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Krein-Rutman Theorem on the Spectrum of Compact Positive Operators on Ordered Banach Spaces

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Ao.Univ.Prof. Dipl.-Ing. Dr.Techn. Michael KALTENBÄCK

durch

Borbala Mercedes GERHAT, B.Sc.
Hötzendorfstraße 131, 2345 Brunn am Gebirge, Austria

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Preface

In the beginning of the 20^{th} century, the German mathematicians Oskar Perron (1880-1975) and Georg Frobenius (1849-1917) were investigating the spectrum of real matrices with positive entries. Independently from one another, they established a collection of statements now referred to as Perron-Frobenius Theorem; Perron published his work [P] in 1907, Frobenius [F] in 1912.

They both proved, that the spectral radius $r(A)$ of an irreducible, non-negative matrix $A \in \mathbb{R}^{n\times n}$ is a positive and algebraically simple eigenvalue of A. Moreover, there exists a positive eigenvector of A associated to $r(A)$. Its multiples are the only possible non-negative eigenvectors, i.e. other eigenvalues cannot have corresponding non-negative eigenvectors. Furthermore, the work of Oskar Perron contains the so-called Perron Theorem with the more stringent requirement of the matrix $A \in \mathbb{R}^{n\times n}$ to be positive. In this case, in addition to the above mentioned properties, the spectral radius $r(A)$ is the only eigenvalue with maximal absolute value.

The obvious question, whether the results of Perron and Frobenius can be generalized to infinite dimensions, was addressed by the Soviet mathematician Mark Grigor'evich Krein (1907-1989) and his student Mark Aronovich Rutman. In the year 1948, they published their work [KR] regarding the spectrum of positive, compact linear operators on real, ordered Banach spaces. It generalizes the statements of the Perron-Frobenius Theorem to infinite dimensions. As its title reveals, this thesis is centered in and around the results in [KR, Th 6.1, Th 6.3], which are referred to as the Krein-Rutman Theorem. Therein, positive and compact linear operators $T \in \mathcal{B}(X)$ on a real, ordered Banach space X replace the positive matrices considered by Perron and Frobenius. As usual, $\mathcal{B}(X)$ denotes the set of all linear and bounded operators on X.

Under the assumption $r(T) > 0$, where $T \in \mathcal{B}(X)$ is considered positive and compact, Krein and Rutman proved the existence of a positive eigenvector of both T and its adjoint $T' \in \mathcal{B}(X')$ associated with the eigenvalue $r(T) \in \sigma(T) = \sigma(T')$. In case that

T is even assumed strongly positive, the spectral radius is a positive and algebraically simple eigenvalue of T and T', with corresponding strictly positive eigenvectors. Moreover, it is the only eigenvalue of maximal absolute value and the only one associated to positive eigenvectors. Remarkably, all statements of Perron and Frobenius can indeed be adapted to infinite dimensions.

This work is set up in six chapters, four of which serve as introduction and preparation for the results of the last two, which contain the actual Krein-Rutman Theorem and the setting it is placed in.

The first chapter is dedicated to the Perron-Frobenius Theorem. It gives the reader a short insight in the finite dimensional setting we aim to generalise. First, the Theorem of Perron for positive matrices is presented, before stating the more general Perron-Frobenius Theorem for non-negative ones.

Since the Krein-Rutman Theorem involves compact operators, some of their properties are recalled in the second chapter. Furthermore, we introduce the ascent, descent and Riesz number of operators. As a later on relevant result, we show that $(T - \lambda I)$ has finite Riesz number, whenever $T \in \mathcal{B}(Y)$ is a compact operator on a Banach space Y and $\lambda \in \mathbb{C}\backslash\{0\}$.

As the Perron-Frobenius Theorem is stated for non-negative, thus real matrices, the Krein-Rutman Theorem is framed for operators on real Banach spaces. Unlike in finite dimensions, extending a given linear and bounded operator on a real Banach space to a $\mathbb{C}$-linear and bounded operator on a complex one, requires some considerations. The third chapter deals with the so-called complexification of real Banach spaces and the extensions of linear and bounded operators onto them. Since we apply common results of spectral theory in the proof of our main theorem, taking this detour over the complexification is necessary.

In the fourth chapter, we investigate the resolvent of a compact operator. As its central result, we show that every non-zero eigenvalue $\lambda \in \sigma(T)\backslash\{0\}$ of a compact linear operator $T \in \mathcal{B}(Y)$ on a complex Banach space Y is a pole of its resolvent. In order to do so, we recall the general concept of a Banach algebra and holomorphic functions with values in such, before stating a powerful functional calculus for them. Applying this calculus to the Banach algebra $\mathcal{B}(Y)$ allows us to relate the poles $\lambda \in \mathbb{C}$ of the resolvent of $T \in \mathcal{B}(Y)$ to the previously introduced Riesz number of the operator $(T - \lambda I)$, which is finite for compact T and $\lambda \neq 0$.

Chapter five develops the setting of the Krein-Rutman Theorem. By introducing order cones $K \subseteq X$, an order relation on the real Banach space X is defined, as well as the notion of positive, strictly and strongly positive operators $T \in \mathcal{B}(X)$. The topological dual space X' is then naturally equipped with an order relation induced by defining the dual order cone $K' \subseteq X'$.

The last chapter contains the core of this thesis. We present the Krein-Rutman Theorem in two parts, corresponding to the Theorems [KR, Th 6.1, Th 6.3]. First, we elaborate the above mentioned statements for positive, compact operators with positive spectral radius, in order to apply them to the case of a strongly positive, compact operator in the second part of the theorem. Finally, we present some applications of the Krein-Rutman Theorem to operator equations, completing this excursion to the spectral theory of positive operators.

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I am grateful to all my friends, who made sure the last years were full of enriching experiences that I will never forget. They naturally put an upper bound to my working hours, which without them would have diverged. I want to thank Felipe for his loving support and for patiently enduring my stress induced mood swings. And finally, my deepest gratitude goes to my loving parents, Marta and Zoltan, whose unmeasurable effort and support allowed me to achieve my goals so far. And of course, to my sister Dora and her husband Ben, who always had an open home and heart for me.

... and to Kitty, for letting me pet her without scratching.

Borbala Mercedes Gerhat Vienna, September 12, 2016

Chapter 1

Motivation in finite dimensions

As a motivation for our main result, the Krein-Rutman Theorem, we introduce the Theorem of Perron and Frobenius. It states properties of the spectra of positive, real matrices. As is well known, real and complex matrices correspond to linear (and bounded) operators on finite dimensional real and complex Banach spaces, respectively. We will exclusively consider $\mathbb{R}^n$ and $\mathbb{C}^n$, $n \in \mathbb{N}$, with an arbitrary norm, knowing that they are isomorphic to any finite dimensional real or complex Banach space. In the framework of the Krein-Rutman Theorem, we will later generalise the results presented in this chapter to the infinite dimensional case.

Let us start by recalling some linear algebra, in particular on the spectra of complex matrices. All the presented facts can be found proven in [Ha].

Consider a complex matrix $A \in \mathbb{C}^{n\times n}$, which represents an endomorphism of $\mathbb{C}^n$. Corresponding to the spectrum of a general linear operator, the *spectrum* of A consists of all $\lambda \in \mathbb{C}$, such that the matrix $(\lambda I - A) \in \mathbb{C}^{n\times n}$ is singular, i.e. not invertible. This is equivalent to $(\lambda I - A)$ having a non-trivial kernel,

$$\sigma(A) := \{\, \lambda \in \mathbb{C} \,:\, \ker(\lambda I - A) \neq \{0\} \,\}\,.$$

Every $x \in \ker(\lambda I - A)\backslash\{0\}$ is an *eigenvector* associated to the *eigenvalue* $\lambda \in \sigma(A)$. As singularity of the matrix $(\lambda I - A)$ is equivalent to $\det(\lambda I - A) = 0$, the eigenvalues of A are given by the zeros of the *characteristic polynomial*

$$p_A(z) := \det(\lambda I - A) \in \mathbb{C}[z].$$

The *algebraic multiplicity* $\mathrm{alg}_A(\lambda)$ of an eigenvalue $\lambda \in \sigma(A)$ is defined as the multiplicity of λ as a zero of the characteristic polynomial, whereas its *geometric multiplicity* is given by

$$\mathrm{geom}_A(\lambda) := \dim\ \ker(\lambda I - A).$$

Note, that in accordance with the infinite dimensional case, the algebraic multiplicity of $\lambda \in \sigma(A)$ equals the smallest number $p \in \mathbb{N}$ satisfying $\ker(\lambda I - A)^p = \ker(\lambda I - A)^{p+1}$; see Remark 2.2.8. Clearly,

$$\sum_{\lambda \in \sigma(A)} \mathrm{alg}_A(\lambda) = n$$

and $\mathrm{geom}_A(\lambda) \leq \mathrm{alg}_A(\lambda) \leq n$ for $\lambda \in \sigma(A)$. The quantity we are going to investigate is the *spectral radius* of A, defined as

$$r(A) := \max_{\lambda \in \sigma(A)} |\lambda|.$$

Consider an arbitrary norm $\|\cdot\|$ on $\mathbb{C}^n$. Recall, that all norms on $\mathbb{C}^n$ are equivalent and equipped with every one of them, $\langle \mathbb{C}^n, \|\cdot\| \rangle$ becomes a Banach space. The norm $\|\cdot\|$ induces a *matrix norm* on $\mathbb{C}^{n \times n}$,

$$\|A\| := \max_{x \in \mathbb{C}^n \setminus \{0\}} \frac{\|Ax\|}{\|x\|},$$

corresponding to the operator norm in infinite dimensions. Independently of the underlying norm $\|\cdot\|$ given on $\mathbb{C}^n$, one easily proves $r(A) \leq \|A\|$ and

$$r(A) = \lim_{m \to \infty} \|A^m\|^{\frac{1}{m}}; \tag{1.0.1}$$

see the corresponding result of Theorem 4.1.1 (ii) in infinite dimensions.

1.1 Perron Theorem for positive matrices

As the predecessor of the Perron-Frobenius Theorem, we start elaborating the Theorem of Perron. Since it concerns the spectrum of a positive matrix, we need to introduce an order on $\mathbb{C}^{n \times n}$.

Consider $A = (a_{ij}) \in \mathbb{C}^{n \times m}$. Then A is called

- *positive*, denoted $A > 0$, if $a_{ij} > 0$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$.
- *non-negative*, denoted $A \geq 0$, if $a_{ij} \geq 0$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$.

In both cases, obviously $A \in \mathbb{R}^{n \times m}$ and $A > 0$ implies $A \geq 0$. Accordingly, with another matrix $B = (B_{ij}) \in \mathbb{C}^{n \times m}$, one writes $A < B$ or $A \leq B$, if $B - A > 0$ or $B - A \geq 0$, respectively. As we will investigate their spectrum, we will only consider square matrices and their eigenvectors, i.e. the cases $m = n$ and $m = 1$.

Note, that in finite dimensions it is easily derived from (1.0.1), that $0 \leq A \leq B$ implies $r(A) \leq r(B)$ for quadratic matrices $A, B \in \mathbb{C}^{n\times n}$. As we will see, the corresponding result in infinite dimensions is a consequence of the Krein-Rutman Theorem.

Given a real matrix $A \in \mathbb{R}^{n\times n}$, it can be considered as a complex one, $A \in \mathbb{C}^{n\times n}$. By this identification, one implicitly extends the endomorphism $A : \mathbb{R}^n \to \mathbb{R}^n$ to an endomorphism $A : \mathbb{C}^n \to \mathbb{C}^n$. The spectrum of $A \in \mathbb{R}^{n\times n}$ is then naturally declared as the spectrum of $A \in \mathbb{C}^{n\times n}$. In the setting of the Krein-Rutman Theorem, a linear and bounded operator on a real Banach space will be given and it will be necessary to associate it with a corresponding operator on a complex Banach space. For matrices $A \in \mathbb{R}^{n\times n} \subseteq \mathbb{C}^{n\times n}$, this identification is obvious, while it is more complex in infinite dimensions; see Chapter 3.

Every non-negative matrix $A \geq 0$ has real entries and thus can be seen as an element of $\mathbb{R}^{n\times n}$. From now on, we will denote positive matrices as elements of $\mathbb{R}^{n\times n}$, in order to underline the parallels to the Krein-Rutman Theorem. Considerations about the spectrum are made via the identification $A \in \mathbb{C}^{n\times n}$.

Let us now state the Perron Theorem, providing a strong characterisation of the spectrum of a positive matrix.

Theorem 1.1.1 [Perron]. *Let $A \in \mathbb{R}^{n\times n}$ be positive, i.e. $A > 0$. The following assertions hold true:*

(i) *The spectral radius is a positive eigenvalue of A, i.e. $r(A) > 0$ and $r(A) \in \sigma(A)$. Moreover, it is* algebraically simple*, i.e.*

$$\mathrm{geom}_A(r(A)) = \mathrm{alg}_A(r(A)) = 1.$$

(ii) *There exists a positive eigenvector $x > 0$ associated to $r(A)$, i.e. $Ax = r(A)x > 0$.*

(iii) *Except the ones associated with the eigenvalue $r(A)$, there exist no other non-negative eigenvectors of A.*

(iv) *Every other eigenvalue has strictly smaller absolute value, i.e. $|\lambda| < r(A)$ for all $\lambda \in \sigma(A) \backslash \{r(A)\}$.*

(v) *The spectral radius can be computed by the* Collatz-Wielandt Formula*:*

$$r(A) = \max_{x\in\mathcal{N}} \min_{x_i\neq 0} \frac{(Ax)_i}{x_i},$$

where $\mathcal{N} := \{ x \in \mathbb{R}^n \backslash \{0\} : x \geq 0 \}$.

Proof. The proof of this theorem can be found e.g. in [M, Ch 8]. □

In the above setting, $r(A) > 0$ is called the *Perron root* of A. Consider the sum norm on $\mathbb{R}^n$,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \text{for} \quad x = (x_1, \dots, n)^T \in \mathbb{R}^n.$$

As the *eigenspace* $\ker(r(A)I - A)$ is one-dimensional, there exists a unique eigenvector

$$p > 0 \qquad \text{with} \qquad Ap = r(A)p \qquad \text{and} \qquad \|p\|_1 = \sum_{i=1}^{n} p_i = 1,$$

called the *right Perron vector* of A. Clearly, A is positive, if and only if the transpose $A^T \in \mathbb{R}^{n\times n}$ is positive, thus Theorem 1.1.1 can be applied to A^T. Since $A^T q = r(A)q$ is equivalent to $q^T A = r(A)q^T$, there exists a unique eigenvector

$$q > 0 \qquad \text{with} \qquad q^T A = r(A)q^T \qquad \text{and} \qquad \|q\|_1 = \sum_{i=1}^{n} q_i = 1,$$

called the *left Perron vector* of A.

Example 1.1.2. Let us give an elementary example for the statement of Theorem 1.1.1 in a two-dimensional setting. Consider the positive matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Then $\sigma(A) = \{0, 2\}$, i.e. $r(A) = 2$ is an algebraically simple eigenvalue. The corresponding eigenspace is given by $\ker(2I - A) = \operatorname{span}\{x\}$ with $x = (1,1)^T > 0$. Since A is symmetric, i.e. $A = A^T$, both the left and the right Perron vector of A are given by $p = q = (\frac{1}{2}, \frac{1}{2})^T$. //

Except the Collatz-Wielandt Formula, the Krein-Rutman Theorem correspondingly recovers all the results of Theorem 1.1.1 for compact and strongly positive operators $T \in \mathcal{B}(X)$ on infinite dimensional, real Banach spaces X; see Theorem 6.2.3.

1.2 Perron-Frobenius Theorem for non-negative matrices

Knowing that the spectrum of a positive matrix has the remarkable properties we presented in the previous section, the question is, whether or not they remain valid, if one weakens the assumptions of Theorem 1.1.1 to A only being non-negative. We will see, that a priori almost all properties obtained for the spectrum of $A > 0$ get lost by passing on to arbitrary non-negative matrices $A \geq 0$. In order to preserve as many statements of the Perron Theorem as possible, the given matrix will be required to have another property in addition.

Example 1.2.1. Let us consider some elementary examples in order to determine, which properties from the Perron Theorem cannot be saved without making further assumptions about the matrix A. Let $A_1, A_2, A_3 \geq 0$ be given by

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $\sigma(A_1) = \{0\}$, $\sigma(A_2) = \{-1, 1\}$ and $\sigma(A_3) = \{1, 2\}$. Thus, the spectral radii $r(A_1) = 0$, $r(A_2) = 1$ and $r(A_3) = 2$ are all eigenvalues of A_1, A_2 and A_3, respectively. Moreover, $\text{alg}_{A_1}(r(A_1)) = 2$ and $\text{geom}_{A_1}(r(A_1)) = 1$. The eigenvalues $r(A_2)$ of A_2 and $r(A_3)$ of A_3 are both algebraically simple. Corresponding eigenvectors of A_1, A_2 and A_3 are given by e.g.

$$x_1 = (1, 0)^T \geq 0, \qquad x_2 = (1, 1)^T \geq 0 \qquad \text{and} \qquad x_3 = (0, 1)^T \geq 0,$$

associated with $r(A_1)$, $r(A_2)$ and $r(A_3)$, respectively.

Let us compare the properties of A_1, A_2 and A_3 with Theorem 1.1.1. Considering A_1, we realize that the spectral radius of a non-negative matrix $A \geq 0$ does not have to be positive, neither does it have to be algebraically simple as an eigenvalue of A, i.e. (i) is partly violated. Regarding A_2, we see that other eigenvalues $\lambda \in \sigma(A)$ can exists with $|\lambda| = r(A)$, thus (iv) is not satisfied. As

$$x = (1, 0)^T \geq 0$$

is a non-negative eigenvalue of A_3 associated with the eigenvalue $1 \in \sigma(A) \backslash \{r(A)\}$, clearly statement (iii) does not hold, either. We observe, that the spectral radius is indeed an eigenvalue of all three considered matrices. Although (ii) is clearly violated, there at least exists a non-negative eigenvector associated with the spectral radius in all three cases. //

The above example gives reason to think, that the properties preserved for the matrices we considered remain valid for arbitrary $A \geq 0$. Indeed, the spectral radius is always an eigenvalue, $r(A) \in \sigma(A)$ for every $A \geq 0$. As we saw above, $r(A) = 0$ is possible. Moreover, there always exists an eigenvector $x \in \mathcal{N} = \{\, x \in \mathbb{R}^n \backslash \{0\} : x \geq 0 \,\}$ of A

associated to $r(A)$. Hence, statement (i) and (ii) are partly preserved, while (iii) and (iv) are completely lost. The Collatz-Wielandt Formula however, remains valid for arbitrary $A \geq 0$, i.e.

$$r(A) = \max_{x \in \mathcal{N}} \min_{x_i \neq 0} \frac{(Ax)_i}{x_i}.$$

In order to recover the other properties, $A \geq 0$ has to be assumed irreducible, a notion we will shortly introduce.

A matrix $A \in \mathbb{C}^{n \times n}$ is called *reducible*, if there exists a permutation matrix $P \in \mathbb{R}^{n \times n}$ with

$$P^T AP = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}, \tag{1.2.1}$$

where $X \in \mathbb{C}^{m \times m}$, $Y \in \mathbb{C}^{m \times k}$ and $Z \in \mathbb{C}^{k \times k}$ with $m, k \in \mathbb{N}$ and $m + k = n$. Recall, that a *permutation matrix* is a matrix $P = (p_{ij})$ with

$$p_{ij} \in \{0, 1\} \qquad \text{and} \qquad \sum_{i=1}^{n} p_{ij} = \sum_{j=1}^{n} p_{ij} = 1 \qquad \text{for} \qquad i, j = 1, \ldots, n,$$

The transformation in (1.2.1) corresponds to a permutation of the rows and columns of the matrix A. Correspondingly, A is called *irreducible*, if A is not reducible.

With the additional assumption of A being irreducible, the Perron-Frobenius Theorem preserves nearly all results of Theorem 1.1.1. Only statement (iv) cannot be saved.

Theorem 1.2.2 [Perron-Frobenius]. *Consider $A \in \mathbb{R}^{n \times n}$ non-negative, i.e. $A \geq 0$, and irreducible. The following assertions hold true:*

(i) *The spectral radius is a positive and algebraically simple eigenvalue of A, i.e. $r(A) > 0$, $r(A) \in \sigma(A)$ and*

$$\operatorname{geom}_A(r(A)) = \operatorname{alg}_A(r(A)) = 1.$$

(ii) *There exists a positive eigenvector $x > 0$ of A associated to the eigenvalue $r(A)$, i.e. $Ax = r(A)x > 0$.*

(iii) *There exist no further non-negative eigenvectors of A, except the ones associated to the spectral radius $r(A)$.*

(iv) *The Collatz-Wielandt Formula holds true:*

$$r(A) = \max_{x \in \mathcal{N}} \min_{x_i \neq 0} \frac{(Ax)_i}{x_i},$$

with $\mathcal{N} = \{\, x \in \mathbb{R}^n \backslash \{0\} \,:\, x \geq 0 \,\}$.

Proof. For the proof of this theorem see for example [M, Ch 8]. ☐

Example 1.2.3. Let us demonstrate the assertions of the above theorem in an example. Consider the irreducible matrix $A \in \mathbb{R}^{2\times 2}$ given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \geq 0.$$

The spectrum of A can be identified as $\sigma(A) = \{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\}$, thus $r(A) = \frac{1+\sqrt{5}}{2} > 0$ is an algebraically simple eigenvalue. Eigenvectors of A associated with the eigenvalues $\lambda_1 := \frac{1-\sqrt{5}}{2}$ and $\lambda_2 := r(A)$ are e.g. given by

$$x_1 = (1-\sqrt{5}, 2)^T \ngeq 0 \qquad \text{and} \qquad x_2 = (2, \sqrt{5}-1)^T > 0.$$

Indeed, all non-negative eigenvectors correspond to $r(A)$, as all other eigenvectors are multiples of x_1. //

Chapter 6 will show, that under the right assumptions, the corresponding statements to Theorem 1.1.1 (i)-(iv) all remain valid in infinite dimensions. Note, that we first stated the Perron Theorem for positive matrices before loosening up the conditions to non-negative, irreducible ones in the Perron-Frobenius Theorem. We will use the reversed approach for the elaboration of the Krein-Rutman Theorem. First we will consider compact, positive operators $T \in \mathcal{B}(X)$ with $r(T) > 0$, before sharpening the assumptions to compact, strongly positive ones in order to attain the complete Krein-Rutman Theorem.

Chapter 2

Compact operators

In order to generalise the results of the first chapter to linear and bounded operators on infinite dimensional Banach spaces, we recall the notion of a compact operator. Indeed, compact operators can be seen as infinite dimensional equivalents of matrices, which represent linear (and bounded) operators on finite dimensional vector spaces. All the given results can be found in [R, Ch 4], including their full proofs.

If not stated differently, Y will denote a complex Banach space throughout the entire chapter. A linear operator $T : Y \to Y$ is called *compact*, if the image of the open unit ball is relatively compact, i.e. if

$$\overline{T(U)} \quad \text{is compact in} \quad Y, \qquad \text{where} \quad U := \{\, x \in Y \,:\, \|x\| < 1 \,\}\,.$$

The compactness of $T : Y \to Z$, where Z is another complex Banach space, is defined analogously. Since we are interested in the structure of the spectra of compact operators, we will only consider endomorphisms of Y. Except the ones concerning spectral properties, all the given results remain valid for $T : Y \to Z$ or $T \in \mathcal{B}(Y, Z)$, respectively.

We list some well known properties of compact operators, which are direct consequences of the above definition:

- Every compact operator is bounded, i.e. $T \in \mathcal{B}(Y)$ for compact $T : Y \to Y$.
- An operator $T \in \mathcal{B}(Y)$ is compact, if and only if $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence in Y for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in Y.
- If $T \in \mathcal{B}(Y)$ with $\dim\ \operatorname{ran} T < \infty$, then T is compact.

Among all linear and bounded operators on an infinite dimensional Banach space Y, compact operators are the ones with the biggest resemblance to linear operators on finite dimensional vector spaces. Moreover, the subspace of compact operators is a closed ideal of $\mathcal{B}(Y)$:

- If the sequence $(T_n)_{n\in\mathbb{N}}$ of compact operators is convergent in $\mathcal{B}(Y)$, its limit

$$T := \lim_{n\to\infty} T_n \in \mathcal{B}(Y)$$

  is compact.

- For arbitrary $S \in \mathcal{B}(Y)$ and compact $T \in \mathcal{B}(Y)$, the compositions $TS \in \mathcal{B}(Y)$ and $TS \in \mathcal{B}(Y)$ are compact.

2.1 The spectrum of a compact operator

Consider a Banach space Y. The aim of this work is, to elaborate a similar result to Theorem 1.2.2 for compact and positive operators $T \in \mathcal{B}(Y)$. Regarding their spectrum, the similarities to linear operators on finite dimensional vector spaces are particularly evident.

In this section, we give an overview on the the spectral properties of compact operators. In order to do so, let us recall the *resolvent set* of an operator $T \in \mathcal{B}(Y)$,

$$\rho(T) := \{ \mu \in \mathbb{C} : (\mu I - T)^{-1} \in \mathcal{B}(Y) \} .$$

More specifically, $\mu \in \rho(T)$ if the operator $(\mu I - T)$ is invertible in the Banach algebra $\mathcal{B}(Y)$, i.e. if it is bijective and the inverse $(\mu I - T)^{-1}$ is bounded. The *resolvent* of T is defined as

$$R_T : \begin{cases} \rho(T) & \to & \mathcal{B}(Y), \\ \mu & \mapsto & (\mu I - T)^{-1}. \end{cases}$$

By the Open Mapping Theorem (e.g. [Yo, p 75f]), $\mu \in \rho(T)$ is equivalent to the bijectivity of $(\mu I - T)$. Like in finite dimensions, the *spectrum* of T is the compact set

$$\sigma(T) := \mathbb{C}\backslash\rho(T) = \{ \lambda \in \mathbb{C} : (\lambda I - T) \text{ is not bijective} \} .$$

For $\lambda \in \sigma(T)$, either $\ker(\lambda I - T) \neq \{0\}$ or $\operatorname{ran}(\lambda I - T) \neq Y$ has to be satisfied. The *point spectrum*

$$\sigma_p(T) := \{ \lambda \in \mathbb{C} : \ker(\lambda I - T) \neq \{0\} \} \subseteq \sigma(T)$$

consists of all *eigenvalues* of T and every $x \in \ker(\lambda I - T)\backslash\{0\}$ is an *eigenvector* corresponding to the eigenvalue λ. Unlike in finite dimensions, the spectrum and the point spectrum do not necessarily coincide. Spectrum, resolvent set and resolvent can equally be defined for elements of general Banach algebras, which will be presented in Section 4.1.

Remarkably, the spectrum of a compact operator $T \in \mathcal{B}(Y)$ has a certain discrete structure. More specifically, for every $r > 0$,

$$\sigma(T) \cap \{\, \lambda \in \mathbb{C} \,:\, |\lambda| > r \,\}$$

is a finite set. In other words, $\sigma(T)$ is at most countable and its only possible accumulation point is 0. Especially, every $\lambda \in \sigma(T)\backslash\{0\}$ is an isolated point of $\sigma(T)$. Note, that if $\dim\ Y = \infty$, then necessarily $0 \in \sigma(T)$, since the bijectivity of T would imply the compactness of the unit ball.

Whenever $\lambda \in \mathbb{C}\backslash\{0\}$, clearly the subspace $\ker(\lambda I - T)$ is closed. It turns out, that whenever T is compact and $\lambda \neq 0$, the subspace $\operatorname{ran}(\lambda I - T)$ is closed, too. Moreover, the dimension of $\ker(\lambda I - T)$ and the codimension of $\operatorname{ran}(\lambda I - T)$ are both finite and equal, i.e.

$$\dim\ \ker(\lambda I - T) = \dim\ \left(Y \big/ \operatorname{ran}(\lambda I - T)\right) < \infty \tag{2.1.1}$$

for compact $T \in \mathcal{B}(Y)$ and $\lambda \neq 0$. Note, that since $\operatorname{ran}(\lambda I - T)$ is closed, the quotient space $Y \big/ \operatorname{ran}(\lambda I - T)$ is a Banach space.

An operator $T \in \mathcal{B}(Y)$ is compact, if and only if the *adjoint operator* $T' \in \mathcal{B}(Y')$ is. Recall, that T' is the uniquely determined operator satisfying

$$\langle T'y', x\rangle = \langle y', Tx\rangle \qquad \text{for all} \quad x \in Y, \quad y' \in Y'.$$

Here $\langle\cdot,\cdot\rangle$ denotes the bilinear form

$$\langle\cdot,\cdot\rangle : \begin{cases} Y' \times Y & \to \quad \mathbb{C}, \\ (y', x) & \mapsto \quad \langle y', x\rangle = y'(x). \end{cases}$$

Remarkably, the quantities in (2.1.1) are equal to

$$\dim\ \ker(\lambda I' - T') = \dim\ \left(Y' \big/ \operatorname{ran}(\lambda I' - T')\right) < \infty.$$

As is well known, the spectrum of $T \in \mathcal{B}(Y)$ and its adjoint $T' \in \mathcal{B}(Y')$ coincide. In fact, if T is compact, every $0 \neq \lambda \in \sigma(T) = \sigma(T')$ is an eigenvalue of both T and T', with the same finite *geometric multiplicity*

$$\begin{aligned}\mathrm{geom}_T(\lambda) &:= \dim\ \ker(\lambda I - T)\\ &= \dim\ \ker(\lambda I' - T') = \mathrm{geom}_{T'}(\lambda).\end{aligned}$$

In summary, the spectrum of a compact operator and its compact adjoint consists of at most countably many eigenvalues of finite geometric multiplicity and, possibly, of their single accumulation point 0. The above results are known as the *Fredholm Alternative*:

Theorem 2.1.1 [Fredholm alternative]. *Let Y be a Banach space, $T \in \mathcal{B}(Y)$ be compact and $\lambda \in \mathbb{C}\backslash\{0\}$. Then*

(i) $\mathrm{ran}(\lambda I - T)$ *is closed.*

(ii) *the quantities*

$$\begin{aligned}\alpha &:= \dim\ \ker(\lambda I - T), & \beta &:= \dim\ \left(Y \big/ \mathrm{ran}(\lambda I - T)\right),\\ \alpha' &:= \dim\ \ker(\lambda I' - T'), & \beta' &:= \dim\ \left(Y' \big/ \mathrm{ran}(\lambda I' - T')\right)\end{aligned}$$

are all finite and equal.

(iii) *each $\lambda \in \sigma(T)\backslash\{0\}$ is an eigenvalue of both T and T' with the same finite geometric multiplicity.*

(iv) $\sigma(T)$ *is at most countably infinite and its only possible accumulation point is* 0.

2.2 Ascent and descent of operators

The proof of the Krein-Rutman Theorem makes use of the fact, that every non-zero eigenvalue of a compact operator is a pole of its resolvent. In Section 4.5, we will recall some properties of the resolvent of an operator and elaborate a characterisation of its poles. The concept we introduce in the present section will turn out to be strongly related to the singularities of the resolvent.

For $T \in \mathcal{B}(Y)$, clearly, the sequence of subspaces

$$\{0\} = \ker T^0 \subseteq \ker T^1 \subseteq \ker T^2 \subseteq \cdots \tag{2.2.1}$$

is ascending. If there exists $n \in \mathbb{N}\cup\{0\}$ with $\ker T^n = \ker T^{n+1}$, then $\ker T^n = \ker T^{n+j}$ for all $j \in \mathbb{N}$. Indeed, assuming $\ker T^n = \ker T^{n+j}$ for $j \in \mathbb{N}$ yields

$$\ker T^n = \ker T^{n+1} = \ker T^{(n+j)+1} = \ker T^{n+(j+1)}.$$

The sequence of subspaces

$$Y = \operatorname{ran} T^0 \supseteq \operatorname{ran} T^1 \supseteq \operatorname{ran} T^2 \supseteq \cdots \tag{2.2.2}$$

is descending and if $\operatorname{ran} T^n = \operatorname{ran} T^{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $\operatorname{ran} T^n = \operatorname{ran} T^{n+j}$ for every $j \in \mathbb{N}$. As before, inductively, $\operatorname{ran} T^n = \operatorname{ran} T^{n+j}$ for $j \in \mathbb{N}$ implies

$$\operatorname{ran} T^n = \operatorname{ran} T^{n+1} = T(\operatorname{ran} T^n) = T(\operatorname{ran} T^{n+j}) = \operatorname{ran} T^{n+(j+1)}.$$

Consequently, either all the inclusions between the subspaces in (2.2.1) or (2.2.2) are proper, or they are proper until a certain index, and then all the subsequent subspaces are equal. This observation motivates the next definition.

Definition 2.2.1. Let Y be a Banach space and $T \in \mathcal{B}(Y)$. By declaring $\min \emptyset := \infty$, the following notions are well defined:

- The *ascent* of T is defined as

$$p(T) := \min \left\{ n \in \mathbb{N} \cup \{0\} \, : \, \ker T^n = \ker T^{n+1} \right\} \in \mathbb{N} \cup \{\infty\}.$$

- We define the *descent* of T as

$$q(T) := \min \left\{ n \in \mathbb{N} \cup \{0\} \, : \, \operatorname{ran} T^n = \operatorname{ran} T^{n+1} \right\} \in \mathbb{N} \cup \{\infty\}.$$

If either $\ker T^n \neq \ker T^{n+1}$ or $\operatorname{ran} T^n \neq \operatorname{ran} T^{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then $p(T) = \infty$ or $q(T) = \infty$, respectively. //

Remark 2.2.2. Obviously $p(T) = 0$, if and only if T is injective and $q(T) = 0$, if and only if T is surjective. //

Assume, that $p(T) \leq m \in \mathbb{N} \cup \{0\}$. Let $n \in \mathbb{N}$ be arbitrary and $y \in \ker T^n \cap \operatorname{ran} T^m$. Then $y = T^m x$ for some $x \in Y$ and $x \in \ker T^{m+n} = \ker T^m$. Hence, $y = T^m x = 0$, i.e. $p(T) \leq m$ implies

$$\ker T^n \cap \operatorname{ran} T^m = \{0\} \tag{2.2.3}$$

for every $n \in \mathbb{N}$. It turns out, that the converse equally holds true.

Theorem 2.2.3. *For a Banach space Y and $T \in \mathcal{B}(Y)$, one has $p(T) \leq m$ for some $m \in \mathbb{N} \cup \{0\}$, if and only if there exists an $n \in \mathbb{N}$ such that* (2.2.3) *holds true. In this case,* (2.2.3) *is satisfied for any $n \in \mathbb{N}$. In particular, if $p(T)$ is finite, then*

$$\ker T^n \cap \operatorname{ran} T^{p(T)} = \{0\} \qquad \text{for all} \quad n \in \mathbb{N}.$$

Proof. Assume, that there exists $n \in \mathbb{N}$ such that (2.2.3) is fulfilled for some $m \in \mathbb{N} \cup \{0\}$. For any $x \in \ker T^{m+1}$,

$$T^m x \in \ker T \cap \operatorname{ran} T^m \subseteq \ker T^n \cap \operatorname{ran} T^m = \{0\},$$

i.e. $x \in \ker T^m$. Hence, $\ker T^m = \ker T^{m+1}$ and $p(T) \leq m$. Above, we showed the other implication for arbitrary $n \in \mathbb{N}$. □

As the finiteness of the ascent is linked to the range of the powers of T, the finiteness of the descent is linked to their kernel. If there exists $n \in \mathbb{N}$ and a subspace $C_n \subseteq \ker T^m$ with $m \in \mathbb{N} \cup \{0\}$, such that (*direct sum*)

$$Y = C_n \dotplus \operatorname{ran} T^n, \tag{2.2.4}$$

then the descent is less or equal to m. Indeed by (2.2.4),

$$\operatorname{ran} T^m = T^m(C_n) + T^m(\operatorname{ran} T^n) = \operatorname{ran} T^{m+n},$$

implying $\operatorname{ran} T^m = \operatorname{ran} T^{m+1}$, and therefore $q(T) \leq m$. The following theorem shows, that the reverse implication also holds true.

Theorem 2.2.4. *For a Banach space Y and $T \in \mathcal{B}(Y)$ one has $q(T) \leq m$ for some $m \in \mathbb{N} \cup \{0\}$, if and only if there exists an $n \in \mathbb{N}$ and a subspace $C_n \subseteq \ker T^m$ with* (2.2.4)*. In this case, a subspace $C_n \subseteq \ker T^m$ satisfying* (2.2.4) *exists for every $n \in \mathbb{N}$. In particular, if $q(T)$ is finite, then*

$$Y = \ker T^{q(T)} + \operatorname{ran} T^n \qquad \text{for all} \quad n \in \mathbb{N}.$$

Proof. Assume, that $q(T) \leq m$ with $m \in \mathbb{N} \cup \{0\}$. Take any $n \in \mathbb{N}$ and let $C \subseteq Y$ be a subspace with $Y = C \dotplus \operatorname{ran} T^n$, i.e.

$$Y = C + \operatorname{ran} T^n \qquad \text{and} \qquad C \cap \operatorname{ran} T^n = \{0\}. \tag{2.2.5}$$

Clearly, such a subspace can always be found. Consider an arbitrary algebraic basis $\mathcal{B} := \{ x_i : i \in I \}$ of C. Then by $\operatorname{ran} T^m = \operatorname{ran} T^{m+n}$, for every $i \in I$ there exists $y_i \in Y$ with $T^m x_i = T^{m+n} y_i$. Define $z_i := x_i - T^n y_i$ for all $i \in I$. Then $T^m z_i = 0$ and

$$C_n := \operatorname{span} \{ z_i : i \in I \} \subseteq \ker T^m.$$

By (2.2.5), every $x \in Y$ can be written as $x = \sum_{i \in I} a_i x_i + T^n y$ with $a_i \in \mathbb{C}$ and $y \in Y$. Hence,

$$x = \sum_{i \in I} a_i (z_i + T^n y_i) + T^n y = \sum_{i \in I} a_i z_i + T^n z$$

for some $z \in Y$. Therefore, $Y = C_n + \operatorname{ran} T^n$. For any $x \in C_n \cap \operatorname{ran} T^n$, there exist $b_i \in \mathbb{C}$ and $v \in Y$ with $x = \sum_{i \in I} b_i z_i = T^n v$. Consequently,

$$\sum_{i \in I} b_i x_i = \sum_{i \in I} b_i T^n y_i + T^n v \in \operatorname{ran} T^n.$$

By (2.2.5) we derive $b_i = 0$ for all $i \in I$, and in turn $x = 0$. Thus, we obtain $Y = C_n \dotplus \operatorname{ran} T^n$. $\square$

The ascent and descent of an operator are strongly related. The next theorem uses the previously developed equivalent conditions for their finiteness in order to correlate them.

Theorem 2.2.5. *Let Y be a Banach space and $T \in \mathcal{B}(Y)$. If $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$. In this case, $p(T) = q(T)$ is called the* Riesz number *of T.*

Proof. Set $p := p(T)$ and $q := q(T)$.

- First assume that $p \leq q$. To obtain equality, we need to show $\operatorname{ran} T^p = \operatorname{ran} T^q$. Since $q = 0$ immediately implies $p = q$, we can assume $q > 0$. Clearly, we have $\operatorname{ran} T^p \supseteq \operatorname{ran} T^q$. Let $y = T^p x \in \operatorname{ran} T^p$ with $x \in Y$ be arbitrary. Theorem 2.2.4 with $n = q$ yields
$$Y = \ker T^q + \operatorname{ran} T^q.$$
Hence, $y = z + T^q w$ for some $z \in \ker T^q$ and $w \in Y$. We obtain
$$z = T^p x - T^q w \in \ker T^q \cap \operatorname{ran} T^p.$$
By Theorem 2.2.3 with $n = q$ we conclude $z = 0$. Thus, $y = T^q w \in \operatorname{ran} T^q$ and in turn, $\operatorname{ran} T^p = \operatorname{ran} T^q$.

– In case that $q \leq p$, we need to show $\ker T^q = \ker T^q$ in order to obtain $p = q$. Again, we can assume $p > 0$. Clearly, $\ker T^q \subseteq \ker T^p$. By Theorem 2.2.4 with $n = p$,

$$Y = \ker T^q + \operatorname{ran} T^p.$$

Hence, every $x \in \ker T^p$ can be written as $x = u + T^p v$ with $u \in \ker T^q$ and $v \in Y$. Since $x \in \ker T^p$ and $u \in \ker T^q \subseteq \ker T^p$, we have $v \in \ker T^{2p} = \ker T^p$. Thus, $x = u \in \ker T^q$, implying $\ker T^q = \ker T^p$.

□

The above theorems can be combined in order to find an equivalent condition for the Riesz number of T to be finite. Assume $p := p(T) = q(T) < \infty$. If $p > 0$, setting $n = p$ in Theorem 2.2.3 and Theorem 2.2.4 yields $\ker T^p \cap \operatorname{ran} T^p = \{0\}$ and $Y = \ker T^p + \operatorname{ran} T^p$, i.e.

$$Y = \ker T^p \dot{+} \operatorname{ran} T^p. \tag{2.2.6}$$

In case that $p = 0$, relation (2.2.6) obviously holds true, since T is bijective by Remark 2.2.2 and $T^p = \mathrm{id}_Y$. In fact, the above condition is not only necessary, but sufficient.

Corollary 2.2.6. *Let Y be a Banach space and $T \in \mathcal{B}(Y)$. If T has finite Riesz number $p := p(T) = q(T)$, then Y has a representation as in* (2.2.6)*. Moreover, the restriction $T|_{\operatorname{ran} T^p} \in \mathcal{B}(\operatorname{ran} T^p)$ is bijective. If, conversely, for some $m \in \mathbb{N}$*

$$Y = \ker T^m \dot{+} \operatorname{ran} T^m \tag{2.2.7}$$

is satisfied, then $p(T) = q(T) \leq m$.

Proof.

– Assume, that there is an $m \in \mathbb{N}$ such that (2.2.7) holds true. Then by Theorem 2.2.3 and 2.2.4 with $n = m$ and Theorem 2.2.5 we obtain $p(T) = q(T) \leq m$. The reverse implication was shown above.

– Let $T_0 := T|_{\operatorname{ran} T^p}$ be the restriction of T to $\operatorname{ran} T^p$. Then

$$\operatorname{ran} T_0 = T(\operatorname{ran} T^p) = \operatorname{ran} T^{p+1} = \operatorname{ran} T^p,$$

thus $T_0 \in \mathcal{B}(\operatorname{ran} T^p)$ is surjective. Moreover, $\ker T_0 \subseteq \ker T \subseteq \ker T^p$ and by (2.2.6),

$$\ker T_0 \subseteq \ker T^p \cap \operatorname{ran} T^p = \{0\}.$$

Hence, T_0 is injective.

□

In Section 4.5 we will see, that $\lambda \in \sigma(T)$ is a pole of the resolvent of $T \in \mathcal{B}(Y)$, if and only if the Riesz number of the operator $(T - \lambda I)$ is finite and positive. The following theorem will enable us to apply this result to an eigenvalue $\lambda \neq 0$ of a compact operator.

Theorem 2.2.7. *Let Y be a Banach space and $T \in \mathcal{B}(Y)$ be compact. Then for every $\lambda \in \mathbb{C}\backslash\{0\}$, the operator $(T - \lambda I)$ has finite Riesz number*

$$p(T - \lambda I) = q(T - \lambda I) < \infty.$$

Proof. We need to show, that the ascent and descent of $(T - \lambda I)$ are both finite. Theorem 2.2.5 yields their equality. To simplify notations, for $n \in \mathbb{N}$ we set

$$K_n := \ker(T - \lambda I)^n,$$
$$R_n := \operatorname{ran}(T - \lambda I)^n.$$

- Assume that $q(T - \lambda I) = \infty$, i.e. all inclusions $R_n \supsetneq R_{n+1}$ are proper and we can choose $y_n \in R_n \backslash R_{n+1}$ for all $n \in \mathbb{N}$. By Theorem 2.1.1 (i), R_{n+1} is closed and consequently, $d(y_n, R_{n+1}) = \inf_{z \in R_{n+1}} \|y_n - z\| > 0$. According to the definition of the infimum, there exists $z_n \in R_{n+1}$ with

$$d(y_n, R_{n+1}) \leq \|y_n - z_n\| \leq 2\, d(y_n, R_{n+1}) \tag{2.2.8}$$

for every $n \in \mathbb{N}$. Define $x_n \in R_n \backslash R_{n+1}$ with $\|x_n\| = 1$ by

$$x_n := \frac{1}{\|y_n - z_n\|}(y_n - z_n), \qquad n \in \mathbb{N}.$$

Since $z_n + \|y_n - z_n\|\, z \in R_{n+1}$ for all $z \in R_{n+1}$, the inequality (2.2.8) implies

$$\|x_n - z\| = \frac{1}{\|y_n - z_n\|}\, \|y_n - (z_n + \|y_n - z_n\|\, z)\| \geq \frac{d(y_n, R_{n+1})}{\|y_n - z_n\|} \geq \frac{1}{2}.$$

and in turn, $d(x_n, R_{n+1}) \geq \frac{1}{2}$. For every $j \in \mathbb{N}$, we have

$$\lambda x_{n+j} - (T - \lambda I)x_n + (T - \lambda I)x_{n+j} \in R_{n+1}.$$

Therefore, the sequence $(Tx_n)_{n \in \mathbb{N}}$ satisfies

$$\|Tx_n - Tx_{n+j}\| = \|\lambda x_n - (\lambda x_{n+j} - (T - \lambda I)x_n + (T - \lambda I)x_{n+j})\| \geq \frac{|\lambda|}{2}$$

for $j \in \mathbb{N}$, thus it cannot contain any convergent subsequence. Since $(x_n)_{n \in \mathbb{N}}$ is bounded, we obtain a contradiction to the compactness of T.

– Let us assume $p(T-\lambda I)=\infty$, i.e. $K_n \neq K_{n+1}$ for all $n \in \mathbb{N}$. As before, one can construct a sequence $(x_n)_{n\in\mathbb{N}}$ with

$$x_{n+1} \in K_{n+1}\backslash K_n, \qquad \|x_{n+1}\| = 1 \qquad \text{and} \qquad d(x_{n+1}, K_n) \geq \frac{1}{2}$$

for $n \in \mathbb{N}$. Clearly, $(T-\lambda I)(K_{n+1}) \subseteq K_n$. Therefore, for every $j \in \mathbb{N}$ we obtain $\lambda x_n + (T-\lambda I)x_n - (T-\lambda I)x_{n+j} \in K_{n+j-1}$. Consequently,

$$\|Tx_n - Tx_{n+j}\| = \|\lambda x_n + (T-\lambda I)x_n - (T-\lambda I)x_{n+j} - \lambda x_{n+j}\| \geq \frac{|\lambda|}{2},$$

and $(Tx_n)_{n\in\mathbb{N}}$ cannot have any convergent subsequence, in contradiction to the compactness of T.

□

Remark 2.2.8. In accordance to the finite dimensional case, the *algebraic multiplicity* of an eigenvalue $\lambda \in \sigma_p(T)$ of the operator $T \in \mathcal{B}(Y)$ is defined as

$$\mathrm{alg}_T(\lambda) := \dim \bigcup_{n=1}^{\infty} \ker(\lambda I - T)^n.$$

In the previous section, we recalled that for compact $T \in \mathcal{B}(Y)$, every $\lambda \in \sigma(T)\backslash\{0\}$ is an eigenvalue of finite geometric multiplicity $\mathrm{geom}_T(\lambda) = \dim\ \ker(\lambda I - T)$. By the previous theorem,

$$\bigcup_{n=1}^{\infty} \ker(\lambda I - T)^n = \ker(\lambda I - T)^p, \tag{2.2.9}$$

where $p \in \mathbb{N}$ denotes the Riesz number of $(\lambda I - T)$. Note, that since λ is an eigenvalue of T, truly $p > 0$. By (2.2.9), the algebraic multiplicity of λ is given by

$$\mathrm{alg}_T(\lambda) = \dim\ \ker(\lambda I - T)^p.$$

For every $n \in \mathbb{N}$, the power $(\lambda I - T)^n$ can be written as $(\lambda I - T)^n = \lambda^n I + q(T)T$ with some polynomial $q(z) \in \mathbb{C}[z]$. By $(q(T)T)' = q(T')T'$ and the compactness of $q(T)T$, Theorem 2.1.1 (ii) implies

$$\dim\ \ker(\lambda I - T)^n = \dim\ \ker(\lambda I' - T')^n < \infty.$$

for every $n \in \mathbb{N}$. Consequently, $p = p(\lambda I - T) = p(\lambda I' - T')$ and the algebraic multiplicities of λ as an eigenvalue of both T and T' are finite and equal, i.e.

$$\mathrm{alg}_T(\lambda) = \mathrm{alg}_{T'}(\lambda) = \dim\ \ker(\lambda I - T)^p \in \mathbb{N}$$

for $\lambda \in \sigma(T)\backslash\{0\} = \sigma(T')\backslash\{0\}$. //

Chapter 3

Complexification of real Banach spaces

Most considerations of spectral theory are made for operators on Banach spaces over the scalar field $\mathbb{C}$. Since their resolvent is holomorphic, tools from complex analysis can be applied and strong results can be found, which in general are not satisfied for operators on real Banach spaces. In the proof of the Krein-Rutman Theorem, which deals with the spectrum of operators on real Banach spaces, we will need the powerful machinery of complex analysis. To be able to use it, we will extend the given operators to a complex Banach space in a canonical way.

3.1 The complexification $X_{\mathbb{C}}$ and $\mathcal{B}(X_{\mathbb{C}})$

In the entire section, let X be a real Banach space. To be able to extend a given operator $T \in \mathcal{B}(X)$, we first need to construct a complex Banach space $X_{\mathbb{C}} \supseteq X$ in a suitable way. The following construction is inspired by how the complex numbers are built from $\mathbb{R}$.

Consider the product space $X \times X$, which naturally becomes a real Banach space with e.g. either $\|\cdot\|_1$ or $\|\cdot\|_\infty$, where

$$\begin{aligned} \|(x,y)\|_1 &= \|x\| + \|y\|, \\ \|(x,y)\|_\infty &= \max(\|x\|, \|y\|), \end{aligned} \qquad \text{for} \quad (x,y) \in X \times X.$$

Both norms induce the product topology of the norm topology on X. For the real Banach space $X \times X$ to become a complex one, we need to equip it with a complex scalar multiplication and a new norm, which is homogeneous with respect to the multiplication not only with real, but with complex scalars.

Definition 3.1.1. On the product space $X \times X$ of a real Banach space X, let the addition be defined as usual,

$$(x,y)+(u,v) := (x+u, y+v) \qquad \text{for} \quad (x,y),(u,v) \in X \times X.$$

Define a complex scalar multiplication on $X \times X$ as

$$(a+ib)\cdot(x,y) := (ax-by, ay+bx) \qquad \text{for} \quad (x,y) \in X \times X,\ a+ib \in \mathbb{C}.$$

For the sake of simplicity we also write $(a+ib)(x,y) := (a+ib)\cdot(x,y)$. We call the vector space $X_{\mathbb{C}} := X \times X$ together with the above defined operations the *complexification* of X. Moreover, we define a new norm

$$\|(x,y)\|_{\mathbb{C}} := \max_{\varphi \in [0,2\pi]} \|\cos\varphi\ x + \sin\varphi\ y\|, \quad (x,y) \in X_{\mathbb{C}}.$$

//

It is elementary to check, that $X_{\mathbb{C}}$ is truly a complex vector space. Hence the product $X \times X$ is equipped with both a real and a complex vector space structure. From now on, $X_{\mathbb{C}}$ will always denote the complex and $X \times X$ the real vector space. It turns out, that $\|\cdot\|_{\mathbb{C}}$ is not only a norm on $X_{\mathbb{C}}$, but also equivalent to $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Thus, it also induces the product topology.

Proposition 3.1.2. *Let X be a real Banach space and let $X_{\mathbb{C}}$ and $\|\cdot\|_{\mathbb{C}}$ be as in Definition 3.1.1. Then $\langle X_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}}\rangle$, together with the operations from Definition 3.1.1, forms a complex Banach space. Its norm $\|\cdot\|_{\mathbb{C}}$ is equivalent to $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $X \times X$. The injection*

$$\iota_1 : \begin{cases} X & \to & X_{\mathbb{C}}, \\ x & \mapsto & (x,0), \end{cases}$$

is isometric and $\mathbb{R}$-linear. Hence, X can be identified canonically with the closed subset and real subspace $\iota_1(X) = X \times \{0\}$ of $X_{\mathbb{C}}$. With this identification, $\|x\|_{\mathbb{C}} = \|ix\|_{\mathbb{C}} = \|x\|$ holds true.

Proof.

- First we show, that $\|\cdot\|_{\mathbb{C}}$ is a norm on $X_{\mathbb{C}}$. Obviously, $\|x+iy\|_{\mathbb{C}} \geq 0$ for all $x+iy \in X_{\mathbb{C}}$. Now consider $x+iy \in X_{\mathbb{C}}$ with $\|x+iy\|_{\mathbb{C}} = 0$. Consequently, $\|\cos\varphi\ x + \sin\varphi\ y\| = 0$ and thus,

$$\cos\varphi\ x + \sin\varphi\ y = 0 \qquad \text{for all} \quad \varphi \in [0,2\pi].$$

Setting $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ yields $x = y = 0$. To show the triangle inequality, let $x + iy,\ u + iv \in X_{\mathbb{C}}$ be arbitrary. We obtain

$$\begin{aligned}\|(x+iy)+(u+iv)\|_{\mathbb{C}} &= \max_{\varphi\in[0,2\pi]} \|\cos\varphi\ (x+u) + \sin\varphi\ (y+v)\| \\ &\leq \max_{\varphi\in[0,2\pi]} \big(\|\cos\varphi\ x + \sin\varphi\ y\| + \|\cos\varphi\ u + \sin\varphi\ v\| \big) \\ &\leq \|x+iy\|_{\mathbb{C}} + \|u+iv\|_{\mathbb{C}}.\end{aligned}$$

It remains to show homogeneity. Consider $x + iy \in X_{\mathbb{C}}$ and $z \in \mathbb{C}$, such that $z = r\exp(i\phi) = r(\cos\phi + i\sin\phi)$ with $r \geq 0$ and $\phi \in [0, 2\pi]$. By the well known Angle Addition Theorem for sinus and cosinus,

$$\|r(\cos\phi + i\sin\phi)(x+iy)\|_{\mathbb{C}} =$$

$$\begin{aligned}&= \max_{\varphi\in[0,2\pi]} \|\cos\varphi\ (r\cos\phi\ x - r\sin\phi\ y) + \sin\varphi\ (r\sin\phi\ x + r\cos\phi\ y)\| \\ &= r \max_{\varphi\in[0,2\pi]} \|(\cos\varphi\cos\phi + \sin\varphi\sin\phi)\ x + (\cos\varphi\sin\phi - \sin\varphi\cos\phi)\ y\| \\ &= r \max_{\varphi\in[0,2\pi]} \|\cos(\phi-\varphi)\ x + \sin(\phi-\varphi)\ y\| = |z|\, \|x+iy\|_{\mathbb{C}}.\end{aligned}$$

– For $x \in X$ we have

$$\|x+i0\|_{\mathbb{C}} = \max_{\varphi\in[0,2\pi]} \|\cos\varphi\ x + \sin\varphi\ 0\| = \max_{\varphi\in[0,2\pi]} |\cos\varphi|\, \|x\| = \|x\|\,.$$

Hence, ι_1 is isometric and therefore, X is closed in $X_{\mathbb{C}}$ with respect to $\|\cdot\|_{\mathbb{C}}$. Moreover, $\|0+ix\|_{\mathbb{C}} = |i|\, \|x+i0\|_{\mathbb{C}} = \|x\|$.

– It is well known, that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are equivalent on $X \times X$. Take $(x,y) \in X \times X$, then

$$\begin{aligned}\|x\| &= \|\cos 0\ x + \sin 0\ y\| \leq \max_{\varphi\in[0,2\pi]} \|\cos\varphi\ x + \sin\varphi\ y\| = \|(x,y)\|_{\mathbb{C}}, \\ \|y\| &= \|\cos\frac{\pi}{2}\ x + \sin\frac{\pi}{2}\ y\| \leq \max_{\varphi\in[0,2\pi]} \|\cos\varphi\ x + \sin\varphi\ y\| = \|(x,y)\|_{\mathbb{C}}.\end{aligned}$$

Consequently,

$$\|(x,y)\|_\infty \leq \|x+iy\|_{\mathbb{C}} \leq \|x\|_{\mathbb{C}} + \|iy\|_{\mathbb{C}} = \|x\| + \|y\| \leq 2\,\|(x,y)\|_\infty\,. \tag{3.1.1}$$

– The completeness of $\langle X_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}}\rangle$ follows from the completeness of $X \times X$ with respect to the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$.

□

Identifying X with $X \times \{0\}$, we have $(0,x) = ix$ and $(x,y) = x + iy$ for $(x,y) \in X_{\mathbb{C}}$, i.e.

$$X_{\mathbb{C}} = X \dot{+}_{\mathbb{R}}\, iX.$$

By using the symbol $\dot{+}_{\mathbb{R}}$ we indicate, that the real Banach space $X \times X$ is the direct sum of $X = X \times \{0\}$ and $iX = \{0\} \times X$. Note, that while X and iX are real subspaces of $X \times X$, they are only subsets of the complex vector space $X_{\mathbb{C}}$.

Remark 3.1.3. As an easy consequence of $\operatorname{span}_{\mathbb{C}}(X) = X_{\mathbb{C}}$ and $X \cap iX = \{0\}$, every algebraic basis of X, if considered as a subset of $X_{\mathbb{C}}$, is also an algebraic basis of $X_{\mathbb{C}}$. Therefore,

$$\dim_{\mathbb{R}} X = \dim_{\mathbb{C}} X_{\mathbb{C}}.$$

//

We are going to extend a given operator $T \in \mathcal{B}(X)$ to the complexification $X_{\mathbb{C}}$. The sought operator $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$ has to be $\mathbb{C}$-linear and an extension of T, i.e. $T_{\mathbb{C}}x = Tx$ and $T_{\mathbb{C}}(ix) = iT_{\mathbb{C}}(x) = iTx$ for $x \in X$. Hence, it has to satisfy

$$T_{\mathbb{C}}(x + iy) = Tx + iTy, \qquad x + iy \in X_{\mathbb{C}}.$$

Definition 3.1.4. Let X be a real Banach space and $T \in \mathcal{B}(X)$. The *complexification* of T is defined as the operator

$$T_{\mathbb{C}} : \begin{cases} X_{\mathbb{C}} & \to \quad X_{\mathbb{C}}, \\ x + iy & \mapsto \quad Tx + iTy, \end{cases}$$

acting on the complexification $X_{\mathbb{C}}$ of X. //

By its definition, $T_{\mathbb{C}}$ is a $\mathbb{C}$-linear extension of T. What might not be evident at first is, that $T_{\mathbb{C}}$ is bounded with $\|T\| = \|T_{\mathbb{C}}\|$. It is even compact, if and only if T is.

Proposition 3.1.5. *Let X be a real Banach space and $T \in \mathcal{B}(X)$. The operator $T_{\mathbb{C}}$ from Definition 3.1.4 satisfies $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$ and $\|T\| = \|T_{\mathbb{C}}\|$. Moreover, T is compact, if and only if the complexification $T_{\mathbb{C}}$ is.*

Proof.

– For $z = x + iy \in X_{\mathbb{C}}$ and every $\varphi \in [0, 2\pi]$ we have

$$\begin{aligned} \|\cos\varphi\, Tx + \sin\varphi\, Ty\| &= \|T(\cos\varphi\, x + \sin\varphi\, y)\| \\ &\le \|T\|\,\|\cos\varphi\, x + \sin\varphi\, y\| \le \|T\|\,\|z\|_{\mathbb{C}}. \end{aligned}$$

Thus, $\|T_{\mathbb{C}}z\|_{\mathbb{C}} = \|Tx + iTy\|_{\mathbb{C}} \le \|T\|\,\|z\|_{\mathbb{C}}$ and in turn, $\|T_{\mathbb{C}}\| \le \|T\|$. In particular, $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$.

For the other inequality, take $x \in X$. Then

$$\|Tx\| = \|T_{\mathbb{C}}(x+i0)\|_{\mathbb{C}} \leq \|T_{\mathbb{C}}\| \, \|x+i0\|_{\mathbb{C}} = \|T_{\mathbb{C}}\| \, \|x\|$$

and consequently, $\|T\| \leq \|T_{\mathbb{C}}\|$.

– Let $T \in \mathcal{B}(X)$ be compact. In order to prove the compactness of $T_{\mathbb{C}}$, we need to show that the image of the unit ball

$$U_{\mathbb{C}} := \{\, x+iy \in X_{\mathbb{C}} \,:\, \|x+iy\|_{\mathbb{C}} < 1 \,\}$$

of $X_{\mathbb{C}}$ is relatively compact with respect to the product topology induced by $\|\cdot\|_{\mathbb{C}}$. By (3.1.1) we obtain $U_{\mathbb{C}} \subseteq U \times U$, where $U := U_1(0)$ denotes the open unit ball in X. By definition, $T_{\mathbb{C}}(U \times U) = T(U) \times T(U)$ and therefore,

$$\overline{T_{\mathbb{C}}(U_{\mathbb{C}})} \subseteq \overline{T(U) \times T(U)} = \overline{T(U)} \times \overline{T(U)}.$$

Since T is compact and since products of compact sets are compact with respect to the product topology, so is $\overline{T_{\mathbb{C}}(U_{\mathbb{C}})}$.

Conversely, assume $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$ to be compact. By $\|x\| = \|x+i0\|_{\mathbb{C}}$ we have $U \times \{0\} \subseteq U_{\mathbb{C}}$ and therefore

$$T(U) \times \{0\} = T_{\mathbb{C}}(U \times \{0\}) \subseteq T_{\mathbb{C}}(U_{\mathbb{C}}).$$

Hence, $\overline{T(U)} \times \{0\} = \overline{T(U) \times \{0\}} \subseteq \overline{T_{\mathbb{C}}(U_{\mathbb{C}})}$ is compact, implying the compactness of $\overline{T(U)}$.

□

We would like to classify the operators $S \in \mathcal{B}(X_{\mathbb{C}})$, which are the complexification of some $T \in \mathcal{B}(X)$. By the previous proposition,

$$\mathcal{B}(X)_{\mathbb{C}} := \{\, T_{\mathbb{C}} \,:\, T \in \mathcal{B}(X) \,\} \subseteq \mathcal{B}(X_{\mathbb{C}}).$$

Consider the real Banach space $\mathcal{B}(X \times X)$. Then $\mathcal{B}(X_{\mathbb{C}}) \subseteq \mathcal{B}(X \times X)$, since every $S \in \mathcal{B}(X_{\mathbb{C}})$ is $\mathbb{R}$-linear and both $X_{\mathbb{C}}$ and $X \times X$ carry the product topology. Because of the special structure of the underlying Banach space, every $S \in \mathcal{B}(X \times X)$ has a unique representation as a block operator matrix,

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \text{with} \quad A,\, B,\, C,\, D \in \mathcal{B}(X). \tag{3.1.2}$$

Recall, that the above notation signifies $S(x,y) = (Ax + By, Cx + Dy)$ for every $(x,y) \in X \times X$. The entries are given by

$$A = \pi_1 \circ S \circ \iota_1, \qquad B = \pi_1 \circ S \circ \iota_2,$$
$$C = \pi_2 \circ S \circ \iota_1, \qquad D = \pi_2 \circ S \circ \iota_2,$$

where $\iota_1, \iota_2 \in \mathcal{B}(X, X \times X)$ denote the canonical embeddings

$$\iota_1 : \begin{cases} X & \to & X \times X, \\ x & \mapsto & (x,0), \end{cases} \qquad \text{and} \qquad \iota_2 : \begin{cases} X & \to & X \times X, \\ y & \mapsto & (0,y), \end{cases}$$

and $\pi_1, \pi_2 \in \mathcal{B}(X \times X, X)$ denote the canonical projections

$$\pi_1 : \begin{cases} X \times X & \to & X, \\ (x,y) & \mapsto & x, \end{cases} \qquad \text{and} \qquad \pi_2 : \begin{cases} X \times X & \to & X, \\ (x,y) & \mapsto & y. \end{cases}$$

Consider $S \in \mathcal{B}(X_{\mathbb{C}})$, then S has a representation like in (3.1.2) and is additionally $\mathbb{C}$-linear. Hence, $S(0,x) = S(i(x,0)) = iS(x,0)$ and in turn, $(Bx, Dx) = (-Cx, Ax)$ for $x \in X$, i.e. $A = D$ and $B = -C$. We obtain

$$\mathcal{B}(X_{\mathbb{C}}) = \left\{ S = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A, B \in \mathcal{B}(X) \right\}. \tag{3.1.3}$$

Clearly, $S = T_{\mathbb{C}}$ with some $T \in \mathcal{B}(X)$, if and only if $B = 0$ and $A = T$, i.e.

$$\mathcal{B}(X)_{\mathbb{C}} = \left\{ T_{\mathbb{C}} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} : T \in \mathcal{B}(X) \right\}.$$

Hence, the complexification of $T \in \mathcal{B}(X)$ is exactly the block diagonal operator matrix with T on its diagonal.

In Section 2.1, we recalled the spectrum $\sigma(S) \subseteq \mathbb{C}$ of an operator $S \in \mathcal{B}(Y)$, where Y denotes a complex Banach space. The *spectral radius* of S is defined by

$$r(S) := \max_{\lambda \in \sigma(S)} |\lambda|.$$

Note, that $\sigma(S)$ is compact and thus, the maximum exists. The spectral radius can be determined by the limit

$$r(S) = \lim_{n \to \infty} \|S^n\|^{\frac{1}{n}}. \tag{3.1.4}$$

We use the complexification in order to define spectrum and spectral radius for operators $T \in \mathcal{B}(X)$, which is crucial for the Krein-Rutman Theorem.

Definition 3.1.6. Let X be a real Banach space and $T \in \mathcal{B}(X)$. The *spectrum* of T is defined as $\sigma(T) := \sigma(T_{\mathbb{C}})$ and the *spectral radius* of T as $r(T) := r(T_{\mathbb{C}})$, where $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$ denotes the complexification of T. //

Formula (3.1.4) remains valid in the real case.

Proposition 3.1.7. *Consider a real Banach space X and $T \in \mathcal{B}(X)$. Then*

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} .$$

Proof. It is immediately clear from the definition, that $(T_{\mathbb{C}})^n = (T^n)_{\mathbb{C}}$ for all $n \in \mathbb{N}$. Hence, $\|T^n\| = \|(T_{\mathbb{C}})^n\|$ and (3.1.4) yields

$$r(T) = \lim_{n\to\infty} \|(T_{\mathbb{C}})^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} .$$

□

Without involving the complexification, the *point spectrum* of $T \in \mathcal{B}(X)$ can be defined similarly to the complex case:

$$\sigma_p(T) := \{\, \lambda \in \mathbb{R} \,:\, \ker(\lambda I - T) \neq \{0\} \,\} .$$

Every $\lambda \in \sigma_p(T)$ is an *eigenvalue* of T and every $x \in \ker(\lambda I - T)\backslash\{0\}$ an *eigenvector* of T corresponding to the eigenvalue λ.

Remark 3.1.8. If the context clarifies, which identity is referred to, we will denote both $I_X \in \mathcal{B}(X)$ and $I_{X_{\mathbb{C}}} \in \mathcal{B}(X_{\mathbb{C}})$ by the symbol I. //

Consider $\lambda \in \mathbb{R}$. Since the operator $(\lambda I - T_{\mathbb{C}}) = (\lambda I - T)_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$ has diagonal structure, its kernel is of the special form

$$\ker(\lambda I - T_{\mathbb{C}}) = \ker(\lambda I - T) \dotplus_{\mathbb{R}} i \ker(\lambda I - T). \tag{3.1.5}$$

Consequently, $\sigma_p(T) \subseteq \sigma_p(T_{\mathbb{C}}) \cap \mathbb{R}$ and $\sigma_p(T) \subseteq \sigma(T)$ in terms of Definition 3.1.6.

Equally to the complex case, the *geometric* and *algebraic multiplicity* of an eigenvalue $\lambda \in \sigma(T)$ are given by

$$\begin{aligned} \mathrm{geom}_T(\lambda) &:= \dim\ \ker(\lambda I - T), \\ \mathrm{alg}_T(\lambda) &:= \dim \bigcup_{n=1}^{\infty} \ker(\lambda I - T)^n. \end{aligned} \tag{3.1.6}$$

Not only is every eigenvalue of T an eigenvalue of $T_{\mathbb{C}}$, but conversely, every real eigenvalue of $T_{\mathbb{C}}$ is an eigenvalue of T and their geometric and algebraic multiplicities coincide.

Proposition 3.1.9. *Let X be a real Banach space and $X_{\mathbb{C}}$ its complexification. The point spectrum of every $T \in \mathcal{B}(X)$ equals the real point spectrum of its complexification $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$,*

$$\sigma_p(T) = \sigma_p(T_{\mathbb{C}}) \cap \mathbb{R}.$$

Moreover, for every $\lambda \in \sigma_p(T)$, the algebraic and geometric multiplicities of λ as an eigenvalue of T and $T_{\mathbb{C}}$ coincide, i.e.

$$\mathrm{geom}_T(\lambda) = \mathrm{geom}_{T_{\mathbb{C}}}(\lambda) \qquad \text{and} \qquad \mathrm{alg}_T(\lambda) = \mathrm{alg}_{T_{\mathbb{C}}}(\lambda).$$

Proof. Recall Remark 3.1.8 concerning the notation.

- By (3.1.5), for $\lambda \in \mathbb{R}$ we have $\ker(\lambda I - T) = \{0\}$, if and only if $\ker(\lambda I - T_{\mathbb{C}}) = \{0\}$, implying $\sigma_p(T) = \sigma_p(T_{\mathbb{C}}) \cap \mathbb{R}$.
- Let $\lambda \in \sigma_p(T)$. By (3.1.5), the eigenspace $\ker(\lambda I - T_{\mathbb{C}})$ is the complexification of the real eigenspace $\ker(\lambda I - T)$. According to Remark 3.1.3, their dimensions are equal, proving $\mathrm{geom}_T(\lambda) = \mathrm{geom}_{T_{\mathbb{C}}}(\lambda)$.
- Consider $\lambda \in \sigma_p(T)$. For every $n \in \mathbb{N}$, the operator

  $$(\lambda I - T_{\mathbb{C}})^n = ((\lambda I - T)_{\mathbb{C}})^n = ((\lambda I - T)^n)_{\mathbb{C}}$$

  has diagonal structure. Hence, similarly to (3.1.5),

  $$\bigcup_{n=1}^{\infty} \ker(\lambda I - T_{\mathbb{C}})^n = \bigcup_{n=1}^{\infty} \ker(\lambda I - T)^n \dot{+}_{\mathbb{R}} i \ker(\lambda I - T)^n.$$

  The union on the right side is an infinite union of ascending proper subspaces, if and only if the union in (3.1.6) is. In this case, $\mathrm{geom}_T(\lambda) = \mathrm{geom}_{T_{\mathbb{C}}}(\lambda) = \infty$ is evident. Otherwise, there exists a $p \in \mathbb{N}$ with

  $$\bigcup_{n=1}^{\infty} \ker(\lambda I - T)^n = \ker(\lambda I - T)^p,$$

  $$\bigcup_{n=1}^{\infty} \ker(\lambda I - T_{\mathbb{C}})^n = \ker(\lambda I - T)^p \dot{+}_{\mathbb{R}} i \ker(\lambda I - T)^p.$$

  By Remark 3.1.3, the dimensions of the above subspaces are equal. Consequently, $\mathrm{alg}_T(\lambda) = \mathrm{alg}_{T_{\mathbb{C}}}(\lambda)$.

□

Hence for $\lambda \in \sigma_p(T) = \sigma_p(T_{\mathbb{C}}) \cap \mathbb{R}$, it makes no difference, whether we refer to the multiplicities of λ as an eigenvalue of T or the complexification $T_{\mathbb{C}}$. Every eigenvector of $T_{\mathbb{C}}$ corresponding to λ is of the form $x + iy$ with $x, y \in \ker(\lambda I - T)$. Thus, both x and y are, if they are not zero, eigenvectors of T. Conversely, every eigenvector of T is obviously one of $T_{\mathbb{C}}$.

3.2 The complex dual space $(X')_{\mathbb{C}} \simeq (X_{\mathbb{C}})'$

Let X denote a real Banach space throughout the present section. The topological dual space $X' = \mathcal{B}(X, \mathbb{R})$ is a real Banach space and has a complexification $(X')_{\mathbb{C}}$. As a complex Banach space, the complexification $X_{\mathbb{C}}$ of X has a topological dual space of its own, $(X_{\mathbb{C}})' = \mathcal{B}(X_{\mathbb{C}}, \mathbb{C})$. In this section, we elaborate an isomorphic connection between the two complex Banach spaces $(X')_{\mathbb{C}}$ and $(X_{\mathbb{C}})'$.

Consider the real Banach space $(X \times X)' = \mathcal{B}(X \times X, \mathbb{R})$. Although it is not a complex Banach space, it will serve as a bridge to build the sought isomorphism between $(X')_{\mathbb{C}}$ and $(X_{\mathbb{C}})'$. We will show, that the three real Banach spaces

$$(X_{\mathbb{C}})' \simeq (X \times X)' \simeq (X')_{\mathbb{C}}$$

are $\mathbb{R}$-isomorphic. The corresponding isomorphism between $(X')_{\mathbb{C}}$ and $(X_{\mathbb{C}})'$ will turn out to be $\mathbb{C}$-linear. From now on, let all product spaces be equipped with $\|\cdot\|_\infty$. Since we are only interested in the continuity of the considered mappings, it is sufficient to limit our considerations to any norm inducing the product topology.

Given $f \in (X_{\mathbb{C}})'$, the functional $u := \operatorname{Re} f$ is $\mathbb{R}$-linear. Moreover, for $(x, y) \in X \times X$,

$$|\operatorname{Re} f(x+iy)| \leq |f(x+iy)| \leq \|f\| \, \|x+iy\|_{\mathbb{C}} \leq 2 \, \|f\| \, \|(x,y)\|_\infty \,, \tag{3.2.1}$$

thus $u \in (X \times X)'$. The mapping

$$\Phi : \begin{cases} (X_{\mathbb{C}})' & \to \quad (X \times X)', \\ \quad f & \mapsto \quad \big[(x,y) \mapsto \operatorname{Re} f(x+iy)\big], \end{cases} \tag{3.2.2}$$

proves to be not only bijective, but $\mathbb{R}$-linear and bounded.

Proposition 3.2.1. *Let X be a real Banach space and $X_{\mathbb{C}}$ its complexification. The mapping $\Phi : (X_{\mathbb{C}})' \to (X \times X)'$ defined in* (3.2.2) *is a bounded $\mathbb{R}$-isomorphism. Its inverse is given by*

$$\Theta : \begin{cases} (X \times X)' & \to \quad (X_{\mathbb{C}})', \\ \quad u & \mapsto \quad \big[x+iy \mapsto u(x,y) + iu(y,-x)\big]. \end{cases} \tag{3.2.3}$$

Proof. It remains to show, that Φ is $\mathbb{R}$-linear and bounded, and that Θ is its well defined inverse.

- The $\mathbb{R}$-linearity of Φ follows directly from the linearity of Re and f. By (3.2.1),

  $$|\Phi f(x,y)| \le 2\,\|f\|\,\|(x,y)\|_\infty .$$

  Thus, $\|\Phi f\| \le 2\,\|f\|$ and in turn, $\|\Phi\| \le 2$.

- Consider $u \in (X \times X)'$ and let $f := \Theta u$ be defined as in (3.2.3). An easy computation verifies, that f is $\mathbb{C}$-linear. Take any $x + iy \in X_{\mathbb{C}}$, then

  $$|f(x+iy)| = |u(x,y) + iu(y,-x)| \le 2\,\|u\|\,\|(x,y)\|_\infty \le 2\,\|u\|\,\|x+iy\|_{\mathbb{C}} .$$

  Hence, $f \in (X_{\mathbb{C}})'$.

- Obviously, $(\Phi \circ \Theta)u = u$ for all $u \in (X \times X)'$. For arbitrary $f \in (X_{\mathbb{C}})'$ we obtain

  $$\begin{aligned}(\Theta \circ \Phi)f(x+iy) &= \Phi f(x,y) + i\Phi f(y,-x) = \operatorname{Re} f(x+iy) + i \operatorname{Re} f(y - ix)\\ &= \operatorname{Re} f(x+iy) + i \operatorname{Im} f(x+iy)\\ &= f(x+iy)\end{aligned}$$

  for all $x + iy \in X_{\mathbb{C}}$. We showed $\Theta \circ \Phi = \mathrm{id}_{(X_{\mathbb{C}})'}$ and $\Phi \circ \Theta = \mathrm{id}_{(X \times X)'}$. Hence, Φ is bijective with the inverse $\Phi^{-1} = \Theta$.

$\square$

The real Banach spaces $(X \times X)'$ and $X' \times X'$ are isomorphic in a canonical manner. Since we want the resulting $\mathbb{R}$-isomorphism between $(X_{\mathbb{C}})'$ and $X' \times X' = (X')_{\mathbb{C}}$ to be $\mathbb{C}$-linear, we need to modify the usual isomorphism slightly.

Consider $u \in (X \times X)'$. Then $u_1 := \big[x \mapsto u(x,0)\big]$ and $u_2 := \big[y \mapsto -u(0,y)\big]$ are clearly $\mathbb{R}$-linear and bounded. Hence, the mapping

$$\Psi : \begin{cases} (X \times X)' & \to \quad X' \times X', \\ \quad u & \mapsto \quad \big(\big[x \mapsto u(x,0)\big], \big[y \mapsto -u(0,y)\big]\big) , \end{cases} \tag{3.2.4}$$

is well defined.

Proposition 3.2.2. *Let X be a real Banach space and $X \times X$ its product space. The real Banach spaces $(X \times X)'$ and $X' \times X'$ are $\mathbb{R}$-isomorphic by virtue of the bounded isomorphism $\Psi : (X \times X)' \to X' \times X'$ defined in* (3.2.4)*. Its inverse is given by*

$$\Xi : \begin{cases} X' \times X' & \to \quad (X \times X)', \\ (u_1, u_2) & \mapsto \quad \big[(x,y) \mapsto u_1(x) - u_2(y)\big]. \end{cases} \tag{3.2.5}$$

Proof.

- The $\mathbb{R}$-linearity of Ψ is obvious. Moreover,

$$\|\Psi u\|_\infty = \max\left(\big\| [x \mapsto u(x,0)] \big\| , \big\| [y \mapsto -u(0,y)] \big\| \right) \le \|u\|$$

for all $u \in (X \times X)'$. Thus, Ψ is bounded with $\|\Psi\| \le 1$.

- For every $(u_1, u_2) \in X' \times X'$, the functional $u := \Xi(u_1, u_2)$ defined in (3.2.5) is obviously $\mathbb{R}$-linear and

$$|u(x,y)| \le |u_1(x)| + |u_2(y)| \le \|u_1\| \, \|x\| + \|u_2\| \, \|y\| \le \|(u_1,u_2)\|_\infty \, \|(x,y)\|_\infty \, .$$

Hence, $u \in (X \times X)'$.

- By linearity of every $u \in (X \times X)'$, we obtain

$$(\Xi \circ \Psi)u = \Xi\big([x \mapsto u(x,0)], [y \mapsto -u(0,y)] \big) = \big[(x,y) \mapsto u(x,0) + u(0,y) \big] = u,$$

thus $\Xi \circ \Psi = \mathrm{id}_{(X \times X)'}$. Moreover, for $(u_1, u_2) \in X' \times X'$,

$$\begin{aligned} (\Psi \circ \Xi)(u_1,u_2) &= \Psi\big[(x,y) \mapsto u_1(x) - u_2(y) \big] \\ &= \big([x \mapsto u_1(x) - u_2(0)], [y \mapsto u_2(y) - u_1(0)] \big) \\ &= (u_1, u_2). \end{aligned}$$

Hence, $\Psi \circ \Xi = \mathrm{id}_{X' \times X'}$ and Ψ is bijective with $\Xi = \Psi^{-1}$.

□

By means of the bounded $\mathbb{R}$-isomorphism $J := \Psi \circ \Phi$, the two real Banach spaces $(X_{\mathbb{C}})'$ and $X' \times X'$ are isomorphic. A priori, $(X_{\mathbb{C}})'$ is a complex Banach space and $(X' \times X')$ can be considered as the complexification $(X')_{\mathbb{C}}$. In fact, it is easy to show that $J : (X_{\mathbb{C}})' \mapsto (X')_{\mathbb{C}}$ is $\mathbb{C}$-linear in addition. Since $\|\cdot\|_{\mathbb{C}}$ induces the product topology on $(X')_{\mathbb{C}} = X' \times X'$, the isomorphism J, considered as mapping between the complex Banach spaces $(X_{\mathbb{C}})'$ and $(X')_{\mathbb{C}}$, is also continuous.

Corollary 3.2.3. *Let X be a real Banach space and $X_{\mathbb{C}}$ be its complexification. The complex Banach spaces $(X_{\mathbb{C}})'$ and $(X')_{\mathbb{C}}$ are $\mathbb{C}$-isomorphic. A bounded isomorphism $J : (X_{\mathbb{C}})' \to (X')_{\mathbb{C}}$ is given by*

$$J : \begin{cases} (X_{\mathbb{C}})' & \to \quad (X')_{\mathbb{C}}, \\ \quad f & \mapsto \quad [x \mapsto \operatorname{Re} f(x + i0)] + i[y \mapsto \operatorname{Im} f(y + i0)], \end{cases}$$

and its inverse by

$$J^{-1} : \begin{cases} (X')_{\mathbb{C}} & \to & (X_{\mathbb{C}})', \\ u_1 + iu_2 & \mapsto & [x + iy \mapsto (u_1(x) - u_2(y)) + i(u_1(y) + u_2(x))]. \end{cases}$$

Proof. With $J = \Psi \circ \Phi$, the claims follow from the previous propositions by an easy computation. $\square$

By composing the isometric injection $\iota_1 : X' \to (X')_{\mathbb{C}}$ with the bounded isomorphism $J^{-1} : (X')_{\mathbb{C}} \to (X_{\mathbb{C}})'$, we obtain a bounded injection

$$J^{-1} \circ \iota_1 : \begin{cases} X' & \to & (X_{\mathbb{C}})', \\ u & \mapsto & u_{\mathbb{C}}, \end{cases}$$

where $u_{\mathbb{C}} := J^{-1}(u + i0) \in (X_{\mathbb{C}})'$ is given by

$$u_{\mathbb{C}}(x + iy) = u(x) + iu(y) \qquad \text{for} \quad x + iy \in X_{\mathbb{C}}.$$

With this injection, we can identify X' with the closed subset

$$J^{-1} \circ \iota_1(X') = \left\{ u_{\mathbb{C}} \in (X_{\mathbb{C}})' \, : \, u \in X' \right\} \subseteq (X_{\mathbb{C}})'.$$

A functional $f \in (X_{\mathbb{C}})'$ is an element of $X' \subseteq (X_{\mathbb{C}})'$, if and only if $Jf \in X' \times \{0\}$. This is equivalent to $\operatorname{Im} f(y + i0) = 0$ for all $y \in X$, i.e. to f being real valued on X. In this case, the corresponding element $u \in X'$ with $f = u_{\mathbb{C}}$ is given by the restriction of f to X:

$$u(x) = \operatorname{Re} f(x + i0) = f(x + i0) \in \mathbb{R} \qquad \text{for all} \quad x \in X.$$

Consider $T \in \mathcal{B}(X)$ and its complexification $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$. With the help of the above constructed bounded isomorphism J, we would like to elaborate a connection between the adjoint of the complexification $(T_{\mathbb{C}})' \in \mathcal{B}((X_{\mathbb{C}})')$ and the complexification of the adjoint $(T')_{\mathbb{C}} \in \mathcal{B}((X')_{\mathbb{C}})$. In the following proposition we will show, that the operator $J \circ (T_{\mathbb{C}})' \circ J^{-1} \in \mathcal{B}((X')_{\mathbb{C}})$ has diagonal structure. More precisely, the diagram

$$\begin{tikzcd} (X')_{\mathbb{C}} \arrow[r, "(T')_{\mathbb{C}}"] & (X')_{\mathbb{C}} \\ (X_{\mathbb{C}})' \arrow[u, "J"] \arrow[r, "(T_{\mathbb{C}})'"'] & (X_{\mathbb{C}})' \arrow[u, "J"'] \end{tikzcd}$$

is commutative, which is equivalent to

$$J \circ (T_{\mathbb{C}})' \circ J^{-1} = \begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix} = (T')_{\mathbb{C}} \qquad \text{on} \quad (X')_{\mathbb{C}}.$$

Hence, the adjoint of the complexification, considered on the isomorphic Banach space $(X')_{\mathbb{C}}$, is exactly the complexification of the adjoint, i.e. $(T_{\mathbb{C}})' \simeq (T')_{\mathbb{C}}$.

Proposition 3.2.4. *Let X be a real Banach space and $X_{\mathbb{C}}$ its complexification. Consider an operator $T \in \mathcal{B}(X)$ and its complexification $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$. Via the isomorphism $J : (X_{\mathbb{C}})' \to (X')_{\mathbb{C}}$, the adjoint of the complexification $(T_{\mathbb{C}})' \in \mathcal{B}((X_{\mathbb{C}})')$ corresponds to the complexification of the adjoint $(T')_{\mathbb{C}} \in \mathcal{B}((X')_{\mathbb{C}})$, i.e.*

$$J \circ (T_{\mathbb{C}})' = (T')_{\mathbb{C}} \circ J \qquad \text{on} \quad (X_{\mathbb{C}})'.$$

Proof. By (3.1.3), the operator $S := J \circ (T_{\mathbb{C}})' \circ J^{-1} \in \mathcal{B}((X')_{\mathbb{C}})$ can be written as a block operator matrix,

$$S = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad \text{with} \quad A,\, B \in \mathcal{B}(X').$$

We need to show $B = 0$ and $A = T'$. Recall, that $B = \pi_1 \circ S \circ \iota_2$ and $A = \pi_1 \circ S \circ \iota_1$, where $\iota_1, \iota_2 \in \mathcal{B}(X', X' \times X')$ denote the canonical embeddings and $\pi_1, \pi_2 \in \mathcal{B}(X' \times X', X')$ the canonical projections. For $u \in X'$ and any $x \in X$ we obtain

$$\begin{aligned}
\langle (\pi_1 \circ J \circ (T_{\mathbb{C}})' \circ J^{-1} \circ \iota_2) u, x \rangle &= \mathrm{Re}\ \langle (T_{\mathbb{C}})' J^{-1}(0 + iu), x + i0 \rangle \\
&= \mathrm{Re}\ \langle J^{-1}(0 + iu), T_{\mathbb{C}}(x + i0) \rangle \\
&= \mathrm{Re}\ \ i \langle u, Tx \rangle = 0,
\end{aligned}$$

and in turn, $B = \pi_1 \circ S \circ \iota_2 = \pi_1 \circ J \circ (T_{\mathbb{C}})' \circ J^{-1} \circ \iota_2 = 0$. Moreover, for $u \in X'$ and arbitrary $x \in X$,

$$\begin{aligned}
\langle (\pi_1 \circ J \circ (T_{\mathbb{C}})' \circ J^{-1} \circ \iota_1) u, x \rangle &= \mathrm{Re}\ \langle (T_{\mathbb{C}})' J^{-1}(u + i0), x + i0 \rangle \\
&= \mathrm{Re}\ \langle u_{\mathbb{C}}, T_{\mathbb{C}}(x + i0) \rangle = \mathrm{Re}\ \langle u, Tx \rangle \\
&= \langle T'u, x \rangle.
\end{aligned}$$

Therefore, $A = \pi_1 \circ S \circ \iota_1 = \pi_1 \circ J \circ (T_{\mathbb{C}})' \circ J^{-1} \circ \iota_1 = T'$. □

Note, that since $(T')_{\mathbb{C}} \in \mathcal{B}((X')_{\mathbb{C}})$ has diagonal structure, it leaves $X' \subseteq (X')_{\mathbb{C}}$ invariant. Hence, $(T_{\mathbb{C}})' \in \mathcal{B}((X_{\mathbb{C}})')$ leaves $X' \simeq J^{-1} \circ \iota_1(X') \subseteq (X_{\mathbb{C}})'$ invariant.

Chapter 4

The resolvent of a compact operator

In the present chapter we investigate the Laurent series representation of the resolvent of a compact operator $T \in \mathcal{B}(Y)$, where Y is a complex Banach space, centered at an isolated spectral point $\lambda \in \sigma(T)$. As we recalled in Theorem 2.1.1, for compact $T \in \mathcal{B}(Y)$, every $\lambda \in \sigma(T)\backslash\{0\}$ is an isolated point of the spectrum. We will show, that it is a pole of the resolvent. In order to do so, we recall the theory of Banach algebras and present a functional calculus for them, which we apply to the Banach algebra $\mathcal{B}(Y)$. In the following, all Banach spaces are considered over $\mathbb{C}$.

4.1 Banach algebras

A *Banach algebra* $\langle Z, \cdot \rangle$ is a Banach space $\langle Z, \|.\| \rangle$, that carries an additional algebraic structure, i.e. an operation

$$\cdot : \begin{cases} Z \times Z & \to \quad Z, \\ (x, y) & \mapsto \quad x \cdot y, \end{cases}$$

which is associative and consistent with the norm,

$$\|x \cdot y\| \leq \|x\| \, \|y\| \, ,$$

for $x, y \in Z$. We denote $xy := x \cdot y$. If in addition, there exists a neutral element e,

$$ex = xe = x \qquad \text{for all} \quad x \in Z,$$

with $\|e\| = 1$, then $\langle Z, \cdot, e \rangle$ is a *unital* Banach algebra with *unit* e. For the sake of simplicity, we will write Z instead of $\langle Z, \cdot, e \rangle$, if the operation and the unit cannot be mistaken. Let us recall some basic results about Banach algebras. Their proofs can be found in most books on functional analysis, e.g [He, Ch XIII].

An element $x \in Z$ of a unital Banach algebra is *invertible*, if $yx = xy = e$ for some $y \in Z$. In this case, the *inverse* y is unique and denoted $x^{-1} := y$. For $x \in Z$, the *resolvent set* of x is the set

$$\rho(x) := \{\, \lambda \in \mathbb{C} \,:\, \lambda e - x \text{ is invertible in } Z \,\} \subseteq \mathbb{C}.$$

The *spectrum* of x is defined as

$$\sigma(x) := \mathbb{C} \backslash \rho(x) = \{\, \lambda \in \mathbb{C} \,:\, \lambda e - x \text{ is not invertible in } Z \,\}.$$

Some properties of the *resolvent*

$$r_x : \begin{cases} \rho(x) & \to & Z, \\ \mu & \mapsto & (\mu e - x)^{-1}, \end{cases}$$

and the spectrum are recalled in the following theorem.

Theorem 4.1.1. *Let $\langle Z, \cdot, e \rangle$ be a unital Banach algebra and $x \in Z$.*

(i) *The resolvent set $\rho(x)$ is open. In particular,*

$$U_{\frac{1}{\|r_x(\mu)\|}}(\mu) \subseteq \rho(x)$$

for every $\mu \in \rho(x)$. The resolvent $r_x(\nu)$ can be written as

$$r_x(\nu) = \sum_{n=0}^{\infty} (-1)^n (\nu - \mu)^n r_x(\mu)^{n+1}$$

for $|\nu - \mu| < \frac{1}{\|r_x(\mu)\|}$, the series being absolutely convergent in Z.

(ii) *The spectrum $\sigma(x)$ is non-empty, compact and*

$$r(x) := \max_{\lambda \in \sigma(x)} |\lambda| = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}},$$

called the spectral radius *of x.*

(iii) *If $|\mu| > r(x)$, then $\mu \in \rho(x)$ and $r_x(\mu)$ is given by*

$$r_x(\mu) = \sum_{n=0}^{\infty} \mu^{-n-1} x^n,$$

in terms of absolute convergence in Z.

Proof. This theorem and its proof can be found in [He, p 467]. □

4.2 Holomorphic functions with values in Banach spaces

In this section we recall some results of Banach space valued complex analysis. Note, that their proofs can easily be adapted from the ordinary complex valued case and are therefore omitted. Full proofs and further explanations can be found in [K, Ch 11].

Consider $f : D \to Z$, where $D \subseteq \mathbb{C}$ is open and Z is a complex Banach space. Then f is called *complex differentiable* in $z \in D$, if the limit

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists in Z. If $f'(z)$ exists for all $z \in D$ and

$$f' : \begin{cases} D & \to \quad Z, \\ z & \mapsto \quad f'(z), \end{cases}$$

is continuous, f is called *holomorphic* in D. As usual, higher derivatives are defined recursively. Clearly, the differentiability of f implies the continuity at the considered point.

Recall the definition of the *complex line integral* of a function $f : D \to Z$ over a *path* γ, i.e. a continuous function $\gamma : [a, b] \to D$:

$$\int_\gamma f(\zeta)\, d\zeta := \lim_{|\mathcal{R}| \to 0} \sum_{j=1}^{n(\mathcal{R})} (\gamma(\zeta_j) - \gamma(\zeta_{j-1})) f(\gamma(\alpha_j)),$$

if the limit exists in Z. Here $\mathcal{R} = \{a = \zeta_0 < \cdots < \zeta_{n(\mathcal{R})} = b\} \subseteq [a, b]$ is a *partition* of the interval $[a, b]$ and $\alpha_j \in (\zeta_j, \zeta_{j+1})$.

$$|\mathcal{R}| := \max_{j=1,..,n} (\zeta_j - \zeta_{j-1})$$

denotes the *norm* of the partition $\mathcal{R}$. The existence of the line integral is assured, if f is continuous and γ is a *rectifiable* path, i.e. the *length*

$$l(\gamma) := \sup_{\mathcal{R}} \sum_{j=1}^{n(\mathcal{R})} |\gamma(\zeta_j) - \gamma(\zeta_{j-1})|$$

of γ is finite. In particular, it exists if f is holomorphic and γ is a piecewise continuously differentiable path. We will limit our considerations to piecewise continuously differentiable paths, although all the given results also hold true for general rectifiable paths. If it exists, the complex line integral has the following properties:

- It is linear, i.e.
$$\int_\gamma (\alpha f(\zeta) + \beta g(\zeta))\ d\zeta = \alpha \int_\gamma f(\zeta)\ d\zeta + \beta \int_\gamma g(\zeta)\ d\zeta$$
for $f, g : D \to Z$ and $\alpha, \beta \in \mathbb{C}$.

- The norm of the integral satisfies the estimate
$$\Big\| \int_\gamma f(\zeta)\ d\zeta \Big\| \leq \sup_{t \in [a,b]} \| f \circ \gamma(t) \|\, l(\gamma). \tag{4.2.1}$$

- With another complex Banach space U and $A \in \mathcal{B}(Z, U)$, we have
$$A\Big(\int_\gamma f(\zeta)\ d\zeta \Big) = \int_\gamma A\left(f(\zeta)\right)\ d\zeta. \tag{4.2.2}$$

Consider two paths $\gamma, \eta : [a, b] \to D$. A continuous mapping $\Gamma : [a, b] \times [c, d] \to D$ satisfying
$$\Gamma(t, c) = \gamma(t) \qquad \text{and} \qquad \Gamma(t, d) = \eta(t) \qquad \text{for all} \quad t \in [a, b]$$
is called a *homotopy* between γ and η. If $\gamma(a) = \eta(a)$, $\gamma(b) = \eta(b)$ and if there exists a homotopy Γ between γ and η with
$$\Gamma(a, s) = \gamma(a) \qquad \text{and} \qquad \Gamma(b, s) = \gamma(b) \qquad \text{for all} \quad s \in [c, d],$$
then γ and η are called *homotopic* in D.

When integrated over homotopic paths, holomorphic functions show a particular behaviour. Indeed, if $f : D \to Z$ is holomorphic and γ, η are piecewise continuously differentiable paths, which are homotopic in D, then the *Cauchy Theorem* states
$$\int_\gamma f(\zeta)\ d\zeta = \int_\eta f(\zeta)\ d\zeta. \tag{4.2.3}$$
In fact, the two paths do not necessarily need to satisfy $\gamma(a) = \eta(a)$ and $\gamma(b) = \eta(b)$ for the above to be true. In the specific case, that $v, w \in D$ and $0 < r_1 < r_2$ such that $K_{r_1}(v) \subseteq K_{r_2}(w)$ and $K_{r_2}(w) \backslash U_{r_1}(v) \subseteq D$, the identity (4.2.3) is satisfied with
$$\gamma(t) = v + r_1 \exp(it) \qquad \text{and} \qquad \eta(t) = w + r_2 \exp(it), \qquad t \in [0, 2\pi].$$
As usual, $K_r(z_0) = \{\, z \in \mathbb{C} : |z - z_0| \leq r \,\} \subseteq \mathbb{C}$ denotes the closed disc with radius $r > 0$ around $z_0 \in \mathbb{C}$ and $U_r(z_0) = K_r(z_0)^\circ$ the open one. The Cauchy Theorem ensures, that all the line integrals we will encounter will only depend on the homotopy class, regardless of which representative will be integrated over.

Holomorphic functions turn out to have the extraordinary property of being continuously differentiable infinitely many times. More specifically, if $f : D \to Z$ is holomorphic on the open subset $D \subseteq \mathbb{C}$, then the nth iterated derivative $f^{(n)} : D \to Z$ exists for every $n \in \mathbb{N}$ and is holomorphic. If $w \in D$ and $\rho > 0$ such that $K_\rho(w) \subseteq D$, then for every $z \in U_\rho(w)$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Here either $\gamma = \gamma_\rho$, where

$$\gamma_\rho : \begin{cases} [0, 2\pi] & \to \quad D, \\ \zeta & \mapsto \quad w + \rho \exp(i\zeta), \end{cases} \tag{4.2.4}$$

or γ is a piecewise continuously differentiable path, which is homotopic to γ_ρ in $D \backslash \{z\}$; see (4.2.3). For $n = 1$, the above formula yields the well-known *Cauchy integral formula*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

for all $z \in U_\rho(w)$ and corresponding γ.

Let us recall an important class of holomorphic functions. Consider the Z-valued *power series*

$$\sum_{n=0}^{\infty} (z - w)^n a_n \qquad \text{with} \quad a_n \in Z, \quad n \in \mathbb{N} \cup \{0\},$$

centered at $w \in \mathbb{C}$. Then the *radius of convergence*

$$R := \sup \left\{ |z - w| \ : \ \sum_{n=0}^{\infty} (z - w)^n a_n \text{ convergent in } Z \right\}$$

satisfies

$$R = \frac{1}{\limsup_{n\to\infty} \|a_n\|^{\frac{1}{n}}}.$$

For $z \in \mathbb{C}$ such that $|z - w| < R$, the series converges absolutely. For $|z - w| > R$, it is divergent. Given $0 < r < R$, the series converges uniformly on $K_r(w)$ and the function

$$z \mapsto f(z) := \sum_{n=0}^{\infty} (z - w)^n a_n$$

is bounded on $K_r(w)$ and continuous on its domain $U_R(w)$. Moreover, f is holomorphic on $U_R(w)$ and $a_n = \frac{f^{(n)}(w)}{n!}$ for every $n \in \mathbb{N} \cup \{0\}$. Importantly, the coefficients a_n of the power series representation of f around w are uniquely determined.

The concept of power series is equivalent to the one of holomorphy. In fact, every holomorphic function can be expanded into a convergent power series around any point of its domain. To be more precise, $f : D \to Z$ is holomorphic, if and only if it is *analytic*, i.e. for every $w \in D$ there exists $\rho_w > 0$, such that on $U_{\rho_w}(w) \subseteq D$ the function f is given by a power series

$$f(z) = \sum_{n=0}^{\infty}(z-w)^n a_n$$

with radius of convergence $R_w \geq \rho_w$. The coefficients $a_n \in Z$ are uniquely determined by f and w. Moreover, f is represented by the above power series expansion on the biggest disc around w, which is still contained in D, i.e.

$$R_w \geq \sup\{\, r > 0 \,:\, U_r(w) \subseteq D \,\}\,.$$

For our considerations about the resolvent, we need a more general series representation of f, where the center of the expansion does not necessarily have to lie in the domain of holomorphy D. In general, the series is not a power series any more, but a power series in z and z^{-1}. Concretely, if $f : D \to Z$ is holomorphic and $w \in \mathbb{C}$, $0 \leq r_w < R_w \leq +\infty$ such that $U_{r_w,R_w}(w) := U_{R_w}(w) \backslash K_{r_w}(w) \subseteq D$, then

$$f(z) = \sum_{n=-\infty}^{\infty}(z-w)^n a_n := \sum_{n=1}^{\infty}\frac{1}{(z-w)^n}a_{-n} + \sum_{n=0}^{\infty}(z-w)^n a_n \tag{4.2.5}$$

for every $z \in U_{r_w,R_w}(w)$, where $a_n \in Z$ for all $n \in \mathbb{Z}$. The radius of convergence of the series $\sum_{n=0}^{\infty}\zeta^n a_n$ is at least R_w and the radius of convergence of $\sum_{n=1}^{\infty}\zeta^n a_{-n}$ is at least $\frac{1}{r_w}$, so the above series converge absolutely for every $z \in U_{R_w}(w)\backslash K_{r_w}(w)$ and uniformly on every $K_R(w)\backslash U_r(w)$ with $r_w < r < R < R_w$. The series (4.2.5) is called the *Laurent series* of f around w and

$$\sum_{n=1}^{\infty}\frac{1}{(z-w)^n}a_{-n}$$

is called its *principal part.* The coefficients $a_n \in Z$ are uniquely determined by f,w and the domain $U_{r_w,R_w}(w)$ and are given by

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-w)^{n+1}}\,d\zeta, \qquad n \in \mathbb{Z}.$$

Either $\gamma = \gamma_\rho$ for some $r_w < \rho < R_w$, or γ is a piecewise continuously differentiable path, which is homotopic to γ_ρ in $U_{r_w,R_w}(w)$; see (4.2.3). Note, that the radii can be chosen in a maximal way,

$$\begin{aligned} r_w &= \inf\{\, r > 0 \,:\, \exists\, R > 0,\ U_{r,R}(w) \subseteq D \,\}\,, \\ R_w &= \sup\{\, R > 0 \,:\, \exists\, r > 0,\ U_{r,R}(w) \subseteq D \,\}\,. \end{aligned} \tag{4.2.6}$$

Consider $f : D \to Z$ holomorphic. An *isolated* point w of $\mathbb{C}\backslash D$, i.e. $U_\varepsilon(w)\backslash\{w\} \subseteq D$ for some $\varepsilon > 0$, is called *isolated singularity* of f. The Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} (z-w)^n a_n$$

of f around w converges for $z \in U_{R_w}(w)\backslash\{w\}$, i.e. $r_w = 0$. Regarding an isolated singularity, three cases can occur:

- If $a_{-n} = 0$ for all $n \in \mathbb{N}$, then w is a *removable singularity* of f.
- If $a_{-N} \neq 0$ for some $N \in \mathbb{N}$ and $a_{-n} = 0$ for all $n > N$, then w is called a *pole* of f of *order* N.
- If $a_{-n} \neq 0$ for infinitely many $n \in \mathbb{N}$, one calls w an *essential singularity* of f.

In case that w is a removable singularity, f can be extended to a holomorphic function $\tilde{f} : D \cup \{w\} \to Z$ by setting $\tilde{f}(z) := a_0$.

We apply the above results to a unital Banach algebra Z, the subset $D = \rho(x) \subseteq \mathbb{C}$ and the function $f = r_x$. We already recalled some properties of the resolvent of an element $x \in Z$ in the previous section. According to Theorem 4.1.1, the domain $\rho(x)$ is open and the resolvent $r_x : \rho(x) \to Z$ is analytic, i.e. holomorphic on $\rho(x)$.

Let $\lambda \in \sigma(x)$ be an isolated singularity of r_x. As a holomorphic function, r_x can be expanded into a Laurent series, which converges at least on $U_\varepsilon(\lambda)\backslash\{\lambda\}$ for some $\varepsilon > 0$. We summarize the above results in the following theorem:

Theorem 4.2.1 [Laurent expansion of the resolvent]. *Let $\langle Z, \cdot, e\rangle$ be a unital Banach algebra and $x \in Z$. Consider an isolated spectral point of x, i.e. $\lambda \in \sigma(x)$ and $U_\varepsilon(\lambda) \cap \sigma(x) = \{\lambda\}$ for some $\varepsilon > 0$. Then r_x can be expanded into a Laurent series centered at λ,*

$$r_x(\mu) = \sum_{n=-\infty}^{\infty} (\mu-\lambda)^n a_n, \qquad a_n \in Z,\ n \in \mathbb{Z}, \quad \mu \in U_{R_\lambda}(\lambda)\backslash\{\lambda\}.$$

The series is absolutely convergent for $\mu \in U_{R_\lambda}(\lambda)\backslash\{\lambda\}$, where $R_\lambda \geq \varepsilon$ is the biggest radius, such that $U_{R_\lambda}(\lambda)\backslash\{\lambda\} \subseteq \rho(x)$, defined in (4.2.6). It converges uniformly on $K_R(\lambda)\backslash U_r(\lambda)$ with $0 < r < R < R_\lambda$. The coefficients $a_n \in Z$ are uniquely determined and given by

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{r_x(\zeta)}{(\zeta-\lambda)^{n+1}}\, d\zeta, \qquad n \in \mathbb{Z}.$$

Here, either $\gamma = \gamma_\rho$ for some $0 < \rho < R_\lambda$, or γ is a piecewise continuously differentiable path, which is homotopic to γ_ρ in $\rho(x)$.

In the setting of general Banach algebras, we are now able to show the first step of the sought result, that an isolated singularity of the resolvent is not removable.

Corollary 4.2.2. *Let $\langle Z, \cdot, e\rangle$ be a unital Banach algebra and $x \in Z$. If $\lambda \in \sigma(x)$ is an isolated point of the spectrum, it is either a pole or an essential singularity of the resolvent r_x.*

Proof. Assume, λ is a removable singularity of r_x. Then the Laurent series of r_x around λ is a regular power series,

$$r_x(\mu) = \sum_{n=0}^{\infty} (\mu - \lambda)^n a_n \qquad \text{for} \quad 0 < |\mu - \lambda| < R_\lambda.$$

As a power series, it is uniformly convergent on $K_r(\lambda)$ for every $0 < r < R_\lambda$ and we obtain

$$\lim_{\mu \to \lambda} r_x(\mu) = \lim_{\mu \to \lambda} \sum_{n=0}^{\infty} (\mu - \lambda)^n a_n = \sum_{n=0}^{\infty} \lim_{\mu \to \lambda} (\mu - \lambda)^n a_n = a_0.$$

Taking the limit $\mu \to \lambda$ in

$$(\mu e - x) r_x(\mu) = r_x(\mu)(\mu e - x) = e$$

yields $(\lambda e - x)a_0 = a_0(\lambda e - x) = e$, in contradiction to $\lambda \in \sigma(x)$. $\square$

In case, that the center of expansion is a pole of the resolvent of order $m \in \mathbb{N}$, the leading coefficient a_{-m} is easily computed.

Proposition 4.2.3. *Let $\langle Z, \cdot, e\rangle$ be a unital Banach algebra and let $\lambda \in \sigma(x)$ be a pole of the resolvent r_x of order $m \in \mathbb{N}$. Consider the Laurent series representation of r_x around λ,*

$$r_x(\mu) = \sum_{n=-m}^{\infty} (\mu - \lambda)^n a_n, \qquad a_n \in Z,\ n \geq -m, \quad \mu \in U_{R_\lambda}(\lambda) \backslash \{\lambda\},$$

with $R_\lambda > 0$ as in Theorem 4.2.1. Then

$$a_{-m} = \lim_{\mu \to \lambda} (\mu - \lambda)^m r_x(\mu) \qquad \text{in} \quad Z.$$

Proof. Multiplying $r_x(\mu)$ with $(\mu - \lambda)^m$, for every $\mu \in U_{R_\lambda}(\lambda) \backslash \{\lambda\}$ we obtain

$$(\mu - \lambda)^m r_x(\mu) = (\mu - \lambda)^m \sum_{n=-m}^{\infty} (\mu - \lambda)^n a_n = \sum_{n=0}^{\infty} (\mu - \lambda)^n a_{n-m}.$$

The series on the right is a regular power series. Thus, it is uniformly convergent on $K_r(\lambda)$ for every $0 < r < R_\lambda$. Taking the limit $\mu \to \lambda$ yields

$$\lim_{\mu\to\lambda}(\mu-\lambda)^m r_x(\mu) = \lim_{\mu\to\lambda}\sum_{n=0}^{\infty}(\mu-\lambda)^n a_{n-m} = \sum_{n=0}^{\infty}\lim_{\mu\to\lambda}(\mu-\lambda)^n a_{n-m} = a_{-m}.$$

□

Remark 4.2.4. The above statement equally holds true for general holomorphic functions and their Laurent series having a pole at its center. We will not make use of the general case though. //

4.3 Functional calculus

In this section, we present a functional calculus, which allows us to define $f(x) \in Z$ for an element x of a unital Banach algebra Z and a holomorphic function f. We will not prove the presented results. They can be found together with their complete proofs in [He, Ch 98,99].

We demonstrate the idea of the calculus in a specific setting. Consider the holomorphic function $f : U_r(0) \to \mathbb{C}$ given as the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \in \mathbb{C}, \quad |z| < r,$$

with radius of convergence $r > 0$. The idea is to define $f(x)$ by naively replacing $z \in \mathbb{C}$ by $x \in Z$ in this expansion. Indeed, if $x \in Z$ satisfies $r(x) < r$, the series

$$f(x) := \sum_{n=0}^{\infty} a_n x^n \tag{4.3.1}$$

converges in Z. Another representation of f is given by the Cauchy formula,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_\rho} f(\zeta)(\zeta - z)^{-1}\, d\zeta, \tag{4.3.2}$$

where $z \in U_\rho(0)$ with $\rho < r$ and γ_ρ is defined as in (4.2.4). If in addition $\rho > r(x)$, then the image of γ_ρ is fully contained in the open set $U_{r(x),r}(0)$, where both f and r_x are holomorphic and the line integral

$$\int_{\gamma_\rho} f(\zeta) r_x(\zeta)\, d\zeta$$

is well defined. It turns out, that once more, naively replacing z by x in (4.3.2), is consistent with (4.3.1):

$$f(x) = \frac{1}{2\pi i}\int_{\gamma_\rho} f(\zeta) r_x(\zeta)\ d\zeta.$$

The previous considerations give reason to define $f(x)$ by a Cauchy integral. For fixed $x \in Z$, we present a calculus for $f(x)$ in dependence of the holomorphic function f. In the following, $Z = \langle Z, \cdot, e\rangle$ will always denote a unital Banach algebra.

We turn our attention to the domain of the calculus. Fix $x \in Z$. For the definition of $f(x)$ we are only interested in the behaviour of the functions in a neighborhood of $\sigma(x)$, which motivates identifying two functions, if they coincide on some neighborhood of the spectrum.

$$H(x) := \{\, f : \Delta_f \to \mathbb{C} \,:\, \Delta_f \supset \sigma(x) \text{ open},\ f \text{ holomorphic on } \Delta_f \,\}$$

is the set of all holomorphic functions with open domains containing $\sigma(x)$. Consider the equivalence relation $\sim$ defined by

$$f \sim g \qquad :\Leftrightarrow \qquad f|_D \equiv g|_D \quad \text{with} \quad D \subseteq \mathbb{C} \text{ open},\ \sigma(x) \subseteq D \subseteq \Delta_f \cap \Delta_g$$

on $H(x)$. Finally, the domain $\mathcal{H}(x) := H(x)/_\sim$ is defined as the quotient set of $H(x)$ with respect to $\sim$.

In order to equip $\mathcal{H}(x)$ with an algebraic structure, the corresponding operations are defined on $H(x)$ first. Since two functions in $H(x)$ do not have a common domain, the domains need to be adjusted appropriately. For $f, g \in H(x)$ and $\lambda \in \mathbb{C}$ the addition and scalar multiplication are defined as

$$\begin{aligned} (\lambda f)(z) &:= \lambda f(z) \qquad && \text{for} \quad z \in \Delta_f, \\ (f+g)(z) &:= f(z) + g(z) \qquad && \text{for} \quad z \in \Delta_f \cap \Delta_g. \end{aligned}$$

Note, that $H(x)$ is not a vector space with these operations, since e.g. $f - g = 0$ does not imply $f = g$. An additional operation $\cdot$, the multiplication of functions, is defined on $H(x)$ as

$$(f \cdot g)(z) := f(z)g(z) \qquad \text{for} \quad z \in \Delta_f \cap \Delta_g.$$

For the sake of simplicity we also write $fg := f \cdot g$. All the above defined operations are consistent with the equivalence relation, i.e. $f_1 \sim f_2$ and $g_1 \sim g_2$ imply

$$\lambda f_1 \sim \lambda f_2, \qquad f_1 + g_1 \sim f_2 + g_2 \qquad \text{and} \qquad f_1 g_1 \sim f_2 g_2.$$

Hence, the operations are well defined on the quotient set $\mathcal{H}(x)$, which becomes a vector space with the multiplication $\cdot$ as an additional binary operation. It is of importance for the calculus, that $\mathcal{H}(x)$ carries the same algebraic structure as Z. Note, that $\mathcal{H}(x)$ is not a Banach algebra though.

We classify the paths of integration used in the calculus. In the motivating situation it was graphically clear, that $\sigma(x)$ lied in the interior of γ_ρ and γ_ρ circled around $\sigma(x)$ in a mathematically positive way. These notions can be defined precisely for a more general class of paths.

For a closed, piecewise continuously differentiable path $\gamma : [a, b] \to \mathbb{C}$ and $\alpha \in \mathbb{C}\backslash\{\gamma\}$, where $\{\gamma\} := \{\, \gamma(t) \, : \, t \in [a, b] \,\}$ denotes the compact *trace* of γ, the *winding number* of γ around α is defined as

$$\mathrm{n}(\gamma, \alpha) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - \alpha} \, d\zeta \ \in \mathbb{Z}.$$

On every connected component of $\mathbb{C}\backslash\{\gamma\}$, the winding number is a constant integer and vanishes on the unbounded connected component. A closed, piecewise continuously differentiable path is called *positively oriented*, if $\mathrm{n}(\gamma, \alpha) \in \{0, 1\}$ for every $\alpha \notin \{\gamma\}$. We define the *inside* and *outside* of such γ as

$$\begin{aligned} \mathrm{ins}(\gamma) &:= \{\, \alpha \in \mathbb{C}\backslash\{\gamma\} \, : \, \mathrm{n}(\gamma, \alpha) = 1 \,\}, \\ \mathrm{out}(\gamma) &:= \{\, \alpha \in \mathbb{C}\backslash\{\gamma\} \, : \, \mathrm{n}(\gamma, \alpha) = 0 \,\}. \end{aligned}$$

For a collection $\Gamma = \{\gamma_1, ..., \gamma_m\}$ of closed, piecewise continuously differentiable paths and $\alpha \notin \{\Gamma\} := \bigcup_{i=1}^m \{\gamma_i\}$, the *winding number* of Γ is defined as

$$\mathrm{n}(\Gamma, \alpha) := \sum_{i=1}^m \mathrm{n}(\gamma_i, \alpha) \ \in \mathbb{Z}.$$

One calls Γ *positively oriented*, if $\{\gamma_i\} \cap \{\gamma_j\} = \emptyset$ for $i \neq j$ and $\mathrm{n}(\Gamma, \alpha) \in \{0, 1\}$ for every $\alpha \notin \{\Gamma\}$. For positively oriented Γ, the *inside* and *outside* of Γ are defined as before,

$$\begin{aligned} \mathrm{ins}(\Gamma) &:= \{\, \alpha \in \mathbb{C}\backslash\{\Gamma\} \, : \, \mathrm{n}(\Gamma, \alpha) = 1 \,\}, \\ \mathrm{out}(\Gamma) &:= \{\, \alpha \in \mathbb{C}\backslash\{\Gamma\} \, : \, \mathrm{n}(\Gamma, \alpha) = 0 \,\}. \end{aligned}$$

If $f : D \to Z$ is (at least) continuous on $D \subseteq \mathbb{C}$ open with $\{\Gamma\} \subseteq D$, the integral of f over Γ is defined as

$$\int_\Gamma f(\zeta) \, d\zeta := \sum_{i=1}^m \int_{\gamma_i} f(\zeta) \, d\zeta.$$

Note, that the above integral exists under the given circumstances. We have now prepared everything to define $f(x)$.

Definition 4.3.1. Let $\langle Z, \cdot, e\rangle$ be a unital Banach algebra and $x \in Z$. For $f \in H(x)$ let Γ be a positively oriented collection of closed, piecewise continuously differentiable paths such that $\{\Gamma\} \subseteq \Delta_f$ and $\sigma(x) \subseteq \mathrm{ins}(\Gamma)$. Define

$$f(x) := \frac{1}{2\pi i} \int_\Gamma f(\zeta) r_x(\zeta)\, d\zeta \ \in Z.$$

//

For the compact subset $\sigma(x) \subseteq \Delta_f$ there always exists a collection Γ of closed, piecewise continuously differentiable paths with the above properties. In fact, they can even be chosen as the border of a covering of $\sigma(x)$ by finitely many squares; see [C, p 195f].

It turns out, that indeed the expression $f(x)$ does not depend on the representative of the equivalence classes of $\mathcal{H}(x)$. If $f \sim g$ and Γ_f, Γ_g are positively oriented collections of closed, piecewise continuously differentiable paths such that

$$\sigma(x) \subseteq \mathrm{ins}(\Gamma_f), \mathrm{ins}(\Gamma_g) \qquad \text{and} \qquad \{\Gamma_f\} \subseteq \Delta_f, \quad \{\Gamma_g\} \subseteq \Delta_g,$$

then applying the Cauchy Theorem (4.2.3) one can show

$$\frac{1}{2\pi i} \int_{\Gamma_f} f(\zeta) r_x(\zeta)\, d\zeta = \frac{1}{2\pi i} \int_{\Gamma_g} g(\zeta) r_x(\zeta)\, d\zeta.$$

This guarantees the functional calculus to be well defined. We state its useful properties.

Theorem 4.3.2 [Functional calculus]. *Let $\langle Z, \cdot, e\rangle$ be a unital Banach algebra and $x \in Z$. The previously elaborated functional calculus*

$$\Phi : \begin{cases} \mathcal{H}(x) & \to & Z, \\ [f]_\sim & \mapsto & f(x), \end{cases}$$

is an algebra homomorphism, i.e. it is compatible with the addition, scalar multiplication and multiplication of elements in $\mathcal{H}(x)$ and Z. In particular, $f(x)g(x) = g(x)f(x)$ for $f, g \in H(x)$. The calculus has the following properties:

(i) *If f is given by the the power series*

$$f(z) := \sum_{n=0}^\infty a_n z^n, \qquad a_n \in \mathbb{C}, \qquad |z| < R,$$

with radius of convergence $R > r(x)$, then $f \in H(x)$ and

$$f(x) = \sum_{n=0}^\infty a_n x^n.$$

(ii) *If* $f \equiv 1$ *on* Δ_f*, then* $f(x) = e$.

(iii) *If* $f(z) = z$ *for* $z \in \Delta_f$*, then* $f(x) = x$.

(iv) *If* $f(z) \neq 0$ *for* $z \in \sigma(x)$*, then* $f(x)$ *is invertible and* $f(x)^{-1} = (\frac{1}{f})(x)$.

Proof. See [He, p 484]. □

One special case is worth to be considered closely. The subset $\sigma_1 \subseteq \sigma(x)$ is called a *spectral set*, if it is *clopen* in $\sigma(x)$, i.e. closed and open with respect to the relative topology on $\sigma(x)$. The two spectral sets σ_1 and $\sigma_2 := \sigma(x) \backslash \sigma_1$ are called *complementary spectral sets*. There exist two open and disjoint sets $\Delta_1, \Delta_2 \subseteq \mathbb{C}$ with $\Delta_1 \supseteq \sigma_1$ and $\Delta_2 \supseteq \sigma_2$. On the open set $\Delta := \Delta_1 \dot\cup \Delta_2 \supseteq \sigma(x)$ we define two holomorphic functions,

$$f_1(z) := \begin{cases} 1 & \text{for } z \in \Delta_1, \\ 0 & \text{for } z \in \Delta_2, \end{cases} \qquad \text{and} \qquad f_2(z) := \begin{cases} 0 & \text{for } z \in \Delta_1, \\ 1 & \text{for } z \in \Delta_2. \end{cases}$$

Then $f_1, f_2 \in H(x)$ and

$$p_1 := f_1(x), \qquad p_2 := f_2(x)$$

are well defined elements of Z. They are called the *idempotents* associated with σ_1 and σ_2, respectively. Note, that p_1 and p_2 do not depend on the choice of the sets Δ_1 and Δ_2.

Corollary 4.3.3. *Let* $\langle Z, \cdot, e\rangle$ *be a unital Banach algebra and* $x \in Z$*. For the complementary spectral sets* $\sigma_1 \dot\cup \sigma_2 = \sigma(x)$*, let* $p_1, p_2 \in Z$ *be the corresponding idempotents. Then* $p_1^2 = p_1$ *and* $p_2^2 = p_2$*. Moreover,*

$$p_1 p_2 = p_2 p_1 = 0 \qquad \text{and} \qquad p_1 + p_2 = e.$$

All the statements remain true, if one of the spectral sets is empty.

Remark 4.3.4. Consider $f \in H(x)$ and the complementary spectral sets $\sigma_1 \dot\cup \sigma_2 = \sigma(x)$. Assume, that $f(z) = 0$ for $z \in \sigma_2$, i.e. there exists an open set $\Delta_2 \supseteq \sigma_2$ with $f|_{\Delta_2} \equiv 0$. Then $f(x) \in Z$ can be written as the integral

$$f(x) = \frac{1}{2\pi i} \int_\Gamma f(\zeta) r_x(\zeta)\, d\zeta,$$

where Γ is a positively oriented collection of closed, piecewise continuously differentiable paths, such that $\sigma_1 \subseteq \mathrm{int}(\Gamma)$ and $\sigma_2 \subseteq \mathrm{out}(\Gamma)$. In particular, the idempotents associated with σ_1 and σ_2 are given by

$$p_j = \frac{1}{2\pi i} \int_{\Gamma_j} r_x(\zeta)\, d\zeta, \qquad j = 1, 2, \tag{4.3.3}$$

where each Γ_j is a positively oriented collection of closed, piecewise continuously differentiable paths such that $\sigma_j \subseteq \operatorname{int}(\Gamma_j)$ and $\sigma_i \subseteq \operatorname{out}(\Gamma_j)$, $i \neq j$. //

4.4 Spectral projections

It is well known, that the Banach space $\mathcal{B}(Y)$, together with the composition $\circ$ of operators and the identity operator I, forms a unital Banach algebra. In the present section, the functional calculus for the said unital Banach algebra $\langle Z, \cdot, e\rangle = \langle \mathcal{B}(Y), \circ, I\rangle$ is examined, where Y is a complex Banach space. We can apply the results of the previous section, in particular Corollary 4.3.3 and relation (4.3.3).

Let $T \in \mathcal{B}(Y)$. Consider a (potentially empty) spectral set $\sigma \subseteq \sigma(T)$ and the corresponding complementary spectral set $\tau := \sigma(T)\backslash\sigma$. According to (4.3.3), the *spectral projector* $P_\sigma \in \mathcal{B}(Y)$ associated with σ is given by

$$P_\sigma := \frac{1}{2\pi i}\int_{\Gamma_\sigma} R_T(\zeta)\ d\zeta, \tag{4.4.1}$$

where Γ_σ is a positively oriented collection of closed, piecewise continuously differentiable paths with $\sigma \subseteq \operatorname{ins}(\Gamma_\sigma)$ and $\tau \subseteq \operatorname{out}(\Gamma_\sigma)$. The complementary spectral projector $P_\tau \in \mathcal{B}(Y)$ is given by

$$I - P_\sigma = P_\tau := \frac{1}{2\pi i}\int_{\Gamma_\tau} R_T(\zeta)\ d\zeta,$$

with a positively oriented collection of closed, piecewise continuously differentiable paths Γ_τ, such that $\tau \subseteq \operatorname{ins}(\Gamma_\tau)$ and $\sigma \subseteq \operatorname{out}(\Gamma_\tau)$. Both P_σ and P_τ are projections and

$$P_\sigma P_\tau = P_\tau P_\sigma = 0.$$

In case that σ is empty, one has $P_\sigma = 0$ and $P_\tau = I$. The subspaces $M_\sigma := \operatorname{ran} P_\sigma$, $N_\sigma := \ker P_\sigma$, $M_\tau := \operatorname{ran} P_\tau$ and $N_\tau := \ker P_\tau$ are closed. They satisfy

$$M_\sigma = N_\tau, \qquad N_\sigma = M_\tau \qquad \text{and} \qquad Y = M_\sigma \dotplus N_\sigma.$$

Before we state some particular properties of this decomposition of Y in the next theorem, we provide a lemma, which will be needed for their proofs.

Lemma 4.4.1. *Consider a Banach space Y and $T \in \mathcal{B}(Y)$. Let $Y = U \dotplus V$ be the direct sum of the closed and T-invariant subspaces $U, V \subseteq Y$. Then*

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2),$$

where $T_1 := T|_U \in \mathcal{B}(U)$ *and* $T_2 := T|_V \in \mathcal{B}(V)$ *denote the corresponding restrictions.*

Proof. We show the equivalent relation $\rho(T) = \rho(T_1) \cap \rho(T_1)$.

- Assume $\mu \in \rho(T)$. Set $I_1 := I_U$ and $I_2 := I_V$, then the operators $(\mu I_1 - T_1)$ and $(\mu I_2 - T_2)$ are injective. For arbitrary $y \in U$, there exists $x = x_1 + x_2 \in U \dot{+} V$ satisfying
$$y = (\mu I - T)x = (\mu I_1 - T_1)x_1 + (\mu I_2 - T_2)x_2 \in U.$$
Hence, $(\mu I_2 - T_2)x_2 \in U \cap V = \{0\}$ and consequently, $y = (\mu I_1 - T_1)x_1$. Thus, $(\mu I_1 - T_1)$ is bijective, i.e. $\mu \in \rho(T_1)$. In the same way we obtain $\mu \in \rho(T_2)$, implying
$$\rho(T) \subseteq \rho(T_1) \cap \rho(T_2).$$
- Let $\mu \in \rho(T_1) \cap \rho(T_2)$. Take any $x = x_1 + x_2 \in U \dot{+} V$, then
$$(\mu I - T)x = (\mu I_1 - T_1)x_1 + (\mu I_2 - T_2)x_2 = 0,$$
implying $(\mu I_1 - T_1)x_1 = -(\mu I_2 - T_2)x_2 \in U \cap V = \{0\}$. Hence, we obtain $x_j \in \ker(\mu I_j - T_j) = \{0\}$ for $j = 1, 2$ and in turn, $x = 0$. Decomposing an arbitrary $y = y_1 + y_2 \in U \dot{+} V$ and solving the equations
$$y_j = (\mu I_j - T_j)x_j \qquad \text{for} \quad j = 1, 2$$
yields the surjectivity of $(\mu I - T)$. Thus, $(\mu I - T)$ is bijective, i.e. $\mu \in \rho(T)$ and we obtain the other inclusion.

□

Theorem 4.4.2 [Spectral decomposition]. *Let* Y *be a Banach space and* $T \in \mathcal{B}(Y)$. *Consider a spectral set* $\sigma \subseteq \sigma(T)$ *and the associated spectral projector* $P_\sigma \in \mathcal{B}(Y)$, *given in* (4.4.1). *Then* P_σ *induces a decomposition*
$$Y = M_\sigma \dot{+} N_\sigma$$
of Y *into the closed subspaces* $M_\sigma = \operatorname{ran} P_\sigma$ *and* $N_\sigma = \ker P_\sigma$. *Both* M_σ *and* N_σ *are invariant under* T *and* $f(T)$ *for every* $f \in H(T)$. *Moreover, the spectra of the corresponding restrictions* $T|_{M_\sigma} \in \mathcal{B}(M_\sigma)$ *and* $T|_{N_\sigma} \in \mathcal{B}(N_\sigma)$ *are given by*
$$\sigma(T|_{M_\sigma}) = \sigma \qquad \textit{and} \qquad \sigma(T|_{N_\sigma}) = \sigma(T) \backslash \sigma.$$

Proof.

- We show the invariance under $f(T)$ with $f \in H(T)$, which in particular implies the invariance under T. According to Theorem 4.3.2, $f(T)$ commutes with P_σ. Hence, for $x \in M_\sigma$ we have

  $$f(T)x = f(T)P_\sigma x = P_\sigma f(T)x \in \operatorname{ran} P_\sigma = M_\sigma.$$

  By the same considerations regarding the complementary spectral projector P_τ, we obtain $f(T)x \in N_\sigma = M_\tau$ for all $x \in N_\sigma$.

- Lemma 4.4.1 applied to $T_1 := T|_{M_\sigma}$ and $T_2 := T|_{N_\sigma}$ yields $\sigma(T_1) \cup \sigma(T_2) = \sigma(T)$. To prove the present theorem, only $\sigma(T_1) \subseteq \sigma$ remains to show. The rest follows by a symmetry argument. For $\mu \notin \sigma$, there exist disjoint open sets

  $$\Delta_\sigma \supseteq \sigma, \quad \mu \notin \Delta_\sigma \qquad \text{and} \qquad \Delta_\tau \supseteq \tau.$$

  We define the two functions $f_\sigma, f \in H(T)$ as

  $$f_\sigma(z) := \begin{cases} 1 & \text{for } z \in \Delta_\sigma, \\ 0 & \text{for } z \in \Delta_\tau, \end{cases} \qquad \text{and} \qquad f(z) := \begin{cases} \frac{1}{\mu - z} & \text{for } z \in \Delta_\sigma, \\ 0 & \text{for } z \in \Delta_\tau. \end{cases}$$

  As $\mu \notin \Delta_\sigma$, truly $f \in H(T)$. We have the relation $(\mu - z)f(z) = f_\sigma(z)$ for all $z \in \Delta_\sigma \dot\cup \Delta_\tau \supseteq \sigma(x)$ and Theorem 4.3.2 yields

  $$(\mu I - T)f(T) = f(T)(\mu I - T) = f_\sigma(T) = P_\sigma.$$

  Since M_σ is invariant under T and $f(T)$, and P_σ is the identity on M_σ, restricting the above equality to M_σ implies $\mu \in \rho(T_1)$. Hence, $\sigma(T_1) \subseteq \sigma$.

□

Theorem 4.4.2 especially applies to the case $\sigma = \{\lambda\}$, where $\lambda \in \sigma(T)$ is an isolated spectral point of $T \in \mathcal{B}(Y)$. In this case, the path of integration in (4.4.1) can be chosen as γ_ρ with $0 < \rho < R_\lambda$, where R_λ is the biggest radius such that $U_{R_\lambda}(\lambda) \cap \sigma(T) = \{\lambda\}$; see (4.2.4). We obtain

$$P_\lambda := P_\sigma = \frac{1}{2\pi i} \int_{\gamma_\rho} R_T(\zeta)\, d\zeta. \tag{4.4.2}$$

The range of P_λ can be characterized in a useful way.

Theorem 4.4.3. *Let Y be a Banach space and $T \in \mathcal{B}(Y)$. Consider the spectral set $\sigma := \{\lambda\}$, where $\lambda \in \sigma(T)$ is an isolated point of the spectrum. The range of the spectral projector P_λ is given by*

$$M_\lambda := M_\sigma = \left\{ x \in Y \,:\, \lim_{n \to \infty} \|(\lambda I - T)^n x\|^{\frac{1}{n}} = 0 \right\}.$$

Proof.

- First let $x \in M_\lambda$. Recall, that $P_\lambda = f_\sigma(x)$ with $f_\sigma \in H(x)$ given by
$$f_\sigma(z) = \begin{cases} 1 & \text{for } z \in \Delta_\sigma, \\ 0 & \text{for } z \in \Delta_\tau. \end{cases}$$
Here $\Delta_\sigma \supseteq \{\lambda\}$ and $\Delta_\tau \supseteq \sigma(x)\backslash\{\lambda\}$ are open and disjoint. Since $(\lambda - z)^n f_\sigma(z) = 0$ for all $z \in \Delta_\tau$, Remark 4.3.4 and Theorem 4.3.2 yield
$$(\lambda I - T)^n P_\lambda = \frac{1}{2\pi i}\int_{\gamma_\rho} (\lambda - \zeta)^n f_\sigma(\zeta) R_T(\zeta)\ d\zeta = \frac{1}{2\pi i}\int_{\gamma_\rho} (\lambda - \zeta)^n R_T(\zeta)\ d\zeta$$
for every $n \in \mathbb{N}$ and $0 < \rho < R_\lambda$. Consider the evaluation mapping
$$\iota_x \in \mathcal{B}(\mathcal{B}(Y), Y), \qquad \iota_x : \begin{cases} \mathcal{B}(Y) & \to & Y, \\ S & \mapsto & Sx. \end{cases}$$
Applying (4.2.2) to ι_x, by $P_\lambda x = x$ we obtain
$$(\lambda I - T)^n x = \frac{1}{2\pi i}\int_{\gamma_\rho} (\lambda - \zeta)^n R_T(\zeta) x\ d\zeta.$$
Consequently, (4.2.1) yields
$$\|(\lambda I - T)^n x\| \leq \frac{1}{2\pi} 2\pi\rho\rho^n \max_{\zeta \in \{\gamma_\rho\}} \|R_T(\zeta)\|\,\|x\|$$
and we conclude $\limsup_{n\to\infty} \|(\lambda I - T)^n x\|^{\frac{1}{n}} \leq \rho$. Taking the limit $\rho \to 0$ implies $\lim_{n\to\infty} \|(\lambda I - T)^n x\|^{\frac{1}{n}} = 0$.

- Conversely, let $x \in Y$ satisfy $\lim_{n\to\infty} \|(\lambda I - T)^n x\|^{\frac{1}{n}} = 0$. For every $0 < \rho < R_\lambda$ and $\mu \in \{\gamma_\rho\}$, the series
$$s(\mu) := \sum_{n=0}^{\infty} \left(\frac{\lambda I - T}{\lambda - \mu}\right)^n x$$
converges absolutely. We obtain
$$s(\mu) = x + \sum_{n=1}^{\infty} \left(\frac{\lambda I - T}{\lambda - \mu}\right)^n x = x + \left(\frac{\lambda I - T}{\lambda - \mu}\right) s(\mu)$$
and therefore, $s(\mu) = (\mu - \lambda) R_T(\mu) x$. Hence, the resolvent can be written as the series
$$R_T(\mu)x = -\sum_{n=0}^{\infty} \frac{1}{(\lambda - \mu)^{n+1}} (\lambda I - T)^n x$$
for $\mu \in \{\gamma_\rho\}$ with $0 < \rho < R_\lambda$. As a Laurent series, the above series converges uniformly on $\{\gamma_\rho\}$ and we obtain
$$\int_{\gamma_\rho} R_T(\zeta) x\ d\zeta = -\sum_{n=0}^{\infty} \int_{\gamma_\rho} \frac{1}{(\lambda - \zeta)^{n+1}} (\lambda I - T)^n x\ d\zeta.$$

Consequently, (4.2.2) applied to ι_x and $\iota_{(\lambda I-T)^n x} \in \mathcal{B}(\mathcal{B}(Y), Y)$ yields

$$P_\lambda x = \frac{1}{2\pi i}\int_{\gamma_\rho} R_T(\zeta)x \, d\zeta = -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\Big(\int_{\gamma_\rho}\frac{1}{(\lambda-\zeta)^{n+1}}\, d\zeta\Big)(\lambda I - T)^n x.$$

The value of the line integral on the right side can be determined by an easy computation. In fact,

$$\int_{\gamma_\rho}\frac{1}{(\lambda-\zeta)^{n+1}}\, d\zeta = \begin{cases} -2\pi i & \text{for } n = 0, \\ 0 & \text{for } n \in \mathbb{N}, \end{cases}$$

which implies $P_\lambda x = x$, i.e. $x \in M_\lambda$.

□

4.5 Isolated singularities of the resolvent

The aim of the present chapter is to prove, that whenever $T \in \mathcal{B}(Y)$ is compact, every $\lambda \in \sigma(T)\backslash\{0\}$ is a pole of its resolvent. In Theorem 4.5.3 an even stronger result will be presented, characterizing the poles of R_T for general $T \in \mathcal{B}(Y)$. In order to do so, we first combine the results of the previous sections to a useful representation of the Laurent series coefficients.

Consider an isolated point $\lambda \in \sigma(T)$, where $T \in \mathcal{B}(Y)$ is a (not necessarily compact) operator on a Banach space Y. Recall the Laurent series expansion of R_T centered at λ from Theorem 4.2.1,

$$R_T(\mu) = \sum_{n=-\infty}^{\infty} (\mu-\lambda)^n P_n,$$

which converges absolutely for $\mu \in U_{R_\lambda}(\lambda)\backslash\{\lambda\}$, where R_λ is the biggest radius such that $U_{R_\lambda}(\lambda) \cap \sigma(T) = \{\lambda\}$. The coefficients $P_n \in \mathcal{B}(Y)$ are given by

$$P_n = \frac{1}{2\pi i}\int_\gamma \frac{R_T(\zeta)}{(\zeta-\lambda)^{n+1}}\, d\zeta, \qquad n \in \mathbb{Z}.$$

Either $\gamma = \gamma_\rho$ as in (4.2.4) for some $0 < \rho < R_\lambda$, or γ is a piecewise continuously differentiable path, which is homotopic to γ_ρ in $U_{R_\lambda}(\lambda)\backslash\{\lambda\}$. We are interested in the principle part of the Laurent series, i.e. in the coefficients

$$P_{-n} = \frac{1}{2\pi i}\int_\gamma (\zeta-\lambda)^{n-1} R_T(\zeta)\, d\zeta$$

for $n \in \mathbb{N}$. By (4.4.2) and Theorem 4.3.2 we immediately obtain

$$P_{-n} = (T - \lambda I)^{n-1} P_\lambda, \qquad n \in \mathbb{N}. \tag{4.5.1}$$

We already showed, that λ cannot be a removable singularity, i.e. it is not possible, that all P_{-n} vanish. It remains to show, that there exists $m \in \mathbb{N}$, such that $P_{-n} = 0$ for all $n > m$. The next theorem will use (4.5.1) to give necessary and sufficient conditions for this to happen.

Before we state the theorem, we present two lemmata, which will be necessary for its proof.

Lemma 4.5.1. *Let Y be a Banach space and $T \in \mathcal{B}(Y)$. If there exists a closed subspace $U \subseteq Y$, such that* $\operatorname{ran} T \cap U = \{0\}$ *and* $\operatorname{ran} T \dotplus U$ *is closed in Y, then* $\operatorname{ran} T$ *is closed.*

Proof. As $U \subseteq Y$ is closed, it is a Banach space and so is the product space $Y \times U$. Consider the operator $S \in \mathcal{B}(Y \times U, Y)$,

$$S : \begin{cases} Y \times U & \to \quad Y, \\ (x, y) & \mapsto \quad Tx + y. \end{cases}$$

Let $\hat{S}$ be the corresponding injection on the quotient space,

$$\hat{S} : \begin{cases} (Y \times U)\big/ \ker S & \to \quad Y, \\ (x, y) + \ker S & \mapsto \quad S(x, y). \end{cases}$$

Since $\ker S$ is closed, $(Y \times U)\big/ \ker S$ is a Banach space and $\hat{S} \in \mathcal{B}((Y \times U)\big/ \ker S, Y)$. The subspace $\operatorname{ran} T \dotplus U = \operatorname{ran} S = \operatorname{ran} \hat{S} \subseteq Y$ is closed and therefore a Banach space itself. By the Open Mapping Theorem, the bijection $\hat{S} \in \mathcal{B}((Y \times U)\big/ \ker S, \operatorname{ran} \hat{S})$ satisfies $\hat{S}^{-1} \in \mathcal{B}(\operatorname{ran} \hat{S}, (Y \times U)\big/ \ker S)$. Equivalently, there exists $C > 0$ such that

$$C \, \|(x, y) + \ker S\| \leq \|\hat{S}((x, y) + \ker S)\| = \|S(x, y)\|$$

for all $(x, y) \in Y \times U$. Since $\operatorname{ran} T \cap U = \{0\}$, we obtain $\ker S = \ker T \times \{0\}$ and therefore,

$$C \, \|(x, y) + (\ker T \times \{0\})\| = C \inf_{u \in \ker T} \|(x + u, y)\| \leq \|Tx + y\|$$

for all $x \in Y$ and $y \in U$. By setting $y = 0$, we conclude

$$C \inf_{u \in \ker T} \|x + u\| \leq \|Tx\| \qquad \text{for all} \quad x \in Y. \tag{4.5.2}$$

Since $\ker T$ is closed, $Y/\ker T$ is a Banach space. Consider the canonical injection

$$\hat{T} : \begin{cases} Y/\ker T & \to & Y, \\ x + \ker T & \mapsto & Tx. \end{cases}$$

Then by (4.5.2), its inverse $\hat{T}^{-1} : \operatorname{ran} \hat{T} \to Y$ is continuous. Hence, $\operatorname{ran} \hat{T} = \operatorname{ran} T \subseteq Y$ is closed. □

Lemma 4.5.2. *Let Y be a Banach space and $T \in \mathcal{B}(Y)$. For an isolated spectral point $\lambda \in \sigma(T)$, let $P_\lambda \in \mathcal{B}(Y)$ be the corresponding spectral projector. Then $T_\lambda := T - \lambda I$ can be written as*

$$T_\lambda = R + S,$$

where R is invertible in $\mathcal{B}(Y)$ and $S = P_\lambda B$ with some $B \in \mathcal{B}(Y)$.

Proof. Let $P_\tau = I - P_\lambda \in \mathcal{B}(Y)$ be the spectral projector associated with the complementary spectral set $\tau = \sigma(T) \backslash \{\lambda\}$. Then $P_\lambda = f_\lambda(T)$ and $P_\tau = f_\tau(T)$ with $f_\lambda, f_\tau \in H(T)$ given by

$$f_\lambda(z) = \begin{cases} 1 & \text{for } z \in \Delta_\lambda, \\ 0 & \text{for } z \in \Delta_\tau, \end{cases} \qquad \text{and} \qquad f_\tau(z) = \begin{cases} 0 & \text{for } z \in \Delta_\lambda, \\ 1 & \text{for } z \in \Delta_\tau, \end{cases}$$

with the open and disjoint sets $\Delta_\lambda \supseteq \{\lambda\}$ and $\Delta_\tau \supseteq \tau$.

- First, assume that $\lambda = 0$. The operator $T_\lambda = T$ can be written as

  $$T = (P_\tau + P_\lambda)T = (P_\tau T + P_\lambda) + P_\lambda(T - I).$$

  Set $B := T - I$ and $R := P_\tau T + P_\lambda$. Then $R = g(T)$ for $g \in H(T)$ with $g(z) = f_\tau(z)z + f_\lambda(z)$ for $z \in \Delta_\lambda \mathbin{\dot\cup} \Delta_\tau$. Since $0 = \lambda \notin \Delta_\tau$, we have $g(z) \neq 0$ for all $z \in \sigma(T)$. Theorem 4.3.2 (iv) implies $R^{-1} = g(T)^{-1} \in \mathcal{B}(Y)$.

- In case that $\lambda \neq 0$, consider the representation

  $$T_\lambda = (P_\tau + P_\lambda)T - \lambda I = (P_\tau T - \lambda I) + P_\lambda T.$$

  With $B := T$ and $R := (P_\tau T - \lambda I)$, we obtain $R = h(T)$ for $h \in H(T)$ with $h(z) = f_\tau(z)z - \lambda$ for $z \in \Delta_\lambda \mathbin{\dot\cup} \Delta_\tau$. By $\lambda \notin \Delta_\tau$, we conclude $h(z) \neq 0$ for $z \in \sigma(T)$. Theorem 4.3.2 (iv) yields the invertibility of $R = h(T)$.

□

Theorem 4.5.3. *Let Y be a Banach space, $T \in \mathcal{B}(Y)$ and $\lambda \in \sigma(T)$. Then λ is a pole of the resolvent R_T, if and only if the Riesz number of $(T - \lambda I)$ is finite and positive. In this case, λ is an eigenvalue of T and its order as a pole equals $p(T - \lambda I) = q(T - \lambda I)$.*

Proof. In case that the Riesz number of $T_\lambda := (T - \lambda I)$ is positive, λ is obviously an eigenvalue of T.

- First let λ be a pole of the resolvent of order $m \in \mathbb{N}$. Then the coefficients $P_n \in \mathcal{B}(Y)$ of the Laurent series expansion of R_T centered at λ satisfy $P_{-m} \neq 0$ and $P_{-n} = 0$ for all $n > m$. Relation (4.5.1) yields

$$\begin{aligned} {T_\lambda}^{m-1} P_\lambda &\neq 0 \\ {T_\lambda}^{n} P_\lambda &= 0 \end{aligned} \tag{4.5.3}$$

for every $n \geq m$ and therefore, $\operatorname{ran} P_\lambda = M_\lambda \neq \ker {T_\lambda}^{m-1}$ and $M_\lambda \subseteq \ker {T_\lambda}^n$. Moreover, $\ker {T_\lambda}^n \subseteq M_\lambda$ by Theorem 4.4.3. Consequently,

$$\ker {T_\lambda}^{m-1} \neq M_\lambda = \ker {T_\lambda}^n$$

for all $n \geq m$. Hence, $p(T_\lambda) = m$; see Definition 2.2.1.

According to Lemma 4.5.2, T_λ can be represented as $T_\lambda = R + S$, where $R \in \mathcal{B}(Y)$ is bijective and $S = P_\lambda B$ for some $B \in \mathcal{B}(Y)$. With (4.5.3) we conclude

$${T_\lambda}^{m+1} = {T_\lambda}^m R + {T_\lambda}^m P_\lambda B = {T_\lambda}^m R.$$

Since R is bijective, we obtain $\operatorname{ran} {T_\lambda}^{m+1} = \operatorname{ran} {T_\lambda}^m$. Hence, the descent of T_λ is finite; see Definition 2.2.1. Theorem 2.2.5 yields $q(T_\lambda) = p(T_\lambda) = m$.

- Conversely, assume that T_λ has Riesz number $p \in \mathbb{N}$. By Corollary 2.2.6, Y can be written as the direct sum

$$Y = \ker {T_\lambda}^p \dotplus \operatorname{ran} {T_\lambda}^p.$$

The subspaces $\ker {T_\lambda}^p$ and $\operatorname{ran} {T_\lambda}^p$ are T_λ-invariant. By Lemma 4.5.1, they are both closed. Consider the restrictions

$$T_1 := T_\lambda|_{\ker {T_\lambda}^p} \in \mathcal{B}(\ker {T_\lambda}^p) \qquad \text{and} \qquad T_2 := T_\lambda|_{\operatorname{ran} {T_\lambda}^p} \in \mathcal{B}(\operatorname{ran} {T_\lambda}^p).$$

Then T_1 is nilpotent and by Corollary 2.2.6, T_2 is bijective. Hence, $\rho(T_1) = \mathbb{C}\backslash\{0\}$ and $\rho(T_2) \supseteq U_r(0)$ with some $r > 0$. Lemma 4.4.1 yields $\rho(T_\lambda) = \rho(T_1) \cap \rho(T_2)$ and thus, $0 \in \sigma(T_\lambda)$ is an isolated spectral point. Consequently, λ is an isolated point of $\sigma(T)$ and we can consider the representation (4.5.1) of the Laurent series coefficients.

Since both M_λ and $\operatorname{ran} {T_\lambda}^p$ are closed and T-invariant, so is the subspace

$$D := M_\lambda \cap \operatorname{ran} {T_\lambda}^p.$$

Therefore, D is a Banach space for itself and $T_0 := T|_D \in \mathcal{B}(D)$. Let us show $\rho(T_0) = \mathbb{C}$, which implies $D = \{0\}$.

- First we show $\lambda \in \rho(T_0)$. Consider $x \in D$ arbitrary. By the bijectivity of T_2, there exists a unique $y \in \operatorname{ran} T_\lambda{}^p$ with $T_\lambda y = x$. Since $x \in M_\lambda$, Theorem 4.4.3 implies $y \in M_\lambda$ and in turn, $y \in D$. Hence, $T_\lambda|_D$ is bijective, i.e. $\lambda \in \rho(T_0)$.
- Let $\mu \in \mathbb{C}\backslash\{\lambda\}$, then $\mu \in \rho(T|_{M_\lambda})$ by Theorem 4.4.2. Therefore, for every $v \in D$ there exists a unique $w \in M_\lambda$ with $v = (\mu I - T)w$. Consider the decomposition $w = w_1 + w_2 \in \ker T_\lambda{}^p \dotplus \operatorname{ran} T_\lambda{}^p$. We obtain

  $$v - (\mu I - T)w_2 = (\mu I - T)w_1 \in \ker T_\lambda{}^p \cap \operatorname{ran} T_\lambda{}^p = \{0\},$$

  implying $w_1 \in \ker(\mu I - T) \cap \ker T_\lambda{}^p$. The complex polynomials $(\mu - z) \in \mathbb{C}[z]$ and $(z-\lambda)^p \in \mathbb{C}[z]$ are coprime. Thus, there exist $f(z), g(z) \in \mathbb{C}[z]$ satisfying

  $$1 = f(z)(\mu - z) + g(z)(z - \lambda)^p.$$

  Replacing z by T and evaluating at w_1 yields

  $$w_1 = f(T)(\mu I - T)w_1 + g(T)T_\lambda{}^p w_1 = 0.$$

  Thus, $w = w_2 \in \operatorname{ran} T_\lambda{}^p$ and in turn, $w \in D$. Hence, $(\mu I - T_0)$ is bijective, i.e. $\mu \in \rho(T_0)$.

Let $u \in M_\lambda$ be arbitrary, then $u = u_1 + u_2 \in \ker T_\lambda{}^p \dotplus \operatorname{ran} T_\lambda{}^p$. Consequently, $T_\lambda{}^n u = T_\lambda{}^n u_2$ for $n \geq p$. Since $u \in M_\lambda$, by Theorem 4.4.3 we conclude $u_2 \in M_\lambda$, and therefore $u_2 \in D = \{0\}$. Hence, $u = u_1 \in \ker T_\lambda{}^p$, implying

$$M_\lambda = \operatorname{ran} P_\lambda \subseteq \ker T_\lambda{}^p.$$

With (4.5.1), we obtain $P_{-n} = 0$ for all $n > p$, i.e. λ is a pole of the resolvent.

□

In the last few sections, we achieved some important results about the resolvent, some of which were stated for general Banach algebras. Let us summarize the results we obtained considering an isolated singularity of the resolvent of an operator $T \in \mathcal{B}(Y)$.

Corollary 4.5.4. *For a Banach space Y and $T \in \mathcal{B}(Y)$, let $\lambda \in \sigma(T)$ be an isolated point of $\sigma(T)$. The following statements hold true:*

(i) *R_T has a representation as a Laurent series centered at λ,*

$$R_T(\mu) = \sum_{n=-\infty}^{\infty} (\mu - \lambda)^n P_n,$$

with uniquely determined $P_n \in \mathcal{B}(Y)$, $n \in \mathbb{Z}$. The series converges absolutely on $U_{R_\lambda}(\lambda)\backslash\{\lambda\}$, where $R_\lambda > 0$ is the biggest radius such that $U_{R_\lambda}(\lambda)\backslash\{\lambda\} \subseteq \rho(T)$. It converges uniformly on $K_R(\lambda)\backslash U_r(\lambda)$, whenever $0 < r < R < R_\lambda$.

(ii) *λ is either a pole or an essential singularity of the resolvent.*

(iii) *λ is a pole of R_T, if and only if the Riesz number of $(T - \lambda I)$ is finite and positive. In this case, the order of the pole λ equals $p(T - \lambda I) = q(T - \lambda I)$.*

The Krein-Rutman Theorem characterises not only the spectrum of $T \in \mathcal{B}(Y)$, but also of $T' \in \mathcal{B}(Y')$. The following proposition will enable us to apply the results we elaborated so far, in order to obtain the corresponding ones for the resolvent $R_{T'}$ of the adjoint T'.

Proposition 4.5.5. *Let Y be a Banach space and $T \in \mathcal{B}(Y)$. Consider the Laurent series expansion of the resolvent R_T around an isolated point $\lambda \in \sigma(T)$ of the spectrum as in Corollary 4.5.4 (i). The Laurent series of the resolvent $R_{T'}$ of $T' \in \mathcal{B}(Y')$ centered at $\lambda \in \sigma(T') = \sigma(T)$ is given by*

$$R_{T'}(\mu) = R_T(\mu)' = \sum_{n=-\infty}^{\infty} (\mu - \lambda)^n P_n'$$

with $P_n' \in \mathcal{B}(Y')$, $n \in \mathbb{Z}$. It has the same convergence properties as the corresponding Laurent series of R_T. More precisely, it is absolutely convergent for $\mu \in U_{R_\lambda}(\lambda)\backslash\{\lambda\}$ and uniformly convergent for $\mu \in K_R(\lambda)\backslash U_r(\lambda)$ with $0 < r < R < R_\lambda$.

Proof. Let $\mu \in \rho(T) = \rho(T')$. Considering the adjoint of the equality

$$(\mu I - T)R_T(\mu) = R_T(\mu)(\mu I - T) = I$$

yields $R_{T'}(\mu) = R_T(\mu)'$. Convergence in $\mathcal{B}(Y)$ is compatible with the adjoint. Thus, we obtain

$$R_T(\mu)' = \left(\sum_{n=-\infty}^{\infty} (\mu - \lambda)^n P_n \right)' = \sum_{n=-\infty}^{\infty} (\mu - \lambda)^n P_n'$$

for $\mu \in U_{R_\lambda}(\lambda)\backslash\{\lambda\}$. The coefficients of the Laurent expansion of $R_{T'}$ around its singularity λ are unique. Hence, the proof of the proposition is complete. □

In particular, the above proposition implies that the Laurent expansions of R_T and $R_{T'}$ have the same structure. This allows us to transfer the obtained results to the adjoint T' and its resolvent.

Corollary 4.5.6. *Let Y be a Banach space and $T \in \mathcal{B}(Y)$. Then $\lambda \in \sigma(T) = \sigma(T')$ is a pole of the resolvent R_T of order $m \in \mathbb{N}$, if and only if it is a pole of the resolvent $R_{T'}$ of $T' \in \mathcal{B}(Y')$ of the same order m. In this case, both $(T - \lambda I)$ and $(T' - \lambda I')$ have Riesz number m.*

In our further considerations, we will only be interested in compact operators. By Theorem 2.2.7, for compact $T \in \mathcal{B}(Y)$ and $\lambda \in \mathbb{C}\backslash\{0\}$, the operator $(T - \lambda I)$ has finite Riesz number. Moreover, Theorem 2.1.1 states that every $\lambda \in \mathbb{C}\backslash\{0\}$ is not only an isolated point of $\sigma(T)$, but an eigenvalue of T, i.e. the Riesz number of $(T - \lambda I)$ is positive. Hence, the previous results can be applied and we obtain the following corollary:

Corollary 4.5.7. *Let Y be a Banach space, $T \in \mathcal{B}(Y)$ compact and $\lambda \in \sigma(T)\backslash\{0\}$. Then λ is a pole of the resolvent R_T of order $m := p(T - \lambda I) = q(T - \lambda I) \in \mathbb{N}$. It can be developed into a Laurent series centered at λ,*

$$R_T(\mu) = \sum_{n=-m}^{\infty} (\mu - \lambda)^n P_n,$$

with uniquely determined $P_n \in \mathcal{B}(Y)$, $n \geq -m$ and $P_{-m} \neq 0$. The series converges absolutely on the biggest punctured disc $U_{R_\lambda}(\lambda)\backslash\{\lambda\}$, which is fully contained in the resolvent set $\rho(T)$. Moreover, it converges uniformly on every closed annulus $K_R(\lambda)\backslash U_r(\lambda)$ with $0 < r < R < R_\lambda$. The corresponding results hold true for the adjoint $T' \in \mathcal{B}(Y')$ and its resolvent $R_{T'}$; see Corollary 4.5.6.

The main aim of this chapter is achieved by the above corollary. In the proof of the Krein-Rutman Theorem, we will use it to construct a positive eigenvector corresponding to the spectral radius of a compact and positive operator $T \in \mathcal{B}(Y)$. Indeed, every $\lambda \in \sigma(T)\backslash\{0\}$ is an eigenvalue of T and a pole of its resolvent. Given the Laurent series expansion of R_T around λ, a corresponding eigenvector can easily be constructed.

Proposition 4.5.8. *For a Banach space Y and $T \in \mathcal{B}(Y)$, let $\lambda \in \sigma(T)$ be an isolated point of the spectrum. If λ is a pole of the resolvent R_T of order $m \in \mathbb{N}$, then*

$$\{0\} \neq \operatorname{ran} P_{-m} \subseteq \ker(\lambda I - T).$$

In particular, λ is an eigenvalue of T and every $x \in \operatorname{ran} P_{-m} \backslash \{0\}$ is an eigenvector of T corresponding to λ. Here

$$R_T(\mu) = \sum_{n=-m}^{\infty} (\mu - \lambda)^n P_n, \qquad 0 < |\mu - \lambda| < R_\lambda,$$

is the Laurent series of R_T centered at λ with $P_{-m} \neq 0$ as in Corollary 4.5.4.

Proof. Consider the Laurent series of the resolvent R_T centered at λ and the representation (4.5.1) of the coefficients of its principal part:

$$P_{-n} = (T - \lambda I)^{n-1} P_\lambda \qquad \text{for} \quad n \in \mathbb{N},$$

where $P_\lambda \in \mathcal{B}(Y)$ is the spectral projector associated with the spectral set $\{\lambda\}$. Since $P_{-n} = 0$ for $n > m$, we obtain $0 = P_{-m-1} = (T - \lambda I) P_{-m}$ and therefore the inclusion $\operatorname{ran} P_{-m} \subseteq \ker(T - \lambda I) = \ker(\lambda I - T)$. $\square$

In the setting of the Krein-Rutman Theorem, the given Banach space X will be a real one. All the results we discussed in the previous sections, will be applied to the complexification $Y = X_\mathbb{C}$ of X. Let us make some considerations about the resolvent of the complexification of an operator $T \in \mathcal{B}(X)$. Remarkably, the diagonal structure of $T_\mathbb{C}$ is passed on to $R_{T_\mathbb{C}}$ as well as to the leading coefficient of its Laurent series, when developed around a real pole.

Given a real Banach space X and its complexification $X_\mathbb{C}$, consider $T \in \mathcal{B}(X)$. Recall, that the complexification of T has block operator matrix structure

$$T_\mathbb{C} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \in \mathcal{B}(X_\mathbb{C}),$$

leaving $X \subseteq X_\mathbb{C}$ invariant, i.e. $T_\mathbb{C}(X) \subseteq X$.

For $\mu \in \rho(T_\mathbb{C}) \cap \mathbb{R}$, consider the resolvent $R_{T_\mathbb{C}}(\mu) = (\mu I - T_\mathbb{C})^{-1} \in \mathcal{B}(X_\mathbb{C})$; see Remark 3.1.8. Making use of its diagonal structure, it is easy to see, that $(\mu I - T_\mathbb{C}) = (\mu I - T)_\mathbb{C}$ is invertible in $\mathcal{B}(X_\mathbb{C})$, if and only if $(\mu I - T)$ is invertible in $\mathcal{B}(X)$ and the inverse is given by

$$(\mu I - T_\mathbb{C})^{-1} = \begin{pmatrix} (\mu I - T)^{-1} & 0 \\ 0 & (\mu I - T)^{-1} \end{pmatrix}.$$

Hence, whenever $\mu \in \rho(T_\mathbb{C}) \cap \mathbb{R}$, the inverse $(\mu I - T)^{-1} \in \mathcal{B}(X)$ exists and the resolvent equals the complexification of $(\mu I - T)^{-1}$,

$$R_{T_\mathbb{C}}(\mu) = ((\mu I - T)^{-1})_\mathbb{C}.$$

Consequently, $R_{T_{\mathbb{C}}}(\mu)$ leaves X invariant.

Proposition 4.5.9. *Let X be a real Banach space and $X_{\mathbb{C}}$ its complexification. Consider an operator $T \in \mathcal{B}(X)$ and its complexification $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$. Then $\mu \in \rho(T_{\mathbb{C}}) \cap \mathbb{R}$, if and only if the operator $(\mu I - T)$ is invertible in $\mathcal{B}(X)$. The resolvent $R_{T_{\mathbb{C}}}(\mu)$ of the complexification has diagonal structure,*

$$R_{T_{\mathbb{C}}}(\mu) = ((\mu I - T)^{-1})_{\mathbb{C}},$$

and therefore $R_{T_{\mathbb{C}}}(\mu)(X) = (\mu I - T)^{-1}(X) \subseteq X$. Moreover, if $\lambda \in \sigma(T_{\mathbb{C}}) \cap \mathbb{R}$ is a pole of $R_{T_{\mathbb{C}}}$ of order $m \in \mathbb{N}$, then the leading coefficient of the unique Laurent series expansion

$$R_{T_{\mathbb{C}}}(\mu) = \sum_{n=-m}^{\infty} (\mu - \lambda)^n P_n$$

from Corollary 4.5.4 has diagonal structure. More precisely,

$$P_{-m} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = A_{\mathbb{C}} \qquad \text{with} \quad A = P_{-m}|_X$$

and $P_{-m}(X) \subseteq X$.

Proof. Set $Y := X_{\mathbb{C}}$ and $S := T_{\mathbb{C}}$. The coefficient $P_{-m} \in \mathcal{B}(Y)$ has a block operator matrix representation like in (3.1.3),

$$P_{-m} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad \text{for some} \quad A,\, B \in \mathcal{B}(X).$$

The entry B is given by $B = -\pi_2 \circ P_{-m} \circ \iota_1$, where $\pi_1, \pi_2 \in \mathcal{B}(X \times X, X)$ denote the canonical projections and $\iota_1, \iota_2 \in \mathcal{B}(X, X \times X)$ the canonical embeddings. By Proposition 4.2.3,

$$\lim_{\mu \to \lambda} (\mu - \lambda)^m R_S(\mu) = P_{-m}$$

in $\mathcal{B}(Y)$. Performing the limit $\mathbb{R} \ni \mu \to \lambda^+$ along the real axis yields

$$\begin{aligned} B &= -\pi_2 \circ P_{-m} \circ \iota_1 = -\pi_2 \circ \big(\lim_{\mu \to \lambda^+} (\mu - \lambda)^m R_S(\mu)\big) \circ \iota_1 \\ &= -\lim_{\mu \to \lambda^+} (\mu - \lambda)^m \big(\pi_2 \circ R_S(\mu) \circ \iota_1\big) \end{aligned}$$

Note, that π_2 is $\mathbb{R}$-linear and $(\mu - \lambda)^m \in \mathbb{R}$. Since $R_S(\mu) = ((\mu I - T)^{-1})_{\mathbb{C}}$ has block diagonal structure, $\pi_2 \circ R_S(\mu) \circ \iota_1 = 0$ and in turn, $B = 0$. Hence, $P_{-m} = \mathrm{diag}(A, A)$, implying $P_{-m}(X) \subseteq X$ and thus, $A = \pi_1 \circ P_{-m} \circ \iota_1 = P_{-m}|_X$. $\square$

Remark 4.5.10. In the setting of the previous proposition, consider $A \in \mathcal{B}(X)$ and its complexification $P_{-m} \in \mathcal{B}(X_{\mathbb{C}})$. By identifying $(X_{\mathbb{C}})'$ with $(X')_{\mathbb{C}}$ via the isomorphism

$J : (X_{\mathbb{C}})' \to (X')_{\mathbb{C}}$ that we presented in Corollary 3.2.3, the adjoint of the complexification corresponds to the complexification of the adjoint,

$$\mathcal{B}((X')_{\mathbb{C}}) \ni (A')_{\mathbb{C}} \simeq (A_{\mathbb{C}})' = P'_{-m} \in \mathcal{B}((X_{\mathbb{C}})').$$

As in Proposition 3.2.4, X' can be identified with the subset $J^{-1} \circ \iota_1(X') \subseteq (X_{\mathbb{C}})'$ and by the diagonal structure of $P'_{-m} \simeq (A')_{\mathbb{C}}$, it leaves $X' \subseteq (X_{\mathbb{C}})'$ invariant,

$$P'_{-m}(X') \subseteq X'.$$

//

Chapter 5

Ordered Banach spaces and positive operators

At this point we have prepared everything we need for the proof of the Krein-Rutman Theorem and we can begin to elaborate its setting. This chapter presents the concept of ordered Banach spaces, where X will always denote a real Banach space. Some of the following results also apply to complex Banach spaces. Since the Krein-Rutman Theorem is placed in a real setting, we will not discuss them though.

5.1 Order cones, ordered Banach spaces

We would like to define a partial order on a real Banach space, inspired by order relations we know from $\mathbb{R}$: Consider the subset $K := \mathbb{R}^+ \cup \{0\}$ of the Banach space $\mathbb{R}$. Evidently, $x \leq y$ is equivalent to $y - x \in K$. Hence, K can be used to define a partial order on the underlying real Banach space $\mathbb{R}$.

In order to generalise this concept to arbitrary real Banach spaces, one observes that the subset $K = \mathbb{R}^+ \cup \{0\}$ is closed, non-empty and $K \neq \{0\}$. Moreover, it satisfies

$$K \cap (-K) = \{0\}, \qquad K + K \subseteq K \qquad \text{and} \qquad \alpha K \subseteq K \quad \text{for} \quad \alpha \geq 0.$$

Necessarily, the set K of non-negative elements should have the above properties.

Definition 5.1.1. Let X be a real Banach space. A subset $K \subseteq X$ is called an *order cone*, if

(i) K is closed, non-empty and $K \neq \{0\}$,

(ii) $K + K \subseteq K$ and $\alpha K \subseteq K$ for $\alpha \geq 0$,

(iii) $K \cap (-K) = \{0\}$.

The pair $\langle X, K \rangle$, where X is a real Banach space and $K \subseteq X$ is an order cone, is called an *ordered Banach space.* //

The above definition does not make use of the fact, that X is a Banach space. Order cones can equally be defined for common topological vector spaces. Note, that although in general it is not clear, if there exists an order cone for every Banach space, in the following we will always assume its existence.

In comparison to an order cone, a subset C of a Banach space X is called a *cone*, if $\alpha C \subseteq C$ for $\alpha > 0$. Clearly, every order cone is a cone but the converse is not true in general.

Remark 5.1.2.

- Property (ii) in Definition 5.1.1 is equivalent to

  (ii)$'$ K is convex and $\alpha K \subseteq K$ for $\alpha \geq 0$.

  Assume, that (ii) holds and let $t \in [0, 1]$ be arbitrary. Then $t, (1 - t) \geq 0$ and therefore
  $$tK + (1 - t)K \subseteq K + K \subseteq K,$$
  i.e. K is convex. If conversely (ii)$'$ is satisfied, then
  $$K + K = \frac{1}{2}\ 2K + \frac{1}{2}\ 2K \subseteq \frac{1}{2}K + \frac{1}{2}K \subseteq K.$$

- Clearly, $0 \in K$ but $0 \notin K^\circ$. Indeed, $0 \in K^\circ$ would imply $U_r(0) \subseteq K$ for some $r > 0$. Let $x \in X \backslash \{0\}$ be arbitrary and define
  $$y := \frac{r}{2\,\|x\|} x \in U_r(0) \subseteq K.$$
  Since equally $-y \in U_r(0) \subseteq K$, we obtain the contradiction $y = 0$.

//

For our purposes, it is not sufficient to consider general order cones. We will need them to have more specific properties.

Definition 5.1.3. Let $\langle X, K\rangle$ be an ordered Banach space. The order cone K is called *generating*, if $X = \mathrm{span}(K)$. If $\mathrm{span}(K)$ is dense in X, i.e. $\overline{\mathrm{span}(K)} = X$, then K is called *total*. //

Obviously, every generating order cone is total. The defining properties of generating and total cones can be simplified. Note, that at this point it is crucial, that the underlying Banach space X is real.

Remark 5.1.4. K is generating (respectively total), if and only if $X = K - K$ (respectively $X = \overline{K - K}$). In fact, for $x = \sum_{i=1}^n a_i x_i \in \mathrm{span}(K)$ with $a_i \in \mathbb{R}$ and $x_i \in K$, set

$$k_1 := \sum_{a_i \geq 0} a_i x_i \qquad \text{and} \qquad k_2 := \sum_{a_i < 0} (-a_i) x_i.$$

Then $k_1, k_2 \in K$ and $x = k_1 - k_2 \in K - K$. //

We already saw $0 \notin K^\circ$ for every order cone. We only know, that $\overline{K} = K \supsetneqq \{0\}$ but possibly, the interior of K is empty. Consider for instance $X = \mathbb{R}^2$, then the positive cone $K = (\mathbb{R}^+ \cup \{0\}) \times \{0\}$ has empty interior. In case that $K^\circ \neq \emptyset$, the order cone proves to be generating.

Proposition 5.1.5. *Let $\langle X, K\rangle$ be an ordered Banach space. If $K^\circ \neq \emptyset$, then K is generating.*

Proof. For $y \in K^\circ$ there exists $r > 0$ with $U_r(y) \subseteq K$. Let $x \in X$ be arbitrary. Then $y \pm \alpha x \in U_r(y) \subseteq K$ for any $0 < \alpha < \frac{r}{\|x\|}$ and we obtain

$$x = \frac{1}{2\alpha}(y + \alpha x) - \frac{1}{2\alpha}(y - \alpha x) \in K - K.$$

Hence, $X = K - K$. $\square$

By means of the order cone, we are able to define order relations on X induced by K.

Definition 5.1.6. For an ordered Banach space $\langle X, K\rangle$ and $x, y \in X$ we define:

- $x \leq y$, if $y - x \in K$.
- $x < y$, if $y - x \in K \backslash \{0\}$, i.e. if $x \leq y$ and $x \neq y$.
- $x \ll y$, if $y - x \in K^\circ$.

//

Obviously, $x \geq 0$, $x > 0$ and $x \gg 0$ are equivalent to $x \in K$, $x \in K \backslash \{0\}$ and $x \in K^\circ$, respectively. Since $0 \notin K^\circ$, clearly $x \ll y$ implies $x < y$, and $x < y$ implies $x \leq y$.

Apart from the above provided examples

$$\begin{aligned}\langle X, K\rangle &= \langle \mathbb{R}, \mathbb{R}^+ \cap \{0\}\rangle, \\ \langle X, K\rangle &= \langle \mathbb{R}^2, (\mathbb{R}^+ \cap \{0\}) \times \{0\}\rangle,\end{aligned}$$

consider the following ordered Banach spaces.

Example 5.1.7.

(i) Let $X = C(M)$ be the Banach space of all continuous, $\mathbb{R}$-valued functions on a compact topological space M and let $K \subseteq X$ be the subset

$$K = \{\, f \in C(M) \,:\, f(x) \geq 0 \ \text{ for all } x \in M \,\}\,.$$

Obviously, K is an order cone and for $f, g \in X$ one has

- $f \leq g$, if $f(x) \leq g(x)$ for all $x \in M$.
- $f < g$, if $f \leq g$ and $f(x_0) < g(x_0)$ for at least one $x_0 \in M$.
- $f \ll g$, if $f(x) < g(x)$ for all $x \in M$.

(ii) Consider the real Banach space $X = \mathbb{R}^n$ and

$$K = \left\{\, (x_1, \ldots, x_n)^T \in \mathbb{R}^n \,:\, x_i \geq 0, \ \ i = 1, \ldots, n \,\right\}.$$

Clearly, K is a generating order cone and for $x, y \in X$ with $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ we have

- $x \leq y$, if $x_i \leq y_i$ for every $i = 1, \ldots, n$.
- $x < y$, if $x \leq y$ and $x_{i_0} < y_{i_0}$ for at least one i_0.
- $x \ll y$, if $x_i < y_i$ for every $i = 1, \ldots, n$.

For $n = 1$, we again obtain $X = \mathbb{R}$ and $K = \mathbb{R}^+ \cup \{0\}$. In this case, $x < y$ if and only if $x \ll y$.

//

Remark 5.1.8. Comparing the notion $\leq$ defined in Section 1.1 to (ii) in the above example, we see that the two definitions of $\leq$ are equivalent. Moreover, $<$ from Section 1.1 corresponds to $\ll$ in Example 5.1.7 (ii). //

Indeed, $\leq$ is a partial order on X. It has all the properties one could expect from an order relation.

Proposition 5.1.9. *Let $\langle X, K\rangle$ be an ordered Banach space. Then with $x, y, z, u, v \in X$ and $a, b \in \mathbb{R}$, the relations $\leq$, $<$ and $\ll$ satisfy:*

(i) *$\leq$ is a partial order on X.*

(ii) $x \leq y$ *and* $a \geq 0 \quad \Rightarrow \quad ax \leq ay, \qquad x \leq y$ *and* $a \leq 0 \quad \Rightarrow \quad ay \leq ax.$

(iii) $x \leq y$ *and* $u \leq v \quad \Rightarrow \quad x + u \leq y + v.$

(iv) *For convergent sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X,*

$$x_n \leq y_n \quad \text{for all} \quad n \in \mathbb{N} \qquad \text{implies} \qquad \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

(v) *Moreover, for $\ll$ we have*

$$x \ll y \text{ and } y \ll z \quad \Rightarrow \quad x \ll z, \qquad x \ll y \text{ and } y \leq z \quad \Rightarrow \quad x \ll z,$$

$$x \leq y \text{ and } y \ll z \quad \Rightarrow \quad x \ll z, \qquad x \ll y \text{ and } a > 0 \quad \Rightarrow \quad ax \ll ay.$$

Proof.

(i) By $x - x = 0 \in K$, we obtain reflexivity. Let $x \leq y$ and $y \leq x$, i.e. $y - x \in K$ and $x - y = -(y - x) \in K$. This implies $y - x \in K \cap (-K) = \{0\}$. Hence, $\leq$ is antisymmetric. Assume $x \leq y$ and $y \leq z$. Then $z - x = (z - y) + (y - x) \in K$ and in turn, $x \leq z$.

(ii) Assume, that $x \leq y$ and $a \geq 0$. By $y - x \in K$, we obtain $ay - ax = a(y - x) \in K$, i.e. $ax \leq ay$. Since $x \leq y$ is equivalent to $-y \leq -x$, we have $ay \leq ax$ for $a \leq 0$.

(iii) Let $x \leq y$ and $u \leq v$. Then $(y + v) - (x + u) = (y - x) + (v - u) \in K$, and therefore $x + u \leq y + v$.

(iv) Consider two convergent sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X with the limits $\lim_{n\to\infty} x_n = x \in X$ and $\lim_{n\to\infty} y_n = y \in X$. Then $x_n \leq y_n$ implies $y_n - x_n \in K$ for all $n \in \mathbb{N}$. Since K is closed,

$$\lim_{n\to\infty} y_n - x_n = y - x \in K,$$

and thus, $x \leq y$.

(v) Let $x \ll y$ and $a > 0$. In particular, $y - x \in K^\circ$. Since multiplication with a fixed scalar is a homeomorphism, aK° is open. We obtain

$$ay - ax = a(y - x) \in aK^\circ \subseteq aK \subseteq K,$$

and in turn $ay - ax \in K^\circ$, i.e. $ax \ll ay$. If $x \ll y$ and $y \leq z$, then $y - x \in K^\circ$ and $z - y \in K$. Since the translation with a fixed element is a homeomorphism, $(z - y) + K^\circ$ is open. We have

$$z - x = (z - y) + (y - x) \in (z - y) + K^\circ \subseteq K + K \subseteq K.$$

Hence, $z - x \in K^\circ$, i.e. $x \ll z$. The remaining results easily follow from what we just showed.

□

According to the above results, the relations $\leq$ and $<$ on X can be handled just like the common order relation on the real numbers. Likewise, an *order interval* can be defined for $x, y \in X$ as

$$[x, y] := \{\, z \in X \;:\; x \leq z \leq y \,\},$$

where as usual $x \leq z \leq y$ is a short notation for $x \leq z$ and $z \leq y$. Unlike one would expect, the order interval does not have to be bounded. If however, the order cone is *normal*, i.e. if there exists $C > 0$ such that

$$x, y \in X, \quad 0 \leq x \leq y \qquad \text{implies} \qquad \|x\| \leq C \, \|y\|, \tag{5.1.1}$$

then $[x, y]$ is bounded for all $x, y \in X$. Indeed, $x \leq z \leq y$ implies $0 \leq z - x \leq y - x$ and by (5.1.1) and the reverse triangle inequality,

$$\|z\| \leq \|z - x\| + \|x\| \leq C \, \|y - x\| + \|x\|$$

for all $z \in [x, y]$.

Example 5.1.10.

(i) Consider X and K from Example 5.1.7 (ii). If we equip X for instance with the maximum norm,

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|, \qquad x = (x_1, \ldots, x_n)^T \in X,$$

then K is a normal cone with $C = 1$.

(ii) Let $\langle X, K\rangle$ be as in Example 5.1.7 (i). By definition of the uniform norm $\|\cdot\|_\infty$ on $C(M)$, the assumption $0 \le f \le g$ immediately implies

$$\|f\|_\infty = \max_{x\in M}|f(x)| \le \max_{x\in M}|g(x)| = \|g\|_\infty .$$

Hence, K is a normal cone with $C = 1$.

(iii) Considering $X = C^1[0,1]$ and $K = \left\{ f \in C^1[0,1] \,:\, f(x) \ge 0,\ x \in [0,1] \right\}$, let X carry the norm

$$\|f\| = \max_{x\in[0,1]}|f(x)| + \max_{x\in[0,1]}|f'(x)|, \qquad f \in X.$$

As before, $0 \le f \le g$ is equivalent to $0 \le f(x) \le g(x)$ for all $x \in [0,1]$. Since this condition does not include any restriction for the derivatives, there cannot exist any constant $C > 0$ such that $0 \le f \le g$ always implies $\|f\| \le C\,\|g\|$. Therefore, K is not normal.

//

Normal order cones and the boundedness of order intervals play an important role in the convergence of iterative methods for solutions of operator equations; see [Z, S 7.4].

5.2 Positive operators

Having defined an order on X, we are able to introduce the notion of positive operators. Compact, positive operators will serve as the generalisation of positive matrices in the Krein-Rutman Theorem. Their definition is intuitive.

Definition 5.2.1. Let $\langle X, K\rangle$, $\langle U, L\rangle$ be real, ordered Banach spaces and $T \in \mathcal{B}(X,U)$.

- We call T *positive*, if $T(K\backslash\{0\}) \subseteq L$.
- T is called *strictly positive*, if $T(K\backslash\{0\}) \subseteq L\backslash\{0\}$.
- If $T(K\backslash\{0\}) \subseteq L^\circ$, then T is called *strongly positive*.

//

For not necessarily linear $T : D(T) \subseteq X \to U$, similar notions can be defined.

- $T : D(T) \subseteq X \to U$ is called *monotone increasing*, if

$$x < y \qquad \text{implies} \qquad T(x) \leq T(y)$$

for all $x, y \in D(T)$. T is called *strictly* or *strongly monotone increasing*, if $\leq$ can be replaced by $<$ or $\ll$, respectively. Similarly, T is called *monotone decreasing*, if $\leq$ can be replaced by $\geq$ and *strictly* or *strongly monotone decreasing*, if $\leq$ can be replaced by $>$ or $\gg$, respectively.

- $T : D(T) \subseteq X \to U$ is called *positive*, if $T(0) \geq 0$ and

$$x > 0 \qquad \text{implies} \qquad Tx \geq 0 \tag{5.2.1}$$

for all $x \in D(T)$. T is called *strictly* or *strongly positive*, if $\geq$ can be replaced by $>$ or $\gg$, respectively.

If $T : D(T) \subseteq X \to U$ is linear, then T being monotone increasing is equivalent to T being positive and these concepts correspond to positivity defined for $T \in \mathcal{B}(X, U)$. The same holds true for T being strictly or strongly monotone increasing and strictly or strongly positive, respectively. Indeed, if T is positive in terms of (5.2.1), then for $x, y \in D(T)$ with $x < y$, i.e. $y - x > 0$, we obtain

$$0 \leq T(y - x) = Ty - Tx,$$

and T proves to be monotone increasing. In case of T that is strictly or strongly positive, $\leq$ can be replaced in the above line by $<$ or $\ll$, respectively. Conversely, every (strictly or strongly) monotone increasing linear operator T is clearly (strictly or strongly) positive.

Since our main result only concerns linear and bounded operators, the non-linear case will not be considered further in this work. Discussing their spectrum, we will almost exclusively work with bounded endomorphisms $T \in \mathcal{B}(X)$, i.e. $X = U$ and $K = L$. Let us consider some examples of positive operators.

Example 5.2.2.

(i) Consider $X = U = \mathbb{R}$ and $K = L = \mathbb{R}^+ \cup \{0\}$. For functions $T : D(T) \subseteq \mathbb{R} \to \mathbb{R}$ the notions (strictly) monotone increasing (resp. decreasing) correspond to their usual definition. Since in this case the relations $<$ and $\ll$ coincide, the terms strictly and strongly monotone increasing (resp. decreasing) are equivalent.

(ii) Let $\langle X, K\rangle = \langle U, L\rangle$ be like in Example 5.1.7 (i) with a compact $M \subseteq \mathbb{R}^n$. Consider the integral operator

$$T : \begin{cases} X & \to & X, \\ f & \mapsto & \displaystyle\int_M A(\cdot, y) f(y)\; dy, \end{cases}$$

where the kernel $A : M \times M \to \mathbb{R}$ is supposed to be continuous. It is easy to show, that $T \in \mathcal{B}(X)$. Additionally, one can prove that T is compact. If we assume $A(x, y) \geq 0$ for $x, y \in M$, then T is clearly positive.

If A is even assumed to satisfy $A(x, y) > 0$ for all $x, y \in M$, then T is strongly positive. In order to show this, consider $f \in K\backslash\{0\}$ and $x_0 \in M$ with $f(x_0) > 0$. There exists a neighborhood U_0 of x_0 and $c > 0$ with $f(x) > c$ for all $x \in U_0$. Moreover, the compactness of M yields

$$d := \min_{x,y \in M} A(x, y) > 0.$$

We obtain

$$Tf(x) \geq \int_{U_0} A(x, y) f(y)\; dy \geq dc\; \lambda^n(U_0) =: c_1 > 0$$

for all $x \in M$, where λ^n denotes the n-dimensional Lebesgue measure. In particular, T is strongly positive.

//

Remark 5.2.3. Let $A \in \mathbb{R}^{n\times n}$. We can consider $A \in \mathcal{B}(X)$ as an operator on the real, ordered Banach space $\langle X, K\rangle$ from Example 5.1.7 (ii). It is easy to verify, that the notion $A \geq 0$ defined in Section 1.1.1 is equivalent to the positivity of A as an operator on X. Moreover, $A > 0$ in terms of Section 1.1.1, if and only if $A \in \mathcal{B}(X)$ is strongly positive. By this equivalence, the statements of the Perron-Frobenius Theorem and the Krein-Rutman Theorem correspond with each other. //

Let us introduce a concept, which plays a central role in the discussion of boundary value problems for semi linear elliptic equations; see [Z, S 7.16]. Since it is not related to the Krein-Rutman Theorem, we will only present a short introduction, proofs and further explanations can be found in [Z, p 279ff].

Consider the real, ordered Banach spaces $\langle X, K\rangle$ and $\langle U, L\rangle$ and fix $e \in L\backslash\{0\}$. As $L \supsetneqq \{0\}$, this can always be done. A linear operator $T : D(T) \subseteq X \to U$ is called e-positive, if for every $x \in D(T) \cap (K\backslash\{0\})$ there exist $\alpha_x, \beta_x > 0$, satisfying

$$\alpha_x e \leq Tx \leq \beta_x e.$$

Example 5.2.4. Let us continue with Example 5.2.2 (ii). If $A(x,y) > 0$ for all $x, y \in M$, then T is not only strongly positive, but e-positive with $e = \mathbb{1}_M > 0$. Consider $f \in K \backslash \{0\}$. Previously, we showed $c_1 \leq Tf(x)$ for $x \in M$. Moreover,

$$Tf(x) \leq \lambda^n(M) \max_{x,y \in M} |A(x,y)| \max_{y \in M} |f(y)| =: c_2.$$

Consequently, $c_1 \mathbb{1}_M(x) \leq Tf(x) \leq c_2 \mathbb{1}_M(x)$ for all $x \in M$ with the constants $c_1, c_2 > 0$, i.e. $c_1 \mathbb{1}_M \leq Tf \leq c_2 \mathbb{1}_M$. //

Based on an ordered Banach space $\langle X, K \rangle$ and $e > 0$, one can define a subspace $X_e \subseteq X$ by

$$X_e := \bigcup_{a>0} [-ae, ae].$$

It becomes a Banach space itself, equipped with the norm $\|\cdot\|_e$ given by

$$\|x\|_e := \inf \{ a > 0 \, : \, x \in [-ae, ae] \}$$

for $x \in X_e$, and the embedding $\langle X_e, \|\cdot\|_e \rangle \hookrightarrow \langle X, \|\cdot\| \rangle$ is continuous. If additionally $e \in K^\circ$, then $X = X_e$ and the norms $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent. Moreover,

$$K_e := K \cap X_e$$

is an order cone for the real Banach space X_e with $e \in K_e{}^\circ$, where $K_e{}^\circ$ denotes the interior of K_e in $\langle X_e, \|\cdot\|_e \rangle$. If a linear operator $T : X \to X$ is e-positive with $T(X) \subseteq X_e$, then $T : X \to X_e$ is strongly positive. It is noteworthy, that the interior of K_e is not empty, even if $K^\circ = \emptyset$ and that strongly positive operators can be constructed in this way.

5.3 Dual cones, ordered dual spaces

After equipping the Banach space X with an order cone K and studying the induced order relation, we would like to do so with its topological dual space X'. Every given order cone K naturally induces a subset $K' \subseteq X'$, which proves to be an order cone for X' under certain conditions.

Definition 5.3.1. Let $\langle X, K \rangle$ be a real, ordered Banach space. We call

$$K' := \left\{ x' \in X' \, : \, \langle x', x \rangle \geq 0 \quad \text{for all} \quad x \in K \right\}$$

the *dual order cone* of K. For $x' \in X'$ we define

- $x' \geq 0, \quad$ if $\quad x' \in K'$.
- $x' > 0, \quad$ if $\quad x' \in K'$ and $\langle x', x_0 \rangle > 0$ for at least one $x_0 \in K$.

Moreover, x' is called *strictly positive*, if $\langle x', x \rangle > 0$ for all $x \in K \backslash \{0\}$. //

In terms of the above definition, $x' \geq 0$ is equivalent to $x' \in \mathcal{B}(X, \mathbb{R})$ being a positive operator regarding Definition 5.2.1. The notion of a functional $x' \in X'$ being strictly positive coincides with x' being strictly positive as an operator $x' \in \mathcal{B}(X, \mathbb{R})$.

Note, that $x' > 0$ implies $x' \in K' \backslash \{0\}$, but the converse does not hold true in general. For instance, $x'|_K \equiv 0$ is possible for $x' \in K' \backslash \{0\}$, if the order cone K is not required to have additional properties. Indeed, $x' > 0$ in general is a stronger condition than $x' \in K' \backslash \{0\}$:

Proposition 5.3.2. *Let $\langle X, K \rangle$ be a real, ordered Banach space and consider $x' \in X'$ with $x' > 0$. Then $\langle x', x \rangle > 0$ for all $x \in K^\circ$.*

Proof. As $x' > 0$, there exists $x_0 \in K$ with $\langle x', x_0 \rangle > 0$. Assume, that $\langle x', y \rangle = 0$ for some $y \in K^\circ$. Choose $r > 0$, such that $U_r(y) \subseteq K$. We obtain

$$y \pm \alpha x_0 \in U_r(y) \subseteq K \qquad \text{for any} \quad \alpha < \frac{r}{\|x_0\|}.$$

From $x' > 0$ we conclude $\pm \alpha \langle x', x_0 \rangle = \langle x', y \pm \alpha x_0 \rangle \geq 0$, which implies the contradiction $\langle x', x_0 \rangle = 0$. Hence, $\langle x', x \rangle > 0$ for all $x \in K^\circ$. □

At first it is not clear, if functionals $x' > 0$ exist for every order cone K. The next proposition shows more than just their existence.

Proposition 5.3.3. *Let $\langle X, K \rangle$ be a real, ordered Banach space. For every $x \in K \backslash \{0\}$, there exists $x' \in K'$ with $\langle x', x \rangle > 0$. In particular, $K' \supsetneqq \{0\}$.*

Proof. For $x \in K \backslash \{0\}$, certainly $-x \notin K$. Since K is closed, there exists an open and convex neighborhood U of $-x$ with $U \subseteq K^c$. The two sets U and K are disjoint, convex and non-empty. Since U is open, we can apply the Separation Theorem of Hahn-Banach (e.g. [R, p 59]) and obtain the existence of $x' \in X'$ and $a \in \mathbb{R}$, such that

$$\langle x', z \rangle < a \leq \langle x', y \rangle \qquad \text{for} \quad z \in U \quad \text{and} \quad y \in K.$$

As $0 \in K$, we conclude $a \leq 0$ and consequently, $\langle x', z \rangle < 0$ for $z \in U$. Assume, that $c := \langle x', x_0 \rangle < 0$ for some $x_0 \in K$. By $nx_0 \in K$ we obtain

$$a \leq \langle x', nx_0 \rangle = n \langle x', x_0 \rangle = nc$$

for all $n \in \mathbb{N}$, which is an obvious contradiction. Hence, $x' \geq 0$. Finally, $-x \in U$ yields $\langle x', x \rangle = -\langle x', -x \rangle > 0$. $\square$

The dual order cone K' is not an order cone in general. Unlike (i) and (ii), the property (iii) from Definition 5.1.1 does not necessarily have to be satisfied. However, if K is assumed to be total, then K' is an order cone for X'.

Proposition 5.3.4. *Consider a real, ordered Banach space $\langle X, K \rangle$. In case that K is total, i.e. $\overline{K - K} = X$, the dual order cone K' is an order cone for X'.*

Proof. We show the defining properties of an order cone from Definition 5.1.1.

(i) Proposition 5.3.3 yields $K' \supsetneq \{0\}$. We need to show, that K' is closed. Consider a sequence $(x_n')_{n\in\mathbb{N}}$ in K' with $\lim_{n\to\infty} x_n' = x' \in X'$. Being convergent in the operator norm, the sequence also converges pointwise. Consequently, $\langle {x_n}', x \rangle \geq 0$ for all $n \in \mathbb{N}$ and $x \in K$ implies

$$\langle x', x \rangle = \lim_{n\to\infty} \langle {x_n}', x \rangle \geq 0$$

and in turn, $x' \in K'$.

(ii) Let $x', y' \in K'$ and $\alpha \geq 0$. For all $x \in K$ we obtain

$$\langle x' + y', x \rangle = \langle x', x \rangle + \langle y', x \rangle \geq 0 \qquad \text{and} \qquad \langle \alpha x', x \rangle = \alpha \langle x' x \rangle \geq 0,$$

i.e. $K + K \subseteq K$ and $\alpha K \subseteq K$.

(iii) Assume, that $x' \in K'$ and $-x' \in K'$ for some $x' \in X'$. Then for $x \in K$ we have

$$\langle x', x \rangle \geq 0 \qquad \text{and} \qquad \langle x', -x \rangle = \langle -x', x \rangle \geq 0.$$

Hence, $\langle x', x \rangle \geq 0$ for all $x \in K - K$. Since K is total and x' is continuous, we conclude $\langle x, x' \rangle \geq 0$ for all $x \in X$, implying $x' = 0$.

$\square$

Remark 5.3.5. If the order cone K is total, $\langle X', K' \rangle$ is an ordered Banach space. The order relation induced by the cone K' coincides with the previously defined relations on K', i.e. $x' \geq 0$ and $x' > 0$ from Definition 5.3.1 are equivalent to $x' \in K'$ and $x' \in K' \backslash \{0\}$, respectively.

We show, that $x' \in K' \backslash \{0\}$ implies $x' > 0$, i.e. $x' \geq 0$ and $\langle x', x_0 \rangle > 0$ for some $x_0 \in K$. Assuming $\langle x', x \rangle = 0$ for all $x \in K$ yields $\langle x', x \rangle = 0$ for all $x \in K - K$. By the totality of K, we obtain the contradiction $x' = 0$. The reverse implication and the other equivalence are obvious. //

Remark 5.3.6. Let us turn our attention to the complexification $X_{\mathbb{C}}$ of a real, ordered Banach space $\langle X, K\rangle$. The order cone K can be considered as a subset of $X_{\mathbb{C}}$:

$$K \subseteq X \subseteq X_{\mathbb{C}}. \tag{5.3.1}$$

Since X is closed in $X_{\mathbb{C}}$, so is K. All the other defining properties of an order cone remain valid considering $K \subseteq X_{\mathbb{C}}$. Equally to the real case, one can define an order relation on the complex Banach space $X_{\mathbb{C}}$ by using the order cone K, i.e. $\langle X_{\mathbb{C}}, K\rangle$ becomes a *complex ordered Banach space* with its order cone contained in the subset $X \subseteq X_{\mathbb{C}}$. Clearly, $x \geq 0$ and $x > 0$ in X are equivalent to $x \geq 0$ and $x > 0$ in $X_{\mathbb{C}}$, respectively. This avoids any confusion in simply writing $x > 0$ for elements $x \in X \subseteq X_{\mathbb{C}}$. The same way as for $T \in \mathcal{B}(X)$, one defines (strict or strong) positivity of operators $S \in \mathcal{B}(X_{\mathbb{C}})$. By the inclusions in (5.3.1), obviously $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$ is (strictly or strongly) positive, if and only if $T \in \mathcal{B}(X)$ is (strictly or strongly) positive.

Like we elaborated in Proposition 3.1.9, the point spectrum of T is exactly the real part of the point spectrum of $T_{\mathbb{C}}$:

$$\sigma_p(T) = \sigma_p(T_{\mathbb{C}}) \cap \mathbb{R}.$$

As a consequence of its diagonal structure, every positive eigenvector $x \in K\backslash\{0\}$ of $T_{\mathbb{C}}$ corresponds to a real eigenvalue and thus has to be an eigenvector of both T and $T_{\mathbb{C}}$; see (3.1.5). If T is assumed to be positive in addition, the corresponding real eigenvalue has to be positive, i.e. $\lambda \in \sigma_p(T_{\mathbb{C}}) \cap \mathbb{R}^+ = \sigma_p(T) \cap \mathbb{R}^+$. Hence, if there exist any positive eigenvectors of the complexification of a positive operator T, they all correspond to positive eigenvalues of both T and $T_{\mathbb{C}}$.

In case, that K is total, $\langle X', K'\rangle$ is a real, ordered Banach space. As was elaborated in Section 3.2, we can consider X' as a closed subset of $(X_{\mathbb{C}})'$ by means of a bounded isomorphism. Consequently,

$$K' \subseteq X' \subseteq (X_{\mathbb{C}})'$$

can be understood as an order cone for the complex Banach space $(X_{\mathbb{C}})'$. We saw, that for $T \in \mathcal{B}(X)$, the complexification of the adjoint $(T')_{\mathbb{C}} \in \mathcal{B}((X'_{\mathbb{C}}))$ corresponds to the adjoint of the complexification $(T_{\mathbb{C}})' \in \mathcal{B}((X_{\mathbb{C}})')$. Hence, by its diagonal structure, $(T_{\mathbb{C}})'$ is (strictly or strongly) positive, if and only if T' is (strictly or strongly) positive.

Note, that if $T \in \mathcal{B}(X)$ is positive, so is $T' \in \mathcal{B}(X')$. Indeed, for $x' \in K' \backslash \{0\}$ and $x \in K \backslash \{0\}$ we have

$$\langle T'x', x\rangle = \langle x', Tx\rangle \geq 0 \tag{5.3.2}$$

by $Tx \in K$, i.e. $T'x' \geq 0$. If T is even strongly positive, then $Tx \in K^{\circ}$ and according to Proposition 5.3.3, we can replace $\geq$ by $>$ in (5.3.2). In this case, $T'x'$ is strictly positive for every $x' > 0$; see Definition 5.3.1. //

Chapter 6

Krein-Rutman Theorem

Recall the Perron-Frobenius Theorem, which was presented in the first chapter. It gives strong results on the spectrum of positive matrices, especially on their spectral radius. We approached the topic by first considering positive matrices and secondly, extending most of the obtained results to non-negative ones. We reverse this approach, first elaborating the Krein-Rutman Theorem for compact, positive operators and after, sharpening the results by additionally assuming them to be strongly positive. In the following, X will always denote a real Banach space.

6.1 The peripheral spectrum of a compact and positive operator

Consider an ordered Banach space $\langle X, K \rangle$ and let $T \in \mathcal{B}(X)$ be compact and positive. Recall, that the spectrum $\sigma(T) = \sigma(T_{\mathbb{C}})$ is compact, where $T_{\mathbb{C}} \in \mathcal{B}(X_{\mathbb{C}})$ is the complexification of T. The spectral radius of T is given by

$$r(T) = r(T_{\mathbb{C}}) = \max_{\lambda \in \sigma(T_{\mathbb{C}})} |\lambda|.$$

The above maximum is attained, i.e. there exists $\lambda \in \sigma(T_{\mathbb{C}})$ with $|\lambda| = r(T)$. It turns out, that it is attained by $\lambda = r(T)$, if we assume the order cone K to be total and $r(T) > 0$, i.e. $\sigma(T) \neq \{0\}$. In this case, $\lambda = r(T)$ is an eigenvalue of both T and T'. In Theorem 6.1.3, the first part of the Krein-Rutman Theorem, we will show the existence of positive eigenvectors of both T and T' corresponding to $r(T)$.

The following two lemmata will be needed in the proof of the theorem. Lemma 6.1.1 remains valid in the setting of an underlying complex Banach space. We will only consider the real case though, which will be relevant for the proof.

Lemma 6.1.1. *Let X be a real Banach space and $T \in \mathcal{B}(X)$. If $T^n u = \lambda^n u$ for some $n \in \mathbb{N}$, $\lambda \in \mathbb{R}\backslash\{0\}$ and $u \in X$, then*

$$x := \sum_{k=0}^{n-1} \lambda^{n-1-k} T^k u \ \in X$$

is an eigenvector of T corresponding to λ, i.e. $Tx = \lambda x$.

Proof. The claim follows by an easy computation using the linearity of T:

$$\begin{aligned} T\sum_{k=0}^{n-1} \lambda^{n-1-k} T^k u &= \sum_{k=0}^{n-1} \lambda^{n-1-k} T^{k+1} u = \lambda \sum_{k=1}^{n} \lambda^{n-1-k} T^k u \\ &= \lambda\Big(\sum_{k=1}^{n-1} \lambda^{n-1-k} T^k u + \lambda^{-1} T^n u \Big) \\ &= \lambda\Big(\sum_{k=1}^{n-1} \lambda^{n-1-k} T^k u + \lambda^{n-1} u \Big) = \lambda \sum_{k=0}^{n-1} \lambda^{n-1-k} T^k u. \end{aligned}$$

□

The next lemma is a basic consideration of the geometry of complex numbers.

Lemma 6.1.2. *Let $M \subseteq \mathbb{C}$ be compact and non-empty. For $\varepsilon \geq 0$ we set*

$$m_\varepsilon := \max_{\lambda \in M} |\lambda + \varepsilon\lambda^2|.$$

Assume, that $\hat{M} := M\backslash M_0$ is compact, where $M_0 := M \cap \{\, \lambda \in \mathbb{C} : |\lambda| = m_0 \,\}$. Let $\lambda_0 \in M_0 \subseteq M$ satisfy $\operatorname{Re}\lambda_0 = \max\{\, \operatorname{Re}\lambda : \lambda \in M_0 \,\} =: c$. Then there exists $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0)$

$$m_\varepsilon = |\lambda_0 + \varepsilon\lambda_0^2| \qquad \text{and} \qquad |\lambda + \varepsilon\lambda^2| < m_\varepsilon \quad \text{for} \quad \lambda \in M\backslash\{\lambda_0, \overline{\lambda_0}\}.$$

Either the element λ_0 is uniquely determined in M, or λ_0 and $\overline{\lambda_0}$ are the only elements of M with the above properties.

Proof. Since $\operatorname{Re} : \mathbb{C} \to \mathbb{R}$ is continuous and $M_0 \neq \emptyset$ is compact, a $\lambda_0 \in M_0 \subseteq M$ with the above properties always exists. Moreover, the two cases

$$\{\, \lambda \in \mathbb{C} : \operatorname{Re}\lambda = c \,\} \cap M_0 = \begin{cases} \{\lambda_0\}, \\ \{\lambda_0, \overline{\lambda_0}\} \quad (\lambda_0 \neq \overline{\lambda_0}), \end{cases}$$

can occur. Obviously, in the first case, λ_0 is the unique element of M having the required properties. In the second case, the only elements of M with the said properties are λ_0 and $\overline{\lambda_0}$. Define

$$\varepsilon_0 := \min_{\lambda \in \hat{M}} 2\, \frac{c - \operatorname{Re}\lambda}{|\lambda|^2 - m_0^2}.$$

Since $\hat{M}$ is assumed to be compact, the above minimum exists, if $\hat{M} \neq \emptyset$. In case that $\hat{M} = \emptyset$, set $\varepsilon_0 := \infty$. Then for every $\varepsilon \in (0, \varepsilon_0)$ we have

$$\varepsilon(|\lambda|^2 - m_0^2) < 2(c - \operatorname{Re}\lambda) \qquad \text{for all} \quad \lambda \in M \backslash \{\lambda_0, \overline{\lambda_0}\}. \tag{6.1.1}$$

Note, that the above relation is obviously satisfied for $\lambda \in M_0 \backslash \{\lambda_0, \overline{\lambda_0}\}$, since then $|\lambda|^2 - m_0^2 = 0$ and $\operatorname{Re}\lambda < c$. By (6.1.1), we have $2\operatorname{Re}\lambda + \varepsilon|\lambda|^2 < 2c + \varepsilon m_0^2$ and therefore,

$$\begin{aligned} |\lambda + \varepsilon\lambda^2| &= |\lambda|^2 + \varepsilon|\lambda|^2(2\operatorname{Re}\lambda + \varepsilon|\lambda|^2) \\ &< m_0^2 + \varepsilon m_0^2\,(2c + \varepsilon m_0^2) = |\lambda_0 + \varepsilon\lambda_0^2| \end{aligned}$$

for arbitrary $\lambda \in M \backslash \{\lambda_0, \overline{\lambda_0}\}$, implying $m_\varepsilon = |\lambda_0 + \varepsilon\lambda_0^2| = |\overline{\lambda_0} + \varepsilon\overline{\lambda_0}^2|$. □

Before presenting the first part of the Krein-Rutman Theorem, let us have a look on the setting it is placed in. If the order cone K of a real, ordered Banach space $\langle X, K\rangle$ is total, the dual space $\langle X', K'\rangle$ becomes a real, ordered Banach space with the dual cone K'; see Proposition 5.3.4. For the proof of the theorem, we will need the complexification of X in order to be able to apply the results that we elaborated about the resolvent in Section 4.5.

The complexifications $X_{\mathbb{C}}$ and $(X')_{\mathbb{C}} \simeq (X_{\mathbb{C}})'$ can be seen as complex, ordered Banach spaces by considering the order cones as their subsets, $K \subseteq X_{\mathbb{C}}$ and $K' \subseteq (X_{\mathbb{C}})'$; see Remark 5.3.6. In the same remark we saw that if $T \in \mathcal{B}(X)$ is positive, so is $T' \in \mathcal{B}(X')$ and by the diagonal structure of the operators $T_{\mathbb{C}}$ and $(T')_{\mathbb{C}} \simeq (T_{\mathbb{C}})'$, they can be considered as positive operators on the complex, ordered Banach spaces $\langle X_{\mathbb{C}}, K\rangle$ and $\langle (X_{\mathbb{C}})', K'\rangle$.

Theorem 6.1.3 [Krein-Rutman I]. *Let $\langle X, K\rangle$ be a real, ordered Banach space with a total order cone K. Let $T \in \mathcal{B}(X)$ be compact and positive with $r(T) > 0$. Then $r(T)$ is an eigenvalue of T and T' and there exist eigenvectors $x_0 \in K\backslash\{0\}$ and $x_0' \in K'\backslash\{0\}$ corresponding to $r(T)$, i.e.*

$$Tx_0 = r(T)x_0 \qquad \text{and} \qquad T'x_0' = r(T)x_0'$$

for some $x_0 > 0$ and $x_0' > 0$.

Proof. Consider the complexification $Y := X_{\mathbb{C}}$ of X and let $S := T_{\mathbb{C}} \in \mathcal{B}(Y)$ denote the complexification of T. Proposition 3.1.5 yields the compactness of the operators $S \in \mathcal{B}(Y)$ and $S' \in \mathcal{B}(Y')$. By Theorem 2.1.1 and Remark 2.2.8, every $\lambda \in \sigma(S)\backslash\{0\}$ is an isolated point of the spectrum and an eigenvalue of finite geometric and algebraic multiplicity of both S and S'. Since the spectrum is compact, there exists at least one $\lambda_0 \in \sigma(S)$ with $|\lambda_0| = r(S)$. In order to show the existence of positive eigenvectors of T and T' corresponding to $r(T) = r(S)$, we will distinguish three cases.

<u>1^{st} case</u>: *$\lambda_0 = r(S)$ is an eigenvalue of S.*

– First we show the positivity of $R_S(\mu) \in \mathcal{B}(Y)$ for $\mu \in \mathbb{R}$ with $\mu > \lambda_0$, i.e.

$$R_S(\mu)(K) \subseteq K \qquad \text{for} \quad \mu > \lambda_0. \tag{6.1.2}$$

Consider the series representation

$$R_S(\mu) = \sum_{n=0}^{\infty} \mu^{-n-1} S^n$$

of the resolvent from Theorem 4.1.1 (iii), which converges absolutely for $|\mu| > \lambda_0$. For $x \in K \subseteq X$, we have

$$\mu^{-n-1} S^n x = \mu^{-n-1} T^n x \in K$$

for all $n \in \mathbb{N} \cup \{0\}$. Convergence in $\mathcal{B}(Y)$ implies pointwise convergence. Since K is closed,

$$R_S(\mu)x = \left(\sum_{n=0}^{\infty} \mu^{-n-1} S^n\right) x = \lim_{N\to\infty} \sum_{n=0}^{N} \mu^{-n-1} S^n x \in K.$$

– We construct a positive eigenvector $x_0 \in K\backslash\{0\} \subseteq Y$ with $Sx_0 = \lambda_0 x_0$, implying $Tx_0 = \lambda_0 x_0$. By Corollary 4.5.7, the resolvent R_S has a pole at λ_0 and its Laurent series expansion centered at λ_0 is given by

$$R_S(\mu) = \sum_{n=-m}^{\infty} (\mu - \lambda_0)^n P_n, \tag{6.1.3}$$

where $P_{-m} \neq 0$, $P_n \in \mathcal{B}(Y)$ for all $n \geq -m$ and $m = p(S - \lambda_0 I)$. The series is absolutely convergent on any punctured disc around λ_0 fully contained in the resolvent set $\rho(S)$. As convergence in $\mathcal{B}(Y)$ implies pointwise convergence, Proposition 4.2.3 yields

$$\lim_{\mu\to\lambda_0} (\mu - \lambda_0)^m R_S(\mu)x = P_{-m}x$$

for arbitrary $x \in K$. In particular, the above equality holds true for $\mu \in \mathbb{R}$ with $\mu \to \lambda_0^+$. For such $\mu > \lambda_0$, by $(\mu - \lambda_0)^m > 0$ and $R_S(\mu)x \geq 0$ we conclude $(\mu - \lambda_0)^m R_S(\mu)x \geq 0$. As K is closed, we derive

$$P_{-m}x = \lim_{\mu\to\lambda_0^+} (\mu - \lambda_0) R_S(\mu)x \in K.$$

We showed $P_{-m}(K) \subseteq K$. Pick $u \in K\backslash\{0\}$ with $x_0 := P_{-m}u > 0$. This is possible, since K is total and $P_{-m}(K) = \{0\}$ would imply

$$P_{-m}(Y) = P_{-m}(\mathbb{C}X) = P_{-m}(\mathbb{C}\overline{(K-K)}) \subseteq \mathbb{C}\overline{(P_{-m}(K) - P_{-m}(K))} = \{0\}$$

contradicting $P_{-m} \neq 0$. Proposition 4.5.8 yields $Sx_0 = \lambda_0 x_0$.

– We construct $x_0' \in K'\backslash\{0\}$ with $T'x_0' = \lambda_0 x_0'$. Let $x_0 \in K\backslash\{0\}$ be the eigenvector of T corresponding to λ_0 whose existence we proved above. According to Proposition 5.3.3, there exists $u' \in K'\backslash\{0\}$ with $\langle u', x_0\rangle > 0$. Consider the corresponding element $u'_{\mathbb{C}} = J^{-1}(u' + i0)$ of Y' given by

$$u'_{\mathbb{C}}(z + iw) = u'(z) + iu'(w) \qquad \text{for} \quad z + iw \in X_{\mathbb{C}}$$

as was discussed in Section 3.2. Set $v' := P'_{-m}u'_{\mathbb{C}} \in Y'$. According to Remark 4.5.10, the operator $P'_{-m} \in \mathcal{B}(Y')$ leaves $X' \subseteq Y'$ invariant, i.e. v' is real valued on X and the corresponding element of X' is given by

$$x_0' := v'|_X \in X'.$$

In order to show $x_0' > 0$, let $x \in K$. Since $P_{-m}(K) \subseteq K \subseteq X$ and $u' > 0$, we obtain

$$\langle x_0', x\rangle = \langle v', x\rangle = \langle P'_{-m}u'_{\mathbb{C}}, x\rangle = \langle u'_{\mathbb{C}}, P_{-m}x\rangle = \langle u', P_{-m}x\rangle \geq 0,$$

i.e. $x_0' \geq 0$. Consider $u > 0$ with $x_0 = P_{-m}u$ from the previous step. As $x_0 \in K \subseteq X$, evaluating at u yields

$$\langle x_0', u\rangle = \langle v', u\rangle = \langle P'_{-m}u'_{\mathbb{C}}, u\rangle = \langle u'_{\mathbb{C}}, P_{-m}u\rangle = \langle u', x_0\rangle > 0,$$

and in turn, $x_0' > 0$. The identity $T'x_0' = \lambda_0 x_0'$ remains to be shown. By Proposition 4.5.5, the Laurent series expansion of $R_{S'}$ around λ_0 is given by

$$R_{S'}(\mu) = \sum_{n=-m}^{\infty} (\mu - \lambda_0)^n P_n',$$

the series being absolutely convergent on every punctured disc around λ_0 fully contained in $\rho(S') = \rho(S)$. Hence, its leading coefficient is given by P'_{-m}. Proposition 4.5.8 applied to $S' \in \mathcal{B}(Y')$ and $v' = P'_{-m}u'_{\mathbb{C}}$ yields $S'v' = \lambda_0 v'$. Consequently,

$$\langle T'x_0', x\rangle = \langle x_0', Tx\rangle = \langle v', Sx\rangle = \langle S'v', x\rangle = \langle \lambda_0 v', x\rangle = \langle \lambda_0 x_0', x\rangle,$$

for all $x \in X$, i.e. $T'x_0' = \lambda_0 x_0'$.

2nd case: There exists $\lambda_0 \in \sigma(S) = \sigma(S')$ with $|\lambda_0| = r(S)$ and $\lambda_0^n > 0$ for some $n \in \mathbb{N}$.

– According to the Spectral Mapping Theorem (e.g. [He, p 485]), λ_0^n is an eigenvalue of the compact and positive operator $S^n = (T_{\mathbb{C}})^n = (T^n)_{\mathbb{C}}$. Obviously, $r(S^n) = r(S)^n = \lambda_0^n > 0$. We can apply the result of the first case and obtain an eigenvector $u \in K\backslash\{0\}$ of T^n corresponding to the eigenvalue $\lambda_0^n > 0$. Clearly,

$$T^n u = \lambda_0^n u = |\lambda_0^n| u = |\lambda_0|^n u.$$

Since inductively, $T^k u \geq 0$ for $k = 0, \ldots, n-1$, the element

$$x_0 := \sum_{k=0}^{n-1} |\lambda_0|^{n-1-k} T^k u \in X$$

satisfies $x_0 \geq |\lambda_0|^{n-1} u > 0$. Applying Lemma 6.1.1 with $\lambda = |\lambda_0|$ yields

$$T x_0 = |\lambda_0| x_0 = r(T) x_0.$$

– Since we showed $r(S) \in \sigma(S)$, we can apply the results of the first case and obtain an eigenvector $x_0' \in K'\backslash\{0\}$ of T' corresponding to the eigenvalue $r(T)$, i.e. $T' x_0' = r(T) x_0'$.

3rd case: $\lambda^n \notin \mathbb{R}^+$ for every $\lambda \in \sigma(S) = \sigma(S')$ with $|\lambda| = r(T)$ and all $n \in \mathbb{N}$.

We will prove this case to be impossible to occur, as we will show $r(T) \in \sigma(S)$.

For $\varepsilon > 0$, define $T_\varepsilon := T + \varepsilon T^2$. Clearly, the operator $T_\varepsilon \in \mathcal{B}(X)$ is compact and positive and its complexification is the compact operator $S_\varepsilon := S + \varepsilon S^2 = (T_\varepsilon)_{\mathbb{C}} \in \mathcal{B}(Y)$. According to the Spectral Mapping Theorem (e.g. [He, p 485]), we have

$$\sigma(S_\varepsilon) = \left\{ \lambda + \varepsilon \lambda^2 \, : \, \lambda \in \sigma(S) \right\}.$$

Lemma 6.1.2 applied to $M = \sigma(S)$ yields the existence of $\lambda_0 \in \sigma(S)$ with $|\lambda_0| = r(S)$ and $\varepsilon_0 > 0$, such that the eigenvalue $\lambda_\varepsilon := \lambda_0 + \varepsilon \lambda_0^2 \in \sigma(S_\varepsilon)$ satisfies $r(S_\varepsilon) = |\lambda_\epsilon|$ and

$$|\lambda| < r(S_\varepsilon) \qquad \text{for} \quad \lambda \in \sigma(S_\varepsilon)\backslash\{\lambda_\varepsilon, \overline{\lambda_\varepsilon}\} \tag{6.1.4}$$

and $\varepsilon \in (0, \varepsilon_0)$. For every $n \in \mathbb{N}$, choose $\varepsilon_n < \min(\frac{1}{n}, \varepsilon_0)$ such that there exist $p_n \in \mathbb{Z}$ and $q_n \in \mathbb{N}$ with

$$\frac{\arg \lambda_{\varepsilon_n}}{2\pi} = \frac{\arg(\lambda_0 + \varepsilon_n \lambda_0^2)}{2\pi} = \frac{p_n}{q_n} \in \mathbb{Q}.$$

Importantly, by the continuity of arg, such ε_n can indeed be found arbitrarily small. We obtain

$$\arg \lambda_{\varepsilon_n}^{q_n} = q_n \arg \lambda_{\varepsilon_n} = p_n 2\pi$$

and consequently, $\lambda_{\varepsilon_n}^{q_n} \in \mathbb{R}^+$. Since T_{ε_n} satisfies all requirements of the present theorem, we can use the previously proven results. The second case applies to $\lambda_{\varepsilon_n} \in \sigma(S_{\varepsilon_n})$. Therefore, $r(T_{\varepsilon_n}) = r(S_{\varepsilon_n}) = |\lambda_{\varepsilon_n}|$ is an eigenvalue of T_{ε_n} and S_{ε_n}. By (6.1.4), we conclude $\lambda_{\varepsilon_n} = \overline{\lambda_{\varepsilon_n}} = |\lambda_{\varepsilon_n}| > 0$. For $\lim_{n\to\infty} \varepsilon_n = 0$ we obtain

$$\lambda_0 = \lim_{n\to\infty} \lambda_0 + \varepsilon_n \lambda_0^2 = \lim_{n\to\infty} \lambda_{\varepsilon_n} \in \mathbb{R}^+.$$

Hence, the previously determined $\lambda_0 \in \sigma(S)$ satisfies $\lambda_0 = |\lambda_0| = r(S)$ and in turn, $r(T) \in \sigma(S)$. $\square$

6.2 Krein-Rutman Theorem for compact, strongly positive operators

In the present section, which is the last one of this work, we will present the second part of the Krein-Rutman Theorem concerning the spectrum of a strongly positive, compact operator. For its proof, we will need a lemma, which is strongly related to a well known concept of functional analysis:

Let X be a real Banach space. For an *absorbing* subset $A \subseteq X$, i.e. $X = \bigcup_{t>0} tA$, the *Minkowski functional* $\mu_A : X \to [0,\infty)$ of A is defined by

$$\mu_A(x) := \inf \{ t > 0 \,:\, x \in tA \}, \qquad \text{for} \quad x \in X.$$

The functional satisfies $\mu_A(0) = 0$ and $\mu_{\frac{1}{r}A}(x) = \mu_A(rx) = r\mu_A(x)$ for $r > 0$ and $x \in X$. If A is convex, then $\mu_A(x+y) \le \mu_A(x) + \mu_A(y)$ for $x, y \in X$ and

$$\{ x \in X \,:\, \mu_A(x) < 1 \} \subseteq A \subseteq \{ x \in X \,:\, \mu_A(x) \le 1 \}.$$

If A is in addition open, then

$$A = \{ x \in X \,:\, \mu_A(x) < 1 \}. \tag{6.2.1}$$

Consider an ordered Banach space $\langle X, K\rangle$ with $K^\circ \neq \emptyset$ and fix $u \in K^\circ$. Then the set $K - u$ contains a neighborhood of 0 and is thus absorbing. Hence, we can consider the Minkowski functional $\mu_{K-u} : X \to [0,\infty)$ of $K - u$. If we set

$$\alpha_u := \frac{1}{\mu_{K-u}} : X \to (0, \infty],$$

then $\alpha_u(x)$ is also given by

$$\begin{aligned}\alpha_u(x) &= \frac{1}{\inf\left\{\, t > 0 \,:\, x \in t(K-u) \,\right\}} = \sup\left\{\, \tfrac{1}{t} > 0 \,:\, x \in t(K-u) \,\right\} \\ &= \sup\left\{\, t > 0 \,:\, x \in \tfrac{1}{t}(K-u) \,\right\} = \sup\left\{\, t > 0 \,:\, u + tx \in K \,\right\}\end{aligned}$$

for $x \in X$. The mapping α_u will play an important role in the proof of Theorem 6.2.3. Its properties are stated in the following lemma.

Lemma 6.2.1. *Let $\langle X, K\rangle$ be a real, ordered Banach space and fix $u \in K^\circ \neq \emptyset$. For $x \in X$ the quantity*

$$\alpha_u(x) := \sup\{\, \alpha \geq 0 \,:\, u + \alpha x \in K \,\}$$

satisfies the following properties:

(i) *If $x \in K$, then $\alpha_u(x) = \infty$. If $x \notin K$, then $0 < \alpha_u(x) < \infty$ and*

$$\alpha_u(x) = \max\{\, \alpha \geq 0 \,:\, u + \alpha x \in K \,\}\,.$$

(ii) *We have $u + \alpha x \in K$, if and only if $0 \leq \alpha \leq \alpha_u(x)$, and $u + \alpha x \notin K$, if and only if $\alpha > \alpha_u(x)$.*

(iii) *For $\alpha \in \mathbb{R}^+$, we have $u + \alpha x \gg 0$, if and only if $\alpha < \alpha_u(x)$. In particular, $u + \alpha_u(x)x \in \partial K$ for $x \notin K$.*

(iv) *The mapping*

$$\alpha_u : \begin{cases} X\backslash K & \to \quad \mathbb{R}^+, \\ \quad x & \mapsto \quad \alpha_u(x), \end{cases}$$

is continuous.

Proof.

(i) – For every $x \in X$ we saw above that

$$\alpha_u(x) = \frac{1}{\mu_{K-u}(x)} \in (0, \infty].$$

If $x \in K$, then for every $\alpha \geq 0$ we have

$$u + \alpha x \in K + \alpha K \subseteq K + K \subseteq K,$$

i.e. $\alpha_u(x) = \infty$. Assume $\alpha_u(x) = \infty$, i.e. $u + \alpha_n x \in K$ for a sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $0 < \alpha_n \to \infty$. Taking the limit $n \to \infty$ in

$$\alpha_n^{-1}u + x \in \alpha_n^{-1}K \subseteq K$$

yields $x \in K$.

- Let $x \in X \backslash K$. According to the definition on the supremum, there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $\lim_{n\to\infty} \alpha_n = \alpha_u(x)$ and $u + \alpha_n x \in K$ for all $n \in \mathbb{N}$. Since K is closed, we obtain

$$u + \alpha_u(x)x = \lim_{n\to\infty} u + \alpha_n x \in K$$

and subsequently, $\alpha_u(x) = \max\{\, \alpha \geq 0 \,:\, u + \alpha x \geq 0 \,\}$.

(ii) By Remark 5.1.2, the order cone K is convex. Therefore, $\alpha \leq \alpha_u(x)$ implies $u + \alpha x \in K$. Obviously, $u + \alpha x \notin K$ for any $\alpha > \alpha_u(x)$.

(iii)
- First let $\alpha \in \mathbb{R}^+$ with $u+\alpha x \gg 0$. There exists $r > 0$ with $U_r(u+\alpha x) \subseteq K$. Clearly,

$$u + \alpha_0 x \in U_r(u + \alpha x) \subseteq K \qquad \text{for any} \quad \alpha_0 \in (\alpha, \alpha + r\,\|x\|^{-1}).$$

Therefore, $\alpha_u(x) \geq \alpha + r\,\|x\|^{-1} > \alpha$.

- Conversely, suppose $0 < \alpha < \alpha_u(x)$. Take $\alpha_0 \in \mathbb{R}^+$ with $\alpha < \alpha_0 < \alpha_u(x)$. Then $u + \alpha x$ can be written as the convex combination

$$u + \alpha x = t(u + \alpha_0 x) + (1-t)u \qquad \text{with} \quad t = \frac{\alpha}{\alpha_0} \in (0,1). \tag{6.2.2}$$

Since $u \in K^\circ$, there exists $r > 0$ such that $U_r(u) \subseteq K$. By $u+\alpha_0 x \in K$ and the convexity of K, we conclude

$$U := t(u + \alpha_0 x) + (1-t)U_r(u) \subseteq tK + (1-t)K \subseteq K.$$

As the operations $+$ and $\cdot$ are homeomorphic, U is open. Moreover, (6.2.2) implies $u + \alpha x \in U \subseteq K$. Hence $u + \alpha x \in K^\circ$, i.e. $u + \alpha x \gg 0$.

(iv) Consider any $x \in X\backslash K$ and $\varepsilon > 0$. By (i) and (iii), we have $u+(\alpha_u(x)-\varepsilon)x \in K^\circ$ and $u+(\alpha_u(x)+\varepsilon)x \in X\backslash K$. Since $+$ and $\cdot$ are homeomorphic and both K° and $X\backslash K$ are open, there exists $\delta > 0$ such that

$$u + (\alpha_u(x) - \varepsilon)U_\delta(x) \subseteq K^\circ \qquad \text{and} \qquad u + (\alpha_u(x) + \varepsilon)U_\delta(x) \subseteq X\backslash K.$$

This implies $\alpha_u(x)-\varepsilon < \alpha_u(w)$ and $\alpha_u(x)+\varepsilon > \alpha_u(w)$, thus $|\alpha_u(x)-\alpha_u(w)| < \varepsilon$ for all $w \in U_\delta(x)$. Hence, α_u is continuous.

□

The next lemma contains an elementary argument in finite dimensions. Together with Lemma 6.2.1, it will provide statement (iii) of Theorem 6.2.3.

Lemma 6.2.2. *Consider a real Banach space X and $T \in \mathcal{B}(X)$. For $\lambda = \sigma + i\tau \in \mathbb{C}$ with $\tau \neq 0$, let $u, v \in X$ satisfy*

$$\begin{aligned} Tu &= \sigma u - \tau v, \\ Tv &= \tau u + \sigma v. \end{aligned} \tag{6.2.3}$$

Then the operator $|\lambda|^{-1}T \in \mathcal{B}(X)$ leaves the set

$$C := \{\, (\cos t)u + (\sin t)v \,:\, t \in [0, 2\pi) \,\}$$

invariant, i.e. $|\lambda|^{-1}Tw \in C$ for every $w = (\cos t_0)u + (\sin t_0)v$, where $t_0 \in [0, 2\pi)$.

Proof. Consider $w = (\cos t_0)u + (\sin t_0)v \in C$. By (6.2.3),

$$|\lambda|^{-1}Tw = \frac{1}{\sqrt{\sigma^2 + \tau^2}}((\sigma \cos t_0 + \tau \sin t_0)u + (\sigma \sin t_0 - \tau \cos t_0)v). \tag{6.2.4}$$

If we set

$$a_0 := \frac{\tau}{\sqrt{\sigma^2 + \tau^2}} \qquad \text{and} \qquad b_0 := \frac{\sigma}{\sqrt{\sigma^2 + \tau^2}},$$

then $|a_0| \leq 1$, $|b_0| \leq 1$ and ${a_0}^2 + {b_0}^2 = 1$. Hence, there exists $t_1 \in [0, 2\pi)$ such that $\sin t_1 = a_0$ and $\cos t_1 = b_0$ and (6.2.4) can be written as

$$|\lambda|^{-1}Tw = (\cos t_1 \cos t_0 + \sin t_1 \sin t_0)u + (\cos t_1 \sin t_0 - \sin t_1 \cos t_0)v.$$

Using the well known Angle Addition Theorem for sinus and cosinus, we obtain

$$|\lambda|^{-1}Tw = \cos(t_0 - t_1)u + \sin(t_0 - t_1)v.$$

Define $\tilde{t} := (t_0 - t_1) \mod 2\pi$, then $\tilde{t} \in [0, 2\pi)$ and $|\lambda|^{-1}Tw = (\cos \tilde{t})u + (\sin \tilde{t})v$, i.e. $|\lambda|^{-1}Tw \in C$.

□

Let us finally state the second part of the Krein Rutman Theorem. The aim is, to recover as many of the statements about spectral properties from Theorem 1.2.2 as possible.

Theorem 6.2.3 [Krein-Rutman II]. *Let $\langle X, K \rangle$ be a real, ordered Banach space with $K^\circ \neq \emptyset$ and let $T \in \mathcal{B}(X)$ be a compact and strongly positive operator. The following assertions hold true:*

(i) *The spectral radius $r(T) > 0$ is an* algebraically simple *eigenvalue of T, i.e.*

$$\mathrm{geom}_T(r(T)) = \mathrm{alg}_T(r(T)) = 1,$$

with a corresponding eigenvector $x_0 \in K^\circ$.

(ii) *No other eigenvalue of T has a corresponding positive eigenvector, i.e.*

$$\ker(\lambda I - T) \cap K = \{0\} \qquad \text{for all} \quad \lambda \in \sigma(T) \backslash \{r(T)\}.$$

(iii) *For every $\lambda \in \sigma(T) \backslash \{r(T)\}$, we have $|\lambda| < r(T)$.*

(iv) *The adjoint $T' \in \mathcal{B}(X')$ has a strictly positive eigenvector $x_0' \in K' \backslash \{0\}$ corresponding to the algebraically simple eigenvalue $r(T) \in \sigma(T) = \sigma(T')$,*

$$\text{geom}_{T'}(r(T)) = \text{alg}_{T'}(r(T)) = 1.$$

Proof. Note that by Proposition 5.1.5, K is total. The first step of the proof will yield $r(T) > 0$. Thus, all requirements of Theorem 6.1.3 are satisfied.

(i) *1st step: We show $r(T) > 0$ and apply Theorem 6.1.3.*

Fix an arbitrary $u \in K^\circ \neq \emptyset$. Then $Tu \in K^\circ$ and $\alpha_{Tu}(-u) \in \mathbb{R}^+$, as $-u \notin K$. Thus, $Tu - \alpha_{Tu}(-u)u \in K$, i.e.

$$\alpha T u \geq u \qquad \text{with} \qquad \alpha := \frac{1}{\alpha_{Tu}(-u)} > 0. \tag{6.2.5}$$

There exists $\delta > 0$, such that $U_\delta(u) \subseteq K$, implying $u - \delta \|y\|^{-1} y \in K$ and therefore

$$y \leq \delta^{-1} \|y\| \, u \qquad \text{for every} \quad y \in X. \tag{6.2.6}$$

For $P := \alpha T \in \mathcal{B}(X)$ by (6.2.5) we obtain inductively $u \leq P^n u$ for $n \in \mathbb{N}$. Applying (6.2.6) with $y = P^n u$, we conclude

$$u \leq P^n u \leq \delta^{-1} \|P^n u\| \, u \leq \delta^{-1} \|P^n\| \, \|u\| \, u \qquad \text{for all} \quad n \in \mathbb{N}.$$

Hence, by Proposition 3.1.7,

$$\|P^n\| \geq \frac{\delta}{\|u\|} \qquad \text{implies} \qquad r(P) = \lim_{n \to \infty} \|P^n\|^{\frac{1}{n}} > 0.$$

By the Spectral Mapping Theorem (e.g. [He, p 485]), $r(T) = \alpha^{-1} r(P) > 0$ and Theorem 6.1.3 can be applied to T. It yields the existence of an eigenvector $x_0 > 0$ of T associated to the eigenvalue $\lambda_0 := r(T) \in \sigma(T)$, i.e. $Tx_0 = \lambda_0 x_0$. The strong positivity of T implies $x_0 \gg 0$.

(i) <u>*2^nd step:*</u> *We prove $\lambda_0 = r(T)$ to be geometrically simple.*

We need to show

$$\ker(\lambda_0 I - T) = \mathrm{span}_{\mathbb{R}}\{x_0\}, \tag{6.2.7}$$

where $x_0 \in K^\circ$ denotes the positive eigenvector constructed in the previous step. Consider $y \in X\backslash\{0\}$ with $Ty = \lambda_0 y$. By $K \cap (-K) = \{0\}$, we have $y \notin K$ or $-y \notin K$. Without loss of generality, assume $y \notin K$, since otherwise we can consider $-y$ instead. Then $\alpha_{x_0}(y) \in \mathbb{R}^+$ and $x_0 + \alpha_{x_0}(y)y \geq 0$. Suppose $x_0 + \alpha_{x_0}(y)y > 0$. By the strong positivity of T,

$$0 \ll T(x_0 + \alpha_{x_0}(y)y) = \lambda_0(x_0 + \alpha_{x_0}(y)y).$$

Since $\lambda_0 > 0$, we obtain $x_0 + \alpha_{x_0}(y)y \gg 0$, in contradiction to Lemma 6.2.1 (iii). Thus, $x_0 + \alpha_{x_0}(y)y = 0$ and in turn, $y \in \mathrm{span}_{\mathbb{R}}\{x_0\}$.

(i) <u>*3^rd step:*</u> *We prove the algebraic simplicity of $\lambda_0 = r(T)$.*

We need to show $\ker(\lambda_0 I - T)^2 = \ker(\lambda_0 I - T)$. Consider $y \in X$ with

$$(\lambda_0 I - T)^2 y = (\lambda_0 I - T)(\lambda_0 I - T)y = 0.$$

By (6.2.7), there exists $\beta \in \mathbb{R}$ satisfying $(\lambda_0 I - T)y = \beta x_0$. As before, $x_0 \in K^\circ$ denotes the positive eigenvector corresponding to λ_0. Suppose that $\beta \neq 0$. We may assume $\beta > 0$, since otherwise $-y$ can be considered.

- If $y \notin K$, then $\alpha_{x_0}(y) \in \mathbb{R}^+$ and $x_0 + \alpha_{x_0}(y)y \geq 0$. Since $x_0 + \alpha_{x_0}(y)y = 0$ would imply $y \in \mathrm{span}_{\mathbb{R}}\{x_0\}$ and thus $\beta = 0$, we conclude $x_0 + \alpha_{x_0}(y)y > 0$. The strong positivity of T implies

$$\begin{aligned} 0 \ll T(x_0 + \alpha_{x_0}(y)y) &= \lambda_0(x_0 + \alpha_{x_0}(y)y) - \alpha_{x_0}(y)\beta x_0 \\ &\ll \lambda_0(x_0 + \alpha_{x_0}(y)y). \end{aligned}$$

  Therefore, $x_0 + \alpha_{x_0}(y)y \gg 0$, contradicting Lemma 6.2.1.

- Assuming $y \in K$, we obtain $y = \lambda_0^{-1}(\beta x_0 + Ty) \in K^\circ$ and since $-x_0 \notin K$, we have $\alpha_y(-x_0) \in \mathbb{R}^+$ with $y - \alpha_y(-x_0)x_0 \geq 0$. Hence,

$$\lambda_0 y - \beta x_0 - \alpha_y(-x_0)\lambda_0 x_0 = T(y - \alpha_y(-x_0)x_0) \geq 0.$$

  Consequently, $y - \alpha_y(-x_0)x_0 \geq r(T)^{-1}\beta x_0 \gg 0$, in contradiction to Lemma 6.2.1.

Assuming both $y \in K$ and $y \notin K$ leads to $\beta = 0$ and therefore, $y \in \ker(\lambda_0 I - T)$.

(ii) Note, that in Remark 5.3.6 we saw, that positive eigenvectors are associated to eigenvalues $\lambda \in \sigma_p(T) \cap \mathbb{R}^+ = \sigma_p(T_\mathbb{C}) \cap \mathbb{R}^+$ of both T and $T_\mathbb{C}$. Hence, the complexification $T_\mathbb{C}$ does not need to be considered. Clearly,

$$Tx = \lambda x \qquad \text{with} \qquad x \in K\backslash\{0\} \quad \text{and} \quad \lambda \neq \lambda_0$$

implies $0 < \lambda < \lambda_0$. Since T is strongly positive, we obtain $x \gg 0$ and $\alpha_x(-x_0) \in \mathbb{R}^+$, $x_0 \in K^\circ$ being the positive eigenvector associated to λ_0. Then $x - \alpha_x(-x_0)x_0 \geq 0$ and consequently,

$$x - \alpha_x(-x_0)\lambda_0\lambda^{-1}x_0 = \lambda^{-1}T(x - \alpha_x(-x_0)x_0) \geq 0.$$

Lemma 6.2.1 implies $\alpha_x(-x_0)\lambda_0\lambda^{-1} \leq \alpha_x(-x_0)$, in contradiction to $\lambda < \lambda_0$. Hence, $\ker(\lambda I - T) \cap K = \{0\}$.

(iii) Consider the complexification $Y := X_\mathbb{C}$ and $S := T_\mathbb{C} \in \mathcal{B}(Y)$. We need to show, that $Sz = \lambda z$ with $z = u + iv \in Y$ and $\lambda \in \sigma(T)\backslash\{r(T)\} = \sigma(T_\mathbb{C})\backslash\{r(T)\}$ implies $|\lambda| < r(T)$.

(iii.a) *Consider the case* $\lambda \in \mathbb{R}$.

It is sufficient to show, that $\lambda = -r(T) = -\lambda_0$ is not an eigenvalue of T (and $T_\mathbb{C}$; see Proposition 3.1.9). Assume $Ty = -\lambda_0 y$ for some $y \in X\backslash\{0\}$. Then $\pm y \notin K$, since $\pm y \in K$ would imply $\mp\lambda_0 y = T(\pm y) \gg 0$ and thus, $y \in K \cap (-K) = \{0\}$. Hence, $\alpha_{x_0}(\pm y) \in \mathbb{R}^+$ and $x_0 \pm \alpha_{x_0}(\pm y)y \in K$, where $x_0 \in K^\circ$ denotes the positive eigenvector corresponding to λ_0. Consequently,

$$\lambda_0 x_0 \mp \lambda_0 \alpha_{x_0}(\pm y)y = T(x_0 \pm \alpha_{x_0}(\pm y)y) \gg 0.$$

Lemma 6.2.1 (iii) yields the contradiction $\alpha_{x_0}(y) < \alpha_{x_0}(-y) < \alpha_{x_0}(y)$.

(iii.b) *Assume, that* $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

Consider $\lambda = \sigma + i\tau$ with $\tau \neq 0$. Then

$$Tu + iTv = Sz = (\sigma + i\tau)(u + iv) = (\sigma u - \tau v) + i(\tau u + \sigma v)$$

and therefore,

$$\begin{aligned} Tu &= \sigma u - \tau v, \\ Tv &= \tau u + \sigma v. \end{aligned} \tag{6.2.8}$$

(iii.b) *1st step: We show* $\operatorname{span}_{\mathbb{R}}\{u,v\} \cap K = \{0\}$.

By (6.2.8), $u, v \neq 0$ are linearly independent and the (real) subspace

$$X_1 := \operatorname{span}_{\mathbb{R}}\{u,v\} \subseteq X.$$

is invariant under T. As X_1 is finite dimensional and K is closed, their intersection

$$K_1 := X_1 \cap K$$

is closed. Suppose, that $K_1 \neq \{0\}$. It is easy to verify, that in this case K_1 is an order cone for the Banach space X_1. Let $x \in K_1 \backslash \{0\}$ be arbitrary. Since X_1 is T-invariant and T is strongly positive, $Tx \in X_1 \cap K^\circ \subseteq {K_1}^\circ$ and thus, ${K_1}^\circ \neq \emptyset$.

We can apply the already proven statement (i) of the present theorem to the real, ordered Banach space $\langle X_1, K_1 \rangle$ and the compact and strongly positive operator $T_1 := T|_{X_1} \in \mathcal{B}(X_1)$ and obtain $r(T_1) > 0$ and the existence of

$$x_1 \in {K_1}^\circ \qquad \text{with} \qquad T_1 x_1 = r(T_1) x_1.$$

Clearly, $x_1 \in K \backslash \{0\}$ is also an eigenvector of T. By statement (ii) we obtain $r(T_1) = \lambda_0 = r(T)$ and $x_1 \in \operatorname{span}_{\mathbb{R}}\{x_0\}$, where $x_0 \in K^\circ$ is the positive eigenvector corresponding to λ_0 that we constructed. We conclude $x_0 \in X_1$, i.e. $x_0 = \alpha u + \beta v$ for some $\alpha, \beta \in \mathbb{R}$. From (6.2.8) and the identity

$$\alpha T u + \beta T v = T x_0 = \lambda_0 x_0 = \lambda_0 \alpha u + \lambda_0 \beta v,$$

one easily derives the contradiction $\alpha = \beta = 0$. Hence, $K_1 = X_1 \cap K = \{0\}$.

(iii.b) *2nd step: We use Lemma 6.2.2 and the continuity of* α_{x_0} *to show* $|\lambda| < \lambda_0$.

Consider $C := \{\, (\cos t) u + (\sin t) v \;:\; t \in [0, 2\pi) \,\} \subseteq X_1$. In the previous step, we saw $X_1 \cap K = \{0\}$. Since $0 \notin C$, we have $C \subseteq X \backslash K$. Hence, $\alpha_{x_0} : X \backslash K \to \mathbb{R}^+$ is continuous on C, where $x_0 \in K^\circ$ with $T x_0 = \lambda_0 x_0$. Obviously, C is compact in X and thus, α_{x_0} attains its maximum on C,

$$\alpha_{x_0}(w_0) = \max_{w \in C} \alpha_{x_0}(w)$$

for some $w_0 \in C$. By Lemma 6.2.2, we obtain $|\lambda|^{-1} T w_0 \in C$ and therefore,

$$\alpha_{x_0}(|\lambda|^{-1} T w_0) \leq \alpha_{x_0}(w_0). \tag{6.2.9}$$

By definition, $x_0 + \alpha_{x_0}(w_0)w_0 \in K$. Assuming $x_0 + \alpha_{x_0}(w_0)w_0 = 0$ yields the contradiction $0 \ll x_0 = -\alpha_{x_0}(w_0)w_0 \in X_1 \cap K = \{0\}$. Consequently, we have $x_0 + \alpha_{x_0}(w_0)w_0 > 0$ and the strong positivity of T implies

$$0 \ll T(x_0 + \alpha_{x_0}(w_0)w_0) = \lambda_0(x_0 + \lambda_0^{-1}\alpha_{x_0}(w_0)|\lambda|(|\lambda|^{-1}Tw_0).$$

Hence, $x_0 + \alpha_{x_0}(w_0)\lambda_0^{-1}|\lambda|(|\lambda|^{-1}Tw_0) \gg 0$ and by Lemma 6.2.1 (iii) and (6.2.9) we obtain

$$\alpha_{x_0}(w_0)\lambda_0^{-1}|\lambda| < \alpha_{x_0}(|\lambda|^{-1}Tw_0) \leq \alpha_{x_0}(w_0).$$

Thus, $\lambda_0^{-1}|\lambda| < 1$ and in turn, $|\lambda| < \lambda_0$.

(iv) According to Theorem 6.1.3, T' has a positive eigenvector $x_0' \in K'\backslash\{0\}$ corresponding to the eigenvalue $\lambda_0 = r(T) \in \sigma(T) = \sigma(T')$, i.e.

$$T'x_0' = \lambda_0 x_0' \qquad \text{and} \qquad \langle x_0', x\rangle \geq 0 \quad \text{for all} \quad x \in K.$$

– We show, that x_0' is even strictly positive, i.e. $\langle x_0', x\rangle > 0$ for $x \in K\backslash\{0\}$. Fix $x > 0$. Since T is strongly positive, we have $Tx \in K^\circ$ and by Proposition 5.3.2, we conclude $\langle x_0', Tx\rangle > 0$. Consequently,

$$\lambda_0\langle x_0', x\rangle = \langle \lambda_0 x_0', x\rangle = \langle T'x_0', x\rangle = \langle x_0', Tx\rangle > 0.$$

– We already showed, that λ_0 is an algebraically simple eigenvalue of T. According to Remark 2.2.8, T and T' have the same eigenvalues, with the same geometric and algebraic multiplicities. Hence, λ_0 is an algebraically simple eigenvalue of T'.

□

This entire work is built around the above theorem, which we achieved to prove completely. It is remarkable, that all statements of Theorem 1.2.2 could indeed be preserved while passing on to general compact and (strictly) positive operators. Having elaborated these results, we will introduce two of their applications. The next corollary applies the Krein Rutman Theorem in order to obtain conditions for the existence of positive solutions of operator equations.

Corollary 6.2.4. *Consider a real, ordered Banach space $\langle X, K\rangle$ with $K^\circ \neq \emptyset$ and a compact, strongly positive operator $T \in \mathcal{B}(X)$. For every $v > 0$ the inhomogeneous equation*

$$\lambda u - Tu = v \qquad \text{with} \qquad \lambda \in \mathbb{R}$$

has a positive solution $u > 0$, if and only if $\lambda > r(T)$. In this case u is unique. Moreover, whenever $\lambda, \mu \in \mathbb{R}$ and $u, v > 0$ such that $\lambda u - Tu = \mu v$, then

$$\operatorname{sgn}(\mu) = \operatorname{sgn}(\lambda - r(T)). \tag{6.2.10}$$

Proof. Theorem 6.2.3 yields the existence of a strictly positive eigenvector $x_0' \in K' \backslash \{0\}$ of T' corresponding to $\lambda_0 = r(T)$, i.e. $T'x_0' = \lambda_0 x_0'$ and $\langle x_0', x\rangle > 0$ for all $x \in K\backslash\{0\}$.

– Suppose, there exists $u > 0$ with $\lambda u - Tu = v > 0$ for some $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}(\lambda - \lambda_0)\langle x_0', u\rangle &= \langle \lambda x_0', u\rangle - \langle T'x_0', u\rangle \\ &= \langle x_0', \lambda u\rangle - \langle x_0', Tu\rangle = \langle x_0', v\rangle > 0\end{aligned}$$

  and in turn, $\lambda > \lambda_0$.

– Let $\lambda > \lambda_0$. For every $\mu > \lambda_0$, we have $R_S(\mu)(K) \subseteq K$, where $R_S(\mu) \in \mathcal{B}(Y)$ with $\mu \in \rho(S)$ is the resolvent of the complexification $S := T_{\mathbb{C}} \in \mathcal{B}(Y)$ on $Y := X_{\mathbb{C}}$; see (6.1.2) in the proof of Theorem 6.1.3. This implies $u := R_S(\lambda)v \in K$ for arbitrary $v \in K\backslash\{0\}$. Clearly, u is the unique solution of the equation $\lambda u - Tu = v$, which also yields $u > 0$.

– To show 6.2.10, consider $u, v \in K\backslash\{0\}$ and $\mu, \lambda \in \mathbb{R}$ satisfying $\lambda u - Tu = \mu v$. We obtain

$$(\lambda - \lambda_0)\langle x_0', u\rangle = \langle x_0', \lambda u - Tu\rangle = \mu\langle x_0', v\rangle.$$

  Since x_0' is strictly positive, we conclude $\operatorname{sgn}(\mu) = \operatorname{sgn}(\lambda - \lambda_0)$.

□

The Krein Rutman Theorem enables us to obtain a relation between the spectral radii of two operators, given the corresponding relation for their values on the order cone.

Corollary 6.2.5. *Let $\langle X, K\rangle$ be a real, ordered Banach space with $K^\circ \neq 0$ and let $T \in \mathcal{B}(X)$ be compact and strongly positive. Moreover, let $P \in \mathcal{B}(X)$ be compact. If $Px \geq Tx$ for all $x \geq 0$, then $r(P) \geq r(T)$. In case, that the stronger condition $Px > Tx$ for all $x > 0$ is satisfied, the strict inequality $r(P) > r(T)$ holds true.*

Proof. Assume $Px \geq Tx$ for all $x \in K$. Obviously, $Px \geq Tx \gg 0$ for $x \in K\backslash\{0\}$ and thus, P is strongly positive. Hence, Theorem 6.2.3 can be applied to both T and P. Denote by $y_0 \in K^\circ$ the positive eigenvector of P corresponding to the eigenvalue

$r(P) > 0$. Let $x_0' \in K' \backslash \{0\}$ be the strictly positive eigenvector of T' corresponding to $r(T) > 0$, i.e. $T'x_0' = r(T)x_0'$ and $\langle x_0', x \rangle > 0$ for $x \in K\backslash\{0\}$. According to our assumptions, $R := P - T$ is positive. Thus $Ry_0 \in K$ and in turn, $\langle x_0', Ry_0 \rangle \geq 0$. Therefore,

$$
\begin{aligned}
r(P)\langle x_0', y_0 \rangle = \langle x_0', Py_0 \rangle = \langle x_0', Ry_0 + Ty_0 \rangle &= \langle x_0', Ry_0 \rangle + \langle T'x_0', y_0 \rangle \\
&= \langle x_0', Ry_0 \rangle + r(T)\langle x_0', y_0 \rangle \geq r(T)\langle x_0', y_0 \rangle.
\end{aligned}
$$

As $y_0 \in K^\circ$, we have $\langle x_0', y_0 \rangle > 0$ and consequently $r(P) \geq r(T)$. If additionally $Py > Ty$ for $y > 0$, then $Ry_0 > 0$ yields $\langle x_0', Ry_0 \rangle > 0$ and therefore,

$$
r(P)\langle x_0', y_0 \rangle > r(T)\langle x_0', y_0 \rangle,
$$

implying $r(P) > r(T)$. □

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