TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DISSERTATION

Algorithmic Properties of Equivalence Relations

Ausgeführt zum Zwecke der Erlangung des akademischen Grades eines Doktor der technischen Wissenschaften

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Wien, am 05. Oktober 2016

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Kurzfassung der Dissertation

Äquivalenzrelationen formalisieren den Begriff der Ähnlichkeit von mathematischen Objekten. Aus Sicht der berechenbaren Strukturtheorie sind Äquivalenzrelationen ein nützliches Mittel, um die Effektivität mathematischer Strukturen zu analysieren. Weiters ist die Untersuchung algorithmischer Eigenschaften von Äquivalenzrelationen ein aktives Forschungsgebiet, das viele offene Fragen hat und die Aufmerksamkeit von Spezialisten auf sich zieht. Wir betrachten nur abzählbare Strukturen in berechenbaren Sprachen. Wir setzen Strukturen mit ihren Atomardiagrammen gleich. Insbesondere ist eine Struktur berechenbar, wenn ihr Atomardiagramm eine berechenbare Teilmenge der natürlichen Zahlen ist. Wir betrachten die obengenannten Richtungen zur Erforschung von Äquivalenzrelationen in berechenbarer Strukturtheorie.

Zuerst betrachten wir Isomorphismen beschränkten Turinggrades. Für einen Turinggrad $\mathbf{d}$ sagen wir, dass eine berechenbare Struktur $\mathcal{A}$ $\mathbf{d}$-kategorisch ist, falls es für jede isomorphe Kopie $\mathcal{B}$ einen $\mathbf{d}$-berechenbaren Isomorphismus zwischen $\mathcal{A}$ und $\mathcal{B}$ gibt. Wir untersuchen die Beziehungen zwischen algebraischen, deskriptiven, und algorithmischen Eigenschaften mathematischer Strukturen mit Hilfe der Begriffe Δ_n^0 Kategorizität und relativer Δ_n^0 Kategorizität. Wir betrachten natürliche Strukturklassen, inklusive verschiedener Arten von Gruppen, Booleschen Algebras, Fraïssé Limits, usw. Diese Ergebnisse erscheinen als Teil des Papers "Computability-Theoretic Categoricity and Scott Families" von E. Fokina, V. Harizanov und D. Turetsky.

Wir beantworten dann die Frage, wie schwierig es ist die Eigenschaft der effektiven Kategorizität für Strukturen mit verschiedenen algorithmischen Einschränkungen zu beschreiben. Der Hauptbegriff hier ist der Begriff der Indexmenge. Wir geben die exakte Abschätzung der Komplexität für die n-entscheidbare Strukturen, die kategorisch bezüglich m-entscheidbarer Präsentationen sind, für verschiedene $m, n \in \omega$. Die Ergebnisse erscheinen in "Index sets of n-decidable structures categorical relative to m-decidable presentations" von E. Fokina, S. Goncharov, V. Harizanov, O. Kudinov und D. Turetsky.

Schließlich verwenden wir Grade von Atomardiagrammen, um die inhärente Komplexität

von Äquivalenzklassen verschiedener Strukturen zu charakterisieren. Wir führen den Begriff des Gradspektrums bezüglich Σ_n-Äquivalenz ein (zwei Strukturen sind Σ_n-äquivalent, falls ihre Σ_n-Diagramme gleich sind). Die Ergebnisse stellen ein Teil des Papers "Degree Spectra of Structures relative to Equivalence Relations" von E. Fokina, P. Semukhin und D. Turetsky dar. Für alle Ergebnisse, die in dieser Dissertation enthalten sind, hat E. Fokina den Hauptbeitrag geleistet, sowohl bei der Erbingung der Beweise als auch beim Verfassen der Publikationen.

Die Einführung enthält den Stoff aus "Computable Model Theory" von E. Fokina, V. Harizanov und A. Melnikov.

Summary

Equivalence relations formalize the idea of resemblance between mathematical objects. In the context of computable structure theory, equivalence relations are a useful tool to study the effectiveness of mathematical structures. On the other hand, the study of algorithmic properties of equivalence relations themselves is an active area of investigation with many open questions attracting a lot of attention of specialists in the field. We only consider countable structures in computable languages. We identify structures with their atomic diagrams.In particular, a structure is computable if its atomic diagram is a computable subset of the natural numbers. We consider the above mentioned directions to study the role of equivalence relations in computable structure theory.

We start by looking at isomorphisms of bounded Turing degree. For a degree $\mathbf{d}$, we say that a computable structure $\mathcal{A}$ is $\mathbf{d}$-categorical if for every its isomorphic computable copy $\mathcal{B}$ there exists a $\mathbf{d}$-computable isomorphims between $\mathcal{A}$ and $\mathcal{B}$. First we concider the connections between algebraic, descriptive and algorithmic properties of mathematical structure through the notions of Δ^0_n categoricity and relative Δ^0_n categoricity. We look at natural classes of structures including various kinds of groups, Boolean algebras, Fraïssé limits, etc. The results form a part of the paper "Computability-Theoretic Categoricity and Scott Families" by E. Fokina, V. Harizanov and D. Turetsky.

We then answer the question how hard it is to describe the property of effective categoricity for structures with various algorithmic restrictions. The main notion here is the notion of index sets. We give the exact estimation of complexity for structures that are n-decidable categorical relative to m-decidable presentations, for various $m, n \in \omega$. The results appear in the paper "Index sets of n-decidable structures categorical relative to m-decidable presentations", joint with S. Goncharov, V. Harizanov, O. Kudinov and D. Turetsky.

Finally, we employ degrees of atomic diagrams of structures to characterize the inherent complexity of equivalence classes of structures, up to various equivalence relations. Here we introduce the notion of degree spectra relative to equivalence relations. We study the degree spectra of structures relative to Σ_n-equivalence (two structures are Σ_n-equivalent if there Σ_n-theories coincide). The presented results form a part of the paper "Degree Spectra of Structures relative to Equivalence Relations" by E. Fokina, P. Semukhin and D. Turetsky. For all the new results provided in this thesis, the main contribution to proving and writing them down was done by E. Fokina.

Introduction uses material from the paper "Computable Model Theory" by E. Fokina, V. Harizanov and A. Melnikov.

Contents

1 Introduction

1.1 Motivation

Computable structure theory uses the tools of computability theory to explore algorithmic content (effectiveness) of notions, theorems, and constructions in various areas of ordinary mathematics. In algebra this investigation based on intuitive notion of effectiveness dates back to van der Waerden who in his 1930 book *Modern Algebra* defined an *explicitly* given field as one the elements of which are uniquely represented by distinguishable symbols with which we can perform the field operations algorithmically. In his pioneering paper [81] on non-factorability of polynomials from 1930, van der Waerden essentially proved that an explicit (i.e., effectively given) field $(F, +, \cdot)$ does not necessarily have an algorithm for splitting polynomials in $F[x]$ into their irreducible factors. A remarkable property of the van der Waerden's example was that the field F had an isomorphic effective copy which at the same time did possess a splitting algorithm. Thus, implicitely van der Waerden gave an example of two isomorphic effectively given structures that were not effectively isomorphic. After rigorous definitions of computable functions appeared, van der Waerden's results were revisited and formalized by Fröhlich and Shepherdson in [29].

Several different notions of effectiveness of structures have been investigated since then. The generalization and formalization of van der Waerden's intuitive notion of an explicitly given field led to the notion of a computable structure, which is one of the main notions in computable structure theory. A structure is *computable* if its domain is computable and its relations and functions are uniformly computable. Further generalization led to a countable structure of a certain Turing degree $\mathbf{d}$. (Computable structures are of degree $\mathbf{0}$.) Henkin's construction of a model for a complete decidable theory is effective and produces a structure $\mathcal{A}$ with a computable domain such that the elementary diagram of $\mathcal{A}$ is decidable. Such a structure is called *decidable*. Thus, in the case of a computable structure, our starting point was semantic, while in the case of a decidable structure, the starting point was syntactic. It is easy to see that not every computable structure is decidable since for computable structures only the atomic (open) diagram has to be decidable. To bridge the gap between computable and decidable structures, one uses the following notion. A structure $\mathcal{A}$ is *n-decidable*, for $n \geqslant 0$, if the Σ_n-diagram of $\mathcal{A}$ is decidable. In particular, a structure is 0-decidable iff it is computable.

We can also assign Turing degrees or some other computability-theoretic degrees to isomorphisms, as well as to various relations on structures. We can also investigate structures, their theories, fragments of diagrams, relations, and isomorphisms within arithmetic and hyperarithmetic hierarchies.

The emphasis of this thesis is on algorithmic properties of equivalence relations between countable structures. In the context of computable structure theory, equivalence relations are a useful tool to study the effectiveness of mathematical structures. Moreover, the study of algorithmic properties of equivalence relations themselves is an active area of investigation with many open questions attracting a lot of attention of specialists in the field. In this work we consider several approaches to study the role of equivalence relations in computable structure theory. We start by looking at isomorphisms of bounded Turing degree, generalising the direction of research started by van der Waerden. We look at the complexity of isomorphisms between effectively given isomorphic copies of structures from natural classes of structures including various kinds of groups, Boolean algebras, Fraïssé limits, etc. Later on, we discuss the complexity of descriptions of structures for which there exist effective isomorphisms between their copies. Finally, we discuss how equivalence relations turn out to be useful when one studies algorithmic properties of non-computable structures.

Computability-theoretic notation in this thesis is standard and as in [78].

1.2 Preliminaries

We will assume that all structures are at most countable and their languages are computable. Clearly, finite structures are computable. Let $\mathbf{d}$ be a Turing degree. An infinite structure $\mathcal{M}$ is $\mathbf{d}$-*computable* if its universe can be identified with the set of natural numbers ω in such a way that the relations and operations of $\mathcal{M}$ are uniformly $\mathbf{d}$-computable.

If an algebraic structure is not computable, then it is natural to ask how close it is to a computable one. This property can be captured by the collection of all Turing degrees relative to which a given structure has a computable isomorphic copy. Thus, we have the following definition. For a countable structure $\mathcal{A}$, its degree spectrum $\mathrm{DgSp}(\mathcal{A})$ was defined by Richter in [75] and consists of the Turing degrees of all isomorphic copies of $\mathcal{A}$. As shown by Knight in [58], in all nontrivial cases, the degree spectrum of a structure is closed upward. Degree spectra of structures with various model-theoretic and algebraic properties have been widely studied; an overview of the current situation can be found, e.g., in [25]. Probably the simplest example of a degree spectrum is a cone above a Turing degree $\mathbf{d}$. On the other hand, no non-degenerate finite or countable union of cones can be a degree spectrum [80]. Slaman and Wehner in [76, 82] gave examples of structures with the degree spectrum consisting of exactly the non-computable degrees. In [55] Kalimullin constructed an example of a structure with its degree spectrum equal to all the non-Δ^0_2 degrees. Greenberg, Montalbán and Slaman showed that non-hyperarithmetical degrees form a spectrum of a structure in [48].

For a theory T, the degree spectrum of T was defined in [2]. It consists of all degrees of countable models of T. Some of the known examples of the spectra of theories include [2]: cones, a non-degenerate union of two cones, exactly the PA degrees, exactly the 1-random degrees. On the other hand, the authors of [2] prove that the collection of non-hyperarithmetical degrees is not the spectrum of a theory. In particular, these examples show that not every spectrum of a structure is a spectrum of a theory and, vice versa, not every spectrum of a theory is a spectrum of a structure.

We suggest the following generalization of these two notions to arbitrary equivalence relations.

Definition 1. The *degree spectrum* of a countable structure $\mathcal{A}$ with universe ω *under an equivalence relation E* is

$$\mathrm{DgSp}(\mathcal{A}, E) = \{\mathbf{d} \mid \text{there exists a } \mathbf{d}\text{-computable } \mathcal{B} \text{ which is } E-\text{equivalent to } \mathcal{A}\}.$$

Then the classical degree spectrum of $\mathcal{A}$ is $\mathrm{DgSp}(\mathcal{A}, \cong)$, the degree spectrum of $\mathcal{A}$ under isomorphism, while the degree spectra of the theory of $\mathcal{A}$ is $\mathrm{DgSp}(\mathcal{A}, \equiv)$, the degree spectrum of $\mathcal{A}$ under elementary equivalence.

If we restrict our attention to computable structures and study equivalence relations on those, then the main notion used in this direction of reserach is that of computable categoricity. The topic dates back to Fröhlich and Shepherdson [29] who revisited van der Waerden's results and produced examples of computable fields that are not computably isomorphic. Mal'cev in [63] studied the question of uniqueness of a constructive enumeration for a model and introduced the notion of a recursively stable model. Later in [64] he built isomorphic computable infinite-dimensional vector spaces that were not computably isomorphic. In the same paper he introduced the notion of an *autostable* model, which is equivalent to that of a computably categorical model. Since then, the definition of computable categoricity has been standardized and relativized to arbitrary Turing degrees **d**, and has been the subject of much study (see, e.g., surveys [35, 25]).

A computable structure $\mathcal{A}$ is called *computably categorical* if for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. For example, Ershov [21] established that a computable algebraically closed field is computably categorical if and only if it has a finite transcendence degree over its prime subfield. Miller and Schoutens [71] constructed a computably categorical field of infinite transcendence degree over the field of rational numbers.

The notion of computable categoricity can be extended to higher levels of hyperarithmetic hierarchy. Let α be a computable ordinal. A computable structure $\mathcal{A}$ is Δ^0_α*-categorical* if for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists a Δ^0_α isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. More generally, a computable structure $\mathcal{A}$ is *relatively* Δ^0_α*-categorical* if for every $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, which is Δ^0_α relative to the atomic diagram of $\mathcal{B}$. Clearly, a relatively Δ^0_α-categorical structure is Δ^0_α-categorical. The converse is not always true.

Relative Δ^0_α-categoricity has a syntactic characterization that involves the existence of certain Scott families of computable formulas. Roughly speaking, computable formulas are infinitary formulas with disjunctions and conjunctions over computable enumerable (c.e.) sets. A *Scott family* for a structure $\mathcal{A}$ is a countable family Φ of $L_{\omega_1\omega}$-formulas with finitely many fixed parameters from A such that:

(*i*) Each finite tuple in $\mathcal{A}$ satisfies some $\psi \in \Phi$;

(*ii*) If $\overline{a}, \overline{b}$ are tuples in $\mathcal{A}$, of the same length, satisfying the same formulas in Φ, then there is an automorphism of $\mathcal{A}$, which maps $\overline{a}$ to $\overline{b}$.

Ash [4] defined computable Σ_α and Π_α formulas of $L_{\omega_1\omega}$, where α is a computable ordinal, recursively and simultaneously and together with their Gödel numbers. The computable Σ_0 and Π_0 formulas are the finitary quantifier-free formulas. The computable $\Sigma_{\alpha+1}$ formulas are of the form

$$\bigvee_{n \in W_e} \exists \overline{y}_n \psi_n(\overline{x}, \overline{y}_n),$$

where for $n \in W_e$, ψ_n is a Π_α formula indexed by its Gödel number n, and $\exists \overline{y}_n$ is a finite block of existential quantifiers. Similarly, $\Pi_{\alpha+1}$ formulas are c.e. conjunctions of $\forall \Sigma_\alpha$ formulas. If α is a limit ordinal, then Σ_α (Π_α, respectively) formulas are of the form $\bigvee\limits_{n \in W_e} \psi_n$ ($\bigwedge\limits_{n \in W_e} \psi_n$, respectively), such that there is a sequence $(\alpha_n)_{n \in W_e}$ of ordinals less than α, given by the ordinal notation for α, and every ψ_n is a Σ_{α_n} (Π_{α_n}, respectively) formula. For a more precise definition see [4].

A *formally* Σ^0_α *Scott family* is a Σ^0_α Scott family consisting of computable Σ_α formulas. It follows that a formally c.e. Scott family is also a c.e. Scott family of finitary existential formulas.

The following equivalence (i)–(ii)–(iii) for a computable structure $\mathcal{A}$ was established by Goncharov [32] for $\alpha = 1$, and by Ash, Knight, Manasse, and Slaman [6] and independently by Chisholm [13] for any computable ordinal α:

(i) The structure $\mathcal{A}$ is relatively Δ^0_α-categorical.

(ii) The structure $\mathcal{A}$ has a formally Σ^0_α Scott family.

(iii) The structure $\mathcal{A}$ has a c.e. Scott family consisting of computable Σ_α formulas.

Infinitary language is essential for Scott families. Cholak, Shore, and Solomon [15] proved the existence of a computably categorical rigid graph that does not have a Scott family of finitary formulas. It follows that this structure is not relatively computably categorical.

Goncharov [33] was the first to show that computable categoricity of a computable structure does not imply relative computable categoricity. The result of Goncharov was lifted to higher levels in the hyperarithmetic hierarchy by Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon for successor ordinals [46], and by Chisholm, Fokina, Goncharov, Harizanov, Knight, and Quinn for limit ordinals [12]. Hence, for every computable ordinal α, there is a Δ^0_α-categorical but not relatively Δ^0_α-categorical structure. If follows from results by Hirschfeldt, Khoussainov, Shore, and Slinko in [50] that there are (computable) computably categorical but not relatively computably categorical structures in the following classes: partial orders, lattices, 2-step nilpotent groups, commutative semigroups, and integral domains of arbitrary characteristic. Hirschfeldt, Kramer, R. Miller, and Shlapentokh [51] showed that there is a computably categorical algebraic field, which is not relatively computably categorical.

Cholak, Goncharov, Khoussainov, and Shore [14] showed that there is a computable structure, which is computably categorical, but ceases to be after naming any element of the structure.

Clearly, this structure is not relatively computably categorical. Khoussainov and Shore [57] proved that there is a computably categorical structure $\mathcal{A}$, which is not relatively computably categorical, but the expansion of $\mathcal{A}$ by any finite number of constants is computably categorical. Previously, T. Millar [68] showed that if a computably categorical structure $\mathcal{A}$ is 1-decidable, then any expansion of $\mathcal{A}$ by finitely many constants remains computably categorical.

Goncharov's graph in [33], which is computably categorical but not relatively computably categorical, is rigid, and hence computably stable but not relatively computably stable. A structure $\mathcal{A}$ is Δ^0_α-*stable* if for every computable copy $\mathcal{B}$ of $\mathcal{A}$, all isomorphisms from $\mathcal{A}$ onto $\mathcal{B}$ are Δ^0_α. Similarly, we define relatively Δ^0_α-stable structures. A *defining family* for a structure $\mathcal{A}$ is a set Φ of $\mathcal{L}_{\omega_1\omega}$ formulas with one free variable and a fixed finite tuple of parameters from $\mathcal{A}$ such that:

(i) Every element of $\mathcal{A}$ satisfies some formula $\psi \in \Phi$;

(ii) No formula of Φ is satisfied by more than one element of $\mathcal{A}$.

The existence of a defining family is equivalent to rigidity relative to a finite set of parameters. A countable structure is rigid if and only if it has a defining family with no parameters. A computable structure $\mathcal{A}$ is relatively Δ^0_α-stable if and only if it has a formally Σ^0_α defining family.

Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky [19] proved that for every computable ordinal α, there is a computably categorical structure, which is not relatively Δ^0_α-categorical. In fact, it follows from their construction that the structure is rigid. Thus, they answered positively the following question from [46, 12]: For a computable ordinal $\alpha > 1$, is there a computable structure $\mathcal{A}$ that is Δ^0_α-stable but not relatively Δ^0_α-stable? On the other hand, a natural open question arising from [19] is whether there is a computably categorical structure that is not relatively hyperarithmetically categorical.

Ash [5] proved that a computable structure $\mathcal{A}$ is Δ^1_1-categorical if and only if $\mathcal{A}$ is Δ^0_α-categorical for some computable ordinal α. It is not known whether every computable Δ^1_1-categorical structure is relatively Δ^1_1-categorical. A similar question has been resolved for relations on structures – intrinsically Δ^1_1 and relatively intrinsically Δ^1_1 relations are the same (see [47]). Namely, it follows from a result by Soskov [79] that for a computable structure $\mathcal{A}$ and a relation R on $\mathcal{A}$, if R is invariant under automorphisms of $\mathcal{A}$, and Δ^1_1, then R is definable in $\mathcal{A}$ by a computable infinitary formula with no parameters. This is used to establish that if R is intrinsically Δ^1_1 on $\mathcal{A}$, then R is relatively intrinsically Δ^1_1 on $\mathcal{A}$.

An injection structure is a structure (A, f) where $f : A \to A$ is a $1-1$ function. For a linear order [31, 74], a Boolean algebra [31, 73], a tree of finite height [62], an abelian p-group [34, 77, 10], an equivalence structure [9], an injection structure [11], and an algebraic field with a splitting algorithm [72], computable categoricity coincides with relative computable categoricity.

For an injection structure $\mathcal{A} = (A, f)$ and $a \in A$, we define the orbit of a:

$$O_f(a) = \{b \in A : (\exists n \in \omega)[f^n(a) = b \vee f^n(b) = a]\}.$$

Cenzer, Harizanov, and Remmel [11] established that a computable injection structure is Δ^0_2-categorical if and only if it has finitely many orbits of type ω or finitely many orbits of type $\mathbb{Z}$. They showed that every Δ^0_2-categorical injection structure is relatively Δ^0_2-categorical. It is not hard to see that every computable injection structure is relatively Δ^0_3-categorical.

Calvert, Cenzer, Harizanov, and Morozov [9] proved that a computable equivalence structure is relatively Δ^0_2-categorical if and only if it either has finitely many infinite equivalence classes, or there is an upper bound on the size of its finite equivalence classes. They also have partial results towards characterizing Δ^0_2-categoricity. First we need some definitions. A function $f : \omega^2 \to \omega$ is a Khisamiev *s-function* if for every i and s, $f(i, s) \leqslant f(i, s+1)$, and the limit $m_i = lim_t f(i, t)$ exists. If, in addition, $m_i < m_{i+1}$ for every i, then we say that f is a Khisamiev s_1*-function*. If an equivalence structure $\mathcal{A}$ has no upper bound on the size of the finite equivalence classes, then Khisamiev s_1-function for $\mathcal{A}$ is such that $\mathcal{A}$ contains an equivalence class of size m_i for every i. If an equivalence structure $\mathcal{A}$ has infinitely many infinite equivalence classes, no upper bound on the size of its finite equivalence classes, and has a computable Khisamiev s_1-function, then $\mathcal{A}$ is not Δ^0_2-categorical (see [9]). Kach and Turetsky [54] showed that there exists a Δ^0_2-categorical equivalence structure $\mathcal{M}$, which is not relatively Δ^0_2-categorical. Their equivalence structure $\mathcal{M}$ has infinitely many infinite equivalence classes and unbounded character, but has no computable Khisamiev's s_1-function, and has only finitely many equivalence classes of size k for any finite k. Every computable equivalence structure is relatively Δ^0_3-categorical.

Goncharov and Dzgoev [31], and independently Remmel [74] proved that a computable linear order is computably categorical (also, relatively computably categorical) if and only if it has only finitely many adjacencies (successor pairs). In [67], McCoy characterized relatively Δ^0_2-categorical linear orders as follows. By ω^* we denote the reverse order of ω, and by η the order type of rationals. A computable linear order is relatively Δ^0_2-categorical if and only if it is a sum of finitely many intervals, each of type m, ω, ω^*, $\mathbb{Z}$ or $n \cdot \eta$, so that each interval of type $n \cdot \eta$ has a supremum and an infimum. McCoy [67] also characterized, after adding certain extra predicates, Δ^0_2-categorical linear orders. However, it still remains open whether there is a Δ^0_2-categorical linear order, which is not relatively Δ^0_2-categorical. In [66], McCoy proved that there are $2^{\aleph_0}$ relatively Δ^0_3-categorical linear orders.

Goncharov and Dzgoev [31], and independently Remmel [73] established that a computable Boolean algebra is computably categorical (also, relatively computably categorical) if and only if it has finitely many atoms (see also LaRoche [61]). In [67], McCoy characterized computable relatively Δ^0_2-categorical Boolean algebras as those that can be expressed as finite direct sums of subalgebras $C_0 \oplus \cdots \oplus C_k$ where each C_k is either atomless, an atom, or a 1-atom. Using McCoy's characterization, Bazhenov [7] showed that for Boolean algebras the notions of Δ^0_2-categoricity and relative Δ^0_2-categoricity coincide. Harris gave another proof in [49]. In [66], McCoy gave a complete description of relatively Δ^0_3-categorical Boolean algebras.

Fokina, Kalimullin, and R. Miller [26] introduced the following notions trying to capture the set of all Turing degrees capable of computing isomorphisms between computable structures. Let $\mathcal{A}$ be a computable structure. The *categoricity spectrum* of $\mathcal{A}$ is the following set of Turing degrees:

$$\mathrm{CatSpec}(\mathcal{A}) = \{\mathbf{x} : \mathcal{A} \text{ is } \mathbf{x}\text{-computably categorical}\}.$$

The *degree of categoricity* of $\mathcal{A}$, if it exists, is the least Turing degree in CatSpec($\mathcal{A}$). If $\mathbf{d}$ is a non-hyperarithmetic degree, then $\mathbf{d}$ cannot be the degree of categoricity of a computable structure. A Turing degree $\mathbf{d}$ is called *categorically definable* if it is the degree of categoricity of some computable structure. Fokina, Kalimullin, and R. Miller [26] investigated which arithmetic degrees are categorically definable. Csima, Franklin, and Shore [16] extended their results to hyperarithmetic degrees. For sets X and Y, we say that Y is *c.e. in and above* (c.e.a. in) X if Y is c.e. relative to X, and $X \leqslant_T Y$. Csima, Franklin, and Shore [16] proved that for every computable ordinal α, $\mathbf{0}^{(\alpha)}$ is categorically definable. They also established that for a computable successor ordinal α, every degree $\mathbf{d}$ that is c.e.a. in $\mathbf{0}^{(\alpha)}$ is categorically definable. There were also negative results in [26, 16]. Anderson and Csima [1] showed that there exists a Σ^0_2 set the degree of which is not categorically definable. They also showed that no noncomputable hyperimmune-free degree is categorically definable. It is an open question whether all Δ^0_2 degrees are categorically definable.

Not every computable structure has the degree of categoricity. The first negative example was built by R. Miller [70]. Examples of rigid structures without the degrees of categoricity were built by Fokina, Frolov, and Kalimullin [23]. It is an open question whether there is a computable structure the categoricity spectrum of which is the set of all noncomputable Turing degrees.

Similarly to the case of computable categoricity, we define *decidable categoricity spectrum* of $\mathcal{A}$ to be the collection of degrees that can compute at least one isomorphism between *decidable* copies of $\mathcal{M}$. In a series of works, including [8, 37, 38, 40], Goncharov and his students studied decidable categoricity of structures. In particular, Goncharov established the following results.

Theorem 2. *([38]) Every c.e. degree* $\mathbf{d}$ *is the degree of decidable categoricity of some decidable almost prime model.*

Theorem 3. *([37]) There exists a decidable Ehrenfeucht theory T such that T has a decidable prime model that is decidably categorical, and T has a decidable almost prime model that is not decidably categorical.*

We can extend the definition also to the case of effective categoricity relative to m-decidable presentations, as follows.

Definition 4. We call a structure *categorical relative to m-decidable presentations* (or *autostable relative to m-constructivizations*) if any two m-decidable copies of $\mathcal{A}$ are computably isomorphic.

In particular, being computably categorical is the same as being categorical relative to 0-decidable presentations.

To estimate the complexity of algorithmic, algebraic and model-theoretic properties on computable structures, one of the approaches is to use the notion of index set. In Chapter 3 we will use this approach to find out how hard it is to say that a computable structure is categorical relative to m-decidable presentations, where $0 \leqslant m \in \omega$. Here we review the necessary definitions as given in [42].

Let K be a class of structures. We denote by K^c the set of computable structures in K. A *computable characterization* of K should separate computable structures in K from all other structures (those not in K, or noncomputable ones). We say that K has a *computable characterization* if K^c is the set of computable models of a computable infinitary sentence.

Proposition 5. *(i) The class of linear orders can be characterized by a single first-order sentence.*

(ii) The class of abelian p-groups is characterized by a single computable Π_2 sentence.

(iii) The classes of well orders and reduced abelian p-groups cannot be characterized by single computable infinitary sentences.

A *computable index* for a structure $\mathcal{A}$ is a number e such that $D(\mathcal{A}) = W_e$, where $D(\mathcal{A})$ is the atomic diagram of $\mathcal{A}$. We denote the structure with index e by $\mathcal{M}_e$. For a class K of structures, the *index set* $I(K)$ is the set of computable indices of members of K^c:

$$I(K) = \{e : W_e = D(\mathcal{A}) \wedge \mathcal{A} \in K\}.$$

Existence of a computable infinitary sentence describing a class K^c is equivalent to hyperarithmetical complexity of $I(K)$, as shown in [42]. In fact, we do not know a better way to estimate the complexity of an index set than by giving a description by a computable infinitary formula.

Proposition 6. *([42]) (i) For the following classes K, the index set $I(K)$ is Π^0_2:*

(a) linear orders,

(b) Boolean algebras,

(c) abelian p-groups,

(d) vector spaces over $\mathbb{Q}$.

(ii) (Kleene, Spector) For the following classes K, the index set $I(K)$ is not hyperarithmetic:

(a) well-orders,

(b) superatomic Boolean algebras,

(c) reduced abelian p-groups.

Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky [19] proved that there is no simple syntactic characterization of computable categoricity. More formally, they showed that the index set of computable categorical structures is Π^1_1-complete. Combining the methods from [19] and from [34], Bazhenov, Goncharov and Marchuk showed that also the index set of computable structures of algorithmic dimension $n > 1$ is Π^1_1 complete [40]. On the other hand, the index set of relatively computably categorical structures is Σ^0_3-complete (see [19]).

More recently, Goncharov introduced the notion of categoricity restricted to decidable structures [36, 37, 38].

Definition 7. A structure $\mathcal{A}$ is called *decidably categorical* (also called *autostable relative to strong constructivizations*) if any two decidable copies of $\mathcal{A}$ are computably isomorphic.

Goncharov and Marchuk in [44] showed that the index set of computable, decidably categorical structures is $\Sigma^0_{\omega+2}$ complete, while for decidable, decidably categorical structures the index set is a complete Σ^0_3 set. Index sets for decidably categorical structures with particular algebraic, model-theoretic and algorithmic properties were further studied in [39, 43, 45, 40].

1.3 Structure of the thesis

In Chapter 2 we study the relations between algorithmic properties of isomorphisms and descriptive properties of natural structures. We present some new examples of structures in natural classes, which are computably categorical but not relatively computably categorical, as well as Δ^0_2-categorical but not relatively Δ^0_2-categorical. We build a 1-decidable structure that is a Fraïssé limit, which is computably categorical but not relatively computably categorical.We build computable Δ^0_2-categorical but not relatively Δ^0_2-categorical trees of finite and infinite heights. Here, a tree can be viewed both as a partial order and as a directed graph. Furthermore, we prove that there is a homogenous completely decomposable abelian group, which is Δ^0_2-categorical but not relatively Δ^0_2-categorical. We also compute the degrees of categoricity for relatively Δ^0_2-categorical abelian p-groups. This parallels Frolov's work in [30] where he computed degrees of categoricity for relatively Δ^0_2-categorical linear orders. We further compute the degrees of categoricity for relatively Δ^0_3-categorical Boolean algebras. This extends Bazhenov's work in [7] where he computed the degrees of categoricity for relatively Δ^0_2-categorical Boolean algebras.

In Chapter 3 we consider n-decidable structures and their categoricity with respect to m-decidable copies, where $m, n \in \omega$. It turns out the the answer depends on the relationship between m and n. For $n \leqslant m$, there exists no nice description of n-decidable structures categorical relative to m-decidable presentation. For $m < n$, there exist descriptions of arithmetical complexity, but they are also different for $m = n-1$ and $m \leqslant n-2$. We give the exact estimation of complexity of index sets for the three cases.

Finally, in Chapter 4 we consider Σ_n-fragments of theories and the corresponding equivalence relations $\equiv_{\Sigma_n}$ (two structures are $\equiv_{\Sigma_n}$-equivalent if their Σ_n-theories coincide). We also write $\mathcal{A} \equiv_{\Sigma_n} \mathcal{B}$ when $\mathcal{A}$ and $\mathcal{B}$ are Σ_n-equivalent. We call $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_n})$ the Σ_n*–spectrum* of $\mathcal{A}$. We study what kinds of spectra are possible with respect to these equivalence relations and how such spectra are related to each other and to the theory spectra of structures, defined above.

2 Effective categoricity and Scott families

In this chapter, we present some new examples of structures in natural classes, which are computably categorical but not relatively computably categorical, as well as Δ_2^0-categorical but not relatively Δ_2^0-categorical. In Section 2.1, we present 1-decidable structure that is a Fraïssé limit, which is computably categorical but not relatively computably categorical. In Section 2.2, we build computable Δ_2^0-categorical but not relatively Δ_2^0-categorical trees of finite and infinite heights. Here, a tree can be viewed both as a partial order and as a directed graph. In Section 2.3, we prove that there is a homogenous completely decomposable abelian group, which is Δ_2^0-categorical but not relatively Δ_2^0-categorical. In Section 2.4, we compute the degrees of categoricity for relatively Δ_2^0-categorical abelian p-groups. This parallels Frolov's work in [30] where he computed degrees of categoricity for relatively Δ_2^0-categorical linear orders. We further compute the degrees of categoricity for relatively Δ_3^0-categorical Boolean algebras. This extends Bazhenov's work in [7] where he computed the degrees of categoricity for relatively Δ_2^0-categorical Boolean algebras.

2.1 Computably categorical but not relatively computably categorical Fraïssé limits

For a computable ordinal α, the notions of Δ_α^0-categoricity and relative Δ_α^0-categoricity of a computable structure $\mathcal{A}$ coincide if $\mathcal{A}$ satisfies certain extra decidability conditions (see Goncharov [32] and Ash [5]). Goncharov [32] proved that if $\mathcal{A}$ is 2-decidable, then computable categoricity and relative computable categoricity of $\mathcal{A}$ coincide. Kudinov [59] showed that the assumption of 2-decidability cannot be weakened to 1-decidability, by giving an example of 1-decidable and computably categorical structure, which is not relatively computably categorical. On the other hand, Downey, Kach, Lempp, and Turetsky [20] showed that any 1-decidable computably categorical structure is relatively Δ_2^0-categorical.

The proofs by Goncharov and by Downey, Kach, Lempp, and Turetsky use the decidability of the structure to determine if certain finitely generated substructures can be extended to various larger finitely generated substructures. Because of the special properties of a Fraïssé limit, one might expect that all such questions would be trivial to determine, and so the decidability condition could be weakened or dropped entirely for such structures. However, this is not the case. Here, we give an example of 1-decidable and computably categorical Fraïssé limit which is not relatively computably categorical.

Let us recall the definition of a Fraïssé limit (see [53, Chapter 6]). The *age* of a structure $\mathcal{M}$ is the class of all finitely generated structures that can be embedded in $\mathcal{M}$. Fraïssé showed that a (nonempty) finite or countable class $\mathbb{K}$ of finitely generated structures is the age of a finite or a countable structure if and only if $\mathbb{K}$ has the hereditary property and the joint embedding property. A class $\mathbb{K}$ has the *hereditary property* if whenever $C \in \mathbb{K}$ and $\mathcal{S}$ is a finitely generated substructure of C, then $\mathcal{S}$ is isomorphic to some structure in $\mathbb{K}$. A class $\mathbb{K}$ has the *joint embedding property* if for every $\mathcal{B}, C \in \mathbb{K}$ there is $\mathcal{D} \in \mathbb{K}$ such that $\mathcal{B}$ and C embed into $\mathcal{D}$. A structure $\mathcal{U}$ is *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\mathcal{U}$ extends to an automorphism of $\mathcal{U}$.

Definition 8. (see [53, Chapter 6]) A structure $\mathcal{A}$ is a *Fraïssé limit* of a class of finitely generated structures $\mathbb{K}$ if $\mathcal{A}$ is countable, ultrahomogeneous, and has age $\mathbb{K}$.

Fraïssé proved that the Fraïssé limit of a class of finitely generated structures is unique up to isomorphism. We say that a structure $\mathcal{A}$ is a Fraïssé limit if for some class $\mathbb{K}$, $\mathcal{A}$ is the Fraïssé limit of $\mathbb{K}$. First we show that every Fraïssé limit is relatively Δ^0_2-categorical.

Theorem 9. *Let $\mathcal{A}$ be a computable structure which is a Fraïssé limit. Then $\mathcal{A}$ is relatively Δ^0_2-categorical.*

Proof. Because of ultrahomogeneity, we can construct isomorphisms between $\mathcal{A}$ and an isomorphic structure $\mathcal{B}$ using a back-and-forth argument, as long as we can determine, for every $\overline{a} \in \mathcal{A}$ and $\overline{b} \in \mathcal{B}$, whether there is an isomorphism from the structure generated by $\overline{a}$ to the structure generated by $\overline{b}$ that maps $\overline{a}$ to $\overline{b}$ in order. This can be determined by $(\mathcal{B})'$, since there is such an isomorphism precisely if there is no atomic formula φ with $\mathcal{A} \models \varphi(\overline{a})$ and $\mathcal{B} \not\models \varphi(\overline{b})$. This is a Π^0_1 condition relative to $\mathcal{A} \oplus \mathcal{B} \equiv_T \mathcal{B}$.

Therefore, we can use $(\mathcal{B})'$ as an oracle to perform the back-and-forth construction of an isomorphism, and so there is a $\Delta^0_2(\mathcal{B})$ isomorphism. □

Note that if the language of $\mathcal{A}$ is finite and relational, then there are only finitely many atomic formulas φ to consider, and the set of such formulas can be effectively determined. Hence, if the language is finite and relational, then a Fraïssé limit is necessarily relatively computably categorical.

Theorem 10. *There is a* 1*-decidable structure* $\mathcal{F}$ *that is a Fraïssé limit and computably categorical, but not relatively computably categorical. Moreover, the language for such* $\mathcal{F}$ *can be finite or relational.*

Proof. The proof is a modification of the first construction in Theorem 3.3 by Downey, Kach, Lempp, and Turetsky [20], where the structure they build is, in particular, 1-decidable, computably categorical but not relatively computably categorical. The only new ingredient we add is to make the resulting structure a Fraïssé limit. We sketch the original constructions and explain the modifications we must make to ensure that the resulting structure is a Fraïssé limit. All the formal details can be easily recovered from the original proof in [20].

The original construction is an undirected graph. To assure that the structure is made not relatively computably categorical, we diagonalize agains all potential Scott families of computable Σ_1 formulas with finitely many parameters. This is done by creating infinitely many connected components that are all accumulation points in the Σ_1 type-space; this is similar to the technique used in Kudinov's construction in [59]. Then for any potential Scott family of Σ_1 formulas, there must be some accumulation point in a component disjoint from the finitely many parameters of the family with the following property. Any Σ_1 formula from the Scott family, which holds of the accumulation point would also need to hold of any other point that is "sufficiently close" in the type space, contradicting the definition of a Scott family.

The original construction created these accumulation points as vertices with loops of various sizes coming out of them. For each accumulation point, there would be a pair of computable sequences $\{n_k\}_{k\in\omega}$ and $\{m_k\}_{k\in\omega}$, chosen exclusively for this accumulation point. For every k, there would be a vertex v_k with attached loops of sizes $n_0, \ldots, n_k$ and a loop of size m_k. The loop of size m_k is meant to identify the component corresponding to v_k, so loops of this size are not used in any other component of the construction. There would also be a vertex v_∞ with attached loops $n_0, n_1, \dots$. Each v_k and v_∞ would also have infinitely many *rays* – non-branching infinite paths originating from the vertex. The Σ_1 type of v_∞ was then the limit of the Σ_1 types of the v_k.

The original construction took place on a tree of strategies, where each accumulation point was created by an individual strategy. Because a strategy might be visited only finitely many times in the construction, not all strategies would create the full set of vertices described above. Each time a strategy was visited, it performed one of the following steps, in alternation:

- Increment k, choose n_{k+1} and attach a loop of size n_{k+1} to v_∞.
- Choose m_k. Create the full v_k component.

Thus, if a strategy was only visited finitely many times, the v_∞-component would have loops of sizes $n_0, \dots, n_{k+1}$, and the components $v_0, \dots, v_{k-1}$ would have all been created, and possibly v_k as well. Numbers n_k and m_k are always chosen larger than the current stage, and two distinct strategies choose completely distinct numbers n_k and m_k. That is, any number is chosen by at most one strategy.

Notice that each time the strategy first chooses a sufficiently large new n_{k+1} and attaches a corresponding loop to v_∞. Only after that it chooses a new m_k and creates the v_k component. This ensures that the resulting structure is computably categorical. The fact that each component has infinitely many infinite rays makes the structure 1-decidable. Finally, the structure is not relatively computably categorical, as the construction destroys any potential Scott family.

We describe now two ways of modifying this construction so that the structure becomes a Fraïssé limit while still being computably categorical, 1-decidable and not relatively computably categorical. The first uses a finite language with function symbols, while the second uses an infinite relational language. Let $\mathcal{L}_1 = \{E, f, g, h\}$, where E is a binary relation symbol and f, g and h are unary function symbols. Let

$$\mathcal{L}_\infty = \{E\} \cup \{U_{i,j} : j < i \wedge i, j \in \omega\} \cup \{V_{i,j} : j \leqslant i \wedge i, j \in \omega\} \cup \{R_i : i \in \omega\} \cup \{S_i : i \in \omega\},$$

where E is a binary relation symbol and each $U_{i,j}$, $V_{i,j}$, R_i and S_i is a unary relation symbol.

The intention is that E is the edge relation of the graph from the original construction. That is, in both cases, the reduct of the structures we make to the language $\{E\}$ will be the original structure in [20]. We will now describe the new functions and relations on the structure.

Suppose that v is one of the v_k or v_∞, and $a_0, \ldots, a_{n_k-1}$ are vertices with vEa_0, a_iEa_{i+1} for all $i < n_k - 1$, and $a_{n_k-1}Ev$; that is, $v, a_0, \ldots, a_{n_k-1}$ is the loop of size n_k attached to v. Suppose also that a_0 has lower Gödel number than a_{n_k-1}, so that we have chosen a particular orientation of the loop. Then we define $f(a_i) = a_{i+1}$, and $f(a_{n_k-1}) = v$. We also define $g(a_{i+1}) = a_i$ and $g(a_0) = v$. So f "walks" along the loop in one direction, and g "walks" along it in the other direction. We also define $U_{n_k,i}(a_i)$ to hold for every $i < n_k$, while $U_{n_k,i}(x)$ fails to hold for any other x.

For v_k, suppose that $a_0, \ldots, a_{m_k-1}$ are vertices as above, so that $v_k, a_0, \ldots, a_{m_k-1}$ is the loop of size m_k attached to v_k, again with a chosen orientation. Then we define $f(a_i) = a_{i+1}$, $f(a_{m_k-1}) = v_k$ and $f(v_k) = a_0$. We also define $g(a_{i+1}) = a_i$, $g(a_0) = v_k$ and $g(v_k) = a_{m_k-1}$. So again f and g walk along the loop in the opposite directions, but the walks continue through v_k. We also define $V_{m_k,i}(a_i)$ to hold, and $V_{m_k,i}(x)$ fails to hold for any other x, for every $i < m_k$. Finally, we define $V_{m_k,m_k}(z)$ to hold for every vertex z in the same component as v_k.

Suppose that v is one of the v_k's or v_∞, and consider a ray of the form $a_0, a_1, \ldots$ with vEa_0 and a_iEa_{i+1} for all $i \in \omega$. For infinitely many of these rays, we define $f(a_i) = a_{i+1}$, $g(a_{i+1}) = a_i$ and $g(a_0) = v$, and for infinitely many rays we define $g(a_i) = a_{i+1}$, $f(a_{i+1}) = a_i$ and $f(a_0) = v$. So for infinitely many rays, f walks away from v, while g walks towards v, and for infinitely many rays the reverse holds. For every ray, we define $R_i(a_i)$ to hold.

For v_∞, we choose some a_0 from some ray with $g(a_0) = v_\infty$ and define $f(v_\infty) = a_0$. We choose some b_0 from some ray with $f(b_0) = v_\infty$ and define $g(v_\infty) = b_0$.

Suppose that v is one of the v_k's or v_∞, and a is part of the loop of size n_0 with $g(a) = v$. Then we define $h(v) = a$. For every other x, we define $h(x) = f(x)$.

For every vertex x in every component created by strategy i from the priority tree, we define $S_i(x)$ to hold.

Claim 10.1. *In both $\mathcal{L}_1$ and $\mathcal{L}_\infty$, if $\overline{x}$ and $\overline{y}$ generate substructures that are isomorphic* via *an isomorphism mapping $\overline{x}$ to $\overline{y}$, then there is an automorphism of the full structure $\mathcal{F}$ mapping $\overline{x}$ to $\overline{y}$.*

Proof. We prove the result for singletons x and y. The general case proceeds similarly. The point is that if $x \neq y$, then they must both be vertices from loops/rays within the same component, and they must be the same length along those loops/rays. Then, loops are identified uniquely and for any two rays, there is an automorphism switching those rays and fixing the remainder of the structure. The argument is slightly longer for $\mathcal{L}_\infty$, because rays come in two sorts, and there are two distinguished rays in the component of v_∞.

In $\mathcal{L}_1$, through f or g, the substructure generated by x contains some vertex v_k or v_∞. The same is true for y. Through h, the substructure also contains the entire loop of size n_0. Since n_0 is unique to some strategy from the priority tree, x and y are both placed by the same strategy.

In $\mathcal{L}_\infty$, there is some i such that $S_i(x)$ and $S_i(y)$ hold. So x and y must again both be placed by the same strategy.

In $\mathcal{L}_1$, if the substructure generated by x contains v_k, then through $f(v_k)$ it also contains the loop of size m_k. If the substructure contains v_∞, then through $f(v_\infty)$ it also contains an infinite ray with $f(v_\infty) = a_0$. The same holds for y. This loop or ray uniquely characterizes the component, so x and y must be part of the same component.

In $\mathcal{L}_\infty$, if the component of x contains v_k, then $V_{m_k,m_k}(x)$ holds. If instead it contains v_∞, then no $V_{m_k,m_k}(x)$ holds for any k. The same is true for y. So x and y must be part of the same component.

In $\mathcal{L}_1$, there are four possibilities: $f^i(x) = v$ and $g^j(x) = v$ for some i and j; $f^i(x) = v$ for some i but $g^j(x) \neq v$ for all j; $g^j(x) = v$ for some j but $f^i(x) \neq v$ for all i; or $x = v$. Note that v is uniquely characterized by having degree greater than 2, even in the substructures generated by x or y. In the first case, x must be a_{j-1} from the loop of size $i + j$. In the second case, x must be a_{i-1} from one of the rays in which f walks towards v. In the third case, x must be a_{j-1} from one of the rays in which g walks towards v. The same holds for y. The first case is unique in the component, so in this case we know that $x = y$. If $v \neq v_\infty$, there is a single orbit containing every instance of the second case, and another containing every instance of the third case, so there must be an automorphism mapping x to y. If $v = v_\infty$, then the second case breaks into two subcases: $g(v) = f^{i-1}(x)$, and $g(v) \neq f^{i-1}(x)$. The first subcase is unique in the component, so $x = y$, while the second subcase again comprises a single orbit. We reason similarly in the third case. The fourth case is again unique in the component.

In $\mathcal{L}_\infty$, if x is part of some loop, then there is some $U_{i,j}$ or $V_{i,j}$ that holds of x and no other point. So $x = y$. If x is part of some ray, then there is some R_i that holds of x and only of the points on rays, which are distance i from v. So y is also a point on a ray, which is distance i from v. So there is an automorphism of the structure switching those two rays, and in particular sending x to y.

In $\mathcal{L}_\infty$, v_k is uniquely characterized by $V_{m_k,m_k}(v_k)$ holding, some $S_i(v_k)$ holding, and no other unary relation holding. So if $x = v_k$, then $y = v_k$. Also, v_∞ is uniquely characterized by some $S_i(v_\infty)$ holding and no other unary relation holding. So if $x = v_\infty$, then $y = v_\infty$. □

It follows that the structures we have described are Fraïssé limits. Observe that they are defined in a computable fashion. Furthermore, our expanded language does not provide an obstacle to 1-decidability, since n_k and m_k are always chosen larger than the current stage. Thus any statement about $f^s(x)$, $g^s(x)$, $h^s(x)$, $U_{s,j}(x)$, $V_{s,j}(x)$, $R_s(x)$ or $S_s(x)$ can be decided by considering the construction up through stage s. From the definition of the additional functions and relations it also follows that the expanded structure is still computably categorical but not relatively computably categorical (as the vertices v_∞ are still accumulation points in the Σ_1-space, allowing us to diagonalize against Scott families). □

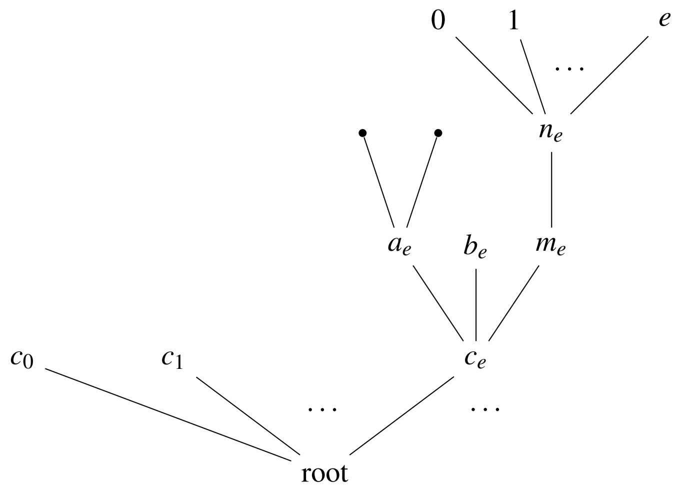
2.2 Δ_2^0-categorical but not relatively Δ_2^0-categorical trees

We consider trees as partial orders. R. Miller [69] established that no computable tree of infinite height is computably categorical. Lempp, McCoy, R. Miller, and Solomon [62] characterized computably categorical trees of finite height, and showed that for these structures, computable categoricity coincides with relative computable categoricity. There is no known characterization of Δ_2^0-categoricity or higher level categoricity for trees of finite height. Lempp, McCoy, R. Miller, and Solomon [62] proved that for every $n \geqslant 1$, there is a computable tree of finite height, which is Δ_{n+1}^0-categorical but not Δ_n^0-categorical. We will establish the following result, which also holds when a tree is presented as a directed graph.

Theorem 11. *There is a computable Δ_2^0-categorical tree of finite height, which is not relatively Δ_2^0-categorical.*

Proof. While building a computable tree $\mathcal{T}$ (with domain ω), we diagonalize against all potential c.e. Scott families of computable Σ_2 formulas with finitely many parameters. Thus, we consider all pairs $(\mathcal{X}, \overline{p})$, where $\mathcal{X}$ is a c.e. family of computable Σ_2 formulas and $\overline{p}$ is a finite tuple of elements from the domain of $\mathcal{T}$, and we must ensure that for each pair $(\mathcal{X}, \overline{p})$, $\mathcal{X}$ with parameters $\overline{p}$ is not a Scott family for $\mathcal{T}$. At the same time, we have to assure that every isomorphic computable tree is $\mathbf{0}'$-isomorphic to $\mathcal{T}$. The construction will be an infinite injury construction where strategies are arranged on a priority tree with the true path defined as usual.

The root of $\mathcal{T}$ will have infinitely many "children," which we label $c_0, c_1, c_2, \ldots$. Each c_e will have 3 children, a_e, b_e and m_e. The purpose of m_e is to uniquely identify c_e. The node m_e will have a child n_e, and n_e will have $e + 1$ many children. See the diagram.



At stage 0, a_e will have 2 children and b_e will have no children. Through the action of some strategy, more children may be added to a_e and b_e at later stages.

Let $(\mathcal{X}_i, \overline{p}_i)_i$ be an enumeration of pairs, where $\mathcal{X}_i$ is a c.e. family of computable Σ_2 formulas, and $\overline{p}_i$ is a tuple drawn from ω, the domain of $\mathcal{T}$. We must meet the following categoricity and isomorphism requirements. Let $M_0, M_1, \ldots$ be an effective enumeration of all computable structures.

R_i : $\mathcal{X}_i$ with parameters $\overline{p}_i$ is not a Scott family for $\mathcal{T}$.

Q_j : If $M_j \cong \mathcal{T}$, then there is a $\mathbf{0}'$-computable isomorphism between M_j and $\mathcal{T}$.

Strategy for R_i

Our strategy will appear on a priority tree. When the strategy is visited, s is always the current stage, and $t < s$ is the last stage at which the strategy took outcome ∞ (or $t = 0$ if the strategy has never before taken outcome ∞). The first time the strategy is visited, we choose a large e to work with. In particular, a_e and b_e must not occur in $\overline{p}_i$, and $e > s$.

We will take advantage of the fact that if $\varphi(\overline{x})$ is a computable Σ_2 formula and $\overline{a} \in \mathcal{T}$, then we have a computable approximation $(\mathcal{T}_s)_s$ to $\mathcal{T}$ and a computable sequence of finitary formulas $(\varphi_s(\overline{x}))_s$ such that $\mathcal{T} \models \varphi(\overline{a})$ if and only if $\mathcal{T}_s \models \varphi_s(\overline{a})$ for co-finitely many stages s, in the future we will simply write $\mathcal{T}_s \models \varphi(\overline{a})$, meaning the corresponding finite formula $\varphi_s(\overline{x})$. We may also define the sequences $(\varphi_s(\overline{x})_s$ in such a way that that $\mathcal{T}_s \nvDash \varphi(\overline{a})$ for any $\overline{a}$ if $\varphi(\overline{x})$ is not one of the first s elements of $\mathcal{X}_i$, and this is what we will assume from now on.

We proceed as follows.

1. Among the first s elements of $\mathcal{X}_i$, locate the $\varphi(\overline{x})$ that minimizes the u such that $\mathcal{T}_r \models \varphi(a_e, \overline{p}_i) \wedge \varphi(b_e, \overline{p}_i)$ for every $r \in (u, s]$. Note that $u = s$ always works. Decide ties by favoring earlier elements of $\mathcal{X}_i$.

2. Wait until there is an $r \in (t, s]$ with $\mathcal{T}_r \nvDash \varphi(a_e, \overline{p}_i) \wedge \varphi(b_e, \overline{p}_i)$.

3. Add a child to both a_e and b_e, ensuring that these children are not elements of $\overline{p}_i$.

4. Return to Step (1).

We perform at most one step at every stage at which the strategy is visited. In particular, we never add more than 1 child to a_e at a single stage. This will be important for interactions with higher priority categoricity requirements. Note also that at every stage, a_e has exactly 2 more children than b_e.

The strategy has infinitely many outcomes: ∞ and $\mathtt{fin}_k$ for $k \in \omega$. Every time we reach Step (4), we take outcome ∞ for a single stage. At all other stages, we take outcome $\mathtt{fin}_k$, where k is the number of previous stages at which we have taken outcome ∞.

Strategy for Q_j

Suppose σ is a strategy for Q_j. This strategy will also appear on the priority tree. When σ is visited, s is always the current stage and $t < s$ is the last stage at which the strategy took outcome ∞ (or $t = 0$ if σ has never before taken outcome ∞).

We construct the isomorphism on c_e and its descendants independently of the isomorphism for all the other $c_{e'}$'s. We begin by searching for a tuple $(r, c, m, n) \in M_j$ with

$$r \lhd_{M_j} c \lhd_{M_j} m \lhd_{M_j} n,$$

and n having $e+1$ many children. When we find such a tuple, we map c_e to c; m_e to m; n_e to n; and the children of n_e to the children of n. Of course, we may later see that the $(e+2)$nd child of n_e appear, in which case we have made a mistake. If this happens, we will discard our mapping and begin again. If $M_j \cong \mathcal{T}$, eventually the tuple in M_j that respects the isomorphism is the Gödel least satisfying the above, and so we will define the correct mapping. The oracle $\mathbf{0}'$ will be able to predict our mistakes, and so can ignore all mappings before the correct one.

Under the assumption that we have correctly mapped c_e, we must map a_e and b_e. This part will not rely on the oracle. We wait until σ is visited and $s > e$. If e has not been chosen by an R_i-strategy by this point, we know by construction that it will be never chosen. In this case, we search for an $a \rhd_{M_j} c$ such that a has two children and map a_e to a. We then search for any child $b \rhd_{M_j} c$ other than m or a, and map b_e to b.

If e has been chosen by an R_i-strategy, and that strategy is incomparable with σ on the tree, then, under the assumption that σ is along the true path, the strategy that chose e will never be visited again. So let p^e be the number of children of a_e. We search for an $a \rhd_{M_j} c$ such that a has p^e children, and map a_e to a. We then search for any $b \rhd_{M_j} c$ which is incomparable with m and a, and, in case $p^e > 2$, itself has children, and map b_e to b.

If e has been chosen by an R_i-strategy τ with $\tau^\frown\infty \subseteq \sigma$, then, under the assumption that σ is along the true path, a_e and b_e are automorphic. So we search for any $a, b \rhd_{M_j} c$ incomparable with m, a and having children, and map a_e to a and b_e to b.

If e has been chosen by an R_i-strategy τ with $\tau^\frown\mathtt{fin}_k \subseteq \sigma$, then, under the assumption that σ is along the true path, a_e and b_e will never gain any more children. So let p^e be the number of children on a_e. We search for an $a \rhd_{M_j} c$ such that a has p^e children, and map a_e to a. We then search for any $b \rhd_{M_j} c$ which is incomparable with m, a, and, in case $p^e > 2$, itself has children, and map b_e to b.

If e has been chosen by an R_i-strategy τ with $\sigma^\frown\mathtt{fin}_k \subseteq \tau$, then we wait until a stage t when σ is accessible and $t > e$. At this stage, we know that τ will never again be accessible (since τ was visited before t, σ had taken outcome ∞ at least k times strictly before t, so at least $k+1$ times by any stage after t, so any future outcomes of σ must be ∞ or $\mathtt{fin}_{k'}$ for $k' > k$). So let p^e be the number of children on a_e. We search for an $a \rhd_{M_j} c$ such that a has p^e children, and map a_e to a. We then search for any $b \rhd_{M_j} c$ which is incomparable with m, a, and, in case $p^e > 2$, has children, and map b_e to b.

If e has been chosen by an R_i-strategy τ with $\sigma\frown\infty \subseteq \tau$, then let p^e_s be the number of children on a_e at the beginning of stage s. We search for an $a \rhd_{M_j} c$ such that a has p^e_s children, and map a_e to a. We then search for any $b \rhd_{M_j} c$ which is incomparable with m, a, and, in case $p^e_2 > 2$, has children, and map b_e to b. Note that, unlike in the other cases, p^e_s may change, which is why we have subscripted it with the stage number.

The strategy has infinitely many outcomes: ∞ and $\mathtt{fin}_k$ for $k \in \omega$. At stage s, if the isomorphism is defined on a_e for every $e < s$, which has been chosen by a τ extending $\sigma\frown\infty$, and further the image of a_e in M_j has p^e_s many children for every such e, then we take outcome ∞. Otherwise, we take outcome $\mathtt{fin}_k$ where k is the number of previous stages at which we have taken outcome ∞.

Construction

Arrange the strategies on a tree in some effective fashion, and at every stage allow strategies to be visited according to the outcome of previous strategies at that stage in the usual fashion.

Verification

Define the true path in the usual fashion for a $\mathbf{0}''$-construction.

Lemma 12. *Suppose that τ is an R_i-strategy along the true path. Then τ ensures R_i is satisfied.*

Proof. Since τ is along the true path, it is visited infinitely often. We have 2 cases to consider.

Case 1. There is some $\varphi(\overline{x}) \in \mathcal{X}_i$ such that $\mathcal{T} \models \varphi(a_e, \overline{p}_i) \wedge \varphi(b_e, \overline{p}_i)$. Choose the least such $\varphi(\overline{x})$. Let u be such that $\mathcal{T}_r \models \varphi(a_e, \overline{p}_i) \wedge \varphi(b_e, \overline{p}_i)$ for every $r \in (u, \infty]$. Then for any $\psi(\overline{x}) \in \mathcal{X}_i$, which is not one of the first $u+1$ elements of $\mathcal{X}_i$, we know that τ will never choose $\psi(\overline{x})$ because it will always prefer $\varphi(\overline{x})$.

So if τ were to take outcome ∞ infinitely many times, by the pigeon hole principle, it would choose one of the first $u+1$ elements of $\mathcal{X}_i$ infinitely many times. But if there are infinitely many r with $\mathcal{T}_r \not\models \psi(a_e, \overline{p}_i) \wedge \psi(b_e, \overline{p})$, then eventually τ will prefer φ over ψ, and so will stop choosing

ψ. Since φ was chosen to be least such, it will eventually be preferred to every other formula, but then once that occurs, we will never again reach Stage 4. Therefore, τ cannot have outcome ∞ infinitely often. So τ has true outcome $\mathtt{fin}_k$ for some k, and a_e and b_e have different finite numbers of children. This means that a_e and b_e are not automorphic, so φ witnesses the failure of $(\mathcal{X}_i, \overline{p}_i)$ as a Scott family.

Case 2. There is no $\varphi(\overline{x}) \in \mathcal{X}_i$ such that $\mathcal{T} \models \varphi(a_e, \overline{p}_i) \wedge \varphi(b_e, \overline{p}_i)$. Then for any φ, there are infinitely many r with $\mathcal{T}_r \not\models \varphi(a_e, \overline{p}_i) \wedge \varphi(b_e, \overline{p}_i)$. So with any chosen φ we eventually reach Step (3), so a_e and b_e have infinitely many children. So a_e and b_e will be automorphic, and in particular there will be an automorphism permuting a_e and b_e and pointwise fixing $\overline{p}_i$. So for any φ with $\mathcal{T} \models \varphi(a_e, \overline{p}_i)$, we know that $\mathcal{T} \models \varphi(b_e, \overline{p}_i)$. Hence there can be no $\varphi \in \mathcal{X}_i$, so that $\mathcal{T} \models \varphi(a_e, \overline{p}_i)$, and thus $\mathcal{X}_i$ fails to be a Scott family. □

Lemma 13. *Suppose that σ is a Q_j-strategy along the true path, that $M_j \cong \mathcal{T}$, and e is chosen by some $\tau \supseteq \sigma^\frown\infty$. Then σ eventually correctly maps a_e and b_e.*

Proof. Certainly, σ eventually correctly maps c_e and m_e, and defines some map for a_e and b_e. If τ has true outcome ∞, then a_e and b_e are automorphic, so this is a correct map.

Suppose instead that τ has true outcome $\mathtt{fin}_k$ (thus a_e has $k+2$ children, and b_e has k children). Let s_0 be the stage at which σ correctly maps c_e, and let t_0 be the final stage at which τ takes outcome ∞. Suppose that $s_0 > t_0$. Then at stage s_0, σ searches for an $a \rhd_{M_j} c$ with $p^e_{s_0} = k+2$ children, and maps a_e to a. By assumption, a_e never gains any more children, so, since $M_j \cong \mathcal{T}$, the correct image of a_e is the only such child of c. The element b_e is correctly mapped by elimination.

If instead $s_0 \leqslant t_0$, then let a be the element to which σ has mapped a_e at stage t_0. (Such an element necessarily exists because σ must have taken outcome ∞ at stage t_0.) Since a_e can gain at most one child during stage t_0, and will gain no children after stage t_0, it has at least $k+1$ children at the start of stage t_0. Since σ has outcome ∞ at stage t_0, a has at least $p^e_{t_0} = k+1$ children. Since $M_j \cong \mathcal{T}$, the correct image of a_e is the only child of c with at least $k+1$ children, so a_e is correctly mapped. The element b_e is correctly mapped by elimination. □

Lemma 14. *Suppose that σ is a Q_j-strategy along the true path, and that $M_j \cong \mathcal{T}$. Then σ has true outcome ∞.*

Proof. Suppose otherwise. Let t_0 be the final stage at which σ takes outcome ∞. Then there are only finitely many e that are chosen by strategies extending $\sigma^\frown\infty$, and, by Lemma 13, σ eventually correctly maps a_e for each of these e's. Since $M_j \cong \mathcal{T}$, σ eventually sees $p^e_{t_0}$ many children below the target of a_e for each e, and so σ will take outcome ∞ at some stage after t_0, contrary to our assumption. □

Lemma 15. *If $M_j \cong \mathcal{T}$, then there is a Δ_2^0 isomorphism between M_j and $\mathcal{T}$.*

Proof. Non-uniformly fix σ that is the Q_j-strategy along the true path. As argued before, σ eventually correctly maps every c_e and m_e, and $\mathbf{0}'$ can determine when this occurs. By Lemma 13, or by the description of σ's action, σ correctly maps a_e and b_e once c_e has been correctly mapped. The only new ingredient is the observation that since σ has true outcome ∞, there is eventually a stage s with $t > e$, thus treating those e's chosen by strategies extending $\sigma^\frown\texttt{fin}_k$.

Once a_e and b_e are mapped, their children can be mapped by a simple back-and-forth argument. Thus $\mathbf{0}'$ can build an isomorphism. □

This completes the proof. Note that every step we have described above can be performed equally well for partial orders and directed graphs. □

We can modify the construction in the proof of the previous theorem to make the tree have infinite height by extending every child of a_e, b_e and n_e to an infinite non-branching path. Once a_e, b_e and n_e are correctly mapped, we then need to use the $\mathbf{0}'$-oracle to correctly map their descendants. Hence we have the following result, which is interesting, in particular, since there is no computably categorical tree of infinite height.

Theorem 16. *There is a computable Δ_2^0-categorical tree of infinite height, which is not relatively Δ_2^0-categorical.*

2.3 Δ_2^0-categorical but not relatively Δ_2^0-categorical homogenous completely decomposable abelian groups

We will now consider certain torsion-free abelian groups. A *homogenous completely decomposable abelian group* is a group of the form $\bigoplus_{i\in\kappa} H$, where H is a subgroup of the additive group of the rationals, $(\mathbb{Q}, +)$. Note that we have only a single H in the sum – any two summands are isomorphic. It is well known that such a group is computably categorical if and only if κ is finite; the proof is similar to the analogous result that a computable vector space is computably categorical if and only if it has finite dimension. In the remainder of this section, we will restrict our attention to groups of infinite rank κ.

For P a set of primes, define $Q^{(P)}$ to be the subgroup of $(\mathbb{Q}, +)$ generated by $\{\frac{1}{p^k} \ : \ p \in P \wedge k \in \omega\}$. Downey and Melnikov [18] showed that a computable homogenous completely decomposable abelian group of infinite rank is Δ^0_2-categorical if and only if it is isomorphic to $\bigoplus_\omega Q^{(P)}$, where P is c.e. and the set (Primes $-$ P) is semi-low. Recall that a set $S \subseteq \omega$ is *semi-low* if the set $H_S = \{e : W_e \cap S \neq \emptyset\}$ is computable from $\emptyset'$. Here, we will first fully characterize the computable relatively Δ^0_2-categorical homogenous completely decomposable abelian groups of infinite rank.

Theorem 17. *A computable homogenous completely decomposable abelian group of infinite rank is relatively Δ^0_2-categorical if and only if it is isomorphic to $\bigoplus_\omega Q^{(P)}$, where P is a computable set of primes.*

Proof. Suppose that G is relatively Δ^0_2-categorical. Since this implies that G is Δ^0_2-categorical, by the above mentioned result of Downey and Melnikov, we know that $G \cong \bigoplus_\omega Q^{(P)}$ for P a c.e. set of primes. We will show that P is also co-c.e.

Fix $\mathcal{X}$, a c.e. Scott family of computable Σ_2 formulas for G, with parameters $\overline{a} \in G^{<\omega}$. By definition, any element of G has all but finitely many coordinates equal to 0. Choose $l \in \omega$ to be a coordinate which equals to 0 for every element of $\overline{a}$. Fix an element $b \in G$, such that the only non-zero coordinate of b is l. Then b is independent of $\overline{a}$. The map $b \mapsto p \cdot b$ can be extended to an automorphism of G fixing $\overline{a}$ if and only if $p \in P$. Fix some formula $\exists \overline{x}\, \theta(\overline{z}, \overline{x}, y) \in \mathcal{X}$, where θ is a computable Π_1 formula and $G \models \exists \overline{x}\, \theta(\overline{a}, \overline{x}, b)$. Fix some tuple $\overline{c} \in G$ such that $G \models \theta(\overline{a}, \overline{c}, b)$.

Now, decompose the elements of $\overline{c}$ as $c_i = d_i + e_i$, where d_i is a rational multiple of b, and b is independent of $\{\overline{a}, \overline{e}\}$. Observe that the map $b \mapsto p \cdot b$ can be extended to an automorphism of G fixing $\overline{a}$ and $\overline{e}$ if and only if $p \in P$, and any such isomorphism would need to map $d_i \mapsto p \cdot d_i$.

Define $\overline{c}^p$ by $c^p_i = p \cdot d_i + e_i$. Note that an isomorphism sending $b \mapsto p \cdot b$ and fixing $\overline{a}$ and $\overline{e}$ would necessarily map $\overline{c} \mapsto \overline{c}^p$. So, if there is such an isomorphism, then $G \models \theta(\overline{a}, \overline{c}^p, p \cdot b)$.

Conversely, if $G \models \theta(\overline{a}, \overline{c}^p, p \cdot b)$ then $G \models \exists \overline{x}\, \theta(\overline{a}, \overline{x}, p \cdot b)$, and, by the definition of Scott family, there must be an isomorphism fixing $\overline{a}$ and mapping $b \mapsto p \cdot b$. Thus,

$$p \in P \Leftrightarrow G \models \theta(\overline{a}, \overline{c}^p, p \cdot b).$$

Since θ is a computable Π_1 formula, and $\overline{c}^p$ can be obtained effectively from p, it follows that P is co-c.e. □

Since there exist co-c.e. sets that are semi-low and noncomputable, we obtain the following categoricity result.

Corollary 18. *There is a computable homogenous completely decomposable abelian group, which is Δ^0_2-categorical but not relatively Δ^0_2-categorical.*

2.4 Degrees of categoricity of certain Boolean algebras and abelian p-groups

Cenzer, Harizanov, and Remmel established in [11] that the degrees of categorictiy of computable injections structures can only be $\mathbf{0}$, $\mathbf{0}'$ and $\mathbf{0}''$. Frolov [30] showed that the degrees of categoricity of relatively Δ^0_2-categorical linear orders can only be $\mathbf{0}$ and $\mathbf{0}'$. Using the characterization of relatively Δ^0_2-categorical Boolean algebras by McCoy in [67], Bazhenov [7] established that the degrees of categoricity of relatively Δ^0_2-categorical (equivalently, Δ^0_2-categorical) Boolean algebras can only be $\mathbf{0}$ and $\mathbf{0}'$. In this section, we will extend Bazhenov's result to relatively Δ^0_3-categorical Boolean algebras.

A Boolean algebra $\mathcal{B}$ is *atomic* if for every $a \in \mathcal{B}$ there is an atom $b \leqslant a$. An equivalence relation $\sim$ on a Boolean algebra $\mathcal{A}$ is defined by:

$$a \sim b \text{ iff each of } a \cap \overline{b} \text{ and } b \cap \overline{a} \text{ is } \emptyset \text{ or a union of finitely many atoms of } \mathcal{A}.$$

A Boolean algebra $\mathcal{A}$ is a 1-*atom* if $\mathcal{A}/\sim$ is a two-element algebra. A Boolean algebra $\mathcal{A}$ is *rank* 1 if $\mathcal{A}/\sim$ is a nontrivial atomless Boolean algebra. McCoy [67] proved that a countable rank 1 atomic Boolean algebra is isomorphic to $I(2 \cdot \eta)$.

In [67], McCoy established that a Boolean algebra is relatively Δ^0_2-categorical if and only if it is a finite direct sum of algebras that are atoms, atomless, or 1-atoms. Furthermore, in [66], McCoy characterized relatively Δ^0_3-categorical Boolean algebras as those computable Boolean algebras that can be expressed as finite direct sums of algebras that are atoms, atomless, 1-atoms, rank 1 atomic, or isomorphic to the interval algebra $I(\omega + \eta)$. In our next theorem, we will use this characterization and the following isomorphism result of Remmel [73] .

Lemma 19 (Remmel)**.** *If $\mathcal{A}$ is a Boolean algebra, $\mathcal{B} \subseteq \mathcal{A}$ is a subalgebra, $\mathcal{B}$ has infinitely many atoms, every atom in $\mathcal{B}$ is a finite join of atoms in $\mathcal{A}$, and $\mathcal{A}$ is generated by $\mathcal{B}$ and the elements below the atoms of $\mathcal{B}$, then $\mathcal{B} \cong \mathcal{A}$.*

Theorem 20. *The degrees of categoricity of relatively Δ^0_3-categorical Boolean algebras can only be* $\mathbf{0}$*,* $\mathbf{0}'$ *and* $\mathbf{0}''$*.*

Proof. Fix a relatively Δ^0_3-categorical Boolean algebra $\mathcal{B}$. If $\mathcal{B}$ is a finite join of atoms, 1-atoms and atomless Boolean algebras, then $\mathcal{B}$ is relatively Δ^0_2-categorical, and so its degree of categoricity is either $\mathbf{0}$ or $\mathbf{0}'$. Otherwise, $\mathcal{B}$ has a summand which is either rank 1 atomic or isomorphic to the interval algebra $I(\omega + \eta)$.

All of the potential summands in the characterization of relatively Δ^0_3-categorical Boolean algebras have computable isomorphic copies in which the set of finite elements (that is, the elements a with $a \sim 0$) is computable. We will show that both the rank 1 atomic algebra and $I(\omega + \eta)$ have computable isomorphic copies where the set of finite elements is Σ^0_2-complete. It will follow that $\mathcal{B}$ has a computable isomorphic copy in which the set of finite elements is computable, and another computable isomorphic copy in which it is Σ^0_2-complete, and so any isomorphism between these two copies will compute $\emptyset''$.

We begin with the rank 1 atomic algebra. Let C be a computable copy of this algebra in which the set of atoms is computable. Let $\{a_i : i \in \omega\}$ be the atoms of C. We will create an algebra $\mathcal{A}$ by extending C. Let $\varphi(i, x)$ be a computable formula such that

$$i \in \emptyset'' \Leftrightarrow \exists^{<\infty} x\, \varphi(i, x).$$

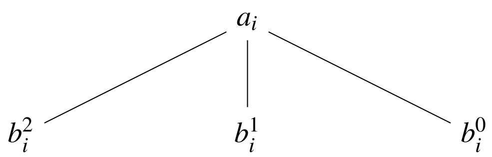
At every step s, we will consider whether $\varphi(i,s)$ holds. The first time $\varphi(i,s)$ holds, we choose three large elements b_i^0, b_i^1 and b_i^2 and use them to partition a_i into three pieces. That is,

$$b_i^0 \wedge b_i^1 = b_i^1 \wedge b_i^2 = b_i^2 \wedge b_i^0 = 0$$

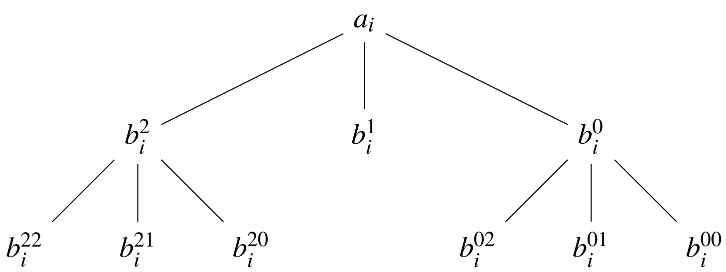
and

$$b_i^0 \vee b_i^1 \vee b_i^2 = a_i.$$

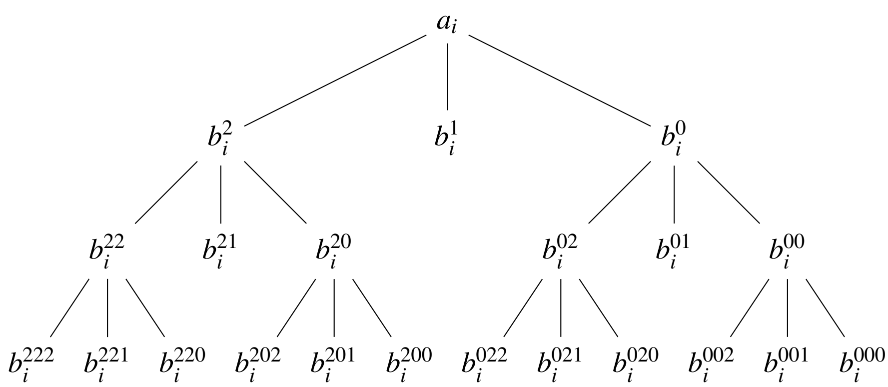
At the second stage at which we see $\varphi(i,s)$ hold, we repeat the process on b_i^0 and b_i^2. See the following diagrams.



Working with rank 1 atomic, the first time we see $\varphi(i,s)$ hold.



Working with rank 1 atomic, the second time we see $\varphi(i,s)$ hold.

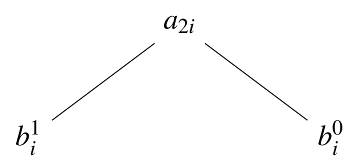


Working with rank 1 atomic, the third time we see $\varphi(i,s)$ hold.

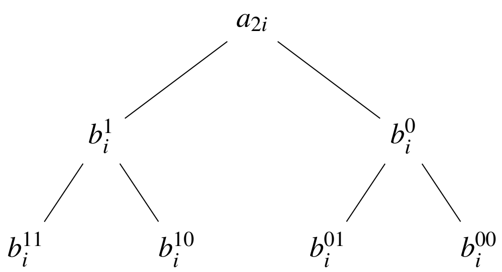
We then let $\mathcal{A}$ be the Boolean algebra generated by C along with these new elements we have added. Note that every element of $\mathcal{A}$ is the join of an element from C and some of these new elements (among b_i^{σ}'s). That is, for all $d \in \mathcal{A}$, $d = c \vee b_{i_0}^{\sigma_0} \vee b_{i_1}^{\sigma_1} \vee \cdots \vee b_{i_k}^{\sigma_k}$ for some $c \in C$ and some $b_{i_0}^{\sigma_0}, \ldots, b_{i_k}^{\sigma_k}$.

Observe that a_i is infinite in $\mathcal{A}$ if and only if $\varphi(i, x)$ holds for infinitely many x, which is if and only if $i \notin \emptyset''$. Also, a_i necessarily bounds an atom in $\mathcal{A}$, e.g., b_i^1. Finally, if a_i is infinite, then it can be partitioned into two infinite elements, e.g., b_i^0 and $b_i^1 \vee b_i^2$. Since every element of C bounds an atom, and every infinite element of C can be partitioned into two infinite elements, it follows that the same holds for every element of $\mathcal{A}$. This characterizes the rank 1 atomic algebra. Thus $\mathcal{A} \cong C$, and $\mathcal{A}$ is as desired.

Next, consider $I(\omega + \eta)$. Again, let C be a computable copy of $I(\omega + \eta)$ in which the set of atoms is computable. Let $\{a_i : i \in \omega\}$ be the atoms of C. We again create $\mathcal{A}$ extending C. Let $\varphi(i, x)$ be as before. At every step s, if $\varphi(i, s)$ holds, we add new elements below a_{2i}. The first time $\varphi(i, s)$ holds, we partition $a_{2i} = b_i^0 \vee b_i^1$. The second time it holds, we partition b_i^0 and b_i^1. See the diagrams.

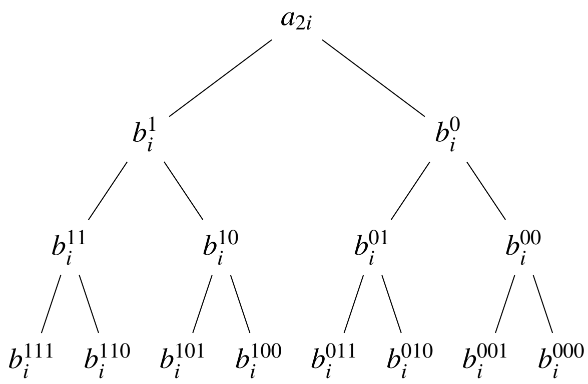


Working with $I(\omega + \eta)$, the first time we see $\varphi(i, s)$ hold.



Working with $I(\omega + \eta)$, the second time we see $\varphi(i, s)$ hold.

We again let $\mathcal{A}$ be the Boolean algebra generated by C along with these new elements. The isomorphism type of $I(\omega + \eta)$ is characterized by three properties: there are infinitely many atoms; any element which bounds infinitely many atoms also bounds an atomless element; and



Working with $I(\omega + \eta)$, the third time we see $\varphi(i, s)$ hold.

no two disjoint elements both bound infinitely many atoms. Since every atom of $\mathcal{A}$ is bounded by an atom of C, every atomless element of C is still atomless in $\mathcal{A}$, and every atom of C is either atomless or finite in $\mathcal{A}$, the second and the third properties are inherited from C to $\mathcal{A}$. Meanwhile, the first property is ensured by the fact that each a_{2i+1} is still an atom of $\mathcal{A}$. Thus $\mathcal{A} \cong C$. Also, a_{2i} is finite if and only if $i \in \emptyset''$, so $\mathcal{A}$ is as desired.

This completes the proof. □

It follows from proofs in [9] that the degrees of categoricity of computable relatively Δ^0_2-categorical equivalence structures can only be $\mathbf{0}$ and $\mathbf{0}'$. Using the characterization of relatively Δ^0_2-categorical abelian p-groups in [10] we can show the following.

Proposition 21. *The categoricity degrees of computable relatively Δ^0_2-categorical abelian p-groups can only be* $\mathbf{0}$ *and* $\mathbf{0}'$.

Proof. Suppose that G is a computable abelian p-group, which is relatively Δ^0_2-categorical but not computably categorical. We will show that G has degree of categoricity $\mathbf{0}'$. From the earlier described classifications of categoricity, it follows that G is of one of the following two forms:

1. $\bigoplus_{\omega} \mathbb{Z}(p^k) \oplus \bigoplus_{\omega} \mathbb{Z}(p^m) \oplus H$, where $0 < k < m \leqslant \omega$; or
2. Every element of G has finite height, but G contains elements of arbitrarily large finite heights.

We will handle the two cases separately.

First Case

Consider elements $x \in G$ with $x \neq 0$, $p \cdot x = 0$ and $ht(x) = k - 1$. Note that $\mathbb{Z}(p^k)$ contains such an element (indeed, $p - 1$ such elements). By the observation that $G \cong \bigoplus_{\omega} \mathbb{Z}(p^k) \oplus G$, we may assume that we have an effective enumeration $\{a_n : n \in \omega\}$ of elements of this sort.

Fix μ the modulus function of $\emptyset'$. We will build a second computable copy A such that the first $\mu(n)$ elements of A contain at most n elements of the desired sort. Then given any isomorphism $f : G \cong A$, the function $n \mapsto f(a_n)$ would necessarily dominate μ. Thus, any isomorphism from G to A would compute $\emptyset'$.

The construction is now straightforward. By $dom(F)$ we denote the domain and by $ran(F)$ the range of a function F. We will build a Δ^0_2 homomorphism $F : G \cong A$ and arrange that $A = ran(F) \oplus \bigoplus_{\omega} \mathbb{Z}(p^m)$. We begin with $F_0 = \emptyset$.

At stage $s + 1$, for every $n \leqslant s$, we consider every $x \in G$ with $n \leqslant x \leqslant s$, $x \neq 0$, $p \cdot x = 0$ and $[ht(x)]^{G_s} < k$. For each such element, if $F_s(x) \leqslant \mu_s(n)$, we define $F_{s+1}(x)$ as some new large element. This requires that we also define $F_{s+1}(y)$ for every y dividing such an x, to be some new large element. We let $F_{s+1}(x) = F_s(x)$ for every other x. We then extend the domain of F_{s+1} to the next element of G. We let F_{s+1} induce the group operation on its range *via* pull-back.

Let $D_{s+1} = ran(F_s) - ran(F_{s+1})$. Note that every elements of D_s has height less than k. We add new elements to extend D_{s+1} to a copy of $\bigoplus_{l} \mathbb{Z}(p^m)$ for some $l < \omega$. Also, for every $a \in A_{s+1} - ran(F_{s+1})$ and every $b \in ran(F_{s+1})$, if A does not yet have an element corresponding to $a + b$, we add an appropriate element now. This completes stage $s + 1$.

Now we argue that F is a total Δ^0_2 function. Fix $x \in G$ with $x \neq 0$ and $p \cdot x = 0$. If $F_{s+1}(x) \neq F_s(x)$, then either our construction was deliberately redefining $F(x)$, or it was required to redefine $F(x)$ because it deliberately redefined $F(z)$ for some z that x divides. The only such z's are of the form $i \cdot x$ for $1 \leqslant i < p$. Let s_0 be such that $\mu_{s_0}(i \cdot x) = \mu(i \cdot x)$ for $1 \leqslant i < p$. Then at any stage $s > s_0$ with $F_{s+1}(x) \neq F_s(x)$, necessarily $F_{s+1}(i \cdot x) > \mu_s(i \cdot x) = \mu(i \cdot x)$, since $F_{s+1}(i \cdot x)$ is chosen to be large. Then at any stage $t > s$, $F_t(i \cdot x) > \mu(i \cdot x) = \mu_t(i \cdot x)$, and so we will have $F_{t+1}(x) = F_t(x)$, and thus $F(x)$ will reach a limit.

Now, consider $y \in G$ with $p^{\alpha+1} \cdot y = 0$. Then $p \cdot (p^\alpha \cdot y) = 0$, and $F_{s+1}(y) \neq F_s(y)$ only when $F_{s+1}(p^\alpha \cdot y) \neq F_s(p^\alpha \cdot y)$. Since we have just argued that $F(p^\alpha \cdot y)$ reaches a limit, it follows that $F(y)$ reaches a limit.

Note that $A = ran(F) \oplus \bigoplus_{\omega} \mathbb{Z}(p^m)$ by construction. It follows that $A \cong G$. It also follows that every $x \in A - ran(F)$ with $p \cdot x = 0$ has height at least $m - 1 \geqslant k$. Finally, our construction ensured that there are at most n elements $x \in G$ with $p \cdot x = 0$, $ht(x) < k$ and $F(x) < \mu(n)$. Thus, there are at most n elements $x \in A$ with $p \cdot x = 0$, $ht(x) < k$ and $x < \mu(n)$, as desired.

Second Case

By a result of Khisamiev [56] and independently of Ash, Knight and Oates [3], we know that

$$G \cong \mathbb{Z}(p^{k_0}) \oplus \mathbb{Z}(p^{k_1}) \oplus \cdots,$$

where the sequence $(k_i)_{i\in\omega}$ is uniformly computable from below. That is, there is a computable function $g : \omega \times \omega \to \omega$ such that for all i and s, $g(i, s) \leqslant g(i, s+1)$, and for all i, $k_i = \lim_s g(i, s)$. Fix such a function g. By our assumptions on G, we know that the k_i's are unbounded.

We will construct a computable function h and a Δ^0_2 function ι such that:

1. For all i and s, $h(i, s) \leqslant h(i, s + 1)$;

2. $\iota : \omega \to \omega$ is a bijection;

3. For all i, $\lim_s h(i, s) = \lim_s g(\iota(i), s)$; and

4. For all n and all $x \in G$ with $x < \mu(n)$ and $x \neq 0$, $ht(x) + 1 < \lim_s h(2n, s)$.

We will then let $A = \mathbb{Z}(p^{\lim_s h(0,s)}) \oplus \mathbb{Z}(p^{\lim_s h(0,s)}) \oplus \cdots$. By the first property above, this is a computable structure. By the second and the third properties, $A \cong G$. By the fourth property, given an isomorphism $f : A \cong G$, for any element x of the $(2n)$th summand of A with $x \neq 0$ and $p \cdot x = 0$, it must be that $f(x) \geqslant \mu(n)$. Thus, f computes $\emptyset'$.

It remains to construct h and ι. We begin with $\iota_0 = \emptyset$ and $h(i, 0) = 0$ for all i.

At stage $s + 1$, if there is an n with $2n \in dom(\iota_s)$ and an $x \in G$ with $x < \mu(n)$, $x \neq 0$ and $[ht(x)]^{G_s} \geqslant h(2n, s)$, we search for a large pair (j, t) with $g(j, t) > h(2n, s)$, and define $\iota_{s+1}(2n) = j$ and $h(2n, s + 1) = g(j, t)$. We then choose a large m and define $\iota_{s+1}(2m + 1) = \iota_s(2n)$. We let $\iota_{s+1}(k) = \iota_s(k)$ for every other k.

We then choose the least $a \notin dom(\iota_{s+1})$ and the least $b \notin ran(\iota_{s+1})$, and define $\iota_{s+1}(a) = b$. Then, for every $i \in dom(\iota_{s+1})$ with $h(i, s + 1)$ not yet defined, we define $h(i, s + 1) =$

$\max\{g(\iota_{s+1}(i), s+1), h(i, s)\}$. For every $i \notin dom(\iota_{s+1})$, we define $h(i, s+1) = 0$. This completes stage $s+1$.

First, note that, by construction, $h(i, s) \leqslant h(i, s+1)$ for every i and s.

Next, we argue that ι is a total Δ^0_2 function. Note that, by construction, for every i, there is eventually a stage s_0 with $\iota_s(i)$ defined for all $s \geqslant s_0$. If i is odd, then $\iota_s(i) = \iota_{s_0}(i)$ for all $s \geqslant s_0$. If instead $i = 2n$, then at every stage s with $\iota_s(i) \neq \iota_s(i+1)$, we have $h(i, s+1) \geqslant h(i, s) + 1$. Let $u = \max\{ht(x) : x \in G \wedge x < \mu(n)\}$. So for sufficiently large s_1, $h(i, s_1) > u$, and then $h(i, s) = h(i, s_1)$ for all $s \geqslant s_1$.

Next, we argue that ι is surjective. If $b = \iota_{s_0}(a)$, then either $b = \iota_s(a)$ for all $s > s_0$, or there is a stage $s_1 > s_0$ with $b = \iota_{s_1}(c)$ for some odd c. By construction, ι never changes on odd inputs, so $b = \iota_s(c)$ for all $s \geqslant s_1$. By construction, every element is eventually added to the range of some ι_s, so every element is in $ran(\iota)$.

By induction on s, $h(i, s) \leqslant \lim_s g(\iota_s(i), s)$ for all i and s, and so in particular, $\lim_s h(i, s)$ exists and equals at most $\lim_s g(\iota(i), s)$. On the other hand, $h(i, s) \geqslant g(\iota_s(i), s)$ for all i and s by construction, and so $\lim_s h(i, s) = \lim_s g(\iota(i), s)$, as desired.

Finally, for all n and all $x \in G$ with $x < \mu(n)$ and $x \neq 0$, $ht(x) + 1 < \lim_s h(2n, s)$, as we deliberately increase $h(2n, s)$ whenever this appears to be false. This completes the proof. $\square$

3 Decidablilty, effective categoricity and complexity of descriptions

In this chapter we consider n-decidable structures and their categoricity with respect to m-decidable copies, where $m, n \in \omega$. Recall the definitions.

Definition 22. A structure is n-decidable if its Σ_n-diagram is a computable subset of ω.

A structure is *categorical relative to m-decidable presentations* (or *autostable relative to m-constructivizations*) if any two m-decidable copies of $\mathcal{A}$ are computably isomorphic.

In particular, being computable is the same as being 0-decidable, and being computably categorical is the same as being categorical relative to 0-decidable presentations.

The index sets for structures with specific algorithmic properties related to decidability and effective categoricity were studied by White [83], Fokina [24], Downey, Kach, Lempp, and Turetsky [20] and Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky [19].

Theorem 23. *(i) ([24]) The index set of decidable structures is Σ^0_3-complete.*

(ii) ([83]) The index set of hyperarithmetically categorical structures is Π^1_1-complete.

(iii) ([20]) The index set of relatively computably categorical structures is Σ^0_3-complete.

(iv) ([19]) The index set of computably categorical structures is Π^1_1-complete.

The question we investigate in this chapter is how complicated the property of being n-decidable and categorical relative to m-decidable presentations is. The goal is to find the exact complexity of index sets of computable structures with the mentioned properties. The answer depends on the relationship between m and n. We summarize the results of this chapter in the following table.

n-decidable $n \geqslant 2$	m-decidably categorical $m \leqslant n-2$	Σ^0_3 complete
n-decidable $n \geqslant 1$	$(n-1)$-decidably categorical	Π^0_4 complete
n-decidable $n \geqslant 0$	m-decidably categorical $m \geqslant n$	Π^1_1 complete

Figure 3.1: Complexity of index sets for n-decidable structures, categorical relative to m-decidable presentations

3.1 Case $0 \leqslant n \leqslant m$

We first consider n-decidable structures that are categorical relative to n-decidable presentations.

Theorem 24. *The index set of n-decidable structures that are categorical relative to n-decidable presentations is Π^1_1 complete.*

Proof. Recall that by the result of Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky [19] the index set of computably categorical structures is Π^1_1 complete. This means that for every Π^1_1 set S there is a uniformly computable sequence of structures $\{\mathcal{A}_i\}_{i\in\omega}$ such that $i \in S \iff \mathcal{A}_i$ is computably categorical.

Marker in [65] defined $\forall$- and $\exists$-extensions, $\mathcal{A}_\forall$ and $\mathcal{A}_\exists$, respectively, of an arbitrary structure $\mathcal{A}$. Fix a finite language L with no function symbols, and let $\mathcal{A} = (A, P_0^{n_0}, \dots, P_m^{n_m})$ be a structure of L. We assume that for every P of this structure the sets P and $A^k \setminus P$ are infinite, where k is the arity of P. For each k-ary predicate P of this structure we define $\exists$- and $\forall$-extensions of P, following the work of Marker in [65].

Marker's $\exists$-extension of P is a $(k+1)$-ary predicate denoted by $P_\exists$ with the following properties. Let X be an infinite set disjoint with A. Then $P_\exists$ satisfies the following conditions:

1. If $P_\exists(a_1, a_2, \dots, a_k, a_{k+1})$ then $P(a_1, \dots, a_k)$ and $a_{k+1} \in X$.

2. For every $a_{k+1} \in X$ there exists a unique tuple $(a_1, \dots, a_k) \in A^k$ such that $P_\exists(a_1, a_2, \dots, a_k, a_{k+1})$.

3. If $P(a_1, \dots, a_k)$ then there exists a unique a such that $P_\exists(a_1, a_2, \dots, a_k, a)$.

Marker's $\forall$-extension of the predicate P is a $(k+1)$-ary predicate $P_\forall$ with the following properties. Let X be an infinite set disjoint with A. Then $P_\forall$ satisfies the following conditions:

1. If $P_\forall(a_1, a_2, \dots, a_k, a_{k+1})$ then $a_1, \dots, a_k \in A$ and $a_{k+1} \in X$.

2. For all $(a_1, \dots, a_k) \in A$ there exists at most one $a_{k+1} \in X$ such that $\neg P_\forall(a_1, a_2, \dots, a_k, a_{k+1})$.

3. $P(a_1, \dots, a_k)$ iff for every $a_{k+1} \in X$ we have $P_\forall(a_1, a_2, \dots, a_k, a_{k+1})$.

4. For every $a_{k+1} \in X$ there exists a unique $(a_1, \dots, a_k) \in A^k$ such that $\neg P_\forall(a_1, a_2, \dots, a_k, a_{k+1})$.

The set X in an $\exists$- or $\forall$-extension is called a **f**ellow of P.

Definition 25. Let $\mathcal{A} = (A, P_0^{n_0}, \dots, P_m^{n_m})$ be a structure.

1. $\mathcal{A}_\exists$ is a structure $(A \cup X_0 \dots \cup X_m, P_0^{n_0+1}, \dots, P_m^{n_m+1}, X_0, \dots, X_m)$, where each $P_i^{n_i+1}$, $i = 0, \dots, m$, is a Marker's $\exists$-extension of $P_i^{n_i}$ such that fellows X_i of distinct predicates are pairwise disjoint sets.

2. $\mathcal{A}_\forall$ is a structure $(A \cup X_0 \dots \cup X_m, P_0^{n_0+1}, \dots, P_m^{n_m+1}, X_0, \dots, X_m)$, where each $P_i^{n_i+1}$, $i = 0, \dots, m$, is a Marker's $\forall$-extension of $P_i^{n_i}$ such that fellows X_i of distinct predicates are pairwise disjoint sets.

The main property of Marker's extensions is that the domain and the basic relations of $\mathcal{A}$ are definable in $\mathcal{A}_\forall, \mathcal{A}_\exists$ by universal or existential formulas, respectively. One can iteratively apply the extensions in the obvious way. Define B_i to be the result of the application of Marker's $(\forall\exists)$-extension n-times. As follows from [2] or [41], if $\mathcal{A}_i$ was computable, then B_i is n-decidable. And from properties of the Marker's extensions proved in [26], A_i is computably categorical iff B_i is categorical relative to n-decidable presentations. The claim follows immediately. □

Corollary 26. *For all $m \geqslant n \geqslant 0$, the index set of n-decidable structures that are categorical relative to m-decidable presentations is Π^1_1 complete.*

3.2 Case $m = n - 1$, where $n \geqslant 1$

We now consider 1-decidable, computably categorical structures, i.e. we do not impose additional effectiveness conditions on the copies of the structure except of being computable.

Theorem 27. *The index set of* 1*-decidable, computably categorical structures is* Π^0_4 *complete.*

Proof. We first show that the index set is Π^0_4. Recall that $\langle \mathcal{M}_i \rangle_{i\in\omega}$ is a fixed effective listing of all partial computable structures.

The relation "$\mathcal{M}_i$ is n-decidable" is Σ^0_3, as it states that there is a partial computable $\{0, 1\}$-valued function f defined on pairs $(\varphi(\overline{x}), \overline{a})$ with $\varphi(\overline{x})$ a Σ_n formula in the language of $\mathcal{M}_i$ and $\overline{a} \in \mathcal{M}_i^{<\omega}$ such that:

- f is total;
- For $\varphi(\overline{x})$ quantifier-free, $f(\varphi(\overline{x}), \overline{a}) = 1 \iff \mathcal{M}_i \models \varphi(\overline{a})$;
- For $\varphi(\overline{x}, \overline{y})$ a Π_{n-1} formula, $f(\exists\overline{y}\, \varphi(\overline{x}, \overline{y}), \overline{a}) = 1 \iff \exists\overline{b}\, f(\neg\varphi(\overline{x}, \overline{y}), \overline{a}\overline{b}) = 0$.

Consider the following relations on pairs (i, j):

$$
\begin{aligned}
(i, j) \in E \quad &\iff \quad \mathcal{M}_i \text{ and } \mathcal{M}_j \text{ are total structures and there is a computable} \\
&\qquad \text{isomorphism between them} \\
(i, j) \in F \quad &\iff \quad \mathcal{M}_i \text{ and } \mathcal{M}_j \text{ are total structures and there is a } \Delta^0_2 \text{ isomorphism} \\
&\qquad \text{between them}
\end{aligned}
$$

It is straightforward to show that E is Σ^0_3, while F is Σ^0_4.

Now consider the following property of a computable structure $\mathcal{A}$:

> For every computable structure $\mathcal{B}$, if there is a Δ^0_2 isomorphism from $\mathcal{A}$ to $\mathcal{B}$, then there is a computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$. (†)

As a relation on i, this can be written as

$$\forall j\, F(i, j) \to E(i, j),$$

and so this is Π^0_4.

Note that property (†) is a weakening of computable categoricity. Downey, Kach, Lempp and Turetsky [20] showed that if a structure is computably categorical and 1-decidable, then it is relatively Δ^0_2-categorical. Inspection of their proof reveals that they did not use the full power of computable categoricity; instead, they only used property (†). Thus they showed the following:

Lemma 28. *If a structure is 1-decidable and has property (†), then it is relatively* Δ^0_2*-categorical.*

Note that a structure which is simultaneously relatively Δ^0_2-categorical and has property (†) is necessarily computably categorical. Thus we have the following: a structure is 1-decidable and computably categorical if and only if it is 1-decidable and has property (†). So the relation "$\mathcal{M}_i$ is 1-decidable and computably categorical" can be written as the conjunction of a Σ^0_3 formula and a Π^0_4 formula, and so is Π^0_4.

To show the completeness at the level Π^0_4, we use a known method to code computable families of functions in 1-decidable unars (for short, S is coded in $\mathcal{M}_S$), as exposed in [22, 59]. The main feature of the construction is the following: S admits exactly one computable numbering up to equivalence iff the unar $\mathcal{M}_S$ is computably categorical. So, the index set of computable families of functions with exactly one computable numbering is m-reducible to required index set. And the first index set was investigated in [60], where its Π^0_4-completeness was proven. The theorem is proven.

□

Using the technique of Marker's extensions, it is not hard to show:

Corollary 29. *For any $n \geqslant 1$, the index set of n-decidable, categorical relative to $(n-1)$-decidable presentations structures is Π^0_4 complete.*

3.3 Case $m \leqslant n-2$, where $n \geqslant 2$

Goncharov [32] proved that a 2-decidable computably categorical structure is relatively computably categorical. Downey, Kach, Lempp and Turetsky [20] showed that the index set of relatively computably categorical structures is Σ^0_3 complete. In fact, they show that the index set of 2-decidable computably categorical structures is Σ^0_3 complete. Applying Marker's extensions, we get the following result.

Proposition 30. *For any $n \geqslant 2$ and $m \leqslant n-2$, the index set of n-decidable, categorical relative to m-decidable presentations structures is Σ^0_3 complete.*

4 Degree spectra with respect to equivalence relations

Recall the definition of degree spectra relative to equivalence relations.

Definition 31. The *degree spectrum* of a countable structure $\mathcal{A}$ with universe ω *relative to the equivalence relation E* is

$$\mathrm{DgSp}(\mathcal{A}, E) = \{\mathbf{d} \mid \text{there exists a } \mathbf{d}\text{-computable } \mathcal{B} \text{ which is } E-\text{equivalent to } \mathcal{A}\}.$$

Then the classical degree spectrum of $\mathcal{A}$ is $\mathrm{DgSp}(\mathcal{A}, \cong)$, the degree spectrum of $\mathcal{A}$ under isomorphism, while the degree spectra of the theory of $\mathcal{A}$ is $\mathrm{DgSp}(\mathcal{A}, \equiv)$, the degree spectrum of $\mathcal{A}$ under elementary equivalence.

In this chapter, instead of considering the full theory of a structure, as for theory spectra, we consider Σ_n-fragments of theories and the corresponding equivalence relations $\equiv_{\Sigma_n}$ (two structures are $\equiv_{\Sigma_n}$-equivalent if their Σ_n-theories coincide). We also write $\mathcal{A} \equiv_{\Sigma_n} \mathcal{B}$ when $\mathcal{A}$ and $\mathcal{B}$ are Σ_n-equivalent. We call $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_n})$ the Σ_n*–spectrum* of $\mathcal{A}$. We will study what kinds of spectra are possible with respect to these equivalence relations.

4.1 Two cones

It is well-known that the degree spectrum of a structure cannot be the union of two cones [80]. On the other hand, the authors of [2] built a theory T with the spectrum of T being a non-degenerate union of two cones. For Σ_n-spectra, the situation depends on n.

We start with a simple observation.

Lemma 32. *Two relational structures $\mathcal{A}$ and $\mathcal{B}$ are Σ_1–equivalent iff they have the same finite substructures (in finite sublanguages).*

Proof. Suppose $\mathcal{A} \equiv_{\Sigma_1} \mathcal{B}$. Choose an arbitrary finite substructure $\mathcal{A}_0$ of $\mathcal{A}$ of a finite sublanguage. As its language is finite, we can write its atomic diagram $D(\mathcal{A}_0)$ as a single first order sentence $\varphi(\overline{a})$ with parameters $\overline{a}$ from A_0. Then $\mathcal{A} \models \exists\overline{x}\varphi(\overline{x})$, where $|\overline{x}| = |\overline{a}|$. By Σ_1–equivalence, $\mathcal{B} \models \exists\overline{x}\varphi(\overline{x})$. Let $\overline{b}$ witness φ in $\mathcal{B}$. Then the finite substructure $\mathcal{B}_0$ of $\mathcal{B}$ with domain $\overline{b}$ and with relation symbols that appear in φ is isomorphic to $\mathcal{A}_0$.

Suppose now that $\mathcal{A}$ and $\mathcal{B}$ have the same finite substructures in finite sublanguages. Assume $\mathcal{A} \models \exists\overline{x}\varphi(\overline{x})$. Let $\overline{a}$ be a witness. Consider the finite substructure $\mathcal{A}_0$ of $\mathcal{A}$ with the universe $\overline{a}$ and the language consisting of the relation symbols used in φ. By assumption, there is a finite substructure $\mathcal{B}_0$ of $\mathcal{B}$ in the same language which is isomorphic to $\mathcal{A}_0$. Then $\mathcal{B}_0 \models \exists\overline{x}\varphi(\overline{x})$, and thus $\mathcal{B} \models \exists\overline{x}\varphi(\overline{x})$. □

Theorem 33. *No Σ_1–spectrum of a structure can be a non-degenerate union of two cones.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ_1–equivalent structures that have degrees $\mathbf{a}$ and $\mathbf{b}$, respectively, where $\mathbf{a}$ and $\mathbf{b}$ are incomparable. For simplicity, we use the standard assumption that the language of the structures is relational. We build a Σ_1–equivalent structure C of degree $\mathbf{c}$, such that $\mathbf{c}$ is neither above $\mathbf{a}$ nor above $\mathbf{b}$.

The universe of C will be ω. At each stage s we define a finite substructure C_s with the universe an initial segment of ω. To make sure that C computes neither $\mathcal{A}$ nor $\mathcal{B}$, we as usually consider the list of requirements of the form $\Phi_e^C \neq \mathcal{A}$ and $\Phi_e^C \neq \mathcal{B}$. Assume that the next requirement is of the form $\Phi_e^C \neq \mathcal{A}$, so we want to diagonalize against C computing $\mathcal{A}$. Let $\{\mathcal{N}_j\}_{j\in\omega}$ be a list of finite structures, such that each $\mathcal{N}_j$:

- extends C_s,
- has the universe an initial segment of ω,
- is isomorphic to a finite substructure of $\mathcal{B}$ in a finite language,
- every such substructure of $\mathcal{B}$ appears in the list.

Obviously, we can construct such a list computable in $\mathcal{B}$. Now we ask if there are n and $\mathcal{N}_j$ such that $\Phi^{\mathcal{N}_j}(n) \downarrow \neq \mathcal{A}(n)$. If the answer is positive, we let C_{s+1} be equal to such $\mathcal{N}_j$. So the requirement $\Phi_e^C \neq \mathcal{A}$ will be satisfied.

On the other hand, if the answer is negative, then for all n and $\mathcal{N}_j$ either $\Phi^{\mathcal{N}_j}(n) \uparrow$ or $\Phi^{\mathcal{N}_j}(n) \downarrow = \mathcal{A}(n)$. Suppose that in the end of the construction Φ_e^C is everywhere defined. Then for every n

there exists an $\mathcal{N}_j$ such that $\Phi^{\mathcal{N}_j}(n) \downarrow= \mathcal{A}(n)$. So we can compute $\mathcal{A}$ from $\mathcal{B}$, which is a contradiction. Therefore, in this case Φ_e^{C} must be partial, and the requirement is again satisfied.

Note that the above construction guarantees that every substructure of C in a finite sublanguage appears in $\mathcal{A}$ and $\mathcal{B}$. To ensure that $C \equiv_{\Sigma_1} \mathcal{A}, \mathcal{B}$, we also add stages where we extend the previously built C_s to include the next finite substructure of $\mathcal{A}$ or $\mathcal{B}$. $\square$

Theorem 34. *There is a structure $\mathcal{A}$ with* $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_2})$ *equal to the union of two non-degenerate cones.*

Proof. If we allow infinite languages, the statement follows directly from the result of Andrews and Miller [2], where they build a theory T with the spectrum of T consisting of exactly two cones. Let $\mathcal{A}$ be a model of T and let $\mathcal{B} \equiv_{\Sigma_2} \mathcal{A}$. The theory T is a complete theory that can be axiomatized using Σ_2- and Π_2-sentences. Thus, $\mathcal{B}$ is also a model of T. In other words, $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_2}) = \mathrm{DgSp}(\mathcal{A}, \equiv)$, which is the union of two cones.

The result is also true for finite languages, for example, using the transformation from [50] of arbitrary structures into graphs. It is not hard to show that the transformation preserves Σ_2-equivalence. A formal proof of a more general fact about preservation of Σ_n-spectra, $n \in \omega$ under effective transformations can be found in [27]. $\square$

4.2 All but computable

According to [76] and [82], there exist structures with the classical degree spectrum containing exactly all the non–computable degrees. Moreover, as the structure from [76] is not elementary equivalent to a computable structure, the built example actually shows that the degree spectrum of the theory of the constructed structure consists of all the non–computable degrees.

The theory of the structure built in [76] is Σ_3- and Π_3-axiomatizable, however minor modifications can make it axiomatizable using Σ_2- and Π_2-sentences.

Theorem 35. *There exists a countable structure $\mathcal{A}$, such that* $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_2})$ *consists of exactly all the non–computable Turing degrees. The same is also true for* $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_n})$*, for all $n \geqslant 2$.*

On the other hand, for Σ_1-spectra this is again not true:

Proposition 36. *No structure $\mathcal{A}$ may have its Σ_1-spectrum consisting of exactly the non-computable degrees.*

Proof. The Σ_1-spectrum of any structure $\mathcal{A}$ has the form $\{\mathbf{d} \mid X \text{ is } \mathbf{d}\text{-c.e.}\}$, where X is the set of Gödel indices of the sentences from the Σ_1-theory of $\mathcal{A}$. As shown in [17], if the collection of oracles that enumerate any set X has positive measure, then X is c.e. So, if $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_1})$ contains all non–computable degrees, then the Σ_1-theory of $\mathcal{A}$ is c.e. It is not hard to show that if a Σ_1-theory is c.e., then it has a computable model (see Theorem 40 below for a more general statement). This completes the proof of the proposition. □

Similar considerations prove the following:

Corollary 37.

1. *If* $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_1})$ *contains all non–computable c.e. degrees, it also contains* $\mathbf{0}$.
2. *If* $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_1})$ *contains all low degrees, it also contains* $\mathbf{0}$.
3. *If* $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_1})$ *contains all high degrees, it also contains* $\mathbf{0}$.
4. *If* $\mathrm{DgSp}(\mathcal{A} \equiv_{,\Sigma_1})$ *contains all PA degrees, it also contains* $\mathbf{0}$.
5. *If* $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_1})$ *contains all degrees above* $\mathbf{a}$*, it also contains* $\mathbf{a}$.

Proposition 36 and Corollary 37 can also be proved by coding a special kind of a minimal pair of degrees into the above collections of degrees.

Definition 38. The sets X and Y form a Σ_1-*minimal pair* if $\Sigma_1(X) \cap \Sigma_1(Y) = \Sigma^0_1$.

For example, if the set of all non-computable degrees were a Σ_1-spectrum, there would exist structures $\mathcal{A}, \mathcal{B}$ of degrees $\mathbf{a}, \mathbf{b}$, respectively, where $\mathbf{a}$ and $\mathbf{b}$ form a Σ_1-minimal pair. As the Σ_1-theory T_{Σ_1} is c.e. in $\mathcal{A}$ and in $\mathcal{B}$, it must be c.e. In this case it must have a computable model, so the Σ_1-spectrum must contain $\mathbf{0}$. Analogously for results from Corollary 37. A similar idea was used in [2] to prove that certain collections of degrees are not structure spectra.

We use Σ_1-minimal pairs to prove that further collections of degrees cannot be Σ_n-degree spectra, for suitable $n \in \omega$. We need the following two facts.

Observation 39. For any C, if $A \oplus B$ is sufficiently generic, then $A \oplus C$ and $B \oplus C$ form a Σ^0_1-minimal pair over C. That is, $\Sigma^0_1(A \oplus C) \cap \Sigma^0_1(A \oplus C) = \Sigma^0_1(C)$.

Theorem 40. *If T is a complete consistent theory in computable language $\mathcal{L}$, and S is the Σ_n-fragment of T (equivalently, S is the Σ_n-theory of a structure), and S is c.e., then S has a computable model.*

Proof. We perform an effective Henkin construction. Let our universe be $\{c_i\}_{i\in\omega}$, and let $\{\exists\overline{x}\varphi_i(\overline{x})\}_{i\in\omega}$ be an enumeration of all Σ_n-sentences in $\mathcal{L}$, where φ_i is a Π_{n-1}-formula. Let $\{\theta_i\}_{i\in\omega}$ be an enumeration of all Σ_{n-1}-sentences in $\mathcal{L}\cup\{c_i\}_{i\in\omega}$. We will compute the $(n-1)$-diagram of our structure.

During the construction, we will have a set of sentences δ_s, which is the fragment of the diagram we have committed to so far. We begin with $\delta_0=\emptyset$. We also keep a stage t_s which is the stage we have enumerated S to. We begin with $t_0=0$.

At stage $s+1$, let $\hat{\delta}_s$ be made from δ_s by replacing the constant for c_i with the new variable y_i, and similarly $\hat{\theta}_s(\overline{y})$ (where the same substitution $c_i\mapsto y_i$ is made).

Define the following:

$$\psi_t^{s,+} = \exists\overline{y}\exists\overline{z}\left[\begin{array}{c}\hat{\theta}_s(\overline{y})\wedge\left(\bigwedge_{\rho\in\hat{\delta}_s}\rho(\overline{y})\right)\wedge\left(\bigwedge_{\exists\overline{x}\tau(\overline{x},\overline{y})\in\hat{\delta}_s}(\exists\overline{w}\in\overline{z})\tau(\overline{w},\overline{y})\right)\\ \wedge\left(\bigwedge_{\substack{i<s\\ \exists\overline{x}\varphi_i(\overline{x})\in S_t}}(\exists\overline{w}\in\overline{yz})\varphi_i(\overline{w})\right)\wedge\left(\bigwedge_{\substack{i<s\\ \exists\overline{x}\varphi_i(\overline{x})\notin S_t}}(\forall\overline{w}\in\overline{yz})\neg\varphi_i(\overline{w})\right)\end{array}\right],$$

$$\psi_t^{s,-} = \exists\overline{y}\exists\overline{z}\left[\begin{array}{c}\neg\hat{\theta}_s(\overline{y})\wedge\left(\bigwedge_{\rho\in\hat{\delta}_s}\rho(\overline{y})\right)\wedge\left(\bigwedge_{\exists\overline{x}\tau(\overline{x},\overline{y})\in\hat{\delta}_s}(\exists\overline{w}\in\overline{z})\tau(\overline{w},\overline{y})\right)\\ \wedge\left(\bigwedge_{\substack{i<s\\ \exists\overline{x}\varphi_i(\overline{x})\in S_t}}(\exists\overline{w}\in\overline{yz})\varphi_i(\overline{w})\right)\wedge\left(\bigwedge_{\substack{i<s\\ \exists\overline{x}\varphi_i(\overline{x})\notin S_t}}(\forall\overline{w}\in\overline{yz})\neg\varphi_i(\overline{w})\right)\end{array}\right].$$

where "$\exists\overline{w}\in\overline{yz}$" means there is a tuple of the appropriate length made from the elements of the tuples $\overline{y}$ and $\overline{z}$, and similarly for "$\forall\overline{w}\in\overline{yz}$". Note that both $\psi_t^{s,+}$ and $\psi_t^{s,-}$ are Σ_n-sentences in $\mathcal{L}$. We enumerate S until we see some $\psi_t^{s,+}$ or $\psi_t^{s,-}$ enumerated with $t>t_s$. We will argue in the verification that this must eventually occur.

Suppose we have seen $\psi_t^{s,+}$ be enumerated. Fix some tuple $\overline{c}\in\{c_i\}_{i\in\omega}$ with $|\overline{c}|=|\overline{z}|$ and none of $\overline{c}$ occurring in δ_s or θ_s. Fix a bijection between $\overline{c}$ and $\overline{z}$. Define the map f such that for $z\in\overline{z}$, $f(z)$ follows this bijection, and for y_j, $f(y_j)=c_j$. Note that this is an injection from the variables occurring in $\overline{yz}$ into $\{c_i\}_{i\in\omega}$.

For every sentence $\exists\overline{x}\varphi_i(\overline{x})\in S_t$, fix a witnessing tuple $\overline{w}_i$. Note that we can identify such $\overline{w}$ effectively: since "$\exists\overline{w}\in\overline{yz}$" is a finite disjunction, we can make more specific versions of $\psi_t^{s,+}$ by retaining only a single disjunct for every φ_i. Eventually, one of these more specific sentences must be enumerated. Similarly, for every sentence $\exists\overline{x}\tau(\overline{x},\overline{y})\in\delta_s$, fix a witnessing tuple $\overline{w}_\tau$.

Define $t_{s+1}=t$ and

$$\begin{aligned}\delta_{s+1} &= \delta_s\cup\{\theta_s\}\cup\{\tau(f(\overline{w}_\tau),f(\overline{y})):\exists\overline{x}\tau(\overline{x},\overline{y})\in\hat{\delta}_s\}\\ &\quad\cup\{\varphi_i(f(\overline{w_i})):i<s\ \&\ \exists\overline{x}\varphi_i(\overline{x})\in S_t\}\\ &\quad\cup\{\neg\varphi_i(f(\overline{w})):i<s\ \&\ \overline{w}\in\overline{yz}\ \&\ \exists\overline{x}\varphi_i(\overline{x})\notin S_t\}.\end{aligned}$$

If instead $\psi_t^{s,-}$ is enumerated, proceed similarly except define δ_{s+1} with $\neg\theta_s$ instead of θ_s. Once t_{s+1} and δ_{s+1} are defined, proceed on to stage $s+2$.

Verification:

Claim 40.1. *For every s, $\exists\overline{y} \bigwedge_{\rho\in\hat{\delta}_s} \rho(\overline{y}) \in S$.*

Proof. Induction. □

In particular, the diagram $D = \{\delta_s\}_{s\in\omega}$ we build is consistent.

Claim 40.2. *For every s, we will eventually see some $\psi_t^{s,+}$ or $\psi_t^{s,-}$ enumerated into S.*

Proof. We know that $\exists\overline{y}\hat{\delta}_s(\overline{y})$ is in S and thus in T. Since T is complete, at least one of $\exists\overline{y}(\hat{\delta}_s(\overline{y}) \wedge \hat{\theta}_s(\overline{y}))$ or $\exists\overline{y}(\hat{\delta}_s(\overline{y}) \wedge \neg\hat{\theta}_s(\overline{y}))$ is in T, and by counting quantifiers, must thus be in S.

Let t be such that $S_t \upharpoonright_i = S \upharpoonright_i$. Then at least one of $\psi_t^{s,+}$ or $\psi_t^{s,-}$ is in T, and thus is in S. □

Claim 40.3. *D is computable.*

Proof. We decide θ_s at stage s. □

Let $\mathcal{M}$ be the structure with universe $\{c_i\}_{i\in\omega}$ determined by the quantifier-free fragment of D.

Claim 40.4. *$\mathcal{M} \models D$.*

Proof. Induction on sentence complexity. For quantifier-free sentences, this is immediate.

Suppose $\exists\overline{x}\tau(\overline{x}, \overline{b}) \in D$. Then at some sufficiently large stage, we act to put $\tau(\overline{c}, \overline{b}) \in D$ for some $\overline{b}$. By the inductive hypothesis, $\mathcal{M} \models \tau(\overline{c}, \overline{b})$, so $\mathcal{M} \models \exists\overline{x}\tau(\overline{x}, \overline{b})$.

Suppose $\forall\overline{x}\tau(\overline{x}, \overline{b}) \in D$. Then for any $\overline{c}$, it cannot be that $\neg\tau(\overline{c}, \overline{b}) \in D$, as that would violate the consistency of D. Since we eventually act to decide $\theta = \tau(\overline{c}, \overline{b})$, it must be that $\tau(\overline{c}, \overline{b}) \in D$. By the inductive hypothesis, $\mathcal{M} \models \tau(\overline{c}, \overline{b})$. Since $\overline{c}$ was arbitrary, $\mathcal{M} \models \forall\overline{x}\tau(\overline{x}, \overline{b})$. □

Claim 40.5. *$\mathcal{M} \models S$.*

Proof. If $\exists\overline{x}\varphi_i(\overline{x}) \in S_t$, then at any stage with $i < s$ and $t < t_s$, we will place the sentence $\varphi_i(\overline{c})$ in D for some $\overline{c}$, and thus $\mathcal{M} \models \exists\overline{x}\varphi_i(\overline{x})$.

If $\exists\overline{x}\varphi_i(\overline{x}) \notin S$, then at every stage with $i < s$, we will place the sentence $\neg\varphi_i(\overline{c})$ in D for every $\overline{c}$ mentioned so far in the construction. Thus $\mathcal{M} \not\models \varphi_i(\overline{c})$ for any $\overline{c}$, and so $\mathcal{M} \not\models \exists\overline{x}\varphi_i(\overline{x})$. □

This completes the proof. □

We now use Observation 39 und Theorem 40 to prove that non-Δ^0_n-degrees cannot be a Σ_n-spectrum.

Theorem 41. *The non-Δ^0_n degrees are not the Σ_n-spectrum of any structure.*

Proof. Suppose there were a structure $\mathcal{M}$ with $\mathrm{Spec}_{\Sigma_n}(\mathcal{M})$ consisting precisely of the non-Δ^0_n degrees. Using Observation 39, fix degrees $\mathbf{a}$ and $\mathbf{b}$ forming a Σ^0_1-minimal pair over $\mathbf{0}^{(n-1)}$, with $\mathbf{a}$ and $\mathbf{b}$ not arithmetical. By jump inversion, there are degrees $\mathbf{c}$ and $\mathbf{d}$ with $\mathbf{c}^{(n-1)} = \mathbf{a}$ and $\mathbf{d}^{(n-1)} = \mathbf{b}$, and neither $\mathbf{c}$ nor $\mathbf{d}$ are arithmetical.

By assumption, $\mathbf{c}, \mathbf{d} \in \mathrm{Spec}_{\Sigma_n}(\mathcal{M})$. Let S be the Σ_n-theory of $\mathcal{M}$. Then $S \in \Sigma^0_n(\mathbf{c}) = \Sigma^0_1(\mathbf{a})$ and also $S \in \Sigma^0_n(\mathbf{d}) = \Sigma^0_1(\mathbf{b})$. Since $\mathbf{a}$ and $\mathbf{b}$ form a Σ^0_1-minimal pair over $\mathbf{0}^{(n-1)}$, $S \in \Sigma^0_1(\mathbf{0}^{(n-1)})$, and thus by Theorem 40 $\mathbf{0}^{(n-1)}$ can compute a model of S. This model has Δ^0_n-degree, contrary to assumption. □

4.3 A non–trivial spectrum for Σ_1–equivalence

In view of the results about Σ_1-spectra from the previous two sections, we study the question of existence of Σ_1–spectra that are not cones.

Theorem 42. *There exists a structure $\mathcal{A}$ such that its Σ_1–spectrum* $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_1})$ *cannot be presented as a cone above a degree* $\mathbf{a}$.

Proof. As we already noted above, Σ_1-spectra must have the form $\{\mathbf{d} \mid X \text{ is } \mathbf{d}\text{-c.e.}\}$, where X is the set of Gödel indices of the sentences from the Σ_1-theory. On the other hand, every set of degrees of the form $\{\mathbf{d} \mid X \text{ is } \mathbf{d}\text{-c.e.}\}$, for some X, is a Σ_1-spectrum of a structure $\mathcal{A}_X$: the structure $\mathcal{A}_X$ contains an ω–chain $x_0, x_1, \dots$ using a binary predicate $P(x_n, x_{n+1})$ (and a constant that fixes x_0 as the first element of the chain). Whenever n is enumerated into X, we define $Q(x_n, y_n)$, where y_n is a new element that from now on witnesses $n \in X$. It is clear that $\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_1}) = \{\mathbf{d} \mid X \text{ is } \mathbf{d}\text{-c.e.}\}$.

Richter studied sets of this form in [75]. She constructed a non-computably enumerable set X, which is computably enumerable in sets B and C forming a minimal pair. Thus, the degrees that enumerate X do not form a cone. The corresponding structure $\mathcal{A}_X$, built as described above, witnesses the statement of the theorem. □

4.4 Relations between Σ_n-spectra

In this section we study relations between Σ_n-spectra, for various n.

Proposition 43. *If S is a Σ_n–spectrum then $\{\mathbf{d} \mid \mathbf{d}' \in S\}$ is a Σ_{n+1}–spectrum.*

Proof. The proof is essentially the same as the proof of Lemma 2.8 in [2] which is based on Marker's construction already discussed in Chapter 3. In that lemma it is proved that if S is a theory spectrum, then so is $\{\mathbf{d} \mid \mathbf{d}' \in S\}$. Recall that the idea of the Marker's construction is to build a new theory T' in such a way that every predicate of the original theory T is interpreted by both Σ_2- and Π_2-formula in T'. Using this, one can make sure that for an arbitrary sentence φ from T, the number of quantifier alternations in its interpretation φ' in T' increases only by one. Therefore, if the original theory is axiomatizable by Σ_n- or Π_n-sentences, then the new theory is axiomatizable by Σ_{n+1}- or Π_{n+1}-sentences. □

This result allows us to prove that some collections of degrees are Σ_n-spectra.

Proposition 44. *Non-low$_n$ degrees form a Σ_{n+2}-spectrum.*

Proof. By Theorem 35, the set of degrees strictly above $\mathbf{0}^{(n)}$ is a Σ_2-spectrum. Applying Proposition 43 n times we get the desired result. □

Proposition 45. *The hign$_n$ degrees form a Σ_{n+1}-spectrum of a structure.*

Proof. We build a structure $\mathcal{A}$ with its Σ_{n+1}-spectrum consisting of exactly the high$_n$ degrees. Let $\mathcal{B}$ be a structure that has the Σ_1-spectrum of the form $\{\mathbf{d} : \mathbf{d} \geqslant_T \mathbf{0}^{(n+1)}\}$. Applying Proposition 43 n times, we get $\mathcal{A}$ with the desired Σ_{n+1} spectrum. □

Recall that by Corollary 37, high degrees do not form a Σ_1–spectrum. We are going to extend this result by showing that high$_n$ degrees never form a Σ_n-spectrum.

Proposition 46. *The high$_n$ degrees do not form a Σ_n-spectrum of a structure.*

To show that, we compare the descriptive complexity of $\{X \in \omega^\omega : X \text{ is high}_n\}$ and $\{X \in \omega^\omega : X \in S\}$, where S is a Σ_n-spectrum.

Proposition 47. *Let T be a Σ_n-fragment of a (complete) theory. Then*

$$\{X : X \textit{ computes (the atomic diagram of) a model of } T\}$$

is a $\mathbf{\Sigma}^0_{n+2}$-class.

Proof. X computes a model of T iff

$$\exists\Phi\forall\varphi \in \Sigma_n[\varphi \in T \iff \Phi^X \models \varphi].$$

Here Φ^X is the X-computable structure computed by Φ with oracle X. Then for a Σ_n sentence φ, the complexity of "$\Phi^X \models \varphi$" is $\Sigma_n^{0,X}$. Considering T as a parameter, we get the desired complexity $\mathbf{\Sigma}^0_{n+2}$. □

Theorem 48. *$\{X \in \omega^\omega \ : \ X$ is high$_n\}$ is not a $\mathbf{\Sigma}^0_{n+2}$-class.*

The proof of the theorem can be found in [28], we omit it in this thesis.

4.5 Σ_n-spectra and theory spectra

We now prove that there is a theory spectrum that is not a Σ_n-spectrum, for any $n \geqslant 1$.

Definition 49. For $n \in \omega$, let $\mathcal{F} = \{X \in 2^\omega \ : \ (\exists\Phi)(\forall n)[\Phi(X^{(n)} \oplus \{n\}) = \emptyset^{(2n)}]\}$.

Theorem 50. *$\mathcal{F}$ is not the Σ_k-spectrum of any structure $\mathcal{M}$ for any $k \in \omega$.*

Proof. Suppose not, and fix witnessing M and k. By a standard Friedberg jump inversion construction, fix $\mathbf{a}$ and $\mathbf{b}$ forming a minimal pair over $\mathbf{0}^{(3k)}$ with $\mathbf{a}' = \mathbf{b}' = \mathbf{0}^{(\omega)}$. By jump inversion again, there are $\mathbf{c}$ and $\mathbf{d}$ both above $\mathbf{0}^{(2k)}$ with $\mathbf{c}^{(k)} = \mathbf{a}$ and $\mathbf{d}^{(k)} = \mathbf{b}$.

Note that $\mathbf{c} \in \mathcal{F}$: for $C \in \mathbf{c}$, if $n \leq k$, $C^{(n)} \geqslant_T C \geqslant_T \emptyset^{(2k)} \geqslant_T \emptyset^{(2n)}$; if $n > k$, $C^{(n)} \geqslant_T C^{(k+1)} = \emptyset^{(\omega)} \geqslant_T \emptyset^{(2n)}$. Further, all of these reductions are uniform. Similarly, $\mathbf{d} \in \mathcal{F}$. Thus there is an $M_\mathbf{c} \in \mathbf{c}$ and an $M_\mathbf{d} \in \mathbf{d}$ with

$$\mathrm{Th}_{\Sigma_k}(\mathcal{M}_\mathbf{c}) = \mathrm{Th}_{\Sigma_k}(\mathcal{M}_\mathbf{d}) = \mathrm{Th}_{\Sigma_k}(\mathcal{M}).$$

Then $\mathrm{Th}_{\Sigma_k}(\mathcal{M}) \in \Sigma^0_k(\mathbf{c}) \subset \Delta^0_1(\mathbf{a})$, and $\mathrm{Th}_{\Sigma_k}(\mathcal{M}) \in \Sigma^0_k(\mathbf{d}) \subset \Delta^0_1(\mathbf{b})$. By our choice of $\mathbf{a}$ and $\mathbf{b}$, $\mathrm{Th}_{\Sigma_k}(\mathcal{M}) \in \Delta^0_1(\mathbf{0}^{(3k)})$, and so there is a $\mathbf{0}^{(3k)}$-computable model of $\mathrm{Th}_{\Sigma_k}(\mathcal{M})$. But clearly no arithmetical degree can be in $\mathcal{F}$, which is a contradiction. □

Theorem 51. *There is a structure $\mathcal{M}$ with* $\mathrm{DgSp}(\mathcal{M}, \cong) = \mathrm{DgSp}(\mathcal{M}, \equiv) = \mathcal{F}$.

Proof. Our structure will be an effective disjoint union $\mathcal{M} = \bigsqcup_{n\in\omega} \mathcal{M}_n$. In $\mathcal{M}_n$, we will code $\emptyset^{(2n)}$ in a manner than can be decoded by the nth jump. Our language for $\mathcal{M}_n$ will be $\{P_i, N_i\}_{i\in\omega} \cup \{\rightarrow\}$, where the P_i and N_i are unary relations, and $\rightarrow$ is a binary relation.

We recall the following trees (in the language of directed graphs), originally due to Hirschfeldt and White [52]:

- A_1 is the tree consisting of only the root;
- E_1 is the tree where the root has infinitely many children, and all of these children are leaves;

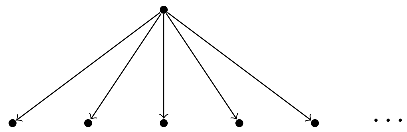


Figure 4.1: The tree E_1.

- A_{k+1} is the tree where the root has infinitely many children all of whose subtrees are a copy of E_k;

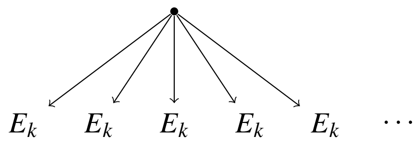


Figure 4.2: The tree A_{k+1}.

- E_{k+1} is the tree where the root has infinitely many children whose subtrees are a copy of E_k, and also has infinitely many children whose subtrees are a copy A_k.

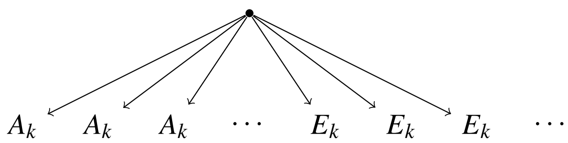


Figure 4.3: The tree E_{k+1}.

Hirschfeldt and White showed that given a Σ^0_k predicate, one can computably construct a tree T which is isomorphic to E_k if the predicate holds, and is isomorphic to A_k if it fails, and further this construction is uniform in an index for the predicate.

Also, there is a first-order Σ_k formula that holds of the root of the E_k tree, but does not hold of the root of the A_k tree. We define these recursively: define $\varphi_1(x) : \exists z[x \to z]$; define $\varphi_{k+1}(x) : \exists z[x \to z \wedge \neg\varphi_k(z)]$.

We now construct $\mathcal{M}_n$ as follows: for each i, there is a unique element x with $\mathcal{M} \models P_i(x)$, and x is the root of a tree of type E_{n+1} if $i \in \emptyset^{(2n)}$ and of type A_{n+1} if $i \notin \emptyset^{(2n)}$; conversely there is a unique element y with $\mathcal{M} \models N_i(y)$, and y is the root of a tree of type A_{n+1} if $i \in \emptyset^{(2n)}$ and of type E_{n+1} if $i \notin \emptyset^{(2n)}$.

We claim that if $X \in \mathcal{F}$, then X uniformly computes a copy of $\mathcal{M}_n$. For $\emptyset^{(2n)} \in \Delta^0_{n+1}(X)$, and thus for the x and y with $P_i(x)$ and $N_i(y)$, we can construct the trees rooted at x and y computably relative to X as described above.

Conversely, we claim that if X uniformly computes structures $(L_n)_{n\in\omega}$ with L_n elementarily equivalent to $\mathcal{M}_n$, then $X \in \mathcal{F}$. For

$$i \in \emptyset^{(2n)} \iff (\exists x \in L_n)[P_i(x) \wedge \varphi_{n+1}(x)] \iff (\forall y \in L_n)[N_i(y) \Rightarrow \neg\varphi_{n+1}(y)].$$

Thus $\emptyset^{(2n)} \in \Delta^0_{n+1}(X)$, and further the code is obtained uniformly. □

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Wien, am 05. Oktober 2016