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D I S S E R T A T I O N

# Growth Estimates for Nevanlinna Matrices

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# Kurzfassung

Wir betrachten ein zweidimensionales kanonisches System, das ist eine Differentialgleichung einer gewissen Gestalt auf einem Intervall, die durch eine lokal integrierbare Funktion $H$, den Hamiltonian, gegeben ist. Dieser nimmt reelle, positiv semidefinite $2 \times 2$ Matrizen als Werte an.

Im Grenzkreisfall, d.h. wenn $H$ bis zum rechten Intervallrand integrierbar ist, kann die Fundamentallösung eines kanonischen Systems dort ausgewertet werden. Man erhält die sogenannte Monodromiematrix, eine Nevanlinna Matrix bestehend aus 4 ganzen Funktionen mit identischem Wachstum.

Wissen über das Wachstum dieser Funktionen liefert das asymptotische Verhalten des Spektrums von selbstadjungierten Realisierungen des kanonischen Systems. Es stellt sich die Aufgabe, das Wachstum für einen gegebenen Hamiltonian möglichst exakt zu bestimmen.

Der Exponentialtyp kann mithilfe der Krein-de Branges Formel als das Integral von der Wurzel der Determinante von $H$ berechnet werden. Falls dieses Integral jedoch Null ist, d.h. falls die Determinante von $H$ fast überall verschwindet, liefert die Krein-de Branges Formel keine signifikante Information.

Nach einem einführenden Teil beschäftigen wir uns zunächst mit allgemeinen kanonischen Systemen und verfeinern zwei Sätze von Roman Romanov. Im zweiten Teil studieren wir den wichtigen Spezialfall eines Hamburger Hamiltonians.

# Contents

# Preface

In this thesis we present some recent results estimating the growth of Nevanlinna matrices, which appear as monodromy matrices of canonical systems on a finite interval $[a, b]$, i.e., as the fundamental solution of the system at the right end point b. Obtaining knowledge of the growth of the monodromy matrix has operator theoretic relevance: it translates to knowledge of the asymptotic behaviour of the spectrum of selfadjoint realisations of the system.

Classical theorems fully solve the question for exponential type. However, for a given Hamiltonian $H$ with $\det H = 0$ a.e. on $[a, b]$, this does not lead to significant assertions about spectral distribution. For such systems more sophisticated results measuring growth with respect to general growth functions rather than exponential type are necessary. However, already determining the order (less than 1) and type of the monodromy matrix is in general a difficult task for which no full solution is known.

We shall mainly focus on the subclass of Hamburger Hamiltonians, i.e. trace-normed Hamiltonians with $\det H = 0$ a.e. on $[a, b]$ which are piecewise constant on a sequence of intervals accumulating only at the right end point. In other words, a Hamburger Hamiltonian is given by two sequences, its lengths and its angles. Hamiltonians of this form are in a one-to-one correspondence with Hamburger moment problems.

Let us outline the content of this thesis. In the introductory chapters 1 and 2, we repeat the definitions of growth functions, Nevanlinna matrices and canonical systems. We recall the classical Hamburger power moment problem and discuss the one-to-one correspondence to Jacobi matrices on the one hand, and (maybe less known) to Hamburger Hamiltonians on the other hand. Moreover, we survey a selection of known results concerning the order of such systems.

In Part I, consisting of chapters 3 and 4, we discuss results for general Hamiltonians. Chapter 3 is devoted to a theorem by R.Romanov, cf. [Rom17, Theorem 1], which gives an upper bound for the order of a general Hamiltonian. We present a refined version of this theorem, which is formulated for growth functions, cf. Theorem 3.3. Some parts of this chapter present ongoing (currently unpublished) research with R.Romanov and H.Woracek.

In Chapter 4 we present another result by R.Romanov, cf. [Rom17, Theorem 2], which determines the order of a diagonal Hamiltonian. We use this result mainly as a tool to construct examples.

In Part II, which is the core of the thesis, we focus on results for Hamburger Hamiltonians. In Chapter 5 we begin with a direct application of the refined version of Romanov's Theorem 1. This yields Theorem 5.1, which is an upper estimate for the $\lambda$-type of a Hamburger Hamiltonian. Unfortunately, like Romanov's Theorem 1, it can be difficult to apply. Next, we introduce measures for the decay of lengths and angle-differences of a Hamburger Hamiltonian $H$, i.e. $\Delta_l(H)$ and $\Delta_\phi(H)$, respectively, and a measure for the quality of possible convergence of angles, i.e. $\mu(H)$. These quantities can easily be read of the parameters of $H$. The knowledge of only these three quantities enables us to apply Theorem 5.1, which gives rise to an accessible upper bound of the order of $H$, cf. Theorem 5.10. Moreover, we present a lower estimate for the order, see Proposition 5.17.

In Chapter 6 we compare the upper and lower estimates developed above. Roughly speaking the situation is as follows: If the parameters of the Hamburger Hamiltonian behave regular, then the upper and lower estimates for the order of $H$ coincide, see Theorem 6.5. On the other hand, the construction of a class of irregular examples, cf. Theorem 6.11, shows that both the upper and lower estimates do not coincide with the actual order in general. On the way we answer a question formulated in [BS14, p.32], whether the lower estimate of the order of a moment sequence by M.S.Livšic, cf. [Liv39], can be strict or not. The answer is no, cf. Corollary 6.14. The content of chapters 5 and 6 has already been published in [PRW16]. Note that the present form is stronger since we use the refined version of Romanov's Theorem 1.

Chapter 7, which has been published in [PW17], contains a completely different approach. We employ the so-called square transform which can be thought of a diagonalisation of a given Hamburger Hamiltonian: The transformed "Hamiltonian" is indeed diagonal, but it is not anymore positive semidefinite. By removing the negativity using another transformation and by keeping track of the order, we get another upper estimate, cf. Theorem 7.18. Combining this result with Kac's formula for the order of a string, i.e. [Kac90, Theorem 1], yields Theorem 7.22.

In the last chapter, which has been submitted in [Pru18], we consider the order of Jacobi matrices with parameters having a power asymptotics. We study asymptotics of solutions of the difference equation by employing, e.g., recent work of R.-J.Kooman [Koo07]. The main result in this chapter is Theorem 8.1.

There are still many question that wait to be answered. A collection of open problems can be found at the end of this thesis.

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# 1. Introduction

## 1.1. Growth functions

The growth of an entire function $f$ can be measured on different scales. Set

$$M(f,R) := \max_{|z|\le R} |f(z)| = \max_{|z|=R} |f(z)|\,.$$

The *order* of an entire function $f$ is given by

$$\rho(f) := \inf\left\{d > 0\ :\ \exists c_1, c_2 > 0 : M(f,R) \le c_1 \exp(c_2 R^d) \text{ for all } R > 0\right\} \in [0,\infty].$$

A finer measure for the growth is the *type* of $f$ with respect to the order $\rho \in (0,\infty)$, which is defined as

$$\tau_\rho(f) := \limsup_{R\to\infty} \frac{\ln^+ M(f,R)}{R^\rho}.$$

We write $\tau(f)$ if the order is clear from the context. If $\tau_\rho(f) = 0$, then the function $f$ is said to be of *minimal* type, if $0 < \tau_\rho(f) < \infty$ of *normal* type, and if $\tau_\rho(f) = \infty$ of *maximal* type. Instead of $\tau_1(f)$, we write $\mathrm{et}(f)$ and speak of the *exponential type* of $f$.

For functions of order zero some authors refine the scale further by considering (double-)logarithmic order and type, cf. [BP07]. The next definition subsumes all these notions.

**1.1 Definition.** A function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *growth function* if the following conditions hold:

*(i)* The limit $\rho := \lim_{R\to\infty} \frac{\log\lambda(R)}{\log R}$ exists, is non-negative and finite.

*(ii)* For sufficiently large values of $R$, the function $\lambda$ is differentiable and

$$\lim_{R\to\infty} \left(R\,\frac{\lambda'(R)}{\lambda(R)} \Big/ \frac{\log\lambda(R)}{\log R}\right) = 1. \tag{1.1}$$

*(iii)* $\log R = \mathrm{o}(\lambda(R))$.

◊

*1.2 Remark.* For $\rho > 0$, condition (1.1) is equivalent to $\lim_{R\to\infty} \frac{R\lambda'(R)}{\lambda(R)} = \rho$, which is more convenient to check. ◊

For an entire function $f$ and a growth function $\lambda$, we set

$$\tau_\lambda(f) := \limsup_{R\to\infty} \frac{\ln^+ M(f,R)}{\lambda(R)},$$

and speak of the $\lambda$-*type of* $f$.

Typical examples of growth functions are functions of the form $\lambda(R) = R^a(\ln R)^{1+b}$, with $a > 0$ and $b \in \mathbb{R}$, or $a = 0$ and $b > 0$. For $b = -1$ we recover the classical notion of type with respect to order $a$, whereas for $a = 0$ and $b > 0$ the expression $\tau_\lambda(f)$ is known as the logarithmic type with respect to logarithmic order $b$.

For more details on growth functions see [LG86, Section I.6] or [Lev80, Section I.12]. Note here that growth functions are exponentials of proximate orders.

## 1.2. Nevanlinna matrices

Let us introduce the main object of this thesis.

**1.3 Definition.** A $2\times 2$-matrix valued function $W(z) = (w_{ij}(z))_{i,j=1}^2$ consisting of real (i.e. $w_{ij}(\bar{z}) = \overline{w_{ij}(z)}$) entire functions is called a *Nevanlinna matrix* if $\det W(z) = 1$ for all $z \in \mathbb{C}$ and if the following matrix is positive semidefinite,

$$\frac{1}{i}\big(W(z)JW(z)^* - J\big) \geq 0, \tag{1.2}$$

for all $z \in \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. ◊

*1.4 Remark.* A matrix valued entire function $W(z)$ is called *$iJ$-inner*, if the following reproducing kernel is positive semidefinite,

$$\frac{W(z)JW(w)^* - J}{z - \bar{w}}, \quad z, w \in \mathbb{C}\setminus\mathbb{R}. \tag{1.3}$$

By choosing $z = w$ we get (1.2). In fact, it can be shown that a Nevanlinna matrix $W(z)$ is automatically $iJ$-inner. ◊

In Definition 1.3 it is equivalent to replace (1.2) by the property that for each $z \in \mathbb{C}^+$ the Möbius transformation,

$$w \mapsto \frac{w_{11}(z)w + w_{12}(z)}{w_{21}(z)w + w_{22}(z)},$$

maps the closed upper plane into itself. This argument has been carried out in [Win93] or [Win95], where this notion has been studied under the name $\rho$-matrix. Clearly, another equivalent replacement is

$$\operatorname{Im}\left(\frac{w_{11}(z)t + w_{12}(z)}{w_{21}(z)t + w_{22}(z)}\right) \geq 0, \quad z \in \mathbb{C}^+, t \in \mathbb{R}\cup\{\infty\}. \tag{1.4}$$

In [BP95] the authors showed that, under the assumption that one (and hence all) entries of $W(z)$ are not polynomials, (1.4) can be replaced by the slightly stronger condition

$$\operatorname{Im}\left(\frac{w_{11}(z)t+w_{12}(z)}{w_{21}(z)t+w_{22}(z)}\right)>0, \quad z\in\mathbb{C}^+, t\in\mathbb{R}\cup\{\infty\}. \tag{1.5}$$

Entire $2\times 2$-matrix valued functions with this property probably first appeared in the description of all solutions of an indeterminate Hamburger power moment problem by Nevanlinna, cf. Theorem 2.1.

We denote by $\mathcal{N}_0$ the set of all *Nevanlinna functions*, i.e., the set of all function $f$ which are analytic on $\mathbb{C}\backslash\mathbb{R}$, satisfy $f(\bar{z})=\overline{f(z)}$ and $\operatorname{Im} f(z)\geq 0$ for $z\in\mathbb{C}^+$.

By (1.4) we have that $(w_{11}t+w_{12})/(w_{21}t+w_{22})$ is a Nevanlinna function for each $t\in\mathbb{R}\cup\{\infty\}$. In particular $w_{11}/w_{21}\in\mathcal{N}_0$ and $w_{12}/w_{22}\in\mathcal{N}_0$. Note that if $W(z)$ is a Nevanlinna matrix, then so is

$$\begin{pmatrix} w_{22}(z) & w_{12}(z)\\ w_{21}(z) & w_{11}(z)\end{pmatrix}.$$

Hence, also $w_{22}/w_{21}$ and $w_{12}/w_{11}$ are Nevanlinna functions. These properties actually characterise Nevanlinna matrices: A matrix $W(z)=(w_{ij}(z))_{i,j=1}^2$ consisting of real entire functions is a Nevanlinna matrix if and only if the quotients $w_{11}/w_{21}$, $w_{12}/w_{22}$ and $w_{22}/w_{21}$ are Nevanlinna functions, cf. [Sod96].

Regarding the growth of the entries of a Nevanlinna matrix we have the following result, which generalises [BP95, Theorem 4.7].

**1.5 Lemma.** *Let $W(z)$ be a Nevanlinna matrix, and let $\lambda$ be a growth function. Then all entries of $W(z)$ have the same $\lambda$-type. In particular, they have the same order and type.*

*Proof.* Just note that for entire functions $A$ and $B$ with $A/B\in\mathcal{N}_0$, [BW06, Proposition 2.3] implies $\tau_\lambda(A)=\tau_\lambda(B)$. ❑

## 1.3. Canonical systems

Let $H$ be a $2\times 2$-matrix valued locally integrable function on an interval $[a,b)$ whose values are almost everywhere real and positive semidefinite matrices. The *canonical system with Hamiltonian $H$* is the differential equation

$$y'(x)=zJH(x)y(x), \quad x\in[a,b), \tag{1.6}$$

where $J$ is the symplectic matrix $J:=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, and $z$ is a complex parameter. After a reparametrisation, we may assume $\operatorname{tr} H=1$ a.e. on $[a,b)$, cf. [WW12]. The *fundamental solution* of the system is the solution of the initial value problem

$$\begin{cases}\frac{d}{dx}W(x,z)J=zW(x,z)H(x), & x\in[a,b),\\ W(a,z)=I.\end{cases} \tag{1.7}$$

In the literature, this equation is sometimes written in the form

$$\begin{cases} J\frac{d}{dx}\tilde{W}(x,z) = zH(x)\tilde{W}(x,z), \quad x \in [a,b), \\ \tilde{W}(a,z) = I. \end{cases} \tag{1.8}$$

This is just a question of notational convention; the relation between these notations is $\tilde{W}(x,z) = W(x,-z)^T$. By classical theory of differential equations it is well-know that the fundamental solution $W(x,z)$ exists, is unique and depends analytically on $z$ for all $x \in [a,b)$, see, e.g., [Atk64, Chapter 9] or [GK67, Chapter VI]. Differentiating $W(t,z)JW(t,w)^*$ with respect to $t$ and integrating over $[a,x]$ for $x \in [a,b)$ gives

$$W(x,z)JW(x,w)^* - J = (z-\bar{w}) \int_a^x W(t,z)JW(t,z)^*\,dt.$$

This yields (1.3) and we conclude that $W(x,z)$ is a Nevanlinna matrix. The exponential type of any entry of $W(x,z) = (w_{ij}(x,z))_{i,j=1}^2$ is given by the Krein-de Branges formula

$$\operatorname{et}(w_{ij}(x,.)) = \int_a^x \sqrt{\det H(t)}\,dt,$$

cf. [Kre51], [Bra61, Theorem X].

In the *limit circle case* (lcc), i.e. if $\int_a^b \operatorname{tr} H(t)\,dt < \infty$, the limit

$$W(b,z) := \lim_{x\to b} W(x,z)$$

exists locally uniformly on $\mathbb{C}$ and is called the *monodromy matrix*. Clearly, it is again a Nevanlinna matrix. When we do not want to emphasize the right end point of the interval, we only write $W(z) = W(b,z)$.

**1.6 Definition.** Let $H$ be a Hamiltonian in the lcc and let $W(z)$ be its monodromy matrix.

We denote by $\rho(H)$ the order of any entry of $W(z)$, and call it the *order* of the canonical system. For a growth function $\lambda$, we write $\tau_\lambda(H)$ for the $\lambda$-type of any entry of $W(z)$, and speak of the $\lambda$-type of the canonical system. ◊

If the *limit point case* (lpc) takes places, i.e. if $\int_a^b \operatorname{tr} H(t)\,dt = \infty$, then for each function $\tau \in \mathcal{N}_0 \cup \{\infty\}$ the limit

$$Q_H(z) := \lim_{x\to b} \frac{w_{11}(x,z)\tau(z) + w_{12}(x,z)}{w_{21}(x,z)\tau(z) + w_{22}(x,z)}$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and does not depend on $\tau$, cf. [HSW00, Theorem 2.1(2.7)]. The function $Q_H$ is a Nevanlinna function and is called the *Titchmarsh-Weyl coefficient* of $H$. Except for chapter 7, we will only consider Hamiltonians in the lcc. It is possible to define the order of $H$ also in the lpc, cf. Definition 7.3.

With $H$ there is associated a Hilbert $L^2(H)$ consisting of 2-vector valued measurable functions satisfying a usual $L^2$-condition and a constancy condition on indivisible intervals, cf. [Kac85; Kac86a] or [HSW00] [1]. In this space a linear relation $T_{\max}(H)$ is given by the differential expression $f' = JHg$ on its natural maximal domain. The adjoint $T_{\min}(H) := T_{\max}(H)^*$ is a completely nonselfadjoint symmetry in $L^2(H)$. There is a rich spectral theory for canonical systems. See, e.g., [GK67] or [HSW00] for the Weyl limit disk construction and the direct spectral problem.

In the lpc, $T_{\min}$ has defect index $(1,1)$ and the spectrum of selfadjoint extensions may be discrete, continuous, or be composed of different types.

In the lcc, $T_{\min}$ has defect index $(2,2)$. The symmetric extension

$$S(H) := \big\{(f;g) \in T_{\max}(H) : (1,0)f(0) = 0, f(L) = 0\big\}$$

has defect index $(1,1)$ and is entire in the sense of M.G. Kreĭn. In particular the spectra of canonical selfadjoint extensions of $S(H)$ are discrete and interlace with the zeros of $w_{21}(z)$, cf. Section 7.1 for more details. Let $(\omega_n^{\pm})_n$ denote the sequences of positive and negative, respectively, eigenvalues of a selfadjoint extension of $S(H)$ arranged according to increasing modulus. By the Krein-de Branges formula we have

$$\lim_{n\to\infty} \frac{n}{|\omega_n^{\pm}|} = \frac{1}{\pi}\,\mathrm{et}(W(L,.)) = \frac{1}{\pi}\int_a^b \sqrt{\det H(t)}\,dt. \tag{1.9}$$

If $\det H > 0$ on a set of positive measure, then $\rho(H) = 1$, $C := \mathrm{et}(W(L,.)) > 0$ and (1.9) gives

$$|\omega_n^{\pm}| = \frac{\pi}{C} n + \mathrm{o}(n), \quad n \in \mathbb{N}.$$

If $\det H = 0$ a.e. on $[a,b)$, then the entries of the monodromy matrix are of minimal exponential type. Thus $\rho(H) \leq 1$ and (1.9) only says that $\lim_{n\to\infty} n/|\omega_n^{\pm}| = 0$. Heuristically, this means that the eigenvalues are sparser than integers. They may behave, for instance, like $n^{\gamma}$ for some $\gamma > 1$, and it is a fundamental question to determine the actual asymptotic behaviour of the eigenvalues. This can be done via growth estimates, since knowledge of the growth of the Nevanlinna matrix leads to knowledge of the asymptotic behaviour of the spectrum.

Let $n_\sigma(R) := \#\{\omega_n^{\pm} : |\omega_n^{\pm}| < R\}$ denote the counting function of the spectrum, and set $\rho := \rho(H)$. By [Boa54, Theorem 2.5.12] we have $n_\sigma(R) = \mathrm{O}(R^{\rho+\epsilon})$ for all $\epsilon > 0$, and consequently

$$\lim_{n\to\infty} \frac{n^{\frac{1}{\rho}-\epsilon}}{|\omega_n^{\pm}|} = 0.$$

In other words, the eigenvalues are sparser than $n^{\frac{1}{\rho}-\epsilon}$ for all $\epsilon > 0$. Additionally, if the type of $H$ with respect to the order $\rho$ is finite, then [Boa54, Theorem 2.5.13] gives

$$\limsup_{n\to\infty} \frac{n^{1/\rho}}{|\omega_n^{\pm}|} < \infty.$$

[1] One word of caution concerning notation: In [HSW00] the space we call $L^2(H)$ is denoted as $L^2_s(H)$.

The following inverse spectral theorem is due to L.de Branges: The assignment $H \mapsto W$ is a bijection between the set of all lcc Hamiltonians (modulo reparameterization) and the set of all Nevanlinna matrices $W(z)$. Moreover, the assignment $H \mapsto Q_H$ is a bijection between the set of all lpc Hamiltonians (modulo reparameterization) and the set of all Nevanlinna functions $Q_H(z)$. This result follows from [Bra68], an explicit deduction from this source in the lpc is given in [Win95].

# 2. The power moment problem

## 2.1. Hamburger moment problem vs. Jacobi matrix vs. Hamburger Hamiltonian

We introduce an important object of this work: A subclass of Hamiltonians, which corresponds one-to-one to Hamburger moment problems and Jacobi matrices.

### 2.1.1. Hamburger moment problem

Recall the formulation of the classical Hamburger moment problem: Given a sequence of real numbers $(s_n)_{n=0}^{\infty}$, does there exist a positive Borel measure $\mu$ on $\mathbb{R}$ such that for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$

$$s_n = \int_{\mathbb{R}} x^n \, d\mu(x)?$$

Set $D_n := \det((s_{j+k})_{j,k=0}^n)$ for $n \in \mathbb{N}_0$. If $D_n \geq 0$ for all $n \in \mathbb{N}_0$, then the answer to the above question is yes, and $(s_n)_{n=0}^{\infty}$ is called a *moment sequence*. If $D_n = 0$ for some $n \in \mathbb{N}_0$, then the moment problem has a unique solution $\mu$, which is a discrete measure with only finitely many mass points. We will always assume that the sequences $(s_n)_{n=0}^{\infty}$ is *positive*, i.e. $D_n > 0$ for all $n \in \mathbb{N}_0$. By normalizing the measure to a probability measure, we may assume $s_0 = 1$.

A positive sequence $(s_n)_{n=0}^{\infty}$ induces an inner product on $\mathbb{C}[z]$ by $(p, q)_s := \sum_{k,l=0}^{\infty} p_k \bar{q}_l s_{k+l}$ for $p(z) = \sum_{k=0}^{\infty} p_k z^k, q(z) = \sum_{l=0}^{\infty} q_l z^l \in \mathbb{C}[z]$. Clearly, the inner product $(.,.)_s$ coincides with the standard $L^2(\mu)$-inner product on polynomials for each solutions $\mu$ of the Hamburger moment problem. The Gram-Schmidt process applied to the sequence $1, z, z^2, \ldots$ generates polynomials $P_n(z)$, $n \in \mathbb{N}_0$, called the *orthonormal polynomials of the first kind*. Those of the second kind are given by

$$Q_n(z) := \Big( w \mapsto \frac{P_n(z) - P_n(w)}{z - w}, 1 \Big)_s, \quad n \in \mathbb{N}_0.$$

We say that the moment problem is *determinate* if it has a unique solution. This is the case if and only if $\sum_{n=0}^{\infty} P_n(0)^2 + Q_n(0)^2 = \infty$.

In the *indeterminate* case, this series converges and we have more than one solutions, in fact, infinitely many. A classical result of R.Nevanlinna describes how they can be parametrized. Consider the following four entire functions,

$$A(z) := z\sum_{k=0}^{\infty} Q_k(0)Q_k(z), \qquad B(z) := -1 + z\sum_{k=0}^{\infty} Q_k(0)P_k(z),$$
$$C(z) := 1 + z\sum_{k=0}^{\infty} P_k(0)Q_k(z), \quad D(z) := z\sum_{k=0}^{\infty} P_k(0)P_k(z).$$

**2.1 Theorem** ([Nev22])**.** *Let $(s_n)_{n=0}^{\infty}$ be an indeterminate moment sequence.*
*A measure $\mu$ is a solution of the Hamburger power moment problem if and only if*

$$\int_{\mathbb{R}} \frac{1}{x-z}d\mu(x) = -\frac{A(z)\phi(z) - C(z)}{B(z)\phi(z) - D(z)}, \tag{2.1}$$

*for some Nevanlinna function $\phi$.*

From (2.1) it follows that for $t \in \mathbb{R} \cup \{\infty\}$

$$\operatorname{Im}\left(-\frac{A(z)t - C(z)}{B(z)t - D(z)}\right) = \operatorname{Im}\left(\frac{C(z)(-\frac{1}{t}) + A(z)}{-D(z)(-\frac{1}{t}) - B(z)}\right) > 0, \quad z \in \mathbb{C}^+.$$

Furthermore, we have $A(z)D(z) - B(z)C(z) = 1$ for all complex $z$, cf. [Akh61, (2.29), p.54]. Hence with our notation, which differs from the traditional one,

$$\begin{pmatrix} C(z) & A(z) \\ -D(z) & -B(z) \end{pmatrix}$$

is a Nevanlinna matrix.

The first results about the growth of Nevanlinna matrices arising in this situation is due to M.Riesz, who showed that all four entries are of minimal exponential type, cf. [Rie23]. Much later, it was noted that these four functions have the same order and type, cf. [BP94, Theorem 3.3, Theorem 4.2]. In fact, by Lemma 1.5, all four functions have the same growth, i.e. the same type with respect to any growth function.

We denote by $\rho((s_n)_{n=0}^{\infty})$ this common order, and speak of the order of the moment problem.

### 2.1.2. Jacobi matrix

A *Jacobi matrix* $J$ is a tridiagonal symmetric semi-infinite matrix

$$J = \begin{pmatrix} q_0 & \rho_0 & & 0 \\ \rho_0 & q_1 & \rho_1 & \\ & \rho_1 & q_2 & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix},$$

with real $q_n$ and positive $\rho_n$. There is a one-to-one correspondence between Jacobi matrices and positive moment sequences, cf. [Akh61] for the classical reference. Let us recall this relation. For a given positive moment sequence $(s_n)_{n=0}^{\infty}$ set

$$D'_n := \det \begin{pmatrix} s_0 & s_1 & \dots & s_{n-1} & s_{n+1} \\ s_1 & & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots \\ s_n & s_{n+1} & \dots & s_{2n-1} & s_{2n+1} \end{pmatrix}, \quad \text{for } n \in \mathbb{N},$$

and $D'_0 := s_1$. With the conventions $D'_{-1} := 0$ and $D_{-1} := 1$, we define for $n \in \mathbb{N}_0$

$$\rho_n := \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}, \quad q_n := \frac{D'_n}{D_n} - \frac{D'_{n-1}}{D_{n-1}}.$$

Obviously $q_n$ is real and $\rho_n$ is positive, and one can show that the orthogonal polynomials of the first and second kind satisfy the three-term recurrence relation ($\rho_{-1} := 1$)

$$zU_n = \rho_{n-1}U_{n-1} + q_nU_n + \rho_nU_{n+1}, \quad n \in \mathbb{N}_0, \tag{2.2}$$

with initial conditions $P_{-1} = 0, P_0 = 1$ and $Q_{-1} = -1, Q_0 = 0$. Clearly, (2.2) determines a Jacobi matrix with diagonal $(q_n)_{n=0}^{\infty}$ and off-diagonal $(\rho_n)_{n=0}^{\infty}$.

On the other hand, for given $(q_n)_{n=0}^{\infty}$ and $(\rho_n)_{n=0}^{\infty}$ one can recover all orthogonal polynomials $P_n(z)$ by solving (2.2) starting with the appropriate initial values. In the sequel, it is possible to find the corresponding positive moment sequence $(s_n)_{n=0}^{\infty}$ by using the orthogonality $(P_n(z), 1)_s = 0$ for $n \in \mathbb{N}$.

Looking at (2.2) reveals that the leading coefficient of $P_n(z) = \sum_{k=0}^{n} b_{k,n}z^k$ is equal to

$$b_{n,n} = \left(\prod_{k=0}^{n-1} \rho_k\right)^{-1}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

There occurs an alternative, cf. [Akh61, Theorem 1.3.2.]: Either all solutions $(U_n)_{n=0}^{\infty}$ of (2.2) are square summable for one (and hence all) non-real $z$ (one speaks of the *limit circle case*, or, in the language of [Akh61], *type* $C$), or there are non-summable solutions of (2.2) for one (and hence all) non-real $z$ (called the *limit point case*, or, synonymously, *type* $D$).

The Jacobi matrix is of type $C$ if and only if the corresponding moment problem is indeterminate, cf. [Akh61, Theorem 2.1.2, Corollary 2.2.4]. We will use the notation $\rho(J)$ for the order of the corresponding moment problem.

In general it is difficult to decide from the parameters $\rho_n, q_n$ whether $J$ is of type C or D. Two classical necessary conditions for type C are Carleman's condition which says that $\sum_{n=0}^{\infty} \rho_n^{-1} = \infty$ implies type D, cf. [Car26], and Wouk's theorem that a dominating diagonal in the sense that either $\sup_{n\geq 0}(\rho_n + \rho_{n-1} - q_n) < \infty$ or $\sup_{n\geq 0}(\rho_n + \rho_{n-1} + q_n) < \infty$ implies type D, cf. [Wou53]. A more subtle result by Yu.M.Berezanskiĭ, which gives a sufficient condition for type C and $\rho(J)$, will be presented below as Theorem 2.5.

Each Jacobi matrix induces a closed symmetric operator $T_J$ on $\ell^2(\mathbb{N})$, namely the closure of the natural action of $J$ on the subspace of finitely supported sequences, see, e.g., [Akh61, Chapter 4.1].

Note that $T_J$ is selfadjoint if and only if $J$ is of type D. In this case the spectrum of $T_J$ may be discrete, continuous, or be composed of different types. There is a vast literature dealing with Jacobi matrices of type D, whose aim is to establish discreteness of the spectrum and investigate spectral asymptotics, e.g., [BZ12; Dei+99; JM07; JN04; Tur03].

If $J$ is of type C, then $T_J$ has defect index $(1,1)$ and is entire in the sense of Krein. In particular the spectra of all selfadjoint extensions of $T_J$ are discrete, and any two are interlacing.

### 2.1.3. Hamburger Hamiltonian

We consider the class of Hamiltonians which consist only of indivisible intervals (i.e. intervals on which $H$ is a.e. equal to a constant singular matrix), which accumulate only at the right end point. The definition is due to [Kac99, §3].

**2.2 Definition.** Let $\vec{l} = (l_n)_{n=1}^{\infty}$ and $\vec{\phi} = (\phi_n)_{n=1}^{\infty}$ be sequences of real numbers with $l_n > 0$ and $\phi_{n+1} \not\equiv \phi_n \mod \pi$, $n \in \mathbb{N}$. Set

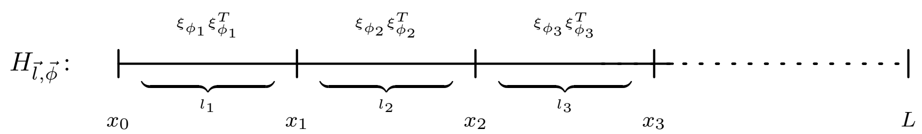
$$x_0 := 0, \qquad x_n := \sum_{k=1}^{n} l_k,\ n \in \mathbb{N}, \qquad L := x_\infty := \sum_{k=1}^{\infty} l_k \in (0,\infty], \tag{2.4}$$

$$\xi_\phi := \big(\cos(\phi), \sin(\phi)\big)^T.$$

Then we call $H_{\vec{l},\vec{\phi}} : [0,L] \to \mathbb{R}^{2\times 2}$ which is piecewise defined as

$$H_{\vec{l},\vec{\phi}}(x) := \xi_{\phi_n}\xi_{\phi_n}^T, \quad x \in [x_{n-1}, x_n),\ n \in \mathbb{N},$$

the *Hamburger Hamiltonian* with *lengths* $\vec{l}$ and *angles* $\vec{\phi}$. ◊




The set of all Hamburger Hamiltonians corresponds (up to normalization) one-to-one with the set of all Hamburger moment problems, cf. [Kac99, Theorem 3.1]. Thereby, the Hamburger moment problem is indeterminate if and only if the corresponding Hamburger Hamiltonian is in the lcc. In this case, the Nevanlinna matrix, which describes all solutions of the Hamburger moment problem, coincides with the monodromy matrix. In particular $\rho(H_{\vec{l},\vec{\phi}}) = \rho((s_n)_{n=0}^{\infty}) = \rho(J)$.

The connection between the parameters of the Hamburger Hamiltonian, the Jacobi matrix and the moment sequence is given by the formulae ($\phi_0 := \frac{\pi}{2}$)

$$\rho_n = \frac{1}{\sqrt{l_n l_{n+1}}|\sin(\phi_{n+1}-\phi_n)|}, \quad n \in \mathbb{N}, \tag{2.5}$$

$$q_n = \frac{-1}{l_n}\big[\cot(\phi_{n+1}-\phi_n)+\cot(\phi_n-\phi_{n-1})\big], \quad n \in \mathbb{N}, \tag{2.6}$$

cf. [Kac99, (3.16),(3.17)], and [Kac99, (3.22)]

$$l_n = P_n(0)^2 + Q_n(0)^2, \quad n \in \mathbb{N}. \tag{2.7}$$

## 2.2. Theorems on growth

We present some known results regarding the growth of Nevanlinna matrices.

### 2.2.1. Livšic's Theorem

The probably first result dealing with growth properties of canonical systems other than the exponential type is due to M.S.Livšic back in 1939.

**2.3 Theorem** ([Liv39]). *Let $(s_n)_{n=0}^\infty$ be an indeterminate moment sequence, and set*

$$L(z) := \sum_{n=0}^\infty \frac{z^{2n}}{s_{2n}}.$$

*Then the order of the Hamburger power moment problem is greater than or equal to the order of the entire function $L(z)$, i.e.*

$$\rho((s_n)_{n=0}^\infty) \geq \rho(L) = \limsup_{n\to\infty} \frac{2n\log n}{\log s_{2n}}.$$

For a long time it was apparently unclear whether there exist moment problems for which the order actually is different from its Livšic estimate, cf. [BS14, p.32].

By constructing a class of examples, we show that equality does not hold in general. In fact, we shall see that the gap between the actual order and its Livšic estimate can be arbitrarily close to 1, cf. Corollary 6.14.

### 2.2.2. Berezanskiĭ's Theorem

The following result about Jacobi matrices goes back to Yu.M.Berezanskiĭ, cf. [Ber56] or [Ber68, VII,Theorem 1.5]. The extension to the log-convex case is due to C.Berg and R.Szwarc, cf. [BS14, Theorem 4.11].

First, let us recall the following classical notion.

**2.4 Definition.** Let $\vec{z} = (z_n)_{n=0}^\infty$ be a sequence of non-zero complex numbers with $\lim_{n\to\infty}|z_n| = \infty$. Then

$$\mathrm{c.e.}(\vec{z}) := \inf\big\{p > 0 : (|z_n|^{-1})_{n=1}^\infty \in \ell^p\big\},$$

is called the *convergence exponent* of $\vec{z}$. ◊

**2.5 Theorem.** *Let $\rho_n > 0$, $q_n \in \mathbb{R}$ be the parameters of a Jacobi matrix. Assume that*

$$\sum_{n=1}^\infty \frac{1}{\rho_n} < \infty \qquad \textit{(Carleman condition)}$$

$$\rho_n^2 \geq \rho_{n-1}\rho_{n+1} \textit{ or } \rho_n^2 \leq \rho_{n-1}\rho_{n+1} \qquad \textit{(log-concave/convex)}$$

$$\sum_{n=1}^\infty \frac{|q_n|}{\rho_n} < \infty \qquad \textit{(small diagonal)}$$

*Then $J$ is of type $C$, i.e. the corresponding moment problem is indeterminate, and the order of $J$ is equal to the convergence exponent of $(\rho_n)_{n=1}^{\infty}$, i.e. $\rho(J) = \mathrm{c.e.}((\rho_n)_{n=1}^{\infty})$.*

*2.6 Remark.* Note that the order is always zero in the log-convex case:

Log-convexity translates to $\sigma_n := \rho_n/\rho_{n-1}$ being monotonically increasing. Due to the Carleman condition, $\sigma_n$ cannot be bounded from above by 1. Therefore, there exists $b > 1$ and $N \in \mathbb{N}$ such that $\sigma_n \geq b$ for all $n > N$, i.e.

$$\rho_n \geq b\rho_{n-1} \geq \ldots \geq b^{n-N}\rho_N, \quad n > N.$$

This implies that the convergence exponent of $(\rho_n)_{n=1}^{\infty}$ is zero. By Theorem 2.5, the order of the corresponding system is zero. $\Diamond$

Below, we prove some results which can be viewed, to some extent, as generalisations of Berezanskiĭ's theorem, cf. Theorems 7.22 and 8.1.

### 2.2.3. Valent's Conjecture

A birth-and-death process is a particular type of stationary Markov process having the non-negative integers as state space, see [KM57; BV94]. The parameters $(\lambda_n)_{n=0}^{\infty}$ and $(\mu_n)_{n=0}^{\infty}$ which determine the transition probabilities are called the *rates* of the birth-and-death process, and satisfy $\lambda_n, \mu_n > 0$ for $n \in \mathbb{N}$, $\lambda_0 > 0$ and $\mu_0 = 0$.

Associated to a birth-and-death process is the Jacobi matrix with parameters

$$q_n = \lambda_n + \mu_n, \quad \rho_n = \sqrt{\lambda_n \mu_{n+1}}, \qquad n \in \mathbb{N}_0.$$

For an integer $p \geq 3$ consider polynomial rates of the form

$$\begin{aligned} \lambda_n &= (pn + B_1) \cdot \ldots \cdot (pn + B_p), \\ \mu_n &= (pn + A_1) \cdot \ldots \cdot (pn + A_p), \quad n \in \mathbb{N}_0. \end{aligned}$$

It is known that the corresponding Jacobi matrix $J$ is in type $C$ if and only if $1 < \frac{1}{p}\sum_{j=1}^{p} B_j - A_j < p-1$. In this case, Valent formulated in [Val99] the following conjecture regarding the order and type of $J$:

$$\rho(J) = \frac{1}{p}, \quad \tau(J) = \int_0^1 \frac{du}{(1-u^p)^{2/p}}.$$

The conjecture about the order was verified in [Rom17, Corollary 6]. Regarding the type, it was shown in [BS17] that $\frac{\pi}{p\sin(\pi/p)} \leq \tau(J) \leq \frac{\pi}{p\sin(\pi/p)\cos(\pi/p)}$. More recently, the type-conjecture has been proved in recent work involving R.Romanov.

In Chapter 8, we consider the situation that $\rho_n$ and $q_n$ are not necessarily polynomials of the special form described above, but have, more generally, the following power asymptotic

$$\rho_n = n^{\beta_1}\left(x_0 + \frac{x_1}{n} + \mathrm{O}(n^{-2})\right), \quad q_n = n^{\beta_2}\left(y_0 + \frac{y_1}{n} + \mathrm{O}(n^{-2})\right).$$

In Theorems 8.1 and 8.2 we show that, under certain conditions, still $\rho(J) = \frac{1}{\beta_1}$.

### 2.2.4. Berg-Szwarc's Theorems

We present two theorems of C.Berg and R.Szwarc. The first one is essentially [BS14, Theorem 1.2] formulated in the setting of Hamburger Hamiltonians using (2.7). It contains an upper bound for the order, which depends on the sequence of lengths only.

**2.7 Theorem.** *Let $H$ be a Hamburger Hamiltonian in the lcc with lengths $\vec{l}$ and angles $\vec{\phi}$. Then the order of $H$ does not exceed the convergence exponent of $(l_n^{-1})_{n=1}^{\infty}$, i.e. $\rho(H) \leq \mathrm{c.\,e.}((l_n^{-1})_{n=1}^{\infty})$.*

The next theorem is [BS14, Theorem 3.1], which evaluates the order of an indeterminate moment problem in terms of its orthonormal polynomials.

**2.8 Theorem.** *Let $(s_n)_{n=0}^{\infty}$ be an indeterminate moment sequence, let $P_n(z) = \sum_{k=0}^{n} b_{k,n} z^k$, $n \in \mathbb{N}_0$, be the orthonormal polynomials of the first kind, and set*

$$\Phi(z) := \sum_{n=0}^{\infty} \left( \sum_{k=n}^{\infty} b_{n,k}^2 \right)^{1/2} z^n.$$

*Then $\Phi$ is entire, and the order and type of $\Phi$ coincide with the order and type of the Hamburger power moment problem.*

Since the order and type of an entire function can be calculated from its power-series coefficients, the applicability of this theorem depends heavily on the availability of the sequence $\left(\sum_{k=n}^{\infty} b_{n,k}^2\right)_{n=0}^{\infty}$.

Knowing this sequence means, however, to know all orthonormal polynomials $P_n(z)$. In other words, one has to solve the direct problem (2.2) for all $z \in \mathbb{C}$, and needs to be able to handle all series $\sum_{k=n}^{\infty} b_{n,k}^2$. This makes it hard to apply Theorem 2.8 in practice.

# Part I.

# Results for arbitrary Hamiltonians

# 3. Romanov's Theorem 1

This chapter is devoted to [Rom17, Theorem 1] which provides an upper bound for the growth of the monodromy matrix of a general Hamiltonian. We give a refined version, cf. Theorem 3.3, and formulate the original version of Romanov's Theorem in a variant for growth functions $\lambda(R)$ instead of powers $R^a$, cf. Theorem 3.8.

## 3.1. Refined version

The next lemma is an elementary fact which enables us to work with the $\lambda$-type of a monodromy matrix using matrix norms.

In the subsequent computations we use the following practical notation:

$$f(x) \asymp g(x) \quad :\Longleftrightarrow \quad \exists\, c_1, c_1 > 0\, \forall\, x : c_1 f(x) \leq g(x) \leq c_2 f(x).$$

The notation $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$ refers to the corresponding one-sided properties.

**3.1 Lemma.** *Let $W(z)$ be a $2 \times 2$-matrix consisting of four entire functions, let $\lambda$ be a growth function and let $\|.\|$ be a matrix norm on $\mathbb{C}^{2\times 2}$.*

*If all entries of $W(z)$ have the same $\lambda$-type $\tau$, then $\tau_\lambda(\|W\|) = \tau$.*

*Proof.* For two equivalent norms $\|.\|_1$ and $\|.\|_2$ we have $M(\|W\|_1, R) \asymp M(\|W\|_2, R)$ and, hence, $\tau_\lambda(\|W\|_1) = \tau_\lambda(\|W\|_2)$. Since all matrix norms on $\mathbb{C}^{2\times 2}$ are equivalent, we may assume $\|W(z)\| = \max_{i,j\in\{1,2\}} |w_{ij}(z)|$. Then

$$M(\|W\|, R) = \max_{|z|=R} \max_{i,j\in\{1,2\}} |w_{ij}(z)| = \max_{i,j\in\{1,2\}} \max_{|z|=R} |w_{ij}(z)| = \max_{i,j\in\{1,2\}} M(w_{ij}, R).$$

Applying $\ln^+$, dividing by $\lambda(R)$ and taking the limes superior on both sides yields

$$\tau_\lambda(\|W\|) = \limsup_{R\to\infty} \frac{\ln^+ M(\|W\|, R)}{\lambda(R)} = \limsup_{R\to\infty} \max_{i,j\in\{1,2\}} \frac{\ln^+ M(w_{ij}, R)}{\lambda(R)} = \tau.$$

Note that the maximum of finitely many functions which have limes superior $\tau$, has again the same limes superior. ❑

The next result uses the multiplicative structure of fundamental solutions together with Grönwall's[1] Lemma.

[1] Thomas Hakon Grönwall, *1877 Dylta Bruk in Sweden, emigrated to the U.S. in 1904

**3.2 Lemma.** *Let $H$ be a lcc Hamiltonian on $[a,b]$. Let $N$ be a natural number, let $a = y_0 < y_1 < \ldots < y_N = b$ be a partition of $[a,b]$, and let $\Omega_j$ be real invertible $2\times 2$-matrices for $j = 1,\ldots,N$.*

*Denote by $W(z)$ the monodromy matrix of $H$, and let $\|.\|$ be a submultiplicative matrix norm. Then*

$$\|W(z)\| \leq \exp\Big(|z|\sum_{j=1}^{N}\int_{y_{j-1}}^{y_j}\|\Omega_j JH(t)\Omega_j^{-1}\|\,dt\Big)\|\Omega_1\|\|\Omega_N^{-1}\|\prod_{j=1}^{N-1}\|\Omega_{j+1}\Omega_j^{-1}\|. \tag{3.1}$$

*Proof.* Because of notational reasons, we work in this proof with monodromy matrices as in (1.8). Note that $\|W(x,z)\| = \|\tilde{W}(x,-z)\|$. Denote by $\tilde{W}_j(z) = \tilde{W}_j(y_j,z)$ the monodromy matrix of $H|_{[y_{j-1},y_j]}$, $j = 1,\ldots,N$. Then,

$$\tilde{W}(z) = \tilde{W}_N(z)\tilde{W}_{N-1}(z)\cdot\ldots\cdot\tilde{W}_1(z).$$

We insert matrices $\Omega_j$ and get

$$\tilde{W}(z) = \Omega_N^{-1}\big(\Omega_N\tilde{W}_N(z)\Omega_N^{-1}\big)\Omega_N\Omega_{N-1}^{-1}\big(\Omega_{N-1}\tilde{W}_{N-1}(z)\Omega_{N-1}^{-1}\big)\cdot\ldots\cdot\big(\Omega_1\tilde{W}_1(z)\Omega_1^{-1}\big)\Omega_1$$

Applying Grönwall's Lemma to the differential equation

$$\frac{d}{dx}\Omega_j\tilde{W}_j(x,z)\Omega_j^{-1} = -z\Omega_j JH(x)\Omega_j^{-1}\Omega_j\tilde{W}_j(x,z)\Omega_j^{-1}, \quad x\in[y_{j-1},y_j],$$

yields for $j = 1,\ldots N$

$$\|\Omega_j\tilde{W}_j(z)\Omega_j^{-1}\| \leq \exp\Big(|z|\int_{y_{j-1}}^{y_j}\|\Omega_j JH(t)\Omega_j^{-1}\|\,dt\Big),$$

and the assertion follows. ❑

The main idea is as follows: For a given Hamiltonian, cut the interval $[a,b]$ into pieces and find on each piece a matrix $\Omega_j$ such that the upper bound in (3.1) is small.

**3.3 Theorem** (Romanov's Theorem; refined)**.** *Let $H$ be a lcc Hamiltonian on $[a,b]$, let $\|.\|$ be any submultiplicative matrix norm and let $\lambda$ be a growth function.*

*Let $N(R)$, $R>1$, be a family of natural numbers, let*

$$a = y_0(R) < y_1(R) < \ldots < y_{N(R)}(R) = b, \quad R>1,$$

*be a family of partitions of $[a,b]$, and let $\big(\Omega_j(R)\big)_{j=1}^{N(R)}$, $R>1$, be a family of sequences of invertible $2\times 2$-matrices. Set*

$$A_1(R) := \sum_{j=1}^{N(R)}\int_{y_{j-1}(R)}^{y_j(R)}\big\|\Omega_j(R)JH(t)\Omega_j^{-1}(R)\big\|\,dt$$

$$A_2(R) := \sum_{j=1}^{N(R)-1}\ln\big\|\Omega_{j+1}(R)\Omega_j^{-1}(R)\big\|$$

$$A_3(R) := \ln\|\Omega_1(R)\| + \ln\big\|\Omega_{N(R)}^{-1}(R)\big\|.$$

*Then we have the following upper bound for the $\lambda$-type of the canonical system*

$$\tau_\lambda(H) \le \limsup_{R\to\infty} \frac{1}{\lambda(R)} \big[ R A_1(R) + A_2(R) + A_3(R) \big].$$

*Proof.* Fix $R > 1$. An application of Lemma 3.2 with the choice $N(R)$, $y_j(R)$ for $j = 0, \dots, N(R)$ and $\Omega_j(R)$ for $j = 1, \dots, N(R)$ gives, for $z \in \mathbb{C}$,

$$\begin{aligned} \ln \|W(z)\| \le |z| \sum_{j=1}^{N(R)} \int_{y_{j-1}(R)}^{y_j(R)} \big\| \Omega_j(R) J H(t) \Omega_j^{-1}(R) \big\| \, dt + \sum_{j=1}^{N(R)-1} \ln \big\| \Omega_{j+1}(R) \Omega_j^{-1}(R) \big\| \\ + \ln \|\Omega_1(R)\| + \ln \big\| \Omega_{N(R)}^{-1}(R) \big\|. \end{aligned}$$

For $|z| = R$ we get $\ln \|W(z)\| \le R A_1(R) + A_2(R) + A_3(R)$, and the statement follows from Lemma 3.1. ❑

In order to apply Theorem 3.3 we have to choose a family of natural numbers $N(R)$, of partitions $(y_j(R))$, of sequences of matrices $(\Omega_j(R))$, and a submultiplicative matrix norm. We get a better estimate of the $\lambda$-type of $H$ if our choice leads to smaller $A_1$, $A_2$ and $A_3$. The best estimate which we can achieve this way can be described as follows.

**3.4 Corollary.** *Let $H$ be a lcc Hamiltonian and let $\lambda$ be a growth function. Denote by $\Xi$ the set of all families of tuples*

$$\big( N(R), (y_j(R))_{j=0}^{N(R)}, (\Omega_j(R))_{j=1}^{N(R)} \big), \quad R > 1,$$

*where $N(R)$ is a natural number, $(y_j(R))_{j=0}^{N(R)}$ is a partition of $[a,b]$, and $(\Omega_j(R))_{j=1}^{N(R)}$ is a sequence of invertible matrices.*

*For each $\xi \in \Xi$ and each submultiplicative matrix norm $\|.\|$ we denote the upper bound of the $\lambda$-type of $H$ arising from Theorem 3.3 by*

$$C(\xi, \|.\|) := \limsup_{R\to\infty} \frac{1}{\lambda(R)} \big[ R A_1(R) + A_2(R) + A_3(R) \big].$$

*Then, the $\lambda$-type of $H$ is not larger than*

$$\tau_\lambda(H) \le \inf \big\{ C(\xi, \|.\|) \,\big|\, \xi \in \Xi, \ \|.\| \text{ is a submultiplicative norm} \big\}.$$

It is an open question whether the upper bound in Corollary 3.4 is always equal to the $\lambda$-type of the canonical system.

Certainly, this upper bound enables us to determine the order for a large class of examples, namely for fairly reasonable Hamburger Hamiltonians, cf. Theorem 6.5.

## 3.2. Original version

Next, we introduce matrices $\Omega(a,b,\psi)$ and calculate the spectral norm of expressions, which we encounter frequently.

**3.5 Definition.** Denote for $a, b \in (0,\infty)$ and $\psi \in \mathbb{R}$

$$D(a,b) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \Omega(a,b,\psi) := D(a,b)\exp(-\psi J).$$

Note that $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and

$$\exp(\psi J) = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}.$$

$\Diamond$

**3.6 Lemma.** *Let $\|.\|$ denote the spectral norm. For $a, b, a_1, b_1, a_2, b_2 \in (0,\infty)$ and $\phi, \psi, \psi_1, \psi_2 \in \mathbb{R}$, we have*

(*i*) $\big\|\Omega(a,b,\psi)\big\| = \max\{a,b\}, \quad \big\|\Omega(a,b,\psi)^{-1}\big\| = \max\left\{\frac{1}{a}, \frac{1}{b}\right\}$,

(*ii*) $\big\|\Omega(a,b,\psi)J\xi_\phi\xi_\phi^T\Omega(a,b,\psi)^{-1}\big\| = \frac{b}{a}\cos^2(\phi-\psi) + \frac{a}{b}\sin^2(\phi-\psi)$,

(*iii*) $\big\|\Omega(a_2,b_2,\psi_2)\Omega(a_1,b_1,\psi_1)^{-1}\big\| \le$

$$\le |\cos(\psi_1-\psi_2)| \max\left\{\frac{a_2}{a_1}, \frac{b_2}{b_1}\right\} + |\sin(\psi_1-\psi_2)| \max\left\{\frac{b_2}{a_1}, \frac{a_2}{b_1}\right\}.$$

*Proof.* Since $\exp(-\psi J)$ is orthogonal, $(i)$ amounts to finding the largest singular value, which can easily be read of the diagonal.

In order to show $(ii)$ we use $J\exp(\phi J) = \exp(\phi J)J$ and $\xi_\phi\xi_\phi^T = \exp(\phi J)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\exp(-\phi J)$, which can easily be verifed. This gives

$$\begin{aligned} B &:= \Omega(a,b,\psi)J\xi_\phi\xi_\phi^T\Omega^{-1}(a,b,\psi) = D(a,b)\exp(-\psi J)J\xi_\phi\xi_\phi^T\exp(\psi J)D\big(a^{-1},b^{-1}\big) \\ &= D(a,b)J\exp(-\psi J)\exp(\phi J)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\exp(-\phi J)\exp(\psi J)D\big(a^{-1},b^{-1}\big) \\ &= D(a,b)J\exp(\sigma J)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\exp(-\sigma J)D\big(a^{-1},b^{-1}\big) = D(a,b)J\xi_\sigma\xi_\sigma^T D\big(a^{-1},b^{-1}\big), \end{aligned}$$

with $\sigma := \phi - \psi$, i.e.

$$B = \begin{pmatrix} -\cos(\sigma)\sin(\sigma) & -\frac{a}{b}\sin^2(\sigma) \\ \frac{b}{a}\cos^2(\sigma) & \cos(\sigma)\sin(\sigma) \end{pmatrix}.$$

A direct computation shows

$$B^T B = \begin{pmatrix} \frac{b^2}{a^2}\cos^4(\sigma) + \cos^2(\sigma)\sin^2(\sigma) & * \\ * & \frac{a^2}{b^2}\sin^4(\sigma) + \cos^2(\sigma)\sin^2(\sigma) \end{pmatrix},$$

and $\operatorname{tr}(B^TB) = \big(\frac{b}{a}\cos^2(\sigma) + \frac{a}{b}\sin^2(\sigma)\big)^2$. Since $B$ is singular, we have $\|B\| = \sqrt{\operatorname{tr}(B^TB)}$.

Now turning to $(iii)$, we have

$$\Omega(a_2,b_2,\psi_2)\Omega(a_1,b_1,\psi_1)^{-1} = D(a_2,b_2)\exp((\psi_1-\psi_2)J)D(a_1^{-1},b_1^{-1}) =$$

$$= \begin{pmatrix} \frac{a_2}{a_1}\cos\sigma & -\frac{a_2}{b_1}\sin\sigma \\ \frac{b_2}{a_1}\sin\sigma & \frac{b_2}{b_1}\cos\sigma \end{pmatrix} = \cos\sigma \begin{pmatrix} \frac{a_2}{a_1} & 0 \\ 0 & \frac{b_2}{b_1} \end{pmatrix} + \sin\sigma \begin{pmatrix} 0 & -\frac{a_2}{b_1} \\ \frac{b_2}{a_1} & 0 \end{pmatrix},$$

with $\sigma := \psi_1 - \psi_2$, and the triangle inequality gives

$$\big\|\Omega(a_2,b_2,\psi_2)\Omega(a_1,b_1,\psi_1)^{-1}\big\| \le |\cos\sigma|\max\left\{\frac{a_2}{a_1},\frac{b_2}{b_1}\right\} + |\sin\sigma|\max\left\{\frac{b_2}{a_1},\frac{a_2}{b_1}\right\}.$$

❑

The original version of Romanov's theorem follows from Theorem 3.3 when we take $\Omega_j(R)$ of the form $\Omega(a^{-1},a,\psi)$ for $a\in(0,1]$ and $\psi\in\mathbb{R}$, and rewrite the expressions by employing the triangle inequality. For this choice, Lemma 3.6 reads as follows:

**3.7 Lemma.** *Let $\|.\|$ denote the spectral norm. For $a,a_1,a_2\in(0,1]$ and $\phi,\psi,\psi_1,\psi_2\in\mathbb{R}$, we have*

*(i)* $\big\|\Omega(a^{-1},a,\psi)\big\| = \big\|\Omega(a^{-1},a,\psi)^{-1}\big\| = \dfrac{1}{a}$,

*(ii)* $\big\|\Omega(a^{-1},a,\psi)J\xi_\phi\xi_\phi^T\Omega(a^{-1},a,\psi)^{-1}\big\| = a^2\cos^2(\phi-\psi) + \dfrac{1}{a^2}\sin^2(\phi-\psi)$,

*(iii)* $\big\|\Omega(a_2^{-1},a_2,\psi_2)\Omega(a_1^{-1},a_1,\psi_1)^{-1}\big\| \le$

$$\le \max\left\{\frac{a_1}{a_2},\frac{a_2}{a_1}\right\}\big|\cos(\psi_1-\psi_2)\big| + \frac{1}{a_1a_2}\big|\sin(\psi_1-\psi_2)\big|.$$

**3.8 Theorem** (Romanov's Theorem 1; [Rom17])**.** *Let $H$ be a lcc Hamiltonian on $[a,b]$, let $\|.\|$ be the spectral norm and let $\lambda$ be a growth function.*

*Let $N(R)$, $R>1$, be a family of natural numbers, let*

$$a = y_0(R) < y_1(R) < \ldots < y_{N(R)}(R) = b, \quad R>1,$$

*be a family of partitions of $[a,b]$, and let $(\psi_j(R))_{j=1}^{N(R)}$ and $(a_j(R))_{j=1}^{N(R)}$, $R>1$, be two* *families of sequences of real numbers with $a_j(R) \in (0,1]$. Set*

$$B_1(R) := \sum_{j=1}^{N(R)} \frac{1}{a_j^2(R)} \int\limits_{y_{j-1}(R)}^{y_j(R)} \left\| H(x) - \xi_{\psi_j}\xi_{\psi_j}^T \right\| dx$$

$$B_2(R) := \sum_{j=1}^{N(R)} a_j^2(R)(y_j(R) - y_{j-1}(R))$$

$$B_3(R) := \sum_{j=1}^{N(R)-1} \ln\left(1 + \frac{\left|\sin\left(\psi_{j+1}(R) - \psi_j(R)\right)\right|}{a_{j+1}(R)a_j(R)}\right)$$

$$B_4(R) := \left|\ln a_1(R)\right| + \left|\ln a_{N(R)}(R)\right| + \sum_{j=1}^{N(R)-1} \left|\ln \frac{a_{j+1}(R)}{a_j(R)}\right|.$$

*Then we have the following upper bound for the $\lambda$-type of the canonical system*

$$\tau_\lambda(H) \leq \limsup_{R\to\infty} \frac{1}{\lambda(R)}\left[R\big(B_1(R) + B_2(R)\big) + B_3(R) + B_4(R)\right].$$

*Proof.* In order to apply Theorem 3.3 set $\Omega_j(R) := \Omega(a_j^{-1}(R), a_j(R), \psi_j(R))$ for $j = 1, \ldots, N(R)$, cf. Definition 3.5. We need to look at the quantities $A_1$,$A_2$ and $A_3$.

Inserting $\xi_{\psi_j(R)}\xi_{\psi_j(R)}^T$ and applying the triangle inequality give

$$\begin{aligned}\|\Omega_j(R)JH(t)\Omega_j^{-1}(R)\| \leq \|\Omega_j(R)J\big(H(t) - \xi_{\psi_j(R)}\xi_{\psi_j(R)}^T\big)\Omega_j^{-1}(R)\| + \\ + \|\Omega_j(R)J\xi_{\psi_j(R)}\xi_{\psi_j(R)}^T\Omega_j^{-1}(R)\|. \quad (3.2)\end{aligned}$$

Using the submultiplicativity of the norm and Lemma 3.7, $(i)$, the first summand on the right-hand side of (3.2) can be estimated from above by

$$\|\Omega_j(R)J\big(H(t) - \xi_{\psi_j(R)}\xi_{\psi_j(R)}^T\big)\Omega_j^{-1}(R)\| \leq \frac{1}{a_j^2(R)}\|H(t) - \xi_{\psi_j(R)}\xi_{\psi_j(R)}^T\|.$$

The second summand in (3.2) is equal to $a_j^2(R)$, by Lemma 3.7, $(ii)$. Integrating over $t \in (y_{j-1}(R), y_j(R))$ and taking the sum from $j = 1$ to $N(R)$ yields $A_1(R) \leq B_1(R)+B_2(R)$.

Item $(iii)$ of Lemma 3.7 gives

$$\begin{aligned}&\|\Omega_{j+1}(R)\Omega_j^{-1}(R)\| \leq \\ &\quad\leq \max\left\{\frac{a_j(R)}{a_{j+1}(R)}, \frac{a_{j+1}(R)}{a_j(R)}\right\}\left|\cos(\psi_{j+1}(R) - \psi_j(R))\right| + \frac{\left|\sin(\psi_{j+1}(R) - \psi_j(R))\right|}{a_{j+1}(R)a_j(R)} \\ &\quad\leq \left(1 + \frac{\left|\sin(\psi_{j+1}(R) - \psi_j(R))\right|}{a_{j+1}(R)a_j(R)}\right)\max\left\{\frac{a_j(R)}{a_{j+1}(R)}, \frac{a_{j+1}(R)}{a_j(R)}\right\}\end{aligned}$$

Passing to logarithms and taking the sum from $j = 1$ to $N(R) - 1$ yields

$$A_2(R) \le B_3(R) + \sum_{j=1}^{N(R)-1} \ln \max\Big\{\frac{a_j(R)}{a_{j+1}(R)}, \frac{a_{j+1}(R)}{a_j(R)}\Big\} = B_3(R) + \sum_{j=1}^{N(R)-1} \Big|\ln \frac{a_{j+1}(R)}{a_j(R)}\Big|$$

Together with the remaining term, which is, by Lemma 3.7, $(i)$, equal to

$$A_3(R) = \ln \|\Omega_1(R)\| + \ln \|\Omega_{N(R)}^{-1}(R)\| = \big|\ln a_1(R)\big| + \big|\ln a_{N(R)}(R)\big|,$$

we get $A_2(R)+A_3(R) \le B_3(R)+B_4(R)$, and the assertion follows from Theorem 3.3. ❑

## 3.3. Optimal choice of $\Omega_j(R)$

Calculating the infimum in Corollary 3.4 for a given Hamiltonian $H$ can be very difficult. Clearly, the task gets easier when we show that the infimum is already obtained in a smaller subset of $\Xi$.

In this subsection we want to focus on the choice of $\Omega_j(R)$.

**3.9 Lemma.** *Let the notation be as in Theorem* 3.3. *Additionally assume that the matrix norm* $\|.\|$ *is unitarily invariant, i.e.* $\|A\| = \|UAV\|$ *for arbitrary matrices* $A$ *and orthogonal matrices* $U, V$.

*Let* $(\mu_j(R))_{i=1}^{N(R)}$, $R > 1$, *be a family of sequences of non-zero real numbers, and let* $(U_j(R))_{i=1}^{N(R)}$, $R > 1$, *be a family of sequences of orthogonal* $2 \times 2$*-matrices.*

*Then, replacing the original family* $\big(\Omega_j(R)\big)_{j=1}^{N(R)}$, $R > 1$, *with*

$$\big(\mu_j(R)U_j(R)\Omega_j(R)\big)_{j=1}^{N(R)}, \quad R > 1,$$

*does not change the value of both* $A_1(R)$ *and* $A_2(R) + A_3(R)$. *In particular, Theorem* 3.3 *gives the same estimate for the* $\lambda$*-type of* $H$.

*Proof.* Multiplying from the left by orthogonal matrices does not change the expressions since the norm is unitarily invariant.

Let $A_i'(R)$ denote those quantities which correspond to the new family. The scalars $\mu_j(R)$ cancel out immediately in the first expression, i.e. $A_1'(R) = A_1(R)$. Regarding the second one, we have

$$\begin{aligned} A_2'(R) &= \sum_{j=1}^{N(R)-1} \ln \big\|\mu_{j+1}(R)\mu_j^{-1}(R)\Omega_{j+1}(R)\Omega_j^{-1}(R)\big\| \\ &= \sum_{j=1}^{N(R)-1} \ln \big\|\Omega_{j+1}(R)\Omega_j^{-1}(R)\big\| + \sum_{j=1}^{N(R)-1} \ln|\mu_{j+1}(R)| - \ln|\mu_j(R)| \\ &= A_2(R) + \ln|\mu_{N(R)}(R)| - \ln|\mu_1(R)|. \end{aligned}$$

Since

$$A_3'(R) = \ln \|\mu_1(R)\Omega_1(R)\| + \ln \left\|\mu_{N(R)}^{-1}(R)\Omega_{N(R)}^{-1}(R)\right\| = \\ = A_3(R) - \ln |\mu_{N(R)}(R)| + \ln |\mu_1(R)|,$$

we see $A_2'(R) + A_3'(R) = A_2(R) + A_3(R)$. ❑

Formally, the restriction to unitarily invariant norms is a transition to the infimum over a smaller subset in Corollary 3.4, and it is not clear if this affects the infimum. But this restriction allows us to further reduce the size of the set:

**3.10 Corollary.** *Let $H$ be a lcc Hamiltonian, let $\lambda$ be a growth function and recall the notation from Corollary* 3.4*.*

*Denote by $\Xi_0 \subseteq \Xi$ the set of all families of tuples*

$$\left(N(R), (y_j(R))_{j=0}^{N(R)}, \left(\Omega(a_j^{-1}(R), a_j(R), \psi_j(R))\right)_{j=1}^{N(R)}\right), \quad R > 1,$$

*where $N(R)$ is a natural number, $(y_j(R))_{j=0}^{N(R)}$ is a partition of $[a,b]$, $(\psi_j(R))_{j=1}^{N(R)}$ is a sequence of real numbers, and $(a_j(R))_{j=1}^{N(R)}$ satisfies $a_j \in (0,1]$. Recall Definition* 3.5 *for the definition of $\Omega(a_j^{-1}(R), a_j(R), \psi_j(R))$.*

*Then,*

$$\tau_\lambda(H) \leq \inf \left\{C(\xi, \|.\|) \mid \xi \in \Xi,\ \|.\| \text{ is a unitarily invariant norm}\right\} \\ = \inf \left\{C(\xi, \|.\|) \mid \xi \in \Xi_0,\ \|.\| \text{ is the spectral norm}\right\}.$$

*Proof.* Let $\|.\|$ denote the spectral norm. For all $2 \times 2$ matrices $A$ and all unitarily invariant norms $\|.\|_*$, we have $\|A\| \leq \|A\|_*$, cf. [Bha97, (IV.38), p.93]. Thus, choosing among all unitarily invariant norms the spectral norm leads to potentially smaller quantities $A_i(R)$ and, consequently, to a better upper bound $C(\xi, \|.\|)$.

For each $R > 1$ and $j \in \mathbb{N}$ with $j \leq N(R)$ let

$$\Omega_j(R) = U_j(R) D(s_{j,1}(R), s_{j,2}(R)) V_j(R)^T$$

be a singular value decomposition of $\Omega_j(R)$. Multiplying $\Omega_j(R)$ by the scalar $\mu_j(R) := (s_{j,1}(R)s_{j,2}(R))^{-1/2}$ gives, with $a_j(R) := (s_{j,2}(R)/s_{j,1}(R))^{1/2} \in (0,1]$,

$$\mu_j(R)\Omega_j(R) = U_j(R) D(a_j(R)^{-1}, a_j(R)) V_j(R)^T.$$

To achieve $\det U_j(R) = \det V_j(R) = 1$ we replace, if necessary, $U_j(R)$ and $V_j(R)$ by $U_j(R) \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $V_j(R) \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, note

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D(a,b) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = D(a,b).$$

Then $V_j(R)$ is a rotation and can be written as $V_j(R) = \exp(\psi_j(R)J)$ for some $\psi_j(R) \in \mathbb{R}$. So

$$\mu_j(R) U_j(R)^T \Omega_j(R) = D(a_j(R)^{-1}, a_j(R)) \exp(-\psi_j(R)J) = \Omega(a_j^{-1}, a_j, \psi_j),$$

and the statement follows form Lemma 3.9. ❑

Corollary 3.10 shows that the choice made by R.Romanov is (among other choices) optimal if the norm is unitarily invariant.

Another obstacle is the following. If we want to compute $A_2$ precisely, we have to know the spectral norm of $\Omega_{j+1}\Omega_j^{-1}$. By Corollary 3.10, we may assume $\Omega_j = \Omega(a_j^{-1}, a_j, \psi_j)$.

**3.11 Lemma.** *Let $a_j \in (0,1]$ and $\psi_j \in \mathbb{R}$ for $j = 1, \dots N$, and consider $\Omega_j := \Omega(a_j^{-1}, a_j, \psi_j)$. Let $\|.\|$ denote the spectral norm and set*

$$t_j := \cos^2(\psi_{j+1} - \psi_j)\Big(\frac{a_{j+1}}{a_j} + \frac{a_j}{a_{j+1}}\Big) + \sin^2(\psi_{j+1} - \psi_j)\Big(a_{j+1}a_j + \frac{1}{a_{j+1}a_j}\Big).$$

*Then*

$$\|\Omega_{j+1}\Omega_j^{-1}\| = \sqrt{\frac{1}{2}\left(t_j + \sqrt{t_j^2 - 4}\right)}.$$

*Proof.* Direct computation. ❑

It turned out that the exact expressions for $\|\Omega_{j+1}\Omega_j^{-1}\|$ are too bulky for later usage. To avoid them we use the upper bound given in Lemma 3.6, (*iii*). Potentially, this will produce larger upper bounds for the $\lambda$-type of $H$.

To recall this upper bound, let $\Omega_j := \Omega(a_j, b_j, \psi_j)$ and set $\sigma_j := \psi_{j+1} - \psi_j$. By Lemma 3.6, (*iii*), we have

$$\begin{aligned} A_2 = &\sum_{j=1}^{N(R)-1} \ln \left\|\Omega_{j+1}\Omega_j^{-1}\right\| \leq \\ \leq &\sum_{j=1}^{N(R)-1} \ln \left( |\cos(\sigma_j)| \max\left\{\frac{a_{j+1}}{a_j}, \frac{b_{j+1}}{b_j}\right\} + |\sin(\sigma_j)| \max\left\{\frac{b_{j+1}}{a_j}, \frac{a_{j+1}}{b_j}\right\}\right). \end{aligned} \tag{3.3}$$

Let $\widetilde{A}_2$ denote the right-hand side of the inequality (3.3). By Lemma 3.9 we know that normalizing $\Omega_j$, e.g. to the form $\Omega(a^{-1}, a, \psi)$, does not change $A_2 + A_3$. But does it have an influence on $\widetilde{A}_2 + A_3$? The short answer is no. More precisely, we have the following

**3.12 Lemma.** *Let $a_j, b_j \in (0, \infty)$ and $\psi_j \in \mathbb{R}$ for $j = 1, \dots N$ be given, and consider $\Omega_j := \Omega(a_j, b_j, \psi_j)$.*

*We can find $\alpha_j \in (0,1]$ and $\phi_j \in \mathbb{R}$ for $j = 1, \dots N$ with the following property: Replacing $\Omega_j$ by $\Omega(\alpha_j^{-1}, \alpha_j, \phi_j)$ does not change the value of both $A_1$ and $\widetilde{A}_2 + A_3$.*

*Proof.* Interchanging $a_j$ with $b_j$ and simultaneously replacing the corresponding $\psi_j$ by $\psi_j + \frac{\pi}{2}$ does not change $A_1$. In fact, this is just a multiplication of $\Omega_j$ from the left by the orthogonal matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Also $\widetilde{A}_2$ and $A_3$ do not change, which can be seen directly form (3.3) and Lemma 3.6, (*i*). Thus, it is possible to assume $a_j \geq b_j$.

Multiplying $\Omega_j$ by $\mu_j := (a_j b_j)^{-1/2}$ we obtain

$$\mu_j \Omega_j = D\Big(\sqrt{a_j/b_j}, \sqrt{b_j/a_j}\Big) \exp(-\psi_j J) = \Omega(\alpha_j^{-1}, \alpha_j, \psi_j),$$

with $\alpha_j := \sqrt{b_j/a_j} \in (0,1]$. We denote by $\widetilde{A}_2$ and $A_3$ the original expressions, and by $\widetilde{A}_2'$ and $A_3'$ the new ones. Then

$$\begin{aligned}
\widetilde{A}_2' &= \widetilde{A}_2 + \sum_{j=1}^{N-1} \ln \frac{\mu_{j+1}}{\mu_j} = \widetilde{A}_2 + \ln \mu_N - \ln \mu_1, \\
A_3' &= \ln \mu_1 \|\Omega_1\| + \ln \mu_N^{-1} \|\Omega_N^{-1}\| = A_3 - \ln \mu_N + \ln \mu_1.
\end{aligned}$$

Hence, $\widetilde{A}_2' + A_3' = \widetilde{A}_2 + A_3$. ❑

# 4. Romanov's Theorem 2

## 4.1. Formulation of the Theorem

A particular class of canonical systems is given by Hamiltonians which are almost everywhere a diagonal matrix, and we refer to such as *diagonal Hamiltonians.*

We present another theorem by R.Romanov which characterises the order of a diagonal Hamiltonian. Observe that, since the real and symmetric $2\times 2$-matrix $H(x)$ satisfies $\operatorname{tr} H(x) = 1$, it holds that $\det H(x) = 0$ and $H(x)$ is diagonal if and only if

$$H(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad H(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

**4.1 Definition.** Consider a nonempty interval $[a,b)$. We denote by $\operatorname{Cov}[a,b)$ the set of all coverings $\Omega$ of $[a,b)$ by finitely many pairwise disjoint left-closed and right-open intervals contained in $[a,b)$. $\Diamond$

Moreover, we denote by $\#F$ the number of elements of a finite set $F$ and by $\lambda$ the Lebesgue-measure on $\mathbb{R}$. A confusion with growth functions $\lambda$ is not likely, since we do not talk about growth functions in this chapter.

**4.2 Theorem** (Romanov's Theorem 2; [Rom17])**.** *Let $H$ be a trace normed Hamiltonian on a finite interval $[a,b)$ with*

$$\det H(x) = 0,\ \ H(x) \text{ is diagonal}, \tag{4.1}$$

*for a.e. $x \in [a,b)$. Set*

$$\begin{aligned} M_1 &:= \Big\{x \in [a,b) : H(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\Big\}, \\ M_2 &:= \Big\{x \in [a,b) : H(x) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\Big\}. \end{aligned} \tag{4.2}$$

*Then $\rho(H)$ is equal to the infimum of all numbers $d \in (0,1]$ for which there exists a family of coverings $\Omega(R) \in \operatorname{Cov}[a,b)$, $R > 1$, such that*

$$\#\Omega(R) = \mathrm{O}\big(R^d\big), \tag{4.3}$$

$$\sum_{\omega\in\Omega(R)} \sqrt{\lambda(\omega\cap M_1)\cdot\lambda(\omega\cap M_2)} = \mathrm{O}\big(R^{d-1}\big). \tag{4.4}$$

For later usage, we formulate a version of this theorem for Hamburger Hamiltonians. Clearly, the sequence of angles of a diagonal Hamburger Hamiltonian has to alternate between the values 0 and $\pi/2$. The statement of Theorem 4.2 remains true when we restrict ourselves to the following subclass of coverings.

**4.3 Definition.** Let $H$ be a diagonal Hamburger Hamiltonian in the lcc with lengths $\vec{l}$, and set $x_n$, for $n \in \mathbb{N}_0 \cup \{\infty\}$, as in (2.4). We write $\Omega \in \operatorname{Cov}(H)$, if

(*i*) $\Omega \in \operatorname{Cov}[0, L)$,

(*ii*) $\forall \omega \in \Omega \, \exists n_-, n_+ \in \mathbb{N}_0 \cup \{\infty\} : \ \omega = [x_{n_-}, x_{n_+})$.

$\Diamond$

**4.4 Lemma.** *Let $H$ be a diagonal Hamburger Hamiltonian. Then $\rho(H)$ is equal to the infimum of all numbers $d \in (0, 1]$ for which there exists a family of coverings $\Omega(R) \in \operatorname{Cov}(H)$, $R > 1$, such that* (4.3) *and* (4.4) *hold.*

*Proof.* It is enough to show that for each number $d \in (0, 1]$ and family $\Omega(R) \in \operatorname{Cov}[0, L)$, $R > 1$, with (4.3) and (4.4), there exists a family $\tilde{\Omega}(R) \in \operatorname{Cov}(H)$, $R > 1$, such that (4.3) and (4.4) still hold.

The coverings $\tilde{\Omega}(R)$ are constructed by modifying $\Omega(R)$ in the following way. For each $\omega \in \Omega(R)$, we make a case distinction:

*Case 1:* Assume that there exists $n \in \mathbb{N}_0$ such that $\omega \subseteq [x_n, x_{n+1})$. Then we include the interval $[x_n, x_{n+1})$ into $\tilde{\Omega}(R)$.

*Case 2:* Assume that Case 1 does not take place. Then there exists $n \in \mathbb{N}$ with $x_n \in \operatorname{Int}\omega$ (here $\operatorname{Int}\omega$ denotes the interior of $\omega$). Set

$$n_- := \min\big\{n \in \mathbb{N} : x_n \in \operatorname{Int}\omega\big\}, \ n_+ := \sup\big\{n \in \mathbb{N} : x_n \in \operatorname{Int}\omega\big\} \in \mathbb{N} \cup \{\infty\}.$$

In the case $n_+ = \infty$ we include the intervals

$$[x_{n_- -1}, x_{n_-}), \quad [x_{n_-}, x_{n_+}),$$

into $\tilde{\Omega}(R)$, note $x_\infty = L$. For $n_+ < \infty$ we take

$$[x_{n_- -1}, x_{n_-}), \quad [x_{n_- -1}, x_{n_+}), \quad [x_{n_+}, x_{n_+ +1}),$$

where the middle interval only appears if $n_- < n_+$.

Then, obviously, $\tilde{\Omega}(R) \in \operatorname{Cov}(H)$ and $\#\tilde{\Omega}(R) \le 3 \cdot \#\Omega(R)$. In particular, (4.3) holds for $\tilde{\Omega}(R)$, $R > 1$. Consider the sum in (4.4) for the covering $\tilde{\Omega}(R)$. Then, only intervals of the form $[x_{n_-}, x_{n_+})$ constructed from some $\omega \in \Omega(R)$ contribute a possibly nonzero summand. However, $[x_{n_-}, x_{n_+}) \subseteq \omega$ and hence

$$\lambda\big([x_{n_-}, x_{n_+}) \cap M_i\big) \le \lambda(\omega \cap M_i), \quad i = 1, 2.$$

We see that

$$\sum_{\tilde{\omega} \in \tilde{\Omega}(R)} \sqrt{\lambda(\tilde{\omega} \cap M_1) \cdot \lambda(\tilde{\omega} \cap M_2)} \le \sum_{\omega \in \Omega(R)} \sqrt{\lambda(\omega \cap M_1) \cdot \lambda(\omega \cap M_2)},$$

and conclude that (4.4) holds. ❑

## 4.2. Restricting and extending Hamiltonians

We present two results which follow from Romanov's Theorem 2. Both give an answer to the following type of question: How does the order of a canonical system change when we modify the Hamiltonian?

Intuitively, "removing parts of a Hamiltonian" should not increase the order. First, let us make precise what we mean with "removing parts of a Hamiltonian".

**4.5 Definition.** Let $L \in (0, \infty)$, let $H$ be a trace normed Hamiltonian on $[0, L]$, let $B \subseteq [0, L]$ be Lebesgue measurable with

$$\forall 0 < x_- < x_+ < L : \Big( \lambda\big(B \cap [x_-, x_+]\big) = 0 \ \Rightarrow \ B \cap [x_-, x_+] = \emptyset \Big) \tag{4.5}$$

and set

$$\tau(x) := \lambda\big([0, x] \cap B\big), \quad x \in [0, L].$$

Moreover, choose a right inverse $\tilde{\tau} : [0, \lambda(B)] \to [0, L]$ of $\tau$. Then we denote

$$H_B(y) := (H \circ \tilde{\tau})(y), \quad y \in [0, \lambda(B)],$$

and speak of the restriction of $H$ to $B$. $\Diamond$

Some remarks are in order. The listed facts follow, e.g., from the arguments in [WW12, Remark 3.3,(*ii*)] and [WW12, Lemma 3.5,(*ii, iii*)].

First, the function $\tilde{\tau}$ is Lebesgue-to-Lebesgue measurable, hence $H_B$ is a trace normed Hamiltonian on $[0, \lambda(B)]$. When we choose a different right inverse $\tilde{\tau}$, $H_B$ changes only on $\tau(B^c)$, which is a set of measure zero.

Second, requiring (4.5) is no loss in generality. Given any Lebesgue measurable subset $B$ of $[0, L]$, one can choose $B_0 \subseteq B$ such that $\lambda(B \setminus B_0) = 0$ and $B_0$ satisfies (4.5). Using $B_0$ instead of $B$ does not change $\tau$, and $H_B$ changes only on a set of zero measure.

**4.6 Proposition.** *Let* $L \in (0, \infty)$*, let* $H$ *be a trace normed diagonal Hamiltonian on* $[0, L]$ *with* $\det H(x) = 0$ *for a.e.* $x \in [0, L]$*, and let* $B \subseteq [0, L]$ *be a Lebesgue measurable set with* (4.5)*. Then*

$$\rho(H_B) \leq \rho(H).$$

*Proof.* It is enough to show that for each number $d \in (0, 1]$ and family $\Omega(R) \in \operatorname{Cov}[0, L)$, $R > 1$, with (4.3) and (4.4) for $H$, there exists a family $\tilde{\Omega}(R) \in \operatorname{Cov}[0, \lambda(B))$, $R > 1$, such that (4.3) and (4.4) still hold for $H_B$.

Let $\omega \in \Omega(R)$ and write $\omega = [x_-, x_+)$. If $\tau(x_-) = \tau(x_+)$, do not consider $\omega$ further. If $\tau(x_-) < \tau(x_+)$, include the interval $[\tau(x_-), \tau(x_+))$ into $\tilde{\Omega}(R)$. Then it is obvious that $\tilde{\Omega}(R) \in \operatorname{Cov}[0, \lambda(B))$ and that $\#\tilde{\Omega}(R) \leq \#\Omega(R)$. In particular, (4.3) holds for $\tilde{\Omega}(R)$, $R > 1$.

Let $\tilde{M}_1$ and $\tilde{M}_2$ be the sets (4.2) for $H_B$. Let $\omega = [x_-, x_+) \in \Omega(R)$ with $\tau(x_-) < \tau(x_+)$, and let $\tilde{\omega} = [\tau(x_-), \tau(x_+))$. If $x \in [0, L]$ with $\tau(x) \in \operatorname{Int} \tilde{\omega}$, then $x \in \operatorname{Int} \omega$. Hence,

$$\mathbb{1}_{\operatorname{Int} \tilde{\omega}} \circ \tau \leq \mathbb{1}_{\operatorname{Int} \omega} \leq \mathbb{1}_{\omega}.$$

For a point $x \in B \subseteq [0,L]$, (4.5) ensures $\tilde{\tau}(\tau(x)) = x$. Then $\tau(x) \in \tilde{M}_1$ if and only if $x \in M_1$, since

$$H_B(\tau(x)) = (H_B \circ \tilde{\tau})(\tau(x)) = H_B(x).$$

Hence, we see that

$$\mathbb{1}_B(\mathbb{1}_{\tilde{M}_1} \circ \tau) = \mathbb{1}_B \mathbb{1}_{M_1} \leq \mathbb{1}_{M_1}.$$

Now we can estimate

$$\begin{aligned}
\lambda(\tilde{\omega} \cap \tilde{M}_1) =& \lambda\big((\operatorname{Int}\tilde{\omega}) \cap \tilde{M}_1\big) = \int_{[0,\lambda(B)]} \mathbb{1}_{\operatorname{Int}\tilde{\omega}}(y) \mathbb{1}_{\tilde{M}_1}(y)\, dy \\
=& \int_{[0,L]} (\mathbb{1}_{\operatorname{Int}\tilde{\omega}} \circ \tau)(x)(\mathbb{1}_{\tilde{M}_1} \circ \tau)(x) \underbrace{\tau'(x)}_{=\mathbb{1}_B(x) \text{ a.e.}}\, dx \\
\leq& \int_{[0,L]} \mathbb{1}_\omega(x) \mathbb{1}_{M_1}(x)\, dx = \lambda(\omega \cap M_1).
\end{aligned}$$

The same argument applies with $\tilde{M}_2$ and we obtain

$$\sum_{\xi \in \tilde{\Omega}(R)} \sqrt{\lambda(\xi \cap \tilde{M}_1) \cdot \lambda(\xi \cap \tilde{M}_2)} \leq \sum_{\omega \in \Omega(R)} \sqrt{\lambda(\omega \cap M_1) \cdot \lambda(\omega \cap M_2)}.$$

This shows that (4.4) holds, and the statement follows from Theorem 4.2. ❑

Another intuition is that inserting sufficiently small intervals into a Hamiltonian should not affect its order. For diagonal Hamburger Hamiltonians we can confirm this, and quantify what "sufficiently small" means.

**4.7 Proposition.** *Let $H$ be a diagonal Hamburger Hamiltonian in the lcc and let $(l_n)_{n=1}^\infty$ and $(\phi_n)_{n=1}^\infty$ be its lengths and angles. Moreover, let $(\varepsilon_n)_{n=1}^\infty$ be a sequence of nonnegative numbers with*

$$\sum_{n=1}^\infty \frac{\varepsilon_n}{\min\{l_n, l_{n+1}\}} < \infty, \tag{4.6}$$

*and let $\tilde{H}$ be the Hamburger Hamiltonian with parameters*

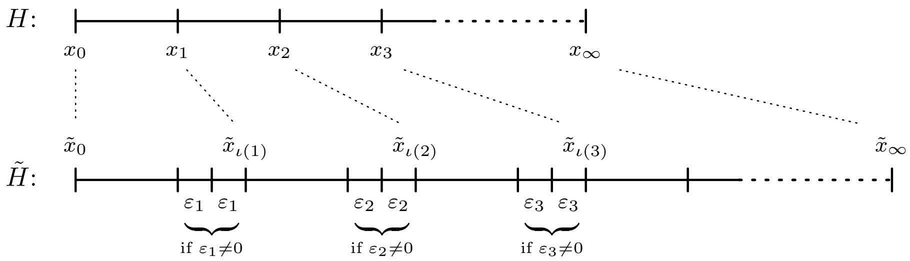
$$\begin{array}{rl}
\textit{lengths:} & (\, l_1\ ,\ \varepsilon_1, \varepsilon_1\ ,\ l_2\ ,\ \varepsilon_2, \varepsilon_2\ ,\ l_3\ ,\ \dots\ ) \\
\textit{angles:} & (\phi_1\ ,\ \underbrace{\phi_2, \phi_1}_{\textit{if } \varepsilon_1 \neq 0}\ ,\ \phi_2\ ,\ \underbrace{\phi_3, \phi_2}_{\textit{if } \varepsilon_2 \neq 0}\ ,\ \phi_3\ ,\ \dots\ )
\end{array}$$

*Then $\tilde{H}$ is again in the lcc and*

$$\rho(\tilde{H}) = \rho(H).$$

*Proof.* In view of Proposition 4.6 and Lemma 4.4 it is enough to show that for each number $d \in (0,1]$ and family $\Omega(R) \in \operatorname{Cov}(H)$, $R > 1$, with (4.3) and (4.4) for $H$, there exists a family $\tilde{\Omega}(R) \in \operatorname{Cov}(\tilde{H})$, $R > 1$, such that (4.3) and (4.4) still hold for $\tilde{H}$.

Before we give the construction, we introduce another notation. Define a map $\iota : \mathbb{N}_0 \cup \{\infty\} \to \mathbb{N}_0 \cup \{\infty\}$ such that ($\tilde{x}_n$ as in (2.4) for $\tilde{H}$) $\tilde{x}_{\iota(n)}$ is the left endpoint of the interval in $\tilde{H}$ which comes from the interval $[x_n, x_{n+1})$ in $H$. Moreover, set $\iota(\infty) := \infty$.




Let $\omega \in \Omega(R)$ and write $\omega := [x_{n_-}, x_{n_+})$. We set

$$\tilde{n}_- := \iota(n_-), \quad \tilde{n}_+ := \iota(n_+) - \begin{cases} 0\,, & n_+ = \infty, \text{ or } \varepsilon_{n_+} = 0, \\ 2\,, & \text{else,} \end{cases}$$

and include the intervals

$$[\tilde{x}_{\tilde{n}_-}, \tilde{x}_{\tilde{n}_+}), \tag{4.7}$$

$$[\tilde{x}_{\tilde{n}_+}, \tilde{x}_{\tilde{n}_+ +1}), [\tilde{x}_{\tilde{n}_+ +1}, \tilde{x}_{\tilde{n}_+ +2}), \qquad \big(\text{only if } n_+ < \infty,\ \varepsilon_{n_+} > 0\big) \tag{4.8}$$

into $\tilde{\Omega}(R)$. Then, clearly, $\tilde{\Omega}(R) \in \operatorname{Cov}(\tilde{H})$ and

$$\#\tilde{\Omega}(R) \leq 3 \cdot \#\Omega(R).$$

In particular, (4.3) holds for $(\tilde{\Omega}(R))_{R>1}$. We need to estimate the sum

$$\sum_{\tilde{\omega} \in \tilde{\Omega}(R)} \sqrt{\lambda(\tilde{\omega} \cap \tilde{M}_1) \cdot \lambda(\tilde{\omega} \cap \tilde{M}_2)} \tag{4.9}$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are the sets (4.2) for $\tilde{H}$.

The intervals in (4.8) do not contribute to the sum (4.9). Moreover, an interval from (4.7) contributes only if $n_+ > n_- + 1$. For an interval $\tilde{\omega}$ of this kind we have

$$\lambda(\tilde{\omega} \cap \tilde{M}_i) = \lambda(\omega \cap M_i) + \sum_{n_- < j < n_+} \varepsilon_j, \quad i = 1, 2.$$

However,

$$\lambda(\omega \cap M_i) \geq \min\{l_j, l_{j+1}\}, \quad n_- < j < n_+,\ i = 1, 2,$$

and hence

$$\lambda(\tilde{\omega} \cap \tilde{M}_i) = \lambda(\omega \cap M_i)\Big(1 + \sum_{n_- < j < n_+} \frac{\varepsilon_j}{\lambda(\omega \cap M_i)}\Big)$$

$$\leq \lambda(\omega \cap M_i)\Big(1 + \sum_{j=1}^{\infty} \frac{\varepsilon_j}{\min\{l_j, l_{j+1}\}}\Big).$$

Thus (4.9) does not exceed

$$\Big(1+\sum_{j=1}^{\infty}\frac{\varepsilon_j}{\min\{l_j,l_{j+1}\}}\Big)\cdot\sum_{\omega\in\Omega(R)}\sqrt{\lambda(\omega\cap M_1)\cdot\lambda(\omega\cap M_2)},$$

and we conclude that (4.4) holds. ❑

# Part II.

# Results for Hamburger Hamiltonians

# 5. Estimates for the order

## 5.1. General upper estimate

We apply Theorem 3.3 to Hamburger Hamiltonians. In this case there is a natural choice for the partition, namely by taking the first $n-1$ indivisible intervals of $H$, and the remaining part as the last interval.

**5.1 Theorem.** *Let $H_{\vec{l},\vec{\phi}}$ be a Hamburger Hamiltonian in the lcc, let $\|.\|$ denote the spectral norm and let $\lambda$ be a growth function.*

*Let $N(R)$, $R>1$, be a family of natural numbers, let $(a_j(R))_{j=1}^{N(R)}$, $R>1$, be a family of sequences of real numbers with $a_j(R)\in(0,1]$, and let $\psi\in\mathbb{R}$. We introduce the notation $\epsilon_j := \phi_{j+1}-\phi_j$ for $j=1,\dots,N(R)-2$ and $\epsilon_{N(R)-1} := \psi-\phi_{N(R)-1}$ and set*

$$\begin{aligned}
C_1(R) &:= \sum_{j=1}^{N(R)-1} l_j a_j^2(R)\\
C_2(R) &:= \sum_{j=N(R)}^{\infty} l_j \Big(a_{N(R)}^2(R)\cos^2(\psi-\phi_j)+a_{N(R)}^{-2}(R)\sin^2(\psi-\phi_j)\Big)\\
C_3(R) &:= \sum_{j=1}^{N(R)-1} \ln\Big(\max\Big\{\frac{a_{j+1}(R)}{a_j(R)},\frac{a_j(R)}{a_{j+1}(R)}\Big\}\big|\cos(\epsilon_j)\big|+\frac{|\sin(\epsilon_j)|}{a_{j+1}(R)a_j(R)}\Big)\\
C_4(R) &:= \ln a_1^{-1}(R)+\ln a_{N(R)}^{-1}(R).
\end{aligned}$$

*Then, we have the following upper bound for the $\lambda$-type of the canonical system*

$$\tau_\lambda(H) \leq \limsup_{R\to\infty}\frac{1}{\lambda(R)}\big[R\big(C_1(R)+C_2(R)\big)+C_3(R)+C_4(R)\big].$$

*Proof.* Recall the notation

$$x_0:=0,\qquad x_n:=\sum_{k=1}^{n} l_k,\ n\in\mathbb{N},\qquad L:=\sum_{k=1}^{\infty} l_k\in(0,\infty).$$

In order to apply Theorem 3.3, we take for $R>1$ the partition $y_j := x_j$ for $j=0,\dots,N(R)-1$ and $y_{N(R)} := L$. Further, set $\Omega_j(R) := \Omega(a_j^{-1}(R),a_j(R),\phi_j)$ for $j=1,\dots,N(R)-1$ and $\Omega_{N(R)}(R) := \Omega(a_{N(R)}^{-1}(R),a_{N(R)}(R),\psi)$.

By Lemma 3.7, $(ii)$, the first part of $A_1(R)$ coincides with

$$\sum_{j=1}^{N(R)-1} \int_{y_{j-1}(R)}^{y_j(R)} \big\|\Omega_j(R)JH(t)\Omega_j^{-1}(R)\big\|\,dt =$$
$$= \sum_{j=1}^{N(R)-1} l_j\big\|\Omega(a_j(R),\phi_j)J\xi_{\phi_j}\xi_{\phi_j}^T\Omega(a_j(R),\phi_j)^{-1}\big\|\,dt = C_1(R).$$

Concerning the remaining summand of $A_1(R)$ we have

$$\int_{y_{N(R)-1}}^{y_{N(R)}} \big\|\Omega_{N(R)}(R)JH(t)\Omega_{N(R)}^{-1}(R)\big\|\,dt =$$
$$= \sum_{j=N(R)}^{\infty} l_j\big\|\Omega(a_{N(R)}(R),\psi)J\xi_{\phi_j}\xi_{\phi_j}^T\Omega(a_{N(R)}(R),\psi)^{-1}(R)\big\| = C_2(R),$$

again due to Lemma 3.7, $(ii)$. This shows $A_1(R) = C_1(R) + C_2(R)$.

By the same means Lemma 3.7, $(i)$, gives $A_3(R) = C_4(R)$, and Lemma 3.7, $(iii)$, gives $A_2(R) \leq C_3(R)$. The assertion follows from Theorem 3.3. ❑

*5.2 Remark.* If the sequences $(a_j(R))_{j=1}^{N(R)}$ are nondecreasing for large enough $R$, we can write the expression $C_3(R)$ as

$$\begin{aligned}
C_3(R) &= \sum_{j=1}^{N(R)-1} \ln\Big(\frac{a_{j+1}(R)}{a_j(R)}\big|\cos(\epsilon_j)\big| + \frac{|\sin(\epsilon_j)|}{a_{j+1}(R)a_j(R)}\Big) \\
&= \sum_{j=1}^{N(R)-1} \ln\frac{a_{j+1}(R)}{a_j(R)}\Big(\big|\cos(\epsilon_j)\big| + \frac{|\sin(\epsilon_j)|}{a_{j+1}^2(R)}\Big) \\
&= \ln a_1^{-1}(R) + \ln a_{N(R)}(R) + \sum_{j=1}^{N(R)-1} \ln\Big(\big|\cos(\epsilon_j)\big| + \frac{|\sin(\epsilon_j)|}{a_{j+1}^2(R)}\Big).
\end{aligned}$$

A similar simplification can be made for nonincreasing $(a_j(R))_{j=1}^{N(R)}$. ◊

## 5.2. An upper estimate via power-growth assumptions

We introduce the following measures for the decay of sequences on the power scale.

**5.3 Definition.** Let $\vec{a} = (a_n)_{n=1}^{\infty}$ be a bounded sequence of positive real numbers. Then we set

$$\Delta^*(\vec{a}) := \sup\big\{\tau \geq 0 : a_n = \mathrm{O}(n^{-\tau})\big\},$$
$$\Delta(\vec{a}) := \sup\Big\{\tau \geq 0 : \frac{1}{n}\sum_{k=n}^{2n-1} a_k = \mathrm{O}(n^{-\tau})\Big\}.$$

◊

*5.4 Remark.* (*i*) Since $\vec{a}$ is bounded, $\tau = 0$ belongs to the sets in the definitions of $\Delta^*(\vec{a})$ and $\Delta(\vec{a})$. Thus these expressions are elements of $[0, \infty]$.

(*ii*) It is easy to see $\Delta^*(\vec{a}) \leq \Delta(\vec{a})$, cf. Lemma 6.1.

(*iii*) Let $\vec{a} = (a_n)_{n=0}^\infty$ be a bounded sequence of positive numbers, let $\alpha \geq 0$, and set $b_n := a_n^\alpha$. Then $\Delta^*(\vec{b}) = \alpha \cdot \Delta^*(\vec{a})$.

(*iv*) Let $\vec{a} = (a_n)_{n=0}^\infty$ and $\vec{b} = (b_n)_{n=0}^\infty$ be bounded sequences of positive numbers. Then

$$\Delta^*\big((a_n \cdot b_n)_{n=1}^\infty\big) \geq \Delta^*(\vec{a}) + \Delta^*(\vec{b}).$$

If $b_n \asymp$[1] $n^{-\tau}$, then equality holds.

(*v*) Clearly, $a_n \asymp b_n$ implies both $\Delta^*(\vec{a}) = \Delta^*(\vec{b})$ and $\Delta(\vec{a}) = \Delta(\vec{b})$.

◊

Next, we introduce measures for the decay of lengths and angle-differences, and a measure for the quality of possible convergence of angles.

**5.5 Definition.** Let $H$ be a Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Set

$$\Delta_l(H) := \Delta^*(\vec{l}), \quad \Delta_l^+(H) := \max\{1, \Delta_l(H)\},$$
$$\Delta_\phi(H) := \Delta((|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty).$$

Provided that $\Delta_l^+(H) < \infty$, set

$$\mu(H) := \sup_{\psi \in [0,\pi)} \sup \left\{ \tau \geq 0 : \sum_{j=n}^\infty l_j \sin^2(\phi_j - \psi) = \mathrm{O}(n^{1-\Delta_l^+-\tau}) \right\} \cup \{0\} \in [0, \infty].$$

When no confusion is possible, we drop explicit notion of $H$. ◊

*5.6 Remark.* When $\phi_n$ modulo $\pi$ has a limit, say $\phi_* \in [0, \pi)$, it is optimal to choose $\psi := \phi_*$ in the definition of $\mu$.

If the angles do not converge, a good choice would be an angle $\psi$ such that $\phi_j \approx \psi$ for those $j$ which belong to the most or the largest lengths $l_j$. ◊

**5.7 Lemma.** *Let $H$ be a Hamburger Hamiltonian in the lcc. Then*

$$2(\Delta_\phi - 1) \leq \mu.$$

*Proof.* The inequality is true if $\Delta_\phi \leq 1$. Thus, assume $\Delta_\phi > 1$. For any $\tau \in (1, \Delta_\phi)$ we have

$$\sum_{n=j}^\infty |\sin(\phi_{n+1} - \phi_n)| = \sum_{k=0}^\infty \sum_{n=2^k j}^{2^{k+1}j-1} |\sin(\phi_{n+1} - \phi_n)| \lesssim \sum_{k=0}^\infty (2^k j)^{1-\tau} \lesssim j^{1-\tau}$$

[1]Recall that $f_n \asymp g_n$ if there are constants $c, d > 0$ s. t. $cg_n \leq f_n \leq dg_n$ for $n$ large enough.

Adding multiples of $\pi$ to any $\phi_j$ does not change both $\Delta_\phi$ and $\mu$. Thus, we can assume without loss of generality $|\phi_{j+1}-\phi_j| \le \frac{\pi}{2}$ such that $|\sin(\phi_{j+1}-\phi_j)| \asymp |\phi_{j+1}-\phi_j|$. Then

$$\sum_{n=j}^{\infty} |\phi_{n+1}-\phi_n| \lesssim j^{1-\tau},$$

and $\vec{\phi}$ has a limit, $\psi$, such that

$$|\sin(\phi_j-\psi)| \le |\phi_j-\psi| \le \sum_{n=j}^{\infty} |\phi_{n+1}-\phi_n| \lesssim j^{1-\tau}.$$

If $\Delta_l^+ = 1$, $\vec{l} \in \ell^1$ gives

$$\sum_{j=n}^{\infty} l_j \sin^2(\phi_j-\psi) \lesssim \sum_{j=n}^{\infty} l_j j^{2(1-\tau)} \lesssim n^{-2(\tau-1)}.$$

In the case $1 < \Delta_l^+ = \Delta_l$, we have for all $\epsilon > 0$

$$\sum_{j=n}^{\infty} l_j \sin^2(\phi_j-\psi) \lesssim \sum_{j=n}^{\infty} j^{-\Delta_l+\epsilon} \cdot j^{2(1-\tau)} \lesssim n^{1-\Delta_l-2(\tau-1)+\epsilon}.$$

In both cases, this shows $2(\tau-1) \le \mu$. ❑

In view of Lemma 5.7 there arises the following question: For fixed $\Delta_\phi$, is it possible that $\mu$ is arbitrary large, or is there some natural upper bound?

Example 5.8 shows that $\mu$ can indeed be arbitrary large, whereas Example 5.9 demonstrates that, in a very regular situation, we have $\mu \le 2\Delta_\phi$. In fact, as we shall see later in Corollary 6.10, the case $\mu > 2\Delta_\phi$ occurs only in somehow irregular situations.

*5.8 Example.* Let $\alpha, \beta > 1$ and $\gamma, \delta > 0$, and set

$$l_n := \begin{cases} n^{-\alpha}, & n \in 2\mathbb{N}, \\ n^{-\beta}, & n \in 2\mathbb{N}-1, \end{cases} \qquad \phi_n := \begin{cases} n^{-\gamma}, & n \in 2\mathbb{N}, \\ n^{-\delta}, & n \in 2\mathbb{N}-1. \end{cases}$$

It is not hard so see $\Delta_l^+ = \min\{\alpha,\beta\}$ and $\Delta_\phi = \min\{\gamma,\delta\}$. Since the angles converge to zero, we should take $\psi := 0$ in order to determine $\mu$. The calculation

$$\sum_{j=n}^{\infty} l_j \sin^2(\phi_j-\psi) \asymp \sum_{j=n}^{\infty} l_j \phi_j^2 = \mathrm{O}(n^{1-\alpha-2\gamma}) + \mathrm{O}(n^{1-\beta-2\delta})$$

results in

$$\mu = \min\{\alpha+2\gamma, \beta+2\delta\} - \min\{\alpha,\beta\}.$$

Let us assume $\alpha > \beta$ and $\gamma < \delta$ such that $\Delta_l^+ = \beta$, $\Delta_\phi = \gamma$ and $\mu = \min\{\alpha-\beta+2\gamma, 2\delta\}$. By choosing $\alpha$ and $\delta$ large enough, $\mu$ will be arbitrary large for fixed $\Delta_\phi$ and $\Delta_l^+$. ◊

*5.9 Example.* Let $\alpha > 1$, $\beta > 0$ and let $(\sigma_n)_{n=1}^{\infty}$ be a sequence of signs, i.e. $\sigma_n = \pm 1$.

Set $l_n := n^{-\alpha}$, $\phi_1 := 0$ and $\phi_{n+1} := \phi_n + \sigma_n n^{-\beta}$ for $n \in \mathbb{N}$. Then $\Delta_l^+ = \alpha$ and, due to $|\phi_{n+1} - \phi_n| = n^{-\beta}$, $\Delta_\phi = \beta$.

First consider the case $\sigma_n = 1$ for $n \in \mathbb{N}$. We claim $\mu = \max\{0, 2(\Delta_\phi - 1)\}$. If $\Delta_\phi \leq 1$, the angles modulo $\pi$ do not converge, and one can show $\mu = 0$. For $\Delta_\phi > 1$ the angles do converge to the limit

$$\psi := \lim_{n\to\infty} \phi_n = \sum_{j=1}^{\infty} (\phi_{j+1} - \phi_j) = \sum_{j=1}^{\infty} \sigma_j j^{-\beta} = \sum_{j=1}^{\infty} j^{-\beta}.$$

Moreover, we have

$$\psi - \phi_n = \sum_{j=n}^{\infty} (\phi_{j+1} - \phi_j) = \sum_{j=n}^{\infty} j^{-\Delta_\phi} = \mathrm{O}(n^{1-\Delta_\phi}),$$

which results in $\mu = 2(\Delta_\phi - 1)$, i.e. equality prevails in Lemma 5.7.

Secondly consider the other extreme case in which the signs alternate, i.e. $\sigma_n = -\sigma_{n+1}$ for $n \in \mathbb{N}$. By Leibniz criterion, $\phi_n$ converges to the limit

$$\psi := \sum_{j=1}^{\infty} (\phi_{j+1} - \phi_j) = \sum_{j=1}^{\infty} \sigma_j j^{-\beta} = \pm \sum_{j=1}^{\infty} (-1)^j j^{-\beta}.$$

In fact,

$$|\psi - \phi_n| = \Big| \sum_{j=n}^{\infty} (-1)^j j^{-\Delta_\phi} \Big| = \mathrm{O}(n^{-\Delta_\phi}),$$

which shows $\mu = 2\Delta_\phi$. $\Diamond$

**5.10 Theorem.** *Let $H$ be a Hamburger Hamiltonian in the lcc with lengths $\vec{l}$ and angles $\vec{\phi}$. Assume that $(\Delta_l^+, \Delta_\phi, \mu) \neq (1,1,0)$.*

(*i*) *If $\mu \leq 2\Delta\phi$, then*

$$\rho(H) \leq \begin{cases} \dfrac{1}{\Delta_l^+ + \Delta_\phi}, & \text{for } \Delta_l^+ + \Delta_\phi \geq 2 \quad (\textit{generic region}) \\[2ex] \dfrac{1 - \Delta_\phi + \frac{\mu}{2}}{\Delta_l^+ - \Delta_\phi + \mu}, & \text{for } \Delta_l^+ + \Delta_\phi < 2 \quad (\textit{critical triangle}). \end{cases}$$

*Note that, for $\mu = 2\Delta_\phi$, this reduces to $\rho(H) \leq \frac{1}{\Delta_l^+ + \Delta_\phi}$.*

(*ii*) *If $\mu > 2\Delta_\phi$, then*

$$\rho(H) \leq \frac{1}{\Delta_l^+ + \frac{\mu}{2}}.$$

*5.11 Remark.* (*i*) The case $\mu > 2\Delta_\phi$ only occurs if the data is somehow irregular. We will discuss this in more detail in Corollary 6.10.

(*ii*) It is easy to verify that larger values of $\Delta_l^+$, $\Delta_\phi$ or $\mu$ do lead to better (in the sense of lower) upper estimates in Theorem 5.10.

$\Diamond$

*Proof of Theorem* 5.10. If $\Delta_l^+ = 1$, set $\Delta_l' := 1$. Otherwise let $\Delta_l' \in (1, \Delta_l^+)$ be arbitrary, and note

$$l_n = \mathrm{O}(n^{-\Delta_l'}).$$

For all $\Delta_l^+$ we have

$$\sum_{j=n}^{\infty} l_j = \mathrm{O}(n^{1-\Delta_l'}). \tag{5.1}$$

If $\Delta_\phi = 0$, set $\Delta_\phi' := 0$. Otherwise let $\Delta_\phi' \in (0, \Delta_\phi)$ be arbitrary. For all $\Delta_\phi$ we have

$$\frac{1}{n}\sum_{j=n}^{2n-1} |\sin(\phi_{j+1} - \phi_j)| = \mathrm{O}(n^{-\Delta_\phi'}). \tag{5.2}$$

If $\mu = 0$, set $\mu' := 0$ and $\psi := 0$. Otherwise let $\mu' \in (0, \mu)$ be arbitrary. Then there exists $\psi \in [0, \pi)$ such that

$$\sum_{j=n}^{\infty} l_j \sin^2(\phi_j - \psi) = \mathrm{O}(n^{1-\Delta_l^+-\mu'}).$$

For all $\mu$ we have

$$\sum_{j=n}^{\infty} l_j \sin^2(\phi_j - \psi) = \mathrm{O}(n^{1-\Delta_l'-\mu'}). \tag{5.3}$$

We will choose $\gamma > 0$ and apply Theorem 5.1 with the choice $N(R) := \lfloor R^\gamma \rfloor$ and

$$a_j^2(R) := \begin{cases} R^{-\gamma(\Delta_l'-1+\frac{\mu'}{2})} j^{\Delta_l'-1}, & j = 1, \dots, N(R)-1 \\ R^{-\frac{\mu'\gamma}{2}}, & j = N(R). \end{cases}$$

Note that $a_j(R) \leq 1$, since

$$a_j^2(R) \leq a_{N(R)-1}^2(R) \leq R^{-\gamma(\Delta_l'-1+\frac{\mu'}{2})+\gamma(\Delta_l'-1)} = R^{-\frac{\gamma\mu'}{2}} \leq 1.$$

We need to estimate expressions $C_1(R), \dots, C_4(R)$. By (5.1)

$$C_1(R) = \sum_{j=1}^{N(R)-1} l_j a_j^2(R) \lesssim R^{-\gamma(\Delta_l'-1+\frac{\mu'}{2})} \sum_{j=1}^{\lfloor R^\gamma \rfloor} j^{-\Delta_l'+\Delta_l'-1} = \mathrm{O}\big(R^{-\gamma(\Delta_l'-1+\frac{\mu'}{2})} \log(R)\big). \tag{5.4}$$

Relation (5.1) and (5.3) give

$$\begin{aligned} C_2(R) &= \sum_{j=N(R)}^{\infty} l_j \left( a^2_{N(R)}(R) \cos^2(\psi - \phi_j) + a^{-2}_{N(R)}(R) \sin^2(\psi - \phi_j) \right) \\ &\leq R^{-\frac{\mu'\gamma}{2}} \sum_{j=N(R)}^{\infty} l_j \; + \; R^{\frac{\mu'\gamma}{2}} \sum_{j=N(R)}^{\infty} l_j \sin^2(\psi - \phi_j) = \\ &\lesssim R^{-\frac{\mu'\gamma}{2}} R^{\gamma(1-\Delta'_l)} + R^{\frac{\mu'\gamma}{2}} R^{\gamma(1-\Delta'_l-\mu')} = \mathrm{O}\big(R^{-\gamma(\Delta'_l - 1 + \frac{\mu'}{2})}\big). \end{aligned} \tag{5.5}$$

Note that $C_4(R) = \mathrm{O}(\log R)$, since both $a_1(R)$ and $a_{N(R)}(R)$ are just a power of $R$. So far, we have

$$R(C_1(R) + C_2(R)) + C_4(R) = \mathrm{O}\big(R^{1-\gamma(\Delta'_l - 1 + \frac{\mu'}{2})} \log R\big). \tag{5.6}$$

Finally, consider $C_3(R)$. The last summand in this expression is an element of $\mathrm{O}(\log R)$. For the remaining part of the sum we can use Remark 5.2 since $(a_j(R))_{j=1}^{N(R)-1}$ is non-decreasing. This gives

$$\begin{aligned} C_3(R) &\leq \sum_{j=1}^{N(R)-2} \ln \left( \left| \cos(\epsilon_j) \right| + \frac{|\sin(\epsilon_j)|}{a^2_{j+1}(R)} \right) \; + \mathrm{O}(\log R) \\ &\leq \sum_{j=1}^{N(R)-2} \ln \left( 1 + R^{\gamma(\Delta'_l - 1 + \frac{\mu'}{2})} |\sin(\phi_{j+1} - \phi_j)| j^{1-\Delta'_l} \right) \; + \mathrm{O}(\log R). \end{aligned} \tag{5.7}$$

First consider the case $\mu \geq 2\Delta_\phi$ where we have to show $\rho(H) \leq (\Delta^+_l + \frac{\mu}{2})^{-1}$. We continue in (5.7) by estimating $|\sin(\phi_{j+1} - \phi_j)| j^{1-\Delta'_l} \leq 1$, which gives $C_3(R) = \mathrm{O}\big(R^\gamma \log R\big)$.

In view of (5.6), we choose $\gamma$ to be the solution of $1 - \gamma(\Delta'_l - 1 + \frac{\mu'}{2}) = \gamma$, i.e.

$$\gamma := d' := \frac{1}{\Delta'_l + \frac{\mu'}{2}},$$

which gives

$$R(C_1(R) + C_2(R)) + C_3(R) + C_4(R) = \mathrm{O}\big(R^{d'} \log R\big).$$

Theorem 5.1 implies that the type of $H$ w.r.t. the growth function $\lambda(R) := R^{d'} \log R$ is finite. In particular, the order of $H$ is at most $d'$. Passing to the upper bounds in $\Delta'_l \to \Delta^+_l$ and $\mu' \to \mu$, shows $\rho(H) \leq (\Delta^+_l + \frac{\mu}{2})^{-1}$.

Finally, consider the case $\mu < 2\Delta_\phi$. We continue to estimate $C_3(R)$ by splitting the sum in (5.7) into two parts. Let $s \in [0, \gamma)$. First we use a rough estimate for the sum from $j = 2$ to $j = \lceil R^s \rceil - 1$,

$$\sum_{j=1}^{\lceil R^s \rceil - 1} \ln \left( 1 + R^{\gamma(\Delta'_l - 1 + \frac{\mu'}{2})} |\sin(\phi_{j+1} - \phi_j)| j^{1-\Delta'_l} \right) = \mathrm{O}\big(R^s \log R\big). \tag{5.8}$$

For the remaining part of the sum we employ $\ln(1+x) \le x$, and get the upper estimate

$$\begin{aligned}
&\sum_{j=\lceil R^s\rceil}^{\lfloor R^\gamma\rfloor-2} \ln\Big(1+R^{\gamma(\Delta_l'-1+\frac{\mu'}{2})}|\sin(\phi_{j+1}-\phi_j)|j^{1-\Delta_l'}\Big) \le \\
&\qquad \le R^{\gamma(\Delta_l'-1+\frac{\mu'}{2})} \sum_{j=\lceil R^s\rceil}^{\lfloor R^\gamma\rfloor-2} |\sin(\phi_{j+1}-\phi_j)|j^{1-\Delta_l'} \\
&\qquad \le R^{\gamma(\Delta_l'-1+\frac{\mu'}{2})} \sum_{k=0}^{\lceil(\gamma-s)\log_2 R\rceil-1} \sum_{j=2^k\lceil R^s\rceil}^{2^{k+1}\lceil R^s\rceil-1} |\sin(\phi_{j+1}-\phi_j)|j^{1-\Delta_l'}
\end{aligned}$$

The fact that $j^{1-\Delta_l'}$ is nonincreasing and (5.2) give

$$\begin{aligned}
&\le R^{\gamma(\Delta_l'-1+\frac{\mu'}{2})+s(1-\Delta_l')} \sum_{k=0}^{\lceil(\gamma-s)\log_2 R\rceil-1} (2^k)^{1-\Delta_l'} \sum_{j=2^k\lceil R^s\rceil}^{2^{k+1}\lceil R^s\rceil-1} |\sin(\phi_{j+1}-\phi_j)| \\
&\le R^{\gamma(\Delta_l'-1+\frac{\mu'}{2})+s(1-\Delta_l')} \sum_{k=0}^{\lceil(\gamma-s)\log_2 R\rceil-1} (2^k)^{1-\Delta_l'}(2^k\lceil R^s\rceil)^{1-\Delta_\phi'} \\
&\lesssim R^{\gamma(\Delta_l'-1+\frac{\mu'}{2})+s[2-(\Delta_l'+\Delta_\phi')]} \sum_{k=0}^{\lceil(\gamma-s)\log_2 R\rceil-1} \big(2^{[2-(\Delta_l'+\Delta_\phi')]}\big)^k. \qquad (5.9)
\end{aligned}$$

Setting $q := 2^{[2-(\Delta_l'+\Delta_\phi')]}$ and combining (5.7) with (5.8) and (5.9) yields

$$C_3(R) = \mathrm{O}\big(R^s\log R\big) + \mathrm{O}\big(R^{\gamma(\Delta_l'-1+\frac{\mu'}{2})+s[2-(\Delta_l'+\Delta_\phi')]}\big) \sum_{k=0}^{\lceil(\gamma-s)\log_2 R\rceil-1} q^k. \qquad (5.10)$$

First, consider the sub-case $\Delta_l^+ + \Delta_\phi > 2$. By taking $\Delta_l', \Delta_\phi'$ and $\mu'$ sufficiently close to $\Delta_l^+, \Delta_\phi$ and $\mu$, it is possible to assume $\Delta_l' + \Delta_\phi' > 2$ and $\Delta_\phi' > \frac{\mu'}{2}$. Thus, $q < 1$ and he sum on the right-hand side of (5.10) converges.

Our estimate of $C_3(R)$ is now optimal if $s$ and $\gamma$ satisfies $s = \gamma(\Delta_l' - 1 + \frac{\mu'}{2}) + s[2 - (\Delta_l' + \Delta_\phi')]$. Balancing with (5.6) yields

$$s := d' := \frac{1}{\Delta_l' + \Delta_\phi'}, \quad \gamma := \frac{\Delta_l' + \Delta_\phi' - 1}{(\Delta_l' + \Delta_\phi')(\Delta_l' - 1 + \frac{\mu'}{2})}.$$

Note that $s < \gamma$ and $\Delta_l' - 1 + \frac{\mu'}{2} > 0$. The case $(\Delta_l', \mu') = (1, 0)$ does not appear here, since Lemma 5.7 would imply $\Delta_\phi' \le \Delta_\phi \le 1$. Thus, we have

$$R(C_1(R) + C_2(R)) + C_3(R) + C_4(R) = \mathrm{O}\big(R^{d'}\log R\big),$$

and Theorem 5.1 implies that the type of $H$ w.r.t. the growth function $\lambda(R) := R^{d'} \log R$ is finite. In particular, the order of $H$ is at most $d'$. Passing to the upper bounds in $\Delta_l' \to \Delta_l^+$ and $\Delta_\phi' \to \Delta_\phi$, leads to $\rho(H) \le (\Delta_l^+ + \Delta_\phi)^{-1}$.

Secondly, assume $\Delta_l^+ + \Delta_\phi \le 2$. If $\Delta_l^+ = 1$ and $\Delta_\phi = 0$, obviously $\Delta_l' + \Delta_\phi' = 1 < 2$. In any other case, we have $\Delta_l' + \Delta_\phi' < \Delta_l^+ + \Delta_\phi \le 2$. Thus, $q > 1$ and the size of the sum in (5.10) can be estimated by

$$\sum_{k=0}^{\lceil(\gamma-s)\log_2 R\rceil - 1} q^k = \frac{q^{\lceil(\gamma-s)\log_2 R\rceil} - 1}{q-1} \lesssim q^{(\gamma-s)\log_2 R} =$$
$$= 2^{[2-(\Delta_l'+\Delta_\phi')](\gamma-s)\log_2 R} = \mathrm{O}\big(R^{(\gamma-s)[2-(\Delta_l'+\Delta_\phi')]}\big).$$

Therefore, (5.10) reduces to

$$C_3(R) = \mathrm{O}\big(R^s \log R\big) + \mathrm{O}\big(R^{\gamma(1-\Delta_\phi'+\frac{\mu'}{2})}\big).$$

In this case set $s := 0$. In view of (5.6), we define $\gamma$ as the solution of $\gamma(1 - \Delta_\phi' + \frac{\mu'}{2}) = 1 - \gamma(\Delta_l' - 1 + \frac{\mu'}{2})$, i.e. set

$$\gamma := \frac{1}{\Delta_l' - \Delta_\phi' + \mu'}, \quad d' := \frac{1 - \Delta_\phi' + \frac{\mu'}{2}}{\Delta_l' - \Delta_\phi' + \mu'}.$$

The assumption $(\Delta_l^+, \Delta_\phi, \mu) \ne (1, 1, 0)$ assures that $\Delta_l^+ - \Delta_\phi + \mu > 0$ and $1 - \Delta_\phi + \frac{\mu}{2} > 0$. Choosing $\Delta_l'$,$\Delta_\phi'$ and $\mu'$ close enough to their original values ensures that $\gamma$ and $d'$ are well-defined and positive. So,

$$R(C_1(R) + C_2(R)) + C_3(R) + C_4(R) = \mathrm{O}\big(R^{d'} \log R\big),$$

and Theorem 5.1 implies that the type of $H$ with respect to the growth function $\lambda(R) := R^{d'} \log R$ is finite. In particular, the order of $H$ is at most $d'$. Passing to the upper bounds in $\Delta_l' \to \Delta_l^+$, $\Delta_\phi' \to \Delta_\phi$ and $\mu' \to \mu$ gives $\rho(H) \le (1 - \Delta_\phi + \frac{\mu}{2})/(\Delta_l - \Delta_\phi + \mu)$. ❑

Let us mention the following direct corollary of Theorem 5.10.

**5.12 Corollary.** *Let $H$ be a lcc Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. If at least one of the quantities $\Delta_l^+, \Delta_\phi$ or $\mu$ is infinite, then the order of $H$ is zero.*

We close this section with a comparison between Theorem 5.10 and its initial version, cf. [PRW16, Theorem 2.7]. The new result has improved in two ways: First, the case $\mu > 2\Delta_\phi$ has now been treated properly. Secondly, the quantity which measures the quality of possible convergence of angles is now larger. The initial version used instead of $\mu(H)$ the quantity $\Lambda(H)$:

**5.13 Definition.** Let H be a Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Provided that $\Delta_l^+(H) < \infty$, set

$$\Lambda(H) := \sup_{\psi\in[0,\pi)} \sup\left\{\tau \geq 0 : \sum_{j=n}^{\infty} l_j |\sin(\phi_j - \psi)| = \mathrm{O}(n^{1-\Delta_l^+-\tau})\right\} \cup \{0\} \in [0,\infty].$$

$\diamondsuit$

The new quantity $\mu(H)$ is up to two times larger than $\Lambda(H)$, which improves the upper bounds for the order of $H$ significantly. This stronger result enables us to determine the order for a new class of examples, cf. Theorem 6.5, (*ii*). This improvement was possible due to the refinement of Romanov's Theorem 1.

**5.14 Lemma.** *We have the inequality*

$$\Lambda(H) \leq \mu(H) \leq 2\Lambda(H).$$

*Proof.* The first inequality is clear. For the second one, take $\psi \in [0,\pi)$ and $\tau \geq 0$ such that

$$\sum_{j=n}^{\infty} l_j \sin^2(\phi_j - \psi) = \mathrm{O}(n^{1-\Delta_l^+-\tau}).$$

By the Cauchy-Schwarz inequality, we have for $\epsilon > 0$

$$\sum_{j=n}^{\infty} l_j |\sin(\phi_j - \psi)| \leq \left(\sum_{j=n}^{\infty} l_j\right)^{1/2} \left(\sum_{j=n}^{\infty} l_j \sin^2(\phi_j - \psi)\right)^{1/2} =$$

$$= \mathrm{O}\big(n^{\frac{1}{2}(1-\Delta_l^++\epsilon+1-\Delta_l^+-\tau)}\big) = \mathrm{O}\big(n^{1-\Delta_l^++(\epsilon-\tau)/2}\big),$$

since $l_n = \mathrm{O}(n^{-\Delta_l+\epsilon})$ implies $\sum_{j=n}^{\infty} l_j = \mathrm{O}(n^{1-\Delta_l^++\epsilon})$. Taking $\epsilon \to 0$ and taking the supremum over $\tau \geq 0$ and $\psi \in [0,\pi)$ gives $\mu(H) \leq 2\Lambda(H)$. ❑

## 5.3. A Lower estimate

We begin with a result which is formulated in terms of moment sequences and contains the lower bound $\rho((s_n)_{n=0}^{\infty}) \geq \rho(F)$. The order of $F(z)$ is quite accessible due to (2.3) and is potentially larger than the Livšic bound $\rho((s_n)_{n=0}^{\infty}) \geq \rho(L)$, cf. Theorem 2.3.

**5.15 Corollary.** *Let $(s_n)_{n=0}^{\infty}$ be an indeterminate moment sequence, and set*

$$F(z) := \sum_{n=0}^{\infty} b_{n,n} z^n, \quad L(z) := \sum_{n=0}^{\infty} \frac{z^{2n}}{s_{2n}}.$$

*Then,*

$$\rho((s_n)_{n=0}^{\infty}) \geq \rho(F) \geq \rho(L). \tag{5.11}$$

*Moreover, when the order of the moment sequence $\rho$ is not zero and coincides with the order of $F(z)$, we have $\tau_\rho((s_n)_{n=0}^{\infty}) \geq \tau_\rho(F)$. Similarly, $0 \neq \rho(F) = \rho(L) =: \rho$ gives $2\tau_\rho(F) \geq \tau_\rho(L)$.*

It is convenient to prove this result using Theorem 2.8, as it has been done already in [BS14, Theorem 7.1]. For the sake of completeness we include it here.

Let us mention that it is also possible to show the substantial inequality $\rho((s_n)_{n=0}^\infty) \geq \rho(F)$ directly, using the multiplicative structure of monodromy matrices. This has been carried out in [PRW15, Proposition 2.15].

*Proof of Corollary* 5.15. Entire functions $g(z) = \sum_{n=0}^\infty a_n z^n$ with non-negative $a_n$ satisfy $M(g,R) = g(R)$. The rough estimate $(\sum_{k=n}^\infty b_{n,k}^2)^{1/2} \geq b_{n,n}$ shows $\Phi(R) \geq F(R)$, and, hence, $M(\Phi,R) \geq M(F,R)$. Theorem 2.8 gives the first inequality in (5.11) and the corresponding statement about the type.

The orthonormality of $P_n$ and the Cauchy-Schwarz inequality give

$$1 = (P_n(z), P_n(z))_s^2 = b_{n,n}^2 (P_n(z), z^n)_s^2 \leq b_{n,n}^2 (z^n, z^n)_s = b_{n,n}^2 s_{2n},$$

i.e. $b_{n,n} \geq s_{2n}^{-1/2}$. Together with the standard formula which calculates the order of an entire function from its Taylor coefficients, cf. [Boa54, Theorem 2.2.2], we get

$$\begin{aligned}\rho(F) = \limsup_{n\to\infty} \frac{n\log(n)}{\log(b_{n,n}^{-1})} &\geq \limsup_{n\to\infty} \frac{n\log(n)}{\log\left(s_{2n}^{1/2}\right)} = \\ &= \limsup_{n\to\infty} \frac{2n\log(2n)}{\log(s_{2n})} = \rho(L).\end{aligned}$$

Assume $\rho(F) = \rho(L) \neq 0$ and denote this value by $\rho$. Then the corresponding formula for the type with respect to order $\rho$ gives

$$2\tau_\rho(F) = 2\Big[\limsup_{n\to\infty}\left(n^{\frac{1}{\rho}} b_{n,n}^{\frac{1}{n}}\right)\Big]^\rho \frac{1}{e\rho} \geq \Big[\limsup_{n\to\infty}\left((2n)^{\frac{1}{\rho}} s_{2n}^{-\frac{1}{2n}}\right)\Big]^\rho \frac{1}{e\rho} = \tau_\rho(L).$$

❑

Let us formulate this result in the language of Hamburger Hamiltonian.

**5.16 Definition.** For a Hamburger Hamiltonian $H$ in the lcc with lengths $\vec{l}$ and angles $\vec{\phi}$, set

$$\delta_{l,\phi}(H) := \liminf_{n\to\infty} \frac{-1}{n\ln n} \ln\left(\sqrt{l_n}\prod_{k=1}^{n-1} l_k |\sin(\phi_{k+1}-\phi_k)|\right) \in [0,\infty].$$

◊

**5.17 Proposition.** *Let $H$ be a Hamburger Hamiltonian in the lcc with lengths $\vec{l}$ and angles $\vec{\phi}$. Then*

$$\rho(H) \geq \frac{1}{\delta_{l,\phi}(H)}.$$

*Proof.* By (2.3) and (2.5), we have

$$\begin{aligned}\rho(F) &= \limsup_{n\to\infty} \frac{n\ln n}{-\ln b_{n,n}} = \limsup_{n\to\infty} \frac{n\ln n}{-\ln\left(\prod_{k=0}^{n-1}\rho_k^{-1}\right)} \\ &= \limsup_{n\to\infty} \frac{n\ln n}{-\ln\left(\sqrt{l_n}\prod_{k=1}^{n-1} l_k|\sin(\phi_{k+1}-\phi_k)|\right)} \\ &= \left[\liminf_{n\to\infty} \frac{-1}{n\ln n}\ln\left(\sqrt{l_n}\prod_{k=1}^{n-1} l_k|\sin(\phi_{k+1}-\phi_k)|\right)\right]^{-1} = \frac{1}{\delta_{l,\phi}(H)}.\end{aligned}$$

Thus, the statement follows from Corollary 5.15. ❑

Obviously $\delta_{l,\phi}$ contains joint information about the lengths and the angles. An attempt can be made to separate this quantity into one part depending only on the lengths and another one depending only on the angles. Unfortunately, this procedure requires certain regularity assumptions, cf. Corollary 5.21.

**5.18 Definition.** Let $\vec{a} = (a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers, and let $\beta \geq 0$. Then we define

$$\delta(\vec{a},\beta) := \liminf_{n\to\infty} G(n;\vec{a},\beta), \qquad \text{where } G(n;\vec{a},\beta) := \frac{-1}{n\ln n}\ln\left(a_n^{\beta}\prod_{k=1}^{n-1} a_k\right).$$

Moreover, for a Hamburger Hamiltonian $H$ with lengths $\vec{l}$ and angles $\vec{\phi}$ set

$$\delta_l(H) := \delta(\vec{l},\tfrac{1}{2}), \quad \delta_\phi(H) := \delta((|\sin(\phi_{k+1}-\phi_k)|)_{k=1}^{\infty},0).$$

◊

*5.19 Remark.*

(*i*) A priori, $\delta(\vec{a},\beta)$ is an element of $[-\infty,\infty]$, but for bounded sequences $\vec{a}$ we have

$$G(n;\vec{a},\beta) \geq \frac{-1}{n\ln n}\ln\left(C^{n+\beta-1}\right) \longrightarrow 0, \text{ for } n\to\infty,$$

i.e. $\delta(\vec{a},\beta) \geq 0$.

(*ii*) Changing finitely many elements of a sequence $\vec{a}$ has no influence on $\delta(\vec{a},\beta)$.

(*iii*) Let $\vec{x} = (x_n)_{n=1}^{\infty}$ be a sequence of positive real numbers, $\alpha \in \mathbb{R}$, and set $a_n := x_n^{\alpha}$. Then for $\beta \geq 0$

$$G(n;\vec{a},\beta) = \alpha \cdot G(n;\vec{x},\beta).$$

In particular, $\delta(\vec{a},\beta)$ exists as a limit if and only if $\delta(\vec{x},\beta)$ has this property. In this case $\delta(\vec{a},\beta) = \alpha\cdot\delta(\vec{x},\beta)$.

Otherwise $\delta(\vec{a},\beta) = \alpha\cdot\delta(\vec{x},\beta)$ holds if $\alpha > 0$.

(*iv*) Let $\vec{x} = (x_n)_{n=1}^\infty$, $\vec{y} = (y_n)_{n=1}^\infty$ be sequences of positive real numbers and set $a_n := x_n \cdot y_n$. For $\beta \geq 0$ we have

$$G(n; \vec{a}, \beta) = G(n; \vec{x}, \beta) + G(n; \vec{y}, \beta),$$

which gives

$$\delta(\vec{a}, \beta) \geq \delta(\vec{x}, \beta) + \delta(\vec{y}, \beta).$$

Equality holds if at least one of $\delta(\vec{x}, \beta)$ and $\delta(\vec{y}, \beta)$ exists as a limit.

(*v*) Observe that $a_n \lesssim b_n$ gives

$$G(n; \vec{a}, \beta) \geq G(n; \vec{b}, \beta) + \mathrm{o}(1),$$

and hence $\delta(\vec{a}, \beta) \geq \delta(\vec{b}, \beta)$.

Consequently $a_n \asymp b_n$ yields

$$G(n; \vec{a}, \beta) = G(n; \vec{b}, \beta) + \mathrm{o}(1).$$

In particular $\delta(\vec{a}, \beta) = \delta(\vec{b}, \beta)$, and $\delta(\vec{a}, \beta)$ exists as a limit if and only if $\delta(\vec{b}, \beta)$ does so.

$\Diamond$

*5.20 Example.* Let $\vec{a} = (a_n)_{n=1}^\infty$ be a sequence of positive real numbers with $a_n \asymp n^{-\alpha}$ for $\alpha \in \mathbb{R}$. Then $\delta(\vec{a}, \beta) = \alpha$ for all $\beta \geq 0$, and $\delta(\vec{a}, \beta)$ exists as a limit.

To verify this, consider at first $x_n := n$. By Stirling's formula we have

$$\ln \left( n^\beta \prod_{k=1}^{n-1} k \right) = \ln(n! n^{\beta-1}) = n \ln n + \mathrm{O}(n),$$

i.e. $\delta(\vec{x}, \beta) = -1$ as a limit. Remark 5.19, (*iii*), gives $\delta((n^{-\alpha})_{n=1}^\infty, \beta) = \alpha$. The statement follows from Remark 5.19, (*v*). $\Diamond$

Coming back to $\delta_{l,\phi}(H)$, note that

$$\delta_{l,\phi}(H) = \liminf_{n\to\infty} \left( G(n; \vec{l}, \tfrac{1}{2}) + G(n; (|\sin(\phi_{k+1} - \phi_k)|)_{k=1}^\infty, 0) \right),$$

which gives $\delta_{l,\phi}(H) \geq \delta_l(H) + \delta_\phi(H)$. In general $(\delta_l(H) + \delta_\phi(H))^{-1}$ is not a lower bound for the order of $H$.

**5.21 Corollary.** *Let $H$ be a Hamburger Hamiltonian in the lcc with lengths $\vec{l}$ and angles $\vec{\phi}$. Assume that at least one of $\delta_l(H)$ and $\delta_\phi(H)$ exists as a limit. Then*

$$\rho(H) \geq \frac{1}{\delta_l(H) + \delta_\phi(H)}.$$

# 6. Regularly vs. irregularly distributed data

## 6.1. Regular behaviour

The goal of this section is to compare the upper bound for the order of a Hamburger Hamiltonian from Theorem 5.10 with the lower bound from Corollary 5.21, and describe classes where these bounds coincide, cf. Theorem 6.5.

We need to overcome the obstacle that the decay of the sequences $\vec{l}$ and $(|\sin(\phi_{n+1}-\phi_n)|)_{n=1}^{\infty}$ is measured in different ways. Hence, we begin with a comparison of these measures for a bounded sequence $\vec{a}=(a_n)_{n=0}^{\infty}$.

To recall the notions $\Delta(\vec{a})$ and $\Delta^*(\vec{a})$ see Definition 5.3; $\delta(\vec{a},\beta)$ is defined in Definition 5.18.

**6.1 Lemma.** *For a bounded sequence of positive real numbers $\vec{a}=(a_n)_{n=1}^{\infty}$ and for $\beta\geq 0$, we have*

$$\Delta^*(\vec{a})\leq\Delta(\vec{a})\leq\delta(\vec{a},\beta).$$

*Proof.* The inequality $\Delta^*(\vec{a})\leq\Delta(\vec{a})$ is clear. We have to show $\Delta(\vec{a})\leq\delta(\vec{a},\beta)$, which is not trivial for $\Delta(\vec{a})>0$. Multiplying the whole sequence $\vec{a}$ by a positive constant does not change both $\Delta(\vec{a})$ and $\delta(\vec{a},\beta)$. Thus, we can assume $a_n\leq 1$, $n\in\mathbb{N}$.

Let $\tau\in(0,\Delta(\vec{a}))$ be arbitrary. Then $\frac{1}{n}\sum_{k=n}^{2n-1}a_k\leq cn^{-\tau}$, $n\in\mathbb{N}$, for some $c\geq 1$. Set $r(n):=n-2^{\lfloor\log_2 n\rfloor}\leq n/2$. The Inequality of arithmetic and geometric means gives

$$\begin{aligned}
a_n^{\beta}\prod_{k=1}^{n-1}a_k &\leq \prod_{k=2^{\lfloor\log_2 n\rfloor}}^{n-1}a_k\cdot\prod_{j=0}^{\lfloor\log_2 n\rfloor-1}\prod_{k=2^j}^{2^{j+1}-1}a_k\\
&\leq\left(\frac{1}{r(n)}\sum_{k=2^{\lfloor\log_2 n\rfloor}}^{n-1}a_k\right)^{r(n)}\cdot\prod_{j=0}^{\lfloor\log_2 n\rfloor-1}\left(\frac{1}{2^j}\sum_{k=2^j}^{2^{j+1}-1}a_k\right)^{2^j}\\
&\leq\left(\frac{c2^{\lfloor\log_2 n\rfloor}}{r(n)}2^{-\tau\lfloor\log_2 n\rfloor}\right)^{r(n)}\cdot\prod_{j=0}^{\lfloor\log_2 n\rfloor-1}\left(c2^{-j\tau}\right)^{2^j}.
\end{aligned}$$

Taking logarithms results in

$$\ln\left(a_n^{\beta}\prod_{k=1}^{n-1}a_k\right)\leq r(n)\ln\left(\frac{c2^{\lfloor\log_2 n\rfloor}}{r(n)}\right)-\tau r(n)\ln\left(2^{\lfloor\log_2 n\rfloor}\right)+\tag{6.1}$$

$$+\ln c\sum_{j=0}^{\lfloor\log_2 n\rfloor-1}2^j-\tau\ln 2\sum_{j=0}^{\lfloor\log_2 n\rfloor-1}j2^j.\tag{6.2}$$

In the first term in (6.1), use $2^{\lfloor \log_2 n \rfloor} \le n$. It is easy to see that the maximum of the function $x \mapsto x \ln(cn/x)$ for $x \in [1, n/2]$ is attained at $x = \min\{c/e, 1/2\}n$, and its value is an element of $\mathrm{O}(n)$.

By standard calculations we get the following expressions for the two sums in (6.2),

$$\sum_{j=0}^{\lfloor \log_2 n \rfloor - 1} 2^j = 2^{\lfloor \log_2 n \rfloor} - 1 = \mathrm{O}(n),$$

$$\sum_{j=0}^{\lfloor \log_2 n \rfloor - 1} j 2^j = 2^{\lfloor \log_2 n \rfloor}(\lfloor \log_2 n \rfloor - 2) + 2 = 2^{\lfloor \log_2 n \rfloor}(\log_2 n) + \mathrm{O}(n).$$

In sum, this yields

$$\begin{aligned} \ln\left(a_n^\beta \prod_{k=1}^{n-1} a_k\right) &\le -\tau\left(r(n)\ln\left(2^{\lfloor \log_2 n \rfloor}\right) + 2^{\lfloor \log_2 n \rfloor}\ln n\right) + \mathrm{O}(n) \\ &= -\tau \ln n\left(r(n) + 2^{\lfloor \log_2 n \rfloor}\right) + \mathrm{O}(n) = -\tau n \ln n + \mathrm{O}(n). \end{aligned}$$

Finally, we get

$$\liminf_{n\to\infty} \frac{-1}{n \ln n} \ln\left(a_n^\beta \prod_{k=1}^{n-1} a_k\right) \ge \liminf_{n\to\infty} \frac{-1}{n \ln n}\left(-\tau n \ln n + \mathrm{O}(n)\right) = \tau,$$

which shows $\delta(\vec{a}, \beta) \ge \Delta(\vec{a})$. ❑

The following rather weak regularity assumption ensures that equality holds in Lemma 6.1, which is essential for the comparison between the upper and lower bound of $\rho(H)$.

**6.2 Definition.** We call a sequence $\vec{a} = (a_n)_{n=1}^\infty$ of positive real numbers *regularly distributed* if

$$\frac{a_n}{\left(\prod_{k=1}^n a_k\right)^{1/n}} = \mathrm{O}(1).$$

◊

*6.3 Remark.* Each monotonically decreasing sequence is regularly distributed. In addition, if $\vec{a}$ is regularly distributed and $a_n \asymp b_n$, then $\vec{b}$ is regularly distributed. ◊

**6.4 Lemma.** *For a bounded and regularly distributed sequence of positive real numbers $\vec{a} = (a_n)_{n=1}^\infty$ and for $0 \le \beta \le 1$, we have*

$$\Delta^*(\vec{a}) = \Delta(\vec{a}) = \delta(\vec{a}, \beta).$$

*Proof.* By Lemma 6.1 it is enough to show $\delta(\vec{a},\beta) \le \Delta^*(\vec{a})$. If $\delta(\vec{a},\beta) = 0$, this is trivial. Otherwise, let $\tau \in (0, \delta(\vec{a},\beta))$ be arbitrary. Since $\delta(\vec{a},\beta)$ is the limes inferior of $G(n;\vec{a},\beta)$ there exists $N \in \mathbb{N}$ such that

$$\tau \le G(n;\vec{a},\beta) = \frac{-1}{n \ln n} \ln \left( a_n^\beta \prod_{k=1}^{n-1} a_k \right), \quad n \ge N.$$

It follows that

$$n^{-\tau} \ge \left( a_n^\beta \prod_{k=1}^{n-1} a_k \right)^{\frac{1}{n}} = a_n^{\frac{\beta-1}{n}} \left( \prod_{k=1}^{n} a_k \right)^{\frac{1}{n}} \gtrsim a_n^{\frac{\beta-1}{n}} a_n.$$

Since $a_n$ is bounded and $\beta - 1 \le 0$, we conclude $a_n \lesssim n^{-\tau}$. ❑

**6.5 Theorem.** *Let $H$ be a lcc Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Assume that $\vec{l}$ is regularly distributed, and that at least one of $\delta_l$ or $\delta_\phi$ exists as a limit. Furthermore, assume that at least one of the following conditions hold true:*

(*i*) $(|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty$ *is regularly distributed,* $\delta_l + \delta_\phi \ge 2$, *and* $(\delta_l, \delta_\phi, \mu) \ne (1,1,0)$.

(*ii*) $(|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty$ *is regularly distributed, and* $\mu = 2\delta_\phi$.

(*iii*) $\delta_\phi = 0$.

*Then,*

$$\rho(H) = \frac{1}{\delta_l + \delta_\phi}.$$

*Proof.* By Lemma 6.4, we have $\delta_l = \Delta_l = \Delta(\vec{l})$. The fact that $\vec{l}$ is summable leads to $\Delta(\vec{l}) \ge 1$, i.e. $\Delta_l = \Delta_l^+$.

If $\delta_\phi = 0$, Lemma 6.1 implies $\Delta_\phi = 0$. Otherwise $(|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty$ is regularly distributed, and Lemma 6.4 gives $\delta_\phi = \Delta_\phi$.

In case (*i*) Theorem 5.10 yields $\rho(H) \le (\delta_l + \delta_\phi)^{-1}$. If (*ii*) holds, this theorem gives

$$\rho(H) \le \frac{1}{\delta_l + \frac{\mu}{2}} = \frac{1}{\delta_l + \delta_\phi}.$$

In case (*iii*) we get $\rho(H) \le (\delta_l + \mu/2)^{-1} \le \delta_l^{-1}$.

In all cases, the upper bound of the order of $H$ coincides with the lower bound from Corollary 5.21. ❑

Next we demonstrate how Theorem 6.5 can be applied in a regular example. In there, one can observe the phenomenon that for $\delta_l + \delta_\phi < 2$ our lower and upper bounds do sometimes not coincide. This is the reason why we speak of this region as the critical triangle. The example also contains a case inside the critical triangle where the order can still be determined.

*6.6 Example.* We come back to Example 5.9. Consider $l_n := n^{-\alpha}$, $\phi_1 := 0$ and $\phi_{n+1} := \phi_n + \sigma_n n^{-\beta}$ for $n \in \mathbb{N}$.

As we already mentioned, we have $\Delta_l^+ = \alpha$ and $\Delta_\phi = \beta$. Moreover, $\vec{l}$ and $(|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty$ are regularly distributed, cf. Remark 6.3, and $\delta_l$ exists as a limit, cf. Example 5.20.

First, assume that $\sigma_n = 1$ for all $n \in \mathbb{N}$, which results in $\mu = \max\{0, 2(\Delta_\phi - 1)\} \le 2\Delta_\phi$. If $\alpha + \beta \ge 2$ or $\beta = 0$, then the assumptions of Theorem 6.5 (namely $(i)$ or $(iii)$, respectively), are satisfied, and we get

$$\rho(H) = \frac{1}{\alpha + \beta}.$$

Inside the critical triangle $\alpha + \beta < 2$ for $\beta > 0$, we have $\beta < 1$ and $\mu = 0$. In this case, the lower bound form Corollary 5.21 and the upper bound from Theorem 5.10 do not coincide. We know that

$$\rho(H) \in \Big[\frac{1}{\alpha + \beta}\,,\,\frac{1 - \beta}{\alpha - \beta}\Big],$$

but the actual order of $H$ is not known in this case.

Secondly, assume $\sigma_n = (-1)^n$ for $n \in \mathbb{N}$, which gives $\mu = 2\Delta_\phi$. Independent of $\alpha + \beta$ being greater or less than 2, Theorem 6.5, $(ii)$, yields that the order is equal to $\frac{1}{\alpha+\beta}$. ◊

Another frequently-used measure for the growth of a sequence is the convergence exponent, cf. Definition 2.4. Note that

$$\frac{1}{\text{c. e.}\left((a_n^{-1})_{n=1}^\infty\right)} = \sup\{p > 0 : \vec{a} \in \ell^{1/p}\},$$

where this expression is understood to be 0 if $\vec{a} \notin \ell^{1/p}$ for all $p > 0$.

**6.7 Lemma.** *For a sequence of positive real numbers $\vec{a} = (a_n)_{n=1}^\infty$ with $\lim_{n\to\infty} a_n = 0$, and for $\beta \ge 0$, we have*

$(i)$ *In general we have the inequality,*

$$\Delta^*(\vec{a}) \le \left[\text{c. e.}\left((a_n^{-1})_{n=1}^\infty\right)\right]^{-1} \le \delta(\vec{a}, \beta).$$

$(ii)$ *Regarding $\Delta(\vec{a})$, it holds that*

$$\Delta(\vec{a}) \le \max\left\{\left[\text{c. e.}\left((a_n^{-1})_{n=1}^\infty\right)\right]^{-1}, 1\right\}.$$

$(iii)$ *If $\vec{a}$ is regularly distributed, then we have*

$$\Delta^*(\vec{a}) = \Delta(\vec{a}) = \left[\text{c. e.}\left((a_n^{-1})_{n=1}^\infty\right)\right]^{-1} = \delta(\vec{a}, \beta). \tag{6.3}$$

*Proof.* add $(i)$: If $\Delta^*(\vec{a}) = 0$, then the first inequality is clear. Otherwise, let $\tau > 0$ be arbitrary with $a_n = \mathrm{O}(n^{-\tau})$, and take $p > 0$. Then

$$\sum_{n=1}^{\infty} a_n^{1/p} \lesssim \sum_{n=1}^{\infty} n^{-\tau/p} < \infty,$$

if $p < \tau$. This gives $\tau \le \left[\text{c.e.}\left((a_n^{-1})_{n=1}^{\infty}\right)\right]^{-1}$, and hence $\Delta^*(\vec{a}) \le \left[\text{c.e.}\left((a_n^{-1})_{n=1}^{\infty}\right)\right]^{-1}$.

The other inequality is clear if $\left[\text{c.e.}\left((a_n^{-1})_{n=1}^{\infty}\right)\right]^{-1} = 0$. So let $p > 0$ be arbitrary such that $\vec{a} \in \ell^{1/p}$. The inequality of arithmetic and geometric means gives

$$\left(\prod_{k=1}^{n} a_k^{1/p}\right)^{1/n} \le \frac{1}{n}\sum_{k=1}^{n} a_k^{1/p} \le C\frac{1}{n}$$

for a constant $C > 1$. Thus $\prod_{k=1}^{n} a_k \le C^{np} n^{-np}$, and together with $a_{n+1}^{\beta} \le 1$ we get

$$G(n+1;\vec{a},\beta) \ge \frac{-1}{n\ln n}\ln\left(C^{np}n^{-np}\right) \overset{n\to\infty}{\longrightarrow} p.$$

This gives $p \le \delta(\vec{a},\beta)$, and hence $\left[\text{c.e.}\left((a_n^{-1})_{n=1}^{\infty}\right)\right]^{-1} \le \delta(\vec{a},\beta)$.

add $(ii)$: The inequality holds trivially in the case $\Delta(\vec{a}) \le 1$. Otherwise, let $\tau > 1$ be arbitrary such that $n^{-1}\sum_{k=n}^{2n-1} a_k = \mathrm{O}(n^{-\tau})$, and take $p \in (1,\tau)$. Jensen's inequality (note that $x \mapsto x^{1/p}$ is concave) yields

$$\sum_{n=1}^{\infty} a_n^{1/p} = \sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1} a_n^{1/p} \le \sum_{k=0}^{\infty} 2^k\left(\frac{1}{2^k}\sum_{n=2^k}^{2^{k+1}-1} a_n\right)^{1/p} \sum_{k=0}^{\infty} 2^k(2^k)^{-\tau/p} = \sum_{k=0}^{\infty}\left(2^{1-\frac{\tau}{p}}\right)^k.$$

This geometric series converges since $1 - \frac{\tau}{p} < 0$, and we note $p \le \left[\text{c.e.}\left((a_n^{-1})_{n=1}^{\infty}\right)\right]^{-1}$. Because $p \in (1,\tau)$ was arbitrary, we get $\tau \le \left[\text{c.e.}\left((a_n^{-1})_{n=1}^{\infty}\right)\right]^{-1}$, and taking the supremum over $\tau$ gives $\Delta(\vec{a}) \le \left[\text{c.e.}\left((a_n^{-1})_{n=1}^{\infty}\right)\right]^{-1}$.

add $(iii)$: This follows from $(i)$ and Lemma 6.4. ❑

Next we present an example of a sequence which is not regularly distributed such that all 4 quantities in (6.3) are distinct. Moreover, this example shows that the maximum in Lemma 6.7, $(ii)$, is necessary.

*6.8 Example.* Let $q$ and $\alpha$ be two positive real numbers with $q \in (0,1)$, and set

$$a_n := \begin{cases} k^{-\alpha} & n = 2^k, k \in \mathbb{N}_0, \\ q^n & \text{else.} \end{cases}$$

We are going to see ($\beta \ge 0$)

$$\begin{aligned} \Delta^*(\vec{a}) &= 0, & \left[\text{c.e.}\left((a_n^{-1})_{n=1}^{\infty}\right)\right]^{-1} &= \alpha, \\ \Delta(\vec{a}) &= 1, & \delta(\vec{a},\beta) &= \infty. \end{aligned}$$

In particular for $\alpha < 1$

$$\big[\text{c. e.}\,\big((a_n^{-1})_{n=1}^{\infty}\big)\big]^{-1} < \Delta(\vec{a}).$$

To verify this, let $\tau \geq 0$ and assume $a_{2^k} = k^{-\alpha} \lesssim 2^{-\tau k}$. Clearly this forces $\tau = 0$, and we see $\Delta^*(\vec{a}) = 0$.

Now let $p > 0$. Clearly $q^{1/p} < 1$, and thus $\sum_{n=0}^{\infty} q^{n/p} < \infty$. Hence $\sum_{n=0}^{\infty} a_n^{1/p} < \infty$ if and only if $\sum_{k=1}^{\infty} k^{-\alpha/p} < \infty$. This is equivalent to $\alpha/p > 1$ and we conclude $\big[\text{c. e.}\,\big((a_n^{-1})_{n=1}^{\infty}\big)\big]^{-1} = \alpha$.

Next let $\tau \geq 0$ and assume $n^{-1}\sum_{j=n}^{2n-1} a_j = \mathrm{O}(n^{-\tau})$. Then

$$2^{-k}k^{-\alpha} = 2^{-k}a_{2^k} \leq \frac{1}{2^k}\sum_{n=2^k}^{2^{k+1}-1} a_n \leq C2^{-k\tau},$$

for a constant $C > 0$. Since $2^{k(\tau-1)}k^{-\alpha}$ can only be bounded if $\tau \leq 1$, we conclude $\Delta(\vec{a}) \leq 1$.

To show the other inequality, let $n \in \mathbb{N}$ be arbitrary. Since $\{n, n+1, \dots, 2n-1\}$ contains exactly one power of two, we have

$$\frac{1}{n}\sum_{j=n}^{2n-1} a_j \leq \frac{1}{n}\Bigg((\log_2 n)^{-\alpha} + \sum_{j=n}^{2n-1} q^j\Bigg) = \mathrm{O}(n^{-1}),$$

which shows $\Delta(\vec{a}) \geq 1$.

Finally, let $M_n$ denote the set of those natural numbers in $\{1, 2, \dots, n-1\}$ which are not of the form $2^k$ for $k \in \mathbb{N}_0$. Using $a_j \leq 1$ for all $j \in \mathbb{N}$, we have

$$\frac{-1}{n\ln n}\ln\Bigg(a_n^{\beta}\prod_{j=1}^{n-1} a_j\Bigg) \geq \frac{-1}{n\ln n}\ln\Bigg(\prod_{j\in M_n} a_j\Bigg) = \frac{-\ln q}{n\ln n}\sum_{j\in M_n} j \gtrsim \frac{-\ln q}{n\ln n}n^2 \overset{n\to\infty}{\longrightarrow} \infty.$$

Hence, $\delta(\vec{a}, \beta) = \infty$. ◊

## 6.2. Irregular behaviour

First, we look at the case $\mu > \delta_\phi$ or $\mu > \Delta_\phi$. A comparison of the upper and lower bound reveals occurrence of some irregularity. More precisely, we have the following statements.

**6.9 Theorem.** *Let $H$ be a Hamburger Hamiltonian in the lcc with lengths $\vec{l}$ and angles $\vec{\phi}$. Assume that $\mu > 2\delta_\phi$.*

*Then, either $\vec{l}$ is not regularly distributed or both $\delta_l$ and $\delta_\phi$ do not exist as a limit.*

*Proof.* Assume on the contrary that $\vec{l}$ is regularly distributed and that at least one of $\delta_l$ and $\delta_\phi$ does exist as a limit.

By Lemma 6.4, we have $\delta_l = \Delta_l = \Delta(\vec{l})$. The summability of $\vec{l}$ gives $\Delta(\vec{l}) \geq 1$, i.e. $\Delta_l = \Delta_l^+$. The upper bound from Theorem 5.10 (note $\mu > 2\delta_\phi \geq 2\Delta_\phi$) and the lower bound from Corollary 5.21 give

$$\frac{1}{\Delta_l^+ + \delta_\phi} = \frac{1}{\delta_l + \delta_\phi} \leq \rho(H) \leq \frac{1}{\Delta_l^+ + \frac{\mu}{2}},$$

which leads to the contradiction $\delta_\phi \geq \frac{\mu}{2} > \delta_\phi$. ❑

**6.10 Corollary.** *Let $H$ be a Hamburger Hamiltonian in the lcc with lengths $\vec{l}$ and angles $\vec{\phi}$. Assume that $\mu > 2\Delta_\phi$.*

*Then, either $\vec{l}$ or $(|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty$ is not regularly distributed, or both $\delta_l$ and $\delta_\phi$ do not exist as a limit.*

*Proof.* The statement is trivial if $(|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty$ is not regularly distributed. Otherwise it follows directly from Theorem 6.9. ❑

In the rest of this section, we construct a class of examples which shows that both the lower bound established in Proposition 5.17 and the upper bound from Theorem 5.10 do not necessarily coincide with the order when lengths and/or angle-differences are not regularly distributed. Such examples can already be found in the class of diagonal Hamburger Hamiltonians.

**6.11 Theorem.** *Let $\rho \in (0,1)$, and let $\delta$ and $\Delta$ be positive real numbers with*

$$\frac{1}{\delta} < \rho < \frac{1}{\Delta}, \quad \Delta \geq \max\Big\{1, \frac{1}{\rho} - 1\Big\}.$$

*Then there exists a summable sequence $\vec{l}$ of positive numbers, such that the diagonal Hamburger Hamiltonian $H$ with lengths $\vec{l}$ (and angles $\phi_n := n\frac{\pi}{2}$) satisfies*

$$\rho(H) = \rho, \quad \delta_l(H) = \delta, \quad \Delta_l^+(H) = \Delta.$$

The conditions $1/\delta < \rho < 1/\Delta$ and $\Delta \geq 1$ are necessary, since we are looking for examples where both the upper and the lower bound are different from the order.

The additional condition $\Delta \geq \rho^{-1} - 1$ is a constraint only in the case $\rho < 1/2$. In this case the upper estimate is at most $1/\Delta \leq \rho/(1-\rho)$, which is for small order asymptotically correct. Hence our method does not produce examples with very small order such that $1/\Delta$ is really bad, say close to one.

The proof uses some general results on the one hand, namely Propositions 4.7 and 4.6, which are corollaries of [Rom17, Theorem 2], and some elementary construction of sequences on the other, namely Lemma 6.12 and Lemma 6.13 below.

**6.12 Lemma.** *Let $\alpha$ and $\Delta$ be positive real numbers with*

$$\alpha > 1, \quad \max\{1, \alpha - 1\} \leq \Delta < \alpha,$$

*and set $l_n := n^{-\alpha}$, $n \in \mathbb{N}$. Then there exists a function $\beta : \mathbb{N}_0 \to 2\mathbb{N}$ and a sequence of natural numbers $(n_j)_{j=1}^\infty$ with*

$$n_{j+1} > n_j + \beta(n_j) + 1, \quad j \in \mathbb{N},$$

*such that the sequence $(\tilde{l}_n)_{n=1}^\infty$ defined by ($n_0 := -1$)*

$$\tilde{l}_n := \begin{cases} l_{n+\sum_{i=1}^{j-1}\beta(n_i)} & \text{for } n = n_{j-1} - \sum_{i=1}^{j-2}\beta(n_i) + 2, \dots, n_j - \sum_{i=1}^{j-1}\beta(n_i) \\ \sum\limits_{\substack{k=1 \\ k \text{ odd}}}^{\beta(n_j)+1} l_{n_j+k} & \text{for } n = n_j - \sum_{i=1}^{j-1}\beta(n_i) + 1 \end{cases}, \quad j \in \mathbb{N},$$

*satisfies*

$$\delta\big((\tilde{l}_n)_{n=1}^\infty, \tfrac{1}{2}\big) \leq \alpha, \quad \Delta^*\big((\tilde{l}_n)_{n=1}^\infty\big) = \Delta.$$

Let $H$ and $\tilde{H}$ be the diagonal Hamburger Hamiltonians with lengths $(l_n)_{n=1}^\infty$ and $(\tilde{l}_n)_{n=1}^\infty$, respectively. The construction of $(\tilde{l}_n)_{n=1}^\infty$ is done in such a way that $\tilde{H}$ is obtained from $H$ by removing some intervals, namely those corresponding to $l_{n_j+k}$ with $k = 2, 4, \dots, \beta(n_j)$, and gluing the contiguous ones to one long interval.

$$(\, l_1, \dots, l_{n_1},\ \underbrace{l_{n_1+1}, \dots, l_{n_1+\beta(n_1)+1}},\ l_{n_1+\beta(n_1)+2}, \dots, l_{n_2},\ \underbrace{l_{n_2+1}, \dots, l_{n_2+\beta(n_2)+1}},\ l_{n_2+\beta(n_2)+2}, \dots \,)$$

$$(\, l_1, \dots, l_{n_1},\ \tilde{l}_{n_1+1},\ l_{n_1+\beta(n_1)+2}, \dots, l_{n_2},\ \tilde{l}_{n_2-\beta(n_1)+1},\ l_{n_2+\beta(n_2)+2}, \dots \,)$$

Observe here that $\beta(n_j)$ is even, and hence angles alternate in $\tilde{H}$.

*Proof.* Set $\gamma := \alpha - \Delta$, then $0 < \gamma \leq 1$. For $n \in \mathbb{N}_0$ let $\beta(n)$ be the smallest integer larger than $n^\gamma$.

We are going to define a sequence $(n_j)_{j=0}^\infty$ recursively. Set $n_0 := -1$. Assume that $j \geq 1$ and that $n_1, \dots, n_{j-1}$ are already defined. Consider the sequence $(l_n^{(j-1)})_{n=1}^\infty$ defined as

$$l_n^{(j-1)} := \begin{cases} \tilde{l}_n & \text{for } n \leq n_{j-1} - \sum\limits_{i=1}^{j-2}\beta(n_i) + 1 \\ l_{n+\sum_{i=1}^{j-1}\beta(n_i)} & \text{for } n > n_{j-1} - \sum\limits_{i=1}^{j-2}\beta(n_i) + 1 \end{cases}$$

Note here that $\tilde{l}_n$ is already well-defined for $n \leq n_{j-1} - \sum_{i=1}^{j-2}\beta(n_i) + 1$. We have

$$\liminf_{m\to\infty} G\big(m; (l_n^{(j-1)})_{n=1}^\infty, \tfrac{1}{2}\big) = \liminf_{m\to\infty} G\big(m; (l_n)_{n=1}^\infty, \tfrac{1}{2}\big) = \alpha.$$

Therefore, we can choose $n_j \in \mathbb{N}$ sufficiently large so that

$$n_j > n_{j-1} + \beta(n_{j-1}) + 1,$$

$$n_j > 2\bigg(\sum_{i=1}^{j-1}\beta(n_i) - 1\bigg), \tag{6.4}$$

$$G\bigg(n_j - \sum_{i=1}^{j-1}\beta(n_i); (l_n^{(j-1)})_{n=1}^\infty, \tfrac{1}{2}\bigg) \leq \alpha + \frac{1}{j}. \tag{6.5}$$

Consider the sequence $(\tilde{l}_n)_{n=1}^{\infty}$ constructed with this choice of $n_j$, $j \in \mathbb{N}_0$. First,

$$\tilde{l}_n = l_n^{(j-1)}, \quad n \le n_j - \sum_{i=1}^{j-1} \beta(n_i), \; j \in \mathbb{N},$$

and hence by (6.5)

$$G\big(m; (\tilde{l}_n)_{n=1}^{\infty}, \tfrac{1}{2}\big) = G\big(m; (l_n^{(j-1)})_{n=1}^{\infty}, \tfrac{1}{2}\big) \le \alpha + \frac{1}{j}, \quad m = n_j - \sum_{i=1}^{j-1} \beta(n_i), \; j \in \mathbb{N}.$$

We conclude $\delta\big((\tilde{l}_n)_{n=1}^{\infty}, \tfrac{1}{2}\big) \le \alpha$.

Second, we estimate the length of the "glued" interval,

$$\tilde{l}_{n_j - \sum_{i=1}^{j-1} \beta(n_i)+1} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\beta(n_j)+1} l_{n_j+k} \asymp \int_{n_j}^{n_j+\beta(n_j)} x^{-\alpha}\, dx \asymp \beta(n_j) n_j^{-\alpha} \asymp n_j^{-\Delta}.$$

Note that $n_j + \beta(n_j) \asymp n_j + n_j^{\gamma} \asymp n_j$, since $\gamma \le 1$. Furthermore, $n_j \asymp n_j - \sum_{i=1}^{j-1} \beta(n_i) + 1$ due to (6.4). Since the other intervals are by construction smaller, we conclude $\Delta^*\big((\tilde{l}_n)_{n=1}^{\infty}\big) = \Delta$. ❑

**6.13 Lemma.** *Let $(l_n)_{n=1}^{\infty}$ be a summable sequence of positive real numbers, and let $\delta > 0$ be a positive real number with $\delta > \delta((l_n)_{n=1}^{\infty}, \frac{1}{2})$.*

*Then there exists a strictly monotonically increasing sequences of natural numbers $(n_j)_{j=0}^{\infty}$ and a sequence of positive real numbers $(\varepsilon_j)_{j=1}^{\infty}$ with*

$$\sum_{j=1}^{\infty} \frac{\varepsilon_j}{\min\{l_{n_{j-1}}, l_{n_{j-1}+1}\}} < \infty, \tag{6.6}$$

*such that the sequence ($n_{-1} := 0$)*

$$\begin{aligned}(\tilde{l}_n)_{n=1}^{\infty} :=& \big(l_1, \dots, l_{n_0}, \, \varepsilon_1, \varepsilon_1 \,, l_{n_0+1}, \dots, l_{n_1}, \, \varepsilon_2, \varepsilon_2 \,, l_{n_1+1}, \dots \big) \\ =& \begin{cases} l_{n-2j} \,, & n_{j-1} + 2j < n \le n_j + 2j, \\ \varepsilon_{j+1} \;, & n_j + 2j < n \le n_j + 2j + 2 \end{cases}, \quad j \in \mathbb{N}_0,\end{aligned}$$

*satisfies*

$$\begin{aligned}&\delta\big((\tilde{l}_n)_{n=1}^{\infty}, \tfrac{1}{2}\big) = \delta, \\ &\Delta^*\big((\tilde{l}_n)_{n=1}^{\infty}\big) = \Delta^*\big((l_n)_{n=1}^{\infty}\big).\end{aligned}$$

*Proof.* If $\Delta^*\big((l_n)_{n=1}^{\infty}\big) = 0$, set $\beta_j := 0$, $j \in \mathbb{N}$. Otherwise, let $(\beta_j)_{j=1}^{\infty}$ be an increasing sequence of positive numbers with $\beta_j < \Delta^*\big((l_n)_{n=1}^{\infty}\big)$ and $\lim_{n\to\infty} \beta_j = \Delta^*\big((l_n)_{n=1}^{\infty}\big)$.

Choose in both cases a sequence of natural numbers $(\mathring{n}_j)_{j=0}^{\infty}$ such that for all $j \in \mathbb{N}$,

$$l_n \le (n + 2j)^{-\beta_j}, \quad n > \mathring{n}_{j-1}, \tag{6.7}$$

which is possible since

$$\lim_{n\to\infty} l_n n^\tau = 0, \quad \tau \in \big[0, \Delta^*((l_n)_{n=1}^\infty)\big).$$

Now we construct inductively a sequence of natural numbers $(n_j)_{j=0}^\infty$ and a sequence of positive real numbers $(\varepsilon_j)_{j=1}^\infty$ such that

$$n_j \geq \mathring{n}_j, \quad j \in \mathbb{N}_0, \tag{6.8}$$

$$n_j > n_{j-1}, \quad j \in \mathbb{N}, \tag{6.9}$$

$$\varepsilon_j \leq \min\left\{ \frac{\min\{l_{n_{j-1}}, l_{n_{j-1}+1}\}}{j^2}, (n_{j-1}+2j)^{-\beta_j} \right\}, \quad j \in \mathbb{N}, \tag{6.10}$$

$$G\big(m; (l_n^{(j)})_{n=1}^\infty, \tfrac{1}{2}\big) \begin{cases} \geq \delta\,, & n_0 \leq m < n_j + 2j \\ = \delta\,, & m = n_j + 2j \end{cases}, \quad j \in \mathbb{N}. \tag{6.11}$$

where

$$(l_n^{(j)})_{n=1}^\infty := \big(\tilde{l}_1, \ldots, \tilde{l}_{n_{j-1}+2(j-1)}, \varepsilon_j, \varepsilon_j, l_{n_{j-1}+1}, l_{n_{j-1}+2}, \ldots\big),$$

depends on $n_0, \ldots, n_{j-1}$ and $\varepsilon_1, \ldots, \varepsilon_j$.

*Induction base "$j = 0$":* Set $n_0 := \mathring{n}_0$. Then (6.8) holds. The conditions (6.9)–(6.11) are void.

*Induction step "$j - 1 \mapsto j$":* Assume that $j \in \mathbb{N}$ and that $n_0, \ldots, n_{j-1}$ and $\varepsilon_1, \ldots, \varepsilon_{j-1}$ are already constructed such that (6.8)–(6.11) hold up to $j - 1$.

(a) Choose $\mathring{\varepsilon}_j > 0$ sufficiently small such that

$$\mathring{\varepsilon}_j \leq \min\left\{ \frac{\min\{l_{n_{j-1}}, l_{n_{j-1}+1}\}}{j^2}, (n_{j-1}+2j)^{-\beta_j} \right\},$$

$$G\big(m; (\mathring{l}_n^{(j)})_{n=1}^\infty, \tfrac{1}{2}\big) \geq \delta, \quad n_{j-1} + 2j - 1 \leq m \leq \max\{\mathring{n}_j, n_{j-1}\} + 2j,$$

where

$$(\mathring{l}_n^{(j)})_{n=1}^\infty := \big(\tilde{l}_1, \ldots, \tilde{l}_{n_{j-1}+2(j-1)}, \mathring{\varepsilon}_j, \mathring{\varepsilon}_j, l_{n_{j-1}+1}, l_{n_{j-1}+2}, \ldots\big).$$

Observe here that the numbers $\tilde{l}_1, \ldots, \tilde{l}_{n_{j-1}+2(j-1)}$ involve only the already defined data.

(b) The set

$$K_j := \big\{ m \geq n_0 : G\big(m; (\mathring{l}_n^{(j)})_{n=1}^\infty, \tfrac{1}{2}\big) < \delta \big\}$$

is nonempty since

$$\liminf_{m\to\infty} G\big(m; (\mathring{l}_n^{(j)})_{n=1}^\infty, \tfrac{1}{2}\big) = \liminf_{m\to\infty} G\big(m; (l_n)_{n=1}^\infty, \tfrac{1}{2}\big) = \delta((l_n)_{n=1}^\infty, \tfrac{1}{2}) < \delta.$$

Moreover, by the inductive hypothesis and our choice of $\mathring{\varepsilon}_j$, we have

$$K_j \subseteq (\max\{\mathring{n}_j, n_{j-1}\} + 2j, \infty). \tag{6.12}$$

Observe here that $\tilde{l}_n = l_n^{(j-1)}$, $n \leq n_{j-1} + 2(j-1)$. Set

$$n_j := \min K_j - 2j.$$

In particular $n_j + 2j \in K_j$, which means $G\big(n_j + 2j; (\mathring{l}_n^{(j)})_{n=1}^\infty, \frac{1}{2}\big) < \delta$.
(c) Choose $\varepsilon_j > 0$ such that $\varepsilon_j \leq \mathring{\varepsilon}_j$ and

$$G\big(n_j + 2j; (l_n^{(j)})_{n=1}^\infty, \tfrac{1}{2}\big) = \delta.$$

Let us check that (6.8)–(6.11) hold for $j$. From (6.12) we obtain $n_j > \max\{\mathring{n}_j, n_{j-1}\}$, which gives (6.8) and (6.9). The condition (6.10) is satisfied by $\mathring{\varepsilon}_j$ and hence also by $\varepsilon_j$. To see (6.11) note that

$$G\big(m; (l_n^{(j)})_{n=1}^\infty, \tfrac{1}{2}\big) \begin{cases} = G\big(m; (\mathring{l}_n^{(j)})_{n=1}^\infty, \frac{1}{2}\big) \geq \delta\,, & n_0 \leq m \leq n_{j-1} + 2(j-1), \\ \geq G\big(m; (\mathring{l}_n^{(j)})_{n=1}^\infty, \frac{1}{2}\big) \quad\quad , & n_{j-1} + 2j - 1 \leq m. \end{cases}$$

Moreover, $m \leq n_j + 2j - 1$ implies $m \notin K_j$, which means $G\big(m; (\mathring{l}_n^{(j)})_{n=1}^\infty, \frac{1}{2}\big) \geq \delta$. This completes the inductive construction.

The required properties of the sequence $(\tilde{l}_n)_{n=1}^\infty$ follow easily. First, remembering that $\tilde{l}_n = l_n^{(j)}$ if $j$ is large enough such that $n \leq n_j + 2j$, we find

$$\begin{aligned} &G\big(m; (\tilde{l}_n)_{n=1}^\infty, \tfrac{1}{2}\big) \geq \delta, \quad m \geq n_0, \\ &G\big(n_j + 2j; (\tilde{l}_n)_{n=1}^\infty, \tfrac{1}{2}\big) = \delta, \quad j \in \mathbb{N}, \end{aligned}$$

and therefore $\liminf_{m\to\infty} G\big(m; (\tilde{l}_n)_{n=1}^\infty, \frac{1}{2}\big) = \delta$. Second, by (6.10) the series in (6.6) converges. This also implies $\sum_{n=1}^\infty \tilde{l}_n < \infty$.

Since $(l_n)_{n=1}^\infty$ is a subsequence of $(\tilde{l}_n)_{n=1}^\infty$ in such a way that at least every third element of $(\tilde{l}_n)_{n=1}^\infty$ belongs to $(l_n)_{n=1}^\infty$, the inequality $\Delta^*\big((\tilde{l}_n)_{n=1}^\infty\big) \leq \Delta^*\big((l_n)_{n=1}^\infty\big)$ holds. If $\Delta^*\big((l_n)_{n=1}^\infty\big) = 0$, then equality follows immediately. Assume that $\Delta^*\big((l_n)_{n=1}^\infty\big) > 0$ and let $\tau \in \big(0, \Delta^*\big((l_n)_{n=1}^\infty\big)\big)$. Choose $j_0 \in \mathbb{N}$ such that $\beta_j > \tau$ for $j \geq j_0$. It holds that

$$\begin{aligned} \tilde{l}_n = &\left\{ \begin{array}{ll} \varepsilon_j \quad , & n \in \{n_{j-1} + 2j - 1, n_{j-1} + 2j\} \\ l_{n-2j}, & n_{j-1} + 2j < n \leq n_j + 2j \end{array} \right\} \\ \leq &n^{-\beta_j} \leq n^{-\tau}, \quad j \geq j_0, \end{aligned}$$

where we used (6.10) to bound $\varepsilon_j$, and (6.7) to bound $l_{n-2j}$ in the case $n_{j-1} + 2j < n \leq n_j + 2j$, note that $n - 2j > n_{j-1} \geq \mathring{n}_{j-1}$. Hence, $\Delta^*\big((\tilde{l}_n)_{n=1}^\infty\big) \geq \Delta^*\big((l_n)_{n=1}^\infty\big)$. ❑

We have collected all necessary ingredients for the proof of Theorem 6.11.

*Proof of Theorem* 6.11. Set $l_n^{(1)} := n^{-\frac{1}{\rho}}$, $n \in \mathbb{N}$. Then

$$\delta\big((l_n^{(1)})_{n=1}^\infty, \tfrac{1}{2}\big) = \Delta^*\big((l_n^{(1)})_{n=1}^\infty\big) = \frac{1}{\rho},$$

cf. Example 5.20. The diagonal Hamburger Hamiltonian $H^{(1)}$ with lengths $(l_n^{(1)})_{n=1}^\infty$ has order $\rho$ by Theorem 6.5.

Lemma 6.12 with $\alpha = \frac{1}{\rho}$ provides us with a sequence $(l_n^{(2)})_{n=1}^\infty$ such that

$$\delta\big((l_n^{(2)})_{n=1}^\infty, \tfrac{1}{2}\big) \le \frac{1}{\rho}, \quad \Delta^*\big((l_n^{(2)})_{n=1}^\infty\big) = \Delta,$$

and so that the diagonal Hamburger Hamiltonian $H^{(2)}$ with lengths $(l_n^{(2)})_{n=1}^\infty$ is obtained by removing certain intervals from $H^{(1)}$. By Proposition 4.6, we obtain

$$\rho \le \frac{1}{\delta\big((l_n^{(2)})_{n=1}^\infty, \frac{1}{2}\big)} \le \rho\big(H^{(2)}\big) \le \rho\big(H^{(1)}\big) = \rho,$$

and, thus, $\rho(H^{(2)}) = \rho$ and $\delta\big((l_n^{(2)})_{n=1}^\infty, \frac{1}{2}\big) = \frac{1}{\rho}$.

Now we apply Lemma 6.13 with $(l_n^{(2)})_{n=1}^\infty$ and $\delta > \delta\big((l_n^{(2)})_{n=1}^\infty, \frac{1}{2}\big)$, which gives a sequence $(l_n^{(3)})_{n=1}^\infty$ such that

$$\delta\big((l_n^{(3)})_{n=1}^\infty, \tfrac{1}{2}\big) = \delta, \quad \Delta^*\big((l_n^{(3)})_{n=1}^\infty\big) = \Delta.$$

Let $H^{(3)}$ be the diagonal Hamburger Hamiltonian with lengths $(l_n^{(3)})_{n=1}^\infty$. Then (6.6) implies (4.6), and Proposition 4.7 gives $\rho(H^{(3)}) = \rho(H^{(2)}) = \rho$. ❑

As a corollary, we get that Livšic's lower bound for the order of an indeterminate moment problem, i.e. $\rho((s_n)_{n=0}^\infty) \ge \rho(L)$, is not always equal to the order. In fact, the gap between $\rho(L)$ and the actual order of the moment sequence can be arbitrarily close to 1.

**6.14 Corollary.** *Let $\rho, \rho' \in (0,1)$ with $\rho > \rho'$. Then there exists an indeterminate moment sequence $(s_n)_{n=0}^\infty$ such that*

$$\rho\big((s_n)_{n=0}^\infty\big) = \rho, \quad \rho' \ge \rho(L).$$

*Proof.* By Lemma 6.4, we can choose a diagonal Hamburger Hamiltonian $H$ with $\rho(H) = \rho$ and $\delta_l(H)^{-1} = \rho'$. Consider the corresponding moment sequence $(s_n)_{n=0}^\infty$. By Corollary 5.15,

$$\rho\big((s_n)_{n=0}^\infty\big) \ge \rho(F) \ge \rho(L),$$

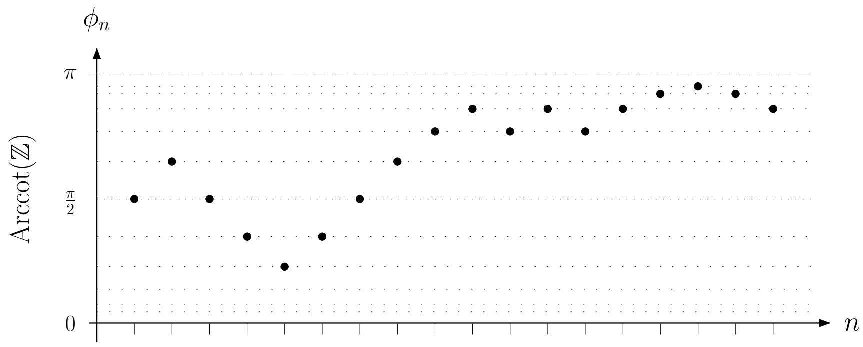
and Proposition 5.17 gives $\rho(F) = \delta_{l,\phi}(H)^{-1} = \delta_l(H)^{-1} = \rho'$. ❑

# 7. Square Transform and the indefinite method

We establish another upper estimate for the order of a Hamburger Hamiltonian, cf. Theorem 7.18 which is the first main result in this chapter. The proof is achieved by associating with the given Hamburger Hamiltonian a certain (singular) Krein-string. During this process, several different types of arguments come into play. Our method relies on an operator theoretic limiting argument (Proposition 7.6), some purely algebraic computations and transformations (Section 7.2), and estimates for canonical products by means of the density of their zeroes. Moreover, on the way, we leave the positive definite scheme and encounter Hamiltonians which may take negative semidefinite matrices as values.

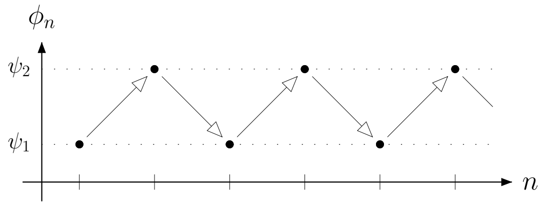
The estimate in Theorem 7.18 is incomparable with the one obtained in Theorem 5.10; in some cases it is better and in some others it is worse, cf. Proposition 7.23 and Example 7.25.

Our second main result in this chapter is Theorem 7.22, where we discuss a class of Hamiltonians whose order can be determined. Consider a Hamburger Hamiltonian $H$ whose angles $\phi_n$ (up to a small deviation) walk on the grid $\operatorname{Arccot}(\mathbb{Z})$. By this we mean that $|\cot\phi_n - \cot\phi_{n-1}|$ is constant equal to 1:




Further, assume that the lengths $l_n$ and angles together decay sufficiently rapidly (the series $\sum_{n=1}^{\infty}[l_n\sin^2\phi_n]^{\frac{1}{2}}\ln n$ should converge) and regularly (the sequence $l_n\sin^2\phi_n$ should be nonincreasing). The conclusion then is that the order of $H$ is equal to the convergence exponent of $([l_n\sin^2\phi_n]^{-1})_{n=1}^{\infty}$. The proof is obtained by evaluating the upper estimate Theorem 7.18 with help of [Kac90], and by combining this with the lower estimate Proposition 5.17.

Theorem 7.22 can be seen as a generalisation for orders $\leq 1/2$ of a theorem of Yu.M.Berezanskiĭ. In the language of Hamburger Hamiltonians the essence of Berezanskiĭ's theorem can be phrased as follows: Consider a Hamburger Hamiltonian $H$ whose angles alternate between two values:




If lengths decay regularly (the sequence $l_{n-2}/l_n$ should be monotone), then the order of $H$ is equal to the convergence exponent of $(l_n^{-1})_{n=1}^\infty$.

A detailed discussion of the connection with Berezanskiĭ's theorem is given in subsection 7.3.3, where we shall also see that the present result actually goes far beyond Berezanskiĭ's result, cf. Example 7.26.

## 7.1. Schatten-class properties and the order

Let $H : [0, L) \to \mathbb{R}^{2\times 2}$ be a positive semidefinite Hamiltonian. As mentioned in the Introduction, the fundamental distinction whether $H$ stays integrable towards $L$ or not is of importance for the study of the corresponding operator.

Recall the construction of the Titchmarsh-Weyl coefficient associated with a lpc Hamiltonian:

*7.1. The Weyl-construction:* Let $H$ be a Hamiltonian in the lpc. Then for each parameter $\tau \in \mathcal{N}_0 \cup \{\infty\}$ the limit

$$Q_H(z) := \lim_{x\to L} \frac{w_{11}(x,z)\tau(z) + w_{12}(x,z)}{w_{21}(x,z)\tau(z) + w_{22}(x,z)} \tag{7.1}$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and does not depend on $\tau$, cf. [HSW00, Theorem 2.1(2.7)]. The function $Q_H$ is called the *Titchmarsh-Weyl coefficient* of $H$. It belongs to the Nevanlinna class and is the $Q$-function of the canonical selfadjoint extension of $T_{\min}(H)$ given as

$$A(H) := \big\{(f; g) \in T_{\max}(H) : (1,0)f(0) = 0\big\}.$$

This means that the Nevanlinna-kernel $\frac{Q(z)-\overline{Q(w)}}{z-\overline{w}}$ of $Q$ is given as the scalar product of a family of defect elements of $T_{\min}(H)$ generated by $A(H)$, cf. [HSW00, Theorem 4.3, (4.8)]. ♢

Let us turn to the case that $H$ is integrable up to $L$:

*7.2. Limit circle case:* In this case $W(x,z)$ converges locally uniformly to the monodromy matrix $W(L,z)$. Therefore, the right side of (7.1) can be evaluated as

$$Q_{H/\tau}(z) := \frac{w_{11}(L,z)\tau(z) + w_{12}(L,z)}{w_{21}(L,z)\tau(z) + w_{22}(L,z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

When $\tau$ runs through $\mathcal{N}_0 \cup \{\infty\}$, the functions $Q_{H/\tau}$ parameterise the family of regularised 1-resolvents of all selfadjoint exit space extensions of the symmetric extension of $T_{\min}(H)$ given as

$$S(H) := \big\{(f;g) \in T_{\max}(H) : (1,0)f(0) = 0, f(L) = 0\big\}.$$

Thereby constant parameters correspond to canonical extensions. This is due to the interpretation of $W(L,z)$ as a resolvent matrix in the sense of M.G.Krein, which follows from [HSW00, Proposition 4.4].

If $\tau \in \mathbb{R} \cup \{\infty\}$, then set $\phi := \operatorname{Arccot} \tau$ and append an indivisible interval with angle $\phi$ and infinite length to $H$. For the resulting lpc Hamiltonian $\tilde{H}$ we have that $Q_{\tilde{H}} = Q_{H/\tau}$ is the $Q$-function of $T_{\min}(\tilde{H})$ induced by its extension

$$A_\tau(H) := \big\{(f;g) \in T_{\max}(H) : (1,0)f(0) = 0,\ \xi_\phi^* f(L) = 0\big\}. \tag{7.2}$$

$\Diamond$

Now we can define the order of a Hamiltonian, which is not necessarily in the lcc.

**7.3 Definition.** Let $H$ be a positive semidefinite Hamiltonian defined on the finite or infinite interval $[0, L)$.

If $H$ is lpc and $Q_H$ is not meromorphic throughout $\mathbb{C}$, set $\rho(H) := \infty$. Otherwise, let $(\omega_n)_{n=1,2,\ldots}$ be the sequence of non-zero poles of $Q_H$ (or $Q_{H/0}$ if $H$ is lcc) arranged according to nondecreasing modulus, and define $\rho(H)$ as the convergence exponent of $(\omega_n)_{n=1,2,\ldots}$, i.e.

$$\rho(H) := \inf\Big\{\alpha > 0 : \sum_{n=1,2,\ldots} |\omega_n|^{-\alpha} < \infty\Big\}.$$

We call $\rho(H)$ the *order of $H$*. $\Diamond$

**7.4 Lemma.** *Let $H$ be a positive semidefinite Hamiltonian in the lcc.*

*Then he poles of $Q_{H/\tau}$ and the zeroes of $w_{21}(L,z)$ are interlacing. Thus Definition* 7.3 *is consistent with Definition* 1.6.

*Proof.* The entries $w_{ij}(L,z)$ are entire functions of bounded type in both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$ and are real along the real axis. Hence, they are canonical products and their order is equal to the convergence exponent of their zeros, cf. [Lev80, Theorem 7].

Let $\tau \in \mathbb{R} \cup \{\infty\}$ be arbitrary. Next we show that the poles of $Q_{H/\tau}$, i.e. the zeros of $f(z) := w_{21}(L,z)\tau + w_{22}(L,z)$, are interlacing with the zeroes of $g(z) := w_{21}(L,z)$:

The entire functions $f(z)$ and $g(z)$ are not identically zero and have no common zeros, because of $\det W(x,z) = 1$. Moreover,

$$\frac{f(z)}{g(z)} = \frac{w_{21}(L,z)\tau + w_{22}(L,z)}{w_{21}(L,z)} = \tau + \frac{w_{22}(L,z)}{w_{21}(L,z)}$$

is a Nevanlinna function, as we already mentioned in section 1.2. By [BP95, Corollary 1.2], the zeros of $f(z)$ and $g(z)$ are real, simple and interlace.

Hence the corresponding convergence exponents coincide, which means that both definitions are equivalent in the lcc. ❑

We use an operator theoretic interpretation of $\rho(H)$. For $p > 0$ denote by $\mathfrak{S}_p$ the Schatten-von Neumann ideal of all compact operators whose sequence of $s$-numbers belongs to $\ell^p$, see, e.g., [GK69].

*7.5 Remark.* Let $H$ be a positive semidefinite Hamiltonian which is either lcc or lpc with $Q_H$ meromorphic throughout $\mathbb{C}$. Then the spectrum of $A(H)$ (or $A_\tau(H)$, respectively) coincides with the set of poles of $Q_H$ (or $Q_{H/\tau}$, respectively). Therefore $A(H)$ (or $A_\tau(H)$, respectively) has compact resolvents and, for arbitrary $z$ in the resolvent set of the operator

$$\rho(H) = \inf\big\{p > 0 : (A(H) - z)^{-1} \in \mathfrak{S}_p\big\},$$

or $\rho(H) = \inf\big\{p > 0 : (A_\tau(H) - z)^{-1} \in \mathfrak{S}_p\big\}$, respectively.

Assume now that $H$ is lpc with $Q_H$ meromorphic throughout $\mathbb{C}$. There exists a unique canonical selfadjoint extension of $T_{\min}(H)$ having 0 in its spectrum, and hence as an eigenvalue. This means that there exists some constant $\xi_{\phi(H)}$ belonging to $L^2(H)$. Since we are in lpc, the angle $\phi(H)$ is uniquely determined (modulo $\pi$). It is related to $Q_H$ by

$$Q_H(0) = -\tan\phi(H), \tag{7.3}$$

cf. [HSW00, Theorem 2.1(2.8)]. $\lozenge$

In our present considerations, we employ the following result which is interesting in its own right.

**7.6 Proposition.** *Let $H : [0, L) \to \mathbb{R}^{2\times 2}$ be a positive semidefinite Hamiltonian in lpc such that $(0, L)$ is not indivisible. Assume that $Q_H$ is meromorphic in $\mathbb{C}$, $Q_H(0) = 0$, and $\sum_n \frac{1}{|\omega_n|} < \infty$, where $(\omega_n)_{n=1,2,\ldots}$ is the sequence of poles of $Q_H$ arranged according to nondecreasing modulus. Then the following statements hold.*

*(i) Denote by $J$ the set of all points $x \in (0, L)$ such that $x$ is not inner point of an indivisible interval and $(0, x)$ is not indivisible. The limits*

$$b(z) := \lim_{\substack{x\to\sup J\\ x\in J}} w_{12}(x, z), \quad d(z) := \lim_{\substack{x\to\sup J\\ x\in J}} w_{22}(x, z),$$

*exist locally uniformly on $\mathbb{C}$.*

*(ii) The functions $b$ and $d$ are real along the real axis, have no common zeroes, are of Polya class and of order $\rho(H)$ (with zero type if $\rho(H) = 1$).*

*(iii) For each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\forall\, x \in J : \quad |w_{ij}(x, z)| \le C_\varepsilon \exp\big(|z|^{\rho(H)+\varepsilon}\big),\ z \in \mathbb{C}, (i, j) \in \{(1, 2), (2, 2)\}.$$

In the proof we exploit the connection Remark 7.5 and use a standard estimate for canonical products[1].

[1] Probably an alternative proof could proceed using [Bra68, Theorem 41, Problem 154] and the "reversing direction transformation" [KW11, Definition 2.6]. However, we did not try to work out details of this approach since we believe that the operator theoretic argument is simple and elegant.

*Proof.* For $x \in J$ we can consider $L^2(H|_{[0,x]})$ as a subspace of $L^2(H)$, namely by identifying a function $f$ from $L^2(H_{[0,x]})$ with its extension $\hat{f}$ defined by $\hat{f}(y) = 0$, $y \in (x, L)$. We shall always tacitly apply this identification.

Set $P_x : f \mapsto \mathbb{1}_{[0,x]} f$, where $\mathbb{1}_{[0,x]}$ denotes the indicator function of the interval $[0, x]$. Then $P_x$ is the orthogonal projection of $L^2(H)$ onto $L^2(H|_{[0,x]})$. Moreover, set

$$T := A(H)^{-1}, \qquad T_x := A_0(H|_{[0,x]})^{-1},\ x \in J.$$

Note here that $0 \in \rho(A(H))$ by assumption and $0 \in \rho(A_0(H|_{[0,x]}))$ by the boundary condition in the definition (7.2). The spectrum of $T$ equals $(\omega_n^{-1})_{n=1,2,\dots}$ with all eigenvalues being simple. Hence, $T \in \mathfrak{S}_1$.

The crucial observation is that

$$T_x = P_x T|_{\operatorname{ran} P_x}, \quad x \in J.$$

To see this, let $g \in \operatorname{ran} P_x$ be given and set $f := Tg$. Then $f'(x) = JH(x)g(x)$, $x \in [0, L)$ a.e., and $(1, 0)f(0) = 0$. Since $g(y) = 0$, $y \in (x, L)$, the function $f|_{[x,L)}$ is constant. It follows from (7.3) that $f|_{[x,L)} \in \operatorname{span}\{\binom{1}{0}\}$, which implies $(0, 1)f(x) = 0$. We see that $T_x g = \mathbb{1}_{[0,x]} f = P_x T g$.

We proceed with establishing the required properties of the right lower entries $w_{22}(x, z)$. Since $P_x \to I$ in the strong operator topology when $x \nearrow \sup J$ and $T \in \mathfrak{S}_1$, we have $P_x T P_x \to T$ in the norm of $\mathfrak{S}_1$. This implies that

$$\lim_{\substack{x \to \sup J \\ x \in J}} \det(I - zP_x T P_x) = \det(I - zT)$$

locally uniformly on $\mathbb{C}$. We have

$$\ker(P_x T P_x - \lambda) = \ker(P_x T|_{\operatorname{ran} P_x} - \lambda), \quad \lambda \neq 0,$$

and hence $\det(I - zP_x T P_x) = \det(I - zP_x T|_{\operatorname{ran} P_x}) = \det(I - zT_x)$.

Let $\omega_n(x)$ be the zeroes of $w_{22}(x, \cdot)$ arranged according to nondecreasing modulus. The spectrum of $T_x$ equals $\{1/\omega_1(x), 1/\omega_2(x), \dots\}$, and all eigenvalues of $T_x$ are simple. Using that $w_{22}(x, \cdot)$ is of bounded type in $\mathbb{C}^+$ and real along the real axis we obtain

$$w_{22}(x, z) = \prod_n \Big(1 - \frac{z}{\omega_n(x)}\Big) = \det(I - zT_x) = \det(I - zP_x T P_x).$$

Thus the limit in $(i)$ exists, in fact $d(z) = \det(I - zT)$. Since $\det(I - zT) = \prod_n \big(1 - \frac{z}{\omega_n}\big)$, the properties of $d$ listed in $(ii)$ follow.

For the proof of the uniform estimate in $(iii)$ consider the counting functions

$$n(x, r) := \#\{n : |\omega_n(x)| \le r\}, \quad n(r) := \#\{n : |\omega_n| \le r\}.$$

Denote by $s_n(\cdot)$ the $n$-th $s$-number of an operator, then

$$|\omega_n(x)|^{-1} = s_n(T_x) = s_n\big(P_x T|_{\operatorname{ran} P_x}\big) = s_n(P_x T P_x) \le s_n(T) = |\omega_n|^{-1},$$

whence $n(x,r) \leq n(r)$, $x \in J$, $r > 0$. Using [Lev80, Lemma I.4.3] we obtain the required bound.

We turn to the function $w_{12}(x,z)$. Let $\tilde{\omega}_n(x)$ be the nonzero zeroes of $w_{12}(x,\cdot)$ arranged according to nondecreasing modulus, and let $\tilde{n}(x,r)$ be the counting function for $\tilde{\omega}_1(x), \tilde{\omega}_2(x), \ldots$. Since the zeroes of $w_{12}(x,\cdot)$ interlace with the zeroes of $w_{22}(x,\cdot)$ and $w_{12}(x,0) = 0$, we have

$$\tilde{n}(x,r) \leq n(x,r) \leq n(r), \quad x \in J, r > 0.$$

Again [Lev80, Lemma I.4.3] applies and yields a uniform estimate for the canonical product $\prod_n \big(1 - \frac{z}{\tilde{\omega}_n(x)}\big)$. The function $w_{12}(x,\cdot)$ is of bounded type in $\mathbb{C}^+$ and real along the real axis, hence admits the representation (a prime denotes differentiation with respect to $z$)

$$w_{12}(x,z) = w_{12}'(x,0) \cdot z \prod_n \Big(1 - \frac{z}{\tilde{\omega}_n(x)}\Big).$$

However,

$$w_{12}'(x,0) = \int_0^x \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* H(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} dy \leq \Big\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big\|^2_{L^2(H)} < \infty,$$

and the bound required in $(iii)$ for $w_{12}(x,\cdot)$ follows.

We have $w_{12}(x,\cdot) w_{22}(x,\cdot)^{-1} \to Q_H$ locally uniformly on $\mathbb{C} \setminus \mathbb{R}$, in particular, $w_{12}(x,z) \to d(z) Q_H(z)$ pointwise on $\mathbb{C} \setminus \mathbb{R}$. Since the functions $w_{12}(x,\cdot)$ form a normal family of entire function and $b := dQ_H$ is entire, this limit is actually assumed locally uniformly on all of $\mathbb{C}$. Using the product representation of $Q_H$ and the fact that the zeroes of $d$ are exactly the poles of $Q_H$, we obtain

$$b(z) = \Big\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big\|^2_{L^2(H)} \cdot z \cdot \prod_n \Big(1 - \frac{z}{\tilde{\omega}_n}\Big)$$

where $\tilde{\omega}_n$ denote the nonzero zeroes of $Q_H$. Thus $b$ has all the properties listed in $(ii)$. ❑

In Proposition 7.6 we assume the normalisation $Q_H(0) = 0$, equivalently, that $\phi(H) = 0$. Passing to arbitrary angles $\phi(H)$ is easily possible by performing a rotation (see, e.g., [KW11, Definition 2.4, Lemma 3.29]). Due to the (annoying) fact that different sources of literature use different normalisations, we need the corresponding result obtained after a rotation by $\pi/2$.

**7.7 Corollary.** *Assume in Proposition* 7.6 *that* $Q_H$ *has a pole at* 0 *instead of the value* 0. *Then the assertion remains true when the functions* $w_{12}(x,z)$ *and* $w_{22}(x,z)$ *are replaced by* $w_{11}(x,z)$ *and* $w_{21}(x,z)$.

*7.8 Remark.* Proposition 7.6 is a natural generalisation of the lcc.

— Assume that $H$ is lcc: The limits $w_{ij}(L,z) = \lim_{x \nearrow L} w_{ij}(x,z)$, $i,j = 1,2$, exist, and the functions $w_{ij}(L,z)$ are real along the real axis and have no common zeroes.

— Assume in addition that $\det H = 0$ a.e.: The functions $w_{ij}(L,z)$, $i,j = 1,2$, are of Polya class and of order $\rho(H)$ (with zero type if $\rho(H) = 1$).

— Uniform estimate: For each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\forall\, x \in [0,L]: \quad |w_{ij}(x,z)| \leq C_\varepsilon \exp\left(|z|^{\rho(H)+\varepsilon}\right),\ z \in \mathbb{C},\ i,j = 1,2.$$

To see the uniform estimate just append an indivisible interval of infinite length and type $\pi/2$ (type 0 for the first column), and apply Proposition 7.6 (Corollary 7.7 for the first column)[2]. ◊

## 7.2. Signed Hamburger Hamiltonians

For an equation (1.6) where $H$ is not anymore positive semidefinite, no comprehensive theory corresponding to what we mentioned in 7.1 and 7.2 is known. Generalisations to some particular indefinite situations have been undertaken in [KL79; KL80; KL85], [Fle96], [LW98], [KW06; KW11; KW10]. Except for [Fle96] all papers deal with a Pontryagin space situation (i.e., finite negative index).

We deal with a class of possibly indefinite Hamiltonians having the very simple form analogous to Hamburger Hamiltonians.

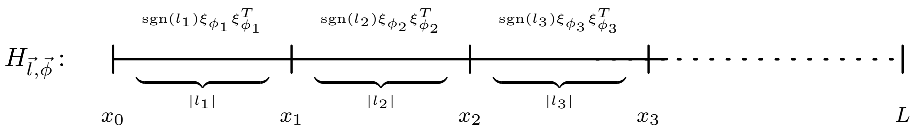
**7.9 Definition.** Let $\vec{l} = (l_n)_{n=1}^\infty$ and $\vec{\phi} = (\phi_n)_{n=1}^\infty$ be sequences of real numbers with $l_n \neq 0$ and $\phi_{n+1} \not\equiv \phi_n \mod \pi$, $n \in \mathbb{N}$, and set

$$x_0 := 0, \qquad x_n := \sum_{k=1}^{n} |l_k|,\ n \in \mathbb{N}, \qquad L := \sum_{k=1}^{\infty} |l_k| \in (0,\infty]. \tag{7.4}$$

Then we call the function $H_{\vec{l},\vec{\phi}} : [0,L) \to \mathbb{R}^{2\times 2}$ defined as

$$H_{\vec{l},\vec{\phi}}(x) := \operatorname{sgn}(l_n)\xi_{\phi_n}\xi_{\phi_n}^T, \quad x \in [x_{n-1}, x_n),\ n \in \mathbb{N},$$

the *signed Hamburger Hamiltonian* with *lengths* $\vec{l}$ and *angles* $\vec{\phi}$. The points $x_n$ are called the *nodes* of $H_{\vec{l},\vec{\phi}}$.




◊

A signed Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$ is a.e. positive semidefinite if and only if all lengths $l_n$ are positive. A signed Hamburger Hamiltonian is associated with an indefinite power moment problem as in [KL79; KL80] if and only if all but finitely many lengths are positive.

[2]A direct proof can be given repeating some of the arguments from the proof of Proposition 7.6.

*7.10 Remark.* The fact mentioned above that a fundamental solution exists and is entire, depends only on local integrability of $H$ and not on definiteness properties.

For a signed Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$ the fundamental solution $W_{\vec{l},\vec{\phi}}$ can easily be computed explicitly. Denote

$$w_\phi(x,z) := \begin{pmatrix} 1 - xz\sin\phi\cos\phi & xz\cos^2\phi \\ -xz\sin^2\phi & 1 + xz\sin\phi\cos\phi \end{pmatrix} = I - zx\xi_\phi\xi_\phi^* J, \qquad x \in \mathbb{R},\ z \in \mathbb{C},\ \phi \in \mathbb{R},$$

then

$$W_{\vec{l},\vec{\phi}}(x,z) = w_{\phi_1}(l_1,z) \cdot \ldots \cdot w_{\phi_{n-1}}(l_{n-1},z) \cdot w_{\phi_n}\big(\operatorname{sgn}(l_n)(x - x_{n-1}), z\big), \qquad x \in [x_{n-1}, x_n),\ n \in \mathbb{N}.$$

$\Diamond$

Analogously to the positive definite case, we call a signed Hamburger Hamiltonian *diagonal* if $H$ is almost everywhere a diagonal matrix, which is the case if and only if $\phi_n \in \{0, \frac{\pi}{2}\}$ (modulo $\pi$), $n \in \mathbb{N}$. Diagonal Hamiltonians (in the positive semidefinite situation) are in many ways easier to treat and a variety of symmetry properties is present, see, e.g., [Bra68, Chapter 47].

## Square-root and square transform

The *Stieltjes class* $\mathcal{S}$ is the subclass of $\mathcal{N}_0$ consisting of all Nevanlinna functions $Q$ which are analytic in $\mathbb{C} \setminus [0,\infty)$ and satisfy $Q(x) \geq 0$, $x \in (-\infty, 0)$. If $Q \in \mathcal{S}$, then the function $Q_d(z) := zQ(z^2)$ also belongs to the Nevanlinna class, cf. [KK68, Lemma S1.5.1]. Hence, for $Q \in \mathcal{S}$, de Branges' inverse theorem gives two positive semidefinite Hamiltonians $H$ and $H_d$, namely those having $Q$ and $Q_d$ as corresponding Titchmarsh-Weyl coefficients. Since $Q_d(-z) = -Q_d(z)$, $H_d$ is a diagonal Hamiltonian. These two Hamiltonians can be transformed into each other by explicit formulae, see, e.g., [KWW07]. We speak of the *square-root transform* turning $H_d$ into $H$, and its converse, the *square transform.* These transformations can also be carried out on the level of fundamental solutions. A systematic discussion on this level including certain indefinite cases is given in [KWW06].

For a positive semidefinite Hamburger Hamiltonian the mentioned transformations are established by explicit algebraic formulae. We use the same formulae to define corresponding transforms for signed Hamburger Hamiltonians.

First, let us introduce a practical abbreviation: for two sequences of real numbers $\vec{x} = (x_n)_{n=1}^\infty$ and $\vec{y} = (y_n)_{n=1}^\infty$, we denote by $\vec{x} : \vec{y}$ the mixed sequence

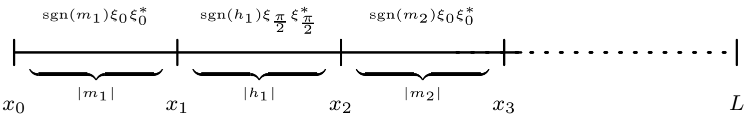
$$\vec{x} : \vec{y} \ := \ (x_1, y_1, x_2, y_2, x_3, \ldots).$$

Moreover, we set for the rest of the chapter

$$\vec{\delta} := \big(0, \frac{\pi}{2}, 0, \frac{\pi}{2}, 0, \ldots\big).$$

**7.11 Definition.** Let $H$ be a diagonal signed Hamburger Hamiltonian, and assume (for normalisation) that its first angle is equal to 0. Denote by $\vec{m}$ and $\vec{h}$ the sequences of odd and even lengths of $H$, respectively. That just means that we write $H$ in the form $H = H_{\vec{m}:\vec{h},\vec{\delta}}$:

$H = H_{\vec{m}:\vec{h},\vec{\delta}}$ :




Set

$$l_n := h_n\Big(1 + \Big(\sum_{k=1}^{n} m_k\Big)^2\Big),\ \phi_n := \operatorname{Arccot}\Big(\sum_{k=1}^{n} m_k\Big), \quad n \in \mathbb{N}. \tag{7.5}$$

Then we call $H_{\vec{l},\vec{\phi}}$ the *square-root transform* of $H$. $\Diamond$

The converse transformation is obtained by simply inverting the relations (7.5).

**7.12 Definition.** Let $H_{\vec{l},\vec{\phi}}$ be a signed Hamburger Hamiltonian, and assume that $\phi_n \not\equiv 0 \mod \pi$, $n \in \mathbb{N}$. Set (with $\phi_0 := \frac{\pi}{2}$)

$$m_n := \cot(\phi_n) - \cot(\phi_{n-1}),\ h_n := l_n \sin^2(\phi_n), \quad n \in \mathbb{N}.$$

Then we call $H_{\vec{m}:\vec{h},\vec{\delta}}$ the *square transform* of $H_{\vec{l},\vec{\phi}}$. $\Diamond$

Inductively applying the computation [KWW06, Proposition 3.6(i)] yields the following fact.

**7.13 Lemma.** *Let $H_{\vec{l},\vec{\phi}}$ be a signed Hamburger Hamiltonian with $\phi_n \not\equiv 0 \mod \pi$, $n \in \mathbb{N}$, and let $H_d$ be its square transform. Denote by $W_{\vec{l},\vec{\phi}}(x,z)$ and $W_d(y,z)$ the corresponding fundamental solutions, and let $x_n$ and $y_n$ be the nodes of $H_{\vec{l},\vec{\phi}}$ and $H_d$, respectively. Then for all $n \in \mathbb{N}$ (a prime denote differentiation with respect to $z$)*

$$W_{\vec{l},\vec{\phi}}(x_n, z^2) = \begin{pmatrix} w_{d,11}(y_{2n}, z) & \frac{w_{d,12}(y_{2n},z)}{z} - w'_{d,12}(y_{2n},0)w_{d,11}(y_{2n},z) \\ zw_{d,21}(y_{2n},z) & w_{d,22}(y_{2n},z) - w'_{d,12}(y_{2n},0)zw_{d,21}(y_{2n},z) \end{pmatrix}. \tag{7.6}$$

Let us now state some immediate properties of these transformations.

*7.14 Remark.*

(*i*) The square-root transform of a diagonal signed Hamburger Hamiltonian $H_d$ is positive semidefinite if and only if all even lengths of $H_d$ are positive. The square transform of a signed Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$ is positive semidefinite if and only if $H_{\vec{l},\vec{\phi}}$ itself is positive semidefinite and the sequence of angles is monotonically decreasing when considered modulo $\pi$ as a sequence in $(0, \pi)$.

(*ii*) Assume that $H_{\vec{l},\vec{\phi}}$ and its square transform $H_d$ are both positive semidefinite. Then

$$\rho(H_d) = 2\rho(H_{\vec{l},\vec{\phi}}).$$

To see this, let $Q_d$ be the function $Q_{H_d}$ or $Q_{H_d/\infty}$ depending whether $H_d$ is lpc or lcc, and let $Q_{\vec{l},\vec{\phi}}$ be defined analogously for $H_{\vec{l},\vec{\phi}}$. Lemma 7.13 shows that $Q_d(z) = zQ_{\vec{l},\vec{\phi}}(z^2)$.

(*iii*) Assume again that $H_{\vec{l},\vec{\phi}}$ and $H_d$ are both positive semidefinite. If $H_{\vec{l},\vec{\phi}}$ is lcc and $H_d$ is lpc, then $\phi(H_d) = \frac{\pi}{2}$. This follows since (denote $\hat{L} := \sum_{n=1}^{\infty}(m_n + h_n)$)

$$\int_0^{\hat{L}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* H_d(y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} dy = \sum_{n=1}^{\infty} h_n \leq \sum_{n=1}^{\infty} l_n < \infty.$$

$\Diamond$

## The modulus transform

A signed Hamburger Hamiltonian can be transformed into a positive semidefinite one simply by taking absolute values of its lengths.

For a sequence $\vec{l}$ of real numbers, we denote

$$|\vec{l}| := (|l_n|)_{n=1}^{\infty}.$$

**7.15 Definition.** Let $H_{\vec{l},\vec{\phi}}$ be a signed Hamburger Hamiltonian. Then we call $H_{|\vec{l}|,\vec{\phi}}$ the *modulus transform* of $H_{\vec{l},\vec{\phi}}$. $\Diamond$

The next result shows that the fundamental solution of a diagonal signed Hamburger Hamiltonian can be estimated from above by the fundamental solution of its modulus transform. Recall that $\vec{\delta} = (0, \pi/2, 0, \pi/2, 0, \ldots)$.

**7.16 Proposition.** *Let $\vec{l}$ be a sequence of nonzero real numbers, and consider the Hamburger Hamiltonians $H_{\vec{l},\vec{\delta}}$ and $H_{|\vec{l}|,\vec{\delta}}$ with corresponding fundamental solutions $W_{\vec{l},\vec{\delta}}$ and $W_{|\vec{l}|,\vec{\delta}}$, respectively. Then (note that the sequences $(x_n)_{n=1}^{\infty}$ defined in* (7.4) *for $\vec{l}$ and $|\vec{l}|$ coincide)*

$$\left|(1,0)W_{\vec{l},\vec{\delta}}(x_{2n}, z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right| \leq (1,0)W_{|\vec{l}|,\vec{\delta}}(x_{2n}, i|z|)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n \in \mathbb{N}, z \in \mathbb{C}. \tag{7.7}$$

The proof follows from a purely algebraic and explicit formula for (the first row of) the fundamental solution of a diagonal signed Hamburger Hamiltonian. We define for each $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$ polynomials $a_{n,k}$ and $b_{n,k}$ in variables $v_1, v_2, \ldots$ by the recursions

$$a_{1,0}(\vec{v}) := 1, \qquad\qquad a_{1,1}(\vec{v}) := v_1 v_2,$$
$$b_{1,0}(\vec{v}) := v_1, \qquad\qquad b_{1,1}(\vec{v}) := 0,$$
$$a_{n+1,k}(\vec{v}) := \begin{cases} 1 & , \quad k = 0, \\ a_{n,k}(\vec{v}) + v_{2n+1}v_{2n+2}a_{n,k-1}(\vec{v}) + v_{2n+2}b_{n,k-1}(\vec{v}) \, , & k = 1, \dots, n, \\ v_{2n+1}v_{2n+2}a_{n,n}(\vec{v}) & , \quad k = n+1, \end{cases}$$
$$b_{n+1,k}(\vec{v}) := \begin{cases} b_{n,k}(\vec{v}) + v_{2n+1}a_{n,k}(\vec{v}) \, , & k = 0, \dots, n, \\ 0 & , \quad k = n+1. \end{cases}$$

Observe that $a_{n,k}$ and $b_{n,k}$ have nonnegative integer coefficients. The polynomial $a_{n,k}$ involves only the variables $v_1, \dots, v_{2n}$, and $b_{n,k}$ only the variables $v_1, \dots, v_{2n-1}$. Moreover,

$$a_{n,0}(\vec{v}) = 1, \; a_{n,n}(\vec{v}) = \prod_{k=1}^{2n} v_k, \quad b_{n,0}(\vec{v}) = \sum_{k=1}^{n} v_{2k-1}, \; b_{n,n}(\vec{v}) = 0,$$

for all $n \in \mathbb{N}$.

**7.17 Lemma.** *Let $\vec{l}$ be a sequence of nonzero real numbers, and let $W_{\vec{l},\vec{\delta}}$ be the fundamental solution of $H_{\vec{l},\vec{\delta}}$. Then*

$$(1,0)W_{\vec{l},\vec{\delta}}(x_{2n}, z) = \sum_{k=0}^{n} \Big(\frac{z}{i}\Big)^{2k} \big(a_{n,k}(\vec{l}), z b_{n,k}(\vec{l})\big), \quad n \in \mathbb{N}. \tag{7.8}$$

*Proof.* We use induction on $n$ where the computation is based on the formula

$$w_0(l,z) w_{\frac{\pi}{2}}(h,z) = \begin{pmatrix} 1 + \big(\frac{z}{i}\big)^2 lh & zl \\ -zh & 1 \end{pmatrix}. \tag{7.9}$$

For $n = 1$ this formula already establishes the required representation of $W_{\vec{l},\delta}(x_2, z)$. Assume (7.8) holds for some $n \in \mathbb{N}$. Then (7.9) yields

$$\begin{aligned}
(1,0)W_{\vec{l},\vec{\delta}}(x_{2n+2}, z) &= (1,0)W_{\vec{l},\vec{\delta}}(x_{2n}, z) \begin{pmatrix} 1 + \big(\frac{z}{i}\big)^2 l_{2n+1}l_{2n+2} & z l_{2n+1} \\ -z l_{2n+2} & 1 \end{pmatrix} \\
&= \sum_{k=0}^{n} \Big(\frac{z}{i}\Big)^{2k} \big(a_{n,k}(\vec{l}) + (\tfrac{z}{i})^2 l_{2n+1}l_{2n+2}a_{n,k}(\vec{l}) - z^2 l_{2n+2} b_{n,k}(\vec{l}), \\
&\qquad\qquad z l_{2n+1} a_{n,k}(\vec{l}) + z b_{n,k}(\vec{l})\big) \\
&= \sum_{k=0}^{n} \Big(\frac{z}{i}\Big)^{2k} \big(a_{n,k}(\vec{l}), z[l_{2n+1}a_{n,k}(\vec{l}) + b_{n,k}(\vec{l})]\big) \\
&\quad + \sum_{k=1}^{n+1} \Big(\frac{z}{i}\Big)^{2k} \big(l_{2n+1}l_{2n+2}a_{n,k-1}(\vec{l}) + l_{2n+2}b_{n,k-1}(\vec{l}), 0\big) \\
&= \sum_{k=0}^{n} \Big(\frac{z}{i}\Big)^{2k} \big(a_{n+1,k}(\vec{l}), z b_{n+1,k}(\vec{l})\big).
\end{aligned}$$

❑

The estimate (7.7) is now nearly obvious.

*Proof of Proposition* 7.16. We use the representation from Lemma 7.17 and the fact that the polynomials $a_{n,k}$ have nonnegative coefficients to estimate

$$\Big|(1,0)W_{\vec{l},\vec{\delta}}(x_{2n},z)\begin{pmatrix}1\\0\end{pmatrix}\Big| \leq \sum_{k=0}^{n}|z|^{2k}\big|a_{n,k}(\vec{l})\big|$$
$$\leq \sum_{k=0}^{n}|z|^{2k}a_{n,k}(|\vec{l}|) = (1,0)W_{|\vec{l}|,\vec{\delta}}(x_{2n},i|z|)\begin{pmatrix}1\\0\end{pmatrix}.$$

❑

## 7.3. An estimate for order

### 7.3.1. Formulation and proof of our two main theorems

The next statement is the first main theorem of this chapter. In order to keep the notation as clean as possible, we assume that $\phi_n \not\equiv 0 \bmod \pi$ for all $n \in \mathbb{N}$. Note that any Hamburger Hamiltonian can be transformed into one with nonzero angles by adding a certain constant offset to the angles, i.e., by performing a rotation as discussed before the statement of Corollary 7.7. The form of the rotation transformation [KW11, Definition 2.4] ensures $\rho(H_{\vec{l},(\phi_n+\alpha)}) = \rho(H_{\vec{l},\vec{\phi}})$.

**7.18 Theorem.** *Let $H_{\vec{l},\vec{\phi}}$ be a positive semidefinite Hamburger Hamiltonian in lcc, and assume that $\phi_n \not\equiv 0 \bmod \pi$, $n \in \mathbb{N}$. Set (with $\phi_0 := \frac{\pi}{2}$)*

$$m_n := \cot(\phi_n) - \cot(\phi_{n-1}),\ \ h_n := l_n \sin^2(\phi_n), \quad n \in \mathbb{N},$$
$$\vec{\delta} := \Big(0, \frac{\pi}{2}, 0, \frac{\pi}{2}, \dots\Big), \qquad |\vec{m}:\vec{h}| := \big(|m_1|, |h_1|, |m_2|, |h_2|, \dots\big).$$

*Then*

$$\rho(H_{\vec{l},\vec{\phi}}) \leq \frac{1}{2}\rho\big(H_{|\vec{m}:\vec{h}|,\vec{\delta}}\big).$$

The main point here is that the Hamiltonian appearing on the right side is *diagonal.* This implies that $\rho(H_{|\vec{m}:\vec{h}|,\vec{\delta}})$ can *in principle* be determined using R.Romanov's Theorem 2 for the order of a diagonal Hamiltonian, cf. Theorem 4.2, or using Kac's formula [Kac86b, Theorems A–C] for the order of a string (unfortunately, a quite bulky expression).

*Proof of Theorem* 7.18. Starting from $H := H_{\vec{l},\vec{\phi}}$ build the following successive transforms:

— $H_d = H_{\vec{m}:\vec{h},\vec{\delta}}$ is the square transform of $H$;

— $H_d^+ = H_{|\vec{m}:\vec{h}|,\vec{\delta}}$ is the modulus transform of $H_d$;

— $H^+$ is the square-root transform of $H_d^+$.

The Hamiltonian $H_d$ will in general carry signs, whereas $H_d^+$ and $H^+$ are positive semi-definite, cf. Remark 7.14, $(i)$.

Denote by $x_n$ the nodes of $H$, by $y_n$ the common nodes of $H_d$ and $H_d^+$, and by $x_n^+$ the nodes of $H^+$. Denote by $W, W_d, W_d^+, W^+$ the fundamental solutions of the respective Hamiltonian $H, H_d, H_d^+, H^+$, let $Q^+$ be either the function $Q_{H^+/\infty}$ if $H^+$ is in the lcc or the Titchmarsh-Weyl coefficient $Q_{H^+}$ if $H^+$ is in the lpc, and let $Q_d^+$ be defined in the same way for $H_d^+$, respectively. Then $Q_d^+(z) = zQ^+(z^2)$ and $\rho(H_d^+) = 2\rho(H^+)$, cf. Remark 7.14, $(ii)$.

Hence the assertion of the theorem is equivalent to $\rho(H) \leq \rho(H^+)$. This is trivially true when $\rho(H^+) \geq 1$. Hence, assume throughout the following that $\rho(H^+) < 1$. In particular, $Q^+$ is meromorphic throughout the plane, and the sequence $(\omega_n)_{n=1,2,\dots}$ of its nonzero poles satisfies $\sum_n \frac{1}{|\omega_n|} < \infty$.

If $H_d^+$ is in the lcc, then the function $Q_d^+$ has a pole at 0 by its definition. If $H_d^+$ is in the lpc, then we have (denoting $\hat{L} := \sum_{n=1}^\infty (m_n + h_n)$)

$$\int_0^{\hat{L}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* H_d^+(y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} dy = \sum_{n=1}^\infty h_n \leq \sum_{n=1}^\infty l_n < \infty,$$

i.e. $\binom{0}{1} \in L^2(H_d^+)$. Again it follows that $Q_d^+$ has a pole at 0. From the relation $Q_d^+(z) = zQ^+(z^2)$ we see that also $Q^+$ has a pole at 0.

Corollary 7.7 and Remark 7.8 provide estimates ($\varepsilon > 0$ arbitrary)

$$\left|w_{11}^+(x_n^+, z)\right| \leq C_\varepsilon \exp\left(|z|^{\rho(H^+)+\varepsilon}\right), \quad n \geq 2, z \in \mathbb{C},$$

and (7.6) and Proposition 7.16 yield

$$\begin{aligned} |w_{11}(x_n, z^2)| &= |w_{d,11}(y_{2n}, z)| \leq w_{d,11}^+(y_{2n}, i|z|) \\ &= w_{11}^+(x_n^+, -|z|^2) \leq C_\varepsilon \exp\left(|z^2|^{\rho(H^+)+\varepsilon}\right), \quad n \geq 2, z \in \mathbb{C}. \end{aligned}$$

Passing to the limit $n \to \infty$ in the leftmost term, which is possible since $H$ is lcc, we obtain that the same estimates hold for $w_{11}(L, z^2)$. We conclude that the order of $w_{11}(L, \cdot)$, which equals $\rho(H)$, does not exceed $\rho(H^+)$. ❑

For the case of a Stieltjes string (translated to the language of Hamiltonians this means for a diagonal Hamburger Hamiltonian) Kac' formula [Kac86b, Theorems A–C] takes the form [Kac90, p.31 (15)]. Still, a complicated expression which hardly allows explicit evaluation. Under some regularity assumptions on the involved data, however, it was shown in [Kac90] that it can be handled. We recall this result in the language of Hamiltonians. The following statement is the direct translation of [Kac90, Theorem 1].

**7.19 Theorem** ([Kac90], Theorem 1)**.** *Let $\vec{M} = (M_n)_{n=1}^\infty$ and $\vec{L} = (L_n)_{n=1}^\infty$ be sequences of positive real numbers such that $\vec{M}$ is nonincreasing and $\vec{L}$ is nondecreasing. Set*

$$\vec{M} : \vec{L} := \left(M_1, L_1, M_2, L_2, \dots\right), \qquad \vec{\Delta} := \left(\frac{\pi}{2}, 0, \frac{\pi}{2}, 0, \dots\right),$$

*and consider the positive definite diagonal Hamburger Hamiltonian $H_{\vec{M}:\vec{L},\vec{\Delta}}$. Then the following statements hold.*

(i) *If $\alpha \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty}(L_n M_{n+1})^{\alpha} < \infty$, then $\rho(H_{\vec{M}:\vec{L},\vec{\Delta}}) \leq 2\alpha$.*

(ii) *If $\sum_{n=1}^{\infty}(L_n M_{n+1})^{\frac{1}{2}} \ln n < \infty$, then $\rho(H_{\vec{M}:\vec{L},\vec{\Delta}}) \leq 1$.*

(iii) *If $\alpha \in (\frac{1}{2}, 1)$ and $\sum_{n=1}^{\infty}(L_n M_{n+1})^{\alpha} n^{2\alpha-1} < \infty$, then $\rho(H_{\vec{M}:\vec{L},\vec{\Delta}}) \leq 2\alpha$.*

*Proof.* The Hamiltonian $H_{\vec{M}:\vec{L},\vec{\Delta}}$ is related to the Stieltjes string with masses $(M_{n+1})_{n=0}^{\infty}$ and lengths $(L_n)_{n=1}^{\infty}$, cf. [KWW07, (4.4),(4.6)].

With the notation from [Kac90], this string is an element of $\mathcal{S}_{\alpha}$, by definition, if $\rho(H_{\vec{M}:\vec{L},\vec{\Delta}}) \leq 2\alpha$. The statement follows from [Kac90, Theorem 1]. ❑

Concerning [Kac90, Theorem 1] one word of caution is in order. This statement contains the a priori assumption that the string under consideration is of trace class, i.e. that

$$\sum_{n=1}^{\infty}\left(\sum_{k=n+1}^{\infty} M_k\right) L_n < \infty$$

or, equivalently, $\sum_{n=1}^{\infty}\big(\sum_{k=1}^{n} L_k\big) M_{n+1} < \infty$. It is said without a proof on p.31 right after Theorem 2 that this assumption is superfluous: convergence of this series can be deduced from convergence of the respective series in (*i*), (*ii*), or (*iii*). In the next result – which is the second main theorem of this chapter – we use this fact for the cases (*i*) and (*ii*). Let us give a proof for these cases.

**7.20 Lemma.** *Let $\vec{M} = (M_n)_{n=1}^{\infty}$ and $\vec{L} = (L_n)_{n=1}^{\infty}$ be sequences of positive real numbers such that $\vec{M}$ is nonincreasing and $\vec{L}$ is nondecreasing, and let $\alpha \in (0, \frac{1}{2}]$. If*

$$\sum_{n=1}^{\infty}(L_n M_{n+1})^{\alpha} < \infty,$$

*then also*

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n} L_k\Big) M_{n+1} < \infty.$$

*Proof.* Set $r_n := \frac{1}{M_{n+1}}$, $n \in \mathbb{N}$, then $r_n$ is positive, nondecreasing and unbounded. Let $\mu$ be the positive measure ($\delta_r$ denotes the unit point mass at $r$)

$$\mu := \sum_{n=1}^{\infty} L_n \delta_{r_n},$$

and choose a decreasing $C^{\infty}$-function $f : [0,\infty) \to (0,\infty)$ with $f(r_n) = L_n^{\alpha-1}$, $n \in \mathbb{N}$. We have

$$\int_0^{\infty} t^{-\alpha} f(t)\, d\mu(t) = \sum_{n=1}^{\infty} r_n^{-\alpha} f(r_n) L_n = \sum_{n=1}^{\infty}\Big(\frac{L_n}{r_n}\Big)^{\alpha} < \infty.$$

Integrating by parts yields that for each $T > 0$

$$\int_0^T t^{-\alpha} f(t)\, d\mu(t) = T^{-\alpha} f(T)\mu([0,T]) - \int_0^T \underbrace{\frac{d}{dt}\big[t^{-\alpha} f(t)\big]}_{<0} \cdot\mu([0,t])\, dt$$
$$\geq T^{-\alpha} f(T)\mu([0,T]),$$

and, choosing $T = r_n$, we obtain the estimate

$$1 \gtrsim r_n^{-\alpha} f(r_n)\mu([0,r_n]) = r_n^{-\alpha} L_n^{\alpha-1}\Big(\sum_{k=1}^{n} L_k\Big).$$

Since $1-\alpha \geq \alpha$ and $\frac{L_n}{r_n} \leq 1$ for large $n$, it follows that

$$\frac{1}{r_n}\Big(\sum_{k=1}^{n} L_k\Big) \lesssim \Big(\frac{L_n}{r_n}\Big)^{1-\alpha} \lesssim \Big(\frac{L_n}{r_n}\Big)^{\alpha}.$$

❑

Combining Theorem 7.18 with Theorem 7.19 leads to the following corollary.

**7.21 Corollary.** *Let $H_{\vec{l},\vec{\phi}}$ be a positive semidefinite Hamburger Hamiltonian in lcc, and let notation $\vec{m}, \vec{h}$, etc. be as in Theorem 7.18. Assume that $|\vec{m}|$ is nondecreasing and $\vec{h}$ is nonincreasing. Then the following statements hold.*

*(i) If $\alpha \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty}(h_n|m_n|)^{\alpha} < \infty$, then $\rho(H_{\vec{l},\vec{\phi}}) \leq \alpha$.*

*(ii) If $\sum_{n=1}^{\infty}(h_n|m_n|)^{\frac{1}{2}} \ln n < \infty$, then $\rho(H_{\vec{l},\vec{\phi}}) \leq \frac{1}{2}$.*

*(iii) If $\alpha \in (\frac{1}{2}, 1)$ and $\sum_{n=1}^{\infty}(h_n|m_n|)^{\alpha} n^{2\alpha-1} < \infty$, then $\rho(H_{\vec{l},\vec{\phi}}) \leq \alpha$.*

*Proof.* Theorem 7.18 gives

$$\rho(H_{\vec{l},\vec{\phi}}) \leq \frac{1}{2}\rho(H_{|\vec{m}:\vec{h}|,\vec{\delta}}).$$

Set $\vec{m_1} := (m_{n+1})_{n=1}^{\infty}$. Removing the first interval of a Hamburger Hamiltonian does not change the order, i.e. $\rho(H_{|\vec{m}:\vec{h}|,\vec{\delta}}) = \rho(H_{|\vec{h}:\vec{m}_1|,\vec{\Delta}})$. Apply Theorem 7.19. ❑

**7.22 Theorem.** *Let $H_{\vec{l},\vec{\phi}}$ be a positive semidefinite Hamburger Hamiltonian in lcc, and assume that $\phi_n \not\equiv 0 \bmod \pi$, $n \in \mathbb{N}$. Set $\phi_0 := \frac{\pi}{2}$, and assume that $(|\cot\phi_n - \cot\phi_{n-1}|)_{n=1}^{\infty}$ is nondecreasing and bounded, $(l_n \sin^2\phi_n)_{n=1}^{\infty}$ is nonincreasing, and*

$$\sum_{n=1}^{\infty}[l_n \sin^2\phi_n]^{\frac{1}{2}} \ln n < \infty. \tag{7.10}$$

*Then*

$$\rho(H_{\vec{l},\vec{\phi}}) = \text{c. e.}\big(([l_n \sin^2\phi_n]^{-1})_{n=1}^{\infty}\big),$$

*i.e. the order of $H_{\vec{l},\vec{\phi}}$ equals the convergence exponent of $([l_n \sin^2\phi_n]^{-1})_{n=1}^{\infty}$.*

Observe that, when $\phi_n$ perform a walk on the grid $\operatorname{Arccot}(\mathbb{Z})$, the assumption on angles is clearly satisfied.

*Proof of Theorem* 7.22. Let $\vec{m}$ and $\vec{h}$ be as in Theorem 7.18, set $M_n := \sum_{k=1}^n m_k$, and set $\gamma := \text{c. e.}\left((h_n^{-1})_{n=1}^\infty\right)$. We have to show that $\rho(H_{\vec{l},\vec{\phi}}) = \gamma$.

By our assumptions $\vec{h}$ is nonincreasing, and $|\vec{m}|$ is nondecreasing and convergent (say $m_\infty := \lim_{n\to\infty} |m_n|$) whence $|m_n| \asymp 1$ and $M_n \lesssim n$.

We start with showing $\rho(H_{\vec{l},\vec{\phi}}) \leq \gamma$. Corollary 7.21, $(ii)$, yields $\rho(H_{\vec{l},\vec{\phi}}) \leq 1/2$. If $\gamma = 1/2$ (note that by (7.10) certainly $\gamma \leq 1/2$), then we are done. For $\gamma < 1/2$, we can apply Corollary 7.21, $(i)$, to obtain the desired inequality.

To establish the other inequality, we use the lower bound established in Proposition 5.17,

$$\rho(H_{\vec{l},\vec{\phi}}) \geq \frac{1}{\delta_{l,\phi}(H_{\vec{l},\vec{\phi}})} = \limsup_{n\to\infty} \frac{-n\ln n}{\ln\left(\sqrt{l_n}\prod_{i=1}^{n-1} l_i|\sin(\phi_{i+1}-\phi_i)|\right)}.$$

To evaluate the product, remember (7.5), which yields

$$l_i|\sin(\phi_{i+1}-\phi_i)| = h_i\cdot(1+M_i^2)\,\big|\sin(\operatorname{Arccot} M_{i+1} - \operatorname{Arccot} M_i)\big|.$$

Now, $\sup_{i\in\mathbb{N}} |\operatorname{Arccot}(M_{i+1}) - \operatorname{Arccot}(M_i)| < \pi$ since $|M_{i+1}-M_i| = |m_i| \leq m_\infty$, and hence

$$\sin\big(|\operatorname{Arccot}(M_{i+1}) - \operatorname{Arccot}(M_i)|\big) \asymp |\operatorname{Arccot}(M_{i+1}) - \operatorname{Arccot}(M_i)|.$$

The mean value theorem provides $\xi_i \in (\min\{M_i, M_{i+1}\}, \max\{M_i, M_{i+1}\})$ with

$$|\operatorname{Arccot}(M_{i+1}) - \operatorname{Arccot}(M_i)| = \frac{1}{1+\xi_i^2}. \tag{7.11}$$

Since the length of the written interval is at most $m_\infty$, it follows that $1+\xi_i^2 \asymp 1+M_i^2$. All together we get $l_i|\sin(\phi_{i+1}-\phi_i)| \asymp h_i$, whence

$$\rho(H_{\vec{l},\vec{\phi}}) \geq \limsup_{n\to\infty} \frac{-n\ln n}{\ln\left(\sqrt{l_n}\prod_{i=1}^{n-1} h_i\right)} \geq \limsup_{n\to\infty} \frac{-n\ln n}{\ln\left(\sqrt{h_n}\prod_{i=1}^{n-1} h_i\right)} = \frac{1}{\delta(\vec{h},\frac{1}{2})}.$$

Since $\vec{h}$ is nonincreasing, it is regularly distributed in the sense of Definition 6.2, and Lemma 6.7, $(ii)$, gives

$$\frac{1}{\delta(\vec{h},\frac{1}{2})} = \gamma.$$

In particular, the lower estimate of $\rho(H_{\vec{l},\vec{\phi}})$ coincides with the upper estimate, i.e. $\rho(H_{\vec{l},\vec{\phi}}) = \gamma$ ❑

### 7.3.2. Relation with previous estimates

Theorem 5.10 states an upper estimate for the order of a Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$, which coincides with the order when lengths and angle-differences are regularly behaving, cf. Theorem 6.5. In Theorem 7.22 we obtained a formula for $\rho(H_{\vec{l},\vec{\phi}})$ when lengths and angles commonly behave regularly, angle-differences are never too large, and the order is at most 1/2. This theorem, however, allows that lengths and angles separately are very irregular. In this subsection we show that these two results are incomparable.

First, we show that for a large class of Hamiltonians Theorem 7.22 is applicable whereas the upper estimate Theorem 5.10 does not coincide with the order.

**7.23 Proposition.** *Let $\vec{h}$ be a nonincreasing sequence of positive real numbers which satisfies*

$$\sum_{n=1}^{\infty} h_n^{\frac{1}{2}} \ln n < \infty, \tag{7.12}$$

*and denote by $\gamma$ the convergence exponent of $(h_n^{-1})_{n=1}^{\infty}$. Let $\delta_\phi^\circ > 0$ and $\delta_l^\circ \geq 1$ be given such that*

$$\delta_\phi^\circ < \frac{1}{\gamma} - \delta_l^\circ < 2.$$

*Then there exists a sequence of angles $\vec{\phi}$ performing a walk on* $\operatorname{Arccot}(\mathbb{Z})$*, such that the Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$ with lengths $l_n := h_n \sin^{-2}\phi_n$, $n \in \mathbb{N}$, and angles $\vec{\phi}$ satisfies*

$$\rho(H_{\vec{l},\vec{\phi}}) = \delta_{l,\phi}(H_{\vec{l},\vec{\phi}})^{-1} = \gamma,$$

*as well as (recall Definitions* 5.16, 5.18, *and* 5.5*)*

$$\delta_l(H_{\vec{l},\vec{\phi}}) = \Delta_l(H_{\vec{l},\vec{\phi}}) = \delta_l^\circ, \quad \delta_\phi(H_{\vec{l},\vec{\phi}}) = \Delta_\phi(H_{\vec{l},\vec{\phi}}) = \delta_\phi^\circ, \quad \mu(H_{\vec{l},\vec{\phi}}) = \frac{1}{\gamma} - \delta_l^\circ.$$

*In particular, the upper bound for the order of $H_{\vec{l},\vec{\phi}}$ given in Theorem* 5.10 *is strictly larger than the order.*

The proof is based on the following elementary construction.

**7.24 Lemma.** *Let $\alpha \in (0,1)$. Then there exists a sequence of signs $\varepsilon_{\alpha,n} \in \{+1,-1\}$, such that the partial sums*

$$s_\alpha(n) := \sum_{i=1}^{n} \varepsilon_{\alpha,i}, \quad n \in \mathbb{N},$$

*satisfy*

$$\lim_{n\to\infty} \frac{s_\alpha(n)}{n^\alpha} = 1. \tag{7.13}$$

*Proof.* We simply make $s_\alpha(n)$ oscillating around $n^\alpha$ as close as possible: Define inductively

$$\varepsilon_{\alpha,1} := 1, \qquad \varepsilon_{\alpha,n+1} := \begin{cases} +1\,, & \frac{s_\alpha(n)}{n^\alpha} \leq 1 \\ -1\,, & \frac{s_\alpha(n)}{n^\alpha} > 1 \end{cases}$$

The sequence $\sigma_n := \frac{s_\alpha(n)}{n^\alpha}$ can be handled easily.

($i$) Monotonicity behaviour: Assume first $\sigma_n \leq 1$. Then (with appropriate $\xi_n \in (n, n+1)$)

$$\sigma_{n+1} - \sigma_n = \frac{-\sigma_n[(n+1)^\alpha - n^\alpha] + 1}{(n+1)^\alpha} = \frac{1 - \sigma_n \alpha \xi_n^{\alpha-1}}{(n+1)^\alpha} \begin{cases} \geq \frac{1-\alpha}{(n+1)^\alpha} > 0 \\ \leq \frac{1}{(n+1)^\alpha} \end{cases}$$

Second, if $\sigma_n > 1$, then $s_\alpha(n+1) < s_\alpha(n)$ and hence trivially $\sigma_{n+1} < \sigma_n$.

($ii$) Convergence: Let $n_k$ be those indices (arranged in increasing order) where $\varepsilon_{\alpha,n}$ changes its sign, i.e., where either $\sigma_n \leq 1 < \sigma_{n+1}$ or $\sigma_n > 1 \geq \sigma_{n+1}$. Note that the first of these cases occurs for all odd $k$ and the second for all even. Then $\limsup_{n\to\infty} \sigma_n = \limsup_{k\to\infty} \sigma_{n_{2k-1}+1}$ and $\liminf_{n\to\infty} \sigma_n = \liminf_{k\to\infty} \sigma_{n_{2k}+1}$. By the previous estimate,

$$\sigma_{n_{2k-1}+1} \leq \sigma_{n_{2k-1}} + \frac{1}{(n+1)^\alpha} \leq 1 + \frac{1}{(n+1)^\alpha} \to 1,$$

whence $\limsup_{k\to\infty} \sigma_n \leq 1$.

In particular, $\sigma_n \leq 2$ for large $n$. Now we estimate for all (sufficiently large) $n$ with $\sigma_n > 1$

$$\sigma_n - \sigma_{n+1} = \frac{1 + \sigma_n \alpha \xi_n^{\alpha-1}}{(n+1)^\alpha} \leq \frac{1+2\alpha}{(n+1)^\alpha}.$$

This shows that

$$\sigma_{n_{2k}+1} \geq \sigma_{n_{2k}} - \frac{1+2\alpha}{(n+1)^\alpha} \geq 1 - \frac{1+2\alpha}{(n+1)^\alpha} \to 1,$$

whence $\liminf_{k\to\infty} \sigma_n \geq 1$.

❑

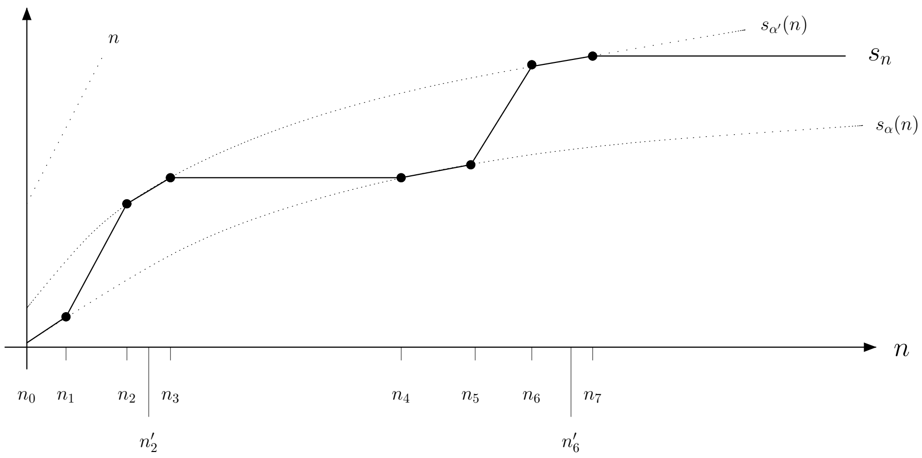
*Proof of Proposition* 7.23. For $\alpha \in (0,1)$ we have $s_\alpha(n) \asymp n^\alpha$ due to relation (7.13). From Example 5.20 we know that for $\beta \geq 0$

$$\delta(\vec{s_\alpha}, \beta) = \lim_{n\to\infty} G(n; \vec{s_\alpha}, \beta) = -\alpha.$$

Set

$$\alpha := \frac{1}{2}\delta_\phi^\circ, \quad \alpha' := \frac{1}{2}\Big(\frac{1}{\gamma} - \delta_l^\circ\Big),$$

and denote by $\vec{s_\alpha}$ and $\vec{s_{\alpha'}}$ the partial sums defined in Lemma 7.24. We construct another sequence of signs whose sequence $\vec{s} = (s_n)_{n=1}^\infty$ of partial sums alternates between $\vec{s_\alpha}$ and $\vec{s_{\alpha'}}$.




For a strictly monotonically increasing sequence of natural numbers $(n_k)_{k=0}^{\infty}$, we define the sequence $\vec{s}$ as follows (here $[x \bmod 2]$ denotes the element of $\{0,1\}$ with the same parity as $x$)

$$s_n := \begin{cases} s_\alpha(n) & , \quad 1 \le n \le n_0 \\ s_\alpha(n) & , \quad n_k < n \le n_{k+1}, \quad k \equiv 0 \bmod 4 \\ s_\alpha(n_k) + (n - n_k) & , \quad n_k < n \le n_{k+1}, \quad k \equiv 1 \bmod 4 \\ s_{\alpha'}(n) & , \quad n_k < n \le n_{k+1}, \quad k \equiv 2 \bmod 4 \\ s_{\alpha'}(n_k) + \big[(n - n_k) \bmod 2\big] & , \quad n_k < n \le n_{k+1}, \quad k \equiv 3 \bmod 4 \end{cases}$$

The switching indices $(n_k)_{k=0}^{\infty}$ will be constructed inductively.

To start with, choose $n_0 > 1$ such that $s_{\alpha'}(n) > s_\alpha(n)$, $n \ge n_0$, and define $s_n$, for $1 \le n \le n_0$, by the first line of the above formula. Now let $k \in \mathbb{N}_0$ and assume that $n_k$ has already been defined (and with it $s_n$ for $n \le n_k$).

(*i*) $k \equiv 0 \bmod 4$: Consider the auxiliary sequence

$$b_{0,n} := \begin{cases} s_n & , \quad n \le n_k \\ s_\alpha(n) \, , & \quad n > n_k \end{cases}$$

Then $G(n; \vec{b_0}, 0) = G(n; \vec{s}, 0)$, $n \le n_k$, and $\lim_{n\to\infty} G(n; \vec{b_0}, 0) = -\alpha$. Choose $n_{k+1} > n_k$ such that

$$G(n_{k+1}; \vec{b_0}, 0) \ge -\alpha - \frac{1}{k}, \qquad \frac{b_{0,n_{k+1}}}{n_{k+1}^{\alpha}} \le 2. \tag{7.14}$$

(*ii*) $k \equiv 1 \bmod 4$: Set

$$n_{k+1} := \min\big\{ n > n_k : s_\alpha(n_k) + (n - n_k) = s_{\alpha'}(n) \big\}.$$

This is well-defined since $s_\alpha(n_k) < s_{\alpha'}(n_k)$ and $s_{\alpha'}(n) = \mathrm{o}(n)$.

(*iii*) $k \equiv 2 \bmod 4$: Consider the auxiliary sequence

$$b_{2,n} := \begin{cases} s_n & , \quad n \leq n_k \\ s_{\alpha'}(n) \, , & \quad n > n_k \end{cases}$$

Then $G(n; \vec{b_2}, \frac{1}{2}) = G(n; \vec{s}, \frac{1}{2})$, $n \leq n_k$, and $\lim_{n\to\infty} G(n; \vec{b_2}, \frac{1}{2}) = -\alpha'$. Choose $n'_k > n_k$ such that for all $n \geq n'_k$

$$G(n; \vec{b_2}, \tfrac{1}{2}) \leq -\alpha' + \frac{1}{k}, \qquad \frac{b_{2,n}}{n^{\alpha'}} \geq \frac{1}{2}. \tag{7.15}$$

Since $\vec{h}$ is nonincreasing, and hence regularly distributed in the sense of Definition 6.2, we have $\liminf_{n\to\infty} G(n; \vec{h}, \frac{1}{2}) = \Delta^*(\vec{h}) = \frac{1}{\gamma}$ by Lemma 6.7, (*ii*). Hence we can choose $n_{k+1} > n'_k$ such that

$$\exists n \in [n'_k, n_{k+1}] : \; G(n; \vec{h}, \tfrac{1}{2}) \leq \frac{1}{\gamma} + \frac{1}{k}, \tag{7.16}$$

$$\exists n \in [n'_k, n_{k+1}] : \; h_n \geq n^{-\frac{1}{\gamma} - \frac{1}{k}}. \tag{7.17}$$

(*iv*) $k \equiv 3 \bmod 4$: Set

$$n_{k+1} := \min\big\{ n > n_k : s_{\alpha'}(n_k) + [(n - n_k) \mod 2] = s_\alpha(n) \big\}.$$

This is well-defined since $s_{\alpha'}(n_k) > s_\alpha(n_k)$ and $\lim_{n\to\infty} s_\alpha(n) = \infty$.

Set $\phi_n := \operatorname{Arccot} s_n$ and $l_n := h_n \sin^{-2} \phi_n$. Then Theorem 7.22 is applicable and yields $\rho(H_{\vec{l},\vec{\phi}}) = \gamma = \delta_{\vec{l},\vec{\phi}}(H_{\vec{l},\vec{\phi}})$.

Remembering (7.11) and the formulae before and after, we have

$$|\sin(\phi_{n+1} - \phi_n)| \asymp \frac{1}{s_n^2} \tag{7.18}$$

and therefore

$$G\big(n; (|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty, 0\big) = -2G(n; \vec{s}, 0) + \mathrm{o}(1),$$

cf. Remark 5.19, (*iii*) and (*v*). For $k \equiv 0 \bmod 4$ it holds that

$$G(n_{k+1}; \vec{s}, 0) \geq -\alpha - \frac{1}{k} = -\frac{1}{2}\delta^\circ_\phi - \frac{1}{k},$$

and we conclude that

$$\delta_\phi(H_{\vec{l},\vec{\phi}}) = \liminf_{n\to\infty} G\big(n; (|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty, 0\big) \leq \delta^\circ_\phi.$$

However, $s_n \geq s_\alpha(n)$ for all $n \in \mathbb{N}$, whence

$$-2G(n; \vec{s}, 0) \geq -2G(n; \vec{s_\alpha}, 0) \to 2\alpha = \delta^\circ_\phi,$$

and this shows $\delta_\phi(H_{\vec{l},\vec{\phi}}) \geq \delta_\phi^\circ$.

Since $\lim_{n\to\infty} s_n = \infty$, we have $\lim_{n\to\infty} \phi_n = 0$ and hence $\sin^2\phi_n \asymp s_n^{-2}$. Thus $l_n \asymp h_n \cdot s_n^2$, and Remark 5.19, $(iii)$ – $(v)$, gives

$$G(n;\vec{l},\tfrac{1}{2}) = G(n;\vec{h},\tfrac{1}{2}) + 2G(n;\vec{s},\tfrac{1}{2}) + \mathrm{o}(1).$$

Let $k \equiv 2 \bmod 4$ and choose $n \in [n_k', n_{k+1}]$ according to (7.16).

$$G(n;\vec{l},\tfrac{1}{2}) \leq \Big(\frac{1}{\gamma} + \frac{1}{k}\Big) + \Big(-2\alpha' + \frac{2}{k}\Big) + \mathrm{o}(1) = \delta_l^\circ + \mathrm{o}(1),$$

which gives $\delta_l(H_{\vec{l},\vec{\phi}}) \leq \delta_l^\circ$. However, $s_n \leq s_{\alpha'}(n)$ for all $n \in \mathbb{N}$, and hence

$$G(n;\vec{s},\tfrac{1}{2}) \geq G(n;\vec{s_{\alpha'}},\tfrac{1}{2}) \to -\alpha' = -\frac{1}{2}\Big(\frac{1}{\gamma} - \delta_l^\circ\Big).$$

This shows that $\delta_l(H_{\vec{l},\vec{\phi}}) \geq \delta(\vec{h},\frac{1}{2}) + 2\delta(\vec{s},\frac{1}{2}) \geq \frac{1}{\gamma} - 2\alpha' = \delta_l^\circ$.

By construction $s_n \gtrsim n^\alpha$. This gives $\Delta^*\big((s_n^{-1})_{n=1}^\infty\big) \geq \alpha$, whereas the right-hand side of (7.14) results in $\Delta^*\big((s_n^{-1})_{n=1}^\infty\big) \leq \alpha$. Together with (7.18) this yields $\Delta^*(|\sin(\phi_{n+1} - \phi_n)|) = 2\alpha = \delta_\phi^\circ$, and Lemma 6.1 gives $\Delta_\phi(H_{\vec{l},\vec{\phi}}) = \delta_\phi^\circ$.

Note that we have $s_n \lesssim n^{\alpha'}$, which results in $\Delta^*(\vec{s}\,) \geq -\alpha'$. Due to $l_n \asymp h_n s_n^2$ we can use Remark 5.4, $(iv)$ and $(v)$, to get

$$\Delta^*(\vec{l}) \geq \Delta^*(\vec{h}) + 2\Delta^*(\vec{s}\,) \geq \frac{1}{\gamma} - 2\alpha' = \delta_l^\circ.$$

Let $k \equiv 2 \bmod 4$ and choose $n \in [n_k', n_{k+1}]$ according to (7.17). Then we have $h_n \geq n^{-\frac{1}{\gamma}-\frac{1}{k}}$ and, by the right-hand side of (7.15), $s_n \geq \frac{1}{2}n^{\alpha'}$. Together $l_n \gtrsim n^{-\frac{1}{\gamma}-\frac{1}{k}+2\alpha'}$ which yields $\Delta^*(\vec{l}) \leq \delta_l^\circ$, i.e. $\Delta_l(H_{\vec{l},\vec{\phi}}) = \delta_l^\circ$.

Next we calculate $\mu(H_{\vec{l},\vec{\phi}})$. Since $\phi_j$ converges to zero, set $\psi := 0$. The calculation

$$\sum_{j=n}^\infty l_j \sin^2\phi_j = \sum_{j=n}^\infty h_j \lesssim n^{1-\Delta^*(\vec{h})} = n^{1-\frac{1}{\gamma}},$$

shows $\mu(H_{\vec{l},\vec{\phi}}) = \frac{1}{\gamma} - \delta_l^\circ$.

Finally, we are able to apply Theorem 5.10. First assume that $\mu \leq 2\Delta_\phi$, which is equivalent to $\frac{1}{\gamma} - \delta_l^\circ \leq 2\delta_\phi^\circ$. If $\Delta_l + \Delta_\phi \geq 2$, then the upper bound from Theorem 5.10 is $(\Delta_l + \Delta_\phi)^{-1}$, which is strictly larger than the order, since

$$\gamma < \frac{1}{\delta_l^\circ + \delta_\phi^\circ} \;\Leftrightarrow\; \delta_\phi < \frac{1}{\gamma} - \delta_l^\circ.$$

In the critical triangle $\Delta_l + \Delta_\phi < 2$, the upper estimate is even larger.

Secondly consider the case $\mu > 2\Delta_\phi$, which takes places if $\frac{1}{\gamma} - \delta_l^\circ > 2\delta_\phi^\circ$. Here Theorem 5.10 gives the upper bound $2/(\delta_l^\circ + \frac{1}{\gamma})$, which is again strictly larger than $\gamma$, since

$$\gamma < \frac{2}{\delta_l^\circ + \frac{1}{\gamma}} \;\Leftrightarrow\; 0 < \frac{1}{\gamma} - \delta_l^\circ.$$

❑

Next, we show that (for arbitrary small orders) it might be possible to compute $\rho(H_{\vec{l},\vec{\phi}})$ with help of Theorem 6.5, but $\rho(H_{\vec{l},\vec{\phi}})$ is not equal to the convergence exponent of $([l_n \sin^2\phi_n]^{-1})_{n=1}^{\infty}$.

*7.25 Example.* Let $\alpha > -1$ and $\beta > 3 + 2\alpha$, set

$$M_n := \sum_{k=1}^{n} k^{\alpha}, \qquad l_n := n^{-\beta}(1 + M_n^2),\ \phi_n := \operatorname{Arccot} M_n,$$

and consider the Hamiltonian $H_{\vec{l},\vec{\phi}}$.

Since $\alpha > -1$, we have $M_n \asymp n^{\alpha+1}$ and hence $l_n \asymp n^{2(\alpha+1)-\beta}$. The assumption on $\beta$ just says that $2(\alpha+1) - \beta < -1$, i.e., that $H_{\vec{l},\vec{\phi}}$ is lcc.

By Example 5.20 we obtain $\delta_l = \beta - 2(\alpha+1)$ and this expression exists as a limit. In order to compute $\delta_\phi$, we use the identity

$$\big|\sin(\operatorname{Arccot} x - \operatorname{Arccot} y)\big| = \Big|\sin\Big(\operatorname{Arccot}\Big(\frac{xy+1}{x-y}\Big)\Big)\Big| = \Big[\Big(\frac{xy+1}{x-y}\Big)^2 + 1\Big]^{-\frac{1}{2}},$$

which holds for arbitrary $x, y \in \mathbb{R}$, $x \neq y$. Clearly, $M_{n+1} - M_n = k^{\alpha}$, and we find

$$|\sin(\phi_{n+1} - \phi_n)| \asymp n^{-(\alpha+2)},$$

whence $\delta_\phi = \alpha + 2$. Since $\delta_l + \delta_\phi = \beta - \alpha > 2$, we can apply Theorem 6.5, $(i)$, to obtain

$$\rho\big(H_{\vec{l},\vec{\phi}}\big) = \frac{1}{\beta - \alpha}.$$

We have $\sin^{-2}\phi_n = 1 + M_n^2$ and hence $l_n \sin^2\phi_n = n^{-\beta}$. The convergence exponent of $([l_n \sin^2\phi_n]^{-1})_{n=1}^{\infty}$ thus equals $\frac{1}{\beta}$. For $\alpha < 0$ this is larger than the order, for $\alpha > 0$ it is smaller.

It is interesting to observe which hypothesis of Theorem 7.22 are violated in this example. Of course, if $\beta < 2$, then already (7.10) fails. If $\alpha \in (-1, 0)$, then the sequence $(|\cot\phi_n - \cot\phi_{n-1}|)_{n=1}^{\infty}$ is decreasing, and if $\alpha > 0$, then it is increasing but unbounded. $\Diamond$

### 7.3.3. Connection to Berezanskiĭ's theorem

Recall Berezanskiĭ's theorem which we formulated already in Theorem 2.5:

Let $J$ be a Jacobi matrix with diagonal $(q_n)_{n=0}^{\infty}$ and off-diagonal $(\rho_n)_{n=0}^{\infty}$. Assume that $\sum_{n=0}^{\infty} \rho^{-1} < \infty$, that $\sum_{n=0}^{\infty} |q_n|/\rho_n < \infty$, and that the off-diagonal parameters are log-concave or log-convex, i.e. $\rho_n^2 \geq \rho_{n-1}\rho_{n+1}$ or $\rho_n^2 \leq \rho_{n-1}\rho_{n+1}$ respectively. Then, $J$ is of type C, and the order is equal to the convergence exponent of $(\rho_n)_{n=0}^{\infty}$.

In this subsection we explain the relation between this Theorem and the main results of this and previous chapters. To this end recall the connection of Jacobi matrices and Hamburger Hamiltonians outlined in the Introduction. In particular, the parameters of a Jacobi matrix are related to the parameters of the corresponding Hamburger Hamiltonian via

$$\begin{aligned} \frac{1}{\rho_n} &= |\sin(\phi_{n+1}-\phi_n)|\sqrt{l_n l_{n+1}}, \\ q_n &= -\frac{1}{l_n}\big[\cot(\phi_{n+1}-\phi_n)+\cot(\phi_n-\phi_{n-1})\big]. \end{aligned}$$

The essence of Theorem 2.5 is the case of a zero-diagonal; adding a small diagonal can be achieved with a perturbation argument. Let us therefore focus on this case, where the above formulae are easy to handle.

First, we see that $q_n = 0$ for all $n$ if and only if the angles $\phi_n$ alternate between two fixed values. Due to common normalisation, these are 0 and $\frac{\pi}{2}$. However, multiplying a Jacobi matrix with a positive scalar or adding an offset to the sequence of angles of a Hamburger Hamiltonian does not influence the respective order. Hence, we are free to choose those two values and work with different ones interchangeably.

Plugging the above formula for $\rho_n$ (with alternating angles) log-concavity or convexity means that

$$\frac{l_{n+1}}{l_{n-1}} \le \frac{l_{n+2}}{l_n} \quad \text{or} \quad \frac{l_{n+1}}{l_{n-1}} \ge \frac{l_{n+2}}{l_n} \text{ resp.,} \tag{7.19}$$

or equivalently,

$$\frac{l_n}{l_{n-1}} \le \frac{l_{n+2}}{l_{n+1}} \quad \text{or} \quad \frac{l_n}{l_{n-1}} \ge \frac{l_{n+2}}{l_{n+1}} \text{ resp..} \tag{7.20}$$

Monotonicity of the quotients (7.19) leads to the distinction of three cases.

(I) $\frac{l_{n+1}}{l_{n-1}} \ge 1$ *for large $n$:* Then $\rho_n \ge \rho_{n+1}$ for those $n$, which contradicts Carleman's condition.

(II) $\frac{l_{n+1}}{l_{n-1}} \le t < 1$ *for large $n$:* Then $l_n, \frac{1}{\rho_n} \lesssim t^n$, whence the convergence exponent of $(\rho_n)_{n=1}^{\infty}$ is zero, and the order is zero by Corollary 5.12, since $\Delta_l^+ = \infty$.

(III) $\frac{l_{n+1}}{l_{n-1}} \nearrow 1$*:* This is the nontrivial case concerning order (note that it appears only when $\rho_n$ are log-concave), and requires some further analysis.

First, since $\frac{l_{n+1}}{l_{n-1}} < 1$, the sequence $\vec{l}$ splits into two decreasing subsequences $(l_{2k-1})_{k=1}^{\infty}$ and $(l_{2k})_{k=1}^{\infty}$. The quotients $\big(l_{2k}/l_{2k-1}\big)_{k=1}^{\infty}$ and $\big(l_{2k+1}/l_{2k}\big)_{k=1}^{\infty}$ are nondecreasing by (7.20), and hence have limits $t_0, t_1 \in (0,\infty]$. However, since $\frac{l_{n+1}}{l_{n-1}}$ tends to 1,

$$\frac{1}{t_0} = \lim_{k\to\infty}\frac{l_{2k-1}}{l_{2k}} = \lim_{k\to\infty}\frac{l_{2k+1}}{l_{2k}} = t_1,$$

in particular, $t_0, t_1 < \infty$. Now we pass to the sequence

$$l_n' := \begin{cases} t_0 l_n\,, & n \text{ odd} \\ l_n \quad, & n \text{ even} \end{cases}$$

Then the quotient sequences $\big(l'_{2k}/l'_{2k-1}\big)_{k=1}^{\infty}$ and $\big(l'_{2k+1}/l'_{2k}\big)_{k=1}^{\infty}$ are still nondecreasing and both tend to 1. Thus $\vec{l'}$ is nonincreasing. Monotonicty implies that the convergence exponents of $({l'_n}^{-1})_{n=1}^{\infty}$ and $([l'_n l'_{n+1}]^{-\frac{1}{2}})_{n=1}^{\infty}$ coincide. Since $l'_n \asymp l_n$, these are the same as the convergence exponents of $(l_n^{-1})_{n=1}^{\infty}$ and of $(\rho_n)_{n=1}^{\infty}$, respectively. Moreover, $\vec{l}$ is regularly distributed in the sense of Definition 6.2, cf. Remark 6.3.

Now we can compute the order from Theorem 6.5, (*iii*). Since angles alternate, we have $\delta_\phi = 0$ as a limit, and hence the order equals $\delta_l^{-1}$. By Lemma 6.7, (*ii*), $\delta_l^{-1}$ coincides with the convergence exponent of $(l_n^{-1})_{n=1}^{\infty}$ and hence with the convergence exponent of $(\rho_n)_{n=1}^{\infty}$.

If the convergence exponent of $(\rho_n)_{n=1}^{\infty}$ is less than $1/2$, we also can compute the order from Theorem 7.22. To this end we pass to the Jacobi matrix $(1/\sqrt{t_0}\,)J$ and add an offset $-\pi/4$ to the sequence of angles. This leads to the Hamburger Hamiltonian with lengths $(l'_n)_{n=1}^{\infty}$ and angles alternating between $\pm\pi/4$. Thus the order equals the convergence exponent of $(\sqrt{2}/l'_n)_{n=1}^{\infty}$ which is equal to the convergence exponent of $(\rho_n)_{n=1}^{\infty}$.

Having seen that Berezanskiĭ's theorem (for orders $< 1/2$) can be deduced from Theorem 7.22, we shall now show that Theorem 7.22 actually goes far beyond the Berezanskiĭ case.

*7.26 Example.* We revisit the Hamiltonians constructed in Proposition 7.23 (so to make sure that order cannot be computed already from Theorem 5.10), and consider the associated Jacobi matrices. To this end let $\vec{h}$ be a decreasing sequence with (7.12) which has the property that $\lim_{n\to\infty} h_n/h_{n+1} = 1$. For instance use $h_n = n^{-\frac{1}{\alpha}}(\ln n)^{-5}$ where $\alpha \in (0, 1/2]$. Let $\vec{s}, \vec{l}, \vec{\phi}$ be the sequences constructed in the proof of Proposition 7.23. Then we know that

$$\lim_{n\to\infty} s_n = \infty, \quad |\sin(\phi_{n+1} - \phi_n)| \asymp \frac{1}{s_n^2}, \quad l_n \asymp h_n \cdot s_n^2,$$

and hence

$$\frac{|q_n|}{\rho_n} = \underbrace{\frac{\sqrt{l_n l_{n+1}}}{l_n}}_{\to 1} \cdot \left| \underbrace{\cos(\phi_{n+1} - \phi_n)}_{\to 1} + \underbrace{\cos(\phi_n - \phi_{n-1})}_{\to 1} \frac{\sin(\phi_{n+1} - \phi_n)}{\sin(\phi_n - \phi_{n-1})} \right|.$$

Since $s_n$ is unbounded but $|s_{n+1} - s_n| = 1$, we find a subsequence $(\phi_{n_k})_{k=1}^{\infty}$ with $\phi_{n_k-1} > \phi_{n_k} > \phi_{n_k+1}$, or equivalent $s_{n_k-1} + 1 = s_{n_k} = s_{n_k+1} - 1$. Along this subsequence

$$\inf_{k\in\mathbb{N}} \frac{\sin(\phi_{n_k+1} - \phi_{n_k})}{\sin(\phi_{n_k} - \phi_{n_k-1})} > 0,$$

and we conclude that $\limsup_{n\to\infty} |q_n|/\rho_n > 1$. This shows that the Jacobi matrix associated with $H_{\vec{l},\vec{\phi}}$ is far from being a small perturbation of the corresponding zero-diagonal matrix in the sense of Theorem 2.5. ◊

# 8. Using power asymptotics

In this section we study the spectrum of Jacobi matrices of type C, whose parameters have a power asymptotics. More precisely, we consider the *upper density* of the spectrum with respect to a power $R^\alpha$, i.e.

$$\limsup_{R\to\infty} \frac{n_\sigma(R)}{R^\alpha} \in [0,\infty],$$

where $n_\sigma(R)$ denotes the counting function, i.e. the number of spectral points of a selfadjoint extension of $T_J$ in the interval $[-R,R]$.

Studying the upper density is natural, since this quantity is accessible via the growth of the canonical product having the spectrum as its zero-set. Passing to a canonical product and applying the theory of entire functions is a common tool in the theory of operators with compact resolvents, e.g., [GK69]. It was applied in various instances to investigate the asymptotic behaviour of the spectrum, e.g., [Fre05].

Once more we recall Berezanskiĭ's theorem, cf. Theorem 2.5: Assume that $\sum_{n=0}^\infty \rho^{-1} < \infty$, that $\sum_{n=0}^\infty \frac{|q_n|}{\rho_n} < \infty$, and that the off-diagonal parameters are log-concave, i.e. $\rho_n^2 \geq \rho_{n-1}\rho_{n+1}$. Then, $J$ is of type C, and the order is equal to the convergence exponent of $(\rho_n)_{n=0}^\infty$. Equivalently one could say that $\tau_\alpha(H) = 0$ for all $\alpha$ greater than the convergence exponent of $(\rho_n)_{n=0}^\infty$, and $\tau_\alpha(H) = \infty$ for $\alpha$ smaller than the convergence exponent of $(\rho_n)_{n=0}^\infty$. It is the same to say that the upper density of the zeros of any entry of the monodromy matrix $W(z)$ is zero or infinity, respectively, cf. [Lev80, Theorem 14]. Since the zeros of $w_{2,1}(z)$ interlace with the spectrum of $T_J$, we get

$$\limsup_{R\to\infty} \frac{n_\sigma(R)}{R^\alpha} = \begin{cases} 0, & \text{for } \alpha > \text{ convergence exponent of } (\rho_n)_{n=0}^\infty, \\ \infty, & \text{for } \alpha < \text{ convergence exponent of } (\rho_n)_{n=0}^\infty. \end{cases} \tag{8.1}$$

## 8.1. The generic case

In this section we study the upper density of the spectrum for Jacobi matrices $J$ whose parameters have the power asymptotics

$$\rho_n = n^{\beta_1}\left(x_0 + \frac{x_1}{n} + \mathrm{O}(n^{-2})\right), \quad q_n = n^{\beta_2}\left(y_0 + \frac{y_1}{n} + \mathrm{O}(n^{-2})\right), \tag{8.2}$$

with $x_0 > 0$, $y_0 \neq 0$. We assume $\beta_1 > 1$, $\delta := \beta_1 - \beta_2 \geq 0$, and $|y_0| \leq 2x_0$ if $\delta = 0$. These conditions are necessary for $J$ being of type $C$ by Carleman and Wouk, cf. subsection 2.1.2.

Having (8.2) implies that $(\rho_n)_{n=0}^{\infty}$ is log-concave. Hence, if $\delta > 1$, Berezanskiĭ's theorem applies and yields (8.1). Observe that the convergence exponent of a sequence $(\rho_n)_{n=0}^{\infty}$ with (8.2) is $\frac{1}{\beta_1}$.

Our main result is the following theorem which, roughly speaking, says that (8.1) remains valid for $\delta \in (0, 1]$, and even in some cases where $\delta = 0$, i.e. where diagonal and off-diagonal parameters are comparable.

**8.1 Theorem.** *Let $J$ be the Jacobi matrix with parameters $\rho_n, q_n$, let $T$ be a selfadjoint extension of $T_J$ in $\ell^2(\mathbb{N})$, and let $n_\sigma$ be the counting function of the spectrum of $T$. Assume that $\rho_n$ and $q_n$ have the asymptotics* (8.2) *where $x_0 > 0$, $y_0 \neq 0$, $\beta_1 > 1$ and $\delta := \beta_1 - \beta_2 \in [0, 1]$. If $\delta = 0$, assume further that $|y_0| < 2x_0$.*

*Then $J$ is of type C and the order is equal to $\frac{1}{\beta_1}$. Moreover, the upper density of the spectrum of $T$ is finite and not zero,*

$$\limsup_{R\to\infty} \frac{n_\sigma(R)}{R^{1/\beta_1}} \in (0, \infty). \tag{8.3}$$

In the proof of this theorem we use the already mentioned fact that the growth of the counting function $n_\sigma$ relates to the growth of the corresponding canonical product, pass from the Jacobi matrix to the corresponding Hamburger Hamiltonian, and apply Theorem 5.1 to estimate the growth of the monodromy matrix of this system. A crucial step is to establish that the power asymptotics (8.2) of the Jacobi parameters give rise to similar power asymptotics for the data determining the canonical system. To achieve this we use recent work of R.-J.Kooman [Koo07] and a discrete Levinson type theorem.

*Proof of Theorem* 8.1. Let a Jacobi matrix $J$ whose parameters $\rho_n$ and $q_n$ have an asymptotic expansion (8.2) be given, and assume that $x_0 > 0$, $y_0 \neq 0$, $\beta_1 > 1$, $\delta \in [0, 1]$, and that $|y_0| < 2x_0$ if $\delta = 0$.

In order to apply Theorem 5.1, we need knowledge about the lengths and angles of the Hamburger Hamiltonian associated with $J$. Since $P_n(0)$ and $Q_n(0)$ form a fundamental system of solutions of the difference equation

$$\rho_{n+1}u_{n+2} + q_{n+1}u_{n+1} + \rho_n u_n = 0, \tag{8.4}$$

we start with studying asymptotics of solutions of this equation.

**Step 1:** Growth of solutions, $\delta \in [0, 1)$

In the case $\delta \in [0, 1)$, we begin with rewriting (8.4). Setting $r_i := \frac{-q_i}{2\rho_i}$ and dividing by $\rho_{n+1}\prod_{i=1}^{n+1} r_i$ gives

$$\frac{u_{n+2}}{\prod_{i=1}^{n+1} r_i} - 2\frac{u_{n+1}}{\prod_{i=1}^{n} r_i} + \frac{\rho_n}{\rho_{n+1}r_n r_{n+1}} \frac{u_n}{\prod_{i=1}^{n-1} r_i} = 0.$$

Introducing the new variable $v_n := u_n\big(\prod_{i=1}^{n-1} r_i\big)^{-1}$ and setting $C_n := 1 - \frac{\rho_n}{\rho_{n+1}r_n r_{n+1}}$ gives

$$v_{n+2} - 2v_{n+1} + (1 - C_n)v_n = 0. \tag{8.5}$$

A computation shows

$$C_n = 1 - 4n^{2\delta}\left(z_0 + \frac{z_1}{n} + \mathrm{O}(n^{-2})\right)$$

with

$$z_0 := \left(\frac{x_0}{y_0}\right)^2, \quad z_1 := \frac{x_0}{y_0^3}\big(2(x_1y_0 - x_0y_1) - \beta_2 x_0 y_0\big).$$

Clearly, we have

$$\lim_{n\to\infty} n^{-2\delta} C_n = \begin{cases} -4z_0 & \delta \in (0,1) \\ 1-4z_0 & \delta = 0 \end{cases}.$$

Our assumptions ensure that this limit is always negative.

We can apply [Koo07, Theorem 1 ($i$)], and get two linearly independent solutions of (8.5), denote them by $(v_n^{(j)})_{n=1}^{\infty}$ for $j = 1,2$, such that

$$v_n^{(1)} = \overline{v_n^{(2)}} = (1+\mathrm{O}(1))n^{-\frac{\delta}{2}} \prod_{k=1}^{n-1}\big(1 + i\sqrt{-C_k}\big).$$

The absolute value of each factor is equal to

$$\begin{aligned}\left|1 + i\sqrt{-C_k}\right| = \sqrt{1 - C_k} &= \sqrt{4k^{2\delta}\left(z_0 + \frac{z_1}{k} + \mathrm{O}(k^{-2})\right)} \\ &= 2\sqrt{z_0}k^{\delta}\sqrt{1 + \frac{z_1}{z_0 k} + \mathrm{O}(k^{-2})},\end{aligned}$$

and we get

$$\begin{aligned}\left|\prod_{k=1}^{n-1}\big(1 + i\sqrt{-C_k}\big)\right| &= (2\sqrt{z_0})^{n-1}\big[(n-1)!\big]^{\delta}\sqrt{\prod_{k=1}^{n-1}\big(1 + \frac{z_1}{z_0 k} + \mathrm{O}(k^{-2})\big)} \\ &= \left(\frac{2x_0}{|y_0|}\right)^{n-1}\big[(n-1)!\big]^{\delta}(c_1 + o(1))n^{\frac{z_1}{2z_0}},\end{aligned}$$

for some $c_1 > 0$, due to [Koo07, Lemma 4] adding a summable perturbation. Thus

$$\begin{aligned}\left|v_n^{(1)}\right| = \left|v_n^{(2)}\right| &= (c_1 + o(1))\Big(\frac{2x_0}{|y_0|}\Big)^{n-1}\big[(n-1)!\big]^{\delta} n^{\frac{z_1}{2z_0} - \frac{\delta}{2}} \\ &= (c_1 + o(1))\Big(\frac{2x_0}{|y_0|}\Big)^{n-1}\big[(n-1)!\big]^{\delta} n^{\frac{x_1}{x_0} - \frac{y_1}{y_0} - \frac{\beta_1}{2}}. \end{aligned} \tag{8.6}$$

Substituting back via $u_n = v_n \prod_{i=1}^{n-1} r_i$ produces two solutions of (8.4), $(u_n^{(j)})_{n=1}^{\infty}$ for

$j = 1, 2$. At first note

$$
\begin{aligned}
\prod_{k=1}^{n-1} r_k &= \prod_{k=1}^{n-1} \frac{-q_k}{2\rho_k} = \prod_{k=1}^{n-1} \frac{-1}{2} k^{-\delta} \frac{y_0 + \frac{y_1}{k} + \mathrm{O}(k^{-2})}{x_0 + \frac{x_1}{k} + \mathrm{O}(k^{-2})} \\
&= \prod_{k=1}^{n-1} \frac{-1}{2} k^{-\delta} \left( \frac{y_0}{x_0} + \frac{1}{k} \frac{y_0}{x_0} \Big( \frac{y_1}{y_0} - \frac{x_1}{x_0} \Big) + \mathrm{O}(k^{-2}) \right) \\
&= \Big( \frac{-y_0}{2x_0} \Big)^{n-1} \big[ (n-1)! \big]^{-\delta} \prod_{k=1}^{n-1} \left( 1 + \frac{1}{k} \Big( \frac{y_1}{y_0} - \frac{x_1}{x_0} \Big) + \mathrm{O}(k^{-2}) \right) \\
&= \Big( \frac{-y_0}{2x_0} \Big)^{n-1} \big[ (n-1)! \big]^{-\delta} (c_2 + o(1)) n^{\frac{y_1}{y_0} - \frac{x_1}{x_0}},
\end{aligned} \tag{8.7}
$$

for some $c_2 \neq 0$, again by [Koo07, Lemma 4]. Combining (8.6) and (8.7) gives the asymptotic behaviour

$$
\big| u_n^{(1)} \big| = \big| u_n^{(2)} \big| = \big| v_n^{(1)} \big| \left| \prod_{k=1}^{n-1} r_k \right| = (c_3 + o(1)) n^{-\frac{\beta_1}{2}}, \tag{8.8}
$$

where $c_3 = c_1 \, |c_2| > 0$. Since these solutions are square-summable, $J$ is of type C.

**Step 2:** Growth of solutions, $\delta = 1$

The case $\delta = 1$ is not covered by [Koo07, Theorem 1], but can be handled with direct computations reduced to Levinson's theorem.

Dividing (8.4) by $\rho_{n+1}$ and shifting the index by 1 gives

$$
u_{n+1} + \underbrace{\frac{q_n}{\rho_n}}_{=:a_n} u_n + \underbrace{\frac{\rho_{n-1}}{\rho_n}}_{=:b_n} u_{n-1} = 0. \tag{8.9}
$$

Clearly $a_n = \frac{y_0}{x_0} n^{-1} + \mathrm{O}(n^{-2})$ and $b_n = 1 - \beta_1 n^{-1} + \mathrm{O}(n^{-2})$. By setting $\vec{u}_n = (u_n, u_{n+1})^T$ and

$$
A_n := \begin{pmatrix} 0 & 1 \\ -b_n & -a_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{1}{n} \begin{pmatrix} 0 & 0 \\ \beta_1 & -\frac{y_0}{x_0} \end{pmatrix} + \mathrm{O}\left( n^{-2} \right),
$$

we write (8.9) as the difference system

$$
\vec{u}_n = A_n \vec{u}_{n-1}. \tag{8.10}
$$

The idea is to diagonalise $A_n$ modulo summable terms. To this end, we set

$$
R_n := \begin{pmatrix} i^n & (-i)^n \\ i^{n+1} & (-i)^{n+1} \end{pmatrix},
$$

and introduce the variable $\vec{v}_n := R_n^{-1} \vec{u}_n$. This leads to the difference system

$$
\vec{v}_n = R_n^{-1} A_n R_{n-1} \vec{v}_{n-1}.
$$

Set $z := \beta_1 + i\frac{y_0}{x_0}$. A calculation shows,

$$R_n^{-1} A_n R_{n-1} = \begin{pmatrix} 1 - \frac{1}{2n}\bar{z} & 0 \\ 0 & 1 - \frac{1}{2n} z \end{pmatrix} + \frac{(-1)^n}{2n} \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} + \mathrm{O}\left(n^{-2}\right).$$

In order to get rid of off-diagonal terms which are not summable we perform one more transformation:

$$S_n := I - \frac{1}{2} \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k} \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}$$

Note that $S_n$ is invertible for sufficiently large $n$, say $n \geq n_0$. Then $\vec{w}_n := S_n^{-1} \vec{v}_n$ satisfies

$$\vec{w}_n = S_n^{-1} R_n^{-1} A_n R_{n-1} S_{n-1} \vec{w}_{n-1}, \tag{8.11}$$

with

$$(R_n S_n)^{-1} A_n R_{n-1} S_{n-1} = \begin{pmatrix} 1 - \frac{1}{2n}\bar{z} & 0 \\ 0 & 1 - \frac{1}{2n} z \end{pmatrix} + \mathrm{O}\left(n^{-2}\right).$$

Note that the diagonal entries are complex conjugated numbers which converge to 1. The discrete version of Levinsons's Fundamental Theorem [BL15, Theorem 3.4] gives a fundamental solution of (8.11) with

$$\begin{aligned} W_n &= (I + \mathrm{O}(1)) \prod_{k=n_0}^{n} \begin{pmatrix} 1 - \frac{1}{2k}\bar{z} & 0 \\ 0 & 1 - \frac{1}{2k} z \end{pmatrix} \\ &= (I + \mathrm{O}(1)) \begin{pmatrix} \prod_{k=n_0}^{n} (1 - \frac{1}{2k}\bar{z}) & 0 \\ 0 & \prod_{k=n_0}^{n} (1 - \frac{1}{2k} z) \end{pmatrix}. \end{aligned}$$

Substituting back yields that

$$\begin{aligned} R_n S_n W_n &= (I + \mathrm{O}(1))\, R_n \begin{pmatrix} \prod_{k=n_0}^{n} (1 - \frac{1}{2k}\bar{z}) & 0 \\ 0 & \prod_{k=n_0}^{n} (1 - \frac{1}{2k} z) \end{pmatrix} \\ &= (I + \mathrm{O}(1)) \begin{pmatrix} i^n \prod_{k=n_0}^{n} (1 - \frac{1}{2k}\bar{z}) & (-i)^n \prod_{k=n_0}^{n} (1 - \frac{1}{2k} z) \\ i^{n+1} \prod_{k=n_0}^{n} (1 - \frac{1}{2k}\bar{z}) & (-i)^{n+1} \prod_{k=n_0}^{n} (1 - \frac{1}{2k} z) \end{pmatrix} \end{aligned}$$

is a fundamental solution of (8.10). By inspecting the first row we get two solutions $u_n^{(1)}, u_n^{(2)}$ of (8.9), or equivalently (8.4), with

$$|u_n^{(1)}|, |u_n^{(2)}| \asymp \left| \prod_{k=n_0}^{n} (1 - \frac{1}{2k}\bar{z}) \right| \asymp n^{-\frac{\operatorname{Re} z}{2}} = n^{-\frac{\beta_1}{2}}. \tag{8.12}$$

In particular, $J$ is of type C.

**Step 3:** Conclusions concerning the spectrum

We have seen in the first and second step, cf. (8.8) and (8.12) respectively, that the difference equation (8.4) has a fundamental system of solutions $u_n^{(1)}, u_n^{(2)}$ with $|u_n^{(j)}| \asymp n^{-\frac{\beta_1}{2}}$ for $j \in \{1,2\}$.

Recall that also $P_n(0)$ and $Q_n(0)$ are linearly independent solutions of (8.4). The quotient $\big(|P_n(0)| + |Q_n(0)|\big)/n^{-\frac{\beta_1}{2}}$ is bounded from above since $P_n(0)$ and $Q_n(0)$ can be written as linear combinations of $u_n^{(1)}$ and $u_n^{(2)}$. It is also bounded away from zero, since $u_n^{(1)}$ is a linear combination of $P_n(0)$ and $Q_n(0)$ and $\big|u_n^{(1)}\big|/n^{-\frac{\beta_1}{2}}$ is bounded away from zero. Thus, we obtain $P_n(0)^2 + Q_n(0)^2 \asymp n^{-\beta_1}$.

Consider the canonical system related to $J$. By (2.7) and (2.5), the lengths and angles of the corresponding Hamburger Hamiltonian $H$ satisfy

$$
\begin{aligned}
&l_n = P_n(0)^2 + Q_n(0)^2 \asymp n^{-\beta_1}, \\
&\big|\sin(\phi_{n+1} - \phi_n)\big| = \big(\rho_n \sqrt{l_n l_{n+1}}\big)^{-1} \asymp n^{\beta_1 - \beta_1} = 1.
\end{aligned}
$$

The lengths and angle-differences are regularly distributed in the sense of Definition 6.2, also note Remark 6.3. Moreover, $\delta_l = \beta_1$, $\delta_\phi = 0$ and both expressions exist as a limit, cf. Example 5.20. Theorem 6.5, (*iii*), states that the order of the canonical system is equal to $\beta_1^{-1}$. We take a closer look at the growth by considering the type of the system, $\tau_{1/\beta_1}(H)$.

First, Corollary 5.15 gives that the type of the entire function $F(z) := \sum_{n=0}^{\infty} b_{n,n} z^n$ does not exceed $\tau_{1/\beta_1}(H)$. Note that the order of both $F(z)$ and the canonical system is equal to $\beta_1^{-1}$. Recall that $b_{n,n} = (\rho_1 \rho_2 \dots \rho_{n-1})^{-1}$ denotes the leading coefficient of $P_n(z)$, cf. (2.3). The asymptotics of $\rho_n$ yields

$$
b_{n,n} = (c + \mathrm{O}(1))\big[n!\big]^{-\beta_1} n^{\beta_1 - \frac{x_1}{x_0}} x_0^{-n+1},
$$

for a constant $c > 0$. Using the standard formula for the type of a power series, [Lev80, Theorem 2], we get that the type with respect to the order $\beta_1^{-1}$ is equal to $\beta_1 x_0^{-1/\beta_1}$. In particular, we get

$$
0 < \tau_{1/\beta_1}(F) \le \tau_{1/\beta_1}(H). \tag{8.13}
$$

Secondly we show that $\tau_{1/\beta_1}(H)$ is finite by applying Theorem 5.1 to $H$ and $\lambda(R) := R^{1/\beta_1}$. To this end, set $N(R) := \lfloor \lambda(R) \rfloor$, $\psi := 0$ and

$$
a_j^2(R) := \begin{cases} \frac{j^{\beta_1}}{R} & j = 1, \dots, N(R) - 1 \\ 1 & j = N(R). \end{cases}
$$

Note that the sequence $a_j(R)$ is increasing and bounded from above by 1. We need to estimate the quantities $C_1(R), \dots, C_4(R)$.

$$
C_1(R) = \sum_{j=1}^{N(R)-1} l_j a_j^2(R) \lesssim \frac{1}{R} \sum_{j=1}^{N(R)-1} j^{-\beta_1 + \beta_1} \le \frac{\lambda(R)}{R}.
$$

Concerning $C_2(R)$, we have

$$\begin{aligned} C_2(R) &= \sum_{j=N(R)}^{\infty} l_j\big(a^2_{N(R)}(R)\cos^2(\psi-\phi_j) + a^{-2}_{N(R)}(R)\sin^2(\psi-\phi_j)\big) \\ &\lesssim \sum_{j=N(R)}^{\infty} j^{-\beta_1}\big(\cos^2(\psi-\phi_j)+\sin^2(\psi-\phi_j)\big) \\ &= \sum_{j=N(R)}^{\infty} j^{-\beta_1} = \mathrm{O}\big(R^{(1-\beta_1)/\beta_1}\big) = \mathrm{O}\Big(\frac{\lambda(R)}{R}\Big). \end{aligned}$$

Due to $\ln a_1^{-1}(R) = \ln R$ and $\ln a^{-1}_{N(R)}(R) = 0$, we get $C_4(R) = \mathrm{O}(\log R)$. Regarding $C_3(R)$, we use the simplification for increasing sequences $a_j(R)$ mentioned in Remark 5.2,

$$\begin{aligned} \sum_{j=1}^{N(R)-1} \ln\Big(\big|\cos(\epsilon_j)\big| + \frac{|\sin(\epsilon_j)|}{a^2_{j+1}(R)}\Big) &\le \sum_{j=1}^{N(R)-1} \ln\Big(1+\frac{1}{a^2_{j+1}(R)}\Big) \\ &\le \sum_{j=1}^{N(R)-1} \ln\Big(\frac{2}{a^2_{j+1}(R)}\Big) = \sum_{j=2}^{N(R)} \ln\big(2Rj^{-\beta_1}\big). \end{aligned}$$

Writing the sum as an integral and substituting $y^{-\beta_1} = Rx^{-\beta_1}$ gives,

$$\sum_{j=2}^{N(R)} \ln\big(2Rj^{-\beta_1}\big) \asymp \int_2^{N(R)} \ln\big(2Rx^{-\beta_1}\big)\,dx \le R^{\frac{1}{\beta_1}} \int_0^1 \ln\big(2y^{-\beta_1}\big)\,dy,$$

i.e. $C_3(R) = \mathrm{O}(\lambda(R))$. Theorem 5.1 yields that the type of the canonical system with respect to the order $\beta_1^{-1}$ is finite.

Together with (8.13) we get $0 < \tau_{1/\beta_1}(H) < \infty$, i.e. all four entries of the monodromy matrix are of normal type. By [Lev80, Theorem 14], the upper density of the zeros of the entry $w_{2,1}(z)$ is finite and not zero, i.e.

$$\limsup_{R\to\infty} \frac{n_{w_{2,1}}(R)}{\lambda(R)} \in (0,+\infty),$$

where

$$n_{w_{2,1}}(R) = \#\{z\in\mathbb{C} : w_{2,1}(z)=0, |z|<R\}.$$

Because the spectrum of $T$ interlaces with the zeros of $w_{2,1}$, we conclude that the upper density of the spectrum is also finite and not zero. ❏

## 8.2. The exceptional case

In this section, we consider the exceptional case $\delta = 0$, i.e. $\beta_1 = \beta_2$, and $|y_0| = 2x_0$. Set $\beta := \beta_1 = \beta_2$. The asymptotics of $\rho_n$ and $q_n$ give

$$\rho_n + \rho_{n-1} - \operatorname{sgn}(y_0)q_n = n^{\beta}\big(d_0 + \frac{d_1}{n} + \mathrm{O}(n^{-2})\big),$$

where

$$d_0 := 2x_0 - |y_0| = 0, \quad d_1 := 2x_1 - \operatorname{sgn}(y_0)y_1 - \beta x_0.$$

If $d_1 < 0$, $J$ is of type D due to Wouk's theorem, which is formulated in subsection 2.1.2. Hence, we assume $d_1 \geq 0$.

**8.2 Theorem.** *In the situation of Theorem* 8.1, *assume that* $\delta = 0$, $|y_0| = 2x_0$ *and* $d_1 = 2x_1 - \beta x_0 - \operatorname{sgn}(y_0)y_1 > 0$.

*If* $\beta \in (1, \frac{3}{2}]$, *then* $J$ *is of type D. In the case* $\beta > \frac{3}{2}$ *we state that* $J$ *is of type C. Concerning the order of* $J$, *we have*

$$\rho(J) \begin{cases} \in [\frac{1}{\beta}, \frac{1}{2(\beta-1)}], & \frac{3}{2} < \beta < 2 \\ = \frac{1}{\beta}, & \beta \geq 2. \end{cases} \tag{8.14}$$

*Furthermore, the upper density of the spectrum of* $T$ *is not zero in the case* $\beta \geq 2$, *i.e.*

$$\limsup_{r\to\infty} \frac{n_\sigma(r)}{r^{\frac{1}{\beta}}} \in (0, \infty].$$

*8.3 Remark.* In the case $\frac{3}{2} < \beta < 2$, we do not know the exact order of $J$. Another open question is whether the upper density of the spectrum can be infinity for $\beta \geq 2$. $\Diamond$

*Proof of Theorem* 8.2. We start, exactly as in the proof of Theorem 8.1, with the difference equation,

$$\rho_{n+1}u_{n+2} + q_{n+1}u_{n+1} + \rho_n u_n = 0. \tag{8.15}$$

**Step 1:** Growth of solutions

As in the proof of Theorem 8.1, we write (8.15) in the form

$$v_{n+2} - 2v_{n+1} + (1 - C_n)v_n = 0, \tag{8.16}$$

with $C_n := 1 - \frac{\rho_n}{\rho_{n+1}r_n r_{n+1}}$. This time we get the following asymptotic expansion of $C_n$,

$$C_n = \frac{z_1}{n} + \mathrm{O}(n^{-2}), \quad z_1 := -\frac{2x_1}{x_0} + \frac{2y_1}{y_0} + \beta,$$

in particular $\lim_{n\to\infty} nC_n = z_1$. Due to

$$z_1 = \frac{-1}{x_0}\left(2x_1 - \frac{2x_0}{y_0}y_1 - \beta x_0\right) = \frac{-d_1}{x_0},$$

and due to the assumption $d_1 > 0$, we get that $z_1$ is negative.

We apply [Koo07, Theorem 1 (*i*)], and get two linearly independent solutions of (8.16), denote them by $(v_n^{(j)})_{n=1}^\infty$ for $j = 1, 2$, such that

$$v_n^{(1)} = \overline{v_n^{(2)}} = (1 + \mathrm{O}(1))n^{1/4} \prod_{k=1}^{n-1} \left(1 + i\sqrt{-C_k}\right).$$

The absolute value of each factor is equal to

$$\left|1 + i\sqrt{-C_k}\right| = \sqrt{1 - C_k} = \sqrt{1 - \frac{z_1}{k} + \mathrm{O}(k^{-2})},$$

and we get

$$\left|\prod_{k=1}^{n-1} \left(1 + i\sqrt{-C_k}\right)\right| = (c_1 + o(1))n^{\frac{-z_1}{2}} = (c_1 + o(1))n^{\frac{x_1}{x_0} - \frac{y_1}{y_0} - \frac{\beta}{2}},$$

for some $c_1 > 0$, due to [Koo07, Lemma 4] adding a summable perturbation. Thus

$$\left|v_n^{(1)}\right| = \left|v_n^{(2)}\right| = (c_1 + o(1))n^{\frac{1}{4} + \frac{x_1}{x_0} - \frac{y_1}{y_0} - \frac{\beta}{2}}. \tag{8.17}$$

Substituting back via $u_n = v_n \prod_{i=1}^{n-1} r_i$ produces two solutions of (8.4), denoted by $(u_n^{(j)})_{n=1}^{\infty}$ for $j = 1, 2$. The calculation made in (8.7) yields in our situation

$$\left|\prod_{k=1}^{n-1} r_k\right| = (c_2 + o(1))n^{\frac{y_1}{y_0} - \frac{x_1}{x_0}}$$

for some $c_2 > 0$. Together with (8.17) this results in the asymptotic behaviour

$$\left|u_n^{(1)}\right| = \left|u_n^{(2)}\right| = \left|v_n^{(1)}\right| \left|\prod_{k=1}^{n-1} r_k\right| = (c_3 + o(1))n^{\frac{1}{4} - \frac{\beta}{2}},$$

where $c_3 = c_1 c_2 > 0$. In particular, $J$ is of type C if and only if $\beta > \frac{3}{2}$.

**Step 2:** Conclusions concerning the spectrum

Let $\vec{l}$, and $\vec{\phi}$ denote the lengths and angles of the corresponding Hamburger Hamiltonian. The argument carried out in the beginning of the 3rd step in the proof of Theorem 8.1 yields that

$$\begin{aligned} l_n &= P_n(0)^2 + Q_n(0)^2 \asymp n^{\frac{1}{2} - \beta}, \\ \left|\sin(\phi_{n+1} - \phi_n)\right| &= \left(\rho_n \sqrt{l_n l_{n+1}}\right)^{-1} \asymp n^{-\frac{1}{2} + \beta - \beta} = n^{-\frac{1}{2}}. \end{aligned}$$

The lengths and angle-differences are regularly distributed in the sense of Definition 6.2, note Remark 6.3. Moreover, $\delta_l = \beta - 1/2$, $\delta_\phi = 1/2$ and both expressions exist as a limit, cf. Example 5.20.

In the case $3/2 < \beta < 2$ we apply Theorem 5.10 (with $\mu = 0$, since we have no knowledge about convergence of angles) and get that the order of the canonical system does not exceed

$$\frac{1 - \Delta_\phi - \frac{\mu}{2}}{\Delta_l - \Delta_\phi + \mu} = \frac{1}{2(\beta - 1)}.$$

By Corollary 5.21, $\rho(J)$ is bounded from below by $(\delta_l + \delta_\phi)^{-1} = \beta^{-1}$. These bounds do not coincide.

For $\beta \geq 2$ the conditions of Theorem 6.5, $(i)$, are satisfied, and we get that the order is equal to $\beta_1^{-1}$. In this case it is possible to consider the type of the system, $\tau_{1/\beta_1}(H)$. In contrary to the situation in Theorem 8.1, we will only show that the type is not zero by employing Corollary 5.15. As we have already seen there, the order of $F(z)$ is $\beta_1^{-1}$ and the type with respect to this order is equal to $\beta x_0^{-1/\beta} > 0$. As a result, $\tau_{1/\beta_1}(H) > 0$.

[Lev80, Theorem 14] yields that the upper density of the zeros of the entry $w_{2,1}(z)$ is not zero. The proof is finished, since the spectrum of $T$ interlaces with the zeros of $w_{2,1}$. ❑

The case $d_1 = 2x_1 - \operatorname{sgn}(y_0)y_1 - \beta x_0 = 0$ is not covered in Theorem 8.2, since then $C_n = \mathrm{O}(n^{-2})$, and the essential condition of Kooman's theorem, the existence of a real number $a$ such that

$$\lim_{n\to\infty} n^{-a} C_n = C \neq 0$$

is not guaranteed.

In order to handle this case, it is necessary to start with longer asymptotics,

$$\rho_n = n^{\beta_1}\left(x_0 + \frac{x_1}{n} + \frac{x_2}{n^2} + \mathrm{O}(n^{-3})\right), \quad q_n = n^{\beta_2}\left(y_0 + \frac{y_1}{n} + \frac{y_2}{n^2} + \mathrm{O}(n^{-3})\right).$$

Then $C_n = \frac{z_2}{n^2} + \mathrm{O}(n^{-3})$ for some real $z_2$, and we can start the above argument once more. Note that the next exceptional case emerges in the form of $z_2 = 0$.

In fact, there will always be exceptional cases, regardless of the length of the asymptotics of $\rho_n, q_n$: If $m \in \mathbb{N}$ and $\rho_n, q_n$ have an asymptotic up to $\mathrm{O}(n^{-m})$ and satisfy $\rho_n^2 = (1 + \frac{1}{n})^{\beta_2} q_n^2$, then $C_n = \mathrm{O}(n^{-m})$.

# Open problems

1. Is it always possible to compute the $\lambda$-type of a lcc Hamiltonian with Theorem 3.3, the refined version of Romanov's Theorem 1? In other words, does equality hold in Corollary 3.4 for all $H$?

2. Is it always possible to compute the order of a lcc Hamiltonian with Theorem 3.3?

3. Consider a Hamburger Hamiltonian in the critical triangle, i.e. $\Delta_l + \Delta_\phi < 2$. Even if the data is regularly distributed, the upper bound for the order in Theorem 5.10 and the lower bound from Corollary 5.21 do not coincide, cf. Example 6.6.

   Either the lower or the upper bound should be improved in this case.

4. For an indeterminate moment sequence, Corollary 5.15 gives

   $$\rho\big((s_n)_{n=0}^\infty\big) \geq \rho(F) \geq \rho(L).$$

   By Corollary 6.14 the first inequality can be strict. Is there a moment sequence such that $\rho(F) > \rho(L)$?

5. Proof Proposition 4.6 for general Hamiltonians.

6. Find a direct proof of Theorem 6.9 and Corollary 6.10, without using Theorem 5.10 and Corollary 5.21.

7. Is it possible to formulate an analogue of Theorem 7.18 for general Hamiltonians?

# Bibliography


[Akh61] N. I. Akhiezer. Классическая проблема моментов и некоторые вопросы анализа, связанные с нею. Russian. English translation: *The classical moment problem and some related questions in analysis*, Oliver & Boyd, Edinburgh, 1965. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961.

[Atk64] F.V. Atkinson. *Discrete and continuous boundary problems.* Mathematics in Science and Engineering, Vol. 8. New York: Academic Press, 1964, pp. xiv+570.

[Ber56] Yu. M. Berezanskiĭ. 'Expansion according to eigenfunction of a partial difference equation of order two'. In: *Trudy Moskov. Mat. Obšč.* 5 (1956), pp. 203–268.

[Ber68] Yu. M. Berezanskiĭ. *Expansions in eigenfunctions of selfadjoint operators.* Translated from the Russian by R. Bolstein, J. M. Danskin, J. Rovnyak and L. Shulman. Translations of Mathematical Monographs, Vol. 17. American Mathematical Society, Providence, R.I., 1968, pp. ix+809.

[Bha97] R. Bhatia. *Matrix analysis.* Vol. 169. Graduate Texts in Mathematics. Springer-Verlag, New York, 1997, pp. xii+347.

[BL15] S. Bodine and D. A. Lutz. *Asymptotic integration of differential and difference equations.* Vol. 2129. Lecture Notes in Mathematics. Springer, Cham, 2015, pp. xii+402.

[Boa54] R.P. Boas Jr. *Entire functions.* New York: Academic Press Inc., 1954, pp. x+276.

[BP07] C. Berg and H. L. Pedersen. 'Logarithmic order and type of indeterminate moment problems'. In: *Difference equations, special functions and orthogonal polynomials.* With an appendix by Walter Hayman. World Sci. Publ., Hackensack, NJ, 2007, pp. 51–79.

[BP94] C. Berg and H.L. Pedersen. 'On the order and type of the entire functions associated with an indeterminate Hamburger moment problem'. In: *Ark. Mat.* 32.1 (1994), pp. 1–11.

[BP95] C. Berg and H.L. Pedersen. 'Nevanlinna matrices of entire functions'. In: *Math. Nachr.* 171 (1995), pp. 29–52.

[Bra61] L. de Branges. 'Some Hilbert spaces of entire functions. II'. In: *Trans. Amer. Math. Soc.* 99 (1961), pp. 118–152.

[Bra68] L. de Branges. *Hilbert spaces of entire functions.* Englewood Cliffs, N.J.: Prentice-Hall Inc., 1968, pp. ix+326.

[BS14] C. Berg and R. Szwarc. 'On the order of indeterminate moment problems'. In: *Adv. Math.* 250 (2014), pp. 105–143.

[BS17] C. Berg and R. Szwarc. 'Symmetric moment problems and a conjecture of Valent'. In: *Mat. Sb.* 208.3 (2017), pp. 28–53.

[BV94] C. Berg and G. Valent. 'The Nevanlinna parametrization for some indeterminate Stieltjes moment problems associated with birth and death processes'. In: *Methods Appl. Anal.* 1.2 (1994), pp. 169–209.

[BW06] A.D. Baranov and H. Woracek. 'Subspaces of de Branges spaces with prescribed growth'. In: *Algebra i Analiz* 18.5 (2006), pp. 23–45.

[BZ12] A. Boutet de Monvel and L. Zielinski. 'Explicit error estimates for eigenvalues of some unbounded Jacobi matrices'. In: *Spectral theory, mathematical system theory, evolution equations, differential and difference equations.* Vol. 221. Oper. Theory Adv. Appl. Birkhäuser/Springer Basel AG, Basel, 2012, pp. 189–217.

[Car26] T. Carleman. *Les Fonctions quasi analytiques.* Collection de monographies sur la théorie des fonctions. Gauthier-Villars et Cie, 1926.

[Dei+99] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides and X. Zhou. 'Strong asymptotics of orthogonal polynomials with respect to exponential weights'. In: *Comm. Pure Appl. Math.* 52.12 (1999), pp. 1491–1552.

[Fle96] A. Fleige. *Spectral theory of indefinite Krein-Feller differential operators.* Vol. 98. Mathematical Research. Akademie Verlag, Berlin, 1996, p. 133.

[Fre05] U. Freiberg. 'Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets'. In: *Forum Math.* 17.1 (2005), pp. 87–104.

[GK67] I. C. Gohberg and M. G. Kreĭn. Теория вол′терровых операторов в гил′бертовом пространстве и ее приложения. Russian. English translation: *Theory and applications of Volterra operators in Hilbert space*, Translations of Mathematical Monographs, Vol.24, American Mathematical Society, Providence, R.I., 1970. Izdat. "Nauka", Moscow, 1967, p. 508.

[GK69] I. C. Gohberg and M. G. Kreĭn. *Introduction to the theory of linear nonselfadjoint operators.* Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969, pp. xv+378.

[HSW00] S. Hassi, H. de Snoo and H. Winkler. 'Boundary-value problems for two-dimensional canonical systems'. In: *Integral Equations Operator Theory* 36.4 (2000), pp. 445–479.

[JM07] J. Janas and M. Malejki. 'Alternative approaches to asymptotic behaviour of eigenvalues of some unbounded Jacobi matrices'. In: *J. Comput. Appl. Math.* 200.1 (2007), pp. 342–356.

[JN04] J. Janas and S. Naboko. 'Infinite Jacobi matrices with unbounded entries: asymptotics of eigenvalues and the transformation operator approach'. In: *SIAM J. Math. Anal.* 36.2 (2004), pp. 643–658.

[Kac85] I.S. Kac. 'Linear relations, generated by a canonical differential equation on an interval with a regular endpoint, and expansibility in eigenfunctions'. Russian. In: *VINITI Deponirovannye Nauchnye Raboty* 195.1 (1985). Deposited in Ukr NIINTI, No. 1453, 1984, 50 pp., b.o. 720.

[Kac86a] I.S. Kac. 'Expansibility in eigenfunctions of a canonical differential equation on an interval with singular endpoints and associated linear relations'. Russian. In: *VINITI Deponirovannye Nauchnye Raboty* 282.12 (1986). Deposited in Ukr NIINTI, No. 2111, 1986, 64 pp., b.o. 1536.

[Kac86b] I.S. Kac. 'Integral estimates for the distribution of the spectrum of a string'. In: *Sibirsk. Mat. Zh.* 27.2 (1986), pp. 62–74, 221.

[Kac90] I.S. Kac. 'Thightness of the spectrum of a singular string'. In: *Izv. Vyssh. Uchebn. Zaved. Mat.* 3 (1990), pp. 23–30.

[Kac99] I.S. Kac. 'Inclusion of the Hamburger power moment problem in the spectral theory of canonical systems'. Russian. In: *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 262.Issled. po Linein. Oper. i Teor. Funkts. 27 (1999). English translation: J. Math. Sci. (New York) 110 (2002), no. 5, 2991–3004, pp. 147–171, 234.

[KK68] I.S. Kac and M.G. Krein. '$R$-functions – Analytic functions mapping the upper half plane into itself'. Russian. In: Addition I in F.V.Atkinson, Дискретные и непрерывные граничные задачи (Russian translation). English translation: Amer. Math. Soc. Transl. (2) 103 (1974), 1–19. Izdat. "Mir", Moscow, 1968, pp. 629–647.

[KL79] M.G. Krein and H. Langer. 'On some extension problems which are closely connected with the theory of Hermitian operators in a space $\Pi_\kappa$. III. Indefinite analogues of the Hamburger and Stieltjes moment problems. Part I'. In: *Beiträge Anal.* 14 (1979), 25–40 (loose errata).

[KL80] M.G. Krein and H. Langer. 'On some extension problems which are closely connected with the theory of Hermitian operators in a space $\Pi_\kappa$. III. Indefinite analogues of the Hamburger and Stieltjes moment problems. Part II'. In: *Beiträge Anal.* 15 (1980), 27–45 (1981).

[KL85] M.G. Krein and H. Langer. 'On some continuation problems which are closely related to the theory of operators in spaces $\Pi_\kappa$. IV. Continuous analogues of orthogonal polynomials on the unit circle with respect to an indefinite weight and related continuation problems for some classes of functions'. In: *J. Operator Theory* 13.2 (1985), pp. 299–417.

[KM57] S. Karlin and J. McGregor. 'The differential equations of birth-and-death processes, and the Stieltjes moment problem'. In: *Trans. Amer. Math. Soc.* 85 (1957), pp. 489–546.

[Koo07] R.-J. Kooman. 'An asymptotic formula for solutions of linear second-order difference equations with regularly behaving coefficients'. In: *J. Difference Equ. Appl.* 13.11 (2007), pp. 1037–1049.

[Kre51] M.G. Krein. 'On the theory of entire matrix functions of exponential type'. In: *Ukrain. Mat. Žurnal* 3 (1951), pp. 164–173.

[KW06] M. Kaltenbäck and H. Woracek. 'Pontryagin spaces of entire functions. IV'. In: *Acta Sci. Math. (Szeged)* 72.3-4 (2006), pp. 709–835.

[KW10] M. Kaltenbäck and H. Woracek. 'Pontryagin spaces of entire functions. VI'. In: *Acta Sci. Math. (Szeged)* 76.3-4 (2010), pp. 511–560.

[KW11] M. Kaltenbäck and H. Woracek. 'Pontryagin spaces of entire functions. V'. In: *Acta Sci. Math. (Szeged)* 77.1-2 (2011), pp. 223–336.

[KWW06] M. Kaltenbäck, H. Winkler and H. Woracek. 'Singularities of generalized strings'. In: *Operator theory and indefinite inner product spaces.* Vol. 163. Oper. Theory Adv. Appl. Basel: Birkhäuser, 2006, pp. 191–248.

[KWW07] M. Kaltenbäck, H. Winkler and H. Woracek. 'Strings, dual strings, and related canonical systems'. In: *Math. Nachr.* 280.13-14 (2007), pp. 1518–1536.

[Lev80] B. Ja. Levin. *Distribution of zeros of entire functions.* Revised. Vol. 5. Translations of Mathematical Monographs. Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. Providence, R.I.: American Mathematical Society, 1980, pp. xii+523.

[LG86] P. Lelong and L. Gruman. *Entire functions of several complex variables.* Vol. 282. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, 1986, pp. xii+270.

[Liv39] M. Livšic. 'On some questions concerning the determinate case of Hamburger's moment problem'. Russian. English summary. In: *Rec. Math. N. S. [Mat. Sbornik]* 6(48) (1939), pp. 293–306.

[LW98] H. Langer and H. Winkler. 'Direct and inverse spectral problems for generalized strings'. In: *Integral Equations Operator Theory* 30.4 (1998). Dedicated to the memory of Mark Grigorievich Krein (1907–1989), pp. 409–431.

[Nev22] R. Nevanlinna. 'Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentenproblem.' German. In: *Ann. Acad. Sci. Fenn., Ser. A* 18 (1922), p. 53.

[Pru18] R. Pruckner. 'Density of the spectrum of Jacobi matrices with power asymptotics'. In: *Asymptot. Anal.* (2018). submitted.

[PRW15] R. Pruckner, R. Romanov and H. Woracek. *Bounds on order of indeterminate moment sequences.* 26 pp., ASC Report 38. http://www.asc.tuwien.ac.at/preprint/2015/asc38x2015.pdf. Vienna University of Technology, 2015.

[PRW16] R. Pruckner, R. Romanov and H. Woracek. 'Bounds on Order of Indeterminate Moment Sequences'. In: *Constructive Approximation* (2016), pp. 1–27.

[PW17] R. Pruckner and H. Woracek. 'Estimates for order of Nevanlinna matrices and a Berezanskii-type theorem'. In: *Proc. Roy. Soc. Edinburgh Sect. A* (2017). to appear.

[Rie23] M. Riesz. 'Sur le problème des moments. III.' French. In: *Ark. f. Mat., Astr. och Fys.* 17.16 (1923), 52 p.

[Rom17] R. Romanov. 'Order problem for canonical systems and a conjecture of Valent'. In: *Trans. Amer. Math. Soc.* 369.2 (2017), pp. 1061–1078.

[Sod96] M. Sodin. 'A remark to the definition of Nevanlinna matrices'. In: *Mat. Fiz. Anal. Geom.* 3.3-4 (1996), pp. 412–422.

[Tur03] È. A. Tur. 'Asymptotics of eigenvalues for a class of Jacobi matrices with a limit point spectrum'. In: *Mat. Zametki* 74.3 (2003), pp. 449–462.

[Val99] G. Valent. 'Indeterminate moment problems and a conjecture on the growth of the entire functions in the Nevanlinna parametrization'. In: *Applications and computation of orthogonal polynomials (Oberwolfach, 1998).* Vol. 131. Internat. Ser. Numer. Math. Birkhäuser, Basel, 1999, pp. 227–237.

[Win93] H. Winkler. 'Zum inversen Spektralproblem für zweidimensionale kanonische Systeme'. PhD thesis. TU Wien, May 1993.

[Win95] H. Winkler. 'The inverse spectral problem for canonical systems'. In: *Integral Equations Operator Theory* 22.3 (1995), pp. 360–374.

[Wou53] A. Wouk. 'Difference equations and $J$-matrices'. In: *Duke Math. J.* 20 (1953), pp. 141–159.

[WW12] H. Winkler and H. Woracek. 'Reparametrizations of non trace-normed Hamiltonians'. In: *Spectral theory, mathematical system theory, evolution equations, differential and difference equations.* Vol. 221. Oper. Theory Adv. Appl. Birkhäuser/Springer Basel AG, Basel, 2012, pp. 667–690.

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