DIPLOMARBEIT AUS FINANZ- UND VERSICHERUNGSMATHEMATIK

Energy Futures

Ausgeführt am Institut für
Stochastik und Wirtschaftsmathematik
der Technischen Universität Wien

unter Anleitung von

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Wien, Dezember 4, 2017

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Ich erkläre hiermit an Eides statt, dass die vorliegende Arbeit nach den anerkannten Grundsätzen für wissenschaftliche Abhandlungen von mir selbstständig erstellt wurde. Alle verwendeten Hilfsmittel, insbesondere die zugrunde gelegte Literatur, sind in dieser Arbeit genannt und aufgelistet.

Das Thema dieser Arbeit wurde von mir bisher weder im In- noch Ausland einer Beurteilerin/einem Beurteiler zur Begutachtung in irgendeiner Form als Prüfungsarbeit vorgelegt.

Wien, Dezember 4, 2017

Abstract

We develop a model of Heath-Jarrow-Morton type for the forward price of energy that is driven by a d-dimensional Lévy process. The specification of the forward yields a representation of the futures price of electricity as well. Moreover, a finite dimensional realization for the forward in Musiela parametrisation is assumed. The model is then fitted using base-load prices of ®PHELIX futures contracts traded on the European Energy Exchange. This calibration is performed using both ordinary and weighted least square methods and the results are compared. Moreover, possible applications of the model are given. Those comprise the simulation from the presented model and the estimation of market covariance structures.

Kurzfassung

Die vorliegende Arbeit beschäftigt sich mit der Entwicklung eines Heath-Jarrow-Morton-Modells für den Forward-Preisprozess von Strom, das von einem d-dimensionalen Lévy-Prozess getrieben wird. Aus der Spezifikation des Forwards resultiert eine Darstellung für den Energie-Futures Preis. Weiters wird der Forward im Sinne von Musiela umparametrisiert und eine angemessene endlich dimensionale Realisierung für diesen Prozesses angenommen. Die darauf folgende Modellkalibrierung geschieht auf Basis von ®PHELIX Base-Load Futures Kontrakten, die an der Europäischen Strombörse (EEX) gehandelt werden. Diese Justierung wird sowohl anhand der gewöhnlichen als auch gewichteten Methode der kleinsten Fehlerquadrate durchgeführt und deren Resultate verglichen. Zudem werden mögliche Anwendungen des vorgestellten Modells diskutiert. Hierzu zählen die Simulation aus dem Modell sowie die Bestimmung von Kovarianzstrukturen im Markt.

Acknowledgements

I would above all like to thank my thesis advisors Univ.Prof. Dipl.-Math. Dr.rer.nat. Thorsten Rheinländer and Univ.Ass. Dipl.-Math. Dr.rer.nat. Paul Krühner for being available at all times to answer my questions. The input I received was very valuable and is greatly appreciated!

A very special gratitude goes out to ©ENERGIEALLIANZ Austria GmbH for showing great interest in the work and providing the necessary data. Moreover, I like to thank Dr. Elke Moser who made the collaboration very simple and successful and never hesitated to offer help.

Contents

Introduction

A convenient way to model forwards and swaps in fixed income markets is to directly specify the forward rate dynamics. This approach, originating from Heath Jarrow and Morton [6] has already been applied to commodity markets as well (see i.a. [2]). In the present thesis, we like to present such a HJM modelling approach in the context of energy markets.
The object of interest in this thesis is the futures price of energy. More precisely, it is the price specified in a futures contract at some time t which is to be paid for the delivery of electricity throughout a future period $[T_1, T_2]$, where $t < T_1 \leq T_2$. We will see however, that the futures price may be obtained integrating over the (weighted) forward price of energy. This forward price is to be understood as the payment agreed upon in time t that is to be made on a fixed future date $T > t$ by some agent in exchange for receiving one MWh of energy. Hence, to obtain a model for the futures price we make use of the HJM methodology and employ such a model for the forward price of electricity. To incorporate possible jumps, its dynamics is modelled as diffusion driven by a d-dimensional Lévy process with independent components.
For the ensuing model calibration, we consider the following: first, we work with the Musiela parametrization of the forward curves to guarantee their comparability in time. Second, in order to adjust the model to finitely many observed prices, we give a suitable vector space as finite dimensional realization for the forward curves. The vector space we suggest exhibits a number of convenient properties. In particular we find that the futures price in Musiela parametrization can be nicely linked to the re-parametrized forward which aids at adapting the model parameters to the observed data.
The fitting in the FDR setting is then performed based on observed base-load prices of ®PHELIX futures contracts traded on the EEX in the period of 02.01.2012 to 14.04.2017. Those (financially settled) futures contracts exhibit the greatest liquidity among futures contracts (of the same settlement type) in Europe. We present two different estimation methods and compare their results. Thereafter, possible applications of the model are discussed. First, we display a method of simulating the forward price. Second, we briefly consider the task of quantifying the market covariance structure. In further consequence, a model as presented in this thesis can be used to price or hedge options on futures contracts. This however is beyond the scope of discussion of the present thesis.

Notation

Throughout this thesis we mainly work on $\mathbb{R}^d$ and write $\mathcal{B}(\mathbb{R}^d)$ for the corresponding Borel σ-algebra. We write $\mathbb{R}_+$ for the non-negative real numbers $[0, \infty)$. Furthermore, we denote by $\langle \cdot | \cdot \rangle$ and $\| \cdot \|$ the scalar product respectively some arbitrary norm on $\mathbb{R}^d$.

Chapter 1

The Model

In this chapter we establish a suitable modelling framework that will serve as basis throughout this master thesis. We do so by first stating the forward price dynamics of power which will then specify the energy futures prices as well. In order to later be able to conveniently calibrate the model that is set up in Sections 1.1-1.2 to observed market prices, we re-parametrize both the forward and futures price-curves and state a fitting finite-dimensional vector space for the former. This will be covered in Section 1.3.

1.1 The forward dynamics

Pursuing a Heath-Jarrow-Morton-approach, we directly model the dynamics of the energy forward. To capture possibly large jumps that characterize the spot and should be mirrored by the forward price of electricity, we decide upon finite dimensional Lévy processes as noise to the forward dynamics. For a short recap on characteristics of Lévy processes and further results on them referred to in this thesis, see appendix Chapter A. Hereafter, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ be the underlying probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ where $\mathbb{Q}$ denotes a risk-neutral measure[1]. Let $L = (L_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Lévy process with characteristic triplet (γ, Σ, ν) and independent components. Assume further that for any $t \geq 0$ L_t has finite first moment, i.e. $\mathbb{E}\left[\|L_t\|\right] < \infty$. Denoting the maximal delivery time in the market by $\tau \in \mathbb{R}_+$, we define the forward price at time t for delivery at the fixed time $T > t$ via

$$f(t,T) = f_0 + \int_0^t \beta(s,T)\,ds + \int_0^t \sigma(s,T)\,dL_s, \qquad 0 \leq t < T \leq \tau$$

or in differential notation

$$df(t,T) = \beta(t,T)\,dt + \sum_{i=1}^{d} \sigma_i(t,T)\,dL_t^i, \qquad 0 \leq t < T \leq \tau \tag{1.1}$$

where

- $f_0 = f(0,T)$ denotes the forward curve currently observed on the market

[1] Note that a risk-neutral measure $\mathbb{Q}$ is, inter alia, characterized by the fact that the price process of any tradeable asset is a local-$\mathbb{Q}$-martingale. Usually in commodities markets, this includes the corresponding spot price process too. However, electricity can not yet be efficiently stored and therefore not be considered a tradeable asset. Hence, up to now, a risk-neutral measure may be any measure equivalent to the market measure.

- $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$
- $\sigma_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ bounded and predictable, $i = 1, \dots, d$

For details on stochastic integrals based on Lévy processes as in (1.1) we refer to [8] Chapter 23. See also appendix Section A.1 for a short overview of the approach to define the stochastic integral for arbitrary semimartingales presented therein.
As mentioned earlier, we directly state the risk-neutral dynamics of the forward rate. In what follows, we pose certain conditions on the drift to ensure the martingale property of the forward rate f for fixed maturity T. This in return will be sufficient for price processes of the energy futures, that is the actually traded objects, $F(t, T_1, T_2) = \int_{T_1}^{T_2} \omega(u, T_1, T_2) f(t, u)\, du$ (cf. Section 1.2) to be martingales under $\mathbb{Q}$.
Recall that by the Lévy-Itô decomposition any Lévy process with characteristic triplet (γ, Σ, ν) can be decomposed into the sum of four independent Lévy processes via

$$L_t = \gamma t + \Sigma B_t + \int_0^t \int_{\|x\| \le 1} x\, (\mu^L - \nu^L)(ds, dx) + \int_0^t \int_{\|x\| > 1} x\, \mu^L(ds, dx)$$

where μ^L is the jump measure of the Lévy process L and ν^L denotes the corresponding compensator measure (see Chapter A.1 of the appendix for details). Here $L^{(1)} := \gamma t$ is a constant drift, $L^{(2)} := \Sigma B_t$ a d-dimensional standard Brownian motion, $L^{(3)} := \int_0^t \int_{\|x\| \le 1} x(\mu^L - \nu^L)(ds, dx)$ a pure jump process, and $L^{(4)} := \int_0^t \int_{\|x\|>1} x \mu^L(ds, dx)$ a compound Poisson process[2]. Both the building block $L^{(2)}$ and the integral w.r.t. compensated small jumps $L^{(3)}$ are martingales. Since in our model we assume L to have finite first moment, we have by Proposition A.6 that $\int_{\|x\| \ge 1} \|x\|\, \nu(dx) < \infty$. Hence big jumps can be compensated as well and the above decomposition simplifies to

$$L_t = \bar{\gamma} t + \Sigma B_t + \int_0^t \int_{\mathbb{R}^d} x\, (\mu^L - \nu^L)(ds, dx)$$

with $\bar{\gamma} := \gamma + \int_{\|x\|>1} x\, \nu(dx)$. For the sake of simplifying notation we henceforth write γ instead of $\bar{\gamma}$. Using this observation, we can rewrite the dynamics of the forward price in (1.1). Moreover, following the above, we pose a condition on the drift $\beta(s, T)$ in order to obtain the martingale property of $f(t, T)_{0 \le t \le T}$.

Lemma 1.1 ***(riskneutral dynamics)*** *Under the drift condition*

$$\beta(t, T) = -\langle \sigma(t, T) | \gamma \rangle$$

the forward price $f(t, T)$ *as introduced in* (1.1) *has the following dynamics for any* $T \le \tau$ *and* $t \le T$

$$df(t, T) = \sigma(t, T) \Sigma\, dB_t + \int_{\mathbb{R}^d} \langle \sigma(t, T) | x \rangle\, (\mu^L - \nu^L)(dt, dx)$$

[2]See [10] for details on the one-dimensional case.

Proof. By (1.1) we have:

$$\begin{aligned} df(t,T) &= \beta(t,T)\,dt + \sigma(t,T)\,dL_t = \beta(t,T)\,dt + \sigma(t,T)\,d\left(\gamma t + \Sigma B_t + \int_0^t \int_{\mathbb{R}^d} x\,(\mu^L - \nu^L)(ds,dx)\right) \\ &= \underbrace{\left(\beta(t,T) + \langle \sigma(t,T) | \gamma \rangle\right)}_{=0} dt + \sigma(t,T)\Sigma\,dB_t + \int_{\mathbb{R}^d} \langle \sigma(t,T) | x \rangle\,(\mu^L - \nu^L)(dt,dx) \\ &= \sigma(t,T)\Sigma\,dB_t + \int_{\mathbb{R}^d} \langle \sigma(t,T) | x \rangle\,(\mu^L - \nu^L)(dt,dx) \end{aligned}$$

□

Remark 1.2 *Following Lemma 1.1, we will from now on assume the underlying risk-neutral dynamics to be*

$$df(t,T) = \sigma(t,T)\,dL_t \tag{1.2}$$

with d-dimensional Lévy process L, whose characteristic triplet is given by $(0, \Sigma, \nu)$ *and* σ *as above. Moreover, we still require L to have independent components and finite first moment.*

Since in the long run, the goal is to adapt the theoretical forward curves to observed futures prices that are available on the market, we need to determine the characteristics of the forward curve $f(t,T)$ under the market-measure that we denote by $\mathbb{P}$. For this purpose we assume that the measure change can be described by an Esscher transform, that is we assume there is some $\theta \in \mathbb{R}^d$ such that $\mathbb{E}[\exp(\langle \theta | L_1 \rangle)] < \infty$ that defines the risk neutral measure $\mathbb{Q} = \mathbb{Q}^\theta \sim \mathbb{P}$ via the density process

$$\left.\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\right|_{F_t} = e^{\langle \theta | L_t \rangle - \kappa(\theta) t}$$

Here κ denotes the cumulant function of the Lévy process L (under $\mathbb{P}$) cf. Definition A.3. This choice seems particularly convenient, since the Esscher transform is structure preserving. However, the measure change we like to conduct goes the opposite way, from the risk-neutral to the market environment, hence we consider

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}^\theta}\right|_{F_t} = e^{\langle -\theta | L_t \rangle + \kappa(\theta) t}$$

to determine the market dynamics of the forward curves. We will in what follows neglect the explicit dependence of the risk neutral measure on the parameter θ from the Esscher transform in the notation of $\mathbb{Q}^\theta$ and simply write $\mathbb{Q}$ instead.

Lemma 1.3 *Let* $L = (L_t)_{0 \le t \le T}$ *be a Lévy process under* $\mathbb{Q}$ *with canonical representation*

$$L_t = \Sigma B_t + \int_0^t \int_{\mathbb{R}^d} x\,(\mu^L - \nu^L)(ds,dx) \tag{1.3}$$

Assume that $\mathbb{P} \sim \mathbb{Q}$ *is defined via the density process*

$$Z_t := \left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{F_t} := e^{\langle -\theta | L_t \rangle + \kappa(\theta) t}$$

Then L remains a Lévy process under $\mathbb{P}$ and its characteristic triplet $(\tilde{\gamma}, \tilde{\Sigma}, \tilde{\nu})$ is given by

$$\begin{aligned}\tilde{\gamma} &= -\Sigma\theta + \int_{\mathbb{R}^d} \left(e^{\langle -\theta|x\rangle} - 1\right) x \mathbb{1}_{\|x\|\leq 1}\, \nu(dx) \\ \tilde{\Sigma} &= \Sigma \\ \tilde{\nu}(dx) &= e^{\langle -\theta|x\rangle}\nu(dx)\end{aligned}$$

Proof. The following proof is mainly along the lines of the one given in [10] for Proposition 12.7. By Bayes' formula for conditional expectations (cf. [11]) we have that Z is a $\mathbb{Q}^\theta$-martingale iff $\frac{d\mathbb{Q}^\theta}{d\mathbb{P}} Z$ is one w.r.t. $\mathbb{P}$. This holds true since $\frac{d\mathbb{Q}^\theta}{d\mathbb{P}} Z \equiv 1$. Moreover, we have $Z_0 = 1$. Therefore, Z functions as density process. As Z is in fact a non-negative process, the measure change it describes is equivalent.
To show that the measure change preserves the Lévy property we show that L has independent and stationary increments under $\mathbb{P}$. Note that property (L1) of Definition A.1 follows directly since $\mathbb{P} \sim \mathbb{Q}$. Let $0 \leq s < t \leq T$ and denote by $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ with $\mathcal{F}_t := \sigma(L_u,\, u \leq t)$ the filtration generated by L. Let further $F_s \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R}^d)$. Then we have

$$\begin{aligned}\mathbb{P}\left[\{L_t - L_s \in B\} \cap F_s\right] &= \mathbb{E}_\mathbb{P}\left[\mathbb{1}_{\{L_t - L_s \in B\}} \mathbb{1}_{F_s}\right] = \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{L_t - L_s \in B\}} \mathbb{1}_{F_s} Z_t\right] \\ &= \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{L_t - L_s \in B\}} \mathbb{1}_{F_s} \frac{Z_t}{Z_s} Z_s\right] \overset{(1)}{=} \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s}\right] \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{F_s} Z_s\right] \\ &= \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s}\right] \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{F_s} Z_s\right] \underbrace{\mathbb{E}_\mathbb{Q}\left[Z_s\right]}_{=1} \overset{(2)}{=} \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{L_t - L_s \in B\}} Z_t\right] \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{F_s} Z_s\right] \\ &= \mathbb{E}_\mathbb{P}\left[\mathbb{1}_{\{L_t - L_s \in B\}}\right] \mathbb{E}_\mathbb{P}\left[\mathbb{1}_{F_s}\right] = \mathbb{P}\left[\{L_t - L_s \in B\}\right] \mathbb{P}\left[F_s\right]\end{aligned}$$

where in (1) and (2) we used that $\mathbb{1}_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s}$ (as measurable function of the increment $L_t - L_s$) is independent of $\mathbb{1}_{F_s} Z_s$ and Z_s respectively. Arguing similarly, the stationarity of the increments follows due to

$$\begin{aligned}\mathbb{P}\left[\{L_t - L_s \in B\}\right] &= \mathbb{E}_\mathbb{P}\left[\mathbb{1}_{\{L_t - L_s \in B\}}\right] = \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{L_t - L_s \in B\}} Z_t\right] \\ &= \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s}\right] \mathbb{E}_\mathbb{Q}\left[Z_s\right] = \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{L_t - L_s \in B\}} \frac{e^{\langle -\theta|L_t - L_s\rangle}}{\mathbb{E}_\mathbb{Q}\left[e^{\langle -\theta|L_t - L_s\rangle}\right]}\right] \\ &= \mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{L_{t-s} \in B\}} \frac{e^{\langle -\theta|L_{t-s}\rangle}}{\mathbb{E}_\mathbb{Q}\left[e^{\langle -\theta|L_{t-s}\rangle}\right]}\right] = \mathbb{E}_\mathbb{P}\left[\mathbb{1}_{\{L_{t-s} \in B\}}\right] \\ &= \mathbb{P}\left[\{L_{t-s} \in B\}\right]\end{aligned}$$

Finally we determine the characteristic triplet of L under the market measure $\mathbb{P}$. Let $\kappa^\mathbb{P}(\cdot)$ be the cumulant function of L under $\mathbb{P}$ (cf. A.3). Then we have for any $z \in \mathbb{C}$ such that $\mathbb{E}\left[e^{Re(z)L_1}\right] < \infty$

$$\begin{aligned}e^{t\kappa^\mathbb{P}(z)} &= \mathbb{E}_\mathbb{P}\left[e^{zL_t}\right] = \mathbb{E}_\mathbb{Q}\left[e^{zL_t} \frac{d\mathbb{P}}{d\mathbb{Q}}\right] = \mathbb{E}_\mathbb{Q}\left[e^{zL_t}\, \mathbb{E}_\mathbb{Q}\left[\frac{d\mathbb{P}}{d\mathbb{Q}} | F_t\right]\right] \\ &= \mathbb{E}_\mathbb{Q}\left[e^{\langle (z-\theta)|L_t\rangle + \kappa^\mathbb{P}(\theta)t}\right] = e^{\kappa^\mathbb{P}(\theta)t} \underbrace{\mathbb{E}_\mathbb{Q}\left[e^{\langle (z-\theta)|L_t\rangle}\right]}_{e^{\kappa^\mathbb{Q}(z-\theta)t}} = e^{t(\kappa^\mathbb{Q}(z-\theta) + \kappa^\mathbb{P}(\theta))}\end{aligned}$$

which is equivalent to $\kappa^{\mathbb{P}}(z) - \kappa^{\mathbb{P}}(\theta) = \kappa^{\mathbb{Q}}(z-\theta)$. By Lévy-Kinchine the r.h.s. can be further extended to

$$\begin{aligned}
\kappa^{\mathbb{Q}}(z-\theta) &= \frac{1}{2}(z-\theta)^T\Sigma(z-\theta) + \int_{\mathbb{R}^d} e^{\langle (z-\theta)|x\rangle} - 1 - \langle (z-\theta)|x\rangle \mathbb{1}_{\|x\|\leq 1}\, \nu(dx) \\
&= \frac{1}{2}z^T\Sigma z + z^T\left(-\Sigma\theta - \int_{\mathbb{R}^d} x\mathbb{1}_{\|x\|\leq 1}\,\nu(dx)\right) \\
&\quad + \frac{1}{2}\theta^T\Sigma\theta + \int_{\mathbb{R}^d} e^{\langle (z-\theta)|x\rangle} - 1 - \langle -\theta|x\rangle \mathbb{1}_{\|x\|\leq 1}\,\nu(dx) \qquad (1.4)
\end{aligned}$$

For the l.h.s. we proceed in the same way, where $(\tilde{\gamma}, \tilde{\Sigma}, \tilde{\nu})$ denotes the characteristic triplet of L under $\mathbb{P}$.

$$\begin{aligned}
\kappa^{\mathbb{P}}(z) - \kappa^{\mathbb{P}}(\theta) &= \tilde{\gamma}^T z + \frac{1}{2}z^T\tilde{\Sigma}z + \int_{\mathbb{R}^d} e^{\langle z|x\rangle} - 1 - \langle z|x\rangle \mathbb{1}_{\|x\|\leq 1}\,\tilde{\nu}(dx) \\
&\quad -\tilde{\gamma}^T\theta - \frac{1}{2}\theta^T\tilde{\Sigma}\theta - \int_{\mathbb{R}^d} e^{\langle \theta|x\rangle} - 1 - \langle \theta|x\rangle \mathbb{1}_{\|x\|\leq 1}\,\tilde{\nu}(dx) \\
&= \frac{1}{2}z^T\tilde{\Sigma}z + z^T\left(\tilde{\gamma} - \int_{\mathbb{R}^d} x\mathbb{1}_{\|x\|\leq 1}\,\tilde{\nu}(dx)\right) \\
&\quad -\theta^T\tilde{\gamma} - \frac{1}{2}\theta^T\tilde{\Sigma}\theta + \int_{\mathbb{R}^d} e^{\langle z|x\rangle} - e^{\langle \theta|x\rangle} + \langle \theta|x\rangle \mathbb{1}_{\|x\|\leq 1}\,\tilde{\nu}(dx) \qquad (1.5)
\end{aligned}$$

Comparing the coefficients of (1.4) and (1.5) yields $\Sigma = \tilde{\Sigma}$. This implies

$$\begin{aligned}
&-\theta^T\tilde{\gamma} - \frac{1}{2}\theta^T\Sigma\theta + \int_{\mathbb{R}^d} e^{\langle z|x\rangle} - e^{\langle \theta|x\rangle} + \langle \theta|x\rangle \mathbb{1}_{\|x\|\leq 1}\,\tilde{\nu}(dx) \\
&= \frac{1}{2}\theta^T\Sigma\theta + \int_{\mathbb{R}^d} e^{\langle (z-\theta)|x\rangle} - 1 - \langle -\theta|x\rangle \mathbb{1}_{\|x\|\leq 1}\,\nu(dx) \\
\Longleftrightarrow\quad & -\frac{1}{2}\theta^T\Sigma\theta + \theta^T\Sigma\theta + \int_{\mathbb{R}^d} e^{\langle z|x\rangle} - e^{\langle \theta|x\rangle} + \langle \theta|x\rangle \mathbb{1}_{\|x\|\leq 1}\,\tilde{\nu}(dx) + \int_{\mathbb{R}^d} \langle -\theta|x\rangle \mathbb{1}_{\|x\|\leq 1}\,\tilde{\nu}(dx) \\
&+ \int_{\mathbb{R}^d} \langle \theta|x\rangle \mathbb{1}_{\|x\|\leq 1}\,\nu(dx) = \frac{1}{2}\theta^T\Sigma\theta + \int_{\mathbb{R}^d} e^{\langle (z-\theta)|x\rangle} - 1 - \langle -\theta|x\rangle \mathbb{1}_{\|x\|\leq 1}\,\nu(dx) \\
\Longleftrightarrow\quad & \int_{\mathbb{R}^d} e^{\langle z|x\rangle} - e^{\langle \theta|x\rangle}\,\tilde{\nu}(dx) = \int_{\mathbb{R}^d} \left(e^{\langle (z-\theta)|x\rangle} - 1\right)\,\nu(dx) \\
\Longleftrightarrow\quad & \int_{\mathbb{R}^d} \left(e^{\langle z-\theta|x\rangle} - 1\right) e^{\langle \theta|x\rangle}\,\tilde{\nu}(dx) = \int_{\mathbb{R}^d} \left(e^{\langle (z-\theta)|x\rangle} - 1\right)\,\nu(dx) \\
\Longrightarrow\quad & \tilde{\nu}(dx) = e^{\langle -\theta|x\rangle}\,\nu(dx)
\end{aligned}$$

Finally we get:

$$\tilde{\gamma} = \int_{\mathbb{R}^d} x\mathbb{1}_{\|x\|\leq 1}\,\tilde{\nu}(dx) - \Sigma\theta - \int_{\mathbb{R}^d} x\mathbb{1}_{\|x\|\leq 1}\,\nu(dx) = -\Sigma\theta + \int_{\mathbb{R}^d} \left(e^{\langle -\theta|x\rangle} - 1\right) x\mathbb{1}_{\|x\|\leq 1}\,\nu(dx)$$

□

Remark 1.4 *The measure transform performed in lemma 1.3 is structure preserving. The corresponding Girsanov parameters (cf. [7] Theorem III3.24) (β, Y) can be obtained from the characteristic triplet under the market measure $\mathbb{P}$ and read as:*

$$\beta_t \equiv -\theta \qquad Y : \mathbb{R}^d \to \mathbb{R}, x \mapsto e^{\langle -\theta|x\rangle}$$

We now have everything at hand to state the dynamics of both the driving noise L and the forward price, which constitute the model that we established in the risk-neutral world, under the market measure $\mathbb{P}$.

Corollary 1.5 ***(Market dynamics)*** *The market dynamics of L are*

$$L_t = \tilde{\gamma} t + \Sigma W_t + \int_0^t \int_{\mathbb{R}^d} x\,(\mu^L - \tilde{\nu}^L)(ds, dx), \qquad 0 \le t \le T \tag{1.6}$$

with

$$\begin{aligned} \tilde{\gamma} &= -\Sigma\theta + \int_{\mathbb{R}^d} \left(e^{\langle -\theta | x\rangle} - 1\right) x\,\nu(dx) \\ W_t &= B_t + \theta t \\ \tilde{\nu}^L &= \tilde{\nu} \otimes dt \ \ \textit{and} \ \ \tilde{\nu}(dx) = e^{\langle -\theta | x\rangle} \nu(dx) \end{aligned}$$

By Girsanov's theorem, $(W_t)_{0 \le t \le T}$ is a $\mathbb{P}$-Brownian motion. Consequently, the market dynamics of the forward curve are given by

$$df(t,T) = \sigma(t,T) dL_t, \qquad 0 \le t \le T \tag{1.7}$$

with L as in (1.6)

Proof. Girsanov's theorem for semimartingales ([7] Theorem III.3.24) applied to the $\mathbb{Q}$-semimartingale B with characteristics $(0, I, 0)$ under $\mathbb{Q}$ yields $(\beta, I, 0)$ as $\mathbb{P}$-characteristics with $\beta = -\theta$ as in Remark 1.4. Hence $W_t := B_t + \theta t$ constitutes a Brownian Motion w.r.t $\mathbb{P}$. Moreover, we have

$$\begin{aligned} L_t &= \Sigma B_t + \int_0^t \int_{\mathbb{R}^d} x\,(\mu^L - \nu^L)(ds, dx) \\ &= -\Sigma\theta t + \Sigma W_t + \int_0^t \int_{\mathbb{R}^d} x\,(\mu^L - \nu^L)(ds, dx) \pm \int_0^t \int_{\mathbb{R}^d} x\,\tilde{\nu}^L(ds, dx) \\ &= -\Sigma\theta t + \Sigma W_t + \int_0^t \int_{\mathbb{R}^d} x\,(\mu^L - \tilde{\nu}^L)(ds, dx) + \int_0^t \int_{\mathbb{R}^d} x \left(e^{\langle -\theta | x\rangle} - 1\right) \nu^L(ds, dx) \\ &= \left(-\Sigma\theta + \int_{\mathbb{R}^d} x \left(e^{\langle -\theta | x\rangle} - 1\right) \nu(dx)\right) t + \Sigma W_t + \int_0^t \int_{\mathbb{R}^d} x\,(\mu^L - \tilde{\nu}^L)(ds, dx) \end{aligned}$$

□

As a final remark in this section, observe that we can in a natural way recover the spot dynamics of the energy price. We obtain the spot price at time t $S(t)$ as the price of a forward contract at the same time with immediate delivery $f(t,t)$. Hence we have for any $0 \le t \le T$

$$S(t) = f(t,t) = f(0,t) + \int_0^t \sigma(s,t)\, dL_s$$

where L as in (1.6) and (1.3) yield the market and risk neutral dynamics of S respectively. This implies that under the risk neutral measure the spot price will, except for specific choice of σ, in general not be a martingale.

1.2 The futures dynamics

Let us now turn to energy futures that constitute the central objects of interest in this thesis. Throughout this section we assume that the market provides a constant risk-free interest rate $r > 0$. Denote for $0 \leq t \leq T_1 < T_2 \leq \tau$ by $F(t, T_1, T_2)$ the price at time t of an energy futures contract that delivers energy over the period $[T_1, T_2]$. Following [2] (Chapter 4) this price is of the form

$$F(t, T_1, T_2) = \int_{T_1}^{T_2} \bar{\omega}(T, T_1, T_2) f(t, T) \, dT \tag{1.8}$$

where $f(t, T)$ is the forward curve as in Section 1.1. The deterministic function $T \mapsto \bar{\omega}(T, T_1, T_2)$ serves as weight function and is defined depending on in what way the settlement procedure is specified by the particular futures contract. It is set to be

$$\bar{\omega}(T, T_1, T_2) := \frac{\omega(T)}{\int_{T_1}^{T_2} \omega(u) \, du}$$

with $\omega(T) \equiv 1$ for futures that are settled at the end of the delivery period and $\omega(T) = exp(-rT)$ for the case when settlement takes place continuously over $[T_1, T_2]$. Thus we have in the former case

$$\bar{\omega}(T, T_1, T_2) = \frac{1}{T_2 - T_1} \tag{1.9}$$

while the weight function is specified to be

$$\bar{\omega}(T, T_1, T_2) = \frac{re^{-rT}}{e^{-rT_1} - e^{-rT_2}} \tag{1.10}$$

in the latter.

1.2.1 No-arbitrage relations

Typically, delivery periods of energy futures may range from days over months or weeks up to years. Thereby, one inevitably observes that contracts with overlapping delivery periods are traded on energy markets. Assume an agent would like to buy a futures contract that guarantees delivery over one year. In most energy markets, she may choose between a series of contracts that meet her demands. There are for instance, monthly or quarterly contracts over one whole year as well as contracts with delivery over one year themselves. Hence, there is some need for a relation between prices of traded futures contracts that ensure no arbitrage opportunity exists.

To that end, consider a futures contract with delivery over $[T_1, T_N]$ with corresponding futures price $F(t, T_1, T_N)$. Assume further that there are N tradeable futures contracts $F(t, T_k, T_{k+1})$ for $k \in \{1, \ldots, N-1\}$ with $T_1 < T_2 < \ldots < T_N$. Then, as can be shown by simple reformulations we have the following relation between futures prices

$$F(t, T_1, T_N) = \sum_{k=1}^{N-1} \omega_k F(t, T_k, T_{k+1}) \tag{1.11}$$

where

$$\omega_k = \frac{\int_{T_k}^{T_{k+1}} \omega(u) \, du}{\int_{T_1}^{T_N} \omega(u) \, du}$$

In concordance with [2] we refer to (1.11) as no-arbitrage relation, since, if (1.11) were violated, this would yield static arbitrage opportunities.
In our framework, we are interested in a continuous no-arbitrage relation that holds for all possible delivery periods. Passing to a limit in (1.11), one obtains for arbitrary maturities $T_1 < T_2 \leq \tau$ (see [2] Chapter 6.3)

$$F(t,T_1,T_2) = \int_{T_1}^{T_2} \bar{\omega}(T,T_1,T_2)F(t,T,T)\,dT \tag{1.12}$$

Using (1.8) together with an application of the fundamental theorem of calculus yields

$$\lim_{T_2\downarrow T_1} F(t,T_1,T_2) = \lim_{T_2\downarrow T_1} \frac{\int_{T_1}^{T_2}\omega(T)f(t,T)\,dT}{\int_{T_1}^{T_2}\omega(u)\,du} = \lim_{T_2\downarrow T_1} \frac{\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\omega(T)f(t,T)\,dT}{\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\omega(u)\,du} = f(t,T_1)$$

Hence in our model, both the relations preventing arbitrage (1.12) and (1.11) hold true.

1.3 Musiela-parametrization and finite dimensional realisation

In order to conveniently adapt the forward curves in our model to observed futures prices, we will consider in what follows a reasonable, finitely parametrized vector space of curves. Moreover, we like to work with the same set of curves *at any time t*, thus we first need to restate the above defined forward curves in so-called Musiela parametrisation. This means that instead of denoting the forward curves as functions of time and maturity, we view them as functions of time t and time-to-maturity $x := T - t$. Hereinafter, we denote the maximal time-to-maturity observed in the market by $\xi \in \mathbb{R}_+$. Naturally, we have $\xi \leq \tau$.

Lemma 1.6 *Let $f(t,T)$ be the forward curve with dynamics as in* (1.7) *and define $g(t,x) := f(t,t+x)$ and $c(t,x) := \sigma(t,t+x)$. If additionally $g(t,x)$ is continuously differentiable w.r.t. x, then for any $x \leq \xi$ the stochastic process $(\omega,t) \mapsto g(\omega,t,x)$ is a solution to the stochastic partial differential equation*

$$dg(t) = \partial_x g(t)dt + c(t)dL_t \tag{1.13}$$

Proof. Denoting with ∂_i for $i = 1,2$ the partial derivative w.r.t. the i'th argument of g, we consider for fixed $\omega \in \Omega$

$$\begin{aligned}
\frac{d}{dt}g(t,x) &= \frac{d}{dt}f(t,x+t) = \partial_1 f(t,t+x)\cdot\frac{\partial}{\partial t}(t) + \partial_2 f(t,t+x)\cdot\frac{\partial}{\partial t}(t+x) \\
&= \partial_1 f(t,t+x) + \partial_2 \underbrace{f(t,t+x)}_{=g(t,x)} \\
\Longrightarrow\ dg(t,x) &= \partial_1 f(t,t+x)\,dt + \partial_x g(t,x)\,dt = df(t,T) + \partial_2 g(t,x)\,dt \\
&= \sigma(t,T)\,dL_t + \partial_x g(t,x)\,dt = \partial_x g(t,x)\,dt + c(t,x)\,dL_t
\end{aligned}$$

□

Note that the above lemma holds true as well in the case where $\partial_x g(t,\cdot)$ is not defined, yielding a weak solution to 1.13. However, we will in what follows consider such re-parametrized forward curves $g(t,\cdot)$ for fixed t that are suitably differentiable.
Moreover, since in our model we demand σ to be bounded, likewise is the re-parametrized diffusion

coefficient c.

As mentioned above, our aim is to adapt the forward curves we model to finitely many observed prices of energy futures. Therefore, we introduce a finite dimensional vector space and postulate that the (re-parametrized) forward curves (up to a $\mathbb{P}$-null-set) stay within this vector space for fixed $t \geq 0$.

Assumption 1.7 *Let $Pol_1 := \{p : [0,\xi] \to \mathbb{R}, x \mapsto a + bx | a, b \in \mathbb{R}\}$ denote the vector space of real-valued polynomials on $[0,\xi] \subset \mathbb{R}_+$ up to degree one and define for $\lambda > \mu > 0$ the following finite-dimensional vector space*

$$V_{\lambda,\mu} := \left\{ p_1(\cdot)e^{-\lambda(\cdot)} + p_2(\cdot)e^{-\mu(\cdot)} \middle| p_1, p_2 \in Pol_1 \right\}$$

Moreover, let g be the solution to (1.13). *We assume that there are positive constants λ and μ such that g satisfies $g(t) \in V_{\lambda,\mu}$ $\mathbb{P}$-almost surely $\forall t \geq 0$ and say f has* finite dimensional realisation $V_{\lambda,\mu}$.

Let us now elaborate why this choice of $V_{\lambda,\mu}$ is sensible. With regard to the historical evolution of market futures prices, we may determine empirically the form of the forward curve. Typically, such analyses show highly volatile prices for contracts with short time to maturity, while those of products looking far into the future remain almost constant at some non-zero level. Figure 1.1 exemplarily shows base-load ®PHELIX futures prices over time to maturity, as observed on 22^{nd} of June 2016 at the European Energy Exchange AG. The length of the dotted lines corresponds to the delivery period on which the traded contracts are written. They comprise monthly, quarterly and yearly power futures.

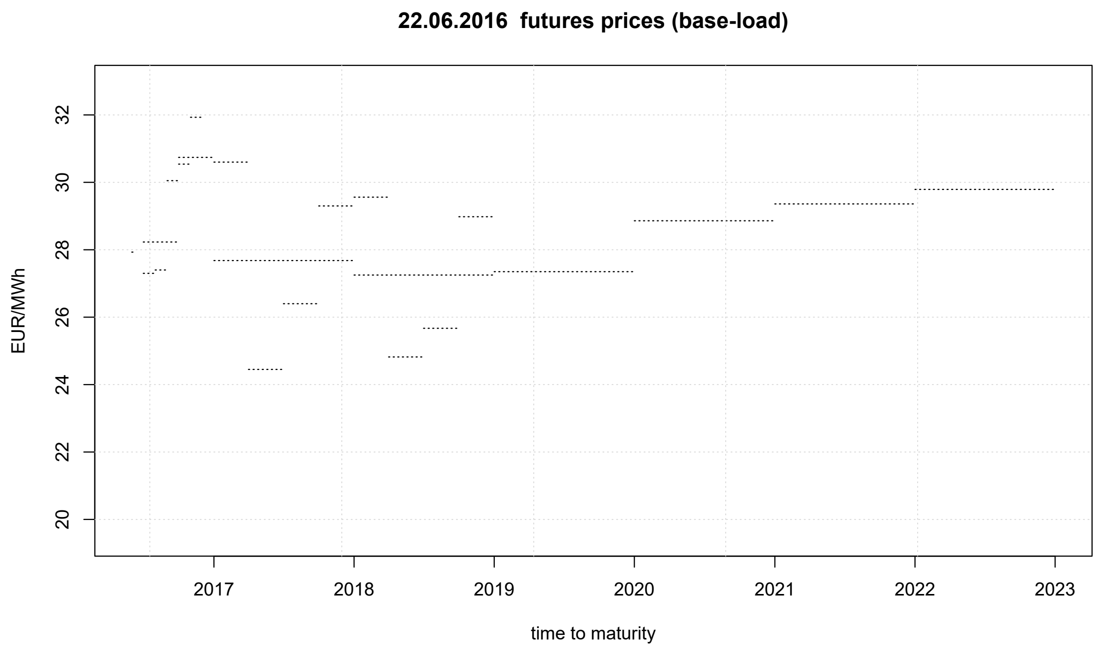

Figure 1.1: Base-load ®PHELIX futures prices observed at EEX on 22^{nd} June 2016.

The particular functional space mentioned in Assumption 1.7 serves well to create curves that fit such a historic development. Moreover, if said assumption holds true, we obtain that the quintupel

encompassing the re-parametrized forward curve itself as well as its four coefficient processes is affine.
The following corollary outlines some characteristics of the vector space $V_{\lambda,\mu}$ that can be easily verified.

Corollary 1.8 ***(properties of*** $V_{\lambda,\mu}$***)***

1. $\partial_x h \in V_{\lambda,\mu}$, $\forall h \in V_{\lambda,\mu}$
2. $V_{\lambda,\mu} \subset C^\infty([0,\xi],\mathbb{R})$
3. *The set* $\{b_1,\ldots,b_4\}$ *with*

$$b_1(x) := e^{-\lambda x} \qquad b_3(x) := e^{-\mu x}$$
$$b_2(x) := xe^{-\lambda x} \qquad b_4(x) := xe^{-\mu x}$$

for $x \in [0,\xi]$ *constitutes a basis of* $V_{\lambda,\mu}$.

By Assumption 1.7 and Corollary 1.8, we obtain that for any $t \geq 0$ there are random variables $a_i(t,\cdot):\Omega \to \mathbb{R}; \omega \mapsto a_i(t,\omega)$, $i=1,\ldots,4$ such that

$$g(t,\cdot,\omega) = \sum_{i=1}^{4} a_i(t,\omega) b_i(\cdot) \qquad \mathbb{P}-a.s \tag{1.14}$$

In fact, we will see that the coefficients $a_i(t,\cdot)$ can be determined more precisely, if the re-parametrized diffusion coefficients c_j, $j \in \{1,\ldots,4\}$ are chosen to be deterministic and homogeneous in time. To do so, we first show that if the forward curve f has finite dimensional realisation $V_{\lambda,\mu}$, for fixed $0 \leq t \leq T$, every component of the diffusion coefficient c of g as a function of x almost surely lies within $V_{\lambda,\mu}$ as well.

Lemma 1.9 *Let the forward curve be given as in* (1.7) *and let the diffusion coefficient* σ *be continuous. Assume further Assumption 1.7 holds true, i.e. that* f *has finite dimensional realisation* $V_{\lambda,\mu}$*. Then we have for* $c(t,\cdot)$ *as in Lemma 1.6:*

$$c_i(t,\cdot) \in V_{\lambda,\mu} \qquad \mathbb{P}\text{-}a.s.\, \forall i=1,\ldots,d, \;\; \textit{for any } 0 \leq t \leq T$$

Proof.
Since (1.14) holds for arbitrary $t \in [0,T]$ we have $g(0) \in V_{\lambda,\mu}$ $\mathbb{P}$-a.s. Moreover, the invariance of $V_{\lambda,\mu}$ w.r.t. ∂_x yields $\partial_x g(s) \in V_{\lambda,\mu}$ $\mathbb{P}$-a.s. This implies that both these functions of x can be represented w.r.t. the basis defined in Corollary 1.8. Thus there are random variables $a_i(0)$ and $\tilde{a}_i(s)$ (for any $s \in [0,t]$), $i=1,\ldots,4$ such that

$$\begin{aligned} g(0,\cdot) &= \sum_{i=1}^{4} a_i(0) b_i(\cdot) \qquad \mathbb{P}-a.s. \\ \int_0^t \partial_x g(s,\cdot)\,ds &= \sum_{i=1}^{4} b_i(\cdot) \int_0^t \tilde{a}_i(s)\,ds \qquad \mathbb{P}-a.s. \end{aligned}$$

Hence, all in all, for any $t \geq 0$ there are random variables $\hat{a}_i(t)$, $i=1,\ldots,4$ such that (1.13) can be restated as

$$\sum_{i=1}^{4} \hat{a}_i(t) b_i(\cdot) = \int_0^t c(s,\cdot)\, dL_s = \sum_{j=1}^{d} \int_0^t c_j(s,\cdot)\, dL_s^j \qquad \mathbb{P} - as \tag{1.15}$$

To show that this implies $c_j(s,\cdot) \in V_{\lambda,\mu}$ $\forall 0 \le s \le t$ $\mathbb{P}$-a.s. we define the linear functional $\varphi_\ell : C([0,\xi],\mathbb{R}) \to \mathbb{R}$ that operates on elements of $V_{\lambda,\mu} \subset C([0,\xi],\mathbb{R})$ in the following way:

$$\varphi_\ell \left(\sum_{i=1}^{4} \alpha_i b_i(\cdot) \right) = \alpha_\ell \qquad \forall \alpha \in \mathbb{R}^4$$

A suitable version of the Hahn-Banach theorems may then be employed to obtain the existence of a continuous extension of φ_ℓ to $C([0,\xi],\mathbb{R})$. More precisely, Theorem 3.36 of [12] applied to the normed (and hence locally convex) space $(C([0,\xi],\mathbb{R}), \|\cdot\|_\infty)$, its finite-dimensional subspace $V_{\lambda,\mu}$ and the thereon defined linear (and hence continuous) functional φ_ℓ, yields the existence of a continuous extension of φ_ℓ to $C([0,\xi],\mathbb{R})$. Moreover we set

$$k_\ell^j(s) := \varphi_\ell\left(c_j(s,\cdot)\right), s \in [0,t] \qquad \forall j \in \{1,\ldots,d\}$$

We then have

$$\sum_{j=1}^{d} \int_0^t k_\ell^j(s)\, dL_s^j \overset{(*)}{=} \varphi_\ell \underbrace{\left(\sum_{j=1}^{d} \int_0^t c_j(s,\cdot)\, dL_s^j \right)}_{\overset{(1.15)}{=} \sum_{i=1}^{4} \hat{a}_i(t) b_i(\cdot)} = \hat{a}_\ell(t) \qquad \mathbb{P} - as \tag{1.16}$$

whereby the first equality $(*)$ shall be shown in more detail: By linearity of φ_ℓ we have

$$\varphi_\ell \left(\sum_{j=1}^{d} \int_0^t c_j(s,\cdot)\, dL_s^j \right) = \sum_{j=1}^{d} \varphi_\ell \left(\int_0^t c_j(s,\cdot)\, dL_s^j \right)$$

and hence it only remains to show that the linear operator φ_ℓ may be interchanged with the stochastic integral, i.e. that the following holds true for any $j \in \{1,\ldots,d\}$:

$$\int_0^t \varphi_\ell\left(c_j(s,\cdot)\right) dL_s^j = \varphi_\ell \left(\int_0^t c_j(s,\cdot)\, dL_s^j \right)$$

Recall that in the present model the re-parametrized diffusion coefficient c is bounded. Thus there exists some constant $C \in \mathbb{R}$ such that for any $(x,\omega) \in ([0,\xi] \times \Omega)$ and all $0 \le s \le t$ $|c_j(s,x,\omega)| < C$. Moreover, due to continuity in time, the real-valued function $c_j(\cdot,x,\omega)$ may for any $(x,\omega) \in ([0,\xi],\Omega)$ be uniformly approximated by step-functions $c_j^n, n \in \mathbb{N}$ specified via $c_j^N(s,x,\omega) = \sum_{n=1}^{N} Y_n(x,\omega) \mathbb{1}_{(a_{n-1},a_n]}(s)$ for some $N \in \mathbb{N}$. Here, the real-valued functions Y_n, $n \in \mathbb{N}$ are themselves bounded. Theorem 23.4 of [8] applied to the $C([0,\xi],\mathbb{R})$-valued bounded processes $V_N := c_j(\cdot) - c_j^N(\cdot)$ yields $\int_0^t c_j^N(s,\cdot)\, dL_s \overset{P}{\to} \int_0^t c_j(s,\cdot)\, dL_s$. For every sequence that converges in probability there exists a subsequence that does so almost surely and to the same limit, thus we have $\lim_{m\to\infty} \int_0^t c_j^{N_m}(s,\cdot)\, dL_s = \int_0^t c_j(s,\cdot)\, dL_s$ $\mathbb{P}$-a.s. for some suitable subsequence

$(c_j^{N_m})_{m\in\mathbb{N}}$. This in turn, as well as continuity of φ_ℓ yields

$$\begin{aligned}\varphi_\ell\left(\int_0^t c_j(s,\cdot)\,dL_s^j\right) &= \varphi_\ell\left(\lim_{m\to\infty}\int_0^t c_j^{N_m}(s,\cdot)\,dL_s^j\right) = \lim_{m\to\infty}\varphi_\ell\left(\int_0^t c_j^{N_m}(s,\cdot)\,dL_s^j\right)\\ &= \lim_{m\to\infty}\varphi_\ell\left(\sum_{n=1}^{N_m} Y_n(\cdot)\left(L_{a_n}^j - L_{a_{n-1}}^j\right)\right) = \lim_{m\to\infty}\sum_{n=1}^{N_m}\varphi_\ell\left(Y_n(\cdot)\right)\left(L_{a_n}^j - L_{a_{n-1}}^j\right)\end{aligned}$$

Being a continuous operator between normed spaces φ_ℓ is bounded as well. Hence we may continue via the definition of the stochastic integral for elementary integrands and apply Theorem 23.4 of [8] once more to obtain

$$= \lim_{m\to\infty}\int_0^t \varphi_\ell(c_j^{N_m}(s,\cdot))\,dL_s^j = \int_0^t \varphi_\ell(c_j(s,\cdot))\,dL_s^j$$

We may then proceed with

$$\begin{aligned}\sum_{j=1}^d\int_0^t c_j(s,\cdot)\,dL_s^j &\overset{(1.15)}{=} \sum_{i=1}^4 \hat{a}_i(t)b_i(\cdot) \overset{(1.16)}{=} \sum_{i=1}^4\left(\sum_{j=1}^d\int_0^t k_i^j(s)\,dL_s^j\right)b_i(\cdot)\\ &= \sum_{j=1}^d\int_0^t\left(\sum_{i=1}^4 k_i^j(s)b_i(\cdot)\right)dL_s^j \qquad \mathbb{P}-as\end{aligned}$$

$$\Longleftrightarrow \sum_{j=1}^d\int_0^t\left(c_j(s,\cdot)-\sum_{i=1}^4 k_i^j(s)b_i(\cdot)\right)dL_s^j = 0 \qquad \mathbb{P}-as$$

We further obtain by [8] Theorem 23.6 (ii) & (v) and the independence of the Lévy components for the quadratic variation of above semimartingale

$$\begin{aligned}\Longrightarrow 0 &= \left[\sum_{j=1}^d\int_0^\cdot\left(c_j(s,\cdot)-\sum_{i=1}^4 k_i^j(s)b_i(\cdot)\right)dL_s^j\right]_t \qquad \mathbb{P}-as\\ &= \sum_{j=1}^d\int_0^t\left(c_j(s,\cdot)-\sum_{i=1}^4 k_i^j(s)b_i(\cdot)\right)^2 d[L^j]_s + 0 \qquad \mathbb{P}-as\\ &= \sum_{j=1}^d\underbrace{\left(\sigma_j^2+\int_{\mathbb{R}}x^2\,\nu_j(dx)\right)}_{=:A_j\geq 0}\int_0^t\underbrace{\left(c_j(s,\cdot)-\sum_{i=1}^4 k_i^j(s)b_i(\cdot)\right)^2 ds}_{\geq 0} \qquad \mathbb{P}-as\end{aligned}$$

For the definition of the quadratic variation process of a semimartingale we refer to [8] as well. With the help some results on it presented therein (see Theorem 23.6 and Corollary 23.15), the quadratic variation of the present (1-dimensional) Lévy process can be determined as

$$\left[L^j\right]_s = \left[\gamma\cdot+\sigma_j^2 W_\cdot + \int_0^\cdot\int_{\mathbb{R}}x\,(\mu^{L^j}-\nu^{L^j})(ds,dx)\right]_s = \sigma_j^2 s + \int_0^s\int_{\mathbb{R}}x^2\,\mu^{L^j}(ds,dx)$$

and hence the final step in above equation follows.

For any $j \in \{1, \dots, d\}$ such that $A_j \neq 0$ we next obtain, due to non-negativity of every summand,

$$\begin{aligned} & \int_0^t \Big(c_j(s,\cdot) - \textstyle\sum_{i=1}^4 k_i^j(s) b_i(\cdot) \Big)^2 \, ds = 0 \qquad \mathbb{P} - as \\ \Longrightarrow \quad & \Big(c_j(s,\cdot) - \textstyle\sum_{i=1}^4 k_i^j(s) b_i(\cdot) \Big)^2 \qquad \text{for } \mathbb{P} \otimes \lambda\text{-a.e. } 0 \le s \le t \\ \Longrightarrow \quad & c_j(s,\cdot) = \textstyle\sum_{i=1}^4 k_i^j(s) b_i(\cdot) \qquad \text{for } \mathbb{P} \otimes \lambda\text{-a.e. } 0 \le s \le t \end{aligned}$$

The degenerate case $A_j = 0$ in which L^j merely consists in a constant drift, shall in fact not be considered, since in the model we set up deterministic components are rather taken account for separately (c.f. (1.1)).
Hence we obtain that there is some t-dependent null set N_t such that $c_j(s, \omega, \cdot) \in V_{\lambda,\mu}$ for all $j \in \{1, \dots, d\}$, $s \in [0, t]$ and any $\omega \in \Omega \backslash N_t$. Due to continuity we arrive at $c_j(s, \cdot) \in V_{\lambda,\mu}$ for all $j \in \{1, \dots, d\}$ and $s \ge 0$ $\mathbb{P}-$almost surely.

□

Let us now turn towards the question of characterizing the coefficient-processes $a_i(t), t \ge 0$, $i \in \{1, \dots, 4\}$ of the re-parametrized forward curve g c.f. (1.14). We have for any $x \in [0, \xi]$

$$\begin{aligned} \partial_x b_1(x) &= -\lambda b_1(x) & \qquad \partial_x b_3(x) &= -\mu b_3(x) \\ \partial_x b_2(x) &= b_1(x) - \lambda b_2(x) & \qquad \partial_x b_4(x) &= b_3(x) - \mu b_4(x) \end{aligned}$$

which implies

$$\begin{aligned} \partial_x g(t,\cdot) = \sum_{i=1}^4 a_i(t) \partial_x b_i(\cdot) &= b_1(\cdot)[a_2(t) - \lambda a_1(t)] + b_2(\cdot)[-\lambda a_2(t)] \\ &+ b_3(\cdot)[a_4(t) - \mu a_3(t)] + b_4(\cdot)[-\mu a_4(t)] \end{aligned}$$

Moreover we get by (1.13) and (1.14) the following two representations for $dg(t, x)$ and $x \in [0, \xi]$.

$$\sum_{i=1}^4 b_i(x)\, da_i(t) \overset{(1.14)}{=} dg(t,x) \overset{(1.13)}{=} \sum_{i=1}^4 a_i(t) \partial_x b_i(x)\, dt + \sum_{j=1}^d c_j(t,x) dL_t^j \qquad \forall x \in [0,\xi] \tag{1.17}$$

In what follows, we choose the (re-parametrized) diffusion coefficient c to be time-homogeneous and deterministic. By Lemma 1.9 this yields $c_j(t, \cdot, \omega) := \sum_{i=1}^4 k_i^j b_i(\cdot)$ for $j = 1, \dots, 4$ and $k^j \in \mathbb{R}^4$. Thus we can reformulate (1.17) for any $x \in [0, \xi]$ as

$$\begin{aligned} \sum_{i=1}^4 b_i(x)\, da_i(t) &= b_1(x) \left[(a_2(t) - \lambda a_1(t))\, dt + \sum_{j=1}^d k_1^j \, dL_t^j \right] + b_2(x) \left[(-\lambda a_2(t))\, dt + \sum_{j=1}^d k_2^j \, dL_t^j \right] \\ &+ b_3(x) \left[(a_4(t) - \mu a_3(t))\, dt + \sum_{j=1}^d k_3^j \, dL_t^j \right] + b_4(x) \left[(-\mu a_4(t))\, dt + \sum_{j=1}^d k_4^j \, dL_t^j \right] \end{aligned}$$

Hence, by comparing coefficients, one arrives at

$$da_1(t) = (a_2(t) - \lambda a_1(t))\,dt + \sum_{j=1}^{d} k_1^j\,dL_t^j \qquad da_3(t) = (a_4(t) - \mu a_3(t))\,dt + \sum_{j=1}^{d} k_3^j\,dL_t^j$$

$$da_2(t) = (-\lambda a_2(t))\,dt + \sum_{j=1}^{d} k_2^j\,dL_t^j \qquad da_4(t) = (-\mu a_4(t))\,dt + \sum_{j=1}^{d} k_4^j\,dL_t^j$$

We make the following observation

Corollary 1.10 *Assume the setting of Lemma 1.13, and let Assumption 1.7 hold true. For c time-homogeneous and deterministic, i.e. $c_j(t,\cdot,\omega) := \sum_{i=1}^4 k_i^j b_i(\cdot)$ with $k^j \in \mathbb{R}^4$, $j = 1,\dots,4$, the process*

$$\begin{aligned} Y_x &: (\Omega, \mathbb{R}_+) \to \mathbb{R}^5 \\ &(\omega,t) \mapsto Y_x(\omega,t) := (a_1(\omega,t), a_2(\omega,t), a_3(\omega,t), a_4(\omega,t), g(\omega,t,x))^T \end{aligned}$$

is affine for any fixed $x \in [0,\xi]$.

Proof. Let $x \in [0,\xi]$ be fixed. From above we have for all $t \in \mathbb{R}_+$:

$$\begin{aligned} g(t,x) &= g(0,x) + \sum_{i=1}^4 b_i(x) \int_0^t da_i(s) = \\ &= g(0,x) + b_1(x)\left[\int_0^t (a_2(s) - \lambda a_1(s))\,ds + \sum_{j=1}^d k_1^j L_t^j\right] + b_2(x)\left[\int_0^t (-\lambda a_2(s))\,ds + \sum_{j=1}^d k_2^j L_t^j\right] = \\ &+ b_3(x)\left[\int_0^t (a_4(s) - \mu a_3(s))\,ds + \sum_{j=1}^d k_3^j L_t^j\right] + b_4(x)\left[\int_0^t (-\mu a_4(s))\,ds + \sum_{j=1}^d k_4^j L_t^j\right] \\ &= g(0,x) + b_1(x)\int_0^t (a_2(s) - \lambda a_1(s))\,ds + b_2(x)\int_0^t (-\lambda a_2(s))\,ds \\ &+ b_3(x)\int_0^t (a_4(s) - \mu a_3(s))\,ds + b_4(x)\int_0^t (-\mu a_4(s))\,ds + b^T K L_t \end{aligned}$$

with

$$b := (b_1(x), b_2(x), b_3(x), b_4(x))^T \in \mathbb{R}^4 \qquad \text{and} \qquad K := \begin{pmatrix} k_1^1 & \cdots & k_1^d \\ \vdots & & \vdots \\ k_4^1 & \cdots & k_4^d \end{pmatrix} \in \mathbb{R}^{4\times d}$$

Since $B := b^T K \in \mathbb{R}^{1\times d}$ we obtain (see [4] THEOREM 4.1) that the process BL is again Lévy (on $\mathbb{R}$). For $L \sim (\gamma, \Sigma, \nu)$ the characteristic triplet $(\gamma^B, \Sigma^B, \nu^B)$ of the linearly transformed process BL is given by

$$\begin{aligned} \gamma^B &= B\gamma + \int_{\mathbb{R}} y\left(\mathbb{1}_{\{|y|\le 1\}}(y) - \mathbb{1}_{\{By:\|y\|\le 1\}}(y)\right)\,\nu^B(dy) \\ \Sigma^B &= B\Sigma B^T \\ \nu^B(A) &= \nu(\{z \in \mathbb{R}^d : Bz \in A\}), \qquad \forall A \in \mathcal{B}(\mathbb{R}) \end{aligned}$$

By Lévy-Itô, the process $A^B(t) := \gamma^B \cdot t$ denotes the finite-variation part of the semimartingale BL. Since $(A^B(t))_{t\in\mathbb{R}_+}$ is predictable, it constitutes a part of the canonical decomposition $BL_t = M^B(t) + A^B(t)$, where $M^B(t)$ denotes the local martingale part.[3]

Next we determine the semimartingale characteristics[4] of $g(t,x)$. Its (canonical) decomposition is of the form $g(t,x) = g(0,x) + A(t) + M(t)$ with local martingale part $M(t) = M^B(t)$ and $A(t)$ a predictable process of finite variation given by

$$A(t) = A^B(t) + \int_0^t b_1(x)(a_2(s) - \lambda a_1(s)) + b_2(x)(-\lambda a_2(s)) + b_3(x)(a_4(s) - \mu a_3(s)) + b_4(x)(-\mu a_4(s))\, ds$$

By [7] (Chapter II Definition 2.6) the distribution of the process $g(t,x), t \in \mathbb{R}_+$ hence is characterized by the triplet (A, C, ν^g) with

- $A = A(t), t \in \mathbb{R}_+$ the predictable process appearing above
- $C = C(t), t \in \mathbb{R}_+$ the continuous process $\langle g^c, g^c\rangle_t, t \in \mathbb{R}_+$, i.e. the predictable quadratic variation process associated with the continuous martinale part of the process $g(\cdot, x)$. Since $g^c(t,x) = B\Sigma B^T W_t$ by the Lévy-Itô decomposition, where W denotes a one-dimensional Brownian Motion, we obtain: $C(t) = (B\Sigma B^T)^2 \cdot t$.
- ν^g the compensator measure associated with the jumps of $g(\cdot, x)$. The jumps of $g(\cdot, x)$ however coincide with those of BL, whose compensator measure is given by $\nu^B \otimes dt$.

To prove our claim, we now similarly to above derive the semimartingale characteristics for Y_x. Since we have

$$Y_x(t) = Y_x(0) + \underbrace{\begin{pmatrix} \int_0^t a_2(s) - \lambda a_1(s)\, ds \\ \int_0^t -\lambda a_2(s)\, ds \\ \int_0^t a_4(s) - \mu a_3(s)\, ds \\ \int_0^t -\mu a_4(s)\, ds \\ \int_0^t \partial_x g(s,x)\, ds \end{pmatrix}}_{=:d^Y(t)} + \begin{pmatrix} K \\ B \end{pmatrix} L_t$$

we may proceed as before, first applying the linear transformation $M := \begin{pmatrix} K \\ B \end{pmatrix} \in \mathbb{R}^{5\times d}$ to the d-dimensional Lévy process L, and then adding d^Y, a predictable term of finite variation. Thus, the canonical decomposition of the semimartingale Y_x reads as

$$Y_x(t) = Y_x(0) + A^Y(t) + M^Y(t)$$

with

$$\begin{aligned} A^Y(t) &= \underbrace{\left(M\gamma + \int_{\mathbb{R}^5} y \left(\mathbb{1}_{\{|y|\le 1\}}(y) - \mathbb{1}_{\{My: \|y\|\le 1\}}(y) \right) \nu^M(dy) \right)}_{\gamma^M} \cdot t + d^Y(t) \\ M^Y(t) &= M\Sigma M^T W_t + \int_0^t \int_{\mathbb{R}^5} y \left(\mu^{ML} - \nu^{ML} \right)(ds, dy) \end{aligned}$$

[3]Note that both the local martingale and finite variation parts of the canonical decomposition depend on the truncation function of choice. Throughout this thesis, we take it to be $h : \mathbb{R}^d \to \mathbb{R}^d$, $h(x) := x\mathbb{1}_{\|x\|\le 1}$ (cf. Appendix Theorem A.4).

[4]The notion of characteristics of a semimartingale serves as generalisation of a Lévy process' characteristic triplet to semimartingales and is nicely characterized in [7] Chapter II.

where $\nu^M(A) := \nu(\{z \in \mathbb{R}^d : Mz \in A\})$, $\forall A \in \mathcal{B}(\mathbb{R}^5)$ and W_t a 5-dim Brownian Motion. Analogously, its semimartingale characteristics (A^Y, C^Y, ν^Y)then is given by

- $A^Y = A^Y(t), t \in \mathbb{R}_+$
- $C^Y = C^Y(t), t \in \mathbb{R}_+$, with $C(t) = (M\Sigma M^T)(M\Sigma M^T)^T \cdot t$
- ν^Y the compensator measure associated with the jumps of $Y_x(\cdot)$ which coincide with those of ML, whose compensator measure ν^{ML} is given by $\nu^M \otimes dt$.

These just derived characteristics correspond to those of a Lévy driven Ornstein-Uhlenbeck process. To see that the just derived characteristics are of the desired affine form we rephrase the drift once more

$$
\begin{aligned}
A^Y(t) &= \int_0^t \gamma^M \cdot s + \begin{pmatrix} a_2(s) - \lambda a_1(s) \\ -\lambda a_2(s) \\ a_4(s) - \mu a_3(s) \\ -\mu a_4(s) \\ b_1(x)(a_2(s) - \lambda a_1(s)) + b_2(x)(-\lambda a_2(s)) + b_3(x)(a_4(s) - \mu a_3(s)) + b_4(x)(-\mu a_4(s)) \end{pmatrix} ds \\
&= \int_0^t \gamma^M \cdot s + \begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & -\mu & 1 & 0 \\ 0 & 0 & 0 & -\mu & 0 \\ -\lambda b_1(x) & b_1(x) - b_2(x)\lambda & -\mu b_3(x) & b_3(x) - \mu b_4(x) & 0 \end{pmatrix} Y_{x_s}\, ds
\end{aligned}
$$

Therefore, Theorem 2.12. of [5] is applicable and it follows that Y_x is affine. □

1.3.1 Futures dynamics in Musiela parametrisation

Concluding this chapter, we state the re-parametrized futures dynamics as well. Consider $F(t, T_1, T_2)$, the price of a futures contract with delivery over $[T_1, T_2]$ as in (1.8). We set $x := T_1 - t > 0$ the time to delivery and denote by $\ell := T_2 - T_1 > 0$ the length of the delivery period of the specific contract and define

$$
\begin{aligned}
G(t, x, \ell) := F(t, t + x, t + x + \ell) &= \int_{t+x}^{t+x+\ell} \bar{\omega}(T, t + x, t + x + \ell) f(t, T)\, dT \\
&= \int_x^{x+\ell} \bar{\omega}(t + u, t + x, t + x + \ell) f(t, t + u)\, du
\end{aligned}
$$

Recall that $f(t, t+x) = g(t, x)$ and set $w(t, u, x, \ell) = \bar{\omega}(t+u, t+x, t+x+\ell)$. Regarding the weight functions introduced in Section 1.2, we obtain $w(t, u, x, \ell) = \frac{1}{\ell}$ for futures contracts with terminal settlement whereas those contracts that determine settlement continuously over the delivery period yield $w(t, u, x, \ell) = \frac{r}{1-\exp(-r\ell)} \cdot \exp(-r(u-x))$. We observe that both these functions do not depend on time. Therefore, we henceforth consider weighting functions to be time-independent and write

$w(u, x, \ell)$.[5] We then arrive at the following re-parametrized futures price

$$G(t, x, \ell) = \int_x^{x+\ell} w(u, x, \ell) g(t, u)\, du \tag{1.18}$$

In order to fit observed futures prices to theoretical ones according to the above formula (1.18), we restrict ourselves to the weighting function associated with futures that settle *at the end* of the delivery period with corresponding futures price

$$G(t, x, \ell) = \frac{1}{\ell} \int_x^{x+\ell} g(t, u)\, du \tag{1.19}$$

This (simple) choice is in spirit of Lucia and Schwartz [9] who show that an arithmetic mean of the forward prices yields sensible approximations to the futures prices in the Nordic Power Exchange. Passing on to the data we have at hand, i.e. futures prices that originate from the European Energy Exchange AG, our next objective will consist in determining the coefficients $a_1, \dots, a_4$ of $g(t, x)$ as well as the parameters λ and μ which co-create the basis of the vector space $V_{\lambda,\mu}$ with regards to the given data. For the sake of facilitating parameter estimation we re-formulate the futures price once more. As before we assume the setting of Lemma 1.13, and let Assumption 1.7 hold true.

Lemma 1.11 *Denote with ∂_x^{-1} the integral operator on $V_{\lambda,\mu}$ with kernel $k(u) \equiv 1$ i.e.*

$$\left(\partial_x^{-1} g\right)(y) := \int g(z)\, dz\, (y) \qquad \forall g \in V_{\lambda,\mu},\ y \in [0, \xi]$$

Then the operator maps to $V_{\lambda,\mu}$, in particular, we have $h_t(\cdot) := \partial_x^{-1} g(t, \cdot) \in V_{\lambda,\mu}$ for any $t \geq 0$. Accordingly, the futures price from (1.19) *can be represented as*

$$G(t, x, \ell) = \frac{1}{\ell} \left(h_t(x + \ell) - h_t(x) \right) \tag{1.20}$$

Proof. Let $a_1, \dots, a_4$ be the coordinates of some arbitrary function $g \in V_{\lambda,\mu}$ with respect to the vector space's basis $\left\{ e^{-\lambda x}, xe^{-\lambda x}, e^{-\mu x}, xe^{-\mu x} \right\}$. Simple integration leads to the following coordinates $\tilde{a}_i$, $i = 1, \dots, 4$ for $\partial_x^{-1} g$:

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \\ \tilde{a}_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\lambda} & -\frac{1}{\lambda^2} & 0 & 0 \\ 0 & -\frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & -\frac{1}{\mu} & -\frac{1}{\mu^2} \\ 0 & 0 & 0 & -\frac{1}{\mu} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

Since the integral operator ∂_x^{-1} assigns a primitive function to every $g \in V_{\lambda,\mu}$, the second result follows by the fundamental theorem of calculus. □

By the above lemma, the futures price as we model it in (1.19) has itself a representation w.r.t. the basis of $V_{\lambda,\mu}$. However, we will not make use of this when fitting the coefficient processes $a_1(t), \dots, a_4(t)$ of $g(t, \cdot)$. Our approach rather consists in first an estimation of the coefficient

[5] In fact, among those two choices of weighting functions, only the latter depends on x and u as well. Moreover this dependency is stationary, meaning that the weight only depends on the time that has passed since the maturity $u - x$.

processes $\tilde{a}_1(t), \ldots, \tilde{a}_4(t)$ of $h_t(x)$ with the help of (1.20). Thereafter, the actual processes of interest can be obtained via a linear transformation. This procedure will be covered in the following chapter.

Chapter 2

Model Calibration

Ensuing the fundamental model description in Chapter 1, the upcoming part aims at a thorough calibration of theoretical expressions to present data. For our analysis we consider the base-load prices of monthly, quarterly and annual ®PHELIX futures contracts traded at the European Energy Exchange AG during the period from 02.01.2012 to 14.04.2017. ®PHELIX futures contracts are financially settled although physical delivery is also possible in the market region of Germany/Austria. More than half of the volume traded at the EEX's electricity forward market stems from such futures contracts. Besides, they feature the greatest liquidity among financial electricity futures products in Europe.

2.1 Fitting of coefficient processes for the re-parametrized forward curve

At first, we consider the forward curve in Musiela parametrisation g cf. Section 1.3 and assume 1.7 to be in place. We are interested in finding a suitable basis representation of the re-parametrized forward curve which includes its coefficient processes $a_1, \dots, a_4$ as well as the parameters λ and μ that shape the vector space $V_{\lambda,\mu}$. To this end, we proceed in two steps:

1. **estimate coefficient processes for $\partial_x^{-1} g(t,\cdot) \in V_{\lambda,\mu}$ $t \geq 0$ and the parameters λ, μ**
   Initially, we fit the theoretical futures prices to observed ones that we denote by $G_O(t,x,\ell)$ for each time-point t in the present observation period. Making use of representation (1.20) of the futures price with regards to $h_t(\cdot) = \partial_x^{-1} g(t,\cdot) \in V_{\lambda,\mu}$, which we educed in Section 1.3.1, this first optimization yields an estimation of the vector space parameters λ and μ as well as the coefficient processes $\tilde{a}_1, \dots, \tilde{a}_4$ of $h_t(\cdot)$.

2. **estimate coefficient processes for $g(t,\cdot) \in V_{\lambda,\mu}$ $t \geq 0$**
   With $\tilde{a}_1, \dots, \tilde{a}_4$ and λ, μ at hand, it only remains to determine the coefficient processes $a_1, \dots, a_4$ of g. For each time-point t, those are obtained by applying the coordinate matrix of the linear mapping $\partial_x : V_{\lambda,\mu} \to V_{\lambda,\mu}$ to the coefficients of $h_t(\cdot)$, that is we have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\mu & 1 \\ 0 & 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \\ \tilde{a}_4 \end{pmatrix} = \begin{pmatrix} -\lambda\tilde{a}_1 + \tilde{a}_2 \\ -\lambda\tilde{a}_2 \\ -\mu\tilde{a}_3 + \tilde{a}_4 \\ -\mu\tilde{a}_4 \end{pmatrix}$$

We resort to two different methods for the fitting in step one. First, a standard non-linear least square (in short: nls) algorithm is exercised. Second, in view of liquidity issues we introduce weights to the nls estimation, that should emphasize the fitting of futures contracts on the short in expense of those on the long end.
In the ensuing Sections 2.1.1 and 2.1.2 we like to discuss each method separately, whereupon Section 2.1.3 encompasses a comparison of said methods.
Regarding both of the non-linear least square algorithms listed below, it should be noted that convergence problems are likely to arise for days where only few distinct products are traded. Furthermore, convergence may fail due to contracts whose delivery period has already started. The latter we exclude in both fitting processes, as such products are rather illiquid anyhow and thus not adding much relevant information about price movements. As for days in the data on which only a small number of different contracts is traded, the coefficients fitted for those time-points are taken to be the mean value of preceding and subsequent days' values, assuming that from one day to another the forward curve does not change significantly.
The results presented in this chapter are obtained from optimisation conducted via the free software R.

2.1.1 Method 1: Non-linear Least Square

We initially focus on calibrating the coefficient processes of $\partial_x^{-1} g$ using a standard nls-approach. Recall from Lemma 1.11 that we have $h_t(\cdot) := \partial_x^{-1} g(t, \cdot) \in V_{\lambda,\mu}$ for any $t \in \mathbb{R}_+$. Hence, with regard to (1.20), we consider for fixed time t the following problem:

$$\min_{\tilde{a}_i(t), i=1,\ldots,4} \sum_{x,\ell} \left[\frac{1}{\ell} \left(h_t(x+\ell) - h_t(x) \right) - G_O(t,x,\ell) \right]^2 \tag{2.1}$$

where $h_t(x) = \sum_{i=1}^4 \tilde{a}_i(t) b_i(x)$, and $b_i(x) \in \left\{ e^{-\lambda x}, x e^{-\lambda x}, e^{-\mu x}, x e^{-\mu x} \right\}$. In order to derive sensible values for the vector space parameters λ and μ, we conduct a six-dimensional optimisation to begin with. Then we perform the four dimensional non-linear least square optimisation (2.1) using the mean values for λ and μ and the previously obtained estimates for the coefficients as starting values. This first calibration yields as vector space parameters:

$$\lambda = 0.01816636 \qquad \mu = 0.00009999$$

The estimated coefficient processes are visualized in figure 2.1. We observe that the vector space parameter μ is significantly smaller than λ. Hence, the basis elements corresponding to the latter $b_1(x) = \exp(-\lambda x)$ and $b_2(x) = x \exp(-\lambda x)$ indeed contribute to the shape of $\partial_x^{-1} g$ on the short end, however, their influence weighs off more quickly than that of the basis vectors associated with the former parameter.
An application of the linear transform outlined in step 2 yields the corresponding coefficient processes $a_1, \ldots, a_4$ for the forward curve as function of time to maturity. Those are depicted in figure 2.2. Once more, as the vector-space parameters remain unchanged, we observe that the fitted functional form of the re-parametrized forward curve is (especially in the long run) mainly determined by the basis elements b_3 and b_4. We observe that, due to its coefficient's comparably large values, basis vector b_3 contributes the main part to the level of the forward price. This particular basis vector could be interpreted as representing the overall price trend.

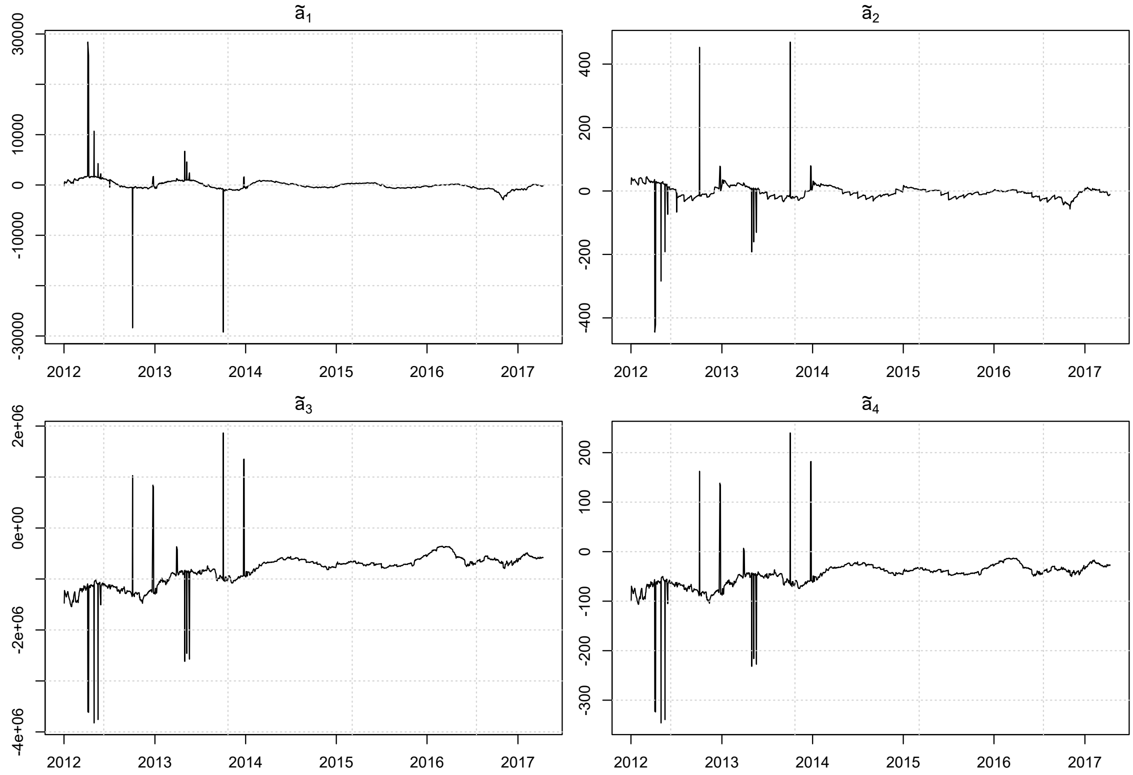

Figure 2.1: Coefficient processes of $\partial_x^{-1} g$ resulting from ordinary nls estimation.

As for the basis elements b_1 and b_2, those could be seen as seasonal adjustments to the short end of the forward. Their seasonal pattern becomes more evident in figure 2.3. Although the fourth basis element shows a slight, recurring pattern as well, it could also be interpreted as the non-foreseeable random effects that contribute to the shape of the forward price.
Moreover, we note that every coefficient process displays some distinct comparably extreme values at similar points in time. Those outliers solely appear before 2014 and may represent reactions to surprising developments in the relatively young market.

2.1.2 Method 2: Weighted non-linear Least Square

The second estimation takes into account the liquidity of the different products disposable for trade. This method involves introducing a weighting function $v : \mathbb{R}_+ \to \mathbb{R}$ to the nls-estimation in (2.1) that - among each group of products with equal length of delivery (in our case: monthly, quarterly and yearly futures contracts) - is decreasing in time to maturity x. This approach rests on the observation that futures on the short side are much more actively traded, compared to those on the long side, that might even be highly illiquid. Moreover, liquidity of a futures contract decreases more severely in time to maturity the longer its specified delivery period lasts. The constructed weighting function should mirror this by putting equally most weight on the monthly, quarterly or yearly products with lowest time to maturity, whereas the futures prices of the remaining contracts should be rated less important in decreasing order. The rate at which the weight decreases with growing time to maturity should hence depend on the length of the delivery period specified by the respective futures contract.
As suitable weighting function we choose in this section for some fixed time-point $t \in \mathbb{R}_+$ and any

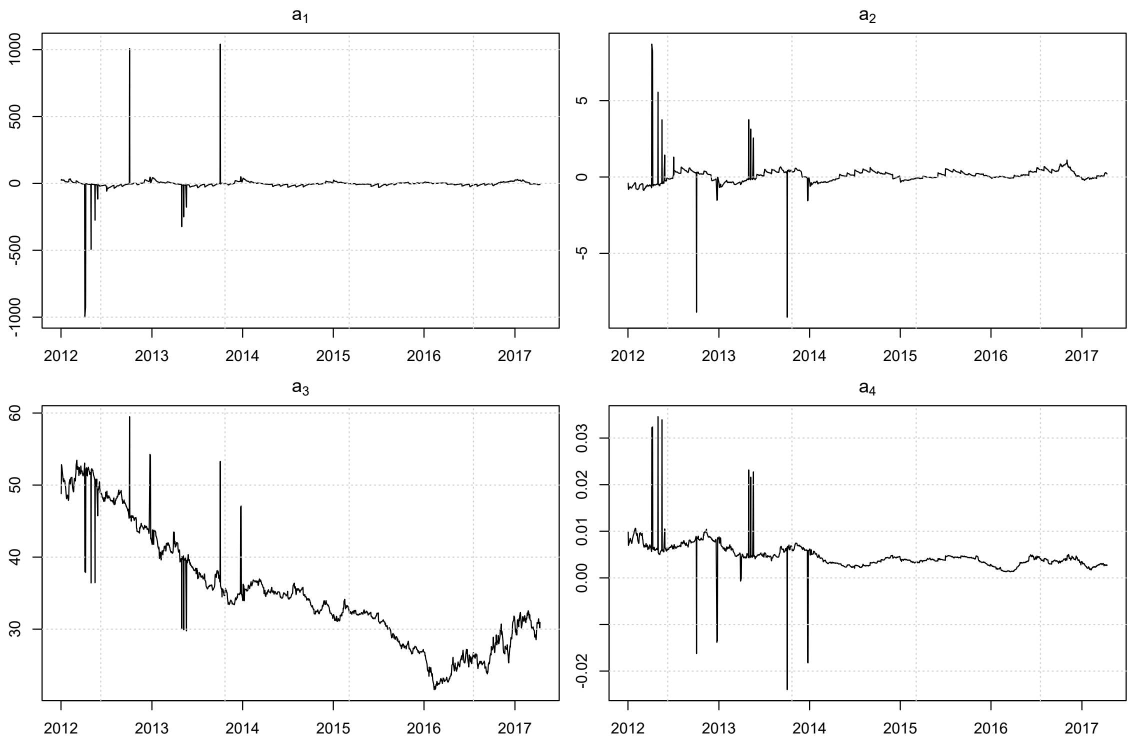

Figure 2.2: Coefficient processes of the re-parametrized forward curve g obtained via ordinary nls estimation.

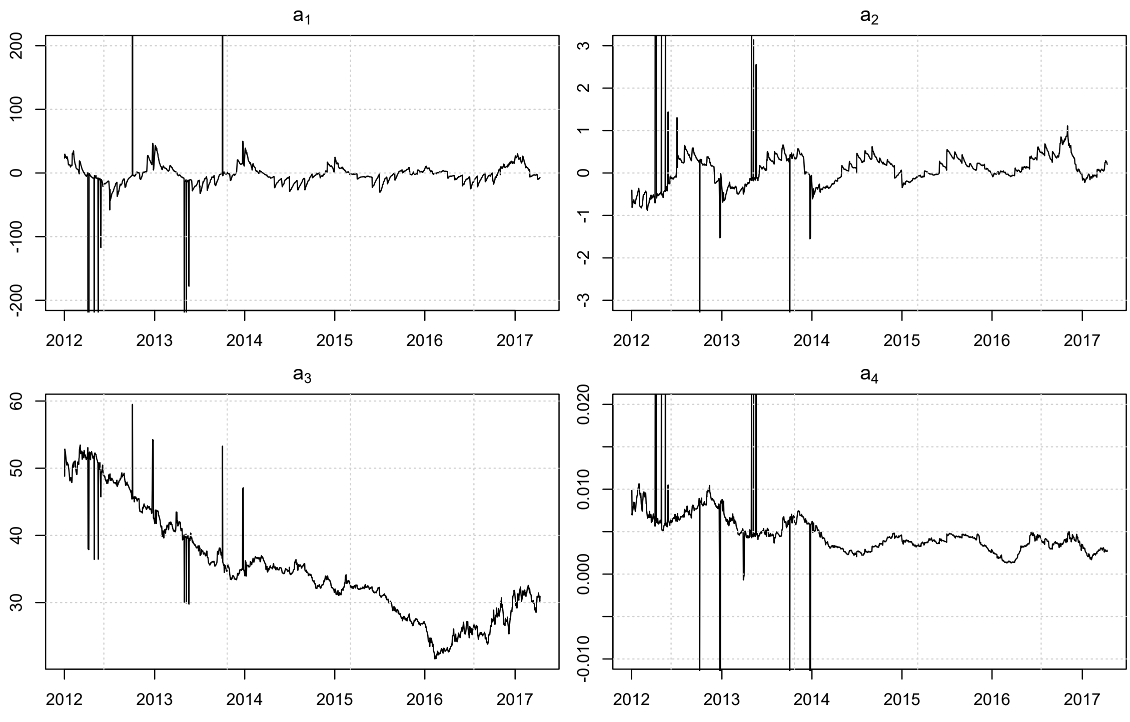

Figure 2.3: Zoom on coefficient processes of the re-parametrized forward curve g obtained via ordinary nls estimation.

finite $O_t \subset [0,\xi] \times \mathbb{R}_+$:

$$v_t(x;\ell) := \frac{1}{\sum\limits_{(x,\ell)\in O_t} \frac{1}{1+\left(\frac{x-x_\ell}{100}\right)^2}} \cdot \frac{1}{1+\left(\frac{x-x_\ell}{100}\right)^2}, \qquad (x,\ell) \in O_t \tag{2.2}$$

In our setting, we have

$$O_t := \{(x,\ell) \in [0,\xi] \times \mathbb{R}_+ : \exists \text{ futures contract at time } t \text{ with time to maturity } x \text{ and length of delivery } \ell\}$$

Moreover, for any futures contract with length of delivery ℓ, x_ℓ (in simplified notation) represents the smallest value of time to maturity among all (currently traded) futures contracts with matching length of delivery period. This weighting function hence puts weight $1/c$ on products on the very short end among their delivery group, where $c := \sum\limits_{(x,\ell)\in O_t} \frac{1}{1+\left(\frac{x-x_\ell}{100}\right)^2}$ denotes the normalizing constant. Calibration of the coefficient processes in this section is then performed solving the problem:

$$\min_{\tilde{a}_i(t), i=1,\ldots,4} \sum_{x,\ell} v_t(x) \left[\frac{1}{\ell}\left(h_t(x+\ell) - h_t(x)\right) - G_O(t,x,\ell)\right]^2 \tag{2.3}$$

The optimisation then is performed analogously to before in Section 2.1.1. Similarly, we obtain as estimates for the parameters that determine the vector space $V_{\lambda,\mu}$

$$\lambda = 0.01395158 \qquad \mu = 0.0001000013$$

Thereafter, we proceed with step 2 and further determine the forward curve's coefficient processes that are visualized in figure 2.4. We again observe that each of those attains few distinct extreme values that may later on be interpreted as jumps. However, compared to the results obtained from estimation 1, the processes' periodic behaviour now shows more clearly. Figure 2.5 provides a more detailed view on the calibrated coefficient process a_1. Once more, we observe a distinct seasonal pattern.

2.1.3 Comparison of estimated coefficient processes

In the following section we will compare the estimated forward curves obtained by the standard nls-approach of Section 2.1.1 to those that result from the weighted optimisation conducted in 2.1.2.

With the calibrated coefficients at hand, we may compare both method's fitted forward curves to observed futures prices at arbitrary time-points. Exemplarily, we consider the power contracts available on June 22$^{\text{nd}}$ 2016 and visualize the adjusted forward curves in figure 2.6. In said figure, the horizontal dotted lines are closing prices of monthly, quarterly and yearly futures contracts, whereby the length of those lines corresponds to the length of the delivery period specified in the particular futures contract. The solid lines show the calibrated forward curves for that day for both the standard and weighted optimisation.

We note that the red forward curve, estimated using weights, in fact emphasizes futures prices on the short end in expense of those on the long end. Indeed, the prices of yearly contracts whose time to maturity is greater than three years do not seem to be considered to a great extend for

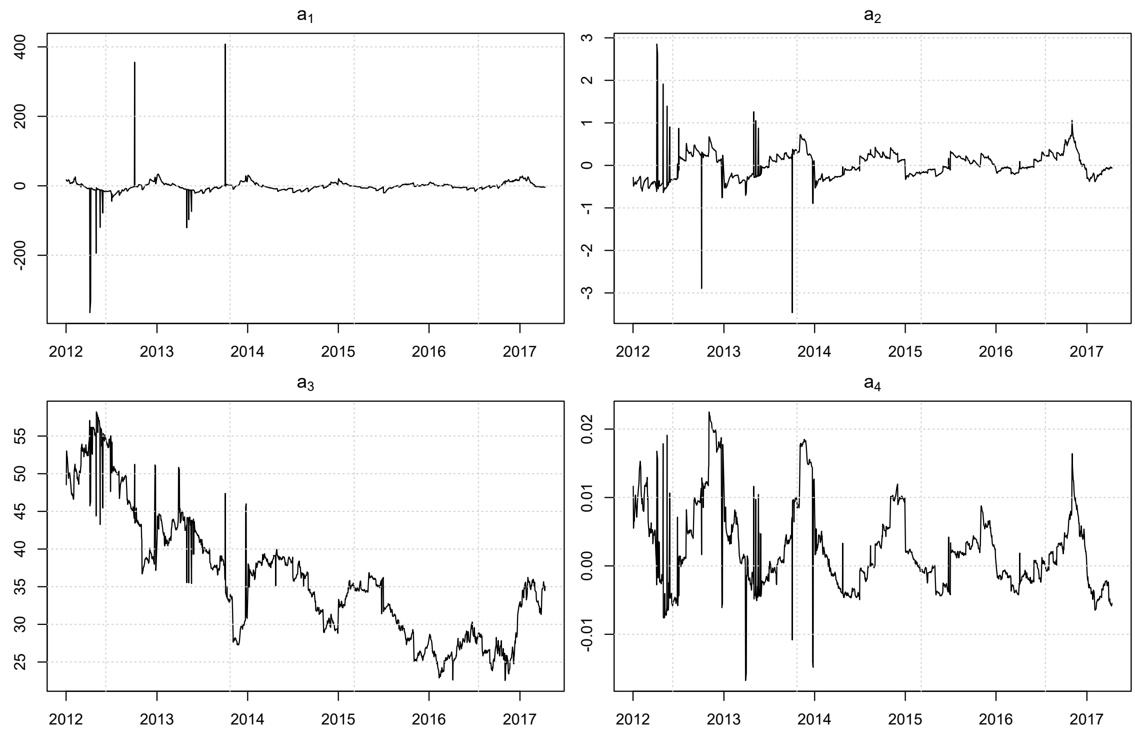

Figure 2.4: Coefficient processes of the re-parametrized forward curve g obtained via weighted nls estimation.

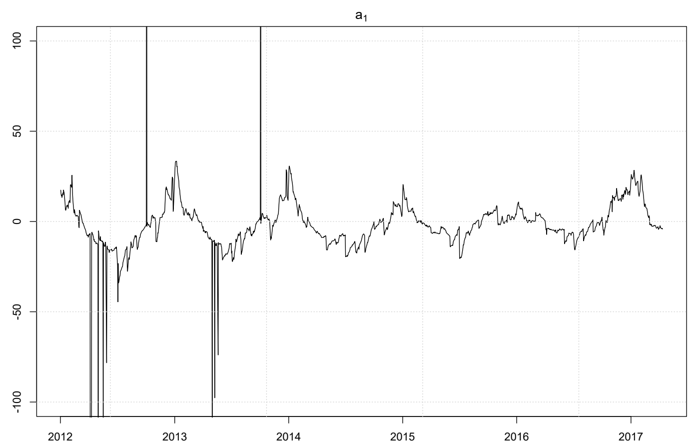

Figure 2.5: Zoom on coefficient process a_1 of the re-parametrized forward curve g obtained via weighted nls estimation.

calibration. Instead, the forward curve resulting from weighted nls is strongly oriented towards capturing correctly the prices of monthly and most of the quarterly products.
In contrast, the blue curve mirrors the fact that every futures price is deemed to be equally important in ordinary nls estimation.
The extent to which this difference in approximations comes into effect shows more clearly in the comparison shown below in figure 2.7. For each class of futures contracts, grouped by their length of delivery period, this figure shows a comparison of both estimated forward curves (as functions

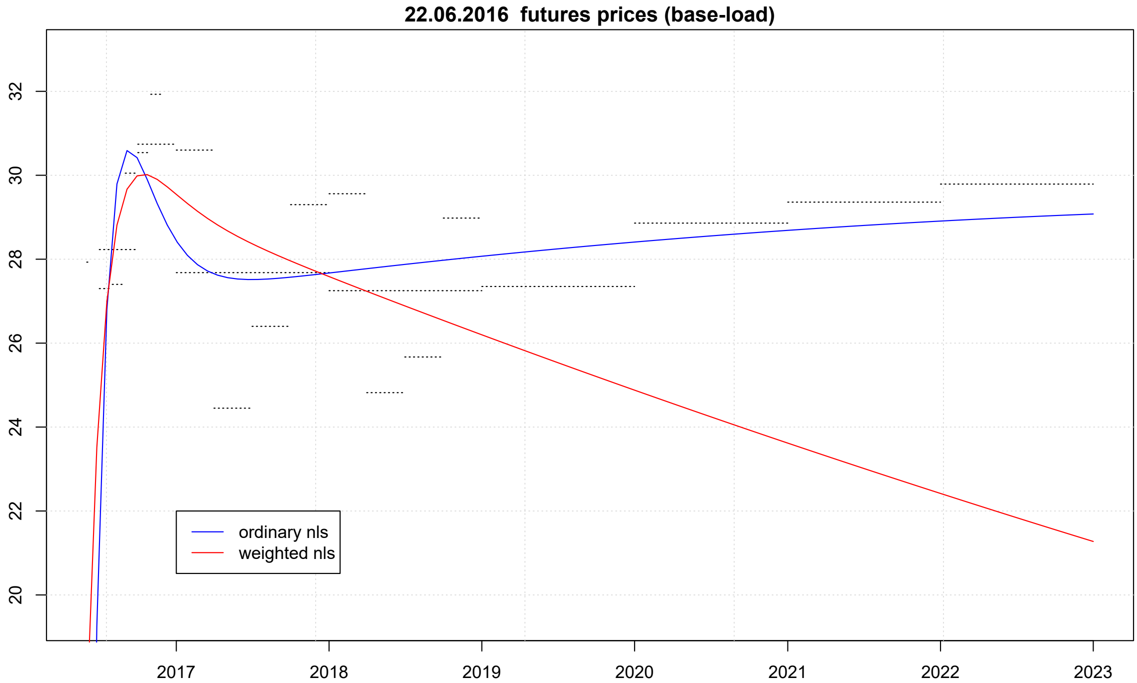

Figure 2.6: Futures prices for power contracts traded on 22$^{\mathrm{nd}}$ of June 2016 and the fitted forward curves for both methods of estimation.

of time to maturity) superimposed on observed prices (in MWh) for the same date as in figure 2.6. We now clearly observe mechanism of the weighting function. While monthly and quarterly prices seem to be fitted more accurately with the weighted estimation, yearly contracts whose maturity lies far in the future are hardly considered in the optimisation process.
Regarding this first comparison, we remark the following: One may be interested in capturing the prices on the short end more correctly while still not deteriorating from given futures prices on the long end, however uninformative they might be. One option to proceed would be to consider a "hybrid-version" of estimated coefficient processes, considering those arising from weighted and ordinary nls-estimation for fitting the short respectively long end of the forward curve. This way, one could capture the (due to increasing uncertainty) typically rising prices for products with large time to maturity.
Although in practice one might opt for such a hybrid form of coefficient processes, for the succeeding analyses we will work with the coefficient processes obtained by the economically reasonable weighted nls-estimation.

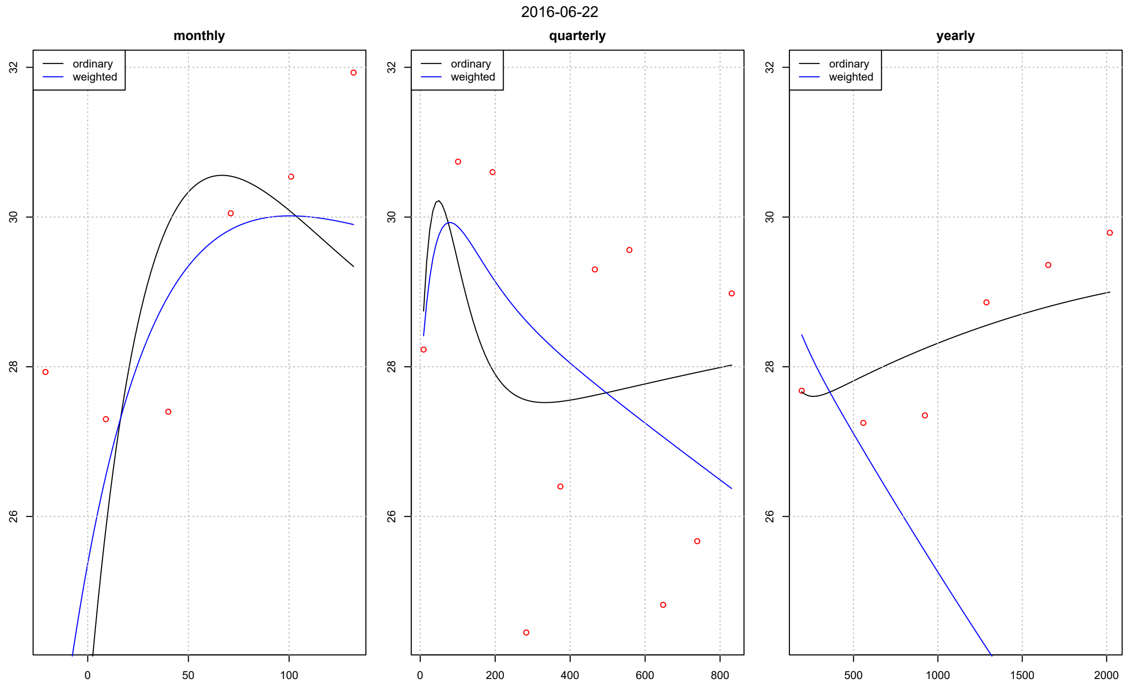

Figure 2.7: Futures prices for monthly, quarterly and yearly power contracts traded on 22^{nd} of June 2016 and the fitted forward curves for both methods of estimation.

Chapter 3

Applications of the model

In this chapter, we discuss two possible applications of the model presented in this thesis: First, we introduce a method to simulate paths of the forward in Section 3.1. Second, we address the topic of quantifying the market volatility. This will be covered in Section 3.2.
We start by building the basis of the succeeding discussion. Recall that the dynamics of the re-parametrized forward are (cf. Lemma 1.13)

$$g(t) = g(0) + \int_0^t \partial_x g(s)\, ds + \int_0^t c(s)\, dL_s \qquad t \geq 0$$

where L is a d-dimensional Lévy process with independent components. For simplicity, we assume that the d-dimensional Lévy process reduces to four dimensions and consists of the sum of a Brownian motion B and compound Poisson process $C_t := \sum_{i=1}^{N_t} Y_i$ for $t \geq 0$, where Y_i are i.i.d. and independent of both B and the Poisson process N[1]. Hence, our ansatz reads as

$$g(t) = g(0) + \int_0^t \partial_x g(s)\, ds + \int_0^t c(s)\, dB_s + \int_0^t c(s)\, dC_t \tag{3.1}$$

Moreover, due to the FDR-Assumption 1.7 we may, for our subsequent analysis, consider the isomorphic coordinate representation of the re-parametrized forward g in the vector space $V_{\lambda,\mu}$, that is we have

$$g(t)(\cdot) \cong (a_1(t), a_2(t), a_3(t), a_4(t))^T =: a^T(t) \qquad \forall t \geq 0$$

Denoting the continuous and jump part of this 4-dimensional coefficient process a by a^c and a^d respectively, we obtain

$$a = a^c + a^d$$

In order to receive observations of both a^c and a^b that will be needed in the following sections, we need to distinguish observations that most likely can be ascribed to the compound Poisson, from those we assume to stem from the continuous part. Thus, there is need to set up a rule that classifies observations in either category. We do so by considering absolute returns of the observed coefficient processes $a_1, \ldots, a_4$ and selecting those which exceed the 97.5% sample-quantile as "jumps". Adjusting the observed process for those jumps we obtain the continuous time-series sample visualized in figure 3.1. The remaining observations of course yield a sample of the

[1]This simplification is well justified in the one dimensional case, where any Lévy process may be approxiamted by the independent sum of a Brownian motion and a "compound Poisson"-like process (see [1] Chapter XII). In the multidimensional setting, it is motivated by the assymptotic result that is presented in [3].

discontinuous part a^d.

With those separate observations at hand, we may in what follows conduct the analyses mentioned in the beginning.

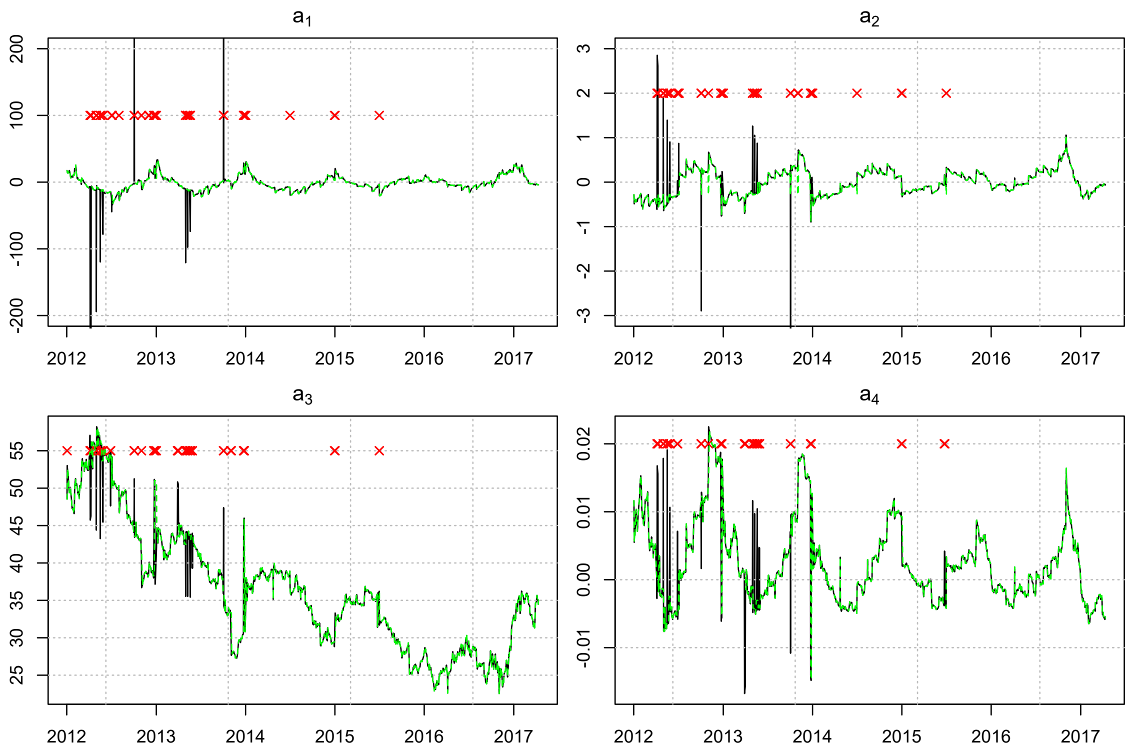

Figure 3.1: Coefficient processes of the re-parametrized forward g estimated via weighted nls (black solid line) and the adjustment for jumps (green dashed line). The red crosses indicate positions where jumps are said to have occured.

3.1 Simulation of forward price

For simulation within the presented model we consider the representation of the forward within the vector space $V_{\lambda,\mu}$, that is we simulate $a = a^c + a^d$. Motivated by the behaviour of the fitted coefficient process, which strongly suggests that the continuous part is governed by a seasonal pattern, we assume the following dynamics of a^c:

$$a_i^c(t) = \ell_i + h_i \cos(\varphi_i(t - t_i)) + s_i t + \tilde{a}_i(t) \qquad t \geq 0, \, \forall i \in \{1, \dots, 4\}$$

where $t \mapsto s(t)_i := \ell_i + h_i \cos(\varphi_i(t - t_i)) + s_i t$ is a deterministic, Lipschitz continuous function capturing the seasonal movements of the coefficient process' i$^{\text{th}}$ component and $\tilde{a}_i$ denotes a remaining centred Brownian noise. Note that we incorporate in this functions a trend component as well. We fit the coefficients that shape this deterministic pattern once more with ordinary least square optimisation. The results are depicted below in figures 3.2 and 3.3, where the fitted seasonality functions are super-imposed on the observed continuous time-series. The remaining centred diffusions are visualised as well.

Strictly speaking, one may observe that although the deterministic function s_3 captures well the trend behaviour of coefficient process a_3, there might still be a recognizable yearly pattern in the

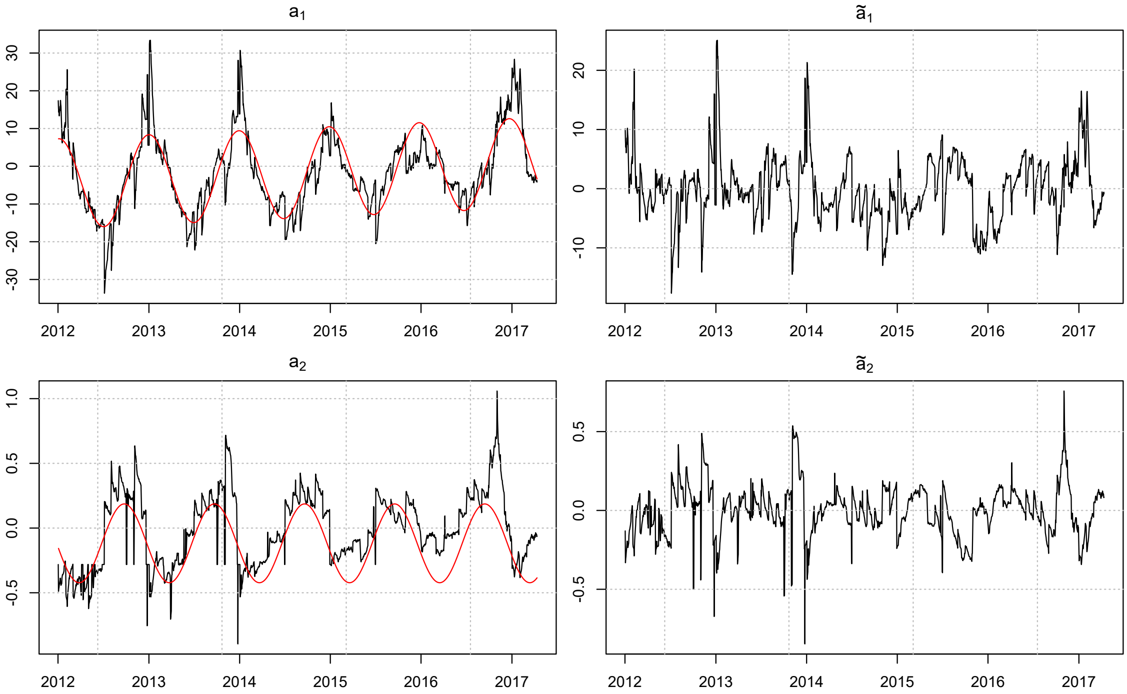

Figure 3.2: The top and bottom left graphs show the coefficient processes a_1 and a_2 of the re-parametrized forward g estimated via weighted nls, adjusted for jumps and the fitted seasonal function super-imposed. To the right, the remaining noise series are visualized.

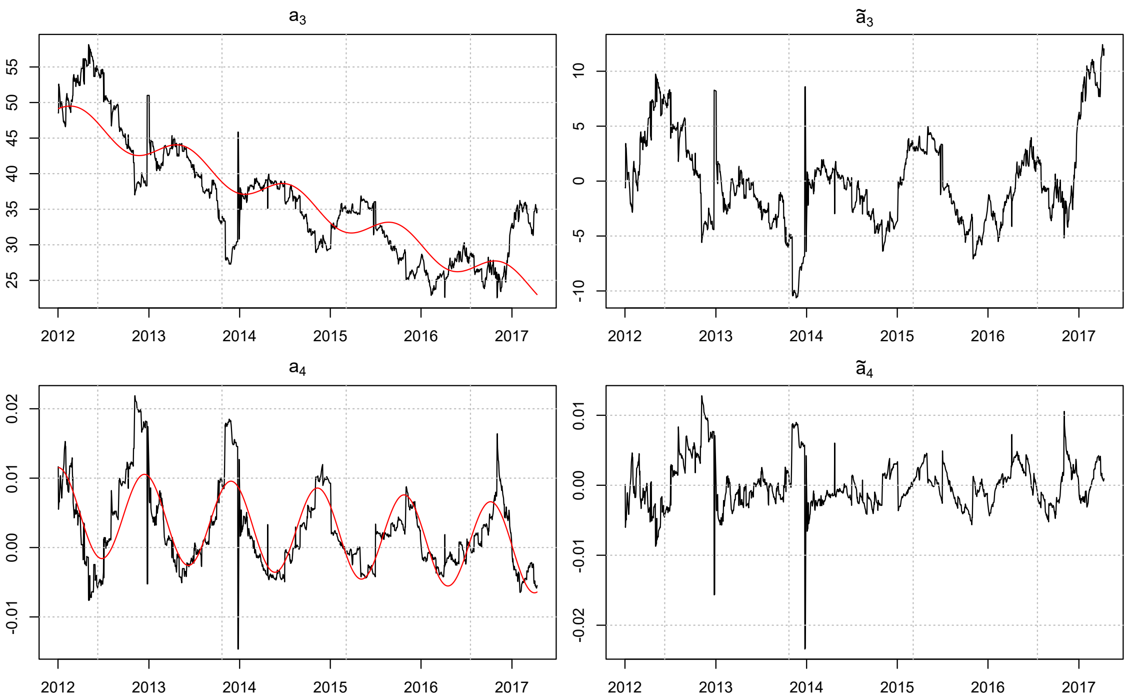

Figure 3.3: The top and bottom left graphs show the coefficient processes a_3 and a_4 of the re-parametrized forward g estimated via weighted nls, adjusted for jumps and the fitted seasonal function super-imposed. To the right, the remaining noise series are visualized.

noise series. However, we consider the sesaonal function a good enough fit for the pursuing analysis.

Once a fit for the seasonal function is obtained, simulation of paths for the forward price is a matter of generating independent Gaussian and compound Poisson variables. Motivated by the fitted coefficient processes, which clearly exhibit jumps at similar times, we consider the following simple model for simulation: We assume that jumps are introduced to all four coefficient processes by one common Poisson shock $N_t, t \geq 0$. The jump sizes are specific to each process and modelled by i.i.d. 4-dimensional random vectors $(Y_j)_{j=1}^{\infty}$. Thus the model we consider reads as

$$a_t^i = s(t)_i + B_t^i + \sum_{j=1}^{N_t} Y_j^i \qquad i = 1, \ldots, 4$$

where $(B_t)_{t \geq 0}$ is a 4-dimensional Brownian motion with covariance matrix Σ. Moreover the dependence structure between the jumps of the four coefficient processes $a^1, \ldots, a^4$ need be specified. If we make the simplifying assumption that $Y_j \sim \mathcal{N}(\mu, \tilde{\Sigma})\ \forall j$, this amounts to determining the mean vector μ and covariance matrix $\tilde{\Sigma}$. Estimates for these two parameters can easily be obtained from the realized jump-sizes of the discontinuous process a^d. In addition, we derive approximate parameter values for the intensity of the Poisson driver via the classical parametric maximum likelihood estimator dividing realized jump-times by overall length of the observation period. Finally, the covariance matrix of the centred Brownian noise is estimated from the realized $\tilde{a}$ processes.
The results of a MC-simulation following the steps outlined above are visualized in figures 3.4-3.6. First, we consider the simulated coefficient processes for a time horizon of approximately three years (1200 days). We observe that the estimated seasonality acts particularly strong on the fourth coefficient process, as the estimated impact of both the corresponding Brownian noise and the variation in the jumps remains rather weak. Moreover, the fitted trend in coefficient a_3 is mirrored as well. Note however, that the typical behaviour of the forward prices to display occasional spikes, which also shows in the fitted coefficient processes (see e.g. figure 2.4), can in general not be captured by the simulation method outlined in this section. As it seems, the underlying Poisson driver is not particularly suited for generating such "kick-backs".
Second, in figure 3.5, we visualize resulting paths of the forward price for two exemplary contracts specifying delivery in approximately one quarter and one year respectively. Moreover, the cross sections indicated by the red dashed lines in each graph correspond to such futures prices at 30 and 120 days ahead. The entire forward curves for those time-points are then depicted in figure 3.6.

Concluding this section, we note once more that the simplification of the jump structure comes at the expense of capturing the typical leptokurtic behaviour of forward prices. Although simple to implement, the presented method can be improved by considering processes with smaller frequency and larger magnitude of the jumps.

3.2 Estimating market volatility

Another objective of a model as presented in this thesis may be to quantify the market covariance structure. We like to do so by specifying its model-free notion of quadratic covariation. Hence, the following section is dedicated to determining the covariation process of the re-parametrized

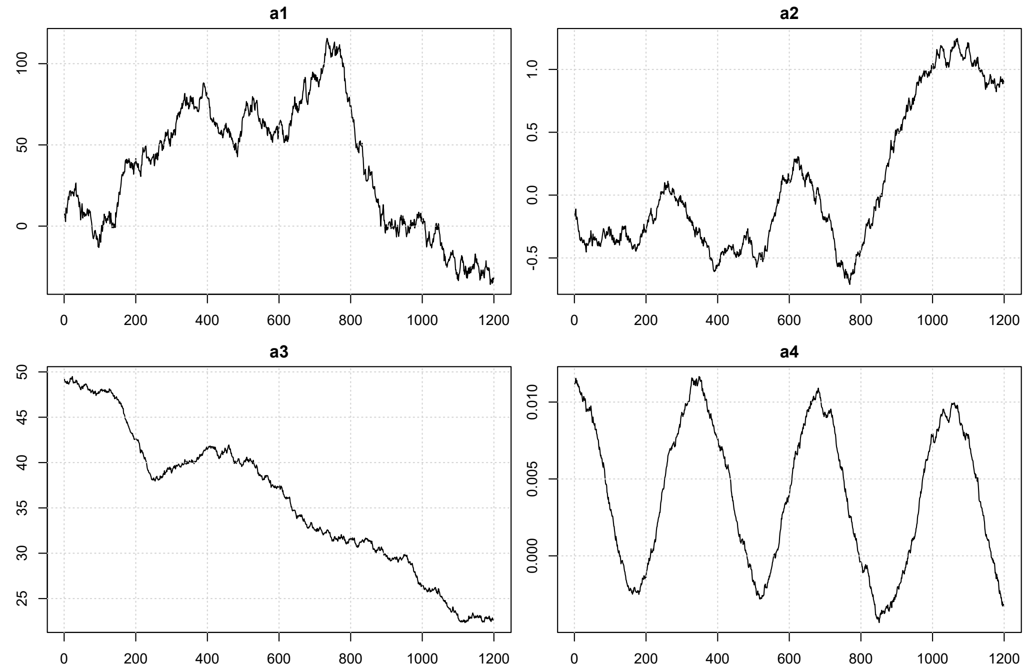

Figure 3.4: Simulated coefficient processes resulting from MC-simulation.

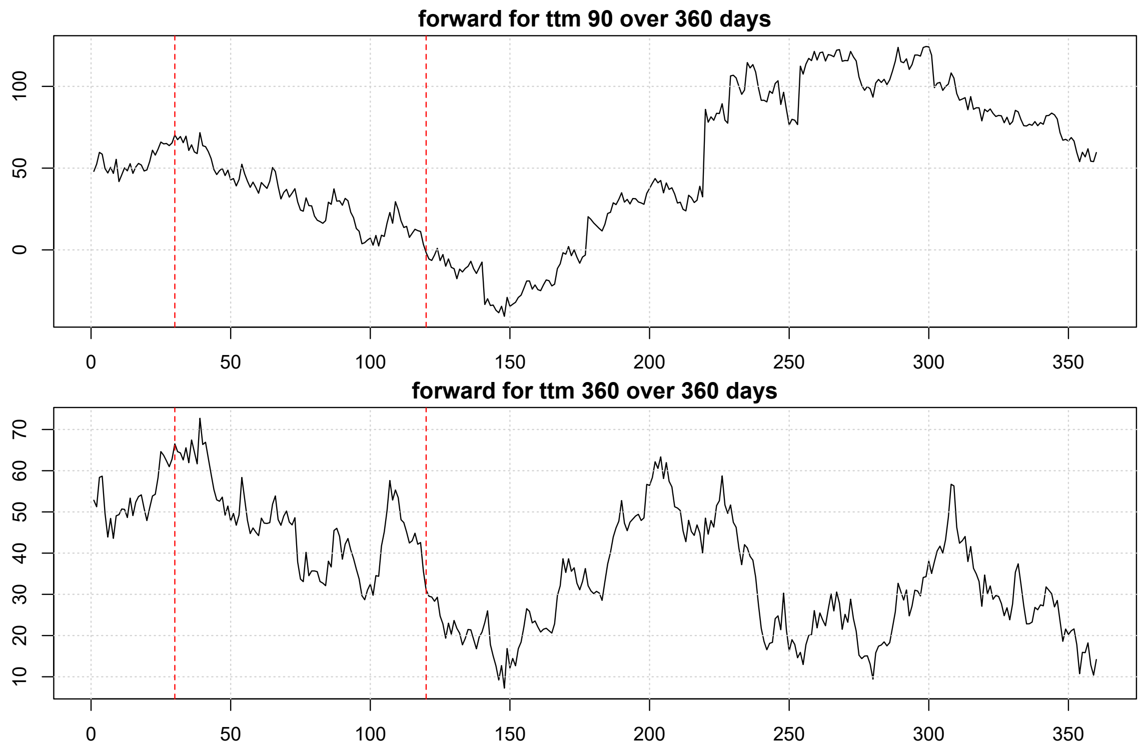

Figure 3.5: Simulated paths for forward prices with time to maturity of 90 respectively 360 days over the period of 360 days. The dashed red lines correspond to such prices at 30 and 120 days from today.

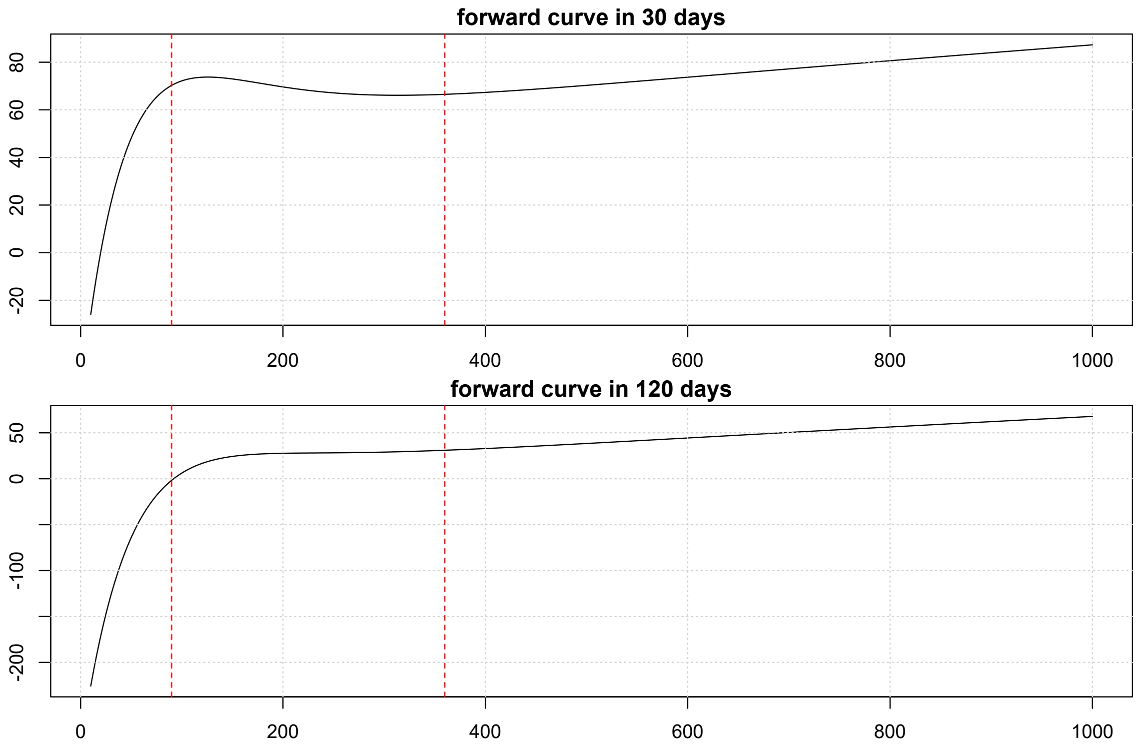

Figure 3.6: Simulated forward curves at 30 respectively 120 days ahead. The dashed red lines indicate forward prices at the respective time-points that specify delivery in 90 and 360 days.

forward, which, with regard to (3.1), is given by

$$[g]_t = \int_0^t c(s)c(s)^T\, ds + \sum_{0<s\leq t} \left(c(s)Y_s\right)\left(c(s)Y_s\right)^T \mathbb{1}_{\Delta C_s \neq 0} \qquad t \geq 0 \tag{3.2}$$

Moreover, due to the FDR-Assumption 1.7 we may determine the quadratic variation in (3.2) via its equivalent representation using the corresponding coefficient processes $a_1, \ldots, a_4$:

$$[a]_t = [a]_t^c + \sum_{\substack{s\in[0,t] \\ \Delta a_s \neq 0}} |\Delta a_s||\Delta a_s|^T \tag{3.3}$$

With the coefficient processes fitted in Section 2.1 at hand, we are now able to determine an estimate for (3.3). In what follows, we will specify the sample covariance for both the jump and continuous part separately using the distinct observations that we obtained in the beginning of this chapter. The discontinuous part can easily be estimated taking the sum of these squared returns of the observations corresponding to a^d.
Estimation of $[a]_t^c \cong \int_0^t c(s)c(s)^T\, ds$ is more elaborate. For the calibration we consider the observed time series adjusted for jumps as depicted by the green line in figure 3.1. We may then proceed by estimating the quadratic variation of the four-dimensional coefficient process. Alternatively, one may as well consider the noise processes adjusted for seasonality $\tilde{a}_1, \ldots, \tilde{a}_4$, that were fit in Section 3.1, since the deterministic and Lipschitz continuous seasonality function s has no effect on the quadratic variation. We opt for the latter, and hence estimate for every time-point $t \leq T$ the quadratic variation of the coefficient processes adjusted for periodicity $[\tilde{a}]_t$.

The classical estimator for this integrated covariance matrix is given by the realized covariance matrix defined as

$$RC(\tilde{a})_t(i,j) = \sum_{n=1}^{N}(\tilde{a}_i(t_n) - \tilde{a}_i(t_{n-1}))(\tilde{a}_j(t_n) - \tilde{a}_j(t_{n-1}))$$

where $t_0 = 0 < \ldots < t_N = t$ and $i, j \in \{1, \ldots, 4\}$. However, this estimator is known to be biased as observations commonly are contaminated by market micro structure noise. Those disturbances are induced by various frictions in the trading process such as the bid-ask spread or asymmetric information of traders. Moreover, the discreteness of observations of continuous price processes may add to such inaccuracies as well. Despite their small size, market micro structure noise accumulates at high frequency and renders the realized covariance strongly biased.
Therefore, we determine the multivariate extension of the two scaled realized variance estimator, an estimate that accounts for a bias-correction using sparse sampling[2]. For some sampling frequency $K \in \mathbb{N}$ it is given by

$$TSRC(\tilde{a})_t(i,j) := \frac{1}{K}\sum_{k=1}^{K} RC(\tilde{a})_t^k - \frac{n-K+1}{nK} RC(\tilde{a})_t\,(i,j) \qquad t \leq T$$

where $RC(\tilde{a})^k$ is obtained by splitting the entire set of observations into K non-overlapping sub-grids (with sampling frequency K) and computing the standard estimate for observations from the k^{th} sub-grid among those.
Below we provide the estimates we obtained for the yearly integrated covariance matrices (normalized to one trading day) using the TSRC-estimate. Table 3.1 shows the average daily integrated volatility per year in the observation period corresponding to the continuous part of the estimated coefficient processes. Compared to the remaining coefficients, the observed absolute change of $\tilde{a}_4$ seems negligible. Moreover, overall volatility mostly tends to decrease over the years, at least by some slight margin. Table 3.2 collects the corresponding mean covariances per trading day observed in the period 2012-2017. Here too, we note that the mutual influence the (continuous) coefficient processes have on each other decreases over the years.

	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$
2012	1.8361	0.0565	0.7086	0.0010
2013	1.8235	0.0560	0.6467	0.0010
2014	1.4968	0.0303	0.3180	0.0004
2015	1.3618	0.0340	0.3956	0.0006
2016	1.1759	0.0404	0.5030	0.0007
2017	1.5662	0.0227	0.4634	0.0004

Table 3.1: Yearly integrated volatility per trading day of the coefficient processes' continuous parts.

Taking into account the quadratic variation of the coefficient processes' jump parts as well, we arrive at the estimated market volatility and covariance (see tables 3.3 and 3.4). Comparing the results in table 3.3 to those in 3.1, we observe that especially in the early years, the variation in the jumps added highly to the volatility observed in the market.
In contrast to the covariance estimated from the continuous part only, we note that the estimates of the overall mean covariance structure rather indicate that there might be positive dependence between some of the coefficients. This mirrors the fact that, judging by the data, jumps occur in

[2] For the univariate case see [13].

	$\tilde{a}_1/\tilde{a}_2$	$\tilde{a}_1/\tilde{a}_3$	$\tilde{a}_2/\tilde{a}_3$	$\tilde{a}_1/\tilde{a}_4$	$\tilde{a}_2/\tilde{a}_4$	$\tilde{a}_3/\tilde{a}_4$
2012	-0.0678	0.6490	-0.0259	-0.0010	0.0000	-0.0007
2013	-0.0647	0.6255	-0.0285	-0.0007	0.0000	-0.0006
2014	-0.0404	0.2726	-0.0060	-0.0005	0.0000	-0.0001
2015	-0.0391	0.2440	-0.0070	-0.0006	0.0000	-0.0001
2016	-0.0173	0.1800	-0.0078	-0.0001	0.0000	-0.0001
2017	-0.0292	0.3376	-0.0051	-0.0002	0.0000	-0.0001

Table 3.2: Yearly integrated covariance per trading day of the coefficient processes' continuous parts.

all four coefficient processes at similar times.

	a1	a2	a3	a4
2012	133.5487	1.3329	2.8876	0.0053
2013	99.9583	1.0002	2.4826	0.0046
2014	2.5858	0.0471	0.3180	0.0004
2015	2.6076	0.0503	0.4036	0.0006
2016	1.1759	0.0404	0.5030	0.0007
2017	1.5662	0.0227	0.4634	0.0004

Table 3.3: Yearly realized volatility per trading day.

	a1/a2	a1/a3	a2/a3	a1/a4	a2/a4	a3/a4
2012	174.8150	316.1266	3.3853	0.5741	0.0063	0.0138
2013	97.9574	198.7920	2.1551	0.3577	0.0040	0.0101
2014	0.0348	0.2726	-0.0060	-0.0005	0.0000	-0.0001
2015	0.0433	0.4162	-0.0042	-0.0005	0.0000	-0.0001
2016	-0.0173	0.1800	-0.0078	-0.0001	0.0000	-0.0001
2017	-0.0292	0.3376	-0.0051	-0.0002	0.0000	-0.0001

Table 3.4: Yearly realized covariances per trading day.

As final note, we remark that the main advantage of models of Heath-Jarrow-Morton type over models of the spot price is that they are better suited to describe options on futures prices. Although models on the spot price of energy nicely fit the initial curve, the market prices of derivatives are mostly inconsistent with those arising from said model. With that in mind, the idea to model the whole curve of instantaneous forward rates proposed by Heath et al. in the interest rate setting has been accommodated in the context of energy markets. The challenge of fitting those infinite dimensional objects to finitely many observed futures prices can, as outlined in the present thesis, be taken up by proposing a suitable finite dimensional realisation for the forward curves. Hence, the present model may well serve as basis for subsequent analyses of options on energy futures.

Appendix A

A recap on Lévy processes

In the following we will outline some results about Lévy processes without proofs that are used throughout this thesis. These subsequent findings are obtained from both [11] and the lecture notes [10]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ henceforth be the underlying probability space. We start by giving the definition of an $\mathbb{R}^d$-valued Lévy process which constitutes a central object in this thesis.

Definition A.1 *We call a stochastic process $X := (X_t)_{t \geq 0}$ with values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ a (d-dimensional)* ***Lévy process****, if it satisfies the following conditions:*

(L1) *X has $\mathbb{P}$-almost surely cádlág paths*

(L2) *X has stationary increments, i.e. $X_t - X_s \overset{d}{=} X_{t-s}$ for any $0 \leq s \leq t$*

(L3) *The increments of X are independent of the past, i.e. $X_t - X_s$ is independent of $\sigma(X_u : u \leq s)$ for any $0 \leq s \leq t$*

Throughout this section, let X be a d-dimensional Lévy process with values in $\mathbb{R}^d$.

Proposition A.2 *Define a set function ν on $\mathbb{R}^d$ by*

$$\nu(A) := \mathbb{E}_{\mathbb{P}}\left[\sum_{0<t\leq 1} \mathbb{1}_A(\Delta X_t)\right] \qquad \text{for} \qquad A \in \mathcal{B}(\mathbb{R}^d), 0 \notin A \tag{A.1}$$

where $\Delta X_t := X_t - \lim_{s \uparrow t} X_s$, and $\nu(\{0\}) := 0$. Then ν is a σ-finite measure on $\mathbb{R}^d \backslash \{0\}$, extended on zero to a measure on $\mathbb{R}^d$. with

$$\int_{\mathbb{R}^d} \|x\|^2 \wedge 1 \, \nu(dx) < \infty$$

It is referred to as the ***Lévy measure****.*

Hence, by Definition (A.1), given some Borel set A that does not contain zero, the Lévy measure ν assesses the expected number of jumps per unit time whose sizes fall into that set A.
Next we provide the fundamental result, that states that any Lévy process can be uniquely characterized by a triple (γ, Σ, ν) where γ is referred to as the drift term, Σ the diffusion coefficient and ν denotes the Lévy measure defined above.

Definition A.3 *We define for $t > 0$ the* ***cumulant function*** *κ as*

$$\begin{aligned} \kappa &: D \to \mathbb{C} \\ e^{\kappa(z)t} &= \mathbb{E}\left[e^{\langle z|X_t\rangle}\right] \end{aligned}$$

where the domain D is specified to be

$$D := \left\{ z \in \mathbb{C}^d \middle| \mathbb{E}\left[e^{\langle Re(z)|X_1\rangle}\right] < \infty \right\}$$

Theorem A.4 ***(Lévy-Khintchine)*** *There exists a triplet (γ, Σ, ν) such that the cumulant function of any Lévy process in terms of Definition A.1 can be written as*

$$\kappa(z) = \gamma z + \frac{1}{2} z^T \Sigma z + \int_{\mathbb{R}^d} \left(e^{\langle z|x\rangle} - 1 - z h(x) \right) \nu(dx) \tag{A.2}$$

where

1. $\gamma \in \mathbb{R}^d$
2. $\Sigma \in \mathbb{R}^d \times \mathbb{R}^d$ *is a positive semi-definite matrix*
3. ν *is the Lévy measure cf.* (A.1)
4. $h(x)$ *is some truncation function, i.e. a bounded measurable function such that $h(x) = x$ in a neighbourhood of zero*

We call (γ, Σ, ν) the characterizing triplet.

Note that the Lévy-Khintchine representation of the cumulant function, or more precisely the drift component γ, may vary depending on the choice of the truncation function. In this thesis, we will choose the commonly used truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$, $h(x) := x\mathbb{1}_{\|x\| \leq 1}$.
For any adapted cádlàg process X, hence in particular for a Lévy process, one can define an integer valued random measure μ^X, called the **jump measure**, for $t \geq 0$ via

$$\mu^X(\omega, t, A) := \sum_{s \leq t} \mathbb{1}_A(\Delta X_s(\omega)) \qquad \text{for} \qquad A \in \mathcal{B}(\mathbb{R}^d) : 0 \notin A \tag{A.3}$$

For (ω, t) fixed $A \to \mu^X(\omega, t, A)$ is a counting measure that determines the number of jumps of the process X of size in A up to time t. Next we define the integral w.r.t. the counting measure $\mu^X(\omega, t, .)$ for a Borel set $A \in \mathcal{B}(\mathbb{R}^d) : 0 \notin A$ and some borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}$ that is finite on A as the real-valued random variable

$$\int_A f(x)\, \mu^X(\omega, t, dx) := \sum_{s \leq t} f(\Delta X_s(\omega)) \mathbb{1}_A(\Delta X_s(\omega))$$

Moreover, we have that the stochastic process

$$\left(\int_0^t \int_A f(x)\, \mu^X(\omega, ds, dx) \right)_{0 \leq t \leq T} =: \left(\left(f(x) * \mu^X \right)_t \right)_{0 \leq t \leq T}$$

defines a compound Poisson process (see [10] Theorem 5.3.).
We denote the *compensator measure* of μ^X by $\nu^X := \nu \otimes dt$.

Theorem A.5 ***(Lévy-Itô decomposition)*** *Any d-dimensional Lévy process X with characterizing triplet (γ, Σ, ν) can be represented as sum of four independent Lévy processes in the following way:*

$$X_t = \gamma t + \Sigma B_t + \int_0^t \int_{\|x\|\leq 1} x\,(\mu^L - \nu^L)(ds, dx) + \int_0^t \int_{\|x\|>1} x\,\mu^L(ds, dx) \qquad \forall t \geq 0 \tag{A.4}$$

Here, B_t is a standard $\mathbb{P}$-Brownian motion and μ^L and ν^L refer to the process' jump and compensator measure respectively.

More precisely, the components of said decomposition consist of

- a constant drift $L_t^{(1)} := \gamma t$, $t \geq 0$
- a d-dimensional standard $\mathbb{P}$-Brownian motion $L_t^{(2)} := \Sigma B_t$, $t \geq 0$
- a pure jump square integrable martingale $L_t^{(3)} := \int_0^t \int_{\|x\|\leq 1} x(\mu^L - \nu^L)(ds, dx)$, $t \geq 0$
- and a compound Poisson process $L_t^{(4)} := \int_0^t \int_{\|x\|>1} x\mu^L(ds, dx)$, $t \geq 0$

Finally we state how the Lévy measure can be linked to the finiteness of the moments of a Lévy process X.

Proposition A.6 *Let X be a d-dimensional Lévy process. X_t possesses finite first moments for any $t \geq 0$ iff*

$$\int_{\|x\|>1} \|x\|\,\nu(dx) < \infty$$

A.1 Integration based on Lévy processes

In the present thesis, we frequently integrate against Lévy processes, hence there is a need to provide some basic theoretical background concerning stochastic integration against such processes. The subsequent section is aimed at providing a brief collection of (some of) the results on stochastic integration against semimartingales in general, as it is presented in [8].
Initially, Kallenberg establishes the notion of a stochastic integral against *continuous* integrators (see [8] Chapter 15). In particular, the integral against a continuous local martingale M^c is defined for progressive integrands V such that $\int_0^t V_s^2\,d[M^c]_s < \infty$ a.s. for every $t > 0$. The class of such processes is denoted by $L(M^c)$. The process $[M^c]_t, t \geq 0$ appearing above denotes the quadratic variation which is a.s. uniquely associated with the continuous local martingale M^c (c.f. [8] Theorem 15.5). Integration against some continuous semimartingale X^c with decomposition $X = M^c + A^c$ is then defined for integrands $V \in L(X^c) := (L(M^c) \cap L(A^c))$, with $L(A^c)$ being the class of all progressive processes U such that $\int_0^\cdot U_s\,dA_s^c$ exists path-wise as Lebesgue-Stieltjes integral. For $V \in L(X^c)$ the corresponding integral process is then set to be

$$\int_0^\cdot V_s\,dX_s^c := \int_0^\cdot V_s\,dM_s^c + \int_0^\cdot V_s\,dA_s^c$$

Those results are then further extended to possibly *discontinuous* semimartingales in [8] Chapter 23. First, for every uniformly square-integrable local martingale M (or put differently for every $M \in \mathcal{M}_{loc}^2$) the predictable quadratic covariation of M is defined to be the predictable compensator of M^2 and is denoted by $\langle M \rangle$. Thereafter, similar to the continuous case, the stochastic integral against $M \in \mathcal{M}^2_{loc}$ is established, feasible integrands being those progressively measurable processes V such that $\int_0^t V_s^2 \, d\langle M\rangle_s < \infty$ a.s. for every $t > 0$. Finally, the stochastic integral is then extended to general semimartingales for locally bounded integrands (c.f. Theorem 23.4). As is shown in Lemma 23.5, due to suitable truncation, an arbitrary semimartingale X may be decomposed into the sum of a local martingale with bounded jumps and a process of locally finite variation that we denote by M and A respectively. The just mentioned integral process then is specified via

$$\int_0^{\cdot} V_s \, dX_s := \int_0^{\cdot} V_s \, dM_s + \int_0^{\cdot} V_s \, dA_s$$

where the first integral on the r.h.s. is defined as before (since every local martingale with bounded jumps is locally square integrable) and the second exists as Lebesgue-Stieltjes integral.

Thus, for an arbitrary $\mathbb{R}^d$-valued Lévy process as defined in A.1 with characteristics (γ, Σ, ν) the stochastic integral is well-defined for locally bounded, predictable integrands.

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List of Figures