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Optimal Contour Choice for Option Pricing by Fourier Transform

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Abstract

This thesis studies the contour choice suggested by Lord and Kahl for the computation of an option price by means of the Carr-Madan representation. In particular, we show that this choice is optimal in the sense that it minimizes the maximal modulus of the integrand associated with the Carr-Madan representation. Furthermore, we prove that the optimization problem characterizing this choice always has a solution. The established results are then applied in the Heston model. For that purpose we also provide a full derivation of the Heston moment generating function. It turns out that once this is done properly there cannot be any worries about discontinuities appearing in the result. Moreover, we derive the critical time in the Heston model and present its connection to the critical moments in detail. Finally, we derive the asymptotic behavoir of the Heston moment generating function when the imaginary part of the argument becomes large. This leads to two formulas for the Heston call price. The first one allows for an easy computation by means of Gauss-Laguerre quadrature and the second contains only an integral of a continuous function on $[0, 1]$. Using the programming language `R` we illustrate the derived results.

Keywords: Fourier transform, option pricing, optimal contour choice, Heston model, moment generating function, critical time, asymptotic behavior

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Chapter 1

Introduction

As for instance mentioned by John C. Hull, [Hul09, Ch. 1], derivative securities have become increasingly important in finance over the last decades. A financial derivative can be shortly yet a bit vaguely described as a financial instrument whose value depends on a more basic security. A concrete example for such a derivative contract would be a European call option on a certain stock. As pointed out in [Hul09, Ch. 1] such a call option gives its holder the right but not the obligation to buy the underlying asset at a certain price and date. That price and date are referred to as strike and maturity respectively. Obviously, when trading these options one may ask how to determine their precise values in a sensible way. At the maturity associated with the call option, which we denote by T, this is easy. Therefore, assume that you consider a European call option with strike K, maturity T whose underlying is a stock with value S_T at maturity. Now distinguish two cases.

- ▷ First suppose that the price S_T of the underlying stock exceeds the strike price K, i. e. $S_T > K$. At maturity T any sensible holder would exercise the option, which means to purchase the stock at the strike price K. Immediately afterwards one would sell the stock at its market price S_T. Because of the assumption $S_T > K$ this generates a profit of $S_T - K > 0$.
- ▷ Now assume that the price S_T of the underlying stock does not exceed the strike price K, i. e. $S_T \leq K$. In this case it would not make to exercise the option, i. e. buying the stock at the strike price K, since the stock is at least as cheap on the market. Accordingly, the option holder does not undertake any action leading to a profit of 0. In a nutshell the option expires worthlessly.

Summing up the value of the call option at maturity T is given by

$$(S_T - K)^+ := \max\{S_T - K, 0\}.$$

Determining the price of this call option prior to its maturity T is not that straightforward. The difficulty comes from the fact that the stock price S_T at maturity is not known before T. Thus, it is usually regarded as random variable. Referring to the current point in time as time zero, i. e. $t = 0$, it even makes sense to model the stock price S in the whole time interval $[0, T]$ by means of a stochastic process $(S_t)_{t\in[0,T]}$.
To the author's knowledge modelling of stock prices and option pricing has first been discussed in the seminal thesis "Théorie de la Spéculation" by Louis Bachelier in 1900, [Bac00]. As mentioned by Walter Schachermayer in [Sch08], Bachelier analyzed Brownian motion mathematically in order to develop a theory on option pricing. In modern terminology Bachelier's model amounts to modelling the discounted stock prices $(\widetilde{S}_t)_{t\in[0,T]}$ by

$$\widetilde{S}_t = S_0(1 + \sigma W_t), \quad 0 \leq t \leq T,$$

where $W = (W_t)_{t\in[0,T]}$ denotes a Brownian motion under the risk neutral measure, S_0 is the known initial stock price and $\sigma > 0$ is a parameter for the stock volatility. Decades after Bachelier's seminal paper, P.

Samuelson proposed in 1965 to consider a financial market made up of two securities B and S. On the one hand, there is a riskless security $B = (B_t)_{t\in[0,T]}$, given by

$$B_t = e^{rt}, \quad 0 \le t \le T,$$

where $r \ge 0$ denotes the riskless interest rate. On the other hand, there is a risky stock S, defined via the discounted stock price process $\widetilde{S}$ by

$$\widetilde{S}_t := S_t B_t^{-1} = S_0 \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right), \quad 0 \le t \le T, \tag{1.1}$$

where $W = (W_t)_{t\in[0,T]}$ is a Brownian motion under the (unique) risk-neutral measure, S_0 is the known initial stock price and $\sigma > 0$ is a volatility parameter. This model became famous as Black-Merton-Scholes model because it was also used in the groundbreaking papers [BS73] and [Mer73], which were published by Fischer Black, Myron Scholes and Robert Merton in 1973. As pointed out by Freddy Delbaen and Walter Schachermayer in [DS06] these papers introduce the key ideas of option pricing, namely the use of trading in continuous time and the notion of arbitrage. In particular within this so-called Black-Merton-Scholes model the option price is determined so that no arbitrage opportunities arise when adding the option to the financial market.

Subsequently, in the late seventies and early eighties, a fascinatingly close link between these no-arbitrage arguments and so-called risk neutral pricing was established. The latter basically refers to pricing an option by means of the corresponding expected discounted payoff at maturity, i. e.

$$\mathbb{E}\big(B_T^{-1}(S_T - K)^+\big), \tag{1.2}$$

where the expectation is taken with respect to a probability measure such that the discounted stock price process is a martingale. A measure with the latter property is then referred to as martingale measure. Evaluating the expectation in (1.2) under the assumptions made in (1.1) leads exactly to the option price that has already been presented by Black, Scholes and Merton in 1973.

Nowadays the Black-Merton-Scholes model is still a very popular tool for pricing an option. However, there are lots of emperical observations suggesting that stock prices cannot be modelled well by geometric Brownian motion, which is assumed in (1.1). Even if one puts aside the empirical knowledge about daily stock returns – by pointing out that only the risk-neutral distribution of the underlying matters for option pricing – major discrepancies remain. For example, one would need different volatilities σ for the same underlying stock in order to reproduce market option prices via the Black-Scholes-Merton model. This empirical observation is commonly referred to as volatility smile and for instance discussed in more detail in [Hul09, Ch. 18].

Such shortcomings of the Black-Merton-Scholes model have led to the introduction of other option pricing models in the literature. One example is a model introduced by Heston in 1993, [Hes93]. The main feature of the so-called Heston model is the inclusion of a stochastic variance process ν. In addition to the riskless security B, which we already know from the Black-Scholes-Merton model, the risky stock S is defined – again via the discounted stock price process $\widetilde{S}$ – as follows.

$$\widetilde{S}_t := S_t B_t^{-1} = S_0 \exp\left(-\frac{1}{2}\int_0^t \nu_u \, du + \int_0^t \sqrt{\nu_u}\, dW_u\right), \quad 0 \le t \le T, \tag{1.3}$$

where $W = (W_t)_{t\in[0,T]}$ is again a Brownian motion under the risk neutral measure, S_0 denotes the known initial stock price and ν is the already mentioned stochastic variance process. Note that with a constant and deterministic variance, given by $\nu_t = \sigma^2$ for $t \in [0,T]$, the process in (1.3) reduces to those presented in (1.1). While the expectation in (1.2) also gives an arbitrage free option price in the Heston model, the actual computation is more demanding than in the Black-Merton-Scholes model. For that purpose one can use a quite general formula relating the call option price, given by (1.2), to the moment generating function (MGF) of the log-discounted underlying $X_T := \ln \widetilde{S}_T$. Since one can derive a closed-form expression for the MGF of X_T in the Heston model that formula is of particular use here. This relation, which was for example already mentioned by Peter Carr and Dilip Madan in [CMS99] as well as by Lewis in [Lew01], reads as follows for every log-discounted strike $k = \ln K - rT$.

$$\mathbb{E}\Big(\big(\widetilde{S}_T - e^k\big)^+\Big) = R_\alpha(k) + \frac{e^k}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{-kz}\frac{M_{X_T}(z)}{z(z-1)}\,dz, \tag{1.4}$$

where M_{X_T} denotes the MGF of X_T, α is basically such that $M_{X_T}(\alpha)$ is finite and $R_\alpha(k)$ is a residue term depending on k and α. An interesting feature of the formula in (1.4) is that a parameter α appears on the right hand side which cannot be found on the left hand side. The only restriction on α is that it must belong to the interior of the set of all real numbers where M_{X_T} is finite. As long as α satifies the latter it can be freely chosen when using (1.4) for the numerical computation of a call option price. However, for the different possible choices of α the integrand in (1.4) can behave very differently. In particular this Fourier-type integrand can get very oscillatory, which is not quite desirable from a numerical point of view. Therefore, it makes sense to think about choosing α optimally in some way. One choice for α is that discussed by Roger Lord and Christian Kahl in [LK07]. They suggest to use an α that minimizes the maximal modulus of the integrand in (1.4). This choice is particularly discussed within this thesis.

This thesis is structured as follows. In Chapter 2, we start by recalling the key mathematical concepts necessary for a precise treatise. This includes the introduction of the very general financial market model suggested by Harrison and Pliska in 1981, [HP81], a short section on stochastic differential equations and results from complex analysis.
Chapter 3 is then dedicated to the derivation of the Fourier call price representation mentioned in (1.4). In the proof of that formula we provide much detail regarding the application of the residue theorem.
Next, we introduce the Heston model in Chapter 4, as we want to apply the eventual results in that model. There we focus on the derivation of MGF of the log-discounted underlying X_T since it plays an important role in (1.4). Even though the MGF of X_T is presented in various papers, such as [Hes93], [LK10] and in Jim Gatheral's well-known book [GT06], the author has decided to provide a full derivation within this thesis. The motivation behind that mainly comes from the following considerations. On the one hand, it can easily be proven that M_{X_T} is analytic on the interior D of its domain. Hence, whenever one states that a certain expression equals $M_{X_T}(z)$, also the continuous dependence of that expression on $z \in D$ is stated. On the other hand, there are discussions in the literature whether an expression for $M_{X_T}(z)$ continuously depends on the argument $z \in D$. For example Lord and Kahl mention this issue for the Heston MGF in [LK10]. Thus it is somehow confusing to question the continuity of expressions that are regarded to equal M_{X_T} because every flawlessly derived expression must automatically be continuous on D. This caused the author to look for a rigorous and detailed proof that a certain expression equals M_{X_T}. Unfortunately, such a derivation turned out to be hard to find, which eventually led to the focus on the MGF of X_T in the Heston model in Chapter 4. In particular that comprises results on the boundaries of the domain of M_{X_T}, which one can use to determine the interval of possible choices for α's in (1.4) in the Heston model.
In Chapter 5, we continue by introducing the usual FFT-method for the computation of call prices by means of (1.4). There we also provide a straightforward error-estimate for that method and illustrate its application. Based on the findings by Lord and Kahl in [LK07] we deal with an optimal choice of α for the numerical evaluation of (1.4). In the Heston model we derive the precise asymptotic behavior of M_{X_T}, which is used to apply the results regarding the optimal choice of α. The latter is then utilized to present formulas for the Heston call option price that can easily be implemented by means of Gauss-Laguerre or Gauss-Legendre quadrature.
Finally, note that at the end of Chapter 3 and Chapter 4 one can find summaries of the proven results. Furthermore, an appendix contains additional proofs to make the treatise complete.

Chapter 2

Mathematical Foundations

2.1 Mathematical Finance

We present the quite general continuous time model suggested by Harrison and Pliska (1981), [HP81]. In particular we also address how to price options within that framework. Throughout this section let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space where $\mathbb{F}$ satisfies the usual conditions.
We define a financial market as a collection of $d+1$ primary assets $S = (S^{(0)}, S^{(1)}, \dots, S^{(d)})$. These assets are assumed to be càdlàg semimartingales on a time interval $[0, T]$. Time $T > 0$ denotes the finite time horizon of the financial market. Furthermore, $S^{(0)}$ is interpreted as riskless savings account and thus assumed to have the following properties

- $S_0^{(0)} = 1$.
- $S^{(0)}$ has continuous and increasing paths.

As a consequence we can assume that $S^{(0)}$ is always of the form

$$S_t^{(0)} = e^{R_t}, \quad 0 \leq t \leq T, \tag{2.1}$$

with a semimartingale $R = (R_t)_{t \geq 0}$ that has increasing paths and satisfies $R_0 = 0$. Because of the monotonicity the paths are always of finite variation on every bounded interval. In the sequel the process $S^{(0)}$ will also be referred to as numeraire.
The other d assets, i. e. $S^{(1)}, \dots, S^{(d)}$, are the risky assets available on the financial market considered. First, we introduce the notation we use when considering the value of the risky assets relative to the riskless savings account.

Definition 2.1.1. The *discounted asset processes* $\widetilde{S}^{(1)}, \dots, \widetilde{S}^{(d)}$ are defined by

$$\widetilde{S}_t^{(i)} = \frac{S_t^{(i)}}{S_t^{(0)}}, \quad 0 \leq t \leq T,$$

for $1 \leq i \leq d$. Since we assume $\widetilde{S}_t^{(0)} > 0$ for $0 \leq t \leq T$ they are well-defined.

Next, we introduce trading strategies, the value process and the gains process of the corresponding portfolio.

Definition 2.1.2. A *trading strategy* is any $\mathbb{R}^{d+1}$-valued process that is predictable and locally bounded. The corresponding *value process* $V(\phi)$ is defined by

$$V_t(\phi) := \sum_{i=0}^{d} \phi_t^i S_t^{(i)}, \quad 0 \leq t \leq T.$$

The corresponding *gains process* $G(\phi)$ is defined by

$$G_t(\phi) := \int_0^t \phi_u \, dS_u := \sum_{i=0}^{d} \int_0^t \phi_u^i \, dS_u^{(i)}, \quad 0 \le t \le T.$$

The corresponding *discounted value process* $\widetilde{V}(\phi)$ and *discounted gains process* $\widetilde{G}(\phi)$ are given by

$$\widetilde{V}_t(\phi) = \frac{V_t(\phi)}{S_t^{(0)}} \quad \text{and} \quad \widetilde{G}_t(\phi) = \frac{G_t(\phi)}{S_t^{(0)}},$$

for $0 \le t \le T$.

Remark 2.1.3. The assumption that ϕ is predictable and locally bounded ensures that the occurring integrals with respect to càdlàg semimartingales are well-defined.

Self-financing and admissible trading strategies are particularly important when talking about arbitrage opportunities later on. These are defined as follows.

Definition 2.1.4. A trading strategy ϕ is called *self-financing* if

$$V_t(\phi) = V_0(\phi) + G_t(\phi) \qquad 0 \le t \le T, \quad a.\ s.$$

A trading strategy ϕ is called *admissible* if there is a constant $c \ge 0$ such that

$$V_t(\phi) \ge -c, \quad 0 \le t \le T, \quad a.\ s.$$

The following lemma shows that for admissible trading strategies also the discounted value process is bounded from below.

Lemma 2.1.5. *If a trading strategy ϕ is admissible then there is a constant $\widetilde{c} \ge 0$ satisfying*

$$\widetilde{V}_t(\phi) \ge -\widetilde{c}, \quad 0 \le t \le T, \quad a.\ s. \tag{2.2}$$

Proof. Assume that ϕ is admissible. Define $\widetilde{c} = c$. Then we a. s. have

$$V_t(\phi) \ge -c \Leftrightarrow \widetilde{V}_t(\phi) \ge -\frac{c}{S_t^{(0)}} \Rightarrow \widetilde{V}_t(\phi) \ge -c = -\tilde{c}, \quad 0 \le t \le T.$$

□

In the next proposition a very important characterization of the self-financing property is derived.

Proposition 2.1.6 ([HP81], Proposition 3.24)**.** *A trading strategy $(\phi)_{t\in[0,T]}$ is self-financing if and only if*

$$\widetilde{V}_t(\phi) = \widetilde{V}_0(\phi) + \sum_{i=1}^{d} \int_0^t \phi_u^i \, d\widetilde{S}_u^{(i)}, \quad 0 \le t \le T, \quad a.\ s.$$

holds.

Proof. See [HP81, Proposition 3.24]. It is essential that $S^{(0)}$ has paths which are continuous and of finite variation. □

Now we can introduce a key concept in mathematical finance, namely those of an arbitrage opportunity.

Definition 2.1.7. A trading strategy ϕ is an *arbitrage opportunity* if it is self-financing, admissible and satisfies

- $P\big(V_0(\phi) = 0\big) = 1$,
- $P\big(V_T(\phi) \geq 0\big) = 1$ and
- $P\big(V_T(\phi) > 0\big) > 0$.

A financial market model is called *arbitrage free* if there are no arbitrage opportunities.

Another key concept is those of martingale measures which is closely linked to the abscence of arbitrage.

Definition 2.1.8. An probability measure Q on $(\Omega, \mathcal{F}_T)$ is called *martingale measure* for S if $\widetilde{S}$ is a Q-local martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ and $Q \sim P$ on $(\Omega, \mathcal{F}_T)$.

The aim now is to derive a connection between martingale measures and the absence of arbitrage. Therefore we need the subsequent lemmas.

Lemma 2.1.9 ([Pro04], Theorem IV.29)**.** *Let $M = (M_t)_{t\geq 0}$ be a local martingale and $H = (H_t)_{t\geq 0}$ a predictable and locally bounded process. Then the process $N = (N_t)_{t\geq 0}$ defined by*

$$N_t = \int_0^t H_u \, dM_u, \quad t \geq 0,$$

is a local martingale.

Proof. See [Pro04, Theorem IV.29]. □

Lemma 2.1.10. *Consider a local martingale $N = (N_t)_{t\in[0,T]}$ satisfying*

$$N_t \geq -c, \quad 0 \leq t \leq T, \quad a.\ s.$$

for a constant $c \in \mathbb{R}$. Then it holds that

$$\mathbb{E}\big(N_t\big) \leq \mathbb{E}\big(N_u\big), \quad 0 \leq u \leq t \leq T.$$

Proof. Assume $0 \leq u \leq t$. With a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ for N we clearly have

$$N_{t\wedge\tau_n} \geq c, \quad n \in \mathbb{N}, \quad a.\ s.$$

Furthermore, the uniform lower bound implies that N_t and N_u are quasi-integrable. Thus we can apply Fatou's lemma for conditional expectations to obtain

$$\mathbb{E}\big(N_t \big| \mathcal{F}_u\big) = \mathbb{E}\Big(\lim_{n\to\infty} N_{t\wedge\tau_n} \big| \mathcal{F}_u\Big) \leq \liminf_{n\to\infty} \mathbb{E}\Big(N_{t\wedge\tau_n} \big| \mathcal{F}_u\Big) = \lim_{n\to\infty} N_{u\wedge\tau_n} = N_u.$$

Taking expectations on both sides yields the result. □

Now we can present a link between martingale measures and the absence of arbitrage.

Theorem 2.1.11. *Assume that there is a martingale measure $Q \sim P$ for $S = \big(S^{(0)}, \ldots, S^{(d)}\big)$. Then the financial market is arbitrage free.*

Proof. Assume that there is a self-financing and admissible trading strategy such that $V_0(\phi) = 0$ and $V_T(\phi) \geq 0$ hold $P-a.\ s.$ Since $Q \sim P$ this also holds $Q-a.\ s.$ Because ϕ is self-financing we know by Proposition 2.1.6 that

$$\widetilde{V}_t(\phi) = \widetilde{V}_0(\phi) + \sum_{i=1}^d \int_0^t \phi_u^i \, d\widetilde{S}_u^{(i)}, \quad 0 \leq t \leq T, \quad P-a.\ s.$$

Again because of $Q \sim P$ the latter also holds $Q - a.\ s.$ Now observe that $\widetilde{V}(\phi)$ can be represented as a sum of integrals with integrators that are local martingales under Q and predictable and locally bounded integrands. By Lemma 2.1.9 we can conclude that $\widetilde{V}(\phi)$ is a local martingale under Q. Furthermore, we know that ϕ is an admissible trading strategy. Together with the fact that $P \sim Q$ we can use Lemma 2.1.5 to see that there is a constant $c \geq 0$ such that

$$\widetilde{V}_t(\phi) \geq -c, \quad 0 \leq t \leq T, \quad Q - a.\ s.$$

Consequently under Q the assumptions of Lemma 2.1.10 are fulfilled with $N = \widetilde{V}(\phi)$. Applying this Lemma with $u = 0,\ t = T$ and recalling $V_0(\phi) = 0$ a. s. gives

$$\mathbb{E}^Q\big(V_T(\phi)\big) \leq \mathbb{E}^Q\big(V_0(\phi)\big) = 0.$$

Since $V_T(\phi)$ was assumed to be non-negative $Q - a.\ s.$ we conclude

$$V_T(\phi) = 0, \quad Q - a.\ s.$$

Because of $P \sim Q$ we see that $V_T(\phi) = 0$ holds also $P - a.\ s.$ Consequently ϕ cannot be an arbitrage opportunity. ☐

Remark 2.1.12. The last theorem is a part of the fundamental theorem of asset pricing (FTAP) which states that the converse is also true in essence. What in essence exactly means is a priori not entirely clear but is adressed in great detail in Delbaen's and Schachermayer's Mathematics of Arbitrage, [DS06].

The last theorem gives us now the tools to include options or other contingent claims in an existing financial market such that the extended market is arbitrage free.

Definition 2.1.13. A *contingent claim* with maturity T is a non-negative $\mathcal{F}_T$-measurable random variable.

Corollary 2.1.14. *Consider a financial market model with assets $S = \big(S^{(0)}, S^{(1)}, \ldots, S^{(d)}\big)$, where $S^{(0)}$ is of the form given in* (2.1)*, and a contingent claim X with maturity T. Define the discounted contingent claim $\widetilde{X}$ by*

$$\widetilde{X} = \frac{X}{S_T^{(0)}}.$$

Let Q be a martingale measure such that

$$\mathbb{E}^Q\big(|\widetilde{X}|\big) < \infty.$$

Define $S^{(d+1)}$ by

$$S_t^{(d+1)} = S_t^{(0)} \mathbb{E}^Q\big(\widetilde{X}|\mathcal{F}_t\big), \quad 0 \leq t \leq T. \tag{2.3}$$

Then the extended financial market with assets $\big(S^{(0)}, S^{(1)}, \ldots, S^{(d+1)}\big)$ is arbitrage free.

Proof. The assets $S^{(0)},\ S^{(1)}, \ldots,\ S^{(d)}$ are local martingales under Q by assumption. Now observe that $\widetilde{S}^{(d+1)}$ as it is defined in (2.3) is a martingale. To show this we have to show the three defining properties of a martingale. First, by definition of the conditional expectation the process $\widetilde{S}^{(d+1)}$ is adapted to $\mathbb{F}$. Furthermore,

$$\mathbb{E}^Q\Big(\big|\widetilde{S}_t^{(d+1)}\big|\Big) = \mathbb{E}^Q\Big(\big|\mathbb{E}^Q\big(\widetilde{X}|\mathcal{F}_t\big)\big|\Big) \leq \mathbb{E}^Q\Big(\mathbb{E}^Q\big(|\widetilde{X}||\mathcal{F}_t\big)\Big) \leq \mathbb{E}^Q\big(|\widetilde{X}|\big) < +\infty,$$

and thus $\widetilde{S}_t^{(d+1)}$ is integrable with respect to Q for $0 \leq t \leq T$. Because of the tower property for conditional expectations we obtain

$$\mathbb{E}^Q\Big(\widetilde{S}_t^{(d+1)}\Big|\mathcal{F}_u\Big) = \mathbb{E}^Q\Big(\mathbb{E}^Q\big(\widetilde{X}|\mathcal{F}_t\big)\Big|\mathcal{F}_u\Big) = \mathbb{E}^Q\big(\widetilde{X}|\mathcal{F}_u\big) = \widetilde{S}_u^{(d+1)}, \quad a.\ s.$$

for $0 \leq u \leq t \leq T$, which finally makes $\widetilde{S}^{(d+1)}$ a martingale under Q. In particular it is a local martingale under Q. This makes $Q \sim P$ also a martingale measure for the extended financial market $\big(S^{(0)}, S^{(1)}, \ldots, S^{(d+1)}\big)$. By Theorem 2.1.11 the extended market is arbitrage free. ☐

2.2 Stochastic Differential Equations

In continuous time financial market models assets are often defined as unique solutions to stochastic differential equations (SDEs). Therefore we give a short introduction on SDEs where we present results basically taken from the well-known book by Ikeda and Watanabe, [IW89].

Definition 2.2.1. Let $b\colon [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ be Borel-measurable functions. The pair (b, σ) is then called a *stochastic differential equation (SDE)* of Markovian type with drift coefficient b and diffusion coefficient σ. Furthermore, if the coefficients do not depend on the first argument, i. e. the time argument, the SDE is called *time-independent.* In the latter case the time argument t will often be omitted in the coefficients b and σ.

Remark 2.2.2. At this point our approach to interpret an SDE as pair is quite formal. However, in the author's opinion this interpretation makes it easier to explain the different solution concepts for SDEs. In particular the equation is not associated with a single Brownian motion or probability space which is also the case for strong solutions as we will see later on.

2.2.1 Solution Concepts for Stochastic Differential Equations

In this subsection we introduce solution concepts for SDEs following [IW89].

Definition 2.2.3. By a *(weak) solution* to the SDE (b, σ) we mean an $\mathbb{R}^d$-valued stochastic process $X = (X_t)_{t\geq 0}$ defined on a filtered probability space $\big(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}\big)$ such that

(i) there exists an r-dimensional $\mathbb{F}$-Brownian motion $W = (W_t)_{t\geq 0}$,

(ii) X is adapted to $\mathbb{F}$ and has continuous paths a. s.

(iii) it holds that

$$\int_0^t |b_i(s, X_s)|\, ds + \int_0^t \sigma_{ij}^2(s, X_s)\, ds < \infty, \quad t \geq 0, \quad a.\ s.$$

for $1 \leq i \leq d$ and $1 \leq j \leq r$.

(iv) Finally, with probability one the process X and the Brownian motion W satisfy

$$X_t - X_0 = \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s, \quad t \geq 0. \tag{2.4}$$

To emphasize the particular role of the $\mathbb{F}$-Brownian motion W we also call X a solution with the Brownian motion W or sometimes even consider the pair (X, W).

Remark 2.2.4. The property that (X, W) is a solution to the SDE (b, σ) is commonly abbreviated by

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \quad t \geq 0.$$

Next, we want to present different concepts for the uniqueness of solutions. To do so we need to introduce the following notation at first.

- C_d denotes the space of all continuous functions from $[0, +\infty)$ to $\mathbb{R}^d$. This space is also equipped with the metric ρ defined by

$$\rho(f, g) = \sum_{k=1}^{\infty} 2^{-k} \Big(1 \wedge \max_{0\leq t\leq k} |f(t) - g(t)|\Big), \quad f,\ g \in C_d.$$

  Together with this metric C_d is even a complete and separable space.

- $C_{d,0}$ denotes the subspace $C_{d,0} := \{f \in C_d \,|\, f(0) = 0\}$ of C_d.
- $\mathcal{B}(S)$, where S is a metric space, denotes the corresponding Borel σ-algebra, i. e. the σ-algebra generated by all open sets with respect to metric considered.
- $\mathcal{B}_t(S)$, where S is a subspace of C_d and $t \in [0, +\infty)$, denotes the σ-algebra generated by the mapping $\rho_t \colon S \to (C_d, \mathcal{B}(C_d))$, given by

$$\rho_t(f) = f(\cdot \wedge t).$$

- $\overline{\mathcal{F}}^{\mu}$ denotes the completion of a σ-algebra $\mathcal{F}$ with respect to a measure μ.

Definition 2.2.5. We say that there is *uniqueness in law* for an SDE (b, σ) if whenever X and $\widetilde{X}$ are two solutions such that X_0 and $\widetilde{X}_0$ have the same law on $\mathbb{R}^d$, then the laws of the processes X and $\widetilde{X}$ coincide on C_d.

Remark 2.2.6. Note that the processes X and $\widetilde{X}$ considered in Definition 2.2.5 may be defined on different probability spaces.

Definition 2.2.7. We say that *pathwise uniqueness* holds for an SDE (b, σ) if whenever (X, W) and $(\widetilde{X}, W)$ are any two solutions, defined on the same filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$ and with the same Brownian motion W, such that $X_0 = \widetilde{X}_0$ a. s. then X and $\widetilde{X}$ are indistinguishable, i. e.

$$P\big(X_t = \widetilde{X}_t, \quad t \geq 0\big) = 1.$$

With the notation introduced before we can now define strong solutions and strong uniqueness.

Definition 2.2.8. A solution $X = (X_t)_{t \geq 0}$ with Brownian motion $W = (W_t)_{t \geq 0}$ is called *strong solution* if there exists a function $F \colon \mathbb{R}^d \times C_{d,0} \to C_d$ with the following properties.

(i) For any probability measure μ on $\mathbb{R}^d$ there exists a function $\widetilde{F}_\mu \colon \mathbb{R}^d \times C_{d,0} \to (C_d, \mathcal{B}(C_d))$ which is measurable with respect to $\overline{\mathcal{B}(\mathbb{R}^d \times C_{d,0})}^{\mu \otimes P_W}$, such that

$$\widetilde{F}_\mu(x, f) = F(x, f)$$

holds for μ-almost all $x \in \mathbb{R}^d$ and P_W-almost all $f \in C_{d,0}$. Here P_W denotes the Wiener measure on $(C_{d,0}, \mathcal{B}(C_{d,0}))$.

(ii) For each $x \in \mathbb{R}^d$ the function $F(x, \cdot)$ is $\overline{\mathcal{B}_t(C_{d,0})}^{P_W}$-$\mathcal{B}(C_d)$-measurable for all $t \geq 0$.

(iii) It holds that

$$X = F(X_0, W), \quad a.\ s. \tag{2.5}$$

Definition 2.2.9. We say that an SDE (b, σ) admits a *unique strong solution* if there exists a function $F \colon \mathbb{R}^d \times C_{d,0} \to C_d$ with properties (i) and (ii) from Definition 2.2.8, such that

(i) for any $\mathbb{F}$-Brownian motion W on a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$ and any $\mathcal{F}_0$-measurable random variable ξ_0 the continuous process X, defined by

$$X = F(\xi_0, W),$$

is a solution to the SDE (b, σ) on that filtered probability space with $X_0 = \xi_0$ a. s.

(ii) for any solution (X, W) to the SDE (b, σ) it holds that

$$X = F(X_0, W), \quad a.\ s.$$

2.2.2 Existence and Uniqueness of Solutions

Now we tackle the question of existence and uniqueness of solutions to an SDE. We start by very fundamental results which are due to Watanabe and Yamada (1971), [WY71].

Theorem 2.2.10 ([IW89], Theorem IV.1.1)**.** *An SDE (b, σ) admits a unique strong solution if and only if the following two conditions are satisfied.*

(i) For any probability measure μ on $\mathbb{R}^d$ a (weak) solution X to (b, σ) exists, such that the law of X_0 coincides with μ.

(ii) Pathwise uniqueness holds for (b, σ).

Proof. See [IW89, Theorem IV.1.1] or [WY71, Corollary 1]. □

Theorem 2.2.11 ([IW89], Corollary IV)**.** *Pathwise uniqueness implies uniqueness in law for an SDE (b, σ).*

Proof. See [IW89, Corollary IV] or [WY71, Proposition 1]. □

The next theorem addresses the existence of a (weak) solution to an SDE with continuous and time-independent coefficients.

Theorem 2.2.12 ([IW89], Theorem IV.2.3 and IV.2.4)**.** *Consider a time-independent SDE (b, σ). Suppose the coefficients b and σ are continuous functions and that there is a constant $C > 0$ such that*

$$\|b(x)\|_2^2 + \|\sigma(x)\|_F^2 \leq C\big(1 + \|x\|_2^2\big),$$

holds for all $x \in \mathbb{R}^d$. Here $\|\cdot\|_F$ denotes the Frobenius norm. Then for any probability measure μ on $\mathbb{R}^d$ there exists a weak solution X such that the law of X_0 coincides with μ.

Proof. For probability measure μ with compact supports this is [IW89, Theorems IV.2.3, IV.2.4]. The extension to general probability measures μ can be done using [IW89, Remark IV.2.1] and [IW89, Proposition IV.2.1]. □

Next, we want to address the question of pathwise uniqueness. Recalling Theorem 2.2.10 and Theorem 2.2.12 observe that this also brings us closer to conditions which ensure the existence of a unique strong solution.

Theorem 2.2.13 ([IW89], Theorem IV.3.1)**.** *Consider a time-independent SDE (b, σ). Suppose that the coefficients b and σ are locally Lipschitz-continuous, i. e. for every integer $n \geq 1$ there exists a constant $K_n > 0$ such that*

$$\|b(x) - b(y)\|_2^2 + \|\sigma(x) - \sigma(y)\|_F^2 \leq K_n \|x - y\|_2^2, \quad x, y \in B_n(0). \tag{2.6}$$

Then pathwise uniqueness holds for the SDE (b, σ).

Proof. See [IW89, Theorem IV.3.1]. □

The Lipschitz condition is essential for the proof of Theorem 2.2.13. However, this assumption can be considerably weakened which is made more precise with a result, due to Watanabe and Yamada (1971), [WY71, Theorem 1]. We state a theorem found in [SA13] where $\mathbb{R}^d$-valued solutions are addressed.

Theorem 2.2.14 ([SA13], Theorem 2.1)**.** *Consider an SDE (b, σ). Assume that there exists a constant $\gamma > 0$ and functions $\kappa, \varrho \colon [0, \gamma] \to [0, +\infty)$ satisfying $\kappa(0) = 0$,*

$$\|b(t, x) - b(t, y)\|_2 \leq \kappa(\|x - y\|_2), \tag{2.7}$$

$$\|\sigma(t, x) - \sigma(t, y)\|_F \leq \varrho(\|x - y\|_2) \tag{2.8}$$

for all $t \geq 0$ and $x,\ y \in \mathbb{R}^d$ with $\|x-y\|_2 \leq \gamma$. Furthermore, assume that ϱ is non-decreasing, $\varrho(u) > 0$ for $u \in (0,\gamma]$ and its square satisfies the Osgood condition, i. e.

$$\int_{(0,\gamma]} \frac{1}{\varrho^2(u)}\, du = +\infty.$$

In addition, assume that there exists a non-decreasing, concave and continuous function $G\colon [0,\gamma] \to [0,+\infty)$ with $G(0) = 0$, strictly positive on $(0,\gamma]$, such that

$$G(u) \geq \kappa(u) + \frac{d-1}{2u}\varrho^2(u), \quad u \in (0,\gamma],$$

which also satisfies the Osgood condition

$$\int_{(0,\gamma]} \frac{1}{G(u)}\, du = +\infty.$$

Then pathwise uniqueness holds for the SDE (b,σ) holds.

Proof. See [SA13, Theorem 2.1]. □

For the one-dimensional case there is the following useful corollary of Theorem 2.2.14.

Corollary 2.2.15. *Consider a one-dimensional SDE (b,σ). Assume that $b\colon [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with respect to the second argument and that $\sigma\colon [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is Hölder continuous with exponent $p \geq \frac{1}{2}$ in the second argument, i. e. there are constants $E, F > 0$ and $p \geq \frac{1}{2}$ such that*

$$|b(t,x) - b(t,y)| \leq E|x-y|,$$
$$|\sigma(t,x) - \sigma(t,y)| \leq F|x-y|^p,$$

for all $x, y \in \mathbb{R}$ and $t \geq 0$. Then pathwise uniqueness holds for the SDE (b,σ).

Proof. We show that Theorem 2.2.14 can be applied. Therefore define $\gamma = 1$ and $\kappa, \varrho\colon [0,\gamma] \to [0,+\infty)$ by

$$\kappa(u) = Eu$$
$$\varrho(u) = Fu^p$$

for $u \in [0,\gamma]$. Then (2.7) and (2.8) are clearly satisfied by the assumptions made. Furthermore, ϱ is non-decreasing and satisfies $\varrho(u) > 0$ for $u \in (0,\gamma]$. Because of $2p \geq 1$ we have

$$\int_{(0,\gamma]} \frac{1}{\varrho^2(u)}\, du = +\infty.$$

With $G := \kappa$ we have

$$\int_{(0,\gamma]} \frac{1}{G(v)}\, dv = -\frac{1}{C} \lim_{\epsilon \to 0} \ln \epsilon = +\infty,$$

and

$$G(u) \geq \kappa(u) + 0 \quad \text{for all } u \in (0,\gamma].$$

Thus all the assumptions of Theorem 2.2.14 are satisfied and pathwise uniqueness holds. □

The following theorem combines the presented existence and uniqueness results.

Theorem 2.2.16. *Consider an SDE (b, σ) with time-independent coefficients. Assume that the coefficients are continuous and satisfy*

$$\|b(x)\|_2^2 + \|\sigma(x)\|_F^2 \le C\big(1 + \|x\|_2^2\big), \quad x \in \mathbb{R}, \tag{2.9}$$

with a constant $C > 0$. Furthermore, assume that the coefficients b and σ are such that Theorem 2.2.14 can be applied. Then the SDE (b, σ) has a unique strong solution.

Proof. The assumptions ensure that Theorem 2.2.12 and Theorem 2.2.14 are applicable. Hence the considered SDE has a (weak) solution for every initial law μ and pathwise uniqueness holds. Consequently, by Theorem 2.2.10, the SDE (b, σ) admits a unique strong solution. ☐

The subsequent corollary gives a simplified result for the one-dimensional case. It is useful when showing that the variance process in the Heston model is well-defined.

Corollary 2.2.17. *Consider a one-dimensional and time-independent SDE (b, σ). Assume that $b\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and that $\sigma\colon \mathbb{R} \to \mathbb{R}$ is Hölder continuous with exponent $p \ge \frac{1}{2}$, i. e. there are constants $E, F > 0$ and $p \ge \frac{1}{2}$ such that*

$$|b(x) - b(y)| \le E|x - y|,$$
$$|\sigma(x) - \sigma(y)| \le F|x - y|^p,$$

for all $x, y \in \mathbb{R}$. Furthermore, assume that the coefficients satisfy the linear growth condition, i. e. there is a constant $C > 0$ such that

$$|b(x)|^2 + |\sigma(x)|^2 \le C(1 + x^2),$$

for $x \in \mathbb{R}$. Then the SDE (b, σ) admits a unique strong solution.

Proof. Firstly, observe that (2.9) is clearly fulfilled by the assumptions made. Moreover, the coefficients b and σ are in particular assumed to be continuous. Thus Theorem 2.2.12 can be applied to see that a (weak) solution exists. Since the assumptions also ensure that Corollary 2.2.15 is applicable for the time-independent SDE (b, σ) we deduce that pathwise uniqueness holds. Hence Theorem 2.2.10 can be applied to obtain that a unique strong solution exists for (b, σ). ☐

Finally, we also want to present a so-called comparison theorem. This turns out to be useful when examining whether a solution takes values beyond a certain threshold. For example we use it to show the non-negativity of the variance process in the Heston model.

Theorem 2.2.18 ([KS91], Theorem 5.2.18)**.** *Consider two one-dimensional SDEs $(b^{(1)}, \sigma)$ and $(b^{(2)}, \sigma)$ with the same real-valued diffusion coefficient. Assume there are solutions $X^{(1)}$ to $(b^{(1)}, \sigma)$ and $X^{(2)}$ to $(b^{(2)}, \sigma)$ on the same filtered probability space $\big(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t \ge 0}\big)$ with the same Brownian motion W. Furthermore, assume the following.*

(i) $b^{(1)}$*,* $b^{(2)}$ *and* σ *are continuous, real-valued functions on* $[0, +\infty) \times \mathbb{R}$*.*

(ii) σ *satisfies*

$$|\sigma(t, x) - \sigma(t, y)| \le \varrho(|x - y|),$$

for every $t \in [0, +\infty)$ and $x, y \in \mathbb{R}$ with a function $\varrho\colon [0, +\infty) \to [0, +\infty)$ which is strictly increasing with $\varrho(0) = 0$ and

$$\int_{(0,\epsilon)} \frac{1}{\varrho^2(u)}\, du = +\infty, \quad \textit{for all } \epsilon > 0.$$

(iii) $X_0^{(1)} \le X_0^{(2)}$ *a. s. and* $b^{(1)}(t,x) \le b^{(2)}(t,x)$ *for all* $(t,x) \in [0,+\infty) \times \mathbb{R}$.

(iv) Either $b^{(1)}$ *or* $b^{(2)}$ *is Lipschitz continuous in the second argument.*

Then

$$P\Big(X_t^{(1)} \le X_t^{(2)}, \quad 0 \le t < +\infty\Big) = 1.$$

Proof. See [KS91, Theorem 5.2.18]. □

Remark 2.2.19. Analogously as in Corollary 2.2.15 one sees that assumption (ii) in Theorem 2.2.18 can be replaced by the condition that the diffusion coefficient σ is Hölder-continuous with exponent $p \ge \frac{1}{2}$ in the second argument, i. e. there is an $F > 0$ and $p \ge \frac{1}{2}$ such that

$$|\sigma(t,x) - \sigma(t,y)| \le F|x-y|^p,$$

for all $x, y \in \mathbb{R}$ and $t \ge 0$.

2.2.3 Feynman-Kac Theorem

In this subsection consider a filtered probability space $\big(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t\ge 0}\big)$ with an $\mathbb{F}$-Brownian motion $W = (W_t)_{t\ge 0}$ on it. Motivated by the drift and diffusion coefficients of an SDE we define infinitesimal generators as follows.

Definition 2.2.20. For some $T > 0$ and Borel sets $D \subseteq G \subseteq \mathbb{R}^d$, where G is also open, consider continuous functions

$$b\colon [0,T] \times G \to \mathbb{R}^d \quad \text{and} \quad \sigma\colon [0,T] \times D \to \mathbb{R}^{d\times r}.$$

Then any differential operator $A\colon \mathcal{C}^2\big([0,T] \times G, \mathbb{R}\big) \to \mathcal{C}\big([0,T] \times G, \mathbb{R}\big)$ given by

$$(Af)(t,x) = \sum_{i=1}^n b_i(t,x)\frac{\partial f}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^n C_{ij}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x), \quad (t,x) \in [0,T] \times G,$$

where $C\colon [0,T] \times G \to \mathbb{R}$ is such that $C|_{[0,T]\times D} := \sigma\sigma^\top$, is referred to as corresponding infinitesimal generator associated with (b, σ).

The following proposition will be useful when computing expectations of diffusions, in particular to derive characteristic functions. Versions of this result can be found in e. g. [KS91, Theorem 5.7.6] or [Oks00, Theorem 8.2.1].

Theorem 2.2.21 (Feynman-Kac)**.** *For* $T > 0$ *and Borel sets* $D \subseteq G \subseteq \mathbb{R}^d$*, where* G *is also open, consider continuous functions*

$$b\colon [0,T] \times G \to \mathbb{R}^d \qquad \sigma\colon [0,T] \times D \to \mathbb{R}^{d\times r}.$$

Assume that an $\mathbb{F}$*-adapted and almost surely* D*-valued and continuous stochastic process* $X = (X_t)_{t\in[0,T]}$ *with initial value* $x_0 \in D$ *satisfies*

$$X_t^i = x_0^i + \int_0^t b_i(s, X_s)\, ds + \sum_{k=1}^r \int_0^t \sigma_{ik}(s, X_s)\, dW_s^k, \quad 0 \le t \le T, \quad 1 \le i \le d,$$

where the equality sign is understood in the way that the left and the right hand side are indistinguishable. Furthermore, consider functions $\phi \in \mathcal{C}(D, \mathbb{R})$ *and* $q \in \mathcal{C}\big([0,T] \times D, \mathbb{R}\big)$*, where the latter is bounded from below. Assume that* $f \in \mathcal{C}^{1,2}\big([0,T] \times G, \mathbb{R}\big)$ *satisfies*

$$f(t,x) \ge c, \quad (t,x) \in [0,T] \times D,$$

for a constant $c \in \mathbb{R}$, and

$$\frac{\partial f}{\partial t}(t,x) + Af(t,x) = q(t,x)f(t,x), \quad (t,x) \in [0,T] \times D \tag{2.10}$$

$$f(T,x) = \phi(x), \quad x \in D, \tag{2.11}$$

where A denotes an infinitesimal generator associated with (b,σ). Then we have

$$\mathbb{E}\Big(\phi(X_T)e^{-\int_0^T q(s,X_s)\,ds}\Big) \le f(0,x_0). \tag{2.12}$$

In particular the expectation on the left hand side cannot be $+\infty$. With the additional assumption that f is bounded on $[0,T] \times D$ even equality holds in (2.12).

Proof. Define $L = (L_t)_{t\in[0,T]}$ by

$$L_t = e^{-\int_0^t q(s,X_s)\,ds}, \quad 0 \le t \le T.$$

Standard calculus shows

$$dL_t = -q(t,X_t)L_t\,dt. \tag{2.13}$$

Since G is open we can use Ito's formula, see [Sep10, Theorem 6.15] for example, to obtain

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t) + Af(t,X_t)\,dt + \sum_{i=1}^d\sum_{k=1}^r \sigma_{ik}(t,X_t)\frac{\partial f}{\partial x_i}(t,X_t)\,dW_t^k \tag{2.14}$$

on $[0,T]$. Now define $(Z_t)_{t\in[0,T]}$ by $Z_t = L_t f(t,X_t)$ for $t \in [0,T]$. Then we get

$$dZ_t = f(t,X_t)\,dL_t + L_t\,df(t,X_t) + d[f(\cdot,X),L]_t = f(t,X_t)\,dL_t + L_t\,df(t,X_t) \tag{2.15}$$

on $[0,T]$, where the latter equality sign holds because L is continuous and of finite variation on $[0,T]$. Plugging (2.13) and (2.14) into (2.15) gives

$$\begin{aligned} dZ_t &= -q(t,X_t)L_t f(t,X_t)\,dt + L_t\Big(\frac{\partial f}{\partial t}(t,X_t) + Af(t,X_t)\Big)\,dt + \sum_{i=1}^d\sum_{k=1}^r L_t\sigma_{ik}(t,X_t)\frac{\partial f}{\partial x_i}(t,X_t)\,dW_t^k = \\ &= L_t\Big(\frac{\partial f}{\partial t}(t,X_t) + Af(t,X_t) - q(t,X_t)f(t,X_t)\Big)\,dt + \sum_{i=1}^d\sum_{k=1}^r L_t\sigma_{ik}(t,X_t)\frac{\partial f}{\partial x_i}(t,X_t)\,dW_t^k = \\ &= \sum_{i=1}^d\sum_{k=1}^r L_t\sigma_{ik}(t,X_t)\frac{\partial f}{\partial x_i}(t,X_t)\,dW_t^k \end{aligned}$$

on $[0,T]$. In particular $(Z_t)_{t\in[0,T]}$ is a local martingale. This means that there exists a localizing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $(Z_{t\wedge\tau_n})_{t\in[0,T]}$ is a true martingale for every n. In particular we have

$$\mathbb{E}\Big[L_{T\wedge\tau_n} f\big(T\wedge\tau_n, X_{T\wedge\tau_n}\big)\Big] = \mathbb{E}\big[Z_{T\wedge\tau_n}\big] = \mathbb{E}\big[Z_{0\wedge\tau_n}\big] = \mathbb{E}\big[Z_0\big] = \mathbb{E}\big[L_0 f(0,X_0)\big] = f(0,x_0).$$

Because q is bounded from below, i. e. for some $C_q \ge 0$ we have $q(t,x) \ge -C_q$ for $(t,x) \in [0,T] \times D$, gives

$$\big|L_t\big| = L_t = \exp\Big(-\int_0^t q(s,X_s)\,ds\Big) \le e^{C_q t} \le e^{C_q T}, \quad 0 \le t \le T.$$

Thus

$$\big|L_{T\wedge\tau_n}\big| \le e^{C_q T}, \quad \text{for } n \in \mathbb{N}.$$

Assuming in addition that $f(t,x) \geq c$ for $(t,x) \in [0,T] \times D$ gives

$$L_{T\wedge\tau_n} f\big(T \wedge \tau_n, X_{T\wedge\tau_n}\big) \geq \; - e^{C_q T}|c|, \quad n \in \mathbb{N}.$$

Thus we can apply Fatou's lemma to get

$$\begin{aligned}\mathbb{E}\Big(\phi(X_T)e^{-\int_0^T q(s,X_s)\,ds}\Big) &= \mathbb{E}\big(L_T f(T,X_T)\big) = \mathbb{E}\Big(\liminf_{n\to\infty} L_{T\wedge\tau_n} f\big(T\wedge\tau_n, X_{T\wedge\tau_n}\big)\Big) \leq \\ &= \liminf_{n\to\infty} \mathbb{E}\Big(L_{T\wedge\tau_n} f\big(T\wedge\tau_n, X_{T\wedge\tau_n}\big)\Big) = \liminf_{n\to\infty} f(0,x_0) = f(0,x_0).\end{aligned}$$

In particular the expectation cannot be $+\infty$.
If for some $C_f \geq 0$ we even have $|f(t,x)| \leq C_f$, for $(t,x) \in [0,T] \times D$, then one can obtain the estimate

$$|L_{T\wedge\tau_n} f\big(T \wedge \tau_n, X_{T\wedge\tau_n}\big)| \leq e^{C_q T} C_f, \quad n \in \mathbb{N}.$$

Consequently we can apply the theorem of dominated convergence instead of Fatou's lemma to get

$$\begin{aligned}\mathbb{E}\Big(\phi(X_T)e^{-\int_0^T q(s,X_s)\,ds}\Big) &= \mathbb{E}\big(L_T f(T,X_T)\big) = \mathbb{E}\Big(\lim_{n\to\infty} L_{T\wedge\tau_n} f\big(T\wedge\tau_n, X_{T\wedge\tau_n}\big)\Big) = \\ &= \lim_{n\to\infty} \mathbb{E}\Big(L_{T\wedge\tau_n} f\big(T\wedge\tau_n, X_{T\wedge\tau_n}\big)\Big) = \lim_{n\to\infty} f(0,x_0) = f(0,x_0).\end{aligned}$$

□

Remark 2.2.22. Furthermore, note that due to the linearity of the PDE in (2.10) Theorem 2.2.21 can also be applied to a complex valued $\phi\colon D \to \mathbb{C}$ and a bounded complex-valued $f \in \mathcal{C}^{1,2}\big([0,T] \times G, \mathbb{C}\big)$ with an open set $G \subseteq \mathbb{R}^d$. To see this just consider the real and imaginary parts of ϕ and f separately.

2.3 Complex Analysis

To derive different formulations of call prices we will need the residue theorem. This is a very fundamental result in complex analysis. Thus we shortly introduce the reader to complex analysis. The terminology and theorems presented in this section are taken from a class on complex analysis I attended in the summer term 2013 at Vienna University of Technology, [Kal13].

Definition 2.3.1. Let $D \subseteq \mathbb{C}$ be open. A function is said to be complex differentiable in a point $z \in D$ if the limit

$$f'(z) := \lim_{w\to z} \frac{f(w) - f(z)}{w - z}$$

exists. A function $f\colon D \to \mathbb{C}$ that is complex differentiable in every $z \in D$ such that the function $z \mapsto f'(z)$ is continuous is called holomorphic.

Definition 2.3.2. Let $D \subset \mathbb{C}$ be an open set and $f\colon D \to \mathbb{C}$ be a continuous function. Consider a continuous and piecewise continuously differentiable path $\gamma\colon [a,b] \to D$. Then we define

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Theorem 2.3.3. *Consider a function $f\colon D \to \mathbb{C}$, where D is an open subset of $\mathbb{C}$. Then the following statements are equivalent.*

(i) f is holomorphic.

(ii) f is complex differentiable at every point $z \in D$.

(iii) f *is analytic, i. e. for every* $w \in D$ *there is an open disk* $B_{\rho_w}(w) \subseteq D$ *with respect to* $|\cdot|$*, such that* f *can be represented by a power series on* $B_{\rho_w}(w)$ *with a radius of convergence not smaller than* ρ_w*.*

(iv) f *is arbitrarily often complex differentiable at every point* $z \in D$*.*

As the previous theorem indicates complex differentiability is a very strong property. This concept has even a lot more consequences as for example the theorems below show.

Theorem 2.3.4 (Liouville's Theorem)**.** *Consider a bounded holomorphic function* $f\colon \mathbb{C} \to \mathbb{C}$*. Then* f *is constant.*

Theorem 2.3.5 (Identity Theorem)**.** *Let* $D \subseteq \mathbb{C}$ *be an open connected set and* $f,\ g\colon D \to \mathbb{C}$ *holomorphic. Assume that* f *and* g *satisfy at least one of the following two properties.*

- *The set* N *of all points where* f *and* g *coincide has a limit point in* D*.*
- $f^{(n)}(w) = g^{(n)}(w)$ *for all* $n \in \mathbb{N}_0$ *and at least one* $w \in D$*.*

Then $f = g$ *on* D*.*

Under suitable conditions complex differentiability also remains after integration such that differential and integral sign can be interchanged as the following theorem shows.

Theorem 2.3.6. *Let* $(\omega, \mathcal{A}, \mu)$ *be a measure space,* $G \subseteq \mathbb{C}$ *open and* $f\colon G \times \Omega \to \mathbb{C}$ *a function such that*

- $x \mapsto f(z,x)$ *is integrable for all* $z \in G$
- $z \mapsto f(z,x)$ *is holomorphic for all* $x \in \Omega \setminus N$ *where* N *is a fixed null set.*
- *For every compact set* $K \subseteq G$ *there exists an integrable function* $g_K\colon \Omega \to \overline{\mathbb{R}}$ *such that for all* $z \in K$ *and* $x \in \Omega \setminus N$ *the estimate*
$$|f(z,x)| \leq g_K(x)$$
holds. Note that N *is the fixed null set from above.*

Then it holds that the function $F(z) := \int_\Omega f(z,x)\, d\mu(x)$ *is holomorphic on* G *and* $\frac{\partial^n f}{\partial z^n}(z,\cdot)$ *is integrable for all* $z \in G$*,* $n \in \mathbb{N}$ *such that*
$$F^{(n)}(z) = \int_\Omega \frac{\partial^n f}{\partial z^n}(z,x)\, d\mu(x).$$

Before we can present the next results on holomorphic functions we need the following terminology.

Definition 2.3.7. Consider an open set $D \subseteq \mathbb{C}$, $w \in D$ and a holomorophic $f\colon D \setminus \{w\} \to \mathbb{C}$. Then w is a

- a *removable singularity* if there is a holomorphic extension to D of f,
- a *pole* if its not a removable singularity and there exists an $N \in \mathbb{N}$ such that w is a removable singularity of the function $z \mapsto (z-w)^N f(z)$. The minimum of all these N is called the order of the pole.
- an *essential singularity* if it is neither a removable singularity nor a pole.

Definition 2.3.8. For a closed, continuous and piecewise continuously differentiable path $\gamma\colon [a,b] \to \mathbb{C}$ and for $w \in \mathbb{C} \setminus \gamma([a,b])$ the winding number $n(\gamma, w)$ is defined by
$$n(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - w}\, d\zeta.$$

Our aim now is to formulate the residue theorem. We start with a special case of this theorem, namely Cauchy's Integral Formula.

Theorem 2.3.9 (Cauchy's Integral Formula)**.** *Let $D \subseteq \mathbb{C}$ be open, $f\colon D \to \mathbb{C}$ holomorphic and $\gamma_k\colon [a_k, b_k] \to D$, $k = 1, \ldots, m$ closed, continuous and piecewise continuously differentiable paths satisfying $\sum_{k=1}^m n(\gamma_k, z) = 0$ for all $z \in \mathbb{C} \setminus D$. Then for every $z \in D \setminus \bigcup_{k=1}^m \gamma_k([a_k, b_k])$ one has*

$$f(z) \sum_{k=1}^{m} n(\gamma_k, z) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{2.16}$$

Furthermore, the right hand side of (2.16) *vanishes for $z \in \mathbb{C} \setminus D$.*

In order to define residues we introduce Laurent series and their connection to complex analysis.

Theorem 2.3.10 (Laurent Series)**.** *Let $D \subseteq \mathbb{C}$ be open and $w \in \mathbb{C}$ such that $B_{R_w}(w) \setminus K_{r_w}(w) \subseteq D$ for some $0 \le r_w < R_w \le +\infty$. Furthermore let $f\colon D \to \mathbb{C}$ holomorphic. Then*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-w)^n, \quad z \in B_{R_w}(w) \setminus K_{r_w}(w), \tag{2.17}$$

where the power series $\sum_{n=0}^{+\infty} a_n (z-w)^n$ has a radius of convergence not smaller than R_w and $\sum_{n=0}^{+\infty} a_{-n}(z-w)^n$ has a radius of convergence not smaller than r_w^{-1}. The coefficients a_n, $n \in \mathbb{Z}$, are uniquely determined and given by

$$a_n = \frac{1}{2\pi i} \int_{\vec{\partial} U_\rho(w)} \frac{f(\zeta)}{(\zeta - w)^{n+1}}\, d\zeta, \quad n \in \mathbb{Z},$$

for each radius $\rho \in (r_w, R_w)$, where $\vec{\partial} U_\rho(w)$ denotes the circle parametrized by $t \mapsto w + \rho e^{it}$ for $t \in [0, 2\pi]$. The series in (2.17) *is referred to as Laurent series.*

Having introduced Laurent series we can define residues.

Definition 2.3.11. Let $D \subseteq \mathbb{C}$ be open, $w \in D$ and $f\colon D \setminus \{w\} \to \mathbb{C}$ holomorphic. With the corresponding Laurent series from Theorem 2.3.10 satisfying

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-w)^n, \quad z \in B_{R_w}(w) \setminus K_{r_w}(w),$$

the residue $\operatorname{Res}(f, w)$ is defined by

$$\operatorname{Res}(f, w) := a_{-1} = \frac{1}{2\pi i} \int_{\vec{\partial} U_\rho(w)} f(\zeta)\, d\zeta.$$

Remark 2.3.12. Assume that $D \subseteq \mathbb{C}$ is open, $w \in D$ and $f\colon D \setminus \{w\} \to \mathbb{C}$ is holomorphic. Furthermore, suppose f has a pole of order m in w. Then one can show that

$$\operatorname{Res}(f, w) = \frac{1}{(m-1)!} \left[\frac{\partial^{m-1} f}{\partial z^{m-1}} \Big((z-w)^m f(z) \Big) \right]_{z=w}.$$

Finally, we can present the residue theorem, which was the aim of this section.

Theorem 2.3.13 (Residue Theorem)**.** *Let $D \subseteq \mathbb{C}$ be open, $w_1, \ldots, w_n \in D$ and $f\colon D \setminus \{w_1, \ldots, w_n\} \to \mathbb{C}$ holomorphic. Furthermore, let $\gamma_k\colon [a_k, b_k] \to D \setminus \{w_1, \ldots, w_n\}$, $k = 1, \ldots, m$, be closed, continuous and piecewise continuously differentiable paths satisfying $\sum_{k=1}^m n(\gamma_k, z) = 0$ for all $z \in \mathbb{C} \setminus D$. Then it holds that*

$$\sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} f(\zeta)\, d\zeta = \sum_{j=1}^{n} \operatorname{Res}(f, w_j) \cdot \sum_{k=1}^{m} n(\gamma_k, w_j).$$

Chapter 3

Option Pricing by Fourier Transform

Consider a financial market with a riskless asset B and a risky asset S as primary securities. Assume you face the task of pricing a European call option C with strike K and maturity T. The terminal payoff of the option is then of course given by $(S_T - K)^+$. The challenge is now to find a price for the option such that the extended model does not admit any arbitrage opportunities. The fundamental theorem of asset pricing states that this is closely connected to finding a probability measure Q such that the stochastic process associated with the discounted risky asset is a martingale with respect to that measure. Using such a measure Q an arbitrage free extended market can be obtained by defining the prices of the call option via

$$C_t = B_t \mathbb{E}_Q\big(B_T^{-1}(S_T - K)^+ | \mathcal{F}_t\big), \quad 0 \leq \ \leq T$$

Note that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ models the information about the primary assets available to the market. Hence pricing a call option can be reduced to the computation of an expectation. Unfortunately the cumulative distribution function or density of S_T is very often not given in closed form which makes an efficient computation very difficult. However, sometimes the moment generating function of the log-underlying can be expressed in closed form. Examples for such models are the Heston model, the Normal Inverse Gaussian model and many more. By means of Fourier transform methods one can use the moment generating function to determine the desired expectation. Thus, following [Lee04] and [CMS99], we introduce this Fourier transform approach here.

3.1 Characteristic and Moment Generating Functions

We start with the definition of moment generating functions and present some of their properties. Throughout this section we consider a probability space $(\Omega, \mathcal{A}, P)$ and a real-valued random variable X on it.

Definition 3.1.1. For a random variable X the moment generating function (MGF) $M_X\colon D \subseteq \mathbb{C} \to \mathbb{C}$ is defined by

$$M_X(z) = \mathbb{E}\big(e^{zX}\big), \quad z \in D, \tag{3.1}$$

where D is the set of all complex numbers $z = z_1 + iz_2$ such that

$$\mathbb{E}\big(e^{z_1 X}\big) < +\infty.$$

Remark 3.1.2. Note that we allow for complex arguments in the moment generating function. Moreover, for $u \in \mathbb{R}$ always a sensible value in $\mathbb{R} \cup \{+\infty\}$ can be assigned to $M_X(u)$, which extends the notation from above. However, by domain of the moment generating function we mean the set D from Definition 3.1.1.

Definition 3.1.3. For a random variable X the characteristic function $\varphi_X\colon \mathbb{R} \to \mathbb{C}$ is defined by

$$\varphi_X(u) = \mathbb{E}\big(e^{iuX}\big), \quad u \in \mathbb{R}.$$

Remark 3.1.4. Because of $\left|e^{itX}\right| = 1$ the characteristic function is always well-defined for all real numbers. Next, we present some results which turn out to be useful when deriving MGFs and their domains.

Proposition 3.1.5. *Assume that there is an $u_0 > 0$ ($u_0 < 0$) such that $\mathbb{E}\big(e^{u_0 X}\big) < \infty$. Then*

$$\mathbb{E}\big(e^{uX}\big) < \infty, \quad \textit{for} \quad 0 < u < u_0 \quad (u_0 < u < 0).$$

Proof. Because of $\left|\frac{u}{u_0}\right| < 1$ we can use Jensen's inequality for concave functions to obtain

$$\mathbb{E}\big(e^{uX}\big) = \mathbb{E}\Big(\big(e^{u_0 X}\big)^{u/u_0}\Big) \le \mathbb{E}\big(e^{u_0 X}\big)^{u/u_0} < \infty.$$

□

Remark 3.1.6. The previous proposition implies that the set of all real numbers where the moment-generating function is finite is an interval. However, it is a priori not clear whether it is open, closed or semi-closed.

Proposition 3.1.7. *Assume that there are real numbers $u_- < u_+$ such that*

$$\mathbb{E}\big(e^{uX}\big) < \infty, \quad u_- < u < u_+.$$

Then the MGF M_X is defined for all complex numbers belonging to $(u_-, u_+) + i\mathbb{R} \subseteq \mathbb{C}$. Moreover, M_X is holomorphic on $(u_-, u_+) + i\mathbb{R}$ and its derivatives are given by

$$\frac{\partial^k M_X}{\partial z^k}(z) = \mathbb{E}\big(X^k e^{zX}\big),$$

for $z \in (u_-, u_+) + i\mathbb{R}$ and $k \in \mathbb{N}$.

Proof. Obviously, we have $\left|e^{zX}\right| = e^{z_1 X}$ for $z = z_1 + iz_2$. Thus M_X is well-defined on $(u_-, u_+) + i\mathbb{R} \subseteq \mathbb{C}$. Since we have

$$M_X(z) = \int_\Omega e^{zX(\omega)}\, dP(\omega), \quad z \in (u_-, u_+) + i\mathbb{R},$$

it remains to show that the assumptions of Theorem 2.3.6 are satisfied. Firstly, $\omega \mapsto e^{zX(\omega)}$ is integrable for every $z \in (u_-, u_+) + i\mathbb{R}$ by assumption. Secondly, $z \mapsto e^{zX(\omega)}$ is holomorphic on $(u_-, u_+) + i\mathbb{R}$ for every $\omega \in \Omega$. Thirdly, on a compact set $K \subseteq (u_-, u_+) + i\mathbb{R}$ the function $z \mapsto \mathrm{Re}(z)$ attains a maximum at $\eta_1 + i\eta_2$ and a minimum at $\xi_1 + i\xi_2$. This gives the estimate

$$\left|e^{zX(\omega)}\right| \le 1 + e^{\eta_1 X(\omega)} + e^{\xi_1 X(\omega)}, \quad z \in K, \quad \omega \in \Omega,$$

where the right hand side is independent of $z \in K$ and integrable. Hence Theorem 2.3.6 can be applied. □

Now we want to use the discovered analyticity to extend moment generating functions in some sense. The next proposition will be useful to determine the whole domain of the MGF.

Proposition 3.1.8. *Consider a random variable X whose MGF is denoted by M_X. Assume that there are real numbers u_-, u_+ such that $0 \in (u_-, u_+)$ and $M_X(u) < +\infty$ for $u \in (u_-, u_+)$. Furthermore, suppose that there exist ϵ, $\delta > 0$, a domain*

$$D = (0, u_+ + \delta) + i(-\epsilon, \epsilon) \subseteq \mathbb{C} \quad \Big(D = (u_- - \delta, 0) + i(-\epsilon, \epsilon) \subseteq \mathbb{C}\Big)$$

and an analytic function $h\colon D \to \mathbb{C}$ satisfying

$$h(u) = M_X(u), \quad 0 < u < u_+ \quad (u_- < u < 0).$$

Then it holds that

$$h(u) = M_X(u) < +\infty, \quad 0 < u < u_+ + \delta \quad (u_- - \delta < u < 0).$$

Proof. See A.1.1 in the appendix. □

Remark 3.1.9. A similar formulation of this proposition was found in [dBRFCU09] and the idea of the proof is from [Wid41], where a similar result is proved for Laplace transforms.

3.2 Fourier Transform

The Fourier transform will be an essential tool to derive an expression for the call option price. Therefore we will shortly introduce and define the Fourier transform of a function $f \in L^1(\mathbb{R})$. This chapter is basically taken from [Kal13, Kapitel 18] and [Rud87, Chapter 9]. At first note that if $f \in L^1(\mathbb{R})$, also $x \mapsto e^{iux}f(x)$ belongs to $L^1(\mathbb{R})$ for every $u \in \mathbb{R}$ because $|f(x)| = |e^{iux}f(x)|$, for $x \in \mathbb{R}$.

Definition 3.2.1. For $f \in L^1(\mathbb{R})$ let the Fourier transform $\hat{f}\colon \mathbb{R} \to \mathbb{C}$ be defined by

$$\hat{f}(u) = \int_{\mathbb{R}} e^{iux} f(x)\, dx.$$

The Fourier transform has the following properties.

Proposition 3.2.2. *Let $f \in L^1(\mathbb{R})$ and $a, b \in \mathbb{C}$, $r, t \in \mathbb{R}$, $r \neq 0$. Then*

1. *If also $g \in L^1(\mathbb{R})$ we have $\widehat{(af + bg)} = a\hat{f} + b\hat{g}$.*
2. *$\hat{f} \in C_0(\mathbb{R})$ and $\|\hat{f}\|_\infty \leq \|f\|_1$.*
3. *If also $g \in L^1(\mathbb{R})$ we have $\widehat{f * g} = \hat{f} \cdot \hat{g}$.*
4. *With $f_t = f(x + t)$, $x \in \mathbb{R}$, we get $f_t \in L^1(\mathbb{R})$, $\|f_t\|_1 = \|f\|_1$ and $\hat{f}_t(u) = e^{-itu}\hat{f}(u)$.*
5. *With $g(x) = e^{itx}f(x)$, $x \in \mathbb{R}$, we get $g \in L^1(\mathbb{R})$, $\|g\|_1 = \|f\|_1$ and $\hat{g}(u) = \hat{f}(u + t)$.*
6. *With $g(x) = |r|f(rx)$, $x \in \mathbb{R}$, we get $g \in L^1(\mathbb{R})$, $\|g\|_1 = \|f\|_1$ and $\hat{g}(u) = \hat{f}\big(\frac{1}{r}u\big)$.*
7. *With $g(x) = \overline{f(-x)}$, $x \in \mathbb{R}$, we get $g \in L^1(\mathbb{R})$, $\|g\|_1 = \|f\|_1$ and $\hat{g}(u) = \overline{\hat{f}(u)}$.*
8. *With $g(x) = -ixf(x)$, $x \in \mathbb{R}$, and $g \in L^1(\mathbb{R})$ we get $\hat{g}(u) = -\hat{f}'(u)$.*
9. *Let $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ such that also the derivative f' belongs to $L^1(\mathbb{R})$ then with $g := f'$ we get $\lim_{|x|\to\infty} f(x) = 0$ and $\hat{g}(u) = -iu\hat{f}(u)$.*

Proof. See [Kal13, Kapitel 18] or [Rud87, Theorem 9.2]. Note that the definition of the Fourier transform varies in the literature. On the one hand sometimes a scaling factor of $1/\sqrt{2\pi}$ appears. On the other hand the sign of the argument in the integral can differ. □

The inversion theorem stated below is a key thing when computing the the damped option price in the next section.

Theorem 3.2.3 (Inversion Theorem)**.** *If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$ the identity*

$$\frac{1}{2\pi}\hat{\hat{f}}(x) = f(-x) \quad \lambda - a.e.$$

holds. If f is continuous on the whole real line we even have

$$\frac{1}{2\pi}\hat{\hat{f}}(x) = f(-x) \quad \forall x \in \mathbb{R}. \tag{3.2}$$

Proof. See [Rud87, Theorem 9.11]. Again take into account that there the definition of the Fourier transform is slightly different. □

3.3 Fourier Transform of the Damped Option Price

Now consider a financial market with time horizon $T > 0$ and two primary assets, namely a riskless asset $B = (B_t)_{t\in[0,T]}$ and a risky asset $S = (S_t)_{t\in[0,T]}$. We assume the riskless asset to be given by

$$B_t = e^{R_t}, \quad 0 \le t \le T,$$

where

$$R_t = \exp\left(\int_0^t r(s)\,ds\right), \quad 0 \le t \le T,$$

for some continuous function $r\colon [0,T] \to [0,+\infty)$. The risky asset is assumed to be a positive càdlàg semimartingale.

Remark 3.3.1. Here we make the assumption that we have deterministic interest rates. This assumption can be relaxed as it is indicated in [Lee04, Section 2].

Within the setting described above we additionally consider a European call option on the underlying S which has strike price $K > 0$ and maturity T. Its payoff C_T at maturity is then given by $C_T = (S_T - K)^+$. Now define the log-discounted underlying by $X_T := \ln S_T - R_T$ and the log-discounted strike by $k := \ln K - R_T$. Then the discounted payoff of the call option can be written as $C_T = G(X_T, k)$, where

$$G(x,k) = \left(e^x - e^k\right)^+ = \left(e^x - e^k\right)^+, \quad (x,k) \in \mathbb{R}^2.$$

Now we can introduce our notation for the time zero option price C and the damped option price C_α depending on the log-discounted strike k. They are given by

$$C(k) := \mathbb{E}\big(G(X_T,k)\big), \quad k \in \mathbb{R} \qquad \text{and} \qquad C_\alpha(k) := e^{(\alpha-1)k}C(k), \quad (k,\alpha) \in \mathbb{R}^2, \tag{3.3}$$

where the expectations are again taken with respect to some martingale measure such that $C(k)$ is finite. We assume that such a measure exists. Then consider the moment generating function M_{X_T} under that martingale measure. By Proposition 3.1.5 there exists a maximal open interval (u_-, u_+) such that $M_{X_T}(u) < +\infty$ for every point belonging to that interval. Moreover, the interior of the domain of M_{X_T} coincides with $(u_-, u_+) + i\mathbb{R}$.

3.3.1 Damped Option Price for $\alpha > 1$

Next, we show that the damped option price has a Fourier transform if $\alpha > 1$ is suitably chosen. Moreover, we compute the Fourier transform using the MGF of X_T.

Theorem 3.3.2. *With u_+ and u_- introduced just prior to Subsection 3.3.1 assume that $u_+ > 1$ and consider $\alpha \in (1, u_+)$. Then $C_\alpha \in L^1(\mathbb{R})$ has a Fourier transform. It is given by*

$$\widehat{C}_\alpha(u) = \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu - 1)(\alpha + iu)}. \tag{3.4}$$

Proof. Using the assumptions for α and u_+ one gets

$$\begin{aligned}
\int_{\mathbb{R}} C_\alpha(k)\,dk &= \int_{\mathbb{R}} e^{(\alpha-1)k}\mathbb{E}\big((e^{X_T} - e^k)^+\big)\,dk = \mathbb{E}\left(\int_{(-\infty,X_T]} e^{(\alpha-1)k}(e^{X_T} - e^k)\,dk\right) \le \\
&\le \mathbb{E}\left(e^{X_T}\int_{(-\infty,X_T]} e^{(\alpha-1)k}\,dk\right) + \mathbb{E}\left(\int_{(-\infty,X_T]} e^{\alpha k}\,dk\right) \le \\
&\le \frac{1}{\alpha-1}\mathbb{E}\big(e^{\alpha X_T}\big) + \frac{1}{\alpha}\mathbb{E}\big(e^{\alpha X_T}\big) \le \frac{2}{\alpha-1}\mathbb{E}\big(e^{\alpha X_T}\big) < +\infty,
\end{aligned}$$

where the latter is finite due to $\alpha \in (1, u_+)$ and Fubini can be used due to the fact that the integrand is non-negative. Thus $C_\alpha(k) \in L^1(\mathbb{R})$ and consequently $\widehat{C}_\alpha$ exists. Using Fubini again, which we can because the above is finite, the Fourier transform can be computed via

$$\widehat{C}_\alpha(u) = \int_{\mathbb{R}} e^{iuk} C_\alpha(k)\, dk = \int_{\mathbb{R}} \mathbb{E}\big(e^{iuk} e^{(\alpha-1)k}(e^{X_T} - e^k)^+\big)\, dk = \mathbb{E}\bigg(\int_{(-\infty, X_T]} e^{(\alpha-1+iu)k}(e^{X_T} - e^k)\, dk\bigg) =$$

$$= \frac{1}{\alpha - 1 + iu}\mathbb{E}\big(e^{(\alpha+iu)X_T}\big) - \frac{1}{\alpha + iu}\mathbb{E}\big(e^{(\alpha+iu)X_T}\big) = \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu - 1)(\alpha + iu)}.$$

□

The next thing we want to do is to ensure that the Fourier transform of the damped option price is again in $L^1(\mathbb{R})$, because this makes recovering the option price possible by using the Fourier inversion theorem.

Lemma 3.3.3. *With the same assumptions as in Theorem 3.3.2 we also have $\widehat{C}_\alpha \in L^1(\mathbb{R})$.*

Proof. From Theorem 3.3.2 we know that the Fourier transform of the damped option price exists. For the denominator in (3.4) we have the following estimate

$$\big|(\alpha - 1 + iu)(\alpha + iu)\big|^2 = |\alpha - 1 + iu|^2|\alpha + iu|^2 = \big((\alpha-1)^2 + u^2\big)(\alpha^2 + u^2) \geq \big((\alpha-1)^2 + u^2\big)^2,$$

for $\alpha \in (1, u_+)$. This yields the following estimate for $\widehat{C}_\alpha$

$$\big|\widehat{C}_\alpha(u)\big| \leq \frac{\big|M_{X_T}(\alpha + iu)\big|}{(\alpha-1)^2 + u^2} \leq \frac{\mathbb{E}\big(e^{\alpha X}\big)}{(\alpha-1)^2 + u^2}.$$

Consequently we obtain

$$\int_{\mathbb{R}} \big|\widehat{C}_\alpha(u)\big|\, du \leq \mathbb{E}\big(e^{\alpha X}\big) \int_{\mathbb{R}} \frac{1}{(\alpha-1)^2 + u^2}\, du =$$

$$= \mathbb{E}\big(e^{\alpha X}\big)\, \frac{1}{\alpha - 1} \arctan \frac{u}{\alpha - 1}\bigg|_{-\infty}^{+\infty} = \mathbb{E}\big(e^{\alpha X}\big)\frac{\pi}{\alpha - 1} < \infty,$$

since $\alpha \in (1, u_+)$ which implies $\widehat{C}_\alpha \in L^1(\mathbb{R})$. □

Applying the Fourier inversion theorem now gives the following formula for the option price. This can be used to compute call option prices by means of the MGF of the log-discounted underlying X_T.

Theorem 3.3.4 ([Lee04], Theorem 4.3)**.** *With u_+ and u_- introduced just prior to Subsection 3.3.1 assume that $u_+ > 1$ and let $\alpha \in (1, u_+)$. Then the option price $C(k)$ as defined in* (3.3) *is given by*

$$C(k) = \frac{e^k}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{-kz} \frac{M_{X_T}(z)}{z(z-1)}\, dz = \frac{e^k}{\pi} \int_0^\infty \mathrm{Re}\left(e^{-k(\alpha+iu)} \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu)(\alpha + iu - 1)}\right) du. \tag{3.5}$$

Proof. With the assumptions made we can directly apply Theorem 3.3.2 and Lemma 3.3.3. Consequently the assumptions of Theorem 3.2.3 are fulfilled. Since C_α is in addition continuous on $\mathbb{R}$ we can use (3.2) to obtain

$$e^{(\alpha-1)k} C(k) = C_\alpha(k) = \frac{1}{2\pi}\widehat{\widehat{C}}_\alpha(-k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iku} \widehat{C}_\alpha(u)\, du.$$

A Multiplication by $e^{-(\alpha-1)k}$ and using (3.4) leads to

$$C(k) = \frac{e^k}{2\pi} \int_{\mathbb{R}} e^{-k(\alpha+iu)} \widehat{C}_\alpha(u)\, du = \frac{e^k}{2\pi} \int_{\mathbb{R}} e^{-k(\alpha+iu)} \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu)(\alpha + iu - 1)}\, du = \frac{e^k}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{-kz} \frac{M_{X_T}(z)}{z(z-1)}\, dz,$$

and thus the first equality in (3.5). Since the left hand side of that equation is real also the right hand side must be real. In particular the integral of the imaginary part of the integrand on the right hand side must be zero. This leads to

$$e^{(\alpha-1)k}C(k) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{Re}\left(e^{-iku}\widehat{C}_\alpha(u)\right) du. \tag{3.6}$$

Furthermore, we know that for a real-valued function f the corresponding Fourier transform $\hat{f}$ has an even real part and an odd imaginary part. For the integrand in (3.6) we obtain

$$\mathrm{Re}\left(e^{-iku}\widehat{C}_\alpha(u)\right) = \mathrm{Re}\left(\widehat{C}_\alpha(u)\right)\cos(ku) + \mathrm{Im}\left(\widehat{C}_\alpha(u)\right)\sin(ku),$$

and thus we see that this is an even function. Hence

$$e^{(\alpha-1)k}C(k) = \frac{1}{\pi}\int_{(0,+\infty)} \mathrm{Re}\left(e^{-iku}\widehat{C}_\alpha(u)\right) du. \tag{3.7}$$

Multiplication by $e^{-(\alpha-1)k}$ and using (3.4) yields the second equality in (3.5). □

3.3.2 Damped Option Price for $\alpha < 1$

The assumption of $\alpha > 1$ was essential to ensure the existence of the Fourier transform of the damped option price. In this subsection we will find expressions for the damped option price also for $\alpha < 1$. This is done by interpreting the integral in (3.5) as a contour integral with the contour depending on α. Shifting this contour results in an expression for an α smaller than one. Therefore we define

$$f(z) = e^{-kz}\frac{M_{X_T}(z)}{z(z-1)}, \quad z \in D \setminus \{0,1\},$$

where $D := (u_-, u_+) + i\mathbb{R}$.

Remark 3.3.5. Clearly f is well-defined on $D \setminus \{0,1\}$. Furthermore, with Proposition 3.1.7 we see that f is holomorphic on $D \setminus \{0,1\}$. If $u_+ > 1$ there is a pole of order one at $z = 1$ and if $0 \in (u_-, u_+)$ there is a pole of order one in $z = 0$.

Now define the path $\gamma_{\alpha,R}$ by

$$\gamma_{\alpha,R}(u) = \alpha + iu, \quad u \in [-R, R],$$

for each $\alpha \in (1, u_+)$ and $R > 0$. Then recalling Theorem 3.3.4 we see that if $u_+ > 1$ the call price in (3.5) can be represented by

$$C(k) = \frac{e^k}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} f(z)\,dz = \frac{e^k}{2\pi i}\lim_{R\to\infty}\int_{\gamma_{\alpha,R}} f(z)\,dz, \tag{3.8}$$

for $\alpha \in (1, u_+)$.

The interpretation of the call price as contour integral as it is given in (3.8) motivates additional representations by choosing another vertical line in the complex plane as path to integrate over. However, one has to take care when shifting the contour to the left as the integrand f has poles in 0 and in 1 and thus residues have to be added.

Theorem 3.3.6. *Assume that $u_+ > 1$. Then for $\alpha \in (0,1)$ we have*

$$C(k) = \mathbb{E}\big(e^{X_T}\big) + \frac{e^k}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} f(z)\,dz = \mathbb{E}\big(e^{X_T}\big) + \frac{e^k}{\pi}\int_0^\infty \mathrm{Re}\Big(f(\alpha+iu)\Big)\,du. \tag{3.9}$$

If additionally $u_- < 0$ holds then for $\alpha \in (u_-, 0)$ we have

$$C(k) = \mathbb{E}\big(e^{X_T}\big) - e^k + \frac{e^k}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} f(z)\,dz = \mathbb{E}\big(e^{X_T}\big) - e^k + \frac{e^k}{\pi}\int_0^\infty \mathrm{Re}\Big(f(\alpha+iu)\Big)\,dz. \tag{3.10}$$

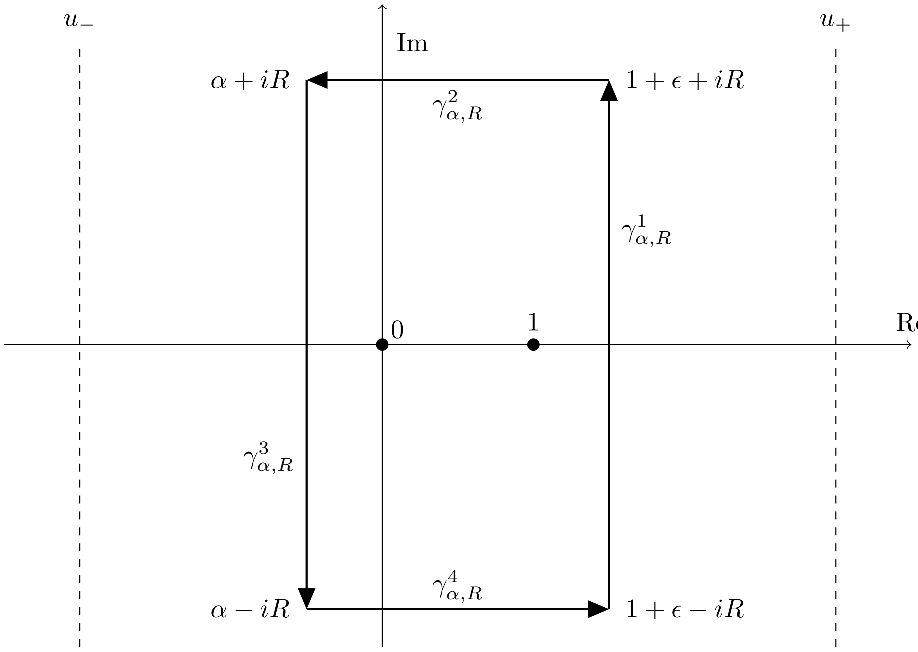


Figure 3.1: The path $\gamma_{\alpha,R}$ for negative α in the complex plane.

Proof. In order to derive the expressions above for the option price we apply the residue theorem (Theorem 2.3.13) as we want to shift the contour over poles. Therefore we define $P := \{0,1\}$ if $u_- < 0$ and $P := \{0\}$ if $u_- = 0$. Furthermore, recall that $D = (u_-, u_+) + i\mathbb{R}$. Since $u_+ > 1$ there is an $\epsilon > 0$ satisfying $1+\epsilon < u_+$. As visualized in Figure 3.1 we use the paths

$$\begin{array}{rcll}
\gamma^1_{\alpha,R}(u) & = & 1+\epsilon+iu & u \in [-R,R] \\
\gamma^2_{\alpha,R}(u) & = & -u+iR & u \in [-1-\epsilon,-\alpha] \\
\gamma^3_{\alpha,R}(u) & = & \alpha-iu & u \in [-R,R] \\
\gamma^4_{\alpha,R}(u) & = & u-iR & u \in [\alpha,1+\epsilon],
\end{array}$$

to define $\gamma_{\alpha,R}$ as the connection of these four paths by

$$\gamma_{\alpha,R} = \gamma^1_{\alpha,R} \oplus \gamma^2_{\alpha,R} \oplus \gamma^3_{\alpha,R} \oplus \gamma^4_{\alpha,R}.$$

Because of $\alpha \notin \{0,1\}$, $1+\epsilon < u_+$ and $u_- < \alpha$ we know that $\gamma_{\alpha,R}$ is a path in $D \setminus \{0,1\}$. For $z \in \mathbb{C} \setminus D$ it holds that

$$n(\gamma_{\alpha,R}, z) = \frac{1}{2\pi i} \int_{\gamma_{\alpha,R}} \frac{1}{\zeta - z}\, d\zeta = 0,$$

because the integrand there is holomorphic in D, which is simply connected as convex set, and $\gamma_{\alpha,R}$ is a closed, continuous and piecewise continuously differentiable path in D. The latter also allows us to apply the residue theorem to the function f we used in (3.8), the open set D and the path $\gamma_{\alpha,R}$. For the winding numbers we get

$$n(\gamma_{\alpha,R}, 1) = \frac{1}{2\pi i} \int_{\gamma_{\alpha,R}} \frac{1}{\zeta}\, d\zeta = 1.$$

Analogously one gets in the case of $u_- < 0$ that

$$n(\gamma_{\alpha,R},0) = \frac{1}{2\pi i}\int_{\gamma_{\alpha,R}} \frac{1}{\zeta}\,d\zeta = \begin{cases} 1 & \alpha \in (u_-,0) \\ 0 & \alpha \in (0,1) \end{cases}.$$

The application of the residue theorem yields then

$$\frac{1}{2\pi i}\int_{\gamma_{\alpha,R}} f(z)\,dz = \sum_{w\in P} n(\gamma_{\alpha,R},w)\operatorname{Res}(f,w) = \begin{cases} \operatorname{Res}(f,1) & \alpha \in (0,1) \\ \operatorname{Res}(f,1)+\operatorname{Res}(f,0) & \alpha \in (u_-,0) \end{cases}.$$

Multiplication by $2\pi i$ and considering the limit for $R \to \infty$ leads to

$$2\pi i \sum_{w\in P} n(\gamma_{\alpha,R},w)\operatorname{Res}(f,w) = \lim_{R\to\infty}\left(\int_{\gamma^1_{\alpha,R}} f(z)\,dz + \int_{\gamma^2_{\alpha,R}} f(z)\,dz + \int_{\gamma^3_{\alpha,R}} f(z)\,dz + \int_{\gamma^4_{\alpha,R}} f(z)\,dz\right). \tag{3.11}$$

For the first integral we obtain

$$\lim_{R\to\infty}\int_{\gamma^1_{\alpha,R}} f(z)dz = \lim_{R\to\infty}\int_{1+\epsilon-iR}^{1+\epsilon+iR} f(z)\,dz = \int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} f(z)\,dz = 2\pi i e^{-k}C(k),$$

where the last equality comes from (3.8) as $1+\epsilon$ is such that it belongs to $(1,u_+)$. The second integral converges to 0 since

$$\begin{aligned}
\lim_{R\to\infty}\left|\int_{\gamma^2_{\alpha,R}} f(z)dz\right| &\le \lim_{R\to\infty}\int_{-1-\epsilon}^{-\alpha}\left|f(-u+iR)\right|du \le \\
&\le \lim_{R\to\infty}\int_{-1-\epsilon}^{-\alpha} e^{|k|(1+\epsilon+|\alpha|)}\left|\frac{M_{X_T}(-u+iR)}{(-u+iR)(-u+iR-1)}\right|du \le \\
&\le \lim_{R\to\infty}\frac{e^{|k|(1+\epsilon+|\alpha|)}}{R^2}\int_{-1-\epsilon}^{-\alpha}\left\|M_{X_T}\big|_{[\alpha,1+\epsilon]}\right\|_\infty du \le \\
&\le \lim_{R\to\infty}\frac{e^{|k|(1+\epsilon+|\alpha|)}}{R^2}(1+\epsilon-\alpha)\left\|M_{X_T}\big|_{[\alpha,1+\epsilon]}\right\|_\infty = 0,
\end{aligned}\tag{3.12}$$

where the norm appearing in the last two expressions is finite since M_{X_T} is continuous on the compact set $[\alpha,1+\epsilon] \subseteq (u_-,u_+)$. For the limit of the fourth integral we also get 0, which is verified similarly as follows.

$$\begin{aligned}
\lim_{R\to\infty}\left|\int_{\gamma^4_{\alpha,R}} f(z)dz\right| \le \lim_{R\to\infty}\int_{\alpha}^{1+\epsilon}\left|f(u-iR)\right|du &\le \lim_{R\to\infty}\int_{\alpha}^{1+\epsilon} e^{|k|(1+\epsilon+|\alpha|)}\left|\frac{M_{X_T}(u-iR)}{(u-iR)(u-iR-1)}\right|du \le \\
&\le \lim_{R\to\infty}\int_{\alpha}^{1+\epsilon} e^{|k|(1+\epsilon+|\alpha|)}\frac{M_{X_T}(u)}{R^2}du \le \lim_{R\to\infty}\frac{e^{|k|(1+\epsilon+|\alpha|)}}{R^2}\int_{\alpha}^{1+\epsilon}\left\|M_{X_T}\big|_{[\alpha,1+\epsilon]}\right\|_\infty du \le \\
&\le \lim_{R\to\infty}\frac{e^{|k|(1+\epsilon+|\alpha|)}}{R^2}(1+\epsilon-\alpha)\left\|M_{X_T}\big|_{[\alpha,1+\epsilon]}\right\|_\infty = 0,
\end{aligned}\tag{3.13}$$

where the assumptions again ensure that the occurring norm is finite. Taking into account that integrals 1, 2 and 4 and the sum of all four integrals converge we conclude that also the limit of the third integral exists. Using the three limits computed above and (3.11) yields

$$2\pi i\Big(\operatorname{Res}(f,1) + \mathbb{1}_{(u_-,0)}(\alpha)\operatorname{Res}(f,0)\Big) - 2\pi i e^{-k}C(k) = \lim_{R\to+\infty}\int_{\gamma^3_{\alpha,R}} f(z)dz = -\int_{\alpha-i\infty}^{\alpha+i\infty} f(z)\,dz. \tag{3.14}$$

Now recall that both poles in P have order one. Consequently by means of Remark 2.3.12 the residues are given by

$$\operatorname{Res}(f,1) = \lim_{z\to 1} e^{-kz}\frac{M_{X_T}(z)}{z} = e^{-k}M_{X_T}(1) = e^{-k}\mathbb{E}\big(e^{X_T}\big) \tag{3.15}$$

$$\operatorname{Res}(f,0) = \lim_{z\to 0} e^{-kz}\frac{M_{X_T}(z)}{z-1} = -M_{X_T}(0) = -1, \tag{3.16}$$

where of course the latter can only be computed in case $u_- < 0$. Plugging in the residues and rearranging terms in (3.14) yields the first equalities in (3.9) and (3.10) respectively. To derive the second equalities in (3.9) and (3.10) one again observes that the option price on the left hand side is real and thus the integral of the imaginary part on the right hand side must vanish. Furthermore, observe that $\overline{f(z)} = f(\overline{z})$ for any $z \in D \setminus \{0,1\}$. Thus another expression for the real part of the integrand is given by

$$\begin{aligned}\operatorname{Re}\big(f(\alpha+iu)\big) &= \frac{1}{2}\Big(f(\alpha+iu)+\overline{f(\alpha+iu)}\Big) = \\ &= \frac{1}{2}\Big(f(\alpha+iu)+f(\alpha-iu)\Big),\end{aligned} \tag{3.17}$$

which is clearly an even function of $u \in \mathbb{R}$. This yields the second equalities in (3.9) and (3.10) respectively. □

3.3.3 Damped Option Price for $\alpha = 1$ and $\alpha = 0$

In the formulas we have derived for the call option price we have always excluded the values the cases $\alpha = 1$ and $\alpha = 0$. It is quite easy to see that the integral in (3.8) does not converge for $\alpha = 1$ or $\alpha = 0$. However, one can compute the principle value of these integrals which eventually leads to an improper Riemann-integral over the positive real line in these cases. Therefore, we recall the definition of the Cauchy principle value of an integral.

Definition 3.3.7. Let $-\infty \le a < c < b \le \infty$ be three distinct numbers in $\mathbb{R}\cup\{\pm\infty\}$ and $\alpha \in \mathbb{R}$. Furthermore, let f be a complex-valued function defined on the set $\{\alpha+iu \,|\, u \neq c,\ u\in\mathbb{R}\}$. Assume that $u \mapsto f(\alpha+iu)$ is improper Riemann-integrable on the intervals $(a, c-\epsilon]$ and $[c+\epsilon, b)$ for every $\epsilon > 0$. Then the Cauchy principle value around $\alpha + ic$ is defined by

$$\text{PV-}\int_{\alpha+ia}^{\alpha+ib} f(z)\,dz := i\lim_{\epsilon\searrow 0}\left(\int_a^{c-\epsilon} f(\alpha+iu)\,du + \int_{c+\epsilon}^{b} f(\alpha+iu)\,du\right),$$

whenever the limit on the right hand side exists.

Theorem 3.3.8. *Assume that $u_+ > 1$. Then we have*

$$C(k) = \frac{\mathbb{E}\big(e^{X_T}\big)}{2} + \frac{e^k}{2\pi i}\ \text{PV-}\int_{1-i\infty}^{1+i\infty} f(z)\,dz = \frac{\mathbb{E}\big(e^{X_T}\big)}{2} + \frac{e^k}{\pi}\int_0^\infty \operatorname{Re}\Big(f(1+iu)\Big)\,du. \tag{3.18}$$

If moreover $u_- < 0$ holds we also have

$$C(k) = \mathbb{E}\big(e^{X_T}\big) - \frac{e^k}{2} + \frac{e^k}{2\pi i}\ \text{PV-}\int_{-i\infty}^{i\infty} f(z)\,dz = \mathbb{E}\big(e^{X_T}\big) - \frac{e^k}{2} + \frac{e^k}{\pi}\int_0^\infty \operatorname{Re}\Big(f(iu)\Big)\,du. \tag{3.19}$$

Proof. The key idea here is again to use the residue theorem. Let $R > 0$ and $\epsilon \in (0,1)$ be such that $1+\epsilon < u_+$. In case of $u_- < 0$ additionally choose ϵ such that $u_- < -\epsilon$. In the sequel let $\alpha \in \{0,1\}$ if $u_- < 0$ and $\alpha = 1$ if $u_- = 0$. Here we use the same notation as in the proof of Theorem 3.3.6 for everything but the paths. The paths we consider here are defined by

$$\begin{array}{rcll} \gamma^1_{\epsilon,R}(u) & = & \alpha + iu & u \in [-R,-\epsilon] \\ \gamma^2_{\epsilon,+}(u) & = & \alpha - i\epsilon e^{-iu} & u \in [0,\pi] \\ \gamma^2_{\epsilon,-}(u) & = & \alpha - i\epsilon e^{iu} & u \in [0,\pi] \\ \gamma^3_{\epsilon,R}(u) & = & \alpha + iu & u \in [\epsilon,R] \\ \gamma^4_{\epsilon,R}(u) & = & u + iR & u \in [\alpha,1+\epsilon] \\ \gamma^5_{\epsilon,R}(u) & = & 1+\epsilon - iu & u \in [-R,R] \\ \gamma^6_{\epsilon,R}(u) & = & -u - iR & u \in [-1-\epsilon,-\alpha]. \end{array}$$

As visualized in Figure 3.2 we connect them to get the paths $\gamma^+_{\epsilon,R}$ and $\gamma^-_{\epsilon,R}$ defined by

$$\gamma^+_{\epsilon,R} := \gamma^1_{\epsilon,R} \oplus \gamma^2_{\epsilon,+} \oplus \gamma^3_{\epsilon,R} \oplus \gamma^4_{\epsilon,R} \oplus \gamma^5_{\epsilon,R} \oplus \gamma^6_{\epsilon,R}$$
$$\gamma^-_{\epsilon,R} := \gamma^1_{\epsilon,R} \oplus \gamma^2_{\epsilon,-} \oplus \gamma^3_{\epsilon,R} \oplus \gamma^4_{\epsilon,R} \oplus \gamma^5_{\epsilon,R} \oplus \gamma^6_{\epsilon,R}.$$

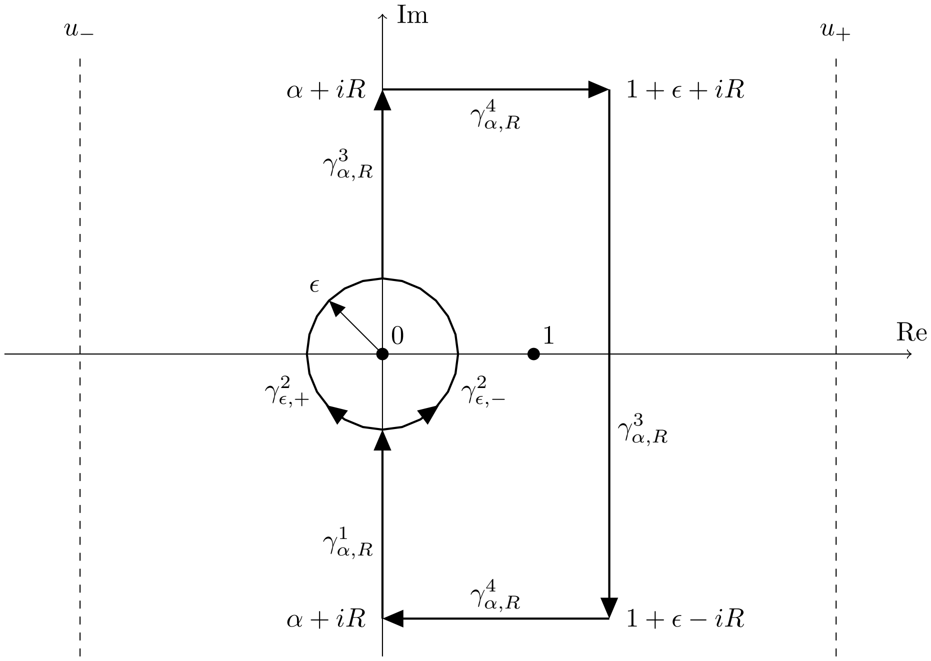


Figure 3.2: The path $\gamma_{\alpha,R}$ for $\alpha = 0$ in the complex plane.

Since we have $1 < 1+\epsilon < u_+$ and $\epsilon < \min(-u_-, 1)$ in any case the images of the paths $\gamma^+_{\epsilon,R}$ and $\gamma^-_{\epsilon,R}$ surely are in $D \setminus \{0,1\}$. We see that for $z \in \mathbb{C} \setminus D$

$$n(\gamma^+_{\epsilon,R}, z) = \frac{1}{2\pi i} \int_{\gamma^+_{\epsilon,R}} \frac{1}{\zeta - z}\, d\zeta = 0.$$

Applying the residue theorem (Theorem 2.3.13) to the function f, the open set D, the Poles P and the path

$\gamma^+_{\epsilon,R}$ and $\gamma^-_{\epsilon,R}$ respectively, we obtain

$$2\pi i \sum_{w\in P} n(\gamma^+_{\epsilon,R}, w)\,\mathrm{Res}(f,w) = \int_{\gamma^+_{\epsilon,R}} f(z)\,dz, \tag{3.20}$$

$$2\pi i \sum_{w\in P} n(\gamma^-_{\epsilon,R}, w)\,\mathrm{Res}(f,w) = \int_{\gamma^-_{\epsilon,R}} f(z)\,dz. \tag{3.21}$$

Furthermore, for the windings numbers of $\gamma^+_{\epsilon,R}$ and $\gamma^-_{\epsilon,R}$ at the poles we get

$$n(\gamma^+_{\epsilon,R}, w) = \begin{cases} -1 & w=\alpha \\ 0 & w=0,\ \alpha=1 \\ -1 & w=1,\ \alpha=0 \end{cases} \quad \text{and} \quad n(\gamma^-_{\epsilon,R}, w) = \begin{cases} 0 & w=0 \\ 0 & w=1,\ \alpha=1 \\ -1 & w=1,\ \alpha=0 \end{cases},$$

for $w \in P$. Now observe that with these winding numbers one obtains

$$\sum_{w\in P} n(\gamma^+_{\epsilon,R}, w)\,\mathrm{Res}(f,w) + \sum_{w\in P} n(\gamma^-_{\epsilon,R}, w)\,\mathrm{Res}(f,w) = \varrho_\alpha,$$

where

$$\varrho_\alpha = \begin{cases} -\mathrm{Res}(f,1) & \alpha=1 \\ -2\,\mathrm{Res}(f,1) - \mathrm{Res}(f,0) & \alpha=0. \end{cases} \tag{3.22}$$

Adding (3.20) and (3.21) and taking the limit $R\to\infty$ on both sides leads to

$$\begin{aligned} 2\pi i\varrho_\alpha = \lim_{R\to\infty} \Bigg(2\int_{\alpha-iR}^{\alpha-i\epsilon} f(z)\,dz + 2\int_{\alpha+i\epsilon}^{\alpha+iR} f(z)\,dz - 2\int_{1+\epsilon-iR}^{1+\epsilon+iR} f(z)\,du + \\ + 2\int_{\gamma^4_{\epsilon,R}} f(z)\,dz + 2\int_{\gamma^6_{\epsilon,R}} f(z)\,dz\Bigg) + \int_{\gamma^2_{\epsilon,+}} f(z)\,dz + \int_{\gamma^2_{\epsilon,-}} f(z)\,dz = \\ = \quad 2\int_{\alpha-i\infty}^{\alpha-i\epsilon} f(z)\,dz + 2\int_{\alpha+i\epsilon}^{\alpha+i\infty} f(z)\,dz - 2\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} f(z)\,dz + \int_{\gamma^2_{\epsilon,+}} f(z)\,dz + \int_{\gamma^2_{\epsilon,-}} f(z)\,dz, \end{aligned}$$

where the integrals over $\gamma^4_{\epsilon,R}$ and $\gamma^6_{\epsilon,R}$ vanish in the limit due to a completely analogous argument as in (3.12) and (3.13). Furthermore, we have $1 < 1+\epsilon < u_+$ and can thus apply (3.8). Hence the third integral converges to

$$\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} f(z)\,dz = 2\pi i e^{-k} C(k).$$

Consequently we get

$$4\pi i e^{-k} C(k) = -2\pi i\varrho_\alpha + 2\Bigg(\int_{\alpha-i\infty}^{\alpha-i\epsilon} f(z)\,dz + \int_{\alpha+i\epsilon}^{\alpha+i\infty} f(z)\,dz\Bigg) + \int_{\gamma^2_{\epsilon,+}} f(z)\,dz + \int_{\gamma^2_{\epsilon,-}} f(z)\,dz. \tag{3.23}$$

It remains to take the limit with respect to ϵ. At first we show that the sum of the latter two integrals converges to 0.

$$\begin{aligned} \lim_{\epsilon\searrow 0} \int_{\gamma^2_{\epsilon,+}} f(z)\,dz &= -\lim_{\epsilon\searrow 0} \int_0^\pi f(\alpha - i\epsilon e^{-iu})\epsilon e^{-iu}\,du = \\ &= -\lim_{\epsilon\searrow 0} \int_0^\pi e^{-k(\alpha-i\epsilon e^{-iu})} \frac{M_{X_T}\left(\alpha - i\epsilon e^{-iu}\right)}{(\alpha - i\epsilon e^{-iu})(\alpha - i\epsilon e^{-iu} - 1)} \epsilon e^{-iu}\,du = \\ &= \lim_{\epsilon\searrow 0} \int_0^\pi e^{-k(\alpha-i\epsilon e^{-iu})} \frac{M_{X_T}\left(\alpha - i\epsilon e^{-iu}\right)}{\epsilon e^{-iu} \pm i}\,du, \end{aligned}$$

where the plus-minus sign in the denominator corresponds to the two cases of $\alpha = 1$ (plus) and $\alpha = 0$ (minus). Since the integrand is a continuous and thus bounded function of (ϵ, u) on $[0, \eta] \times [0, \pi]$, where η is sufficiently small, we can use dominated converge to interchange the limit and the integral. We obtain

$$\lim_{\epsilon \searrow 0} \int_{\gamma^2_{\epsilon,+}} f(z)\, dz = \int_0^\pi \lim_{\epsilon \searrow 0} e^{-k(\alpha - i\epsilon e^{-iu})} \frac{M_{X_T}\big(\alpha - i\epsilon e^{-iu}\big)}{\epsilon e^{-iu} \pm i}\, du = \mp i e^{-k\alpha} M_{X_T}(\alpha).$$

A completely analogous computation leads to

$$\begin{aligned}\lim_{\epsilon \searrow 0} \int_{\gamma^2_{\epsilon,-}} f(z)\, dz &= \lim_{\epsilon \searrow 0} \int_0^\pi f\big(\alpha - i\epsilon e^{iu}\big)\epsilon e^{iu}\, du = \\ &= -\lim_{\epsilon \searrow 0} \int_0^\pi e^{-k(\alpha - i\epsilon e^{iu})} \frac{M_{X_T}\big(\alpha - i\epsilon e^{iu}\big)}{\epsilon e^{iu} \pm i}\, du = \pm i e^{-k\alpha} M_{X_T}(\alpha)\end{aligned}$$

where one sees with the same reasoning as above that the integrand is bounded before applying the theorem of dominated convergence. In particular we now have

$$\lim_{\epsilon \searrow 0} \left(\int_{\gamma^2_{\epsilon,+}} g(z)\, dz + \int_{\gamma^2_{\epsilon,-}} g(z)\, dz \right) = \mp i e^{-k\alpha} M_{X_T}(\alpha) \pm i e^{-k\alpha} M_{X_T}(\alpha) = 0. \tag{3.24}$$

Recall that the residues occurring in (3.22) have already been computed in (3.15) and (3.16) and are given by

$$\mathrm{Res}(f, 1) = e^{-k}\mathbb{E}(e^{X_T}) \quad \text{and} \quad \mathrm{Res}(f, 0) = -1,$$

where the residue at 0 is only defined if $u_- < 0$. Using (3.24) when taking the limit $\epsilon \searrow 0$ in (3.23) and dividing by $4\pi i e^{-k}$ and plugging in ϱ_α yields then the first equalities in (3.18) and (3.19). Note that the integrals here denote the Cauchy principal value.
For the remaining equalities observe that the imaginary part of the right hand side (of the first equality) can be omitted since the left hand side is real. Moreover, it also holds that

$$\overline{f(z)} = f(\overline{z}), \quad \mathrm{Im}(z) \neq 0, \quad z \in D.$$

Hence we see that $u \mapsto \mathrm{Re}\big(g(\alpha + iu)\big)$ for $u \in \mathbb{R} \setminus \{0\}$ is also an even function in the cases for α considered here. This gives

$$\begin{aligned}\mathrm{Re}\left(\frac{e^k}{2\pi i}\ \text{PV-}\int_{-\infty}^\infty f(\alpha + iu) i\, du\right) &= \frac{e^k}{2\pi} \lim_{\epsilon \searrow 0} \left(\int_{-\infty}^{-\epsilon} \mathrm{Re}\big(f(\alpha + iu)\big)\, du + \int_\epsilon^\infty \mathrm{Re}\big(f(\alpha + iu)\big)\, du \right) = \\ &= \frac{e^k}{\pi} \lim_{\epsilon \searrow 0} \int_\epsilon^\infty \mathrm{Re}\big(f(\alpha + iu)\big)\, du = \frac{e^k}{\pi} \int_0^\infty \mathrm{Re}\big(f(\alpha + iu)\big)\, du.\end{aligned}$$

Using this after taking real parts on both sides of the first equalities in (3.18) and (3.19) concludes the proof. □

3.4 Call Option Price – Summary

Towards the end of this chapter we want to summarize our results. Recall that the financial market model we consider has a time horizon $T > 0$ and consists of two primary assets. On the one hand we have a riskless savings account $(B_t)_{t \in [0,T]}$ given by

$$B_t = e^{R_t}, \quad 0 \le t \le T,$$

where

$$R_t = \exp\bigg(\int_0^t r(s)\,ds\bigg), \quad 0 \le t \le T,$$

for some continuous function $r\colon [0,T] \to [0,+\infty)$. On the other hand the risky underlying $S = (S_t)_{t\in[0,T]}$ is assumed to be a positive càdlàg semimartingale such that a martingale measure exists. We fix a martingale measure Q and in following corollary every expectation is meant with respect to that measure.

Corollary 3.4.1. *In the financial market (B,S) define*

$$X_T := \ln S_T - R_T,$$
$$k = \ln K - R_T.$$

Then an arbitrage free price $C(k)$ of a call option with maturity T and strike K is given by

$$C(k) = \mathbb{E}_Q\big(e^{X_T} - e^k\big)^+ = \mathbb{E}_Q\big(R_T^{-1}(S_T - K)^+\big).$$

Now consider the moment generating function

$$M_{X_T}(z) = \mathbb{E}_Q(e^{zX_T}),$$

for complex arguments z such that $\mathbb{E}\big(|e^{zX_T}|\big) < +\infty$. Then the interior of the domain of M_{X_T} is always of the form $(u_-, u_+) + i\mathbb{R}$, where $u_- \le 0 \le u_+$. If we even have $u_+ > 1$ then with

$$R_\alpha := \frac{S_0}{2}\mathbb{1}_{\{1\}}(\alpha) + S_0\mathbb{1}_{(-\infty,1)}(\alpha) - \frac{K}{2}e^{-R_T}\mathbb{1}_{\{0\}}(\alpha) - Ke^{-R_T}\mathbb{1}_{(-\infty,0)}(\alpha), \quad \alpha \in \mathbb{R},$$

the call price identity

$$C(k) = R_\alpha + \frac{e^k}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{-kz}\frac{M_{X_T}(z)}{z(z-1)}\,dz, \tag{3.25}$$

holds for every $\alpha \in (u_-, u_+)$. Note that in case $\alpha \in \{0,-1\}$ the integral on the right hand side denotes the Cauchy principal value around α. This integral representation can then be simplified to

$$C(k) = R_\alpha + \frac{e^k}{\pi}\int_0^\infty \mathrm{Re}\left(e^{-k(\alpha+iu)}\frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha+iu-1)}\right)du, \tag{3.26}$$

with an improper Riemann integral on the right hand side.

Proof. Combining Remark 3.1.6, Proposition 3.1.7 and Theorems 3.3.4, 3.3.6 and 3.3.8 and using $e^k = Ke^{-R_T}$ as well as

$$\mathbb{E}_Q(e^{X_T}) = \mathbb{E}_Q(R_T^{-1}S_T) = S_0,$$

which is due to the fact that the expectation is taken under a martingale measure with respect to $(S_t)_{t\in[0,T]}$, leads to the result. □

Remark 3.4.2. Note that the condition of u_+ being strictly greater than 1 is equivalent to the existence of an $\epsilon > 0$ such that

$$\mathbb{E}_Q({S_T}^{1+\epsilon}) < +\infty.$$

Chapter 4

Heston Model

A standard model used for option pricing which goes beyond the Black-Merton-Scholes framework is the Heston Model. It was introduced by Steven Heston (1993), [Hes93]. One of the main features in the Heston model is that the volatility of the underlying asset is also stochastic.
Throughout this chapter we work on a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ supporting a two-dimensional $\mathbb{F}$-Brownian motion $W = (W^1, W^2)$. The Heston model is an example for a financial market as defined in Section 2.1. In that setting we have two primary assets. On the one hand there is a risk-free savings account B given by

$$B_t = \exp\left(\int_0^t r_s\, ds\right), \quad 0 \leq t \leq T,$$

where $r\colon [0,T] \to [0,+\infty)$ is a continuous function and T denotes the time horizon. On the other hand we want to define a risky asset S by means of a stochastic variance process ν. Furthermore, we want to allow for correlation between the noise driving the variance process and those driving the underlying process. Therefore we define the correlated Brownian motions W^S and W^ν by

$$W^S = \sqrt{1-\rho^2}W^1 + \rho W^2 \quad \text{and} \quad W^\nu = W^2,$$

where $\rho \in [-1,1]$ is the correlation parameter. Using Lévy's characterization one can easily show that W^S and W^ν are Brownian motions indeed. For the quadratic covariation we clearly get

$$[W^S, W^\nu]_t = \rho t, \quad t \geq 0.$$

Now, the risky asset S and the variance ν should uniquely solve the SDE

$$\begin{array}{rcrcl} dS_t & = & r_t S_t\, dt & + & \sqrt{\nu_t} S_t\, dW_t^S \\ d\nu_t & = & \kappa(\theta - \nu_t)\, dt & + & \eta\sqrt{\nu_t}\, dW_t^\nu \end{array}, \qquad 0 \leq t \leq T, \tag{4.1}$$

with positive and deterministic initial values $S_0 = s_0$ and $\nu_0 = v_0$. As variance parameters we have the mean reversion speed $\kappa > 0$, the mean reversion level $\theta > 0$ and the volatility parameter $\eta > 0$. Of course it is of key importance to show that the SDE in (4.1) has a unique solution for the given initial values. Otherwise the whole discussion of the model would be worthless as the object of the discussion would not be well-defined. Consequently our first goal is to show that this SDE has a unique solution.

4.1 Variance and Underlying Process

At first we want to show that the SDE for the variance process ν has a solution on any given filtered probability space. This is the hard part when showing that the risky asset in the Heston model is well-defined. More precisely, we want to ensure that on a given filtered probability space with Brownian motion

W there is an a. s. unique non-negative process $\nu = (\nu)_{t\in[0,T]}$ solving

$$d\nu_t = \kappa(\theta - \nu_t)\,dt + \eta\sqrt{\nu_t}\,dW_t, \quad \nu_0 = v_0. \tag{4.2}$$

That is derived in the following proposition. There we need quite some tools from the theory on SDEs introduced in Section 2.2.

Proposition 4.1.1. *Consider positive constants $\kappa, \theta, \eta, v_0 > 0$ as parameters for the variance process. Then on any given filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ supporting an $\mathbb{F}$-Brownian motion W, where $\mathbb{F}$ satisfies the usual conditions, there exists an up to indistinguishability unique $\mathbb{F}$-adapted process $\nu = (\nu_t)_{t\in[0,T]}$ that has continuous and non-negative paths a. s. and satisfies*

$$\nu_t = v_0 + \int_0^t \kappa(\nu_s - \theta)\,ds + \eta\int_0^t \sqrt{\nu_s}\,dW_s, \quad 0 \leq t \leq T, \quad a.\ s. \tag{4.3}$$

Proof. We consider the SDE with drift coefficient $b\colon \mathbb{R} \to \mathbb{R}$ and diffusion coefficent $\sigma\colon \mathbb{R} \to \mathbb{R}$ defined by

$$b(v) = \kappa(\theta - v) \quad \text{and} \quad \sigma(v) = \eta\sqrt{\max(v, 0)},$$

for $v \in \mathbb{R}$. We now check that the assumptions of Corollary 2.2.17 are fulfilled for the SDE (b, σ). Firstly, observe that b is as affine function Lipschitz-continuous. Next, we prove that σ is Hölder continuous with exponent $\frac{1}{2}$. We distinguish a few cases to show

$$|\sigma(v) - \sigma(w)| \leq \eta\sqrt{|v - w|}, \quad v, w \in \mathbb{R}.$$

- For $v, w \geq 0$ it holds that

  $$|\sigma(v) - \sigma(w)|^2 = \eta^2\big|\sqrt{v} - \sqrt{w}\big|^2 = \eta^2\big|\sqrt{v} - \sqrt{w}\big|\big|\sqrt{v} - \sqrt{w}\big| \leq \eta^2\big|\sqrt{v} - \sqrt{w}\big|\big|\sqrt{v} + \sqrt{w}\big| = \eta^2|v - w|.$$

  Thus we have $|\sigma(v) - \sigma(w)| \leq \eta\sqrt{|v - w|}$ if $v, w \geq 0$.
- If $v \geq 0$ and $w < 0$ we have

  $$\big|\sigma(v) - \sigma(w)\big| = \eta\sqrt{v} \leq \eta\sqrt{v - w} = \eta\sqrt{|v - w|}.$$

- The case $v < 0$, $w \geq 0$ is covered by the previous case due to symmetry.
- If $v < 0$ and $w < 0$ it holds that

  $$|\sigma(v) - \sigma(w)| = 0 \leq \eta\sqrt{|v - w|}.$$

Furthermore, because of

$$|b(v)|^2 + |\sigma(v)|^2 \leq \kappa^2(\theta - v)^2 + \eta^2|v| \leq 2\kappa^2(\theta^2 + v^2) + \eta^2(1 + v^2) \leq (2\kappa^2\theta^2 + \eta^2 + 2\kappa^2)(1 + v^2), \quad v \in \mathbb{R},$$

the coefficients also satisfy the linear growth condition. Hence Corollary 2.2.17 can be applied and thus (b, σ) admits a unique strong solution. Consequently there is an almost surely unique adapted process $(\nu_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P, \mathbb{F})$ that has continuous sample paths and satisfies

$$\nu_t = v_0 + \int_0^t \kappa(\nu_s - \theta)\,ds + \eta\int_0^t \sqrt{\max(\nu_s, 0)}\,dW_s, \tag{4.4}$$

for $t \in [0, +\infty)$, a. s.

Next, we show that the process $(\nu_t)_{t\in[0,T]}$ is also up to indistinguishability the only process satisfying (4.4) on the time interval $[0,T]$. Therefore, define the time-dependent SDE (α,β) by

$$\alpha(t,v) = \begin{cases} b(v) & 0 \le t \le T \\ 0 & t > T \end{cases} \quad \text{and} \quad \beta(t,v) = \begin{cases} \sigma(v) & 0 \le t \le T \\ 0 & t > T \end{cases},$$

for $t \ge 0$ and $v \in \mathbb{R}$. Clearly, α is Lipschitz-continuous in the second argument and β is Hölder-continuous with exponent $\frac{1}{2}$ in the second argument for every $t \ge 0$. Thus Corollary 2.2.15 can be applied to the SDE (α,β) and consequently pathwise uniqueness holds for (α,β). Furthermore, for every process solving (4.4) one can define $v = (v_t)_{t\ge 0}$ by

$$v_t = \begin{cases} \nu_t & 0 \le t \le T \\ \nu_T & t > T \end{cases}$$

Clearly, v is adapted to $\mathbb{F}$, has continuous sample paths and the integrability condition (iii) in Definition 2.2.3 is fulfilled. Moreover, on $[0,T]$ we have

$$v_t = \nu_t = v_0 + \int_0^t b(\nu_s)\,ds + \int_0^t \sigma(\nu_s)\,dW_s = v_0 + \int_0^t \alpha(s,v_s)\,ds + \int_0^t \beta(s,\nu_s)\,dW_s,$$

for $0 \le t \le T$ a. s. On $(T,+\infty)$ we have

$$v_t = v_T = \nu_T = v_0 + \int_0^T b(\nu_s)\,ds + \int_0^T \sigma(\nu_s)\,dW_s = v_0 + \int_0^t \alpha(s,v_s)\,ds + \int_0^t \beta(s,v_s)\,dW_s,$$

for $t > T$ a. s. Thus for every solution of (4.4) the constructed v is a solution to (α,β) on $(\Omega,\mathcal{F},P,\mathbb{F})$. Now consider any other solution $\widetilde{\nu}$ to (4.4) on $[0,T]$ and $(\Omega,\mathcal{F},P,\mathbb{F})$. Since pathwise uniqueness holds for (α,β) we obtain

$$P\big(\widetilde{\nu}_t = \nu_t, \quad 0 \le t \le T\big) = P\big(\widetilde{v}_t = v_t, \quad 0 \le t \le T\big) = P\big(\widetilde{v}_t = v_t, \quad 0 \le t < +\infty\big) = 1.$$

Hence $(\nu_t)_{t\in[0,T]}$ is also the unique solution to (4.4) on the time interval $[0,T]$.
It remains to show the non-negativity of ν because then (4.4) can be simplified to (4.3). Therefore we want to apply Theorem 2.2.18 together with Remark 2.2.19. This we can do with $b^{(1)}$ defined by $b^{(1)}(v) = -\kappa v$, $v \in \mathbb{R}$, $b^{(2)} = b$ and the diffusion coefficient σ considered above and $X^{(1)} = 0$, $X^{(2)} = \nu$. This yields

$$P(0 \le \nu_t, \quad t \ge 0) = P\Big(X_t^{(1)} \le X_t^{(2)}, \quad t \ge 0\Big) = 1,$$

which concludes the proof. □

With an additional restriction on the parameters one can even prove that the variance process ν, defined by (4.2), has strictly positive paths a. s.

Proposition 4.1.2 (Feller-Condition)**.** *Consider the variance process ν defined by* (4.2)*. If the parameters additionally satisfy the so-called Feller-condition, i. e.*

$$2\kappa\theta > \eta^2, \tag{4.5}$$

then the process ν even satisfies

$$P(\nu_t > 0, \quad 0 \le t \le T) = 1.$$

Proof. This proof is basically taken from [Gik11]. Let $n_0 \in \mathbb{N}$ be large enough to satisfy

$$\frac{1}{n_0} \le v_0.$$

Using the process ν obtained in Proposition 4.1.1 define the following stopping times for integers $n \ge n_0$

$$\tau_n := \inf\left\{t \ge 0 \,\middle|\, \nu_t \le 1/n\right\}.$$

Note that τ_n takes the value $+\infty$ if ν_t is greater than $1/n$ on the entire interval $[0,T]$. Additionally for every $n \ge n_0$ define $(\nu_{t,n})_{t\in[0,T]}$ by

$$\nu_{t,n} := \nu_{t\wedge\tau_n}, \quad 0 \le t \le T. \tag{4.6}$$

Elementary properties of the Ito-integral show

$$\begin{aligned}\nu_{t,n} &= \nu_0 + \int_0^{t\wedge\tau_n} \kappa(\theta - \nu_s)\, ds + \int_0^{t\wedge\tau_n} \eta\sqrt{\nu_s}\, dW_s = \\ &= \nu_0 + \int_0^t \mathbb{1}_{[0,\tau_n]}(s)\, \kappa(\theta-\nu_s)\, ds + \int_0^t \mathbb{1}_{[0,\tau_n]}(s)\, \eta\sqrt{\nu_s}\, dW_s.\end{aligned}$$

By the definition of the stopping time τ_n and because $\nu_{\cdot,n}$ has a. s. continuous paths it holds that

$$\left|\nu_{t,n}\right| \ge \frac{1}{n} > 0. \tag{4.7}$$

Now define $q > 0$ by

$$q := \frac{2\kappa\theta}{\eta^2} - 1,$$

where the positivity of q is equivalent to (4.5). Furthermore, we have

$$\frac{1+q}{2}\eta^2 q - \kappa\theta q = \kappa\theta q - \kappa\theta q = 0. \tag{4.8}$$

Now apply Ito's formula to the process $(\nu_{t,n})_{t\in[0,T]}$ and the function $f\colon (0,+\infty) \to (0,+\infty)$ defined by

$$f(v) = v^{-q}.$$

Note that due to (4.7) the image of $\nu_{\cdot,n}$ a. s. lies in the domain of f. We obtain

$$\begin{aligned}(\nu_{t,n})^{-q} = f(\nu_{t,n}) &= \nu_0^{-q} - q\int_0^t (\nu_{s,n})^{-q-1}\, d\nu_{s,n} + \frac{q(q+1)}{2}\int_0^t (\nu_{s,n})^{-q-2}\, d\big[\nu_{\cdot,n}\big]_s = \\ &= \nu_0^{-q} - q\int_0^t \mathbb{1}_{[0,\tau_n]}(s)\, \kappa(\theta-\nu_s)(\nu_{s,n})^{-q-1}\, ds - q\int_0^t \mathbb{1}_{[0,\tau_n]}(s)\, \eta\sqrt{\nu_s}(\nu_{s,n})^{-q-1}\, dW_s + \\ &\quad + q\frac{1+q}{2}\int_0^t \mathbb{1}_{[0,\tau_n]}(s)\, \eta^2\nu_s(\nu_{s,n})^{-q-2}\, ds.\end{aligned}$$

For every $s \ge 0$ and $\alpha \in \{0.5, 1\}$ it holds that

$$\mathbb{1}_{[0,\tau_n]}(s)\,(\nu_s)^\alpha = \mathbb{1}_{[0,\tau_n]}(s)\,(\nu_{s\wedge\tau_n})^\alpha = \mathbb{1}_{[0,\tau_n]}(s)\,(\nu_{s,n})^\alpha.$$

Thus using the previous expression we get

$$
\begin{aligned}
(\nu_{t,n})^{-q} = \nu_0^{-q} &+ \int_0^t \mathbb{1}_{[0,\tau_n]}(s) \underbrace{\left(\frac{1+q}{2}\eta^2 q - \kappa\theta q\right)}_{=\,0 \text{ by (4.8)}} (\nu_{s,n})^{-q-1}\, ds + \\
&+ \kappa q \int_0^t \mathbb{1}_{[0,\tau_n]}(s)\,(\nu_{s,n})^{-q}\, ds - \eta q \int_0^t \mathbb{1}_{[0,\tau_n]}(s)\,(\nu_{s,n})^{-q-0.5}\, dW_s \\
= \nu_0^{-q} &+ \kappa q \int_0^t \mathbb{1}_{[0,\tau_n]}(s)\,(\nu_{s,n})^{-q}\, ds - \eta q \int_0^t \mathbb{1}_{[0,\tau_n]}(s)\,(\nu_{s,n})^{-q-0.5}\, dW_s.
\end{aligned}
$$

It holds that

$$
\mathbb{E}\left(\int_{[0,T]} \left(\mathbb{1}_{[0,\tau_n]}(s)\,(\nu_{s,n})^{-q-0.5} \right)^2 ds \right) \le n^{2q+1} T < \infty, \quad n \ge n_0,
$$

hence the occurring Ito-integral is a martingale. Using this martingale property when taking expectations and then applying Fubini's theorem, which we can since we deal with a non-negative integrand, gives

$$
\mathbb{E}\Big((\nu_{t,n})^{-q} \Big) = \nu_0^{-q} + \kappa q \int_0^t \mathbb{E}\Big(\mathbb{1}_{[0,\tau_n]}(s)\,(\nu_{s,n})^{-q} \Big)\, ds \le \nu_0^{-q} + \kappa q \int_0^t \mathbb{E}\Big((\nu_{s,n})^{-q} \Big)\, ds.
$$

Using Gronwall's inequality gives the estimate

$$
\mathbb{E}\Big((\nu_{t,n})^{-q} \Big) \le \nu_0^{-q} e^{q\kappa t} = \left(\frac{e^{\kappa t}}{\nu_0} \right)^q.
$$

This can be used to get the following for every $n \ge n_0$.

$$
\begin{aligned}
P\Big(\bigcap_{k \ge n_0} \{\tau_k \le T\} \Big) &\le P\big(\tau_n \le T\big) \le P\big(\nu_{\tau_n \wedge T} \le 1/n\big) \le P\big(\nu_{T,n} \le 1/n\big) = \mathbb{E}\Big(\mathbb{1}_{\{\nu_{T,n} \le 1/n\}} \Big) \\
&\le \frac{1}{n^q} \mathbb{E}\Big((\nu_{T,n})^{-q} \mathbb{1}_{\{\nu_{T,n} \le 1/n\}} \Big) \le \frac{1}{n^q} \mathbb{E}\Big((\nu_{T,n})^{-q} \Big) \le \left(\frac{e^{\kappa T}}{n\nu_0} \right)^q \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

In the last estimate $q > 0$ was again very essential. Since

$$
N := \bigcup_{t \in [0,T]} \{\nu_t = 0\} \subseteq \bigcap_{k \ge n_0} \{\tau_k \le T\},
$$

we see that N is a Null-set and thus measurable since $\mathbb{F}$ is complete. This yields

$$
P(\nu_t > 0, \quad 0 \le t \le T) = 1 - P\Big(\big\{ \nu_t > 0, \quad 0 \le t \le T \big\}^c \Big) = 1.
$$

□

Now that we know that the variance process is well-defined on a given probability space we can define the underlying S. Therefore recall the setting at the beginning of this chapter. Following Proposition 4.1.1 from now on $\nu = (\nu_t)_{t\in[0,T]}$ denotes the non-negative unique $\mathbb{F}$-adapted process with continuous paths satisfying

$$
d\nu_t = \theta(\kappa - \nu_t)\, dt + \eta\sqrt{\nu_t}\, dW_t^{\nu}, \quad 0 \le t \le T. \tag{4.9}
$$

By means of the variance process one can now derive the existence of a unique solution to the SDE for the underlying S in (4.1). This is done in the following proposition.

Proposition 4.1.3. *Consider the variance process $\nu = (\nu_t)_{t\geq 0}$ defined in* (4.9) *whose existence is guaranteed by Proposition 4.1.1. Define $S = (S_t)_{t\in[0,T]}$ by*

$$S_t = s_0 \exp\Big(\int_0^t r_u - \frac{\nu_u}{2}\,du + \int_0^t \sqrt{\nu_u}\,dW_u^S\Big), \quad 0 \leq t \leq T. \tag{4.10}$$

Then S is the up to indistinguishability unique $\mathbb{F}$-adapted solution with continuous paths to the SDE

$$dS_t = r_t S_t\,dt + \sqrt{\nu_t} S_t\,dW_t^S, \quad S_0 = s_0. \tag{4.11}$$

Proof. First observe that S, defined by (4.10), is clearly $\mathbb{F}$-adapted and has continuous paths and satisfies $S_0 = s_0$. Furthermore, an application of Ito's formula with $f\colon \mathbb{R} \to (0,+\infty)$, defined by $f(x) = s_0 e^x$, and the Ito-process $Y = (Y_t)_{t\geq 0}$, defined by

$$Y_t = \int_0^t r_u - \frac{\nu_u}{2}\,du + \int_0^t \sqrt{\nu_u}\,dW_u^S, \quad t \geq 0,$$

gives

$$\begin{aligned} dS_t = df(Y_t) &= f'(Y_t)\,dY_t + \frac{1}{2} f''(Y_t)\,d[Y]_t = S_t\,dY_t + \frac{1}{2} S_t\,d[Y]_t = \\ &= \Big(S_t r_t - \frac{1}{2} S_t \nu_t\Big)\,dt + S_t\sqrt{\nu_t}\,dW_t^\nu + \frac{1}{2} S_t \nu_t\,dt = r_t S_t\,dt + S_t\sqrt{\nu_t}\,dW_t^\nu. \end{aligned}$$

Hence S solves (4.11). To show the uniqueness consider another $\mathbb{F}$-adapted process $(Z_t)_{t\in[0,T]}$ with continuous paths a. s. solving (4.11). Applying Ito's formula to the function $g\colon (0,+\infty)\times\mathbb{R} \to \mathbb{R}$ with $g(x,y) = \frac{y}{x}$ and the process $(S_t, Z_t)_{t\in[0,T]}$ – which can be done because we know that S has strictly positive paths – yields

$$\begin{aligned} \frac{Z_t}{S_t} = g(S_t, Z_t) = g(S_0, Z_0) &- \int_0^t \frac{Z_u}{S_u^2}\,dS_u + \int_0^t \frac{1}{S_u}\,dZ_u + \int_0^t \frac{Z_u}{S_u^3}\,d[S]_u - \int_0^t \frac{1}{S_u^2}\,d[S,Z]_u = \\ = g(S_0, Z_0) &- \int_0^t \frac{Z_u}{S_u} r_u\,du - \int_0^t \frac{Z_u}{S_u}\sqrt{\nu_u}\,dW_u^S + \int_0^t \frac{Z_u}{S_u} r_u\,du + \int_0^t \frac{Z_u}{S_u}\sqrt{\nu_u}\,dW_u^S + \\ &+ \int_0^t \frac{Z_u}{S_u}\nu_u\,du - \int_0^t \frac{Z_u}{S_u}\nu_u\,du = g(S_0, Z_0), \end{aligned}$$

where every equality holds a. s. Since

$$g(S_0, Z_0) = \frac{Z_0}{S_0} = \frac{s_0}{s_0} = 1$$

we get

$$P(S_t = Z_t) = 1, \quad t \geq 0.$$

Since both Z and S have continuous paths a. s. they are also indistinguishable. □

Together with the savings account $B = (B_t)_{t\in[0,T]}$ given by

$$B_t = \exp\Big(\int_0^t r(u)\,du\Big), \quad 0 \leq t \leq T,$$

the pair (B,S) forms a well-defined financial market with two primary assets.
The next theorem states that the dynamics in (4.9) and (4.11) are even such that P is a martingale measure for S and thus the market (B, S) is arbitrage free.

Proposition 4.1.4. *The discounted asset price process* $\widetilde{S}$ *satisfies*

$$\widetilde{S}_t = \widetilde{S}_0 + \int_0^t \sqrt{\nu_u}\widetilde{S}_u \, dW_u^S, \quad 0 \le t \le T.$$

In particular $\widetilde{S}$ *is a local martingale under* P *and thus the Heston model we consider is arbitrage free.*

Proof. First observe that

$$\widetilde{S}_t = e^{-\int_0^t r_u \, du} S_t = s_0 \exp\left(-\int_0^t \frac{\nu_u}{2} \, du + \int_0^t \sqrt{\nu_u} \, dW_u^S \right), \quad 0 \le t \le T. \tag{4.12}$$

An application of Ito's formula with $f\colon \mathbb{R} \to (0, +\infty)$, defined by $f(x) = s_0 e^x$, and the Ito-process $Y = (Y_t)_{t \ge 0}$, defined by

$$Y_t = -\int_0^t \frac{\nu_u}{2} \, du + \int_0^t \sqrt{\nu_u} \, dW_u^S, \quad t \ge 0,$$

gives

$$\begin{aligned} d\widetilde{S}_t = df(Y_t) = f'(Y_t) \, dY_t + \frac{1}{2} f''(Y_t) \, d[Y]_t = \widetilde{S}_t \, dY_t + \frac{1}{2} \widetilde{S}_t \, d[Y]_t = \\ = -\frac{1}{2} \widetilde{S}_t \nu_t \, dt + \widetilde{S}_t \sqrt{\nu_t} \, dW_t^\nu + \frac{1}{2} \widetilde{S}_t \nu_t \, dt = \widetilde{S}_t \sqrt{\nu_t} \, dW_t^\nu. \end{aligned}$$

Since ν and $\widetilde{S}$ have continuous paths a. s. it holds that

$$P\left(\int_0^t \nu_u \widetilde{S}_u^2 \, du < \infty, \quad 0 \le t \le T \right) = 1,$$

and consequently $\widetilde{S}$ is a local martingale under P. By Theorem 2.1.11 the considered model is arbitrage free. $\square$

Remark 4.1.5. One can easily show by means of Ito's formula that the log-discounted underyling $X_t = \ln \widetilde{S}_t = \ln S_t - R_t$ is given by

$$\begin{array}{rclcc} dX_t & = & -\nu_t/2 \, dt & & \sqrt{\nu_t} \, dW_t^S \\ d\nu_t & = & \kappa(\theta - \nu_t) \, dt & + & \eta\sqrt{\nu_t} \, dW_t^\nu \end{array}, \qquad 0 \le t \le T. \tag{4.13}$$

When considering (4.13) by means of the two-dimensional Brownian motion W instead of the correlated Brownian motions W^S and W^ν the SDE is transformed to

$$d(X_t, \nu_t)^\top = b(\nu_t) \, dt + \sigma(\nu_t) \, dW_t, \quad (X_0, \nu_0) = (x_0, v_0), \qquad 0 \le t \le T, \tag{4.14}$$

where

$$b(v) = \begin{pmatrix} -v/2 \\ \kappa(\theta - v) \end{pmatrix} \qquad \text{and} \qquad \sigma(v) = \sqrt{v} \begin{pmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & \eta \end{pmatrix} \tag{4.15}$$

for $x \in \mathbb{R}$, $v \ge 0$.

4.2 Moment Generating Function

In this section we determine the moment generating function of $X_T = \ln S_T - R_T$ in the Heston model. By means of the Feynman-Kac theorem this can be done by solving a partial differential equation. Therefore we consider the differential operator $A\colon \mathcal{C}^2\big([0,T]\times\mathbb{R}^2,\mathbb{C}\big) \to \mathcal{C}\big([0,T]\times\mathbb{R},\mathbb{C}\big)$, defined by

$$Af(t,x,v) = -\frac{v}{2}\frac{\partial f}{\partial x}(t,x,v) + \kappa(\theta - v)\frac{\partial f}{\partial v}(t,x,v) + \frac{v}{2}\left(\frac{\partial^2 f}{\partial x^2}(t,x,v) + 2\eta\rho\frac{\partial^2 f}{\partial v \partial x}(t,x,v) + \eta^2\frac{\partial^2 f}{\partial v^2}(t,x,v)\right), \tag{4.16}$$

for $0 \le t \le T$ and $x, v \in \mathbb{R}$, where $\kappa, \theta, \eta > 0$ are the parameters for the variance process and $\rho \in [-1,1]$ is the correlation parameter. The key role of that differential operator is made more precise in the following proposition.

Proposition 4.2.1. *For $T > 0$ consider the Heston model with parameters $\kappa, \theta, \eta > 0$, $\rho \in [-1,1]$ and the log-discounted underlying $X_T = \ln S_T - R_T$ at time T. Furthermore, denote the state space of (X,ν) by $U := \mathbb{R}\times[0,+\infty)$. For $z \in \mathbb{C}$ let $f_z \in \mathcal{C}^{1,2}\big([0,T]\times\mathbb{R}^2,\mathbb{C}\big)$ be a function satisfying the partial differential equation*

$$\frac{\partial f_z}{\partial t}(t,x,v) + Af_z(t,x,v) = 0, \quad 0 \le t \le T, \quad (x,v) \in \mathbb{R}^2, \tag{4.17}$$

where A is defined in (4.16)*. Moreover, assume that the terminal condition*

$$f_z(T,x,v) = e^{zx}, \quad (x,v) \in U, \tag{4.18}$$

is fulfilled. Then the following two statements are true.

(i) If $z \in \mathbb{R}$, $f_z \in \mathcal{C}^{1,2}\big([0,T]\times\mathbb{R}^2,\mathbb{R}\big)$ and there is a constant $c \in \mathbb{R}$ such that $f_z \ge c$ on $[0,T]\times U$ then

$$M_{X_T}(z) = \mathbb{E}(e^{zX_T}) \le f_z(0,x_0,v_0) < +\infty.$$

(ii) If $z \in \mathbb{C}$ and f_z is bounded on $[0,T]\times U$ then

$$M_{X_T}(z) = \mathbb{E}\big(e^{zX_T}\big) = f_z(0,x_0,v_0).$$

Proof. This lemma is basically just an application of Theorem 2.2.21 (Feynman-Kac) to the U-valued process $(X_t,\nu_t)_{t\in[0,T]}$. Recall from (4.14) that (X,ν) is a two-dimensional diffusion whose coefficients b and σ are given by (4.15). To obtain a corresponding infinitesimal generator A we compute

$$\sigma(v)\sigma(v)^\top = v\begin{pmatrix}\sqrt{1-\rho^2} & \rho \\ 0 & \eta\end{pmatrix}\begin{pmatrix}\sqrt{1-\rho^2} & 0 \\ \rho & \eta\end{pmatrix} = v\begin{pmatrix}1 & \eta\rho \\ \eta\rho & \eta^2\end{pmatrix}, \quad (x,v) \in U, \quad 0 \le t \le T.$$

Hence an infinitesimal generator A for the domain $G = \mathbb{R}^2$ is given by (4.16). Next, we apply the Feynman-Kac Theorem (Theorem 2.2.21) with $U = \mathbb{R}\times[0,+\infty)$, $G = \mathbb{R}^2$, $q = 0$ and $\phi\colon U \to \mathbb{C}$ defined by

$$\phi(x,v) = e^{zx}, \quad (x,v) \in U.$$

Now distinguish the two cases considered.

(i) By assumption f_z is real-valued. Because of $z \in \mathbb{R}$ also ϕ is real-valued. Because of $f_z \ge c$ on U, for a constant $c \in \mathbb{R}$, Theorem 2.2.21 (Feynman-Kac) can be applied to obtain

$$\mathbb{E}\big(e^{zX_T}\big) = \mathbb{E}\big(\phi(X_T,\nu_T)\big) \le f_z(0,x_0,v_0) < +\infty.$$

(ii) First observe that because of $z \in \mathbb{C}$ the function ϕ is potentially complex-valued. Recall Remark 2.2.22 to see that Theorem 2.2.21 (Feynman-Kac) can also be applied to bounded complex-valued functions. Since we indeed assume f_z to be bounded we thus obtain the equality

$$\mathbb{E}\big(e^{zX_T}\big) = f_z(0, x_0, v_0).$$

□

Now the aim is to solve (4.17) together with (4.18). One step towards the solution is the next lemma which relates the PDE from the previous proposition to an ODE. This transformation can also for instance be found in Heston's original paper in 1993, [Hes93].

Lemma 4.2.2. *For $T > 0$, $z \in \mathbb{C}$ consider a function $\Psi_z \in \mathcal{C}^1\big([0,T],\mathbb{C}\big)$ satisfying*

$$\Psi_z'(t) = -\frac{\eta^2}{2}\Psi_z(t)^2 + (\kappa - z\eta\rho)\Psi_z(t) - \frac{z(z-1)}{2}, \quad 0 \le t \le T \qquad \text{and} \qquad \Psi_z(T) = 0. \tag{4.19}$$

Then with $\Phi_z \colon [0,T] \to \mathbb{C}$ given by

$$\Phi_z(t) = \kappa\theta \int_t^T \Psi_z(s)\, ds, \quad 0 \le t \le T, \tag{4.20}$$

which is well defined due to the continuity of Ψ_z, a solution to (4.17) and (4.18) is given by

$$f_z(t,x,v) = \exp\Big(zx + \Phi_z(t) + v\Psi_z(t)\Big), \quad (t,x,v) \in [0,T] \times \mathbb{R}^2. \tag{4.21}$$

Note that at this point it remains open whether f_z is bounded.

Proof. For notation convenience we omit the subscript $z \in \mathbb{C}$, which shows the dependence of the solution on the fixed $z \in \mathbb{C}$. To prove this lemma we first note that

$$f(T,x,v) = \exp\Big(zx + \Phi(T) + v\Psi(T)\Big) = e^{zx}, \quad \forall (x,v) \in U, \tag{4.22}$$

is satisfied since $\Phi(T) = \Psi(T) = 0$ holds. Hence we have shown (4.18). Furthermore, we see that $f \in \mathcal{C}^{1,2}\big([0,T] \times \mathbb{R}^2, \mathbb{C}\big)$.
By means of the fundamental theorem of calculus, which can be applied due to the continuity of Ψ, we obtain

$$\Phi'(t) = -\kappa\theta\Psi(t), \quad 0 \le t \le T.$$

Using that and (4.19) we get

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,x,v) &= f(t,x,v)\big(\Phi'(t) + v\Psi'(t)\big) = \\
&= f(t,x,v)\bigg(-\kappa\theta\Psi(t) - v\frac{\eta^2}{2}\Psi(t)^2 + v(\kappa - z\eta\rho)\Psi(t) - v\frac{z(z-1)}{2}\bigg) = \\
&= f(t,x,v)\bigg(\frac{zv}{2} - \kappa\theta\Psi(t) + \kappa v\Psi(t) - \frac{vz^2}{2} - zv\eta\rho\Psi(t) - \frac{v\eta^2}{2}\Psi(t)^2\bigg)
\end{aligned}$$

and

$$\begin{aligned}
Af(t,x,v) = & -\frac{v}{2}zf(t,x,v) + \kappa(\theta - v)\Psi(t)f(t,x,v) + \\
& + \frac{v}{2}\Big(z^2 f(t,x,v) + 2z\eta\rho\Psi(t)f(t,x,v) + \eta^2\Psi(t)^2 f(t,x,v)\Big) = \\
= & \, f(t,x,v)\bigg(-\frac{zv}{2} + \kappa\theta\Psi(t) - \kappa v\Psi(t) + \frac{vz^2}{2} + zv\eta\rho\Psi(t) + \frac{v\eta^2}{2}\Psi(t)^2\bigg),
\end{aligned}$$

for all $(t,x,v) \in (0,T) \times U$. Adding the latter two equations yields

$$\frac{\partial f}{\partial t}(t,x,v) + Af(t,x,v) = 0.$$

for $(t,x,v) \in (0,T) \times U$, which is (4.17). $\square$

Now we want to solve the ODE presented in Lemma 4.2.2 for z belonging to a complex neighborhood of 0. When doing so the following quadratic polynomial will appear quite frequently. We define $D\colon \mathbb{C} \to \mathbb{C}$ by

$$D(z) = (\kappa - z\eta\rho)^2 - z(z-1)\eta^2, \quad z \in \mathbb{C}. \tag{4.23}$$

The following two lemmas address the key properties of the function D. We start with a result regarding the monotonicity properties of D when restricting it to the real axis.

Lemma 4.2.3. *Consider D as defined in* (4.23) *and define $u_{\max}$ by*

$$u_{\max} = \begin{cases} \frac{1}{2}\frac{\eta - 2\kappa\rho}{\eta(1-\rho^2)} & \rho \in (-1,1) \\ +\infty & \rho \in \{-1,1\} \quad \text{and} \quad \eta > 2\kappa\rho \\ -\infty & \rho = 1 \quad \text{and} \quad \eta < 2\kappa\rho \\ 0 & \rho = 1 \quad \text{and} \quad \eta = 2\kappa\rho \end{cases}. \tag{4.24}$$

Furthermore, assume that either $\rho \in [-1,1)$ or $\eta \neq 2\kappa$ holds. Then D is strictly increasing on $(-\infty, u_{\max})$ and strictly decreasing on $(u_{\max}, +\infty)$.

Proof. This is proven together with Lemma 4.2.4 in Subsection A.2.1 in the appendix. $\square$

Additional useful properties of D are presented in the next lemma.

Lemma 4.2.4. *Consider D as defined in* (4.23) *and restrict it to the real axis. The behavior of $D|_{\mathbb{R}}$ can then be described as follows.*

(i) $\rho \in (-1,1)$

Then $D|_{\mathbb{R}}$ is a quadratic polynomial with a negative leading coefficient and roots $u_r > 0$ and $u_l < 0$ given by

$$u_{r,l} = \frac{\eta - 2\kappa\rho \pm \sqrt{(\eta - 2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2\eta(1-\rho^2)}.$$

(ii) $\rho = -1$

Then $D|_{\mathbb{R}}$ is an affine function with a positive slope and a unique root at

$$u_l = -\frac{\kappa^2}{\eta(\eta + 2\kappa)}.$$

(iii) $\rho = 1$

- $\triangleright$ $\eta > 2\kappa$

  Then $D|_{\mathbb{R}}$ is an affine function with a positive slope and a unique root at

$$u_l = -\frac{\kappa^2}{\eta(\eta - 2\kappa)}.$$

▷ $\eta < 2\kappa$
Then $D|_{\mathbb{R}}$ is an affine function with a negative slope and a unique root at

$$u_r = \frac{\kappa^2}{\eta(2\kappa - \eta)}.$$

▷ $\eta = 2\kappa$
Then $D(u) = \kappa^2 > 0$ for $u \in \mathbb{R}$.

Proof. See Subsection A.2.1 in the appendix. ☐

The properties of D we have just presented can now be used to discuss where D touches the negative real axis or zero. This is of particular interest when one desires to smoothly apply a square root or a logarithm to D.

Lemma 4.2.5. *Consider D as it is defined in (4.23) with complex argument $z = z_1 + iz_2 \in \mathbb{C}$. If $z_2 \neq 0$ then $D(z) \notin (-\infty, 0]$. If $z_2 = 0$ then*

$$D(z) \notin (-\infty, 0] \Leftrightarrow z_1 \in (u_l, u_r),$$

where

$$u_l = \begin{cases} \frac{\eta - 2\kappa\rho - \sqrt{(\eta-2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2\eta(1-\rho^2)} & \rho \in (-1,1) \\ -\frac{\kappa^2}{\eta(\eta+2\kappa)} & \rho = -1 \\ -\frac{\kappa^2}{\eta(\eta-2\kappa)} & \rho = 1,\ \eta > 2\kappa \\ -\infty & \rho = 1 \text{ and } \eta \leq 2\kappa \end{cases}, \tag{4.25}$$

and

$$u_r = \begin{cases} \frac{\eta - 2\kappa\rho + \sqrt{(\eta-2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2\eta(1-\rho^2)} & \rho \in (-1,1) \\ +\infty & \rho = -1 \\ +\infty & \rho = 1,\ \eta \geq 2\kappa \\ \frac{\kappa^2}{\eta(2\kappa-\eta)} & \rho = 1,\ \eta < 2\kappa \end{cases}. \tag{4.26}$$

Furthermore, it always holds that $u_l < 0$ and $u_r \geq 1$.

Proof. First consider the case where $z_2 \neq 0$. Plugging the complex number $z = z_1 + iz_2$ into (4.23) gives

$$\begin{aligned} D(z) &= \kappa^2 - (z_1 + iz_2)^2\eta^2(1-\rho^2) + (z_1 + iz_2)(\eta - 2\kappa\rho)\eta = \\ &= \kappa^2 - (z_1^2 - z_2^2)\eta^2(1-\rho^2) + z_1(\eta - 2\kappa\rho)\eta - iz_2\eta\Big(2z_1\eta(1-\rho^2) - (\eta - 2\kappa\rho)\Big) \end{aligned}$$

If $\rho \notin \{-1, 1\}$ and $z_2 \neq 0$ then $\operatorname{Im}\big(D(z)\big) = 0$ implies

$$z_1 = \frac{\eta - 2\kappa\rho}{2\eta\big(1-\rho^2\big)}.$$

Using this one then gets

$$\begin{aligned} \operatorname{Re}\big(D(z)\big) &= \kappa^2 - (z_1^2 - z_2^2)\eta^2(1-\rho^2) + z_1(\eta - 2\kappa\rho)\eta = \kappa^2 + z_2^2\eta^2(1-\rho^2) - \frac{(\eta-2\kappa\rho)^2}{4\big(1-\rho^2\big)} + \frac{(\eta-2\kappa\rho)^2}{2\big(1-\rho^2\big)} = \\ &= \kappa^2 + z_2^2\eta^2(1-\rho^2) + \frac{(\eta-2\kappa\rho)^2}{4\big(1-\rho^2\big)} > 0, \end{aligned}$$

where the latter holds because of $\kappa > 0$. If $\rho \in \{-1, 1\}$ then $\operatorname{Im}\big(D(z)\big) = 0$ implies

$$\eta - 2\kappa\rho = 0.$$

This then yields

$$\operatorname{Re}\big(D(z)\big) = \kappa^2 + z_1 \underbrace{(\eta - 2\kappa\rho)}_{=0} \eta = \kappa^2 > 0.$$

Hence, we have shown that $z_2 \neq 0$ implies $D(z) \notin (-\infty, 0]$ for any $\rho \in [-1, 1]$. Now consider the case where $z_2 = 0$. Then $D(z) = D(z_1)$. Applying Lemma 4.2.4 case by case yields that $D(z_1) > 0$ is equivalent to $z_1 \in (u_l, u_r)$. Moreover, u_l is always clearly negative. To see that $u_r \geq 1$ holds consider the following. If $u_r = +\infty$ the statement is trivial and if $u_r < +\infty$ then $u = u_r$ is the unique positive root of D. Because of $D(0) > 0$ and

$$D(1) = (\kappa - \eta\rho)^2 \geq 0,$$

we see that $u_r \geq 1$ must hold. □

Remark 4.2.6. Note that if $z_1 = 0$ then D never touches the negative real axis or zero.

Next, we define the functions b and H which play a key role when deriving the moment generating function of the log-discounted underlying in the Heston model. In particular they are used to solve the ODE in (4.19). Since only functions which continuously depend on their arguments will be of use when solving an ODE we restrict the domain such that the complex logarithm is only applied to complex numbers not belonging to $(-\infty, 0]$.

With the domain U_H defined by

$$U_H = (u_l, u_r) \cup \mathbb{C} \setminus \mathbb{R}, \tag{4.27}$$

where u_l and u_r are given by (4.25) and (4.26), the functions $b\colon \mathbb{C} \to \mathbb{C}$ and $H\colon U_H \to \mathbb{C}$ are defined by

$$b(z) = \frac{1}{2}(z\eta\rho - \kappa), \quad z \in \mathbb{C} \qquad \text{and} \qquad H(z) = \frac{1}{2}\exp\left(\frac{1}{2}\log D(z)\right), \quad z \in U_H. \tag{4.28}$$

Now we can tackle the derivation of the desired moment generating function in a neighborhood of 0. The proposition below turns out to be important to find a complex anti-derivative of $\zeta \mapsto \zeta^{-1}$ along the path $\gamma_{z,0}$ presented below.

Proposition 4.2.7. *Assume that $T > 0$. For $z \in U_H$ define the path $\gamma_{z,0}\colon [0, T] \to \mathbb{C}$ by*

$$\gamma_{z,0}(s) = \cosh\big(H(z)(T - s)\big) - \frac{b(z)}{H(z)}\sinh\big(H(z)(T - s)\big), \quad 0 \leq s \leq T. \tag{4.29}$$

Then there is a $\delta > 0$ such that

$$\gamma_{z,0}([0, T]) \subseteq (0, +\infty) + i\mathbb{R}, \quad z \in B_\delta(0),$$

and $B_\delta(0) \subseteq U_H$.

Proof. The proof is based on a compactness argument. Furthermore, all balls considered in this proof are meant with respect to the $\|\cdot\|_\infty$-norm. First observe that $H(0) = -b(0) = \frac{\kappa}{2}$ leads to

$$\gamma_{0,0}(t) = \cosh\big(H(0)(T - t)\big) - \underbrace{\frac{b(0)}{H(0)}}_{=-1}\sinh\big(H(0)(T - t)\big) = e^{\frac{\kappa}{2}(T - t)}, \quad t \in [0, T].$$

In particular we have $\mathrm{Re}(\gamma_{0,0}(t)) > 0$ for every $t \in [0,T]$. Since $(z,t) \mapsto \gamma_{z,0}(t)$ continuously depends on $(z,t) \in U_H \times [0,T]$ the following holds. For every $t \in [0,T]$ there is a $\delta_t > 0$ such that on the one hand $B_{\delta_t}(0) \subseteq U_H$ holds and on the other hand

$$\mathrm{Re}(\gamma_{z,0}(s)) > 0, \quad (z,s) \in B_{\delta_t}(0,t), \tag{4.30}$$

is satisfied. Obviously we have

$$\bigcup_{t\in[0,T]} B_{\delta_t}(t) \supseteq [0,T].$$

Since $[0,T]$ is compact there are $t_1,\dots,t_n$ and $\delta_1,\dots,\delta_n$ such that

$$\bigcup_{i=1}^{n} B_{\delta_i}(t_i) \supseteq [0,T].$$

Next, define $\delta > 0$ by

$$\delta := \min_{i=1,\dots,n} \delta_i > 0.$$

Now we show that the statement of this proposition holds for that $\delta > 0$. Therefore consider a fixed but arbitrary $z \in B_\delta(0)$. For every $s \in [0,T]$ there is a $j \in \{1,\dots,n\}$ such that $s \in B_{\delta_j}(t_j)$. Because of $\delta \le \delta_j$ the considered z also belongs to $B_{\delta_j}(0)$. Consequently we have

$$(z,s) \in B_{\delta_j}(0) \times B_{\delta_j}(t_j) = B_{\delta_j}(0,t_j),$$

where the latter equality is due to the use of balls with respect to the $\|\cdot\|_\infty$-norm. Since (4.30) in particular holds for $t = t_j$ we have

$$\mathrm{Re}(\gamma_{z,0}(s)) > 0.$$

Since $z \in B_\delta(0)$ and $s \in [0,T]$ were arbitrary the first statement is proven. Since $B_{\delta_j}(0) \subseteq U_H$ also $B_\delta(0) \subseteq U_H$ must hold. ☐

With the previous lemma we can now write down the solutions to the ODEs in (4.19) and (4.20) for z belonging to a neighborhood of 0. Note that we do not derive the solution here. We just present it and verify that it actually solves the ODE. Since the considered ODE is a Riccati equation with constant (complex) coefficients the solution can be derived using standard routines. However, one has to watch out that occurring denominators never reach zero as that would cause the solution to explode.

Lemma 4.2.8. *Consider the functions b and H defined in* (4.28) *and the path $\gamma_{z,0}$ from* (4.29) *for $z \in U_H$. Furthermore, take a $\delta > 0$ such that $B_\delta(0) \subseteq U_H$ and*

$$\gamma_{z,0}([0,T]) \subseteq \mathbb{C} \setminus \{0\}, \quad z \in B_\delta(0).$$

Then for every $z \in B_\delta(0)$ the function $\Psi_z \in \mathcal{C}^1\big([0,T],\mathbb{C}\big)$ given by

$$\Psi_z(t) = \frac{z(z-1)}{2} \frac{\sinh\big((T-t)H(z)\big)}{H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)}, \quad t \in [0,T]. \tag{4.31}$$

is well-defined (i. e. the denominator is never zero) and a solution to (4.19).

Proof. First of all note that we have

$$H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big) = H(z)\gamma_{z,0}(t), \quad 0 \le t \le T,$$

for $z \in U_H$. Because of $\gamma_{z,0}([0,T]) \subseteq \mathbb{C} \setminus \{0\}$ for $z \in B_\delta(0)$ and $H(z) \neq 0$ for $z \in U_H$ the right hand side cannot be zero. Thus we conclude that the denominator in (4.31) cannot be zero for $z \in B_\delta(0)$ and $t \in [0,T]$ and hence Ψ_z is well-defined.

The statement that Ψ_z belongs to $\mathcal{C}^1([0,T],\mathbb{C})$ and satisfies the ODE in (4.19) can be proved by taking the derivative of Ψ_z with respect to t and plugging the result into the ODE in (4.19). This is done in detail in Subsection A.2.2 in the appendix. Because of $\sinh 0 = 0$ also $\Psi_z(T) = 0$ is satisfied. Thus Ψ_z solves (4.19). ☐

Now recall that our aim is to determine the moment generating function of X_T by means of Lemma 4.2.2. Clearly with Lemma 4.2.8 we have a candidate for Ψ_z. It remains to determine the corresponding Φ_z which is the next step.

Lemma 4.2.9. *Consider the path $\gamma_{z,0}$ defined in* (4.29) *for $z \in U_H$ and take a $\delta > 0$ such that $B_\delta(0) \subseteq U_H$ and*

$$\gamma_{z,0}([0,T]) \subseteq \mathbb{C} \setminus \{0\}, \quad z \in B_\delta(0).$$

Then for every $z \in B_\delta(0)$ the function Ψ_z satisfies

$$\Psi_z(s) = \frac{2}{\eta^2}\left(\frac{\gamma'_{z,0}(s)}{\gamma_{z,0}(s)} - b(z)\right), \quad s \in [0,T]. \tag{4.32}$$

Proof. Because δ has the properties postulated, the statements from Lemma 4.2.8 hold for $z \in B_\delta(0)$. Furthermore, for the right hand side in (4.32) we get

$$\begin{aligned}\frac{2}{\eta^2}\left(\frac{\gamma'_{z,t}(s)}{\gamma_{z,t}(s)} - b(z)\right) &= \frac{2}{\eta^2}\left(- H(z)\frac{H(z)\sinh\big(H(z)(T-s)\big) - b(z)\cosh\big(H(z)(T-s)\big)}{H(z)\cosh\big(H(z)(T-s)\big) - b(z)\sinh\big(H(z)(T-s)\big)} - b(z)\right) = \\ &= 2\frac{b^2(z) - H^2(z)}{\eta^2}\frac{\sinh\big(H(z)(T-s)\big)}{H(z)\cosh\big(H(z)(T-s)\big) - b(z)\sinh\big(H(z)(T-s)\big)}.\end{aligned}$$

The proof is concluded by showing

$$2\frac{b^2(z) - H^2(z)}{\eta^2} = \frac{(z\eta\rho - \kappa)^2 - D(z)}{2\eta^2} = \frac{z(z-1)\eta^2}{2\eta^2} = \frac{z(z-1)}{2}.$$

☐

Now we can compute the integral associated with the corresponding Φ_z.

Lemma 4.2.10. *Consider the path $\gamma_{z,0}$ defined in* (4.29) *for $z \in U_H$ and take a $\delta > 0$ such that $B_\delta(0) \subseteq U_H$ and*

$$\gamma_{z,0}([0,T]) \subseteq \mathbb{C} \setminus (-\infty, 0], \quad z \in B_\delta(0).$$

Then for $z \in B_\delta(0)$ the corresponding Φ_z, as defined in (4.20), *is given by*

$$\Phi_z(t) = \kappa\theta \int_t^T \Psi_z(s)\, ds = -\frac{2\kappa\theta}{\eta^2}\Big(b(z)(T-t) + \log\gamma_{z,0}(t)\Big), \quad t \in [0,T]. \tag{4.33}$$

Proof. First for $t \in [0,T]$ define $\gamma_{z,t} \colon [0,T] \to \mathbb{C}$ by

$$\gamma_{z,t} := \gamma_{z,0}|_{[t,T]}, \quad z \in U_H.$$

Using the path defined in (4.29) and identity (4.32) we can express the integral over Ψ_z as follows.

$$\kappa\theta \int_t^T \Psi_z(s)\,ds = \frac{2\kappa\theta}{\eta^2}\int_t^T \frac{\gamma_{z,t}'(s)}{\gamma_{z,t}(s)} - b(z)\,ds = -\frac{2\kappa\theta}{\eta^2}\Big(b(z)(T-t) - \int_{\gamma_{z,t}} \frac{1}{\zeta}\,d\zeta\Big).$$

Now note that we assume that $\gamma_{z,t}\big([t,T]\big)$ fully lies in $\mathbb{C}\setminus(-\infty,0]$ for $z \in B_\delta(0)$. On $\mathbb{C}\setminus(-\infty,0]$ the logarithm is a complex anti-derivative of $\zeta \mapsto \zeta^{-1}$ and thus the occurring contour integral simplifies as follows.

$$\int_{\gamma_{z,t}} \frac{1}{\zeta}\,d\zeta = \log\gamma_{z,t}(T) - \log\gamma_{z,t}(t) = -\log\gamma_{z,t}(t), \quad z \in B_\delta(0),$$

where we used $\gamma_{z,t}(T) = 1$ for the last equality. □

With the derived Ψ_z and Φ_z we are very close to a proof of how the moment generating function looks like in a neighborhood of 0. To see that let $\delta > 0$ be such that $B_\delta(0) \subseteq U_H$ and

$$\gamma_{z,0}([0,T]) \subseteq \mathbb{C}\setminus(-\infty,0], \quad z \in B_\delta(0).$$

Such a $\delta > 0$ exists due to Lemma 4.2.7. For $z \in B_\delta(0)$ Lemma 4.2.2 tells us that a solution f_z to (4.17) and (4.18) is given by

$$f_z(t,x,v) = \exp\Big(zx + \Phi_z(t) + v\Psi_z(t)\Big), \quad (t,x,v) \in [0,T]\times\mathbb{R}^2,$$

where Ψ_z and Φ_z are defined in (4.31) and (4.33). In order to conclude the equality

$$M_{X_T}(z) = \mathbb{E}\big(e^{zX_T}\big) = f_z(0,x_0,v_0), \quad z \in B_\delta(0),$$

by applying Proposition 4.2.1 we need to ensure that f is bounded on $[0,T]\times U$. Unfortunately this is not true in general. However, we can show this for arguments of the form $z = iu$ where u lies in a real neighborhood of zero. It turns out that this is enough because then one can use the identity theorem for analytic functions (Theorem 2.3.5) to derive the moment generating function for other arguments. Note that for arguments $z = iu$, where $u \in \mathbb{R}$ and $z \in B_\delta(0)$, one has

$$f_{iu}(t,x,v) = \exp\Big(iux + \Phi_{iu}(t) + v\Psi_{iu}(t)\Big), \quad (t,x,v) \in [0,T]\times\mathbb{R}^2.$$

Thus the boundedness of f_{iu} on $[0,T]\times U$ is equivalent to

$$\operatorname{Re}\big(\Psi_{iu}(t)\big) \le 0. \tag{4.34}$$

For the proof that (4.34) holds for u belonging to a real neighborhood of 0 we need the following lemmas.

Lemma 4.2.11. *Define $C \colon [0,+\infty) \to \mathbb{R}$ by*

$$C(\tau) = (\kappa - \rho\eta)\big(\cosh(\kappa\tau) - 1\big) + \kappa\sinh(\kappa\tau) + \frac{\eta(\eta - 2\kappa\rho)}{2\kappa}\big(\sinh(\kappa\tau) - \kappa\tau\big), \quad \tau \ge 0. \tag{4.35}$$

Then $C(\tau) > 0$ holds for $\tau > 0$.

Proof. See Subsection A.2.3 in the appendix. □

Lemma 4.2.12. *Let $t \in [0,T)$. Then*

$$\operatorname{Re}\big(\Psi_{iu}(s)\big) \sim \; -u^2 \frac{C(T-t)}{2\kappa^2} e^{-\kappa(T-t)}, \quad \textit{for} \quad (s,u) \to (t,0), \tag{4.36}$$

where C is defined in (4.35)*. Furthermore, also*

$$\operatorname{Re}\big(\Psi_{iu}(s)\big) \sim \; -\frac{u^2}{2}(T-s), \quad \textit{for} \quad (s,u) \to (T,0), \tag{4.37}$$

holds.

Proof. See Subsection A.2.4 in the appendix. $\square$

With the following lemma we can summarize the derived properties of $\gamma_{z,0}$ and $\Psi_{z,0}$ when z belongs to a neighborhood of 0.

Lemma 4.2.13. *Let $T > 0$. Then there is a $\delta > 0$ such that all of the three statements*

(i) $z \in U_H$

(ii) $\operatorname{Re}(\gamma_{z,0}(t)) > 0, \qquad 0 \le t \le T$

(iii) $\operatorname{Re}\big(\Psi_{iu}(t)\big) \le 0, \qquad 0 \le t \le T$

hold for $z \in B_\delta(0)$ and $u \in (-\delta, \delta)$.

Proof. By Lemma 4.2.7 we know that there is an $\widetilde{\delta} > 0$ such that $z \in U_H$ and $\operatorname{Re}(\gamma_{z,0})\big([0,T]\big) \subseteq (0,+\infty)$ hold for $z \in B_{\widetilde{\delta}}(0)$. Our aim is to find a $\delta \in (0,\widetilde{\delta})$ such that the third statement holds as we are then done. The first step is to show that for each $t \in [0,T]$ there exists a $\delta_t > 0$ such that

$$\operatorname{Re}\big(\Psi_{iu}(s)\big) \le 0, \quad (s,u) \in B_{\delta_t}(t,0),$$

where the balls are with respect to the $\|\cdot\|_\infty$ in $[0,T] \times \mathbb{R}$. Therefore, we distinguish the following two cases.

- ▷ $t = T$

  By relation (4.37) in Lemma 4.2.12 we have that

  $$\operatorname{Re}\big(\Psi_{iu}(s)\big) \sim \; -\frac{u^2}{2}(T-s), \quad \text{for} \quad (s,u) \to (T,0).$$

  In particular there is a $\delta_T \in (0,\widetilde{\delta})$ such that

  $$\frac{\operatorname{Re}\big(\Psi_{iu}(s)\big)}{-\frac{u^2}{2}(T-s)} \ge \frac{1}{2}, \quad s \in (T-\delta_T, T), \quad u \in (-\delta_T, \delta_T) \setminus \{0\}.$$

  This however implies

  $$\operatorname{Re}\big(\Psi_{iu}(s)\big) \le \; -\frac{u^2}{4}(T-s) \le 0, \qquad (s,u) \in B_{\delta_T}(T,0).$$

- ▷ $t \in [0,T)$

  By relation (4.36) in Lemma 4.2.12 we have that

  $$\operatorname{Re}\big(\Psi_{iu}(s)\big) \sim \; -u^2 \frac{C(T-t)}{2\kappa^2} e^{-\kappa(T-t)}, \quad \text{for} \quad (s,u) \to (t,0).$$

This again implies that there is a $\delta_t \in (0, \widetilde{\delta})$ such that

$$\frac{\mathrm{Re}\left(\Psi_{iu}(s)\right)}{-u^2 \frac{C(T-t)}{2\kappa^2} e^{-\kappa(T-t)}} \geq \frac{1}{2}, \quad s \in (t - \delta_t, t + \delta_t) \setminus \{t\}, \quad u \in (-\delta_t, \delta_t) \setminus \{0\}.$$

Because of $C(T-t) > 0$ this also leads to

$$\mathrm{Re}\left(\Psi_{iu}(s)\right) \leq \; -u^2 \frac{C(T-t)}{4\kappa^2} e^{-\kappa(T-t)} \leq 0, \quad B_{\delta_t}(t, 0).$$

Therefore we have that for every $t \in [0, T]$ there is a $\delta_t \in (0, \widetilde{\delta})$ such that

$$\mathrm{Re}\left(\Psi_{iu}(s)\right) \leq 0, \quad (s, u) \in B_{\delta_t}(t, 0). \tag{4.38}$$

Now observe that we clearly have

$$\bigcup_{t \in [0,T]} B_{\delta_t}(t) \supseteq [0, T].$$

Since $[0, T]$ is compact there are $t_1, \ldots, t_n$ and $\delta_1, \ldots, \delta_n$ such that

$$\bigcup_{j=1}^{n} B_{\delta_j}(t_j) \supseteq [0, T]. \tag{4.39}$$

Now define

$$\delta := \min_{j=1,\ldots,n} \delta_j \in (0, \widetilde{\delta}).$$

and consider $u \in (-\delta, \delta)$. It remains to show that

$$\mathrm{Re}\left(\Psi_{iu}(t)\right) \leq 0,$$

holds for every $t \in [0, T]$. Because of (4.39) we see that for every $t \in [0, T]$ there is an $j \in \{1, \ldots, n\}$ such that $t \in B_{\delta_j}(t_j)$. Since $\delta \leq \delta_j$ we thus have

$$(t, u) \in (t_j - \delta_j, t_j + \delta_j) \times (-\delta, \delta) \subseteq B_{\delta_j}(t_j, 0).$$

Because of (4.38) we obtain

$$\mathrm{Re}\left(\Psi_{iu}(t)\right) \leq 0,$$

for the $u \in (-\delta, \delta)$ considered. Since $t \in [0, T]$ and $u \in (-\delta, \delta)$ were arbitrary we are done. □

Now we can eventually use the functions Ψ_z and Φ_z to derive the moment generating function of X_T in a neighborhood of 0.

Proposition 4.2.14. *Consider the moment-generating function of $X_T = \ln S_T - R_T$ in the Heston-model. Let $\delta > 0$ be such that (i), (ii) and (iii) in Lemma 4.2.13 are satisfied. Then we have*

$$\mathbb{E}\left(e^{uX_T}\right) < +\infty, \quad u \in (-\delta, \delta).$$

and

$$\mathbb{E}\left(e^{iuX_T}\right) = \exp\Big(iux_0 + \Phi_{iu}(0) + v_0 \Psi_{iu}(0)\Big), \quad u \in (-\delta, \delta).$$

Proof. The assumptions ensure that $\delta > 0$ is such that the three statements

(i) $z \in U_H$

(ii) $\gamma_{z,0}\big([0,T]\big) \subseteq (0,+\infty) + i\mathbb{R}$

(iii) $\mathrm{Re}\,\big(\Psi_{iu}(t)\big) \leq 0, \qquad 0 \leq t \leq T$

hold for $z \in B_\delta(0)$ and $u \in (-\delta,\delta)$. Using that δ we can apply Lemma 4.2.8 and Lemma 4.2.10 to have solutions Ψ_z and Φ_z to (4.19) and (4.20) for $z \in B_\delta(0)$. By Lemma 4.2.2 we see that f_z, defined by,

$$f_z(t,x,v) = \exp\Big(zx + \Phi_z(t) + v\Psi_z(t)\Big), \quad (t,x,v) \in [0,T] \times \mathbb{R}^2,$$

is a solution to (4.17) and (4.18) on $[0,T] \times \mathbb{R}^2$ for $z \in B_\delta(0)$. Now distinguish two cases for the two statements.

- ▷ $z = u \in (-\delta,\delta)$

  Then f_u is also real-valued and positive. Thus we can apply part (i) of Proposition 4.2.1 to get

  $$M_{X_T}(u) = \mathbb{E}\big(e^{uX_T}\big) \leq f_u(0,x_0,v_0) < \infty.$$

  for $u \in (-\delta,\delta)$.

- ▷ $z = iu$ with $u \in (-\delta,\delta)$

  Since Φ_{iu} is continuous there is a constant C such that $|\Phi_{iu}(t)| \leq C$ for $0 \leq t \leq T$. Furthermore, we have $\mathrm{Re}\,\big(\Psi_{iu}(t)\big) \leq 0$ for $0 \leq t \leq T$ and thus we get

  $$|f_{iu}(t,x,v)| = \Big|\exp\Big(iux + \Phi_{iu}(t) + v\Psi_{iu}(t)\Big)\Big| \leq e^C, \quad (t,x,v) \in [0,T] \times \mathbb{R} \times [0,+\infty).$$

  Consequently we can apply part (ii) of Proposition 4.2.1 to get

  $$\varphi_{X_T}(u) = \mathbb{E}\big(e^{iuX_T}\big) = f_{iu}(0,x_0,v_0),$$

  for arbitrary $u \in (-\delta,\delta)$.

□

The previous results can now be used to derive the moment generating function of the log-underlying in the Heston model in a neighborhood of 0.

Proposition 4.2.15. *There is a $\delta > 0$ such that the moment-generating function M_{X_T} of the* log *-discounted-underlying X_T at time T in the Heston-model exists on $(-\delta,\delta) + i\mathbb{R}$ and $(-\delta,\delta) \subseteq (u_l,u_r)$ holds. For that $\delta > 0$ we have*

$$M_{X_T}(z) = \exp\Big(zx_0 + A(z) + v_0B(z)\Big), \quad z \in B_\delta(0), \tag{4.40}$$

where with

$$P_T(z) = \cosh\big(H(z)T\big) - b(z)\frac{\sinh\big(H(z)T\big)}{H(z)}, \quad z \in B_\delta(0),$$

the coefficients A and B are given by

$$A(z) = -\frac{2\kappa\theta}{\eta^2}\Big(b(z)T + \log P_T(z)\Big) \tag{4.41}$$

$$B(z) = \frac{z(z-1)}{2}P_T(z)^{-1}\frac{\sinh\big(H(z)T\big)}{H(z)}, \tag{4.42}$$

for $z \in B_\delta(0)$.

Proof. According to Lemma 4.2.13 there exists a $\delta > 0$ such that

(i) $z \in U_H$

(ii) $\gamma_{z,0}\big([0,T]\big) \subseteq (0,+\infty) + i\mathbb{R}$

(iii) $\operatorname{Re}\big(\Psi_{iu}(t)\big) \leq 0, \qquad 0 \leq t \leq T$

are satisfied for $z \in B_\delta(0)$ and $u \in (-\delta,\delta)$. Because of (i) we particularly have $(-\delta,\delta) \subseteq (u_l,u_r)$. Due to Proposition 4.2.14 we have

$$\mathbb{E}\big(e^{uX_T}\big) < +\infty, \quad u \in (-\delta,\delta).$$

Consequently by means of Proposition 3.1.7 we see that M_{X_T} is well-defined and holomorphic on $(-\delta,\delta)+i\mathbb{R}$. The way to go now is to show that the right hand side of (4.40) is a holomorphic function that coincides with the moment generating function on $i(-\delta,\delta)$. The identity theorem of complex analysis then implies that equation (4.40) must hold for arguments z belonging to $B_\delta(0)$.
Now define $A\colon B_\delta(0) \to \mathbb{C}$ by

$$A(z) := \Phi_z(0) = -\frac{2\kappa\theta}{\eta^2}\Big(b(z)T + \log\gamma_{z,0}(0)\Big), \quad z \in B_\delta(0), \tag{4.43}$$

where the last equality sign is due to Lemma 4.2.10. Plugging in the definition of the path $\gamma_{z,0}$ from (4.29) clearly gives

$$\gamma_{z,0}(0) = P_T(z),$$

for $z \in B_\delta(0)$. As $\delta > 0$ is in particular such that property (ii) of Lemma 4.2.13 is fulfilled, we have

$$P_T(z) = \gamma_{z,0}(0) \notin (-\infty,0], \quad z \in (-\delta,\delta)+i\mathbb{R}.$$

Hence the function A defined by (4.43) is holomorphic and coincides with the right hand side of (4.41).
Next, define $B\colon B_\delta(0) \to \mathbb{C}$ by

$$\begin{aligned} B(z) := \Psi_z(0) &= \frac{z(z-1)}{2}\frac{\sinh\big(H(z)T\big)}{H(z)\cosh\big(H(z)T\big) + b(z)\sinh\big(H(z)T\big)} \\ &= \frac{z(z-1)}{2}\left(\cosh\big(H(z)T\big) + b(z)\frac{\sinh\big(H(z)T\big)}{H(z)}\right)^{-1}\frac{\sinh\big(H(z)T\big)}{H(z)} \\ &= \frac{z(z-1)}{2}P_T(z)^{-1}\frac{\sinh\big(H(z)T\big)}{H(z)}, \end{aligned} \tag{4.44}$$

for $z \in B_\delta(0)$. By Lemma 4.2.8 we do not divide by zero in (4.44) and B is well-defined. As composition of holomorphic functions B is itself holomorphic and it coincides with the right hand side of (4.42).
Now define

$$h(z) = e^{zx_0 + A(z) + v_0B(z)}, \quad z \in B_\delta(0).$$

Since A and B are holomorphic also $h\colon B_\delta(0) \to \mathbb{C}$ is. Furthermore, since the $\delta > 0$ we consider is such that (i)-(iii) of Lemma 4.2.13 hold an application of Proposition 4.2.14 gives

$$h(iu) = e^{iux_0 + A(iu) + v_0B(iu)} = e^{iux_0 + \Phi_{iu}(0) + v_0\Psi_{iu}(0)} = \mathbb{E}\big(e^{iuX_T}\big) = M_{X_T}(iu), \quad u \in (-\delta,\delta).$$

From the beginning of the proof we also know that $M_{X_T}\colon (-\delta,\delta)+i\mathbb{R} \to \mathbb{C}$ is holomorphic. Consequently $h\colon B_\delta(0) \to \mathbb{C}$ and $M_{X_T}|_{B_\delta(0)}\colon B_\delta(0) \to \mathbb{C}$ are two holomorphic functions defined on a connected set which coincide on a set with limit point, namely $i(-\delta,\delta)$. The identity theorem of complex analysis (Theorem 2.3.5) gives then that they coincide at every point in their domain, i. e. on $B_\delta(0)$. Hence we have

$$M_{X_T}(z) = h(z) = e^{zx_0 + A(z) + v_0B(z)}, \quad z \in B_\delta(0),$$

which is the statement. ∎

Next, we want to determine the maximum domain of the moment generating function. It is not a surprise that this has a close connection to seeking extensions of the function P_T presented in Lemma 4.2.15. Another key issue is the analysis of the function H which represents one half of the complex square root of D. By Lemma 4.2.5 we know that we have

$$D(z) \in (-\infty, 0], \quad z \in (-\infty, u_l] \cup [u_r, \infty).$$

Consequently we cannot find an analytic extension of H for real arguments not belonging to (u_l, u_r) in a straightforward way. The key observation here is that for arguments sufficiently close to the real line we can work with the function D only instead of H. Then one does not need to care whether the complex logarithm is applied smoothly to D when evaluating H. The next aim is to derive that expression where we only need D instead of H. Therefore, following del Bano Rollin et al. in [dBRFCU09, Section 3.2], we use the entire functions $R_1 \colon \mathbb{C} \to \mathbb{C}$ and $R_2 \colon \mathbb{C} \to \mathbb{C}$, given by

$$R_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{(2k)!}, \quad z \in \mathbb{C} \qquad \text{and} \qquad R_2(z) = \sum_{k=0}^{\infty} \frac{z^k}{(2k+1)!}, \quad z \in \mathbb{C},$$

to define $P_T \colon \mathbb{C} \to \mathbb{C}$ and $Q_T \colon \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned} P_T(z) &= R_1\left(\frac{1}{4}D(z)T^2\right) - b(z)R_2\left(\frac{1}{4}D(z)T^2\right)T, \quad z \in \mathbb{C} \\ Q_T(z) &= R_2\left(\frac{1}{4}D(z)T^2\right)T, \quad z \in \mathbb{C}. \end{aligned} \tag{4.45}$$

By means of the ratio test one can easily see that the series defining R_1 and R_2 converge for every $z \in \mathbb{C}$. Consequently Q_T and P_T are analytic functions on $\mathbb{C}$. Now extend the function H we have considered so far by

$$H(z) = \begin{cases} \sqrt{|D(z)|} & z \in \mathbb{R} \\ e^{\frac{1}{2}\log D(z)} & z \in \mathbb{C} \setminus \mathbb{R}. \end{cases} \tag{4.46}$$

The latter extension of H can be used to find alternative expressions for the functions P_T and Q_T defined in (4.45).

Lemma 4.2.16. *Consider u_l and u_r from* (4.25) *and* (4.26)*, U_H as defined in* (4.27) *and H from* (4.46)*. Then we have the following identities for P_T and Q_T.*

$$Q_T(z) = \begin{cases} \frac{\sinh H(z)T}{H(z)} & z \in U_H \\ T & z \in \{u_l, u_r\} \\ \frac{\sin H(z)T}{H(z)} & z \in (-\infty, u_l) \cup (u_r, \infty) \end{cases},$$

and

$$P_T(z) = \begin{cases} \cosh H(u)T - b(z)\frac{\sinh H(z)T}{H(z)} & z \in U_H \\ 1 - b(z)T & z \in \{u_l, u_r\} \\ \cos H(z)T - b(z)\frac{\sin H(z)T}{H(z)} & z \in (-\infty, u_l) \cup (u_r, \infty) \end{cases},$$

for $z \in \mathbb{C}$.

Proof. See Subsection A.2.5 in the appendix. ∎

The real roots of P_T neighboring 0 will play an important role to determine the maximal domain of the moment generating function. Therefore for every $T \geq 0$ we define $u_+(T) \in \mathbb{R}_+ \cup \{\infty\}$ and $u_-(T) \in \mathbb{R}_- \cup \{-\infty\}$ by

$$u_-(T) = \sup\left\{u \leq 0 \colon P_T(u) = 0\right\} \qquad \text{and} \qquad u_+(T) = \inf\left\{u \geq 0 \colon P_T(u) = 0\right\}. \tag{4.47}$$

Before we can analyze the connection of $u_+(T)$ and $u_-(T)$ to the maximal domain of M_{X_T} we need the following quite technical lemma.

Lemma 4.2.17. *Let $T > 0$. Consider $u_+(T)$ and $u_-(T)$ as defined in* (4.47)*. Then $u_+(T) > 1$ always holds. Furthermore, the following statements are true.*

(i) If $u_+(T) < +\infty$ then $Q_T\big(u_+(T)\big) > 0$ must hold.

(ii) If $u_-(T) > -\infty$ then $Q_T\big(u_-(T)\big) > 0$ must hold.

Proof. See Subsection A.2.6 in the appendix. ☐

In the following two theorems we present the moment generating function of X_T and show that $M_{X_T}(u) < +\infty$ holds if and only if the argument u belongs to the interval $\big(u_-(T), u_+(T)\big)$. Hence (4.47) defines the boundaries of the domain where M_{X_T} takes finite values.

Theorem 4.2.18. *The moment generating function M_{X_T} of the log-discounted underlying at time $T > 0$ in the Heston model is finite for real arguments belonging to $\big(u_-(T), u_+(T)\big)$, i. e.*

$$\mathbb{E}\big(e^{uX_T}\big) < +\infty, \quad u \in \big(u_-(T), u_+(T)\big),$$

where $u_+(T)$ and $u_-(T)$ are defined in (4.47)*. Furthermore, one always has $u_+(T) > 1$ and the moment generating function is given by*

$$M_{X_T}(u) = P_T(u)^{-\frac{2\kappa\theta}{\eta^2}} \exp\left(ux_0 + \frac{u(u-1)}{2}\frac{Q_T(u)}{P_T(u)}v_0 - \frac{2\kappa\theta}{\eta^2}b(u)T\right), \quad u \in \big(u_-(T), u_+(T)\big),$$

where the functions P_T and Q_T are given in Lemma 4.2.16.

Proof. Start with the $\delta > 0$ presented in Proposition 4.2.15. In particular we have $(-\delta, \delta) \subseteq (u_l, u_r)$. Obviously by Lemma 4.2.16 the function P_T we introduced in (4.45) and those used in Proposition 4.2.15 coincide on $B_\delta(0)$. Since Proposition 4.2.15 particularly states that the expressions are well-defined we know that $P_T(u) \neq 0$ for $u \in (-\delta, \delta)$. As P_T is continuous and real-valued on $(-\delta, \delta)$ and since $P_T(0) > 0$ we know that

$$P_T(u) \notin (-\infty, 0], \quad u \in (-\delta, \delta),$$

holds. We thus know that $u_+(T) \geq \delta$ and $u_-(T) \leq -\delta$. Recall that A, defined by (4.41), is given by

$$A(u) = -\frac{2\kappa\theta}{\eta^2}\Big(b(u)T + \log P_T(u)\Big)$$

for $u \in (-\delta, \delta)$. Furthermore, for the function B, given in (4.42), we get

$$B(u) = \frac{u(u-1)}{2}P_T(u)^{-1}\frac{\sinh\big(H(u)T\big)}{H(u)} = \frac{u(u-1)}{2}\frac{Q_T(u)}{P_T(u)}, \quad u \in (-\delta, \delta) \subseteq \big(u_+(T), u_-(T)\big),$$

due to Lemma 4.2.16. Using Proposition 4.2.15 we see that

$$M_{X_T}(u) = e^{ux_0 + A(u) + v_0 B(u)} = \exp\left(ux_0 - \frac{2\kappa\theta}{\eta^2}\Big(b(u)T + \log P_T(u)\Big) + v_0\frac{u(u-1)}{2}\frac{Q_T(u)}{P_T(u)}\right), \tag{4.48}$$

holds for $u \in (-\delta, \delta)$. To show the expression of the moment generating function in (4.48) can be extended $\big(u_-(T), u_+(T)\big)$ we want to apply Proposition 3.1.8. Therefore we basically need to extend the right hand side of (4.48) to a strip $\big(u_-(T), u_+(T)\big) + i(-\epsilon, \epsilon)$ in the complex plane. Since b, P_T and Q_T are entire functions the crucial thing is to ensure that the extension of P_T maps to $\mathbb{C} \setminus (-\infty, 0]$.

Recall that $u_+(T)$ and $u_-(T)$ are the first roots of P_T neighboring zero. Since P_T is continuous we thus have

$$P_T(u) > 0, \quad u_-(T) < u < u_+(T).$$

Without loss of generality we assume $u_-(T) < -\delta$ and $u_+(T) > \delta$. Now fix real numbers v_1 and v_2 satisfying

$$u_-(T) < v_1 < -\delta < 0 < \delta < v_2 < u_+(T).$$

For $u \in [v_1, v_2] \subseteq \big(u_-(T), u_+(T)\big)$ observe the following. Due the continuity of P_T and $P_T(u) > 0$ there is an $\epsilon_u > 0$ such that

$$P_T\big(B_{\epsilon_u}(u)\big) \subseteq (0, +\infty) + i\mathbb{R}. \tag{4.49}$$

Note that the ball $B_{\epsilon_u}(u)$ is meant with respect to $\|\cdot\|_\infty$-norm in the complex plane. We clearly have

$$\bigcup_{u\in[v_1,v_2]} B_{\epsilon_u}(u) \supseteq [v_1, v_2],$$

hence the union on the left hand side is an open cover of the compact set $[v_1, v_2]$. Thus there is a finite subcover, i. e. there is $\epsilon_1, \ldots, \epsilon_n$ and $u_1, \ldots, u_n$ such that

$$\bigcup_{j=1}^{n} B_{\epsilon_j}(u_j) \supseteq [v_1, v_2].$$

Now define

$$\epsilon := \min_{j=1,\ldots,n} \epsilon_j > 0.$$

Then consider an arbitrary $z \in [v_1, v_2] + i(-\epsilon, \epsilon)$. Obviously there is an $j \in \{1, \ldots, n\}$ such that $\mathrm{Re}(z) \in B_{\epsilon_j}(u_j)$. Because of $\epsilon \le \epsilon_j$ we also have $\mathrm{Im}(z) \in (-\epsilon_j, \epsilon_j)$. Consequently $z \in B_{\epsilon_j}(u_j)$. Because of (4.49) we then have $\mathrm{Re}\big(P_T(z)\big) > 0$. Since $z \in [v_1, v_2] + i(-\epsilon, \epsilon)$ was arbitrary we obtain

$$\mathrm{Re}\big(P_T(z)\big) > 0, \quad z \in [v_1, v_2] + i(-\epsilon, \epsilon).$$

As a result the function $h\colon (v_1, v_2) + i(-\epsilon, \epsilon) \to \mathbb{C}$, defined by

$$h(z) := \exp\left(zx_0 - \frac{2\kappa\theta}{\eta^2}\Big(b(z)T + \log P_T(z)\Big) + v_0\frac{z(z-1)}{2}\frac{Q_T(z)}{P_T(z)}\right), \quad z \in (v_1, v_2) + i(-\epsilon, \epsilon),$$

is holomorphic. Since (4.48) holds for $u \in (-\delta, \delta)$ we in addition have

$$h(u) = M_{X_T}(u), \quad u \in (-\delta, \delta).$$

Applying Proposition 3.1.8 yields

$$M_{X_T}(u) = h(u) < +\infty, \quad u \in (v_1, v_2).$$

Since $v_1 \in \big(u_-(T), -\delta\big)$ and $v_2 \in \big(\delta, u_+(T)\big)$ were arbitrary we even have $M_{X_T}(u) < +\infty$ for $u \in \big(u_-(T), u_+(T)\big)$ and

$$M_{X_T}(u) = \exp\left(ux_0 - \frac{2\kappa\theta}{\eta^2}\Big(b(u)T + \log P_T(u)\Big) + v_0\frac{u(u-1)}{2}\frac{Q_T(u)}{P_T(u)}\right), \quad u \in \big(u_-(T), u_+(T)\big).$$

Rearranging terms leads then to

$$\begin{aligned} M_{X_T}(u) =& \exp\left(-\frac{2\kappa\theta}{\eta^2}\log P_T(u)\right)\exp\left(ux_0 - \frac{2\kappa\theta}{\eta^2}b(u)T + v_0\frac{u(u-1)}{2}\frac{Q_T(u)}{P_T(u)}\right) \\ =& P_T(u)^{-\frac{2\kappa\theta}{\eta^2}}\exp\left(ux_0 + \frac{u(u-1)}{2}\frac{Q_T(u)}{P_T(u)}v_0 - \frac{2\kappa\theta}{\eta^2}b(u)T\right), \end{aligned} \tag{4.50}$$

for $u \in \big(u_-(T), u_+(T)\big)$. Moreover, the statement about $u_+(T)$ being strictly greater than 1 directly follows from Lemma 4.2.17 and thus the proof is concluded. □

Theorem 4.2.19. *The moment generating function M_{X_T} of the log-discounted underlying at time $T > 0$ in the Heston model is infinite for every real argument not belonging to $\big(u_-(T), u_+(T)\big)$, i. e.*

$$\mathbb{E}\big(e^{uX_T}\big) = +\infty, \quad u \in \big(-\infty, u_-(T)\big] \cup \big[u_+(T), +\infty\big),$$

where $u_+(T)$ and $u_-(T)$ are defined in (4.47).

Proof. First note that by Theorem 4.2.18 the moment generating function M_{X_T} on $\big(u_-(T), u_+(T)\big)$ is given by

$$M_{X_T}(u) = \exp\left(ux_0 - \frac{2\kappa\theta}{\eta^2}b(u)T - \frac{2\kappa\theta}{\eta^2}\log P_T(u) + v_0\frac{u(u-1)}{2}\frac{Q_T(u)}{P_T(u)}\right), \quad u \in \big(u_-(T), u_+(T)\big), \tag{4.51}$$

where $u_+(T) > 1$ and $u_-(T) < 0$ always hold. To show the statement of this theorem for $u \geq u_+(T)$ consider the limit for $u \nearrow u_+(T)$ of the right hand side of (4.51) if $u_+(T) < +\infty$. First observe that by Lemma 4.2.17 we also have $Q_T\big(u_+(T)\big) > 0$ in addition to $u_+(T) > 1$. Because of $P_T\big(u_+(T)\big) = 0$ that leads to

$$\begin{aligned}\lim_{u\nearrow u_+(T)} -\frac{2\kappa\theta}{\eta^2}\log P_T(u) + \frac{u(u-1)Q_T(u)}{2P_T(u)} &= \lim_{u\nearrow u_+(T)} \frac{-\frac{2\kappa\theta}{\eta^2}P_T(u)\log P_T(u) + \frac{u(u-1)}{2}Q_T(u)}{P_T(u)} = \\ &= \left(\frac{u_+(T)\big(u_+(T)-1\big)}{2}Q_T\big(u_+(T)\big)\right)\lim_{u\nearrow u_+(T)} P_T(u)^{-1} = +\infty.\end{aligned}$$

Because all other arguments of the exponential describing M_{X_T} in (4.51) have finite limits for $u \nearrow u_+(T)$ we get

$$\lim_{u\nearrow u_+(T)} M_{X_T}(u) = +\infty.$$

This can be now used to show

$$\begin{aligned}\mathbb{E}\big(e^{u_+(T)X_T}\big) &= \mathbb{E}\Big(\lim_{u\nearrow u_+(T)} e^{uX_T}\Big) = \mathbb{E}\Big(\lim_{u\nearrow u_+(T)} \mathbb{1}_{\{X_T\geq 0\}}e^{uX_T} + \lim_{u\nearrow u_+(T)} \mathbb{1}_{\{X_T<0\}}e^{uX_T}\Big) = \\ &= \mathbb{E}\Big(\lim_{u\nearrow u_+(T)} \mathbb{1}_{\{X_T\geq 0\}}e^{uX_T}\Big) + \mathbb{E}\Big(\lim_{u\nearrow u_+(T)} \underbrace{\mathbb{1}_{\{X_T<0\}}e^{uX_T}}_{|\cdot|\leq 1}\Big) = \\ &= \lim_{u\nearrow u_+(T)} \mathbb{E}\Big(\mathbb{1}_{\{X_T\geq 0\}}e^{uX_T}\Big) + \lim_{u\nearrow u_+(T)} \mathbb{E}\Big(\mathbb{1}_{\{X_T<0\}}e^{uX_T}\Big) = \\ &= \lim_{u\nearrow u_+(T)} \Big(\mathbb{E}\Big(\mathbb{1}_{\{X_T\geq 0\}}e^{uX_T}\Big) + \mathbb{E}\Big(\mathbb{1}_{\{X_T<0\}}e^{uX_T}\Big)\Big) \\ &= \lim_{u\nearrow u_+(T)} \mathbb{E}\big(e^{uX_T}\big) = \lim_{u\nearrow u_+(T)} M_{X_T}(u) = +\infty.\end{aligned}$$

Note that all occurring expectations exist due to non-negativity and all limits exist due to monotonicity. By means of Proposition 3.1.5 we also see that $\mathbb{E}\big(e^{uX_T}\big) = +\infty$ holds for $u > u_+(T)$.
If $u_-(T) > -\infty$ we have

$$\frac{u_-(T)\big(u_-(T)-1\big)}{2} > 0,$$

due to $u_-(T) < 0$, and Lemma 4.2.17 gives $Q_T(u_-) > 0$. Consequently one can analogously deduce

$$\mathbb{E}\big(e^{u_-(T)X_T}\big) = +\infty,$$

leading by means of Proposition 3.1.5 again to $\mathbb{E}\big(e^{uX_T}\big) = +\infty$ for $u \leq u_-(T)$. This concludes the proof. ☐

4.3 Critical Time in the Heston Model

The object of our analysis in this section is the critical time $T^*\colon \mathbb{R} \to (0,+\infty]$, which is defined by

$$T^*(u) = \inf\{T \geq 0\colon P_T(u) \leq 0\}, \quad u \in \mathbb{R}, \tag{4.52}$$

where the infimum of the empty set is understood to be $+\infty$. Furthermore, observe that because of

$$P_0(u) = 1 > 0, \quad u \in \mathbb{R},$$

and since $P_T(u)$ continuously depends on $T \in [0,+\infty)$ the critical time T^* maps to $(0,+\infty]$ indeed. The name critical time is motivated by a result presented in this section. That result basically tells us that for a given $u \in \mathbb{R}$ the time $T^*(u)$ is the smallest maturity T such that the moment of order u of the underlying X_T does not exist, i. e.

$$\mathbb{E}\big({X_T}^u\big) = +\infty.$$

As the computations in this section repeatedly depend on the sign of the function b we derive the following quite useful lemma.

Lemma 4.3.1. *Consider the function b, as defined in* (4.28)*. Then the following statements holds.*

(i) *For $u \in [u_l, 0] \cap \mathbb{R}$ we always have $b(u) < 0$.*

(ii) *If $\kappa > \eta\rho$ then $b(u) < 0$ also holds for $u \in (0, u_r] \cap \mathbb{R}$.*

(iii) *If $\kappa = \eta\rho$ then $u_r = 1$ and $b(u_r) = 0$. Furthermore, we have $b(u) < 0$ for $u \in [0,1)$ and $b(u) > 0$ for $u \in (1,+\infty)$.*

(iv) *If $\kappa < \eta\rho$ then $\frac{\kappa}{\eta\rho} \in (0,1)$. Moreover, b is strictly increasing and we have*

$$b(u) < 0, \quad \textit{if} \quad -\infty < u < \frac{\kappa}{\eta\rho} \qquad \textit{and} \qquad b(u) > 0, \quad \textit{if} \quad \frac{\kappa}{\eta\rho} < u < +\infty.$$

Proof. See Subsection A.2.7 in the appendix. □

In the following lemma we present the first explicit results for the critical time T^* and particularly address the question where the value $+\infty$ is taken.

Lemma 4.3.2. *Consider $T^*\colon \mathbb{R} \to (0,+\infty]$ as defined in* (4.52)*. Then we have*

$$T^*(u) = +\infty, \quad u \in [u_l, 1].$$

If in addition $\kappa \geq \eta\rho$ holds also

$$T^*(u) = +\infty, \quad u \in (1, u_r],$$

is true.

Proof. We distinguish two cases and start with that where u belongs to the interval $[0,1]$.

(i) $u \in [0,1]$

Recall that by the very definition of $u_+(T)$ in (4.26) we have

$$P_T(u) > 0, \quad u \in \big[0, u_+(T)\big), \tag{4.53}$$

for any maturity $T > 0$. Due to Lemma 4.2.17 we know that $u_+(T) > 1$ holds for any maturity $T > 0$. Thus we have $[0,1] \subseteq \big[0, u_+(T)\big)$ for every $T > 0$ and hence (4.53) implies

$$P_T(u) > 0, \quad u \in [0,1],$$

for any maturity $T > 0$. As $P_0(u) = 1$ always holds we even have $P_T(u) > 0$ for $T \geq 0$. Consequently we obtain $T^*(u) = +\infty$ in this case.

(ii) $u \in [u_l, 0)$ or $\kappa \geq \eta\rho$ and $u \in (1, u_r]$

Under each of the two possible assumptions we have $b(u) \leq 0$ by Lemma 4.3.1. Furthermore $D(u)$ is always non-negative for $u \in [u_l, u_r] \cap \mathbb{R}$. Now observe that since

$$R_1(z) \geq 1, \quad z \in [0, +\infty), \qquad \text{and} \qquad R_2(z) \geq 1, \quad z \in [0, +\infty),$$

holds the non-positivity of $b(u)$ implies

$$P_T(u) = \underbrace{R_1\left(\frac{D(u)T^2}{4}\right)}_{\geq 1} - \underbrace{b(u)R_2\left(\frac{D(u)T^2}{4}\right)T}_{\leq 0} \geq 1 > 0,$$

for any maturity $T \geq 0$. Consequently we obtain

$$T^*(u) = \inf\{T \geq 0 \colon P_T(u) \leq 0\} = \inf \emptyset = +\infty.$$

for $u \in [u_l, 0)$ and for $u \in (1, u_r]$ if $\kappa \geq \eta\rho$.

□

What have seen so far in this section already indicates that the critical time T^* behaves quite differently in the cases $\kappa \geq \eta\rho$ and $\kappa < \eta\rho$. Our next step is the analysis of the the critical time $T^*(u)$ for $u \in (1, u_r]$ if $\kappa < \eta\rho$ holds. For that purpose we need the following lemma.

Lemma 4.3.3. *If $2\kappa \leq \eta\rho$ holds then*

$$\frac{b(v)}{H(v)} < \frac{b(u)}{H(u)}, \quad 1 < v < u < u_r. \tag{4.54}$$

Proof. See Subsection A.2.8 in the appendix. □

Now we have the tools to derive T^* on $(1, u_r]$ for the case which was excluded up to now, namely when $\kappa < \eta\rho$ holds.

Lemma 4.3.4. *Assume $\kappa < \eta\rho$. Then for $u \in \mathbb{R}$ we have*

$$T^*(u) = \begin{cases} H(u)^{-1} \operatorname{areacoth} \frac{b(u)}{H(u)} & u \in (1, u_r) \\ b(u)^{-1} & u = u_r \end{cases}.$$

Furthermore, T^* is continuous and strictly decreasing on $(1, u_r] \cap \mathbb{R}$.

Proof. First observe that due to Lemma 4.3.1 the assumption $\kappa < \eta\rho$ implies that $b(u)$ is positive for $u \in (1, +\infty)$. To prove the analytic representation of T^* distinguish the cases $u \in (1, u_r)$ and $u = u_r$.

▷ $u \in (1, u_r)$

First observe that $H(u) > 0$ holds. Using Lemma 4.2.16 and the fact that $b(u) > 0$ we know that $P_T(u) \leq 0$ is equivalent to

$$\coth\big(H(u)T\big) \leq \frac{b(u)}{H(u)}, \tag{4.55}$$

for $T > 0$. Furthermore, we have

$$\frac{b(u)}{H(u)} = \frac{2|b(u)|}{\sqrt{D(u)}} = \frac{2|b(u)|}{\sqrt{4b(u)^2 + u(1-u)\eta^2}} > 1, \quad u \in (1, u_r).$$

Hence we can apply the strictly decreasing inverse hyperbolic cotangent to both sides of (4.55). Followed by a multiplication of $H(u)^{-1} > 0$ on both sides we obtain that the inequality

$$T \geq H(u)^{-1} \operatorname{areacoth} \frac{b(u)}{H(u)}, \tag{4.56}$$

is equivalent to $P_T(u) \leq 0$ if $T > 0$. As the right hand side of (4.56) is positive we consequently get

$$T^*(u) = \inf\{T \geq 0 \colon P_T(u) \leq 0\} = \inf\{T > 0 \colon P_T(u) \leq 0\} = H(u)^{-1} \operatorname{areacoth} \frac{b(u)}{H(u)}.$$

▷ $u = u_r$

In this case we can assume $u_r < +\infty$ and we also have $u_r > 1$. Due to $b(u_r) > 0$ the inequality $P_T(u_r) \leq 0$ is by means of Lemma 4.2.16 equivalent to

$$T \geq b(u_r)^{-1},$$

for $T > 0$. Since the right hand side of the latter is positive we deduce

$$T^*(u_r) = \inf\{T \geq 0 \colon P_T(u_r) \leq 0\} = \inf\{T > 0 \colon P_T(u_r) \leq 0\} = b(u_r)^{-1}.$$

Next, we prove the continuity statement. The continuity is obvious on $(1, u_r)$. Hence we only need to show it at $u = u_r$ if $u_r < +\infty$. Then we have $H(u_r) = 0$ and $b(u_r) > 0$ and we thus get

$$\lim_{u \nearrow u_r} T^*(u) = \lim_{u \nearrow u_r} H(u)^{-1} \operatorname{areacoth} \frac{b(u)}{H(u)} = b(u_r)^{-1} \lim_{u \nearrow u_r} \frac{b(u)}{H(u)} \operatorname{areacoth} \frac{b(u)}{H(u)} = b(u_r)^{-1},$$

where we used $\lim_{x \to +\infty} x \cdot \operatorname{areacot} x = 1$. Hence T^* is continuous on $(1, u_r]$ if $u_r < +\infty$.
Now we prove the monotonicity statement. Recall that $u_{\max}$, as defined in (4.24), is given by

$$u_{\max} = \begin{cases} \frac{1}{2}\frac{\eta - 2\kappa\rho}{\eta(1-\rho^2)} & \rho \in (-1, 1) \\ +\infty & \rho \in \{-1, 1\} \quad \text{and} \quad \eta > 2\kappa\rho \\ -\infty & \rho = 1 \quad \text{and} \quad \eta < 2\kappa\rho \\ 0 & \rho = 1 \quad \text{and} \quad \eta = 2\kappa\rho \end{cases}.$$

Since T^* is continuous on $(1, u_r]$ it suffices to show that T^* is strictly decreasing on the intervals $(1, u_{\max})$ and $(u_{\max}, u_r) \cap (1, u_r)$. We treat these two domains separately. To see that T^* is strictly decreasing on $(1, u_{\max})$ distinguish the following two subcases.

▷ $\rho \in \left(\frac{\kappa}{\eta}, \frac{2\kappa}{\eta}\right]$

We show that $u_{\max} \leq 1$ holds in this case.

- $\rho = 1$
  Then also $1 \leq \frac{2\kappa}{\eta}$ holds which gives $2\kappa \geq \eta$. The latter together with $\rho = 1$ implies $u_{\max} \leq 0$ and hence $u_{\max}$ is clearly smaller than 1.
- $\rho < 1$
  As we also have $\rho > 0$ the assumption $2\kappa \geq \eta\rho > 0$ leads to

$$u_{\max} = \frac{1}{2}\frac{\eta - 2\kappa\rho}{\eta(1-\rho^2)} \leq \frac{1}{2}\frac{\eta - \eta\rho^2}{\eta(1-\rho^2)} = \frac{1}{2} \leq 1.$$

Thus $u_{\max} \leq 1$ always holds in this case. Consequently the interval $(1, u_{\max})$ is empty and hence T^* is trivially strictly decreasing on $(1, u_{\max})$.

▷ $\rho \in \left(\frac{2\kappa}{\eta}, 1\right]$
In this case we have $2\kappa \leq \eta\rho$ and hence we can apply Lemma 4.3.3 to get

$$\frac{b(v)}{H(v)} < \frac{b(u)}{H(u)}, \quad 1 < v < u < u_{\max},$$

as $u_{\max} \leq u_r$ always holds. Applying the strictly decreasing inverse hyperbolic cotangent to both sides, which we can as we already know that T^* has the form presented in this lemma for v and u belonging to $(1, u_r)$, gives

$$\operatorname{areacoth}\frac{b(v)}{H(v)} > \operatorname{areacoth}\frac{b(u)}{H(u)}, \quad 1 < v < u < u_{\max}.$$

Since H is increasing on $(1, u_{\max})$ we have $H(v) < H(u)$ if $1 < v < u < u_{\max}$. Together with the fact that the inverse hyperbolic cotangent is positive that leads to

$$T^*(v) = H(v)^{-1}\operatorname{areacoth}\frac{b(v)}{H(v)} > H(u)^{-1}\operatorname{areacoth}\frac{b(u)}{H(u)} = T^*(u), \quad 1 < v < u < u_{\max}.$$

Hence T^* is strictly decreasing on $(1, u_{\max})$ in this case.

It remains to show that T^* is also strictly decreasing on $(u_{\max}, u_r) \cap (1, u_r)$. This can be seen as follows. Since H is decreasing on $(u_{\max}, u_r)$ we have

$$H(v) \geq H(u) > 0, \quad u_{\max} < v < u < u_r.$$

Because of $b(v) > 0$ if $v > 1$ the latter implies

$$y_1 := \frac{b(v)}{H(v)} \leq \frac{b(v)}{H(u)} =: y_2, \quad \max(1, u_{\max}) < v < u < u_r.$$

Since $y \mapsto y \cdot \operatorname{areacoth} y$ is decreasing on $(1, +\infty)$ that gives

$$\frac{b(v)}{H(v)}\operatorname{areacoth}\frac{b(v)}{H(v)} = y_1 \cdot \operatorname{areacoth} y_1 \geq y_2 \cdot \operatorname{areacoth} y_2 = \frac{b(v)}{H(u)}\operatorname{areacoth}\frac{b(v)}{H(u)},$$

if $\max(1, u_{\max}) < v < u < u_r$. Multiplication of both sides by $b(v)^{-1} > 0$ leads to

$$H(v)^{-1}\operatorname{areacoth}\frac{b(v)}{H(v)} \geq H(u)^{-1}\operatorname{areacoth}\frac{b(v)}{H(u)}, \quad \max(1, u_{\max}) < v < u < u_r.$$

Now note that we have $b(v) < b(u)$ because $\rho > 0$ holds. Since the inverse hyperbolic cotangent is strictly decreasing the latter inequality can be used to get

$$T^*(v) = H(v)^{-1}\operatorname{areacoth}\frac{b(v)}{H(v)} \geq H(u)^{-1}\operatorname{areacoth}\frac{b(v)}{H(u)} > H(u)^{-1}\operatorname{areacoth}\frac{b(u)}{H(u)} = T^*(u),$$

if $\max(1, u_{\max}) < v < u < u_r$. Consequently T^* is also strictly decreasing on $(u_{\max}, u_r) \cap (1, u_r)$ and thus the proof is concluded. □

Next, we want to tackle the critical time for arguments belonging to $(-\infty, u_l) \cup (u_r, +\infty)$. Before doing so we need the following lemma.

Lemma 4.3.5. *Consider the function defined by*

$$u \mapsto \frac{b(u)}{H(u)}, \quad u \in (-\infty, u_l) \cup (u_r, +\infty), \tag{4.57}$$

whenever the set $(-\infty, u_l) \cup (u_r, +\infty)$ is non-empty. Then the following monotonicity statements hold.

(i) If $\kappa \geq \eta\rho$ and $u_r < +\infty$ then the function defined in (4.57) *is strictly increasing on* $(u_r, +\infty)$.

(ii) If $2\kappa \geq \eta\rho$ and $u_l > -\infty$ the function defined in (4.57) *is strictly decreasing on* $(-\infty, u_l)$.

Proof. See Subsection A.2.9 in the appendix. □

The next result gives an analytic representation of $T^*(u)$ for the remaining arguments u, i. e. for $u \in (-\infty, u_l) \cup (u_r, +\infty)$.

Lemma 4.3.6. *Consider $T^* \colon \mathbb{R} \to (0, +\infty]$ as defined in* (4.52). *For $u \in \mathbb{R}$ we then have*[1]

$$T^*(u) = H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)}, \quad u \in (-\infty, u_l) \cup (u_r, +\infty). \tag{4.58}$$

Furthermore, if $u_r < +\infty$ then $T^*|_{(u_r,+\infty)}$ is strictly decreasing. If additionally $2\kappa \geq \eta\rho$ holds then $T^*|_{(-\infty,u_l)}$ is strictly increasing.

Proof. Using the expressions presented in Lemma 4.2.16 for $P_T(u)$ when $u \in (-\infty, u_l) \cup (u_r, +\infty)$ we see that $P_T(u) \leq 0$ is equivalent to

$$\cos\big(H(u)T\big) - \frac{b(u)}{H(u)} \sin\big(H(u)T\big) \leq 0,$$

for $u \in (-\infty, u_l) \cup (u_r, +\infty)$ and $T \geq 0$. From now on fix an arbitrary $u \in (-\infty, u_l) \cup (u_r, +\infty)$ and consider $T \in \big(0, H(u)^{-1}\pi\big)$. For such maturities T we have $\sin\big(H(u)T\big) > 0$ and hence $P_T(u) \leq 0$ is equivalent to

$$\cot\big(H(u)T\big) \leq \frac{b(u)}{H(u)}.$$

Applying the strictly decreasing inverse cotangent which maps $\mathbb{R}$ to $(0, \pi)$ and multiplying by $H(u)^{-1}$ afterwards leads to $P_T(u) \leq 0$ being equivalent to

$$T \geq H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)},$$

for $T \in \big(0, H(u)^{-1}\pi\big)$. Since the right hand side of the latter inequality clearly belongs to $(0, H(u)^{-1}\pi)$ we get

$$T^*(u) = \inf\{T \geq 0 \colon P_T(u) \leq 0\} = \inf\{T > 0 \colon P_T(u) \leq 0\} = H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)},$$

for $u \in (-\infty, u_l) \cup (u_r, +\infty)$. Now we prove the stated strict monotonicity. Therefore distinguish the following cases.

- $\triangleright$ $\kappa \geq \eta\rho$

  Here we have two subcases as we prove the stated monotonicity on $(u_r, +\infty)$ and $(-\infty, u_l)$ separately.

  - T^* is strictly decreasing on $(u_r, +\infty)$

    By Lemma 4.3.5 we see that

$$\frac{b(v)}{H(v)} < \frac{b(u)}{H(u)}, \quad u_r < v < u < +\infty,$$

[1] The function $\operatorname{arccot} \colon \mathbb{R} \to (0, \pi)$ is defined as the inverse of $\cot|_{(0,\pi)}$.

holds. Since the inverse cotangent is strictly decreasing we obtain

$$\operatorname{arccot}\frac{b(v)}{H(v)} > \operatorname{arccot}\frac{b(u)}{H(u)}, \quad u_r < v < u < +\infty.$$

Finally, as $0 < H(v) \leq H(u)$ holds if $u_r < v < u < +\infty$ and since the inverse cotangent maps to $(0, \pi)$ we get

$$T^*(v) = H(v)^{-1}\operatorname{arccot}\frac{b(v)}{H(v)} > H(u)^{-1}\operatorname{arccot}\frac{b(u)}{H(u)} = T^*(u), \quad u_r < v < u < +\infty.$$

– T^* is strictly increasing on $(-\infty, u_l)$
  As $\kappa \geq \eta\rho$ implies $2\kappa \geq \eta\rho$ an application of Lemma 4.3.5 yields

$$\frac{b(v)}{H(v)} < \frac{b(u)}{H(u)}, \quad -\infty < u < v < u_l.$$

  Since the inverse cotangent is strictly decreasing we get

$$\operatorname{arccot}\frac{b(v)}{H(v)} > \operatorname{arccot}\frac{b(u)}{H(u)}, \quad u_r < v < u < +\infty.$$

  Furthermore, because $0 < H(v) \leq H(u)$ holds if $-\infty < u < v < u_l$ we finally obtain

$$T^*(v) = H(v)^{-1}\operatorname{arccot}\frac{b(v)}{H(v)} > H(u)^{-1}\operatorname{arccot}\frac{b(u)}{H(u)} = T^*(u), \quad -\infty < u < v < u_l,$$

  in the case considered.

▷ $\kappa < \eta\rho$

We again distinguish between arguments u belonging to $(u_r, +\infty)$ and $(-\infty, u_l)$.

– T^* is strictly decreasing on $(u_r, +\infty)$
  Consider positive real numbers v and u such that $u_r < v < u < +\infty$. Furthermore, recall that H is increasing on $(u_r, +\infty)$ and due to Lemma 4.3.1 we know that b is positive and strictly increasing on $(u_r, +\infty)$. With $y_1 = \frac{b(u)}{H(u)}$ and $y_2 = \frac{b(u)}{H(v)}$ we thus have $0 < y_1 \leq y_2 < +\infty$. Then observe that the function $y \mapsto y \cdot \operatorname{arccot} y$ is increasing on $\mathbb{R}$, which leads to

$$\frac{b(u)}{H(u)}\operatorname{arccot}\frac{b(u)}{H(u)} = y_1 \operatorname{arccot} y_1 \leq y_2 \operatorname{arccot} y_2 = \frac{b(u)}{H(v)}\operatorname{arccot}\frac{b(u)}{H(v)}.$$

  Multiplying both sides of the latter inequality by $b(u)^{-1} > 0$ gives

$$H(u)^{-1}\operatorname{arccot}\frac{b(u)}{H(u)} \leq H(v)^{-1}\operatorname{arccot}\frac{b(u)}{H(v)}. \tag{4.59}$$

  Since b is positive and strictly increasing on $(u_r, +\infty)$ if $\kappa < \rho\eta$ we get

$$0 < \frac{b(v)}{H(v)} < \frac{b(u)}{H(v)}, \quad u_r < v < u < +\infty.$$

  Using the latter together with the fact that the inverse cotangent is strictly decreasing and the estimate obtained in (4.59) gives then

$$T^*(u) = H(u)^{-1}\operatorname{arccot}\frac{b(u)}{H(u)} \leq H(v)^{-1}\operatorname{arccot}\frac{b(u)}{H(v)} < H(v)^{-1}\operatorname{arccot}\frac{b(v)}{H(v)} = T^*(v).$$

  Since u and v were arbitrary real numbers satisfying $u_r < v < u < +\infty$ we are done in this case.

– T^* is strictly increasing on $(-\infty, u_l)$
  In this case we can assume $2\kappa \geq \eta\rho$ and $u_l > -\infty$. By Lemma 4.3.5 we know that

$$\frac{b(v)}{H(v)} < \frac{b(u)}{H(u)}, \qquad -\infty < u < v < u_l,$$

holds. Applying the strictly decreasing inverse cotangent gives

$$\operatorname{arccot}\frac{b(v)}{H(v)} > \operatorname{arccot}\frac{b(u)}{H(u)}, \qquad -\infty < u < v < u_l.$$

Because of $H(v) \leq H(u)$ if $-\infty < u < v < u_l$ and since the inverse cotangent maps to $(0, \pi)$ we get

$$T^*(v) = H(v)^{-1}\operatorname{arccot}\frac{b(v)}{H(v)} > H(u)^{-1}\operatorname{arccot}\frac{b(u)}{H(u)} = T^*(u), \qquad -\infty < u < v < u_l,$$

and hence $T^*|_{(-\infty,u_l)}$ is strictly increasing if $u_l > -\infty$ and $2\kappa \geq \eta\rho$.

$\square$

Before we can derive the main theorems of this section we need a last lemma, which deals with the limiting behavior of T^* and reads as follows.

Lemma 4.3.7. *Consider the critical time $T^*\colon \mathbb{R} \to (0, +\infty]$ as defined in (4.52). If $u_l > -\infty$ we have*

$$\lim_{u \nearrow u_l} T^*(u) = +\infty \qquad \text{and} \qquad \lim_{u \searrow -\infty} T^*(u) = 0.$$

Furthermore, the following statements hold for the cases $\kappa \geq \eta\rho$ and $\kappa < \eta\rho$ respectively.

(i) $\kappa \geq \eta\rho$
  Then $T^*(u) < +\infty$ if and only if $u \in (-\infty, u_l) \cup (u_r, \infty)$. Furthermore, if $u_r < +\infty$ we have

$$\lim_{u \searrow u_r} T^*(u) = +\infty \qquad \text{and} \qquad \lim_{u \nearrow +\infty} T^*(u) = 0.$$

(ii) $\kappa < \eta\rho$
  Then $T^*(u) < +\infty$ if and only if $u \in (-\infty, u_l) \cup (1, +\infty)$. Moreover, T^* is continuous on $(-\infty, u_l) \cup (1, +\infty)$ and we have

$$\lim_{u \searrow 1} T^*(u) = +\infty \qquad \text{and} \qquad \lim_{u \nearrow +\infty} T^*(u) = 0.$$

Proof. The if and only if statements about T^* directly follow from Lemma 4.3.2, Lemma 4.3.4 and Lemma 4.3.6. Next, we prove the limit results for $u \nearrow u_l$ if $u_l > -\infty$. Because of $u_l > -\infty$ we have $H(u_l) = 0$. By Lemma 4.3.1 we know that $b(u_l) < 0$ always holds. Hence b is negative in a real neighborhood of u_l which gives

$$\lim_{u \nearrow u_l} T^*(u) = \lim_{u \nearrow u_l} H(u)^{-1}\operatorname{arccot}\frac{b(u)}{H(u)} \geq \lim_{u \nearrow u_l} H(u)^{-1}\operatorname{arccot} 0 = \lim_{u \nearrow u_l} H(u)^{-1}\frac{\pi}{2} = +\infty.$$

Moreover, if $u_l > -\infty$ we get $\lim_{u \searrow -\infty} H(u) = +\infty$ which implies

$$0 \leq \lim_{u \searrow -\infty} T^*(u) = \lim_{u \searrow -\infty} H(u)^{-1}\operatorname{arccot}\frac{b(u)}{H(u)} \leq \lim_{u \searrow -\infty} H(u)^{-1}\pi = 0.$$

Next, assume $u_r < +\infty$. Then we have $\lim_{u\searrow+\infty} H(u) = +\infty$ which implies

$$0 \le \lim_{u\nearrow+\infty} T^*(u) = \lim_{u\nearrow+\infty} H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)} \le \lim_{u\nearrow+\infty} H(u)^{-1}\pi = 0,$$

hence the statements about $u \nearrow +\infty$ are proven except for the case where $\kappa < \eta\rho$ and $u_r = +\infty$ hold simultaneously. For the remaining statements distinguish the cases (i) and (ii).

(i) $\kappa \ge \eta\rho$

From Lemmas 4.3.2 and 4.3.6 it directly follows that $T^*(u) < +\infty$ holds if and only if $u \in (-\infty, u_l) \cup (u_r, +\infty)$. To prove the limits stated distinguish the following subcases.

- $\kappa > \eta\rho$

  Due to $u_r < +\infty$ we have $H(u_r) = 0$. By Lemma 4.3.1 we then have $b(u_r) < 0$. We get

  $$\lim_{u\searrow u_r} T^*(u) = \lim_{u\searrow u_r} H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)} \ge \lim_{u\searrow u_r} H(u)^{-1} \operatorname{arccot} 0 = \lim_{u\searrow u_r} H(u)^{-1} \frac{\pi}{2} = +\infty,$$

  as b is then even negative in a real neighborhood of u_r and since $\operatorname{arccot}\colon \mathbb{R} \to (0, \pi)$ is decreasing.
- $\kappa = \eta\rho$

  Following Lemma 4.3.1 we see that $u_r = 1$ and $b(u_r) = b(1) = 0$ hold then. It also implies that $b(u) > 0$ for $u \in (u_r, +\infty)$. Furthermore, because of $\kappa = \eta\rho$ we obtain

  $$\lim_{u\searrow u_r} \frac{b(u)}{H(u)} = \lim_{u\searrow 1} \frac{b(u)}{H(u)} = \lim_{u\searrow 1} \frac{\kappa(u-1)}{\sqrt{u(u-1)\eta^2 - \kappa^2(u-1)^2}} = \lim_{u\searrow 1} \kappa \left(\eta^2 \frac{u}{u-1} - \kappa^2\right)^{-\frac{1}{2}} = 0.$$

  Due to $b(u) > 0$ for $u \in (u_r, +\infty)$ and $H(u_r) = 0$ we get

  $$\lim_{u\searrow u_r} T^*(u) = \lim_{u\searrow u_r} H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)} \ge \lim_{u\searrow u_r} H(u)^{-1} \operatorname{arccot} 1 = \lim_{u\searrow u_r} H(u)^{-1} \frac{\pi}{4} = +\infty.$$

(ii) $\kappa < \eta\rho$

In this case we start by proving the stated continuity. Clearly by the representations given in Lemma 4.3.6 and Lemma 4.3.4 the critical time T^* is continuous on $(-\infty, u_l) \cup (1, u_r) \cup (u_r, +\infty)$. Hence it remains to show the continuity of T^* at $u = u_r$, whenever $u_r < +\infty$. Due to Lemma 4.3.1 we have $b(u_r) > 0$ in this case. Furthermore, because of $u_r < +\infty$ also $H(u_r) = 0$ holds. Using $\lim_{x\to+\infty} x \cdot \operatorname{arccot} x = 1$ we get the following for the limit from the right.

$$\lim_{u\searrow u_r} T^*(u) = \lim_{u\searrow u_r} H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)} = b(u_r)^{-1} \lim_{u\searrow u_r} \frac{b(u)}{H(u)} \operatorname{arccot} \frac{b(u)}{H(u)} = b(u_r)^{-1} = T^*(u_r).$$

For the limit from the left we obtain

$$\lim_{u\nearrow u_r} T^*(u) = b(u_r)^{-1},$$

as we already know that T^* is continuous on $(1, u_r] \cap \mathbb{R}$ from Lemma 4.3.4. Thus also the limit from the left equals $b(u_r)^{-1} = T^*(u_r)$ and hence we proved the continuity of T^* on $(-\infty, u_l) \cup (1, +\infty)$.

Next, we prove the limit for $u \nearrow +\infty$ when $u_r = +\infty$ and $\kappa < \eta\rho$. Due to Lemma 4.2.5 we see that this is the case if and only if $\rho = 1$ and $\eta \ge 2\kappa$. By Lemma 4.2.4 we get

$$\lim_{u\to+\infty} D(u) = \begin{cases} +\infty & \eta > 2\kappa \\ \kappa^2 & \eta = 2\kappa \end{cases} \quad \text{leading to} \quad \lim_{u\to+\infty} H^{-1}(u) \le \frac{2}{\kappa}. \tag{4.60}$$

Moreover, due to $\rho = 1$ and $b(u_r) > 0$ we have

$$\lim_{u \nearrow +\infty} \frac{b(u)}{H(u)} = \lim_{u \nearrow +\infty} \sqrt{\frac{b(u)^2}{H(u)^2}} = \lim_{u \nearrow +\infty} \sqrt{\frac{(\eta u - \kappa)^2}{\kappa^2 + \eta(\eta - 2\kappa)u}} = \lim_{u \nearrow +\infty} \sqrt{\frac{\kappa^2 - 2\kappa\eta u + \eta^2 u^2}{\kappa^2 + \eta(\eta - 2\kappa)u}} = +\infty.$$

Combining the latter with (4.60), using Lemma 4.3.4 and $\lim_{x \to +\infty} \operatorname{areacoth} x = 0$ leads to

$$0 \leq \lim_{u \nearrow +\infty} T^*(u) = \lim_{u \nearrow +\infty} H(u)^{-1} \operatorname{areacoth} \frac{b(u)}{H(u)} \leq \frac{2}{\kappa} \lim_{u \nearrow +\infty} \operatorname{areacoth} \frac{b(u)}{H(u)} = 0.$$

It remains to prove the limit of $T^*(u)$ for $u \searrow 1$ if $\kappa < \eta\rho$. Because of $\kappa < \eta\rho$ we have $u_r > 1$, which leads to $H(1) > 0$ and $b(1) > 0$. Observe that

$$\lim_{u \searrow 1} \frac{b(u)}{H(u)} = \frac{\eta\rho - \kappa}{\sqrt{(\kappa - \eta\rho)^2}} = 1,$$

holds. Because of $\lim_{x \searrow 1} \operatorname{areacoth} x = +\infty$ the expression for $T^*(u)$ from Lemma 4.3.4 leads to

$$\lim_{u \searrow 1} T^*(u) = \lim_{u \searrow 1} H(u)^{-1} \operatorname{areacoth} \frac{b(u)}{H(u)} = \underbrace{H(1)^{-1}}_{>0} \lim_{u \searrow 1} \operatorname{areacoth} \frac{b(u)}{H(u)} = +\infty,$$

which concludes the proof.

□

Now we have all the tools to derive the main theorems of this chapter. The results given are similar to those presented by Anderson and Piterbarg in [AP07]. However, instead of distinguishing the cases according to the sign of $b(u)$ we emphasize the different behavior of the criticial time in the cases $\kappa \geq \eta\rho$ and $\kappa < \eta\rho$. A similar analysis pointing out the differences between these parameter sets was also presented by del Bano Rollin et al. in [dBRFCU09]. However, we reduce the cases distinguished by using the inverse cotangent instead of the inverse tangent and provide a detailed and rigorous proof. The following theorem explains why we call T^* the critical time.

Theorem 4.3.8. *For a maturity $T \geq 0$ consider the underlying S_T in the Heston model. Furthermore, let u_l and u_r be as in* (4.25) *and* (4.26) *respectively. Then the critical time $T^*(u)$, which is defined in* (4.52), *has the following analytical representation for $u \in \mathbb{R}$.*

(i) $\kappa < \eta\rho$

$$T^*(u) = \begin{cases} +\infty & u \in [u_l, 1] \\ \frac{1}{2} H(u)^{-1} \ln\left(\frac{b(u)+H(u)}{b(u)-H(u)}\right) & u \in (1, u_r) \\ b(u)^{-1} & u = u_r \\ H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)} & u \in (-\infty, u_l) \cup (u_r, +\infty) \end{cases}.$$

(ii) $\kappa \geq \eta\rho$

$$T^*(u) = \begin{cases} +\infty & u \in [u_l, u_r] \\ H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)} & u \in (-\infty, u_l) \cup (u_r, +\infty) \end{cases}.$$

Furthermore, if one of the statements

- ▷ $u \in [0, +\infty)$
- ▷ $u \in (-\infty, 0)$ *and* $2\kappa \geq \eta\rho$

is true we have

$$\mathbb{E}\big(S_T{}^u\big) < +\infty, \quad T \in \big[0, T^*(u)\big) \qquad \textit{and} \qquad \mathbb{E}\big(S_T{}^u\big) = +\infty, \quad T \in [T^*(u), +\infty).$$

Proof. In order to show the analytic representation of $T^*(u)$ we distinguish between the cases $\kappa < \eta\rho$ and $\kappa \geq \eta\rho$.

(i) $\kappa < \eta\rho$

The expression for $u \in [u_l, 1]$ we get by applying Lemma 4.3.2. If $u \in (1, u_r]$ then the statement follows from using

$$\operatorname{areacoth} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right), \quad x \in (1, +\infty),$$

when applying Lemma 4.3.4. The result for the case where $u \in (-\infty, u_l) \cup (u_r, +\infty)$ is already stated in Lemma 4.3.6.

(ii) $\kappa \geq \eta\rho$

The expression for $u \in [u_l, u_r]$ directly follows from Lemma 4.3.2. If $u \in (-\infty, u_l) \cup (u_r, +\infty)$ the representation of $T^*(u)$ has already been given in Lemma 4.3.6.

For the second statement we distinguish two cases according to the assumption which is satisfied.

▷ $u \geq 0$

To see the second statement under the assumption $u \geq 0$ consider the following. According to Lemma 4.3.2, Lemma 4.3.4 and Lemma 4.3.6 it holds that $0 \leq v \leq u < +\infty$ leads to $T^*(v) \geq T^*(u)$ in every case. Consequently $T \in \big[0, T^*(u)\big)$ implies $T \in \big[0, T^*(v))\big)$ for $v \in [0, u]$. By means of the definition of T^* in (4.52) the latter yields

$$P_T(v) > 0, \quad v \in [0, u].$$

Since $T \in \big[0, T^*(u)\big)$ was arbitrary the latter implies

$$u_+(T) > u, \quad T \in \big[0, T^*(u)\big).$$

Using Theorem 4.2.18 one then particularly sees that

$$\mathbb{E}\big(S_T{}^u\big) = e^{uR_T}\mathbb{E}\big(e^{uX_T}\big) < +\infty, \quad T \in \big[0, T^*(u)\big)$$

is true, as $0 \leq u < u_+(T)$ holds for every $T \in \big[0, T^*(u)\big)$. Next, consider a fixed but arbitrary $T \in [T^*(u), +\infty)$. Then observe that due to the monotonicity properties presented in Lemma 4.3.4 and Lemma 4.3.6 and because of the limit behavior given in Lemma 4.3.7 the set $\{u \geq 0 \colon T^*(u) < +\infty\}$ is mapped to $(0, +\infty)$ in a strictly decreasing manner. Hence there is a unique $v^* \in [0, u]$ such that $T^*(v^*) = T$. The latter however implies that there is a $v^* \in [0, u]$ such that

$$P_T(v^*) = P_{T^*(v^*)}(v^*) = 0,$$

and thus

$$u_+(T) \leq v^* \leq u,$$

has to hold. By Theorem 4.2.19 this implies

$$\mathbb{E}\big(S_T{}^u\big) = e^{uR_T}\mathbb{E}\big(e^{uX_T}\big) = +\infty.$$

Since $T \in [T^*(u), +\infty)$ was arbitrary we are done.

▷ $u < 0$ and $2\kappa \geq \eta\rho$

To see the second statement under the assumption $u < 0$ and $2\kappa \geq \eta\rho$ observe following. Because of $2\kappa \geq \eta\rho$ we can apply Lemma 4.3.6 together with Lemma 4.3.2 to see that $-\infty < u \leq v \leq 0$ leads to $T^*(v) \geq T^*(u)$. Consequently $T \in \big[0, T^*(u)\big)$ implies $T \in \big[0, T^*(v)\big)\big)$ for $v \in [u, 0]$. By means of the definition of T^* in (4.52) the latter yields

$$P_T(v) > 0, \quad v \in [u, 0].$$

Since $T \in \big[0, T^*(u)\big)$ was arbitrary the latter implies

$$u_-(T) < u, \quad T \in \big[0, T^*(u)\big).$$

Applying Theorem 4.2.18 particularly leads to

$$\mathbb{E}\big({S_T}^u\big) = e^{uR_T}\mathbb{E}\big(e^{uX_T}\big) < +\infty, \quad T \in \big[0, T^*(u)\big),$$

as $u_-(T) < u \leq 0$ holds for every $T \in \big[0, T^*(u)\big)$. Next, observe that due to the monotonicity property presented in Lemma 4.3.6 for the case $2\kappa \geq \eta\rho$ and because of the limit behavior given in Lemma 4.3.7 the set $\{u \leq 0 \colon T^*(u) < +\infty\}$ is mapped to to $(0, +\infty)$ in a strictly increasing manner. Hence for every $T \in [T^*(u), +\infty)$ there is a unique $v^* \in [u, 0]$ such that $T^*(v^*) = T$. This however implies that for every $T \in [T^*(u), +\infty)$ there is a $v^* \in [u, 0]$ such that

$$P_T(v^*) = P_{T^*(v^*)}(v^*) = 0,$$

and hence

$$u \leq v^* \leq u_-(T).$$

Using the result given in Theorem 4.2.19 the latter yields

$$\mathbb{E}\big({S_T}^u\big) = e^{uR_T}\mathbb{E}\big(e^{uX_T}\big) = +\infty.$$

□

The following theorem connects the critical time $T^*(u)$ more precisely to the critical moments $u_+(T)$ and $u_-(T)$. In particular it provides an efficient procedure to determine the critical moments of the discounted underlying in the Heston model numerically.

Theorem 4.3.9. *For a maturity $T > 0$ consider the underlying S_T in the Heston model. Recall the corresponding critical moments $u_+(T)$ and $u_-(T)$ from (4.47), which are characterized by*

$$\mathbb{E}\big({S_T}^u\big) < +\infty \quad \textit{if and only if} \quad u \in \big(u_-(T), u_+(T)\big).$$

For the critical time $T^* \colon \mathbb{R} \to (0, +\infty]$, which is given in Theorem 4.3.8, the following holds.

(i) $\rho \in (-1, 1]$

Then $u = u_+(T)$ is the unique positive solution to

$$T^*(u) = T.$$

(ii) $\rho \in [-1, 1)$ *and* $2\kappa \geq \eta\rho$

Then $u = u_-(T)$ is the unique negative solution to

$$T^*(u) = T.$$

Furthermore, if $\rho = 1$ and $2\kappa \geq \eta$ hold simultaneously then $u_-(T) = -\infty$. If $\rho = -1$ then we have $u_+(T) = +\infty$.

Proof. First note that the presented characterization of $u_-(T)$ and $u_+(T)$ is due to Theorem 4.2.18, Theorem 4.2.19 and the identity

$$\mathbb{E}\big({S_T}^u\big) = e^{uR_T}\mathbb{E}\big(e^{uX_T}\big).$$

We start by proving statement (i). First note that $\rho > -1$ implies that the set $\{u \geq 0\colon T^*(u) < +\infty\}$ is non-empty. To see this distinguish two cases.

- ▷ $\kappa \geq \eta\rho$

  In particular $2\kappa > \eta\rho$ holds in this case for $\rho \in (-1, 1]$. A glance at the definition of u_r in (4.26) gives $u_r < +\infty$ for every $\rho \in (-1, 1]$ in this case. Consequently

  $$\{u \geq 0\colon T^*(u) < +\infty\} = (u_r, +\infty) \neq \emptyset.$$

- ▷ $\kappa < \eta\rho$

  Applying Theorem 4.3.8 leads to

  $$\{u \geq 0\colon T^*(u) < +\infty\} \supseteq (1, +\infty) \neq \emptyset.$$

Hence by means of Lemma 4.3.4, Lemma 4.3.6 and Lemma 4.3.7 the assumption $\rho > -1$ implies that under T^* the non-empty set $\{u \geq 0\colon T^*(u) < +\infty\}$ is mapped to $(0, +\infty)$ in a strictly decreasing manner. Thus for a given $T > 0$ there is a unique $u > 0$ such that $T^*(u) = T$. The latter already gives

$$P_T(u) = P_{T^*(u)}(u) = 0,$$

and hence $u_+(T) \leq u$. Furthermore, since T^* is either infinite or strictly decreasing on $[0, u]$ we get $T^*(v) > T^*(u) = T$ for $v \in [0, u)$. That however implies

$$P_T(v) > 0, \quad v \in [0, u),$$

which gives $u_+(T) \geq u$. Consequently we have just proven that if $\rho > -1$ the unique positive u satisfying $T^*(u) = T$ must be equal to $u_+(T)$.

Next, we prove statement (ii). Recalling the definition of u_l in (4.25) we see that $u_l > -\infty$ holds if $\rho < 1$. Due to Theorem 4.3.8 this implies that $\{u \leq 0\colon T^*(u) < +\infty\}$ is non-empty. Because of $2\kappa \geq \eta\rho$ we can apply Lemma 4.3.6 and Lemma 4.3.7 to see that under T^* the non-empty set $\{u \leq 0\colon T^*(u) < +\infty\}$ is mapped to $(0, +\infty)$ in a strictly increasing manner. Consequently for a given $T > 0$ there is a unique $u < 0$ such that $T^*(u) = T$. That gives

$$P_T(u) = P_{T^*(u)}(u) = 0,$$

which implies $u_-(T) \geq u$. Moreover, since T^* is either infinite or strictly increasing on $[u, 0]$, we obtain $T^*(v) > T^*(u) = T$ for $v \in (u, 0]$. However, the latter leads to

$$P_T(v) > 0, \quad v \in (u, 0],$$

which implies $u_-(T) \leq u$. Hence $u = u_-(T)$ must hold and $u_-(T)$ is the unique negative solution to $T^*(u) = T$ if $\rho < 1$ and $2\kappa \geq \eta\rho$ hold simultaneously.

We conclude the proof by considering the special cases addressed at the end of this theorem. If $\rho = 1$ and $2\kappa \geq \eta\rho$ we have $u_l = -\infty$. Due to Theorem 4.3.8 this implies $T^*(u) = +\infty$ for $u \in (u_l, 0] = (-\infty, 0]$ and thus we must have

$$P_T(u) > 0,$$

for any $u \in (-\infty, 0]$ and any $T \in [0, +\infty)$. Consequently $u_-(T) = -\infty$ has to hold for any maturity $T \geq 0$ if $\rho = 1$ and $2\kappa \geq \eta$. If $\rho = -1$ we have $\kappa \geq \eta\rho$. By means of Theorem 4.3.8 the latter implies $T^*(u) = +\infty$ for $u \in [0, u_r)$. On the other hand $\rho = -1$ yields $u_r = +\infty$ by (4.26) and thus we have $T^*(u) = +\infty$ for $u \in [0, +\infty)$. Consequently we have

$$P_T(u) > 0,$$

for any $u \in [0, +\infty)$ and any $T \in [0, +\infty)$. Hence $u_+(T) = +\infty$ must hold for any maturity $T \geq 0$. $\square$

Remark 4.3.10. The result presented in Theorem 4.3.9 offers a nice way to compute the critical moments $u_+(T)$ and $u_-(T)$ numerically. For instance consider a Heston model with parameters such that $\rho > -1$. With a pair (v_1, u_1) of initial values satisfying $0 < v_1 < u_1 < +\infty$ such that

$$0 < T^*(u_1) < T < T^*(v_1) < +\infty,$$

the positive critical moment $u_+(T)$ can be computed using the bisection method. This can be done due to the facts that T^* is continuous and strictly decreasing on (v_1, u_1) and since $u_+(T)$ is the unique positive real number being mapped to the maturity T under T^*. Analogously the negative critical moment $u_-(T)$ can be determined numerically if the parameters are such that they satisfy $2\kappa \geq \eta\rho$.

The following example points out how an implementation can look like in the programming language `R`.

Example 4.3.11. Consider the Heston model with the parameters.

κ	θ	S_0	η	v_0	ρ	r
2	0.1	10	0.25	0.05	-0.75	0

and the maturity $T = 10$. We obviously have $\kappa \geq \eta\rho$. By Theorem 4.3.8 the critical time $T^* \colon \mathbb{R} \to (0, +\infty]$ can be described by

$$T^*(u) = \begin{cases} +\infty & u \in [u_l, u_r] \\ H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)} & u \in (-\infty, u_l) \cup (u_r, +\infty) \end{cases},$$

where u_l and u_r are given by (4.25) and (4.26). Using the identity

$$\operatorname{arccot}(x) = \frac{\pi}{2} - \arctan x, \quad x \in \mathbb{R},$$

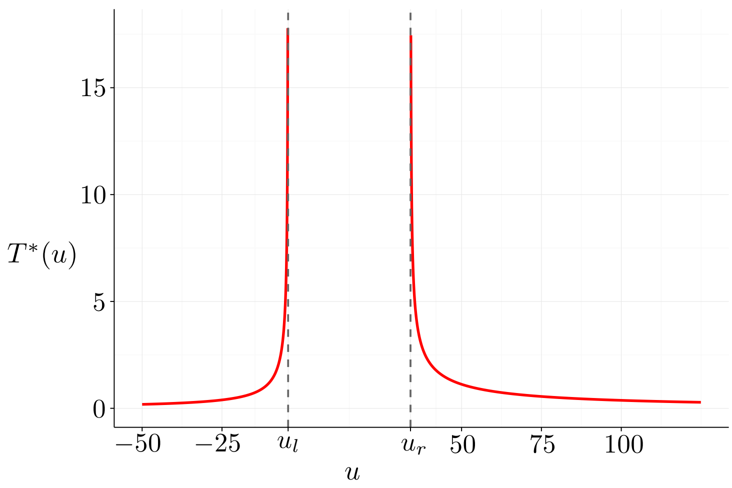
a function for the critical time T^* can very easily be implemented in `R` as follows.

```
## Parameters
kappa<-2; theta<-0.1; S0<-10; eta<-0.25; v0<-0.05; rho<--0.75; r<-0; T<-10;

## Function b, D and H (real arguments) and inverse cotangent
b<-function(u) 0.5*(eta*rho*u-kappa)
D<-function(u) (eta*rho*u-kappa)^2-u*(u-1)*eta^2
H<-function(u) 0.5*sqrt(abs(D(u)))
acot<-function(x) pi/2-atan(x)

## ul and ur
ul<-0.5*(eta-2*kappa*rho-sqrt((eta-2*kappa*rho)^2+
4*kappa^2*(1-rho^2)))/(eta*(1-rho^2))
ur<-0.5*(eta-2*kappa*rho+sqrt((eta-2*kappa*rho)^2+
4*kappa^2*(1-rho^2)))/(eta*(1-rho^2))

## Critical time Tstar
Tstar<-function(u) ifelse(u>ur|u<ul,H(u)^-1*acot(b(u)/H(u)),Inf)
```

Rounding the values for u_l and u_r to three digits gives

$$u_l = -4.301 \quad \text{and} \quad u_r = 34.015.$$

Plotting the critical Time T^* yields the following figure.



We clearly see that the behavior of T^* is as described in Lemma 4.3.7. In order to find the critical moments $u_+(10)$ and $u_-(10)$ we need to find the negative and the positive solution to

$$T^*(u) = 10.$$

This can also be easily done in `R` as follows.

```
## Computation of the critical moments u_-(T) and u_+(T)
uminus<-uniroot(f=function(u) Tstar(u)-T,interval=c(ul-0.01,ul-25))$root
uplus<-uniroot(f=function(u) Tstar(u)-T,interval=c(ur+0.01,ur+50))$root
```

Rounding the values obtained for $u_-(10)$ and $u_+(10)$ to three digits gives

$$u_-(10) = -4.577 \quad \text{and} \quad u_+(10) = 34.371.$$

4.4 Moment Generating Function with Complex Arguments

When pricing a European call option by means of the Fourier transform approach, i. e. by using the result presented and derived in Corollary 3.4.1, the moment generating function of the log-discounted underlying is of particular interest for complex arguments. The goal of this section is to derive an explicit expression for that in the Heston model. However, compared to the case where only real arguments of the moment generating function are considered additional problems arise. On the one hand a derivation similar to those using the functions Ψ_z and Φ_z presented in the last section can be done. On the other hand when deriving an explicit expression for Φ_z things get more tricky. To get an idea why the latter is the case assume for a moment (i. e. for this paragraph) that $\gamma_{z,0}$, which is the path defined in (4.29), satisfies $\gamma_{z,0}\big([0,T]\big) \subseteq \mathbb{C}\setminus\{0\}$[2]

[2]This might not be true in general but as the results in this section will show at least if $\kappa \leq \eta\rho$. However, this assumption makes it possible to shortly and nicely demonstrate the main challenges of this section.

for $z \in U_H$. Analogously to the last section one can then see that Ψ_z is of the form presented in (4.31) for $z \in U_H$. Furthermore, note that Lemma 4.2.9 would also hold for $z \in U_H$ and hence we would have

$$\Psi_z(s) = \frac{2}{\eta^2}\left(\frac{\gamma'_{z,0}(s)}{\gamma_{z,0}(s)} - b(z)\right), \quad s \in [0,T], \tag{4.61}$$

for $z \in U_H$. Next, following Lemma 4.2.2 the function Φ_z would be basically given as integral over Ψ_z. Plugging in the expression from (4.61) for Ψ_z would give

$$\Phi_z(t) = \kappa\theta \int_t^T \Psi_z(s)\, ds = \frac{2\kappa\theta}{\eta^2}\left(\int_t^T \frac{\gamma'_{z,0}(s)}{\gamma_{z,0}(s)}\, ds - b(z)(T-t)\right).$$

Hence it would remain to compute the integral

$$\int_t^T \frac{\gamma'_{z,0}(s)}{\gamma_{z,0}(s)}\, ds = \int_{\gamma_{z,0}} \frac{1}{\zeta}\, d\zeta \tag{4.62}$$

for $z \in U_H$, which is the critical step. For the evaluation of the integral there are the following two obvious approaches. Note that the first evaluates the left and the second the right hand side of (4.62).

1. One could use the classic fundamental theorem of calculus to compute the integral. If we have a continuously differentiable function $F_z\colon [0,T] \to \mathbb{C}$ for $z \in U_H$ such that

   $$F'_z(s) = \frac{\gamma'_{z,0}(s)}{\gamma_{z,0}(s)}, \quad s \in [0,T],$$

   we have

   $$\int_t^T \frac{\gamma'_{z,0}(s)}{\gamma_{z,0}(s)}\, ds = F_z(T) - F_z(t),$$

   for $z \in U_H$. With the complex $\log : \mathbb{C} \setminus \{0\} \to (0,+\infty) + i(-\pi,\pi]$ a choice of F_z could be

   $$F_z(s) = \log \gamma_{z,0}(s), \quad s \in [0,T].$$

   Since the logarithm is discontinuous at every point belonging to $(-\infty,0)$ the function F_z defined via the latter equation is very clearly discontinuous if the path $\gamma_{z,0}$ crosses the negative real axis on the interval $[0,T]$. In that case the fundamental theorem is obviously not applicable to the function F_z as it cannot be differentiable either. Thus in order to conclude

   $$\int_t^T \frac{\gamma'_{z,0}(s)}{\gamma_{z,0}(s)}\, ds = \log \gamma_{z,0}(T) - \log \gamma_{z,0}(t)$$

   using this approach one would need to show that

   $$\gamma_{z,0}\big([0,T]\big) \subseteq \mathbb{C} \setminus (-\infty,0],$$

   holds for $z \in U_H$.

2. Using the right hand side of (4.62) one could also proceed as follows. Observe that the complex logarithm $\log : \mathbb{C} \setminus (-\infty,0] \to (0,+\infty) + i(-\pi,\pi)$ is a complex anti-derivative of the holomorphic function $\zeta \mapsto \zeta^{-1}$ on the simply connected set $\mathbb{C} \setminus (-\infty,0]$. If we show that also

   $$\gamma_{z,0}\big([0,T]\big) \subseteq \mathbb{C} \setminus (-\infty,0],$$

   holds that observation gives

   $$\int_{\gamma_{z,t}} \frac{1}{\zeta}\, d\zeta = \log \gamma_{z,0}(T) - \log \gamma_{z,0}(t).$$

Hence in both approaches considered it is essential to ensure that

$$\gamma_{z,0}\big([0,T]\big) \subseteq \mathbb{C} \setminus (-\infty, 0], \tag{4.63}$$

holds. Unfortunately with the way $\gamma_{z,0}$ is defined in (4.29) this does not hold in general as the following example shows.

Example 4.4.1. Consider the Heston model with the parameters.

κ	θ	S_0	η	v_0	ρ	r
2	0.1	10	0.25	0.05	-0.75	0

and the maturity $T = 10$. Following Theorem 4.3.9 the critical moments $u_+(10)$ and $u_-(10)$ are the two solutions of $T^*(u) = T$. From Example 4.3.11 we know that the critical moments are given by

$$u_-(10) = -4.577 \quad \text{and} \quad u_+(10) = 34.371,$$

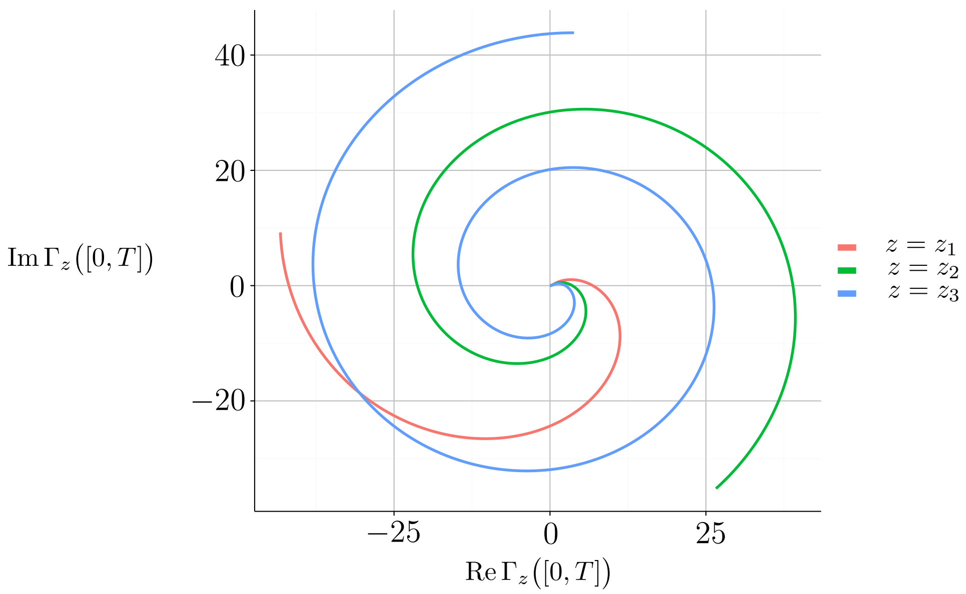
where we rounded to three digits. This model is an example where the path $\gamma_{z,0}$ crosses the negative real axis on $[0, 10]$ for some $z \in \big(u_-(10), u_+(10)\big) + i\mathbb{R}$. For instance this happens for $z \in \{z_1, z_2, z_3\}$, where

$$z_1 = 20 + 50i \quad z_2 = 25 + 50i \quad z_3 = 30 + 50i.$$

To visualize that crossing we use the function $\Gamma_z \colon [0,T] \to \mathbb{C}$ defined by

$$\Gamma_z(s) = \gamma_{z,0}(s)\left|\frac{\ln|\gamma_{z,0}(s)|}{\gamma_{z,0}(s)}\right|, \quad s \in [0,T].$$

Plotting $\Gamma_z \colon [0,T] \to \mathbb{C}$ gives the following figure.



Clearly we see that Γ_z crosses the negative real axis here and hence also $\gamma_{z,0}$ must do that. Consequently (4.63) cannot be satisfied in this case and hence neither of the approaches 1. and 2. can be successful in order to compute Φ_z.

Unfortunately, the author could not find any *derivation* of the moment generating function (MGF) or characteristic function (CF) of the underlying in the Heston model in the literature, where the problem discussed in Example 4.4.1 is addressed explicitly. However, in the literature every expression for the moment generating function M_{X_T} the author has encountered makes use of the complex logarithm when allowing for complex arguments. Equivalently the author also has only found expressions for the CF of X_T that contain the complex logarithm when allowing for arguments which are not purely real. For instance the complex logarithm is used in the original paper by Heston [Hes93, page 331], a paper by Lord and Kahl [LK10, page 674-675], Jim Gatheral's well-known book [GT06, page 20-21] and a paper by Lee [Lee04, page 31]. Moreover since non-integer powers of complex numbers are defined via the complex logarithm, i. e.

$$z^\alpha = e^{\alpha \log z}, \quad z \in \mathbb{C},$$

for $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, also the representation given for example by del Bano Rollin et al. in [dBRFCU09, page 45] makes implicitly use of the complex logarithm. Unfortunately as far as the author knows the papers discussing the MGF (or CF) of the underlying in the Heston model do not clearly point out which definition of the complex logarithm they use. On the one hand, it is possible that the complex logarithm is meant as set-valued[3] function, i. e.

$$\log \colon \mathbb{C} \setminus \{0\} \to \{(0, +\infty) + i(-\pi, \pi] + 2k\pi i \colon k \in \mathbb{Z}\}.$$

This however would mean that every equation for the MGF (or CF) where the complex logarithm occurs is meant to be understood in the way that the value of the MGF (or CF) for a given argument $z \in \mathbb{C}$ belongs to a specified set with countably many values. The author thinks that it is highly unlikely that the equations given in the literature for the MGF (or CF) of the underlying in the Heston model are meant to be understood that way. Moreover, then the statements would not be of great use when one actually wants to evaluate the MGF (or CF) of the underlying in the Heston model.
On the other hand, the usual convention would be that the complex logarithm is meant to be a certain branch, namely the principal branch and then one has

$$\log \colon \mathbb{C} \setminus \{0\} \to (0, +\infty) + i(-\pi, \pi].$$

Unfortunately it is well-known that the principal branch of the complex logarithm is discontinuous at every point belonging to $(-\infty, 0)$. Consequently every argument of the complex logarithm that crosses the negative real line is a potential source of discontinuity when using that definition of the complex logarithm.
Moreover, as pointed out by Lord and Kahl in [LK10, page 675] it indeed happens in the literature that discontinuous expressions are presented to equal the MGF (or CF) of the underlying in the Heston model. In particular Lord and Kahl point out that the expressions presented in [Hes93] and [Lee04] are discontinuous. This is strange as due to Proposition 3.1.7 we know that moment generating functions are even analytic on the interior of their maximal domain (as subset of $\mathbb{C}$). Consequently, as soon as one states that a certain expression equals the MGF (or CF) the continuity is of course also stated. Thus, all discontinuous representations and the derivations of that expressions must be flawed. Not taking into account the fact that (4.63) must hold to evaluate the integral via the suggested two approaches may be a cause.
Furthermore, there are papers which address the issue of continuity of a given representation of the moment generating function. An example for such a paper is one by Lord and Kahl, namely [LK10]. This is also strange because if the continuity of such an expression is an open question also the question whether the moment generating function can be represented that way is open. Interestingly, in such papers condition (4.63) appears when claiming that a certain expression does not cross the negative real axis.

Our approach to derive the moment generating function for complex arguments is different to that of the moment generating function. The idea is to use analytic continuation principles like the identity theorem of complex analysis (Theorem 2.3.5). However, it will turn out that the condition that a certain expression never crosses the negative real axis is also crucial. We start with the following proposition.

[3] In this context also the term multi-valued is used instead of set-valued. The author prefers the latter as a multi-valued function can by definition never be a function which might be a bit confusing.

Proposition 4.4.2. *Let $M_{X_T} \colon \big(u_-(T), u_+(T)\big) + i\mathbb{R} \to \mathbb{C}$ be the moment generating function of the log-discounted underlying in the Heston model. Furthermore, consider a connected and open domain $U \subseteq \big(u_-(T), u_+(T)\big) + i\mathbb{R}$ and an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq U$. Moreover, assume that $h \colon U \to \mathbb{C}$ is a holomorphic function satisfying*

$$h(u) = P_T(u)^{-\frac{2\kappa\theta}{\eta^2}} \exp\left(ux_0 + \frac{u(u-1)}{2} \frac{Q_T(u)}{P_T(u)} v_0 - \frac{2\kappa\theta}{\eta^2} b(u)T \right), \quad u \in (-\epsilon, \epsilon),$$

where P_T and Q_T are given by Lemma 4.2.16. Then we have

$$M_{X_T}(z) = h(z), \quad z \in U.$$

Proof. By Proposition 3.1.7 we know that M_{X_T} is holomorphic on $\big(u_-(T), u_+(T)\big) + i\mathbb{R}$. Furthermore, due to Theorem 4.2.18 we know that

$$M_{X_T}(u) = P_T(u)^{-\frac{2\kappa\theta}{\eta^2}} \exp\left(ux_0 + \frac{u(u-1)}{2} \frac{Q_T(u)}{P_T(u)} v_0 - \frac{2\kappa\theta}{\eta^2} b(u)T \right), \quad u \in (-\epsilon, \epsilon),$$

where we used $(-\epsilon, \epsilon) \subseteq \big(u_-(T), u_+(T)\big)$. Thus $M_{X_T}|_U$ and h are two holomorphic functions on the connected set U such that

$$M_{X_T}|_U(u) = h(u), \quad u \in (-\epsilon, \epsilon) \subseteq U.$$

By Theorem 2.3.5 these two functions coincide on U and we thus have

$$M_{X_T}|_U(z) = h(z), \quad z \in U.$$

□

The last proposition shows that the task is to analytically extend the expression we have found for real arguments of M_{X_T} to $\big(u_-(T), u_+(T)\big) + i\mathbb{R}$. Under the crucial assumption that a certain expression does not touch the negative real line we can derive an expression for the moment generating function in the following theorem.

Theorem 4.4.3. *Let $T > 0$ and assume that*

$$P_T(z)e^{-H(z)T} \notin (-\infty, 0], \quad z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\},$$

holds. Then the moment generating function of the log-discounted-underlying X_T satisfies

$$M_{X_T}(z) = \exp\Big(zx_0 + A(z) + v_0 B(z) \Big), \quad z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\}.$$

The coefficients A and B are given by

$$\begin{aligned} A(z) &= -\frac{2\kappa\theta}{\eta^2}\bigg(\big(b(z) + H(z)\big)T + \log\Big(P_T(z)e^{-H(z)T}\Big) \bigg) \\ B(z) &= \frac{z(z-1)}{2} \frac{Q_T(z)}{P_T(z)}, \end{aligned}$$

for $z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\}$, where P_T and Q_T are given in Lemma 4.2.16.

Proof. For the proof we will apply Proposition 4.4.2. Define

$$U := (u_l, 1) \cup \Big(\big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\} \Big),$$

which is clearly a open and non-empty set. Next, observe that by Theorem 4.3.8 we have

$$T^*(u) = +\infty, \quad u \in [u_l, 1].$$

By the very definition of T^* in (4.52) this means that

$$P_T(u) > 0, \quad (u, T) \in [u_l, 1] \times [0, +\infty).$$

Consequently, we have $(u_l, 1) \subseteq \big(u_-(T), u_+(T)\big)$ and thus $U \subseteq \big(u_-(T), u_+(T)\big) + i\mathbb{R}$ on the one hand. On the other hand, together with the assumption

$$P_T(z)e^{-H(z)T} \notin (-\infty, 0], \quad z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\},$$

we obtain

$$P_T(z)e^{-H(z)T} \notin (-\infty, 0], \quad z \in U. \tag{4.64}$$

Now define $h\colon U \to \mathbb{C}$ by

$$h(z) = \exp\Big(zx_0 + A(z) + v_0 B(z)\Big), \quad z \in U,$$

where

$$\begin{aligned} A(z) &= -\frac{2\kappa\theta}{\eta^2}\bigg(\big(b(z) + H(z)\big)T + \log\Big(P_T(z)e^{-H(z)T}\Big)\bigg) \\ B(z) &= \frac{z(z-1)}{2}\frac{Q_T(z)}{P_T(z)}, \end{aligned}$$

for $z \in U$. Since the complex logarithm $\log\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{R} + i(-\pi, \pi)$ is holomorphic the function A is also holomorphic (as composition of holomorphic functions) on U. Furthermore, (4.64) implies

$$P_T(z) \neq 0, \quad z \in U.$$

Thus as composition of holomorphic functions also B is holomorpic on U. With A and B also h is a holomorphic function on U. Furthermore, due to $P_T(u) > 0$ and $e^{-H(u)T} > 0$ for $u \in (u_l, 1)$ we have

$$\begin{aligned} A(u) &= -\frac{2\kappa\theta}{\eta^2}\bigg(\big(b(u) + H(u)\big)T + \log\Big(P_T(u)e^{-H(u)T}\Big)\bigg) \\ &= -\frac{2\kappa\theta}{\eta^2}\bigg(\big(b(u) + H(u)\big)T + \log P_T(u) + \log\Big(e^{-H(u)T}\Big)\bigg) = -\frac{2\kappa\theta}{\eta^2}\Big(b(u)T + \log P_T(u)\Big), \end{aligned}$$

for $u \in (u_l, 1)$. That leads to

$$\begin{aligned} h(u) &= \exp\left(ux_0 - \frac{2\kappa\theta}{\eta^2}\Big(b(u)T + \log P_T(u)\Big) + v_0\frac{u(u-1)}{2}\frac{Q_T(u)}{P_T(u)}\right) \\ &= P_T(u)^{-\frac{2\kappa\theta}{\eta^2}} \exp\left(ux_0 + v_0\frac{u(u-1)}{2}\frac{Q_T(u)}{P_T(u)} - \frac{2\kappa\theta}{\eta^2}b(u)T\right), \end{aligned}$$

for $u \in (u_l, 1)$. By Lemma 4.2.5 we know $u_l < 0 < 1$ and thus U is connected. Furthermore, there is an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq (u_l, 1) \subseteq U$. Hence with h and U as defined in this proof the assumptions of Proposition 4.4.2 are fulfilled. Hence we have

$$M_{X_T}(z) = h(z), \quad z \in U.$$

Clearly $\big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\} \subseteq U$ and since h is exactly the expression which is stated to equal the moment generating function on $\big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\}$ the proof is concluded. $\square$

In order to derive an explicit expression for the moment generating function in the Heston model we aim for applying Theorem 4.4.3. Hence we want to show that

$$P_T(z)e^{-H(z)T} \notin (-\infty, 0], \tag{4.65}$$

holds for $z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\}$. This will be proven only for the case where the parameters satisfy $\kappa \geq \eta\rho$. Unfortunately the way we show this is not really productive when $\kappa < \eta\rho$ holds. The derivation is quite technical and is based on the ideas presented in [LK06]. We need the following notation for the formulations and proofs of the subsequent lemmas which can also be found in [LK06]. Define

$$\begin{aligned} b_1(z_1) &= z_1\eta\rho - \kappa \\ b_2(z_2) &= -z_2\rho\eta \\ D_1(z_1, z_2) &= \kappa^2 + \eta^2(1-\rho^2)(z_2^2 - z_1^2) + \eta(\eta - 2\kappa\rho)z_1 \\ D_2(z_1, z_2) &= \eta z_2\big(\eta - 2\kappa\rho - 2\eta(1-\rho^2)z_1\big), \end{aligned} \tag{4.66}$$

for $z = z_1 + iz_2 \in \mathbb{C}$. This gives

$$b_1(z_1) + ib_2(z_2) := 2b(\overline{z}) \qquad \text{and} \qquad D_1(z_1) + iD_2(z_2) := D(z),$$

for $z \in \mathbb{C}$.

Lemma 4.4.4. *For $w = w_1 + iw_2 \in \mathbb{C} \setminus (-\infty, 0]$ one has*

$$\sqrt{2}e^{\frac{1}{2}\log(w)} = \sqrt{|w| + w_1} + i\,\mathrm{sgn}(w_2)\sqrt{|w| - w_1},$$

and

$$\mathrm{Im}\left(e^{\frac{1}{2}\log(w)}\right) = \frac{w_2}{2\,\mathrm{Re}\left(e^{\frac{1}{2}\log(w)}\right)}.$$

Proof. See Subsection A.2.10 in the appendix. □

Lemma 4.4.5. *Assume $\rho \neq 0$. For $z = z_1 + iz_2 \in \mathbb{C}$ we have*

$$2\,\mathrm{Re}\big(b(\overline{z})D(z)\big) = b_1(z_1)\rho^{-2}f(\rho z_1) - \eta^2 z_2^2\Big(2\kappa - \rho\eta + (1-\rho^2)b_1(z_1)\Big),$$

where

$$f(x) = \kappa^2\rho^2 - \eta^2(1-\rho^2)x^2 + \eta\rho(\eta - 2\kappa\rho)x, \quad x \in \mathbb{R}.$$

Furthermore, if additionally $\kappa \geq \eta\rho$ holds one has $f(x) \leq 0$ for $x \geq \frac{\kappa}{\eta}$.

Proof. See Subsection A.2.11 in the appendix. □

Lemma 4.4.6. *Assume $\rho \neq 0$. For $z = z_1 + iz_2 \in \mathbb{C}$ we then have*

$$4\,\mathrm{Re}\big(b(\overline{z})D(z)\big)^2 - b_1(z_1)^2|D(z)|^2 = -\frac{\eta^2}{\rho^2}z_2\rho D_2(z_1, z_2)g(\rho z_1, \rho z_2),$$

where $g\colon \mathbb{R}^2 \to \mathbb{R}$ is given by

$$g(w_1, w_2) = \kappa^2(\rho - 2w_1) + (w_1^2 + w_2^2)\eta(2\kappa - \rho\eta), \quad (w_1, w_2) \in \mathbb{R}^2. \tag{4.67}$$

Proof. See Subsection A.2.12 in the appendix. □

Lemma 4.4.7. *Consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ and assume $2\kappa > \rho\eta$. Then one has*

$$g(w_1, w_2) \geq g(w_1, 0) =: g_1(w_1), \quad (w_1, w_2) \in \mathbb{R}^2,$$

Furthermore, $g_1 : \mathbb{R} \to \mathbb{R}$ has a global minimum at

$$w_1^* = \frac{\kappa^2}{\eta(2\kappa - \rho\eta)},$$

such that g_1 is decreasing on $(-\infty, w_1^*]$ and increasing on $[w_1^*, \infty)$.

Proof. See Subsection A.2.13 in the appendix. $\square$

The next proposition will be important to show (4.65) when the Heston model parameters satisfy $\kappa \geq \eta\rho$. Proposition 4.4.8 is due to Lord and Kahl (2006), [LK06], as is the idea of the proof. However, in the proof given here only three different cases need to be considered.

Proposition 4.4.8. *Let $2\kappa > \eta\rho$ and $z \in U_H$. Furthermore, assume that at least one of the following two conditions is satisfied.*

(i) $\kappa \geq \eta\rho$

(ii) $\kappa \geq \eta\rho\,\mathrm{Re}(z)$

Then it holds that

$$|H(z) + b(z)| \leq |H(z) - b(z)|.$$

Proof. First note that $\overline{b(z)} = b(\overline{z})$ and $\overline{H(z)} = H(\overline{z})$ hold for $z \in U_H$. Next, observe the equivalence

$$\begin{aligned}
|H(z) + b(z)| \leq |H(z) - b(z)| &\Leftrightarrow |H(z) + b(z)|^2 \leq |H(z) - b(z)|^2 \\
&\Leftrightarrow |H(z)|^2 + |b(z)|^2 + 2\,\mathrm{Re}(b(\overline{z})H(z)) \leq |H(z)|^2 + |b(z)|^2 - 2\,\mathrm{Re}(b(\overline{z})H(z)) \\
&\Leftrightarrow 4\,\mathrm{Re}(b(\overline{z})H(z)) \leq 0,
\end{aligned}$$

for every $z \in U_H$. Now define

$$H_1 := 2\,\mathrm{Re}(H) \qquad H_2 := 2\,\mathrm{Im}(H).$$

Because of $z \in U_H$ and Lemma 4.2.5 we have $H_1(z) > 0$ and thus a multiplication of the latter inequality by $2H_1(z)$ and the use of Lemma 4.4.4 yields the following equivalent statement.

$$\begin{aligned}
0 \geq 8H_1(z)\,\mathrm{Re}(b(\overline{z})H(z)) &= 2b_1(z_1)H_1(z)^2 - 2b_2(z_2)H_1(z)H_2(z) \\
&= b_1(z_1)\big(|D(z)| + D_1(z_1, z_2)\big) - b_2(z_2)D_2(z_1, z_2) \\
&= b_1(z_1)|D(z)| + 2\,\mathrm{Re}\left(b(\overline{z})D(z)\right),
\end{aligned}$$

for $z = z_1 + iz_2 \in U_H$. Hence it suffices to show

$$-b_1(z_1)|D(z)| \geq 2\,\mathrm{Re}\left(b(\overline{z})D(z)\right), \quad z = z_1 + iz_2 \in U_H. \tag{4.68}$$

To prove (4.68) we distinguish the following three cases.

(i) $\rho \neq 0$ and $b_1(z_1) \leq 0$ and $z_2\rho D_2(z_1, z_2) \geq 0$

Because of $b(z_1) \leq 0$ it is sufficient to show

$$b_1(z_1)^2|D(z)|^2 \geq 4\,\mathrm{Re}\left(b(\overline{z})D(z)\right)^2,$$

which we obtained by squaring both sides in (4.68). By Lemma 4.4.6 this is equivalent to

$$\frac{\eta^2}{\rho^2} z_2 \rho D_2(z_1, z_2) g(\rho z_1, \rho z_2) \geq 0, \tag{4.69}$$

where $g\colon \mathbb{R}^2 \to \mathbb{R}$ is given by (4.67). If $z_2 = 0$ we are obviously done. If $|\rho| = 1$ observe that

$$z_2 \rho D_2(z_1, z_2) = \begin{cases} -\eta z_2^2(\eta + 2\kappa) & \rho = -1 \\ \eta z_2^2(\eta - 2\kappa) & \rho = 1. \end{cases} \tag{4.70}$$

Since $\eta\rho - 2\kappa \leq 0$ we see that the expressions in (4.70) are clearly non-positive. Consequently we are also done if $|\rho| = 1$. For the rest of this case assume now that $|\rho| < 1$ and $|z_2| > 0$. Recall that we assumed $z_2 \rho D_2(z_1, z_2) \geq 0$ in this case and hence in order to show (4.69) it suffices to verify

$$g(\rho z_1, \rho_2 z_2) \geq 0. \tag{4.71}$$

That one can see as follows. By Lemma 4.4.7 we know that g_1, defined by $g_1(w_1) = g(w_1, 0)$ for $w_1 \in \mathbb{R}$, is decreasing on $(-\infty, w_1^*]$, where

$$w_1^* = \frac{\kappa^2}{\eta(2\kappa - \rho\eta)}.$$

However, we have

$$\begin{aligned} w_1^* - \frac{\rho(\eta - 2\kappa\rho)}{2\eta(1-\rho^2)} &= \frac{2\kappa^2(1-\rho^2) - \rho(\eta - 2\kappa\rho)(2\kappa - \eta\rho)}{2\eta(2\kappa - \rho\eta)(1-\rho^2)} = \frac{2\kappa^2 - 2\kappa^2\rho^2 - 2\kappa\eta\rho + 4\kappa^2\rho^2 + \eta^2\rho^2 - 2\kappa\eta\rho^3}{2\eta(2\kappa - \rho\eta)(1-\rho^2)} \\ &= \frac{\kappa^2 + \kappa^2 - 2\kappa\eta\rho + \eta^2\rho^2 + 2\kappa^2\rho^2 - 2\kappa\eta\rho^3}{2\eta(2\kappa - \rho\eta)(1-\rho^2)} = \frac{\kappa^2 + (\kappa - \eta\rho)^2 + 2\kappa\rho^2(\kappa - \eta\rho)}{2\eta(2\kappa - \rho\eta)(1-\rho^2)} \\ &= \frac{\left(\kappa - \eta\rho + \kappa\rho^2\right)^2 + \kappa^2(1-\rho^4)}{2\eta(2\kappa - \rho\eta)(1-\rho^2)} \geq 0, \end{aligned}$$

where the latter is true due to $2\kappa > \rho\eta$. Because of $z_2 \rho D_2(z_1, z_2) \geq 0$ we get

$$\eta z_2^2\big(\rho(\eta - 2\kappa\rho) - 2\eta\rho(1-\rho^2)z_1\big) \geq 0 \Leftrightarrow \rho z_1 \leq \frac{\rho(\eta - 2\kappa\rho)}{2\eta(1-\rho^2)},$$

where we used that we are already done for $z_2 = 0$ and can assume $|z_2| > 0$. In particular we have

$$\rho z_1 \leq \frac{\rho(\eta - 2\kappa\rho)}{2\eta(1-\rho^2)} \leq w_1^*.$$

Since g_1 is decreasing on $(-\infty, w_1^*]$ we conclude that

$$\begin{aligned} g_1(\rho z_1) \geq g_1\left(\frac{\rho(\eta - 2\kappa\rho)}{2\eta(1-\rho^2)}\right) &= \kappa^2\rho\left(1 - \frac{\eta - 2\kappa\rho}{\eta(1-\rho^2)}\right) + \rho^2\frac{(\eta - 2\kappa\rho)^2(2\kappa - \rho\eta)}{4\eta(1-\rho^2)^2} \\ &= \kappa^2\rho^2\frac{2\kappa - \rho\eta}{\eta(1-\rho^2)} + \rho^2\frac{(\eta - 2\kappa\rho)^2(2\kappa - \rho\eta)}{4\eta(1-\rho^2)^2} \\ &= \frac{\rho^2(2\kappa - \eta\rho)}{\eta(1-\rho^2)}\left(\kappa^2 + \frac{(\eta - 2\kappa\rho)^2}{4(1-\rho^2)}\right) > 0. \end{aligned}$$

Applying Lemma 4.4.7 yields

$$g(\rho z_1, \rho z_2) \geq g(\rho z_1, 0) = g_1(\rho z_1) \geq 0.$$

Hence we verified (4.71) and are done with this case.

(ii) $\quad b_1(z_1) \leq 0 \quad$ and $\quad z_2\rho D_2(z_1,z_2) \leq 0$

First note that here also the case $\rho = 0$ is covered. To see that (4.68) is true consider the following. Observe that

$$b_2(z_2)D_2(z_1,z_2) = -\eta\rho z_2 D_2(z_1,z_2) \geq 0,$$

since $\rho z_2 D_2(z_1,z_2) \leq 0$. Because we also have $b_1(z_1) \leq 0$ we obtain

$$-b_1(z_1)|D(z)| \geq \left|b_1(z_1)D_1(z)\right| \geq b_1(z_1)D_1(z_1,z_2) - b_2(z_2)D_2(z_1,z_2) = 2\operatorname{Re}\big(b(\overline{z})D(z)\big).$$

Hence this case is also done.

(iii) $\quad \rho \neq 0 \quad$ and $b_1(z_1) > 0$

In the previous cases we only needed the assumption that $2\kappa > \eta\rho$. Note that $b_1(z_1) > 0$ is equivalent to $\kappa < \eta\rho z_1 = \eta\rho\operatorname{Re}(z)$. Hence in this case we only need to show that (4.68) holds if $\kappa \geq \eta\rho$. Observe that (4.68) is equivalent to

$$b_1(z_1)|D(z)| \leq \; -2\operatorname{Re}\big(b(\overline{z})D(z)\big). \tag{4.72}$$

The right hand side is now non-negative. To see that consider the following. Use Lemma 4.4.5, $b_1(z_1) \geq 0$ and $2\kappa \geq \rho\eta$ to obtain

$$-2\operatorname{Re}\big(b(\overline{z})D(z)\big) = \eta^2 z_2^2\Big(2\kappa - \rho\eta + (1-\rho^2)b_1(z_1)\Big) - b_1(z_1)\rho^{-2}f(\rho z_1) \geq \; - b_1(z_1)\rho^{-2}f(\rho z_1). \tag{4.73}$$

Because we have

$$\rho z_1 > \frac{\kappa}{\eta},$$

and we assume $\kappa \geq \eta\rho$ one can apply Lemma 4.4.5 to see that $f(\rho z_1) \leq 0$. Since also $b_1(z_1) > 0$ holds we conclude from (4.73) that

$$-2\operatorname{Re}\big(b(\overline{z})D(z)\big) \geq 0.$$

Hence in order to show (4.72) it suffices to proof

$$b_1(z_1)^2|D(z)|^2 \leq 4\operatorname{Re}\big(b(\overline{z})D(z)\big)^2, \tag{4.74}$$

which we obtained by squaring both sides in (4.72). By Lemma 4.4.6 this is equivalent to proving

$$\frac{\eta^2}{\rho^2} z_2\rho D_2(z_1,z_2) g(\rho z_1, \rho z_2) \leq 0, \tag{4.75}$$

where $g\colon \mathbb{R}^2 \to \mathbb{R}$ is again given by (4.67). Because of $\eta\rho z_1 - \kappa = 2b_1(z_1) \geq 0$ and $2\kappa > \eta\rho$ we get

$$2\eta\rho z_1(1-\rho^2) \geq 2\kappa(1-\rho^2) \geq \eta\rho - 2\kappa\rho^2 = \rho(\eta - 2\kappa\rho),$$

which yields

$$z_2\rho D_2(z_1,z_2) = \eta z_2^2\big(\rho(\eta-2\kappa\rho) - 2\eta\rho(1-\rho^2)z_1\big) \leq 0.$$

Hence to show (4.75) it suffices to prove

$$g(\rho z_1, \rho z_2) \geq 0. \tag{4.76}$$

Consider again the function g_1 from Lemma 4.4.7 which is increasing on $[w_1^*, +\infty)$, where

$$w_1^* = \frac{\kappa^2}{\eta(2\kappa - \rho\eta)}.$$

Because of

$$\frac{\kappa}{\eta} - w_1^* = \frac{\kappa}{\eta} - \frac{\kappa^2}{\eta(2\kappa - \rho\eta)} = \frac{\kappa}{\eta}\left(1 - \frac{\kappa}{(2\kappa - \rho\eta)}\right) = \frac{\kappa(\kappa - \eta\rho)}{\eta(2\kappa - \rho\eta)} \geq 0,$$

where we again used $\kappa \geq \eta\rho$, we see that g_1 is also increasing on $[\frac{\kappa}{\eta}, \infty)$. Because of $\rho z_1 \geq \frac{\kappa}{\eta}$ we consequently obtain

$$g(\rho z_1, \rho z_2) \geq g_1(\rho z_1) \geq g_1\left(\frac{\kappa}{\eta}\right) = \kappa^2\left(\rho - 2\frac{\kappa}{\eta}\right) + \frac{\kappa^2}{\eta^2}\eta(2\kappa - \eta\rho) = \frac{\kappa^2}{\eta}(2\kappa - \eta\rho)(1 - 1) = 0,$$

where the first estimate is by means of Lemma 4.4.7. Hence we proved (4.76) and thus also this case is done.

$\square$

Now we can prove that the crucial assumption of Theorem 4.4.3, i. e. (4.65), is fulfilled if $\kappa \geq \eta\rho$ holds.

Proposition 4.4.9. *Assume that $\kappa \geq \eta\rho$ and $T > 0$. With the function P_T, defined in (4.45), we have*

$$P_T(z)e^{-H(z)T} \notin (-\infty, 0],$$

for every $z \in U_H$.

Proof. First observe that by Lemma 4.2.16 we obtain

$$\begin{aligned} P_T(z)e^{-H(z)T} &= e^{-H(z)T}\left(\cosh(H(z)T) - b(z)\frac{\sinh(H(z)T)}{H(z)}\right) \\ &= \frac{1}{2H(z)}e^{-H(z)T}\Big(H(z)\big(e^{H(z)T} + e^{-H(z)T}\big) - b(z)\big(e^{H(z)T} - e^{-H(z)T}\big)\Big) \\ &= \frac{1}{2H(z)}\Big(H(z) - b(z) + \big(H(z) + b(z)\big)e^{-2H(z)T}\Big), \end{aligned}$$

for $z \in U_H$. Now assume that we had $P_T(z)e^{-H(z)T} = -\alpha$ for a non-negative $\alpha \in \mathbb{R}$ and $z \in U_H$. We distinguish the following two cases.

- ▷ $H(z) \neq b(z)$

  Because of

  $$-\alpha = P_T(z)e^{-H(z)T} = \frac{1}{2H(z)}\Big(H(z) - b(z) + \big(H(z) + b(z)\big)e^{-2H(z)T}\Big),$$

  we can rearrange terms such that

  $$-\big(H(z) + b(z)\big)\big(\alpha + e^{-2H(z)T}\big) = \big(H(z) - b(z)\big)(\alpha + 1)$$

  Taking absolute values would yield

  $$\big|H(z) + b(z)\big|\big|\alpha + e^{-2H(z)T}\big| = \big|H(z) - b(z)\big||\alpha + 1|. \tag{4.77}$$

  On the other hand because of $\kappa \geq \eta\rho$ we can apply Proposition 4.4.8 to get

  $$\big|H(z) + b(z)\big|\Big|\alpha + e^{-2H(z)T}\Big| \leq \big|H(z) - b(z)\big|\Big(\alpha + e^{-2\operatorname{Re} H(z)T}\Big) < \big|H(z) - b(z)\big|\big(\alpha + 1\big).$$

  Note that for the latter estimate we used $H(z) \neq b(z)$ and $T > 0$. This contradicts (4.77) and hence $P_T(z)e^{-H(z)T} \notin (-\infty, 0]$.

▷ $H(z) = b(z)$

This yields

$$H(z) = b(z) \Rightarrow H(z)^2 = b(z)^2 \Rightarrow D(z)^2 = (\kappa - \eta\rho z)^2 \Rightarrow \eta z(1-z) = 0 \Rightarrow z = 0 \text{ or } z = 1.$$

In particular $z \in \mathbb{R}$ has to hold. Then also $H(z) \in \mathbb{R}$ and we get

$$P_T(z)e^{-H(z)T} = \frac{1}{2H(z)}\Big(2H(z)e^{-2H(z)T}\Big) = e^{-2H(z)T} \in (0, +\infty).$$

Consequently $P_T(z)e^{-H(z)T} \notin (-\infty, 0]$ also holds in this case.

□

By applying Lemma 4.4.9 and Theorem 4.4.3 we get the following corollary which finally presents the moment generating function for arguments z which are not purely real, i. e.

$$z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\},$$

for the case where $\kappa \geq \eta\rho$.

Corollary 4.4.10. *Assume $\kappa \geq \eta\rho$ and let $T > 0$. Then the moment generating function of the log-discounted underlying X_T satisfies*

$$M_{X_T}(z) = \exp\Big(zx_0 + A(z) + v_0 B(z)\Big), \quad z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\},$$

where $u_+(T)$ and $u_-(T)$ are defined in (4.47). The coefficients A and B are given by

$$A(z) = -\frac{2\kappa\theta}{\eta^2}\left(\big(b(z) + H(z)\big)T + \log\Big(P_T(z)e^{-H(z)T}\Big)\right)$$
$$B(z) = \frac{z(z-1)}{2}\frac{Q_T(z)}{P_T(z)},$$

for $z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\}$, where P_T and Q_T are given in Lemma 4.2.16.

Proof. Because of the assumption $\kappa \geq \eta\rho$ we can apply Proposition 4.4.9 to see that

$$P_T(z)e^{-H(z)T} \notin (-\infty, 0], \quad z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\} \subseteq U_H. \tag{4.78}$$

Consequently we can apply Theorem 4.4.3 which gives the statement. □

Clearly we have not proven (4.65) if the parameters satisfy $\kappa < \eta\rho$. Hence in the latter case we cannot be sure whether the expressions given in Theorem 4.4.3 equal the moment genearting function. In particular note that our proof for the case $\kappa \geq \eta\rho$ heavily relies on Proposition 4.4.8. Unfortunately, the latter proposition does not necessarily need to hold when $\kappa < \eta\rho$. Consequently (4.65) cannot be shown analogously to our procedure if $\kappa < \eta\rho$ is fulfilled.

Fortunately, there are papers addressing the question whether (4.65) holds in the case $\kappa < \eta\rho$. For instance Albrecher et al. discuss in [AMST06] a very similar problem. There they use a parameter[4] α whose possible values remain unclear. However, they also refer to the Fourier pricing paper [CMS99] by Carr and Madan where $\alpha > 0$ is used. Thus an educated guess would be that this is also the restriction on α in [AMST06]. That would prove (4.65) for a large set of arguments z. Another paper discussing and giving a solution to the problem of proving (4.65) is presented by Lord and Kahl in [LK10]. As far as the author can tell the ideas presented in [AMST06] and [LK10] can also be used to prove (4.65) in our setting if the parameters satisfy $\kappa < \eta\rho$. However, as it goes beyond the scope of this thesis we do not give a proof here.

[4] The parameter α in [AMST06] corresponds to $\mathrm{Re}(z) - 1$ in our notation.

4.5 Heston Model – Summary

Finally, we want to summarize the results of this chapter. Therefore let $W = (W^1, W^2)$ be a two-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$. With a correlation parameter $\rho \in [-1, 1]$ define the correlated Brownian motions

$$W^S = \sqrt{1-\rho^2}W^1 + \rho W^2 \quad \text{and} \quad W^\nu = W^2.$$

Next, we fix a maturity $T > 0$ and consider a continuous function $r\colon [0,T] \to [0,+\infty)$ to define a risk-free savings account by

$$B_t = \exp\left(\int_0^t r_s \, ds\right), \quad 0 \leq t \leq T.$$

Furthermore, with a mean reversion speed $\kappa > 0$, a mean reversion level $\theta > 0$, a volatility parameter $\eta > 0$, an initial stock value $s_0 > 0$ and an initial volatility value v_0 consider the following system of SDEs.

$$\begin{array}{rcrcl} dS_t & = & r_t S_t \, dt & + & \sqrt{\nu_t} S_t \, dW_t^S \\ d\nu_t & = & \kappa(\theta - \nu_t) \, dt & + & \eta\sqrt{\nu_t} \, dW_t^\nu \end{array}, \quad (S_0, \nu_0) = (s_0, v_0), \tag{4.79}$$

on $[0,T]$. By Proposition 4.1.1 and Proposition 4.1.3 for every interval $[0,T]$ there is up to indistinguishability a unique continuous process $(S_t, \nu_t)_{t\in[0,T]}$ solving (4.79). In particular that solution satisfies

$$P\big(S_t > 0, \quad 0 \leq t \leq T\big) = 1 \quad \text{and} \quad P\big(\nu_t \geq 0, \quad 0 \leq t \leq T\big) = 1.$$

If the so-called Feller condition is fulfilled we even have the following result.

Proposition 4.5.1 (Feller Condition)**.** *Consider the variance process ν defined by* (4.79)*. If the parameters additionally satisfy the so-called Feller-condition, i. e.*

$$2\kappa\theta > \eta^2,$$

then the process ν even satisfies

$$P(\nu_t > 0, \quad 0 \leq t \leq T) = 1.$$

Proof. See Proposition 4.1.2. □

In particular following the notion in Section 2.1 the pair (B, S) from above defines a financial market. By means of Proposition 4.1.4 we know that this financial market is arbitrage free.
In order to price options we consider the log-discounted underlying X_T, which is defined by

$$X_T = \ln S_T - \int_0^T r_s \, ds.$$

A nice feature of the Heston model is that the corresponding moment generating function M_{X_T}, defined by

$$M_{X_T}(z) = \mathbb{E}\big(e^{zX_T}\big),$$

where $z \in \mathbb{C}$ is such that $\mathbb{E}\big(|e^{zX_T}|\big) < +\infty$, can be expressed in closed form. To see this define the following functions.

$$\begin{aligned} b(z) &= \frac{1}{2}(\eta\rho z - \kappa), \quad z \in \mathbb{C} \\ D(z) &= (\kappa - \eta\rho z)^2 - z(z-1)\eta^2, \quad z \in \mathbb{C} \\ H(z) &= \begin{cases} \frac{1}{2}\sqrt{|D(z)|} & z \in \mathbb{R} \\ \frac{1}{2}\exp\left(\frac{1}{2}\log D(z)\right) & z \in \mathbb{C}\setminus\mathbb{R} \end{cases} \end{aligned}$$

Furthermore, we also need the roots u_l and u_r of D, which are given by

$$u_l = \begin{cases} \frac{\eta-2\kappa\rho-\sqrt{(\eta-2\kappa\rho)^2+4\kappa^2(1-\rho^2)}}{2\eta(1-\rho^2)} & \rho \notin \{-1,+1\} \\ -\frac{\kappa^2}{\eta(\eta+2\kappa)} & \rho = -1 \\ -\frac{\kappa^2}{\eta(\eta-2\kappa)} & \rho = 1,\ \eta > 2\kappa \\ -\infty & \rho = 1 \text{ and } \eta \leq 2\kappa \end{cases}$$

and

$$u_r = \begin{cases} \frac{\eta-2\kappa\rho+\sqrt{(\eta-2\kappa\rho)^2+4\kappa^2(1-\rho^2)}}{2\eta(1-\rho^2)} & \rho \notin \{-1,+1\} \\ +\infty & \rho = -1 \\ +\infty & \rho = 1,\ \eta \geq 2\kappa \\ \frac{\kappa^2}{\eta(2\kappa-\eta)} & \rho = 1,\ \eta < 2\kappa \end{cases}.$$

With u_l and u_r define the functions P_T and Q_T by

$$P_T(z) = \begin{cases} \cos H(z)T - b(z)\frac{\sin H(z)T}{H(z)} & z \in (-\infty, u_l) \cup (u_r, \infty) \\ 1 - b(z)T & z \in \{u_l, u_r\} \\ \cosh H(u)T - b(z)\frac{\sinh H(z)T}{H(z)} & \text{else} \end{cases},$$

and

$$Q_T(z) = \begin{cases} \frac{\sin H(z)T}{H(z)} & z \in (-\infty, u_l) \cup (u_r, \infty) \\ T & z \in \{u_l, u_r\} \\ \frac{\sinh H(z)T}{H(z)} & \text{else} \end{cases},$$

for $z \in \mathbb{C}$. Next, we define $u_+(T)$ and $u_-(T)$ by

$$u_-(T) = \sup\big\{u \leq 0 \colon P_T(u) = 0\big\} \quad \text{and} \quad u_+(T) = \inf\big\{u \geq 0 \colon P_T(u) = 0\big\}. \tag{4.80}$$

With all the expressions we have just written down one can now explicitly describe the moment generating function M_{X_T}. We start by presenting a result for real arguments.

Theorem 4.5.2. *The moment generating function M_{X_T} of the log-discounted underlying at time $T > 0$ in the Heston model is finite for real arguments belonging to $\big(u_-(T), u_+(T)\big)$, i. e.*

$$\mathbb{E}\big(e^{uX_T}\big) < +\infty, \quad u \in \big(u_-(T), u_+(T)\big).$$

Furthermore, the moment generating function is given by

$$M_{X_T}(u) = P_T(u)^{-\frac{2\kappa\theta}{\eta^2}} \exp\left(ux_0 + v_0\frac{u(u-1)}{2}\frac{Q_T(u)}{P_T(u)} - \frac{2\kappa\theta}{\eta^2}b(u)T\right), \quad u \in \big(u_-(T), u_+(T)\big),$$

and it always holds that $u_+(T) > 1$. Moreover, we have

$$\mathbb{E}\big(e^{uX_T}\big) = +\infty, \quad u \in \big(-\infty, u_-(T)\big] \cup \big[u_+(T), +\infty).$$

Hence the strip $\big(u_-(T), u_+(T)\big) + i\mathbb{R}$ forms the maximum domain of M_{X_T} in the complex plane.

Proof. See Theorem 4.2.18 and Theorem 4.2.19. □

Very naturally the question arises how one should compute the critical moments $u_+(T)$ and $u_-(T)$. One step towards a procedure to determine these is the following theorem.

Theorem 4.5.3. *Define $T^*\colon \mathbb{R} \to (0,+\infty]$ as follows.*

(i) $\kappa < \eta\rho$

$$T^*(u) = \begin{cases} +\infty & u \in [u_l, 1] \\ \frac{1}{2H(u)} \ln\left(\frac{b(u)+H(u)}{b(u)-H(u)}\right) & u \in (1, u_r) \\ b(u)^{-1} & u = u_r \\ H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)} & u \in (-\infty, u_l) \cup (u_r, +\infty) \end{cases}.$$

(ii) $\kappa \geq \eta\rho$

$$T^*(u) = \begin{cases} +\infty & u \in [u_l, u_r] \\ H(u)^{-1} \operatorname{arccot} \frac{b(u)}{H(u)} & u \in (-\infty, u_l) \cup (u_r, +\infty) \end{cases}.$$

Then if one of the statements

- ▷ $u \in [0, +\infty)$
- ▷ $u \in (-\infty, 0)$ *and* $2\kappa \geq \eta\rho$

is true we have

$$\mathbb{E}\big({S_T}^u\big) < +\infty, \quad T \in \big[0, T^*(u)\big) \qquad \text{and} \qquad \mathbb{E}\big({S_T}^u\big) = +\infty, \quad T \in [T^*(u), +\infty).$$

Proof. See Theorem 4.3.8. □

The next theorem finally provides a very practical way to compute the critical moments $u_+(T)$ and $u_-(T)$.

Theorem 4.5.4. *Let $T > 0$. Recall the critical moments $u_+(T)$ and $u_-(T)$ from (4.80), which are characterized by*

$$\mathbb{E}\big({S_T}^u\big) < +\infty \quad \text{if and only if} \quad u \in \big(u_-(T), u_+(T)\big).$$

For $T^*\colon \mathbb{R} \to (0, +\infty]$, which is described in Theorem 4.5.3, the following holds.

(i) $\rho \in (-1, 1]$

Then $u = u_+(T)$ is the unique positive solution to

$$T^*(u) = T.$$

(ii) $\rho \in [-1, 1)$ *and* $2\kappa \geq \eta\rho$

Then $u = u_-(T)$ is the unique negative solution to

$$T^*(u) = T.$$

Furthermore, if $\rho = 1$ and $2\kappa \geq \eta$ hold simultaneously then $u_-(T) = -\infty$. If $\rho = -1$ then we have $u_+(T) = +\infty$.

Proof. See Theorem 4.2.18, Theorem 4.2.19 and Theorem 4.3.9. □

Finally, we can also present an explicit expression for the moment generating function for arguments belonging to $\mathbb{C} \setminus \mathbb{R}$ if the parameters in the Heston model satisfy $\kappa \geq \eta\rho$.

Theorem 4.5.5. *Assume $\kappa \geq \eta\rho$ and let $T > 0$. Then the moment generating function of the log-discounted-underlying X_T satisfies*

$$M_{X_T}(z) = \exp\Big(zx_0 + A(z) + v_0 B(z)\Big), \quad z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\}.$$

The coefficients A and B are given by

$$\begin{aligned} A(z) &= -\frac{2\kappa\theta}{\eta^2}\left(\big(b(z) + H(z)\big)T + \log\left(P_T(z)e^{-H(z)T}\right)\right) \\ B(z) &= \frac{z(z-1)}{2}\frac{Q_T(z)}{P_T(z)}, \end{aligned}$$

for $z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\}$.

Proof. See Corollary 4.4.10. □

Chapter 5

Optimal Contour Choice

When pricing a European call option we usually have a financial market (B, S) with a time horizon T, where $S = (S_t)_{t\in[0,T]}$ denotes the risky asset and $B = (B_t)_{t\in[0,T]}$ is the riskless savings account, given by

$$B_t = e^{R_t}, \quad 0 \leq t \leq T,$$

where $R\colon [0, T] \to [0, +\infty)$ is absolutely continuous and increasing such that $R_0 = 0$. Under a martingale measure Q for the financial market (B, S) an arbitrage free price of a European call option at time $t = 0$ is given by

$$C_K = \mathbb{E}_Q\big(B_T^{-1}(S_T - K)^+\big),$$

where $K > 0$ denotes the strike price and $T > 0$ the maturity of the call option. Using the Fourier approach introduced in Chapter 3, in particular Corollary 3.4.1, computing the price $C_K = C(k)$ means to evaluate one the two integrals appearing in

$$C(k) = R_\alpha(k) + \frac{e^k}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{-kz}\frac{M_{X_T}(z)}{z(z-1)}\,dz = R_\alpha(k) + \frac{e^k}{\pi}\int_0^\infty \operatorname{Re}\left(e^{-k(\alpha+iu)}\frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha+iu-1)}\right)du. \tag{5.1}$$

These expressions for the European call option price make use of the *moment generating function*[1]

$$M_{X_T}(z) := \mathbb{E}_Q\big(e^{zX_T}\big), \quad z \in (u_-, u_+) + i\mathbb{R},$$

of the *log-discounted underlying* $X_T := \ln S_T - R_T$, the *log-discounted strike* $k = \ln K - R_T$, the *damping factor* $\alpha \in (u_-, u_+)$ and the *residue term* $R_\alpha(k)$, which is given by

$$R_\alpha(k) := \big(S_0 - e^k\big)\mathbb{1}_{(-\infty,0)}(\alpha) + \left(S_0 - \frac{e^k}{2}\right)\mathbb{1}_{\{0\}}(\alpha) + S_0\mathbb{1}_{(0,1)}(\alpha) + \frac{S_0}{2}\mathbb{1}_{\{1\}}(\alpha), \quad \alpha \in (u_-, u_+). \tag{5.2}$$

The careful reader might have noticed that up to now the only restriction on α was that it must belong to the interval (u_-, u_+). Even though for every possible α the presented equation for the call price $C(k)$ holds, the behavior of the integrand can heavily depend on the choice of α. The latter observation is not negligible as integrands which show heavy oscillations might make the numerical computation of the corresponding integral very demanding if not even impossible. This problem is illustrated and shortly discussed in Section 1 of this chapter. Next, observe that the interpretation of $C(k)$ as a Fourier transform quite naturally suggests the use of the Fast Fourier Transformation (FFT) to determine the call price numerically. However, the power of the FFT lies in the fact that the simultaneous evaluation of call prices associated with a certain

[1] Following Chapter 3 the interval $(u_-, u_+) \supseteq (0, 1]$ denotes the interior of all real numbers u where $\mathbb{E}\big(e^{uX_T}\big) < +\infty$.

grid of log-discounted strikes can be conducted very fast. In terms of precision the suggested FFT-method to evaluate (5.1) is not particularly superior. On the one hand, the integration domain $[0,+\infty)$ is truncated to a finite interval $[0,M]$ leading already to an error of the numerical approximation. On the other hand, the integral on the remaining interval is then computed using the trapezoidal rule. The latter method is usually considerably less accurate than standard Gaussian quadrature methods. This FFT-approach is discussed in Section 2 of this chapter.
In Section 3 we focus on the question which α one should or could choose for the numerical computation of $C(k)$. There we discuss the choice of α suggested by Lord and Kahl in [LK07] which basically means to minimize the maximum modulus of the integrand in the middle in (5.1). There we provide rigorous proofs for the statements made and eventually connect that choice of α to saddlepoints.
Finally, in Section 4 of this chapter we apply the obtained results in the Heston model. We start with the derivation of asymptotic results for the moment generating function when the imaginary part of the arguments becomes large. These are then used to actually compute call prices. On the one hand, this is done by means of Gauss-Laguerre quadrature where we exploit the fact that the integrand in (5.1) decays exponentially. On the other hand, we suitably transform the infinite interval $(0,+\infty)$ to the finite interval $(0,1)$, such that the integrand associated with the corresponding substitution has existing limits at 0 and 1.

5.1 Why Can the Choice of α Make a Difference?

From a numerical point of view one would intuitively like to avoid highly oscillatory integrands. For that purpose note that the right hand side of (5.1) is the same for all values α belonging to the interval (u_-,u_+). However, as the following example shows the behavior of the integrand can heavily depend on the choice of α. Therefore, it is sensible to think about which α one should choose.

Example 5.1.1. Consider a European call option with maturity $T=0.5$ and log-strike $k=\ln 10$ in the Heston model specified by the parameters

κ	η	ρ	θ	r	S_0	v_0
1	2	-0.99	1	0	10	5

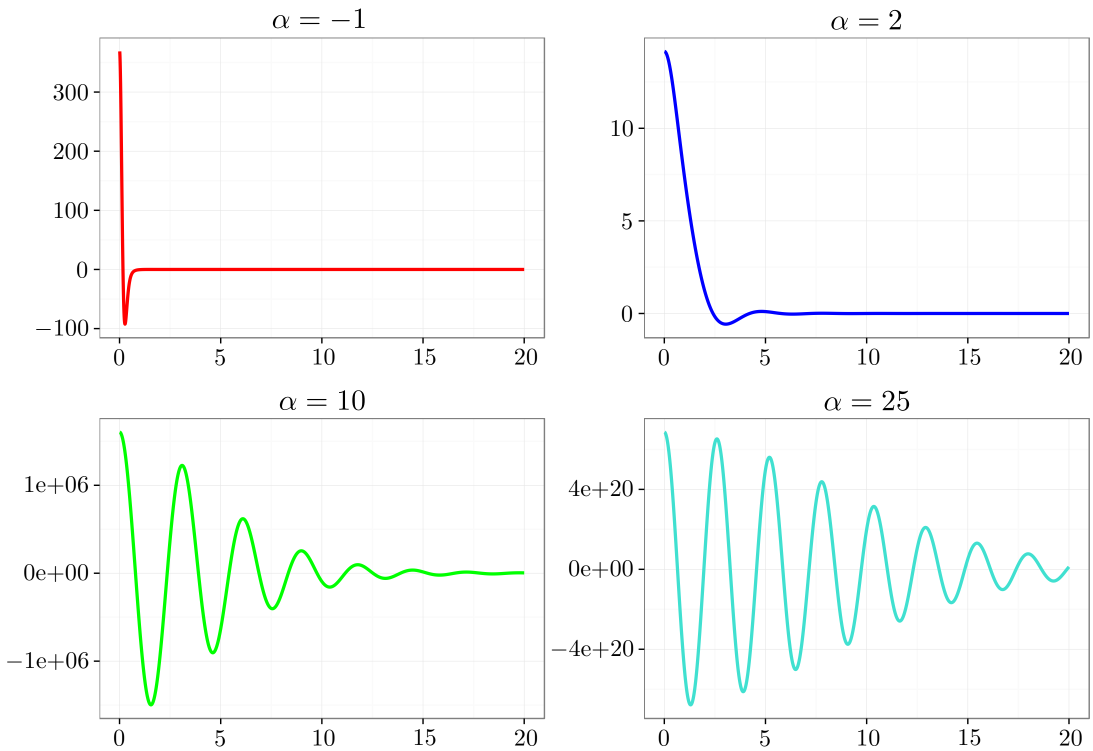
For the time horizon $T=0.5$ we obtain

$$u_-(T)=-1.885 \quad \text{and} \quad u_+(T)=116.556,$$

where we rounded to three digits when implementing this in R. Now recall that for $\alpha\in\big(u_-(T),u_+(T)\big)$ the function

$$u\mapsto \operatorname{Re}\left(e^{-k(\alpha+iu-1)}\frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha+iu-1)}\right),\quad u\in(0,+\infty),$$

represents the integrand used to describe the call price via the right hand side of (5.1). Different values of α lead to very different behaviors of that integrand as the following figure illustrates.



Note the different scales used in the plots to observe that on the one hand the integrand seems to behave very nicely when $\alpha = 2$ (blue graph). On the other hand, the integrand looks very oscillatory when $\alpha = 25$ (turquoise graph). Thus it is a priori quite unclear which values of α lead to a nice behavior of the integrand and thus make numerical integration easier.

5.2 Fast Fourier Transform Method

The first method used to evaluate the integrals associated with $C(k)$ via (5.1) is the Fast Fourier Transform Method (FFT-Method). For instance this is suggested by Carr and Madan in [CMS99] and Lee in [Lee04]. Before we introduce the FFT-method we need to define the discrete Fourier transform.

Definition 5.2.1. For a positive integer n let $x = (x_0, \dots, x_{n-1}) \in \mathbb{C}^n$. Then the discrete Fourier transform (DFT) of x is defined as the vector $y = (y_0, \dots, y_{n-1}) \in \mathbb{C}^n$, given by

$$y_l = \sum_{j=0}^{n-1} e^{-2\pi i \frac{jl}{n}} x_j, \quad l \in \{0, \dots, n-1\}.$$

The discrete Fourier transform of x is also denoted by $\mathcal{F}_n(x)$.

The following establishes a result regarding the DFT which eventually leads to the FFT-Method.

Theorem 5.2.2. *For $p \in \mathbb{N}$, $n = 2^p$ and $m = \frac{n}{2}$ let $\omega_n = e^{-2\pi i/n}$ be the n-th unit root. Define the permutation $\sigma_n \colon \mathbb{C}^n \to \mathbb{C}^n$ by*

$$\sigma_n(x) = (x_1, x_3, \dots, x_{n-1}, x_2, x_4, \dots, x_n).$$

With the vectors $a, b \in \mathbb{C}^{n/2}$, defined by

$$a_j = x_j + x_{j+m}, \quad b_j = (x_j - x_{j+m})\omega_n^{j-1}, \quad j \in \{1, \ldots, m\},$$

we have

$$\sigma_n\big(\mathcal{F}_n(x)\big) = \big(\mathcal{F}_m(a), \mathcal{F}_m(b)\big) \in \mathbb{C}^n.$$

In particular the evaluation of $\mathcal{F}_n$ can be reduced to evaluating $\mathcal{F}_{n/2}$ twice.

Proof. This is taken from a class on Numerical Analysis at the Vienna University of Technology, [Pra11]. ☐

Definition 5.2.3. When $n = 2^p$ for a positive integer p the recursive computation of F_n by means of Theorem 5.2.2 is referred to as *Fast Fourier Transform (FFT).*

Next, we want to connect the DFT to the computation of the integral in (5.1) for an equidistant sequence of log-discounted strikes. Before doing so we need to recall the trapezoidal rule as possibility to approximate integrals numerically.

Proposition 5.2.4. *Let $a, b \in \mathbb{R}$ and $f \in \mathcal{C}^2\big([a,b], \mathbb{C}\big)$. Define*

$$Q_1 f := (b-a)\frac{f(a)+f(b)}{2} \quad \text{and} \quad Qf := \int_a^b f(x)\,dx.$$

Then the estimate

$$\big|Q_1 f - Qf\big| \le \sqrt{2}\frac{(b-a)^3}{12}\left\|f''\right\|_\infty,$$

holds.

Proof. This is taken from a class on Numerical Analysis at the Vienna University of Technology, [Pra11]. ☐

With Proposition 5.2.4 one can derive the approximation formula T_n presented in the following proposition.

Proposition 5.2.5 (Trapezoidal Rule)**.** *Let $a, b \in \mathbb{R}$ such that $a < b$ and $f \in \mathcal{C}^2\big([a,b], \mathbb{C}\big)$. For a positive integer n define the nodes $\xi^n = \big(\xi_j^n\big)_{j=0,\ldots,n}$ by*

$$\xi_j^n = a + j\frac{b-a}{n}, \quad j = 0, \ldots, n.$$

Use the nodes ξ^n to define $T_n f$ by

$$T_n f := \frac{b-a}{2n}\left(f(a) + f(b) + 2\sum_{j=1}^{n-1} f\big(\xi_j^n\big)\right),$$

for $n \in \mathbb{N}$. Then we have the estimate

$$\left|T_n f - \int_a^b f(x)\,dx\right| \le \sqrt{2}\frac{(b-a)^3}{12n^2}\left\|f''\right\|_\infty.$$

In particular we have

$$\left|T_n f - \int_a^b f(x)\,dx\right| = \mathcal{O}\big(n^{-2}\big), \quad \text{for} \quad n \to +\infty.$$

Proof. The statements quite easily follow from Proposition 5.2.4. First observe

$$T_n f = \frac{b-a}{2n}\sum_{j=0}^{n-1}\Big(f\big(\xi_j^n\big)+f\big(\xi_{j+1}^n\big)\Big) = \sum_{j=0}^{n-1}\big(\xi_{j+1}^n-\xi_j^n\big)\frac{f\big(\xi_j^n\big)+f\big(\xi_{j+1}^n\big)}{2}.$$

Applying Proposition 5.2.4 leads to

$$\begin{aligned}\left|T_n f-\int_a^b f(x)\,dx\right| &= \left|T_n f-\sum_{j=0}^{n-1}\int_{\xi_j^n}^{\xi_{j+1}^n} f(x)\,dx\right| \le \sum_{j=0}^{n}\left|\big(\xi_{j+1}^n-\xi_j^n\big)\frac{f\big(\xi_j^n\big)+f\big(\xi_{j+1}^n\big)}{2}-\int_{\xi_j^n}^{\xi_{j+1}^n} f(x)\,dx\right|\\ &\le \sum_{j=0}^{n-1}\sqrt{2}\frac{\big(\xi_{j+1}^n-\xi_j^n\big)^3}{12}\left\|f''\right\|_\infty = \sqrt{2}\frac{(b-a)^3}{12n^2}\left\|f''\right\|_\infty,\end{aligned}$$

for $n\in\mathbb{N}$ and $f\in\mathcal{C}^2\big([a,b],\mathbb{C}\big)$. The big $\mathcal{O}$ statement of this proposition then directly follows from the definition

$$f(n)=\mathcal{O}\big(g(n)\big)\quad\text{for}\quad n\to+\infty\quad :\Leftrightarrow\quad \limsup_{n\to+\infty}\left|\frac{f(n)}{g(n)}\right|<+\infty.$$

□

The following result, based on the thoughts by Carr and Madan in [CMS99], brings us pretty close to a procedure to compute call prices for an equidistant grid of log-discounted strikes in a given interval $\big[\kappa_1,\kappa_2\big]$, where $\kappa_1\le\kappa_2$.

Proposition 5.2.6. *Consider a function $f\in L^1\big([0,+\infty)\big)\cap\mathcal{C}^2\big([0,+\infty)\big)$ such that f, f' and f'' are bounded on $[0,+\infty)$. For positive integers M and N define*

$$F_{M,N}(k) := \frac{M}{2N}\left(f(0)+e^{-ikM}f(M)+2\sum_{j=1}^{N-1}e^{-ik\frac{jM}{N}}f\Big(\tfrac{jM}{N}\Big)\right), \tag{5.3}$$

for every $k\in\mathbb{R}$. For $M,N\in\mathbb{N}$ and $k\in\mathbb{R}$ we then have

$$\left|F_{M,N}(k)-\int_0^\infty e^{-iuk}f(u)\,du\right|\le\int_M^\infty|f(u)|\,du+\sqrt{2}C_k\frac{M^3}{12N^2},$$

where

$$C_k = k^2\left\|f\right\|_\infty+2|k|\left\|f'\right\|_\infty+\left\|f''\right\|_\infty,\quad k\in\mathbb{R}. \tag{5.4}$$

Proof. For the proof define $f_k\colon[0,+\infty)\to\mathbb{R}$ by

$$f_k(u)=e^{-iuk}f(u),\quad u\in[0,+\infty),$$

for every $k\in\mathbb{R}$ and $(u_j)_{j=0,\dots,N}$ by

$$u_j=\frac{jM}{N},\quad 0\le j\le N.$$

Note that the sequence $(u_j)_{j=0,\dots,N}$ depends on M and N even though this is not indicated by the notation used. Recalling the trapezoidal rule introduced in Proposition 5.2.5 we clearly have

$$T_N f_k = F_{M,N}(k),$$

where T_N is associated with the interval $[0, M]$. Thus by applying Proposition 5.2.5 we get the estimate

$$\left|F_{M,N}(k) - \int_0^M e^{-iuk} f(u)\, du\right| = \left|T_N f_k - \int_0^M f_k(u)\, du\right| \le \sqrt{2}\frac{M^3}{12N^2} \left\| f_k''|_{[0,M]} \right\|_\infty \tag{5.5}$$

Because of

$$\left|f_k''(u)\right| = \left| - k^2 e^{-iuk} f(u) - 2ike^{-iuk} f'(u) + e^{-iuk} f''(u)\right| \le k^2 \left|f(u)\right| + 2|k| \left|f'(u)\right| + \left|f''(u)\right|,$$

for $u \in \mathbb{R}$, we consequently obtain

$$\left|F_{M,N}(k) - \int_0^M e^{-iuk} f(u)\, du\right| \le \sqrt{2}\frac{M^3}{12N^2}\Big(k^2 \left\|f\right\|_\infty + 2|k| \left\|f'\right\|_\infty + \left\|f''\right\|_\infty\Big) = \sqrt{2}\frac{M^3}{12N^2} C_k,$$

for every $M, N \in \mathbb{N}$ and $k \in \mathbb{R}$. Note that due to the finiteness assumed for $\|f\|_\infty$, $\|f'\|_\infty$ and $\|f''\|_\infty$ the constant C_k is finite and independent of M. A simple application of the triangle inequality finally gives

$$\left|F_{M,N}(k) - \int_0^\infty e^{-iuk} f(u)\, du\right| - \int_M^\infty |f(u)|\, du \le \left|F_{M,N}(k) - \int_0^M e^{-iuk} f(u)\, du\right| \le \sqrt{2}\frac{M^3}{12N^2} C_k.$$

□

Now we want to apply Proposition 5.2.6 when pricing a European call option. Therefore we introduce the notation

$$\zeta(u, \alpha) = \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu)(\alpha + iu - 1)}, \quad u \in \mathbb{R}, \tag{5.6}$$

and

$$F_\alpha^{M,N}(k) = R_\alpha(k) + \frac{e^{-k(\alpha-1)}}{\pi}\frac{M}{2N} \operatorname{Re}\left(\zeta(0, \alpha) + e^{-ikM}\zeta(M, \alpha) + 2\sum_{j=1}^{N-1} e^{-ik\frac{jM}{N}}\zeta\Big(\tfrac{jM}{N}, \alpha\Big)\right), \quad k \in \mathbb{R}, \tag{5.7}$$

for every $\alpha \in \big(u_-(T), u_+(T)\big) \setminus \{0, 1\}$.

Lemma 5.2.7. *Assume that* $\alpha \in \big(u_-(T), u_+(T)\big) \setminus \{0, 1\}$ *and consider* ζ *as defined in* (5.6). *Then* $\zeta(\cdot, \alpha) \in L^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R})$ *such that* $\zeta(\cdot, \alpha)$, $\zeta'(\cdot, \alpha)$ *and* $\zeta''(\cdot, \alpha)$ *are bounded on* $\mathbb{R}$.

Proof. See Subsection A.3.1 in the appendix. □

Proposition 5.2.8. *For positive integers* M, N *and* $\alpha \in \big(u_-(T), u_+(T)\big) \setminus \{0, 1\}$ *consider* $F_\alpha^{M,N}$ *as defined in* (5.7). *Furthermore, let* $[\kappa_1, \kappa_2]$ *be a given interval of log-discounted strikes. Recalling the price* $C(k)$ *of a European call option with maturity* $T > 0$ *and log-discounted strike* $k \in \mathbb{R}$, *given by* (5.1), *we have*

$$\sup_{k \in [\kappa_1, \kappa_2]} \left|F_\alpha^{M,N}(k) - C(k)\right| \le \gamma_\alpha \int_M^\infty \left|\zeta(u, \alpha)\right| du + \mathcal{O}\left(\frac{M^3}{N^2}\right), \quad \text{for} \quad M, N \to +\infty,$$

where

$$\gamma_\alpha = \frac{e^{(1-\alpha)\kappa_2}}{\pi} \mathbb{1}_{(-\infty,1]}(\alpha) + \frac{e^{(1-\alpha)\kappa_1}}{\pi} \mathbb{1}_{(1,+\infty)}(\alpha).$$

Proof. For a fixed but arbitrary α define f_α by

$$f_\alpha(u) = \zeta(u, \alpha), \quad u \ge 0.$$

Because of Lemma 5.2.7 we know that f_α satisfies the assumptions of Proposition 5.2.6. The $F_{M,N}$ from (5.3) we associate with f_α satisfies

$$R_\alpha(k) + \frac{e^{-k(\alpha-1)}}{\pi} \operatorname{Re}\Big(F_{M,N}(k)\Big) = F_\alpha^{M,N}(k),$$

for every $k \in \mathbb{R}$. Using (5.1) and Proposition 5.2.6 we then get

$$\begin{aligned}
\left|F_\alpha^{M,N}(k) - C(k)\right| &= \left|\frac{e^{-k(\alpha-1)}}{\pi} \operatorname{Re}\Big(F_{M,N}(k)\Big) - \frac{e^k}{\pi}\int_0^\infty \operatorname{Re}\left(e^{-k(\alpha+iu)} \frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha+iu-1)}\right) du\right| \\
&\leq \frac{e^{-k(\alpha-1)}}{\pi}\left|F_{M,N}(k) - \int_0^\infty e^{-iku} \frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha+iu-1)}\, du\right| \\
&= \frac{e^{-k(\alpha-1)}}{\pi}\left|F_{M,N}(k) - \int_0^\infty e^{-iku}\zeta(u,\alpha)\, du\right| \\
&\leq\ = \frac{e^{-k(\alpha-1)}}{\pi}\left(\int_M^\infty |\zeta(u,\alpha)|\, du + \sqrt{2}C_k \frac{M^3}{12N^2}\right),
\end{aligned}$$

where C_k taken from (5.4) is in this case given by

$$C_k = k^2 \left\|\zeta(\cdot,\alpha)\right\|_\infty + 2|k| \left\|\zeta'(\cdot,\alpha)\right\|_\infty + \left\|\zeta''(\cdot,\alpha)\right\|_\infty, \quad k \in \mathbb{R}.$$

Since C_k continuously depends on $k \in \mathbb{R}$ there exsits a $\gamma > 0$ such that

$$e^{-k(\alpha-1)}C_k \leq \gamma, \quad k \in [\kappa_1, \kappa_2].$$

This clearly gives the estimate

$$\left|F_\alpha^{M,N}(k) - C(k)\right| \leq \gamma_\alpha \int_M^\infty |\zeta(u,\alpha)|\, du + \frac{\sqrt{2}\gamma}{12\pi}\frac{M^3}{N^2}, \quad k \in [\kappa_1, \kappa_2],$$

which implies the statement. □

Remark 5.2.9. When we consider the subsequence F_α^{M,M^2} in Proposition 5.2.8 we particularly obtain

$$\sup_{k\in[\kappa_1,\kappa_2]} \left|F_\alpha^{M,M^2}(k) - C(k)\right| \leq \gamma_\alpha \int_M^\infty \big|\zeta(u,\alpha)\big|\, du + \mathcal{O}\big(M^{-1}\big), \quad \text{for} \quad M \to +\infty.$$

With the estimate

$$\int_M^\infty \big|\zeta(u,\alpha)\big|\, du \leq \int_M^\infty \frac{M_{X_T}(\alpha)}{u^2}\, du = \frac{M_{X_T}(\alpha)}{M},$$

for $\alpha \in \big(u_-(T), u_+(T)\big) \setminus \{0,1\}$, one even gets

$$\sup_{k\in[\kappa_1,\kappa_2]} \left|F_\alpha^{M,M^2}(k) - C(k)\right| \leq \mathcal{O}\big(M^{-1}\big), \quad \text{for} \quad M \to +\infty,$$

when $\alpha \in \big(u_-(T), u_+(T)\big) \setminus \{0,1\}$.

For a certain grid of log-discounted strikes the sum in (5.7) is just the discrete Fourier transform of a certain sequence of complex numbers. This is made more precisely in the following corollary.

Corollary 5.2.10. *For $\alpha \in \big(u_-(T), u_+(T)\big) \setminus \{0,1\}$ and positive integers M, N consider $F_\alpha^{M,N}$ as defined in* (5.7)*. In addition let $[\kappa_1, \kappa_2]$ be a given interval of log-discounted strikes and define $(k_l^{M,N})_{l\in\mathbb{Z}}$ by*

$$k_l^{M,N} = \frac{\pi}{M}(2l - N + 1), \quad l \in \mathbb{N}_0.$$

Then $F_\alpha^{M,N}\left(k_l^{M,N}\right)$ *equals*

$$R_\alpha\left(k_l^{M,N}\right) + \frac{e^{-(\alpha-1)k_l^{M,N}}}{\pi}\frac{M}{2N}\operatorname{Re}\left(\zeta(0,\alpha) + (-1)^{N-1}\zeta(M,\alpha) + 2\sum_{j=1}^{N-1} e^{-2\pi i\frac{jl}{N}} e^{i\pi j\frac{N-1}{N}}\zeta\left(\tfrac{jM}{N},\alpha\right)\right), \quad (5.8)$$

for $l \in \mathbb{N}_0$ *and*

$$\max_{\substack{k_l^{M,N}\in[k_-,k_+]\\ l=0,\ldots,N-1}} \left|F_\alpha^{M,N}\left(k_l^{M,N}\right) - C\left(k_l^{M,N}\right)\right| \le \gamma_\alpha \int_M^\infty |\zeta(u,\alpha)|\,du + \mathcal{O}\left(\frac{M^3}{N^2}\right), \quad \text{for} \quad M,N \to +\infty,$$

where

$$\gamma_\alpha = \frac{e^{(1-\alpha)\kappa_2}}{\pi}\mathbb{1}_{(-\infty,1]}(\alpha) + \frac{e^{(1-\alpha)\kappa_1}}{\pi}\mathbb{1}_{(1,+\infty)}(\alpha).$$

Proof. With the definition of $F_\alpha^{M,N}$ in (5.7) the observations

$$e^{-ik_l^{M,N}M} = e^{-i\pi(2l-N+1)} = (-1)^{N-1} \qquad \text{and} \qquad e^{-ik_l^{M,N}\frac{jM}{N}} = e^{-2\pi i\frac{jl}{N}} e^{i\pi j\frac{N-1}{N}},$$

for $M,N \in \mathbb{N}$ and $l,j \in \mathbb{Z}$, lead to $F_\alpha^{M,N}\left(k_l^{M,N}\right)$ being equal to

$$R_\alpha\left(k_l^{M,N}\right) + \frac{e^{-(\alpha-1)k_l^{M,N}}}{\pi}\frac{M}{2N}\operatorname{Re}\left(\zeta(0,\alpha) + (-1)^{N-1}\zeta(M,\alpha) + 2\sum_{j=1}^{N-1} e^{-2\pi i\frac{jl}{N}} e^{i\pi j\frac{N-1}{N}}\zeta\left(\tfrac{jM}{N},\alpha\right)\right),$$

which is the first statement of the corollary. The stated estimate then directly follows from Proposition 5.2.8. ☐

Corollary 5.2.10 makes it possibly to evaluate call prices for an equidistant grid of log-discounted strikes using the FFT. Therefore, note that the sum in (5.8) is exactly of the form that is considered in Definition 5.2.1. This is of particular use if one seeks to evaluate call prices for different log-discounted strikes simultaneously. In order to to be able to apply the FFT these log-discounted strikes must lie on the equidistant grid defined in Corollary 5.2.10. Note however that the trapezoidal rule as underlying numerical integration procedure is usually not as efficient as Gaussian quadrature methods (regarding the degree of accuracy for example). Nevertheless the following recipe can be used for the evaluation of European call option prices associated with the log-discounted strikes lying on the equidistant grid described in Corollary 5.2.10.

1. Let $\alpha \in (u_-,u_+)$ be such that $\alpha \notin \{0,1\}$. Next, using for example the estimates obtained in Corollary 5.2.10, choose positive integers M and $N = 2^p$, $p \in \mathbb{N}$, sufficiently large. That particularly means that $\frac{M^3}{N^2}$ should be close to zero.
2. Use the function ζ from (5.6) to define $x_\alpha \in \mathbb{C}^N$ by

$$x_{\alpha,j} = \begin{cases} \zeta(0,\alpha) & j=0 \\ 2e^{i\pi j\frac{N-1}{N}}\zeta\left(\frac{jM}{N},\alpha\right) & 1 \le j \le N-1 \end{cases}.$$

3. Compute the discrete Fourier transform $\mathcal{F}_N(x_\alpha)$ using the FFT-method presented in Theorem 5.2.2.
4. By Corollary 5.2.10 we then have the approximation

$$C\left(\tfrac{(2l-N+1)\pi}{M}\right) \approx R_\alpha\left(\tfrac{(2l-N+1)\pi}{M}\right) + e^{-\frac{(2l-N+1)\pi}{M}(\alpha-1)}\frac{1}{2\pi}\frac{M}{N}\operatorname{Re}\left(\left(\mathcal{F}_N(x_\alpha)\right)_l - \zeta(M,\alpha)\right),$$

for $l = 0,\ldots,N-1$.

Note that in the recipe above it remains open how to choose α. The following example implements the introduced recipe in `R`.

Example 5.2.11. We again consider a call option with maturity $T = 0.5$ and log-strike $k = \ln K$ in the Heston model specified by the parameters

κ	η	ρ	θ	r	S_0	v_0
1	2	-0.99	1	0	10	5

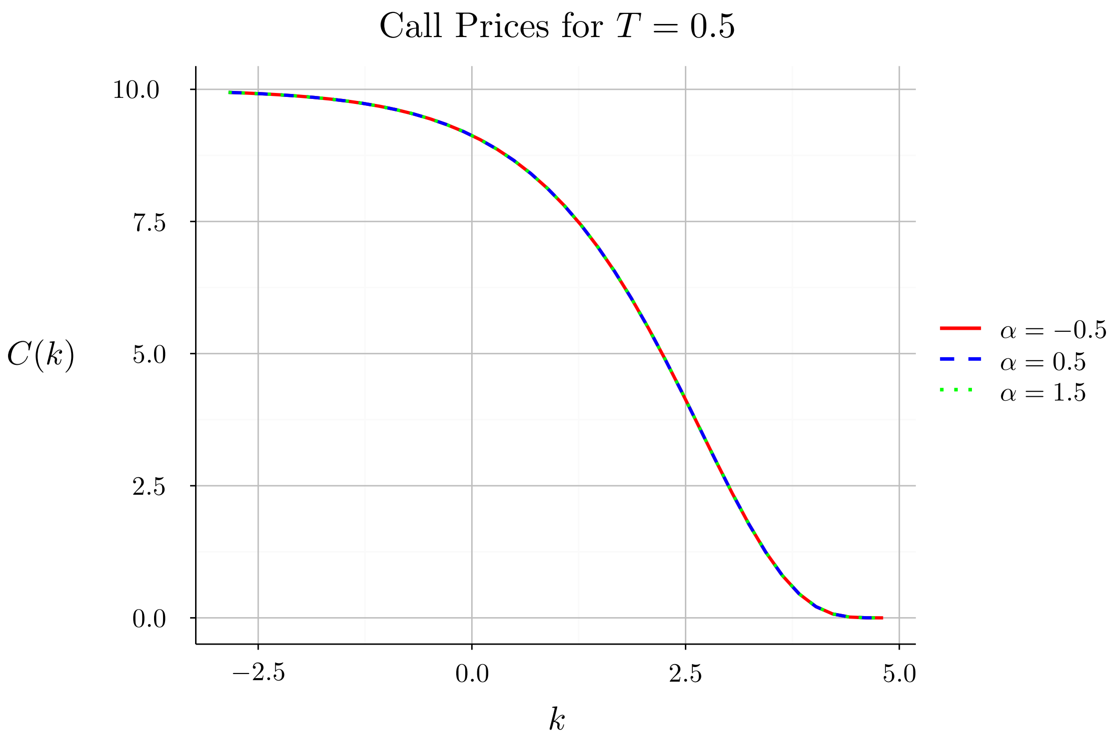
Recall from Example 5.1.1 that

$$u_-(T) = -1.885 \quad \text{and} \quad u_+(T) = 116.556,$$

for the time horizon $T = 0.5$. We implementing the presented recipe in `R` with

- ▷ $\alpha \in \{-0.5, 0.5, 1.5\}$
- ▷ $M = 32$ and $N = 2^{10}$.

A visualization of the results yields the following for the call price $C(k)$ as function of the log-discounted strike k.



We observe that the results are very similar for the different values of α considered. Furthermore, note that the distance between two neighboring log-discounted strikes $k_l^{M,N}$ and $k_{l+1}^{M,N}$ is given by

$$k_{l+1}^{M,N} - k_l^{M,N} = \frac{\pi}{M}(2l + 2 - N + 1) - \frac{\pi}{M}(2l - N + 1) = \frac{2\pi}{M}.$$

Hence M determines the spacing in between the log-discounted strikes we compute call prices for. In particular with $M = 32$ we have a spacing of

$$\frac{\pi}{16} \approx 0.1963.$$

In order to obtain a finer grid of log-discounted strikes we must increase M. But then we have to take into account the following two considerations.

(i) Note that an increase in M causes the the first log-discounted strike $k_0^{M,N}$ to rise and the last log-discounted strike $k_{N-1}^{M,N}$ to fall when holding N constant. Because of

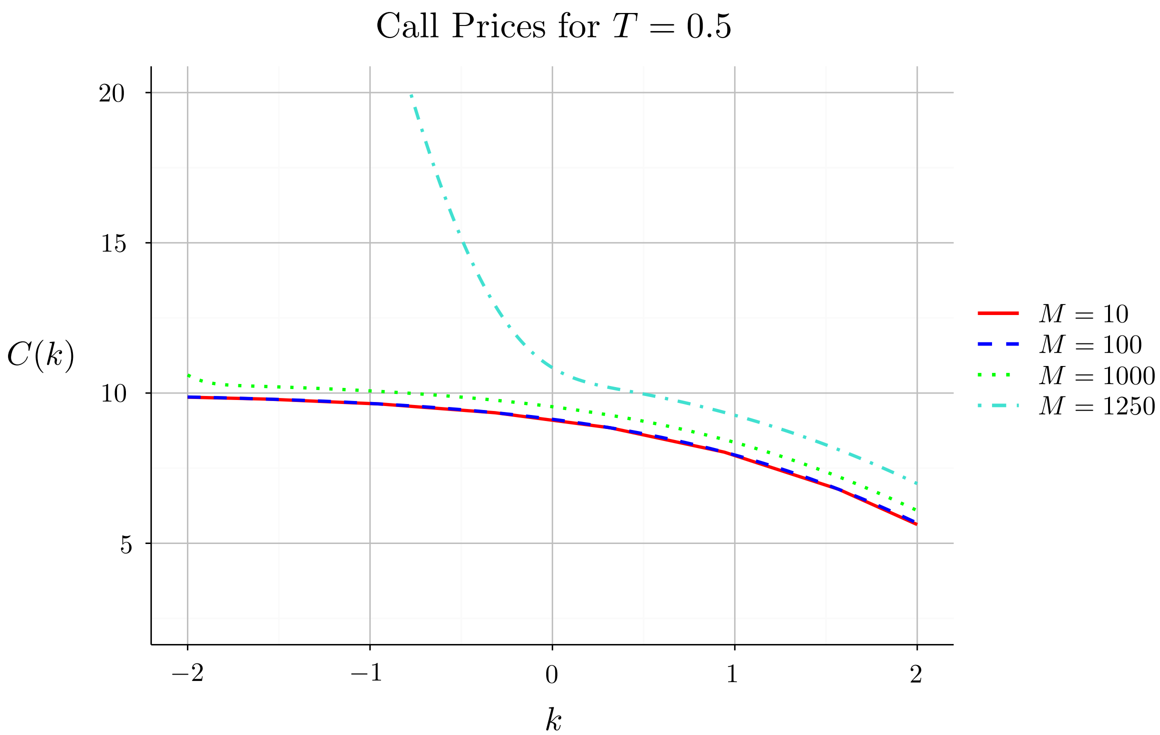
$$k_0^{M,N} = -k_{N-1}^{M,N} = -\frac{\pi}{M}(N-1),$$

one should thus increase N at least as much as M in order to ensure that the range between minimum and the maximum log-discounted strike stays the same.

(ii) In the error estimate derived in Corollary 5.2.10 we have a $\mathcal{O}\big(\frac{M^3}{N^2}\big)$-term. Consequently when increasing M one should also increase N in order to keep the error term small. The increase in N should be even greater than in M as it is desirable that $\frac{M^3}{N^2}$ is close to zero. One possibility is to always choose $N = M^2$ as the error term is then definitely at least $\mathcal{O}\big(M^{-1}\big)$. However, in order to compute the call prices of $M+1$ equidistant strikes in the interval $[-\pi, \pi]$ (or equivalently log-discounted strikes with spacing $2\pi M^{-1}$) we need the Fourier transform of a vector of length $N = M^2$. But then for $M+1$ equidistant strikes in the interval $[-\pi, \pi]$ the number of operations is of order

$$\mathcal{O}(N \ln N) = \mathcal{O}(M^2 \ln M^2) = \mathcal{O}(M^2 \ln M).$$

Finally we want to illustrate that things can indeed go wrong when increasing M without adapting N. With $N = 2^{10}$, $\alpha = 1.5$ and $T = 0$ we obtain the following graph.



Because of $S_0 = 10$ we clearly see that the green and turquoise line cannot represent sensible call prices.

As the previous example has shown the FFT-method offers a nice way to compute call prices for different strikes simultaneously. However, it only allows for log-discounted strikes k lying on an equidistant grid. Moreover, a finer grid on a given interval $[\kappa_1, \kappa_2]$ of log-discounted strikes requires also a higher order N for the trapezoidal rule. Unfortunately that thwarts the efficiency of the FFT-procedure as the number of buckets used for the numerical integration determines the length of the vector we apply the FFT to. Furthermore, standard Gaussian quadrature methods are usually considerably more accurate than the simple trapezoidal rule used in the FFT-method. Therefore, we will also discuss other methods to evaluate the desired integrals in (5.1) numerically. Before we do so we address the issue of which α one should choose for the numerical integration. This question already arises in the first step of the recipe introduced in this section.

5.3 Choice of α Suggested by Lord and Kahl

In Example 5.1.1 we have seen that the integrand in (5.1) can behave very differently for different values of α. In this section we discuss the suggestion by Lord and Kahl in [LK07] how one could choose the damping parameter α.

Throughout this section we consider the log-discounted underlying X_T whose moment generating function is denoted by M_{X_T}. Furthermore, define the associated critical moments $\alpha_- \in [-\infty, 0]$ and $\alpha_+ \in [0, +\infty]$ by

$$\alpha_- := \inf\left\{\alpha \leq 0 \colon \mathbb{E}\left(e^{\alpha X_T}\right) < +\infty\right\} \quad \text{and} \quad \sup\left\{\alpha \geq 0 \colon \mathbb{E}\left(e^{\alpha X_T}\right) < +\infty\right\}. \tag{5.9}$$

With Proposition 3.1.5 we see that

$$\mathbb{E}\left(e^{\alpha X_T}\right) < +\infty, \quad \alpha \in (\alpha_-, \alpha_+),$$

must hold for.

For a given log-discounted strike k Lord and Kahl simply suggest to choose α_k^* from the interval $[\alpha_-, \alpha_+] \cap \mathbb{R}$, such that the maximal modulus of the integrand in

$$C(k) = R_\alpha + \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-k(\alpha + iu - 1)} \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu)(\alpha + iu - 1)} \, du, \tag{5.10}$$

is minimized. Under the assumption

$$\begin{aligned} P(X_T < k) > 0 \quad &\text{if} \quad \alpha_- = -\infty \quad \text{and} \\ P(X_T > k) > 0 \quad &\text{if} \quad \alpha_+ = +\infty, \end{aligned}$$

it turns out that this is equivalent to choosing α_k^* such that it solves

$$\min_{\alpha \in [\alpha_-, \alpha_+] \cap \mathbb{R} \setminus \{0,1\}} \ln M_{X_T}(\alpha) - \frac{1}{2} \ln \alpha^2 (\alpha - 1)^2 - \alpha k.$$

Before we can establish the results we have just indicated we need the following lemma that presents a simpler expression for the maximal modulus of the integrand in (5.10).

Lemma 5.3.1. *For a log-discounted underlying X_T with moment generating function M_{X_T} let $\alpha \in \mathbb{R}$ be such that $M_{X_T}(\alpha) < +\infty$. If $\alpha \notin \{0, 1\}$ holds we have*

$$\sup_{u \in (0, +\infty)} \left| e^{-k(\alpha + iu - 1)} \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu)(\alpha + iu - 1)} \right| = e^{-k(\alpha - 1)} \frac{M_{X_T}(\alpha)}{|\alpha||\alpha - 1|}, \tag{5.11}$$

and if $\alpha \in \{0, 1\}$ we have

$$\sup_{u \in (0, +\infty)} \left| e^{-k(\alpha + iu - 1)} \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu)(\alpha + iu - 1)} \right| = +\infty.$$

Proof. First assume $\alpha \notin \{0,1\}$. Then we clearly have

$$\left| e^{-k(\alpha+iu-1)} \frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha+iu-1)} \right| \leq e^{-k(\alpha-1)} \frac{M_{X_T}(\alpha)}{|\alpha||\alpha-1|}.$$

Since M_{X_T} is continuous on its domain $(\alpha_-, \alpha_+) + i\mathbb{R}$ we get

$$\lim_{u \searrow 0} \left| e^{-k(\alpha+iu-1)} \frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha+iu-1)} \right| = e^{-k(\alpha-1)} \frac{M_{X_T}(\alpha)}{|\alpha||\alpha-1|},$$

and hence the first equation stated in this lemma holds. If $\alpha \in \{0,1\}$ we have $\alpha(\alpha-1) = 0$ which leads to

$$\lim_{u \searrow 0} e^{-k(\alpha+iu-1)} \left| \frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha+iu-1)} \right| = e^{-k(\alpha-1)} M_{X_T}(\alpha) \lim_{u \searrow 0} \frac{1}{\left|(\alpha+iu)(\alpha+iu-1)\right|} = +\infty,$$

and thus also the second equality stated in this lemma is true. $\square$

As we have already indicated the aim will be to find an α_k^* that minimizes the expression found in (5.11). The following lemma turns out to be useful for the analysis of that expression.

Lemma 5.3.2. *For $\mu \in \mathbb{R}$ define the function f_μ by*

$$f_\mu(x) = \frac{e^{\mu x}}{\left|x^2 - x\right|}, \quad x \in \mathbb{R} \setminus \{0,1\}.$$

Then

$$f_\mu''(x) > 0, \quad x \in \mathbb{R} \setminus \{0,1\}.$$

Hence on each of the intervals $(-\infty, 0)$, $(0,1)$ and $(1, +\infty)$ the function f_μ is strictly convex.

Proof. See Subsection A.3.2 in the appendix. $\square$

Before we can prove the main result of this section we have to prove the following important lemma.

Lemma 5.3.3. *For a log-discounted underlying X_T with moment generating function M_{X_T} consider the interval $[\alpha_-, \alpha_+]$, whose boundaries are defined by (5.9). If at least one of the boundaries is not finite additionally assume*

$$\begin{aligned} P(X_T < k) > 0 \quad &\text{if} \quad \alpha_- = -\infty \quad \text{and} \\ P(X_T > k) > 0 \quad &\text{if} \quad \alpha_+ = +\infty. \end{aligned}$$

Furthermore, let $A \subseteq [\alpha_-, \alpha_+] \setminus \{0,1\}$ be one of the three intervals listed below.

- $[\alpha_-, 0) \cap (-\infty, 0)$
- $(0, \alpha_+] \cap (0,1)$
- $(1, \alpha_+] \cap (1, +\infty)$

Then the minimization problem

$$\min_{\alpha \in A} e^{-k(\alpha-1)} \frac{M_{X_T}(\alpha)}{|\alpha||\alpha-1|}. \tag{5.12}$$

has a strictly convex objective function and a unique solution in A exists.

Proof. First note that for each interval A and $\omega \in \Omega$ the function $f_{X_T(\omega)-k}$, defined by

$$f_{X_T(\omega)-k}(\alpha) = \frac{e^{(X_T(\omega)-k)\alpha}}{|\alpha||\alpha-1|} \quad \alpha \in A,$$

is strictly convex by Lemma 5.3.2. Consequently for each $\lambda \in (0,1)$ and $\alpha_1, \alpha_2 \in A$ one has

$$f_{X_T(\omega)-k}\big(\lambda\alpha_1 + (1-\lambda)\alpha_2\big) < \lambda f_{X_T(\omega)-k}(\alpha_1) + (1-\lambda) f_{X_T(\omega)-k}(\alpha_2).$$

Taking expectations on both sides yields

$$e^{-k(\lambda\alpha_1+(1-\lambda)\alpha_2)} \frac{M_{X_T}\big(\lambda\alpha_1 + (1-\lambda)\alpha_2\big)}{|\lambda\alpha_1 + (1-\lambda)\alpha_2||\lambda\alpha_1 + (1-\lambda)\alpha_2 - 1|} < \lambda e^{-k\alpha_1} \frac{M_{X_T}(\alpha_1)}{|\alpha_1||\alpha_1-1|} + (1-\lambda)e^{-k\alpha_2} \frac{M_{X_T}(\alpha_2)}{|\alpha_2||\alpha_2-1|},$$

for $\lambda \in (0,1)$ when $\alpha_1, \alpha_2 \in A$. Multiplying the latter inequality by the positive factor e^k then shows that the objective function J, given by

$$J(\alpha) = e^{-k(\alpha-1)} \frac{M_{X_T}(\alpha)}{|\alpha||\alpha-1|}, \quad \alpha \in A, \tag{5.13}$$

is strictly convex. Since strictly convex functions have at most exactly one minimum it remains to show that the objective function has a minimum on each of the possible choices for A. Therefore distinguish the following cases.

▷ $A = [\alpha_-, 0) \cap (-\infty, 0)$

The statement is only non-trivial if $\alpha_- < 0$. Then we always have

$$\lim_{\alpha \nearrow 0} J(\alpha) = +\infty. \tag{5.14}$$

In order to complete the argument distinguish the following three subcases.

(i) $\alpha_- > -\infty$ and $M_{X_T}(\alpha_-) = +\infty$

By Fatou's lemma we get

$$\liminf_{\alpha \searrow \alpha_-} M_{X_T}(\alpha) = \liminf_{\alpha \searrow \alpha_-} \mathbb{E}\big(e^{\alpha X_T}\big) \geq \mathbb{E}\big(\liminf_{\alpha \searrow \alpha_-} e^{\alpha X_T}\big) = \mathbb{E}\big(e^{\alpha_- X_T}\big) = +\infty.$$

This implies that the objective function J satisfies

$$\lim_{\alpha \searrow \alpha_-} J(\alpha) = +\infty.$$

Together with (5.14) the continuity of J on $(\alpha_-, 0) \subseteq A$ then guarantees the existence of a minimum in $(\alpha_-, 0)$. Because of $M(\alpha_-) = +\infty$ this is also a minimum in $[\alpha_-, 0)$.

(ii) $\alpha_- > -\infty$ and $M_{X_T}(\alpha_-) < +\infty$

Then J is continuous on $[\alpha_-, 0)$. Together with (5.14) this observation leads to the existence of a minimum of J in $[\alpha_-, 0)$.

(iii) $\alpha_- = -\infty$

Fatou's lemma and the assumption $P(X_T < k) > 0$ lead to

$$\liminf_{\alpha \searrow \alpha_-} e^{-k} J(\alpha) \geq \mathbb{E}\left(\liminf_{\alpha \searrow \alpha_-} \frac{e^{(X_T-k)\alpha}}{|\alpha(\alpha-1)|}\right) \geq \mathbb{E}\left(\mathbb{1}_{\{X_T<k\}} \liminf_{\alpha \searrow \alpha_-} \frac{e^{(X_T-k)\alpha}}{|\alpha(\alpha-1)|}\right) = +\infty.$$

Consequently the objective function J satisfies

$$\lim_{\alpha \searrow \alpha_-} J(\alpha) = +\infty.$$

Together with (5.14) the continuity of J on $(\alpha_-, 0) \subseteq A$ implies the existence of a minimum in $(\alpha_-, 0)$.

▷ $A = (0, \alpha_+] \cap (0, 1)$

Here the statement is only non-trivial if $\alpha_+ > 0$. In this case we have

$$\lim_{\alpha \searrow 0} J(\alpha) = +\infty. \tag{5.15}$$

Furthermore, we distinguish the following three subcases.

(i) $M_{X_T}(1) < +\infty$ and $\alpha_+ \geq 1$
Then one clearly has $A = (0, 1)$ and

$$\lim_{\alpha \nearrow 1} J(\alpha) = \lim_{\alpha \nearrow 1} e^{-k(\alpha-1)} \frac{M_{X_T}(\alpha)}{|\alpha||\alpha - 1|} = M_{X_T}(1) \lim_{\alpha \nearrow 1} \frac{1}{|\alpha - 1|} = +\infty.$$

Using (5.15) and the continuity of J we see that the objective function has a minimum in $(0, 1)$.

(ii) $M_{X_T}(\alpha_+) = +\infty$ and $\alpha_+ \leq 1$
Then we have $(0, \alpha_+) \subseteq A \subseteq (0, \alpha_+]$. Because of $x(1 - x) \leq \frac{1}{4}$ for $x \in [0, 1]$ and Fatou's lemma we also get

$$\liminf_{\alpha \nearrow \alpha_+} J(\alpha) \geq 4e^{-|k|} \liminf_{\alpha \nearrow \alpha_+} M_{X_T}(\alpha) \geq 4e^{-|k|} \mathbb{E}\Big(\liminf_{\alpha \nearrow \alpha_+} e^{\alpha X_T} \Big) = +\infty.$$

Combining that with (5.15) and the continuity of J shows that J has a minimum in $(0, \alpha_+)$.

(iii) $M_{X_T}(\alpha_+) < +\infty$ and $\alpha_+ < 1$
Then we have $A = (0, \alpha_+]$. Since J is also continuous on $(0, \alpha_+]$ the observation in (5.15) implies the existence of a minimum in $(0, \alpha_+]$.

▷ $A = (1, \alpha_+] \cap (1, +\infty)$

Again the statement is trivial if $\alpha_+ = 1$ and hence we assume $\alpha_+ > 1$. Then we always have

$$\lim_{\alpha \searrow 1} J(\alpha) = +\infty. \tag{5.16}$$

Similarly to the cases before we distinguish the following three subcases.

(i) $\alpha_+ < +\infty$ and $M_{X_T}(\alpha_+) = +\infty$
Using Fatou's lemma we get

$$\liminf_{\alpha \nearrow \alpha_+} M_{X_T}(\alpha) = \liminf_{\alpha \nearrow \alpha_+} \mathbb{E}\big(e^{\alpha X_T}\big) \geq \mathbb{E}\big(\liminf_{\alpha \nearrow \alpha_+} e^{\alpha X_T} \big) = \mathbb{E}\big(e^{\alpha_+ X_T}\big) = +\infty.$$

Consequently we have

$$\lim_{\alpha \nearrow \alpha_+} J(\alpha) = +\infty.$$

As also (5.16) holds the continuity of J on $(1, \alpha_+) \subseteq A$ implies that a minimum of J exists in $(1, \alpha_+)$.

(ii) $\alpha_+ < +\infty$ and $M_{X_T}(\alpha_+) < +\infty$
Then J is continuous on $(1, \alpha_+]$. Together with (5.16) that observation leads to the existence of a minimum of J in $(1, \alpha_+]$.

(iii) $\alpha_+ = +\infty$
With Fatou's lemma and because of the assumption $P(X_T > k) > 0$ we obtain

$$\liminf_{\alpha \nearrow \alpha_+} e^{-k} J(\alpha) \geq \mathbb{E}\bigg(\liminf_{\alpha \nearrow \alpha_+} \frac{e^{(X_T - k)\alpha}}{|\alpha(\alpha - 1)|} \bigg) \geq \mathbb{E}\bigg(\mathbb{1}_{\{X_T > k\}} \liminf_{\alpha \nearrow \alpha_+} \frac{e^{(X_T - k)\alpha}}{|\alpha(\alpha - 1)|} \bigg) = +\infty.$$

Hence the objective function J satisfies

$$\lim_{\alpha \nearrow \alpha_+} J(\alpha) = +\infty.$$

Together with (5.14) the continuity of J on $(1, \alpha_+) \subseteq A$ implies the existence of a minimum in $(1, \alpha_+)$.

Thus for every choice of A a minimum exists for J. Due to the strict convexity on A we observe that such a minimum is also unique which concludes the proof. $\square$

Now observe that for every $k \in \mathbb{R}$ the function

$$x \mapsto \ln x - k, \quad x \in (0, +\infty),$$

is strictly increasing. Applying this transformation to the results obtained in this section yields the following proposition.

Proposition 5.3.4. *For a log-discounted underlying X_T with moment generating function M_{X_T} let α_+ and α_- be as defined in* (5.9)*. Then $\alpha_k^* \in \{\alpha \in \mathbb{R} \colon M_{X_T}(\alpha) < +\infty\}$ minimizes*

$$\sup_{u \in (0,+\infty)} \left| e^{-k(\alpha+iu-1)} \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu)(\alpha + iu - 1)} \right|, \tag{5.17}$$

if and only if $\alpha_k^* \in [\alpha_-, \alpha_+] \setminus \{0, 1\}$ is a solution to

$$\min_{\alpha \in [\alpha_-, \alpha_+] \cap \mathbb{R} \setminus \{0,1\}} \ln M_{X_T}(\alpha) - \frac{1}{2} \ln \alpha^2 (\alpha - 1)^2 - k\alpha. \tag{5.18}$$

Now additionally assume[2]

$$\begin{aligned} &P(X_T < k) > 0 \quad \textit{if} \quad \alpha_- = -\infty \quad \textit{and} \\ &P(X_T > k) > 0 \quad \textit{if} \quad \alpha_+ = +\infty. \end{aligned}$$

Then the objective function of the minimization problem in (5.18) *has a unique minimum in each of the intervals*

- ▷ $[\alpha_-, 0) \cap (-\infty, 0)$,
- ▷ $(0, \alpha_+] \cap (0, 1)$ *and*
- ▷ $(1, \alpha_+] \cap (1, +\infty)$.

Proof. First observe that by Lemma 5.3.1 we know that α_k^* minimizes (5.17) if and only if $\alpha_k^* \notin \{0, 1\}$ and α_k^* minimizes

$$\alpha \mapsto e^{-k(\alpha - 1)} \frac{M_{X_T}(\alpha)}{|\alpha||\alpha - 1|}. \tag{5.19}$$

Applying the strictly increasing function $x \mapsto \ln x - k$ to the latter expression leads to the fact that α_k^* minimizing (5.17) is equivalent to $\alpha_k^* \notin \{0, 1\}$ and $\alpha_k^* \in$ solving

$$\min_{\alpha \in \{\alpha \in \mathbb{R} \setminus \{0,1\} \colon M_{X_T}(\alpha) < +\infty\}} \ln M_{X_T}(\alpha) - \frac{1}{2} \ln \alpha^2 (\alpha - 1)^2 - k\alpha.$$

Now observe the following.

[2]If $\alpha_- > -\infty$ and $\alpha_+ < +\infty$ there are no additional assumptions.

▷ If $\alpha_+ < \infty$ and $M_{X_T}(\alpha_+) = +\infty$ then the objective function also takes the value $+\infty$.

▷ Analogously if $\alpha_- > -\infty$ and $M_{X_T}(\alpha_-) = +\infty$ then the objective function also takes the value $+\infty$.

These two observations lead to the fact that α_k^* minimizing (5.17) is equivalent to $\alpha_k^* \notin \{0,1\}$ and $\alpha_k^* \in [\alpha_-, \alpha_+] \cap \mathbb{R} \setminus \{0,1\}$ solving (5.18).

For the second statement note that from Lemma 5.3.3 we know that the function defined in (5.19) has a minimum in each of the intervals $[\alpha_-, 0) \cap (-\infty, 0)$, $(0, \alpha_+] \cap (0,1)$ and $(1, \alpha_+] \cap (1, +\infty)$. As the objective function in (5.18) can be obtained by applying the strictly increasing transformation $x \mapsto \ln x + k$ to (5.19) the last statement then directly follows.

□

Remark 5.3.5. The proposition we have just proven also helps us to see that there always exists an $\alpha_k^* \in [\alpha_-, \alpha_+] \cap \mathbb{R} \setminus \{0,1\}$ that minimizes the expressions in (5.17) and (5.18). To see that first consider $L_k \colon [\alpha_-, \alpha_+] \cap \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$, which is defined as the objective function of (5.18), i. e.

$$L_k(\alpha) = \ln M_{X_T}(\alpha) - \frac{1}{2} \ln \alpha^2 (\alpha - 1)^2 - k\alpha, \quad \alpha \in [\alpha_-, \alpha_+] \cap \mathbb{R} \setminus \{0,1\}.$$

Now recall Proposition 5.3.4 and let

▷ $\alpha_{k,1}^*$ be the unique minimum of $L_k|_{[\alpha_-,0)\cap(-\infty,0)}$,

▷ $\alpha_{k,2}^*$ be the unique minimum of $L_k|_{(0,\alpha_+]\cap(0,1)}$ and

▷ $\alpha_{k,3}^*$ be the unique minimum of $L_k|_{(1,\alpha_+]\cap(1,+\infty)}$.

Next, define L_k^* by

$$L_k^* := \min \Big\{ L_k\big(\alpha_{k,1}^*\big), L_k\big(\alpha_{k,2}^*\big), L_k\big(\alpha_{k,3}^*\big) \Big\}.$$

Obviously every $\alpha_k^* \in \{\alpha_{k,1}^*, \alpha_{k,2}^*, \alpha_{k,2}^*\}$ satisfying

$$L_k(\alpha_k^*) = L_k^*,$$

minimizes the function L_k. Consequently such an α_k^* solves (5.18) and thus it also minimizes (5.17).

Proposition 5.3.4 and Remark 5.3.5 lead to the following definition of the so-called payoff-dependent α.

Definition 5.3.6. Consider the moment generating function M_{X_T} and let α_+, α_- be as defined in (5.9). An $\alpha_k^* \in [\alpha_-, \alpha_+] \cap \mathbb{R} \setminus \{0,1\}$ solving

$$\min_{\alpha \in [\alpha_-,\alpha_+]\cap\mathbb{R}\setminus\{0,1\}} \ln M_{X_T}(\alpha) - \frac{1}{2} \ln \alpha^2 (\alpha - 1)^2 - k\alpha, \tag{5.20}$$

is referred to as *payoff-dependent* α.

Finally, we give an example that illustrates the results obtained in this section.

Example 5.3.7. Recall the Heston model with parameters

κ	η	ρ	θ	r	S_0	v_0
1	2	-0.99	1	0	10	5

,

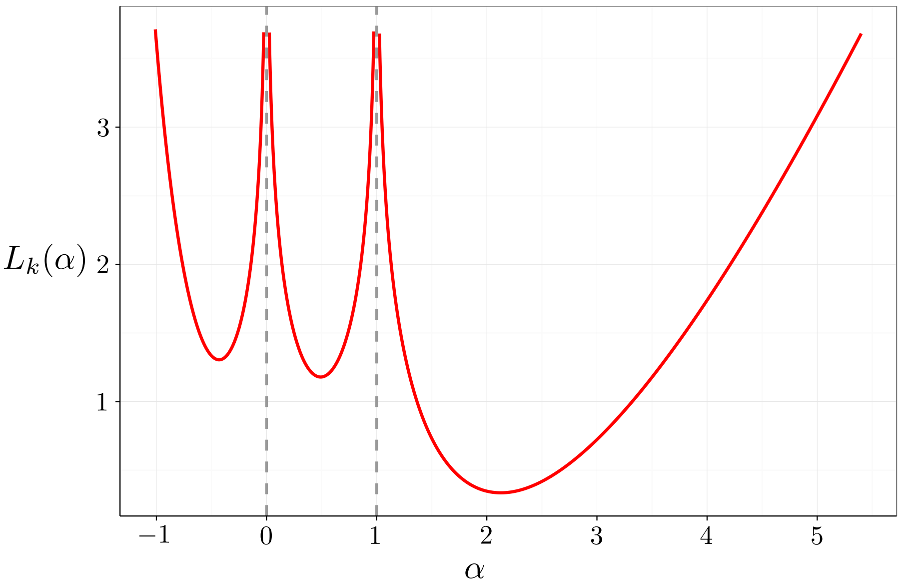
from Example 5.1.1 and the critical moments

$$\alpha_- = u_-(T) = -1.885 \quad \text{and} \quad \alpha_+ = u_+(T) = 116.556,$$

where $T = 0.5$. Now we want to compute a payoff-dependent α_k^* for $k = \ln 10$ in this setting and compare it to the choices of α presented in Example 5.1.1. Now consider the objective function L_k associated with α_k^* from Definition 5.3.6, i. e.

$$L_k(\alpha) = \ln M_{X_T}(\alpha) - \frac{1}{2}\ln \alpha^2(\alpha-1)^2 - k\alpha, \quad \alpha \in (\alpha_-, \alpha_+) \setminus \{0, 1\}.$$

Plotting the objective funciton L_k leads to



where one can very nicely observe the behavior described in Proposition 5.3.4. Using the programming language `R` we obtain the following table of minima and objectives associated with L_k

Interval	$(\alpha_-, 0)$	$(0, 1)$	$(1, \alpha_+)$
Minimum	-0.431	0.491	2.127
Objective	1.304	1.179	0.334

,

where we rounded to three digits. Consequently we have a unique payoff-dependent α here and it is given by $\alpha_k^* = 2.127$. We see that $\alpha = 2$, which we preferred in Example 5.1.1, is pretty close to the computed $\alpha_k^* = 2.127$, which motivates the nice behavior observed back then.

Lord and Kahl also relate the choice of α suggested in Definition 5.3.6 to so-called saddlepoint approximations. We conclude this section with a short comment on the relation to saddlepoints and why the suggested choice of α can also be referred to as laying the contour through a saddlepoint. Lord and Kahl refer to a paper by Daniels, [Dan54], where saddlepoint approximations are discussed. In that paper Daniels considers the density f_n of the sample mean $\overline{X_n} = \frac{1}{n}\sum_{j=1} X_j$, where $X_1, \ldots, X_n$ are i. i. d. random variables such that the distribution of X_1 has a density f with respect to the Lebesuge measure. Furthermore, Daniels assumes the moment generating function to be of the form

$$M(z) = e^{K(z)} = \int_{-\infty}^{+\infty} e^{zx} f(x)\, dx, \quad z \in (-c_1, c_2) + i\mathbb{R},$$

where $c_1, c_2 \geq 0$ such that $c_1 + c_2 > 0$ and K is an analytic function on $(-c_1, c_2) + i\mathbb{R}$. For $x \in \mathbb{R}$ a solution $\alpha = \alpha^*$ to the equation

$$K'(\alpha^*) = x,$$

is used to define the approximation g_n to f_n by

$$g_n(x) := \left(\frac{n}{2\pi K''(\alpha^*)} \right)^{\frac{1}{2}} e^{n(K(\alpha^*) - \alpha^* x)},$$

which is referred to as saddlepoint approximation of f_n. Daniels also presents the estimate

$$\left| \frac{f_n(x)}{g_n(x)} - 1 \right| \leq \frac{A(x)}{n},$$

for $n \to +\infty$. Now recall that we are working with the random variable X_T, which can be interpreted as sample mean of size $n = 1$. Consequently it remains unclear how one can actually sensibly apply these asymptotic results for $n \to +\infty$ in our situation.
However, when the suggested α_k^* even belongs to (α_-, α_+) one can sensibly refer to it as saddlepoint. To see this recall from (5.1) the call price representation

$$C(k) = R_\alpha(k) + \frac{e^k}{2\pi} \int_{-\infty}^{\infty} e^{-k(\alpha + iu)} \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu)(\alpha + iu - 1)} \, du,$$

where $k \in \mathbb{R}$ and $\alpha \in (\alpha_-, \alpha_+) \setminus \{0, 1\}$ is assumed. Now we fix $k \in \mathbb{R}$ and introduce the notation

$$I_\alpha(u) := e^{-k(\alpha + iu)} \frac{M_{X_T}(\alpha + iu)}{(\alpha + iu)(\alpha + iu - 1)}, \quad u \in \mathbb{R},$$

for the integrand, when $\alpha \in (\alpha_-, \alpha_+) \setminus \{0, 1\}$. Next, make the following two observations.

- ▷ On the one hand we have

$$|I_\alpha(u)| \leq e^{-k\alpha} \frac{M_{X_T}(\alpha)}{|\alpha||\alpha - 1|} = |I_\alpha(0)|, \quad u \in \mathbb{R}.$$

  for $\alpha \in (\alpha_-, \alpha_+) \setminus \{0, 1\}$ and thus the modulus of I_α is maximal at $u = 0$ for every $\alpha \in [\alpha_-, \alpha_+] \cap \mathbb{R} \setminus \{0, 1\}$.

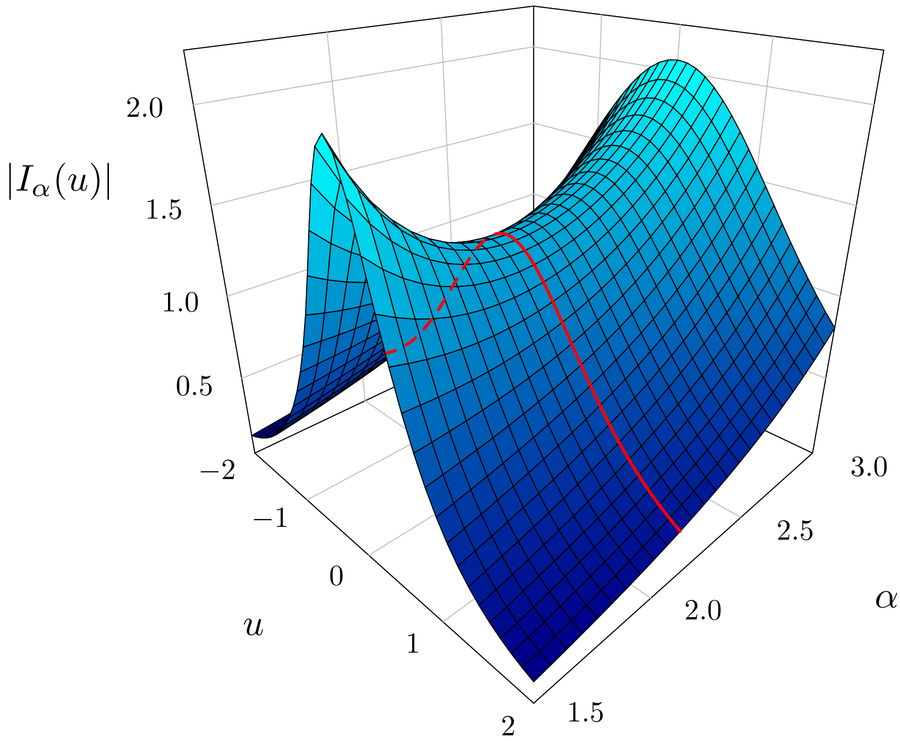
- ▷ On the other hand $|I_\alpha(0)|$ is minimal if and only if α solves the minimization problem

$$\min_{\alpha \in [\alpha_-, \alpha_+] \cap \mathbb{R} \setminus \{0,1\}} \ln M_{X_T}(\alpha) - k\alpha - \frac{1}{2} \ln \alpha^2 (\alpha - 1)^2.$$

  As α_k^* particularly solves the latter we hence also have

$$|I_\alpha(0)| \geq |I_{\alpha_k^*}(0)|, \quad \alpha \in [\alpha_-, \alpha_+] \cap \mathbb{R} \setminus \{0, 1\}.$$

Under the assumption that $\alpha_k^* \in (\alpha_-, \alpha_+)$ holds the following graph visualizes these two observations for a call option with $T = 0.5$ and $K = 10$ in the Heston model from Example 5.3.7.



The red line represents the modulus of integrand for $\alpha_k^* = 2.127$ and clearly the shape of the surface motivates the name saddlepoint for $\alpha_k^* \in (\alpha_-, \alpha_+)$.

5.4 Heston Model

In this section we apply the established results in the Heston model. Therefore recall from Corollary 4.4.10 the moment generating function M_{X_T} of the log-discounted underlying $X_T = \ln S_T - R_T$ in the Heston model. Assuming $\kappa \geq \eta\rho$ and when arguments do not lie on the real axis the moment generating function M_{X_T} is given by

$$M_{X_T}(z) = \exp\Big(zx_0 + A(z) + v_0 B(z)\Big), \quad z \in \big(u_-(T), u_+(T)\big) + i\mathbb{R} \setminus \{0\}, \tag{5.21}$$

where $u_+(T)$ and $u_-(T)$ are the critical moments defined in (4.47) and

$$\begin{aligned} A(z) &= -\frac{2\kappa\theta}{\eta^2}\Big(\big(b(z) + H(z)\big)T + \log\Big(P_T(z)e^{-H(z)T}\Big)\Big), \quad z \in \mathbb{C} \setminus \mathbb{R} \\ B(z) &= \frac{z(z-1)}{2}\frac{Q_T(z)}{P_T(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \end{aligned} \tag{5.22}$$

with

$$
\begin{aligned}
b(z) &= \frac{1}{2}(\eta\rho z - \kappa), \quad z \in \mathbb{C} \\
D(z) &= (\kappa - z\eta\rho)^2 - z(z-1)\eta^2, \quad z \in \mathbb{C} \\
H(z) &= \frac{1}{2}\exp\Big(\frac{1}{2}\log H(z)\Big), \quad z \in \mathbb{C}\setminus\mathbb{R} \\
P_T(z) &= \cosh H(z)T - b(z)\frac{\sinh H(z)T}{H(z)}, \quad z \in \mathbb{C}\setminus\mathbb{R} \\
Q_T(z) &= \frac{\sinh H(z)T}{H(z)}, \quad z \in \mathbb{C}\setminus\mathbb{R}.
\end{aligned}
\tag{5.23}
$$

We start with a derivation of the asymptotic behavior of the moment generating function when the imaginary part of the argument becomes large. The results obtained can be used to compute the call price $C(k)$ via (5.1) in at least two ways. On the one hand the exponential decay we observe makes it possible to sensibly apply Gauss-Laguerre quadrature for the evaluation of the desired integral. On the other hand the asymptotic results allow us to introduce a substitution which transforms the unbounded interval $(0,+\infty)$ to the bounded interval $(0,1)$ such that the integrand associated with the corresponding substitution has limits at 0 and 1.

5.4.1 Asymptotics of the Moment Generating function

Now we focus on the asymptotics of $M_{X_T}(\alpha+iu)$ for $u \to +\infty$. The derivation is based on results given by Lord and Kahl in [LK06]. Clearly therefore one has to examine the behavior of the coefficients A and B. The following three lemmas turn out to be particularly useful.

Lemma 5.4.1. *Let $\alpha \in \mathbb{R}$ and consider the functions defined in* (5.23)*. Then we have*

$$
\lim_{u\to+\infty}\frac{b(\alpha+iu)}{u} = i\frac{\eta\rho}{2} \quad \text{and} \quad \lim_{u\to+\infty}\frac{D(\alpha+iu)}{u^2} = \eta^2(1-\rho^2).
$$

If in addition $\rho \in (-1,1)$ holds we also have

$$
\lim_{u\to+\infty}\frac{H(\alpha+iu)}{u} = \frac{1}{2}\eta\sqrt{1-\rho^2}.
$$

Proof. The statements are simple to show. First, we have

$$
2\lim_{u\to+\infty}\frac{b(\alpha+iu)}{u} = \lim_{u\to+\infty}\frac{\eta\rho(\alpha+iu)-\kappa}{u} = \lim_{u\to+\infty}\eta\rho\alpha u^{-1} + i\eta\rho - \kappa u^{-1} = i\eta\rho.
$$

A division by 2 yields the first statement. Next, we have

$$
\begin{aligned}
\lim_{u\to+\infty}\frac{D(\alpha+iu)}{u^2} &= \lim_{u\to+\infty} u^{-2}\Big(\big(\kappa-(\alpha+iu)\eta\rho\big)^2 - (\alpha+iu)(\alpha+iu-1)\eta^2\Big) \\
&= \lim_{u\to+\infty}\Big(\big(\kappa u^{-1}-(\alpha u^{-1}+i)\eta\rho\big)^2 - (\alpha u^{-1}+i)(\alpha u^{-1}+i-u^{-1})\eta^2\Big) \\
&= -\rho^2\eta^2 - i^2\eta^2 = \eta^2(1-\rho^2).
\end{aligned}
$$

Because of $\rho \in (-1,1)$ the complex logarithm is continuous at $\eta^2(1-\rho^2)$ and we thus obtain

$$
2\lim_{u\to+\infty}\frac{H(\alpha+iu)}{u} = \lim_{u\to+\infty}\exp\Big(\frac{1}{2}\log\big(D(\alpha+iu)u^{-2}\big)\Big) = \exp\Big(\frac{1}{2}\log\big(\eta^2(1-\rho^2)\big)\Big) = \eta\sqrt{1-\rho^2},
$$

which clearly implies the last limit stated in this lemma. □

Lemma 5.4.2. *Let $\alpha \in \mathbb{R}$ and consider the functions defined in* (5.23)*. Then we have*

$$\lim_{u\to+\infty} b(\alpha+iu) - i\frac{\eta\rho}{2}u = b(\alpha) \quad \text{and} \quad \lim_{u\to+\infty} H(\alpha+iu) - \frac{1}{2}\eta\sqrt{1-\rho^2}u = -i\frac{\alpha}{2}\eta\sqrt{1-\rho^2} - i\frac{2\kappa\rho-\eta}{4\sqrt{1-\rho^2}},$$

where we also assumed $\rho \in (-1,1)$ for the latter.

Proof. The first statement can easily be seen as follows.

$$\lim_{u\to+\infty} b(\alpha+iu) - i\frac{\eta\rho}{2}u = \frac{1}{2}\lim_{u\to+\infty} \eta\rho(\alpha+iu) - \kappa - i\eta\rho u = \frac{1}{2}(\eta\rho\alpha-\kappa) = b(\alpha).$$

For the second statement first rearrange terms in the definition of D to obtain

$$\begin{aligned}
D(\alpha+iu) &= (\kappa-\eta\rho\alpha-i\eta\rho u)^2 - \eta^2(\alpha+iu)(\alpha+iu-1) \\
&= (\kappa-\eta\rho\alpha)^2 - 2(\kappa-\eta\rho\alpha)i\eta\rho u - \eta^2\rho^2u^2 - \eta^2\big(\alpha(\alpha-1)+iu(2\alpha-1)-u^2\big) \\
&= \eta^2u^2(1-\rho^2) - iu\eta\big(2(\kappa-\eta\rho\alpha)\rho+\eta(2\alpha-1)\big) + (\kappa-\eta\rho\alpha)^2 - \eta^2\alpha(\alpha-1) \\
&= \eta^2u^2(1-\rho^2) - iu\eta\big(2\alpha\eta(1-\rho^2)+2\kappa\rho-\eta\big) + (\kappa-\eta\rho\alpha)^2 - \eta^2\alpha(\alpha-1),
\end{aligned}$$

for $u > 0$. Furthermore, observe that by Lemma 4.2.5 we know that $D(\alpha+iu) \notin (-\infty, 0]$ holds for $u > 0$. This implies that $H(\alpha+iu)$ has a positive real part when $u > 0$. Hence the following rearrangements can be done.

$$\begin{aligned}
H(\alpha+iu) - \frac{1}{2}\eta\sqrt{1-\rho^2}u &= \frac{1}{2}\frac{4H(\alpha+iu)^2-\eta^2(1-\rho^2)u^2}{2H(\alpha+iu)+\eta\sqrt{1-\rho^2}u} = \frac{1}{2}\frac{D(\alpha+iu)-\eta^2(1-\rho^2)u^2}{2H(\alpha+iu)+\eta\sqrt{1-\rho^2}u} \\
&= \frac{1}{2}\frac{(\kappa-\eta\rho\alpha)^2u^{-1}-\eta^2\alpha(\alpha-1)u^{-1}-i\eta\big(2\alpha\eta(1-\rho^2)+2\kappa\rho-\eta\big)}{2H(\alpha+iu)u^{-1}+\eta\sqrt{1-\rho^2}}.
\end{aligned}$$

Because of $\rho \in (-1,1)$ we can apply the results of Lemma 5.4.1 when letting $u \to +\infty$ to obtain

$$\lim_{u\to+\infty} H(\alpha+iu) - \frac{1}{2}\eta\sqrt{1-\rho^2}u = -i\frac{\eta}{2}\frac{2\eta(1-\rho^2)\alpha+2\kappa\rho-\eta}{\eta\sqrt{1-\rho^2}+\eta\sqrt{1-\rho^2}} = -i\frac{\alpha}{2}\eta\sqrt{1-\rho^2} - i\frac{2\kappa\rho-\eta}{4\sqrt{1-\rho^2}}$$

□

Lemma 5.4.3. *Let $\alpha \in \mathbb{R}$, $\rho \in (-1,1)$, $T > 0$ and consider the functions defined in* (5.23)*. Then we have*

$$\lim_{u\to+\infty} P_T(\alpha+iu)e^{-H(\alpha+iu)T} = \frac{1}{2}\left(1-i\frac{\rho}{\sqrt{1-\rho^2}}\right) \quad \text{and} \quad \lim_{u\to+\infty} ue^{-H(\alpha+iu)T}Q_T(\alpha+iu) = \frac{1}{\eta\sqrt{1-\rho^2}}.$$

Proof. First, observe that rearranging terms leads to

$$P_T(\alpha+iu)e^{-H(\alpha+iu)T} = \frac{1}{2}\left(\big(1+e^{-2H(\alpha+iu)T}\big) - \frac{b(\alpha+iu)u^{-1}}{H(\alpha+iu)u^{-1}}\big(1-e^{-2H(\alpha+iu)T}\big)\right),$$

for $u > 0$. Applying the results from Lemma 5.4.1 and noting that $\eta\sqrt{1-\rho^2} > 0$ for $\rho \in (-1,1)$ leads then to

$$\lim_{u\to+\infty} P_T(\alpha+iu)e^{-H(\alpha+iu)T} = \frac{1}{2}\left(1-i\frac{\eta\rho}{\eta\sqrt{1-\rho^2}}\right) = \frac{1}{2}\left(1-i\frac{\rho}{\sqrt{1-\rho^2}}\right).$$

Furthermore, one has

$$ue^{-H(\alpha+iu)T}Q_T(\alpha+iu) = \frac{1}{2}\frac{u}{H(\alpha+iu)}\left(1-e^{-2H(\alpha+iu)T}\right),$$

for $u > 0$. Applying again the results from Lemma 5.4.1 thus gives

$$\lim_{u\to+\infty} ue^{-H(\alpha+iu)T} Q_T(\alpha+iu) = \frac{1}{2}\lim_{u\to+\infty} \frac{u}{H(\alpha+iu)} = \frac{1}{\eta\sqrt{1-\rho^2}}.$$

$\square$

Now we have collected tools to precisely determine the behavior of $A(\alpha+iu)$ and $B(\alpha+iu)$ when $u > 0$ becomes large. This is done in the two subsequent propositions.

Proposition 5.4.4. *Let $\alpha \in \mathbb{R}$, $\rho \in (-1,1)$, $T > 0$ and consider the function A from (5.22). Define the complex constants A_∞ and A^* by*

$$\begin{aligned} A_\infty &= \frac{\kappa\theta T}{\eta}\Big(\sqrt{1-\rho^2} + i\rho\Big) \\ A^* &= \kappa T - \eta\rho\alpha T + i\frac{\sqrt{1-\rho^2}}{2}\left(2\alpha\eta + \frac{2\kappa\rho-\eta}{1-\rho^2}\right)T - 2\log\left(\frac{1}{2} - i\frac{\rho}{2\sqrt{1-\rho^2}}\right). \end{aligned} \tag{5.24}$$

Then we have

$$\lim_{u\to+\infty} A(\alpha+iu) + A_\infty u = \frac{\kappa\theta}{\eta^2}A^*.$$

Proof. First recall the expression for A

$$A(\alpha+iu) = -\frac{2\kappa\theta}{\eta^2}\left(b(\alpha+iu)T + H(\alpha+iu)T + \log\Big(P_T(\alpha+iu)e^{-H(\alpha+iu)T}\Big)\right), \quad u > 0.$$

In order to compute the desired limit we will treat each term of A separately. Using Lemma 5.4.2 we get

$$L_1 := \lim_{u\to+\infty} -\frac{2\kappa\theta}{\eta^2}b(\alpha+iu)T + i\frac{\kappa\theta T}{\eta}\rho u = \lim_{u\to+\infty} -\frac{2\kappa\theta T}{\eta^2}\left(b(\alpha+iu) - i\frac{\eta\rho}{2}u\right) = -\frac{2\kappa\theta T}{\eta^2}b(\alpha).$$

Applying again Lemma 5.4.2 yields

$$\begin{aligned} L_2 &:= \lim_{u\to+\infty} -\frac{2\kappa\theta}{\eta^2}H(\alpha+iu)T + \frac{\kappa\theta T}{\eta}\sqrt{1-\rho^2}u = \lim_{u\to+\infty} -\frac{2\kappa\theta T}{\eta^2}\left(H(\alpha+iu) - \frac{1}{2}\eta\sqrt{1-\rho^2}u\right) \\ &= -\frac{2\kappa\theta T}{\eta^2}\left(-i\frac{\alpha}{2}\eta\sqrt{1-\rho^2} - i\frac{2\kappa\rho-\eta}{4\sqrt{1-\rho^2}}\right) = i\frac{\kappa\theta T}{\eta^2}\left(\alpha\eta\sqrt{1-\rho^2} + \frac{2\kappa\rho-\eta}{2\sqrt{1-\rho^2}}\right). \end{aligned}$$

Furthermore, by Lemma 5.4.3 and due to the continuity of log on $\mathbb{C}\setminus(-\infty,0]$ we clearly get

$$L_3 := \lim_{u\to+\infty} -\frac{2\kappa\theta}{\eta^2}\log\Big(P_T(\alpha+iu)e^{-H(\alpha+iu)T}\Big) = -\frac{2\kappa\theta}{\eta^2}\log\left(\frac{1}{2} - i\frac{\rho}{2\sqrt{1-\rho^2}}\right)$$

Next, abbreviate the desired limit by

$$L := \lim_{u\to+\infty} A(\alpha+iu) + A_\infty u = \lim_{u\to+\infty} A(\alpha+iu) + \frac{\kappa\theta T}{\eta}\Big(\sqrt{1-\rho^2} + i\rho\Big)u.$$

Now we clearly obtain

$$\begin{aligned} L = L_1 + L_2 + L_3 &= -\frac{2\kappa\theta T}{\eta^2}b(\alpha) + i\frac{\kappa\theta T}{\eta^2}\left(\alpha\eta\sqrt{1-\rho^2} + \frac{2\kappa\rho-\eta}{2\sqrt{1-\rho^2}}\right) - \frac{2\kappa\theta}{\eta^2}\log\left(\frac{1}{2} - i\frac{\rho}{2\sqrt{1-\rho^2}}\right) \\ &= \frac{\kappa\theta}{\eta^2}\left(\kappa T - \eta\rho\alpha T + i\frac{\sqrt{1-\rho^2}}{2}\left(2\alpha\eta + \frac{2\kappa\rho-\eta}{1-\rho^2}\right)T - 2\log\left(\frac{1}{2} - i\frac{\rho}{2\sqrt{1-\rho^2}}\right)\right) = \frac{\kappa\theta}{\eta^2}A^*, \end{aligned}$$

which concludes the proof. $\square$

Proposition 5.4.5. *Let $\alpha \in \mathbb{R}$, $\rho \in (-1,1)$, $T > 0$ and consider the function B from (5.22). Define B_∞ and B^* by*

$$\begin{aligned} B_\infty &= \frac{\sqrt{1-\rho^2}+i\rho}{\eta} \\ B^* &= \frac{\kappa-\eta\rho\alpha}{\eta^2} + i\frac{\sqrt{1-\rho^2}}{2\eta}\left(2\alpha-1+\frac{\rho}{\eta}\frac{2\kappa-\eta\rho}{1-\rho^2}\right). \end{aligned} \tag{5.25}$$

Then we have

$$\lim_{u\to+\infty} B(\alpha+iu)+B_\infty u = B^*.$$

Proof. We start by introducing the following abbreviation for the desired limit.

$$L := \lim_{u\to+\infty} B(\alpha+iu)+B_\infty u = \lim_{u\to+\infty} B(\alpha+iu)+\frac{\sqrt{1-\rho^2}+i\rho}{\eta}u.$$

Next, using the expression for B in (5.22) and rearranging terms leads to

$$L = \lim_{u\to+\infty} \frac{1}{2}\left(\frac{\alpha(\alpha-1)}{u}+i(2\alpha-1)\right)\frac{Q_T(\alpha+iu)u}{P_T(\alpha+iu)} - \frac{u^2}{2}\frac{Q_T(\alpha+iu)}{P_T(\alpha+iu)} + \frac{\sqrt{1-\rho^2}+i\rho}{\eta}u,$$

for $u > 0$. By means of Lemma 5.4.3 we have

$$\lim_{u\to+\infty}\frac{Q_T(\alpha+iu)u}{P_T(\alpha+iu)} = \lim_{u\to+\infty}\frac{ue^{-H(\alpha+iu)T}Q_T(\alpha+iu)}{e^{-H(\alpha+iu)T}P_T(\alpha+iu)} = \frac{2}{\eta}\left(\sqrt{1-\rho^2}-i\rho\right)^{-1},$$

and hence

$$L = -\frac{2\alpha-1}{\eta\left(\rho+i\sqrt{1-\rho^2}\right)} - \frac{1}{2}\underbrace{\lim_{u\to+\infty}\left(u^2\frac{Q_T(\alpha+iu)}{P_T(\alpha+iu)}-2\frac{\sqrt{1-\rho^2}+i\rho}{\eta}u\right)}_{L_1:=},$$

holds. Consequently the latter limit, which we denote by L_1, is of particular interest. For the remaining part of the proof we introduce the notation

$$\omega := 2\frac{\sqrt{1-\rho^2}+i\rho}{\eta}.$$

Then, by plugging in the definitions for P_T and Q_T and multiplying the numerator and denominator by $e^{-H(\alpha+iu)T}$, we obtain

$$\begin{aligned} u^2\frac{Q_T(\alpha+iu)}{P_T(\alpha+iu)}-\omega u &= \frac{uQ_T(\alpha+iu)-\omega P_T(\alpha+iu)}{u^{-1}P_T(\alpha+iu)} \\ &= \frac{u-ue^{-2H(\alpha+iu)}-\omega\big(H(\alpha+iu)-b(\alpha+iu)\big)}{H(\alpha+iu)u^{-1}-b(\alpha+iu)u^{-1}+\big(H(\alpha+iu)u^{-1}+b(\alpha+iu)u^{-1}\big)e^{-2H(\alpha+iu)T}} \\ &\quad - \frac{\omega\big(H(\alpha+iu)+b(\alpha+iu)\big)e^{-2H(\alpha+iu)T}}{H(\alpha+iu)u^{-1}-b(\alpha+iu)u^{-1}+\big(H(\alpha+iu)u^{-1}+b(\alpha+iu)u^{-1}\big)e^{-2H(\alpha+iu)T}}, \end{aligned} \tag{5.26}$$

for $u > 0$. Now recall from Lemma 5.4.1 that we have

$$\lim_{u\to+\infty}\frac{b(\alpha+iu)}{u} = i\frac{\eta\rho}{2} \quad \text{and} \quad \lim_{u\to+\infty}\frac{H(\alpha+iu)}{u} = \frac{1}{2}\eta\sqrt{1-\rho^2}.$$

Consequently because of $\eta > 0$, $T > 0$ and $\rho \in (-1, 1)$ there is always an $\epsilon > 0$ and $M > 0$ such that

$$\left|ue^{-2H(\alpha+iu)T}\right| + \left|e^{-2H(\alpha+iu)T}\right| \leq ue^{-\epsilon u} + e^{-\epsilon u}, \quad u \geq M.$$

Consequently we have

$$\lim_{u\to+\infty} ue^{-2H(\alpha+iu)T} = \lim_{u\to+\infty} e^{-2H(\alpha+iu)T} = 0.$$

Using the latter observation when taking the limit $u \to +\infty$ in (5.26) yields

$$\begin{aligned} L_1 = \lim_{u\to+\infty} u^2 \frac{Q_T(\alpha+iu)}{P_T(\alpha+iu)} - \omega u &= 2\omega \lim_{u\to+\infty} \frac{\omega^{-1}u - \big(H(\alpha+iu) - b(\alpha+iu)\big)}{\eta\sqrt{1-\rho^2} - i\eta\rho} \\ &= \frac{4}{\eta^2}\big(\sqrt{1-\rho^2} + i\rho\big)^2 \underbrace{\lim_{u\to+\infty}\Big(\omega^{-1}u - H(\alpha+iu) + b(\alpha+iu)\Big)}_{L_2:=}. \end{aligned}$$

Consequently we also need to determine the latter limit which we denote by L_2. Therefore, observe

$$\omega^{-1} = \frac{\eta}{2}(\sqrt{1-\rho^2} + i\rho)^{-1} = \frac{1}{2}\eta\sqrt{1-\rho^2} - i\frac{\eta\rho}{2}.$$

Hence an application of Lemma 5.4.2 yields

$$\begin{aligned} L_2 = \lim_{u\to+\infty}\Big(\omega^{-1}u - H(\alpha+iu) + b(\alpha+iu)\Big) &= \frac{1}{2}\eta\sqrt{1-\rho^2}u - H(\alpha+iu) + b(\alpha+iu) - i\frac{\eta\rho}{2}u \\ &= i\frac{\alpha}{2}\eta\sqrt{1-\rho^2} + i\frac{2\kappa\rho - \eta}{4\sqrt{1-\rho^2}} + b(\alpha). \end{aligned}$$

Before we can finally put together what we have shown so far we rearrange terms in the expression for L_2 as follows.

$$\begin{aligned} L_2 &= i\frac{\alpha}{2}\eta\sqrt{1-\rho^2} + i\frac{2\kappa\rho - \eta}{4\sqrt{1-\rho^2}} + b(\alpha) = i\frac{\alpha\eta}{2}\sqrt{1-\rho^2} + i\frac{2\kappa\rho - \eta}{4\sqrt{1-\rho^2}} + \frac{\eta\rho\alpha - \kappa}{2} \\ &= i\frac{\alpha\eta}{2}\Big(\sqrt{1-\rho^2} - i\rho\Big) - \frac{\kappa}{2\sqrt{1-\rho^2}}\Big(\sqrt{1-\rho^2} - i\rho\Big) - i\frac{\eta}{4\sqrt{1-\rho^2}}. \end{aligned}$$

This can now be used to simplify the expression for L_1 as follows.

$$\begin{aligned} \frac{\eta^2}{4}L_1 &= \big(\sqrt{1-\rho^2} + i\rho\big)^2 L_2 = \Big(\sqrt{1-\rho^2} + i\rho\Big)\bigg(i\frac{\alpha\eta}{2} - \frac{\kappa}{2\sqrt{1-\rho^2}} - i\frac{\eta}{4\sqrt{1-\rho^2}}\Big(\sqrt{1-\rho^2} + i\rho\Big)\bigg) \\ &= \Big(\sqrt{1-\rho^2} + i\rho\Big)\bigg(i\frac{\alpha\eta}{2} - i\frac{\eta}{4} + \frac{\eta\rho}{4\sqrt{1-\rho^2}} - \frac{\kappa}{2\sqrt{1-\rho^2}}\bigg) \\ &= \frac{\eta\rho - 2\kappa}{4} + \frac{\eta}{4}(\rho - 2\alpha\rho) + i\frac{\eta}{4}\bigg((2\alpha - 1)\sqrt{1-\rho^2} + \frac{\rho^2}{\sqrt{1-\rho^2}} - \frac{2\kappa\rho}{\eta\sqrt{1-\rho^2}}\bigg) \end{aligned}$$

The desired limit L is thus given by

$$\begin{aligned}
L &= -\frac{2\alpha-1}{\eta\big(\rho+i\sqrt{1-\rho^2}\big)} - \frac{1}{2}L_1 = \frac{\rho-2\alpha\rho}{\eta} + i\frac{2\alpha-1}{\eta}\sqrt{1-\rho^2} - \frac{1}{2}L_1 \\
&= \frac{\rho-2\alpha\rho}{\eta} - \frac{\eta\rho-2\kappa}{2\eta^2} - \frac{\rho-2\alpha\rho}{2\eta} + i\frac{2\alpha-1}{\eta}\sqrt{1-\rho^2} - \frac{1}{2}i\,\mathrm{Im}\,(L_1) \\
&= \frac{\kappa-\alpha\eta\rho}{\eta^2} + i\left(\frac{2\alpha-1}{\eta}\sqrt{1-\rho^2} - \frac{1}{2}\,\mathrm{Im}\,(L_1)\right) \\
&= \frac{\kappa-\alpha\eta\rho}{\eta^2} + i\left(\frac{2\alpha-1}{\eta}\sqrt{1-\rho^2} - \frac{2\alpha-1}{2\eta}\sqrt{1-\rho^2} + \rho\frac{2\kappa-\eta\rho}{2\eta^2\sqrt{1-\rho^2}}\right) \\
&= \frac{\kappa-\alpha\eta\rho}{\eta^2} + \frac{i}{2\eta}\left((2\alpha-1)\sqrt{1-\rho^2} + \rho\frac{2\kappa-\eta\rho}{\eta\sqrt{1-\rho^2}}\right) \\
&= \frac{\kappa-\alpha\eta\rho}{\eta^2} + i\frac{\sqrt{1-\rho^2}}{2\eta}\left(2\alpha-1+\frac{\rho}{\eta}\frac{2\kappa-\eta\rho}{1-\rho^2}\right)
\end{aligned}$$

□

Now the desired asymptotic behavior easily follows in the following corollary. Note that the function M_{X_T} discussed below only coincides with the moment generating function of the log-discounted underlying X_T if α belongs to $(u_-(T), u_+(T))$, where the boundaries of the latter intervals are the critical moments defined in (4.26).

Corollary 5.4.6. *Let $\kappa, \eta, \theta, T, v_0 > 0$, $\rho \in (-1,1)$, $x_0 \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and consider the functions A and B from* (5.22)*. Furthermore, use A^* and B^* from* (5.24) *and* (5.25) *to define the complex constants*

$$C_\infty = \exp\left(\alpha x_0 + \frac{\kappa\theta}{\eta^2}A^* + B^* v_0\right) \quad \text{and} \quad \omega_\infty = \frac{\sqrt{1-\rho^2}+i\rho}{\eta}\big(\kappa\theta T + v_0\big) - i x_0. \tag{5.27}$$

Then the function M_{X_T}, defined by

$$M_{X_T}(z) = e^{z x_0 + A(z) + v_0 B(z)}, \quad z \in \mathbb{C}\setminus\mathbb{R},$$

satisfies

$$M_{X_T}(\alpha+iu) \sim C_\infty e^{-\omega_\infty u}, \quad \text{for} \quad u \to +\infty,$$

where ω_∞ has a positive real part.

Proof. Recall A_∞ and B_∞ from (5.24) and (5.25) to observe

$$A_\infty + B_\infty v_0 = \frac{\sqrt{1-\rho^2}+i\rho}{\eta}\kappa\theta T + \frac{\sqrt{1-\rho^2}+i\rho}{\eta}v_0 = \frac{\sqrt{1-\rho^2}+i\rho}{\eta}\big(\kappa\theta T + v_0\big) = \omega_\infty + i x_0.$$

Plugging in the expression for M_{X_T} and using the results from Proposition 5.4.4 and Proposition 5.4.5 we thus get

$$\begin{aligned}
\lim_{u\to+\infty}\frac{M_{X_T}(\alpha+iu)}{C_\infty e^{-\omega_\infty u}} &= C_\infty^{-1}\lim_{u\to+\infty}\exp\Big((\alpha+iu)x_0 + A(\alpha+iu) + v_0 B(\alpha+iu) + \omega_\infty u\Big) \\
&= C_\infty^{-1}\lim_{u\to+\infty}\exp\Big(\alpha x_0 + A(\alpha+iu) + v_0 B(\alpha+iu) + (A_\infty + B_\infty v_0)u\Big) \\
&= C_\infty^{-1}\lim_{u\to+\infty}\exp\Big(\alpha x_0 + A(\alpha+iu) + A_\infty u + v_0\big(B(\alpha+iu) + B_\infty u\big)\Big) \\
&= C_\infty^{-1}\exp\left(\alpha x_0 + \frac{\kappa\theta}{\eta^2}A^* + v_0 B^*\right) = 1.
\end{aligned}$$

Furthermore, because of $\rho \in (-1,1)$ and $\kappa, \eta, \theta, T, v_0 > 0$ the real part of ω_∞ is obviously positive. □

It is worth mentioning that the constant ω_∞ presented in Corollary 5.4.6 does not depend on α. Hence the order of the exponential decay of $M_{X_T}(\alpha+iu)$ when $u \to +\infty$ is the same for all choices of α. However, the choice of α plays a significant role in the constant C_∞.

The following example illustrates the asymptotic behavior we derived in Corollary 5.4.6.

Example 5.4.7. We consider the Heston model specified by the parameters

κ	η	ρ	θ	r	S_0	v_0
1	2	-0.99	1	0	0.7	0.1

,

and a European call option with maturity $T = 0.5$ and log-strike $k = \ln 0.6$. Again we use the programming language `R` for our implementations. As in Example 5.1.1, up to an accuracy of three digits, we get

$$\alpha_- = u_-(T) = -1.885 \quad \text{and} \quad \alpha_+ = u_+(T) = 116.556,$$

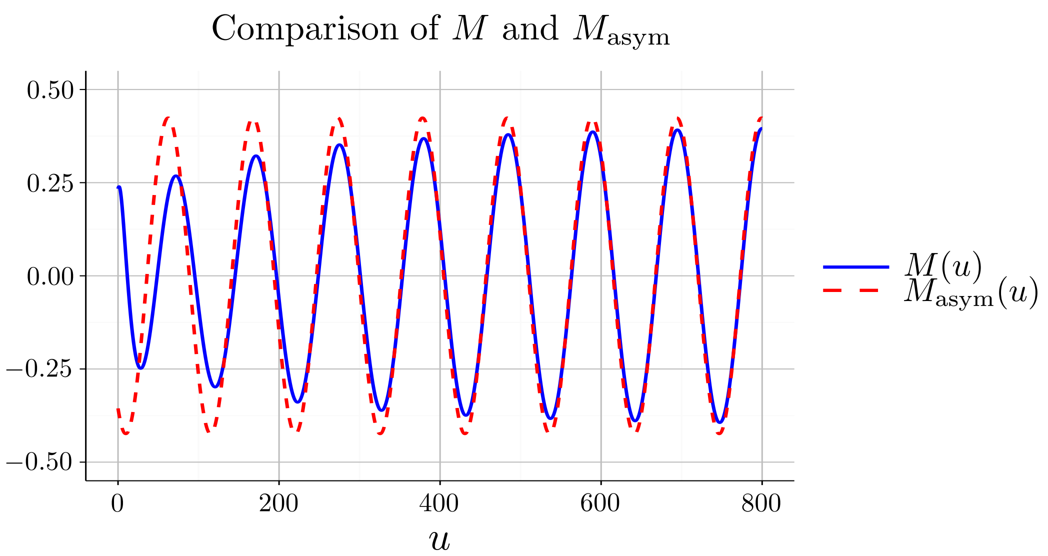
for the critical moments. Furthermore, with an analogous approach as in Example 5.3.7, we obtain $\alpha_k^* = 6.253$ for the payoff-dependent α from Definition 5.3.6. For the rest of the example we fix $\alpha = 6.253$. In order to illustrate the asymptotic behavior we eliminate the exponential decay in the moment generating function. Therefore we introduce the functions $M\colon (0,+\infty) \to \mathbb{C}$ and $M_{\text{asym}}\colon (0,+\infty) \to \mathbb{C}$ by

$$M(u) = \exp\Big(\operatorname{Re}(\omega_\infty)u + (\alpha+iu)x_0 + A(\alpha+iu) + v_0 B(\alpha+iu)\Big), \quad u > 0,$$

and

$$M_{\text{asym}}(u) = C_\infty e^{-i\operatorname{Im}(\omega_\infty)u}, \quad u > 0.$$

The behavior of their real parts is visualized in the following graph.



We can clearly observe that the real part of M approaches those of M_{asym}. For the imaginary part one obtains a similar plot.

With the illustrations we have just presented one might be tempted to conclude that the real parts of $M_{X_T}(\alpha+iu)$ and $C_\infty e^{-\omega_\infty u}$ should also be asymptotically equivalent. Unfortunately one has to take care as the real part of $C_\infty e^{-\omega_\infty u}$ is given by

$$\operatorname{Re}\left(C_\infty e^{-\omega_\infty u}\right) = e^{-\operatorname{Re}(\omega_\infty)u}\Big(\operatorname{Re}(C_\infty)\cos\left(\operatorname{Im}(\omega_\infty)u\right) + \operatorname{Im}(C_\infty)\sin\left(\operatorname{Im}(\omega_\infty)u\right)\Big), \quad u > 0.$$

Thus $\operatorname{Re}\left(C_\infty e^{-\omega_\infty u}\right)$ is a linear combination of a sine- and a cosine-term. Consequently we expect it to have a root on any interval $[M,+\infty)$ for an arbitrary $M>0$. Then the quotient

$$\frac{\operatorname{Re} M_{X_T}(\alpha+iu)}{\operatorname{Re}\left(C_\infty e^{-\omega_\infty u}\right)}$$

is not well-defined on any interval $[M,+\infty)$ for an arbitrary $M>0$. Stating that $\operatorname{Re} M_{X_T}(\alpha+iu)$ and $\operatorname{Re}\left(C_\infty e^{-\omega_\infty u}\right)$ are asymptotically equivalent for $u\to+\infty$ would then be of course also wrong. Finally, the following example is a simple illustration of why the real parts of two asymptotically equivalent functions need not be asymptotically equivalent themselves.

Example 5.4.8. Let $f\colon(0,+\infty)\to\mathbb{C}$ and $g\colon(0,+\infty)\to\mathbb{C}$ be defined by

$$f(x)=ix \quad \text{and} \quad g(x)=\frac{1}{x}+ix, \quad x>0.$$

Then one clearly has

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{ix}{x^{-1}+ix}=\lim_{x\to+\infty}\frac{i}{x^{-2}+i}=1.$$

However, obviously we also have

$$\lim_{x\to+\infty}\frac{\operatorname{Re} f(x)}{\operatorname{Re} g(x)}=\lim_{x\to+\infty}\frac{0}{x^{-1}}=0.$$

Thus the statement

$$\operatorname{Re} f(x)\sim \operatorname{Re} g(x), \quad \text{for} \quad x\to+\infty$$

is clearly wrong.

5.4.2 Computing Call Prices with Gauss-Laguerre Quadrature

In the previous subsection we have seen that the moment generating function of the log-discounted underlying decays exponentially when the imaginary part of the argument becomes large. Recall from (5.1) that for a given log-discounted strike k and maturity $T>0$ an arbitrage free European call price is given by

$$C(k)=R_\alpha+\frac{1}{\pi}\int_0^\infty \operatorname{Re}\left(e^{-k(\alpha-1+iu)}\frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha-1+iu)}\right)du.$$

For European call option prices in the Heston model use the constant ω_∞ from Corollary 5.4.6 to define

$$K_{X_T}(z):=z^{-1}(z-1)^{-1}\exp\Big(-k(z-1)+zx_0+A(z)+B(z)v_0+\operatorname{Re}(\omega_\infty)\operatorname{Im}(z)\Big), \quad z\in\mathbb{C}\setminus\mathbb{R}, \tag{5.28}$$

where A and B are from (5.22). The following proposition relates the integral used to determine the call price $C(k)$ to Gauss-Laguerre quadrature.

Proposition 5.4.9. *Assume that $\kappa\ge\eta\rho$[3]. Furthermore, when $\rho\in(-1,1)$ consider ω_∞ from (5.27) and let K_{X_T} be as defined in (5.28). When α belongs to $\big(u_-(T),u_+(T)\big)$ we then have*

$$C(k)=R_\alpha+\frac{1}{\pi\operatorname{Re}(\omega_\infty)}\int_0^\infty \operatorname{Re}\Big(K_{X_T}\big(\alpha+iu\operatorname{Re}(\omega_\infty)^{-1}\big)\Big)e^{-u}\,du, \tag{5.29}$$

for the price of a European call option in the Heston model. Furthermore, for some constant $D>0$ the estimate

$$\left|K_{X_T}\big(\alpha+iu\operatorname{Re}(\omega_\infty)^{-1}\big)\right|\le\frac{D}{u^2}, \quad v>0,$$

holds.

[3] This assumption comes from the fact that Corollary 4.4.10 was only proven if $\kappa\ge\eta\rho$ holds.

Proof. By Corollary 3.4.1 and Corollary 4.4.10 we have

$$\begin{aligned} C(k) &= R_\alpha + \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left(e^{-k(\alpha-1+iu)}\frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha-1+iu)}\right)du \\ &= R_\alpha + \frac{1}{\pi}\int_0^\infty \operatorname{Re}\Big(K_{X_T}(\alpha+iu)\Big)e^{-\operatorname{Re}(\omega_\infty)u}\,du \\ &= R_\alpha + \frac{1}{\pi\operatorname{Re}(\omega_\infty)}\int_0^\infty \operatorname{Re}\Big(K_{X_T}\big(\alpha+iu\operatorname{Re}(\omega_\infty)^{-1}\big)\Big)e^{-u}\,du, \end{aligned}$$

when α belongs to $\big(u_-(T), u_+(T)\big)$. Note that the boundaries of the latter interval are the critical moments. In order to prove the stated inequality observe

$$\begin{aligned} K_{X_T}\big(\alpha+iv\big) &= \frac{e^{-k(\alpha-1)}}{(\alpha+iv)(\alpha+iv-1)}\frac{M_{X_T}(\alpha+iv)}{e^{-\operatorname{Re}(\omega_\infty)v}}e^{-ikv} \\ &= \frac{e^{-k(\alpha-1)}}{(\alpha+iv)(\alpha+iv-1)}\frac{M_{X_T}(\alpha+iv)}{e^{-\omega_\infty v}}e^{iv(\operatorname{Im}(\omega_\infty)-k)}, \end{aligned}$$

for $v > 0$. Because of Corollary 5.4.6 we have

$$\lim_{v\to+\infty}\left|\frac{M_{X_T}(\alpha+iv)}{e^{-\omega_\infty v}}\right| = \big|C_\infty\big|.$$

Furthermore, the estimate

$$\left|\frac{e^{-k(\alpha-1)}}{(\alpha+iv)(\alpha+iv-1)}e^{iv(\operatorname{Im}(\omega_\infty)-k)}\right| \le \frac{e^{-k(\alpha-1)}}{v^2}, \quad v>0,$$

and the fact that all occurring expressions are continuous lead to the existence of a $\widetilde{D} > 0$ such that

$$\big|K_{X_T}\big(\alpha+iv\big)\big| \le \frac{\widetilde{D}}{v^2}, \quad v > 0.$$

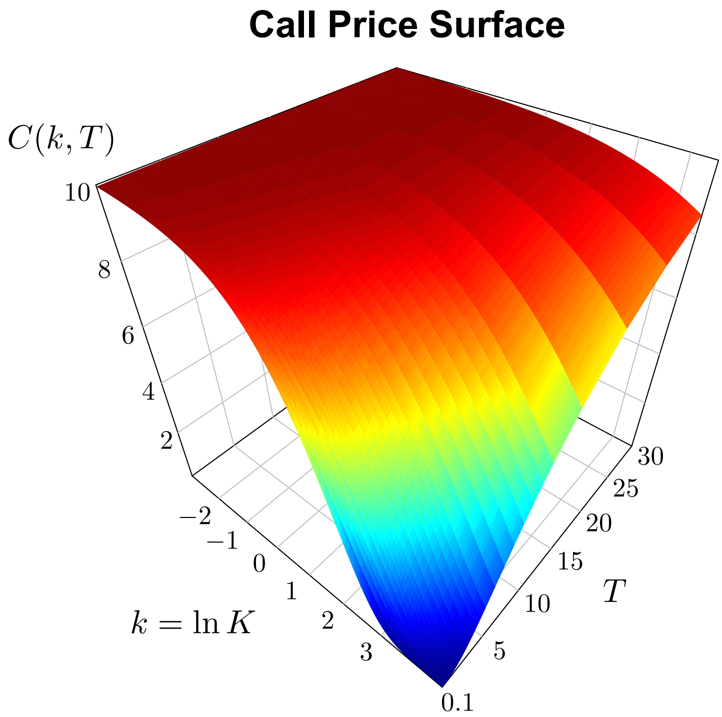
With $D = \operatorname{Re}(\omega_\infty)^2\widetilde{D}$ the stated inequality must then hold. □

Hence the integrand of the call price is represented by a function that decays quadratically times the Laguerre weighting function. Therefore it seems natural use Gauss-Laguerre quadrature to compute call prices in the Heston model. This idea is implemented in `R` in the following example.

Example 5.4.10. We again consider the Heston model specified by the parameters

κ	η	ρ	θ	r	S_0	v_0
1	2	-0.99	1	0	10	5

Now we use the result from Proposition 5.4.9 and Gauss-Laguerre quadrature to compute call prices for various strikes and maturities in `R`. For each log-strike k and maturity T we choose the integrand in (5.29) associated with the corresponding payoff-dependent α_k^*. In order to compute the Gauss-Laguerre nodes and weights we can use the `R`-package `gaussquad`. An implementation of this procedure in `R` results in the surface depicted below.



Hence this procedure works very well here.

5.4.3 Computing Call Prices After Transforming the Domain

The asymptotic behavior of the Heston moment generating function we have derived allows for another way to evaluate the integral associated with the price of a European call option. This method is based on the findings by Kahl and Jäckel in [Kah06]. Recall from (5.1) that we want to evaluate the integral on the right hand side of

$$C(k) = R_\alpha + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(e^{-k(\alpha-1+iu)}\frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha-1+iu)}\right)du.$$

Clearly this is an integral over the unbounded domain $(0,+\infty)$. The idea now is to find a transformation of $(0,+\infty)$ to $(0,1)$ such that the corresponding substitution in the integral leads to an integrand with finite limits at 0 and 1. Following Kahl and Jäckel in [Kah06] we define the transformation

$$v(y) = -\frac{\ln y}{\mathrm{Re}(\omega_\infty)}, \quad y \in (0,+\infty), \tag{5.30}$$

whenever ρ belongs to $(-1,1)$, and where ω_∞ is from (5.27). As the following proposition shows it turns out that this transformation suits our needs.

Proposition 5.4.11. *Assume that $\kappa \geq \eta\rho^4$ and that α belongs to $\big(u_-(T), u_+(T)\big)$. Furthermore, consider ω_∞ from (5.27) and let v be as defined in (5.30). If ρ belongs to the interval $(-1,1)$ define $J_{k,\alpha}\colon (0,1)\to\mathbb{R}$ by*

$$J_{k,\alpha}(y) = \mathrm{Re}\left(e^{-k(\alpha-1)-ikv(y)}\frac{M_{X_T}\big(\alpha+iv(y)\big)}{y\,\mathrm{Re}(\omega_\infty)\big(\alpha+iv(y)\big)\big(\alpha-1+iv(y)\big)}\right), \quad y \in (0,1), \tag{5.31}$$

[4]This assumption comes from the fact that Corollary 4.4.10 was only proven if $\kappa \geq \eta\rho$ holds.

for every log-discounted strike $k \in \mathbb{R}$. Then we have

$$C(k) = R_\alpha + \frac{1}{\pi}\int_0^1 J_{k,\alpha}(y)\,dy, \tag{5.32}$$

and

$$J_{k,\alpha}(0) := \lim_{y\searrow 0} J_{k,\alpha}(y) = 0.$$

If additionally $\alpha \notin \{0,1\}$ holds we also have

$$J_{k,\alpha}(1) := \lim_{y\searrow 1} J_{k,\alpha}(y) = \frac{e^{-k(\alpha-1)}}{\operatorname{Re}(\omega_\infty)}\frac{M_{X_T}(\alpha)}{\alpha(\alpha-1)}.$$

Proof. Using Corollary 3.4.1 and the one-dimensional substitution rule for Riemann integrals we obtain

$$\begin{aligned}
C(k) &= R_\alpha + \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left(e^{-k(\alpha-1+iu)}\frac{M_{X_T}(\alpha+iu)}{(\alpha+iu)(\alpha-1+iu)}\right)du\\
&= R_\alpha - \frac{1}{\pi}\int_0^1 \operatorname{Re}\left(e^{-k(\alpha-1)-ikv(y)}\frac{M_{X_T}\big(\alpha+iv(y)\big)}{\big(\alpha+iv(y)\big)\big(\alpha-1+iv(y)\big)}\right)v'(y)\,dy\\
&= R_\alpha + \frac{1}{\pi}\int_0^1 J_{k,\alpha}(y)\,dy.
\end{aligned}$$

For the second statement first note that because of $\rho \in (-1,1)$ we can use $\lim_{y\searrow 0} v(y) = +\infty$ and apply Corollary 5.4.6 to get

$$\begin{aligned}
\lim_{y\searrow 0} J_{k,\alpha}(y) &= \operatorname{Re}(\omega_\infty)^{-1}\operatorname{Re}\left(\lim_{y\searrow 0}\frac{e^{-k(\alpha-1)-ikv(y)}}{\big(\alpha+iv(y)\big)\big(\alpha-1+iv(y)\big)}\underbrace{\frac{M_{X_T}\big(\alpha+iv(y)\big)}{C_\infty e^{-\omega_\infty v(y)}}}_{\to 1\text{ by Corollary 5.4.6}}C_\infty y^{-1}e^{-\omega_\infty v(y)}\right)\\
&= \operatorname{Re}(\omega_\infty)^{-1}\operatorname{Re}\left(\lim_{y\searrow 0}\underbrace{\frac{e^{-k(\alpha-1)-ikv(y)}}{\big(\alpha+iv(y)\big)\big(\alpha-1+iv(y)\big)}}_{\to 0}C_\infty \underbrace{e^{i\frac{\operatorname{Im}(\omega_\infty)}{\operatorname{Re}(\omega_\infty)}\ln y}}_{|\cdot|\le 1}\right) = 0.
\end{aligned}$$

For the last statement of this proposition first note that $\lim_{y\nearrow 1} v(y) = 0$. Furthermore, observe that because of $\alpha \notin \{0,1\}$ none of the denominators in the expression defining $J_{k,\alpha}$ becomes zero in the limit $y \nearrow 1$. That leads to

$$\begin{aligned}
\lim_{y\nearrow 1} J_{k,\alpha}(y) &= \lim_{y\nearrow 1}\operatorname{Re}\left(e^{-k(\alpha-1)-ikv(y)}\frac{M_{X_T}\big(\alpha+iv(y)\big)}{y\operatorname{Re}(\omega_\infty)\big(\alpha+iv(y)\big)\big(\alpha-1+iv(y)\big)}\right)\\
&= \operatorname{Re}\left(e^{-k(\alpha-1)}\frac{M_{X_T}(\alpha)}{\operatorname{Re}(\omega_\infty)\alpha(\alpha-1)}\right) = \frac{e^{-k(\alpha-1)}}{\operatorname{Re}(\omega_\infty)}\frac{M_{X_T}(\alpha)}{\alpha(\alpha-1)}.
\end{aligned}$$

□

Note, that for the evaluation of the integral in (5.32) we would like to choose α such that $J_{\alpha,k}$ behaves nicely. Clearly when comparing the results of Proposition 5.3.4 and Proposition 5.4.11 we see that the payoff-dependent α_k^* from Definition 5.3.6 just minimizes $J_{k,\alpha}(1)$. As the following example shows this might not lead the integrand we would intuitively consider the least oscillatory.

Example 5.4.12. Recall the Heston model from Example 5.1.1 which is specified by the parameters

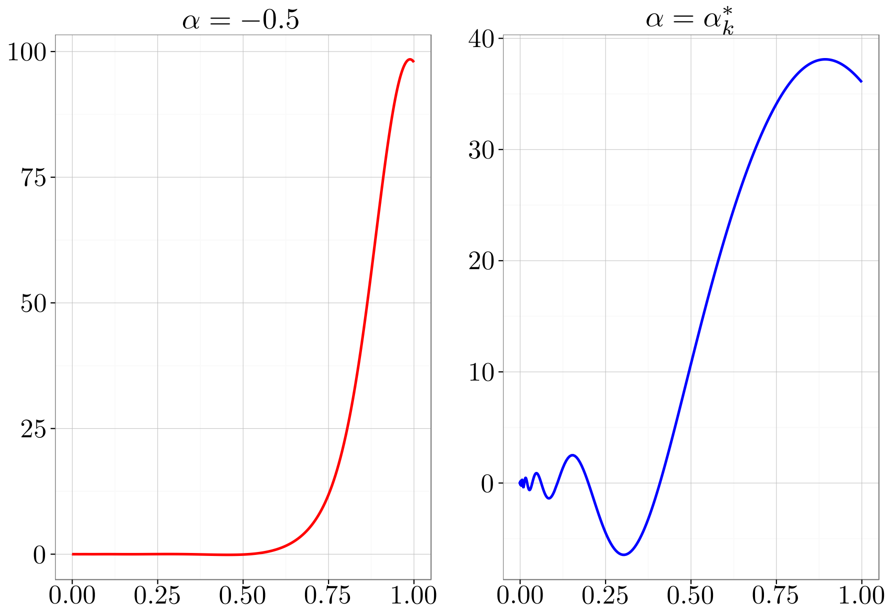
κ	η	ρ	θ	r	S_0	v_0
1	2	-0.99	1	0	10	5

.

For an option with maturity $T = 0.5$ and a log-strike $k = \ln 10$ we have seen in Example 5.3.7 that the corresponding pay-off-dependent α is given by $\alpha_k^* = 2.127$. Since we know from Example 5.1.1 that

$$u_-(T) = -1.885 \quad \text{and} \quad u_+(T) = 116.556,$$

holds also $\alpha = -0.5$ is a possible choice. The following figure now compares the integrands $J_{k,-0.5}$ and J_{k,α_k^*}.



Clearly J_{k,α_k^*} shows a more oscillatory behavior than $J_{k,-0.5}$. However, when looking at the scales of the y-axes we see that the values associated with J_{k,α_k^*} are smaller than those of $J_{k,-0.5}$. This is not surprising as $\alpha = \alpha_k^*$ is such that $J_{k,\alpha}(1)$ is minimal. Consequently, even though $J_{k,\alpha}$ can show more oscillatory behavior for $\alpha = \alpha_k^*$ than for other choices of α, it is a sensible choice as extreme function values can be avoided.

Finally, we create another call surface in the Heston model using the result obtained in Proposition 5.4.11.

Example 5.4.13. In the Heston model specified by the parameters

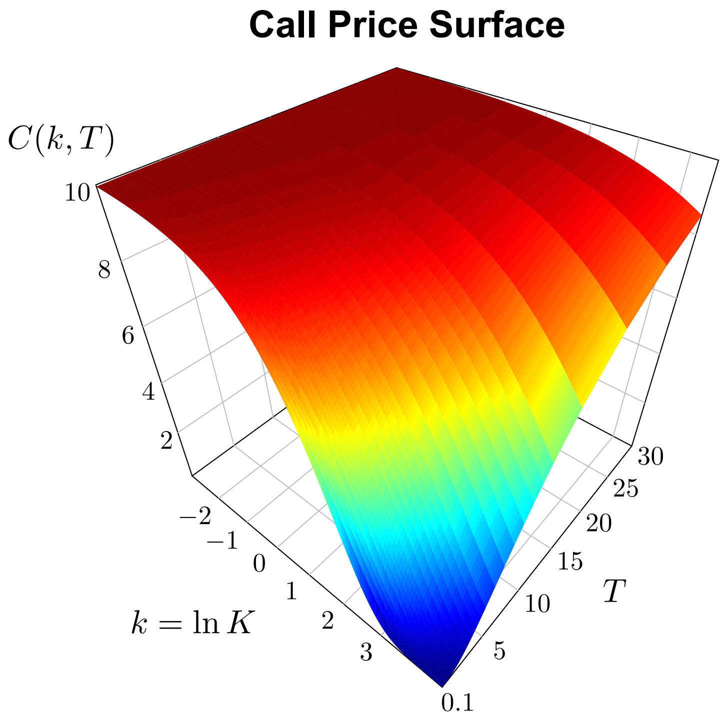
κ	η	ρ	θ	r	S_0	v_0
1	2	-0.99	1	0	10	5

,

we compute a call price surface using the result from Proposition 5.4.11. For each log-strike k and maturity T we compute the corresponding payoff-dependent $\alpha = \alpha_k^*$ and evaluate the integral in

$$C(k) = R_\alpha + \frac{1}{\pi} \int_0^1 J_{k,\alpha}(y)\, dy = R_\alpha + \frac{1}{2\pi} \int_{-1}^1 J_{k,\alpha}(0.5 + 0.5y)\, dy,$$

by means of Gauss-Legendre quadrature. In order to compute the weights and nodes of the latter we use the R-package `gaussquad`. An implementation in R then yields the following call price surface.



This surface is consistent with the one obtained in the previous subsection.

Chapter 6

Conclusion

Based on the findings by Lord and Kahl in [LK07] we discuss an optimal contour choice α in the Carr-Madan representation of the call price. In particular we provide a rigorous proof that this choice, denoted by α_k^*, minimizes the maximal modulus of the integrand associated with the Carr-Madan representation. Moreover, we show that the optimization problem characterizing α_k^* always has a solution and thus α_k^* is well-defined. This is followed by a short illustration depicting why this choice can be regarded as saddlepoint. The obtained results are then applied in the Heston model. In order to do so we need the moment generating function M_{X_T} of the log-discounted underlying in the Heston model. Since there are representations of M_{X_T} in the literature that are discontinuous we decided to provide a full derivation of it within this thesis. It turns out that – once this is properly done – there cannot be any worries about discontinuities. For that derivation it is essential to ensure that a certain complex integrand does not cross the negative real axis. Such considerations appear in a paper by Lord and Kahl were complex logarithms in the Heston model are discussed. However, these observations already need to be made during the derivation of the closed-form expression for M_{X_T}. Moreover, we also provide an analysis of the critical time T^* in the Heston model. This leads to a proof of why the critical moments for a maturity T are characterized by solving $T^*(u) = T$, where we need the additional assumption $2\kappa \geq \eta\rho$ when $u < 0$. For the application of the results on the optimal contour choice α_k^* we derive the precise asymptotic behavior of M_{X_T} in the Heston model when the imaginary part of its argument becomes large and $\rho \in (-1, 1)$. This asymptotic behavior allows us to derive and present two formulas for the call option price in the Heston model. The first formula makes use of the fact that M_{X_T} decays exponentially and can easily be implemented using Gauss-Laguerre quadrature. Together with the suggested choice of $\alpha = \alpha_k^*$ one can reliably compute Heston call prices – even for short maturities and far out-the-money options. The second formula we present for the Heston call option price only contains an integral of a continuous function on $[0, 1]$. Finally, we would like to point out that similar formulas can be derived for option pricing models where the moment generating function of the log-discounted underlying also decays exponentially when the imaginary part of the argument becomes large.

Appendix A

Additional Proofs

A.1 Chapter 2

A.1.1 Proof of Proposition 3.1.8

Proof. Define

$$I := (0, u_+ + \delta)$$
$$A := \Big\{ u \in I \,\Big|\, M_X(u) < \infty \text{ and } M_X(u) = h(u) \Big\}.$$

By a part of our assumptions we have $(0, u_+) \subseteq A$ and due to $u_+ > 0$ we clearly have $A \neq \emptyset$. It is also obvious that $A \subseteq I$ holds. Since the interval I is connected we know that I is the only non-empty subset of I which is open and closed with respect to the relative topology in I. Thus to obtain $A = I$ it suffices to show that A is open and closed.

To see that A is closed consider a sequence $(u_n)_{n\in\mathbb{N}}$ in A converging to some $u \in I$. By Fatou's lemma we have

$$\mathbb{E}\big(e^{uX}\big) = \mathbb{E}\Big(\liminf_{n\to\infty} e^{u_n X}\Big) \leq \liminf_{n\to\infty} \mathbb{E}\big(e^{u_n X}\big) = \liminf_{n\to\infty} h(u_n) = h(u) < +\infty. \tag{A.1}$$

Hence e^{uX} is integrable. Now we distinguish two cases.

(i) There is an $n \in \mathbb{N}$ such that $u_n > u$.

In this case, because of $\lim_{n\to\infty} u_n = u$, there is an n_0 such that $u_{n_0} \geq u_n$ for $n \in \mathbb{N}$. Consequently one has

$$\big|e^{u_n X}\big| \leq 1 + \mathbb{1}_{\{X>0\}} e^{u_{n_0} X} \leq 1 + e^{u_{n_0} X}.$$

Because of $u_{n_0} \in A$ the right hand side of the latter inequality is integrable. Hence dominated convergence can be applied to get

$$\mathbb{E}\big(e^{uX}\big) = \mathbb{E}\Big(\lim_{n\to\infty} e^{u_n X}\Big) = \lim_{n\to\infty} \mathbb{E}\big(e^{u_n X}\big) = \lim_{n\to\infty} h(u_n) = h(u).$$

(ii) For all $n \in \mathbb{N}$ we have $u_n \leq u$.

Here we have

$$\big|e^{u_n X}\big| \leq 1 + \mathbb{1}_{\{X>0\}} e^{uX} \leq 1 + e^{uX},$$

where the latter is integrable due to (A.1). Hence dominated convergence can be applied again to get

$$\mathbb{E}\big(e^{uX}\big) = \mathbb{E}\Big(\lim_{n\to\infty} e^{u_n X}\Big) = \lim_{n\to\infty} \mathbb{E}\big(e^{u_n X}\big) = \lim_{n\to\infty} h(u_n) = h(u).$$

Consequently in any case $u \in A$.
Now we show that A is also open which is the key part of the proof. Therefore consider $u^* \in (0, u_+ + \delta)$ such that $M_X(u^*) = h(u^*) < \infty$. If $u^* \in (0, u_+)$ then there is clearly an open neighborhood of u^* that fully lies in A by the assumptions made. Thus we can restrict ourselves to the case where $u^* \geq u_+$. Then, whenever $0 < u < u^*$ holds, we have

$$\mathbb{E}\big(e^{uX}\big) \leq 1 + \mathbb{E}\big(e^{u^* X}\big) < \infty.$$

Consequently the domain of M_X at least contains the set $(0, u^*) + i\mathbb{R}$. By Proposition 3.1.7 we know that M_X is even analytic on $(0, u^*) + i\mathbb{R}$. By the assumptions made we also know that h is analytic on the connected set $(0, u^*) + i(-\epsilon, \epsilon) \subseteq \mathbb{C}$. In addition M_X and h coincide on the set $(0, u_+)$ which has a limit point. Thus by the identity theorem of complex analysis (Theorem 2.3.5) we know that h and M_X have to coincide on the entire set $(0, u^*) + i(-\epsilon, \epsilon)$. In particular we have

$$M_X(u) = h(u), \quad 0 < u < u^*. \tag{A.2}$$

Now fix $\eta \in \big(0, |u_-|\big)$ and $u_0 \in (0, u^*)$ such that $u_0 + \eta \in (u^*, u_+ + \delta)$ and $B_\eta(u_0) \subseteq D$, where

$$D = (u_-, u_+) + i(-\epsilon, \epsilon).$$

For example η and u_0 such that

$$0 < \eta < \min\left(\epsilon, |u_-|, u_+ + \delta - u^*\right) \quad \text{and} \quad \max(u^* - \eta, 0) < u_0 < u^*$$

do the job. Since h is analytic on D we know that it has the following representation

$$h(z) = \sum_{k=0}^{\infty} \frac{h^{(k)}(u_0)}{k!}(z - u_0)^k, \quad \forall z \in B_\eta(u_0). \tag{A.3}$$

Now consider a fixed but arbitrary $u_1 \in [u^*, u_+ + \delta) \cap B_\eta(u_0)$. Such a u_1 exists by construction of η and u_0. Using Proposition 3.1.7 we see that the function M_X is arbitrarily often differentiable at u_0 with k-th derivative

$$M_X^{(k)}(u_0) = \mathbb{E}\big(X^k e^{u_0 X}\big), \quad \forall k \in \mathbb{N}_0,$$

and in particular we have $\mathbb{E}\big(|X|^k e^{u_0 X}\big) < +\infty$ for every $k \in \mathbb{N}_0$. Using Fubini's theorem for non-negative integrands we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} \int_\Omega \left| \frac{(u_1 - u_0)^k}{k!} X^k e^{u_0 X} \right| dP &= \sum_{k=0}^{\infty} \frac{(u_1 - u_0)^k}{k!} \left(\int_\Omega \mathbb{1}_{\{X \geq 0\}} X^k e^{u_0 X}\, dP + \int_\Omega \mathbb{1}_{\{X<0\}} (-X)^k e^{u_0 X}\, dP \right) \\
&= \sum_{k=0}^{\infty} \frac{(u_1 - u_0)^k}{k!} \left(\int_\Omega X^k e^{u_0 X}\, d\mathbb{P} + \int_\Omega \mathbb{1}_{\{X<0\}} e^{u_0 X} \big((-X)^k - X^k\big)\, dP \right) \leq \\
&\leq \sum_{k=0}^{\infty} \frac{(u_1 - u_0)^k}{k!} \left(\int_\Omega X^k e^{u_0 X}\, dP + 2 \int_\Omega \mathbb{1}_{\{X<0\}} e^{u_0 X} (-X)^k\, dP \right) \leq \\
&\leq \sum_{k=0}^{\infty} \frac{(u_1 - u_0)^k}{k!} \int_\Omega X^k e^{u_0 X}\, dP + 2 \sum_{k=0}^{\infty} \int_{\{X<0\}} \underbrace{\frac{(u_1 - u_0)^k}{k!} (-X)^k}_{\geq 0}\, dP = \\
&= \sum_{k=0}^{\infty} \frac{(u_1 - u_0)^k}{k!} M_X^{(k)}(u_0) + 2 \int_{\{X<0\}} \sum_{k=0}^{\infty} \frac{(u_1 - u_0)^k}{k!} (-X)^k\, dP = \\
&= \sum_{k=0}^{\infty} \frac{(u_1 - u_0)^k}{k!} h^{(k)}(u_0) + 2 \int_{\{X<0\}} e^{-(u_1 - u_0)X}\, dP \leq \\
&\leq h(u_1) + 2 M_X\big(-(u_1 - u_0)\big) < +\infty,
\end{aligned}$$

because $u_1 \in (0, u_+ + \delta)$ and $u_- < -\eta < -(u_1 - u_0) < 0$ by construction. Thus we can apply Fubini's theorem to show

$$h(u_1) = \sum_{k=0} \frac{(u_1 - u_0)^k}{k!} h^{(k)}(u_0) = \sum_{k=0} \frac{(u_1 - u_0)^k}{k!} M_X^{(k)}(u_0) = \sum_{k=0}^{\infty} \int_\Omega \frac{(u_1 - u_0)^k}{k!} X^k e^{u_0 X} \, dP =$$
$$= \int_\Omega \sum_{k=0}^{\infty} \frac{(u_1 - u_0)^k}{k!} X^k e^{u_0 X} \, dP = \int_\Omega e^{(u_1 - u_0)X} e^{u_0 X} \, dP = \int_\Omega e^{u_1 X} \, dP = \mathbb{E}\big(e^{u_1 X}\big).$$

In particular $\mathbb{E}\big(e^{u_1 X}\big) = h(u_1)$ is also finite. Since $u_1 \in [u^*, u_+ + \delta) \cap B_\eta(u_0)$ was arbitrary we thus have $\mathbb{E}\big(e^{uX}\big) = h(u)$ for any $u \in [u^*, u_+ + \delta) \cap B_\eta(u_0)$ additionally to (A.2). Hence the open neighborhood $(0, u_+ + \delta) \cap B_\eta(u_0)$ of u^* fully lies in A. Consequently A is also open. Hence $A = (0, u_+ + \delta)$. $\square$

A.2 Chapter 4

A.2.1 Proof of Lemma 4.2.3 and Lemma 4.2.4

Proof (of Lemma 4.2.3 and 4.2.4). For the proof distinguish the following cases.

(i) $\rho \in (-1, 1)$

For real arguments u we have

$$D(u) = \kappa^2 - u^2\eta^2(1 - \rho^2) + u(\eta - 2\kappa\rho)\eta.$$

Because of $\rho \in (-1, 1)$ the latter is a second order polynomial whose roots are given by

$$u_{r,l} = \frac{\eta - 2\kappa\rho \pm \sqrt{(\eta - 2\kappa\rho)^2 + 4\kappa^2(1 - \rho^2)}}{2\eta(1 - \rho^2)}.$$

Obviously we have u_r is positive and u_l is negative. Every quadratic polynomial with a negative leading coefficient has a global maximum given by the arithmetic mean of its roots. If $\rho \in (-1, 1)$ we just have

$$\frac{u_l + u_r}{2} = \frac{1}{2} \frac{2\eta - 4\kappa\rho}{2\eta(1 - \rho^2)} = \frac{\eta - 2\kappa\rho}{2\eta(1 - \rho^2)} = u_{\max},$$

and thus $D|_{\mathbb{R}}$ is strictly increasing on $(-\infty, u_{\max})$ and strictly decreasing on $(u_{\max}, +\infty)$. Thus Lemma 4.2.3 and Lemma 4.2.4 are proven in this case.

(ii) $\rho = -1$ or $\rho = 1$ and $\eta > 2\kappa$

Then we have

$$D(u) = \kappa^2 - u^2\eta^2(1 - \rho^2) + u(\eta - 2\kappa\rho)\eta = \kappa^2 + u(\eta - 2\kappa\rho)\eta,$$

which is an affine function with positive slope under the assumptions made. Furthermore, it has a unique root at $u_l = -\frac{\kappa^2}{(\eta - 2\kappa\rho)\eta} < 0$. Moreover, due to $u_{\max} = +\infty$ in this case we have $(-\infty, u_{\max}) = \mathbb{R}$ and $(u_{\max}, +\infty) = \emptyset$. Obviously $D|_{\mathbb{R}}$ is then strictly increasing on $(-\infty, u_{\max})$ and strictly decreasing on $(u_{\max}, +\infty)$.

(iii) $\rho = 1$ and $\eta < 2\kappa$

We again have

$$D(u) = \kappa^2 - u^2\eta^2(1 - \rho^2) + u(\eta - 2\kappa\rho)\eta = \kappa^2 + u(\eta - 2\kappa)\eta,$$

which is an affine function with negative slope under the assumptions made. Furthermore, it has a unique root at $u_r = \frac{\kappa^2}{(2\kappa - \eta)\eta} > 0$. Moreover, due to $u_{\max} = -\infty$ in this case we have $(-\infty, u_{\max}) = \emptyset$ and $(u_{\max}, +\infty) = \mathbb{R}$. Obviously $D|_{\mathbb{R}}$ is then strictly increasing on $(-\infty, u_{\max})$ and strictly decreasing on $(u_{\max}, +\infty)$.

(iv) $\rho = 1$ and $\eta = 2\kappa$

Then we have

$$D(u) = \kappa^2 - u^2\eta^2(1-\rho^2) + u(\eta - 2\kappa\rho)\eta = \kappa^2 > 0,$$

for every $u \in \mathbb{R}$.

$\square$

A.2.2 Proof of Lemma 4.2.8 – Remaining Part

Proof. Taking the derivative of Ψ_z with respect to t yields

$$\begin{aligned}
\Psi_z'(t) = {} & \frac{z(z-1)}{2}\Bigg(\frac{-H(z)\cosh\big((T-t)H(z)\big)\Big(H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)\Big)}{\Big(H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)\Big)^2} \\
& + \frac{\sinh\big((T-t)H(z)\big)H(z)\Big(H(z)\sinh\big((T-t)H(z)\big) - b(z)\cosh\big((T-t)H(z)\big)\Big)}{\Big(H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)\Big)^2}\Bigg) \\
= {} & \frac{z(z-1)}{2}H(z)^2\frac{\sinh^2\big((T-t)H(z)\big) - \cosh^2\big((T-t)H(z)\big)}{\Big(H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)\Big)^2} = \\
= {} & -\frac{z(z-1)}{2}H(z)^2\Big(H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)\Big)^{-2}.
\end{aligned}$$

Since the denominator is never zero Ψ_z clearly belongs to $\mathcal{C}^1([0,T],\mathbb{C})$. On the other hand we get for the right hand side of (4.19)

$$\begin{aligned}
\text{R.h.s} = {} & -\frac{\eta^2}{2}\Psi_z(t)^2 - 2b(z)\Psi_u(t) - \frac{z(z-1)}{2} \\
= {} & -\frac{\eta^2}{2}\frac{z^2(z-1)^2}{4}\frac{\sinh^2\big((T-t)H(z)\big)}{\Big(H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)\Big)^2} \\
& - 2b(z)\frac{z(z-1)}{2}\frac{\sinh\big((T-t)H(z)\big)}{H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)} - \frac{z(z-1)}{2} = \\
= {} & \frac{z(z-1)}{2}\Big(H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)\Big)^{-2} \\
& \Bigg(-\frac{\eta^2}{4}z(z-1)\sinh^2\big((T-t)H(z)\big) - 2b(z)\sinh\big((T-t)H(z)\big)\Big(H(z)\cosh\big((T-t)H(z)\big) \\
& - b(z)\sinh\big((T-t)H(z)\big)\Big) - \Big(H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)\Big)^2\Bigg) = \\
= {} & \frac{z(z-1)}{2}\frac{\Big(b(z)^2 - \frac{\eta^2}{4}z(z-1)\Big)\sinh^2\big((T-t)H(z)\big) - H(z)^2\cosh^2\big((T-t)H(z)\big)}{\Big(H(z)\cosh\big((T-t)H(z)\big) - b(z)\sinh\big((T-t)H(z)\big)\Big)^2}.
\end{aligned}$$

This together with

$$b(z)^2 - \frac{\eta^2}{4}z(z-1) = \frac{1}{4}\Big(4b(z)^2 - z(z-1)\eta^2\Big) = \frac{1}{4}D(z) = H(z)^2,$$

yields

$$\text{R.h.s} = -\frac{z(z-1)}{2} H(u)^2 \Big(H(z) \cosh\big((T-t)H(z)\big) - b(z) \sinh\big((T-t)H(z)\big)\Big)^{-2},$$

which coincides with the derivative of Ψ_z at every point $t \in [0,T]$. ☐

A.2.3 Proof of Lemma 4.2.11

Proof. First, we want to show that the derivative of C with respect to τ is positive to conclude monotonicity. Observe that one has

$$\begin{aligned}
4e^{\kappa\tau} C'(\tau) &= 4e^{\kappa\tau} \Big(\kappa(\kappa - \rho\eta) \sinh(\tau\kappa) + \kappa^2 \cosh(\tau\kappa) + \frac{\eta(\eta - 2\kappa\rho)}{2} \big(\cosh(\kappa\tau) - 1 \big) \Big) \\
&= \Big(2\kappa(\kappa - \rho\eta) + 2\kappa^2 + \eta(\eta - 2\kappa\rho) \Big) e^{2\kappa\tau} - 2\eta(\eta - 2\kappa\rho) e^{\kappa\tau} + 2\kappa^2 + \eta(\eta - 2\kappa\rho) - 2\kappa(\kappa - \rho\eta) \\
&= \Big(4\kappa^2 - 4\kappa\rho\eta + \rho^2\eta^2 + \eta^2(1-\rho^2) \Big) e^{2\kappa\tau} - 2\eta(\eta - 2\kappa\rho) e^{\kappa\tau} + \eta^2 \\
&= \Big((2\kappa - \rho\eta)^2 + \eta^2(1-\rho^2) \Big) e^{2\kappa\tau} - 2\eta(\eta - 2\kappa\rho) e^{\kappa\tau} + \eta^2 \\
&= p(e^{\kappa\tau}),
\end{aligned}$$

where $p(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$ with

$$\begin{aligned}
a &= (2\kappa - \rho\eta)^2 + \eta^2(1-\rho^2) \\
b &= -2\eta(\eta - 2\kappa\rho) \\
c &= \eta^2.
\end{aligned}$$

Now we want to show that there is an $\epsilon > 0$ such that

$$\begin{aligned}
p(x) &> 0, \quad x \in [1, 1+\epsilon] \\
p(x) &\geq 0, \quad x \in (1+\epsilon, +\infty).
\end{aligned} \tag{A.4}$$

Therefore, distinguish three cases for $\rho \in [-1,1]$.

- ▷ $\rho \in (-1,1)$

  Because of $|\rho| < 1$ we see that $a > 0$. The quadratic polynomial p has then real roots if and only if

  $$\begin{aligned}
  0 \leq b^2 - 4ac &= 4\eta^2 \Big((\eta - 2\kappa\rho)^2 - (2\kappa - \rho\eta)^2 - \eta^2(1-\rho^2) \Big) \\
  &= 4\eta^2 \Big(\eta^2 - 4\kappa\eta\rho + 4\kappa^2\rho^2 - 4\kappa^2 + 4\kappa\rho\eta - \rho^2\eta^2 - \eta^2 + \eta^2\rho^2 \Big) \\
  &= -16\eta^2\kappa^2(1-\rho^2).
  \end{aligned}$$

  Hence because of $\rho \in (-1,1)$ the polynomial p cannot have any real roots. Since also $p(0) = \eta^2 > 0$ we even get $p(x) > 0$ for $x \in \mathbb{R}$ in this case.

- ▷ $\rho = -1$

  In this case we have $\eta - 2\kappa\rho = \eta + 2\kappa > 0$ and thus $a > 0$. The polynomial p is then of the form

  $$p(x) = (2\kappa + \eta)^2 x^2 - 2\eta(\eta + 2\kappa)x + \eta^2 = \big((\eta + 2\kappa)x - \eta \big)^2,$$

  which is zero if and only if

  $$x = \frac{\eta}{\eta + 2\kappa} < 1.$$

  Hence $p(x) > 0$ for $x \geq 1$.

▷ $\rho = 1$
The polynomial p is then of the form

$$p(x) = (2\kappa - \eta)^2 x^2 + 2\eta(2\kappa - \eta)x + \eta^2 = \big((2\kappa - \eta)x + \eta\big)^2.$$

We again distinguish two cases.

- $2\kappa \geq \eta$
  Because of $2\kappa \geq \eta$ we have

$$p(x) \geq \eta^2, \quad x \geq 0$$

  and in particular $p(x) > 0$ for $x \geq 1$.
- $2\kappa < \eta$
  Then $p(x) = 0$ if and only if

$$x = \frac{\eta}{\eta - 2\kappa} > 1$$

  Here p has a unique root beyond 1 but clearly an $\epsilon > 0$ satisfying (A.4) exists.

Thus we also now that there exists an $\epsilon > 0$ such that C' is positive on $[0, \epsilon]$ and non-negative on (ϵ, ∞). Hence due to the resulting monotonicity we can conclude

$$\begin{array}{ll} C(\tau) > C(0) = 0 & \tau \in (0, \epsilon] \\ C(\tau) \geq C(\epsilon) > 0 & \tau \in [\epsilon, +\infty) \end{array}.$$

Consequently we have $C(\tau) > 0$ for all $\tau > 0$.

□

A.2.4 Proof of Lemma 4.2.12

For the proof we need the following notation and lemma.

$$\begin{aligned} H_1(z) &= 2\,\mathrm{Re}\big(H(z)\big) \\ H_2(z) &= 2\,\mathrm{Im}\big(H(z)\big) \\ Y_\tau(z) &= H(z)\cosh\big(\tau H(z)\big) - b(z)\sinh\big(\tau H(z)\big), \quad 0 \leq \tau \leq T, \end{aligned}$$

for $z \in U_H$.

Lemma A.2.1. *Let $\epsilon > 0$ be such that*

$$Y_\tau(iu) \neq 0, \quad 0 \leq \tau \leq T,$$

for every $u \in (-\epsilon, \epsilon)$. With the notation we have just introduced one then has

$$-8|Y_\tau(u)|^2\,\mathrm{Re}\big(\Psi_{iu}(T - \tau)\big) = h_\tau(u), \quad u \in \mathbb{R} \quad \textit{and} \quad 0 \leq \tau \leq T,$$

where

$$\begin{aligned} h_\tau(u) = u^2\Big(&2(\kappa - \rho\eta)\big|\sinh\big(\tau H(iu)\big)\big|^2 + H_1(iu)\sinh\big(\tau H_1(iu)\big) + H_2(iu)\sin\big(\tau H_2(iu)\big)\Big) \\ &+ u\Big(H_2(iu)\sinh\big(\tau H_1(iu)\big) - H_1(iu)\sin\big(\tau H_2(iu)\big)\Big), \end{aligned}$$

for $u \in (-\epsilon, \epsilon)$.

Proof. First observe that because of $Y_\tau(iu) \neq 0$ we know that $\Psi_{iu}(T-\tau)$ is well-defined for $\tau \in [0,T]$ and $u \in (-\epsilon, \epsilon)$. Then recall the identity

$$\sinh z \cosh \overline{z} = \frac{1}{2}\Big(\sinh\big(2\operatorname{Re}(z)\big) + i \sin\big(2\operatorname{Im}(z)\big)\Big), \quad z \in \mathbb{C}.$$

Using the latter leads to the following expression for $-8|Y_\tau(u)|^2 \Psi_{iu}(T-\tau)$.

$$\begin{aligned}
-8|Y_\tau(u)|^2 \Psi_{iu}(T-\tau) &= 4(u^2+iu)\sinh\big(\tau H(iu)\big)\overline{Y_\tau(u)} \\
&= 4(u^2+iu)\sinh\big(\tau H(iu)\big)\Big(\overline{H(iu)}\cosh\big(\overline{\tau H(iu)}\big) - b(-iu)\sinh\big(\overline{\tau H(iu)}\big)\Big) \\
&= (u^2+iu)\big(H_1(iu) - iH_2(iu)\big)\Big(\sinh\big(\tau H_1(iu)\big) + i\sin\big(\tau H_2(iu)\big)\Big) \\
&\quad + 2(u^2+iu)(iu\eta\rho+\kappa)\big|\sinh\big(\tau H(iu)\big)\big|^2 \\
&= \Big(u^2H_1(iu) + uH_2(iu) + i\big(uH_1(iu) - u^2H_2(iu)\big)\Big)\Big(\sinh\big(\tau H_1(iu)\big) \\
&\quad + i\sin\big(\tau H_2(iu)\big)\Big) + 2\Big(u^2(\kappa-\eta\rho) + iu(\kappa+u^2\eta\rho)\Big)\big|\sinh\big(\tau H(iu)\big)\big|^2.
\end{aligned}$$

Expanding the last expression and taking real parts afterwards gives then

$$\begin{aligned}
-8|Y_\tau(u)|^2 \operatorname{Re}\big(\Psi_{iu}(T-\tau)\big) &= \Big(u^2H_1(iu) + uH_2(iu)\Big)\sinh\big(\tau H_1(iu)\big) \\
&\quad - \Big(uH_1(iu) - u^2H_2(iu)\Big)\sin\big(\tau H_2(iu)\big) + 2u^2(\kappa-\eta\rho)\big|\sinh\big(\tau H(iu)\big)\big|^2 \\
&= u^2\Big(2(\kappa-\eta\rho)\big|\sinh\big(\tau H(iu)\big)\big|^2 \\
&\quad + H_1(iu)\sinh\big(\tau H_1(iu)\big) + H_2(iu)\sin\big(\tau H_2(iu)\big)\Big) \\
&\quad + u\Big(H_2(iu)\sinh\big(\tau H_1(iu)\big) - H_1(iu)\sin\big(\tau H_2(iu)\big)\Big),
\end{aligned}$$

which concludes the proof. □

Now we can prove the Lemma 4.2.12.

Proof (of Lemma 4.2.12). First, note that by Proposition 4.2.7 there is an $\epsilon > 0$ such that

$$Y_\tau(iu) = H(iu)\cosh\big(\tau H(iu)\big) - b(iu)\cosh\big(\tau H(iu)\big) = H(iu)\gamma_{iu,0}(T-\tau) \neq 0, \quad 0 \le \tau \le T,$$

for $u \in (-\epsilon, \epsilon)$. Hence for that $\epsilon > 0$ Lemma A.2.1 is applicable. Next, define the analytic functions $\operatorname{sinc}\colon \mathbb{C} \to \mathbb{C}$ and $\operatorname{sinhc}\colon \mathbb{C} \to \mathbb{C}$ by

$$\operatorname{sinc} z = \begin{cases} \frac{\sin z}{z} & z \in \mathbb{C}\setminus\{0\} \\ 1 & z = 0 \end{cases} \qquad \operatorname{sinhc} z = \begin{cases} \frac{\sinh z}{z} & z \in \mathbb{C}\setminus\{0\} \\ 1 & z = 0 \end{cases}.$$

For $\sigma \in (0,T]$ and $u \in (-\epsilon, \epsilon)$ consider now

$$\begin{aligned}
\frac{h_\sigma(u)}{\sigma u^2} &= 2(\kappa-\rho\eta)\sigma\frac{\big|\sinh\big(\sigma H(iu)\big)\big|^2}{\sigma^2} + H_1(iu)\frac{\sinh\big(\sigma H_1(iu)\big)}{\sigma} + H_2(iu)\frac{\sin\big(\sigma H_2(iu)\big)}{\sigma} \\
&\quad + u^{-1}\Big(H_2(iu)\frac{\sinh\big(\sigma H_1(iu)\big)}{\sigma} - H_1(iu)\frac{\sin\big(\sigma H_2(iu)\big)}{\sigma}\Big) \\
&= 2(\kappa-\rho\eta)\sigma\big|\operatorname{sinhc}\big(\sigma H(iu)\big)\big|^2|H(iu)|^2 + H_1(iu)^2\operatorname{sinhc}\big(\sigma H_1(iu)\big) + H_2(iu)^2\operatorname{sinc}\big(\sigma H_2(iu)\big) \\
&\quad + u^{-1}\Big(H_2(iu)H_1(iu)\operatorname{sinhc}\big(\sigma H_1(iu)\big) - H_1(iu)H_2(iu)\operatorname{sinc}\big(\sigma H_2(iu)\big)\Big).
\end{aligned}$$

Furthermore, because of $4H(iu)^2 = D(iu)$ we get

$$H_2(iu)H_1(iu) = \frac{1}{2}\operatorname{Im}\big(D(iu)\big) = \frac{1}{2}u\eta(\eta - 2\kappa\rho).$$

Plugging the latter into the equation for $\frac{h_\sigma(u)}{\sigma u^2}$ above yields

$$\begin{aligned}\frac{h_\sigma(u)}{\sigma u^2} &= 2(\kappa-\rho\eta)\sigma\big|\operatorname{sinhc}\big(\sigma H(iu)\big)\big|^2|H(iu)|^2 + H_1(iu)^2\operatorname{sinhc}\big(\sigma H_1(iu)\big) + H_2(iu)^2\operatorname{sinc}\big(\sigma H_2(iu)\big)\\ &\quad + \frac{\eta(\eta-2\kappa\rho)}{2}\Big(\operatorname{sinhc}\big(\sigma H_1(iu)\big) - \operatorname{sinc}\big(\sigma H_2(iu)\big)\Big).\end{aligned} \tag{A.5}$$

Now define $\tau := T - t$ and observe that

$$\begin{aligned}&\lim_{u\to 0} H(iu) = H(0) = \frac{\kappa}{2}\\ &\lim_{u\to 0} H_1(iu) = 2\operatorname{Re}(H(0)) = \kappa\\ &\lim_{u\to 0} H_2(iu) = 2\operatorname{Im}(H(0)) = 0.\end{aligned} \tag{A.6}$$

Using (A.6) and $\operatorname{sinc}(0) = 1$ for taking the limit $(\sigma, u) \to (\tau, 0)$ in (A.5) gives then

$$\lim_{(\sigma,u)\to(\tau,0)} \frac{h_\sigma(u)}{\sigma u^2} = 2(\kappa-\rho\eta)\tau\operatorname{sinhc}^2\Big(\frac{\kappa\tau}{2}\Big)\frac{\kappa^2}{4} + \kappa^2\operatorname{sinhc}(\kappa\tau) + \frac{\eta(\eta-2\kappa\rho)}{2}\Big(\operatorname{sinhc}(\kappa\tau) - 1\Big). \tag{A.7}$$

Now distinguish the following cases.

▷ $\tau = 0$
Because of $\operatorname{sinhc}(0) = 1$ equation (A.7) simplifies to

$$\lim_{(\sigma,u)\to(0,0)} \frac{h_\sigma(u)}{\sigma u^2} = \kappa^2.$$

Together with

$$\lim_{(\sigma,u)\to(0,0)} |Y_\sigma(u)|^2 = \big|H(0)\cosh(0) - b(0)\sinh(0)\big|^2 = \frac{\kappa^2}{4},$$

this gives

$$\begin{aligned}\lim_{(s,u)\to(T,0)} \frac{\operatorname{Re}\big(\Psi_{iu}(s)\big)}{(T-s)u^2} &= \lim_{(\sigma,u)\to(0,0)} \frac{\operatorname{Re}\big(\Psi_{iu}(T-\sigma)\big)}{\sigma u^2} = -\lim_{(\sigma,u)\to(0,0)} \frac{h_\sigma(u)}{8|Y_\sigma(u)|^2\sigma u^2} =\\ &= -\lim_{(\sigma,u)\to(0,0)} \frac{1}{8|Y_\sigma(u)|^2}\frac{h_\sigma(u)}{\sigma u^2} = -\frac{1}{2\kappa^2}\kappa^2 = -\frac{1}{2}.\end{aligned}$$

Thus we have

$$\operatorname{Re}\big(\Psi_{iu}(s)\big) \sim -\frac{u^2}{2}(T-s), \qquad (s,u)\to(T,0).$$

▷ $\tau \in (0, T]$
Then (A.7) and

$$\sinh^2 z = \frac{1}{2}\big(\cosh 2z - 1\big), \quad z \in \mathbb{C},$$

lead to

$$\begin{aligned}
\lim_{(\sigma,u)\to(\tau,0)} \frac{h_\sigma(u)}{u^2} &= \tau \lim_{(\sigma,u)\to(\tau,0)} \frac{h_\sigma(u)}{\sigma u^2} \\
&= 2(\kappa-\rho\eta)\sinh^2\left(\frac{\kappa\tau}{2}\right) + \kappa\sinh\big(\kappa\tau\big) + \frac{\eta(\eta-2\kappa\rho)}{2}\left(\frac{\sinh(\kappa\tau)}{\kappa} - \tau\right) \\
&= (\kappa-\rho\eta)\big(\cosh(\kappa\tau)-1\big) + \kappa\sinh\big(\kappa\tau\big) + \frac{\eta(\eta-2\kappa\rho)}{2\kappa}\big(\sinh(\kappa\tau)-\kappa\tau\big) = C(\tau).
\end{aligned}$$

Furthermore, observe

$$\lim_{(\sigma,u)\to(\tau,0)} |Y_\sigma(u)|^2 = \left|\frac{\kappa}{2}\cosh\left(\frac{\kappa\tau}{2}\right) + \frac{\kappa}{2}\sinh\left(\frac{\kappa\tau}{2}\right)\right|^2 = \frac{\kappa^2}{4}\big|e^{\frac{\kappa\tau}{2}}\big|^2 = \frac{\kappa^2}{4}e^{\kappa\tau}.$$

A combination of the latter two limits gives then

$$\lim_{(s,u)\to(t,0)} \frac{\operatorname{Re}\big(\Psi_{iu}(s)\big)}{u^2} = -\lim_{(\sigma,u)\to(\tau,0)} \frac{h_\sigma(u)}{8|Y_\sigma(u)|^2u^2} = -\frac{e^{-\kappa\tau}}{2\kappa^2}\lim_{(\sigma,u)\to(\tau,0)} \frac{h_\sigma(u)}{u^2} = -\frac{e^{-\kappa\tau}}{2\kappa^2}C(\tau).$$

By Lemma 4.2.11 we have $C(\tau) > 0$ and thus

$$\operatorname{Re}\big(\Psi_{iu}(s)\big) \sim -u^2\frac{C(T-t)}{2\kappa^2}e^{-\kappa(T-t)}, \qquad (s,u)\to(t,0).$$

□

A.2.5 Proof of Lemma 4.2.16

Proof. For the proof we distinguish to the following three cases.

▷ $z \in U_H$

Note that $U_H = (u_l, u_r) \cup \mathbb{C}\setminus\mathbb{R}$. On that set we have

$$H(z)^2 = \left(\frac{1}{2}e^{\frac{1}{2}\log D(z)}\right)^2 = \frac{1}{4}e^{\log D(z)} = \frac{1}{4}D(z), \quad z \in U_H.$$

Recalling the power series of the hyperbolic cosine we thus get

$$\cosh\big(H(z)T\big) = \sum_{k=0}^{\infty} \frac{\big(H(z)T\big)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{T^{2k}}{4^k(2k)!}H(z)^{2k} = \sum_{k=0}^{\infty} \frac{T^{2k}}{4^k(2k)!}D(z)^k = R_1\left(\frac{1}{4}D(z)T^2\right),$$

for $z \in U_H$. Using the power series for the hyperbolic sine yields

$$\begin{aligned}
\frac{\sinh\big(H(z)T\big)}{H(u)} &= H(z)^{-1}\sum_{k=0}^{\infty} \frac{\big(H(z)T\big)^{2k+1}}{(2k+1)!} = T\sum_{k=0}^{\infty} \frac{T^{2k}}{(2k+1)!}H(z)^{2k} \\
&= T\sum_{k=0}^{\infty} \frac{T^{2k}}{4^k(2k+1)!}D(z)^k = R_2\left(\frac{1}{4}D(z)T^2\right)T,
\end{aligned}$$

for $z \in U_H$. In particular we get

$$P_T(z) = R_1\left(\frac{1}{4}D(z)T^2\right) - b(z)R_2\left(\frac{1}{4}D(z)T^2\right)T = \cosh\big(H(z)T\big) - b(z)\frac{\sinh\big(H(z)T\big)}{H(z)},$$

and

$$Q_T(z) = R_2\left(\frac{1}{4}D(z)T^2\right)T = \frac{\sinh\big(H(z)T\big)}{H(z)},$$

for $z \in U_H$.

▷ $z \in (-\infty, u_l) \cup (u_r, \infty)$
Now we have $D(z) < 0$. Hence we have $H(z) > 0$ and

$$H(z)^2 = -\frac{D(z)}{4}.$$

Recalling the power series for the cosine leads to

$$\cos\big(H(z)T\big) = \sum_{k=0}^{\infty}(-1)^k \frac{\big(H(z)T\big)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{T^{2k}}{(2k)!}\big(-H(z)^2\big)^k = \sum_{k=0}^{\infty} \frac{T^{2k}}{4^k(2k)!} D(z)^k = R_1\Big(\frac{1}{4}D(z)T^2\Big).$$

The power series of the sine leads to

$$\begin{aligned}\frac{\sin\big(H(z)T\big)}{H(z)} &= H(z)^{-1}\sum_{k=0}^{\infty}(-1)^k \frac{\big(H(z)T\big)^{2k+1}}{(2k+1)!} = T\sum_{k=0}^{\infty} \frac{T^{2k}}{(2k+1)!}\big(-H(z)^2\big)^k \\ &= T\sum_{k=0}^{\infty} \frac{T^{2k}}{4^k(2k+1)!}\big(D(z)\big)^k = R_2\Big(\frac{1}{4}D(z)T^2\Big)T.\end{aligned}$$

This then implies

$$P_T(z) = R_1\Big(\frac{1}{4}D(z)T^2\Big) - b(z)R_2\Big(\frac{1}{4}D(z)T^2\Big)T = \cos\big(H(z)T\big) - b(z)\frac{\sin\big(H(z)T\big)}{H(z)},$$

and

$$Q_T(z) = TR_1\Big(\frac{1}{4}D(z)T^2\Big) = \frac{\sin\big(H(z)T\big)}{H(z)},$$

for $z \in (-\infty, u_l) \cup (u_r, \infty)$.

▷ $z \in \{u_l, u_r\}$
In this case we have $D(z) = 0$. Since $R_1(0) = R_2(0) = 1$ we obtain

$$P_T(z) = R_1(0) - b(z)R_2(0)T = 1 - b(z)T,$$

and

$$Q_T(z) = R_2(0)T = T,$$

for $z \in \{u_l, u_r\}$.

□

A.2.6 Proof of Lemma 4.2.17

Proof. For the first statement we have to show that

$$P_T(u) > 0, \quad 0 \le u \le 1. \tag{A.8}$$

First we show an estimate for the case where $u \in [0, 1)$ as the statement for $u \in [0, 1]$ will then also easily follow by a continuity argument. Due to Lemma 4.2.5 we know that we always have of $u_r \ge 1$. By means of Lemma 4.2.16 we deduce

$$P_T(u) = \cosh\big(H(u)T\big) - \frac{b(u)}{H(u)}\sinh\big(H(u)T\big), \quad 0 \le u \le 1 \le u_r.$$

Then observe that we have

$$b(u)^2 + 0.25u(1-u)\eta^2 > 0, \quad u \in [0,1),$$

which implies

$$\left| \frac{b(u)}{H(u)} \right| = \left| \frac{b(u)}{\sqrt{b(u)^2 + 0.25u(1-u)\eta^2}} \right| \leq 1, \quad u \in [0,1).$$

Because of $\sinh\big(H(u)T\big) \geq 0$ we thus obtain

$$P_T(u) = \cosh\big(H(u)T\big) - \frac{b(u)}{H(u)} \sinh\big(H(u)T\big) \geq \cosh\big(H(u)T\big) - \sinh\big(H(u)T\big) = e^{-H(u)T} > 0, \quad u \in [0,1). \tag{A.9}$$

Hence $P_T(u) > 0$ for $u \in [0,1)$. Since P_T and H are continuous on $\mathbb{R}$ the estimate in (A.9) implies

$$P_T(1) = \lim_{u \nearrow 1} P_T(u) \geq \lim_{u \nearrow 1} e^{-H(u)T} = e^{-H(1)T} > 0.$$

Consequently (A.8) holds.
Next, we prove the second statement. Therefore, suppose that we simultaneously have $u_+(T) < +\infty$ and $Q\big(u_+(T)\big) \leq 0$. We show that this leads to a contradiction. Due to Lemma 4.2.16 we have

$$Q_T(u) = \begin{cases} \frac{\sinh H(u)T}{H(u)} & u \in (u_l, u_r) \\ T & u \in \{u_l, u_r\} \\ \frac{\sin H(u)T}{H(u)} & u \in (-\infty, u_l] \cup [u_r, +\infty) \end{cases}. \tag{A.10}$$

Then recall that $H(u) > 0$ for $u \in \mathbb{R} \setminus \{u_l, u_r\}$. Because of $T > 0$ we have $Q_T(0) > 0$. Since we assumed $Q\big(u_+(T)\big) \leq 0$ there is consequently a smallest $u^* \in \big(0, u_+(T)\big]$ such that $Q_T(u^*) = 0$ holds. Furthermore, a glance at (A.10) yields that $Q_T(u) > 0$ holds for $u \in (0, u_r)$. We also see that $Q_T(u_r) = T > 0$. Thus $u^* > u_r$. Consequently $Q_T(u^*) = 0$ is equivalent to

$$\sin\big(H(u^*)T\big) = 0.$$

With $u_{\max}$ defined in (4.24) we always have $u_{\max} \leq u_r$. Consequently due to Lemma 4.2.3 we know that D is decreasing on $(u_r, +\infty)$. Because of $H(u) = \sqrt{-D(u)}$, for $u \in (u_r, +\infty)$, we thus see that H is positive and increasing on $(u_r, +\infty)$. Furthermore, $H(u_r) = 0$ holds. Because of $T > 0$ and $u^* > u_r$ being the smallest positive root of Q_T we have

$$H(u^*)T = \pi.$$

However, because of $u^* > u_r$ plugging into the representation of P_T given in Lemma 4.2.16 would yield

$$P_T(u^*) = \cos\big(H(u^*)T\big) = \cos\pi = -1.$$

Since also P_T is continuous and $P_T(0) > 0$ this would imply the existence of a $\overline{u} \in (0, u^*) \subseteq \big(0, u_+(T)\big)$ satisfying $P_T(\overline{u}) = 0$. This contradicts the definition of $u_+(T)$. Thus $Q_T\big(u_+(T)\big) > 0$ must hold in case $u_+(T) < +\infty$. Analogously one can argue that $Q_T\big(u_-(T)\big) > 0$ must hold in case $u_-(T) > -\infty$. ☐

A.2.7 Proof of Lemma 4.3.1

Proof. We consider the following four cases to prove the statement.

(i) $u \in [u_l, 0] \cap \mathbb{R}$

To show that $b(u) < 0$ holds in this case we distinguish between non-negative and negative $\rho \in [-1, 1]$.

- $\rho \geq 0$

  Since we also have $u \leq 0$ we obviously get

$$b(u) = \frac{1}{2}(\eta\rho u - \kappa) \leq -\frac{\kappa}{2} < 0.$$

- $\rho < 0$

  Then $\frac{\kappa}{\eta\rho} < 0$ and

$$D\left(\frac{\kappa}{\eta\rho}\right) = \frac{\kappa}{\eta\rho}\left(1 - \frac{\kappa}{\eta\rho}\right) = \frac{\eta\rho - \kappa}{\eta^2\rho^2} = -\frac{\kappa + \eta|\rho|}{\eta^2\rho^2} < 0$$

  As $D(0) > 0$ and since $u_l > -\infty$ is the unique negative root of D we must have $\frac{\kappa}{\eta\rho} < u_l$. Because of $\rho < 0$ this leads to

$$b(u) \leq b(u_l) = \frac{1}{2}(u_l\eta\rho - \kappa) < 0, \quad u \in [u_l, 0].$$

(ii) $\kappa > \eta\rho$ and $u \in (0, u_r] \cap \mathbb{R}$

We differentiate between positive and non-positive $\rho \in [-1, 1]$.

- $\rho \leq 0$

  Since particularly $u \geq 0$ holds we obtain

$$b(u) = \frac{1}{2}(\eta\rho u - \kappa) \leq -\frac{\kappa}{2} < 0.$$

- $\rho > 0$

  Then we have $\frac{\kappa}{\eta\rho} > 0$. Furthermore, it holds that

$$D\left(\frac{\kappa}{\eta\rho}\right) = \frac{\kappa}{\eta\rho}\left(1 - \frac{\kappa}{\eta\rho}\right) = -\frac{\kappa - \eta\rho}{\eta^2\rho^2} < 0.$$

  As $D(0) > 0$ and since $u_r < +\infty$ is the unique positive root of D we must have $\frac{\kappa}{\eta\rho} > u_r$. Because of $\rho > 0$ the latter then implies

$$b(u) \leq b(u_r) = \frac{1}{2}(\eta\rho u_r - \kappa) < 0, \quad u \in (0, u_r].$$

(iii) $\kappa = \eta\rho$

Then $\rho > 0$ must hold and $\frac{\kappa}{\eta\rho} = 1$. This leads to

$$D(1) = (\kappa - \kappa)^2 + (1 - 1)\eta^2 = 0.$$

Since $u_r < +\infty$ is the unique positive root of D we obtain $u_r = 1$. Consequently $b(u_r) = b(1) = 0$ holds due to $\kappa = \eta\rho$. Furthermore, we have $b(u) < b(1) = 0$ for $u \in [0, 1)$ and $b(u) > b(1) = 0$ for $u \in (1, +\infty)$.

(iv) $\kappa < \eta\rho$

In this case clearly $\rho > 0$ has to hold. Obviously b is then a strictly increasing affine function with a root at $\frac{\kappa}{\eta\rho}$. This gives the stated properties in this case.

$\square$

A.2.8 Proof of Lemma 4.3.3

Proof. We need to show that

$$v \mapsto \frac{b(v)}{H(v)}, \quad v \in (1, u_r), \tag{A.11}$$

is strictly increasing if $2\kappa \leq \eta\rho$. Observe that $\kappa - \eta\rho < 2\kappa - \eta\rho \leq 0$ holds and thus $b(v) > 0$ for $v > 1$ by Lemma 4.3.1. Hence the statement that (A.11) is strictly increasing is equivalent to

$$v \mapsto \frac{b^2(v)}{H^2(v)}, \quad v \in (1, u_r),$$

being strictly increasing. Rearranging terms leads to

$$\frac{b^2(v)}{H^2(v)} = \frac{(\eta\rho v - \kappa)^2}{(\eta\rho v - \kappa)^2 - v(v-1)\eta^2} = \frac{1}{1 - \frac{v(v-1)}{(\eta\rho v - \kappa)^2}}, \quad v \in (1, u_r).$$

Hence it suffices to show that

$$v \mapsto \frac{v(v-1)}{(\eta\rho v - \kappa)^2}, \quad v \in (1, u_r), \tag{A.12}$$

is strictly increasing. Taking the derivative of the latter function gives

$$\begin{aligned} \left(\frac{v(v-1)}{(\eta\rho v - \kappa)^2}\right)' &= \frac{(2v-1)(\eta\rho v - \kappa) - 2\eta\rho(v^2 - v)}{(\eta\rho v - \kappa)^3} = \frac{2\eta\rho v^2 - 2\kappa v - \eta\rho v + \kappa - 2\eta\rho v^2 + 2\eta\rho v}{(\eta\rho v - \kappa)^3} \\ &= \frac{\kappa - 2\kappa v + \eta\rho v}{(\eta\rho v - \kappa)^3} = \frac{\kappa + (\eta\rho - 2\kappa)v}{(\eta\rho v - \kappa)^3}. \end{aligned} \tag{A.13}$$

Now we have $(\eta\rho v - \kappa) = 2b(v) > 0$ for $v > 1$ and $\eta\rho - 2\kappa \geq 0$. Thus we see that the latter expression in (A.13) is positive. Consequently also the derivative of (A.12) is positive for $v \in (1, u_r)$. Thus the function in (A.12) is strictly increasing and thus (4.54) is proven. □

A.2.9 Proof of Lemma 4.3.5

Proof. Define the sets B_- and B_+ by

$$B_- := \{u \in \mathbb{R} : b(u) < 0\} = \begin{cases} (-\infty, \frac{\kappa}{\eta\rho}) & \rho > 0 \\ \mathbb{R} & \rho = 0 \\ (\frac{\kappa}{\eta\rho}, +\infty) & \rho < 0 \end{cases} \quad \text{and} \quad B_+ := \{u \in \mathbb{R} : b(u) > 0\} = \begin{cases} (\frac{\kappa}{\eta\rho}, +\infty) & \rho > 0 \\ \emptyset & \rho = 0 \\ (-\infty, \frac{\kappa}{\eta\rho}) & \rho < 0 \end{cases}.$$

If $\frac{\kappa}{\eta\rho} \in (-\infty, u_l) \cup (u_r, +\infty)$ we have

$$\frac{b(v)}{H(v)} > 0 = \frac{b\left(\frac{\kappa}{\eta\rho}\right)}{H\left(\frac{\kappa}{\eta\rho}\right)} = 0 > \frac{b(u)}{H(u)},$$

for $u \in B_-$ and $v \in B_+$ and thus it suffices to show the stated monotonicity properties on $B_+ \cap (u_r, +\infty)$, $B_- \cap (u_r, +\infty)$, $B_+ \cap (-\infty, u_l)$ and $B_- \cap (-\infty, u_l)$. In these cases consider the square of the function defined in (4.57) to see that the following holds.

$$\frac{b(u)^2}{H(u)^2} = \frac{(\eta\rho u - \kappa)^2}{u(u-1)\eta^2 - (\eta\rho u - \kappa)^2} = \frac{1}{\frac{u(u-1)}{(\eta\rho u - \kappa)^2}\eta^2 - 1}, \quad u \in \Big((-\infty, u_l) \cup (u_r, +\infty)\Big) \cap \Big(B_- \cup B_+\Big).$$

Note that it is essential to assume $u \in (-\infty, u_l) \cup (u_r, +\infty)$ and $b(u) \neq 0$ to make the rearrangements we have just done. Because of the latter we observe that it suffices to show the following for the statement of this lemma.

$$u \mapsto \frac{u(u-1)}{(\eta\rho u - \kappa)^2} \quad \text{is} \quad \begin{cases} \text{strictly increasing on } B_- \cap (u_r, +\infty) \\ \text{strictly decreasing on } B_+ \cap (u_r, +\infty) \\ \text{strictly decreasing on } B_- \cap (-\infty, u_l) \\ \text{strictly increasing on } B_+ \cap (-\infty, u_l) \end{cases}, \tag{A.14}$$

where in case $u \in (u_r, +\infty)$ we can assume $\kappa \geq \eta\rho$ and in case $u \in (-\infty, u_l)$ we can assume $2\kappa \geq \eta\rho$. Taking the derivative with respect to u of the latter function gives

$$\begin{aligned}\left(\frac{u(u-1)}{(\eta\rho u - \kappa)^2}\right)' &= \frac{(2u-1)(\eta\rho u - \kappa)^2 - 2\eta\rho(u^2-u)(\eta\rho u - \kappa)}{(\eta\rho u - \kappa)^4} = \frac{(2u-1)(\eta\rho u - \kappa) - 2\eta\rho(u^2-u)}{(\eta\rho u - \kappa)^3} \\ &= \frac{\kappa + \eta\rho u - 2\kappa u}{(\eta\rho u - \kappa)^3} = -\frac{\kappa(u-1) + (\kappa - \eta\rho)u}{(\eta\rho u - \kappa)^3}.\end{aligned}$$

Since we always have $u_r \geq 1$ we clearly see that if $\kappa \geq \eta\rho$ holds the latter derivative is positive for $u \in (u_r, +\infty) \cap B_-$ and negative for $u \in (u_r, +\infty) \cap B_+$. Consequently the statement for $u \in (u_r, +\infty)$ in (A.14) is shown and thus we have proven statement (i) of this lemma. To see statement (ii) note that the expression we computed for the derivative of $u \mapsto \frac{u(u-1)}{(\eta\rho u - \kappa)^2}$ can be rearranged as follows.

$$\left(\frac{u(u-1)}{(\eta\rho u - \kappa)^2}\right)' = -\frac{\kappa(u-1) + (\kappa - \eta\rho)u}{(\eta\rho u - \kappa)^3} = \frac{\kappa - (2\kappa - \eta\rho)u}{(\eta\rho u - \kappa)^3}, \quad u \in B_- \cup B_+$$

Because of $u_l < 0$ we see that the assumption $2\kappa \geq \eta\rho$ implies that the derivative of the latter function is positive for $u \in (-\infty, u_l) \cap B_+$ and negative for $u \in (-\infty, u_l) \cap B_-$. This implies the monotonicity stated in (A.14) for $u \in (-\infty, u_l)$ if $2\kappa \geq \eta\rho$ and hence statement (ii) of this lemma is shown. ◻

A.2.10 Proof of Lemma 4.4.4

Proof. Since $w \notin (-\infty, 0]$ it holds that $\sqrt{2}e^{\frac{1}{2}\log(w)}$ is the unique complex number with positive real part whose square equals $2w$. On the other hand we have $\sqrt{|w| + w_1} > 0$ and

$$\left(\sqrt{|w| + w_1} + i\,\mathrm{sgn}(w_2)\sqrt{|w| - w_1}\right)^2 = 2w_1 + 2i\,\mathrm{sgn}(w_2)\sqrt{w_2^2} = 2(w_1 + iw_2).$$

Thus the left and the right hand side of the first equation must be equal. Furthermore, we have

$$\mathrm{Im}\left(\sqrt{2}e^{\frac{1}{2}\log(w)}\right) = \mathrm{sgn}(w_2)\sqrt{|w| - w_1} = \mathrm{sgn}(w_2)\frac{\sqrt{|w| - w_1}\sqrt{|w| + w_1}}{\sqrt{|w| + w_1}} = \frac{w_2}{\sqrt{|w| + w_1}} = \frac{w_2}{\mathrm{Re}\left(\sqrt{2}e^{\frac{1}{2}\log(w)}\right)}.$$

Division by $\sqrt{2}$ yields the second equation. ◻

A.2.11 Proof of Lemma 4.4.5

Proof. Plugging in by means of the identities in (4.66) gives

$$
\begin{aligned}
2\,\mathrm{Re}\,\big(b(\overline{z})D(z)\big) &= b_1(z_1)D_1(z_1,z_2) - b_2(z_2)D_2(z_1,z_2) \\
&= b_1(z_1)\Big(\kappa^2 + \eta^2(1-\rho^2)(z_2^2 - z_1^2) + \eta(\eta - 2\kappa\rho)z_1\Big) + z_2^2\rho\eta^2\Big(\eta - 2\kappa\rho - 2\eta(1-\rho^2)z_1\Big) \\
&= b_1(z_1)\Big(\kappa^2 - \eta^2(1-\rho^2)z_1^2 + \eta(\eta - 2\kappa\rho)z_1\Big) \\
&\quad + \eta^2 z_2^2\Big(b_1(z_1)(1-\rho^2) + \rho\eta - 2\kappa\rho^2 - 2\eta\rho(1-\rho^2)z_1\Big) \\
&= b_1(z_1)\Big(\kappa^2 - \eta^2(1-\rho^2)z_1^2 + \eta(\eta - 2\kappa\rho)z_1\Big) - \eta^2 z_2^2\Big(z_1(1-\rho^2)\rho\eta - \kappa(1-\rho^2) + 2\kappa - \rho\eta\Big) \\
&= b_1(z_1)\Big(\kappa^2 - \eta^2(1-\rho^2)z_1^2 + \eta(\eta - 2\kappa\rho)z_1\Big) - \eta^2 z_2^2\Big(2\kappa - \rho\eta + (1-\rho^2)b_1(z_1)\Big) \\
&= \rho^{-2}b_1(z_1)\Big(\kappa^2\rho^2 - \eta^2(1-\rho^2)(\rho z_1)^2 + \eta\rho(\eta - 2\kappa\rho)\rho z_1\Big) - \eta^2 z_2^2\Big(2\kappa - \rho\eta + (1-\rho^2)b_1(z_1)\Big) \\
&= \rho^{-2}b_1(z_1)f(\rho z_1) - \eta^2 z_2^2\Big(2\kappa - \rho\eta + (1-\rho^2)b_1(z_1)\Big),
\end{aligned}
$$

for $z \in \mathbb{C}$.
Now assume $\kappa \geq \eta\rho$. To see that then $f(x) \leq 0$ holds for $x \geq \frac{\kappa}{\eta}$ we distinguish the following three cases.

- $\rho \in (-1, 1)$
  Then f is a quadratic polynomial with a negative leading coefficient. Thus f has a global maximum x^* such that f is increasing on $(-\infty, x^*]$ and decreasing on $[x^*, +\infty)$. The maximum is given by the first order condition
  $$0 = f'(x^*) = -2\eta^2(1-\rho^2)x^* + \eta\rho(\eta - 2\kappa\rho).$$
  Rearranging terms gives
  $$x^* = \frac{\rho(\eta - 2\kappa\rho)}{2\eta(1-\rho^2)}.$$
  Now observe
  $$\frac{\kappa}{\eta} - x^* = \frac{\kappa}{\eta} - \frac{\rho(\eta - 2\kappa\rho)}{2\eta(1-\rho^2)} = \frac{2\kappa - 2\kappa\rho^2 - \rho\eta + 2\kappa\rho^2}{2\eta(1-\rho^2)} = \frac{2\kappa - \rho\eta}{2\eta(1-\rho^2)} \geq 0,$$
  which yields $\frac{\kappa}{\eta} \geq x^*$. Hence f is decreasing on $[\frac{\kappa}{\eta}, +\infty)$ and we obtain
  $$f(x) \leq f\left(\frac{\kappa}{\eta}\right) = \kappa^2\rho^2 - \eta^2(1-\rho^2)\frac{\kappa^2}{\eta^2} + \eta\rho(\eta - 2\kappa\rho)\frac{\kappa}{\eta} = -\kappa(\kappa - \rho\eta) \leq 0,$$
  for $x \geq \frac{\kappa}{\eta}$. For the latter inequality we needed $\kappa \geq \rho\eta$.
- $\rho = -1$
  Then $\frac{\kappa}{\eta} \leq x$ leads to
  $$f(x) = \kappa^2 - \eta x(\eta + 2\kappa) \leq \kappa^2 - (\eta + 2\kappa)\kappa = -\kappa^2 - \eta\kappa = -\kappa(\kappa + \eta) < 0.$$
- $\rho = 1$
  First, observe that because of $\kappa \geq \eta\rho$ we have
  $$\eta - 2\kappa = \rho\eta - 2\kappa < \rho\eta - \kappa \leq 0.$$
  Together with $\frac{\kappa}{\eta} \leq x$ this leads to
  $$f(x) = \kappa^2 + (\eta - 2\kappa)\eta x \leq \kappa^2 + (\eta - 2\kappa)\kappa = -\kappa^2 + \eta\kappa = -\kappa(\kappa - \eta) = -\kappa(\kappa - \rho\eta) \leq 0.$$

□

A.2.12 Proof of Lemma 4.4.6

Proof. For the proof define

$$\beta(z_1,z_2) = 4\,\mathrm{Re}\,\big(b(\overline{z})D(z)\big)^2 - b_1(z_1)^2|D(z)|^2, \quad z = z_1 + iz_2 \in \mathbb{C},$$

which is the left hand side of the equation we have to prove. Then observe that

$$\begin{aligned}
\beta(z_1,z_2) &= \Big(b_1(z_1)D_1(z_1,z_2) - b_2(z_2)D_2(z_1,z_2)\Big)^2 - b_1(z_1)^2\Big(D_1(z_1,z_2)^2 + D_2(z_1,z_2)^2\Big)\\
&= b_2(z_2)^2D_2(z_1,z_2)^2 - 2b_1(z_1)b_2(z_2)D_1(z_1,z_2)D_2(z_1,z_2) - b_1(z_1)^2D_2(z_1,z_2)^2\\
&= D_2(z_1,z_2)\Big(b_2(z_2)^2D_2(z_1,z_2) - 2b_1(z_1)b_2(z_2)D_1(z_1,z_2) - b_1(z_1)^2D_2(z_1,z_2)\Big)\\
&= D_2(z_1,z_2)\Big(\underbrace{D_2(z_1,z_2)\big(b_2(z_2)^2 - b_1(z_1)^2\big) - 2b_1(z_1)b_2(z_2)D_1(z_1,z_2)}_{\lambda(z_1,z_2):=}\Big),
\end{aligned}$$

for $(z_1,z_2)\in\mathbb{R}^2$. By plugging in the expressions for b_1, b_2, D_1 and D_2 in $\lambda(z_1,z_2)$ we get

$$\begin{aligned}
\lambda(z_1,z_2) &= \eta z_2\Big(\eta - 2\kappa\rho - 2\eta(1-\rho^2)z_1\Big)\Big(z_2^2\rho^2\eta^2 - (z_1\rho\eta - \kappa)^2\Big)\\
&\quad + 2z_2\eta\rho(z_1\rho\eta - \kappa)\Big(\kappa^2 + \eta^2(1-\rho^2)(z_2^2 - z_1^2) + \eta(\eta - 2\kappa\rho)z_1\Big)\\
&= \kappa^3\Big(2\rho\eta z_2 - 2\rho\eta z_2\Big) + \kappa^2\Big(2\eta^2 z_1z_2(1-\rho^2) - \eta^2 z_2 - 4z_1z_2\rho^2\eta^2 + 2\rho^2\eta^2z_1z_2 + 4\eta^2\rho^2z_1z_2\Big)\\
&\quad + \kappa\Big(2\eta^3z_1^2z_2\rho^3 - 2\eta^3z_2^3\rho^3 + 2\eta^3z_1z_2\rho + 4\eta^3z_1^2z_2\rho^3 - 4\eta^3z_1^2z_2\rho + 2\eta^3z_2^3\rho^3 - 2\eta^3z_2^3\rho\\
&\quad + 2\eta^3z_1^2z_2\rho - 2\eta^3z_1^2z_2\rho^3 - 2\eta^3z_1z_2\rho - 4\eta^3z_1^2z_2\rho^3\Big) + \eta^4z_2^3\rho^2 - 2\eta^4z_1z_2^3\rho^2(1-\rho^2) - \eta^4z_1^2z_2\rho^2\\
&\quad + 2\eta^4z_1^3z_2\rho^2(1-\rho^2) - 2\eta^4z_1^3z_2\rho^2(1-\rho^2) + 2\eta^4z_1z_2^3\rho^2(1-\rho^2) + 2\eta^4z_1^2z_2\rho^2\\
&= \eta^2z_2\kappa^2(2z_1 - 1) - 2\eta^3z_2\rho\kappa(z_1^2 + z_2^2) + \eta^4z_2\rho^2\Big(z_2^2 - 2z_1z_2^2 + 2z_1z_2^2\rho^2 - z_1^2\\
&\quad + 2z_1^3 - 2z_1^3\rho^2 - 2z_1^3 + 2z_1^3\rho^2 + 2z_1z_2^2 - 2z_1z_2^2\rho^2 + 2z_1^2\Big)\\
&= \eta^2z_2\Big(\kappa^2(2z_1 - 1) - 2\eta\rho\kappa(z_1^2 + z_2^2) + \eta^2\rho^2(z_1^2 + z_2^2)\Big)\\
&= \eta^2z_2\Big(\kappa^2(2z_1 - 1) - \eta\rho(2\kappa - \eta\rho)(z_1^2 + z_2^2)\Big)\\
&= -\rho^{-1}\eta^2z_2\Big(\kappa^2(\rho - 2\rho z_1) + \eta(2\kappa - \eta\rho)\big((\rho z_1)^2 + (\rho z_2)^2\big)\Big) = -\frac{\eta^2}{\rho^2}\rho z_2 g(\rho z_1, \rho z_2),
\end{aligned}$$

for every $(z_1,z_2)\in\mathbb{R}^2$. Consequently we have

$$\beta(z_1,z_2) = D_2(z_1,z_2)\lambda(z_1,z_2) = -\frac{\eta^2}{\rho^2}\rho z_2 D_2(z_1,z_2)g(\rho z_1,\rho z_2), \quad (z_1,z_2)\in\mathbb{R}^2,$$

which concludes the proof. □

A.2.13 Proof of Lemma 4.4.7

Proof. Because of $2\kappa - \rho\eta \geq 0$ we clearly have

$$g(w_1,w_2) = \kappa^2(\rho - 2w_1) + \eta(2\kappa - \eta\rho)\big(w_1^2 + w_2^2\big) \geq \kappa^2(\rho - 2w_1) + \eta(2\kappa - \eta\rho)w_1^2 = g(w_1,0) = g_1(w_1),$$

for $(w_1,w_2)\in\mathbb{R}^2$. Since we assume $2\kappa - \rho\eta > 0$ the second order polynomial g_1 has a positive leading coefficient and thus a global minimum w_1^*. Furthermore, g_1 is decreasing on $(-\infty, w_1^*]$ and increasing on

$[w_1^*, \infty)$. The minimum w_1^* is determined by the first order condition

$$0 = g_1'(w_1^*) = -2\kappa^2 + 2\eta(2\kappa - \eta\rho)w_1^*.$$

Rearranging terms gives

$$w_1^* = \frac{\kappa^2}{\eta(2\kappa - \eta\rho)}.$$

□

A.3 Chapter 5

A.3.1 Proof of Lemma 5.2.7

Proof. First note that due Proposition 3.1.7 we know that $z \mapsto M_{X_T}(z)$ is holomorphic on $(u_-(T), u_+(T)) + i\mathbb{R}$. Furthermore, the function h, defined by

$$h(z) = \frac{M_{X_T}(z)}{z(z-1)}, \quad z \in \Big((u_-(T), u_+(T)) + i\mathbb{R}\Big) \setminus \{0,1\}, \tag{A.15}$$

is holomorphic. In addition we have

$$h(\alpha + iu) = \zeta(u, \alpha), \quad u \in \mathbb{R},$$

for $\alpha \in \big(u_-(T), u_+(T)\big) \setminus \{0,1\}$. Since h is holomorphic the function $\zeta(\cdot, \alpha)$ must be arbitrarily often differentiable. Furthermore, we have

$$|\zeta(u,\alpha)| \leq \frac{M_{X_T}(\alpha)}{|\alpha + iu||\alpha - 1 + iu|} \leq \frac{M_{X_T}(\alpha)}{u^2}, \quad |u| > 0.$$

Consequently we have $\lim_{|u|\to+\infty} \zeta(u,\alpha) = 0$ which yields the boundedness of $\zeta(\cdot,\alpha)$ on $\mathbb{R}$. In addition we obtain

$$\int_{\mathbb{R}} |\zeta(u,\alpha)|\, du \leq \int_{[-1,1]} |\zeta(u,\alpha)|\, du + \int_{[-1,1]^C} u^{-2}\, du = \int_{[-1,1]} |\zeta(u,\alpha)|\, du + 2 < +\infty,$$

where we used $\alpha \notin \{0,1\}$ to conclude the latter finiteness, and thus $\zeta(\cdot,\alpha) \in L^1(\mathbb{R})$. Hence it remains to prove the boundedness of the first and second derivative of $\zeta(\cdot,\alpha)$.

▷ $\zeta'(\cdot,\alpha)$

For the holomorphic function h we get

$$\begin{aligned} h'(z) &= \frac{M_{X_T}'(z)z(z-1) - M_{X_T}(z)(2z-1)}{z^2(z-1)^2} = \frac{M_{X_T}'(z)}{z(z-1)} - \frac{2M_{X_T}(z)}{z^2(z-1)} - \frac{M_{X_T}(z)}{z^2(z-1)^2} \\ &= \frac{\mathbb{E}\big(X_T e^{zX_T}\big)}{z(z-1)} - \frac{2\mathbb{E}\big(e^{zX_T}\big)}{z^2(z-1)} - \frac{\mathbb{E}\big(e^{zX_T}\big)}{z^2(z-1)^2} \leq \frac{\mathbb{E}\big(|X_T| e^{\mathrm{Re}(z)X_T}\big)}{|z||z-1|} + \frac{2\mathbb{E}\big(e^{\mathrm{Re}(z)X_T}\big)}{|z|^2|z-1|} + \frac{\mathbb{E}\big(e^{\mathrm{Re}(z)X_T}\big)}{|z|^2|z-1|^2}, \end{aligned}$$

whenever $z \notin \{0,1\}$ belongs to $\big(u_-(T), u_+(T)\big) + i\mathbb{R}$. Thus we obtain

$$\big|\zeta'(u,\alpha)\big| = \big|h'(\alpha + iu)i\big| \leq \frac{\mathbb{E}\big(|X_T| e^{\alpha X_T}\big)}{u^2} + \frac{2\mathbb{E}\big(e^{\alpha X_T}\big)}{|u|^{3/2}} + \frac{\mathbb{E}\big(e^{\alpha X_T}\big)}{u^4}, \quad |u| > 0$$

where we used Proposition 3.1.7 to compute the derivative of M_{X_T}. Consequently we have $\lim_{|u|\to+\infty} \zeta'(u,\alpha) = 0$ which yields the boundedness of $\zeta'(\cdot,\alpha)$ on $\mathbb{R}$.

▷ $\zeta''(\cdot,\alpha)$

For the second derivative of h we get

$$\begin{aligned} h''(z) = & \frac{M''_{X_T}(z)z(z-1) - M'_{X_T}(z)(2z-1)}{z^2(z-1)^2} - 2\frac{M'_{X_T}(z)z^2(z-1) - M_{X_T}(z)(3z^2-2z)}{z^4(z-1)^2} \\ & - \frac{M'_{X_T}(z)z^2(z-1)^2 - M_{X_T}(z)(4z^3-6z^2+2z)}{z^4(z-1)^4}, \end{aligned}$$

whenever $z \notin \{0,1\}$ belongs to $\big(u_-(T), u_+(T)\big) + i\mathbb{R}$. Because of Proposition 3.1.7 we have

$$\begin{aligned} M_{X_T}(\alpha+iu) &\le \mathbb{E}\big(e^{\alpha X_T}\big) \\ M'_{X_T}(\alpha+iu) &\le \mathbb{E}\big(|X_T|e^{\alpha X_T}\big) \\ M''_{X_T}(\alpha+iu) &\le \mathbb{E}\big(X_T^2 e^{\alpha X_T}\big), \end{aligned}$$

for $u \in \mathbb{R}$. Consequently we get

$$|\zeta''(u,\alpha)| = |h''(\alpha+iu)| \le \frac{p\big(|\alpha+iu|\big)}{q\big(|\alpha+iu|\big)},$$

where p and q are complex polynomials such that $\deg p < \deg q$. The latter implies $\lim_{|u|\to+\infty} \zeta''(u,\alpha) = 0$ and hence $\zeta''(\cdot,\alpha)$ must also be bounded on $\mathbb{R}$.

□

A.3.2 Proof of Lemma 5.3.2

Proof. With $g(x) := |x^2 - x|$ for $x \in \mathbb{R}$ we have

$$f_\mu(x) = \exp\big(\mu x - \ln g(x)\big), \quad x \in \mathbb{R}\setminus\{0,1\}.$$

We compute the second derivative. For the first derivative we get

$$f'_\mu(x) = f_\mu(x)\left(\mu - \frac{g'(x)}{g(x)}\right), \quad x \in \mathbb{R}\setminus\{0,1\}.$$

Hence

$$f''_\mu(x) = f_\mu(x)\left(\mu - \frac{g'(x)}{g(x)}\right)^2 - f_\mu(x)\frac{g''(x)g(x) - g'(x)^2}{g(x)^2}, \quad x \in \mathbb{R}\setminus\{0,1\}.$$

Now we have

$$g'(x)^2 = (2x-1)^2, \quad x \in \mathbb{R}\setminus\{0,1\} \quad \textit{and} \quad g(x)g''(x) = 2(x^2-x) \quad x \in \mathbb{R}\setminus\{0,1\}.$$

Plugging the latter into the derived expression for the second derivative of f_μ leads to

$$f''_\mu(x) = \frac{f_\mu(x)}{g(x)^2}\Big(\big(g(x)-g'(x)\big)^2 - 2x^2 + 2x + (2x-1)^2\Big) = \frac{f_\mu(x)}{g(x)^2}\Big(\big(g(x)-g'(x)\big)^2 + x^2 + (x-1)^2\Big),$$

for $x \in \mathbb{R}\setminus\{0,1\}$. Thus we clearly have $f''_\mu(x) > 0$ for $x \in \mathbb{R}\setminus\{0,1\}$. The last statement of this lemma directly follows from the fact that on an interval a strictly positive second derivative always implies strict convexity. □

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