ-complete ultrafilter U.*[2]

Proof. List the required properties of U as a set of propositional sentences in $\mathcal{L}_\kappa$ (a propositional language allowing conjunctions and disjunctions of any size $<\kappa$), using atomic formulas coding $b \in U$ and $b \notin U$ for $b \in B$. □

Assumption 3.23. *U is a κ-complete ultrafilter on B.*

Lemma 3.24. *There is a maximal antichain A_0 in B of size κ such that $A_0 \cap U = \emptyset$. In other words, U is not κ^+-complete.*

Proof. Let A_0 be a maximal antichain in the open dense set $B \setminus U$. As B is κ^+-cc, A_0 has size $\leq\kappa$. It cannot have size $<\kappa$, as U is κ-complete and therefore meets every antichain of size $<\kappa$. □

The Boolean algebra B can be used as forcing notion. As usual, V denotes the universe we start with, sometimes called the ground model. In the following, we will not actually force with B (or any other p.o.); we always remain in V, but we still use forcing notation. In particular, we call the usual B-names "forcing names".

Definition 3.25. *A **BUP-name** (or: labeled antichain) x is a function $A \to V$ whose domain is a maximal antichain. We may write $A(x)$ to denote A.*

Each BUP-name corresponds to a forcing-name[3] for an element of V. We will identify the BUP-name and the corresponding forcing-name. In turn, every forcing name τ for an element of V has a forcing-equivalent BUP-name.

In particular, we can calculate, for two BUP-names x and y, the Boolean value $[\![x = y]\!]$.[4]

Definition 3.26.

- *Two BUP-names x and y are **equivalent**, if $[\![x = y]\!] \in U$.*
- *For $v \in V$, let $\check{v}$ be a BUP-name-version of the standard name for v (unique up to equivalence).*

[2]For this, neither κ^+-cc nor atomless is required, and it is sufficient that B is κ-complete.

[3]More specifically, to the forcing-name $\{(\widetilde{x(a)}, a) : a \in A(x)\}$.

[4]We can calculate $[\![x = y]\!]$ more explicitly as follows: Pick some common refinement A' of $A(x)$ and $A(y)$. This defines in an obvious way BUP-names x' and y' both with domain A': For $a \in A'$ we set $x'(a) = x(\tilde{a})$ for $\tilde{a}$ the unique element of $A(x)$ above a. Then $[\![x = y]\!]$ is $\bigvee\{a \in A' : x'(a) = y'(a)\}$ (which is independent of the refinement A').

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- *The* ***Boolean ultrapower*** M^- *consists of the equivalence classes* $[x]$ *of BUP-names* x*; and we define* $[x] \in^- [y]$ *by* $[\![x \in y]\!] \in U$.
- $j^- : V \to M^-$ *maps* v *to* $[\check{v}]$.

We are interested in the $\in$-structure $(M^-, \in^-)$.

Given BUP-names $x_1, \dots, x_n$ and an $\in$-formula φ, the truth value $[\![\varphi^V(x_1, \dots, x_n)]\!]$ is well defined (it is the weakest element of B forcing that in the ground model $\varphi(x_1, \dots, x_n)$ holds, which makes sense as $x_1, \dots, x_n$ are guaranteed to be in the ground model).[5]

Lemma 3.27.

- *Łoś's theorem:* $(M^-, \in^-) \vDash \varphi([x_1], \dots, [x_n])$ *iff* $[\![\varphi^V(x_1, \dots, x_n)]\!] \in U$.
- $j^- : (V, \in) \to (M^-, \in^-)$ *is an elementary embedding.*
- *In particular,* $(M^-, \in^-)$ *is a ZFC model.*

Proof. Straightforward by the definition of equivalence and of $[x] \in^- [y]$, and by induction (using that U is a filter for $\varphi \wedge \psi$ and for $\exists v\, \varphi(v)$, and that it is an ultrafilter for $\neg\varphi$). For elementarity, note that $M^- \vDash \varphi([\check{x}_1], \dots, [\check{x}_n])$ iff $[\![\varphi^V(\check{x}_1, \dots, \check{x}_n)]\!] \in U$ iff $V \vDash \varphi(x_1, \dots, x_n)$. □

Lemma 3.28. $(M^-, \in^-)$ *is wellfounded.*

Proof. This is the standard argument, using the fact that U is σ-complete:

Assume $[x_{n+1}] \in^- [x_n]$ for $n \in \omega$. Choose a common refinement A of the antichains $A(x_n)$, Again, let x'_n be the BUP-names with domain A equivalent to x_n. So, by our assumption, $u_n := [\![x_{n+1} \in x_n]\!] = \bigvee\{a \in A : x'_{n+1}(a) \in x'_n(a)\}$ is in U for each n. As U is σ-complete, there is some $u \in U$ stronger than all u_n. This implies: If $a \in A$ is compatible with u, then a is compatible with u_n (for all n), and therefore $x'_{n+1}(a) \in x'_n(a)$ for all n, a contradiction. □

Definition 3.29. *Let* M *be the transitive collapse of* $(M^-, \in^-)$*, and let* $j : V \to M$ *be the composition of* j^- *with the collapse. We denote the collapse of* $[x]$ *by* x^U*. So in particular* $\check{v}^U = j(v)$.

[5] Equivalently, we can explicitly calculate $[\![\varphi^V(x_1, \dots, x_n)]\!]$ as follows: Chose a common refinement A' of $A(x_1), \dots, A(x_n)$, and set $[\![\varphi^V(x_1, \dots, x_n)]\!]$ to be $\bigvee\{a \in A' : \varphi(x'_1(a), \dots, x'_n(a))\}$; where again the BUP-names x'_i are the canonically defined BUP-names with domain A' that are equivalent to x_i.

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Lemma 3.30.

- *$M \vDash \varphi(x_1^U, \dots, x_n^U)$ iff $[\![\varphi^V(x_1, \dots, x_n)]\!] \in U$. In particular, $j : V \to M$ is an elementary embedding.*
- *If $|Y| < \kappa$, then $j(Y) = j''Y$. In particular, j restricted to κ is the identity. M is closed under $<\kappa$-sequences.*
- *$j(\kappa) \neq \kappa$. I.e., $\kappa = \mathrm{cr}(j)$.*

Proof. If $[x] \in j^-(Y)$, then we can refine the antichain $A(x)$ to some A' such that each $a \in A'$ either forces $x = y$ for some $y \in Y$, or $x \notin Y$. Without loss of generality (by taking suprema), we can assume different elements a of A' giving different values $y(a)$; i.e., A' has size $|Y| + 1 < \kappa$. So U selects an element a of A', and as $[\![x \in Y]\!] \in U$, this element a proves that $[x] = j^-(y(a))$.

We have already mentioned that there is a maximal antichain $A_0 = \{a_i : i \in \kappa\}$ of size κ such that $A_0 \cap U = \emptyset$. The BUP-name x with $A(x) = A_0$ and $x(a_i) = i$ satisfies $[x] \in^- j^-(\kappa)$, but is not equivalent to any $\check{v}$; so $\kappa \leq x^U < j(\kappa)$. □

As we have already mentioned, an arbitrary forcing-name for an element of V has a forcing-equivalent BUP-name, i.e., a maximal antichain labeled with elements of V. If τ is a forcing-name for an element of Y ($Y \in V$), then without loss of generality τ corresponds to a maximal antichain labeled with elements of Y. We call such an object y a "BUP-name for an element of $j(Y)$" (and not "for an element of Y", for the obvious reason: unlike in the case of a forcing extension, y^U is generally not in Y, but, by definition of $\in^-$, it is in $j(Y)$).

3.2.2 The algebra and the filter

We will now define the concrete Boolean algebra we are going to use:

Definition 3.31. *Assume GCH, let κ be strongly compact, and $\theta > \kappa$ regular.*

$P_{\kappa,\theta}$ is the forcing notion adding θ Cohen subsets of κ. More concretely: $P_{\kappa,\theta}$ consists of partial functions from θ to κ with domain of size $<\kappa$, ordered by extension. Let $f^* : \theta \to \kappa$ be the name of the generic function.

$\mathcal{B}_{\kappa,\theta}$ is the complete Boolean algebra generated by $P_{\kappa,\theta}$.

Clearly $\mathcal{B}_{\kappa,\theta}$ is κ^+-cc and κ-distributive, as $P_{\kappa,\theta}$ is even κ-closed.

Lemma 3.32. *There is a κ-complete ultrafilter U on $B = \mathcal{B}_{\kappa,\theta}$ such that:*

(a) The Boolean ultrapower gives an elementary embedding $j : V \to M$. M is closed under $<\kappa$-sequences.

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(b) *The elements x^U of M are exactly (the collapses of equivalence classes of) B-names x for elements of V; more concretely, a function from an antichain (of size κ) to V. We sometimes say "x^U is a mixture of κ many possibilities".*

Similarly, for $Y \in V$, the elements x^U of $j(Y)$ correspond to the B-names x of elements of Y, i.e., antichains labeled with elements of Y.

(c) *If $|A| < \kappa$, then $j''A = j(A)$. In particular, j restricted to κ is the identity.*

(d) *j has critical point κ, $\mathrm{cf}(j(\kappa)) = \theta$, and $\theta \leq j(\kappa) \leq \theta^+$.*

(e) *If $\lambda > \kappa$ is regular, then $\max(\theta, \lambda) \leq j(\lambda) < \max(\theta, \lambda)^+$.*

(f) *If S is a $<\lambda$-directed partial order, and $\kappa < \lambda$, then $j''S$ is cofinal in $j(S)$.*

(g) *If $\mathrm{cf}(\alpha) \neq \kappa$, then $j''\alpha$ is cofinal in $j(\alpha)$, so in particular $\mathrm{cf}(j(\alpha)) = \mathrm{cf}(\alpha)$.*

Proof. We have already seen (a)–(c).

((d)): For each $\delta \in \theta$, $f^*(\delta)$ is a forcing-name for an element of κ, and thus a BUP-name for an element of $j(\kappa)$. Let x be some other BUP-name for an element of $j(\kappa)$, i.e., an antichain A of size κ labeled with elements of κ. Let $\delta \in \theta$ be bigger than the supremum of $\mathrm{supp}(a)$ for each $a \in A$. We call such a pair (x, δ) "suitable", and set $b_{x,\delta} := [\![f^*(\delta) > x]\!]$. We claim that all these elements form a basis for a κ-complete filter. To see this, fix suitable pairs (x_i, δ_i) for $i < \mu$ where $\mu < \kappa$; we have to show that $\bigwedge_{i\in\mu} b_{x_i,\delta_i} \neq$. Enumerate $\{\delta_i : i \in \mu\}$ increasing (and without repetitions) as δ^j for $j \in \gamma \leq \mu$. Set $A_j = \{i : \delta_i = \delta^j\}$. Given q_j, define $q_{j+1} \in P_{\kappa,\theta}$ as follows: $q_{j+1} \leq q_j$; $\delta^j \in \mathrm{supp}(q_{j+1}) \subseteq \delta^j \cup \{\delta^j\}$; and $q_{j+1} \restriction \delta^j$ decides for all $i \in A_j$ the values of x_i to be some α_i; and $q_{j+1}(\delta^j) = \sup_{i\in A_j}(\alpha_i) + 1$. For $j \leq \gamma$ limit, let q_j be the union of $\{q_k : k < j\}$. Then q_γ is stronger than each b_{x_i,δ_i}.

As κ is strongly compact, we can extend the κ-complete filter generated by all b_{x_i,δ_i} to a κ-complete ultrafilter U. Then the sequence $(f^*(\delta)^U)_{\delta\in\theta}$ is strictly increasing (as $(f^*(\delta), \delta')$ is suitable for all $\delta < \delta'$) and cofinal in $j(\kappa)$ (as we have just seen); so $\mathrm{cf}(j(\kappa)) = \theta$.

((e)): We count all BUP-names for elements of $j(\lambda)$. As we can assume that the antichains are subsets of $P_{\kappa,\theta}$, which has size θ, and as λ is regular and GCH holds, we get $|j(\lambda)| \leq [\theta]^\kappa \times \lambda^\kappa = \max(\theta, \lambda)$.

((f)): An element x^U of $j(S)$ is a mixture of κ many possibilities in S. As $\kappa < \lambda$, there is some $t \in S$ above all the possibilities. Then $j(t) > x^U$.

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((g)): Set $\mu = \mathrm{cf}(\alpha)$, and pick an increasing cofinal sequence $\bar{\beta} = (\beta_i)_{i\in\mu}$ in α. $j(\bar{\beta})$ is increasing cofinal in $j(\alpha)$ (as this is absolute between M and V). If $\mu < \kappa$, then $j''\bar{\beta} = j(\bar{\beta})$, otherwise use (f). □

3.2.3 The ultrapower of a forcing notion

We now investigate the relation of a forcing notion $P \in V$ and its image $j(P) \in M$, which we use as a forcing notion over V. (Think of P as being one of the forcings of Section 3.1; it has no relation with the Boolean algebra B.)

Note that as $j(P) \in M$ and M is transitive, every $j(P)$-generic filter G over V is trivially generic over M as well, and we will use absoluteness between $M[G]$ and $V[G]$ to prove various properties of $j(P)$.

Lemma 3.33. *If P is κ-cc, then j gives a complete embedding from P into $j(P)$. I.e., $j''P$ is a complete subforcing of $j(P)$, and j is an isomorphism from P to $j''P$.*

Proof. It is clear that j is an isomorphism onto $j''P$: By definition the order $<_{j(P)}$ on $j(P)$ is $j(<_P)$, and by elementarity $p \leq_P q$ iff $j(q) <_{j(P)} j(p)$. Also, $p \perp q$ is preserved: $M \vDash p \perp_{j(P)} q$ by elementarity, so $p \perp_{j(P)} q$ holds in V (as $j(P) \in M$ and M is transitive).

It remains to be shown that each maximal antichain A of P is preserved, i.e., $j''A \subseteq j(P)$ is predense.

By our assumption, $|A| < \kappa$, so $j''A = j(A)$ (by Lemma 3.32((c))), which is maximal in M (by elementarity) and thus maximal in V (by absoluteness). □

Accordingly, we can canonically translate P-names into $j(P)$-names, etc.

For later reference, let us make this a bit more explicit: Let g be a P-name for a real (i.e., an element of ω^ω). Each $g(n)$ is decided by a maximal antichains A_n, where $a \in A_n$ forces $g(n) = g_{n,a} \in \omega$. Then the $j(P)$-name $j(g)$ corresponds to the antichains

$$j(A_n) = j''A_n, \text{ and } j(a) \text{ forces } j(g)(n) = g_{n,a} \text{ for each } a \in A_n. \tag{3.34}$$

Lemma 3.35. *If $P = (P_\alpha, Q_\alpha)_{\alpha<\delta}$ is a finite support (FS) ccc iteration of length δ, then $j(P)$ is a FS ccc iteration of length $j(\delta)$ (more formally: it is canonically equivalent to one).*

Proof. M certainly thinks that $j(P) = (P^*_\alpha, Q^*_\alpha)_{\alpha<j(\delta)}$ is a FS iteration of length $j(\delta)$.

By induction on α we define the FS ccc iteration $(\tilde{P}_\alpha, \tilde{Q}_\alpha)_{\alpha<j(\delta)}$ and show that P^*_α is a dense subforcing of $\tilde{P}_\alpha$: Assume this is already the case for P^*_α. M thinks that

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Q^*_α is a P^*_α-name, so we can interpret it as a $\tilde{P}_\alpha$-name and use it as $\tilde{Q}_\alpha$. Assume that (p, q) is an element (in V) of $\tilde{P}_\alpha * \tilde{Q}_\alpha$. So p forces that q is a name in M; we can increase p to some p' that decides q to be the name $q' \in M$. By induction we can further increase p' to $p'' \in P^*_\alpha$, then $(p'', q') \in P^*_{\alpha+1}$ is stronger than (p, q). (At limits there is nothing to do, as we use FS iterations.)

$j(P)$ is ccc, as any $A \subseteq j(P)$ of size $\aleph_1$ is in M (and M thinks that $j(P)$ is ccc). □

Similarly, we get:

- If $\tau = x^U$ is in M a $j(P)$-name for an element of $j(Z)$, then τ is a mixture of κ many P-names for an element of Z (i.e., the BUP-name x consists of an antichain $A \subseteq B$ labeled, without loss of generality, with P-names for elements of Z).

  (This is just the instance of "each $x^U \in j(Y)$ is a mixture of elements of Y", where we set Y to be the set[6] of P-names for elements of Z.)

- A $j(P)$-name τ for an element of $M[G]$ has an equivalent $j(P)$-name in M.

  (There is a maximal antichain A of $j(P)$ labeled with $j(P)$-names in M. As M is countably closed, this labeled antichain is in M, and gives a $j(P)$-name in M equivalent to τ.)

- In $V[G]$, $M[G]$ is closed under $<\kappa$ sequences.

  (We can assume the names to be in M and use $<\kappa$-closure.)

- In particular, every $j(P)$-name for a real, a Borel-code, a countable sequence of reals, etc., is in M (more formally: has an equivalent name in M).

- If each iterand is forced to consist of reals, then $j(P)$ forces the continuum to have size at most $|2 + j(\delta)|^{\aleph_0}$.

  (This follows from Lemma 3.17 as $j(P)$ also satisfies that each iterand consists of reals.)

3.2.4 Preservation of values of characteristics

Lemma 3.36. *Let λ be a regular uncountable cardinal and P a ccc forcing.*

[6]Formally: We set Y to be some set that contains representatives of each equivalence class of P-names of elements of Z.

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1. *Let* $\mathfrak{x}$ *be either* $\mathrm{add}(\mathcal{N})$ *or* $\mathfrak{b}$. *If* $P \Vdash \mathfrak{x} = \lambda$ *and* $\kappa \neq \lambda$, *then* $j(P) \Vdash \mathfrak{x} = \lambda$.

2. *Let* $\mathfrak{y}$ *be either* $\mathrm{cof}(\mathcal{N})$ *or* $\mathfrak{d}$. *If* $P \Vdash \mathfrak{y} \geq \lambda$ *and* $\kappa < \lambda$, *then* $j(P) \Vdash \mathfrak{y} \geq \lambda$.

3. *Let* $(\mathfrak{x}, \mathfrak{y})$ *be either* $(\mathfrak{b}, \mathfrak{d})$ *or* $(\mathrm{add}(\mathcal{N}), \mathrm{cof}(\mathcal{N}))$. *Then we get:*
   If $P \Vdash (\kappa < \mathfrak{x} \,\&\, \mathfrak{y} \leq \lambda)$ *then* $j(P) \Vdash \mathfrak{y} \leq \lambda$.

Proof. **(a)** We formulate the proof for $\mathrm{add}(\mathcal{N})$; the proof for $\mathfrak{b}$ is the same.

Let $\bar{N} = (N_i)_{i<\lambda}$ be P-names for an increasing sequence of null sets such that $\bigcup_{i<\lambda} N_i$ is not null. So in particular for every P-name N of a null set: $(\exists i_0 \in \lambda)(\forall i \in \lambda \setminus i_0)\, P \Vdash N_i \nsubseteq N$. (We can choose the i_0 in V due to ccc.)

Therefore M thinks that the same holds for the sequence $j(\bar{N})$ of $j(P)$-names of length $j(\lambda)$. So whenever N is a $j(P)$-name of a null set, we can assume without loss of generality that $N \in M$, so M thinks that from some i_0 on it is forced that $N_i \nsubseteq N$, which is absolute.

As $\kappa \neq \lambda$, we know that $j''\lambda$ is cofinal in $j(\lambda)$. So (since the sequence $j(\bar{N})$ is increasing) we can use $(j(N_i))_{i\in\lambda}$ and get the same property.

This shows that $j(P) \Vdash \mathrm{add}(\mathcal{N}) \leq \lambda$

For the other inequality, fix some $\chi < \lambda$, and $(N_i)_{i<\chi}$ a family of $j(P)$-names for null sets (without loss of generality each name is in M), and $p \in j(P)$.

- Case 1: $\kappa \geq \lambda$. Then the sequence $(N_i)_{i<\chi}$ (as well as p) is in M, and $M \vDash \big(p \Vdash \bigcup N_i \text{ null}\big)$; which is absolute.

- Case 2: $\kappa < \lambda$. Every N_i is a "mixture" of κ many P-names for null sets, so there is a single P-name N_i' such that P forces N_i' is superset of all the names involved. Therefore, $j(P)$ forces that $j(N_i') \supseteq N_i$. And P forces that $\bigcup_{i<\chi} N_i'$ is null, i.e., covered by some null set N^*. Then $j(P)$ forces that $j(N^*)$ covers $\bigcup_{i<\chi} N_i$.

(b) We show that a small set cannot be dominating: Fix a sequence $(f_i)_{i<\chi}$ of $j(P)$-names of reals, with $\chi < \lambda$. Each f_i corresponds to $\kappa < \lambda$ many possible P-names. As $\chi < \lambda$, there is a P-name g unbounded by all $\chi \times \kappa < \lambda$ many possible P-names. So if f is any of the possibilities, then P forces $g \nleq^* f$; and thus $j(P)$ forces $j(g) \nleq^* f_i$ for all i. So $j(P)$ forces $\mathfrak{d} \geq \lambda$.

The same proof works for $\mathrm{cof}(\mathcal{N})$ (using "the null set g is not a subset of any of the possible null sets").

(c) For $(\mathfrak{x}, \mathfrak{y}) = (\mathfrak{b}, \mathfrak{d})$: Fix a P-name of a dominating family $\bar{f} = (f_i)_{i\in\lambda}$.

We claim that $j(P)$ forces that $j''\bar{f} = (j(f_i))_{i<\lambda}$ is dominating. Let r be a $j(P)$-name of a real, i.e., a mixture of κ many possibilities (each possibility corresponding to a P-name for a real). As $P \Vdash \kappa < \mathfrak{b}$, P forces that these reals cannot be

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unbounded, i.e., there is a P-name $\alpha \in \lambda$ such that f_α is forced to dominate all the possibilities. By absoluteness, $j(P) \Vdash j(f_\alpha) >^* r$.

It remains to be shown that $j(P) \Vdash j(f_\alpha) \in j''\bar{f}$. (Note that α is just a P-name.) Fix a maximal antichain A in P deciding α, i.e., $a \in A$ forces $\alpha = \alpha(a)$. As j maps P completely into $j(P)$, $j''A$ is a maximal antichain in $j(P)$. So $j(P)$ forces that exactly on $j(a)$ for $a \in A$ is in the generic filter, cf. (3.34). Accordingly $j(f_\alpha) = j(f_{\alpha(a)}) \in j''\bar{f}$.

The proof for $\mathrm{cof}(\mathcal{N})$ is the same. □

For the other direction of the invariants, and the pair $(\mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{N}))$, we use the following two lemmas, which are reformulations of results of Shelah.[7]

Recall Definition 3.7 (which is useful because of Lemma 3.8 and satisfied for the inital forcing according to Lemma 3.18).

Lemma 3.37. *Assume $\circledcirc_{\mathrm{i}}(P, \lambda)$. Then $\circledcirc_{\mathrm{i}}(j(P), \mathrm{cf}(j(\lambda)))$.*
So if $\kappa \neq \lambda$, then $\circledcirc_{\mathrm{i}}(j(P), \lambda)$, and if $\kappa = \lambda$, then $\circledcirc_{\mathrm{i}}(j(P), \theta)$.

Proof. Let $\bar{y} = (y_\alpha)_{\alpha<\lambda}$ be the sequence of P-names witnessing $\circledcirc_{\mathrm{i}}(P, \lambda)$. Note that $j(\bar{y})$ is a sequence of length $j(\lambda)$; we denote the β-th element by $(j(\bar{y}))_\beta$. So M thinks: For every $j(P)$-name r of a real $(\exists \alpha \in j(\lambda))\,(\forall \beta \in j(\lambda) \setminus \alpha)\,\neg (j(\bar{y}))_\beta \,\mathrm{R_i}\, r$. This is absolute. In particular, pick in V a cofinal subset A of $j(\lambda)$ of order type $\mathrm{cf}(j(\lambda)) =: \mu$. Then $j(\bar{y}) \restriction A$ witnesses that $\circledcirc_{\mathrm{i}}(j(P), \mu)$ holds. □

We have seen in Lemma 3.20 that $\boxplus_2(P^4, \lambda_2, \lambda_4)$ holds and implies that P^4 forces $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathrm{non}(\mathcal{N}) \leq \lambda_4$ (the latter being trivial in the case of P^4).

Lemma 3.38. *Assume $\boxplus_2(P, \lambda, \mu)$. If $\kappa > \lambda$, then $\boxplus_2(j(P), \lambda, |j(\mu)|)$; if $\kappa < \lambda$, then $\boxplus_2(j(P), \lambda, \mu)$.*

Proof. Let $(S, \prec)$ and $\bar{r}$ witness $\boxplus_2(P, \lambda, \mu)$.

M thinks that

for each $j(P)$-name N of a null set

$$(\exists s \in j(S))\,(\forall t \in j(S))\, t \succ s \to j(P) \Vdash (j(\bar{r}))_t \notin N, \quad (*)$$

which is absolute.

If $\kappa > \lambda$, then $j(\lambda) = \lambda$, and $j(S)$ is λ-directed in M and therefore in V as well, and so we get $\boxplus_2(j(P), \lambda, |j(\mu)|)$.

[7] S. Shelah, personal communication.

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So assume $\kappa < \lambda$. We claim that $j''S$ and $j''\bar{r}$ witness $\boxplus_2(j(P), \lambda, \mu)$. $j''S$ is isomorphic to S, so directedness is trivial. Given a $j(P)$-name N, without loss of generality in M, there is in M a bound $s \in j(S)$ as in $(*)$. As $j''S$ is cofinal in $j(S)$ (according to Lemma 3.32((f))), there is some $s' \in S$ such that $j(s') \succ s$. Then for all $t' \succ s'$, i.e., $j(t') \succ j(s')$, we get $j(P) \Vdash j(r_t) \notin N$. □

3.2.5 The main theorem

We now have everything required for the main result:

Theorem 3.39. *Assume GCH and that* $\aleph_1 < \kappa_7 < \lambda_1 < \kappa_6 < \lambda_2 < \kappa_5 < \lambda_3 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7$ *are regular,* κ_i *strongly compact for* $i = 5, 6, 7$. *Then there is a ccc order* P^7 *forcing*

$$\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathrm{cov}(\mathcal{N}) = \lambda_2 < \mathfrak{b} = \lambda_3 < \\ < \mathfrak{d} = \lambda_4 < \mathrm{non}(\mathcal{N}) = \lambda_5 < \mathrm{cof}(\mathcal{N}) = \lambda_6 < 2^{\aleph_0} = \lambda_7.$$

Proof. Let $j_i : V \to M_i$ be the Boolean ultrapower embedding with $\mathrm{cf}(j(\kappa_i)) = \lambda_i$ (for $i = 5, 6, 7$). Recall that P^4 is an iteration of length δ_4. We set $P^5 := j_5(P^4)$, $P^6 := j_6(P^5)$, and $P^7 := j_7(P^6)$; and $\delta_5 := j_5(\delta_4)$, $\delta_6 := j_6(\delta_5)$ and $\delta_7 := j_7(\delta_6)$.

It is enough to show the following:

(a) P^i is a FS ccc iteration of length δ_i and forces $2^{\aleph_0} = \lambda_i$ for $i = 4, 5, 6, 7$.

(b) $P^i \Vdash (\,\mathrm{add}(\mathcal{N}) = \lambda_1 \,\&\, \mathfrak{b} = \lambda_3 \,\&\, \mathfrak{d} = \lambda_4\,)$ for $i = 4, 5, 6, 7$.

(c) $P^i \Vdash \mathrm{non}(\mathcal{N}) \geq \lambda_5$ for $i = 5, 6, 7$.
$P^i \Vdash \mathrm{cof}(\mathcal{N}) \geq \lambda_6$ for $i = 6, 7$.
$P^i \Vdash \mathrm{cov}(\mathcal{N}) \leq \lambda_2$ for $i = 4, 5, 6, 7$.

(d) $P^i \Vdash \mathrm{cof}(\mathcal{N}) = \lambda_6$ for $i = 6, 7$.

(e) $P^i \vDash (\,\mathrm{cov}(\mathcal{N}) \geq \lambda_2 \,\&\, \mathrm{non}(\mathcal{N}) \leq \lambda_5\,)$ for $i = 4, 5, 6, 7$.

((a)) was shown in Section 3.2.3.

((b)): For P^4 this is Theorem 3.19. For P^5 use Lemma 3.36 (using for $\mathfrak{d}$ that $\kappa_5 < \lambda_3$). Using the same lemma again we get the result for P^6 and P^7 (using that $\kappa_i < \lambda_3$ for $i = 6, 7$ as well.)

((c)): As $\kappa_5 > \lambda_2$, we have $\circledcirc_2(P^4, \kappa_5)$ (by Lemma 3.18), and thus $\circledcirc_2(P^5, \lambda_5)$ (by Lemma 3.37, as $\mathrm{cf}(j_5(\kappa_5)) = \lambda_5$), so $P^5 \Vdash \mathrm{non}(\mathcal{N}) \geq \lambda_5$ (Lemma 3.8). Repeating the same argument we get $\circledcirc_2(P^i, \lambda_5)$ for $i = 6, 7$ (as $\kappa_i \neq \lambda_5$ for $i = 6, 7$).

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Analogously, as $\kappa_6 > \lambda_1$, we start with $\circledcirc_1(P^4, \kappa_6)$, get $\circledcirc_1(P^5, \kappa_6)$ (as $\kappa_5 \neq \kappa_6$) and then $\circledcirc_1(P^6, \lambda_6)$ (as $\mathrm{cf}(j_6(\kappa_6)) = \lambda_6$) and $\circledcirc_1(P^7, \lambda_6)$ (again as $\kappa_7 \neq \lambda_6$). So we get thus $P^i \Vdash \mathrm{cof}(\mathcal{N}) \geq \lambda_6$ for $i = 6, 7$.

Similarly, $\circledcirc_2(P^4, \lambda_2)$ holds, which is preserved by all embeddings, so we get $\mathrm{cov}(\mathcal{N}) \leq \lambda_2$.

((d)): As P^6 forces the continuum to have size λ_6, the previous item implies $P^6 \Vdash \mathrm{cof}(\mathcal{N}) = \lambda_6$. And as in ((b)), this implies the same for P^7 (as $\kappa_7 < \lambda_1$, the value of $\mathrm{add}(\mathcal{N})$).

((e)): $\boxplus_2(P^4, \lambda_2, \lambda_4)$ holds (cf. Lemma 3.20). So by Lemma 3.38 for the case $\kappa > \lambda$, and as $|j_5(\lambda_4)| = \lambda_5$, according to Lemma 3.32((e)), $\boxplus_2(P^5, \lambda_2, \lambda_5)$ holds. I.e., P^5 forces $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathrm{non}(\mathcal{N}) \leq \lambda_5$ (the latter being trivial as the continuum has size λ_5). For $i = 6, 7$, the same lemma, now for the case $\kappa < \lambda$, gives $\boxplus_2(P^i, \lambda_2, \lambda_5)$, i.e., P^i forces $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathrm{non}(\mathcal{N}) \leq \lambda_5$. □

An alternative

In the same way we can prove the consistency of

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

(I.e., we can replace $\mathfrak{b}$ and $\mathfrak{d}$ by $\mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$, respectively.)

For this, we use the following relation as R_3:

$$f \,\mathrm{R}_3\, g, \text{ if } f, g \in \omega^\omega \text{ and } (\forall^* n \in \omega)\, f(n) \neq g(n).$$

By a result of [Mil82, Bar87] (cf. [BJ95, 2.4.1 and 2.4.7]) we have

$$\mathrm{non}(\mathcal{M}) = \mathfrak{b}_3 \text{ and } \mathrm{cov}(\mathcal{M}) = \mathfrak{d}_3.$$

As before, we use that an iteration where each iterand has size $<\lambda_3$ is $(\mathrm{R}_3, \lambda_3)$-good.

To define P^4, we use partial eventually different (instead of partial Hechler) forcings.

Unlike for $(\mathfrak{b}, \mathfrak{d})$, we do not know whether $\mathrm{non}(\mathcal{M}) = \lambda$ is generally preserved if $\kappa \neq \lambda$ and $\mathrm{cov}(\mathcal{M}) = \lambda$ is preserved if κ is small; but we can use the same argument for $(\mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}))$ that we have used for $(\mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{N}))$. So we can get the analog of Lemma 3.20 that proves that $\mathrm{non}(\mathcal{M})$ is large and $\mathrm{cov}(\mathcal{M})$ small; and $\circledcirc_3$ implies that $\mathrm{non}(\mathcal{M})$ is small and $\mathrm{cov}(\mathcal{M})$ large.

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Tutorial for introduction to mathematical analysis for physicists.

Organisational Activities

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