FAKULTÄT
FÜR !NFORMATIK
Faculty of Informatics

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
Bibliothek Your knowledge hub

Sequent-Type Calculi for Variants of Default Logic

MASTERARBEIT

zur Erlangung des akademischen Grades

Master of Science

im Rahmen des Studiums

Computational Logic

eingereicht von

Sopo Pkhakadze, B.Sc.
Matrikelnummer 01328435

an der Fakultät für Informatik
der Technischen Universität Wien

Betreuung: Ao.Univ.-Prof. Dr.techn. Hans Tompits

Wien, 12. August 2019

Sopo Pkhakadze	Hans Tompits

TU WIEN Bibliothek Your knowledge hub
Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Erklärung zur Verfassung der Arbeit

Sopo Pkhakadze, B.Sc.
Promenadegasse 25/7, 1170 Wien

Hiermit erkläre ich, dass ich diese Arbeit selbständig verfasst habe, dass ich die verwendeten Quellen und Hilfsmittel vollständig angegeben habe und dass ich die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken oder dem Internet im Wortlaut oder dem Sinn nach entnommen sind, auf jeden Fall unter Angabe der Quelle als Entlehnung kenntlich gemacht habe.

Wien, 12. August 2019

Sopo Pkhakadze

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU WIEN Bibliothek Your knowledge hub

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

"Anything worth doing is worth doing slowly.
— Mae West

TU Bibliothek WIEN Your knowledge hub
Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Acknowledgements

Herewith, I would like to thank everybody who was involved directly and indirectly in the process of finishing my master study.

First of all, I want to show my sincerest gratitude to my supervisor, Ao.Univ.-Prof. Dr. Hans Tompits, for his indispensable help and guidance for all the work related to my master's thesis. I want to thank him for helping me finding a suitable topic, for teaching me so much about logic and far beyond it, and for all interesting discussions and experiences I gained from participating in conferences where I had the privilege to present the results based on this thesis. I could not have achieved any of this without his help.

I would like to thank the Joint Commission of EMCL for accepting my application, providing funding, and giving me all the time I needed for the completion of my studies. Especially, I want to thank Sylvia Wünsch for years of her help in organisational and bureaucratical aspects, and for sending me so many reminders.

Above all, I thank my family for their unlimited support, patience, and love, and for always encouraging and believing in me, especially in the difficult moments when I needed them the most.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU WIEN Bibliothek Your knowledge hub

Bibliothek, Your knowledge hub
Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Kurzfassung

Sequenzenkalküle sind eine wichtige Beweismethode in der automatischen Deduktion. In dieser Arbeit führen wir solche Kalküle für zwei Versionen von Default Logik ein, nämlich einerseits für eine dreiwertige Form der Default Logik, eingeführt von Radzikowska, und der Disjunktiven Default Logik nach Gelfond, Lifschitz, Przymusinska und Truszczyński. Die erste Variante von Default Logik verwendet die bekannte dreiwertige Logik von Łukasiewicz als zugrundeliegenden logischen Apparat, während in der Disjunktiven Default Logik verallgemeinerte Default Regeln verwendet werden, die eine Auswahl von Konklusionen erlauben. Beide Kalküle wurden eingeführt um gewisse Probleme der üblichen Default Logik zu adressieren. Die Kalküle die wir beschreiben axiomatisieren das sogenannte *Brave Reasoning* und folgen der Methode von Bonatti, der solche Kalküle im Bereich des Nichtmonotonen Schließens postulierte. Ein besonderes Merkmal der Kalküle von Bonatti ist die Verwendung eines *komplementären Kalküls* der Ungültigkeit formalisiert und der für die Axiomatisierung der Konsistenzbedingungen von Defaults zuständig ist.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Abstract

Sequent-type proof systems constitute an important and widely-used class of calculi well-suited for analysing proof search. In this thesis, we introduce sequent-type calculi for two variants of default logic, viz., on the one hand, for three-valued default logic due to Radzikowska, and, on the other hand, for disjunctive default logic, due to Gelfond, Lifschitz, Przymusinska, and Truszczyński. The first variant of default logic employs Łukasiewicz's three-valued logic as the underlying base logic and the second variant generalises defaults by allowing a selection of consequents in defaults. Both versions have been introduced to address certain representational shortcomings of standard default logic. The calculi we introduce axiomatise brave reasoning for these versions of default logic, following the sequent method first introduced in the context of nonmonotonic reasoning by Bonatti, which employs a *complementary calculus* for axiomatising invalid formulas, taking care of expressing the consistency condition of defaults.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Contents

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Wien Bibliothek Your knowledge hub

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

CHAPTER 1

Introduction

"I deny that there is such a thing as originality.
All the artist can do is to bring his personality to bear.
If he is true to himself, he can't help but be different,
even unique, for no two persons are alike.
I do not strive to be different for the sake of
being different, but do not mind being different
if my difference is a result of my being myself."

Moondog (1969)

Sequent-type proof systems, first introduced in the 1930s by Gerhard Gentzen [23] for classical and intuitionistic logic, are among the basic calculi used in automated deduction for analysing proof search. In the area of nonmonotonic reasoning, Bonatti [12] introduced in the early 1990s sequent-style systems for default logic [51] and autoepistemic logic [42], and a few years later together with Olivetti [14] also for circumscription [39]. A distinguishing feature of these calculi is the usage of a *complementary calculus* for axiomatising invalid formulas, i.e., of non-theorems, taking care of formalising consistency conditions, which makes these calculi arguably particularly elegant and suitable for proof-complexity elaborations as, e.g., recently undertaken by Beyersdorff, Meier, Thomas, and Vollmer [7]. In a complementary calculus, the inference rules formalise the propagation of refutability instead of validity and establish invalidity by deduction and thus in a purely syntactic manner. Complementary calculi are also referred to as *refutation calculi* or *rejection calculi* and the first axiomatic treatment of rejection was done by Łukasiewicz in his formalisation of Aristotle's syllogistic [32].

In this thesis, we introduce sequent-type calculi for brave reasoning in the style of Bonatti [12] for two variants of default logic, viz., on the one hand, for three-valued default logic due to Radzikowska [50], and, on the other hand, for disjunctive default

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

logic, due to Gelfond, Lifschitz, Przymusinska, and Truszczyński [22]. The first variant of default logic employs Łukasiewicz's three-valued logic [31] as the underlying base logic and the second variant generalises defaults by allowing a selection of consequents in defaults, closely related to the answer-set semantics of disjunctive logic programs [21]. Both versions have been introduced to address certain representational shortcomings of standard default logic.

Our calculi consist of three parts, similar to Bonatti's calculus for standard default logic [51], viz. a sequent calculus for axiomatising validity in the underlying base logics, complementary *anti-sequent* calculi for axiomatising *non-theorems* of these logics, taking care of formalising the consistency conditions of defaults, and specific default inference rules. As far as three-valued logics are concerned, different kinds of sequent-style systems exist in the literature, like systems based on (two-sided) sequents [8, 4] in the style of Gentzen [23] employing additional non-standard rules, or using *hypersequents* [2], which are tuples of Gentzen-style sequents. In our sequent and anti-sequent calculi for Łukasiewicz's three-valued logic, we adopt the approach of Rousseau [56], which is a natural generalisation for many-valued logics of the classical two-sided sequent formulation of Gentzen. The respective calculi are obtained from a systematic construction for many-valued logics as described by Zach [74] and Bogojeski [10].

For the case of disjunctive default logic, the calculus we define employs the well-known sequent-type calculus as introduced by Gentzen [23] and an anti-sequent calculus due to Bonatti [12].

The thesis is organised as follows. In Chapter 2 we recapitulate the necessary background from classical logic and Łukasiewicz's three-valued logic. Afterwards, in Chapter 3, we review the different default-logic formalisms, viz. Reiter's original version of default logic [51], Radzikowska's three-valued default logic [50], and disjunctive default logic [22]. Then, in Section 4, we develop our calculi.

The results of thesis have been presented and published at international conferences. The result on the calculus for three-valued default logic appears in the proceedings of the *15th International Conference on Logic Programming and Nonmonotonic Reasoning* [47], and the calculus for disjunctive default logic was presented at the conference *Kurt Gödel's Legacy: Does Future lie in the Past?* [46].

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU WIEN Bibliothek Your knowledge hub

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

CHAPTER 2

Basic Concepts from Sentential Logics

*"It doesn't matter whether a cat is black or white,
if it catches mice it is a good cat."*

Deng Xiaoping

In this chapter, we recapitulate the elementary facts about the underlying logics employed in the default-reasoning formalisms discussed in this thesis. In particular, the logics needed in what follows are classical propositional logic and Łukasiewicz's three-valued logic. We first deal with classical propositional logic and afterwards, in Section 2.2, we continue with Łukasiewicz's three-valued logic [31].

2.1 Classical Propositional Logic

In this section, we introduce syntax, semantics, and a Hilbert-style proof theory for classical (two-valued) propositional logic, **PL**. Our exposition follows the presentation given by Smullyan [68] and Tompits [70].

2.1.1 Syntax

The alphabet, $\mathcal{L}_{\mathbf{PL}}$, of **PL** consists of the following pairwise disjoint classes of symbols:

(i) a countable set $\mathcal{P}$ of *propositional constants*;

(ii) the *truth constants* "$\top$" ("truth") and "$\bot$" ("falsehood");

(iii) the *primitive logical connectives* "$\neg$" ("negation") and "$\supset$" ("implication"), and

(iv) the *punctuation symbols* "(" ("left parenthesis") and ")" ("right parenthesis").

Definition 1 A *formula* (*of* **PL**) is given according to the following rules:

F_0: Any propositional constant from $\mathcal{P}$ and any truth constant is a formula.

F_1: If A is a formula, then $(\neg A)$ is a formula.

F_2: If A and B are formulas, then $(A \supset B)$ is a formula.

F_3: The only formulas are those given by $F_0 - F_2$. □

This definition can be made explicit as follows: A *formation sequence* is a finite sequence of strings from $\mathcal{L}_{\mathbf{PL}}$ such that each term of the sequence is either

- a propositional constant,
- a truth constant,
- of the form $(\neg A)$, where A is an earlier term of the sequence, or
- is of the form $(A \supset B)$, where A and B are earlier terms of the sequence.

Then, A is a formula iff there is a formation sequence whose last term is A. Such a sequence may also be called a formation sequence *for* A.

A formula formed according to clause F_0 is called an *atom*, or an *atomic formula*. A formula formed according to clauses F_1 or F_2 is called a *composite* formula, or a *molecule*.

In what follows, we will use the letters "P", "Q", "R",... (possibly appended with subscripts and/or with primes) or words from everyday English to refer to propositional constants, and we use the letters "A", "B", "C",... (again possibly appended with subscripts and/or with primes) to refer to arbitrary formulas (distinct such letters need not represent distinct formulas).

Besides the primitive connectives $\neg$ and $\supset$, we also make use of the standard connectives "$\vee$" ("disjunction"), "$\wedge$" ("conjunction"), and "$\equiv$" ("equivalence"), defined in the following way:

$$(A \vee B) := ((\neg A) \supset B));$$
$$(A \wedge B) := \neg((\neg A) \vee (\neg B)); \text{ and}$$
$$(A \equiv B) := ((A \supset B) \wedge (B \supset A)).$$

The connectives $\supset$ and $\equiv$ are often also referred to as *material implication* and *material equivalence*, respectively.[1] Alternatively, they are also sometimes respectively called *conditional* and *biconditional*.

[1] The term "material" refers to the fact that these connectives are defined for elements of the object language—in contradistinction to *logical* implication and equivalence (to be defined later on), which are elements of the *metalanguage*.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

For any formula A, we call $(\neg A)$ the *negation* of A, and for all formulas A and B, we refer to $(A \wedge B)$, $(A \vee B)$, and $(A \supset B)$ as the *conjunction*, *disjunction*, and *conditional* of A and B, respectively. In a conditional $(A \supset B)$, A is called the *antecedent* and B the *consequent* of that formula.

Now, since we are using only one kind of parentheses, the question may arise whether the association of a formula is unambiguously fixed by its parentheses. The following result shows that this is indeed the case (for a proof, cf., e.g., Kleene [27]).

Theorem 1 (Unique Decomposition Theorem) *For every formula A, exactly one of the following conditions holds:*

(i) *A is an atomic formula.*

(ii) *There is a unique formula B such that $A = (\neg B)$.*

(iii) *There is a unique pair of formulas B and C such that $A = (B \supset C)$.*

As customary, we will drop parentheses in formulas as long as no ambiguity arises. In particular, we will drop outermost parentheses.

We may use "$\circ$" to stand for any of the binary connectives $\wedge$, $\vee$, or $\supset$. Accordingly, "$(A \circ B)$" then means $(A \wedge B)$, $(A \vee B)$, or $(A \supset B)$ given that $\circ$ denotes $\wedge$, $\vee$, or $\supset$, respectively.

Strictly speaking, one must distinguish between a logical symbol *per se* and as being a member in a sequence of *occurrences* of logical symbols. The term "occurrence" is used in order to refer to the elements of the sequence in their status of being members thereof and to emphasise that different members may refer to the same symbol. That is to say, occurrences point to *specific members* in a sequence of formal objects. However, for simplicity, we will not always make this distinction explicit.

Definition 2 The *scope* of an occurrence of a connective in a formula is defined explicitly as follows:

Sc_1: The scope of an occurrence of the negation sign $\neg$ in a formula is the formula immediately to the right of that occurrence of $\neg$.

Sc_2: The scope of an occurrence of a binary connective $\circ \in \{\wedge, \vee, \supset, \equiv\}$ in a formula are the formulas immediately to the left and immediately to the right of that occurrence of $\circ$. □

Definition 3 The notion of an *immediate subformula* is given by the following conditions:

I_0: Atomic formulas have no immediate subformulas.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

I_1: $\neg A$ has A as immediate subformula and no others.

I_2: For a binary connective $\circ$, the immediate subformulas of $(A \circ B)$ are A and B (we refer to A and B as the *left immediate subformula* and the *right immediate subformula* of $(A \circ B)$, respectively). □

Definition 4 A formula B is a *subformula* of a formula A if there is a finite sequence of formulas whose first element is B, whose last element is A, and such that each element of the sequence (except the last) is an immediate subformula of the next.

If B is a subformula of A and if B is distinct from A, then B is a *proper* subformula of A. □

Hence, the subformula-relation enjoys the following properties:

S_1: If B is an immediate subformula of A or if B is identical with A, then B is a subformula of A.

S_2: If A is a subformula of B and B is a subformula of C, then A is a subformula of C.

For a formula C, we write "C_A" to indicate that C has a specified (consecutive) part A. Under this usage, C_B is the result of replacing this occurrence of A in C_A by a formula B.

Finally, the *logical complexity* (or *logical degree*) of a formula A, denoted by $d(A)$, is the number of logical connectives and quantifiers occurring in A. Thus:

(i) $d(A) = 0$ if A is an atomic formula;

(ii) $d(\neg A) = d(A) + 1$; and

(iii) $d(A \circ B) = d(A) + d(B) + 1$, where $\circ$ is a binary connective.

2.1.2 Semantics

Definition 5 A (*two-valued*) *interpretation* is a mapping I assigning to each propositional constant from $\mathcal{P}$ an element from $\{\mathbf{t}, \mathbf{f}\}$. □

The elements of $\{\mathbf{t}, \mathbf{f}\}$ are referred to as *truth values*, where $\mathbf{t}$ represents *truth* and $\mathbf{f}$ represents *falsity*. Intuitively, an atom P is considered true under an interpretation I if $I(P) = \mathbf{t}$ and false under I if $I(P) = \mathbf{f}$. Accordingly, $I(P)$ is the *truth value of P under I*. The notion of a truth value is then extended to arbitrary formulas as follows:

Definition 6 Let A be a formula of **PL** and I an interpretation. Then, the *truth value of A under I*, denoted by $\mathrm{V}^I(A)$, is determined as follows:

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

(a) if $A = \top$, then $\mathrm{V}^I(A) = \mathbf{t}$;

(b) if $A = \bot$, then $\mathrm{V}^I(A) = \mathbf{f}$;

(c) if A is an atomic formula, then $\mathrm{V}^I(A) = I(A)$;

(d) if $A = \neg B$, for some formula B, then

$$\mathrm{V}^I(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{V}^I(B) = \mathbf{f}, \\ \mathbf{f}, & \text{if } \mathrm{V}^I(B) = \mathbf{t}; \end{cases}$$

and

(e) if $A = (B \supset C)$, for some formulas B and C, then

$$\mathrm{V}^I(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{V}^I(B) = \mathbf{f} \text{ or } \mathrm{V}^I(C) = \mathbf{t}, \\ \mathbf{f}, & \text{otherwise.} \end{cases}$$

$\square$

From conditions (a)–(e), corresponding conditions for the defined connectives readily follow:

(f) if $A = (B \vee C)$, then

$$\mathrm{V}^I(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{V}^I(B) = \mathbf{t} \text{ or } \mathrm{V}^I(C) = \mathbf{t}, \\ \mathbf{f}, & \text{otherwise;} \end{cases}$$

(g) if $A = (B \wedge C)$, then

$$\mathrm{V}^I(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{V}^I(B) = \mathbf{t} \text{ and } \mathrm{V}^I(C) = \mathbf{t}, \\ \mathbf{f}, & \text{otherwise;} \end{cases}$$

and

(h) if $A = (B \equiv C)$, then

$$\mathrm{V}^I(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{V}^I(B) = \mathrm{V}^I(C), \\ \mathbf{f}, & \text{otherwise.} \end{cases}$$

Figure 2.1 compactly summarises the truth conditions for all sentential connectives in a usual truth-table form.

Given an interpretation I, a formula A is *true under* I iff $\mathrm{V}^I(A) = \mathbf{t}$, and *false under* I if $\mathrm{V}^I(A) = \mathbf{f}$. If A is true under I, then I is said to be a *model* of A, and if A is false under I, then I is a *countermodel* of A. If I is a countermodel of A, then we also say

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

$\neg$	
t	**f**
f	**t**

$\supset$	**t**	**f**
t	**t**	**f**
f	**t**	**t**

$\vee$	**t**	**f**
t	**t**	**t**
f	**t**	**f**

$\wedge$	**t**	**f**
t	**t**	**f**
f	**f**	**f**

$\equiv$	**t**	**f**
t	**t**	**f**
f	**f**	**t**

Figure 2.1: Truth tables for the connectives of **PL**.

that I *refutes* A. We call A *satisfiable* (*in* **PL**) if it has some model, and *falsifiable* (*in* **PL**), or *refutable* (*in* **PL**), if it has some countermodel. Moreover, A is *unsatisfiable* (*in* **PL**) if it has no model. Finally, A is a *tautology*, symbolically $\models_2 A$, if it is true in every interpretation, and *refutable* (*in* **PL**), symbolically $\not\models_2 A$, otherwise.

Clearly, A is a tautology iff $\neg A$ is unsatisfiable, and A is refutable iff $\neg A$ is satisfiable.

Below some well-known tautologies are listed:

Principle of identity: $A \supset A$.

Transitivity of implication: $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$.

Interchange of premisses: $(A \supset (B \supset C)) \equiv (B \supset (A \supset C))$.

Importation: $(A \supset (B \supset C)) \supset ((A \wedge B) \supset C))$.

Exportation: $((A \wedge B) \supset C) \supset (A \supset (B \supset C))$.

Reduction to disjunction: $(A \supset B) \equiv (\neg A \vee B)$.

Modus ponendo ponens: $(A \wedge (A \supset B)) \supset B$.

Modus tollendo tollens: $((A \supset B) \wedge \neg B) \supset \neg A$.

Ex falso sequitur quodlibet: $\neg A \supset (A \supset B)$. (Or, equivalently: $\bot \supset B$.)

Verum sequitur ex quodlibet: $A \supset (B \supset A)$. (Or, equivalently: $B \supset \top$.)

Tertium non datur: $A \vee \neg A$.

Principium contradictionis: $\neg(A \wedge \neg A)$.

Law of double negation: $\neg\neg A \equiv A$.

De Morgan's Laws:

$\neg(A \wedge B) \equiv (\neg A \vee \neg B)$.

$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

Distributive Laws:

$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.

A set of formulas is also referred to as a *theory*. An interpretation I is a model of a theory T if I is a model of all elements of T, otherwise I is a countermodel of T. If a theory T has a model, then T is satisfiable, and if T has a countermodel, then T is falsifiable. A theory is unsatisfiable if it has no model.

A formula A is a *valid consequence* of a theory T (*in* **PL**), or T *entails* A (*in* **PL**), in symbols $T \models_2 A$, iff A is true in any model of T. Two formulas, A and B, are (*logically*) *equivalent* (*in* **PL**) iff $\models_2 (A \equiv B)$. In general, two theories are (logically) equivalent iff they have the same models.

As customary, we will write expressions like "$T \cup \{A\} \models_2 B$" as "$T, A \models_2 B$", and similarly for finite sets of form $\{A_1, \ldots, A_n\}$ instead of a singleton set $\{A\}$. Note that, for any formula B, it holds that $\models_2 B$ iff $\emptyset \models_2 B$, i.e., tautologies are precisely those formulas which are consequences of the empty set.

We next summarise some basic properties of entailment.

Theorem 2 (Replacement Theorem) *Let C_A be a formula containing a specific occurrence of a formula A, let C_B be the result of replacing that occurrence of A in C_A by B, and let T be a theory.*

If $T \models_2 (A \equiv B)$, then $T \models_2 (C_A \equiv C_B)$.

Theorem 3 *Let T, T' be theories and A, B formulas.*

(i) *$T \models_2 A$ iff $T \cup \{\neg A\}$ is unsatisfiable.*

(ii) *$T, A \models_2 B$ iff $T \models_2 (A \supset B)$.* (*"Deduction Theorem."*)

(iii) *If $T \subseteq T'$, then $\{A \mid T \models_2 A\} \subseteq \{A \mid T' \models_2 A\}$.* (*"Monotonicity of Valid Consequence."*)

2.1.3 A Hilbert-Type Proof System

"Deductive reasoning, that's the name of the game."

Lex Luthor (Superman: The Movie, 1978)

Having so far introduced the syntax and semantics of classical propositional logic, we now continue with discussing the proof theory of it. Many different proof methods exist in the literature, among them Hilbert-type systems, resolution methods, tableau systems,

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU WIEN Bibliothek Your knowledge hub

natural deduction, and sequent-style methods. Here, we present a Hilbert calculus for **PL** following Łukaszewicz [35]. In Chapter 4, we will discuss Gentzen-type systems for **PL** and Łukasiewicz's three-valued logic as part of our calculi for the default logics considered in this thesis.

Preparatory for introducing our Hilbert-type proof system, let us first recapitulate some basic facts about axiom systems in general, following Smullyan [68].

Definition 7 An *axiom system*, $\mathcal{A}$, is a triple $\langle D, A, R \rangle$, where

(i) D is a set whose elements are called *formal objects*,

(ii) A is a subset of D whose elements are called *axioms*, and

(iii) R is a set of relations on D called *inference rules*.

If $I \in R$ is an inference rule and if $\langle X_1, \ldots, X_n, Y \rangle \in I$ holds, then $\langle X_1, \ldots, X_n, Y \rangle$ is called an *application* of I, and that Y is a *direct consequence* of $X_1, \ldots, X_n$ under I, or that Y is *directly derivable* from $X_1, \ldots, X_n$ under I. Moreover, in any application $\langle X_1, \ldots, X_n, Y \rangle$ of I, the elements $X_1, \ldots, X_n$ are the *premisses* of the application, and Y is the *conclusion* of the application.

A (*formal*) *proof* in $\mathcal{A}$ is a finite sequence $X_1, \ldots, X_n$ of formal objects in D such that each element in the sequence is either

- an axiom of $\mathcal{A}$ or
- is directly derivable from one or more earlier elements of the sequence under one of the inference rules of $\mathcal{A}$.

A proof $X_1, \ldots, X_n$ is also called a proof *of* X_n. A formal object X is *provable* in $\mathcal{A}$, or is a (*formal*) *theorem* of $\mathcal{A}$, symbolically $\vdash_{\mathcal{A}} X$, iff there is a proof of X in $\mathcal{A}$. □

In a Hilbert-type axiom system, the formal objects are formulas of propositional logic while in a sequent-style system, the formal objects are somewhat more involved. In particular, forestalling our discussion in Chapter 4, a sequent for propositional logic in the style of Gentzen [23] is a pair $\Gamma \to \Delta$, where Γ and Δ are finite sets of formulas of **PL**, having the intuitive meaning that whenever all formulas in Γ are true, then at least one formula in Δ must be true as well.

Returning to axiom systems in general, following Kleene [27], we use the term "postulate" collectively for axioms and inference rules. It is customary to represent an infinite set of axioms by means of a so-called *axiom scheme* which specifies the set of all concrete formulas of a specific form. Inference rules, on the other hand, are usually displayed in the form of a figure in which a horizontal line is drawn, the premisses are written above the line and the conclusion below the line.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

For instance, the rule of *modus ponens*, which states that B is a direct consequence of A and $A \supset B$, is displayed thus:

$$\frac{A \quad (A \supset B)}{B}.$$

We are now ready to introduce the Hilbert-type proof system which we employ for our purposes.

Definition 8 The Hilbert-type axiom system H_2 for **PL** is given by the triple

$$\langle D_{\mathbf{PL}}, A_{\mathbf{PL}}, R_{\mathbf{PL}} \rangle,$$

where

(i) $D_{\mathbf{PL}}$ is the set of formulas of **PL**,

(ii) $A_{\mathbf{PL}}$ is given by the set of all axioms specified by the following axiom schemas:

(A_0) $\top$,

(A_1) $A \supset (B \supset A)$,

(A_2) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$,

(A_3) $(\neg A \supset \neg \mathsf{B}) \supset ((\neg A \supset B) \supset B)$,

and

(iii) $R_{\mathbf{PL}}$ consists of modus ponens as the single inference rule. $\square$

If a formula A is provable in H_2, we denote this by writing "$\vdash_2 A$" instead of the more cumbersome notation $\vdash_{\mathsf{H}_2} A$.

Let T be a theory. A *derivation from* T (*in* H_2) is a formal proof in the axiom system $\langle D_{\mathbf{PL}}, A_{\mathbf{PL}} \cup T, R_{\mathbf{PL}} \rangle$, i.e., elements of T may be used as additional axioms (called *premisses*) in a derivation from T. If A is the last element in a derivation from T, then A is *derivable from* T, or is a *syntactic consequence from* T, symbolically $T \vdash_2 A$. Clearly, as in the case of semantic consequence, it holds that

$$\vdash_2 A \quad \text{iff} \quad \emptyset \vdash_2 A,$$

i.e., a formula is provable in H_2 iff it is derivable from the empty set in H_2.

If, for a theory T and a formula A, $T \vdash_2 A$ does not hold, then we indicate this by writing $T \nvdash_2 A$, and similarly for a formula not provable in H_2. In Chapter 4, we will define an axiom system whose provable formal objects exactly correspond to formulas which are *not* provable in H_2. Such axiom systems are accordingly also referred to as

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

complementary calculi as they axiomatise the *complement* of the provable formulas of a logic.

The *deductive closure operator*, $\mathrm{Th}_2(\cdot)$, of **PL** is given by

$$\mathrm{Th}_2(T) := \{A \mid T \vdash_2 A\},$$

where T is a theory. A theory T is called *deductively closed* iff $T = \mathrm{Th}_2(T)$.

The next result summarises the usual closure properties the operator $\mathrm{Th}_2(\cdot)$ satisfies.

Theorem 4 *For any theory T, the following conditions hold:*

(i) $T \subseteq \mathrm{Th}_2(T)$; (*"Inflationaryness."*)

(ii) $\mathrm{Th}_2(\mathrm{Th}_2(T)) = \mathrm{Th}_2(T)$; (*"Idempotency."*)

(iii) $T \subseteq T'$ *implies* $\mathrm{Th}_2(T) \subseteq \mathrm{Th}_2(T')$. (*"Monotonicity."*)

The adequacy of our Hilbert-type axiom system is reflected by the following well-known result:

Theorem 5 *For any theory T and any formula A, the following conditions hold:*

(i) *If $T \vdash_2 A$, then $T \models_2 A$.* (*"Soundness Theorem."*)

(ii) *If $T \models_2 A$, then $T \vdash_2 A$.* (*"Completeness Theorem."*)

We say that a theory T is *consistent* if there is a formula A such that $T \nvdash_2 A$, otherwise T is *inconsistent*. Moreover, a formula A is *consistent with* T iff $T \nvdash_2 \neg A$. It holds that consistency is the syntactic counterpart of satisfiability, i.e., we have that T is consistent iff T is satisfiable. Moreover, the soundness and completeness of our axiom system for **PL** also yields syntactic counterparts of the properties of Theorems 2 and 3:

Theorem 6 *Under the circumstances of Theorems 2 and 3, the following conditions hold:*

(i) *If $T \vdash_2 (A \equiv B)$, then $T \vdash_2 (C_A \equiv C_B)$.* (*"Replacement Theorem."*)

(ii) *$T \vdash_2 A$ iff $T \cup \{\neg A\}$ is inconsistent.*

(iii) *$T, A \vdash_2 B$ iff $T \vdash_2 (A \supset B)$.* (*"Deduction Theorem."*)

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

2.2 Łukasiewicz's Three-Valued Logic

We now turn to present the basic elements of the three-valued logic of Łukasiewicz [31] for the propositional case, henceforth denoted by $\mathbf{Ł_3}$.

The idea of dealing with a logic transcending a two-valued setting can be traced back already to Aristotle. In *De Interpretatione* ("On Interpretation"), he suggests to assign future contingent propositions like "there will be a sea battle tomorrow"—which cannot be evaluated by truth and falsity alone—a third logical status. Future contingencies were further elaborated in medieval times, e.g., by Duns Scotus (ca. 1266–1308), William of Ockham (ca. 1287–1347), and Peter de Rivo (ca. 1420–1499), which mostly followed the discussion by Thomas of Aquino (1225–1274). More recent discussions about three-valued logical constructions arose at the end of the 19th century, put forward, e.g., by Hugh MacColl, Charles Sanders Peirce, and Nicolai Alexandrovich Vasiliev.

The first appearance of many-valuedness in modern logic can be ascribed to the farewell lecture of Jan Łukasiewicz given in the Assembly Hall of the University of Warsaw University on March 7, 1918. This was then followed by his paper *O logice trójwartościowej* ("On Three-valued logic") [29] in 1920. Almost simultaneously, Emil Post [49] proposed finite-valued propositional systems too while subsequent three-valued systems were introduced by Kleene [26, 27] and Bochvar [9].[2]

In what follows, we provide the necessary basics of Łukasiewicz's logic $\mathbf{Ł_3}$, following the discussion given by Radzikowska [50].

2.2.1 Syntax

The alphabet, $\mathcal{L}_{\mathbf{Ł_3}}$, of $\mathbf{Ł_3}$ consists of the alphabet $\mathcal{L}_{\mathbf{PL}}$ of $\mathbf{PL}$ along with the additional truth constant $\sqcup$ ("undetermined"). Again, we assume $\mathcal{P}$ as a countable set of propositional constants.

The class of formulas of $\mathbf{Ł_3}$ is built similarly to the formulas of $\mathbf{PL}$, except that $\sqcup$ is counted as an additional atomic formula.

A difference to the syntax of $\mathbf{PL}$ concerns the defined connectives: while conjunction, $\wedge$, and material equivalence, $\equiv$, are defined as in propositional logic, disjunction in $\mathbf{Ł_3}$ is defined differently:

$$(A \mathbin{\bar{\vee}} B) := ((A \supset B) \supset B).$$

Furthermore, there are also additional unary defined operators, viz.

- the connective "$\sim$" ("weak negation"), given by

$$\sim A := (A \supset \neg A),$$

[2] For a thorough discussion of the different many-valued approaches and their development, we refer to the well-known textbook by Malinowski [36] and his overview article in the *Handbook of the History of Logic* [37].

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

and

- the unary operators "L" ("certainty operator") and "M" ("possibility operator"), defined by

$$\mathrm{L}A := \neg(A \supset \neg A) \text{ and}$$
$$\mathrm{M}A := (\neg A \supset A).$$

which, according to Łukasiewicz [31], were first formalised in 1921 by Tarski, and

- the operator "I", given by

$$\mathrm{I}A := (\mathrm{M}A \wedge \neg \mathrm{L}A).$$

Intuitively, LA expresses that A is certain, whilst MA means that A is possible. These operators will be used subsequently to distinguish between certain knowledge and defeasible conclusions. Furthermore, IA expresses that A is *contingent* or *modally indifferent*.

The notions of scope, subformula, and degree are defined similarly as in **PL**.

2.2.2 Semantics

A (*three-valued*) *interpretation* is a mapping m assigning to each propositional constant from $\mathcal{P}$ an element from $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$. Here, besides the truth values $\mathbf{t}$ and $\mathbf{f}$, the symbol $\mathbf{u}$ represents a truth value standing for "undetermined" or "indeterminacy". Again, an atom P is considered true under an interpretation I if $m(P) = \mathbf{t}$, false under m if $m(P) = \mathbf{f}$, and has undetermined truth value if $m(P) = \mathbf{u}$. As well, $m(P)$ is the *truth value of* P *under* m. In what follows, we presume a total order $\leq$ over the truth values such that $\mathbf{f} \leq \mathbf{u} \leq \mathbf{t}$ holds.

The notion of a truth value is now extended to arbitrary formulas as follows:

Definition 9 Let A be a formula of $\mathbf{Ł_3}$ and m a three-valued interpretation. Then, the *truth value of* A *under* m, denoted by $\mathrm{V}^m(A)$, is determined as follows:

(a) if $A = \top$, then $\mathrm{V}^m(A) = \mathbf{t}$;

(b) if $A = \sqcup$, then $\mathrm{V}^m(A) = \mathbf{u}$;

(c) if $A = \bot$, then $\mathrm{V}^m(A) = \mathbf{f}$;

(d) if A is an atomic formula, then $\mathrm{V}^m(A) = m(A)$;

(e) if $A = \neg B$, for some formula B, then

$$\mathrm{V}^m(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{V}^m(B) = \mathbf{f}, \\ \mathbf{u}, & \text{if } \mathrm{V}^m(B) = \mathbf{u}, \\ \mathbf{f}, & \text{if } \mathrm{V}^m(B) = \mathbf{t}; \end{cases}$$

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Wien Bibliothek Your knowledge hub

(f) if $A = (B \supset C)$, for some formulas B and C, then

$$\mathrm{V}^m(A) = \begin{cases} \mathbf{t}, & \text{if either } \mathrm{V}^m(B) = \mathbf{f},\ \mathrm{V}^m(C) = \mathbf{t}, \text{ or } \mathrm{V}^m(B) = \mathrm{V}^m(C) = \mathbf{u}, \\ \mathbf{f}, & \text{if } \mathrm{V}^m(B) = \mathbf{t} \text{ and } \mathrm{V}^m(C) = \mathbf{f}, \\ \mathbf{u}, & \text{otherwise.} \end{cases}$$

□

From conditions (a)–(f), corresponding conditions for the remaining operators readily follow:

(g) if $A = (B \bar{\vee} C)$, then $\mathrm{V}^m(A) = \max(\mathrm{V}^m(B), \mathrm{V}^m(C))$;

(h) if $A = (B \wedge C)$, then $\mathrm{V}^m(A) = \min(\mathrm{V}^m(B), \mathrm{V}^m(C))$;

(i) if $A = (B \equiv C)$, then $\mathrm{V}^m(A) = \min(\mathrm{V}^m(B \supset C), \mathrm{V}^m(C \supset B))$;

(j) if $A = \sim B$, then

$$\mathrm{V}^m(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{V}^m(B) \in \{\mathbf{u}, \mathbf{f}\}, \\ \mathbf{f}, & \text{otherwise;} \end{cases}$$

(k) if $A = \mathrm{L}B$, then

$$\mathrm{V}^m(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{V}^m(B) = \mathbf{t}, \\ \mathbf{f}, & \text{otherwise;} \end{cases}$$

(l) if $A = \mathrm{M}B$, then

$$\mathrm{V}^m(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{V}^m(B) \in \{\mathbf{t}, \mathbf{u}\}, \\ \mathbf{f}, & \text{otherwise;} \end{cases}$$

and

(m) if $A = \mathrm{I}B$, then

$$\mathrm{V}^m(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{V}^m(B) = \mathbf{u}, \\ \mathbf{f}, & \text{otherwise.} \end{cases}$$

In Figure 2.2, above truth conditions are compactly represented in a usual truth-table form.

If $\mathrm{V}^m(A) = \mathbf{t}$, then A is *true under m*, if $\mathrm{V}^m(A) = \mathbf{u}$, then A is *undetermined under m*, and if $\mathrm{V}^m(A) = \mathbf{f}$, then A is *false under m*. If A is true under m, then m is a *model* of A, otherwise m is a *countermodel* of A. The notions of a formula being *satisfiable*, *falsifiable*, *refutable*, and *unsatisfiable*, is defined analogously in **Ł3** as for **PL**. Furthermore, A is *valid (in* **Ł3**), symbolically $\models_3 A$, if it is true in every three-valued interpretation, and *refutable (in* **Ł3**), symbolically $\not\models_3 A$, otherwise.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

$\neg$	
t	**f**
u	**u**
f	**t**

$\supset$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**t**	**t**	**u**
f	**t**	**t**	**t**

$\bar{\vee}$	**t**	**u**	**f**
t	**t**	**t**	**t**
u	**t**	**u**	**u**
f	**t**	**u**	**f**

$\wedge$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**f**
f	**f**	**f**	**f**

$\equiv$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**t**	**u**
f	**f**	**u**	**t**

$\sim$	
t	**f**
u	**t**
f	**t**

L	
t	**t**
u	**f**
f	**f**

M	
t	**t**
u	**t**
f	**f**

I	
t	**f**
u	**t**
f	**f**

Figure 2.2: Truth tables for the connectives of $\mathbf{Ł_3}$.

Clearly, the classically valid principle of *tertium non datur*, i.e., the law of excluded middle, $A \vee \neg A$, as well as the corresponding law of noncontradiction, $\neg(A \wedge \neg A)$, are not valid in $\mathbf{Ł_3}$. However, their three-valued pendants, viz., the principle of *quartum non datur*, i.e., the law of excluded fourth,

$$A \vee \mathrm{I}A \vee \neg A,$$

and the corresponding extended non-contradiction principle,

$$\neg(A \wedge \neg \mathrm{I}A \wedge \neg A),$$

are valid in $\mathbf{Ł_3}$.

In classical logic, two formulas are logically equivalent iff they have the same models, where logical equivalence between formulas A and B is defined by the condition that $\models_2 (A \equiv B)$ holds. However, such a relation between logical equivalence and equality of models does not hold in general in the three-valued logic case. Indeed, following Radzikowska [50], let us define that two formulas A and B are *strongly equivalent*, symbolically $A \Leftrightarrow_s B$, iff $\models_3 (A \equiv B)$. That is, A and B are strongly equivalent iff, for any three-valued interpretation m, $\mathrm{V}^m(A) = \mathrm{V}^m(B)$. Furthermore, let us call A and B *equivalent* (*in* $\mathbf{Ł_3}$), symbolically $A \Leftrightarrow B$, iff A and B have the same models. Clearly, strong equivalence implies equivalence, but in general not vice versa. For instance, P and $\mathrm{L}P$, for an atom P, are equivalent but not strongly equivalent. As well, strong equivalence is an equivalence relation (i.e., reflexive, symmetric, and transitive) and enjoys a substitution principle, similar to the one of classical logic, i.e., if a formula C_A contains a subformula A, and C_B is the result of substituting at least one occurrence of A in C_A by a formula B, then $A \Leftrightarrow_s B$ implies $C_A \Leftrightarrow_s C_B$.

Let us also note some strong equivalences which hold in $\mathbf{Ł_3}$:

(a) $(A \supset B) \Leftrightarrow_s (\mathrm{M}\neg A \,\bar{\vee}\, B) \wedge (\mathrm{M}B \,\bar{\vee}\, \neg A)$;

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

(b) $O(A \circ B) \Leftrightarrow_s (OA \circ OB)$, for $O \in \{\mathrm{L}, \mathrm{M}\}$ and $\circ \in \{\wedge, \vee\}$;

(c) $OO'A \Leftrightarrow_s O'A$, for $O, O' \in \{\mathrm{L}, \mathrm{M}\}$;

(d) $\sim A \Leftrightarrow_s \mathrm{M}\neg A$;

(e) $\neg \mathrm{L}A \Leftrightarrow_s \mathrm{M}\neg A$;

(f) $\neg \mathrm{M}A \Leftrightarrow_s \mathrm{L}\neg A$;

(g) $((A \wedge B) \bar{\vee} C \Leftrightarrow_s (A \bar{\vee} C) \wedge (B \bar{\vee} C)$;

(h) $((A \bar{\vee} B) \wedge C \Leftrightarrow_s (A \wedge C) \bar{\vee} (B \wedge C)$;

The notion of a *theory* in **Ł3** is defined as in **PL**, i.e., a theory is a set of formulas. Likewise, the notion of a *model* of a theory, of a theory being *satisfiable* or *unsatisfiable*, of two theories being *equivalent* are defined analogously as in **PL**. As before, we write $T \Leftrightarrow T'$ to denote that T and T' are equivalent.

A theory T is said to *entail* a formula A (*in* **Ł3**), or A is a *valid consequence* of T (*in* **Ł3**), symbolically $T \models_3 A$, iff every model of T is also a model of A.

The following properties hold for entailment in **Ł3**:

Theorem 7 *Let T, T' be theories and A, B formulas.*

(i) *$T \models_3 A$ iff $T \cup \{\mathrm{M}\neg A\}$ is unsatisfiable.*

(ii) *$T, A \models_3 B$ iff $T \models_3 (\mathrm{L}A \supset B)$.* (*"Deduction Theorem."*)

(iii) *If $T \subseteq T'$, then $\{A \mid T \models_3 A\} \subseteq \{A \mid T' \models_3 A\}$.* (*"Monotonicity."*)

(iv) *If $T \Leftrightarrow T'$, then $\{A \mid T \models_3 A\} = \{A \mid T' \models_3 A\}$.*

2.2.3 Hilbert-Type Axiomatisation of Ł3

We now give a Hilbert-style axiomatisation of **Ł3**. The system we present is an adaption of the one given by Wajsberg [71], who was the first to discuss a sound and complete proof system for **Ł3**. Actually, our system differs from the original one from Wajsberg in that we use axiom schemas instead of finitely many concrete axioms together with a substitution rule. Also, we have the truth constant $\top$ as an additional axiom, which is not present in Wajsberg's language.

Definition 10 The Hilbert-type axiom system H_3 for **Ł3** is given by the triple

$$\langle D_{\text{Ł}_3}, A_{\text{Ł}_3}, R_{\text{Ł}_3} \rangle,$$

where

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

(i) $D_{\mathbf{Ł_3}}$ is the set of formulas of $\mathbf{Ł_3}$,

(ii) $A_{\mathbf{Ł_3}}$ is given by the set of all axioms specified by the following axiom schemas:

(A_0) $\top$,
(A_1) $A \supset (B \supset A)$,
(A_2) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$,
(A_3) $(\neg A \supset \neg B) \supset (B \supset A)$,
(A_4) $((A \supset \neg A) \supset A) \supset A$,

and

(iii) $R_{\mathbf{Ł_3}}$ consists of modus ponens as the single inference rule:

$$\frac{A \quad (A \supset B)}{B}.$$

□

We write $\vdash_3 A$ to indicate that A is provable in H_3. As before, we call, for a theory T, a formal proof in the axiom system $\langle D_{\mathbf{Ł_3}}, A_{\mathbf{Ł_3}} \cup T, R_{\mathbf{Ł_3}} \rangle$, a *derivation from* T (*in* H_3), and we say that a formula A is *derivable from* T, or that A is a *syntactic consequence from* T, symbolically $T \vdash_3 A$, if A is the last element in a derivation from T. Clearly, we again have

$$\vdash_3 A \quad \text{iff} \quad \emptyset \vdash_3 A,$$

i.e., a formula is provable in H_3 iff it is derivable from the empty set in H_3.

Also, we write "$T \nvdash_3 A$" if $T \vdash_3 A$ does not hold (and similarly "$\nvdash_3 A$" expresses that $\vdash_3 A$ does not hold).

We define the *deductive closure operator*, $\mathrm{Th}_3(\cdot)$, of $\mathbf{Ł_3}$ by

$$\mathrm{Th}_3(T) := \{A \mid T \vdash_3 A\},$$

where T is a theory. Again, a theory T is *deductively closed* iff $T = \mathrm{Th}_3(T)$.

The operator $\mathrm{Th}_3(\cdot)$ satisfies also the usual properties of inflationaryness, idempotency, and monotonicity as its classical counterpart:

Theorem 8 *For any theory T, the following conditions hold:*

(i) $T \subseteq \mathrm{Th}_3(T)$*;*

(ii) $\mathrm{Th}_3(\mathrm{Th}_3(T)) = \mathrm{Th}_3(T)$*; and*

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

(iii) $T \subseteq T'$ implies $\mathrm{Th}_3(T) \subseteq \mathrm{Th}_3(T')$.

The adequacy of our Hilbert-type axiom system H_3 follows readily from the soundness and completeness of Wajsberg's [71] original version:

Theorem 9 *For any theory T and any formula A, the following conditions hold:*

(i) *If $T \vdash_3 A$, then $T \models_3 A$.* (*"Soundness Theorem."*)

(ii) *If $T \models_3 A$, then $T \vdash_3 A$.* (*"Completeness Theorem."*)

A theory T is *consistent* (*in* $\mathbf{Ł_3}$) iff there is a formula A such that $T \nvdash_3 A$. We again have that consistency is the syntactic counterpart of satisfiability, i.e., a theory T is consistent in $\mathbf{Ł_3}$ iff it is satisfiable in $\mathbf{Ł_3}$. Moreover, a formula A is *consistent with* T (*in* $\mathbf{Ł_3}$) iff $T \nvdash_3 \neg A$.

The pendant of Theorem 6 also holds for the case of $\vdash_3$:

Theorem 10 *Under the circumstances of Theorems 2 and 3, the following conditions hold:*

(i) *If $T \vdash_3 (A \equiv B)$, then $T \vdash_3 (C_A \equiv C_B)$.* (*"Replacement Theorem."*)

(ii) *$T \vdash_3 A$ iff $T \cup \{\mathrm{M}\neg A\}$ is inconsistent (in $\mathbf{Ł_3}$).*

(iii) *$T, A \vdash_3 B$ iff $T \vdash_3 (\mathrm{L}A \supset B)$.* (*"Deduction Theorem."*)

Note that the consistency of a formula A with a theory T implies the consistency of the theory $T \cup \{\mathrm{M}A\}$, but not necessarily of the theory $T \cup \{A\}$. For instance, $\neg P$ is consistent with $\{\mathrm{M}P\}$, so $\{\mathrm{M}P, \mathrm{M}\neg P\}$ is consistent, but $\{\mathrm{M}P, \neg P\}$ is not.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

CHAPTER 3

Variants of Default Logic

"Music is the one incorporeal entrance into the higher world of knowledge—which comprehends mankind. But which mankind cannot comprehend."

Ludwig van Beethoven

The logics we introduced so far are *monotonic*, i.e., their associated inference relation satisfy the monotonicity condition, stating that once a proposition is derived from a set of premisses, it cannot be invalidated by additional information. However, in human commonsense reasoning, we usually draw conclusions from *incomplete information*, and such conclusions may be invalidated by new, more specific information. This type of reasoning is therefore inherently *nonmonotonic*, and formalisms for nonmonotonic reasoning play an important role in logic-based artificial intelligence. The term of referring to a logical system as being "nonmonotonic" was first introduced by Marvin Minsky in 1975 [41].

One of the central formalisms for nonmonotonic reasoning is *default logic*, introduced by Raymond Reiter in 1980 [51]. Its key feature is that nonmonotonic inferences are sanctioned by so-called *default rules*, which generalise ordinary inference rules by having an additional consistency condition. Such rules model the commonsense reasoning patterns of concluding a certain statement A on the basis that there is no evidence to the contrary, i.e., that there is no information that $\neg A$ holds (or, in other words, that A can be *consistently assumed*). Other important nonmonotonic reasoning formalisms which have been proposed in the artificial intelligence literature are, e.g., autoepistemic logic [42], circumscription [39], logic programming under the answer-set semantics [21], and equilibrium logic [44].

Besides Reiter's default logic, different variants of it have been defined in order to address certain shortcomings of the original approach. In what follows, we introduce two such

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU WIEN Bibliothek Your knowledge hub

approaches, viz. three-valued default logic due to Radzikowska [50] and disjunctive default logic due to Gelfond, Lifschitz, Przymusinska, and Truszczyński [22]. Other variants include, e.g., *justified default logic* [34], *constrained default logic* [57, 18], *rational default logic* [40], and *general default logic* [75] (an overview about different versions of default logic is given by Antoniou and Wang [1]).

We start our discussion on default-logic formalisms with the original approach by Reiter [51]. Note that we deal here with propositional versions of the formalisms as our subsequent calculi are defined for the propositional case only, similar to the undertaking of Bonatti [11, 12] and Bonatti and Olivetti [14].

3.1 Reiter's Default Logic

We introduce Reiter's default logic, henceforth denoted by **DL**, by first discussing the basic syntactic elements and afterwards some relevant characterisations.

3.1.1 Default Theories and their Extensions

Definition 11 Let $A, B_1, \ldots, B_n, C$ be formulas from **PL**. A *default*, d, is an expression of the form

$$\frac{A : B_1, \ldots, B_n}{C}. \tag{3.1}$$

A is the *prerequisite*, $B_1, \ldots, B_n$ are the *justifications*, and C is the *consequent* of the default d. □

Informally, the default d has the following meaning: If A is believed and $B_1, \ldots, B_n$ are *consistent to what is believed* (i.e., there is no evidence that $\neg B_1, \ldots, \neg B_n$ hold), then C is to be believed.

We use the following notation for defaults: for a default d of form (3.1), let

- $\mathsf{p}(d)$ denote the prerequisite of d, i.e., $\mathsf{p}(d) = A$,
- $\mathsf{j}(d)$ denote the set consisting of the justifications of d, i.e., $\mathsf{j}(d) = \{B_1, \ldots, B_n\}$, and
- $\mathsf{c}(d)$ denote the consequent of d, i.e., $\mathsf{c}(d) = C$.

For a set S of formulas, we also will write $\neg S$ to denote the set $\{\neg A \mid A \in S\}$. Hence, if $\mathsf{j}(d) = \{B_1, \ldots, B_n\}$, then $\neg\mathsf{j}(d) = \{\neg B_1, \ldots, \neg B_n\}$.

If $\mathsf{p}(d) = \top$, the default is said to be *prerequisite-free*; if $\mathsf{j}(d) = \emptyset$, the default is *justification-free*; and if $\mathsf{j}(d) = \{\mathsf{c}(d)\}$, then the default is *normal*. We also will write prerequisite-free defaults in the form

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

$$\frac{: B_1, \ldots, B_n}{C}$$

and justification-free defaults as

$$\frac{A :}{C}$$

Sometimes we also will write defaults in the form $(A : B_1, \ldots, B_n/C)$.

It will also be convenient to use the notation $\mathrm{CONS}(D)$, where D is a set of defaults, to denote the set of consequents of the defaults in D, i.e.,

$$\mathrm{CONS}(D) \; = \; \{\mathsf{c}(d) \mid d \in D\}\,.$$

Definition 12 A *default theory* is an ordered pair $T = \langle W, D\rangle$, where W is a set of closed formulas from **PL**, called the *premisses* of T, and D is a set of defaults. We say that T is *finite* if both W and D are finite. Furthermore, T is *normal* if all defaults in D are normal. □

Intuitively, for a default theory $T = \langle W, D\rangle$, W represents certain (yet in general incomplete) knowledge about the world whilst D represents *defeasible* knowledge. Now, given T as the initial knowledge of an agent's beliefs, the next definition specifies what totality of beliefs is determined on the basis of the default theory T.

Definition 13 Let $T = \langle W, D\rangle$ be a default theory.

(i) The operator $\Gamma_T(S)$, assigning a set S of formulas to a set of formulas, is given as the smallest set K of formulas satisfying the following conditions:

  (a) $K = \mathrm{Th}_2(K)$,

  (b) $W \subseteq K$, and

  (c) if $(A : B_1, \ldots, B_n/C) \in D$, $A \in K$ and $\{\neg B_1, \ldots, \neg B_n\} \cap S = \emptyset$, then $C \in K$.

(ii) A set E of formulas is an *extension of* T if $\Gamma_T(E) = E$, i.e., if E is a fixed point of the operator Γ_T. □

A default theory may have none, one, or several extensions. A default theory of the form $\langle W, \emptyset\rangle$ has exactly one extension, viz. $\mathrm{Th}_2(W)$. In general, an extension of a default theory $T = \langle W, D\rangle$ is always of the form $\mathrm{Th}_2(W \cup C)$, for some set C of formulas such that $C \subseteq \mathrm{CONS}(D)$ (cf. Theorem 13 below).

Let A be a formula and T a default theory. Then, A is said to be a *brave consequence* of T if there is an extension E of T such that $A \in E$. Similarly, A is a *skeptical consequence* of T if $A \in E$ for all extensions E of T.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Bonatti [11, 12] axiomatised brave reasoning for Reiter's default logic **DL** in terms of a sequent-type calculus and later Bonatti and Olivetti [14] gave an alternative axiomatisation of brave reasoning and also of skeptical reasoning. We will adapt the method of Bonatti [11, 12] to provide sequent-style axiomatisations of brave reasoning for the variants of default logic introduced below in Sections 3.2 and 3.3.

3.1.2 Alternative Characterisations of Extensions

We now provide some alternative ways to characterise extensions. The first one is given by Reiter himself [51] and uses a semi-inductive way to formulate a necessary and sufficient condition for a set to be an extension. This characterisation is somewhat more explicit than Definition 13 which specifies extensions non-constructively.

Theorem 11 *Let $T = \langle W, D\rangle$ be a default theory and E a set of formulas. Define a sequence of sets of formulas as follows:*

$$E_0 := W; \text{ and}$$
$$E_{i+1} := \mathrm{Th}_2(E_i) \cup \{\mathsf{c}(d) \mid d \in D,\, E_i \vdash_2 \mathsf{p}(d),\, \neg\mathsf{j}(d) \cap E = \emptyset\}, \text{ for } i \geq 0.$$

Then, E is an extension of T iff $E = \bigcup_{i\geq 0} E_i$.

The second, and arguably more intuitive, alternative characterisation of extensions is due to Marek and Truszczyński [38] and relies on a proof-theoretical description of the fixed-point operator Γ_T. This kind of characterisation will actually be central for our later purposes and we will therefore provide similar characterisations for three-valued default logic and disjunctive default logic as well.

To formulate the method of Marek and and Truszczyński, let us introduce some notation which is useful in this regard. In fact, the concept introduced next is relevant for identifying those defaults which contribute to the actual construction of an extension.

Definition 14 Let E be a set of formulas. A default $d = (A : B_1, \ldots, B_n/C)$ is *active* in E iff $E \vdash_2 A$ and $\{\neg B_1, \ldots, \neg B_n\} \cap E = \emptyset$. □

Next, we introduce a set of inference rules which are obtained from defaults satisfying the consistency condition relative to a given set of formulas.

Definition 15 Let D be a set of defaults and E a set of formulas. Then, the *reduct of D with respect to E*, denoted by D_E, is the set consisting of the following inference rules:

$$D_E := \left\{ \frac{A}{C} \;\middle|\; \frac{A : B_1, \ldots, B_n}{C} \in D \text{ and } \{\neg B_1, \ldots, \neg B_n\} \cap E = \emptyset \right\}.$$

An inference rule A/C is called *residue* of a default $(A : B_1, \ldots, B_n/C)$. □

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

Definition 16 Let R be a set of inference rules. By the proof system H_2^R we understand the Hilbert-type system H_2 augmented with the inference rules from R, i.e.,

$$\mathsf{H}_2^R = \langle D_{\mathbf{PL}}, A_{\mathbf{PL}}, R_{\mathbf{PL}} \cup R \rangle.$$

$\square$

We denote the provability relation in H_2^R by $\vdash_2^R$. The corresponding set of formulas which are deducible in H_2^R given a set W of formulas is denoted by $\mathrm{Th}_2^R(W)$. Clearly, $\mathrm{Th}_2^{\emptyset}(W) = \mathrm{Th}_2(W)$.

The following result is shown by Marek and Truszczyński [38]:

Theorem 12 *Let $T = \langle W, D \rangle$ be a default theory, E a set of formulas, and D_E the reduct of D with respect to E.*

Then,

$$\Gamma_T(E) = \mathrm{Th}_2^{D_E}(W).$$

Corollary 1 *Let $T = \langle W, D \rangle$ be a default theory. Then, a set E of formulas is an extension of T iff $\mathrm{Th}_2^{D_E}(W) = E$.*

The next theorem states a necessary condition of extensions and is therefore useful in identifying candidates for being extensions of a given default theory. To formulate the result let us introduce the following notation first:

Definition 17 Let $T = \langle W, D \rangle$ be a default theory and E an extension of T. Then, the set of *generating defaults for E with respect to T* is given by the set

$$\mathrm{GD}(E, T) \; := \; \{d \in D \mid d \text{ is active in } E\}.$$

$\square$

Theorem 13 *If E is an extension of a default theory $T = \langle W, D \rangle$, then,*

$$E = \mathrm{Th}_2(W \cup \mathrm{CONS}(\mathrm{GD}(E, T))).$$

A proof of this result can be found in the paper of Reiter [51] or the textbook of Łukaszewicz [35].

As a consequence of Theorem 13, the candidates for being extensions of a default theory $T = \langle W, D \rangle$ are given by the collection of all sets of the form $\mathrm{Th}_2(W \cup C)$, where $C \subseteq \mathrm{CONS}(D)$.

Example 1 Consider the default theory $T = \langle W, D \rangle$, where

$$W = \emptyset \qquad \text{and} \qquad D = \left\{ \frac{:P}{\neg P} \right\},$$

for an atomic formula P.

There are two possible candidates for being extensions of T, viz.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

$$E_1 := \mathrm{Th}_2(\emptyset) \quad \text{and} \quad E_2 := \mathrm{Th}_2(\{\neg P\}).$$

Now, the reducts for E_1 and E_2 are

$$D_{E_1} = \left\{ \frac{\top}{\neg P} \right\} \quad \text{and} \quad D_{E_2} = \emptyset.$$

We obtain:

$$\Gamma_T(E_1) = \mathrm{Th}_2^{D_{E_1}}(W) \; = \; \mathrm{Th}_2(\{\neg P\}) \; = \; E_2;$$
$$\Gamma_T(E_2) = \mathrm{Th}_2^{D_{E_2}}(W) \; = \; \mathrm{Th}_2(\emptyset) \; = \; E_1.$$

Since neither E_1 nor E_2 is a fixed point of Γ_T, from Theorem 13 it follows that T has no extension. □

Example 2 Let $T = \langle W, D \rangle$ be the default theory consisting of

$$W := \{Q, R\} \qquad \text{and} \qquad D := \left\{ \frac{Q : P}{P}, \frac{R : \neg P}{\neg P} \right\}.$$

Intuitively, this default theory represents the commonsense beliefs that

- Quakers are normally pacifists,
- Republicans are normally *not* pacifists, and
- Nixon is both a Quaker and a Republican.

This example is commonly known in the literature as the "Nixon diamond" [52, 53, 54]: if we know that Nixon is both a Quaker and a Republican, should we believe about him whether he is a pacifist or not?

In default logic, we get two extensions; one, where Nixon is a pacifist, and another, where he is not—representing two "stable beliefs" on the basis of T.

Formally, there are four candidates for being an extension of T:

$$E_1 := \mathrm{Th}_2(\{Q, R\}); \qquad E_3 := \mathrm{Th}_2(\{Q, R, \neg P\});$$
$$E_2 := \mathrm{Th}_2(\{Q, R, P\}); \qquad E_4 := \mathrm{Th}_2(\{Q, R, P, \neg P\}).$$

Indeed, only E_2 and E_3 are extensions of T, in view of the following relations:

$$\Gamma_T(E_1) := E_4; \qquad \Gamma_T(E_3) := E_3;$$
$$\Gamma_T(E_2) := E_2; \qquad \Gamma_T(E_4) := E_1.$$

□

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

3.2 Three-Valued Default Logic

Radzikowska's three-valued default logic [50], which in what follows we will denote by $\mathbf{DL_3}$, differs from Reiter's default logic **DL** in two aspects: not only is in $\mathbf{DL_3}$ the deductive machinery of classical logic replaced with $\mathbf{Ł_3}$, but there is also a modified consistency check for default rules employed, in which the consequent of a default is taken into account as well. The latter feature is somewhat reminiscent to the consistency checks as used in justified default logic [34] and constrained default logic [57, 18], where a default may only be applied if it does not lead to a contradiction *a posteriori*.

3.2.1 Three-Valued Default Theories and their Extensions

Analogously to Reiter's default logic, a default rule in $\mathbf{DL_3}$ is an expression of the form

$$\frac{A : B_1, \ldots, B_n}{C} \ ,$$

but where now A, B_1,...,B_n, and C are formulas from $\mathbf{Ł_3}$. The notions of a prerequisite, justification, and consequent of a default are defined as before. However, the intuitive meaning of a default differs now: a default of the above form expresses that

> if A is believed and $B_1, \ldots, B_n$, and LC are consistent with what is believed, then MC is asserted.

Note that under this reading, by applying a default of the above form, it is assumed that C cannot be false, but it is not assumed that C is true in all scenarios. It is only assumed that C must be true in at least one such scenario. This reflects the intuition that accepting a default conclusion, we are prepared to rule out all scenarios where it is false, but we can imagine at least one such scenario in which it is true. As a consequence, we cannot conclude both MC and M$\neg C$ simultaneously.

In what follows, formulas of the form MC obtained by applying defaults will be referred to as *default assumptions*. We retain our convention of allowing to write defaults in the form $(A : B_1, \ldots, B_n/C)$.

A *default theory in* $\mathbf{DL_3}$, or a (*three-valued*) *default theory*, is a pair $T = \langle W, D\rangle$, where W is a set of formulas (i.e., a theory) in $\mathbf{Ł_3}$ and D is a set of defaults built from formulas in $\mathbf{Ł_3}$.

An extension of a default theory $T = \langle W, D\rangle$ in $\mathbf{DL_3}$ is now defined thus:

Definition 18 Let $T = \langle W, D\rangle$ be a three-valued default theory.

(i) For a set S of formulas, let $\Lambda_T(S)$ be the smallest set K of formulas obeying the following conditions:

(a) $K = \mathrm{Th}_3(K)$,

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

(b) $W \subseteq K$, and

(c) if $(A : B_1, \ldots, B_n/C) \in D$, $A \in K$, $\{\neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C\} \cap S = \emptyset$, then $\mathrm{M}C \in K$.

(ii) A set E of formulas in $\mathbf{Ł_3}$ is an *extension* of T iff $\Lambda_T(E) = E$. □

Note that the criterion of the applicability of a default in $\mathbf{DL_3}$ makes the two defaults

$$d = \frac{A : B_1, \ldots, B_n}{C} \quad \text{and} \quad d' = \frac{A : \mathrm{M}B_1, \ldots, \mathrm{M}B_n}{C}$$

equivalent in the sense that the application of d implies the application of d' and vice versa. Thus, in a default theory $T = \langle W, D \rangle$, we can replace all $d \in D$ with their corresponding version d' without changing extensions.

The two basic reasoning tasks we introduced in the previous section for Reiter's default logic, brave and skeptical reasoning, are defined accordingly for three-valued default logic as well.

3.2.2 Some Representational Aspects

Let us now discuss the differences and representational advantages of $\mathbf{DL_3}$ compared to standard default logic **DL**, following the discussion of Radzikowska [50].

First of all, consider the following commonsense statements:

$$\text{“If } A\text{, then normally } B\text{.”} \tag{3.2}$$

$$\text{“If normally } A\text{, then normally } B\text{.”} \tag{3.3}$$

In $\mathbf{DL_3}$, these statements can be represented as follows: Sentence (3.2) can be represented by the default rule

$$\frac{\mathrm{L}A : B}{B}, \tag{3.4}$$

which is equivalent to

$$\frac{A : B}{B}, \tag{3.5}$$

while (3.3) would be expressed by

$$\frac{\mathrm{M}A : B}{B}. \tag{3.6}$$

Default (3.4), as well as (3.5), can be applied if A is known to be true, whilst the application of (3.6) requires A to be a default assumption only. In the standard default logic **DL**, both these rules have the same default representation, however, as classical two-valued logic **PL** lacks the possibility to distinguish between certainty and possibility. Hence, in **DL**, sometimes the derivation of undesired conclusions may be forced. This can be illustrated as follows:

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

Example 3 ([19]) Consider the following two statements:

(i) Normally, a person may be assumed honest.

(ii) If a person is known to be honest, then normally he or she can be safely lent money.

Both in **DL** and **DL₃**, these statements can be represented by the following defaults:

$$\frac{\mathit{person} : \mathit{honest}}{\mathit{honest}} \quad \text{and} \quad \frac{\mathit{honest} : \mathit{lent\_money}}{\mathit{lent\_money}}.$$

Given an arbitrary person, say Jason, in **DL** we infer that he can be safely lent money. However, arguably this conclusion is not acceptable. In $\mathbf{DL_3}$, on the other hand, this conclusion cannot be drawn: For Jason, the default assumption obtained by applying the first default is now too weak to make the second default applicable. Thus, we can now only conclude that the considered person may be honest—something we intuitively would expect. □

Using the representation (3.6), the transitivity of defaults may be permitted, as shown next.

Example 4 Consider now the following information:

(i) Teenagers usually go to school.

(ii) Typically, schoolchildren are not employed.

Given a particular teenager, we wish to conclude that he or she is normally not employed. In $\mathbf{DL_3}$ we can represent the above statements by

$$\frac{\mathit{teenager} : \mathit{schoolchild}}{\mathit{schoolchild}} \quad \text{and} \quad \frac{\mathrm{M}\mathit{schoolchild} : \neg\mathit{employed}}{\neg\mathit{employed}}.$$

Clearly, $\mathrm{M}\neg\mathit{employed}$ can be derived. □

In standard default logic, many default rules, which considered in isolation, are naturally expressed by normal defaults, but when they are used in a wider context, then they must be reformulated in order to avoid unintuitive results. The problem is that in such cases the application of some defaults needs to be blocked. The next example shows that in $\mathbf{DL_3}$, employing defaults of form (3.4) may be useful to block the transitivity of default rules in a straightforward way.

Example 5 ([53]) Consider the following situation:

- Typically, high school dropouts are adults.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

- Typically, adults are employed.

Since high-school dropouts are atypical adults with respect to employment, the transitive inference from being a dropout to concluding being adult and then further deriving being employed is unjustified. In standard default logic, a common way of suppressing unwanted transitive conclusions is to employ so-called *semi-normal defaults*, which are like normal defaults but having an additional exception condition. In our case, this would be represented in the following way:

$$\frac{dropout : adult}{adult} \quad \text{and} \quad \frac{adult : employed \wedge \neg dropout}{employed}.$$

However, this remedy is somewhat unsatisfactory as it requires every default with a possibly large number of conceivable exceptions, each time a new default is added, to be revised, which is arguably ad hoc.

In $\mathbf{DL_3}$, however, the above statements can be represented simply by

$$\frac{dropout : adult}{adult} \quad \text{and} \quad \frac{adult : employed}{employed}$$

since, as can be easily verified, only the first default is applicable for a dropout. □

Consider a default rule of the form

$$\frac{A : B}{B}, \tag{3.7}$$

whose application leads to the conclusion MB. Now, as we know, in $\mathbf{Ł_3}$, the formula MLB is strongly equivalent to LB, so we may always obtain a stronger conclusion, viz. LB, provided that default (3.7) is rewritten in the form

$$\frac{A : B}{\mathrm{L}B},$$

or, equivalently, to

$$\frac{A : \mathrm{L}B}{\mathrm{L}B}.$$

The possibility to distinguish in this way between *weak* and *strong default conclusions* gives the formal means to separately represent rules expressing *causal rules* (“expectation-evoking rules”) and *evidential rules* (“explanation-evoking rules”) [45]. An example of the first kind of rules is “fire usually causes smoke” whilst “smoke usually suggests fire” is an instance of the second kind. As argued, e.g., by Pearl [45], an evidential rule should not be applied if its prerequisite is derived by applying at least one causal rule. To illustrate this, consider the following example:

Example 6 ([45]) Consider the default rules $d_1 = (P : Q/Q)$ and $d_2 = (Q : R/R)$, where P, Q, and R stand for the following propositions:

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

- P: "Tony recites passages from Shakespeare."
- Q: "Tony can read and write."
- R: "Tony is over seven years old."

Obviously, common sense suggests that, given P, there are perfect reasons to apply both defaults to infer that Tony is over seven years old. Suppose now that we add the default rule

$$d_3 = \frac{S : Q}{Q},$$

where S stands for "Tony is a child prodigy". Given S, it is reasonable to infer that Tony can read and write, but the inference that Tony is over seven years old seems to be unjustified.

Like in Example 5 above, in standard default logic one would use instead of default d_2 the semi-normal default

$$d_2' = \frac{Q : R \wedge \neg S}{R}.$$

However, as we argued above, this is somewhat unsatisfactory. Now, in $\mathbf{DL_3}$, on the other hand, this can easily be accommodated by using the defaults

$$\frac{P : \mathrm{L}Q}{Q} \quad \text{and} \quad \frac{Q : \mathrm{L}R}{\mathrm{L}R}$$

instead of d_1 and d_2, and

$$\frac{\mathrm{M}S : Q}{Q}$$

instead of d_3. □

To conclude our discussion on comparing **DL** and $\mathbf{DL_3}$, consider the following example illustrating a situation with incoherent default-rule applications.

Example 7 ([35]) Consider the default theory $T = \langle W, D \rangle$, where

$$W = \{\mathit{Summer}, \neg \mathit{Sun\_Shining}\} \quad \text{and} \quad D = \left\{ \frac{\mathit{Summer} : \neg \mathit{Rain}}{\mathit{Sun\_Shining}} \right\}.$$

The only default of this theory is inapplicable since $W \vdash_3 \neg \mathrm{L}\mathit{Sun\_Shining}$. Hence, T has one extension, $E = \mathrm{Th}_3(W)$. However, T has no extension in Reiter's default logic due to the weaker consistency check which yields a vicious circle where the application of the default violates its justification for applying it. □

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

3.2.3 Characterisations of Extensions

We now discuss pendants of the characterisations given in Section 3.1.2 for **DL** for the case of $\mathbf{DL_3}$.

To begin with, the semi-recursive characterisation of Theorem 11 reads now as follows:

Theorem 14 *Let $T = \langle W, D\rangle$ be a default theory in $\mathbf{DL_3}$ and E a set of formulas of $\mathbf{Ł_3}$. Define a sequence of sets of formulas as follows:*

$$\begin{aligned} E_0 &:= W; \text{ and} \\ E_{i+1} &:= \mathrm{Th}_3(E_i) \cup \{\mathrm{M}C \mid (A : B_1, \ldots, B_n/C) \in D,\, E_i \vdash_3 A, \\ &\qquad\qquad \{\neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C\} \cap E = \emptyset\}, \text{ for } i \geq 0. \end{aligned}$$

Then, E is an extension of T iff $E = \bigcup_{i\geq 0} E_i$.

For a proof, see Radzikowska [50].

We next adapt the characterisation of Marek and Truszczyński [38] of the fixed-point operator Γ_T (Theorem 12) to the three-valued default-logic case. We start with adapting the notions of an active default and of a reduct (cf. Definitions 14 and 15).

Definition 19 Let E be a set of formulas. A default $(A : B_1, \ldots, B_n/C)$ is $\mathbf{DL_3}$-*active* in E iff $E \vdash_3 A$ and $\{\neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C\} \cap E = \emptyset$. □

Definition 20 Let D be a set of defaults and E a set of formulas. The $\mathbf{DL_3}$-*reduct of D with respect to E*, denoted by $\mathcal{R}^*(D, E)$, is the set consisting of the following inference rules:

$$\mathcal{R}^*(D, E) \quad := \quad \left\{ \frac{A}{\mathrm{M}C} \;\middle|\; \frac{A : B_1, \ldots, B_n}{C} \in D \text{ and } \{\neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C\} \cap E = \emptyset \right\}.$$

An inference rule $A/\mathrm{M}C$ is called $\mathbf{DL_3}$-*residue* of a default $(A : B_1, \ldots, B_n/C)$. □

Whenever it is clear from the context, we will allow ourselves to drop the prefix "$\mathbf{DL_3}$-" in "$\mathbf{DL_3}$-active", "$\mathbf{DL_3}$-reduct", and "$\mathbf{DL_3}$-residue" to ease notation.

For a set R of inference rules, let $\vdash_3^R$ be the inference relation obtained from $\vdash_3$ by augmenting the postulates of the Hilbert-type calculus H_3 underlying $\vdash_3$ with the inference rules from R. The corresponding deductive closure operator for $\vdash_3^R$ is given by

$$\mathrm{Th}_3^R(W) := \{A \mid W \vdash_3^R A\}.$$

Clearly, $\mathrm{Th}_3^{\emptyset}(W) = \mathrm{Th}_3(W)$.

We then obtain the analogue of Theorem 12, characterising the operator Λ_T. The proof of it is a an easy adaption of the proof for the case of **DL**.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Theorem 15 *Let $T = \langle W, D\rangle$ be a three-valued default theory, E a set of formulas of $\mathbf{Ł_3}$, and $\mathcal{R}^*(D,E)$ the $\mathbf{DL_3}$-reduct of D with respect to E.*

Then,

$$\Lambda_T(E) = \mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W).$$

Corollary 2 *Let $T = \langle W, D\rangle$ be a three-valued default theory. Then, a set E of formulas is an extension of T iff*

$$\mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W) = E.$$

The concept of a generating default in the context of $\mathbf{DL_3}$ is defined similarly to the case of standard default logic, where the activeness of a default of course is understood in the sense of $\mathbf{DL_3}$. More formally, given a three-valued default theory T and a set E of formulas, the set of generating defaults is given by

$$\mathrm{GD}(E,T) := \{d \in D \mid d \text{ is } \mathbf{DL_3}\text{-active in } E\}.$$

Moreover, for a set D of defaults, let us define

$$\mathrm{CONS}^*(D) = \{\mathrm{M}C \mid (A : B_1, \ldots, B_n/C) \in D\}.$$

We then obtain:

Theorem 16 *If E be an extension of a three-valued default theory $T = \langle W, D\rangle$, then*

$$E = \mathrm{Th}_3(W \cup \mathrm{CONS}^*(\mathrm{GD}(E,T))).$$

The proof is a straightforward adaption of the proof for the case of **DL**, as given by Reiter [51] or Łukaszewicz [35].

As for the case of **DL**, we again obtain from the above result that all candidates for being extensions of a given three-valued default theory $T = \langle W, D\rangle$ are given by the collection of all sets of the form $\mathrm{Th}_2(W \cup C)$, where $C \subseteq \mathrm{CONS}^*(D)$.

Example 8 Reconsider the default theory $T = \langle W, D\rangle$ from Example 2 dealing with the Nixon diamond where

$$W = \{Q, R\} \quad \text{and} \quad D = \left\{\frac{Q : P}{P}, \frac{R : \neg P}{\neg P}\right\}.$$

As shown before, T has two extensions in **DL**, viz.

$$E := \mathrm{Th}_2(\{Q, R, P\}) \quad \text{and} \quad E' := \mathrm{Th}_2(\{Q, R, \neg P\}).$$

For $\mathbf{DL_3}$, we similarly obtain two extensions, viz.

$$F := \mathrm{Th}_3(\{Q, R, \mathrm{M}P\}) \quad \text{and} \quad F' := \mathrm{Th}_2(\{Q, R, \mathrm{M}\neg P\}).$$

□

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU WIEN Bibliothek Your knowledge hub

3.3 Disjunctive Default Logic

We now discuss the basics of disjunctive default logic, introduced by Gelfond, Lifschitz, Przymusinska, and Truszczyński[22], henceforth referred to as $\mathbf{DL_D}$.

The main motivation for introducing disjunctive default logics was to address a difficulty encountered when using defaults in the presence of disjunctive information, a problem which was first observed by David Poole [48]. More specifically, the difficulty lies in the difference between a default theory having two extensions, one containing a formula A and the other a formula B, and a theory with a single extension, containing the disjunction $A \vee B$. This problem was also noted by Lin and Shoham [28], who gave an example of a theory in a modal-logic language, containing disjunctive information, and observed that no default theory exists which corresponds to this theory.

Another nice feature of disjunctive default logic is that it provides a one-to-one correspondence between answer-sets of disjunctive logic programs [21] and extensions of a corresponding disjunctive default theory. Such a correspondence does likewise not directly hold for standard default logic—and again the key problem lies in the presence of disjunctive information. More specifically, viewing $P \vee Q$ as a rule in a logic program under the answer-set semantics, the default naturally corresponding to this rule would be the default rule

$$d = \frac{\top :}{P \vee Q}.$$

Now, while the program consisting of the single rule $P \vee Q$ has two answer sets, viz. $\{P\}$ and $\{Q\}$, the default theory $\langle \emptyset, \{d\} \rangle$ has only one extension, $\mathrm{Th}_2(\{P \vee Q\})$. As long as only programs without disjunctions are considered, such a natural translation of program rules into defaults gives rise to a one-to-one correspondence between answer sets of the given program and the extensions of its translation.

3.3.1 Disjunctive Default Theories and their Extensions

Definition 21 Let A, B_1,…,B_n, C_1,…, C_m be formulas from **PL**. A *disjunctive default rule*, or simply a *disjunctive default*, d, is an expression of the form

$$\frac{A : B_1, \ldots, B_n}{C_1 | \cdots | C_m} \ ,$$

where A is the *prerequisite*, $B_1, \ldots, B_n$ are the *justifications*, and $C_1, \ldots, C_n$ are the *consequents* of d. □

Following Baumgartner and Gottlob [5], we refer to the symbol "|" as *effective disjunction*.

The intuitive meaning of such a default is:

> if A is believed and $B_1, \ldots, B_n$ are consistent with what is believed, then one of $C_1, \ldots, C_m$ is asserted.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Similar to the previous conventions, if the prerequisite of a default d is $\top$, then we will omit it from d. If, additionally, d has no justifications, then d is simply written as

$$C_1|\cdots|C_m,$$

where $C_1, \ldots, C_m$ are the consequents of d. For convenience, disjunctive defaults will also be written in the form $(A : B_1, \ldots, B_n \,/\, C_1|\cdots|C_m)$.

Definition 22 A *disjunctive default theory*, T, is a pair $\langle W, D \rangle$, where W is a set of formulas of **PL** (referred to as the *premisses* of T) and D is a set of disjunctive defaults. □

Intuitively, given a disjunctive default theory $T = \langle W, D \rangle$, W represents certain, yet incomplete information about the world whilst D allows to extend certain knowledge by plausible conclusions.

For defining extensions of disjunctive default theories, we need some further notation: Let us call a set S of formulas to be *closed under propositional consequence* if, whenever $S \vdash_2 A$, then $A \in S$. Clearly, the deductive closure of a set S, $\mathrm{Th}_2(S)$, is the smallest set of formulas closed under propositional consequence containing S.

Moreover, for a family F of sets, $\min(F)$ denotes the minimal elements of F, where minimality is defined with respect to set inclusion, i.e.,

$$\min(F) = \{X \mid X \in F \text{ and there is no } Z \in F \text{ such that } Z \subset X\}.$$

Definition 23 Let $T = \langle W, D \rangle$ be a disjunctive default theory.

(i) Given a set S of formulas of **PL**, let $Cl_T(S)$ be the collection of all sets K satisfying the following conditions:

   (a) $K = \mathrm{Th}_2(K)$,

   (b) $W \subseteq K$, and

   (c) if $(A : B_1, \ldots, B_n \,/\, C_1|\cdots|C_m) \in D$, $A \in K$ and $\{\neg B_1, \ldots, \neg B_n\} \cap S = \emptyset$, then $C_i \in K$, for some $i \in \{1, \ldots, m\}$.

   Moreover, let $\Delta_T(S) = \min(Cl_T(S))$, i.e., $\Delta_T(S)$ consists of all minimal sets satisfying conditions (a)-(c).

(ii) A set E of formulas of **PL** is an *extension* of T if $E \in \Delta_T(E)$. □

The notion of a brave and a skeptical consequence given a disjunctive default theory is defined as before *mutatis mutandis*.

Let us now discuss some examples showing the differences between disjunctive default logic and standard default logic.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU WIEN Bibliothek Your knowledge hub

Example 9 ([51]) Consider the default theory $T = \langle W, D \rangle$ for

$$W = \{P \vee Q\} \quad \text{and} \quad D = \left\{ \frac{P : R}{R}, \frac{Q : S}{S} \right\},$$

where P, Q, R, and S are atomic formulas. Intuitively, given the disjunctive information $P \vee Q$, we would expect to derive $R \vee S$, because, in case P holds, we could apply the first default, and in case Q holds, we could accordingly apply the second default. However, in **DL**, neither of the two defaults is applicable and the single extension of T is $\mathrm{Th}_2(W)$.

Now, in disjunctive default logic, we can represent the information expressed by T in terms of a disjunctive default theory T' containing the three defaults

$$P|Q, \quad \frac{P : R}{R}, \quad \text{and} \quad \frac{Q : S}{S}.$$

In contrast to the situation in **DL**, T' possesses two extensions in $\mathbf{DL_D}$, viz. $\mathrm{Th}_2(\{P, R\})$ and $\mathrm{Th}_2(\{Q, S\})$, and $R \vee S$ is contained in both, which is in accordance to our expectations. □

We next discuss the example by Poole [48].

Example 10 Let us assume the following commonsense information: By default, a person's left arm is usable, the exception being when it is broken, and similarly for the right arm.

In standard default logic, we can express this by the following two prerequisite-free semi-normal defaults:

$$d_1 := \frac{: \mathit{lh\_usable} \wedge \neg \mathit{lh\_broken}}{\mathit{lh\_usable}} \qquad \text{and} \qquad d_2 := \frac{: \mathit{rh\_usable} \wedge \neg \mathit{rh\_broken}}{\mathit{rh\_usable}},$$

where "*lh*" and "*rh*" refer to the left and right hand, respectively.

If there is no further information about one's hands, then one can conclude that both hands are usable. Indeed, the default theory $T = \langle \emptyset, \{d_1, d_2\} \rangle$ has a single extension in **DL**, containing both *lh-usable* and *rh-usable*.

However, if it is now known that the left arm is broken, i.e., *lh_broken* is asserted, then the application of d_1 is blocked and the extended default theory

$$T' = \langle \{\mathit{lh\_broken}\}, \{d_1, d_2\} \rangle$$

has again one extension, containing *rh_usable*.

But let us assume now that we only know that one arm is broken, but we do not remember exactly which one. So, what we can assert now is the formula

$$\mathit{lh\_broken} \vee \mathit{rh\_broken}.$$

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Considering now the extensions of the default theory

$$T'' = \langle \{lh\_broken \vee rh\_broken\}, \{d_1, d_2\} \rangle,$$

this default theory has still one extension, but unfortunately it contains both *lh_usable* and *rh_usable*, which is contrary to our intuition.

Using $\mathbf{DL_D}$, on the other hand, we can represent the information of T'' by a disjunctive default theory containing

$$lh\_broken|rh\_broken,$$

together with the two defaults d_1 and d_2. The resulting theory has two extensions, viz.

$$\mathrm{Th}_2(\{lh\_broken, rh\_usable\}) \quad \text{and} \quad \mathrm{Th}_2(\{rh\_broken, lh\_usable\}),$$

both containing

$$lh\_usable \vee rh\_usable,$$

which corresponds with our intuition.

Note that the difference between a formula $A \vee B$ and a disjunctive default $A|B$ amounts to the difference between the assertions "A or B is known" and "A is known or B is known". □

3.3.2 Reducts for Disjunctive Defaults

Extensions in disjunctive default logic can be characterised in terms of a reduct similar to standard and three-valued default logic.

To this end, we introduce the following terminology: By a *disjunctive inference rule*, or simply a *disjunctive rule*, r, we understand an expression of the form

$$\frac{A}{C_1|\cdots|C_m}. \tag{3.8}$$

We say that a set S of formulas is *closed under* r if, whenever $A \in S$, then $C_i \in S$, for some $i \in \{1, \ldots, m\}$. Moreover, for a set R of disjunctive rules, we say that S is closed under R if S is closed under each $r \in R$.

Definition 24 Let D be a set of disjunctive defaults and E a set of formulas. The $\mathbf{DL_D}$*-reduct of* D *with respect to* E, denoted by $\mathcal{R}^\circ(D, E)$, is the set consisting of the following disjunctive inference rules:

$$\mathcal{R}^\circ(D,E) := \left\{ \frac{A}{C_1|\cdots|C_m} \;\middle|\; \frac{A : B_1, \ldots, B_n}{C_1|\cdots|C_m} \in D \text{ and } \{\neg B_1, \ldots, \neg B_n\} \cap E = \emptyset \right\}.$$

A disjunctive rule

$$\frac{A}{C_1|\cdots|C_m}$$

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

is called $\mathbf{DL_D}$-*residue* of a default

$$\frac{A : B_1, \ldots, B_n}{C_1|\cdots|C_m}.$$

□

We again allow ourselves to drop the prefix "$\mathbf{DL_D}$-" from "$\mathbf{DL_D}$-reduct" and "$\mathbf{DL_D}$-residue" if no ambiguity can arise.

Towards our characterisation of extensions of disjunctive default theories, we introduce the following notation:

Definition 25 For a set W of formulas and a set R of disjunctive rules, let $C^R(W)$ be the collection of all sets which

(i) are closed under propositional consequence,

(ii) contain W, and

(iii) are closed under R.

Furthermore, define $Cn^R(W) := \min(C^R(W))$, i.e., $Cn^R(W)$ contains all minimal sets satisfying (i)–(iii). □

Note that, for a disjunctive default theory $T = \langle W, D\rangle$ and a set E of formulas, we obviously have that

$$Cl_T(E) = C^{\mathcal{R}^\circ(D,E)}(W) \quad \text{and} \quad \Delta_T(E) = Cn^{\mathcal{R}^\circ(D,E)}(W).$$

From this, the following result is immediate:

Theorem 17 *Let $T = \langle W, D\rangle$ be a disjunctive default theory.*

Then, E is an extension of T iff $E \in Cn^{\mathcal{R}^\circ(D,E)}(W)$.

Note furthermore that, if $T = \langle W, D\rangle$ is a standard default theory, i.e., if D contains no proper disjunctive defaults, then clearly $\mathcal{R}^\circ(D, E) = D_E$, for any set E of formulas, and moreover

$$\Delta_T(E) = \{\Gamma_T(E)\} \quad \text{and} \quad Cn^{\mathcal{R}^\circ(D,E)}(W) = \{\mathrm{Th}_2^{D_E}(W)\}.$$

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek Your knowledge hub WIEN

CHAPTER 4

Sequent-Type Calculi for Brave Default Reasoning

"To achieve great things, two things are needed:
a plan, and not quite enough time."

Leonard Bernstein

We now introduce our sequent calculi for brave reasoning in $\mathbf{DL_3}$ and $\mathbf{DL_D}$. Our calculi adapt the approach of Bonatti [13, 12], originally defined for standard default logic **DL**, for the considered default-logic variants.

Analogous to Bonatti's system, our calculi comprise three kinds of sequents each:

(i) assertional sequents for axiomatising validity in the respective underlying monotonic base logic,

(ii) *anti-sequents* for axiomatising *invalidity* for the underlying monotonic logics, i.e., non-theorems of these logics, taking care of the consistency check of defaults, and

(iii) proper default sequents.

Although it would be possible to use just one kind of sequents, this would be at the expense of losing clarity of the sequents' structure. As well, the current separation of types of sequents also reflects the interactions between the underlying monotonic proof machinery the nonmonotonic inferences in a much clearer manner.

The distinguishing feature of Bonatti's approach is the use of a complementary calculus for axiomatising non-tautologies, which makes this approach particularly elegant. However,

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU WIEN Bibliothek Your knowledge hub

a sound and complete axiomatisation of non-theorems is only possible for logics which are decidable or at least where the set of non-theorems is semi-decidable. Hence, Bonatti considered only propositional versions of the nonmonotonic logics for which he developed sequent calculi. The same applies for the nonmonotonic reasoning calculi introduced subsequently by Bonatti and Olivetti [14].

The field of complementary calculi is less studied than other areas of logic but nevertheless interesting. The history of it goes back to Aristotle who not only analysed correct reasoning in his system of syllogisms but studied also invalid arguments, referred to as *fallacies*, in the last part of the Organon, entitled *De Sophisticis Elenchis* ("Sophistical Refutations"), where, in particular, he rejects arguments by reducing them to other already rejected ones. The first usage of the term "rejection" in modern logic was done by Jan Łukasiewicz in his 1921 paper *Logika dwuwartościowa* ("Two-valued logic") in which he states that by doing so he follows Brentano [30]. An axiomatic treatment of rejection was then discussed in his treatment of Aristotle's syllogistic [32, 33] where he introduced a Hilbert-type rejection system. This was then further elaborated by his student Jerzy Słupecki [63] and eventually extended to a theory of rejected propositions [65, 72, 15, 66, 67].[1] Furthermore, complementary calculi where not only studied for classical logic [61, 69, 11] but also for different types of logics, like intuitionistic logic [58, 20, 60], modal logics [24, 62], many-valued logics [16, 59, 43, 10], and others [6].

In what follows, we first introduce our calculus for three-valued default logic, which we denote by B_3, and afterwards the corresponding calculus for disjunctive default logic, denoted by B_D.

4.1 A Sequent Calculus for Three-Valued Default Logic

As pointed out above, and following the general design of the approach of Bonatti [13, 12], the calculus B_3 we are now going to introduce involves three kinds of sequents, viz. assertional sequents for axiomatising validity in $\mathbf{Ł_3}$, anti-sequents for axiomatising non-tautologies of $\mathbf{Ł_3}$, and special default sequents representing brave reasoning in **DL**.

We start with laying down the postulates of B_3 and then, in Section 4.1.2, we show soundness and completeness.

4.1.1 Postulates of the Calculus

As far as sequent-type calculi for three-valued (monotonic) logics are concerned—or, more generally, many-valued logics—, different techniques have been discussed in the literature [17, 2, 74, 3, 8, 25]. Here, we use an approach due to Rousseau [56], which is a natural generalisation for many-valued logics of the classical two-sided sequent formulation as pioneered by Gentzen [23]. In Rousseau's approach, a sequent for a

[1] We refer to the paper by Wybraniec-Skardowska [73] for an excellent survey on the development of refutation systems.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

$$\frac{\Gamma \mid \Delta \mid \Pi, A \qquad \Gamma, B \mid \Delta \mid \Pi}{\Gamma, A \supset B \mid \Delta \mid \Pi}\,(\supset : \mathbf{f})$$

$$\frac{\Gamma \mid \Delta, A, B \mid \Pi \qquad \Gamma, B \mid \Delta \mid \Pi, A}{\Gamma \mid \Delta, A \supset B \mid \Pi}\,(\supset : \mathbf{u})$$

$$\frac{\Gamma, A \mid \Delta, A \mid \Pi, B \qquad \Gamma, A \mid \Delta, B \mid \Pi, B}{\Gamma \mid \Delta \mid \Pi, A \supset B}\,(\supset : \mathbf{t})$$

$$\frac{\Gamma \mid \Delta \mid \Pi, A}{\Gamma, \neg A \mid \Delta \mid \Pi}\,(\neg : \mathbf{f}) \qquad \frac{\Gamma \mid \Delta, A \mid \Pi}{\Gamma \mid \Delta, \neg A \mid \Pi}\,(\neg : \mathbf{u}) \qquad \frac{\Gamma, A \mid \Delta \mid \Pi}{\Gamma \mid \Delta \mid \Pi, \neg A}\,(\neg : \mathbf{t})$$

$$\frac{\Gamma \mid \Delta \mid \Pi}{\Gamma, A \mid \Delta \mid \Pi}\,(w : \mathbf{f}) \qquad \frac{\Gamma \mid \Delta \mid \Pi}{\Gamma \mid \Delta, A \mid \Pi}\,(w : \mathbf{u}) \qquad \frac{\Gamma \mid \Delta \mid \Pi}{\Gamma \mid \Delta \mid \Pi, A}\,(w : \mathbf{t})$$

Figure 4.1: Rules of the sequent calculus $\mathsf{SŁ}_3$.

three-valued logic is a triple of sets of formulas where each component of the sequent represents one of the three truth values.

A Sequent Calculus for $\mathbf{Ł_3}$

Formally, we introduce sequents for $\mathbf{Ł_3}$ as follows:

Definition 26

(i) A (*three-valued*) *sequent* is a triple of the form $\Gamma_1 \mid \Gamma_2 \mid \Gamma_3$, where each Γ_i ($i \in \{1, 2, 3\}$) is a finite set of formulas, called *component* of the sequent.

(ii) For an interpretation m, a sequent $\Gamma_1 \mid \Gamma_2 \mid \Gamma_3$ is *true under* m if, for at least one $i \in \{1, 2, 3\}$, Γ_i contains some formula A such that $V^m(A) = \mathsf{v}_i$, where $\mathsf{v}_1 = \mathbf{f}$, $\mathsf{v}_2 = \mathbf{u}$, and $\mathsf{v}_3 = \mathbf{t}$. Furthermore, a sequent is *valid* if it is true under each interpretation. □

Note that a standard classical sequent in the sense of Gentzen [23], $\Gamma \vdash \Delta$, corresponds to a pair $\Gamma \mid \Delta$ under the usual two-valued semantics of **PL**.

As customary for sequents, we write sequent components comprised of a singleton set $\{A\}$ simply as "A" and similarly $\Gamma \cup \{A\}$ is written as "Γ, A".

For obtaining the postulates of a many-valued logic in Rousseau's approach, the conditions of the logical connectives of a given logic are encoded in two-valued logic by means of a so-called *partial normal form* [55] and expressed by suitable inference rules.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

The calculus we employ for $\mathbf{Ł_3}$, which we denote by $\mathsf{SŁ}_3$, is taken from Zach [74], which is obtained from a systematic construction of sequent-style calculi for many-valued logics and by applying some optimisations of the corresponding partial normal forms.

The postulates of $\mathsf{SŁ}_3$ are as follows:

(i) axioms of $\mathsf{SŁ}_3$ consist of sequents of the form $A \mid A \mid A$, where A is a formula, and

(ii) the inference rules of $\mathsf{SŁ}_3$ are comprised of all rules depicted in Figure 4.1.

Note that from the inference rules of $\mathsf{SŁ}_3$ we can easily obtain derived rules for the defined connectives of $\mathbf{Ł_3}$. Furthermore, the last three rules in Figure 4.1 are also referred to as *weakening rules*.

Soundness and completeness of $\mathsf{SŁ}_3$ follows directly from the method described by Zach [74]:

Theorem 18 *A sequent* $\Gamma \mid \Delta \mid \Pi$ *is valid iff it is provable in* $\mathsf{SŁ}_3$.

Note that sequents in the style of Rousseau are *truth functional* rather than formalising entailment directly, but, as follows directly from a general result for many-valued logics shown by Zach [74], the latter can be expressed simply as follows:

Theorem 19 *For a theory T and a formula A, $T \vdash_3 A$ iff the sequent $T \mid T \mid A$ is provable in* $\mathsf{SŁ}_3$.

An Anti-Sequent Calculus for $\mathbf{Ł_3}$

As for axiomatising non-theorems of $\mathbf{Ł_3}$, a systematic construction of refutation calculi for many-valued logics has been developed by Bogojeski [10] by adapting the approach of Zach [74]. The refutation calculus we introduce now for axiomatising invalid sequents in $\mathbf{Ł_3}$, denoted by $\mathsf{RŁ}_3$, is taken from Bogojeski [10].

Definition 27

(i) A (*three-valued*) *anti-sequent* is a triple of form $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$, where each Γ_i ($i \in \{1,2,3\}$) is a finite set of formulas, called *component* of the anti-sequent.

(ii) For an interpretation m, an anti-sequent $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$ is *refuted by m*, or *m refutes* $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$, if, for every $i \in \{1,2,3\}$ and every formula $A \in \Gamma_i$, $\mathrm{V}^m(A) \neq \mathsf{v}_i$, where v_i is defined as in Definition 26.

(iii) If m refutes $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$, then m is also said to be a *countermodel* of $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$. An anti-sequent $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$ is *refutable*, if there is at least one interpretation that refutes $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$. $\square$

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

$$\frac{\Gamma \nmid \Delta \nmid \Pi, A}{\Gamma, A \supset B \nmid \Delta \nmid \Pi}\ (\supset : \mathbf{f}^1)^r \qquad \frac{\Gamma, B \nmid \Delta \nmid \Pi}{\Gamma, A \supset B \nmid \Delta \nmid \Pi}\ (\supset : \mathbf{f}^2)^r$$

$$\frac{\Gamma \nmid \Delta, A, B \nmid \Pi}{\Gamma \nmid \Delta, A \supset B \nmid \Pi}\ (\supset : \mathbf{u}^1)^r \qquad \frac{\Gamma, B \nmid \Delta \nmid \Pi, A}{\Gamma \nmid \Delta, A \supset B \nmid \Pi}\ (\supset : \mathbf{u}^2)^r$$

$$\frac{\Gamma, A \nmid \Delta, A \nmid \Pi, B}{\Gamma \nmid \Delta \nmid \Pi, A \supset B}\ (\supset : \mathbf{t}^1)^r \qquad \frac{\Gamma, A \nmid \Delta, B \nmid \Pi, B}{\Gamma \nmid \Delta \nmid \Pi, A \supset B}\ (\supset : \mathbf{t}^2)^r$$

$$\frac{\Gamma \nmid \Delta \nmid \Pi, A}{\Gamma, \neg A \nmid \Delta \nmid \Pi}\ (\neg : \mathbf{f})^r \qquad \frac{\Gamma \nmid \Delta, A \nmid \Pi}{\Gamma \nmid \Delta, \neg A \nmid \Pi}\ (\neg : \mathbf{u})^r \qquad \frac{\Gamma, A \nmid \Delta \nmid \Pi}{\Gamma \nmid \Delta \nmid \Pi, \neg A}\ (\neg : \mathbf{t})^r$$

$$\frac{\Gamma, A \nmid \Delta \nmid \Pi}{\Gamma \nmid \Delta \nmid \Pi}\ (w : \mathbf{f})^r \qquad \frac{\Gamma \nmid \Delta, A \nmid \Pi}{\Gamma \nmid \Delta \nmid \Pi}\ (w : \mathbf{u})^r \qquad \frac{\Gamma \nmid \Delta \nmid \Pi, A}{\Gamma \nmid \Delta \nmid \Pi}\ (w : \mathbf{t})^r$$

Figure 4.2: Rules of the anti-sequent calculus $\mathsf{R}Ł_3$.

Clearly, an anti-sequent $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$ is refutable iff the corresponding sequent $\Gamma_1 \mid \Gamma_2 \mid \Gamma_3$ is not valid.

The postulates of $\mathsf{R}Ł_3$ are as follows:

(i) the axioms of $\mathsf{R}Ł_3$ are anti-sequents whose components are sets of propositional constants such that no constant appears in all components, and

(ii) the inference rules of $\mathsf{R}Ł_3$ are those given in Figure 4.2.

Note that, in contrast to $\mathsf{S}Ł_3$, the inference rules of $\mathsf{R}Ł_3$ have only single premisses. Indeed, this is a general pattern in sequent-style rejection calculi: if an inference rule for standard (assertional) sequents for a connective has n premisses, then there are usually n corresponding unary inference rules in the associated rejection calculus. Intuitively, what is exhaustive search in a standard sequent calculus becomes nondeterminism in a rejection calculus.

Again, soundness and completeness of $\mathsf{R}Ł_3$ follow from the systematic construction as described by Bogojeski [10]. Likewise, non-entailment in $\mathbf{Ł_3}$ is expressed similarly as for $\mathsf{S}Ł_3$.

Theorem 20 *An anti-sequent $\Gamma \nmid \Delta \nmid \Pi$ is refutable iff it is provable in* $\mathsf{R}Ł_3$.

Theorem 21 *For a theory T and a formula A, $T \not\models_3 A$ iff $T \nmid T \nmid A$ is provable in* $\mathsf{R}Ł_3$.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

$$\frac{\Gamma \mid \Gamma \mid A}{\Gamma; \emptyset \ \Rightarrow \ A; \emptyset} \ (l_1) \qquad \frac{\Gamma \nmid \Gamma \nmid A}{\Gamma; \emptyset \ \Rightarrow \ \emptyset; A} \ (l_2)$$

$$\frac{\Gamma; \emptyset \ \Rightarrow \ \Sigma_1; \Theta_1 \quad \Gamma; \emptyset \ \Rightarrow \ \Sigma_2; \Theta_2}{\Gamma; \emptyset \ \Rightarrow \ \Sigma_1, \Sigma_2; \Theta_1, \Theta_2} \ (mu)$$

$$\frac{\Gamma; \Delta \ \Rightarrow \ \Sigma; \Theta, A}{\Gamma; \Delta, (A : B_1, \ldots, B_n / C) \ \Rightarrow \ \Sigma; \Theta} \ (d_1) \qquad \frac{\Gamma; \Delta \ \Rightarrow \ \Sigma, \neg B; \Theta}{\Gamma; \Delta, (A : \ldots, B, \ldots / C) \ \Rightarrow \ \Sigma; \Theta} \ (d_2)$$

$$\frac{\Gamma; \Delta \ \Rightarrow \ \Sigma, \neg \mathrm{L}C; \Theta}{\Gamma; \Delta, (A : B_1, \ldots, B_n / C) \ \Rightarrow \ \Sigma; \Theta} \ (d_3)$$

$$\frac{\Gamma; \emptyset \ \Rightarrow \ A; \emptyset \qquad \Gamma, \mathrm{M}C; \Delta \ \Rightarrow \ \Sigma; \Theta, \neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C}{\Gamma; \Delta, (A : B_1, \ldots, B_n / C) \ \Rightarrow \ \Sigma; \Theta} \ (d_4)$$

Figure 4.3: Rules for default sequents of the calculus B_3.

The Default-Sequent Calculus B_3

We are now in a position to specify our calculus B_3 for brave reasoning in $\mathbf{DL_3}$.

Definition 28

(i) A (*brave*) *default sequent* is an ordered quadruple of the form $\Gamma; \Delta \ \Rightarrow \ \Sigma; \Theta$, where Γ, Σ, and Θ are finite sets of formulas and Δ is a finite set of defaults.

(ii) A default sequent $\Gamma; \Delta \ \Rightarrow \ \Sigma; \Theta$ is *true* if there is an extension E of the default theory $\langle \Gamma, \Delta \rangle$ such that $\Sigma \subseteq E$ and $\Theta \cap E = \emptyset$; E is called a *witness* of $\Gamma; \Delta \ \Rightarrow \ \Sigma; \Theta$. □

The default sequent calculus B_3 consists of three-valued sequents, anti-sequents, and default sequents. It incorporates the systems $\mathsf{SŁ}_3$ for three-valued sequents and $\mathsf{RŁ}_3$ for anti-sequents. Additionally, it has axioms of the form

$$\Gamma; \emptyset \ \Rightarrow \ \emptyset; \emptyset,$$

where Γ is a finite set of formulas, and the inference rules as depicted in Figure 4.3.

The informal meaning of the inference rules for the default sequents is the following:

- First of all, rules (l_1) and (l_2) combine three-valued sequents and anti-sequents with default sequents, respectively.
- Rule (mu) is the rule of "monotonic union"—it allows the joining of information in case that no default is present.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

- Rules (d_1)–(d_4) are the default introduction rules: rules (d_1), (d_2), and (d_3) take care of introducing non-active defaults, whilst rule (d_4) allows to introduce an active default.

Let us give an example to illustrate the functioning of the calculus.

Example 11 Consider the default theory $T = \langle W, D \rangle$ from Example 7, where

$$W = \{\mathit{Summer}, \neg \mathit{Sun\_Shining}\} \quad \text{and} \quad D = \left\{ \frac{\mathit{Summer} : \neg \mathit{Rain}}{\mathit{Sun\_Shining}} \right\}.$$

As we saw, the single default of this theory is inapplicable since $W \vdash_3 \neg \mathrm{L}\mathit{Sun\_Shining}$ and $E = \mathrm{Th}_3(W)$ is therefore the only extension of T. Consequently, $\mathit{Sun\_Shining} \notin E$ also holds. Hence, the default sequent

$$\mathit{Summer}, \neg \mathit{Sun\_Shining}; \frac{\mathit{Summer} : \neg \mathit{Rain}}{\mathit{Sun\_Shining}} \Rightarrow \neg \mathrm{L}\mathit{Sun\_Shining}; \mathit{Sun\_Shining}$$

is true. We will give a proof of this sequent in B_3.

The proof of this sequent, depicted below and denoted by β, uses the proof α as subpart. For brevity, we will use "S" for "*Summer*", "R" for "*Rain*", and "H" for *Sun_Shining*".

- Proof α:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg H \mid \neg H \mid \neg H}{\neg H \mid S, \neg H \mid \neg H}\,(w:\mathbf{u})}{\neg H, H \mid S, \neg H \mid \neg H}\,(w:\mathbf{f})}{\neg H, H \mid S, \neg H, H \mid \neg H}\,(w:\mathbf{u})}{S, \neg H, H \mid S, \neg H, H \mid \neg H}\,(w:\mathbf{f}) \qquad \dfrac{\dfrac{\dfrac{\neg H \mid \neg H \mid \neg H}{\neg H \mid S, \neg H \mid \neg H}\,(w:\mathbf{u})}{\neg H, H \mid S, \neg H \mid \neg H}\,(w:\mathbf{f})}{S, \neg H, H \mid S, \neg H \mid \neg H}\,(w:\mathbf{f})}{S, \neg H \mid S, \neg H \mid H \supset \neg H}\,(\supset:\mathbf{t})}{S, \neg H, \neg(H \supset \neg H) \mid S, \neg H \mid \emptyset}\,(\neg:\mathbf{f})}{S, \neg H \mid S, \neg H \mid \neg\neg(H \supset \neg H)}\,(\neg:\mathbf{t})}{S, \neg H; \emptyset \;\Rightarrow\; \neg \mathrm{L}H; \emptyset}\,(l_1),\ \text{definition of L}$$

- Proof β:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{S \nmid S, H \nmid H}{S \nmid S, \neg H \nmid H}\,(\neg:\mathbf{u})^r}{S, \neg H \nmid S, \neg H \nmid H}\,(\neg:\mathbf{f})^r}{S, \neg H; \emptyset \;\Rightarrow\; \emptyset; H}\,(l_2) \qquad \alpha}{S, \neg H; \emptyset \;\Rightarrow\; \neg \mathrm{L}H; H}\,(mu)}{S, \neg H; (S : \neg R / H) \;\Rightarrow\; \neg \mathrm{L}H; H}\,(d_3)$$

□

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

4.1.2 Adequacy of the Calculus

We now show soundness and completeness of B_3. To this end, we need some auxiliary results first dealing with properties of extensions with respect to active and non-active defaults.

Preparatory Properties

We start with two obvious results whose proofs are straightforward.

Lemma 1 *Let R, R' be sets of inference rules, and W, W' sets of formulas. Then, the following properties hold:*

(i) $W \subseteq \mathrm{Th}_3^R(W)$;

(ii) $\mathrm{Th}_3^R(W) = \mathrm{Th}_3^R(\mathrm{Th}_3^R(W))$;

(iii) *if* $R \subseteq R'$, *then* $\mathrm{Th}_3^R(W) \subseteq \mathrm{Th}_3^{R'}(W)$; *and*

(iv) *if* $W \subseteq W'$, *then* $\mathrm{Th}_3^R(W) \subseteq \mathrm{Th}_3^R(W')$.

Lemma 2 *Let A and B be formulas, W a set of formulas, and R a set of inference rules. Then:*

(i) *if* $A \notin \mathrm{Th}_3^R(W)$, *then* $\mathrm{Th}_3^R(W) = \mathrm{Th}_3^{R\cup\{A/B\}}(W)$;

(ii) *if* $A \in \mathrm{Th}_3^{R\cup\{A/B\}}(W)$, *then* $\mathrm{Th}_3^{R\cup\{A/B\}}(W) = \mathrm{Th}_3^R(W \cup \{B\})$.

Recall our notation $\mathsf{p}(d)$, $\mathsf{j}(d)$, and $\mathsf{c}(d)$, where d is a default, which we introduced in Section 3.1.1 for standard default logic **DL**. We adapt this notation for $\mathbf{DL_3}$ as follows: let us now write

$$\mathsf{j}^*(d) := \{B_1, \ldots, B_n, \mathrm{L}C\} \quad \text{and} \quad \mathsf{c}^*(d) := \mathrm{M}C$$

for a default $d = (A : B_1, \ldots, B_n/C)$ (we retain the notation $\mathsf{p}(d) = A$). Recall also that, for a set S of formulas, $\neg S$ stands for $\{\neg A \mid A \in S\}$.

Theorem 22 *Let $T = \langle W, D\rangle$ be a default theory, E a set of formulas, and d a default not active in E. Then, E is an extension of $\langle W, D\rangle$ iff E is an extension of $\langle W, D \cup \{d\}\rangle$.*

Proof. If $\neg\mathsf{j}^*(d) \cap E \neq \emptyset$, then $\mathcal{R}^*(D \cup \{d\}, E) = \mathcal{R}^*(D, E)$. So,

$$\mathrm{Th}_3^{\mathcal{R}^*(D\cup\{d\},E)}(W) = \mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W)$$

and the statement of the lemma holds quite trivially by Corollary 2.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

For the rest of the proof, assume thus $\neg\mathsf{j}^*(d) \cap E = \emptyset$. Since d is not active in E, $E \nvdash_3 \mathsf{p}(d)$ must then hold. Furthermore, $\mathcal{R}^*(D \cup \{d\}, E) = \mathcal{R}^*(D, E) \cup \{\mathsf{p}(d)/\mathsf{c}^*(d)\}$ holds.

Suppose E is an extension of $T = \langle W, D\rangle$, i.e., $E = \mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W)$. Since $E \nvdash_3 \mathsf{p}(d)$ and E is deductively closed, we obtain $\mathsf{p}(d) \notin E$, and so $\mathsf{p}(d) \notin \mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W)$. By Lemma 2(i),

$$\mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W) = \mathrm{Th}_3^{\mathcal{R}^*(D,E) \cup \{\mathsf{p}(d)/\mathsf{c}^*(d)\}}(W).$$

But

$$\mathcal{R}^*(D, E) \cup \{\mathsf{p}(d)/\mathsf{c}^*(d)\} = \mathcal{R}^*(D \cup \{d\}, E),$$

hence

$$\mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W) = \mathrm{Th}_3^{\mathcal{R}^*(D \cup \{d\},E)}(W).$$

Since $E = \mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W)$, we obtain $E = \mathrm{Th}_3^{\mathcal{R}^*(D \cup \{d\},E)}(W)$ and E is extension of $\langle W, D \cup \{d\}\rangle$.

This proves the "only if" direction; the "if" direction follows by essentially the same arguments, but employing additionally Lemma 1(iii). □

Theorem 23 *Let E be a set of formulas and d a default. If E is an extension of $\langle W, D \cup \{d\}\rangle$ and d is active in E, then E is an extension of $\langle W \cup \{\mathsf{c}^*(d)\}, D\rangle$.*

Proof. Suppose E is an extension of $\langle W, D \cup \{d\}\rangle$ and d is active in E. Then,

$$E = \mathrm{Th}_3^{\mathcal{R}^*(D \cup \{d\},E)}(W)$$

and, since d is active in E, $\neg\mathsf{j}^*(d) \cap E = \emptyset$. Therefore,

$$\mathcal{R}^*(D \cup \{d\}, E) = \mathcal{R}^*(D, E) \cup \{\mathsf{p}(d)/\mathsf{c}^*(d)\}$$

and thus

$$E = \mathrm{Th}_3^{\mathcal{R}^*(D,E) \cup \{\mathsf{p}(d)/\mathsf{c}^*(d)\}}(W).$$

But $E \vdash_3 \mathsf{p}(d)$ also holds (since d is active in E), and so

$$\mathsf{p}(d) \in \mathrm{Th}_3^{\mathcal{R}^*(D,E) \cup \{\mathsf{p}(d)/\mathsf{c}^*(d)\}}(W).$$

Therefore, by Lemma 2(ii),

$$\mathrm{Th}_3^{\mathcal{R}^*(D,E) \cup \{\mathsf{p}(d)/\mathsf{c}^*(d)\}}(W) = \mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W \cup \{\mathsf{c}^*(d)\}).$$

Thus, $E = \mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W \cup \{\mathsf{c}^*(d)\})$, and so E is an extension of $\langle W \cup \{\mathsf{c}^*(d)\}, D\rangle$. □

Theorem 24 *Let E be a set of formulas and d a default. If (i) E is an extension of the default theory $\langle W \cup \{\mathsf{c}^*(d)\}, D\rangle$, (ii) $W \vdash_3 \mathsf{p}(d)$, and (iii) $\neg\mathsf{j}^*(d) \cap E = \emptyset$, then E is an extension of $\langle W, D \cup \{d\}\rangle$.*

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Proof. Assume that the preconditions of the theorem hold. Since E is an extension of $\langle W \cup \{\mathsf{c}^*(d)\}, D\rangle$,

$$E = \mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W \cup \{\mathsf{c}^*(d)\}).$$

Furthermore, by the hypothesis $W \vdash_3 \mathsf{p}(d)$, we have

$$\mathsf{p}(d) \in \mathrm{Th}_3^{\mathcal{R}^*(D,E)\cup\{\mathsf{p}(d)/\mathsf{c}^*(d)\}}(W).$$

We thus get

$$\mathrm{Th}_3^{\mathcal{R}^*(D,E)\cup\{\mathsf{p}(d)/\mathsf{c}^*(d)\}}(W) = \mathrm{Th}_3^{\mathcal{R}^*(D,E)}(W \cup \{\mathsf{c}^*(d)\})$$

in view of Lemma 2(ii), and therefore

$$E = \mathrm{Th}_3^{\mathcal{R}^*(D,E)\cup\{\mathsf{p}(d)/\mathsf{c}^*(d)\}}(W).$$

By observing that the assumption $\neg\mathsf{j}^*(d) \cap E = \emptyset$ implies

$$\mathcal{R}^*(D,E) \cup \{\mathsf{p}(d)/\mathsf{c}^*(d)\} = \mathcal{R}^*(D \cup \{d\}, E),$$

the result follows. □

Soundness and Completeness

We are now in a position to prove soundness and completeness of B_3.

Theorem 25 (Soundness) *If* $\Gamma; \Delta \Rightarrow \Sigma; \Theta$ *is provable in* B_3*, then it is true.*

Proof. We show that all axioms are true, and that the conclusions of all inference rules are true whenever its premisses are true (resp., valid or refutable in case of rules (l_1) and (l_2)).

First of all, an axiom $\Gamma; \emptyset \Rightarrow \emptyset; \emptyset$ is trivially true, because $\mathrm{Th}_3(\Gamma)$ is the unique extension of the default theory $\langle \Gamma, \emptyset\rangle$ and hence the unique witness of $\Gamma; \emptyset \Rightarrow \emptyset; \emptyset$.

Suppose $\Gamma \mid \Gamma \mid A$ is the premiss of rule (l_1) and assume it is valid. Hence, $\Gamma \vdash_3 A$ and therefore $A \in \mathrm{Th}_3(\Gamma)$. But $\mathrm{Th}_3(\Gamma)$ is the unique extension of $\langle \Gamma, \emptyset\rangle$, so $\mathrm{Th}_3(\Gamma)$ is the unique witness of $\Gamma; \emptyset \Rightarrow A; \emptyset$. Likewise, if the premiss $\Gamma \nmid \Gamma \nmid A$ of rule (l_2) is refutable, then $A \notin \mathrm{Th}_3(\Gamma)$, and therefore $\mathrm{Th}_3(\Gamma)$ is the (unique) witness of $\Gamma; \emptyset \Rightarrow \emptyset; A$.

If the two premisses $\Gamma; \emptyset \Rightarrow \Sigma_1; \Theta_1$ and $\Gamma; \emptyset \Rightarrow \Sigma_2; \Theta_2$ of rule (mu) are true, then they must have the same witness $E = \mathrm{Th}_3(\Gamma)$. Hence, E is also the (unique) witness of $\Gamma; \emptyset \Rightarrow \Sigma_1, \Sigma_2; \Theta_1, \Theta_2$.

As for the soundness of the rules d_1, d_2, and d_3, we only show the case for d_3; the other two are similar. Let E be a witness of $\Gamma; \Delta \Rightarrow \Sigma, \neg \mathrm{L}C; \Theta$. Then, E is an extension of $\langle \Gamma, \Delta\rangle$, $\Sigma \cup \{\neg \mathrm{L}C\} \subseteq E$, and $\Theta \cap E = \emptyset$. So, $\neg \mathrm{L}C \in E$ and thus the default $(A : B_1, \ldots, B_n / C)$ is not active in E. By Theorem 22, it follows that E is an extension

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

of $\langle \Gamma, \Delta \cup \{(A : B_1, \ldots, B_n/C)\}\rangle$. Moreover, since $\Sigma \subseteq E$ and $\Theta \cap E = \emptyset$, E is a witness of $\Gamma; \Delta, (A : B_1, \ldots, B_n/C) \Rightarrow \Sigma; \Theta$.

Finally, assume that the premisses of rule (d_4) are true. Let E_1 be a witness of

$$\Gamma; \emptyset \Rightarrow A; \emptyset$$

and E_2 a witness of

$$\Gamma, \mathrm{M}C; \Delta \Rightarrow \Sigma; \Theta, \neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C.$$

Thus, E_2 is an extension of $\langle \Gamma \cup \{\mathrm{M}C\}, \Delta\rangle$ and $\{\neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C\} \cap E_2 = \emptyset$. So, E_1 is an extension of $\langle \Gamma, \emptyset\rangle$ with $A \in E_1$, and therefore $\Gamma \vdash_3 A$. Hence, by Theorem 24, E_2 is an extension of

$$\langle \Gamma, \Delta \cup \{(A : B_1, \ldots, B_n/C)\}\rangle.$$

Clearly, $\Sigma \subseteq E_2$ and $\Theta \cap E_2 = \emptyset$, so E_2 is a witness of

$$\Gamma; \Delta, (A : B_1, \ldots, B_n/C) \Rightarrow \Sigma; \Theta.$$

□

Theorem 26 (Completeness) *If* $\Gamma; \Delta \Rightarrow \Sigma; \Theta$ *is true, then it is provable in* B_3.

Proof. Suppose $S = \Gamma; \Delta \Rightarrow \Sigma; \Theta$ is true, with E as its witness. The proof proceeds by induction on the cardinality $|\Delta|$ of Δ.

Induction Base. Assume $|\Delta| = 0$. If $\Sigma = \Theta = \emptyset$, then S is an axiom and hence provable in B_3. So suppose that either $\Sigma \neq \emptyset$ or $\Theta \neq \emptyset$. Since $\mathrm{Th}_3(\Gamma)$ is the unique extension of $\langle \Gamma, \emptyset\rangle$, we have $E = \mathrm{Th}_3(\Gamma)$. Furthermore, $\Sigma \subseteq E$ and $\Theta \cap E = \emptyset$. It follows that for any $A \in \Sigma$, the sequent $\Gamma \mid \Gamma \mid A$ is provable in $\mathsf{SŁ}_3$, and for any $B \in \Theta$, the anti-sequent $\Gamma \nmid \Gamma \nmid B$ is provable in $\mathsf{RŁ}_3$. Repeated applications of rules (l_1), (l_2), and (mu) yield a proof of S in B_3.

Induction Step. Assume $|\Delta| > 0$, and let the statement hold for all default sequents $\Gamma'; \Delta' \Rightarrow \Sigma'; \Theta'$ such that $|\Delta'| < |\Delta|$. We distinguish two cases: (i) there is some default in Δ which is active in E, or (ii) none of the defaults in Δ is active in E.

If (i) holds, then there must be some default $d = (A : B_1, \ldots, B_n/C)$ in Δ such that d is active in E and $\Gamma \vdash_3 A$. Consider $\Delta_0 := \Delta \setminus \{d\}$. Then, $|\Delta_0| = |\Delta| - 1$ and $\Delta_0 \cup \{d\} = \Delta$. By Theorem 23, E is an extension of $\langle \Gamma \cup \{\mathrm{M}C\}, \Delta_0\rangle$. Since d is active in E, $\{\neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C\} \cap E = \emptyset$; and since E is a witness of $S = \Gamma; \Delta \Rightarrow \Sigma; \Theta$, $\Sigma \subseteq E$ and $\Theta \cap E = \emptyset$. So, E is a witness of

$$S' = \Gamma, \mathrm{M}C; \Delta_0 \Rightarrow \Sigma; \Theta, \neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C.$$

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Since $|\Delta_0| < |\Delta|$, by induction hypothesis there is some proof α in B_3 of S'. Furthermore, $\Gamma \vdash_3 A$, so there is some proof β of the sequent $\Gamma \mid \Gamma \mid A$ in $\mathsf{S}Ł_3$. The following figure is a proof of S in B_3:[2]

$$\dfrac{\dfrac{\overset{\beta}{\Gamma \mid \Gamma \mid A}}{\Gamma; \emptyset \;\Rightarrow\; A; \emptyset}\,(l_1) \qquad \overset{\alpha}{\Gamma, \mathrm{M}C; \Delta_0 \;\Rightarrow\; \Sigma; \Theta, \neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C}}{\Gamma; \Delta \;\Rightarrow\; \Sigma; \Theta}\,(d_4)$$

Now assume that (ii) holds, i.e., no default in Δ is active in E. Since $|\Delta| > 0$, there is some default $d = (A : B_1, \ldots, B_n / C)$ in Δ such that $\Delta = \Delta_0 \cup \{d\}$ with $\Delta_0 := \Delta \setminus \{d\}$. Since d is not active in E, according to Theorem 22, E is an extension of $\langle \Gamma, \Delta_0 \rangle$. Furthermore, either

- $E \nvdash_3 A$,
- there is some $B_{i_0} \in \{B_1, \ldots, B_n\}$ such that $\neg B_{i_0} \in E$, or
- $\neg \mathrm{L}C \in E$.

Consequently, E is either a witness of

- $\Gamma; \Delta_0 \;\Rightarrow\; \Sigma; \Theta, A$,
- $\Gamma; \Delta_0 \;\Rightarrow\; \Sigma, \neg B_{i_0}; \Theta$, or
- $\Gamma; \Delta_0 \;\Rightarrow\; \Sigma, \neg \mathrm{L}C; \Theta$.

Since $|\Delta_0| < |\Delta|$, by induction hypothesis there is either

- a proof α in B_3 of $\Gamma; \Delta_0 \;\Rightarrow\; \Sigma; \Theta, A$,
- a proof β in B_3 of $\Gamma; \Delta_0 \;\Rightarrow\; \Sigma, \neg B_{i_0}; \Theta$, or
- a proof γ in B_3 of $\Gamma; \Delta_0 \;\Rightarrow\; \Sigma, \neg \mathrm{L}C; \Theta$.

Therefore, one of the three figures below constitutes a proof of S:[3]

$$\dfrac{\overset{\alpha}{\Gamma; \Delta_0 \;\Rightarrow\; \Sigma; \Theta, A}}{\Gamma; \Delta \;\Rightarrow\; \Sigma; \Theta}\,(d_1) \qquad \dfrac{\overset{\beta}{\Gamma; \Delta_0 \;\Rightarrow\; \Sigma, \neg B_{i_0}; \Theta}}{\Gamma; \Delta \;\Rightarrow\; \Sigma; \Theta}\,(d_2) \qquad \dfrac{\overset{\gamma}{\Gamma; \Delta_0 \;\Rightarrow\; \Sigma, \neg \mathrm{L}C; \Theta}}{\Gamma; \Delta \;\Rightarrow\; \Sigma; \Theta}\,(d_3)$$

□

[2] Note that in this figure, the endsequents of α and β have been displayed explicitly for better clarity.
[3] Again, the respective endsequents of α, β, and γ are explicitly shown.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

4.2 A Sequent Calculus for Disjunctive Default Logic

We now introduce our sequent calculus for brave reasoning for disjunctive default logic which we denote by $\mathsf{B_D}$. Again, the calculus will comprise three kinds of sequents:

(i) sequents for expressing validity in **PL**,

(ii) anti-sequents for expressing non-tautologies, and

(iii) special default inference rules reflecting brave reasoning in $\mathbf{DL_D}$.

As sequents for propositional logic, we use standard two-sided sequents in the sense of Gentzen [23] and a corresponding calculus, LK, which is a slight simplification of the one originally introduced by Gentzen. As a calculus for anti-sequents, we use the one due to Bonatti [11] which he introduced in connection of his calculus for brave reasoning for standard default logic [13, 12]; we will denote this calculus by LK^{r}.

4.2.1 Postulates of the Calculus

We start with defining the sequent calculus LK for classical sequents.

The Sequent Calculus LK

Definition 29

(i) A (*classical*) *sequent* S is an ordered pair of the form $\Gamma \to \Sigma$, where Γ and Σ are finite sets of formulas. Γ is the *antecedent* of S and Σ is the *succedent* of S.

(ii) For a two-valued interpretation I, a sequent $\Gamma \to \Delta$ is *true under* I if, whenever all formulas in Γ are true under I, then at least one formula in Δ is true under I. Furthermore, a sequent is *valid* if it is true under each interpretation. $\square$

As customary for sequents, we write sequents of the form $\Gamma \cup \{A\} \to \Delta$ simply as "$\Gamma, A \to \Delta$", and if the antecedent or succedent of a sequent is the empty set, then it is omitted from the sequent.

The postulates of LK are as follows:

(i) axioms of LK are sequents of the form

  (a) $\to \top$,

  (b) $\bot \to$, and

  (c) $A \to A$, and

(ii) the inference rules of LK are those given in Figure 4.4.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

$$\frac{\Gamma \to \Sigma, A \quad \Lambda, B \to \Pi}{\Gamma, \Lambda, (A \supset B) \to \Sigma, \Pi}\ (\supset \to) \qquad \frac{\Gamma, A \to \Sigma, B}{\Gamma \to \Sigma, (A \supset B)}\ (\to \supset)$$

$$\frac{\Gamma \to \Sigma, A}{\Gamma, \neg A \to \Sigma}\ (\neg \to) \qquad \frac{\Gamma, A \to \Sigma}{\Gamma \to \Sigma, \neg A}\ (\to \neg)$$

$$\frac{\Gamma \to \Sigma}{\Gamma, A \to \Sigma}\ (wl) \qquad \frac{\Sigma \to \Gamma}{\Sigma \to \Gamma, A}\ (wr)$$

Figure 4.4: Rules of the sequent calculus LK.

Note that the last two rules in Figure 4.4 are referred to as *weakening rules*. Moreover, from the rules of LK, we can easily obtain derived rules for the defined connectives $\wedge$, $\vee$, and $\equiv$. For instance, the derived rules for $\wedge$ and $\vee$ are as follows:

$$\frac{\Gamma, A \to \Sigma}{\Gamma, (A \wedge B) \to \Sigma}\ (\wedge \to)_1 \qquad \frac{\Gamma, B \to \Sigma}{\Gamma, (A \wedge B) \to \Sigma}\ (\wedge \to)_2$$

$$\frac{\Gamma \to \Sigma, A \quad \Lambda \to \Pi, B}{\Gamma, \Lambda \to \Sigma, \Pi, (A \wedge B)}\ (\to \wedge) \qquad \frac{\Gamma, A \to \Sigma \quad \Lambda, B \to \Pi}{\Gamma, \Lambda, (A \vee B) \to \Sigma, \Pi}\ (\vee \to)$$

$$\frac{\Gamma \to \Sigma, A}{\Gamma \to \Sigma, (A \vee B)}\ (\to \vee)_1 \qquad \frac{\Gamma \to \Sigma, B}{\Gamma \to \Sigma, (A \vee B)}\ (\to \vee)_2$$

Soundness and completeness of LK is well known:

Theorem 27 *A sequent $\Gamma \to \Sigma$ is valid iff it is provable in* LK.

In particular, the following relation follows immediately:

Corollary 3 *For every formula A,*

$$\models_2 A \text{ iff the sequent } \to A \text{ is provable in } \mathsf{LK}.$$

The Anti-Sequent Calculus LK^{r}

Now we introduce our complementary calculus LK^{r} for axiomatising invalidity in propositional logic.

Definition 30

(i) An *anti-sequent* is an ordered pair of the form $\Gamma \nrightarrow \Theta$, where Γ and Θ are finite sequences of formulas.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

$$\frac{\Gamma \nrightarrow \Theta, A}{\Gamma, (A \supset B) \nrightarrow \Theta}\ (\supset \nrightarrow)^r_1 \qquad \frac{\Gamma, B \nrightarrow \Theta}{\Gamma, (A \supset B) \nrightarrow \Theta}\ (\supset \nrightarrow)^r_2$$

$$\frac{\Gamma, A \nrightarrow \Theta, B}{\Gamma \nrightarrow \Theta, (A \supset B)}\ (\nrightarrow \supset)^r$$

$$\frac{\Gamma \nrightarrow \Theta, A}{\Gamma, \neg A \nrightarrow \Theta}\ (\neg \nrightarrow)^r \qquad \frac{\Gamma, A \nrightarrow \Theta}{\Gamma \nrightarrow \Theta, \neg A}\ (\nrightarrow \neg)^r$$

Figure 4.5: Rules of the anti-sequent calculus LK^r.

(ii) For a two-valued interpretation I, an anti-sequent $\Gamma \nrightarrow \Theta$ is *refuted by* I, or I *refutes* $\Gamma \nrightarrow \Theta$, if every formula in Γ is true under I and every formula in Θ is false under I. If I refutes $\Gamma \nrightarrow \Theta$, then I is also said to be a *countermodel* of $\Gamma \nrightarrow \Theta$. An anti-sequent $\Gamma \nrightarrow \Theta$ is *refutable* if there is at least one interpretation that refutes $\Gamma \nrightarrow \Theta$. □

Hence, the anti-sequent $\Gamma \nrightarrow \Theta$ is refutable iff the classical sequent $\Gamma \rightarrow \Theta$ is invalid.

In accordance to the convention for classical sequents, we write "$\nrightarrow \Theta$" and "$\Gamma \nrightarrow$" whenever Γ or Θ is the empty set.

The postulates of LK^r are as follows:

(i) The axioms of LK^r are anti-sequents of the form $\Phi \nrightarrow \Psi$, where Φ and Ψ are disjoint finite sets of atomic formulas such that $\bot \notin \Phi$ and $\top \notin \Psi$, and

(ii) the inference rules of LK^r are those depicted in Figure 4.5.

Note that, following the general pattern of complementary calculi, the inference rules of LK^r have only single premisses.

We again can obtain corresponding derived rules for the defined connectives; here are the ones for $\wedge$ and $\vee$:

$$\frac{\Gamma, A, B \nrightarrow \Theta}{\Gamma, (A \wedge B) \nrightarrow \Theta}\ (\wedge \nrightarrow)^r$$

$$\frac{\Gamma \nrightarrow \Theta, A}{\Gamma \nrightarrow \Theta, (A \wedge B)}\ (\nrightarrow \wedge)^r_1 \qquad \frac{\Gamma \nrightarrow \Theta, B}{\Gamma \nrightarrow \Theta, (A \wedge B)}\ (\nrightarrow \wedge)^r_2$$

$$\frac{\Gamma, A \nrightarrow \Theta}{\Gamma, (A \vee B) \nrightarrow \Theta}\ (\vee \nrightarrow)^r_1 \qquad \frac{\Gamma, B \nrightarrow \Theta}{\Gamma, (A \vee B) \nrightarrow \Theta}\ (\vee \nrightarrow)^r_2$$

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

$$\frac{\Gamma \nRightarrow \Theta, A, B}{\Gamma \nRightarrow \Theta, (A \vee B)} \; (\nRightarrow \vee)^r$$

Soundness and completeness for LK^r was shown by Bonatti [11] (and, independently, by Goranko [24]):

Theorem 28 *An anti-sequent $\Gamma \nRightarrow \Theta$ is refutable iff it is provable in LK^r.*

For formulas, we have then the following immediate corollary:

Corollary 4 *For every formula A,*

$$\not\models_2 A \text{ iff the anti-sequent } \nRightarrow A \text{ is provable in } \mathsf{LK}^\mathsf{r}.$$

The Default-Sequent Calculus $\mathsf{B_D}$

We are now in a position to specify our calculus $\mathsf{B_D}$ for brave reasoning in disjunctive default logic.

Definition 31

(i) By a (*brave*) *disjunctive default sequent* we understand an ordered quadruple of the form $\Gamma; \Delta \Rightarrow \Sigma; \Theta$, where Γ, Σ, and Θ are finite sets of formulas and Δ is a finite set of disjunctive defaults.

(ii) A disjunctive default sequent $\Gamma; \Delta \Rightarrow \Sigma; \Theta$ is *true* iff there is an extension E of the disjunctive default theory $\langle \Gamma, \Delta \rangle$ such that $\Sigma \subseteq E$ and $\Theta \cap E = \emptyset$; E is called a *witness* of $\Gamma; \Delta \Rightarrow \Sigma; \Theta$. $\square$

The default sequent calculus $\mathsf{B_D}$ consists of sequents, anti-sequents, and default sequents. It incorporates the systems LK for sequents and LK^r for anti-sequents. Additionally, it has axioms of the form

$$\Gamma; \emptyset \Rightarrow \emptyset; \emptyset,$$

where Γ is a finite set of formulas, and the inference rules as depicted in Figure 4.6.

The informal meaning of the nonmonotonic inference rules is similar to the meaning of the rules in $\mathsf{B_3}$:

- Rules $(l_1)^\circ$ and $(l_2)^\circ$ combine classical sequents and anti-sequents with disjunctive default sequents, respectively.
- Rule $(mu)^\circ$ again allows the joining of information in case that no default is present.
- Rules $(d_1)^\circ$, $(d_2)^\circ$, and $(d_3)^\circ$ are the default introduction rules: rules $(d_1)^\circ$ and $(d_2)^\circ$ take care of introducing non-active defaults, whilst rule $(d_3)^\circ$ allows to introduce an active default.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

$$\frac{\Gamma \rightarrow A}{\Gamma;\emptyset \ \Rightarrow\ A;\emptyset}\ (l_1)^\circ \qquad\qquad \frac{\Gamma \not\rightarrow A}{\Gamma;\emptyset \ \Rightarrow\ \emptyset;A}\ (l_2)^\circ$$

$$\frac{\Gamma;\emptyset \ \Rightarrow\ \Sigma_1;\Theta_1 \quad \Gamma;\emptyset \ \Rightarrow\ \Sigma_2;\Theta_2}{\Gamma;\emptyset \ \Rightarrow\ \Sigma_1,\Sigma_2;\Theta_1,\Theta_2}\ (mu)^\circ$$

$$\frac{\Gamma;\Delta \ \Rightarrow\ \Sigma;\Theta,A}{\Gamma;\Delta,(A : B_1,\ldots,B_n\,/\,C_1|\cdots|C_m) \ \Rightarrow\ \Sigma;\Theta}\ (d_1)^\circ$$

$$\frac{\Gamma;\Delta \ \Rightarrow\ \Sigma,\neg B;\Theta}{\Gamma;\Delta,(A : \ldots,B,\ldots\,/\,C_1|\cdots|C_m) \ \Rightarrow\ \Sigma;\Theta}\ (d_2)^\circ$$

$$\frac{\Gamma;\emptyset \ \Rightarrow\ A;\emptyset \qquad \Gamma,C_i;\Delta \ \Rightarrow\ \Sigma;\Theta,\neg B_1,\ldots,\neg B_n}{\Gamma;\Delta,(A : B_1,\ldots,B_n\,/\,C_1|\cdots|C_i|\cdots|C_m) \ \Rightarrow\ \Sigma;\Theta}\ (d_3)^\circ$$

Figure 4.6: Additional rules of the calculus $\mathsf{B_D}$.

4.2.2 Adequacy of the Calculus

Soundness and completeness of $\mathsf{B_D}$ can be shown by similar arguments as in the case of $\mathsf{B_3}$. We sketch the relevant details.

First of all, we need the following pendant to activeness as defined earlier:

Definition 32 Let E be a set of formulas. A disjunctive default

$$\frac{A : B_1,\ldots,B_n}{C_1|\cdots|C_m}$$

is *active* in E iff $E \vdash A$ and $\{\neg B_1,\ldots,\neg B_n\} \cap E = \emptyset$. □

We again employ our notation $\mathsf{p}(d)$ and $\mathsf{j}(d)$ for a default d as in case of **DL**, but now we define

$$\mathsf{c}^\circ(d) := \{C_1,\ldots,C_m\},$$

for $d = (A : B_1,\ldots,B_n/C_1|\cdots|C_M)$.

We obtain the following results corresponding to Lemma 2 and Theorems 22, 23, and 24:

Lemma 3 *Let $r = A/B_1|\cdots|B_n$ be a disjunctive inference rule, W and E sets of formulas, and R a set of disjunctive inference rules. Then:*

(i) *if $A \notin E$ and $E \in Cn^R(W)$, then $E \in Cn^{R\cup\{r\}}(W)$;*

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

(ii) *if* $A \in E$ *and* $E \in Cn^{R \cup \{r\}}(W)$*, then* $E \in Cn^{R}(W \cup \{B\})$*, for some formula* $B \in \{B_1, \ldots, B_n\}$.

Theorem 29 *Let* $T = \langle W, D \rangle$ *be a disjunctive default theory,* E *a set of formulas, and* d *a disjunctive default not active in* E*. Then,* E *is an extension of* $\langle W, D \rangle$ *iff* E *is an extension of* $\langle W, D \cup \{d\} \rangle$.

Theorem 30 *Let* E *be a set of formulas and* d *a disjunctive default.*

(i) *If* E *is an extension of* $\langle W, D \cup \{d\} \rangle$ *and* d *is active in* E*, then* E *is an extension of* $\langle W \cup \{C\}, D \rangle$*, for some* $C \in \mathsf{c}^{\circ}(d)$.

(ii) *If* E *is an extension of the disjunctive default theory* $\langle W \cup \{C\}, D \rangle$*, for some* $C \in \mathsf{c}^{\circ}(d)$*,* $W \vdash \mathsf{p}(d)$*, and* $\neg \mathsf{j}(d) \cap E = \emptyset$*, then* E *is an extension of* $\langle W, D \cup \{d\} \rangle$.

From this, soundness and completeness of $\mathsf{B_D}$ can be shown.

Theorem 31 *A disjunctive default sequent* $\Gamma; \Delta \Rightarrow \Sigma; \Theta$ *is provable in* $\mathsf{B_D}$ *iff it is true.*

To conclude, let us give us an example of a disjunctive default sequent provable in $\mathsf{B_D}$.

Example 12 Let us consider the disjunctive default theory from Example 10 dealing with Poole's broken arms scenario [48], which contains the defaults

$$\frac{:\, lh\_usable \land \neg lh\_broken}{lh\_usable} \quad \text{and} \quad \frac{:\, rh\_usable \land \neg rh\_broken}{rh\_usable},$$

together with the disjunctive default

$$lh\_broken | rh\_broken.$$

This disjunctive default theory has the two extensions

$$\mathrm{Th}_2(\{lh\_broken, rh\_usable\}) \quad \text{and} \quad \mathrm{Th}_2(\{rh\_broken, lh\_usable\}),$$

Let us for brevity use the following abbreviations:

B_l: *lh_broken*;

B_r: *rh_broken*;

U_l: *lh_usable*;

U_r: *rh_usable*.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek Your knowledge hub WIEN

Then, the following disjunctive default sequent is true:

$$\emptyset; (:\emptyset/B_l|B_r), (:(U_l \wedge \neg B_l)/U_l), (:(U_r \wedge \neg B_r)/U_r) \Rightarrow B_l, U_r, \neg(U_l \wedge \neg B_l); U_l.$$

A proof in $\mathsf{B_D}$ is given below; it uses the two proofs α and β:

- Proof α:

$$\dfrac{\dfrac{\dfrac{\dfrac{U_r, B_l \nrightarrow U_l, B_r}{U_r, B_l, U_r, \neg B_r \nrightarrow U_l}\,(\neg\nrightarrow)^r}{U_r, B_l, (U_r \wedge \neg B_r) \nrightarrow U_l}\,(\wedge\nrightarrow)^r}{U_r, B_l \nrightarrow U_l, \neg(U_r \wedge \neg B_r)}\,(\nrightarrow\neg)^r}{U_r, B_l; \emptyset \Rightarrow \emptyset; U_l, \neg(U_r \wedge \neg B_r)}\,(l_2)^\circ$$

- Proof β:

$$\dfrac{\dfrac{\dfrac{\dfrac{B_l \rightarrow B_l}{B_l \rightarrow U_r, B_l}\,(wr)}{U_r, B_l \rightarrow U_r, B_l}\,(wl)}{U_r, B_l; \emptyset \Rightarrow U_r, B_l; \emptyset}\,(l_1)^\circ \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B_l \rightarrow B_l}{U_r, B_l \rightarrow B_l}\,(wl)}{U_r, B_l, \neg B_l \rightarrow \emptyset}\,(\neg\rightarrow)}{U_r, B_l, (U_l \wedge \neg B_l) \rightarrow \emptyset}\,(\wedge\rightarrow)_2}{U_r, B_l \rightarrow \neg(U_l \wedge \neg B_l)}\,(\rightarrow\wedge)}{U_r, B_l; \emptyset \Rightarrow \neg(U_l \wedge \neg B_l); \emptyset}\,(l_1)^\circ \quad \alpha}{U_r, B_l; \emptyset \Rightarrow \neg(U_l \wedge \neg B_l); U_l, \neg(U_r \wedge \neg B_r)}\,(mu)^\circ}{U_r, B_l; \emptyset \Rightarrow B_l, U_r, \neg(U_l \wedge \neg B_l); U_l, \neg(U_r \wedge \neg B_r)}\,(mu)^\circ$$

- Proof γ:

$$\dfrac{\dfrac{\dfrac{\emptyset \rightarrow \top}{\emptyset; \emptyset \Rightarrow \top; \emptyset}\,(l_1)^\circ \quad \dfrac{\dfrac{\dfrac{\emptyset \rightarrow \top}{U_r \rightarrow \top}\,wl}{U_r; \emptyset \Rightarrow \top; \emptyset}\,(l_1)^\circ \quad \beta}{U_r; (:\emptyset/B_l|B_r) \Rightarrow B_l, U_r, \neg(U_l \wedge \neg B_l); \neg(U_r \wedge \neg B_r), U_l}\,(d_3)^\circ}{\emptyset; (:\emptyset/B_l|B_r), (:(U_r \wedge \neg B_r)/U_r) \Rightarrow B_l, U_r, \neg(U_l \wedge \neg B_l), \neg(U_l \wedge \neg B_l); U_l}\,(d_3)^\circ}{\emptyset; (:\emptyset/B_l|B_r), (:(U_l \wedge \neg B_l)/U_l), (:(U_r \wedge \neg B_r)/U_r) \Rightarrow B_l, U_r, \neg(U_l \wedge \neg B_l); U_l}\,(d_2)^\circ$$

$\square$

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

Bibliography

[1] Grigoris Antoniou and Kewen Wang. Default logic. In *Handbook of the History of Logic, Volume 8: The Many Valued and Nonmonotonic Turn in Logic*, pages 517–632. North-Holland, 2007.

[2] Arnon Avron. Natural 3-valued logics - Characterization and proof theory. *Journal of Symbolic Logic*, 56 (1):276–294, 1991.

[3] Arnon Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In *Logic: From Foundations to Applications*, pages 1–32. Clarendon Press, 1996.

[4] Arnon Avron. Classical Gentzen-type methods in propositional many-valued logics. In *Theory and Applications in Multiple-Valued Logics*, pages 113–151. Springer, 2002.

[5] Robert Baumgartner and Georg Gottlob. Propositional default logics made easier: Computational complexity of model checking. *Theoretical Computer Science*, 289:591–627, 2002.

[6] Gerald Berger and Hans Tompits. On axiomatic rejection for the description logic $\mathcal{ALC}$. In *Declarative Programming and Knowledge Management–Declarative Programming Days (KDPD 2013), Revised Selected Papers*, volume 8439 of *Lecture Notes in Computer Science*, pages 65–82. Springer, 2014.

[7] Olaf Beyersdorff, Arne Meier, Michael Thomas, and Heribert Vollmer. The complexity of reasoning for fragments of default logic. *Journal of Logic and Computation*, 22(3):587–604, 2012.

[8] Jean-Yves Béziau. A sequent calculus for Lukasiewicz's three-valued logic based on Suszko's bivalent semantics. *Bulletin of the Section of Logic*, 28(2):89–97, 1999.

[9] Dmitrij Anatoljewitsch Bochvar. On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus (in Russian). *Matematicheskii Sbornik*, 4:287–308, 1938.

[10] Mihail Bogojeski. Gentzen-type refutation systems for finite-valued logics. Bachelor's Thesis, Technische Universität Wien, Institut für Informationssysteme, 2014.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU WIEN Bibliothek Your knowledge hub

[11] Piero A. Bonatti. A Gentzen system for non-theorems. Technical Report CD-TR 93/52, Christian Doppler Labor für Expertensysteme, Technische Universität Wien, 1993.

[12] Piero A. Bonatti. Sequent calculi for default and autoepistemic logic. In *Proceedings of the 5th International Workshop on Theorem Proving with Analytic Tableaux and Related Methods (TABLEAUX '96)*, volume 1071 of *Lecture Notes in Computer Science*, pages 127–142. Springer, 1993.

[13] Piero A. Bonatti. Sequent calculi for default and autoepistemic logic. Technical Report CD-TR 93/53, Christian Doppler Labor für Expertensysteme, Technische Universität Wien, 1993.

[14] Piero A. Bonatti and Nicola Olivetti. Sequent calculi for propositional nonmonotonic logics. *ACM Transactions on Computational Logic*, 3(2):226–278, 2002.

[15] Grzegorz Bryll. Kilka uzupelnień teorii zdań odrzuconych. *Zeszyty Naukowe Wyższej Szkoły Pedagogigicznej w Opolu, Seria B, Studia i Monografie*, 22:133–154, 1969.

[16] Grzegorz Bryll and Marian Maduch. Aksjomaty odrzucone dla wielowartościowych logik Łukasiewieza. *Zeszyty Naukowe Wyższej Szkoły Pedagogicznej w Opolu, Matematyka*, VI:3–19, 1969.

[17] Walter A. Carnielli. On sequents and tableaux for many-valued logics. *Journal of Non-Classical Logic*, 8(1):59–76, 1991.

[18] James P. Delgrande, Torsten Schaub, and W. Ken Jackson. Alternative approaches to default logic. *Artificial Intelligence*, 70(1–2):167–237, 1994.

[19] Patrick Doherty and Witold Łukaszewicz. NML3—A non–monotonic logic with explicit defaults. *Journal of Applied Non-Classical Logics*, 2(1):9–48, 1992.

[20] Rafał Dutkiewicz. The method of axiomatic rejection for the intuitionistic propositional logic. *Studia Logica*, 48(4):449–459, 1989.

[21] Michael Gelfond and Vladimir Lifschitz. Classical negation in logic programs and disjunctive databases. *New Generation Computing*, 9:365–385, 1991.

[22] Michael Gelfond, Vladimir Lifschitz, Halina Przymusinska, and Mirosław Truszczyński. Disjunctive defaults. In *Proc. KR '91*, pages 230–237, 1991.

[23] Gerhard Gentzen. Untersuchungen über das logische Schließen I. *Mathematische Zeitschrift*, 39(1):176–210, 1935.

[24] Valentin Goranko. Refutation systems in modal logic. *Studia Logica*, 53(2):299–324, 1994.

[25] Rainer Hähnle. Tableaux for many-valued logics. In *Handbook of Tableaux Methods*, pages 529–580. Kluwer, 1999.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

[26] Stephen C. Kleene. On a notation for ordinal numbers. *The Journal of Symbolic Logic*, 3:150–155, 1938.

[27] Stephen C. Kleene. *Introduction to Metamathematics*. Van Nostrand, New York, 1952.

[28] Fangzhen Lin and Yoav Shoham. Epistemic semantics for fixed-points non-monotonic logics. In Rohit Parikh, editor, *Proceedings of the Third Conference on Theoretical Aspects of Reasoning about Knowledge (TARK '90)*, pages 111–120. Morgan Kaufmann, 1990.

[29] Jan Łukasiewicz. O logice trójwartościowej. *Ruch filozoficzny*, 5:170–171, 1920.

[30] Jan Łukasiewicz. Logika dwuwartościowa. *Przegląd Filozoficzny*, 23:189–205, 1921.

[31] Jan Łukasiewicz. Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls. *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie Cl. III*, 23:51–77, 1930.

[32] Jan Łukasiewicz. O sylogistyce Arystotelesa. *Sprawozdania z Czynności i Posiedzeń Polskiej Akademii Umiejętności*, 44, 1939.

[33] Jan Łukasiewicz. *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*. Clarendon Press, Oxford, 2nd edition, 1957.

[34] Witold Łukaszewicz. Considerations on default logic – An alternative approach. *Computational Intelligence*, 4:1–16, 1988.

[35] Witold Łukaszewicz. *Non-Monotonic Reasoning: Formalization of Commonsense Reasoning*. Ellis Horwood Series in Artificial Intelligence, Chichester, 1990.

[36] Grzegorz Malinowski. *Many-Valued Logics*. Clarendon Press, Oxford, 1993.

[37] Grzegorz Malinowski. Many-valued logic and its philosophy. In *Handbook of the History of Logic, Volume 8: The Many Valued and Nonmonotonic Turn in Logic*, pages 13–94. North-Holland, 2007.

[38] Wiktor Marek and Mirosław Truszczyński. *Nonmonotonic Logic: Context-Dependent Reasoning*. Springer, Berlin, 1993.

[39] John McCarthy. Circumscription – A form of non-monotonic reasoning. *Artificial Intelligence*, 13:27–39, 1980.

[40] Artur Mikitiuk and Mirosław Truszczyński. Rational default logic and disjunctive logic programming. In *Proceedings of the 2nd International Workshop on Logic Programming and Non-monotonic Reasoning (LPNMR '93)*, pages 283–299. MIT Press, 1993.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

[41] Marvin Minsky. A framework for representing knowledge. In J. Haugeland, editor, *Mind Design*, pages 95–128. MIT Press, Cambridge, MA, 1975.

[42] Robert C. Moore. Semantical considerations on nonmonotonic logic. *Artificial Intelligence*, 25:75–94, 1985.

[43] Johannes Oetsch and Hans Tompits. Gentzen-type refutation systems for three-valued logics with an application to disproving strong equivalence. In *Proceedings of the 11th International Conference on Logic Programming and Nonmonotonic Resoning* (*LPNMR 2011*), volume 6645 of *Lecture Notes in Computer Science*, pages 254–259, 2011.

[44] David Pearce. Equilibrium logic. *Annals of Mathematics and Artificial Intelligence*, 47(1-2):3–41, 2006.

[45] Judea Pearl. *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Morgan Kaufmann Publishers, San Mateo, CA, 1988.

[46] Sopo Pkhakadze and Hans Tompits. A Gentzen-type calculus for disjunctive default logic. In *Gödel's Legacy: Does Future lie in the Past?*, 2019.

[47] Sopo Pkhakadze and Hans Tompits. A sequent-type calculus for three-valued default logic, or: Tweety meets quartum non datur. In *Proceedings of the 15th International Conference on Logic Programming and Nonmonotonic Reasoning* (*LPNMR 2019*), volume 11481 of *Lecture Notes in Computer Science*, pages 163–177. Springer, 2019.

[48] David Poole. What the lottery paradox tells us about default reasoning. In Ronald J. Brachman, Hector J. Levesque, and Raymond Reiter, editors, *Proceedings of the First International Conference on Principles of Knowledge Representation and Reasoning* (*KR '89*), pages 333–340. Morgan Kaufmann, 1989.

[49] Emil Post. Introduction to a general theory of elementary propositions. *Bulletin of the American Mathematical Society*, 26:437, 1920.

[50] Anna Radzikowska. A three-valued approach to default logic. *Journal of Applied Non-Classical Logics*, 6(2):149–190, 1996.

[51] Raymond Reiter. A logic for default reasoning. *Artificial Intelligence*, 13:81–132, 1980.

[52] Raymond Reiter and Giovanni Criscuolo. Some representational issues in default reasoning. Technical Report 80-7, Deptartment of Computer Science, The University of British Columbia, Vancouver, B.C., Canada, 1980.

[53] Raymond Reiter and Giovanni Criscuolo. On interacting defaults. In Patrick J. Hayes, editor, *Proceedings of the 7th International Joint Conference on Artificial Intelligence* (*IJCAI '81*), pages 270–276. William Kaufmann, 1981.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.

[54] Raymond Reiter and Giovanni Criscuolo. Some representational issues in default reasoning. *International Journal on Computers and Mathematics*, 9:1–13, 1983.

[55] J. Barkley Rosser and Atwell R. Turquette. *Many-valued logics*. North-Holland, Amsterdam, 1952.

[56] George Sebastian Rousseau. Sequents in many valued logic I. *Fundamenta Mathematicae*, 60:23–33, 1967.

[57] Torsten Schaub. On constrained default theories. Technical Report AIDA-92-2, FG Intellektik, FB Informatik, TH Darmstadt, 1992.

[58] Tomasz Skura. A complete syntactic characterization of the intuitionistic logic. *Reports on Mathematical Logic*, 23, 1989.

[59] Tomasz Skura. Some results concerning refutation procedures. *Acta Universitatis Wratislaviensis, Logika*, 15, 1993.

[60] Tomasz Skura. Aspects of refutation procedures in the intuitionistic logic and related modal systems. *Acta Universitatis Wratislaviensis, Logika*, 20, 1999.

[61] Tomasz Skura. Refutation systems in propositional logic. In *Handbook of Philosophical Logic*, pages 115–157. Springer, 2011.

[62] Tomasz Skura. *Refutation Methods in Modal Propositional Logic*. Semper, Warszawa, 2013.

[63] Jerzy Słupecki. Z badań nad sylogistyką Arystotelesa. *Prace Wrocławskiego Towarzystwa Naukowego* (B), 6, 1948. English translation: [64].

[64] Jerzy Słupecki. On Aristotelian syllogistic. *Studia Philosophica, Posnaniae*, 4:275–300, 1951.

[65] Jerzy Słupecki. Funkcja Łukasiewieza. *Zeszyty Naukowe Uniwersytetu Wrocławskiego, Seria A*, 3:33–40, 1959.

[66] Jerzy Słupecki, Grzegorz Bryll, and Urszula Wybraniec-Skardowska. Theory of rejected propositions I. *Studia Logica*, 29(1):75–115, 1971.

[67] Jerzy Słupecki, Grzegorz Bryll, and Urszula Wybraniec-Skardowska. Theory of rejected propositions II. *Studia Logica*, 30(1):97–139, 1972.

[68] Raymond Smullyan. *First-Order Logic*, volume 43 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, New York, 1968.

[69] Michael L. Tiomkin. Proving unprovability. In *Symposium on Logic in Computer Science*, pages 22–26. IEEE Computer Society, 1988.

Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.
TU Bibliothek WIEN Your knowledge hub

[70] Hans Tompits. *On Proof Complexities of First-Order Nonmonotonic Logics*. PhD thesis, Technische Universität Wien, Institut für Informationssysteme, 1998.

[71] Mordchaj Wajsberg. Aksjomatyzacja trójwartościowego rachunku zdań. *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie Cl. III*, 24:136–148, 1931.

[72] Urszula Wybraniec-Skardowska. Teoria zdań odrzuconych. *Zeszyty Naukowe Wyższej Szkoły Pedagogigicznej w Opolu, Seria B, Studia i Monografie*, 22:5–131, 1969.

[73] Urszula Wybraniec-Skardowska. On the notion and function of the rejection of propositions. *Acta Universitatis Wratislaviensis, Logika*, 23:179–202, 2005.

[74] Richard Zach. Proof theory of finite-valued logics. Master's thesis, Technische Universität Wien, Institut für Computersprachen, 1993.

[75] Yi Zhou, Fangzhen Lin, and Yan Zhang. General default logic. *Annals of Mathematics and Artificial Intelligence*, 57(2):125–160, 2009.

TU WIEN Bibliothek Your knowledge hub
Die approbierte gedruckte Originalversion dieser Diplomarbeit ist an der TU Wien Bibliothek verfügbar.
The approved original version of this thesis is available in print at TU Wien Bibliothek.