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Different Approaches for Pricing Derivatives in a Multi-Curve Framework

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Rem tene, verba sequentur.

Marcus Porcius Cato, 234 BC - 149 BC

Abstract

About ten years after the financial crisis, the awareness of counterparty credit risk still influences derivatives markets in terms of pricing. This thesis presents the impact on pricing derivatives using different valuation curves. Especially, the correct discounting curve should be chosen corresponding to the funding costs of a financial institution and is not necessarily the same as the curve used to calculate forward rates. Furthermore, different curve construction approaches in a multi-curve world are introduced.

Keywords: derivatives, swap pricing, curve construction, market evolution, collateral, interest rate swaps, multi-curve world

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Finally, I would like to give credit to my advisor Stefan Gerhold, who allowed this thesis to be my own.

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Notation

t	today, initiation date
$t \leq T_0, T_1, \ldots, T_M$	contractual stipulated dates (in the future)
$R(t, T)$	interest rate to maturity T
$P(t, T_i)$	zero coupon bond with maturity T_i
B_t	riskless bank account
$D(t, T_i) = \frac{B_t}{B_{T_i}}$	riskless discount factor
$P(t, T_i) = \mathbb{E}_t^{\mathcal{Q}}[D(t, T_i)]$	no-arbitrage condition
$\delta_{i-1,i} := \delta(T_{i-1}, T_i)$	year fraction between T_{i-1} and T_i
$L(t, T_i)$	Libor rate
$F(t, T_{i-1}, T_i)$	forward interest rate
$Z(t, T_i)$	basis spread
$X(t, T_i)$	currency spread
C, c_i	netted payoff or cash flow (at time T_i)
Π_t	price (or value) at time t

Table 1: Notation overview

Chapter 1

Introduction

When I started working in finance in the middle of my financial and actuarial mathematics studies, pricing a swap, or a swap itself, was a mystery. It took me a while to get into the world of curves, discounting, present values, collateral, terms, tenors, forward rates, cash flows, ...

In this master's thesis, I try to explain, in a very simple way, the impact of the financial crisis on swap pricing. In my calculations I do not assume an underlying market model.

To get a general idea of the financial world before the crisis, all important financial instruments and basic definitions are presented in Chapter 2. We get to know forward rate agreements, bonds and a variety of interest rate swaps. Furthermore, we define arbitrage-free in a single-curve world.

One consequence of the financial crisis is the awareness that counterparty credit risk can be reduced to a minimum by collateralizing OTC derivatives. In Chapter 3, we see that cash collateral is a simple way to avoid a loss, by observing the fluctuation of a swap portfolio's market value over time.

As daily collateral exchange became best practice and curves, which used to behave very close, diverged, the selection of the right curve, in terms of discounting and calculating forward rates, was challenged. In Chapter 4, we discuss definitions of financial instruments in a multi-curve world and explain increasing spreads between curves with different tenors. Consequently, we re-define the no-arbitrage condition.

In Chapter 5, we introduce approaches to calculate discount factors and forward rates depending on tenors and currencies, respectively. On the one hand we consider trades in a multi-curve environment without a CSA in place and on the other hand we have a look at financial products under a standard CSA. In particular, both approaches demand to choose the correct discounting curve depending on funding.

How much is the pricing difference between the different approaches? We compare the pricing of uncollateralized and collateralized interest rate swaps in Chapter 6. Indeed, we calculate the actual costs incurred by switching to another discounting curve without adapting the contractual coupons or floating rates indexed to a market curve. In addition, we show the necessary interest rate adjustment to price the swap fair at initiation.

About ten years after the financial crisis and a lot of discussions about choosing the correct curve for pricing a derivative, the financial market is facing new challenges. In Chapter 7, we have a look at ongoing discussions regarding the end of Libor, which is a very important reference rate. Moreover, the market standard for discounting is questioned.

Chapter 2

Before the Crisis

In this chapter is presented a market environment as it was known before the financial crisis in 2007/08. We introduce riskless rates and corresponding bonds. Furthermore, we have a look at fixed income securities and their derivatives as they were defined in a single-curve world. Most of the general definitions in this chapter are inspired from [5] and [7].

We consider a set of stipulated contractual (future) dates $(t \leq) T_0 < T_1 < \cdots < T_M$. The spot riskless interest rate at time t with maturity T_i is denoted as $R(t, T_i)$. It is the rate counterparty A charges as interest for lending a unit of money to counterparty B from today t until a fixed maturity T_i. We denote today's price of a riskless zero coupon bond with maturity T_i as $P(t, T_i)$. We assume

- $P(T_i, T_i) = 1 \qquad \forall T_i, i \in \{0, \ldots, M\}$ and
- $P(t, T_i)$ is continuously differentiable in T_i.

The latter assumption implicates that the term structure of zero coupon bond prices $T_i \mapsto P(t, T_i)$ is a smooth curve. At this point, we note that $t \mapsto P(t, T_i)$ is a stochastic process.

Further, we suppose a riskless bank account B_t and define

$$D(t, T_i) = \frac{B_t}{B_{T_i}}$$

as the riskless discount factor from T_i to t. In standard no-arbitrage context, the price of a riskless zero coupon bond is

$$P(t, T_i) = \mathbb{E}_t^{\mathcal{Q}}[D(t, T_i)],$$

where $\mathbb{E}_t^{\mathcal{Q}}[.] := \mathbb{E}^{\mathcal{Q}}[.|\mathcal{F}_t]$ denotes the expectation under a risk neutral probability measure $\mathcal{Q}$ with information until today t. We denote (market) information until t as a set of filtration $(\mathcal{F}_t)_{t \geq 0}$.

Buying a riskless bond $P(t, T_i)$ equals lending $P(t, T_i)$ units of money until T_i to a riskless counterparty, consequently generates the same costs at t and the same return at maturity T_i in an arbitrage-free market.

Using simple compounding spot interest rates and denoting the year fraction between today t and T_i, related to the day count convention[1] of the underlying instrument as $\delta(t, T_i)$, we get

$$\begin{aligned} P(t, T_i)[1 + \delta(t, T_i) R(t, T_i)] &= 1, \\ R(t, T_i) &= \frac{1}{\delta(t, T_i)} \left[\frac{1}{P(t, T_i)} - 1 \right]. \end{aligned} \tag{2.1}$$

[1] See Appendix A for a list of day count conventions in the market.

Analogously, we denote the continuously compounded spot interest rate as

$$R(t, T_i) = -\frac{1}{\delta(t, T_i)} \log P(t, T_i).$$

To answer the upcoming question of when to use simple or continuously compounded rates, we consider investing one monetary unit for one year at R (p.a.). After one year one monetary unit is worth $1 + R$. If the interest rate R is compounded, e.g. every quarter, our investment of one monetary unit values $\left(1 + \frac{R}{4}\right)^4$ at the end of one year. Let m be the compounding frequency, e.g. 1 month, 2 months, 3 months, ..., 1 week, ..., 1 day (overnight), so that

$$\left(1 + \frac{R}{m}\right)^m \to e^R \qquad \text{as } m \to \infty. \tag{2.2}$$

As a market example of interest rates we consider Libor rates[2]. The Libor rate $L(t, T_i)$ with maturity T_i is used as a reference rate for derivatives and other financial contracts. In a single-curve world we assume Libor to be risk-free

$$R(t, T_i) = L(t, T_i), \tag{2.3}$$

and the price of a bond indexed to a Libor rate is denoted as

$$\begin{aligned} P(t, T_i) &= \frac{1}{1 + R(t, T_i)\,\delta(t, T_i)} \\ &= \frac{1}{1 + L(t, T_i)\,\delta(t, T_i)} \\ &=: P_L(t, T_i). \end{aligned}$$

The term structure of zero coupon bonds also implies forward interest rates. We consider the following investment strategy to represent forward interest rates intuitively by no-arbitrage arguments. We denote today as t and two fixed dates in the future as $(t \leq)\, T_{i-1} < T_i$. The transactions of the investment strategy are

at time t — sell one zero coupon bond with maturity T_{i-1} and buy $\frac{P(t,T_{i-1})}{P(t,T_i)}$ bonds with maturity T_i, which means a netted zero investment

$$-P(t, T_{i-1}) + \frac{P(t, T_{i-1})}{P(t, T_i)} \cdot P(t, T_i) = 0$$

at time T_{i-1} — pay one monetary unit from the maturing bond $P(t, T_{i-1})$ which was sold at time t

at time T_i — receive $\frac{P(t,T_{i-1})}{P(t,T_i)}$ monetary units from the maturing bond held

To put it in another way: we did a T_{i-1}-forward investment (an investment at a stipulated date in the future) to obtain $\frac{P(t,T_{i-1})}{P(t,T_i)}$ at certainty.

[2]See Appendix B.

Using simple compounding interest rates and denoting the year fraction between today T_{i-1} and T_i as $\delta_{i-1,i} := \delta(T_{i-1}, T_i)$, we get the simple compounding forward rate fixed today for the future period T_{i-1} to T_i

$$\begin{aligned} \frac{P(t, T_{i-1})}{P(t, T_i)} &= [1 + \delta_{i-1,i} F(t, T_{i-1}, T_i)], \\ F(t, T_{i-1}, T_i) &= \frac{1}{\delta_{i-1,i}} \left[\frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right]. \end{aligned} \tag{2.4}$$

In the same way, the continuously compounded forward interest rate is given by

$$\begin{aligned} \frac{P(t, T_{i-1})}{P(t, T_i)} &= e^{\delta_{i-1,i} F(t, T_{i-1}, T_i)}, \\ F(t, T_{i-1}, T_i) &= -\frac{1}{\delta_{i-1,i}} [\log P(t, T_i) - \log P(t, T_{i-1})]. \end{aligned}$$

In this set-up we define a financial instrument in the next section, which is later used to represent other plain vanilla derivatives.

2.1 Forward Rate Agreement

Most of the ideas in this section are from [10].

A forward rate agreement (FRA) is a contract between two counterparties. The agreement defines an interest rate for a specified future term based on a notional amount. The notional amount is not exchanged under an FRA contract but used to calculate the interest rate cash flows. At the end of the future period one of the counterparties has to pay the differential of the FRA rate and the current reference rate, such as Libor, which is fixed at the beginning of the future period. The amount due is actually the netted pay off of receiving the fixed rate and concurrently paying the reference rate (or vice versa).

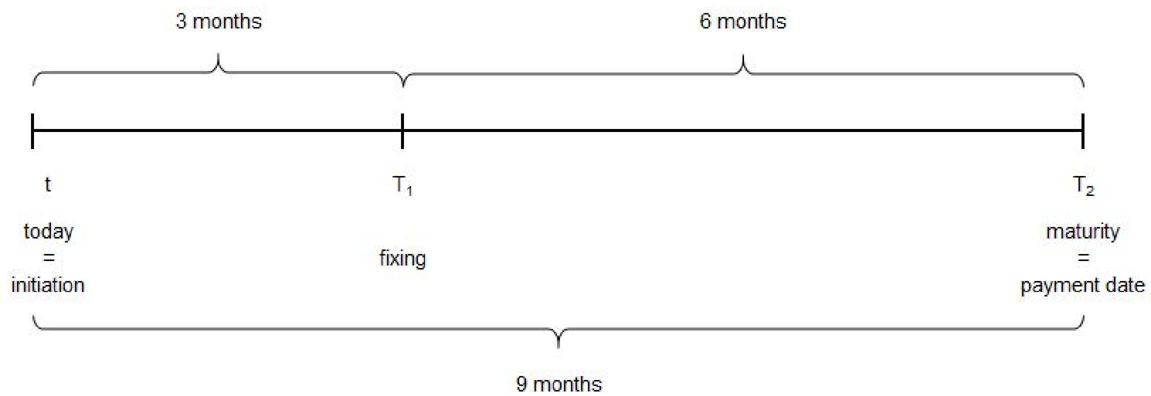

Figure 2.1: Simplified time-line of a 3x9 FRA

On this understanding, from the perspective of the counterparty who pays the fixed rate K, the netted payoff C of an FRA with maturity T_i is

$$C = \delta_{i-1,i} [L(T_{i-1}, T_i) - K], \tag{2.5}$$

with reference rate $L(T_{i-1}, T_i)$, the Libor rate as defined in Equation (2.3). T_{i-1} is the FRA's fixing date and T_i is the maturity date where the cash flow is exchanged. Examples of 3×9 FRAs are presented in Figure 2.2 and Figure 2.3.

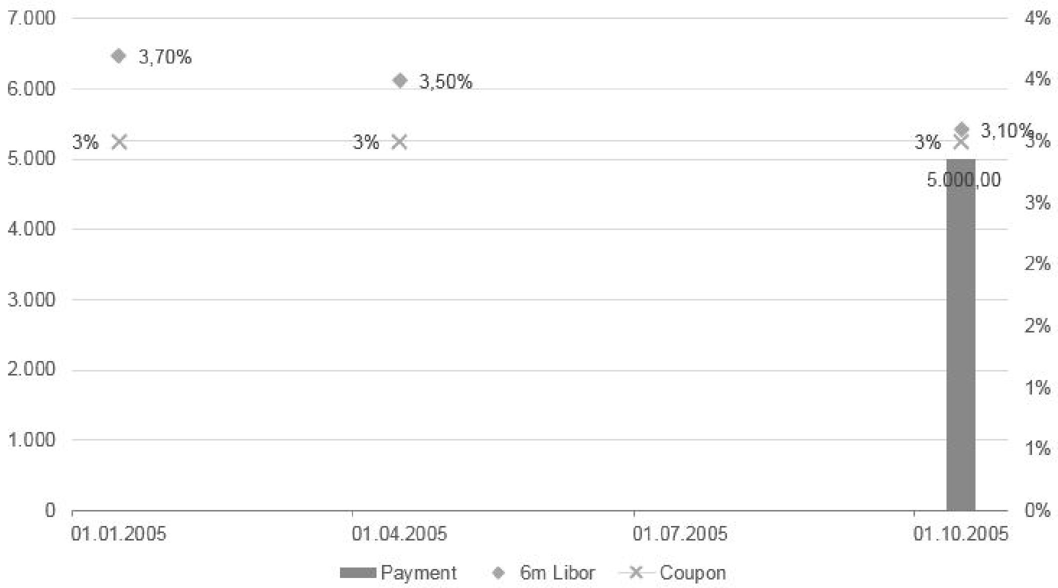

Figure 2.2: Simplified netted cash flow of an 3x9 FRA, where $K = 3\%$, $L(T_{i-1}, T_i) = 3.5\%$ and a notional amount of $1,000,000$; perspective of the fixed rate payer.

To price FRAs fair at initiation t the FRA equilibrium rate $F^{FRA}(t, T_{i-1}, T_i)$ is calculated. Under no-arbitrage conditions and assumption (2.3) the forward interest rate $F(t, T_{i-1}, T_i)$ can be quoted based on Libor spot rates. Under these premises Libor is both, the reference rate of the FRA and the rate to build a discount factors from.

The replication of the FRA's payoff can be done by

- *at time t*
  borrowing $[1 + \delta_{i-1,i}K] P_L(t, T_i)$ with maturity T_i

$$+ [1 + \delta_{i-1,i}K] P_L(t, T_i)$$

  and at the same time lending $P_L(t, T_{i-1})$ with maturity T_{i-1}

$$-P_L(t, T_{i-1})$$

- *at time T_{i-1}*
  obtain one unit

$$+1$$

  from maturing $P_L(t, T_{i-1})$ and reinvest at Libor from T_{i-1} up to T_i

$$-1 \cdot \frac{1}{P_L(T_{i-1}, T_i)}$$

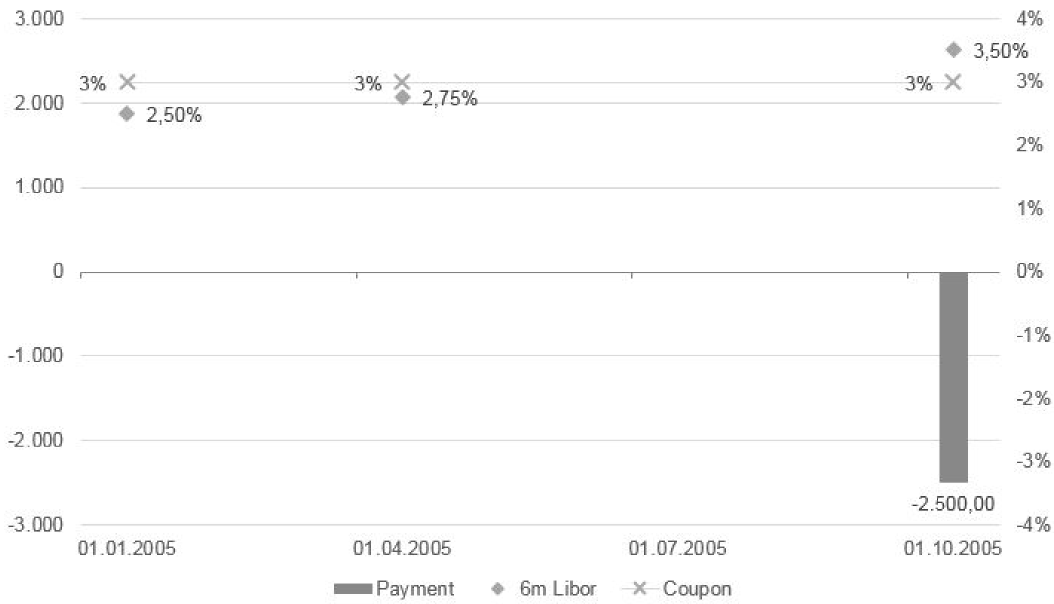

Figure 2.3: Simplified netted cash flow of an 3x9 FRA, where $K = 3\%$, $L(T_{i-1}, T_i) = 2.75\%$ and a notional amount of $1,000,000$; perspective of the fixed rate payer.

Moreover, the FRA's netted payoff (2.5) can be rewritten to

$$\begin{aligned} C =&\delta_{i-1,i}\left[L(T_{i-1}, T_i) - K\right] \\ =&\delta_{i-1,i}L(T_{i-1}, T_i) - \delta_{i-1,i}K \\ =&1 + \delta_{i-1,i}L(T_{i-1}, T_i) - 1 - \delta_{i-1,i}K \\ =&\underbrace{\left[1 + \delta_{i-1,i}L(T_{i-1}, T_i)\right]}_{=:C_A} - \underbrace{\left[1 + \delta_{i-1,i}K\right]}_{=:C_B}. \end{aligned}$$

The deterministic part C_B can be replicated by selling an amount of $[1 + \delta_{i-1,i}K]$ a T_i-maturing bond at time t.

$$-\overbrace{\left[1 + \delta_{i-1,i}K\right]}^{\text{amount}} \cdot \overbrace{P_L(t, T_i)}^{\text{bond price}}$$

At T_{i-1} part C_A

$$C_A = 1 + \delta_{i-1,i}L(T_{i-1}, T_i) = \frac{1}{P_L(T_{i-1}, T_i)}$$

can be replicated by buying bonds with maturity T_i

$$+\overbrace{\frac{1}{P_L(T_{i-1}, T_i)}}^{\text{amount}} \cdot \overbrace{P_L(T_{i-1}, T_i)}^{\text{bond price}} = +1.$$

Applying this strategy leads to costs of one unit at T_{i-1} and to the price Π_t of the replication strategy at t

$$\Pi_t^{FRA}(T_{i-1}, T_i, K) = P_L(t, T_{i-1}) - \left[1 + \delta_{i-1,i}K\right] P_L(t, T_i). \tag{2.6}$$

To price the FRA fair - which means value zero at time t - we solve (2.6) for the fixed rate K:

$$\begin{aligned}
\Pi_t^{FRA} &= 0 \\
0 &= P_L(t,T_{i-1}) - [1+\delta_{i-1,i}K]\, P_L(t,T_i) \\
0 &= \frac{P_L(t,T_{i-1})}{P_L(t,T_i)} - [1+\delta_{i-1,i}K] \\
0 &= \frac{P_L(t,T_{i-1})}{P_L(t,T_i)} - 1 - \delta_{i-1,i}K \\
\delta_{i-1,i}K &= \frac{P_L(t,T_{i-1})}{P_L(t,T_i)} - 1 \\
K &= \frac{1}{\delta_{i-1,i}}\left[\frac{P_L(t,T_{i-1})}{P_L(t,T_i)} - 1\right].
\end{aligned}$$

Hence, the fair forward rate under an FRA is

$$F^{FRA}(t,T_{i-1},T_i) = \frac{1}{\delta_{i-1,i}}\left[\frac{P_L(t,T_{i-1})}{P_L(t,T_i)} - 1\right]. \tag{2.7}$$

Equation (2.7) is also referred to as Libor standard replication forward rate. The rate is established at t for the period T_{i-1} to T_i, and perfectly coincides with the simple compounding forward rate (2.4)

$$\begin{aligned}
F^{FRA}(t,T_{i-1},T_i) &= \frac{1}{\delta_{i-1,i}}\left[\frac{P(t,T_{i-1})}{P(t,T_i)} - 1\right] \\
&= F(t,T_{i-1},T_i), \qquad (2.8)
\end{aligned}$$

depending on the underlying (Libor) reference rate.

2.2 Bond

In the bond markets most bonds include coupons. This section is inspired by [7].

2.2.1 Fixed Coupon Bond

We consider a set of contractual fixed future dates, $T_0 < T_1 < \cdots < T_M$, denoting the due dates and the maturity T_M of the fixed coupon bond $P(t,T_M)$ with $t \leq T_i \;\; \forall i \in \{0,\dots M\}$. Further, we denote the deterministic coupon as K and the notional amount as N. Usually the fixed dates where the interest is paid are equidistant $T_i - T_{i-1} \equiv \delta \;\; \forall i \in \{0,\dots M\}$[3].

Today's price of the bond is the sum of the discounted future cash flows ($t \leq T_0$)

$$\begin{aligned}
\Pi_t^{Bond} &= N \cdot P(t,T_M) + \sum_{i=0}^{M} \delta N K P(t,T_i) \\
&= N \cdot \left[P(t,T_M) + \delta K \sum_{i=0}^{M} P(t,T_i)\right].
\end{aligned}$$

[3]Disregarding day-count conventions as described in Appendix A and leaving aside modified periods due to week-ends or bank holidays.

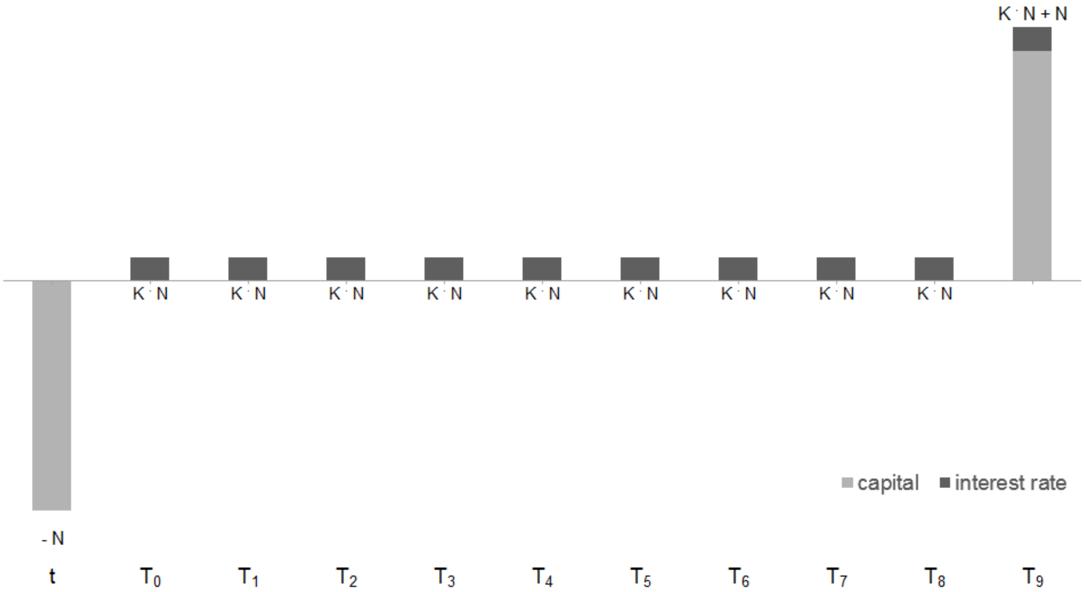

Figure 2.4: Cash flow of a fixed coupon bond

2.2.2 Floating Rate Note

Bonds paying interest rates that are floating in reference to a benchmark yield curve, such as Libor, are called floaters or floating rate notes (FRN). Analogously to a bond with fixed coupons, we consider contractual due dates $(t \leq) T_0 < T_1 < \cdots < T_M$ of the T_M-maturing floater $P(t, T_M)$. Other than the fixed coupon bond the contract of an FRN requires to reset the interest rates for every period $T_i - T_{i-1}$ $(\equiv \delta \quad \forall i \in \{0, \ldots M\})$ at the beginning of the period T_{i-1}, whereas the cash flow is exchanged at the end of the period T_i. A cash flow c_i, exchanged at time T_i, is denoted as

$$c_i = N \cdot \delta R(T_{i-1}, T_i), \tag{2.9}$$

where N denotes the principal of the floater and $R(T_{i-1}, T_i)$ is the simple compounding spot reference rate, which is set at T_{i-1}.

As per definition of the simple spot interest rate (2.1), we can easily rewrite the cash flow c_i of a floating rate note

$$\begin{aligned} c_i &= N \cdot (T_i - T_{i-1}) R(T_{i-1}, T_i) \\ &= N \cdot \delta R(T_{i-1}, T_i) \\ &= N \cdot \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right) \end{aligned}$$

Without loss of generality, we set $N = 1$ and investigate the replication of one single cash flow by

- *at time t*
  buying a T_{i-1}-maturing bond

$$-P(t, T_{i-1})$$

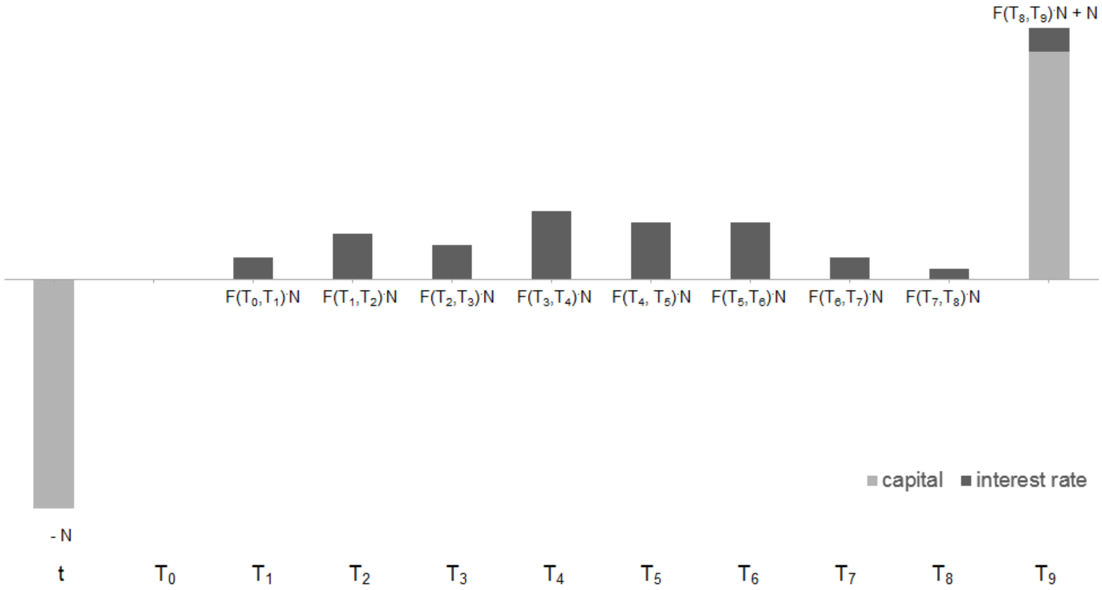

Figure 2.5: Cash flow of a floating rate note

- *at time* T_{i-1}
  receiving one unit from the maturing bond

  $$+1$$

  and buying $\frac{1}{P(T_{i-1},T_i)}$ of T_i-maturing bonds

  $$-\frac{1}{P\left(T_{i-1},T_i\right)} \cdot P\left(T_{i-1},T_i\right) = -1$$

  which is a zero net investment

- *at time* T_i
  we obtain $+\frac{1}{P(T_{i-1},T_i)}$ units

Hence, $\frac{1}{P(T_{i-1},T_i)}$ at time T_i is worth $P\left(t,T_{i-1}\right)$ today. Furthermore, we know, as per definition, that 1 at time T_i is worth $P\left(t,T_i\right)$ today. As the cash flow c_i exchanged at time T_i is

$$c_i = \left[\frac{1}{P\left(T_{i-1},T_i\right)} - 1\right],$$

today's value of a c_i results in

$$\Pi_t\left(c_i\right) = P\left(t,T_{i-1}\right) - P\left(t,T_i\right). \tag{2.10}$$

If we sum up the single cash flows, we get the value for the FRN at time t

$$\begin{aligned}\Pi_t^{FRN} &= P(t,T_M) + \sum_{i=1}^{M} \left[P(t,T_{i-1}) - P(t,T_i)\right] \\ &= P(t,T_M) + \left[P(t,T_0) - P(t,T_1)\right] + \sum_{i=2}^{M} \left[P(t,T_{i-1}) - P(t,T_i)\right] \\ &= P(t,T_M) + \left[P(t,T_0) - P(t,T_1)\right] + \left[P(t,T_1) - P(t,T_2)\right] + \sum_{i=3}^{M} \left[P(t,T_{i-1}) - P(t,T_i)\right] \\ &= P(t,T_M) + P(t,T_0) - P(t,T_2) + \sum_{i=3}^{M} \left[P(t,T_{i-1}) - P(t,T_i)\right] \\ &\;\vdots \\ &= P(t,T_0)\,.\end{aligned}$$

2.3 Overview: Interest Rate Swaps

An interest rate swap (IRS) is a contract between two counterparties to exchange different, specified interest rate cash flows based on a notional amount (the streams of a swap are called legs). Regarding the different kinds of interest rate payments we distinguish swaps.

Coupon swap A coupon swap (also called fixed rate interest swap, par swap, or plain vanilla swap) is the exchange of a fixed versus a floating interest rate.

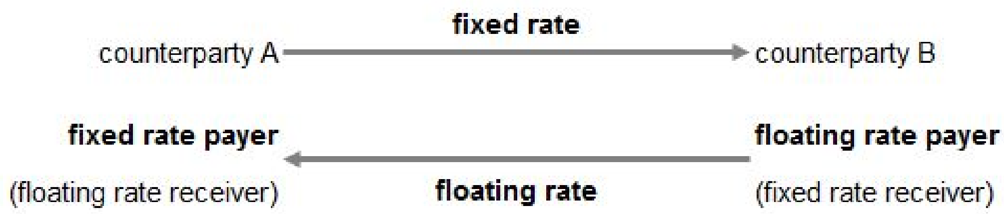

Figure 2.6: Simplified figure of a coupon swap

Basis swap A basis swap is the exchange of two different floating interest rates (in the same currency).

Cross currency interest rate swap A cross currency interest rate swap is the exchange of two interest rates in different currencies.

Notional amount Interest rate payments are based on a notional amount, which is not exchanged under an interest rate swap.

Payment frequency For both legs the frequency of the cash flow exchange is specified in the contract. The floating payment frequency and the fixed-rate payment frequency must not be necessarily the same.

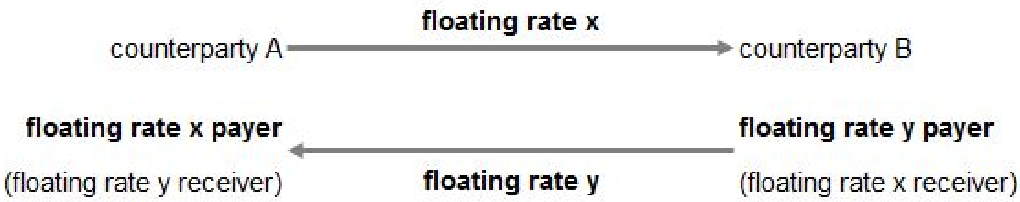

Figure 2.7: Simplified figure of a basis swap

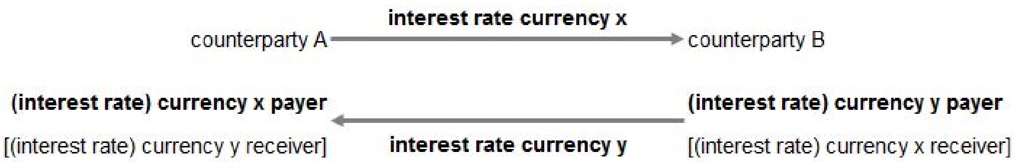

Figure 2.8: Simplified figure of a cross currency interest rate swap

Fixed interest rate (coupon) The coupon (fixed interest rate) is fixed on the trade date and holds for the term of the swap.

Floating index The floating rate is based on a market reference swap rate. It is fixed on certain reset dates for the following interest period. The reset frequency is specified in the contract.

2.4 Fixed Rate Interest Swap

The general definitions of interest rate swaps are based on [5] and [7].

We consider a set of contractual stipulated dates in the future $(t \leq) T_0 < T_1 \cdots < T_M$ where the dates are usually equidistant $T_i - T_{i-1} \equiv \delta \ \ \forall i \in \{0, \dots M\}$. These dates denote the interest rate payment dates and the swap maturity T_M. Further, we denote K as the fixed rate (coupon) and $R(T_{i-1}, T_i)$ as the floating rate for the period from T_{i-1} to T_i, reset at every beginning of the period T_{i-1}. The nominal amount is denoted as N, which is typically used to calculate the interest rates payment, but not physically exchanged under a swap. The actual cash flows are exchanged at the end of a payment period, which is assumed to be the same as an interest period in our set-up. To keep the swap as simple as possible, we assume that the fixed and the floating interest payments are at the same time. From the perspective of the fixed rate payer (floating receiver) the netted cash flow c_i at T_i is

$$c_i = \delta N \left[R(T_{i-1}, T_i) - K \right],$$

which is the obvious reason why swaps are often seen as a portfolio of FRAs, cf. (2.5).
Using result (2.10) the present value at time t of one cash flow c_i is

$$\Pi_t(c_i) = N \left[P(t, T_{i-1}) - P(t, T_i) - \delta K P(t, T_i) \right]. \tag{2.11}$$

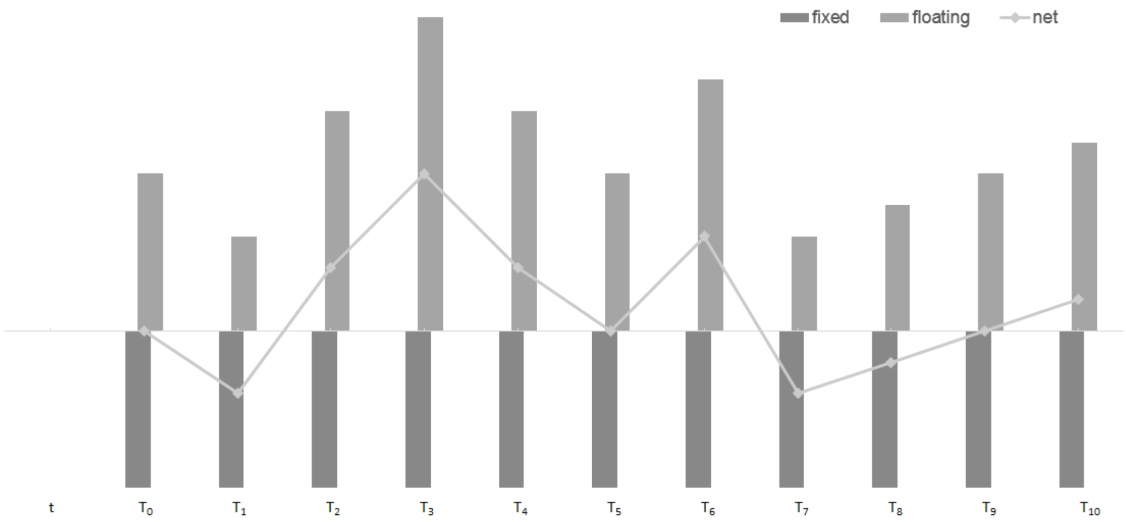

Figure 2.9: Cash flow of a coupon swap

Thus, the value of the swap at time t is

$$\begin{aligned}
\Pi_t^{Swap} &= \sum_{i=1}^{M} N \left[P(t, T_{i-1}) - P(t, T_i) - \delta K P(t, T_i) \right] \\
&= N \left[\sum_{i=1}^{M} \left(P(t, T_{i-1}) - P(t, T_i) \right) - \delta K \sum_{i=1}^{M} P(t, T_i) \right] \\
&= N \left[P(t, T_0) - P(t, T_1) + \sum_{i=2}^{M} \left(P(t, T_{i-1}) - P(t, T_i) \right) - \delta K \sum_{i=1}^{M} P(t, T_i) \right] \\
&= N \left[P(t, T_0) - P(t, T_1) + P(t, T_1) - P(t, T_2) + \sum_{i=3}^{M} \left(P(t, T_{i-1}) - P(t, T_i) \right) - \delta K \sum_{i=1}^{M} P(t, T_i) \right] \\
&= N \left[P(t, T_0) - P(t, T_2) + \sum_{i=3}^{M} \left(P(t, T_{i-1}) - P(t, T_i) \right) - \delta K \sum_{i=1}^{M} P(t, T_i) \right] \\
&\vdots \\
&= N \left[P(t, T_0) - P(t, T_M) - \delta K \sum_{i=1}^{M} P(t, T_i) \right],
\end{aligned}$$

from the perspective of the fixed rate payer. To price a swap fair, the value on the trade date t has to be zero $\Pi_t = 0$. The floating rate is indexed to a market rate, which is given. Hence, we solve for K

$$\begin{aligned}
\Pi_t^{Swap} &= 0 \\
0 &= N\left[P(t,T_0) - P(t,T_M) - \delta K \sum_{i=1}^{M} P(t,T_i)\right] \\
0 &= P(t,T_0) - P(t,T_M) - \delta K \sum_{i=1}^{M} P(t,T_i) \\
\delta K \sum_{i=1}^{M} P(t,T_i) &= P(t,T_0) - P(t,T_M) \\
K &= \frac{P(t,T_0) - P(t,T_M)}{\delta \sum_{i=1}^{M} P(t,T_i)}.
\end{aligned}$$

Thus, we denote the fair fixed interest rate for a swap at time $t\,(\leq T_0)$ as the swap rate R_t^{Swap} with a matching maturity of T_M

$$R_t^{Swap} := \frac{P(t,T_0) - P(t,T_M)}{\delta \sum_{i=1}^{M} P(t,T_i)}. \tag{2.12}$$

We consider an alternative representation of the swap rate R_t^{Swap}, by rewriting the value at time t of the cash flow c_i using (2.4) and (2.11) to get

$$\begin{aligned}
\Pi_t(c_i) &= N\left[P(t,T_{i-1}) - P(t,T_i) - \delta K P(t,T_i)\right] \\
&= N\delta P(t,T_i)\left[\frac{1}{\delta}\cdot\frac{P(t,T_{i-1})}{P(t,T_i)} - 1 - K\right] \\
&= N\delta P(t,T_i)\left[F(t,T_{i-1},T_i) - K\right].
\end{aligned}$$

Summing up all cash flows leads to the value of the swap at time t

$$\Pi_t^{Swap} = \delta N \sum_{i=1}^{M} P(t,T_i)\left[F(t,T_{i-1},T_i) - K\right], \tag{2.13}$$

which results, by solving for K, in

$$\begin{aligned}
0 &= \delta N \sum_{i=1}^{M} P(t,T_i)\left[F(t,T_{i-1},T_i) - K\right] \\
0 &= \delta N\left[\sum_{i=1}^{M} P(t,T_i)\,F(t,T_{i-1},T_i) - \sum_{i=1}^{M} P(t,T_i)\,K\right] \\
0 &= \delta N\left[\sum_{i=1}^{M} P(t,T_i)\,F(t,T_{i-1},T_i) - K\sum_{i=1}^{M} P(t,T_i)\right] \\
0 &= \sum_{i=1}^{M} P(t,T_i)\,F(t,T_{i-1},T_i) - K\sum_{i=1}^{M} P(t,T_i) \\
K\sum_{j=1}^{M} P(t,T_j) &= \sum_{i=1}^{M} P(t,T_i)\,F(t,T_{i-1},T_i) \\
K &= \frac{\sum_{i=1}^{M} P(t,T_i)\,F(t,T_{i-1},T_i)}{\sum_{j=1}^{M} P(t,T_j)}.
\end{aligned}$$

We interpret the latter equation as an alternative representation of the swap rate as a weighted average of simple forward rates

$$R_t^{Swap} = \sum_{i=1}^{M} \omega_i(t) F(t, T_{i-1}, T_i),$$

with random weights ω_i depending on $F(t, T_{i-1}, T_i)$

$$\omega_i(t) = \frac{P(t, T_i)}{\sum_{j=0}^{M} P(t, T_j)}.$$

This representation is useful for further approximations, see [5].

2.4.1 Overnight Indexed Swap (OIS)

An overnight indexed swap is a special case of a plain vanilla fixed vs floating swap as described in Section 2.4. At initiation the coupon is fixed for the length of the term or a specified payment period T_{i-1} to T_i, which is usually short-term (money market). The floating rate is indexed to an overnight rate according to the currency market, e.g.

- EUR $\mapsto$ EONIA,
- GBP $\mapsto$ SONIA[4].

In contradiction to a capital market coupon swap where the floating rate is paid at the end each interest period, the netted interest payment in an OIS is only exchanged at maturity of the swap or at the end of a defined payment period, e.g. one month, a quarter, etc. We consider an OIS with a payment period between T_{i-1} and T_i the floating rate r for this period is

$$r = \left(\prod_{s=T_{i-1}}^{T_i} [1 + \delta_s r(s)] \right) - 1, \tag{2.14}$$

where δ_s denotes the daily accrual factor calculated on the relevant day count convention, $r(s)$ the relevant overnight rate and where s runs daily steps within the time interval T_{i-1} and T_i.

2.5 Basis Swap

As in the previous section, the following general definitions are basically inspired by [5] and [7]. Moreover, [10] and [12] were used.

A basis swap (or tenor swap) is a bilateral agreement, where two floating cash flows, indexed to a (Libor) reference rate, are exchanged. Each leg is tied to a different tenor m or $\tilde{m}$ (e.g. 3-months or 6-months), where $m < \tilde{m}$. In our set-up, we assume the compounding frequency and the payment frequency to be the same. Take, for example, counterparty B as the $\tilde{m}$-Libor payer, set $\tilde{m} = 6 - months$ and counterparty A as the m-Libor payer with $m = 3months$. Assuming the same reset frequency as the payment frequency in each leg, counterparty B pays every 6 months interest at the 6m-Libor, while counterparty A pays every 3 months interest at the 3m-Libor plus a spread Z.

[4] See Appendix B.

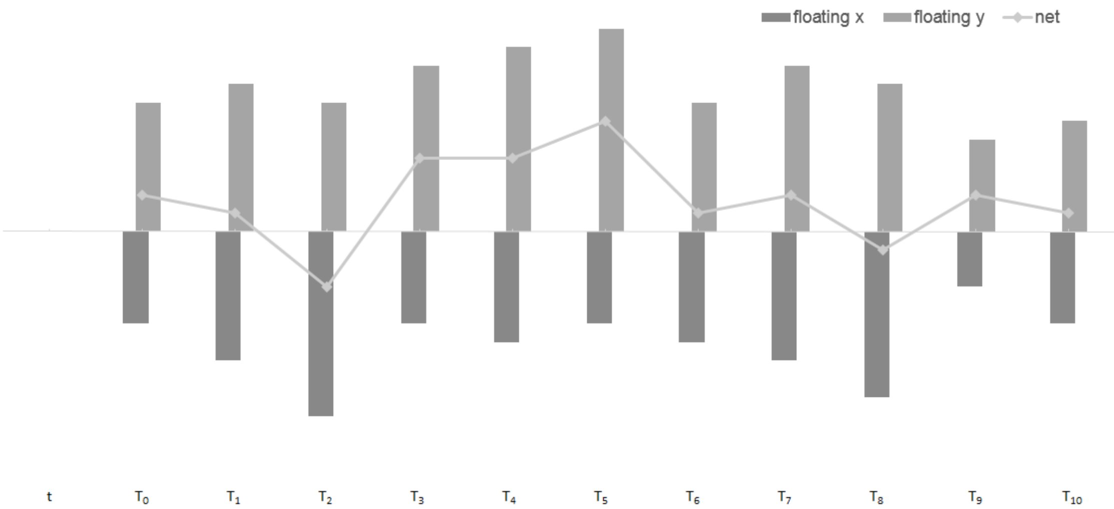

Figure 2.10: Cash flow of a basis swap

We consider an arbitrage-free market and two riskless counterparties. Regardless of the tenor or payment frequency we can replicate a floating cash flow which is indexed to Libor and derive the valuation at time t. The fixing and payment dates are $(t \leq) T_0 < T_1 < \dots T_{M-1}$ and the last payment date is T_M. We consider only the floating leg of a fixed rate interest swap from the previous Section 2.4. The value at t of a floating leg c as presented in (2.11) is

$$\Pi_t(c) = P(t, T_0) - P(t, T_M). \tag{2.15}$$

Using representation (2.13) results in the same price of a floating leg c

$$
\begin{aligned}
\Pi_t(c) =& \sum_{i=1}^{M} \delta_{i-1,i} P(t,T_i) F(t,T_{i-1},T_i) \\
=& \sum_{i=1}^{M} \delta_{i-1,i} P(t,T_i) \frac{1}{\delta_{i-1,i}} \left[\frac{P(t,T_{i-1})}{P(t,T_i)} - 1 \right] \\
=& \sum_{i=1}^{M} P(t,T_i) \left[\frac{P(t,T_{i-1})}{P(t,T_i)} - 1 \right] \\
=& \sum_{i=1}^{M} \left[P(t,T_i) \frac{P(t,T_{i-1})}{P(t,T_i)} - P(t,T_i) \right] \\
=& \sum_{i=1}^{M} \left[P(t,T_{i-1}) - P(t,T_i) \right] \\
=& \left[P(t,T_0) - P(t,T_1) \right] + \sum_{i=2}^{M} \left[P(t,T_{i-1}) - P(t,T_i) \right] \\
=& \left[P(t,T_0) - P(t,T_1) \right] + \left[P(t,T_1) - P(t,T_2) \right] + \sum_{i=3}^{M} \left[P(t,T_{i-1}) - P(t,T_i) \right] \\
=& P(t,T_0) - P(t,T_2) + \sum_{i=3}^{M} \left[P(t,T_{i-1}) - P(t,T_i) \right] \\
& \vdots \\
=& P(t,T_0) - P(t,T_M), \qquad (2.16)
\end{aligned}
$$

what we expected from (2.15).

In conclusion, to price a basis swap fair, the two floating legs each tied to a different tenor must be equal. Assuming Leg 1 c_1 as the cash flow with tenor $m\,(< \tilde{m})$ and taking (2.3) into account, results in

$$
\begin{gathered}
\Pi_t^{BasisSwap} = 0 \\
\Leftrightarrow \\
\Pi_t(c_1) = \Pi_t(c_2) \\
\sum_{i=1}^{M} \left[P_{L_m}(t,T_{i-1}) - P_{L_m}(t,T_i) \right] = \sum_{i=1}^{M} \left[P_{L_{\tilde{m}}}(t,T_{i-1}) - P_{L_{\tilde{m}}}(t,T_i) \right].
\end{gathered}
$$

Consequently, to make the price of a basis swap fair, which means it is worth zero at time t, the spread Z between L_m and $L_{\tilde{m}}$ must be set to zero $Z = 0$. Although the rates are indexed to different tenors, e.g. $3m$ and $6m$, they are worth par in a default-free world.

2.6 Cross Currency Swap

The introduction of cross currency swaps is mainly inspired by [12].

To introduce a (constant notional) cross currency swap (CCY), we define forward outright exchange rates by no-arbitrage arguments.
We denote the (spot) exchange rate of the local currency per unit of foreign currency at time t (today) as f_t, and the forward exchange rate at time $T\,(\geq t)$ as $f\,(t,T)$ which is today's price of one unit foreign currency delivered at time T. Further we denote the risk-free interest rate as $R^{\xi}\,(t,T)$ where ξ tags the currency market, e.g. $R^{\text{USD}}\,(t,T)$ is the T-year USD default free spot interest rate. Analogous, we have the currency-market depending year fraction $\delta^{\xi}_{t,T}$. To derive the forward outright exchange rate, the following transactions are conducted:

- *at time t*
  - borrow f_t of local currency
  - sell f_t of local currency and buy one unit of foreign currency
  - invest one unit of foreign currency
- *at time T*
  - pay back the local currency $-f_t\left[1+\delta^{local}_{t,T}R^{local}\,(t,T)\right]$
  - receive from foreign investment $+1\left[1+\delta^{foreign}_{t,T}R^{foreign}\,(t,T)\right]$
  - sell foreign receipt $-1\left[1+\delta^{foreign}_{t,T}R^{foreign}\,(t,T)\right]$, which means changing to local currency at time T with valid exchange rate $+f\,(t,T)\left[1+\delta^{foreign}_{t,T}R^{foreign}\,(t,T)\right]$ (receive local)

For no-arbitrage, this strategy implies that the forward outright exchange rate at time T (fixed at t, today) must hold

$$-f_t\left[1+\delta^{local}_{t,T}R^{local}\,(t,T)\right]+f\,(t,T)\left[1+\delta^{foreign}_{t,T}R^{foreign}\,(t,T)\right]=0.$$

Hence, solving for the forward outright exchange rate, this results in

$$f\,(t,T)=f_t\frac{1+\delta^{local}_{t,T}R^{local}\,(t,T)}{1+\delta^{foreign}_{t,T}R^{foreign}\,(t,T)}.$$

If the cross currency swap is a par value cross currency swap the capital exchange at initiation and at maturity are happening at the same exchange rate which is also fixed at time t.

Example We consider a USD vs EUR cross currency basis swap (fixed-fixed) in the following set-up.

	USD	EUR
Notional	$10,000,000.00$	$8,000,000.00$
Exchange rate	USDEUR 0.8	EURUSD 1.25
Start	$t=0=31.12.2017$	
Maturity	$T=5Y=31.12.2022$	
Coupon	$R^{USD}\,(5Y)=1.7\%$	$R^{EUR}\,(5Y)=-0.3\%$
Payment Frequency	semi annual	annual
Day Count	ACT/360	ACT/360
	receive	pay

For

$$f_t = 0.8 \text{ and}$$
$$\delta_{t,T}^{USD} = \delta_{t,T}^{EUR} = \frac{1826}{360}$$

the equation

$$-f_t\left[1+\delta_{t,T}^{local}R^{local}(t,T)\right] + f(t,T)\left[1+\delta_{t,T}^{foreign}R^{foreign}(t,T)\right] = 0$$
$$-f_t^{USDEUR}\left[1+\delta_{t,T}^{USD}R^{USD}(t,T)\right] + f^{USDEUR}(t,T)\left[1+\delta_{t,T}^{EUR}R^{EUR}(t,T)\right] = 0$$
$$-0.8\left[1+\frac{1826}{360}0.017\right] + f^{USDEUR}(t,T)\left[1+\frac{1826}{360}\cdot(-0.003)\right] = 0$$

must hold to price the swap fair. Hence, solving for the forward outright exchange rate results in

$$f(t,T) = f_t\frac{1+\delta_{t,T}^{local}R^{local}(t,T)}{1+\delta_{t,T}^{foreign}R^{foreign}(t,T)}$$
$$f^{USDEUR}(t,T) = f_t^{USDEUR}\frac{1+\delta_{t,T}^{USD}R^{USD}(t,T)}{1+\delta_{t,T}^{EUR}R^{EUR}(t,T)}$$
$$f(t,T) = 0.8\frac{1+\frac{1826}{360}0.017}{1+\frac{1826}{360}\cdot(-0.003)}$$
$$f(t,T) = 0.88240955$$

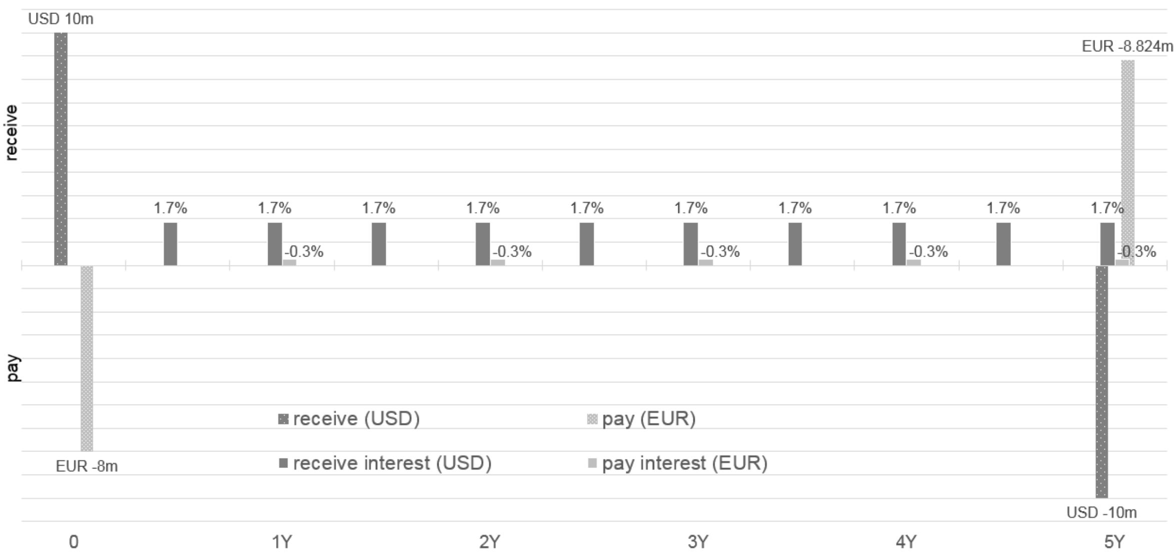

Figure 2.11: Cash flow of the USD vs EUR cross currency swap

We only reviewed a constant notional cross currency swap with two fixed legs as a simple example to get a feeling for a multi currency world. Of course, there exist fixed vs floating and floating vs floating cross currency swaps on which we will a look in Section 4.5 and Chapter 5.

Chapter 3

The Crisis

> As the basis between Libor and overnight index swap rates ballooned during the credit crisis, banks were forced to reassess methods for pricing collateralised and uncollateralised derivative trades. The result is a move towards a new market standard in pricing derivatives backed by collateral.
>
> *The price is wrong, Christopher Whittall, Risk Magazine March 2010*

One of the consequences of the financial crisis 2007/08 is the spreading of collateralization of OTC[1] derivatives, see Figure 3.1. According to the ISDA[2] Margin Survey 2015 89% of fixed income derivatives were under a CSA as of December 31, 2014. The main goal of collateralization is to reduce counterparty credit risk to a minimum.

The risk of an OTC derivative, like a swap, is that the counterparty defaults before maturity. During the term of contract, the transaction has a market value (or present value). This value is calculated by summing up all discounted future cash flows. If this is done for the streams of both counterparties (the payer and the receiver) this equates to the net present value (NPV). The latter at a certain time is the amount one counterparty would have to pay to the other if the contract was terminated at the time of valuation. To eliminate the risk, that one counterparty could not pay, they exchange collateral regularly.

As a market standard collateral is exchanged under a CSA (credit support annex) which is an add-on to the ISDA Master Agreement regulating OTC transactions. The CSA is a bilingual standardized or individualized contract setting a framework for collateral exchange. The annex specifications contain the type of collateral (e.g. cash, government bonds, etc.), thresholds, minimum transfer amount, rounding, valuation and timing, interest rates, etc. Usually two counterparties have more than one active transaction, which means the valuation and therefore the collateral exchange is rather for a portfolio than one single contract.

Example of collateral movements under a CSA

To illustrate how a perfectly cash-collateralized portfolio looks like, I decided to give an example. I want to visualize the netted present value over time, besides, the corresponding cash movements. Assuming one counterparty defaults at the end of the period under observation, it

[1] Over-The-Counter; An OTC product refers to bilateral (only two counterparties) agreements where no third party is involved, in terms of an intermediary or an exchange.

[2] International Swaps and Derivatives Association, http://www.isda.org

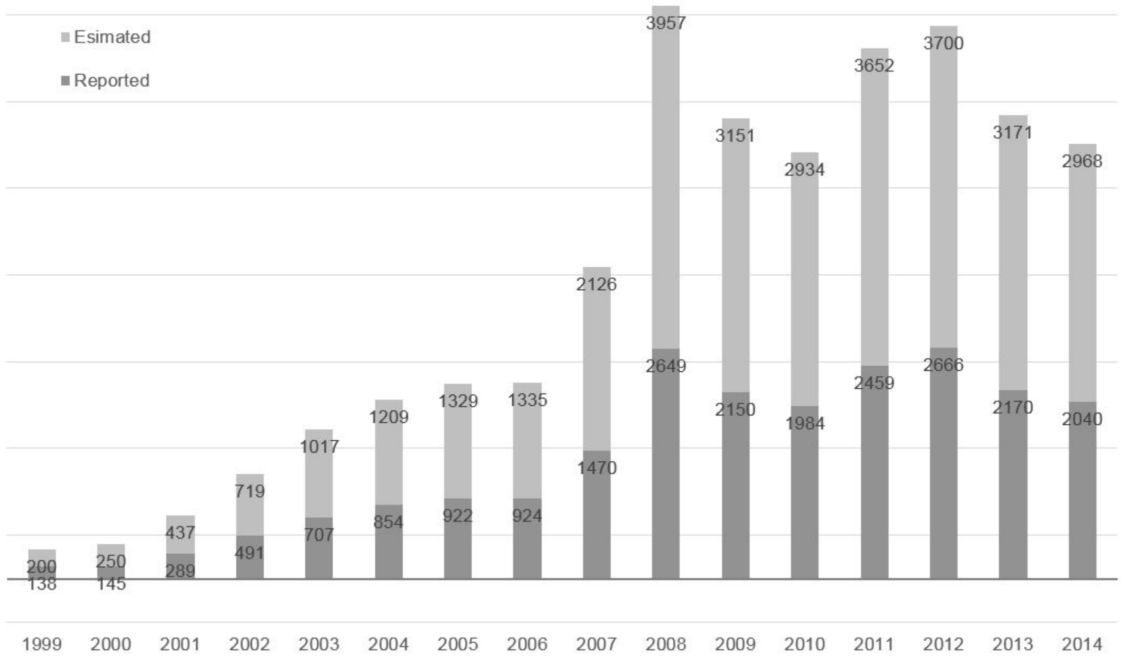

Figure 3.1: Value of reported and estimated collateral in USD billions, as of December 31, 2014; *Source: ISDA Market Survey 2015*

makes sense to reduce the time gap between valuations of the swap exposures and the collateral settlements to a minimum (same day or overnight). Furthermore, it emphasizes that daily collateral exchange minimizes counterparty credit risk.

As a simple example, we consider two counterparties A and B having a portfolio of active transactions under a CSA. We assume that USD cash collateral is exchanged on a daily basis with zero threshold for both counterparties, a minimum transfer amount of USD $100,000.00$, a rounding to a multiple of USD $100,000.00$ and a same day settlement. In this set-up, the collateral cash exchange could be as presented in Table 3.1 from perspective of counterparty A, visualized in Figure 3.2. The collateral balance should stay as close as rounding allows to the net present value of the portfolio. If counterparty B in this example defaulted on March 23, 2040, counterparty A would be slightly over-collateralized (due to rounding).

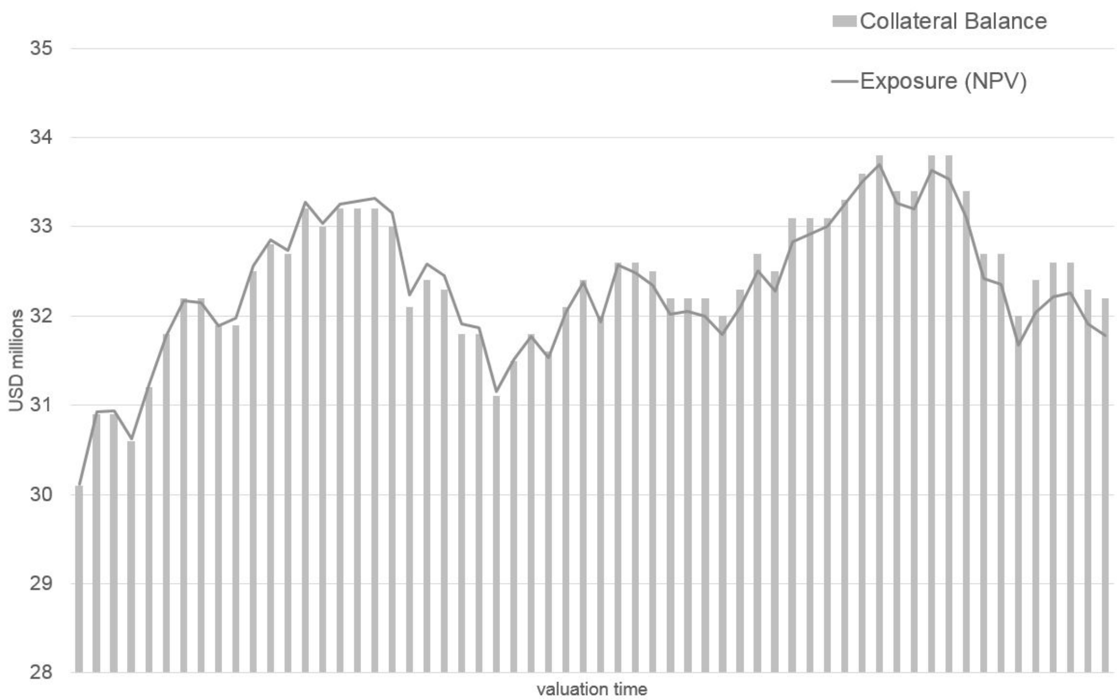

Figure 3.2: Example of exposure (NPV) and cash collateral movement over time (in USD millions), perspective of counterparty A

Valuation Time	Exposure (NPV)	Variation	Rounded	Collateral Balance	Party A	Party B
⋮	⋮	⋮	⋮	⋮	⋮	⋮
01/02/2040	30,112,374.00	⋮	⋮	30,100,000.00	⋮	⋮
01/03/2040	30,921,762.76	809,388.76	800,000.00	30,900,000.00	receive collateral	post collateral
01/04/2040	30,935,758.58	13,995.83	0.00	30,900,000.00	no movement	no movement
01/05/2040	30,625,651.36	-310,107.23	-300,000.00	30,600,000.00	post collateral	receive collateral
01/06/2040	31,227,659.05	602,007.69	600,000.00	31,200,000.00	receive collateral	post collateral
01/09/2040	31,779,467.57	551,808.52	600,000.00	31,800,000.00	receive collateral	post collateral
01/10/2040	32,169,248.82	389,781.25	400,000.00	32,200,000.00	receive collateral	post collateral
01/11/2040	32,148,329.76	-20,919.06	0.00	32,200,000.00	no movement	no movement
01/12/2040	31,884,775.65	-263,554.11	-300,000.00	31,900,000.00	post collateral	receive collateral
01/13/2040	31,976,220.32	91,444.67	0.00	31,900,000.00	no movement	no movement
01/16/2040	32,562,197.84	585,977.51	600,000.00	32,500,000.00	receive collateral	post collateral
01/17/2040	32,848,436.98	286,239.14	300,000.00	32,800,000.00	receive collateral	post collateral
01/18/2040	32,734,005.21	-114,431.77	-100,000.00	32,700,000.00	post collateral	receive collateral
01/19/2040	33,278,797.58	544,792.36	500,000.00	33,200,000.00	receive collateral	post collateral
01/20/2040	33,032,864.19	-245,933.38	-200,000.00	33,000,000.00	post collateral	receive collateral
01/23/2040	33,250,136.62	217,272.43	200,000.00	33,200,000.00	receive collateral	post collateral
01/24/2040	33,289,176.20	39,039.58	0.00	33,200,000.00	no movement	no movement
01/25/2040	33,316,798.08	27,621.88	0.00	33,200,000.00	no movement	no movement
01/26/2040	33,155,012.62	-161,785.46	-200,000.00	33,000,000.00	post collateral	receive collateral
01/27/2040	32,236,572.45	-918,440.17	-900,000.00	32,100,000.00	post collateral	receive collateral
⋮	⋮	⋮	⋮	⋮	⋮	⋮
02/27/2040	32,279,603.74	-226,444.66	-200,000.00	32,500,000.00	post collateral	receive collateral
02/28/2040	32,831,569.76	551,966.03	600,000.00	33,100,000.00	receive collateral	post collateral
02/29/2040	32,922,695.36	91,125.60	0.00	33,100,000.00	no movement	no movement
03/01/2040	33,010,201.51	87,506.15	0.00	33,100,000.00	no movement	no movement
03/02/2040	33,249,241.99	239,040.48	200,000.00	33,300,000.00	receive collateral	post collateral
03/05/2040	33,500,625.86	251,383.87	300,000.00	33,600,000.00	receive collateral	post collateral
03/06/2040	33,697,114.87	196,489.01	200,000.00	33,800,000.00	receive collateral	post collateral
03/07/2040	33,259,548.23	-437,566.64	-400,000.00	33,400,000.00	post collateral	receive collateral
03/08/2040	33,200,381.46	-59,166.76	0.00	33,400,000.00	no movement	no movement
03/09/2040	33,627,978.20	427,596.74	400,000.00	33,800,000.00	receive collateral	post collateral
03/12/2040	33,530,635.80	-97,342.40	0.00	33,800,000.00	no movement	no movement
03/13/2040	33,101,627.31	-429,008.49	-400,000.00	33,400,000.00	post collateral	receive collateral
03/14/2040	32,422,569.62	-679,057.69	-700,000.00	32,700,000.00	post collateral	receive collateral
03/15/2040	32,351,801.00	-70,768.62	0.00	32,700,000.00	no movement	no movement
03/16/2040	31,674,570.17	-677,230.83	-700,000.00	32,000,000.00	post collateral	receive collateral
03/19/2040	32,040,200.20	365,630.03	400,000.00	32,400,000.00	receive collateral	post collateral
03/20/2040	32,216,130.66	175,930.46	200,000.00	32,600,000.00	receive collateral	post collateral
03/21/2040	32,259,673.56	43,542.91	0.00	32,600,000.00	no movement	no movement
03/22/2040	31,911,553.10	-348,120.46	-300,000.00	32,300,000.00	post collateral	receive collateral
03/23/2040	31,777,818.18	-133,734.92	-100,000.00	32,200,000.00	post collateral	receive collateral
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 3.1: Example of exposure (NPV) and cash collateral movement (in USD), perspective of counterparty A

Chapter 4

After the Crisis

In this chapter, we consider direct consequences of the crisis. The notation as well as the ideas are based on [2], [3], [4], [8] and [9].

As during the financial crisis spreads between curves, which were supposed to be the same or at least as close as the difference was negligible, widened pricing formulas of derivatives had to be re-visioned by having a closer look at underlying tenor or currency, see Figure 4.1. Furthermore, counterparty credit risk, consequently collateralization and its consequences in derivative pricing, gained importance. As a result, the curves either used for discounting or calculating a forward rate are to be distinguished.

Figure 4.1: USD Libor Overnight Rate vs 6 Months; *Source: Federal Reserve Bank of St. Louis*

From now on, the reference to a certain market in terms of characteristics such as currency and tenor, is denoted as $x \in \{d, f_1, f_2, \dots f_n\}$. We refer to either

- d, a discount curve

or

- f, a forward curve.

Analogously to Chapter 2, we define a bank account B_x indexed to x and the x-dependent discount factor

$$D(t, T_i) = \frac{B_x(t)}{B_x(T_i)}.$$

Further, we denote T_i-maturing zero-coupon bonds as $P_x(t, T_i)$ according to the market x and assume $P_x(T_i, T_i) = 1$. As in the previous Chapter 2, we denote the risk neutral measure as $\mathcal{Q}$ and the information up to time t as $\mathcal{F}_t$. Analogously, we assume no-arbitrage

$$P_x(t, T_i) = \mathbb{E}_t^{\mathcal{Q}}[D(t, T_i)] = \mathbb{E}_t^{\mathcal{Q}}\left[\frac{B_x(t)}{B_x(T_i)}\right].$$

If $T \to P_x(t, T)$ is defined as the numeraire, we understand $\mathbb{E}_t^{\mathcal{Q}_x^T}$ as the conditional expectation under the T-forward measure $\mathcal{Q}_x^T$

$$\mathbb{E}_t^{\mathcal{Q}_x^T}[.] := \mathbb{E}^{\mathcal{Q}_x^T}[.|\mathcal{F}_t].$$

The simple compounded spot interest rate related to x is denoted as

$$\begin{aligned} P_x(t, T_i)[1 + \delta(t, T_i) R_x(t, T_i)] = 1, \\ R_x(t, T_i) = \frac{1}{\delta(t, T_i)}\left[\frac{1}{P_x(t, T_i)} - 1\right]. \end{aligned}$$

In the same way, the continuously compounded spot interest rate tied to x is denoted as

$$R_x(t, T_i) = -\frac{1}{\delta(t, T_i)} \log P_x(t, T_i).$$

4.1 Link Between CSA and Discounting

In this section, we concentrate on the implications of a financial contract under a CSA, in particular the correct discounting curve.

We assume a simple OTC trade with only one payment Π_T at maturity T under a CSA with margination dates $T_0 < T_1 < \cdots < T_M (=: T)$. We introduce a collateral bank account B_c refered to the collateral rate, assuming that

$$B_c(T_i) = B_c(T_{i-1})[1 + \delta_{i-1,i} R_c(T_{i-1})], \tag{4.1}$$

where $R_c(T_i)$ denotes the simply compounded collateral rate at T_i. It has become common practice to use an overnight rate for collateral interest, such as Eonia or Sonia[1]. Assuming a perfect collateral process, as in the example of Chapter 3, it holds that for each T_i

$$B_c(t) = \Pi_t \quad \forall t \leq T_i. \tag{4.2}$$

Leaving aside the effect of rounding and time gaps due to settlement, a perfect collateral bank account consistently covers the present value of the trade.

To simplify our set-up even further, we assume only two margination dates T_0 and T. We consider counterparty A to receive $\Pi_T (\geq 0)$ at maturity. At time T_0 the amount $\Pi_T (\geq 0)$ is

[1] See Appendix B.

worth Π_{T_0} (≥ 0). Consequently, the first collateral cash flow exchange at margination date T_0 is a post from counterparty B. Under assumption (4.2), an amount of

$$B_c(T_0) = \Pi_{T_0}$$

is sent to counterparty A. Assumption (4.1) results in

$$B_c(T) = B_c(T_0)\left[1 + \delta(T_0, T)\, R_c(T_0)\right]$$

and by no-arbitrage arguments

$$\Pi_{T_0} = P_d(T_0, T)\, \Pi_T, \tag{4.3}$$

where

$$P_d(T_0, T) = \frac{1}{1 + \delta(T_0, T)\, R_c(T_0)}.$$

In conclusion, an OTC deal under a CSA is discounted at the collateral rate.

As of now, we consider financial contracts in a world where banks can default. In contrast to (2.3), we cannot assume Libor as riskless anymore. The results from above are an explanation why market best practice is to use the collateral rate, and therefore the overnight rate, respectively, as the closest-to-risk-free rate for discounting trades under a CSA.

To use the funding curve for discounting alternatively means to fall back to a risky Libor curve. The concept of CSA discounting, as previously presented, can also be applied, if the introduced collateral acccount is interpreted as a funding account with an underlying convenient funding curve, such as Libor. We come back to the choice of discounting curve in Chapter 5.

In the next sections we primarily concentrate on the fact that forward curves and discounting curves are not to be assumed as identical anymore.

4.2 Forward Rate Agreement

In this section, we reconsider a traditional FRA defined in Section 2.1, inspired by [4].

We consider the netted FRA payoff C from the perspective of the fixed rate payer as in (2.5)

$$C = \delta_{i-1,i}\left[L(T_{i-1}, T_i) - K\right],$$

with contractual dates T_{i-1} and T_i, which define the fixing and term for the Libor reference rate $L(T_{i-1}, T_i)$, as well as the payment date and the maturity of the contract, ($T_{i-1} < T_i$). In Section 2.1, we assumed the reference rate of the FRA as Libor and further used the same rate to build discount factors associated to Libor (2.6). As we now regard Libor as a risky rate, it is not the appropriate rate for discounting anymore.

In a multi-curve set-up under no-arbitrage (4.3), we assume $P_d(t, T_i)$ as the discount factor associated to an applicable curve d, usually the overnight curve. In the new framework, the present value of an FRA at time $t \leq T_{i-1}$ is

$$\Pi_t^{FRA} = P_d(t, T_i)\, \mathbb{E}_t^{\mathcal{Q}_d^{T_i}}\left[\delta_{i-1,i}\left(L(T_{i-1}, T_i) - K\right)\right],$$

where $\mathcal{Q}_d^{T_i}$ denotes the T_i forward measure associated to numeraire $P_d(t, T_i)$.

As in Section 2.1, to find the new equilibrium rate for an FRA, we solve for K

$$\begin{aligned}
\Pi_t^{FRA} &= 0 \\
0 &= P_d(t, T_i)\, \mathbb{E}_t^{\mathcal{Q}_d^{T_i}} \left[\delta_{i-1,i}\left(L(T_{i-1}, T_i) - K\right)\right] \\
0 &= P_d(t, T_i)\, \delta_{i-1,i} \mathbb{E}_t^{\mathcal{Q}_d^{T_i}} \left[L(T_{i-1}, T_i) - K\right] \\
0 &= P_d(t, T_i)\, \delta_{i-1,i} \left(\mathbb{E}_t^{\mathcal{Q}_d^{T_i}} \left[L(T_{i-1}, T_i)\right] - K\right) \\
K &= \mathbb{E}_t^{\mathcal{Q}_d^{T_i}} \left[L(T_{i-1}, T_i)\right]
\end{aligned}$$

Hence, the FRA equilibrium rate is

$$\tilde{F}^{FRA}(t, T_{i-1}, T_i) = \mathbb{E}_t^{\mathcal{Q}_d^{T_i}} \left[L(T_{i-1}, T_i)\right]. \tag{4.4}$$

As per definition, the equilibrium rate and the Libor rate coincide

$$\begin{aligned}
\tilde{F}^{FRA}(T_{i-1}, T_{i-1}, T_i) &= \mathbb{E}_{T_{i-1}}^{\mathcal{Q}_d^{T_i}} \left[L(T_{i-1}, T_i)\right] \\
\tilde{F}^{FRA}(T_{i-1}, T_{i-1}, T_i) &= L(T_{i-1}, T_i)
\end{aligned}$$

at the fixing date T_{i-1}. The FRA equilibrium rate is a martingale under $\mathcal{Q}_d^{T_i}$

$$\begin{aligned}
\tilde{F}^{FRA}(t, T_{i-1}, T_i) &\overset{(4.4)}{=} \mathbb{E}_t^{\mathcal{Q}_d^{T_i}} \left[L(T_{i-1}, T_i)\right] \\
&= \mathbb{E}_t^{\mathcal{Q}_d^{T_i}} \left[\tilde{F}^{FRA}(T_{i-1}, T_{i-1}, T_i)\right] \quad \forall t \leq T_{i-1}.
\end{aligned}$$

Reconsidering the single-curve FRA rate (2.6) we get

$$\begin{aligned}
\tilde{F}^{FRA}(t, T_{i-1}, T_i) &= \mathbb{E}_t^{\mathcal{Q}_d^{T_i}} \left[L(T_{i-1}, T_i)\right] \\
&= \mathbb{E}_t^{\mathcal{Q}^{T_i}} \left[L(T_{i-1}, T_i)\right] \\
&= \mathbb{E}_t^{\mathcal{Q}^{T_i}} \left[F(T_{i-1}, T_{i-1}, T_i)\right] \\
&= F(t, T_{i-1}, T_i)
\end{aligned}$$

due to the martingale property of the forward rate $F(t, T_{i-1}, T_i)$ under the forward measure $\mathcal{Q}^{T_i}$.

4.3 Fixed Rate Interest Swap

In this section, we index Libor rates to a certain tenor m which leads to a further distinction of rates, as we already did by defining a basis swap in Section 2.5. A $3m$-Libor does simply mean that the interest period is three months. Without loss of generality, the payment dates and the reset dates are supposed to be the same.

As introduced in Section 2.4, a fixed rate interest rate swap is a contract between two counterparties to exchange cash flows based on a principal amount N. One cash flow is tied to a coupon K, which is fixed on the trade date and the floating cash flow is indexed to a market reference rate. We consider two sets of contractual payment (and reset) dates $\{T_0, T_1 \ldots, T_M\}$ for the floating leg and $\{S_0, S_1 \ldots, S_M\}$ for the fixed leg where $T_0 = S_0$ and $T_M = S_M$, the other stipulated dates must not necessarily be the same. The year fraction between T_i and T_{i-1} (or S_j and S_{j-1}) is denoted as

$$\begin{aligned}\delta_{i-1,i} &:= \delta\left(T_{i-1}, T_i\right) := T_i - T_{i-1} \text{ for } i \in \{1, \ldots, M\} \text{ and}\\ \delta_{j-1,j} &:= \delta\left(S_{j-1}, S_j\right) := S_j - S_{j-1} \text{ for } j \in \{1, \ldots, M\}.\end{aligned}$$

The tenor, which the Libor rate is indexed to, is denoted as m. The floating rate is fixed at the beginning of the period T_{i-1} and the actual payment is at the end T_i, e.g. the first interest rate

$$L_m\left(T_0, T_1\right) \text{ is determined at } T_0, \text{and paid at } T_1.$$

In this setting, the price of the fixed rate payer swap at time t is

$$\Pi_t^{Swap} = N \cdot \left[\sum_{i=1}^{M} \delta_{i-1,i} P_d\left(t, T_i\right) \mathbb{E}_t^{\mathcal{Q}_d^{T_i}}\left[L_m\left(T_{i-1}, T_i\right)\right] - K \sum_{j=1}^{M} \delta_{j-1,j} P_d\left(t, S_j\right)\right], \tag{4.5}$$

adapted from (2.13). We already showed in (4.4) that the FRA forward rate is

$$F^{FRA}\left(t, T_{i-1}, T_i\right) = \mathbb{E}_t^{\mathcal{Q}_d^{T_i}}\left[L\left(T_{i-1}, T_i\right)\right],$$

which is now linked to a tenor m

$$F_m^{FRA}\left(t, T_{i-1}, T_i\right) = \mathbb{E}_t^{\mathcal{Q}_d^{T_i}}\left[L_m\left(T_{i-1}, T_i\right)\right].$$

Solving for the equilibrium swap rate at time t results in

$$\begin{aligned}\Pi_t^{Swap} =& 0\\ 0 =& \sum_{i=1}^{M} \delta_{i-1,i} P_d\left(t, T_i\right) \mathbb{E}_t^{\mathcal{Q}_d^{T_i}}\left[L_m\left(T_{i-1}, T_i\right)\right] - K \sum_{j=1}^{M} \delta_{j-1,j} P_d\left(t, S_j\right)\\ 0 =& \sum_{i=1}^{M} \delta_{i-1,i} P_d\left(t, T_i\right) F_m^{FRA}\left(t, T_{i-1}, T_i\right) - K \sum_{j=1}^{M} \delta_{j-1,j} P_d\left(t, S_j\right)\\ K \sum_{j=1}^{M} \delta_{j-1,j} P_d\left(t, S_j\right) =& \sum_{i=1}^{n} \delta_{i-1,i} P_d\left(t, T_i\right) F_m^{FRA}\left(t, T_{i-1}, T_i\right)\\ K =& \frac{\sum_{i=1}^{M} \delta_{i-1,i} P_d\left(t, T_i\right) F_m^{FRA}\left(t, T_{i-1}, T_i\right)}{\sum_{j=1}^{M} \delta_{j-1,j} P_d\left(t, S_j\right)}.\end{aligned}$$

Hence, the fair T_M-swap rate for a fixed vs floating swap, where the floating leg is tied to a (Libor) reference rate with tenor m, is

$$R_t^{Swap} = \frac{\sum_{i=1}^{M} \delta_{i-1,i} P_d\left(t, T_i\right) F_m^{FRA}\left(t, T_{i-1}, T_i\right)}{\sum_{j=1}^{M} \delta_{j-1,j} P_d\left(t, S_j\right)}.$$

Assuming $T_M = 5years$ and the tenor $m = 3months$, the fair swap rate R_t at execution day $t = 0$ is the "5-year-swap rate vs 3-months".

4.4 Basis Swap

A basis swap (or tenor swap) is an exchange of two floating cash flows each tied to a Libor with different tenors. As in a multi-curve world Libors with different tenors are not equal anymore, there must be added a basis spread (or tenor spread) $Z \neq 0$ to the shorter tenor to make the swap fair at initiation, in contradistinction to Section 2.5. The following ideas are from [4] as well as [9].

We consider a m vs $\tilde{m}$ basis swap, where $m < \tilde{m}$ (e.g 3-months vs 6-months). The basis spread $Z(t, T_M)$, which is fixed at initiation t for the length of the swap T_M, must hold

$$\sum_{i=1}^{M} \delta_{i-1,i} P_d(t, T_i) \left(\mathbb{E}_t \left[L_m(T_{i-1}, T_i) \right] + Z(t, T_M) \right) = \sum_{j=1}^{\tilde{M}} \delta_{j-1,j} P_d(t, T_j) \, \mathbb{E}_t \left[L_{\tilde{m}}(T_{j-1}, T_j) \right], \quad (4.6)$$

where $T_M = T_{\tilde{M}}$ denote the contractual termination and the i, j denote the (contingently) different payment dates. A simple explanation of the need adding a (positive) basis spread to the leg tied to the reference rate with the shorter tenor, is to consider the spread as the difference of two investment strategies in the following set-up.

We assume two financial institutions A, B, which are supposed to have a good rating and are therefore reliable counterparties, both with a collateral agreement in place. As an example, we consider a 3m-Libor vs a 6m-Libor swap, maturing at 6 months, based on a notional N. Hence, we have three contractually stipulated dates t (initiation), $T_1 = t + 3m$ and $T_2 = t + 6m$ to take into account.

- *floating leg indexed to 6m-Libor*
  At time t, we lend counterparty A, under a standard CSA, the notional N for a period of 6 months at 6m-Libor $L(t, T_2)$, fixed at t. After 6 months (at time T_2) we regain the notional N plus the accrued interest from counterparty A. If A defaults during this period of 6 months, we still recover full redemption due to the collateral agreement.

- *floating leg indexed to 3m-Libor*
  At t we lend a counterparty A, under a collateral agreement, a notional N for 3 months at 3m-Libor $L(t, T_1)$, fixed at t. At maturity T_1 of the lending, A pays back the notional plus interest. If A is still a reliable counterparty A, we lend again N to A for a 3-months-period at 3m-Libor $L(T_1, T_2)$, set at T_1. If A is not a credit-worthy anymore, we lend the notional N to the other institution B, under a collateral agreement. In either instance, we regain the notional N plus interest, from counterparty A or B at maturity T_2. If one of the counterparties defaults within the period of 6 months, we fully get back the money due to the CSA.

It is more likely that a bank does not default within a period of 3 months than in a period of 6 months, see Figure 4.2. Although, the counterparty risk is very small thanks to the collateral agreement, there must be added a positive spread to the leg with the shorter tenor to obtain equilibrium with respect to the 6m-leg.

Before the credit crunch, floating cash flows with the same maturity tied to different tenors could have been replicated one with each others because implied liquidity and credit risk was disregarded, since the observable spread between curves was negligible.

Figure 4.3 represents floating cash flows, a floating swap leg, tied to a specified tenor $m \in \{1m, 3m, 6m, 12m\}$, where $L_m\left(T_{i-1}^m, T_i^m\right)$ denotes the Libor rate fixed at T_{i-1} with maturity

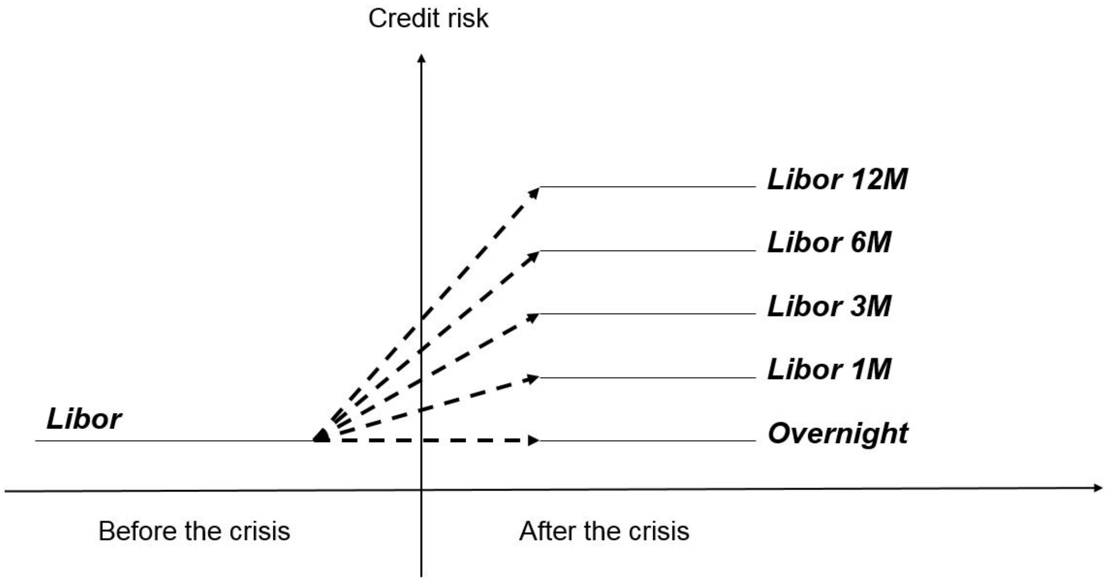

Figure 4.2: Representation of credit risk depending on the term of lending; [4]

T_i tied to tenor m with $m = T_i - T_{i-1}$ and $\delta^m_{i-1,i}$, the year fraction according to the day count convention. As a consequence, the present values Π_t of the floating cash flows tied to different tenors are not equal anymore as assumed in Section 2.5.

$$
\begin{aligned}
\Pi_t^{12m} &= \delta^{12m}_{t,T_{12m}} P_d\left(t, T_{12m}\right) L_{12m}\left(t, T_{12m}\right) \\
\neq \Pi_t^{6m} &= \sum_{i=1}^{2} \delta^{6m}_{i-1,i} P_d\left(t, T_i^{6m}\right) F^{6m}\left(t, T_{i-1}^{6m}, T_i^{6m}\right) \\
\neq \Pi_t^{3m} &= \sum_{i=1}^{4} \delta^{3m}_{i-1,i} P_d\left(t, T_i^{3m}\right) F^{3m}\left(t, T_{i-1}^{3m}, T_i^{3m}\right) \\
\neq \Pi_t^{1m} &= \sum_{i=1}^{12} \delta^{1m}_{i-1,i} P_d\left(t, T_i^{1m}\right) F^{1m}\left(t, T_{i-1}^{1m}, T_i^{1m}\right) \\
&\neq 1 - P_d\left(t, T_{12m}\right)
\end{aligned}
$$

Table 4.1 shows an overview of the formulas for interest swaps before and after the crisis to be easily compared.

4.5 Cross Currency Swap

From the cross currency swap definition in Section 2.6 and the definition of a basis spread Z we can intuitively derive an equation for a cross currency swap by taking a cross currency spread $X(t,T)$ into account. This section is inspired by [8] and [9].

We reconsider a single currency ξ fixed rate interest swap from Section 4.3 as presented in (4.5), with maturity $T_M\,(=S_M)$ and a fair swap rate R. The contractual dates of the fixed leg are denoted as S_i whereas the dates of the floating leg are denoted as T_i. The tenor of the floating leg is denoted as m, δ denotes the year fraction corresponding to the relevant day count convention. Hence, we come to the condition

$$R^{\xi}(t,S_M)\sum_{j=1}^{M}\delta^{\xi}_{j-1,j}P^{\xi}_{d}(t,S_j)=\sum_{i=1}^{M}\delta^{\xi}_{i-1,i}P^{\xi}_{d}(t,T_i)\,\mathbb{E}^{\xi}_{t}\left[L^{\xi}_{m}(T_{i-1},T_i)\right].$$

We understand $\mathbb{E}^{\xi}_{t}[.] := \mathbb{E}^{\xi}[.|\mathcal{F}_t]$ as the expectation under the risk neutral measure $\mathcal{Q}^{T}_{d}$ where $P^{\xi}_{d}(t,T)$ is used as numeraire. In this set-up, d refers to an appropriate discounting curve, which will be specified and discussed in Chapter 5. As we know from (2.16),

$$P^{\xi}_{d}(t,T_0)-P^{\xi}_{d}(t,T_M)=\sum_{i=1}^{M}\delta^{\xi}_{i-1,i}P^{\xi}_{d}(t,T_i)\,\mathbb{E}^{\xi}_{t}\left[L^{\xi}_{m}(T_{i-1},T_i)\right],$$

the ξ-floating leg's present value at time t is

$$\Pi^{\xi}_{t}=-P^{\xi}_{d}(t,T_0)+P^{\xi}_{d}(t,T_M)+\sum_{i=1}^{M}\delta^{\xi}_{i-1,i}P^{\xi}_{d}(t,T_i)\,\mathbb{E}^{\xi}_{t}\left[L^{\xi}_{m}(T_{i-1},T_i)\right].$$

The analogous result for currency η is

$$\Pi^{\eta}_{t}=-P^{\eta}_{d}(t,T_0)+P^{\eta}_{d}(t,T_M)+\sum_{i=1}^{M}\delta^{\eta}_{i-1,i}P^{\eta}_{d}(t,T_i)\,\mathbb{E}^{\eta}_{t}\left[L^{\eta}_{m}(T_{i-1},T_i)\right].$$

Under the assumption both legs are tied to the same tenor m, e.g. 3 months, and taking a cross currency spread $X^{(\xi,\eta)}(t,T_M)$, referring to maturity T_M, into account, we get

$$\begin{aligned}&\frac{N^{\xi}}{f^{(\eta,\xi)}_{t}}\left[-P^{\xi}_{d}(t,T_0)+P^{\xi}_{d}(t,T_M)+\sum_{i=1}^{M}\delta^{\xi}_{i-1,i}P^{\xi}_{d}(t,T_i)\left(\mathbb{E}^{\xi}_{t}\left[L^{\xi}_{m}(T_{i-1},T_i)\right]+X^{(\xi,\eta)}(t,T_M)\right)\right]\\&=-P^{\eta}_{d}(t,T_0)+P^{\eta}_{d}(t,T_M)+\sum_{i=1}^{M}\delta^{\eta}_{i-1,i}P^{\eta}_{d}(t,T_i)\,\mathbb{E}^{\eta}_{t}\left[L^{\eta}_{m}(T_{i-1},T_i)\right],\end{aligned}\tag{4.7}$$

where N^{ξ} is the notional in currency ξ per η and $f^{(\eta,\xi)}_{t}$ is the exchange rate, e.g.

$$\begin{aligned}\xi&=\text{EUR}\\\eta&=\text{USD}\\N^{\xi}&=N^{EUR}=8,000,000.00\\f^{(\eta,\xi)}_{t}&=f^{USDEUR}_{t}=1.25\end{aligned}$$

Further on, we reconsider a m vs $\tilde{m}$ basis swap in a single currency ξ where $Z(t, T_M)$ denotes the basis spread as in Section 4.4

$$\sum_{i=1}^{M} \delta^{\xi}_{i-1,i} P^{\xi}_{d}(t, T_i) \left(\mathbb{E}^{\xi}_{t} \left[L^{\xi}_{m}(T_{i-1}, T_i) \right] + Z(t, T_M) \right) = \sum_{j=1}^{\tilde{M}} \delta^{\xi}_{j-1,j} P^{\xi}_{d}(t, T_j) \, \mathbb{E}^{\xi}_{t} \left[L^{\xi}_{\tilde{m}}(T_{j-1}, T_j) \right].$$

If the legs of the cross currency swap are each tied to different tenors m and $\tilde{m}$ we have to add the corresponding basis spread as well, $X^{(\xi,\eta)}(t, T_M) =: X^{(\xi,\eta)}, Z^{(m,\tilde{m})}(t, T_M) =: Z^{(m,\tilde{m})}$,

$$\frac{N^{\xi}}{f_t^{(\eta,\xi)}} \left[-P^{\xi}_{d}(t, T_0) + P^{\xi}_{d}\left(t, T_{\tilde{M}}\right) + \sum_{i=1}^{\tilde{M}} \delta^{\xi}_{i-1,i} P^{\xi}_{d}(t, T_i) \left(\mathbb{E}^{\xi}_{t} \left[L^{\xi}_{\tilde{m}}(T_{i-1}, T_i) \right] + X^{(\xi,\eta)} + Z^{(m,\tilde{m})} \right) \right]$$

$$= - P^{\eta}_{d}(t, T_0) + P^{\eta}_{d}(t, T_M) + \sum_{j=1}^{M} \delta^{\eta}_{j-1,j} P^{\eta}_{d}(t, T_j) \, \mathbb{E}^{\eta}_{t} \left[L^{\eta}_{m}(T_{j-1}, T_j) \right].$$

Hence, there is needed more than one curve for a floating-floating cross currency swap valuation

- a discounting curve for the currency ξ-leg,
- a discounting curve for the currency η-leg,
- a forward curve, such as Libor, for currency ξ, and
- a forward curve, such as Libor, for currency η.

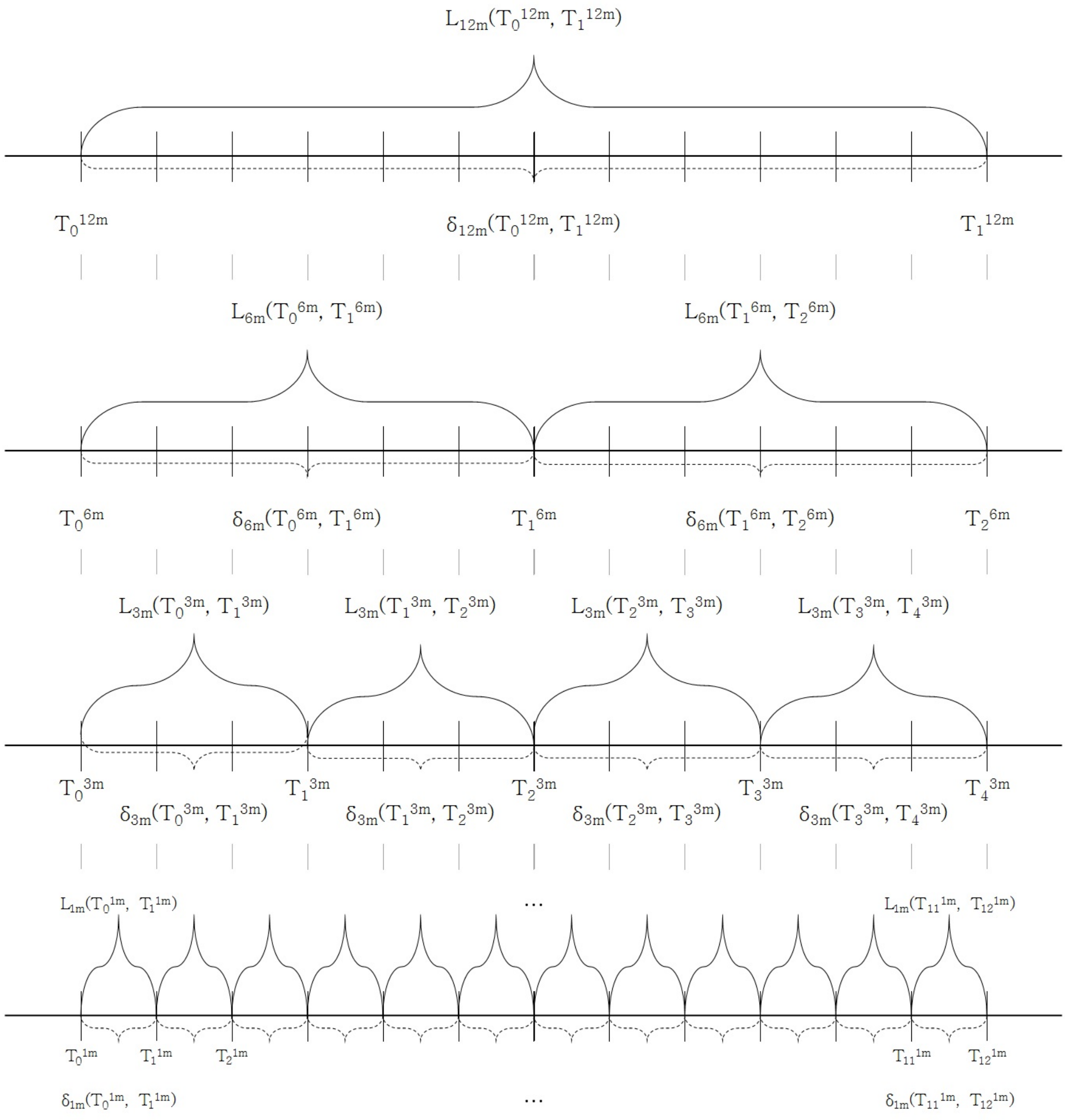

Figure 4.3: Simplified floating cash flows tied to different tenors of 1 month, 3 months, 6 months and 12 months; [4].

Before the Crisis

(Single Curve Approach)

Forward Rate Agreement (FRA) (Section 2.1)

$$\Pi_t^{FRA}(T_{i-1},T_i) = N \cdot \delta_{i-1,i} \cdot P(t,T_i)\left[F(t,T_{i-1},T_i) - K\right]$$

where

$$F(t,T_{i-1},T_i) = \mathbb{E}_t^{\mathcal{Q}^{T_i}}\left[L(T_{i-1},T_i)\right]$$

Fixed Rate Interest Swap (FXFL) (Section 2.4)

$$\Pi_t^{FXFL}(K) = N \cdot \left[\sum_{i=1}^{M} \delta_{i-1,i} P(t,T_i) F(t,T_{i-1},T_i) - K \sum_{j=1}^{M} \delta_{j-1,j} P(t,S_j)\right]$$

where

$$K = R^{FXFL}(t,T_M) = \frac{\sum_{i=1}^{M} \delta_{i-1,i} P(t,T_i) F(t,T_{i-1},T_i)}{\sum_{j=1}^{M} \delta_{j-1,j} P(t,S_j)}$$

Basis Swap (FLFL) (Section 2.5)

$$\Pi_t^{FLFL}(m,\tilde{m},Z) = N \cdot \left[\sum_{i=1}^{M} \delta_{i-1,i} P(t,T_i)\left(F_m(t,T_{i-1},T_i) + Z(t,T_M)\right) - \sum_{j=1}^{\tilde{M}} \delta_{j-1,j} P(t,T_j) F_{\tilde{m}}(t,T_{j-1},T_j)\right]$$

where

$$Z(t,T_M) = \frac{\sum_{j=1}^{\tilde{M}} \delta_{j-1,j} P(t,T_j) F_{\tilde{m}}(t,T_{j-1},T_j) - \sum_{i=1}^{M} \delta_{i-1,i} P(t,T_i) F_m(t,T_{i-1},T_i)}{\sum_{i=1}^{M} \delta_{i-1,i} P(t,T_i)} = 0$$

After the Crisis

(Multiple Curve Approach)

Forward Rate Agreement (FRA) (Section 4.2)

$$\Pi_t^{FRA}(T_{i-1},T_i) = N \cdot \delta_{i-1,i} \cdot P_d(t,T_i)\left[F_m(t,T_{i-1},T_i) - K\right]$$

where

$$F_m(t,T_{i-1},T_i) = \mathbb{E}_t^{\mathcal{Q}_d^{T_i}}\left[L_m(T_{i-1},T_i)\right]$$

Fixed Rate Interest Swap (FXFL) (Section 4.3)

$$\Pi_t^{FXFL}(K) = N \cdot \left[\sum_{i=1}^{M} \delta_{i-1,i} P_d(t,T_i) F_m(t,T_{i-1},T_i) - K \sum_{j=1}^{M} \delta_{j-1,j} P_d(t,S_j)\right]$$

where

$$K = R_m^{FXFL}(t,T_M) = \frac{\sum_{i=1}^{M} \delta_{i-1,i} P_d(t,T_i) F_m(t,T_{i-1},T_i)}{\sum_{j=1}^{M} \delta_{j-1,j} P_d(t,S_j)}$$

Basis Swap (FLFL) (Section 4.4)

$$\Pi_t^{FLFL}(m,\tilde{m},Z) = N \cdot \left[\sum_{i=1}^{M} \delta_{i-1,i} P_d(t,T_i)\left(F_m(t,T_{i-1},T_i) + Z(t,T_M)\right) - \sum_{j=1}^{\tilde{M}} \delta_{j-1,j} P_d(t,T_j) F_{\tilde{m}}(t,T_{j-1},T_j)\right]$$

where

$$Z(t,T_M) = \frac{\sum_{j=1}^{\tilde{M}} \delta_{j-1,j} P_d(t,T_j) F_{\tilde{m}}(t,T_{j-1},T_j) - \sum_{i=1}^{M} \delta_{i-1,i} P_d(t,T_i) F_m(t,T_{i-1},T_i)}{\sum_{i=1}^{M} \delta_{i-1,i} P_d(t,T_i)}$$

Table 4.1: Overview of the derivative's formulas before and after the crisis; [3].

Chapter 5

Curve Construction

The ideas introduced in this chapter are primarily based on [2] and [3].

In a multi-curve framework, we distinguish the discounting curves $\mathcal{C}_d$ from the forward curves $\mathcal{C}_f$, as introduced in Chapter 4.

Forward Curves $\mathcal{C}_f$ These curves are the curves to calculate the forward rates. The curve to use is chosen in accordance with the rate index specified in the contract. If the interest payment cash flow of the contract is tied to a 6m-Euribor, the 6m-Euribor curve is used to calculate the forward rates.

Discounting Curves $\mathcal{C}_d$ These are the curves which are used to discount future cash flows. To select the correct curve for discounting we also have to take the contractual characteristics into account.

- In a standard CSA with daily cash collateral exchange, the OIS curve is used for discounting because the counterparty risk is minimized, so we can use the riskless OIS curve.
- If an agreement is not collateralized, a curve comparable to the funding curve is used.

Both, in a single-curve world and in a multi-curve world the following steps described in detail in Table 5.1, are needed for pricing a derivative:

- Build one (or more) yield curve(s) as needed
- Calculate forward rates using the relevant yield curve
- Use the relevant yield curve to compute discount factors
- Price the derivative by summing up all discounted future cash flows

5.1 Multiple Curve No-CSA Approach

This section, as well as the following section, are mainly inspired by [3]. Furthermore, several ideas are from [9].

The first "quick fix" of curve construction methods after the credit crunch was to distinguish between discounting and forward curve by calculating discount factors, $P_d(t, T_i)$, and FRA forward rates, F^{FRA}, based on different curves under the assumption of No-CSA. The discounting curve is calculated based on a curve which reflects the funding cost of the institution, whereas the curve used for the forward rates is linked to the corresponding swap curve.

Typically the curve with the "natural" tenor corresponding to the currency is used for discounting, e.g. the USD swap rate is the equilibrium fixed rate of a fixed vs floating swap where the fixed rate has a semi-annual payment frequency and the floating rate is payed quarterly tied to the 3m-indexed Libor.

5.1.1 Uncollateralized Fixed Rate Interest Swap

We consider a fixed rate interest rate swap as in (4.5), which is priced fair

$$R\left(t,T_M\right)\sum_{j=1}^{M}\delta_{j-1,j}P_d\left(t,S_j\right)=\sum_{i=1}^{M}\delta_{i-1,i}P_d\left(t,T_i\right)\mathbb{E}_t^{\mathcal{Q}^{T_i}}\left[L_m\left(T_{i-1},T_i\right)\right].$$

$\mathbb{E}_t^{\mathcal{Q}^{T_i}}[.] := \mathbb{E}^{\mathcal{Q}^{T_i}}[.|\mathcal{F}_t]$ is the conditional expectation under the T_i-forward measure $\mathcal{Q}^{T_i}$ with respect to numeraire $P_d\left(t,T_i\right)$. $R\left(t,T_M\right)$ is the fair swap rate at time $t\left(\leq T_i\right)$ for a T_M-maturing swap. $\delta_{i-1,i}$ is the year fraction from T_{i-1} to T_i, where T_i denotes the dates for the floating leg and S_j those for the fixed leg. The Libor rate $L_m\left(T_{i-1},T_i\right)$ is linked to tenor m. Since we use the same curve linked to tenor m for discounting in a non-CSA world, we set

$$P_m\left(t,T_i\right)=P_d\left(t,T_i\right).$$

Consequently, the no-arbitrage condition results in

$$\mathbb{E}_t^{\mathcal{Q}^{T_i}}\left[L_m\left(T_{i-1},T_i\right)\right]=\frac{1}{\delta_{i-1,i}}\left[\frac{P_m\left(t,T_{i-1}\right)}{P_m\left(t,T_i\right)}-1\right]. \tag{5.1}$$

Using (5.1), taking into account that the discounting curve is also linked to tenor m and denoting $P := P_m = P_d$ for simplicity, we get to

$$\begin{aligned}
R\left(t,T_M\right)\sum_{j=1}^{M}\delta_{j-1,j}P\left(t,S_j\right)=&\sum_{i=1}^{M}\delta_{i-1,i}P\left(t,T_i\right)\mathbb{E}_t^{\mathcal{Q}^{T_i}}\left[L_m\left(T_{i-1},T_i\right)\right]\\
=&\sum_{i=1}^{M}\delta_{i-1,i}P\left(t,T_i\right)\frac{1}{\delta_{i-1,i}}\left[\frac{P\left(t,T_{i-1}\right)}{P\left(t,T_i\right)}-1\right]\\
=&\sum_{i=1}^{M}P\left(t,T_i\right)\left[\frac{P\left(t,T_{i-1}\right)}{P\left(t,T_i\right)}-1\right]\\
=&\sum_{i=1}^{M}P\left(t,T_{i-1}\right)-P\left(t,T_i\right)\\
=&P\left(t,T_0\right)-P\left(t,T_1\right)+\sum_{i=2}^{M}P\left(t,T_{i-1}\right)-P\left(t,T_i\right)\\
=&P\left(t,T_0\right)-P\left(t,T_1\right)+P\left(t,T_1\right)-P\left(t,T_2\right)+\sum_{i=3}^{M}P\left(t,T_{i-1}\right)-P\left(t,T_i\right)\\
=&P\left(t,T_0\right)-P\left(t,T_2\right)+\sum_{i=3}^{M}P\left(t,T_{i-1}\right)-P\left(t,T_i\right)\\
&\vdots\\
=&P\left(t,T_0\right)-P\left(t,T_M\right),
\end{aligned}$$

where $\mathbb{E}_t^{\mathcal{Q}^{T_i}}[.] := \mathbb{E}^{\mathcal{Q}^{T_i}}[.|\mathcal{F}_t]$ is the expectation under the T_i-forward measure $\mathcal{Q}^{T_i}$ with respect to numeraire $P(t, T_i)$.

The equation above allows us to get a simple representation of $P(t, T_M)$

$$
\begin{aligned}
R(t,T_M) \sum_{i=1}^{M} \delta_{i-1,i} P(t,T_i) =& P(t,T_0) - P(t,T_M) \\
R(t,T_M) \left[\delta_{M-1,M} P(t,T_M) + \sum_{i=1}^{M-1} \delta_{i-1,i} P(t,T_i) \right] =& P(t,T_0) - P(t,T_M) \\
R(t,T_M)\, \delta_{M-1,M} P(t,T_M) + R(t,T_M) \sum_{i=1}^{M-1} \delta_{i-1,i} P(t,T_i) =& P(t,T_0) - P(t,T_M) \\
R(t,T_M)\, \delta_{M-1,M} P(t,T_M) + P(t,T_M) =& P(t,T_0) - R(t,T_M) \sum_{i=1}^{M-1} \delta_{i-1,i} P(t,T_i) \\
P(t,T_M) \left[R(t,T_M)\, \delta_{M-1,M} + 1 \right] =& P(t,T_0) - R(t,T_M) \sum_{i=1}^{M-1} \delta_{i-1,i} P(t,T_i) \\
P(t,T_M) =& \frac{P(t,T_0) - R(t,T_M) \sum_{i=1}^{M-1} \delta_{i-1,i} P(t,T_i)}{1 + \delta_{M-1,M} R(t,T_M)}.
\end{aligned}
$$

Consequently, the discounting factors $P(t, T_k) \quad \forall k \in \{1, \ldots M\}$ can now be calculated sequentially by solving

$$
P(t,T_k) = \frac{P(t,T_0) - R(t,T_k) \sum_{i=1}^{k-1} \delta_{k-1,k} P(t,T_i)}{1 + \delta_{k-1,k} R(t,T_k)}. \tag{5.2}
$$

5.1.2 Uncollateralized Basis Swap

Further, we consider a basis swap as presented in (4.6), the exchange of two floating legs each tied to a different tenor (e.g. 3m and 6m). Let the two different tenors be $m < \tilde{m}$. Then, the analogous formula for a fair priced basis swap at time t is

$$
\sum_{i=1}^{M} \delta_{i-1,i}^{m} \left\{ \mathbb{E}_t^{\mathcal{Q}^{T_i}} \left[L_m(T_{i-1}, T_i) \right] + Z^{(m,\tilde{m})}(t, T_M) \right\} P(t, T_i) = \sum_{j=1}^{M} \delta_{j-1,j}^{\tilde{m}} \mathbb{E}_t^{\mathcal{Q}^{T_i}} \left[L_{\tilde{m}}(T_{j-1}, T_j) \right] P(t, T_j),
$$

where $\delta_{i-1,i}$ denotes the year fraction of the interval $[T_{i-1}, T_i]$ and $Z^{(m,\tilde{m})}(t, T_M)$ denotes the m vs $\tilde{m}$ basis spread fixed at time t for the length of the swap, which is added to the leg tied to the lower tenor. $\mathbb{E}_t^{\mathcal{Q}^{T_i}}$ denotes the conditional expectation under the T_i-forward measure $\mathcal{Q}^{T_i}$, where $P(t, T_i)$ is used as numeraire, as in Section 5.1.1. As the discount factors linked to tenor m are computed by solving Equation (5.2), derived from a fixed rate IRS, we can easily calculate the corresponding discount factors from the leg tied to the other tenor $\tilde{m}$.

5.1.3 Uncollateralized Cross Currency Swap

Up to this point of the chapter we only have had a single currency market in mind. In this section we want to broaden the range.

We consider a constant notional cross currency basis swap as in (4.7). As before, the currencies are ξ and η and the floating legs are each tied to the same tenor m.

$$\begin{aligned}
&\Pi_t^{\xi} \\
&= \Pi_t^{\eta} \\
&\frac{N^{\xi}}{f_t^{(\eta,\xi)}}\left[-P_d^{\xi}(t,T_0)+P_d^{\xi}(t,T_M)+\sum_{i=1}^{M}\delta_{i-1,i}^{\xi}P_d^{\xi}(t,T_i)\left(\mathbb{E}_t^{\xi}\left[L_m^{\xi}(T_{i-1},T_i)\right]+X^{(\xi,\eta)}(t,T_M)\right)\right] \\
&= -P_d^{\eta}(t,T_0)+P_d^{\eta}(t,T_M)+\sum_{i=1}^{M}\delta_{i-1,i}^{\eta}P_d^{\eta}(t,T_i)\,\mathbb{E}_t^{\eta}\left[L_m^{\eta}(T_{i-1},T_i)\right],
\end{aligned} \tag{5.3}$$

where $\mathbb{E}_t^{\xi} := \mathbb{E}_t^{\mathcal{Q}^{T_i},\xi}$ denotes the expectation under the T_i-forward measure $\mathcal{Q}^{T_i}$ with respect to numeraire $P_d^{\xi}(t,T_i)$, this also applies for η, respectively.

We assume the funding currency is η, thus the discounting curve is the η-Libor curve tied to tenor m, L_m^{η}, respectively. We denote the corresponding zero coupon bond as

$$P_m(t,T_i) := P_{L_m}(t,T_i).$$

The present value of the η-leg is zero at initiation t

$$\begin{aligned}
\Pi_t^{\eta} &= -P_m^{\eta}(t,T_0)+P_m^{\eta}(t,T_M)+\sum_{i=1}^{M}\delta_{i-1,i}^{\eta}P_m^{\eta}(t,T_i)\,\mathbb{E}_t^{\eta}\left[L_m^{\eta}(T_{i-1},T_i)\right] \\
&\overset{(5.1)}{=} -P_m^{\eta}(t,T_0)+P_m^{\eta}(t,T_M)+\sum_{i=1}^{M}\delta_{i-1,i}^{\eta}P_m^{\eta}(t,T_i)\frac{1}{\delta_{i-1,i}^{\eta}}\left[\frac{P_m^{\eta}(t,T_{i-1})}{P_m^{\eta}(t,T_i)}-1\right] \\
&= -P_m^{\eta}(t,T_0)+P_m^{\eta}(t,T_M)+\sum_{i=1}^{M}\left[P_m^{\eta}(t,T_{i-1})-P_m^{\eta}(t,T_i)\right] \\
&= -P_m^{\eta}(t,T_0)+P_m^{\eta}(t,T_M)+P_m^{\eta}(t,T_0)-P_m^{\eta}(t,T_M) \\
&= 0.
\end{aligned} \tag{5.4}$$

We reconsider a fixed rate interest swap in a single currency ξ as presented in (4.5)[1]

$$R^{\xi}(t,T_M)\sum_{i=1}^{M}\delta_{i-1,i}^{\xi,R}P_d^{\xi}(t,T_i) = \sum_{i=1}^{M}\delta_{i-1,i}^{\xi}P_d^{\xi}(t,T_i)\,\mathbb{E}_t^{\xi}\left[L_m^{\xi}(T_{i-1},T_i)\right]. \tag{5.5}$$

[1] The day count conventions for the year fraction δ are (possibly) different according to the interest rate behavior of the leg. Hence, we denote the year fraction linked to the fixed rate R as δ^R.

Bringing together (5.3) and (5.5) results in

$$
\begin{aligned}
-P_d^{\xi}(t,T_0) + P_d^{\xi}(t,T_M) + \sum_{i=1}^{M} \delta_{i-1,i}^{\xi} P_d^{\xi}(t,T_i)\left(\mathbb{E}_t^{\xi}\left[L_m^{\xi}(T_{i-1},T_i)\right] + X^{(\xi,\eta)}(t,T_M)\right) &= 0 \\
R^{\xi}(t,T_M)\sum_{i=1}^{M} \delta_{i-1,i}^{\xi,R} P_d^{\xi}(t,T_i) - \sum_{i=1}^{M} \delta_{i-1,i}^{\xi} P_d^{\xi}(t,T_i)\,\mathbb{E}_t^{\xi}\left[L_m^{\xi}(T_{i-1},T_i)\right] &= 0
\end{aligned}
$$

$$
\begin{aligned}
-P_d^{\xi}(t,T_0) + P_d^{\xi}(t,T_M) + R^{\xi}(t,T_M)\sum_{i=1}^{M} \delta_{i-1,i}^{\xi,R} P_d^{\xi}(t,T_i) + \sum_{i=1}^{M} \delta_{i-1,i}^{\xi} P_d^{\xi}(t,T_i)\, X^{(\xi,\eta)}(t,T_M) &= 0 \\
-P_d^{\xi}(t,T_0) + P_d^{\xi}(t,T_M) + \sum_{i=1}^{M} P_d^{\xi}(t,T_i)\left[\delta_{i-1,i}^{\xi,R} R^{\xi}(t,T_M) + \delta_{i-1,i}^{\xi} X^{(\xi,\eta)}(t)\right] &= 0
\end{aligned}
$$

Analogous to Section 5.1.1, we can now calculate the discount factors sequentially from equation

$$
\sum_{i=1}^{M} \delta_{i-1,i}^{\xi} P_d^{\xi}(t,T_i)\left[R^{\xi}(t,T_M) + X^{(\xi,\eta)}(t,T_M)\right] = P_d^{\xi}(t,T_0) - P_d^{\xi}(t,T_M)\,, \tag{5.6}
$$

and can consequently determine the ξ-Libor forward rates from (5.5).

Assuming $X^{(\xi,\eta)}(t,T_M)$ is a non-zero currency spread, the ξ-discount factors $P_d^{\xi}(t,T_i)$ are not only depending on the corresponding fair swap rate $R^{\xi}(t,T_M)$ but also on the additional cross currency spread. Thus, the discounting curve related to currency ξ does not coincide with the ξ-index curve. As we interpret the modified corresponding fair coupon in the fixed rate swap as

$$
\tilde{K}^{\xi} := R^{\xi}(t,T_M) + \delta_{i-1,i}^{\xi} X^{(\xi,\eta)}(t,T_M)\,,
$$

we see that the discounting curve for the ξ-leg in the cross currency basis swap is lower than the ξ-Libor curve. As a result we constructed four different curves for the ξ vs η cross currency swap, as already mentioned at the end of Section 4.5,

- a discounting curve for the currency ξ-leg, from Equation (5.6), which is not the ξ-Libor curve due to a non-zero cross currency spread $X^{(\xi,\eta)}$
- a discounting curve for the currency η-leg, which is assumed to be the η-Libor curve
- a forward curve, for the ξ-leg, which is ξ Libor, and
- a forward curve, for the η-leg, which is η-Libor.

5.1.4 Different Funding Curves

In the previous Section 5.1.3, we considered an uncollateralized cross currency basis swap as seen from the perspective of counterparty A whose funding currency is η and the funding curve is η-Libor flat, accordingly. Hence, the discounting curve is assumed to be the η-Libor curve.

We assume now, that counterparty A trades the same cross currency basis swap but its funding currency is ξ. The choice of the discounting curve would be the ξ-Libor curve, respectively. Consequently, the value of the ξ-leg distinguishes from the value calculated before. Since uncollateralized derivatives are free of choice in discounting the no-arbitrage condition does not hold any longer.

As an example we set the currencies in our cross currency basis swap to

$$\eta = USD \text{ and}$$
$$\xi = EUR.$$

We consider A as a credit-worthy financial institution who is generally funded in USD. Thus, the adequate interest rate for lending and borrowing a loan is the USD Libor curve. If the financial institution issues a USD floating rate note tied to USD 3m-Libor flat, the value Π_t^{USD} on trade date is zero by using the USD 3m-Libor curve as the appropriate discounting curve, see (5.4).

We question now, what the costs are for company A to get an equivalent EUR loan. We consider two possibilities

1. issue a USD bond and swap the notional into EUR via a cross currency swap, or
2. issue a EUR bond.

Entering a cross currency basis swap after borrowing USD implies funding costs for the final EUR loan of EUR 3m-Libor plus a CCY spread (or any other EUR reference curve), what we have already shown in the previous Section 5.1.3.

Alternatively, we assume company A has access to the EUR market and issues directly in EUR at EUR 3m-Libor flat. Since in the current market the EUR swap curve is below the USD curve, company A would benefit choosing the second option, see also Figure 5.1.

As a consequence, there co-exist two funding curves for one institution.This explains the result in Section 5.1.3, that the discounting curve linked to currency ξ does not coincide with the ξ-Libor curve.

5.2 Multiple Curve CSA Approach

As mentioned at the beginning of Section 5.1, this section is primarily inspired by [3] and [9]. We now concentrate on derivatives under a standard CSA. We assume trades as perfectly collateralized as described in Chapter 3 and Section 4.1.

We consider a standard CSA assuming a daily cash collateral exchange. Accordingly, the collateral rate is the overnight rate related to the collateral currency. Furthermore, we assume that mark-to-market and cash collateral posting (exchange) is made continuously and covers the net present value of the contract (zero threshold) on a daily basis. As in Section 5.1, the overnight rate as discounting rate represents the funding costs for daily collateral posting. If the exposure is negative, the amount must be funded and is provided to the counterparty for one day (or vice versa), see also Chapter 3. Hence, the existence of a collateral agreement reduces counterparty credit risk in a swap to almost zero.

As introduced in Section 4.1, under these assumptions the value Π_t of a T-maturing derivative at time t, where $r(t)$ denotes the rate linked to collateral, in terms of currency and tenor, at time t, is given by

$$\Pi_t = \mathbb{E}_t^{\mathcal{Q}} \left[e^{-\int_t^T r(s)ds} \Pi_T \right],$$

where $\mathcal{Q}$ denotes the risk neutral measure where the bank account corresponding to the collateral (4.1) is used as numeraire.

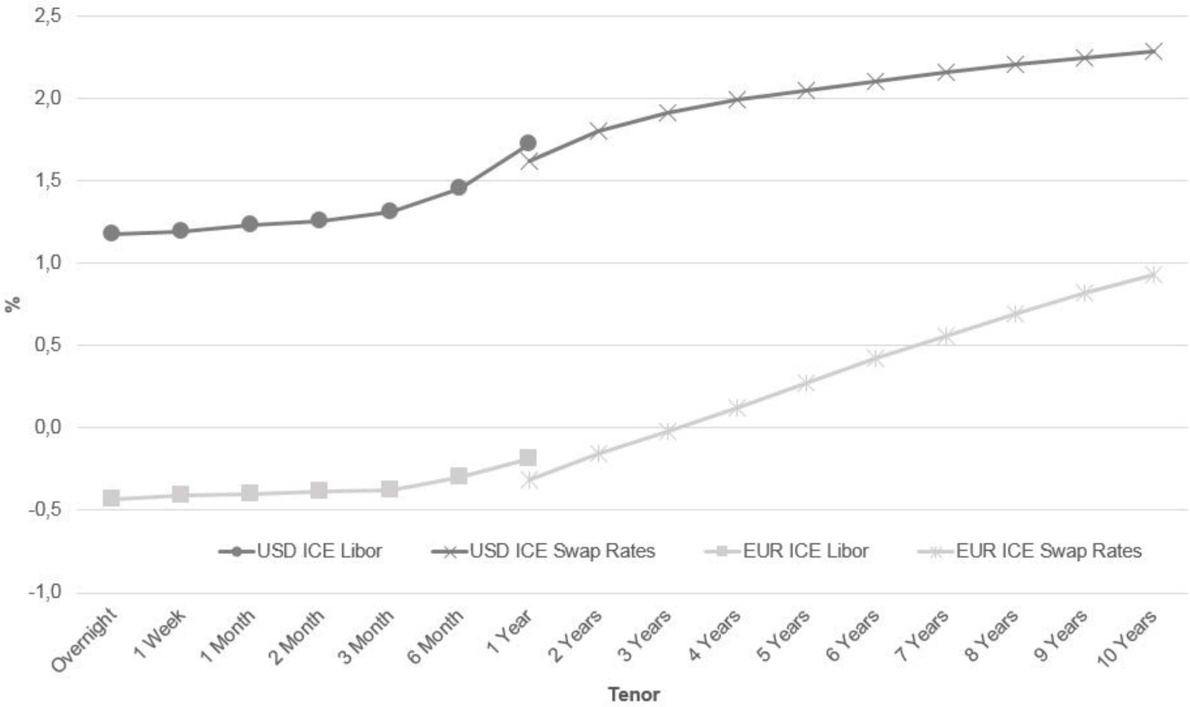

Figure 5.1: Curve Graph of EUR and USD Benchmark Curve, 01-Aug-2017,
Source: https://www.theice.com/marketdata/reports

5.2.1 The OIS (Discounting) Curve

We construct the riskless curve from plain vanilla interest rate instruments, which are indexed to the overnight rate depending on the currency. For bootstrapping this curve we use the Overnight Indexed Swap (OIS) as described in Section 2.4.1.

Using (2.2) we approach the continuous compounding with the daily compounding from Equation (2.14)

$$\left(\prod_{s=T_{i-1}}^{T_i} \left[1+\delta_s r\left(s\right)\right]\right) - 1 \approx e^{\int_{T_{i-1}}^{T_i} r(s)ds} - 1,$$

denoting the overnight rate at time s as $r\left(s\right)$. As we assume a T_M-maturing OIS to be cash-collateralized on a daily basis with zero threshold. As we adapt a traditional coupon swap (4.5), the following equation holds

$$
\begin{aligned}
S\left(t,T_M\right)\sum_{j=1}^{M}\delta_{j-1,j}\mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^{T_j} r(s)ds}\right] &= \sum_{i=1}^{M}\mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^{T_i} r(s)ds}\left(e^{\int_{T_{i-1}}^{T_i} r(s)ds}-1\right)\right]\\
&= \sum_{i=1}^{M}\mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^{T_i} r(s)ds+\int_{T_{i-1}}^{T_i} r(s)ds}-e^{-\int_t^{T_i} r(s)ds}\right]\\
&= \sum_{i=1}^{M}\mathbb{E}_t^{\mathcal{Q}}\left[e^{-\left(\int_t^{T_i} r(s)ds-\int_{T_{i-1}}^{T_i} r(s)ds\right)}-e^{-\int_t^{T_i} r(s)ds}\right]\\
&= \sum_{i=1}^{M}\mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^{T_{i-1}} r(s)ds}-e^{-\int_t^{T_i} r(s)ds}\right]\\
&= \sum_{i=1}^{M}\left(\mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^{T_{i-1}} r(s)ds}\right]-\mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^{T_i} r(s)ds}\right]\right).
\end{aligned}
$$

$\mathbb{E}_t^{\mathcal{Q}}\left[.\right] := \mathbb{E}^{\mathcal{Q}}\left[.|\mathcal{F}_t\right]$ is the conditional expectation under the risk neutral measure $\mathcal{Q}$, where we use the riskless bank account corresponding to $r\left(t\right)$ as a numeraire. $S\left(t,T_M\right)$ denotes the fair swap rate at time t of a T_M-maturing OIS. By rewriting the discount factors linked to the collateral rate $r\left(s\right)$, which is the OIS-rate, to

$$
D\left(t,T_i\right) = \mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^{T_i} r(s)ds}\right],
$$

we come to the equation

$$
\begin{aligned}
S\left(t,T_M\right)\sum_{j=1}^{M}\delta_{j-1,j}D\left(t,T_j\right) &= \sum_{i=1}^{M}\left[D\left(t,T_{i-1}\right)-D\left(t,T_i\right)\right]\\
&= \left[D\left(t,T_0\right)-D\left(t,T_1\right)\right]+\sum_{i=2}^{M}\left[D\left(t,T_{i-1}\right)-D\left(t,T_i\right)\right]\\
&= D\left(t,T_0\right)-D\left(t,T_1\right)+\left[D\left(t,T_1\right)-D\left(t,T_2\right)\right]+\sum_{i=3}^{M}\left[D\left(t,T_{i-1}\right)-D\left(t,T_i\right)\right]\\
&= D\left(t,T_0\right)-D\left(t,T_2\right)+\sum_{i=3}^{M}\left[D\left(t,T_{i-1}\right)-D\left(t,T_i\right)\right]\\
&\vdots\\
&= D\left(t,T_0\right)-D\left(t,T_M\right),
\end{aligned}
$$

where we can derive the discount factors sequentially for each T_k as in the previous section

$$
D\left(t,T_k\right) = \frac{D\left(t,T_0\right)-S\left(t,T_k\right)\sum_{j=1}^{k}\delta_{j-1,j}D\left(t,T_j\right)}{1+\delta_{k-1,k}S\left(t,T_k\right)}.
$$

5.2.2 Collateralized Fixed Rate Interest Swap

Assuming an underlying CSA to an OTC-derivative, the conditions from Section 5.1 have to be adapted as follows.

We consider a fixed rate interest swap under a standard CSA with daily collateral exchange. Thus, we can use OIS discounting for swap valuation. Taking the results from above into account, Equation (4.5) adapts to

$$R\left(t,T_M\right)\sum_{i=1}^{M}\delta_{i-1,i}D\left(t,T_i\right)=\sum_{i=1}^{M}\delta_{i-1,i}D\left(t,T_i\right)\mathbb{E}_t^{\mathcal{Q}}\left[L_m\left(T_{i-1},T_i\right)\right],$$

where $\mathbb{E}_t^{\mathcal{Q}}[.]$ denotes the expectation under the risk neutral measure $\mathcal{Q}$ where $D\left(t,T_i\right)$ is used as numeraire. Assuming discount factors $D\left(t,T_i\right)$ are given from the OIS-market for all terms $T_i \;\; i\in\{1,\dots M\}$, we can easily calculate the Libor forward rates analogous to Section 5.1.1, Equation (5.2).

5.2.3 Collateralized Basis Swap

We consider a basis swap assuming an underlying CSA with daily collateral exchange. By using OIS discounting, Equation (4.6) results in

$$\sum_{i=1}^{n}\delta_{i-1,i}^{m}\left\{\mathbb{E}_t^{\mathcal{Q}}\left[L_m\left(T_{i-1},T_i\right)\right]+Z_{m,\tilde{m}}\left(t,T_n\right)\right\}D\left(t,T_i\right)=\sum_{j=1}^{n}\delta_{j-1,j}^{\tilde{m}}\mathbb{E}_t^{\mathcal{Q}}\left[L_{\tilde{m}}\left(T_{j-1},T_j\right)\right]D\left(t,T_j\right),$$

where $\mathbb{E}_t^{\mathcal{Q}}[.]$ denotes the expectation under the risk neutral measure $\mathcal{Q}$ where $D\left(t,T_i\right)$ is used as numeraire. Again, we assume discount factors $D\left(t,T_i\right)$ to be given from the OIS-market $\forall T_i \;\; i\in\{1,\dots M\}$, so we can easily calculate the Libor forward rates analogous to Section 5.1.2.

5.2.4 Collateralized Cross Currency Swap

Up to this point, we have not asked the question in which currency the collateral is paid. In a single currency swap, it is very common that collateral is exchanged in the same currency the swap is based on.

We consider a constant notional cross currency basis swap as introduced in Section 4.4, with currencies ξ and η, where the floating legs are each tied to the same tenor m. We assume currency η as the collateral currency of this swap, which leads us to

$$D^{\eta}\left(t,T_i\right)=\mathbb{E}_t^{\mathcal{Q}^{\eta}}\left[e^{-\int_t^{T_i}r^{\eta}(s)ds}\right]=P_d^{\eta}\left(t,T_i\right), \tag{5.7}$$

where $r^{\eta}\left(t\right)$ denotes the risk-free OIS-rate at time t, which coincides with the collateral rate, respectively. $\mathcal{Q}^{\eta}$ denotes the risk neutral measure where the corresponding η-collateral bank account is used as numeraire. Applying (5.7) to the results for an OIS described in Section 5.2.1, Equations (4.5) and (4.6), we get the following conditions

$$\begin{aligned}
S^{\eta}\left(t,T_M\right)\sum_{j=1}^{M}\delta_{j-1,j}^{\eta}P_d^{\eta}\left(t,T_j\right)=&P_d^{\eta}\left(t,T_0\right)-P_d^{\eta}\left(t,T_M\right),\\
R^{\eta}\left(t,T_M\right)\sum_{j=1}^{M}\delta_{j-1,j}^{\eta}P_d^{\eta}\left(t,T_j\right)=&\sum_{i=1}^{M}\delta_{i-1,i}^{\eta}P_d^{\eta}\left(t,T_i\right)\mathbb{E}_t^{\eta}\left[L_m^{\eta}\left(T_{i-1},T_i\right)\right].
\end{aligned}$$

We set $\mathbb{E}_t^{\eta}:=\mathbb{E}_t^{\mathcal{Q}_d^{T_i},\eta}$, where $\mathcal{Q}_d^{T_i}$ denotes the T_i-forward measure with numeraire $P_d^{\eta}\left(t,T_i\right)$.

As in the previous section, we have the discounting factors $P_d^{\eta}(t,T_i)$ given from the OIS-market for all relevant terms $T_i \quad i \in \{1 \ldots M\}$ and we can further calculate the corresponding η-Libor forward rates.

Having a look at the ξ-leg of the cross currency swap leads us to the following conditions

$$\begin{aligned} S^{\xi}(t,T_M) \sum_{j=1}^{M} \delta_{j-1,j}^{\xi} D^{\xi}(t,T_j) =& D^{\xi}(t,T_0) - D^{\xi}(t,T_M), \\ R^{\xi}(t,T_M) \sum_{j=1}^{M} \delta_{j-1,j}^{\xi} D^{\xi}(t,T_j) =& \sum_{i=1}^{M} \delta_{i-1,i}^{\xi} D^{\xi}(t,T_i) \mathbb{E}_t^{\mathcal{Q}} \left[L_m^{\xi}(T_{i-1},T_i) \right], \end{aligned}$$

where $\mathcal{Q}$ denotes the risk neutral measure with numeraire $D^{\xi}(t,T_i)$, where we can derive the discount factors $D^{\xi}(t,T_i)$ and consequently the ξ-forward Libors, analogous to the previous sections.

We consider a constant notional cross currency basis swap applying daily η-collateralization.

$$\begin{aligned} -P_d^{\xi}(t,T_0) + P_d^{\xi}(t,T_M) + \sum_{i=1}^{M} \delta_{i-1,i}^{\xi} P_d^{\xi}(t,T_i) \left(\mathbb{E}_t^{\xi} \left[L_m^{\xi}(T_{i-1},T_i) \right] + X^{(\xi,\eta)}(t,T_M) \right) \\ = \underbrace{\frac{f_t^{(\eta,\xi)}}{N^{\xi}}}_{=\frac{1}{N^{\eta}}} \left\{ -P_d^{\eta}(t,T_0) + P_d^{\eta}(t,T_M) + \sum_{i=1}^{M} \delta_{i-1,i}^{\eta} P_d^{\eta}(t,T_i) \mathbb{E}_t^{\eta} \left[L_m^{\eta}(T_{i-1},T_i) \right] \right\} \end{aligned} \tag{5.8}$$

Despite the fact that by the previous results, the right hand side of the above Equation (5.8), linked to η, is already known, it is not possible to calculate the riskless zero bond prices $P^{\xi}(t,T_i)$ or the forward Libors linked to ξ.

There are two approaches determining the η-collateralized ξ interest rates. At first, we consider an η-collateralized ξ-interest rate market, where we obtain the fair maturity matching swap rate, which does not coincide with the traditional fair swap rate linked to ξ. We denote the η-collateralized rate as

$$\tilde{R}^{\xi}(t,T_M) \left(\neq R^{\xi}(t,T_M) \right).$$

Hence,

$$\tilde{R}^{\xi}\left(t,T_{\tilde{M}}\right) \sum_{j=1}^{\tilde{M}} \delta_{j-1,j}^{\xi} P_d^{\xi}(t,T_j) = \sum_{i=1}^{\tilde{M}} \delta_{i-1,i}^{\xi} P_d^{\xi}(t,T_i) \mathbb{E}_t^{\mathcal{Q}} \left[L_m^{\xi}(T_{i-1},T_i) \right] \tag{5.9}$$

To simplify the further equations we denote the given η-part from Equation (5.8) as

$$\Pi_t^{\eta} := \frac{1}{N^{\eta}} \left\{ -P_d^{\eta}(t,T_0) + P_d^{\eta}(t,T_M) + \sum_{i=1}^{M} \delta_{i-1,i}^{\eta} P_d^{\eta}(t,T_i) \mathbb{E}_t^{\eta} \left[L_m^{\eta}(T_{i-1},T_i) \right] \right\}$$

From Equations (5.8) and (5.9) we consequently obtain

$$
\begin{array}{r}
-P_d^{\xi}(t,T_0)+P_d^{\xi}(t,T_M)+\sum_{i=1}^{M}\delta_{i-1,i}^{\xi}P_d^{\xi}(t,T_i)\left(\mathbb{E}_t^{\xi}\left[L_m^{\xi}(T_{i-1},T_i)\right]+X^{(\xi,\eta)}(t,T_M)\right)-\Pi_t^{\eta}=0\\
\tilde{R}^{\xi}\left(t,T_{\tilde{M}}\right)\sum_{j=1}^{\tilde{M}}\delta_{j-1,j}^{\xi,R}P_d^{\xi}(t,T_j)-\sum_{i=1}^{\tilde{M}}\delta_{i-1,i}^{\xi}P_d^{\xi}(t,T_i)\,\mathbb{E}_t^{\mathcal{Q}}\left[L_m^{\xi}(T_{i-1},T_i)\right]=0\\
\hline
-P_d^{\xi}(t,T_0)+P_d^{\xi}(t,T_M)+\tilde{R}^{\xi}\left(t,T_{\tilde{M}}\right)\sum_{j=1}^{\tilde{M}}\delta_{j-1,j}^{\xi,R}P_d^{\xi}(t,T_j)+\sum_{i=1}^{M}\delta_{i-1,i}^{\xi}P_d^{\xi}(t,T_i)\,X^{(\xi,\eta)}(t,T_M)-\Pi_t^{\eta}=0
\end{array}
$$

Hence, we can calculate the set of discount factors $P_d^{\xi}(t,T_i)$ from

$$
\tilde{R}^{\xi}\left(t,T_{\tilde{M}}\right)\sum_{j=1}^{\tilde{M}}\delta_{j-1,j}^{\xi,R}P_d^{\xi}(t,T_j)+\sum_{i=1}^{M}\delta_{i-1,i}^{\xi}P_d^{\xi}(t,T_i)\,X^{(\xi,\eta)}(t,T_M)-\Pi_t^{\eta}=P_d^{\xi}(t,T_0)-P_d^{\xi}(t,T_M),
$$

in the same way as we did in the previous sections, and consequently derive the corresponding forward Libors, respectively.

If, for any reason, there are no relevant η-collateralized ξ-interest rates observable in the market, Equation (5.9) was condemned to be impracticable for curve construction.

The second approach we consider is the assumption that the change of numeraire from $D^{\xi}(t,T_i)$ makes a negligible difference and set

$$
\mathbb{E}_t^{\mathcal{Q}}\left[L_m^{\xi}(T_{i-1},T_i)\right]=\mathbb{E}_t^{\xi}\left[L_m^{\xi}(T_{i-1},T_i)\right].
$$

This assumption further implies the similarity of the ξ-risk-free rate and the corresponding overnight rate. As we know assume

$$
P_d^{\xi}(t,T_i)=D^{\xi}(t,T_i)
$$

we determine the discount factors from Equation (5.8).

Single Curve	**Multiple Curves**
Yield Curve Construction	
Select a set of liquid vanilla interest rate instruments to build one single curve $\mathcal{C}$ using a bootstrapping procedure.	Select a set of liquid vanilla interest rate instruments to build a discount curve $\mathcal{C}_d$ using a bootstrapping procedure. Select sets of vanilla interest rate instruments, each homogeneous in the underlying tenor m to build several forward curves $\mathcal{C}_{f_1}, \mathcal{C}_{f_2}, \dots \mathcal{C}_{f_n}$ using a bootstrapping procedure.
Future Cash Flows	
To compute the relevant forward rates for each interest rate coupon the yield curve $\mathcal{C}$ is used $F\left(t, T_{i-1}, T_i\right) = \frac{1}{\delta_{i-1,i}} \left(\frac{P\left(t, T_{i-1}\right)}{P\left(t, T_i\right)} - 1 \right),$ where $\delta_{i-1,i}$ is the year fraction of the time interval T_{i-1} and T_i. The cash flow $\mathbf{c}_i$ for the period (T_{i-1}, T_i) at time t is the expectation of the corresponding future coupon pay off c_i $\mathbf{c}_i\left(t\right) = \mathbb{E}_t^{\mathcal{Q}^{T_i}}\left[c_i\right]$ where $\mathcal{Q}^{T_i}$ is the T_i-forward measure associated to numeraire $P\left(t, T_i\right)$.	For each interest rate coupon compute the corresponding forward rate F_f depending on the underlying reference rate $F_f\left(t, T_{i-1}, T_i\right) = \frac{1}{\delta_{i-1,i}} \left(\frac{P_f\left(t, T_{i-1}\right)}{P_f\left(t, T_i\right)} - 1 \right),$ where $\delta_{i-1,i}$ is the year fraction of the time interval T_{i-1} and T_i. The cash flow $\mathbf{c}_i$ for the period (T_{i-1}, T_i) at time t is the expectation of the corresponding future coupon pay off c_i $\mathbf{c}_i\left(t\right) = \mathbb{E}_t^{\mathcal{Q}_d^{T_i}}\left[c_i\right]$ where $\mathcal{Q}_d^{T_i}$ is the T_i-forward measure associated to numeraire $P_d\left(t, T_i\right)$ according to the discounting curve $\mathcal{C}_d$.
Discount Factors	
By using the same yield curve $\mathcal{C}$ as defined in the first step we calculate the needed discount factors $P\left(t, T_i\right)$.	By using the discount yield curve $\mathcal{C}_d$ defined in the first step we calculate the discount factors $P_d\left(t, T_i\right)$.
Pricing the Derivative	
The derivative's value at time t is the sum of all discounted expected cash flows $\Pi_t = \sum_{i=1}^{M} P\left(t, T_i\right) \mathbf{c}_i\left(t\right) = \sum_{i=1}^{M} P\left(t, T_i\right) \mathbb{E}_t^{\mathcal{Q}^{T_i}}\left[c_i\right]$	The derivative's price at time t is the sum of all discounted expected cash flows $\Pi_t = \sum_{i=1}^{M} P_d\left(t, T_i\right) \mathbf{c}_i\left(t\right) = \sum_{i=1}^{M} P_d\left(t, T_i\right) \mathbb{E}_t^{\mathcal{Q}_d^{T_i}}\left[c_i\right]$

Table 5.1: Comparison of curve construction before and after the crisis; [2], [3].

Chapter 6

Pricing

In this chapter we want to price swaps with the different approaches as introduced in the previous chapter and compare the results.

6.1 EUR Fixed Interest Rate Swap

In this section we price a fixed vs floating swap in a single currency (EUR).

Uncollateralized EUR Fixed Interest Rate Swap

We assume an uncollateralized fixed interest rate (receiver) swap as presented in Section 4.3, with a maturity of 10 years.

The floating leg is indexed to a EUR Swap Curve with a 6-months tenor. Since the 6m-tenor is the natural EUR-tenor it is also used for discounting, see Table 6.2. The corresponding reset and payment dates are denoted as $T_0, \ldots, T_M$. The day count convention is ACT / 360. The notional amount is EUR 10,000,000.

The set-up for the fixed leg of the swap is illustrated in Table 6.1. We denote the annual payment dates with respect to the fixed leg as $S_0, \ldots, S_M$. The day count conventions are supposed to be the same in both legs.

As we assume the EUR Swap Curve vs 6m as given, the first step is to calculate the fair 10-year swap rate $R(t, T_M)$. The relevant terms with the corresponding swap rates and the discount factors, respectively, are presented in Table 6.3. The floating cash flow is presented in Table 6.4, where the floating payments are calculated based on simple compounded forward rates (2.4). Hence, the corresponding fair swap rate can be calculated

$$
\begin{aligned}
R(t, T_M) &= \frac{\sum_{i=1}^{M} \delta_{i-1,i}^{6m} P_{6m}(t, T_i) F_{6m}(t, T_{i-1}, T_i)}{\sum_{j=1}^{M} \delta_{j-1,j}^{1Y} P_{6m}(t, S_j)} \\
&= 0.8769\%.
\end{aligned}
$$

As we set the coupon $K = 0.8769\%$, we come to a fair priced swap as presented in Table 6.5.

Collateralized EUR Fixed Interest Rate Swap

To compare the effect of a standard CSA on the pricing of a swap, we consider the same fixed interest rate (receiver) swap as presented in the previous section. The difference between the previous set-up and the set-up under a CSA is the discounting curve. We consider our EUR

Uncollateralized EUR Fixed Interest Rate Swap	
Receive	**Fix**
Notional	10,000,000 EUR
Start	$t = 0$
Maturity	$T_M = 10$ years
Coupon	K
Payment Frequency	Annual
Day Count	ACT / 360
Discount Curve	EUR Swap Curve vs 6M
Forward Curve	None

Table 6.1: Fixed Leg

Uncollateralized EUR Fixed Interest Rate Swap	
Pay	**Float**
Notional	10,000,000 EUR
Start	$t = 0$
Maturity	$T_M = 10$ years
Index	6M EUR Swap Curve
Spread	0 bp
Reset Frequency	Semi-annual
Payment Frequency	Semi-annual
Day Count	ACT / 360
Discount Curve	EUR Swap Curve vs 6M
Forward Curve	EUR Swap Curve vs 6M

Table 6.2: Floating Leg

fixed vs floating swap as a daily EUR cash collateralized swap and consequently switch to OIS discounting, see Table 6.6 and Table 6.7.

As we assume the EUR Swap Curve vs 6m, as in Table 6.3, and the EUR OIS Curve as given. The relevant terms and interpolated rates of the EUR OIS Curve and the corresponding discounting factors are presented in Table 6.8. As already mentioned in Chapter 4, visualized in Figure 4.2, the EUR OIS is below the EUR Swap Curve vs 6m, see Figure 6.1.

Furthermore, we keep the coupon as the fair swap rate of the uncollateralized swap

$$K = 0.8769\%,$$

and only change the discount factors which are now calculated with respect to the EUR OIS Curve. The OIS discounted cash flows are presented in Table 6.9. In the column *Discounted Netted Payment* of Table 6.9, we can see that the swap is not priced fair if we only change to another discounting curve. There are three possibilities to trade in this set-up, either the floating counterparty pays an amount of EUR 4,097 up-front or the floating counterparty must add a negative spread on top of the floating index or the coupon has to be adapted.

If we solve for a new fair swap rate, analogously to the previous section, we come to

$$R(t, S_M) = 0.8810\%,$$

which is 0.41bp higher.

Uncollateralized EUR Fixed Interest Rate Swap

Term in years	**EUR Swap Curve vs 6m in % (linear interpolated)**	**Discount Factor**
0.5	-0.273	1.0014
1.0	-0.255	1.0026
1.5	-0.217	1.0033
2.0	-0.155	1.0032
2.5	-0.094	1.0024
3.0	-0.032	1.0010
3.6	0.041	0.9986
4.1	0.114	0.9954
4.6	0.187	0.9915
5.1	0.261	0.9869
5.6	0.334	0.9817
6.1	0.408	0.9758
6.6	0.479	0.9694
7.1	0.552	0.9623
7.6	0.619	0.9551
8.1	0.687	0.9471
8.6	0.753	0.9390
9.1	0.816	0.9306
9.6	0.875	0.9222
10.1	0.933	0.9135

Table 6.3: Relevant terms for the swap, corresponding interest rates and discount factors

6.2 EUR Basis Swap

In this section we price a floating vs floating swap in a single currency (EUR).

Uncollateralized EUR Basis Swap

We assume a basis swap (4.6) as an exchange of a 3-months vs a 6-months index in EUR without a CSA in place. Our set-up is shown in Table 6.10 and Table 6.11.

As before, we take the market data as given. Hence, the floating leg 1 (receiver) tied to a EUR Swap Curve vs 6M is known from Table 6.3 and leg 2 tied to a EUR Swap Curve vs 3m. The relevant interpolated rates of the EUR Swap Curve vs 3m are given in Table 6.12. Both interpolated curves are visualized in Figure 6.2.

In Table 6.13, both cash flows are given as we assume the exchange as flat (no spread added). Since the present value is positive (EUR 94,315), the trade is not priced fair. Hence, we calculate

Period Start	**Float End**	**Days**	δ	**Payment Float (Pay)**	**Payment Float Discounted**
t	T_0	181	181/360	13,747	13,766
T_0	T_1	184	184/360	12,161	12,193
T_1	T_2	181	181/360	7,055	7,078
T_2	T_3	186	186/360	-1,325	-1,329
T_3	T_4	182	182/360	-7,761	-7,779
T_4	T_5	182	182/360	-14,272	-14,286
T_5	T_6	182	182/360	-24,088	-24,053
T_6	T_7	183	183/360	-31,593	-31,448
T_7	T_8	181	181/360	-39,009	-38,679
T_8	T_9	184	184/360	-46,967	-46,352
T_9	T_{10}	181	181/360	-53,085	-52,114
T_{10}	T_{11}	184	184/360	-60,851	-59,376
T_{11}	T_{12}	182	182/360	-65,982	-63,961
T_{12}	T_{13}	186	186/360	-73,761	-70,978
T_{13}	T_{14}	179	179/360	-75,519	-72,125
T_{14}	T_{15}	185	185/360	-83,621	-79,201
T_{15}	T_{16}	182	182/360	-86,245	-80,988
T_{16}	T_{17}	182	182/360	-90,334	-84,067
T_{17}	T_{18}	182	182/360	-91,395	-84,284
T_{18}	$T_{19} = T_M$	183	183/360	-10,094,745	-9,222,017
					-10,000,000

Table 6.4: Cash flow of the floating leg

the spread for the floating leg with the lower tenor

$$\sum_{i=1}^{M} \delta_{i-1,i}^{3m} P_{6m}(t,T_i)\left[F_{3m}(t,T_{i-1},T_i) + Z(t,T_M)\right] = \sum_{j=1}^{M} \delta_{j-1,j}^{6m} P_{6m}(t,T_j) F_{6m}(t,T_{i-1},T_i)$$

$$\sum_{i=1}^{M} \delta_{i-1,i}^{3m} P_{6m}(t,T_i) F_{3m}(t,T_{i-1},T_i)$$

$$+\sum_{i=1}^{M} \delta_{i-1,i}^{3m} P_{6m}(t,T_i) Z(t,T_M) = \sum_{j=1}^{M} \delta_{j-1,j}^{6m} P_{6m}(t,T_j) F_{6m}(t,T_{i-1},T_i)$$

$$Z(t,T_M) = \frac{\sum_{j=1}^{M} \delta_{j-1,j}^{6m} P_{6m}(t,T_j) F_{6m}(t,T_{i-1},T_i) - \sum_{i=1}^{M} \delta_{i-1,i}^{3m} P_{6m}(t,T_i) F_{3m}(t,T_{i-1},T_i)}{\sum_{i=1}^{M} \delta_{i-1,i}^{3m} P_{6m}(t,T_i)}.$$

To set the present value of our trade to zero, we must add a positive spread

$$Z(t,T_M) = 9.5bp.$$

Collateralized EUR Basis Swap

We consider the same EUR basis swap as in the previous section but with a collateral agreement in place. We assume a standard EUR cash-CSA and consequently switch to OIS discounting.

Uncollateralized EUR Fixed Interest Rate Swap

Period Float Start	Period Float End	Period Fix Start	Period Fix End	Payment Float (Pay)	Discounted Payment Float	Payment Fix (Rec)	Discounted Payment Fix	Netted Payment	Discounted Netted Payment
t	T_0	t		13,747	13,766			13,747	13,766
T_0	T_1		S_0	12,161	12,193	88,908	89,139	101,070	101,332
T_1	T_2	S_0		7,055	7,078			7,055	7,078
T_2	T_3		S_1	-1,325	-1,329	89,395	89,679	88,070	88,350
T_3	T_4	S_1		-7,761	-7,779			-7,761	-7,779
T_4	T_5		S_2	-14,272	-14,286	88,665	88,750	74,393	74,464
T_5	T_6	S_2		-24,088	-24,053			-24,088	-24,053
T_6	T_7		S_3	-31,593	-31,448	88,908	88,500	57,316	57,053
T_7	T_8	S_3		-39,009	-38,679			-39,009	-38,679
T_8	T_9		S_4	-46,967	-46,352	88,908	87,745	41,941	41,392
T_9	T_{10}	S_4		-53,085	-52,114			-53,085	-52,114
T_{10}	T_{11}		S_5	-60,851	-59,376	88,908	86,753	28,057	27,377
T_{11}	T_{12}	S_5		-65,982	-63,961			-65,982	-63,961
T_{12}	T_{13}		S_6	-73,761	-70,978	89,639	86,257	15,878	15,278
T_{13}	T_{14}	S_6		-75,519	-72,125			-75,519	-72,125
T_{14}	T_{15}		S_7	-83,621	-79,201	88,665	83,977	5,044	4,777
T_{15}	T_{16}	S_7		-86,245	-80,988			-86,245	-80,988
T_{16}	T_{17}		S_8	-90,334	-84,067	88,665	82,514	-1,669	-1,553
T_{17}	T_{18}	S_8		-91,395	-84,284			-91,395	-84,284
T_{18}	T_M		S_M	-10,094,745	-9,222,017	10,088,908	9,216,685	-5,837	-5,332
					$\sum$ -10,000,000		$\sum$ 10,000,000		$\sum$ -

Table 6.5: Netted cash flow

The collateralized set-up of our basis swap is illustrated in Table 6.14 and Table 6.15.

All relevant curves are known from the previous sections. As we now change the discounting curve to EUR OIS, the spread Z which must be added to make the swap fair at initiation raises to

$$Z\left(t, T_M\right) = 9.6bp.$$

Collateralized EUR Fixed Interest Rate Swap	
Receive	**Fix**
Notional	10,000,000 EUR
Start	$t = 0$
Maturity	$T_M = 10$ years
Coupon	K
Payment Frequency	Annual
Day Count	ACT / 360
Discount Curve	EUR OIS
Forward Curve	None
Collateral Currency	EUR

Table 6.6: Fixed Leg

Collateralized EUR Fixed Interest Rate Swap	
Pay	**Float**
Notional	10,000,000 EUR
Start	$t = 0$
Maturity	$T_M = 10$ years
Index	6M EUR Swap Curve
Spread	0 bp
Reset Frequency	Semi-annual
Payment Frequency	Semi-annual
Day Count	ACT / 360
Discount Curve	EUR OIS
Forward Curve	EUR Swap Curve vs 6M
Collateral Currency	EUR

Table 6.7: Floating Leg

Term in Y	**EUR OIS Curve in % (linear interpolated)**	**Discount Factor**
0.5	-0.355	1.0018
1.0	-0.347	1.0035
1.5	-0.326	1.0050
2.0	-0.286	1.0058
2.5	-0.239	1.0061
3.0	-0.188	1.0057
3.6	-0.126	1.0045
4.1	-0.064	1.0026
4.6	0.000	1.0000
5.1	0.065	0.9967
5.6	0.130	0.9928
6.1	0.197	0.9881
6.6	0.268	0.9827
7.1	0.340	0.9764
7.6	0.406	0.9700
8.1	0.474	0.9629
8.6	0.539	0.9556
9.1	0.603	0.9478
9.6	0.664	0.9399
10.1	0.724	0.9316

Table 6.8: Relevant terms, corresponding interest rates and discount factors

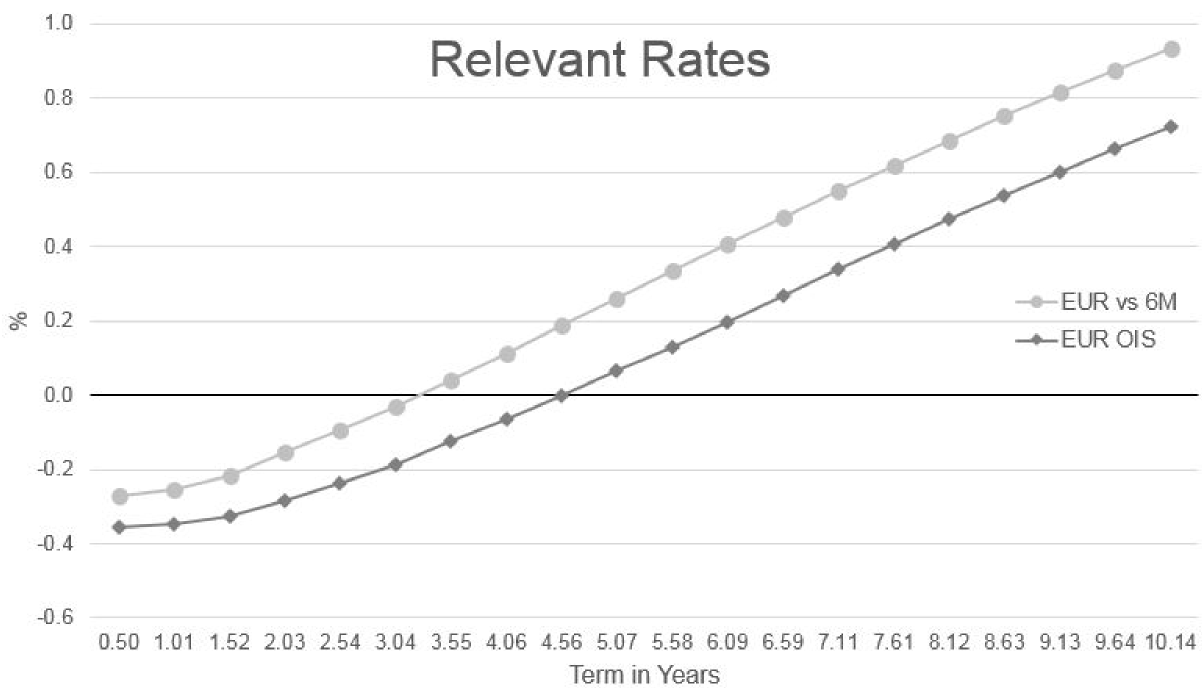

Figure 6.1: Comparison of the interpolated natural EUR Swap Curve vs EUR OIS Curve

Collateralized EUR Fixed Interest Rate Swap

Period Float Start	End	Period Fix Start	End	Payment Float (Pay)	Discounted Payment Float	Payment Fix (Rec)	Discounted Payment Fix	Netted Payment	Discounted Netted Payment
t	T_0	t		13,747	13,772			13,747	13,772
T_0	T_1		S_0	12,161	12,204	88,908	89,222	101,070	101,427
T_1	T_2	S_0		7,055	7,090			7,055	7,090
T_2	T_3		S_1	-1,325	-1,333	89,395	89,918	88,070	88,585
T_3	T_4	S_1		-7,761	-7,808			-7,761	-7,808
T_4	T_5		S_2	-14,272	-14,354	88,665	89,174	74,393	74,820
T_5	T_6	S_2		-24,088	-24,196			-24,088	-24,196
T_6	T_7		S_3	-31,593	-31,674	88,908	89,138	57,316	57,464
T_7	T_8	S_3		-39,009	-39,008			-39,009	-39,008
T_8	T_9		S_4	-46,967	-46,812	88,908	88,615	41,941	41,803
T_9	T_{10}	S_4		-53,085	-52,703			-53,085	-52,703
T_{10}	T_{11}		S_5	-60,851	-60,130	88,908	87,855	28,057	27,725
T_{11}	T_{12}	S_5		-65,982	-64,838			-65,982	-64,838
T_{12}	T_{13}		S_6	-73,761	-72,023	89,639	87,527	15,878	15,503
T_{13}	T_{14}	S_6		-75,519	-73,256			-75,519	-73,256
T_{14}	T_{15}		S_7	-83,621	-80,520	88,665	85,377	5,044	4,857
T_{15}	T_{16}	S_7		-86,245	-82,414			-86,245	-82,414
T_{16}	T_{17}		S_8	-90,334	-85,621	88,665	84,039	-1,669	-1,582
T_{17}	T_{18}	S_8		-91,395	-85,899			-91,395	-85,899
T_{18}	T_M		S_M	-10,094,745	-9,404,412	10,088,908	9,398,975	-5,837	-5,438
					$\sum$ -10,193,936		$\sum$ 10,189,839		$\sum$ -4,097

Table 6.9: Netted cash flow

Uncollateralized EUR Basis Swap

Receive	**Float**
Notional	10,000,000 EUR
Start	$t = 0$
Maturity	$T_M = 10$ years
Index	6M EUR Swap Curve
Spread	0 bp
Reset Frequency	Semi-annual
Payment Frequency	Semi-annual
Day Count	ACT / 360
Discount Curve	EUR Swap Curve vs 6M
Forward Curve	EUR Swap Curve vs 6M

Table 6.10: Floating Leg (Receive)

Uncollateralized EUR Basis Swap	
Pay	**Float**
Notional	10,000,000 EUR
Start	$t = 0$
Maturity	$T_M = 10$ years
Index	3M EUR Swap Curve
Spread	Z bp
Reset Frequency	Quarterly
Payment Frequency	Quarterly
Day Count	ACT / 360
Discount Curve	EUR Swap Curve vs 6M
Forward Curve	EUR Swap Curve vs 3M

Table 6.11: Floating Leg (Pay)

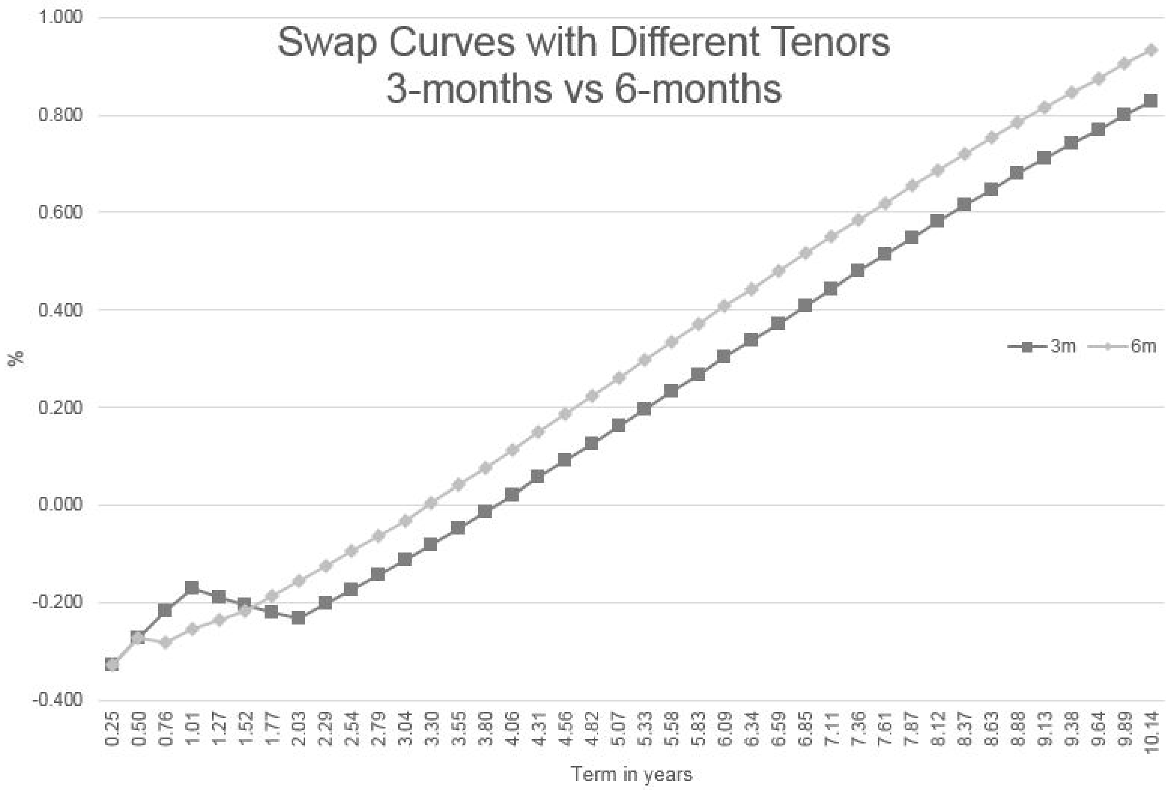

Figure 6.2: Comparison of the interpolated rates relevant for the uncollateralized EUR basis swap

Uncollateralized EUR Basis Swap

Term in years	EUR Swap Curve vs 3m in % (linear interpolated)	Forward Rates
0.25	-0.328	-0.328
0.50	-0.272	-0.216
0.76	-0.218	-0.112
1.01	-0.173	-0.038
1.27	-0.189	-0.256
1.52	-0.206	-0.289
1.77	-0.222	-0.322
2.03	-0.233	-0.308
2.29	-0.204	0.035
2.54	-0.174	0.094
2.79	-0.144	0.153
3.04	-0.114	0.221
3.30	-0.081	0.325
3.55	-0.047	0.392
3.80	-0.013	0.459
4.06	0.021	0.531
4.31	0.056	0.616
4.56	0.090	0.685
4.82	0.126	0.753
5.07	0.161	0.824
5.33	0.197	0.899
5.58	0.232	0.967
5.83	0.267	1.034
6.09	0.303	1.098
6.34	0.338	1.156
6.59	0.373	1.221
6.85	0.408	1.286
7.11	0.444	1.348
7.36	0.478	1.408
7.61	0.512	1.469
7.87	0.548	1.530
8.12	0.581	1.562
8.37	0.614	1.587
8.63	0.647	1.642
8.88	0.679	1.697
9.13	0.711	1.707
9.38	0.741	1.716
9.64	0.771	1.765
9.89	0.801	1.813
10.14	0.830	1.802

Table 6.12: Relevant terms and corresponding (forward) rates of the EUR Swap Curve tied to 3m-tenor

Uncollateralized EUR Basis Swap

Days	Acc. Days	Rec	Disc. Rec	Pay	Disc. Pay	Netted	Disc. Netted
91	91	0	0	8,301	8,308	8,301	8,308
90	181	-13,747	-13,766	5,400	5,407	-8,347	-8,359
92	273	0	0	2,870	2,876	2,870	2,876
92	365	-12,161	-12,193	970	973	-11,191	-11,220
91	456	0	0	6,463	6,483	6,463	6,483
90	546	-7,055	-7,078	7,215	7,239	160	161
92	638	0	0	8,223	8,251	8,223	8,251
94	732	1,325	1,329	8,051	8,076	9,376	9,406
91	823	0	0	-873	-876	-873	-876
91	914	7,761	7,779	-2,375	-2,381	5,386	5,399
91	1005	0	0	-3,876	-3,883	-3,876	-3,883
91	1096	14,272	14,286	-5,590	-5,595	8,682	8,690
91	1187	0	0	-8,209	-8,207	-8,209	-8,207
91	1278	24,088	24,053	-9,903	-9,889	14,185	14,165
91	1369	0	0	-11,592	-11,559	-11,592	-11,559
92	1461	31,593	31,448	-13,562	-13,500	18,031	17,948
91	1552	0	0	-15,579	-15,478	-15,579	-15,478
90	1642	39,009	38,679	-17,117	-16,973	21,892	21,706
92	1734	0	0	-19,247	-19,041	-19,247	-19,041
92	1826	46,967	46,352	-21,068	-20,792	25,899	25,560
91	1917	0	0	-22,731	-22,376	-22,731	-22,376
90	2007	53,085	52,114	-24,172	-23,730	28,913	28,384
92	2099	0	0	-26,435	-25,875	-26,435	-25,875
92	2191	60,851	59,376	-28,063	-27,383	32,788	31,993
91	2282	0	0	-29,230	-28,430	-29,230	-28,430
91	2373	65,982	63,961	-30,871	-29,925	35,111	34,036
92	2465	0	0	-32,864	-31,744	-32,864	-31,744
94	2559	73,761	70,978	-35,205	-33,876	38,557	37,102
91	2650	0	0	-35,588	-34,117	-35,588	-34,117
88	2738	75,519	72,125	-35,902	-34,288	39,617	37,837
94	2832	0	0	-39,953	-37,998	-39,953	-37,998
91	2923	83,621	79,201	-39,486	-37,399	44,135	41,802
91	3014	0	0	-40,104	-37,824	-40,104	-37,824
91	3105	86,245	80,988	-41,512	-38,981	44,734	42,007
91	3196	0	0	-42,902	-40,105	-42,902	-40,105
91	3287	90,334	84,067	-43,149	-40,156	47,185	43,912
91	3378	0	0	-43,373	-40,183	-43,373	-40,183
91	3469	91,395	84,284	-44,607	-41,137	46,787	43,147
91	3560	0	0	-45,825	-42,060	-45,825	-42,060
92	3652	10,094,745	9,222,017	-10,046,056	-9,177,537	48,689	44,480
		$\sum$	10,000,000	$\sum$	-9,905,685	$\sum$	94,315

Table 6.13: Netted cash flow

Collateralized EUR Basis Swap

Receive	**Float**
Notional	10,000,000 EUR
Start	$t = 0$
Maturity	$T_M = 10$ years
Index	6M EUR Swap Curve
Spread	0 bp
Reset Frequency	Semi-annual
Payment Frequency	Semi-annual
Day Count	ACT / 360
Discount Curve	EUR OIS
Forward Curve	EUR Swap Curve vs 6M

Table 6.14: Floating Leg (Receive)

Collateralized EUR Basis Swap

Pay	**Float**
Notional	10,000,000 EUR
Start	$t = 0$
Maturity	$T_M = 10$ years
Index	3M EUR Swap Curve
Spread	Z bp
Reset Frequency	Quarterly
Payment Frequency	Quarterly
Day Count	ACT / 360
Discount Curve	EUR OIS
Forward Curve	EUR Swap Curve vs 6M

Table 6.15: Floating Leg (Pay)

Chapter 7

Market Evolution

In Chapter 5 several (theoretical) (benchmark) curves observable in the market were assumed to be given. Currently, there are discussions ongoing what happens if benchmark curves are discontinued. This chapter is mainly based on [1] and [6].

> Libor is a sick man.
>
> *What to do about Libor?, Darrel Duffie, Risk.net, April 2017*

Since April 2013 UK's Financial Conduct Authority (FCA)[1] is responsible for the regulation of Libor. Currently, there are trillions of dollars of derivatives and other financial contracts, with maturities up to 30 years or more, tied to this important reference rate.

In July 2017, Chief Executive Andrew Bailey announced the discontinuation of Libor by the end of 2021. The panel banks are struggling with the Libor calculation based on real trades, which leads to Libor's transformation into an expert judgment rather than an actual benchmark. The interbank unsecured lending is not anymore adequately used by market participants. Inherently, benchmark curves should be based on market quotes to preserve against manipulation.

Libor previously was discontinued for certain currencies. Hence, there already exist experiences and consequently recommendations how to deal with contractual amendments or terminations. The long-term plan to end Libor and the parallel development of a successor which is supposed to be calculated based on actual transactions mitigates disruptions in the financial market. Nevertheless, the transition will demand regulatory changes and further guidelines.

As the whole financial system is rather dependent on Libor, market participants and regulators plan to move ties to other benchmarks. As the most likely contenders are seen overnight interest rates, which is a logical consequence since the trading volumes of USD overnight swaps has increased. From recent discussions in the UK, two different traded financial instruments, that would underlie the future benchmark, emerge. On the one hand unsecured overnight borrowing and lending rates and on the other hand government bond collateralized repos[2]. In the meantime, the Alternative Reference Rates Committee (ARRC)[3] rather tends to the Secured Overnight Reference Rate as a solid substitute for USD Libor. This new USD derivative reference rate is supposed to be published by the Federal Reserve Bank of New York and the Office of Financial Research.

[1]https://www.fca.org.uk/

[2]See Appendix D.

[3]https://www.newyorkfed.org/arrc/

Although, the size of the Euribor panel is currently shrinking, the discussions replacing it are in arrears. At the moment Euro-zone authorities favor Euribor's maintenance.

In contrast with the volume-growth of USD OIS, the volumes of transactions on which the calculation of Eonia is based have decreased, see Figure 7.1. Also in Europe, unsecured lending and borrowing in the interbank market experienced diminishing liquidity as a result of regulatory revision and ECB's monetary policies. In September 2017, ECB announced the development of a EUR unsecured overnight interest rate. It will be calculated based on trades in EUR. The transactions will be taken from the already established money market statistical reporting (MMSR) database. ECB's new risk-free benchmark is going to be provided officially by 2020 and is meant to be seen as a supplement to existing reference rates. Furthermore, the European Money Markets Institution (EMMI) plans to roll out a EUR repo based benchmark rate in 2018. EMMI reports that market participants think a repo based benchmark rate should be used as discounting curve for bond collateralized swaps, while they prefer Eonia for cash collateralized EUR swaps.

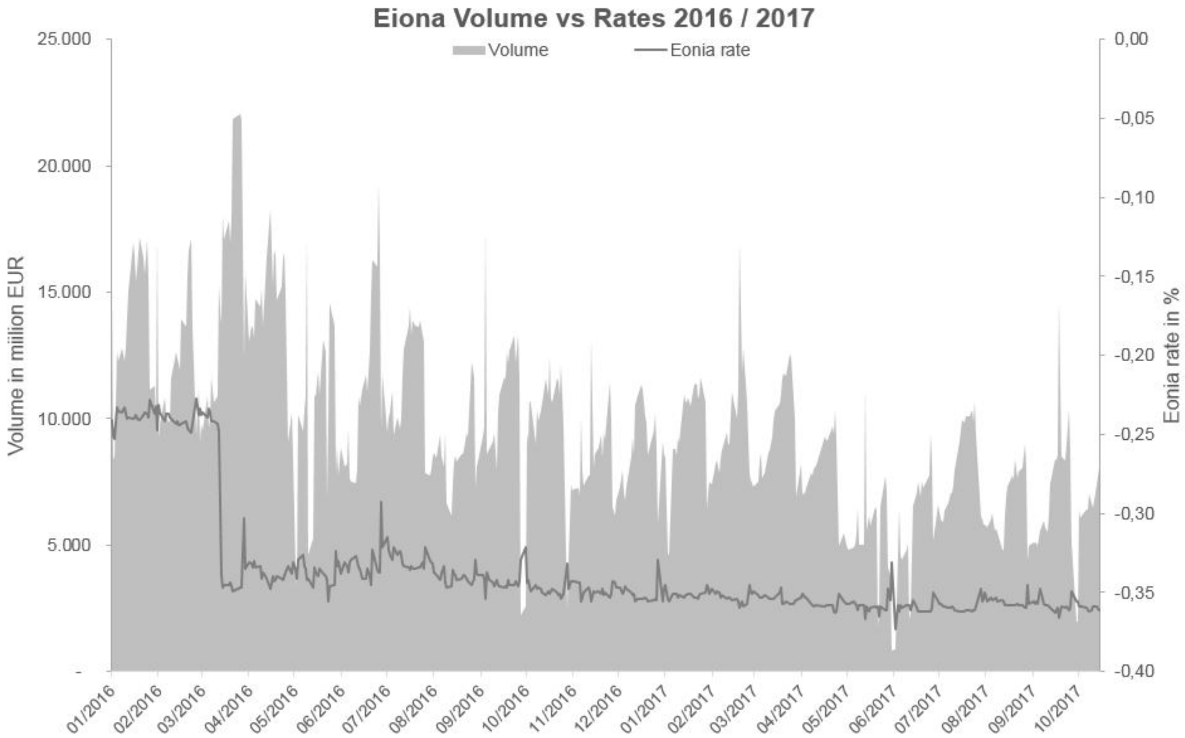

Figure 7.1: *Source: EMMI*; the decline of around 10bps in March 2016 is associated to the decision of the Governing Council to cut of the deposit facility rate.

The OIS Discount Rate Challenged

As already discussed, the choice of the discounting curve has an impact on the price of financial instruments. In Section 5.2 was mentioned, that also the change to OIS as discount rate reflects the costs of funding in terms of daily collateral posting, [11].

> In theory, the discount rate - a key valuation input on swaps trades - should match the interest rate a collateral receiver pays to the poster.
>
> *The price is still wrong: banks tackle bond CSA discounting*
> *Nazneen Sherif, Risk Magazine July 2017*

In the previous chapters collateralized swaps were always regarded as cash collateralized swaps.

Considering a trade which is bond-collateralized under a CSA, costs arise from receiving a security. If a swap counterparty has a positive exposure and is therefore receiving bonds, there exist two possibilities

- reuse the bonds and/or
- repo out the bonds.

Assuming a positive repo rate, the bank may sell the bond at the repo rate (which equals OIS plus the repo spread) to receive cash. Hence, the cash is now in the money market at the OIS rate. This interest rate is transfered as costs to the bond-collateral posting counterparty in the swap. From the perspective of the bond-collateral receiver the netted costs of receiving a bond are

pay	$-$ (OIS + repo spread)	Repo Market
receive	+ (OIS)	Money Market
netted	$-$ repo spread	Costs

the repo spread. The discounting rate in a swap should represent the collateral rate. It has been previously assumed as the OIS rate and now raises by the repo spread from the OIS rate to the repo rate. As the difference between the OIS rate and repo rate had been very small for a very long period in the past, market participants sticked to OIS discounting under a bond-CSA.

Since the ECB started their QE programme (Quantitative Easing) in 2015, European government bonds turned inaccessible for cash investors and as a result, the European government repo rate drifted into negativity and currently ranges below Eonia, see Figure 7.2.

Furthermore, the demand for repos decreased due to regulatory requirements of the Basel Committee on Banking Supervision. This reduction combined with ECB's monetary policy causes that government bond repos trade far below Eonia. Consequently, pricing differences definitely exist.

Applying this understanding to a fair interest rate swap pricing under a bond CSA leads to the need of a maturity matching repo rate. In other words, if the swap has a maturity of 20 years, there must be calculated a 20-year-repo rate for discounting the last cash flow. The fact repo rates are only liquid enough for building a curve for terms of up to a few months requires extrapolation. Indeed, to repo out a bond to maturities over a year is unfeasible in the market. This assumption would result in non-significant extrapolated rates, respectively.

Current Topics Not Covered In This Master's Thesis

In this thesis are not mentioned the following current challenges, (European) derivatives markets participants are facing.

Clearing under EMIR European Markets Infrastructure Regulation (EMIR) is a policy introduced by the European Securities and Markets Authorities (ESMA). It regulates which classes of OTC derivative contracts have to be centrally cleared through Central Counterparty Clearing (CCP) and defines risk mitigation techniques for not centrally cleared contracts. Furthermore, EMIR governs the requirements for clearing houses / CCPs.

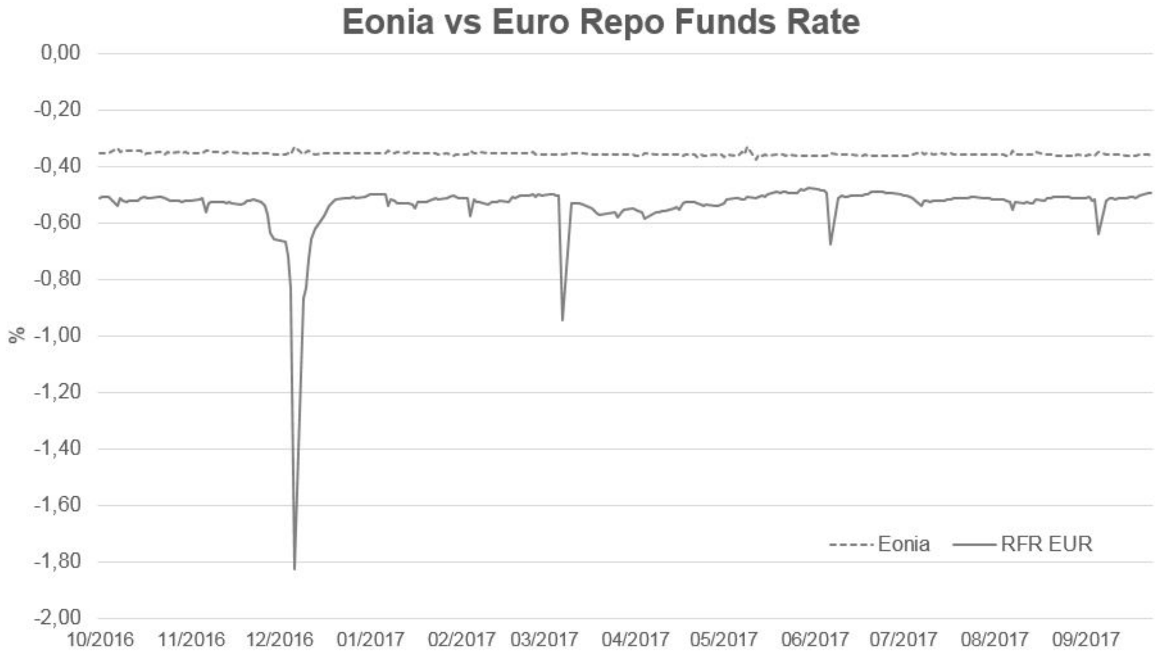

Figure 7.2: *Source EMMI and RepoFundsRate*

Brexit After the referendum, London is questioned to be headquarter of global banks. Some of them are already preparing to move, in particular, one reason is that it is very likely London based CCPs lose the ability to clear EUR swaps. As a consequence, there could happen a market fragmentation. This might lead to lower liquidity in derivatives markets. Moreover, the European Banking Authority (EBA), as part of the EU supervision system, has to leave London, for obvious reasons.

MiFID II/MiFIR Markets in Financial Instruments Directive / Markets in Financial Instruments Regulation (MiFID II/MiFIR) is effective from 3 January, 2018. The regulation includes, among other things, trading obligations for OTC derivatives, which are related to clearing under EMIR. In particular, MiFID II requires price transparency of swaps.

IFRS 9/IAS 39 The accounting standard IFRS 9, which is effective for business years beginning on or after January 1, 2018, replaces the old standard IAS 39 which was criticized for its complexity. Derivatives are classified as "fair value through profit and loss" financial assets under IFRS 9 and have to be measured accordingly.

Valuation Adjustments (XVA) Regulation challenges the calculation of XVA. Besides credit valuation adjustment (CVA) and debit valuation adjustment (DVA), the measure for counterparty credit risk and the own default risk, new measures such as funding valuation adjustment (FVA), margin valuation adjustment (MVA) and capital valuation adjustment (KVA) have to calculated and reported, as well as charged.

For further research on the above mentioned topics, I highly recommend the official ESMA homepage and the Risk Magazine.

Chapter 8

Conclusion

My intention to write about the multi-curve framework, was to understand where exactly the differences between single-curve and multi-curve pricing formulas are. I did not expect, that in fact the most important factor for pricing is the discounting curve, as we can see in the direct comparison in Table 4.1 at the end of Chapter 4.

Why does it make sense to use the OIS curve for discounting, besides the fact that it is the closest to risk-free rate observable in the market? The surprisingly simple explanation is the funding-curve approach as described in Section 4.1. Assuming a daily cash collateral exchange perfectly explains that funding is linked to an overnight rate since the collateral amount must be funded over night.

In Chapter 3, I present a swap exposure and the corresponding collateral cash flows under a CSA. I assumed that the portfolio is perfectly collateralized. Moreover, I introduced the event of default.

As we have seen in Chapter 6, the choice of discounting curve makes a difference in terms of actual money.

We considered a fixed interest rate swap in a single currency, where activating the OIS discounting makes a marginal difference of less than one basis point, in our example.

Furthermore, we calculated a spread which must be added to the leg tied to the lower tenor in an uncollateralized single currency basis swap. In terms of real money, we came to an amount of over EUR 90,000 from a swap notional of EUR 10mm, which is, obviously, a non-negligible amount.

Bibliography

[1] Andrew Bailey *The Future of LIBOR*
Speech delivered on July 27, 2017
https://www.fca.org.uk/news/speeches/the-future-of-libor

[2] Marco Bianchetti *Two Curves, One Price*
Risk Magazine August 2010
https://www.risk.net/derivatives/interest-rate-derivatives/1724866/two-curves-one-Price

[3] Marco Bianchetti and Mattia Carlicchi *Interest Rates After the Credit Crunch: Multiple-Curve Vanilla Derivatives and SABR*
Version March 2012
https://arxiv.org/ftp/arxiv/papers/1103/1103.2567.pdf

[4] Marco Bianchetti and Mattia Carlicchi *Markets Evolution After the Credit Crunch*
Version December 2012
https://arxiv.org/ftp/arxiv/papers/1301/1301.7078.pdf

[5] Damiano Brigo and Fabio Mercurio *Interest Rate Models - Theory and Practice*
2nd edition, Springer, corrected 3rd printing, 2007

[6] Darrel Duffie *What to do about Libor?*
Article published on April 24, 2017
https://www.risk.net/comment/5047281/what-to-do-about-libor

[7] Damir Filipovic *Term Structure Models*
Springer Finance, 2009

[8] Masaaki Fujii, Yasufumi Shimada and Akihiko Takahashi *A Market Model of Interest Rates with Dynamic Basis Spreads in the Presence of Collateral and Multiple Currencies*
Version December 2009
https://ssrn.com/abstract=1520618

[9] Masaaki Fujii, Yasufumi Shimada and Akihiko Takahashi *A Note on Construction of Multiple Swap Curves with and without Collateral*
FSA Research Review March 2010
http://www.fsa.go.jp/frtc/nenpou/2009/07-1.pdf

[10] Massimo Morini *Solving the Puzzle in the Interest Rate Market (Part 1 & 2)*
Version October, 2009
http://ssrn.com/abstract=1506046

[11] Nazneen Sherif *The price is still wrong: banks tackle bond CSA discounting*
Risk Magazine July 2017
https://www.risk.net/derivatives/5299256/the-price-is-still-wrong-banks-tackle-bond-csa-discounting

[12] Bruce Tuckman and Pedro Porfirio *Interest Rate Parity, Money Market Basis Swaps, and Cross-Currency Basis Swaps*
Lehman Brothers, Fixed Income Liquid Markets Research, LMR Quarterly, vol. 2004-Q2, June 2003

Appendix A

Day-Count Convention

Depending on country and the type of the financial instrument conventions for the calculation of interest vary. A day-count convention in a certain market defines the number of days between two dates and apply to swaps, FRAs and bonds. In general the accrued interest of a period can be calculated as

$$\text{interest amount} = \text{notional amount} \cdot \text{interest rate} \cdot \frac{\text{number of days of the term}}{\text{day basis}}$$

There are three different methods to determine the number of days:

1. *ACT-method* (actual-method), where the actual number of days within the interest period are counted

$$\begin{array}{lcl} \text{1 May to 30 May} & \Longrightarrow & \text{29 days } (= 30 - 1) \\ \text{1 May to 31 May} & \Longrightarrow & \text{30 days } (= 31 - 1) \\ \text{1 May to 1 June} & \Longrightarrow & \text{31 days } (= \text{ the actual number of days in May}) \end{array}$$

2. *30-method*, where each month counts 30 days; if the period is shorter than one month, the remaining days to the end-of-month are subtracted from 30

$$\begin{array}{lcl} \text{1 May to 30 May} & \Longrightarrow & \text{29 days } (= 30 - 1) \\ \text{1 May to 31 May} & \Longrightarrow & \text{30 days } (= 31 - 1) \\ \text{1 May to 1 June} & \Longrightarrow & \text{30 days } (= \text{one month counts 30 days}) \end{array}$$

3. *30E-method*, where each month is treated as a 30-days-month; if the term of interest is shorter than one month, the remaining days to the end of the month are subtracted from 30

$$\begin{array}{lcl} \text{1 May to 30 May} & \Longrightarrow & \text{29 days } (= 30 - 1) \\ \text{1 May to 31 May} & \Longrightarrow & \text{29 days } (= (31 \mapsto 30) - 1) \\ \text{1 May to 1 June} & \Longrightarrow & \text{30 days } (= \text{May treated as one 30-day-month counts 30 days}) \end{array}$$

To set the day basis there are also three methods:

1. *360-method*, where one year (contractual annual term) is assumed to have 360 days

$$\begin{array}{lcl} \text{Monday, 1 July 2041 to Tuesday, 1 July 2042 are 365 days} & \to & \text{360 days} \\ \text{Friday, 1 July 2044 to Monday, 3 July 2045 are 367 days} & \to & \text{360 days} \end{array}$$

2. *365-method*, where one year (contractual annual term) is assumed to have 365 days - analoguous to the 360-method

Monday, 1 July 2041 to Tuesday, 1 July 2042 are 365 days	$\rightarrow$	365 days
Friday, 1 July 2044 to Monday, 3 July 2045 are 367 days	$\rightarrow$	365 days

3. *ACT-method*

   (a) *ISDA-method*: in the money market the actual days per year are counted; if a one year contract runs over two calender years, the interest period is split into two parts, e.g. Monday, 5 January 2015 to Tuesday, 5. January 2016 applying ACT/ACT-method

$$\begin{aligned}\text{interest amount} &= \text{notional amount} \cdot \text{interest rate} \cdot \frac{\text{number of days of the term}}{\text{day basis}} \\ &= \text{notional amount} \cdot \text{interest rate} \cdot \left(\frac{360}{365} + \frac{5}{366}\right)\end{aligned}$$

   (b) *ISMA-method*: in the capital market one is the counted as the actual days of the interest period; e.g. semi-annual interest payments, where the interest period is 184 days (1 May to 1 November) the day basis is 368 days (= $2 \cdot 184$ days)

$$\text{interest amount} = \text{notional amount} \cdot \text{interest rate} \cdot \frac{184}{368}$$

Nine combinations of the day count conventions are possible, whereof five are used in practice

- 30/360
- 30E/360
- ACT/360
- ACT/365
- ACT/ACT

In the tables below are the some important conventions listed, although they can vary dependent on certain specifications of the financial instrument, domestic market, ...

Money market	
Australia	ACT/360
Euro	ACT/360
Great Britain	ACT/365
Hong Kong / Singapure	ACT/365
Japan	ACT/360
Norway	ACT/360
Poland	ACT/365
Sweden	ACT/360
Switzerland	ACT/360
USA	ACT/360

Capital market	
Euro	ACT/ACT
Great Britain	ACT/ACT semi-annual
Japan	30/360 or ACT/ACT
Sweden	30/360 or 30E/360
Switzerland	30/360 or 30E/360
USA	30/360 or ACT/ACT

Appendix B

Benchmark Curves

To get a feeling of benchmark curves used in the market, there is listed a selection.

London Interbank Offered Rate (Libor)

The London Interbank Offered Rate (Libor) is an interest rate which is fixed on a daily basis by the world's most credit-worthy banks. The Libor provides an indication of the average rate at which a Libor contributor bank can obtain unsecured funding in the London interbank market for a given period, in a given currency[1]. Currently, Libor rates are published daily at approximately 11:55 a.m. (London local time) for currencies

- CHF (Swiss Franc),
- EUR (Euro),
- GBP (Pound Sterling),
- JPY (Japanese Yen),
- USD (US Dollar),

with tenors

- Overnight,
- 1 Week,
- 1 Month,
- 2 Months,
- 3 Months,
- 6 Months,
- 1 Year.

Libor is used as a benchmark rate for short term interest rates.

[1] https://www.theice.com/iba/libor

Euro Interbank Offered Rate (Euribor)

The Euro Interbank Offered Rate (Euribor) is an interest rate which is calculated as an average of European banks' borrow funds quotes. The Euribor rates are important reference rates for all kinds of financial products in the European market[2]. Euribor and Libor are comparable in terms of that they both are benchmark rates. Euribor rates are published at approximately 11:00 a.m. (CET). Currently the following maturities are provided:

- 1 Week
- 2 Weeks
- 1 Month
- 2 Months
- 3 Months
- 6 Months
- 9 Months
- 1 Year

Euro OverNight Index Average (Eonia)

The Eonia rate is an average of the Euro zone banks' provided rates for a lending term of one day in the interbank market. It can be interpreted as the Overnight Euribor rate. The calculation agent for Eonia is the European Central Bank (ECB)[3]. The officially publisher is European Money Market Institute (EMMI)[4].

Sterling OverNight Index Average (Sonia)

Similar to the Eonia rate, Sonia represents bank and building societies' funding rates for a period of one day in the Sterling market. It is used as a benchmark rate for Sterling financial products and was referred to as the near risk-free interest rate benchmark[5].

[2] http://www.euribor-rates.eu/
[3] https://www.ecb.europa.eu/
[4] https://www.emmi-benchmarks.eu/
[5] Press Release 28 April 2017 (Bank of England)

Appendix C

Measure and Numeraire

For completeness, we define forward measures and the association to a numeraire, what we take as given in this thesis. The following proposition as well as the proof is from [5].

The measure $\mathcal{Q}$ associated to a T-maturing bond $P(t,T)$ is denoted as $\mathcal{Q}^T$ the T-forward measure. We denote the price of a derivative at time t as Π_t and Π_T the corresponding payoff at maturity T. For $t \leq T$ the following equation holds

$$\Pi_t = P(t,T)\,\mathbb{E}^{\mathcal{Q}^T}\left[\Pi_T | \mathcal{F}_t\right],$$

where $\mathcal{F}_t$ denotes the market information up to time t.

Proposition The simple compounded forward rate $F(t,T_{i-1},T_i)$ is a martingale under the forward measure $\mathcal{Q}^{T_i}$.

$$F(t_0,T_{i-1},T_i) = \mathbb{E}^{\mathcal{Q}^{T_i}}\left[F(t,T_{i-1},T_i)\,|\mathcal{F}_{t_0}\right],$$

where $t_0 \leq t \leq T_{i-1} < T_i$. The expectation under $\mathcal{Q}^{T_i}$ of the future simple compounded spot interest rate R for the time interval $(t \leq)\,T_{i-1}$ to T_i is the forward rate at time t

$$F(t,T_{i-1},T_i) = \mathbb{E}^{\mathcal{Q}^{T_i}}\left[R(T_{i-1},T_i)\,|\mathcal{F}_t\right],$$

where $t \leq T_{i-1} < T_i$.

Proof As per definition the simple compounded forward rate is

$$F(t,T_{i-1},T_i) = \frac{1}{\delta_{i-1,i}}\left[\frac{P(t,T_{i-1})}{P(t,T_i)} - 1\right],$$

where $\delta_{i-1,i} := \delta(T_{i-1},T_i)$ denotes the year fraction between the future dates $T_{i-1} < T_i$. Hence, we see that

$$F(t,T_{i-1},T_i)\,P(t,T_i) = \frac{1}{\delta_{i-1,i}}\left[P(t,T_{i-1}) - P(t,T_i)\right],$$

is the price at time t of a bond represented as the difference of two bonds with maturities T_{i-1} and T_i. We see that the forward rate is a martingale under the T_i-forward measure, which is, as per definition, associated to numeraire $P(t, T_i)$

$$\begin{aligned}
\frac{F(t, T_{i-1}, T_i)}{P(t, T_i)} &= \mathbb{E}^{\mathcal{Q}^{T_i}}\left[\frac{F(t, T_{i-1}, T_i)}{P(t, T_i)}|\mathcal{F}_t\right] \\
&= \frac{F(t, T_{i-1}, T_i)}{P(t, T_i)} \\
F(t, T_{i-1}, T_i) &= \frac{F(t, T_{i-1}, T_i)\, P(t, T_i)}{P(t, T_i)}.
\end{aligned}$$

Since, $F(T_{i-1}, T_{i-1}, T_i) = R(T_{i-1}, T_i)$, we proofed the second equation of the proposition. $\square$

Appendix D

Repo

A repo (Sale and Repurchase agreement) is a money market instrument which can be understood as a secured loan, basically. Under the agreement the seller does not only repo out securities (which is the wording to describe the act of selling a security under a repo) but also agrees to repurchase (in other words to buy back) the same (or a similar) security from the buyer at a stipulated date in the future. The seller, who uses the cash over the term of the repo, pays back the original sum of money plus interest. There is consequently an initial and a final transaction under an agreement.

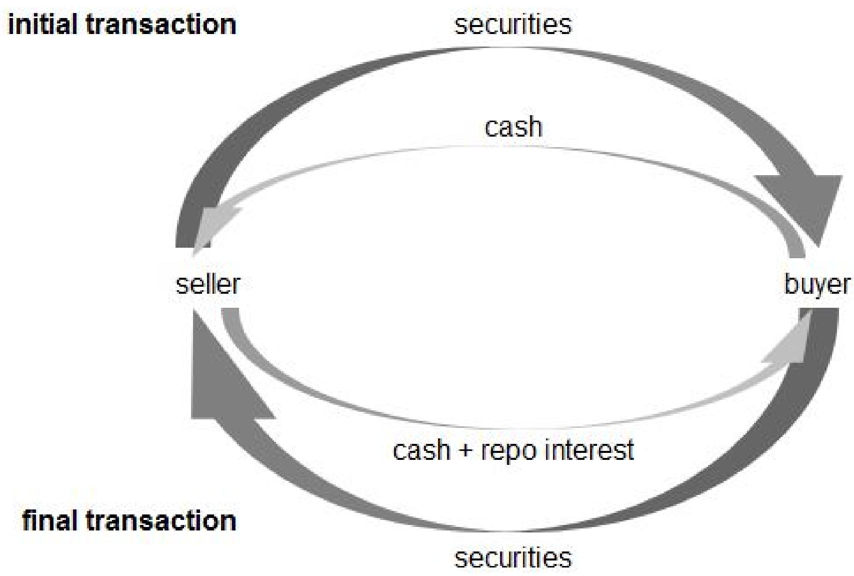

Figure D.1: Simplified representation of a repo

Hence, a repo can be regarded as either

- borrowing / lending a loan on a secured basis (cash-driven repo) or
- borrowing / lending securities against cash (security-driven repo).