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Kurzfassung

Diese Arbeit befasst sich mit der rigorosen Herleitung eines Kreuzdiffusions-Systems ausgehend von einem stochastischen Vielteilchen-Systems. Die Idee und die grundsätzliche Vorgehensweise stammt aus L. Chens, E. Daus's und A. Jüngels gleichnamiger Fachpuplikation "Rigorous Mean-Field Limit and Cross Diffusion". Das Ziel dieser Diplomarbeit ist es die dort gefundenen Resultate auf den Fall nicht-konstanter Diffusion zu erweitern. Sie enthält neben der eigentlichen Herleitung auch die Existenzbeweise für die auftretenden deterministischen und stochatischen Systeme, sowie Resultate betreffend Eindeutigkeit und globaler Existenz. Des weiteren wird vorgeführt wie die Dichtefunktionen der Lösungen der stochastischen Systeme als schwache Lösung der deterministischen Systeme identifiziert werden können. Zu guter letzt enthält diese Arbeit Fehlerabschätzungen bezüglich der Differenz unterschiedlicher stochastischer Prozesse, beziehungsweise Funktionen, die als Lösungen der unterschiedlichen Differentialgleichungen auftreten.

Abstract

This work focuses on the derivation of a special cross-diffusion system starting from a stochastic many-particle system. The general idea can be found in L. Chens, E. Daus's and A. Jüngels same-named publication "Rigorous Mean-Field Limit and Cross Diffusion". The aim of this thesis is to expand the results found in the aforementioned paper to the case of a non-constant diffusion term. Hence it contains, beside the main derivation, the proof of existence for the occurring stochastic and deterministic systems and some results regarding uniqueness and global existence. Furthermore an idea is given how to identify the probability density function of the solutions of the stochastic systems with certain weak solutions of the deterministic systems. The latter part of this work focuses on error estimates regarding the difference of certain stochastic processes, respectively functions, which occur as solutions of the various differential equations.

Eidesstattliche Erklärung

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Wien, am

Markus Fellner

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1. Introduction

Cross-diffusion models consist of several coupled reaction-diffusion equations, whose diffusion matrices are often non-diagonal. These systems are widely used to model the interaction between certain agents and appear in many applications of mathematical biology or thermodynamics, to name a few.
This thesis will concentrate on the derivation and analysis of the n-species cross-diffusion system

$$\begin{aligned} \partial_t u_i - \operatorname{div}\left(\sigma(x,t)_i \nabla u_i\right) &= \operatorname{div}\left(\sum_{j=1}^n u_i a_{ij} \nabla u_j + \sum_{l=1}^d u_i \frac{\partial}{\partial x_l}(\sigma_i)_{\cdot,l}\right) \quad \text{in } \mathbb{R}^d \times [0,T) \\ u_i \mid_{t=0} &= u_i^0, \quad i \in \{1,2,\dots,n\}, \end{aligned} \tag{1.1}$$

where n, d are natural numbers, $T > 0$ and a_{ij} are real numbers and $\sigma_i : \mathbb{R}^d \times [0,T) \to \mathbb{R}^{d\times d}$ are (weakly) differentiable, matrix-valued functions. This system will be derived, via the mean field limit, from the stochastic many-particle-system

$$\begin{aligned} dX_{\eta,i}^{k,N_i}(t) &= -\sum_{j=1}^n \frac{1}{N_j} \sum_{l=1}^{N_j} \nabla V_{ij}^\eta (X_{\eta,i}^{k,N_i}(t) - X_{\eta,j}^{l,N_j}(t))dt + \sqrt{2}\sqrt{\sigma_i}(X_{\eta,i}^{k,N_i}(t),t)dW_i^k(t), \\ X_{\eta,i}^{k,N_i}(0) &= \xi_i^k, \ 1 \le i,j \le n, \ 1 \le k \le N_i \end{aligned} \tag{1.2}$$

where u_i^0 is the probability-density function of the random variable ξ_i^k. The stochastic differential equations above describe the path taken by one of $N_i \in \mathbb{N}$ particles from n different sub-populations. They are driven by the stochastic processes W_i^k, which are d-dimensional, independent Brownian motions. The interaction between particles is regulated by the interaction potential $V_{ij}^\eta : \mathbb{R}^d \to \mathbb{R}$.

This thesis is based on the work [5] and is an attempt to generalize its findings. It is separated into eight chapters. The first contains the prerequisites and some motivation for the argumentation later on. The second and third chapter deal with the existence theory concerning the occurring stochastic systems and how the systems of deterministic equations (PDE I) and (PDE II) originate from them. The prove of the existence of a weak solution of (1.1) is one of the main results of chapter 6. As there are no compactness results for $\mathbb{R}^d$ regarding Sobolev-spaces, at least to the best knowledge of the author, more general approaches are used. These contain, for example, the application of Banach's fix-point theorem and evolutional-operators, which are a generalisation of C_0-semi-groups. Chapter 6 also contains error-estimates for the difference of the solution u of (1.1) and the weak solution u_η, $\eta > 0$, of the intermediate system

$$\begin{aligned} \partial_t u_{\eta,i} - \operatorname{div}\left(\sigma(x,t)_i \nabla u_{\eta,i}\right) &= \operatorname{div}\left(\sum_{j=1}^n u_{\eta,i} \nabla V_{ij}^\eta * u_{\eta,j} + \sum_{l=1}^d u_{\eta,i} \frac{\partial}{\partial x_l}(\sigma_i)_{\cdot,l}\right) \quad \textit{in } \mathbb{R}^d \times [0,T), \\ u_{\eta,i} \mid_{t=0} &= u_i^0, \quad i \in \{1,2,\dots,n\}. \end{aligned} \tag{1.3}$$

How a solution for this intermediate system can be found is the essence of chapter 4 and 5. Chapter 7 and 8 focus on how the probability density functions relate to the solution of (1.1) and (1.3). Additionally, it contains some error-estimates for the difference of the solution of the stochastic many particle system (1.2) and the stochastic process governed by u, which can be understood as a probability density.

1.1. Setting of the Problem and Prerequisites

This section is based on [5, Section 1.1]. As hinted in the introduction, we assume that n-sub-populations of particles exist. Each of these populations consists out of $N_i \in \mathbb{N}$, $i = 1, \dots, n$, individuals, which move through the whole space $\mathbb{R}^d$. The paths taken by these individuals is a realization of the stochastic processes $X^{k,N_i}_{\eta,i}$, which solve the stochastic differential equations

$$(SDE\,0) \quad \begin{cases} dX^{k,N_i}_{\eta,i}(t) = -\sum_{j=1}^{n} \frac{1}{N_j} \sum_{l=1}^{N_j} \nabla V^{\eta}_{ij}(X^{k,N_i}_{\eta,i}(t) - X^{l,N_j}_{\eta,j}(t))dt + \sqrt{2}\sqrt{\sigma_i}(X^{k,N_i}_{\eta,i}(t), t)dW^k_i(t), \\ X^{k,N_i}_{\eta,i}(0) = \xi^k_i, \ 1 \le i, j \le n, \ 1 \le k \le N_i. \end{cases}$$

Here, as mentioned in the section before, W^k_i are independent, d-dimensional, Brownian motions, and ξ^k_i, $k = 1, \dots, N_i$, are independent, identically distributed, square integrable random variables with the common probability density function u^0_i. For the diffusion matrix σ_i we assume symmetry and uniform coercivity, that is to say

$$x^T \sigma(x,t) x \ge \epsilon \quad , \forall x \in \mathbb{R}^d, \ t \in [0, T)$$

for a constant $\epsilon > 0$. As σ_i is thereby a normal and positive operator, a square-root of σ_i exists. It is not strictly necessary to use the square-root $\sqrt{\sigma_i}$ from a mathematical point of view. Instead, one can also assume a more general diffusion term τ_i, as long as $\tau_i^T \tau_i = \sigma_i$, without needing to change the proofs in this work.
The interaction potential V^{η}_{ij} in (1.2) satisfies

$$V^{\eta}_{ij}(x) = \frac{1}{\eta^d} V_{ij}(\frac{x}{\eta}), \quad \forall x \in \mathbb{R}^d$$

where $V_{ij} \in C^2(\mathbb{R}^d)$, supp $V_{ij} \subset B_\eta(0) := \{x \in \mathbb{R}^d : \ |x| \le \eta\}$ and $V_{ij}(x) = V_{ij}(y)$ whenever $|x| = |y|$. The scalar $\eta > 0$ is a scaling parameter. Notice that the L^1-norm of V^{η}_{ij} is invariant regarding changes of the parameter η. Furthermore, it is chosen in such a way that V^{η}_{ij} converges in the sense of distributions for $\eta \to 0$ towards $a_{ij}\delta$. Here δ denotes the Dirac-delta distribution.

For the next (formal) argument assume the processes $X^{k,N_i}_{\eta,i}$ are absolute continuous with respect to the Lebesgue measure λ^d on $\mathbb{R}^d$. That is to say, there exists a probability density function $u_{\eta,i}$ of $X^{k,N_i}_{\eta,i}$. Additionally let the particle numbers N_j, $j = 1 \dots, n$ be big enough such that the influence of one particle onto another is infinitesimal small. In other words, let the stochastic processes $X^{k,N_j}_{\eta,j}$ be practically independent. Then the law of large numbers would imply

$$\begin{aligned}\frac{1}{N_j}\sum_{l=1}^{N_j}\nabla V^\eta_{ij}(X^{k,N_i}_{\eta,i}(t)-X^{l,N_j}_{\eta,j}(t)) \simeq & \mathbb{E}\left[\nabla V^\eta_{ij}(X^{k,N_i}_{\eta,i}(t)-X^{l,N_j}_{\eta,j}(t))\right]\\ =&\int_{\mathbb{R}^d}\nabla V^\eta_{ij}(X^{k,N_i}_{\eta,i}(t)-y)u_{\eta,j}(y,t)dy\\ =&(\nabla V^\eta_{ij}*u_{\eta,j})(X^{k,N_i}_{\eta,i}(t),t).\end{aligned}$$

This motivates the study of the following intermediate system for fixed $\eta>0$ and large $N_i,\ i=1\dots,n$

$$(SDE\ I)\qquad \begin{cases} d\bar{X}^k_{\eta,i}(t)=-\sum_{j=1}^n\nabla V^\eta_{ij}*u_{\eta,j}(\bar{X}^k_{\eta,i}(t),t)dt+\sqrt{2}\sqrt{\sigma_i}(\bar{X}^k_{\eta,i}(t),t)dW^k_i(t),\\ \bar{X}^k_{\eta,i}(0)=\xi^k_i,\ 1\le i,j\le n,\ 1\le k\le N_i\end{cases}$$

in order to approximate the many particle model (SDE 0). Here $u_{\eta,i}$ is the probability density function of the process $d\bar{X}^k_{\eta,i}$, which can be shown, via an application of Ito's lemma, to satisfy the non-local diffusion system

$$(PDE\ I)\qquad \begin{cases}\partial_t u_{\eta,i}-\operatorname{div}(\sigma(x,t)_i\nabla u_{\eta,i})=\operatorname{div}\left(\sum_{j=1}^n u_{\eta,i}\nabla V^\eta_{ij}*u_{\eta,j}+\sum_{l=1}^d u_{\eta,i}\frac{\partial}{\partial x_l}(\sigma_i)_{\cdot,l}\right)\\ u_{\eta,i}\mid_{t=0}=u^0_i,\quad i\in\{1,2,\dots,n\}.\end{cases}$$

in $\mathbb{R}^d\times[0,T]$. By theorem 9.10 of [8], the term $\nabla V^\eta_{ij}*u_{\eta,j}$ converges for $\eta\to0$ towards $a_{ij}u_j$ regarding the $L^2(0,T;L^2)$-norm, where $a_{ij}=\int_{\mathbb{R}^d}V_{ij}(x)dx$ and u is the weak limit of u_η. This leads to the cross diffusion system (1.1)

$$(PDE\ II)\qquad \begin{cases}\partial_t u_i-\operatorname{div}(\sigma(x,t)_i\nabla u_i)=\operatorname{div}\left(\sum_{j=1}^n u_i a_{ij}\nabla u_j+\sum_{l=1}^d u_i\frac{\partial}{\partial x_l}(\sigma_i)_{\cdot,l}\right)\\ u_i\mid_{t=0}=u^0_i,\quad i\in\{1,2,\dots,n\},\end{cases}$$

in $\mathbb{R}^d\times[0,T)$ and the stochastic system

$$(SDE\ II)\qquad \begin{cases} d\widehat{X}^k_i(t)=-\sum_{j=1}^n a_{ij}\nabla u_j(\widehat{X}^k_i(t),t)dt+\sqrt{2}\sqrt{\sigma_i}(\widehat{X}^k_i(t),t)dW^k_i(t),\\ \widehat{X}^{k,N}_i(0)=\xi^k_i,\ 1\le i,j\le n,\ 1\le k\le N_i.\end{cases}$$

Let now $N=\min_{1\le i\le n}N_i$ and $\nu\log(N)\ge\frac{1}{\eta^{2d+4}}$ with ν sufficiently small. Then, for the difference of the solution of (SDE II) and the solution of (SDE 0), the estimate

$$\sup_{1\le k\le N}\mathbb{E}\left(\sum_{i=1}^n\sup_{0<s<t}\left|X^{k,N}_{\eta,i}-\widehat{X}^k_i\right|\right)\le C(t)\eta,\quad t\in(0,T]$$

can be established (see theorem 8.1.1). This is one of the main results of this work and has been already shown for constant diffusion $\sigma>0$ in [5].

Remark 1.1.1. The most commonly used results on the existence of solutions of stochastic differential equations demand the coefficients of said SDE's to be Lipschitz-continuous. So to ensure existence for the systems (SDE I) and (SDE II), we are searching for solutions u, u_η of (PDE I) and (PDE II) which are elements of $L(0, T; H^s)$, where $s > \frac{d}{2} + 2$. Due to the embedding $H^s \hookrightarrow W^{2,\infty}$ and $H^s \hookrightarrow C^2$, these are then uniformly Lipschitz-continuous. For the same reason we also demand $\sigma_i \in L^\infty(0, T; H^s)$ whenever we study a diffusion-matrix which is dependent on the space-variable x.

2. The Stochastic Systems

This chapter focuses on the stochastic systems introduced at the end of the previous chapter. In the first part, the existence and uniqueness of processes solving the corresponding stochastic differential equations are established. The later section concentrates on proving the existence of probability density functions, characterizing the aforementioned processes.

2.1. Existence of the Stochastic Processes

All findings presented in this section build upon the existence result [10, Theorem 5.2.1] for stochastic differential equations (see also theorem B.1.17). We therefore assume $\sqrt{\sigma_i}$ to be measurable and uniformly Lipschitz-continuous on the considered time-interval $[0,T]$, additional to the assumptions made in section 1.1. That is to say, there exists a constant $C > 0$ such that

$$|\sqrt{\sigma_i}(x,t) - \sqrt{\sigma_i}(y,t)| \leq C|x-y|, \quad \text{for all } x,y \in \mathbb{R}^d \text{ and } t \in [0,T].$$

Here the absolute value of a matrix $\sigma \in \mathbb{R}^{d\times d}$ is understood to be $|\sigma| = \sqrt{\sum_{j,l=1}^{d} \sigma_{jl}^2}$. Additionally we assume that $\sqrt{\sigma_i}$ fulfils a linear growth condition:

$$|\sqrt{\sigma_i}(x,t)| \leq C(1+|x|), \quad \text{for all } x \in \mathbb{R}^d \text{ and } t \in [0,T].$$

We will prove the existence of unique solutions of the stochastic differential equations (SDE 0)-(SDE II) regarding the filtrations $\mathcal{F}_t^0$-$\mathcal{F}_t^{II}$. These are the natural filtrations induced by the Brownian motions W_i^k and the initial random variables ξ_i^k, occurring in the corresponding stochastic differential equations. Here we assume all these stochastic processes and random variables to be independent from each other.

Theorem 2.1.1. *The System*

$$(SDE\ 0) \quad \begin{cases} dX_{\eta,i}^{k,N_i}(t) = -\sum_{j=1}^{n} \frac{1}{N_j} \sum_{l=1}^{N_j} \nabla V_{ij}^{\eta}(X_{\eta,i}^{k,N_i}(t) - X_{\eta,j}^{l,N_j}(t))dt + \sqrt{2}\sqrt{\sigma_i}(X_{\eta,i}^{k,N_i}(t),t)dW_i^k(t) \\ X_{\eta,i}^{k,N_i}(0) = \xi_i^k, \ 1 \leq i,j \leq n, \ 1 \leq k \leq N_i, \end{cases}$$

has a unique, $\mathcal{F}_t^0$-adapted, strong solution on the interval $[0,T]$ for any fixed choice of $\eta > 0$.

Proof. To simplify the notation, we omit the parameter η and N_i, i.e. $X_i^k \equiv X_{\eta,i}^{k,N_i}$ and write σ instead of $\sqrt{\sigma}$. We want to use theorem B.1.17. As the system (SDE 0) consists of interdependent equations, we need to consider the stochastic processes $X_i^k, \ 1 \leq i \leq n, \ 1 \leq k \leq N_i$ as one $d \cdot \sum_{j=1}^{n} N_j$- dimensional process instead of several separate ones. We therefore interpret the Brownian motions W_i^k as part of a $d \cdot \sum_{j=1}^{n} N_j$-dimensional Brownian motion. This is possible as we assume that they are independent in a stochastic sense. To organize the processes X_1^k into one stochastic vector, we use the mapping $ind(\cdot,\cdot)$ defined by

$$ind(j,l) = \tilde{N}_{j-1} + l, \ 1 \leq j \leq n, \ 1 \leq l \leq N_j$$

where $\tilde{N}_i = \sum_{j=1}^{i} N_j$. Let X be the stochastic process which satisfies[1] $X_{ind(j,l)} = X_j^l$ and W the corresponding ($d \cdot \tilde{N}_n$-dimensional) Brownian motion. Intuitively these are the vectors one obtains by writing all the processes X_i^k, respectively W_i^k, among one another. Then problem (SDE I) reads as

$$dX_m(t) = -\sum_{j=1}^{n} \frac{1}{N_j} \sum_{l=1}^{N_j} \nabla V_{ij}(X_m(t) - X_{ind(j,l)}(t))dt + \sqrt{2}\sigma_{g(m)}(X_m(t), t)dW_m(t),$$

$$X_m(0) = \xi_{g(m)}^{m-\tilde{N}_{g(m)-1}}$$

where $1 \le m \le \tilde{N}_n$ and $g(m) := \min\{1 \le i \le n : \ m \le \tilde{N}_i\}$. We define

$$b_m(X) := -\sum_{j=1}^{n} \frac{1}{N_j} \sum_{l=1}^{N_j} \nabla V_{ij}(X_m(t) - X_{ind(j,l)}(t))$$

and

$$\sigma(X,t) := \begin{pmatrix} \sigma_1(X_1,t) & & & & & & 0 \\ & \ddots & & & & & \\ & & \sigma_1(X_{N_1},t) & & & & \\ & & & \ddots & & & \\ & & & & \sigma_n(X_{\tilde{N}_{n-1}+1},t) & & \\ & & & & & \ddots & \\ 0 & & & & & & \sigma_n(X_{\tilde{N}_n},t) \end{pmatrix}.$$

In order to apply theorem 5.2.1 of [10], we need to show the existence of a constant $C > 0$ such that

$$|b(x,t)| + |\sigma(x,t)| \le C(1 + |x|), \ x \in \mathbb{R}^{d \cdot \tilde{N}_n}, \ t \in [0,T] \tag{2.1}$$

and

$$|b(x,t) - b(y,t)| + |\sigma(x,t) - \sigma(y,t)| \le C|x - y|, \ x, y \in \mathbb{R}^{d \cdot \tilde{N}_n}, \ t \in [0,T]. \tag{2.2}$$

We start by showing that $b(x,t)$ is uniformly Lipschitz on $[0,T]$. Let therefore $x, y \in \mathbb{R}^{d \cdot \tilde{N}_n}$ and $t \in [0,T]$ be arbitrary but fixed. Then we have

$$|b(x,t) - b(y,t)|^2 = \sum_{m=1}^{\tilde{N}_n} |b_m(x,t) - b_m(y,t)|^2$$

$$\le \sum_{m=1}^{\tilde{N}_n} \left(\sum_{j=1}^{n} \frac{1}{N_j} \sum_{l=1}^{N_j} \left| \nabla V_{ij}(x_m - x_{ind(j,l)}) - \nabla V_{ij}(y_m - y_{ind(j,l)}) \right| \right)^2.$$

[1]The indexing organizes the process in terms of d-dimensional vectors.

By assumption $\nabla V_{ij} \in C_0^1(\mathbb{R}^d)$ with supp $V_{ij} \subset B_\eta(0)$. In particular ∇V_{ij} is Lipschitz[2] with the Lipschitz-constant L_{ij}:

$$|b(x,t) - b(y,t)|^2 = \sum_{m=1}^{\tilde{N}_n} |b_m(x,t) - b_m(y,t)|^2$$

$$\leq \sum_{m=1}^{\tilde{N}_n} \left(\sum_{j=1}^{n} \frac{1}{N_j} \sum_{l=1}^{N_j} L_{ij} \big| x_m - y_m - x_{ind(j,l)} + y_{ind(j,l)}) \big| \right)^2$$

$$\leq \sum_{m=1}^{\tilde{N}_n} \left(\sum_{j=1}^{n} \frac{2L_{ij}}{N_j} \sum_{l=1}^{N_j} |x-y| \right)^2 \leq K|x-y|^2.$$

Taking the square root on either side of the inequality proves the Lipschitz continuity of b. For σ we proceed similarly. Due to the identity

$$|\sigma(x,t) - \sigma(y,t)|^2 = \sum_{k,l=1}^{\tilde{N}_n} |\sigma_{kl}(x,t) - \sigma_{kl}(y,t)|^2$$

it suffices to show $|\sigma_1(x_1, t - \sigma_1(y_1,t))|^2 \leq K|x-y|^2$. But the σ_i are uniformly Lipschitz by assumption and condition (2.2) is therefore satisfied.

Similarly we get

$$|\sigma(x,t)| \leq C(1+|x|).$$

Thus the remaining condition (2.1) follows therefore after a short calculation, from (2.2):

$$\begin{aligned} |b(x,t)| + |\sigma(x,t)| &\leq |b(x,t) - b(0,t)| + |b(0,t)| + C(1+|x|) \\ &\leq C|x-0| + |b(0,t)| + C(1+|x|) \leq (2C + |b(0,t)|)(1+|x|). \end{aligned} \tag{2.3}$$

Notice that b does not depend on the parameter t. □

The systems (SDE I) and (SDE II) depend, besides V_{ij} and σ_i, on the functions u and u_η (see for example section 1.1). These are defined as the respective solutions of (PDE I) and (PDE II). We do not have any existence and regularity results regarding these sets of equations as of now. We therefore assume, for the sake of the proof of existence for the stochastic systems, that they are arbitrary elements of $L^\infty(0,T; L^2(\mathbb{R}^d, \mathbb{R}^n))$, with u satisfying a similar uniform Lipschitz-condition as σ_i. We thus suppose the existence of a constant $C > 0$ such that

$$\begin{aligned} |\nabla u(x,t) - \nabla u(y,t)| &\leq C|x-y| \\ |\nabla u(x,t)| &\leq C(1+|x|), \end{aligned} \tag{2.4}$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0,T]$.

[2]This can be seen, for example, by applying the mean value theorem.

Lemma 2.1.2. *Let $u_\eta \in L^\infty(0,T;L^2(\mathbb{R}^d,\mathbb{R}^n))$ for a fixed choice of $\eta > 0$. Then there exists a strong, $\mathcal{F}_t^I$-adapted, unique solution of*

$$(SDE\ I) \quad \begin{cases} d\bar{X}_{\eta,i}^k(t) = -\sum_{j=1}^n \nabla V_{ij}^\eta * u_{\eta,j}(\bar{X}_{\eta,i}^k(t),t)dt + \sqrt{2}\sqrt{\sigma_i}(\bar{X}_{\eta,i}^k(t),t)dW_i^k(t), \\ \bar{X}_{\eta,i}^k(0) = \xi_i^k,\ 1 \le i,j \le n,\ 1 \le k \le N_i \end{cases}$$

on the interval $[0,T]$. *Furthermore, let u fulfil condition (2.4). Then the system*

$$(SDE\ II) \quad \begin{cases} d\widehat{X}_i^k(t) = -\sum_{j=1}^n a_{ij}\nabla u_j(\widehat{X}_i^k(t),t)dt + \sqrt{2}\sqrt{\sigma_i}(\widehat{X}_i^k(t),t)dW_i^k(t), \\ \widehat{X}_i^k(0) = \xi_i^k,\ 1 \le i,j \le n,\ 1 \le k \le N_i \end{cases}$$

has a unique strong solution on the interval [0,T], which is $\mathcal{F}_t^{II}$-adapted.

Proof. Let $1 \le i \le n$ be arbitrary but fixed. Similar to the proof of theorem 2.1.1, we want to utilize theorem B.1.17. Recall that $\sqrt{\sigma_i}$ is uniformly Lipschitz-continuous by assumption. Thus, we only have to show the uniform Lipschitz-continuity of b_η and b, which we define by

$$b_\eta(x,t) := -\sum_{j=1}^n \nabla V_{ij}^\eta * u_{\eta,j}(x,t)$$

and

$$b(x,t) := -\sum_{j=1}^n a_{ij}\nabla u_i(x,t)$$

where $(x,t) \in \mathbb{R}^d \times [0,T]$. Due to $V_{ij}^\eta \in C_0^2$, with supp $V_{ij}^\eta \subset B_\eta(0)$, $b_\eta(\cdot,t)$ is continuously differentiable and we have for any choice of j, $1 \le j \le d$

$$\begin{aligned} \left|\frac{\partial}{\partial x_j} b_\eta\right| &= \left|\sum_{j=1}^n \frac{\partial}{\partial x_j}\nabla V_{ij}^\eta * u_{\eta,j}\right| = \left|\sum_{j=1}^n \left(\nabla \frac{\partial}{\partial x_j} V_{ij}^\eta\right) * u_{\eta,j}\right| \\ &\le \sum_{j=1}^n \left|\left(\nabla \frac{\partial}{\partial x_j} V_{ij}^\eta\right) * u_{\eta,j}\right| \le \sum_{j=1}^n \left\|\nabla \frac{\partial}{\partial x_j} V_{ij}^\eta\right\|_{L^2} \|u_{\eta,j}\|_{L^2} \\ &\le \sum_{j=1}^n \left\|\nabla \frac{\partial}{\partial x_j} V_{ij}^\eta\right\|_{L^2} \|u_{\eta,j}\|_{L^\infty(0,T;L^2)} \\ &\le \sqrt{\text{meas}(B_\eta(0))}\sum_{j=1}^n \left\|\nabla \frac{\partial}{\partial x_j} V_{ij}^\eta\right\|_{L^\infty} \|u_\eta\|_{L^\infty(0,T;L^2)} \le C. \end{aligned} \tag{2.5}$$

Thus, $b_\eta(\cdot,t) \in C_b^1(\mathbb{R}^d)$. In particular $b_\eta(.,t)$ is Lipschitz continuous. Notice that (2.5) does not depend on t anymore and that the constant $C > 0$ in estimate (2.5) can be chosen independently from the parameter j. Hence we can find a Lipschitz constant of $b(\cdot,t)$ which is independent from t. As u is already assumed to satisfy the condition (2.4), the Lipschitz continuity of b is a simple conclusion.

For the linear growth condition of theorem B.1.17, see (2.3) as $b_\eta(0,t)$ can be shown to be bounded on [0,T] with the same arguments we used to derive estimate (2.5). □

2.2. Absolute Continuity

In this section we establish the existence of the probability density functions of the processes $\bar{X}^k_{\eta,i}$ and $\widehat{X}^k_i$, solving (SDE I) and (SDE II). These densities are needed later on to make the transition from a stochastic differential equation to a deterministic one.

A probability density function of a random variable X is nothing more then the Radon-Nikodym density of the probability law of $X \in \mathbb{R}^d$, regarding the Lebesgue measure λ^d. The Radon-Nikodym density itself exists if and only if the law of X is absolutely continuous with respect to λ^d. Proving absolute continuity of $\bar{X}^k_{\eta,i}$ and $\widehat{X}^k_i$ is therefore sufficient to justify the existence of a probability density.

Lemma 2.2.1. *Let $u_\eta \in L^\infty(0,T;L^2(\mathbb{R}^d,\mathbb{R}^n))$ for a fixed choice of η. Then the solution $\bar{X}^k_{\eta,i}$ of (SDE II) is absolutely continuous with respect to the Lebesgue measure and thus possesses a probability density function.*

Proof. Due to lemma 2.1.2 the stochastic process $\bar{X}^k_{\eta,i}$ solving (SDE II) exists. Let $S := \inf\{0 < t < T : \int\limits_0^t \chi_{\{0<s<T:\ \det\ \sqrt{\sigma_i(\bar{X}^k_{\eta,i}(s),s)}\neq 0\}} ds > 0\}$ where χ is the characteristic function. Theorem 2.3.1 in [11] implies the absolute continuity of the probability law of $\bar{X}^k_{\eta,i}$ in respect to the Lebesgue-measure λ^d on $\mathbb{R}^d$, as long as $S = 0$. This condition is equivalent to the denseness of $\{0 < s < \delta :\ \det\ \sqrt{\sigma_i(\bar{X}^k_{\eta,i}(s),s)} \neq 0\}$ in $[0,\delta]$ for an arbitrary $\delta > 0$. By assumption $\sigma = \sqrt{\sigma_i}^T\sqrt{\sigma_i}$, is uniformly coercive and in particular positive definite. Hence

$$\left(\det\ \sqrt{\sigma_i(x,t)}\right)^2 = \det\ \sigma_i(x,t) \neq 0$$

for all $x \in \mathbb{R}^d$ and $0 < t < T$. This implies $\{0 < s < T :\ \det\ \sqrt{\sigma_i(\bar{X}^k_{\eta,i}(s),s)} \neq 0\} = [0,T]$ and so the law of $\bar{X}^k_{i,\eta}$ is absolute continuous regarding the Lebesgue measure and thus has a probability density function. □

Repeating the argumentation above, we can prove the same result for $\widehat{X}^k_i$ and (SDE I):

Lemma 2.2.2. *Let u satisfy the Lipschitz-condition (2.4). Then the solution $\widehat{X}^k_i$ of (SDE I) is absolutely continuous with respect to the Lebesgue measure and thus possesses a probability density function.*

Proof. See the proof of lemma 2.2.1. □

3. Derivation of the Cross-Diffusion Systems

In this short chapter it is shown, how the intermediate system (PDE I) and the cross diffusion system (PDE II) come about as determining equations for the probability density functions of the solutions of problem (SDE I) and (SDE II). This is established by an application of Ito's lemma.

3.1. Derivation of the Intermediate System (PDE I)

Let $u_\eta \in L^\infty(0,T;L^2(\mathbb{R}^d,\mathbb{R}^n))$. Due to lemma 2.2.1 and theorem 2.1.2, problem (SDE II) is guaranteed to have a solution $\bar{X}^k_{\eta,i}$, which in turn has a probability density $\tilde{u}_\eta$. Let $\phi \in C^2_b(\mathbb{R}^d \times [0,T])$. We apply Ito's lemma to $\phi(\bar{X}^k_{\eta,i}(t),t)$:

$$\begin{aligned}\phi(\bar{X}^k_{\eta,i}(t),t) =&\phi(\xi^k_i,0)\\ &+\int_0^t \frac{\partial\phi}{\partial s}(\bar{X}^k_{\eta,i}(s),s)ds - \int_0^t \sum_{j=1}^n \nabla\phi(\bar{X}^k_{\eta,i}(s),s)\cdot(\nabla V^\eta_{ij} * u_{\eta,j})(\bar{X}^k_{\eta,i}(s),s)ds\\ &+\int_0^t \sum_{k,l=1}^d \frac{\partial\phi}{\partial x_k\partial x_l}(\bar{X}^k_{\eta,i}(s),s)\sigma_i(\bar{X}^k_{\eta,i}(s),s)_{kl}ds\\ &+\sqrt{2}\int_0^t \nabla\phi(\bar{X}^k_{\eta,i}(s),s)\sqrt{\sigma_i(\bar{X}^k_{\eta,i}(s),s)}dW^k_i(s).\end{aligned}$$

To arrive at a deterministic statement, we apply the expectation on both sides of the last equation and use the absolute continuity of the law of $\bar{X}^k_{\eta,i}$:

$$\begin{aligned}\int_{\mathbb{R}^d}\phi(x,t)\tilde{u}_\eta(x,t)dx =& \int_{\mathbb{R}^d}\phi(x,0)\tilde{u}_\eta(x,0)dx + \int_0^t\int_{\mathbb{R}^d}\frac{\partial\phi}{\partial s}(x,s)\tilde{u}_\eta(x,s)dxds\\ &-\int_0^t\int_{\mathbb{R}^d}\sum_{j=1}^n\nabla\phi(x,s)\cdot(\nabla V^\eta_{ij} * u_{\eta,j})(x,s)\tilde{u}_\eta(x,s)dxds \qquad (3.1)\\ &+\int_0^t\int_{\mathbb{R}^d}\sum_{k,l=1}^d\frac{\partial\phi}{\partial x_k\partial x_l}(x,s)\sigma_i(x,s)_{kl}\tilde{u}_\eta(x,s)dxds.\end{aligned}$$

(3.1) can be interpreted as the weak formulation of a linear partial differential equation. To show this, and to make our next steps rigorous, we assume u is a smooth function. Then we have

$$\int\limits_{\mathbb{R}^d} \phi(x,t)\tilde{u}_\eta(x,t)dx = \int\limits_{\mathbb{R}^d} \phi(x,0)\tilde{u}_\eta(x,0)dx + \int\limits_{\mathbb{R}^d} \phi(x,t)\tilde{u}_\eta(x,t)dx - \int\limits_{\mathbb{R}^d} \phi(x,0)\tilde{u}_\eta(x,0)dx$$
$$- \int\limits_0^t \int\limits_{\mathbb{R}^d} \phi(x,s)\frac{\partial \tilde{u}_\eta}{\partial s}(x,s)dxds$$
$$+ \int\limits_0^t \int\limits_{\mathbb{R}^d} \sum_{j=1}^n \phi(x,s)\mathrm{div}\left(\nabla V_{ij}^\eta * u_{\eta,j}(x,s)\tilde{u}_\eta(x,s)\right) dxds$$
$$+ \int\limits_0^t \int\limits_{\mathbb{R}^d} \phi(x,s) \sum_{k,l=1}^d \frac{\partial}{\partial x_k \partial x_l} \left(\sigma_i(x,s)_{kl}\tilde{u}_\eta(x,s)\right) dxds$$
$$= \int\limits_{\mathbb{R}^d} \phi(x,t)\tilde{u}_\eta(x,t)dx$$
$$+ \int\limits_0^t \int\limits_{\mathbb{R}^d} \phi(x,s) \left(-\frac{\partial \tilde{u}_\eta}{\partial s}(x,s) + \sum_{j=1}^n \mathrm{div}\left(\nabla V_{ij}^\eta * u_{\eta,j}(x,s)\tilde{u}_\eta(x,s)\right)\right) dxds$$
$$+ \int\limits_0^t \int\limits_{\mathbb{R}^d} \phi(x,s) \sum_{k,l=1}^d \frac{\partial}{\partial x_k \partial x_l} \left(\sigma_i(x,s)_{kl}\tilde{u}_\eta(x,s)\right) dxds.$$

Rearranging and cancelling some terms, the above equality reads as

$$\int\limits_0^t \int\limits_{\mathbb{R}^d} \left(-\frac{\partial \tilde{u}_\eta}{\partial s}(x,s) + \sum_{j=1}^n \mathrm{div}\left(\nabla V_{ij}^\eta * u_{\eta,j}(x,s)\tilde{u}_\eta(x,s)\right) + \sum_{k,l=1}^d \frac{\partial}{\partial x_k \partial x_l} \left(\sigma_i(x,s)_{kl}\tilde{u}_\eta(x,s)\right)\right) \times$$
$$\phi(x,s)dxds = 0$$

Because ϕ was arbitrary, we apply the fundamental lemma of calculus of variations. This results in the following partial differential equation

$$-\frac{\partial u}{\partial s}(x,s) + \sum_{j=1}^n \mathrm{div}\left(\nabla V_{ij}^\eta * u_{\eta,j}(x,s)u(x,s)\right) + \sum_{k,l=1}^d \frac{\partial}{\partial x_k \partial x_l} \left(\sigma_i(x,s)_{kl}u(x,s)\right) = 0.$$

for $s \in [0,t]$. We rearrange the third term on the left hand side, to obtain a diffusion term:

$$\sum_{k,l=1}^d \frac{\partial}{\partial x_k \partial x_l} \left((\sigma_i)_{kl}\tilde{u}_\eta\right) = \sum_{k,l=1}^d \frac{\partial}{\partial x_k} \left(\frac{\partial (\sigma_i)_{kl}}{\partial x_l}\tilde{u}_\eta + (\sigma_i)_{kl}\frac{\partial \tilde{u}_\eta}{\partial x_l}\right)$$
$$= \sum_{k=1}^d \frac{\partial}{\partial x_k} \sum_{l=1}^d \left(\frac{\partial (\sigma_i)_{kl}}{\partial x_l}\tilde{u}_\eta\right) + \sum_{k=1}^d \frac{\partial}{\partial x_k} \sum_{l=1}^d (\sigma_i)_{kl}\frac{\partial \tilde{u}_\eta}{\partial x_l}$$
$$= \mathrm{div}\left(\sum_{l=1}^d \frac{\partial (\sigma_i)_{\cdot l}}{\partial x_l}\tilde{u}_\eta\right) + \mathrm{div}\left(\sigma_i \nabla \tilde{u}_\eta\right).$$

Using the identity above, this then yields the following, more familiar, parabolic equation:

$$\frac{\partial \tilde{u}_\eta}{\partial t} - \operatorname{div}(\sigma_i \nabla \tilde{u}_\eta) = \operatorname{div}\left(\sum_{j=1}^{n} \tilde{u}_\eta \nabla V_{ij}^\eta * u_{\eta,j} + \sum_{l=1}^{d} \tilde{u}_\eta \frac{\partial (\sigma_i)_{\cdot,l}}{\partial x_l}\right) \quad \text{in } \mathbb{R}^d \times [0,T]. \tag{3.2}$$

Thus, we have proven:

Lemma 3.1.1. *Let $u_\eta \in L^\infty(0,T;L^2(\mathbb{R}^d,\mathbb{R}^n))$. Then the probability density function $\tilde{u}_\eta$ of the solution $\bar{X}_{\eta,i}^k$ of (SDE I) is a weak solution of the linear evolution equation (3.2), in the sense that*

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(x,t)\tilde{u}_\eta(x,t)dx = \int_{\mathbb{R}^d} \phi(x,0)\tilde{u}_\eta(x,0)dx + \int_0^t \int_{\mathbb{R}^d} \frac{\partial \phi}{\partial s}(x,s)\tilde{u}_\eta(x,s)dxds \\ - \int_0^t \int_{\mathbb{R}^d} \sum_{j=1}^{n} \nabla \phi(x,s) \cdot (\nabla V_{ij}^\eta * u_{\eta,j})(x,s)\tilde{u}_\eta(x,s)dxds \\ + \int_0^t \int_{\mathbb{R}^d} \sum_{k,l=1}^{d} \frac{\partial \phi}{\partial x_k \partial x_l}(x,s)\sigma_i(x,s)_{kl}\tilde{u}_\eta(x,s)dxds\end{aligned}$$

for all $\phi \in C_b^2(\mathbb{R}^d \times [0,T])$.

3.2. Derivation of the Cross-Diffusion System (PDE II)

One can prove a similar result to lemma 3.1.1 in the case of the stochastic differential equation (SDE II). To be more precise, one can derive a weak formulation of the equation

$$\frac{\partial \tilde{u}}{\partial t} - \operatorname{div}(\sigma_i \nabla \tilde{u}) = \operatorname{div}\left(\sum_{j=1}^{n} \tilde{u} a_{ij} \nabla u_j + \sum_{l=1}^{d} \tilde{u} \frac{\partial (\sigma_i)_{\cdot,l}}{\partial x_l}\right) \quad \text{in } \mathbb{R}^d \times [0,T] \tag{3.3}$$

which is satisfied by the probability density function $\tilde{u}$ of $\widehat{X}_i^k$. To ensure the existence of the solution $\widehat{X}_i^k$ of (SDE II) and its density function, we assume u to be again uniformly Lipschitz-continuous. Then we can apply theorem 2.1.2 and lemma 2.2.2. Using the same techniques as in the previous section, we arrive at the following result:

Lemma 3.2.1. *Let u satisfy the Lipschitz-condition (2.4). Then the probability density function $\tilde{u}$ of the solution $\widehat{X}_i^k$ of (SDE II) is a weak solution of the linear evolution equation (3.3) in the sense that*

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(x,t)\tilde{u}(x,t)dx = \int_{\mathbb{R}^d} \phi(x,0)\tilde{u}(x,0)dx \\ + \int_0^t \int_{\mathbb{R}^d} \frac{\partial \phi}{\partial s}(x,s)\tilde{u}(x,s)dxds - \int_0^t \int_{\mathbb{R}^d} \sum_{j=1}^{n} a_{ij}\tilde{u}(x,s)\nabla u_j(x,s) \cdot \nabla \phi(x,s)dxds \\ + \int_0^t \int_{\mathbb{R}^d} \sum_{k,l=1}^{d} \frac{\partial \phi}{\partial x_k \partial x_l}(x,s)\sigma_i(x,s)_{kl}\tilde{u}(x,s)dxds,\end{aligned}$$

for all $\phi \in C_b^2(\mathbb{R}^d \times [0,T])$.

4. The Linear Advection-Diffusion Problem

Due to the non-linearity of the occurring systems (PDE I) and (PDE II), this chapter focuses on the proof of existence for a linear problem, which was deduced from the intermediate system (PDE I). Using an evolution operator, a weak solution is constructed on the subinterval $[\delta, T]$ for every $0 < \delta \leq T$. These solutions are then extended to the whole interval $[0, T]$ and shown to converge towards the solution of the linear problem for $\delta \to 0$. The final parts of this chapter deals with the problem of achieving higher regularity of this solution, assuming more regular coefficients.

4.1. The Linear Advection-Diffusion Problem

In preparation for the proof of existence for our non-local diffusion system (PDE I), we first concentrate on the following linear parabolic equation:

$$\begin{aligned} \partial_t u - \operatorname{div}(\sigma(x,t)\nabla u) =& \operatorname{div}(uf(x,t)) \quad \text{in } \mathbb{R}^d \times (0,T], \\ u|_{t=0} =& u_0(x), \ \text{in } \mathbb{R}^d \end{aligned} \tag{4.1}$$

where $\sigma_{(i)}$ is an element of $\in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^{d\times d}))$, in addition to the assumptions made in section 1.1. Furthermore, $f \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d))$ and u_0 is an element of $L^2(\mathbb{R}^d,\mathbb{R})$.

By multiplying (4.1) with $\phi \in \mathcal{D}(\mathbb{R}^d \times [0,T])$ and integrating over the whole space $\mathbb{R}^d \times [0,T]$, one can derive the subsequent weak formulation by partial integration, assuming u is smooth enough:

$$\int_0^T \langle \partial_t u, \phi \rangle_{H^{-1}} dt + \int_0^T \int_{\mathbb{R}^d} \nabla u^T \sigma(x,t) \nabla \phi dx dt = \int_0^T \int_{\mathbb{R}^d} \operatorname{div}(uf(x,t))\, \phi dx dt. \tag{4.2}$$

Here, H^{-1} denotes the topological dual of $H^1(\mathbb{R}^d,\mathbb{R})$.

Definition 4.1.1 (Weak solution)**.** We say u is a *weak solution* of (4.1) if $u \in W^{1,2}(0,T;H^1(\mathbb{R}^d,\mathbb{R}),L^2(\mathbb{R}^d,\mathbb{R}))$, u satisfies (4.2) for all $\phi \in L^2(0,T,H^1(\mathbb{R}^d,\mathbb{R}))$ and $u|_{t=0} = u_0$ almost everywhere.

Going forward, we will omit to state the underlying sets of our functional spaces, if it is obvious from context.

Remark 4.1.2. As $u \in L^2(0,T;H^1)$, one might consider the fulfillment of the initial condition in definition 4.1.1 as too harsh but due to the continuous embedding of $W^{1,2}(0,T;H^1,L^2)$ into $C([0,T],L^2)$, written as $W^{1,2}(0,T;H^1,L^2) \hookrightarrow C([0,T],L^2)$, the initial condition will be fulfilled by the *weak solution* in the L^2 sense, i.e. almost everywhere.

4.1.1. A-priori estimates and uniqueness

Let $u \in W^{1,2}(0,T;H^1,L^2)$ be a weak solution of (4.1). For $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $\psi \in \mathcal{D}([0,T])$, $\phi(x)\cdot\psi(t)$ is an element of $\mathcal{D}(\mathbb{R}^d \times [0,T])$. It therefore holds that:

$$\int_0^T \langle \partial_t u, \phi \rangle_{H^{-1}} \psi dt + \int_0^T \int_{\mathbb{R}^d} \nabla u^T \sigma \nabla \phi dx \psi dt = \int_0^T \int_{\mathbb{R}^d} \operatorname{div}(uf)\, \phi dx \psi dt.$$

As ψ was an arbitrary element of $\mathcal{D}[0,T]$, one can deduce from the fundamental lemma of calculus of variations

$$\langle \partial_t u, \phi \rangle_{H^{-1}} + \int_{\mathbb{R}^d} \nabla u \cdot \sigma \cdot \nabla \phi dx = \int_{\mathbb{R}^d} \operatorname{div}(u \cdot f)\, \phi dx \tag{4.3}$$

for all $\phi \in \mathcal{D}(\mathbb{R}^d)$ and almost every $t \in [0,T]$. This holds also true for $\phi \in H^1(\mathbb{R}^d)$, as one can easily deduce by approximating ϕ via a sequence of test- functions $(\phi_n)_{n\in\mathbb{N}}$, due to the fact that $\mathcal{D}(\mathbb{R}^d)$ lies dense in $H^1(\mathbb{R}^d)$.

The equation (4.3) serves as basis for a number of a-priori estimates that can be derived for *weak solutions* of the system (4.1). Considering that we want to interpret these solutions as density functions of stochastic processes, the first of the results we are presenting here, regarding the positivity of u, is of utmost significance. The second theorem however, provides us with some norm-estimates, which will come in handy for the proof of existence in the later part of this chapter. Furthermore, the uniqueness of *weak solutions* will be a direct conclusion from it.

Theorem 4.1.3 (Positivity)**.** *Let u be a weak solution of (4.1)and $u_0 \geq 0$ almost everywhere. Then $u \geq 0$ almost everywhere.*

Proof. As mentioned at the beginning of this section u solves equation (4.3) almost everywhere for all $\phi \in H^1(\mathbb{R}^d)$. By using u^- as test function and applying Gauss's theorem for Sobolev-functions on the term on the right-hand side of the equation, we get

$$\langle \partial_t u, u^- \rangle_{H^{-1}} + \int_{\mathbb{R}^d} \nabla u^T \sigma \nabla u^- dx = -\int_{\mathbb{R}^d} u\left(f \nabla u^-\right) dx.$$

This calculation is rigorous as $u \in H^1(\mathbb{R}^d)$ implies $u^- \in H^1(\mathbb{R}^d)$, by a well known theorem from Stampacchia, see A.2.7. Furthermore, it holds that $\nabla u^- = \chi_{\{u\leq 0\}} \nabla u$, which leaves us with

$$\langle \partial_t u, u^- \rangle_{H^{-1}} + \int_{\mathbb{R}^d} \nabla (u^-)^T \sigma \nabla u^- dx = -\int_{\mathbb{R}^d} u^- \left(f \cdot \nabla u^-\right) dx.$$

It can be shown (see A.3.22) that $\|u^-\|_{L^2}^2$ is absolutely continuous and

$$\langle \partial_t u, u^- \rangle_{H^{-1}} = \frac{d}{dt} \|u^-\|_{L^2}^2.$$

Additionally, via Youngs inequality for products (see E.0.1), we get the estimate

$$|\int_{\mathbb{R}^d} u^- \left(f \cdot \nabla u^-\right) dx| \leq \int_{\mathbb{R}^d} |u^-||f \cdot \nabla u^-| dx \leq \int_{\mathbb{R}^d} \frac{1}{2\delta}|u^-|^2 + \frac{\delta}{2}|f \cdot \nabla u^-|^2 dx$$

$$\leq \frac{1}{2\delta}\int_{\mathbb{R}^d} |u^-|^2 dx + \frac{\delta}{2}\int_{\mathbb{R}^d} |f|^2|\nabla u^-|^2 dx \leq \frac{1}{2\delta}\|u^-\|_{L^2}^2 + \frac{\delta\|f\|^2_{L^\infty(\mathbb{R}^d\times[0,T])}}{2}\|\nabla u^-\|_{L^2}^2$$

for all $\delta > 0$. Due to the uniform coercivity of σ (with constant ϵ), these results amount to

$$\frac{d}{dt}\|u^-\|_{L^2}^2 + \epsilon\|\nabla u^-\|_{L^2}^2 \leq \frac{1}{2\delta}\|u^-\|_{L^2}^2 + \frac{\delta\|f\|^2_{L^\infty(\mathbb{R}^d\times[0,T])}}{2}\|\nabla u^-\|_{L^2}^2.$$

Choosing δ such that $\delta\|f\|^2_{L^\infty(\mathbb{R}^d\times[0,T])} \leq \epsilon$, leads therefore to

$$\frac{d}{dt}\|u^-\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla u^-\|_{L^2}^2 \leq \frac{1}{2\delta}\|u^-\|_{L^2}^2$$

but most importantly to

$$\frac{d}{dt}\|u^-\|_{L^2}^2 \leq \frac{1}{2\delta}\|u^-\|_{L^2}^2$$

as $\frac{\epsilon}{2}\|\nabla u^-\|_{L^2}^2 \geq 0$. By applying Gronwalls lemma to the last inequality, we obtain

$$\|u^-(t)\|_{L^2}^2 \leq \|u^-(0)\|_{L^2}^2 \exp\left(\int_0^t \frac{1}{2\delta} ds\right) = 0$$

for all $t \in [0, T]$, as we assumed $u_0 \geq 0$. □

Theorem 4.1.4 (Rate of growth). *Let u be a weak solution of (4.1). Then there exists a constant $C > 0$, which only depends on f, such that*

$$\|u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 \exp(Ct)$$

and

$$\|u(t)\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla u\|^2_{L^2(0,t,L^2)} \leq \|u_0\|_{L^2}^2 (\exp(Ct) + 1)$$

for all $t \in [0, T]$.

Proof. We start again with equation (4.3) and use $\phi = u$ as test function to get

$$\langle \partial_t u, u\rangle_{H^{-1}} + \int_{\mathbb{R}^d} \nabla u^T \sigma \nabla u dx = -\int_{\mathbb{R}^d} u \left(f \cdot \nabla u\right) dx$$

by the integration-by-parts formula for Sobolev functions. We will use the identity

$$\langle \partial_t u, u\rangle_{H^{-1}} = \frac{d}{dt}\|u\|_{L^2}^2 \quad a.e.$$

and will argue as in the proof of theorem (4.1.3), to show that

$$\frac{d}{dt}\|u\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla u\|_{L^2}^2 \le \frac{1}{2\delta}\|u\|_{L^2}^2 \tag{4.4}$$

for all $\delta \le \epsilon/\|f\|_{L^\infty(\mathbb{R}^d\times[0,T])}^2$ on the interval $[0,T]$. Let δ from now on be fixed. Integrating (4.4) from 0 to t yields the inequality

$$\|u(t)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla u\|_{L^2(0,t,L^2)}^2 \le \frac{1}{2\delta}\int_0^t \|u(s)\|_{L^2}^2 ds. \tag{4.5}$$

Furthermore, from (4.4) we can also derive

$$\frac{d}{dt}\|u\|_{L^2}^2 \le \frac{1}{2\delta}\|u\|_{L^2}^2$$

which leads, according to Gronwalls lemma, to

$$\|u(t)\|_{L^2}^2 \le \|u(0)\|_{L^2}^2 \exp\left(\int_0^t \frac{1}{2\delta} ds\right) = \|u(0)\|_{L^2}^2 \exp\left(\frac{t}{2\delta}\right)$$

for all $t \in [0,T]$. The right hand side of (4.5) can now be bounded by

$$\int_0^t \|u(s)\|_{L^2}^2 ds \le \int_0^t \|u(0)\|_{L^2}^2 \exp\left(\frac{s}{2\delta}\right) ds = \|u(0)\|_{L^2}^2 2\delta \exp\left(\frac{t}{2\delta}\right).$$

This, together with (4.5), amounts to

$$\|u(t)\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla u\|_{L^2(0,t,L^2)}^2 \le \|u(0)\|_{L^2}^2 \exp\left(\frac{t}{2\delta}\right) + \|u(0)\|_{L^2}^2.$$

Choosing $C := \frac{1}{2\delta}$ completes the proof. □

Corollary 4.1.5 (Uniqueness)**.** *There exists at most one weak solution u of the linear PDE (4.1) with the initial condition $u|_{t=0} = u_0 \in L^2$.*

Proof. Let u_1,u_2 be two *weak solutions* with the same initial condition u_0. Then for each $i \in \{1,2\}$ the equation

$$\int_0^T \langle \partial_t u_i, \phi\rangle_{H^{-1}} dt + \int_0^T\int_{\mathbb{R}^d} \nabla u_i^T \sigma \nabla \phi dx dt = \int_0^T\int_{\mathbb{R}^d} \operatorname{div}(u_i f)\,\phi dx dt$$

is satisfied for all $\phi \in L^2(0,T;H^1)$. Taking the difference of these two equations leads to

$$\int_0^T \langle \partial_t(u_1-u_2), \phi\rangle_{H^{-1}} dt + \int_0^T\int_{\mathbb{R}^d} \nabla(u_1-u_2)^T \sigma \nabla \phi dx dt$$

$$= \int_0^T\int_{\mathbb{R}^d} \operatorname{div}((u_1-u_2)f(x,t))\,\phi dx dt.$$

As $u_1(0)-u_2(0) = 0$ a. e., $w := u_1 - u_2$ is a *weak solution* of (4.1) with initial condition $w_0 = 0$. By theorem 4.1.4 $w \equiv 0$, which implies $u_1(t) = u_2(t)$ a. e. for all $t \in [0,T]$. □

A-priori Estimates of higher Order

In the case of smoother coefficients of our partial differential equation and if the weak solution u has higher regularity, we can prove the following result, similar to theorem 4.1.4:

Theorem 4.1.6. *Let u be a weak solution of (4.1) with initial condition $u_0 \in H^s(\mathbb{R}^d)$ and $u \in L^2(0,T;H^{s+1}(\mathbb{R}^d))$, for $s > d/2+1$. Furthermore, let $f \in L^\infty(0,T;W^{s+1,\infty}(\mathbb{R}^d,\mathbb{R}^d)) \cap L^\infty(0,T;H^s(\mathbb{R}^d,\mathbb{R}^d))$ and $\sigma \in L^\infty(0,T;W^{s,\infty}(\mathbb{R}^d,\mathbb{R}^{d\times d}))$. Then there exist constants $K, C, c > 0$ such that*

i) $\frac{d}{dt}\|u\|^2_{H^s} + \epsilon\|\nabla u\|^2_{H^s} \le K\|u\|^2_{H^s}$ *almost everywhere.*

ii) $\|u(t)\|^2_{H^s} \le \|u_0\|^2_{H^s}\exp(Kt)$, *for* $0 \le t \le T$.

iii) $\|u(t)\|^2_{H^s} + \epsilon\|\nabla u\|^2_{L^2(0,t;H^s)} \le \|u_0\|^2_{H^s}(\exp(Kt)-1)$, *for* $0 \le t \le T$.

iv) $K = 2C\left(1 + (\|f\|_{L^\infty} + c\|D^s f\|_{L^2})^2\right)$.

Proof. The idea of this proof is based partially on the proof of [5, Lemma 4].

The weak solution u fulfils the equation

$$\int_0^T \langle \partial_t u, \phi\rangle_{H^{-1}} dt + \int_0^T\int_{\mathbb{R}^d} \nabla u^T \sigma \nabla\phi dx dt = \int_0^T\int_{\mathbb{R}^d} \operatorname{div}(uf)\,\phi dx dt$$

for all $\phi \in \mathcal{D}([0,T]\times\mathbb{R}^d)$. Let α be a multi-index with $|\alpha| \le s$ and $\phi = D^\alpha\psi$ with $\psi \in \mathcal{D}([0,T]\times\mathbb{R}^d)$. Then we get:

$$\int_0^T \langle \partial_t u, D^\alpha\psi\rangle_{H^{-1}} dt = -\int_0^T \langle u, \partial_t D^\alpha\psi\rangle_{L^2} dt + \langle u(T), D^\alpha\psi(T)\rangle_{L^2} - \langle u(0), D^\alpha\psi(0)\rangle_{L^2}$$

$$= (-1)^{|\alpha|}\left(-\int_0^T \langle D^\alpha u, \partial_t\psi\rangle_{L^2} dt + \langle D^\alpha u(T), \psi(T)\rangle_{L^2} - \langle D^\alpha u(0), \psi(0)\rangle_{L^2}\right) \tag{4.6}$$

$$= -\int_0^T\int_{\mathbb{R}^d} \nabla u^T\sigma\nabla D^\alpha\psi dx dt + \int_0^T\int_{\mathbb{R}^d}\operatorname{div}(uf)\,D^\alpha\psi dx dt$$

$$= (-1)^{|\alpha|}\left(-\int_0^T\int_{\mathbb{R}^d} D^\alpha\left(\nabla u^T\sigma\right)\nabla\psi dx dt + \int_0^T\int_{\mathbb{R}^d} D^\alpha \operatorname{div}(uf)\,\psi dx dt\right).$$

The functions $D^\alpha(\nabla u\cdot\sigma)$ and $D^\alpha\operatorname{div}(u\cdot f)$ are both elements of $L^2(0,T;L^2)$. This can be seen, for example, with the product rule. Expression (4.6), as function of ψ, can therefore be interpreted as element of $L^2(0,T;H^{-1})$. Thus, there exists $v \in L^2(0,T;H^{-1})$ such that for all $\psi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$,

$$-\int_0^T \langle D^\alpha u, \partial_t\psi\rangle_{L^2} dt = -\int_0^T \langle v,\psi\rangle_{H^{-1}} dt.$$

This implies[1] $\partial_t D^\alpha u = v$. Thus, $D^\alpha u \in W^{1,2}(0,T;H^1,L^2)$ and we can deduce, similar to equation (4.3),

$$\langle \partial_t D^\alpha u, \psi\rangle_{H^{-1}} + \int_{\mathbb{R}^d} D^\alpha \left(\nabla u^T \sigma\right) \nabla\psi dx = \int_{\mathbb{R}^d} D^\alpha \mathrm{div}\,(uf)\,\psi dx \tag{4.7}$$

for arbitrary $\psi \in H^1(\mathbb{R}^d)$. Now by writing

$$D^\alpha(\sigma\nabla u) = \sigma D^\alpha \nabla u + \sum_{|\beta|\le|\alpha|-1, 1\le|\gamma|\le|\alpha|} c_{\beta,\gamma} D^\gamma \sigma D^\beta \nabla u \tag{4.8}$$

which holds almost surely for suitable $c_{\beta,\gamma} \in \{0,1\}$, and combining (4.7) with (4.8), we arrive at

$$\langle \partial_t D^\alpha u, \psi\rangle_{H^{-1}} + \int_{\mathbb{R}^d} D^\alpha \nabla u^T \sigma \nabla\psi dx = -\int_{\mathbb{R}^d} \left(\sum_{|\beta|\le|\alpha|-1, 1\le|\gamma|\le|\alpha|} c_{\beta,\gamma} D^\gamma \sigma D^\beta \nabla u \right) \cdot \nabla\psi dx \tag{4.9}$$

$$+ \int_{\mathbb{R}^d} D^\alpha \mathrm{div}\,(uf)\,\psi dx.$$

We choose $\psi = D^\alpha u$ and use Youngs inequality and the coercivity of σ to get the estimate

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha u\|_{L^2}^2 + \epsilon\|D^\alpha \nabla u\|_{L^2}^2 \le \frac{1}{2\delta} \left\| \sum_{|\beta|\le|\alpha|-1, 1\le|\gamma|\le|\alpha|} c_{\beta,\gamma} D^\gamma \sigma D^\beta \nabla u \right\|_{L^2}^2 + \frac{\delta}{2}\|D^\alpha \nabla u\|_{L^2}^2$$

$$+\frac{1}{2\delta}\|D^\alpha(uf)\|_{L^2}^2 + \frac{\delta}{2}\|D^\alpha \nabla u\|_{L^2}^2.$$

The first term on the right hand side can be further estimated:

$$\left\| \sum_{|\beta|\le|\alpha|-1, 1\le|\gamma|\le|\alpha|} c_{\beta,\gamma} D^\gamma \sigma D^\beta \nabla u \right\|_{L^2} \le \sum_{|\beta|\le|\alpha|-1, 1\le|\gamma|\le|\alpha|} \left\| D^\gamma \sigma D^\beta \nabla u \right\|_{L^2}$$

$$= \sum_{|\beta|\le|\alpha|-1, 1\le|\gamma|\le|\alpha|} \left(\int_{\mathbb{R}^d} \left| D^\gamma \sigma D^\beta \nabla u \right|^2 dx \right)^{1/2}$$

[1]Strictly speaking, we can identify the distributional time derivative of $D^\alpha u$ with v.

$$\leq \sum_{|\beta|\leq|\alpha|-1, 1\leq|\gamma|\leq|\alpha|} \left(\int_{\mathbb{R}^d} \|D^\gamma \sigma\|^2 \left| D^\beta \nabla u \right|^2 dx \right)^{1/2}$$

$$\leq \sum_{|\beta|\leq|\alpha|-1, 1\leq|\gamma|\leq|\alpha|} \|D^\gamma \sigma\|_{L^\infty} \left(\int_{\mathbb{R}^d} \left| D^\beta \nabla u \right|^2 dx \right)^{1/2}$$

$$= \sum_{|\beta|\leq|\alpha|-1, 1\leq|\gamma|\leq|\alpha|} \|D^\gamma \sigma\|_{L^\infty} \left\| D^\beta \nabla u \right\|_{L^2} \leq C(s,d) \|\sigma\|_{W^{s,\infty}} \|u\|_{H^s}. \tag{4.10}$$

Altogether we have

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha u\|_{L^2}^2 + \epsilon \|D^\alpha \nabla u\|_{L^2}^2 \leq \underbrace{\frac{1}{2\delta} C(s,d) \|\sigma\|_{W^{s,\infty}}^2}_{=:C(s,d,\sigma(t))\in L^\infty([0,T])} \|u\|_{H^s}^2 + \frac{\delta}{2}\|D^\alpha \nabla u\|_{L^2}^2$$

$$+ \frac{1}{2\delta}\|D^\alpha (uf)\|_{L^2}^2 + \frac{\delta}{2}\|D^\alpha \nabla u\|_{L^2}^2.$$

We now choose $\delta = \epsilon/4$. Then the second and fourth term on the right-hand side can be absorbed by the corresponding term on the left hand side of the inequality.

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha u\|_{L^2}^2 + \frac{\epsilon}{2}\|D^\alpha \nabla u\|_{L^2}^2 \leq C(s,d,\sigma(t))\|u\|_{H^s}^2 + \frac{2}{\epsilon}\|D^\alpha (uf)\|_{L^2}^2$$

We use the Moser-type-calculus inequality (see A.2.12) on the last term on the right-hand side. Notice that because of $s > d/2+1$, we have $H^s \hookrightarrow L^\infty$, due to the Sobolev embedding theorem. All in all we get

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha u\|_{L^2}^2 + \frac{\epsilon}{2}\|D^\alpha \nabla u\|_{L^2}^2 \leq C(s,d,\sigma(t))\|u\|_{H^s}^2 + \frac{2}{\epsilon}\left(\|u\|_{L^\infty}\|D^s f\|_{L^2} + \|f\|_{L^\infty}\|D^s u\|_{L^2}\right)^2$$

$$\leq \left(C(s,d,\sigma(t)) + \frac{2}{\epsilon}\left(\|f\|_{L^\infty} + c\|D^s f\|_{L^2}\right)^2 \right) \|u\|_{H^s}^2$$

$$\leq C\left(1 + \left(\|f\|_{L^\infty} + c\|D^s f\|_{L^2}\right)^2\right) \|u\|_{H^s}^2$$

almost everywhere, where $c > 0$ is the respective constant of the Sobolev embedding and $C > 0$ depends on $s, d, \frac{2}{\epsilon}$ and $\|\sigma\|_{L^\infty(0,T;W^{s,\infty})}$. We sum over all multi-indices α, $|\alpha| \leq s$:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 + \frac{\epsilon}{2}\|\nabla u\|_{H^s}^2 \leq \underbrace{C\left(1 + \left(\|f\|_{L^\infty} + c\|D^s f\|_{L^2}\right)^2\right)}_{=:\frac{1}{2}K} \|u\|_{H^s}^2. \tag{4.11}$$

This especially implies

$$\frac{d}{dt}\|u\|_{H^s}^2 \leq K\|u\|_{H^s}^2 \tag{4.12}$$

almost everywhere. We can now use Gronwalls lemma to obtain

$$\|u(t)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 \exp(Kt)$$

for $0 \leq t \leq T$. We furthermore use (4.12) to control the term on the right-hand side of inequality (4.11) and integrate (4.11) over $[0,t]$:

$$\|u(t)\|_{H^s}^2 + \epsilon\|\nabla u\|_{L^2(0,t;H^s)}^2 \leq \|u_0\|_{H^s}^2(\exp(Kt) - 1)$$

which completes the proof. □

4.1.2. The abstract Cauchy problem

Let $A(t) : D(A(t)) = H^2 \to L^2$ be the family of unbounded linear operators satisfying

$$A(t)u = -\text{div}\,(\sigma(x,t)\nabla u)$$

for all $u \in H^2$, and $g(u,t) : H^1 \times [0,T] \to L^2$ defined by

$$g(u,t) := \text{div}\,(f(x,t)u)\,.$$

Then the linear parabolic problem

$$\partial_t u = \text{div}\,(\sigma(x,t)\nabla u) + \text{div}\,(f(x,t)u) \quad \text{in } \mathbb{R}^d \times (0,T],$$

$$u|_{t=0} = u_0(x), \text{ in } \mathbb{R}^d,$$

can be written as:

$$du/dt + A(t)u = g(u,t), \quad t \in [0,T], \quad u(0) = u_0 \in L^2 \tag{4.13}$$

where $u = u(t)$.

This strongly resembles a so called abstract (semilinear) Cauchy problem. These are initial value problems of the form

$$du/dt + A(t)u = g(t), \quad t \in (0,T], \quad u(0) = u_0 \in X \tag{4.14}$$

where X is some arbitrary Banach space, $A(t) : D(A(t)) \to X$ is a family of linear operators with $t \in [0,T]$ and $g(t)$ is a function from $[0,T]$ to X. The standard procedure to solve such a problem, would be to find the corresponding *evolution operator* and to prove the existence of the so called *mild solution* (see e.g. [3]). Which is, under some rather strict regularity assumptions on u_0 and g, even a strong solution of (4.14).
Our approach will be quite similar. We will show the existence of an evolution operator and of a mild solution on the time interval $[\delta,T]$ for every $0 < \delta < T$. For $\delta \to 0$ we will then obtain the desired weak solution on $[0,T]$.

From now on we will use the following definitions:

Definition 4.1.7 (Evolution operator[2])**.** Let X be a Banach space and $A(t) : D(A(t)) \subseteq X \to X,\ t \in [0,T]$ be a family of not necessarily bounded, linear operators. We call $U(t,s) \in \mathbb{B}[X]$, defined for all $0 \leq s \leq t \leq T$, an evolution operator regarding $A(t)$, if the following conditions are held:

i) $U(t,s)u$ is a continuous function from $\{(t,s) : 0 \leq s \leq t \leq T\}$ to X for every $u \in X$.

ii) $U(t,r)U(r,s) = U(t,s)$ for all $0 \leq s \leq r \leq t \leq T$.

iii) $U(t,t) = Id_X$ for $0 \leq t \leq T$, where $Id_X \in \mathbb{B}[X]$ is the identity on X.

iv) There exists a constant $C > 0$ such that for $s < t$

$$A(t)U(t,s) \in \mathbb{B}[X] \quad \textit{and} \quad \|A(t)U(t,s)\| \leq C(t-s)^{-1}.$$

In particular $U(t,s)(X) \subseteq D(A(t))$ for all $s < t \leq T$.

v) $U(t,s)$ is differentiable in t for $s < t$ and it holds that

$$\partial_t U(t,s) + A(t)U(t,s) = 0.$$

Remark 4.1.8. Condition $i)$ of definition 4.1.7 implies the existence of a constant C_u such that

$$\sup_{0 \leq s \leq t \leq T} \|U(t,s)u\|_X \leq C_u$$

for every fixed $u \in X$. In other words the operator- family $U(t,s),\ 0 \leq s \leq t \leq T$, is bounded point-wise. With the uniform boundedness principle we therefore conclude the existence of a constant $C > 0$ such that

$$\sup_{0 \leq s \leq t \leq T} \|U(t,s)\|_{\mathbb{B}(X)} \leq C.$$

Definition 4.1.9 (Mild solution)**.** Let X be a Banach space, $A(t) : D(A(t)) \subseteq X \to X,\ t \in [0,T]$, a family of linear operators and $U(t,s)$ the corresponding evolution operator. We call u a mild solution of the problem (4.14), if it can be written as

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)g(s)ds, \quad t \in [0,T]$$

where the integral is to be understood as a Bochner- integral in X.

Existence of the evolution operator

Most results of functional analysis, regarding the spectrum of operators or spectral theory in general, demand a complex Hilbert space. In this section we will therefore use the space $(\mathbb{C}, \|\cdot\|_2)$ as target space for our functions $u \in L^2(\mathbb{R}^d)$ and the space $(\mathbb{C}^d, \langle\cdot,\cdot\rangle_{\mathbb{C}^d})$ whenever we have to deal with vectors. For the linear operators we are interested in most properties like positivity, symmetry, etc. will survive the transition into this extension.

[2]In the literature evolution operators often have even more properties ascribed to them, e.g. differentiability in s. We tried to restrict ourselves in this definition to the most necessary ones for us. See e.g. [1] for more details.

Example 4.1.10 (Uniform coercivity).

Let $u \in \mathbb{C}^d$ and A be a real, symmetric and coercive matrix, i.e. $y^T Ay \geq \epsilon |y|^2$ for some $\epsilon > 0$ and arbitrary $y \in \mathbb{R}^d$. We want to show $u^T A\overline{u} \geq \epsilon |u|^2$. We therefore represent u by $u = v + iw$ with $v, w \in \mathbb{R}^d$. Then the desired result follows directly by calculating

$$u^T A\overline{u} = (v + iw)^T A\overline{(v + iw)} = v^T Av + v^T A\bar{i}w + iw^T Av + iw^T A\bar{i}w$$

$$= v^T Av - \underbrace{iv^T Aw}_{=iw^T Av} + iw^T Av + w^T Aw \geq \epsilon |v|^2 + \epsilon |w|^2 = \epsilon |u|^2.$$

Hence A is coercive regarding $(\mathbb{C}^d, \langle \cdot, \cdot \rangle_{\mathbb{C}^d})$.

To obtain an evolution operator for our specific problem, we would like to apply the following result by Tosio Kato [1]:

Theorem 4.1.11 (Existence of an evolution operator)**.** *Let X be a Banach space, $A(t) : D(A(t)) \subseteq X \to X,\ t \in [0, T]$ be a family of linear operators. Furthermore, there exist positive constants $\theta, C, C_0, C_1, C_2, \beta$ such that for all $t \in [0, T]$,*

i) *$A(t)$ is a densely defined and closed operator with its spectrum $\sigma(A(t))$ contained in the sector $S_\theta := \{w = |w|e^{i(arg\ w)} \in \mathbb{C} : \ |arg\ w| < \theta < \frac{\pi}{2}\}$.*

ii) *For every $\lambda \notin S_\theta$ and $t \in [0, T]$, the inequality*

$$\left\| [A(t) - \lambda Id_X]^{-1} \right\|_{\mathbb{B}[X]} \leq \frac{C}{|\lambda|}$$

is satisfied.

iii) *$0 \notin \sigma(A(t))$ and*

$$\left\| A(t)^{-1} \right\|_{\mathbb{B}[X]} \leq C_0$$

for all $t \in [0, T]$.

iv) *The domain $D(A(t)^\alpha)$ is independent from t for some $\alpha = \frac{1}{m}$ with $m \in \mathbb{N}$, and it holds that*

$$\left\| A(t)^\alpha A(s)^{-\alpha} \right\|_{\mathbb{B}[X]} \leq C_1$$

as well as

$$\left\| A(t)^\alpha A(s)^{-\alpha} - Id_X \right\|_{\mathbb{B}[X]} \leq C_2 |t - s|^\beta, \quad s, t \in [0, T].$$

Additionally, the constants α, β fulfil the relation $1 - \alpha < \beta \leq 1$.

Then there exists an evolution operator regarding $\{A(t)\}_{t \in [0,T]}$.

The prerequisites *i)* and *ii)* of theorem 4.1.11 will likely look very familiar for someone who already has extensively dealt with semigroup theory, as they are often demanded from an operator A to obtain a so called analytic semigroup with infinitesimal generator $-A$. In this case, they are also adequate to define the powers of these operators e.g. A^α with $0 \leq \alpha < 1$, as in condition iv). But we wont need a deeper understanding of them here. For more information, as well as for the definition and some properties of closed linear operators, see [3] or A.5.

For arbitrary Banach spaces X, the requirements for theorem 4.1.11 can be quite difficult to prove for a larger class of operators, especially in respect to point iv), as the fractional powers of an operator are defined in a very abstract way. But much is known of the spectrum of operators in the case where X is a Hilbert space, which allows us to prove existence of an evolution operator in a more practicable setting.

Theorem 4.1.12. *Let H be a complex Hilbert space, $\{A(t)\}_{t\in[0,T]}$ be a family of closed, densely defined, self-adjoint, linear operators with $A : D(A(t)) \subseteq H \to H$ and $\{a(t)\}_{t\in[0,T]}$ be a family of sesquilinear-forms with*

$$a(t)[u, v] = \langle A(t)u, v\rangle_H$$

for all $v \in D(a(t))$ and $u \in D(A(t)) \subset D(a(t))$. If it holds that:

a) For a fixed $\epsilon > 0$ and $0 < \theta < \frac{\pi}{2}$, the spectrum of $A(t)$ is contained in the 'shifted' sector $S_{\theta,\epsilon} := \{w = |w|e^{i(arg\ w)} \in \mathbb{C} : \ |arg\ (w - \epsilon)| < \theta\}$, for all $t \in [0, T]$.

b) $a(t)$ is a regular sesquilinear-form and its domain is independent of t, i.e. $D(a(t)) = D$ for all $t \in [0, T]$.

c) There exist constants $C > 0$ and $\frac{2}{3} < \beta \leq 1$ such that

$$|a(t)[u, u] - a(s)[u, u]| \leq C|t - s|^{\beta}|Re\ a(t)[u, u]|$$

for all $u \in D$ and $s, t \in [0, T]$.

Then there exists an evolution operator regarding $\{A(t)\}_{t\in[0,T]}$ on H.

Proof. We will show that the prerequisites $i) - iv)$ of theorem 4.1.11 are fulfilled:

i): As $S_{\theta,\epsilon} \subset S_\theta$, this condition is already satisfied due to assumption a).

ii): It is known from functional analysis that a densely defined and closed self-adjoint operator $A : D(A) \subseteq H \to H$ on a Hilbert space H, gives rise to a projection-valued measure (see theorem A.5.15). Let A be an arbitrary operator with $A \in \{A(t)\}_{t\in[0,T]}$ and E be its assigned projection-value measure.
We choose some fixed θ_1 with $\theta < \theta_1 < \frac{\pi}{2}$ and calculate for $\lambda \in \mathbb{C}$ with $\lambda \notin S_{\theta_1}$:

$$\left\|(A - \lambda Id_H)^{-1}u\right\|_H^2 = \int\limits_{\sigma(A)} \left|\frac{1}{x-\lambda}\right|^2 dE_{u,u}(dx) \leq \left(\frac{1}{\text{dist}(\lambda, \sigma(A))}\right)^2 \int\limits_{\sigma(A)} 1 dE_{u,u}(dx)$$

$$= \left(\frac{1}{\text{dist}(\lambda, \sigma(A))}\right)^2 \|u\|_H^2$$

for $u \in H$, where $\text{dist}(\lambda, \sigma(A))$ is the shortest distance, regarding the euclidean norm in $\mathbb{C}$, between λ and the set $\sigma(A)$.

As $\lambda \in \mathbb{C}$, it has the unique representation $\lambda = |\lambda| e^{i\phi}$, where $\phi \in (-\pi, \pi] \setminus (-\theta_1, \theta_1)$. One can now show via a geometric argument that

$$\operatorname{dist}(\lambda, S_\theta) \geq |\lambda| \sin(|\phi| - \theta).$$

This leads to the following chain of inequalities if we choose $\theta_1 \leq 2\theta$:

$$\operatorname{dist}(\lambda, \sigma(A)) \geq \operatorname{dist}(\lambda, S_{\theta,\epsilon}) \geq \operatorname{dist}(\lambda, S_\theta) \geq \inf_{|\phi| \geq \theta_1} |\lambda| \sin(|\phi| - \theta) \geq |\lambda| \sin(\theta_1 - \theta).$$

All in all we get

$$\left(\frac{1}{\operatorname{dist}(\lambda, \sigma(A))} \right)^2 \|u\|_H^2 \leq \left(\frac{1}{|\lambda| \sin(\theta_1 - \theta)} \right)^2 \|u\|_H^2$$

and therefore

$$\left\| (A - \lambda Id_H)^{-1} \right\|_{\mathbb{B}(H)} \leq \frac{1}{|\lambda| \sin(\theta_1 - \theta)}.$$

Defining $C := \frac{1}{\sin(\theta_1 - \theta)}$, the operator A fulfils condition $ii)$ of theorem 4.1.11 regarding the sector S_{θ_1}. As A was an arbitrary operator from the family $\{A(t)\}_{t \in [0,T]}$, this holds true for all of them.

iii) From the definition of $S_{\theta,\epsilon}$ we can conclude that $B_\epsilon(0) \cap S_{\theta,\epsilon} = \varnothing$, where $B_\epsilon(0)$ is the ball containing all the elements of $\mathbb{C}$, whose absolute value is smaller then ϵ. Let for this reason $\lambda \in \mathbb{C}$ be an element of $B_\epsilon(0)$. Then $\operatorname{Re}(\lambda) < \epsilon$ and thus $\operatorname{Re}(\lambda - \epsilon) < 0$. This implies $|arg(\lambda - \epsilon)| > \frac{\pi}{2}$ as

$$\operatorname{Re}(\lambda - \epsilon) = |\lambda - \epsilon| \cos(arg(\lambda - \epsilon))$$

which is negativ if and only if $arg(\lambda - \epsilon) \in (\frac{\pi}{2}, \pi]$ or $arg(\lambda - \epsilon) \in (-\pi, -\frac{\pi}{2})$. Either way $|arg(\lambda - \epsilon)| > \frac{\pi}{2}$ and hence $B_\epsilon(0) \cap S_{\theta,\epsilon} = \varnothing$. In particular $B_\epsilon(0)$ is part of the resolvent set of $A(t)$, as $\sigma(A(t)) \subseteq S_{\theta,\epsilon}$.
For an arbitrary $A \in \{A(t)\}_{t \in [0,T]}$ and $u \in H$ it is therefore justified to calculate

$$\left\| A^{-1} u \right\|_H = \int_{\sigma(A)} \left| \frac{1}{x} \right|^2 dE_{u,u}(dx) \leq \left(\frac{1}{\epsilon} \right)^2 \int_{\sigma(A)} 1 dE_{u,u}(dx) = \left(\frac{1}{\epsilon} \right)^2 \|u\|_H^2$$

which implies $\left\| A^{-1} \right\|_{\mathbb{B}(H)} \leq \frac{1}{\epsilon}$.

iv) This part of the proof demands some knowledge of definitions, terminology and results we won't need later on. The interested reader can find the mentioned definitions and relations in [1],[3] and [4] or A.5.

Let $u \in H$, $A \in \{A(t)\}_{t \in [0,T]}$, $n \in \mathbb{N}$ and $\lambda > 0$. Then we have

$$\left\| (\lambda I + A)^{-n} u \right\|_H^2 = \int_{\sigma(A)} |\lambda + x|^{-2n} dE_{u,u}(dx) \leq \frac{1}{\lambda^{2n}} \int_{\sigma(A)} 1 dE_{u,u}(dx) = \frac{1}{\lambda^{2n}} \|u\|_H^2$$

which implies $\|(\lambda I-(-A))^{-n}\|_{\mathbb{B}(H)} \leq \frac{1}{\lambda^n}$. Hence, by the theorem of Hille-Yoshida (see theorem A.5.16), $-A$ is the infinitesimal generator of a contraction semigroup[3] for any $A \in \{A(t)\}_{t\in[0,T]}$. This is equivalent to $-A$ being *maximal dissipative*, respectively A being *maximal accretive* (see proposition A.5.17). Due to theorem A.5.11, $A(t)$ is the unique closed maximal accretive operator associated with the regular sesquilinearform $a(t)$. By theorem A.5.12, this implies the existence of constants C_1, C_2 such that

$$\left\|A(t)^{\alpha}A(s)^{-\alpha}\right\|_{\mathbb{B}[H]} \leq C_1$$

and

$$\left\|A(t)^{\alpha}A(s)^{-\alpha} - Id_H\right\|_{\mathbb{B}[H]} \leq C_2|t-s|^{\beta}, \quad s,t \in [0,T]$$

for all $0 \leq \alpha < \frac{1}{2}$. Hence for $\alpha = \frac{1}{3}$ condition *iv)* of theorem 4.1.11 is satisfied.

□

Before we again turn towards our convection- diffusion equation (4.1) we are going to show some more general results for symmetric operators[4].

Lemma 4.1.13 (Spectrum)**.** *Let H be a Hilbert space and $A: D(A) \subseteq H \to H$ a linear, closed and densely defined operator. If A is symmetric and positive then $(-\infty, 0) \subseteq \rho(A)$, where $\rho(A)$ denotes the resolvent set of A.*

Proof. We choose a fixed but arbitrary $\lambda < 0$ and calculate

$$\|(A-\lambda Id_H)\,u\|_H = \|Au\|_H^2 - 2\lambda \underbrace{\langle Au, u\rangle_H}_{\geq 0} + |\lambda|^2\|u\|_H^2 \geq |\lambda|^2\|u\|_H^2,$$

for $u \in D(A)$, where we used the positivity of A to deduce the last inequality. From this we can conclude that $R_\lambda^{-1} := A - \lambda Id_H$ is one to one and

$$\|R_\lambda u\| \leq \frac{1}{|\lambda|}\|u\|_H \tag{4.15}$$

for $u \in \operatorname{ran} R_\lambda^{-1}$.
We want to show that ran R_λ^{-1} equals the whole Hilbert space H. Then R_λ would be a well defined, bounded linear operator from H to $D(A)$. This would furthermore imply $\lambda \in \rho(A)$.
We notice that as long as A is symmetric R_λ^{-1} is as well. One can assure oneself of this fact by writing

$$\langle R_\lambda^{-1}x, y\rangle = \langle Ax - \lambda Id_H x, y\rangle = \langle Ax, y\rangle - \langle \lambda Id_H x, y\rangle = \langle x, Ay\rangle - \lambda\langle x, Id_H y\rangle = \langle x, R_\lambda^{-1}y\rangle$$

where $x, y \in D(A)$. Let now v be an arbitrary element of $D(A) \cap [\operatorname{ran} R_\lambda^{-1}]^{\perp}$. As $D(A)$ is dense in H and for every $u \in D(A)$ the equation

$$0 = \langle R_\lambda^{-1}u, v\rangle_H = \langle u, R_\lambda^{-1}v\rangle_H$$

[3]Contraction semigroups are semigroups which fulfil the condition $\|S(t)\|_{\mathbb{B}(X)} \leq 1$.
[4]Notice that $A(t) = \operatorname{div}(\sigma(x,t)\nabla u)$ is symmetric in respect to the L^2- scalar product if $\sigma(x,t)$ is a symmetric matrix.

is fulfilled, it follows that $R_\lambda(v)^{-1} = 0$. But R_λ^{-1} is one to one, which implies $v = 0$ and furthermore $D(A) \cap [\text{ran } R_\lambda^{-1}]^\perp = \{0\}$. Hence $D(A) \subseteq \overline{\text{ran } R_\lambda^{-1}}$. We use the assumption that $D(A)$ is dense in H to conclude $\overline{\text{ran } R_\lambda^{-1}} = H$. One can therefore approximate any arbitrary $y \in H$ with elements $\{y_n\}_{n\in\mathbb{N}} \in \text{ran } R_\lambda^{-1}$, such that $\lim_{n\to\infty} y_n = y$ in H. Due to the convergence, $\{y_n\}_{n\in\mathbb{N}}$ is also a Cauchy-sequence in H. Consequently, by using (4.15), we get for any $n, m \in \mathbb{N}$

$$\|R_\lambda(y_n - y_m)\|_H \leq \frac{1}{|\lambda|}\|y_n - y_m\|_H.$$

So $R_\lambda y_n$ is a Cauchy-sequence as well and has to converge to some $x \in H$. If now R_λ were closed, the proof would be completed, as y was arbitrary and then $y \in D(R_\lambda)$, $R_\lambda y = x$ would hold. From functional analyses we know a linear densely defined operator, which is injective and has dense range, is closed exactly when its inverse is as well (see A.5). But R_λ^{-1} is closed as it is the sum of a closed operator and an element of $\mathbb{B}(H)$, which completes the proof. □

Lemma 4.1.14 (Self-adjoint operator)**.** *Let H be a Hilbert space and $A : D(A) \subseteq H \to H$ a linear, closed and densely defined operator. If A is symmetric and $0 \in \rho(A)$ then A is self-adjoint.*

Proof. $v \in H$ is an element of $D(A^*)$ if and only if there is a $g \in H$ such that

$$\langle Au, v\rangle_H = \langle u, g\rangle_H, \quad \forall u \in D(A).$$

In this case we write[5] $A^*v = g$. Due to the symmetry of A

$$\langle Au, v\rangle_H = \langle u, Av\rangle_H, \quad \forall u, v \in D(A)$$

we deduce $Av = A^*v$ for $v \in D(A)$ and $D(A) \subseteq D(A^*)$ in particular. We now want to show the converse statement $D(A^*) \subseteq D(A)$, to get $A^* = A$.
Let therefore $v \in D(A^*)$ be chosen arbitrarily. Then, for all $u \in D(A)$, we can write

$$\langle Au, v\rangle_H = \langle u, A^*v\rangle_H = \langle u, AA^{-1}A^*v\rangle_H = \langle Au, A^{-1}A^*v\rangle_H \tag{4.16}$$

because $0 \in \rho(A)$ implies $A^{-1} \in \mathbb{B}(H)$. From ran $A = H$ and (4.16) we conclude $v = A^{-1}A^*v$, which completes the proof as we have shown $v \in D(A)$. □

We will now show that the operator-family $A(t) := -\text{div}\,(\sigma(x,t)\nabla u)$ gives rise to an evolution operator corresponding to definition 4.1.12. We will therefore check if the conditions for theorem 4.1.11 are satisfied and if not, adjust our problem accordingly. The first step will be to demonstrate that $A(t)$ is densely defined and closed. We defined $A(t)$ on the Hilbert space $H^2(\mathbb{R}^d)$, which is already dense in $L^2(\mathbb{R}^d)$, because both these spaces are the closure of the set $C_c^\infty(\mathbb{R}^d)$ in their respective norms. To prove that $A(t)$ is closed, by the definition of the closure of an linear operator, we will have to show that the graph of $A(t)$ is closed in respect to the 'graph-norm' defined by $u \to \|u\|_{L^2} + \|A(t)u\|_{L^2}$ for $u \in H^2$. For this endeavour it will be very helpful to know the equivalence of the norms $\|u\|_{H^1} + \|A(t)u\|_{L^2}$ and $\|u\|_{H^2}$.

Lemma 4.1.15. *Let the assumptions at the beginning of this chapter hold. Then there exists a constant $C > 0$, such that*

$$\|u\|_{H^2} \leq C\left(\|u\|_{H^1} + \|A(t)u\|_{L^2}\right), \quad \forall u \in H^2(\mathbb{R}^d),\ \forall t \in [0,T]. \tag{4.17}$$

[5]This definition of the adjoint of an operator A is well defined as long as $D(A)$ is dense in H.

Proof. We will show (4.17) for smooth test- functions and then obtain the more general case via a density argument.
Let therefore be $u, \phi \in C_c^\infty(\mathbb{R}^d)$ and $A = A(t)$, $\sigma(x,t) = \sigma(x)$ for some fixed but arbitrary t. Then u satisfies the equation

$$-\int_{\mathbb{R}^d} \nabla u \sigma \nabla \overline{\phi} dx = \int_{\mathbb{R}^d} \operatorname{div}(\sigma \nabla u)\, \overline{\phi} dx = \langle Au, \phi \rangle_{L^2}, \tag{4.18}$$

for any choice of ϕ. In particular for $\phi = \partial_{x_i}\partial_{x_i} u$, where $i \in \{1, \dots, d\}$. Let i now be fixed. Via integration by parts, or in other words using the definition of a weak derivative, we get

$$-\int_{\mathbb{R}^d} \nabla u \sigma \overline{\nabla \partial_{x_i}\partial_{x_i} u} dx = \int_{\mathbb{R}^d} \nabla \partial_{x_i} u \sigma \overline{\nabla \partial_{x_i} u} dx + \int_{\mathbb{R}^d} \nabla u \left(\partial_{x_i}\sigma\right) \overline{\nabla \partial_{x_i} u} dx. \tag{4.19}$$

Combining (4.18) and (4.19) and shifting some terms around results in

$$\int_{\mathbb{R}^d} \nabla \partial_{x_i} u \sigma \overline{\nabla \partial_{x_i} u} dx = \int_{\mathbb{R}^d} Au \cdot \overline{\partial_{x_i}\partial_{x_i} u} dx - \int_{\mathbb{R}^d} \nabla u \left(\partial_{x_i}\sigma\right) \overline{\nabla \partial_{x_i} u} dx.$$

From there we can use Youngs inequality for products and the uniform coercivity[6] of σ to obtain

$$\epsilon \|\nabla \partial_{x_i} u\|_{L^2}^2 \leq \frac{1}{2\delta}\|Au\|_{L^2}^2 + \frac{\delta}{2}\|\partial_{x_i}\partial_{x_i} u\|_{L^2}^2 + \frac{1}{2\delta}\|\partial_{x_i}\sigma\|_{L^\infty}\|\nabla u\|_{L^2}^2 + \frac{\delta}{2}\|\nabla \partial_{x_i} u\|_{L^2}, \tag{4.20}$$

for any $\delta > 0$. If we choose $\delta \leq \epsilon/4$ and subtract the terms with second order derivatives on the right-hand side of (4.20), we end up with the following:

$$\frac{\epsilon}{2}\|\nabla \partial_{x_i} u\|_{L^2}^2 \leq \frac{2}{\epsilon}\|Au\|_{L^2}^2 + \frac{2}{\epsilon}\|\partial_{x_i}\sigma\|_{L^\infty}\|\nabla u\|_{L^2}^2.$$

The sum over all the (squared) L^2-norms of the weak derivatives of second order of u can therefore be bounded by

$$\sum_{\alpha=2}\|D^\alpha u\|_{L^2}^2 \leq \sum_{i\in\{1,\dots,d\}} \|\partial_{x_i}\nabla u\|_{L^2}^2 \leq \underbrace{\frac{4d}{\epsilon^2}\left(1 + \|\partial_{x_i}\sigma\|_{L^\infty(0,T;L^\infty)}\right)}_{=:C(\epsilon,\sigma)} \left(\|Au\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right). \tag{4.21}$$

Furthermore for the whole H^2- norm of u we obtain via (4.21) the upper bounds

$$\|u\|_{H^2}^2 = \|u\|_{H^1}^2 + \sum_{\alpha=2}\|D^\alpha u\|_{L^2}^2 \leq (1 + C(\epsilon,\sigma))\left(\|Au\|_{L^2}^2 + \|u\|_{H^1}^2\right). \tag{4.22}$$

Carrying on we wind up with

$$\|u\|_{H^2} \leq \sqrt{(1 + C(\epsilon,\sigma))\left(\|Au\|_{L^2}^2 + \|u\|_{H^1}^2\right)} \leq C_{\max}\sqrt{1 + C(\epsilon,\sigma)}\left(\|Au\|_{L^2} + \|u\|_{H^1}\right)$$

where we took the square root on both sides of (4.22) and used the equivalence of norms on $\mathbb{R}^2$ (notice the new constant $C_{\max}$).
Hence we have proven (4.17) for $u \in C_c^\infty(\mathbb{R}^d)$. But these functions are dense in $H^2(\mathbb{R}^d)$ and the right- and left-hand side of (4.17) consist of H^2- continuous functionals. Therefore the desired statement follows by a simple approximation argument. □

[6] See exercise 4.1.10.

Lemma 4.1.16. *Let the assumptions at the beginning of this chapter hold. Then $A(t)$ is closed for all $t \in [0,T]$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \in H^2(\mathbb{R}^d)$ be a sequence which is converging in the graph norm, i.e. there exist $u, v \in L^2(\mathbb{R}^d)$ such that $u_n \to u$ and $A(t)u_n \to v$ with respect to the L^2- norm. In order to prove the closure property of $A(t)$ we need to show $u \in D(A(t)) = H^2(\mathbb{R}^d)$ and $A(t)u = v$.

We notice that the H^1- norm of u_n is uniformly bounded due to

$$\epsilon\|\nabla u_n\|_{L^2}^2 = \int_{\mathbb{R}^d} \epsilon \nabla u \cdot \nabla u dx \leq \int_{\mathbb{R}^d} \nabla u \sigma(x,t) \nabla u dx \int_{\mathbb{R}^d} u_n \cdot A(t) u_n dx$$

$$= \langle u_n, A(t)u_n\rangle_{L^2} \leq \|u_n\|_{L^2}\|A(t)u_n\|_{L^2}$$

and the convergence of $(u_n, A(t)u_n)$. This implies the existence of a constant $C > 0$ so as to $\|u_n\|_{H^2} \leq C$ for all $n \in \mathbb{N}$, on account of lemma 4.1.15. As $H^2(\mathbb{R}^d)$ is separable and reflexive, the (bounded) sequence u_n has a subsequence u_{n_k} which is weakly converging in H^2 to some $\widehat{u} \in H^2(\mathbb{R}^d)$. But the weak convergence in H^2 implies the weak convergence in L^2 and due to the uniqueness of the limit, $u = \widehat{u}$ almost everywhere. We have therefore shown $u \in H^2(\mathbb{R}^d)$.
To identify $A(t)u$ and v, we use the fundamental lemma of calculus of variations. Let therefore ϕ be an arbitrary element of $\mathcal{D}(\mathbb{R}^d)$. Then the following equality holds:

$$\langle A(t)u, \phi\rangle_{L^2} = \int_{\mathbb{R}^d} \nabla u \sigma(x,t) \nabla \phi dx = \lim_{k\to\infty} \int_{\mathbb{R}^d} \nabla u_{n_k} \sigma(x,t) \nabla \phi dx = \lim_{k\to\infty} \langle A(t)u_{n_k}, \phi\rangle_{L^2} = \langle v, \phi\rangle_{L^2} \tag{4.23}$$

where we used the the weak convergence of u_{n_k} on the left-hand side and the strong convergence of $A(t)u_{n_k}$ on the right-hand side of 4.23. As ϕ was arbitrary, $A(t)u = v$ almost everywhere. □

It is an easy exercise to show that $A(t)$ is symmetric, i.e.

$$\langle A(t)u, v\rangle_{L^2} = \langle u, A(t)v\rangle_{L^2}, \quad u, v \in H^2(\mathbb{R}^d)$$

and positive, i.e.

$$\langle A(t)u, u\rangle_{L^2} \geq 0, \quad u \in H^2(\mathbb{R}^d).$$

The later follows directly from the uniform coercivity of $A(t)$. We can therefore apply lemma 4.1.13 to obtain $(-\infty, 0) \subseteq \rho(A(t))$, $t \in [0,T]$. Note that 0 may not be included in the resolvent set but it is an crucial condition for theorem 4.1.11 and lemma 4.16. We therefore use an arbitrary but fixed real constant $c > 0$ to define a new operator

$$B(t)u := -\text{div}\,(\sigma(x,t)\nabla u) + cu = A(t) - (-c)u = R_{-c}^{-1} \tag{4.24}$$

for all $t \in [0,T]$.
$B(t)$ still keeps all the important attributes, namely coercivity and symmetry, and it is still closed and densely defined on $L^2(\mathbb{R}^d, \mathbb{C})$. Furthermore, the interval $(-\infty, c)$ is now contained in the resolvent set of $B(t)$. In particular $0 \in \rho(B(t))$, which by lemma 4.16 implies that $B(t)$ is self- adjoint. Its spectrum is therefore contained in $\mathbb{R}$. This is a well known fact from functional calculus, which is valid for all self- adjoint operators which are closed and densely defined on a complex Hilbert space (see [24, Theorem 9.17]).
Another important attribute of $B(t)$ is the equivalence of the norm $\|u\|_{H^2}$ and $\|B(t)u\|_{L^2}$ on $H^2(\mathbb{R}^d, \mathbb{C})$:

Lemma 4.1.17. *There exists a constant $C > 0$, independent of t, such that*

$$\|u\|_{H^2} \le C\|B(t)u\|_{L^2}, \quad \forall u \in H^2(\mathbb{R}^d, \mathbb{C}).$$

Proof. From lemma 4.17 we already know for $u \in H^2(\mathbb{R}^d)$,

$$\|u\|_{H^2} \le C\left(\|u\|_{H^1} + \|A(t)u\|_{L^2}\right).$$

or in terms of $B(t)$,

$$\|u\|_{H^2} \le C\left(\|u\|_{H^1} + \|B(t)u - cu\|_{L^2}\right) \le C\left(\|u\|_{H^1} + \|B(t)u\|_{L^2} + c\|u\|_{L^2}\right). \tag{4.25}$$

The sesquilinerform $\langle\sigma(t)\nabla u, \nabla v\rangle_{L^2} + c\langle u, v\rangle_{L^2}$ is coercive regarding H^1, with constant $\kappa = \min\{\epsilon, c\}$. Hence due to the theorem of Lax-Milgram

$$\|u\|_{H^1} \le \frac{1}{\kappa}\|B(t)u\|_{L^2}. \tag{4.26}$$

Combining now (4.25) and (4.26) leads to

$$\|u\|_{H^2} \le C\left(\frac{1}{\kappa}\|B(t)u\|_{L^2} + \|B(t)u\|_{L^2} + c\frac{1}{\kappa}\|B(t)u\|_{L^2}\right) = C\left(\frac{1+c}{\kappa} + 1\right)\|B(t)u\|_{L^2}.$$

□

Summing up we have established:

Corollary 4.1.18.

i) $B(t)$ is positive, closed, densely defined and self- adjoint.

ii) $\sigma(B(t)) \subseteq S_{\theta,c}$ for any $0 < \theta < \frac{\pi}{2}$.

iii) $\|B(t)\cdot\|_{L^2}$ is equivalent to the norm $\|\cdot\|_{H^2}$.

We notice that $B(t)$ already fulfils condition a) of theorem 4.1.12. It turns out we can go all the way:

Theorem 4.1.19. *Let the assumptions made at the beginning of this chapter hold and let $B(t)$ be defined as in (4.24). If additionally the condition*

$$\left|y^T\sigma(x,t)y - y^T\sigma(x,s)y\right| \le C|t-s|^\beta\left|y^T\sigma(x,s)y\right|, \quad \forall s,t \in [0,T], \forall x,y \in \mathbb{R}^m \tag{4.27}$$

is satisfied for some $C > 0$ and $\frac{2}{3} < \beta \le 1$, then $B(t)$ gives rise to an evolution operator.

Proof. We want to apply theorem (4.1.12). We therefore verify its conditions a-c regarding the Hilbert space $H = L^2(\mathbb{R}^d, \mathbb{C})$:

a): We already know that the family $\{B(t)\}_{t\in[0,T]}$ fulfils point a, as shown in corollary 4.1.18.

b): Let $b(t)[u,v] := \langle \sigma(t)\nabla u, \nabla v\rangle_{L^2} + c\langle u, v\rangle_{L^2}$ for $u, v \in H^1(\mathbb{R}^d, \mathbb{C})$. Additionally let $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\mathbb{R}^d, \mathbb{C})$ be a sequence with $u_n \to_{L^2} u$ for some $u \in L^2$ and $b(t)[u_n - u_m, u_n - u_m] \to_{\mathbb{R}} 0$ for $m, n \to \infty$.

Due to the uniform coercivity of $\sigma(x,t)$, and because of

$$\text{Re } \underbrace{b(t)[u,u]}_{\in\mathbb{R}^+} = \langle \sigma(t)\nabla u, \nabla u\rangle_{L^2} + c\langle u, u\rangle_{L^2} \geq \epsilon\langle \nabla u, \nabla u\rangle_{L^2} + c\|u\|_{L^2}^2 \geq \min\{\epsilon, c\}\|u\|_{H^1}^2$$

the convergence of Re $b(t)[u_n - u_m, u_n - u_m]$ against 0 implies $u_n \to_{H^1} u$ for $n \to \infty$, and in particular $u \in H^1$. As $b(t)[\cdot,\cdot]$ is continuous with respect to the H^1- norm,

$$\lim_{n\to\infty} b(t)[u_n, u_n] = b(t)[u,u]$$

follows immediately. Hence Re $b(t)$ is a closed sesquilinearform and since

$$b(t)[u,v] = \langle B(t)u, v\rangle_{L^2} \qquad \forall u \in H^2(\mathbb{R}^d, \mathbb{C}),\ v \in H^1(\mathbb{R}^d, \mathbb{C}),$$

condition $b)$ is satisfied by choosing $a(t) = b(t)$.

c): For symmetric matrices $A \in \mathbb{R}^{d\times d}$ one can prove

$$x^T A\overline{x} = (\text{Re } x)^T A\,(\text{Re } x) + (\text{Im } x)^T A\,(\text{Im } x)$$

quite similar to exercise 4.1.10. Hence for $x, y \in \mathbb{C}^n$ and $s, t \in [0,T]$ the symmetry of σ leads to the following chain of inequalities:

$$\left|y^T\sigma(x,t)\overline{y} - y^T\sigma(x,s)\overline{y}\right| =$$

$$\left|(\text{Re } y)^T\sigma(x,t)\,(\text{Re } y) + (\text{Im } y)^T\sigma(x,t)\,(\text{Im } y) - (\text{Re } y)^T\sigma(x,s)\,(\text{Re } y) - (\text{Im } y)^T\sigma(x,s)\,(\text{Im } y)\right|$$

$$\leq \left|(\text{Re } y)^T\sigma(x,t)\,(\text{Re } y) - (\text{Re } y)^T\sigma(x,s)\,(\text{Re } y)\right|$$

$$+\left|(\text{Im } y)^T\sigma(x,t)\,(\text{Im } y) - (\text{Im } y)^T\sigma(x,s)\,(\text{Im } y)\right|$$

$$\leq C|t-s|^\beta\left(\left|\underbrace{(\text{Re } y)^T\sigma(x,s)\,(\text{Re } y)}_{\in\mathbb{R}_0^+}\right| + \left|\underbrace{(\text{Im } y)^T\sigma(x,s)\,(\text{Im } y)}_{\in\mathbb{R}_0^+}\right|\right)$$

$$= C|t-s|^\beta\left|y^T\sigma(x,s)y\right|.$$

Notice that in the last step we used the positivity of σ. We have shown

$$\left|y^T\sigma(x,t)y - y^T\sigma(x,s)y\right| \leq C|t-s|^\beta y^T\sigma(x,s)y. \tag{4.28}$$

For $u \in H^1(\mathbb{R}^d, \mathbb{C})$ it holds that,

$$\left|\int_{\mathbb{R}^d} \nabla u\sigma(x,t)\nabla u dx - \int_{\mathbb{R}^d} \nabla u\sigma(x,s)\nabla u dx\right| \leq \int_{\mathbb{R}^d} |\nabla u\sigma(x,t)\nabla u - \nabla u\sigma(x,s)\nabla u| dx.$$

Therefore setting $y = u$ and integrating over (4.28) results in

$$|b(t)[u,u] - b(s)[u,u]| \leq C|t-s|^\beta b(s)[u,u] = C|t-s|^\beta|\text{Re } b(s)[u,u]|.$$

This completes the proof.

□

Remark 4.1.20. As σ is uniform coercive, we have

$$C|t-s|^\beta|y|^2 \leq \frac{C}{\epsilon}|t-s|^\beta\left|y^T\sigma(s)y\right|$$

for all $y \in \mathbb{R}^d$ and $s, t \in [0,T]$. The condition 4.27 can therefore be weakened to

$$\left|y^T\sigma(t)y - y^T\sigma(s)y\right| \leq C|t-s|^\beta|y|^2, \quad \forall s,t \in [0,T], \forall y \in \mathbb{R}^d.$$

Because $B(t)$ is more manageable we rewrite problem (4.13) in terms of $B(t)$:

$$du/dt + B(t)u = \widehat{g}(u,t), \quad t \in [0,T], \quad u(0) = u_0 \in L^2 \tag{4.29}$$

where $\widehat{g}(u,t) = \operatorname{div}(f(x,t)u) + cu$. Every $u \in L^2(0,T;H^1)$ that satisfies (4.29) also clearly satisfies (4.13).

Remark 4.1.21 (Real-valued evolution operator). Let $V(t,s) \in \mathbb{B}(L^2(\mathbb{R}^d,\mathbb{C}))$ for $0 \leq s \leq t$ be the operator defined by

$$V(t,s) = \text{Re } U(t,s),$$

where $U(s,t)$ is the evolution operator regarding $B(t)$.
For $V(s,t)$ the points $i), iv), v)$ of definition (4.1.7) hold. This is because a function $u : \mathbb{R}^d \to \mathbb{C}$ is weakly differentiable if and only if the real and imaginary part of u is as well. The linearity of our differential operators, and the fact that $\sigma(x,t) \in \mathbb{R}^{d\times d}$, leads therefore to

$$\text{Re } B(t)u = B(t)\text{Re } u, \quad u \in H^2(\mathbb{R}^d,\mathbb{C}).$$

With this identity, point $v)$ follows immediately and point $iv)$ of definition (4.1.7) is a direct conclusion from

$$C^2|t-s|^{2\beta}\|u\|^2_{L^2(\mathbb{R}^d,\mathbb{C})} \geq \|B(t)U(t,s)u\|^2_{L^2(\mathbb{R}^d,\mathbb{C})} = \|B(t)\left(\text{Re } U(t,s)u + i\text{Im } U(t,s)u\right)\|^2_{L^2(\mathbb{R}^d,\mathbb{C})}$$

$$= \|\text{Re } B(t)U(t,s)u + i\text{Im } B(t)U(t,s)u\|^2_{L^2(\mathbb{R}^d,\mathbb{C})}$$

$$= \|\text{Re } B(t)U(t,s)u\|^2_{L^2(\mathbb{R}^d,\mathbb{C})} + \|\text{Im } B(t)U(t,s)u\|^2_{L^2(\mathbb{R}^d,\mathbb{C})}$$

$$= \|B(t)\text{Re } U(t,s)u\|^2_{L^2(\mathbb{R}^d,\mathbb{C})} + \|B(t)\text{Im } U(t,s)u\|^2_{L^2(\mathbb{R}^d,\mathbb{C})}.$$

Furthermore on the space $L^2(\mathbb{R}^d,\mathbb{R})$, $V(t,t) = Id_{L^2}$ is obviously true and point $i)$ holds due to the equivalence of converging in $\mathbb{C}$ and the simultaneously convergence of the real and imaginary part in $\mathbb{R}$.

4.1.3. Existence for the linear advection-diffusion problem

Let $0 < \delta < T$ and $u_\delta(t)$ solve the fix- point equation

$$u_\delta(t) = V(t,0)u_0 + \int_\delta^t V(t,s)\widehat{g}(u(s),s)ds, \quad t \in [\delta, T] \tag{4.30}$$

where $\widehat{g}$ is as in (4.29) and $V(t,s) = \mathrm{Re}\, U(t,s)$ is the real part of the evolution operator[7] of $B(t)$, as shown in(4.24).
We will show that such a u_δ exists and solves PDE (4.1) in a weak sense on the interval $[\delta, T]$ with the initial condition $u_\delta(\delta) = V(\delta, 0)u_0$.

Theorem 4.1.22. *Let the assumptions of theorem 4.1.19 hold. Then there exists $0 < T^* \leq T$ such that for every $\delta < T^*$ and every $u_0 \in L^2(\mathbb{R}^d)$ there is an $u_\delta \in \mathcal{X} = L^\infty(\delta, T^*; H^1(\mathbb{R}^d))$, which has the representation (4.30) for $t \in [\delta, T^*]$.*

Proof. Let $0 < \delta < T^*$ for some arbitrary $T^* \leq T$ and let us define the operator $F : \mathcal{X} \to F(\mathcal{X})$ by

$$F[x](t) := V(t,0)u_0 + \int_\delta^t V(t,s)\widehat{g}(x(s),s)ds.$$

We would like to apply Banachs fix- point theorem. For this purpose we need to show $F(\mathcal{X}) \subseteq \mathcal{X}$ and that F is in fact a contraction for the right choice of T^*.

$F(\mathcal{X}) \subseteq \mathcal{X}$:

Let $x \in \mathcal{X}$. First of all remember $\widehat{g}(x(s),s) = \mathrm{div}\,(f(s)x(s)) + cx(s)$. We can therefore find a constant $K > 0$ satisfying

$$\|\widehat{g}(x(s),s)\|_{L^2} \leq \|\mathrm{div}\ f(s)\|_{L^\infty}\|x(s)\|_{L^2} + \|f(s)\|_{L^\infty}\|\nabla x(s)\|_{L^2} + c\|x(s)\|_{L^2} \leq K\|x\|_{L^\infty(\delta,T;H^1)},$$

for all $s \in [\delta, T^*]$. Because of $f \in L^\infty(0,T; W^{1,\infty})$ we can choose K independently of δ.
We now have for $t \in [\delta, T^*]$,

$$\|F[x](t)\|_{H^1} \leq \|V(t,0)u_0\|_{H^1} + \int_\delta^t \|V(t,s)\widehat{g}(x(s),s)\|_{H^1} ds.$$

Using the Gagliardo- Nirenberg inequality, see A.2.9, with $\theta = 1/2, j = r = p = q = 2$ and $k = 1$ we get

$$\|F[x](t)\|_{H^1} \leq c\|V(t,0)u_0\|_{H^2}^\theta \|V(t,0)u_0\|_{L^2}^{1-\theta} + c\int_\delta^t \|V(t,s)\widehat{g}(x(s),s)\|_{H^2}^\theta \|V(t,s)\widehat{g}(x(s),s)\|_{L^2}^{1-\theta} ds.$$

Together with the uniform boundedness of $V(t,s)$ (see remark 4.1.8), the estimates for $B(t)V(t,s)$ (see remark 4.1.21) and the norm equivalence of $\|\cdot\|_{H^2}$ and $\|B(t)\cdot\|_{L^2}$ we have

$$\|F[x](t)\|_{H^1}$$

[7] For some properties of $V(t,s)$ see remark 4.1.21.

$$\leq c\|B(t)V(t,0)u_0\|_{L^2}^{\theta}\|V(t,0)u_0\|_{L^2}^{1-\theta} + c\int_{\delta}^{t}\|B(t)V(t,s)\widehat{g}(x(s),s)\|_{L^2}^{\theta}\|V(t,s)\widehat{g}(x(s),s)\|_{L^2}^{1-\theta}ds$$

$$\leq C\frac{1}{|t|^{\theta}}\|u_0\|_{L^2}^{\theta}\|u_0\|_{L^2}^{1-\theta} + C\int_{\delta}^{t}\frac{1}{|t-s|^{\theta}}\|\widehat{g}(x(s),s)\|_{L^2}^{\theta}\|\widehat{g}(x(s),s)\|_{L^2}^{1-\theta}ds$$

$$\leq C\left(\frac{1}{|t|^{\theta}}\|u_0\|_{L^2} + \underbrace{\int_{\delta}^{t}\frac{1}{|t-s|^{\theta}}ds}_{=(t-\delta)^{1-\theta}}\right)\|\widehat{g}(x(s),s)\|_{L^{\infty}(\delta,T^*;L^2)}$$

$$\leq C_1\left(\frac{1}{\delta^{\theta}}\|u_0\|_{L^2} + (T-\delta)^{1-\theta}\right)\|x\|_{L^{\infty}(\delta,T^*;H^1)}. \tag{4.31}$$

The right side of the last inequality does not depend on t anymore, hence $F[x] \in \mathcal{X}$.

F is a contraction:

Let $x, y \in \mathcal{X}$, then we have

$$\|F[x](t) - F[y](t)\|_{H^1} \leq \int_{\delta}^{t}\|V(t,s)\widehat{g}(x(s)-y(s),s)\|_{H^1}ds$$

where we now used the linearity of $\widehat{g}(x,s)$ regarding x. If we repeat the steps we took to arrive at the above estimate (4.31), we get

$$\|F[x](t) - F[y](t)\|_{H^1} \leq c\int_{\delta}^{t}\|V(t,s)\widehat{g}(x(s),s)\|_{H^2}^{\theta}\|V(t,s)\widehat{g}(x(s)-y(s),s)\|_{L^2}^{1-\theta}ds$$

$$\leq c\int_{\delta}^{t}\|B(t)V(t,s)\widehat{g}(x(s)-y(s),s)\|_{L^2}^{\theta}\|V(t,s)\widehat{g}(x(s)-y(s),s)\|_{L^2}^{1-\theta}ds$$

$$\leq C\int_{\delta}^{t}\frac{1}{|t-s|^{\theta}}\|\widehat{g}(x(s)-y(s),s)\|_{L^2}^{\theta}\|\widehat{g}(x(s)-y(s),s)\|_{L^2}^{1-\theta}ds$$

$$\leq C\int_{\delta}^{t}\frac{1}{|t-s|^{\theta}}ds\|\widehat{g}(x(s)-y(s),s)\|_{L^{\infty}(\delta,T^*;L^2)} \leq C\,(T-\delta)^{1-\theta}\,\|x-y\|_{L^{\infty}(\delta,T^*;H^1)}$$

$$\leq \underbrace{C(T^*)^{1-\theta}}_{:=L(T^*)} \|x-y\|_{L^\infty(\delta,T^*;H^1)}.$$

If we now choose T^* such that $L(T^*) = C(T^*)^{1-\theta} < 1$ and $0 \leq T^* \leq T$, $F[x]$ is a contraction on the space $L^\infty(\delta, T^*; H^1)$ as long as $\delta < T^*$. Hence we can apply Banachs fix- point theorem to obtain the required result. □

We can now show

Theorem 4.1.23. *Let $u_\delta \in L^\infty(\delta, T; H^1(\mathbb{R}^d))$ be as in (4.30) and let the assumptions of theorem 4.1.19 hold. Then u_δ is a weak solution of (4.1) on the interval $[\delta, T-\nu]$ with the initial condition $u_\delta(\delta) = V(\delta, 0)u_0$, for every $0 < \nu < T$.*

Proof. Let $u = u_\delta$, $\phi \in L^2(\delta, T-\nu; H^1)$ and $0 < h < \nu < T$. Then we have

$$I_0^h := \int_\delta^{T-\nu} \langle \frac{u(t+h)-u(t)}{h}, \phi\rangle_{L^2} dt = \int_\delta^{T-\nu} \langle \frac{1}{h} \left(V(t+h,0) - V(t,0)\right) u_0, \phi\rangle_{L^2} dt$$

$$+ \int_\delta^{T-\nu} \langle \int_\delta^{t} \frac{1}{h} \left(V(t+h,s) - V(t,s)\right) \widehat{g}(u(s),s) ds, \phi\rangle_{L^2} dt + \int_\delta^{T-\nu} \langle \frac{1}{h} \int_t^{t+h} V(t+h,s)\widehat{g}(u(s),s) ds, \phi\rangle_{L^2} dt$$

$$=: I_1^h + I_2^h + I_3^h.$$

i)I_1^h:

Notice that $V(t,0)u_0$ is differentiable in t (see remark (4.1.21)). Hence we get

$$\int_\delta^{T-\nu} \langle \frac{1}{h} \left(V(t+h,0) - V(t,0)\right) u_0, \phi\rangle_{L^2} dt = \int_\delta^{T-\nu} \langle \frac{1}{h} \int_t^{t+h} \frac{d}{d\tau} V(\tau,0) u_0 d\tau, \phi\rangle_{L^2} dt$$

$$= \int_\delta^{T-\nu} \langle \frac{1}{h} \int_t^{t+h} -B(\tau)V(\tau,0)u_0 d\tau, \phi\rangle_{L^2} dt.$$

The inner integral is a Bochner- integral in $H^1(\mathbb{R}^d)$ (see i.e. the proof of theorem 4.1.22). In particular this Bochner integral is also well defined in $L^2(\mathbb{R}^d)$ and we can therefore exchange the Bochner integral and the inner product of the L^2- space. Via integrations by parts this leads to

$$\int_\delta^{T-\nu} \frac{1}{h} \int_t^{t+h} \langle -B(\tau)V(\tau,0)u_0, \phi\rangle_{L^2} d\tau dt$$

$$= \int_{\delta}^{T-\nu} \frac{1}{h} \int_{t}^{t+h} \langle -\sigma(x,\tau)\nabla V(\tau,0)u_0, \nabla\phi\rangle_{L^2} d\tau dt - \int_{\delta}^{T-\nu} \frac{1}{h} \int_{t}^{t+h} \langle cV(\tau,0)u_0, \phi\rangle_{L^2} d\tau dt$$

$$= \int_{\delta}^{T-\nu} \langle \frac{1}{h} \int_{t}^{t+h} -\sigma(x,\tau)\nabla V(\tau,0)u_0 d\tau, \nabla\phi\rangle_{L^2} dt - \int_{\delta}^{T-\nu} \langle \frac{1}{h} \int_{t}^{t+h} cV(\tau,0)u_0 d\tau, \phi\rangle_{L^2} dt. \tag{4.32}$$

Remember that $V(t,s)$ is uniform bounded regarding the operator norm (see remark 4.1.8). Therefore the last step is rigorous as one can show Bochner- integrability of the term

$$\sigma(x,\tau)\nabla V(\tau,0)u_0$$

regarding the space $L^2(\mathbb{R}^d)$. This can be accomplished by using the Galiardo- Nirenberg inequality, choosing i.e. $\theta = 1/2$, and writing

$$\frac{1}{h}\int_{t}^{t+h} \|\sigma(x,\tau)\nabla V(\tau,0)u_0\|_{L^2} d\tau \leq \|\sigma\|_{L^\infty(0,T)\times\mathbb{R}^d)} \frac{1}{h}\int_{t}^{t+h} \|V(\tau,0)u_0\|_{H^1} d\tau$$

$$\leq C(\sigma)\frac{1}{h}\int_{t}^{t+h} \|V(\tau,0)u_0\|_{H^2}^{\theta} \|V(\tau,0)u_0\|_{L^2}^{1-\theta} d\tau \leq C(\sigma)\frac{1}{h}\int_{t}^{t+h} \|B(\tau)V(\tau,0)u_0\|_{L^2}^{\theta} \|V(\tau,0)u_0\|_{L^2}^{1-\theta} d\tau$$

$$\leq C(\sigma)\frac{1}{h}\int_{t}^{t+h} \frac{1}{\tau^\theta}\|u_0\|_{L^2} d\tau \leq C(\sigma)\frac{1}{t^\theta}\|u_0\|_{L^2} \leq C(\sigma)\frac{1}{\delta^\theta}\|u_0\|_{L^2} < \infty. \tag{4.33}$$

The Bochner- space version of the fundamental theorem of calculus tells us that the term

$$\frac{1}{h}\int_{t}^{t+h} -\sigma(x,\tau)\nabla V(\tau,0)u_0 d\tau$$

converges in $L^2(\mathbb{R}^d)$ for $h \to 0$ against $-\sigma(x,t)\nabla V(t,0)u_0$ for almost every t.
With similar (simpler) arguments we also infer the Bochner integrability of the term $cV(\tau,0)u_0$ and the convergence of $\frac{1}{h}\int_{t}^{t+h} cV(\tau,0)u_0 d\tau$ against $cV(t,0)u_0$ almost everywhere. Because of the uniform bound 4.31 we can now use the dominated convergence theorem and (4.32) to obtain

$$\lim_{h\to 0}\int_{\delta}^{T-\nu} \langle \frac{1}{h}\left(V(t+h,0) - V(t,0)\right)u_0, \phi\rangle_{L^2} dt = \int_{\delta}^{T-\nu} \langle -\sigma(x,t)\nabla V(t,0)u_0, \nabla\phi\rangle_{L^2} dt$$
$$- \int_{\delta}^{T-\nu} \langle cV(t,0)u_0, \phi\rangle_{L^2} dt.$$

ii)I_2^h**:**

We continue in a similar fashion with the term I_2^h:

$$\int_\delta^{T-\nu} \langle \int_\delta^t \frac{1}{h} \left(V(t+h,s) - V(t,s) \right) \widehat{g}(u(s),s) ds, \phi \rangle_{L^2} dt$$

$$= \int_\delta^{T-\nu} \langle \int_\delta^t \frac{1}{h} \int_t^{t+h} \frac{d}{d\tau} V(\tau,s) \widehat{g}(u(s),s) d\tau ds, \phi \rangle_{L^2} dt$$
$$= \int_\delta^{T-\nu} \langle \int_\delta^t \frac{1}{h} \int_t^{t+h} -B(\tau) V(\tau,s) \widehat{g}(u(s),s) d\tau ds, \phi \rangle_{L^2} dt$$
$$= \int_\delta^{T-\nu} \int_\delta^t \frac{1}{h} \int_t^{t+h} \langle -B(\tau) V(\tau,s) \widehat{g}(u(s),s), \phi \rangle_{L^2} d\tau ds dt$$
$$= \int_\delta^{T-\nu} \int_\delta^t \frac{1}{h} \int_t^{t+h} \langle -\sigma(x,\tau) \nabla V(\tau,s) \widehat{g}(u(s),s), \nabla \phi \rangle_{L^2} d\tau ds dt$$
$$- \int_\delta^{T-\nu} \int_\delta^t \frac{1}{h} \int_t^{t+h} \langle c V(\tau,s) \widehat{g}(u(s),s), \phi \rangle_{L^2} d\tau ds dt.$$

As before we will concentrate on the first term on right hand side of the last equation, as the computations for it are more complex, because of the loss of regularity due to the differential operator. Similar results hold also for the second term and can be proven in a similar fashion.

We need to show that the portion of the inner integrand on the left of the scalar product

$$\sigma(x,\tau) \nabla V(\tau,s) \widehat{g}(u(s),s) \tag{4.34}$$

is L^2- Bochner integrable in s and τ. Then we can again exchange the inner product with the Bochner integral. We therefore proceed by

$$\frac{1}{h} \int_t^{t+h} \|\sigma(x,\tau) \nabla V(\tau,s) \widehat{g}(u(s),s)\|_{L^2} d\tau \leq \|\sigma\|_{L^\infty((0,T)\times\mathbb{R}^d)} \frac{1}{h} \int_t^{t+h} \|V(\tau,s) \widehat{g}(u(s),s)\|_{H^1} d\tau$$

$$\leq C(\sigma) \frac{1}{h} \int_t^{t+h} \|V(\tau,s) \widehat{g}(u(s),s)\|_{H^2}^{\theta} \|V(\tau,s) \widehat{g}(u(s),s)\|_{L^2}^{1-\theta} d\tau$$

$$\leq C(\sigma) \frac{1}{h} \int_t^{t+h} \|B(\tau) V(\tau,s) \widehat{g}(u(s),s)\|_{L^2}^{\theta} \|V(\tau,s) \widehat{g}(u(s),s)\|_{L^2}^{1-\theta} d\tau$$

$$\leq C(\sigma)\frac{1}{h}\int_t^{t+h}\frac{1}{(\tau-s)^\theta}\|\widehat{g}(u(s),s)\|_{L^2}d\tau \leq C(\sigma)\frac{1}{(t-s)^\theta}\|\widehat{g}(u(s),s)\|_{L^\infty(\delta,T;L^2)}. \tag{4.35}$$

We now know that (4.34) is Bochner integrable in respect to τ. Hence with the estimate (4.35) we can also show Bochner integrability regarding s:

$$\int_\delta^t\left\|\frac{1}{h}\int_t^{t+h}\sigma(x,\tau)\nabla V(\tau,s)\widehat{g}(u(s),s)d\tau\right\|_{L^2}ds \leq \int_\delta^t\frac{1}{h}\int_t^{t+h}\|\sigma(x,\tau)\nabla V(\tau,s)\widehat{g}(u(s),s)\|_{L^2}d\tau ds$$

$$\leq C(\sigma)\int_\delta^t\frac{1}{(t-s)^\theta}ds\|\widehat{g}(u(s),s)\|_{L^\infty(\delta,T;L^2)} \leq C(\sigma,g,u)(t-\delta)^{1-\theta}$$

$$\leq C(\sigma,g,u)T^{1-\theta}. \tag{4.36}$$

Exchanging the inner product with the inner integrals leads to

$$\int_\delta^{T-\nu}\int_\delta^t\frac{1}{h}\int_t^{t+h}\langle-\sigma(x,\tau)\nabla V(\tau,s)\widehat{g}(u(s),s),\nabla\phi\rangle_{L^2}d\tau ds dt$$

$$=\int_\delta^{T-\nu}\langle\int_\delta^t\frac{1}{h}\int_t^{t+h}-\sigma(x,\tau)\nabla V(\tau,s)\widehat{g}(u(s),s)d\tau ds,\nabla\phi\rangle_{L^2}dt.$$

With regard to the uniform estimates (4.35) we can use the dominated convergence theorem to infer

$$\lim_{h\to 0}\int_\delta^t\frac{1}{h}\int_t^{t+h}-\sigma(x,\tau)\nabla V(\tau,s)\widehat{g}(u(s),s)d\tau ds$$

$$=\int_\delta^t-\sigma(x,t)\nabla V(t,s)\widehat{g}(u(s),s)ds$$

where the convergence is to be understood in the L^2- sense. Remember here that $\frac{1}{h}\int_t^{t+h}-\sigma(x,\tau)\nabla V(\tau,s)\widehat{g}(u(s),s)d\tau$ converges to $-\sigma(x,t)\nabla V(t,s)\widehat{g}(u(s),s)$ in $L^2(\mathbb{R}^d)$ for almost all t and that therefore the estimate (4.35) holds for the limit as well. Similar with the uniform estimate (4.36) and the dominated convergence theorem we get

$$\lim_{h\to 0}\int_\delta^{T-\nu}\langle\int_\delta^t\frac{1}{h}\int_t^{t+h}-\sigma(x,\tau)\nabla V(\tau,s)\widehat{g}(u(s),s)d\tau ds,\nabla\phi\rangle_{L^2}dt$$

$$= \int_{\delta}^{T-\nu} \langle \int_{\delta}^{t} -\sigma(x,t)\nabla V(t,s)\widehat{g}(u(s),s)ds, \nabla\phi\rangle_{L^2} dt$$

and if we repeat the steps above for the remaining terms we finally get

$$\lim_{h\to 0} \int_{\delta}^{T-\nu} \langle \int_{\delta}^{t} \frac{1}{h}\left(V(t+h,s) - V(t,s)\right)\widehat{g}(u(s),s)ds, \phi\rangle_{L^2} dt$$

$$= \int_{\delta}^{T-\nu} \langle \int_{\delta}^{t} -\sigma(x,t)\nabla V(t,s)\widehat{g}(u(s),s)ds, \nabla\phi\rangle_{L^2} dt - \int_{\delta}^{T-\nu} \langle \int_{\delta}^{t} cV(t,s)\widehat{g}(u(s),s)ds, \phi\rangle_{L^2} dt$$

$$= \int_{\delta}^{T-\nu} \langle -\sigma(x,t)\nabla \int_{\delta}^{t} V(t,s)\widehat{g}(u(s),s)ds, \nabla\phi\rangle_{L^2} dt - \int_{\delta}^{T-\nu} \langle c \int_{\delta}^{t} V(t,s)\widehat{g}(u(s),s)ds, \phi\rangle_{L^2} dt.$$

The exchange of the integral and the ∇- operator in the last step is justified because the respective Bochner- integral converges in $H^1(\mathbb{R}^d)$. Because $\sigma(x,t) \in L^\infty(\mathbb{R}^d)$ does not depend on s, the matrix $\sigma(x,t)$ can be taken out of the integral as well.

iii)I_3^h:

Let $G(d,h)$ with $0 < h \leq d < \nu$ be defined by

$$G(d,h)[t] := \frac{1}{h} \int_{t}^{t+h} V(t+d,s)\widehat{g}(u(s),s)ds.$$

Then $I_3^h = G(h,h)$ and for fixed d it holds that

$$G(d,h) \to_{L^2} V(t+d,t)\widehat{g}(u(t),t)$$

almost everywhere if $h \to 0$. Furthermore, if $d \to 0$,

$$V(t+d,t)\widehat{g}(u(t),t) \to_{L^2} \widehat{g}(u(t),t)$$

due to the continuity of $V(t,s)$.
Our intention is to compare I_3^h with $G(d,h)$ and then passing to the limit $h \to h$ and $d \to 0$ to obtain the desired result. We therefore start with arbitrary h, d satisfying $0 < h \leq d$ by computing

$$\int_{\delta}^{T-\nu} \langle G(d,h) - G(h,h), \phi\rangle_{L^2} dt = \int_{\delta}^{T-\nu} \langle \frac{1}{h} \int_{t}^{t+h} \left(V(t+d,h) - V(t+h,s)\right)\widehat{g}(u(s),s), \phi\rangle_{L^2} ds dt$$

$$= \int_{\delta}^{T-\nu} \langle \frac{1}{h} \int_{t}^{t+h} \int_{t+h}^{t+d} \frac{d}{d\tau} V(\tau,s)\widehat{g}(u(s),s)d\tau, \phi\rangle_{L^2} ds dt$$

$$= \int_\delta^{T-\nu} \langle \frac{1}{h} \int_t^{t+h} \int_{t+h}^{t+d} -B(\tau)V(\tau,s)\widehat{g}(u(s),s)d\tau, \phi\rangle_{L^2} ds dt$$

$$= \int_\delta^{T-\nu} \frac{1}{h} \int_t^{t+h} \int_{t+h}^{t+d} \langle -B(\tau)V(\tau,s)\widehat{g}(u(s),s), \phi\rangle_{L^2} d\tau ds dt$$

$$= \int_\delta^{T-\nu} \frac{1}{h} \int_t^{t+h} \int_{t+h}^{t+d} \langle -\sigma(x,\tau)\nabla V(\tau,s)\widehat{g}(u(s),s), \nabla\phi\rangle_{L^2} d\tau ds dt. \tag{4.37}$$

Taking the the absolute value of (4.37), using the triangle inequality and the inequality of Cauchy- Schwarz, we get

$$\left| \int_\delta^{T-\nu} \langle G(d,h) - G(h,h), \phi\rangle_{L^2} dt \right| \le \int_\delta^{T-\nu} \frac{1}{h} \int_t^{t+h} \int_{t+h}^{t+d} |\langle -\sigma(x,\tau)\nabla V(\tau,s)\widehat{g}(u(s),s), \nabla\phi\rangle_{L^2}| d\tau ds dt$$

$$\le \|\sigma\|_{L^\infty((0,T)\times\mathbb{R}^d)} \int_\delta^{T-\nu} \frac{1}{h} \int_t^{t+h} \int_{t+h}^{t+d} \|\nabla V(\tau,s)\widehat{g}(u(s),s)\|_{L^2} \|\nabla\phi\|_{L^2} d\tau ds dt. \tag{4.38}$$

For the term $\|\nabla V(\tau,s)\widehat{g}(u(s),s)\|_{L^2}$ we find an estimate in the same way as in the case of I_1 and I_2 with the help of the Gagliardo Nirenberg inequality ($\frac{1}{2} < \theta < 1$):

$$\|\nabla V(\tau,s)\widehat{g}(u(s),s)\|_{L^2} \le \|V(\tau,s)\widehat{g}(u(s),s)\|_{H^1} \le C\|V(\tau,s)\widehat{g}(u(s),s)\|_{H^2}^\theta \|V(\tau,s)\widehat{g}(u(s),s)\|_{L^2}^{1-\theta}$$

$$\le C\|B(\tau)V(\tau,s)\widehat{g}(u(s),s)\|_{L^2}^\theta \|V(\tau,s)\widehat{g}(u(s),s)\|_{L^2}^{1-\theta}$$

$$\le \frac{C}{(\tau-s)^\theta} \|\widehat{g}(u(s),s)\|_{L^2}^\theta \underbrace{\|V(\tau,s)\widehat{g}(u(s),s)\|_{L^2}^{1-\theta}}_{\le K^{1-\theta}\|\widehat{g}(u(s),s)\|_{L^2}^{1-\theta}}$$

$$\le \frac{C}{(\tau-s)^\theta} \|\widehat{g}(u(s),s)\|_{L^2} \le \frac{C}{(\tau-s)^\theta} \|\widehat{g}(u(s),s)\|_{L^\infty(\delta,T;L^2)}. \tag{4.39}$$

Combining (4.39) and (4.38) leads to

$$\left| \int_\delta^{T-\nu} \langle G(d,h) - G(h,h), \phi\rangle_{L^2} dt \right| \le C(\sigma,g,u) \int_\delta^{T-\nu} \frac{1}{h} \int_t^{t+h} \int_{t+h}^{t+d} \frac{1}{(\tau-s)^\theta} d\tau ds \|\nabla\phi\|_{L^2} dt$$

$$= C(\sigma,g,u) \int_\delta^{T-\nu} \frac{1}{h} \int_t^{t+h} \left(\frac{1}{1-\theta}(t+d-s)^{1-\theta} - \underbrace{\frac{1}{1-\theta}(t+h-s)^{1-\theta}}_{\ge 0} \right) ds \|\nabla\phi\|_{L^2} dt$$

$$\leq C(\sigma,g,u,\theta)\int\limits_{\delta}^{T-\nu}\frac{1}{h}\int\limits_{t}^{t+h}\underbrace{(t+d-s)^{1-\theta}}_{\leq d^{1-\theta}}ds\|\nabla\phi\|_{L^2}dt$$

$$\leq d^{1-\theta}C(\sigma,g,u,\theta)\int\limits_{\delta}^{T-\nu}\|\nabla\phi\|_{L^2}dt\leq d^{1-\theta}C(\sigma,g,u,\theta)\|\nabla\phi\|_{L^2(\delta,T-\nu;L^2)}\sqrt{T-\nu-\delta}. \qquad (4.40)$$

Notice that in the last step we used Cauchy- Schwarz's inequality again.
The next step is to show that $G(d,h)$ and $V(t+d,t)\widehat{g}(u(t),t)$ converge in $L^2(\delta,T-\nu;L^2)$. It is an easy exercise to show that the L^2-norm of $G(d,h)$ is uniformly bounded by

$$C\|g(u(t),t)\|_{L\infty(\delta,T;L^2)} \qquad (4.41)$$

for some constant C. In particular, this constant does not depend on t, hence this bound can be used as majorant for the dominated convergence theorem to show

$$G(d,h)\longrightarrow_{L^2(\delta,T-\nu;L^2)} V(t+d,t)\widehat{g}(u(t),t)$$

for $h\to 0$ and

$$V(t+d,t)\widehat{g}(u(t),t)\longrightarrow_{L^2(\delta,T-\nu;L^2)} V(t,t)\widehat{g}(u(t),t)=g(u(t),t)$$

for $d\to 0$.
Choosing a fixed but arbitrary $\epsilon>0$, we can therefore find two positive constants h_ϵ,d_ϵ such that the following inequalities hold for all $0<h\leq h_\epsilon$:

a) $d_\epsilon^{1-\theta}C(\sigma,g,u,\theta)\|\nabla\phi\|_{L^2(\delta,T-\nu;L^2)}\sqrt{T-\nu-\delta}\leq\frac{\epsilon}{3}$ (see (4.40)).

b) $\left|\langle V(t+d_\epsilon,t)\widehat{g}(u(t),t)-g(u(t),t),\phi\rangle_{L^2(\delta,T-\nu,L^2)}\right|\leq\frac{\epsilon}{3}$.

c) $\left|\langle G(d_\epsilon,h)-V(t+d_\epsilon,t)\widehat{g}(u(t),t),\phi\rangle_{L^2(\delta,T-\nu,L^2)}\right|\leq\frac{\epsilon}{3}$.

Now we can use the triangle inequality and (4.40) to compute

$$\left|\langle I_3^h-\widehat{g}(u(t),t),\phi\rangle_{L^2(\delta,T-\nu,L^2)}\right|=\left|\langle G(h,h)-\widehat{g}(u(t),t),\phi\rangle_{L^2(\delta,T-\nu,L^2)}\right|$$

$$\leq\left|\langle G(h,h)-G(d_\epsilon,h),\phi\rangle_{L^2(\delta,T-\nu,L^2)}\right|+\left|\langle G(d_\epsilon,h)-V(t+d_\epsilon,t)\widehat{g}(u(t),t),\phi\rangle_{L^2(\delta,T-\nu,L^2)}\right|$$

$$+\left|\langle V(t+d_\epsilon,t)g(u(t),t)-\widehat{g}(u(t),t),\phi\rangle_{L^2(\delta,T-\nu,L^2)}\right|\leq\epsilon.$$

Because ϵ was arbitrary we have shown

$$\langle I_3^h,\phi\rangle_{L^2(\delta,T-\nu,L^2)}\to\langle\widehat{g}(u(t),t),\phi\rangle_{L^2(\delta,T-\nu,L^2)}$$

if $h\to 0$, for any $\phi\in L^2(\delta,T-\nu;H^1)$.

$iv) I_0^h$**:**

Due to the computations above now we can write

$$I_0^h = \int\limits_{\delta}^{T-\nu} \langle \frac{u(t+h)-u(t)}{h}, \phi \rangle_{L^2} dt = I_1^h + I_2^h + I_3^h$$

$$= \int\limits_{\delta}^{T-\nu} \langle \frac{1}{h} \int\limits_{t}^{t+h} -\sigma(x,\tau)\nabla V(\tau,0)u_0 d\tau, \nabla\phi \rangle_{L^2} dt - \int\limits_{\delta}^{T-\nu} \langle \frac{1}{h} \int\limits_{t}^{t+h} cV(\tau,0)u_0 d\tau, \phi \rangle_{L^2} dt$$

$$+ \int\limits_{\delta}^{T-\nu} \int\limits_{\delta}^{t} \frac{1}{h} \int\limits_{t}^{t+h} \langle -\sigma(x,\tau)\nabla V(\tau,s)\widehat{g}(u(s),s), \nabla\phi \rangle_{L^2} d\tau ds dt \tag{4.42}$$

$$- \int\limits_{\delta}^{T-\nu} \int\limits_{\delta}^{t} \frac{1}{h} \int\limits_{t}^{t+h} \langle cV(\tau,s)\widehat{g}(u(s),s), \phi \rangle_{L^2} d\tau ds dt$$

$$+ \int\limits_{\delta}^{T-\nu} \langle \frac{1}{h} \int\limits_{t}^{t+h} V(t+h,s)\widehat{g}(u(s),s) ds, \phi \rangle_{L^2} dt.$$

Using Cauchy- Schwarz's inequality and the estimates (4.41), (4.36) and (4.33) we find an estimate for $\left\| \frac{u(t+h)-u(t)}{h} \right\|_{L^2(\delta,T-\nu);H^{-1}}$ by

$$\left| \int\limits_{\delta}^{T-\nu} \langle \frac{u(t+h)-u(t)}{h}, \phi \rangle_{L^2} dt \right|$$

$$\leq \sqrt{T-\nu-\delta} \left(C(\sigma)\frac{1}{\delta^\theta} \|u_0\|_{L^2} \|\nabla\phi\|_{L^2(\delta,T-\nu;L^2)} + C(c)\|u_0\|_{L^2}\|\phi\|_{L^2(\delta,T-\nu);L^2} \right)$$

$$+\sqrt{T-\nu-\delta} \left(C(\sigma,g,u)T^{1-\theta}\|\nabla\phi\|_{L^2(\delta,T-\nu;L^2)} + C(c)\|\widehat{g}(u(s),s)\|_{L^\infty(\delta,T;L^2)}\|\phi\|_{L^2(\delta,T-\nu;L^2)} \right)$$

$$+\sqrt{T-\nu-\delta} C \|\widehat{g}(u(s),s)\|_{L^\infty(\delta,T;L^2)}\|\phi\|_{L^2(\delta,T-\nu;L^2)}$$

$$\leq C(c,\sigma,g,u,T,\nu,\theta,\delta)\|\phi\|_{L^2(\delta,T-\nu;H^1)}. \tag{4.43}$$

Because (4.43) does not depend on h, i.e. the estimate is uniform, we can find a subsequence h_k and an element v of $L^2(\delta,T-\nu;H^{-1})$ such that

$$D_t^{h_k}(u) := \frac{u(t+h_k)-u(t)}{h_k} \rightharpoonup v$$

in $L^2(\delta, T-\nu; H^{-1})$. But $D_t^{h_k}(u)$ converges, in the sense of distributions, to $\partial_t u$ which implies $v = \partial_t u$. In particular $\partial_t u \in L^2(\delta, T-\nu; H^{-1})$ holds true.

Putting all the pieces together and passing to the limit in (4.42) with $h = h_k$, we get

$$\int_\delta^{T-\nu} \langle \partial_t u, \phi \rangle_{H^{-1}} dt = \int_\delta^{T-\nu} \langle -\sigma(x,t)\nabla V(t,0)u_0, \nabla\phi \rangle_{L^2} dt - \int_\delta^{T-\nu} \langle cV(t,0)u_0, \phi \rangle_{L^2} dt$$

$$+ \int_\delta^{T-\nu} \langle -\sigma(x,t)\nabla \int_\delta^t V(t,s)\widehat{g}(u(s),s)ds, \nabla\phi \rangle_{L^2} dt - \int_\delta^{T-\nu} \langle c \int_\delta^t V(t,s)\widehat{g}(u(s),s)ds, \phi \rangle_{L^2} dt$$

$$+ \int_\delta^{T-\nu} \langle \widehat{g}(u(t),t), \phi \rangle_{L^2} dt$$

$$= - \int_\delta^{T-\nu} \langle \sigma(x,t)\nabla u, \nabla\phi \rangle_{L^2} dt - \int_\delta^{T-\nu} \langle cu, \phi \rangle_{L^2} dt + \int_\delta^{T-\nu} \langle \mathrm{div}\,(f(x,t)u) + cu, \phi \rangle_{L^2} dt$$

$$= - \int_\delta^{T-\nu} \langle \sigma(x,t)\nabla u, \nabla\phi \rangle_{L^2} dt + \int_\delta^{T-\nu} \langle \mathrm{div}\,(f(x,t)u)\,, \phi \rangle_{L^2} dt.$$

This is exactly the weak formulation of problem (4.1) on the Interval $[\delta, T-\nu]$. As ϕ was arbitrary, u is therefore a weak solution. □

For small enough $\nu > 0$, theorem 4.1.23 now states the existence of a weak solution on the interval $[\delta, T^* - \nu]$ where T^* is determined by theorem 4.1.22. With our a- priori estimates (see theorem 4.1.4), we conclude the existence of a constant $C > 0$ such that

$$\|u_\delta(t)\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla u_\delta\|_{L^2(\delta,t,L^2)}^2 \le \|V(\delta,0)u_0\|_{L^2}^2 \left(\exp\left(Ct\right) + 1\right) \tag{4.44}$$

for all $t \in [\delta, T^*]$ and $\delta < T^*$. From estimate (4.44) we get in particular an upper- bound for $\|u_\delta\|_{L^2(\delta,T^*;H^1)}$ as long as δ is bounded. To obtain a function on the whole interval $[0, T^*]$, we redefine u_δ by writing

$$u_{\delta,new} = \begin{cases} u_{\delta,old}(t) & ,t \ge \delta \\ V(\delta,0)u_0 & ,t < \delta \end{cases}. \tag{4.45}$$

We like to remind the reader here that $V(\delta,0)u_0$ is continuous in δ and hence has a maximum on $[0,T]$, regarding the L^2- norm, which is finite. In regards to (4.44), $u_{\delta,new}$ is therefore uniform bounded in $L^2(0,T^*;H^1)$ as

$$\|u_\delta(t)\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla u_\delta\|_{L^2(0,t,L^2)}^2 \le \|V(\delta,0)u_0\|_{L^2}^2 \left(\exp\left(CT^*\right) + 1\right) \le C(u_0)\left(\exp\left(CT^*\right) + 1\right). \tag{4.46}$$

Due to the above (uniform) estimate we can find a sequence δ_n for which u_{δ_n} converges weakly to some u in $L^2(0,T^*;H^1)$. This function u will be our candidate for a weak solution on $[0,T^*]$. Per definition a weak solution has a, in a sense regular, weak time derivative. To show that is also the case for u, we need some more estimates for $\partial_t u_\delta$:

Theorem 4.1.24. *Let u_δ be defined as (4.45),where $u_{\delta,old}$ is a weak solution on the interval $[\delta, T^*)$ of (4.1). Then $u_\delta \in W^{1,2}(0,T^*; H^1(\mathbb{R}^d), L^2(\mathbb{R}^d))$. Furthermore, there exists a constant $C > 0$ such that*

$$\|\partial_t u_\delta\|_{L^2(0,T^*;H^{-1})} \leq C \tag{4.47}$$

for all small enough δ, whereby T^* is as determined by theorem 4.1.22.

Proof. Let $u = u_\delta$, $T^* = T$, $0 < h < \nu < T - \delta$ and $D_t^h u$ be the discrete time derivative defined as

$$D_t^h u := \frac{u(t+h) - u(t)}{h}.$$

Then we get the equality

$$\int_0^{T-\nu} \langle D_t^h u, \phi\rangle_{L^2} dt = \int_0^{\delta-h} \langle D_t^h u, \phi\rangle_{L^2} dt + \int_{\delta-h}^{\delta} \langle D_t^h u, \phi\rangle_{L^2} dt + \int_{\delta}^{T-\nu} \langle D_t^h u, \phi\rangle_{L^2} dt \tag{4.48}$$

$$=: I_1^h + I_2^h + I_3^h,$$

for any $\phi \in L^2(0, T^*; H^1)$. Per definition (see (4.45)), u is constant on the interval $[0, \delta]$. $I_1^h = 0$ follows therefore immediately. Regarding the term I_2^h, we can rewrite it as

$$I_2^h = \int_{\delta-h}^{\delta} \langle D_t^h u, \phi\rangle_{L^2} dt = \frac{1}{h}\int_{\delta-h}^{\delta} \langle u(t+h) - u(t), \phi\rangle_{L^2} dt.$$

Because of the embedding $W^{1,2}(\delta, T-\nu; H^1, L^2) \hookrightarrow C(\delta, T-\nu, L^2)$ the integrand of the last integral is a continuous function in t. The fundamental theorem of calculus hence states that

$$\frac{1}{h}\int_{\delta-h}^{\delta} \langle u(t+h) - u(t), \phi\rangle_{L^2} dt \to \langle u(t) - u(t), \phi\rangle_{L^2} = 0$$

for $h \to 0$, or in other words $\lim_{h\to 0} I_2^h = 0$. From the proof of theorem 4.1.23 we already know that there exists a sequence h_k such that $D_t^{h_k} u$ converges weakly in $L^2(\delta, T-\nu; H^{-1})$ to $\partial_t u_{old}$, as $h_k \to 0$. Hence we get

$$\lim_{h_k \to 0} \int_0^{T-\nu} \langle D_t^{h_k} u, \phi\rangle_{L^2} dt = \int_{\delta}^{T-\nu} \langle \partial_t u_{old}, \phi\rangle_{H^{-1}} dt.$$

Meaning, the sequence $D_t^{h_k} u$ converges weakly to[8] $\chi_{[\delta,T)}\partial_t u_{old}$ because ϕ was arbitrary. But $D_t^{h_k} u$ converges also to $\partial_t u$ in the sense of distributions, therefore $\partial_t u = \chi_{[\delta,T)}\partial_t u_{old}$ and in particular $\partial_t u \in L^2(0, T-\nu; H^{-1})$ for every $\nu > 0$.

[8] Here we use the convention 'undefined'$\times$ $0 = 0$.

In order to obtain the estimate 4.47, and in the further course passing to the limit $\nu \to 0$, we remember the fact that u is a weak solution on $[\delta, T-\nu]$:

$$\int_{\delta}^{T-\nu} \langle \partial_t u, \phi \rangle_{H^{-1}} dt = - \int_{\delta}^{T-\nu} \langle \sigma(x,t) \nabla u, \nabla \phi \rangle_{L^2} dt + \int_{\delta}^{T-\nu} \langle \operatorname{div}\left(f(x,t) u \right), \phi \rangle_{L^2} dt. \tag{4.49}$$

Because the functions $\nabla f, f$ and σ are elements of $L^{\infty}((0,T) \times \mathbb{R}^d)$ and $u \in L^2(0, T-\nu; H^1)$, we can find, via Cauchy- Schwarz's inequality, a constant $C(f, \sigma)$ such that

$$\left| \int_{\delta}^{T-\nu} \langle \partial_t u, \phi \rangle_{H^{-1}} dt \right| \leq C(f,\sigma) \|u\|_{L^2(0,T-\nu,H^1)} \|\phi\|_{L^2(0,T-\nu,H^1)}.$$

This leads directly to the upper bound

$$\|\partial_t u\|_{L^2(0,T-\nu,H^{-1})} \leq C(f,\sigma) \|u\|_{L^2(0,T-\nu,H^1)} \tag{4.50}$$

where $C(f, \sigma)$ is independent of ν. In view of (4.46) u is even an element of $L^2(0,T,H^1)$, which in turn implies $\partial_t u \in L^2(0,T,H^{-1})$. Combining (4.50) (with $\nu \to 0$) with estimate (4.46) now completes the proof. □

Remark 4.1.25. As u_δ is an element of $W^{1,2}(0, T^*; H^1, L^2)$, we can also pass to the limit $\nu \to 0$ in equation (4.49), obtaining:

$$\int_{\delta}^{T^*} \langle \partial_t u_\delta, \phi \rangle_{H^{-1}} dt = - \int_{\delta}^{T^*} \langle \sigma(x,t) \nabla u_\delta, \nabla \phi \rangle_{L^2} dt + \int_{\delta}^{T^*} \langle \operatorname{div}\left(f(x,t) u_\delta \right), \phi \rangle_{L^2} dt, \tag{4.51}$$

for all $\phi \in L^2(0, T; H^1)$.

Due to (4.47) and (4.46), the family $\{u_\delta\}_{0<\delta<T^*}$ is uniformly bounded in the space $W^{1,2}(0, T^*; H^1, L^2)$. Hence we can find a sequence, say $\{u_n\}_{n \in \mathbb{N}}$ with $u_n = u_{\delta_n} \in \{u_\delta\}_{0<\delta<T^*}$, so that u_n converges weakly towards u in $L^2(0,T;H^1)$ and $\partial_t u_n$ converges weakly to $\partial_t u$ in $L^2(0,T;H^{-1})$. Additionally, we can assume[9] $\delta_n \to 0$, for $n \to \infty$.
Taking (4.51) into account, we get

$$I_0^n := \int_{0}^{T^*} \langle \partial_t u_n, \phi \rangle_{H^{-1}} dt = \int_{\delta_n}^{T^*} \langle \partial_t u_n, \phi \rangle_{H^{-1}} dt$$

$$= - \int_{\delta_n}^{T^*} \langle \sigma(x,t) \nabla u_n, \nabla \phi \rangle_{L^2} dt + \int_{\delta_n}^{T^*} \langle \operatorname{div}\left(f(x,t) u_n \right), \phi \rangle_{L^2} dt$$

$$= - \int_{0}^{T^*} \langle \sigma(x,t) \nabla u_n, \nabla \phi \rangle_{L^2} dt + \int_{0}^{T^*} \langle \operatorname{div}\left(f(x,t) u_n \right), \phi \rangle_{L^2} dt$$

[9]Take e.g. a weakly convergent subsequence of $u_{\frac{1}{n}}$.

$$+\int_0^{\delta_n} \langle \sigma(x,t)\nabla u_n, \nabla\phi\rangle_{L^2} dt - \int_0^{\delta_n} \langle \operatorname{div}\left(f(x,t)u_n\right), \phi\rangle_{L^2} dt$$

$$:= I_1^n + I_2^n + I_3^n + I_4^n.$$

Because u_n are uniformly bounded in $L^2(0,T;H^1)$ and $\delta_n \to 0$, the terms I_3^n and I_4^n converge towards 0 as n rises to infinity. In the case of the remaining three terms the weak convergence is enough to pass to the limit. We therefore have

$$\int_0^{T^*} \langle \partial_t u, \phi\rangle_{H^{-1}} dt = -\int_0^{T^*} \langle \sigma(x,t)\nabla u, \nabla\phi\rangle_{L^2} dt + \int_0^{T^*} \langle \operatorname{div}\left(f(x,t)u\right), \phi\rangle_{L^2} dt, \tag{4.52}$$

for any $\phi \in L^2(0,T;H^1)$. Due to the continuous embedding $W^{1,2}(0,T^*;H^1,L^2) \hookrightarrow C([0,T^*],L^2)$ we also have $u \in C([0,T^*],L^2)$.

The last thing we need to show in order to prove that u is a weak solution is the fulfilment of the initial condition $u_0 = u_0$. For this purpose we make use of the identity

$$\int_a^b \langle \partial_t v, \phi\rangle_{H^{-1}} dt = \langle v(b), \phi(b)\rangle_{L^2} - \langle v(a), \phi(a)\rangle_{L^2} - \int_a^b \langle v, \partial_t\phi\rangle_{L^2} dt \tag{4.53}$$

which holds for every pair v, ϕ, where $v \in W^{1,2}(a,b;H^1,L^2)$ and $\phi \in \mathcal{D}([a,b]\times\mathbb{R}^d)$. Let therefore be $\phi \in \mathcal{D}([0,T^*)\times\mathbb{R}^d)$. Then we get

$$\int_0^{T^*} \langle \partial_t u, \phi\rangle_{H^{-1}} dt = -\langle u(0), \phi(0)\rangle_{L^2} - \int_0^{T^*} \langle u, \partial_t\phi\rangle_{L^2} dt = \lim_{n\to\infty}\left(-\langle u_n(0), \phi(0)\rangle_{L^2} - \int_0^{T^*} \langle u_n, \partial_t\phi\rangle_{L^2} dt\right)$$

$$= \lim_{n\to\infty} \int_0^{T^*} \langle \partial_t u_n, \phi\rangle_{H^{-1}} dt.$$

Due to $u_n(0) = V(\delta_n, 0)u_0 \to_{L^2} u_0$ and

$$\lim_{n\to\infty} \int_0^{T^*} \langle u_n, \partial_t\phi\rangle_{L^2} dt = \int_0^{T^*} \langle u, \partial_t\phi\rangle_{L^2} dt$$

this leaves us with

$$\langle u(0), \phi(0)\rangle_{L^2} = \langle u_0, \phi(0)\rangle_{L^2}.$$

Because the last equality holds for all $\mathcal{D}([0,T^*)\times\mathbb{R}^d$, it holds in particular for test- functions of the form $\phi(x,t) = \zeta(x)\cdot\eta(t)$, where $\zeta \in \mathcal{D}(\mathbb{R}^d)$, $\eta \in \mathcal{D}([0,T^*))$ and $\eta(0) \neq 0$. Hence we get

$$\langle u(0), \zeta(0)\rangle_{L^2} = \langle u_0, \zeta(0)\rangle_{L^2}$$

for all $\zeta \in \mathcal{D}(\mathbb{R}^d)$. The fundamental lemma of calculus of variations now implies $u(0) = u_0$ almost everywhere.

We therefore have proven:

Theorem 4.1.26. *Let the assumptions of theorem 4.1.19 hold. Then there exists* $0 < T^* \leq T$ *and* $u \in W^{1,2}(0, T^*; H^1(\mathbb{R}^d), L^2(\mathbb{R}^d))$ *such that* u *is a weak solution of* (4.1) *on the interval* $[0, T^*]$.

Corollary 4.1.27. *Let the assumptions of theorem 4.1.19 hold. Then there exists a weak solution of* (4.1)*.*

Proof. The determining factor for how large T^* can be, is theorem 4.1.22. A closer inspection of the proof shows that T^* can be chosen independently of the initial condition[10] u_0 and the point of time t_0 from where we start from, as long as $t_0 \in [0, T]$. In other words for every $t_0 \in [0, T]$ and every initial condition $u_{t_0} \in L^2$, we can find a weak solution which is well defined on the interval $[t_0, \min\{t_0 + T^*, T\}]$, see theorem 4.1.26 and theorem 4.1.24.
Let now u be the weak solution on $[0, T^*]$. Because $u \in C(0, T^*; L^2)$, $u(T^*) \in L^2$ is well defined. But as we argued earlier, we can now use $u(T^*)$ as new initial condition at T^*, to extend our weak solution to the interval $[0, 2T^*]$. This step can now be repeated with $u(2T^*)$, etc., until we extended to the whole interval $[0, T]$. □

4.2. Solutions with higher Regularity for the Linear Advection-Diffusion Problem

In this chapter we are interested in solutions $u \in C(0, T; H^s) \cap L^2(0, T; H^{s+1})$ of the linear problem

$$\partial_t u = \operatorname{div}(\sigma(x,t) \cdot \nabla u) + \operatorname{div}(u \cdot f(x,t)) \quad \text{in } \mathbb{R}^d \times (0, T],$$

$$u|_{t=0} = u_0(x), \text{ in } \mathbb{R}^d$$

for $s \geq \frac{d+2}{2}$. This is partly due to the embedding $H^{s+1} \hookrightarrow C^2$, which holds true for s big enough, and will be essential to prove the results of the next chapter.
To find such a solution, we will need more regular coefficients. We therefore assume from now on $f \in L^\infty(0, T; W^{s+1,\infty}(\mathbb{R}^d, \mathbb{R}^d))$, as well as $u_0 \in H^s$. For σ we will concentrate on two cases:

i) σ depends only on the time ('space independent diffusion-matrix'), i.e. $\sigma \in L^\infty(0, T; \mathbb{R}^{d \times d})$. This case is in particular of interest, as it entails the important case of a σ which is constant.

ii) σ is an element of $C(0, T; H^s)$ and $\partial_t \sigma$ of $C(0, T; H^{s-2}) \cap L^2(0, T; H^{s-1})$.

4.2.1. Space independent diffusion-matrix

In this section we assume $\sigma(x,t)$ is independent of $x \in \mathbb{R}^d$, i.e. $\sigma(x,t) = \sigma(t)$. Furthermore, $\sigma(t)$ is uniformly coercive, symmetric and $f \in L^\infty(0, T; W^{s+1,\infty}) \cap L^\infty(0, T; H^s)$.
We can find a solution with the desired regularity in a similar fashion as in section 4.1.3.
We will therefore discuss the following key-steps:

1) Finding an evolution operator $U(s,t) : H^s \to H^s$ regarding the operator family $B(t)u := \operatorname{div}(\sigma(t)\nabla u) + cu, t \in [0, T]$, for which we restrict the domain to $D(B(t)) = H^{s+2}$. Here c is some positive constant.

[10] It still has to be an element of $L^2(\mathbb{R}^d)$.

2) Showing the existence of a 'mild' δ-solution $u_\delta \in L^\infty(\delta, T, H^{s+1})$, which satisfies

$$u_\delta(t) = V(t,0)u_0 + \int_\delta^t V(t,s)\widehat{g}(s)ds$$

for $0 < \delta \le t$ and $\widehat{g}(t) = \widehat{g}(x,t,u) = \operatorname{div}(f(x,t)u(x,t)) + cu(x,t)$ (see (4.30)).

3) Showing u_δ is a weak solution on the subinterval $[\delta, T]$.

4) Finding uniform estimates regarding u_δ and hence establishing the existence of a subsequence u_n converging weakly to some u in $W^{1,2}(0,T;H^1,L^2)$ and $L^2(0,T;H^{s+1})$.

5) Proving u is a weak solution.

Most of the proofs of section 4.1.3 only need minor adjustments. We will therefore only go into more detail if we deem it necessary and will content ourselves with only pointing out the main differences if the remaining part of the proof is analogue to the case of lower regularity (H^1).

1) Evolution operator

To show the existence of an evolution operator, we want to utilize theorem 4.1.12. We therefore need to show self- adjointness and the closure property of our operator $B(t)$, defined as in (4.24), in the Hilbert space H^s, regarding its corresponding scalar product and norm. Let now D^α be a differential operator (in x) with $|\alpha| \le s$. Then we have for[11] $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$:

$$\langle D^\alpha B(t)\phi, D^\alpha\psi\rangle_{L^2} = \langle D^\alpha - \operatorname{div}(\sigma(t)\nabla\phi), D^\alpha\psi\rangle_{L^2} + c\langle D^\alpha\phi, D^\alpha\psi\rangle_{L^2} \tag{4.54}$$

and

$$D^\alpha \operatorname{div}(\sigma(t)\nabla\phi) = D^\alpha\left(\sum_{i=1}^d \partial_{x_i}(\sigma(t)\nabla\phi)_i\right) = D^\alpha\left(\sum_{i=1}^d \partial_{x_i}\left(\sum_{j=1}^d \sigma(t)_{i,j}(\nabla\phi)_j\right)\right)$$

$$= \sum_{i=1}^d\sum_{j=1}^d \sigma(t)_{i,j}D^\alpha\partial_{x_i}(\nabla\phi)_j = \sum_{i=1}^d\sum_{j=1}^d \sigma(t)_{i,j}\partial_{x_i}D^\alpha(\nabla\phi)_j = \operatorname{div}(\sigma(t)D^\alpha\nabla\phi). \tag{4.55}$$

Combining the tow equations (4.54) and (4.55) gets us

$$\langle D^\alpha B(t)\phi, D^\alpha\psi\rangle_{L^2} = \langle -\operatorname{div}(\sigma(t)D^\alpha\nabla\phi), D^\alpha\psi\rangle_{L^2} + c\langle D^\alpha\phi, D^\alpha\psi\rangle_{L^2}$$

$$= \langle\sigma(t)D^\alpha\nabla\phi, D^\alpha\nabla\psi\rangle_{L^2} + c\langle D^\alpha\phi, D^\alpha\psi\rangle_{L^2} = \langle D^\alpha\nabla\phi, \sigma(t)D^\alpha\nabla\psi\rangle_{L^2} + c\langle D^\alpha\phi, D^\alpha\psi\rangle_{L^2}$$

$$= \langle D^\alpha\phi, B(t)D^\alpha\psi\rangle_{L^2} = \langle D^\alpha\phi, D^\alpha B(t)\psi\rangle_{L^2}.$$

Hence we have proven the identity

$$\langle D^\alpha B(t)\phi, D^\alpha\psi\rangle_{L^2} = \langle D^\alpha\phi, D^\alpha B(t)\psi\rangle_{L^2}. \tag{4.56}$$

[11] It suffices to show the following statements for elements of $\mathcal{D}(\mathbb{R}^d)$, as these are dense in H^{s+2} and we are dealing only with (H^{s+2}-) continuous functionals here.

If we now take the sum over all $|\alpha| \leq s$ in (4.56), we obtain the symmetry of $B(t)$ regarding the H^s- scalar product. Positivity can be shown by using the coercivity of $\sigma(t)$ and writing

$$\langle D^\alpha B(t)\phi, D^\alpha \phi\rangle_{L^2} = \langle D^\alpha \nabla\phi, \sigma(t) D^\alpha \nabla\phi\rangle_{L^2} + c\langle D^\alpha\phi, D^\alpha\phi\rangle_{L^2}$$

$$\geq \min\{\epsilon, c\}\left(\langle D^\alpha\nabla\phi, D^\alpha\nabla\phi\rangle_{L^2} + \langle D^\alpha\phi, D^\alpha\phi\rangle_{L^2}\right).$$

Again summing over all $|\alpha| \leq s$, we get

$$\langle B(t)\phi, \phi\rangle_{H^s} \geq \min\{\epsilon, c\}\|\phi\|^2_{H^{s+1}}.$$

To apply lemma 4.1.14 and lemma 4.1.13 and hence proving that $B(t)$ is selfadjoint, it is sufficient to show the closure property of $B(t)$.

Lemma 4.2.1. *There exists a constant $C > 0$, independent of t, such that*

$$\|u\|_{H^{s+2}} \leq C\|B(t)u\|_{H^s}, \quad \forall u \in H^{s+2}(\mathbb{R}^d, \mathbb{C}).$$

Furthermore, $B(t)$ is a closed operator on the Hilbert-space $H^s(\mathbb{R}^d, \mathbb{C})$.

Proof. Let $v \in H^{s+2}$ and α a (d-dimensional) multi-index with $|\alpha| \leq s$. Then $D^\alpha v \in H^2$ and we can apply theorem 4.1.17:

$$\|D^\alpha v\|_{H^2} \leq C\|B(t)D^\alpha v\|_{L^2} = C\|D^\alpha B(t)v\|_{L^2} \leq C\|B(t)v\|_{H^s}.$$

Because α was arbitrary for every multi-index γ, with $|\gamma| \leq s+2$, the inequality

$$\|D^\gamma v\|_{L^2} \leq C\|B(t)v\|_{H^s}$$

is satisfied. Hence there exists a constant, which we again name C, such that $C > 0$ and

$$\|v\|_{H^{s+2}} \leq C\|B(t)v\|_{H^s} \tag{4.57}$$

holds for every $v \in H^{s+2}$.

Let now $u_n \in H^{s+2}$, $n \in \mathbb{N}$. Furthermore let $u, w \in H^s$ satisfy the following conditions:

i) $u_n \to_{H^s} u$ for $n \to \infty$.

ii) $B(t)u_n \to_{H^s} w$ for $n \to \infty$.

To show the closure property of $B(t)$, we need to proof $B(t)u = w$ almost everywhere.
Due to the convergence, the sequences u_n and $B(t)u_n$ are bounded subsets of H^s. Therefore (4.57) implies

$$\|u_n\|_{H^{s+2}} \leq K, \quad \forall n \in \mathbb{N}$$

for a constant $K > 0$. So u_n converges weakly towards some $\widehat{u}$ in H^{s+2} and because of the uniqueness of limits, $\widehat{u} = u \in H^{s+2}$. Let now $\phi \in D(\mathbb{R}^d)$:

$$\langle B(t)u, \phi\rangle_{L^2} = \lim_{n\to\infty}\langle B(t)u_n, \phi\rangle_{L^2} = \langle w, \phi\rangle_{L^2}.$$

The first equality holds true because $u_n \rightharpoonup_{H^{s+2}} u$, and $\langle B(t)\psi, \phi\rangle_{L^2}$ is a continuous linear functional for $\psi \in H^{s+2}$. The second equality holds true by assumption ii). Hence $B(t)u = w$ almost everywhere, due to the fundamental- lemma of calculus of variations. □

We apply lemma 4.1.14 and 4.1.13 to $B(t)$. All in all we have shown:

Corollary 4.2.2. *Let the assumptions made at the beginning of section 4.2 hold. The operator* $B(t) : D(B(t)) = H^{s+2}(\mathbb{R}^d, \mathbb{C}) \subset H^s(\mathbb{R}^d, \mathbb{C}) \to H^s(\mathbb{R}, \mathbb{C}^d)$*, defined as (4.24), satisfies the following:*

i) $B(t)$ *is a densely defined, closed, linear operator and is selfadjoint.*

ii) $\sigma(B(t)) \subseteq (c, \infty)$.

We prove the existence of an evolution operator regarding $B(t),\ 0 \leq t \leq T$ on H^s.

Theorem 4.2.3. *Let the assumptions made at the beginning of section 4.2 hold. Then the operator family* $B(t),\ 0 \leq t \leq T$*, gives rise to an evolution operator on* $H^s(\mathbb{R}^d, \mathbb{C})$*, if the condition*

$$\left|y^T\sigma(t)y - y^T\sigma(s)y\right| \leq C|t-s|^\beta |y|^2, \quad \forall s,t \in [0,T], \forall y \in \mathbb{R}^m \tag{4.58}$$

is satisfied for some $C > 0$ *and* $\frac{2}{3} < \beta \leq 1$.

Proof. Let $b(t)[u,v] := \langle \sigma\nabla u, \nabla v\rangle_{H^s} + c\langle u, v\rangle_{H^s}$ for $u, v \in H^{s+1}(\mathbb{R}, \mathbb{C})$. Due to lemma 4.2.1 and as

$$\langle u, v\rangle_{H^s} = \sum_{|\alpha|\leq s} \langle D^\alpha u, D^\alpha v\rangle_{L^2}$$

we can reduce the problem to the L^2-case, i.e. we can apply the same methods and steps as in the proof of theorem (4.1.19). Applying and repeating these shows Re $b(t)$ is a closed sesquilinear- form with (time- independent) domain H^{s+1}. Furthermore, it satisfies

$$|b(t)[u,u] - b(s)[u,u]| \leq C|t-s|^\beta |\text{Re } b(s)[u,u]|.$$

Together with corollary 4.2.2 the prerequisites of theorem 4.1.12 are therefore satisfied. □

2) Mild solution

Let $V(t,s) := \text{Re } U(t,s)$, where $U(t,s)$ is the evolution operator regarding $B(t)$ (see theorem 4.2.3). For further traits of $V(t,s)$, look at remark 4.1.21, as these generalize to the Hilbert space H^s as well.

Theorem 4.2.4. *Let the assumptions of theorem 4.2.3 hold. Then there exists* $0 < T^* \leq T$ *such that for every* $\delta < T^*$ *and every* $u_0 \in H^s$ *there is an* $u_\delta \in \mathcal{X} = L^\infty(\delta, T^*; H^{s+1}(\mathbb{R}^d))$*, which has the representation*

$$u_\delta(t) = V(t,0)u_0 + \int_\delta^t V(t,s)\widehat{g}(s)ds \tag{4.59}$$

for $t \in [\delta, T^*]$.

Remark 4.2.5. Recall that $\widehat{g}(t) = \widehat{g}(x,t,u) = \text{div}\,(f(x,t)u(x,t)) + cu(x,t)$ for some $c > 0$.

Proof of theorem 4.2.4.
Let $v \in L^\infty(\delta, T; H^{s+1})$. We want to show $\|\widehat{g}(\cdot, t, v(\cdot, t))\|_{H^s} \leq K\|v\|_{L^\infty(\delta,T;H^{s+1})}$, for some constant $K > 0$. Then the proof of theorem 4.1.22 can be reused by simply replacing the spaces L^2 with H^s, H^1 with H^{s+1} and choosing[12] θ such that $\frac{s+1}{s+2} < \theta < 1$.

Let α be a multi- index with $|\alpha| \leq s$. Because the functions f and v are solely weakly differentiable up to order $s+1$, classical results of differential calculus, as for instance the product rule and the like, hold almost everywhere:

$$D^\alpha(\mathrm{div}(fv)) = \mathrm{div}(D^\alpha(fv)) = \mathrm{div}(D^\alpha f v + f D^\alpha v)$$

$$= \mathrm{div}(D^\alpha f)v + D^\alpha f \nabla v + \mathrm{div}(f) D^\alpha v + f\nabla D^\alpha v$$

$$= \sum_{i=1}^{d} (\partial_{x_i} D^\alpha f)v + D^\alpha f \nabla v + \sum_{i=1}^{d} (\partial_{x_i} f) D^\alpha v + f \nabla D^\alpha v, \quad a.e. \tag{4.60}$$

The identity (4.60) can be used to estimate the corresponding L^2- norm:

$$\|D^\alpha(\mathrm{div}(fv))\|_{L^2} \leq d\|f\|_{W^{s+1,\infty}}\|v\|_{L^2} + \|f\|_{W^{s,\infty}}\|v\|_{H^1} + d\|f\|_{W^{1,\infty}}\|v\|_{H^s} + \|f\|_{L^\infty}\|v\|_{H^{s+1}}$$

$$\leq d\|f\|_{L^\infty(0,T;W^{s+1,\infty})}\|v\|_{L^\infty(\delta,T;H^{s+1})}.$$

As α was arbitrary, there exists a constant $K > 0$ such that

$$\|\mathrm{div}(fv)\|_{H^s} \leq K\|f\|_{L^\infty(0,T;W^{s+1,\infty})}\|v\|_{L^\infty(\delta,T;H^{s+1})}$$

and therefore

$$\|\widehat{g}(s)\|_{H^s} = \|\mathrm{div}\,(f(x,s)u(x,s)) + cu(x,s)\|_{H^s} \leq \left(K\|f\|_{L^\infty(0,T;W^{s+1,\infty})} + c\right)\|v\|_{L^\infty(\delta,T;H^{s+1})},$$

where K does only depend on d and s. □

3) u_δ is a weak solution on $[\delta,\ T^*]$:

Let $u_\delta \in L^\infty(\delta, T^*; H^{s+1})$ satisfy representation (4.59) (see theorem 4.2.4). H^{s+1} is continuously embedded in H^1. Hence the integrand of the occurring Bochner-integral in (4.59) is also Bochner- integrable regarding H^1. Furthermore, $V(t, s)$ satisfies similar estimates regarding the L^2-norm, as shown in section 4.1.3 and the example below.

Example 4.2.6.

$$\|B(t)V(t,s)u\|_{L^2} \leq \|B(t)V(t,s)u\|_{H^s} \leq C\frac{1}{|t-s|}\|u\|_{H^s}, \quad u \in H^s.$$

As such the proof of theorem 4.1.23 only needs minor adjustments. In particular one has solely to replace L^2 with H^s at certain estimates. We therefore wont prove the next theorem as we deem it not necessary.

Theorem 4.2.7. *Let $u_\delta \in L^\infty(\delta, T; H^{s+1}(\mathbb{R}^d))$ be as in (4.59) and let the assumptions of theorem 4.2.3 hold. Then u_δ is a weak solution of (4.1) on the interval $[\delta, T-\nu]$ with the initial condition $u_\delta(\delta) = V(\delta, 0)u_0$ for every $0 < \nu < T$. In particular $u_\delta \in W^{1,2}(\delta, T-\nu; H^1(\mathbb{R}^d); L^2(\mathbb{R}^d))$.*

[12]See A.2.9 for the Gagliardo- Nirenberg inequality.

4) Uniform estimates

Let u_n be defined as

$$u_n := \begin{cases} u_{1/n}(t) & ,t \geq 1/n \\ V(1/n,0)u_0 & ,t < 1/n \end{cases} \tag{4.61}$$

where $u_{1/n}$ is as in theorem 4.2.4 with $\delta = 1/n$ for $n \in \mathbb{N}$. Theorem 4.1.24 and theorem 4.1.6 now imply the existence of constants $C, K > 0$ such that

$$\|\partial_t u_n\|_{L^2(0,T^*;H^{-1})} \leq C,$$

$$\|u(t)_n\|_{H^s}^2 + \epsilon\|\nabla u_n\|_{L^2(0,t;H^s)}^2 \leq \|V(1/n,0)u_0\|_{H^s}^2(\exp(Kt) - 1)$$

for $t \in [0, T^*)$. $V(t,0)u_0$ is continuous as a function from $[0,T]$ to H^s. Hence $\|V(t,0)u_0\|_{H^s}$ assumes a maximum on the compact set $[0,T]$. Therefore the $L^2(0,T^*,H^{-1})$- and $L^2(0,T^*,H^{s+1})$-norms of the series $\partial_t u_n$ and u_n, $n \in \mathbb{N}$ respectively, are uniformly bounded.

Hence there exists a sub-series $1/n_k$ and a function $u \in L^2(0,T^*,H^{-1}) \cap L^2(0,T^*,H^{s+1})$ for which hold:

i) $\partial_t u_{n_k} \rightharpoonup_{L^2(0,T^*,H^{-1})} \partial_t u$.

ii) $u_{n_k} \rightharpoonup_{L^2(0,T^*,H^{s+1})} u$.

In particular $u \in W^{1,2}(0,T;H^{s+1};H^s) \subseteq C(0,T;H^s)$, due to $H^{-1} \subseteq H^{-(s+1)}$.

5) u is a weak solution:

The function u is a weak solution of 4.1, as the calculations[13] we have done in the end of the previous chapter hold here as well. The weak solution can then be extended to the whole interval $[0,T]$ as in corollary 4.1.27. We therefore have shown:

Theorem 4.2.8. *Let $f \in L^\infty(0,T;W^{s+1,\infty}(\mathbb{R}^d,\mathbb{R}^d))$, $u_0 \in H^s(\mathbb{R}^d)$ and $\sigma \in L^\infty(0,T;\mathbb{R}^{d\times d})$. Furthermore let σ be uniformly coercive and satisfy*

$$\left|y^T\sigma(t)y - y^T\sigma(s)y\right| \leq C|t-s|^\beta|y|^2, \quad \forall s,t \in [0,T], \forall y \in \mathbb{R}^d \tag{4.62}$$

for some $C > 0$ and $\frac{2}{3} < \beta \leq 1$. Then there exists a weak solution $u \in C(0,T;H^s(\mathbb{R}^d)) \cap L^2(0,T;H^{s+1}(\mathbb{R}^d))$ of

$$\begin{aligned} \partial_t u - \operatorname{div}(\sigma(x,t)\cdot\nabla u) =& \operatorname{div}(u \cdot f(x,t)) \quad \text{in } \mathbb{R}^d \times (0,T], \\ u|_{t=0} =& u_0(x), \text{ in } \mathbb{R}^d. \end{aligned}$$

4.2.2. Space dependent diffusion-matrix

Let again $\sigma(x,t) \in L^\infty(0,T;W^{s,\infty}(\mathbb{R}^d,\mathbb{R}^{d\times d}))$ and let it additionally satisfy the following:

i) $\sigma(x,t)$ is symmetric for all $x \in \mathbb{R}^d$ and $t \in [0,T]$.

[13] See the proof of theorem 4.1.26.

ii) σ is uniformly coercive. This means, there exists a constant $\epsilon > 0$ such that $y^T \sigma(x,t) y \geq \epsilon |y|^2$ for all $x, y \in \mathbb{R}^d$ and $t \in [0,T]$.

iii) There exist constants $\beta, C > 0$ such that $\frac{2}{3} < \beta \leq 1$ and

$$\left| y^T \sigma(x,t) y - y^T \sigma(x,s) y \right| \leq C |t-s|^{\beta} |y|^2$$

for all $x, y \in \mathbb{R}^d$ and $s, t \in [0,T]$.

iv) $\sigma \in C(0,T; H^s)$ and $\partial_t \sigma \in C(0,T; H^{s-2}) \cap L^2(0,T; H^{s-1})$ for a fixed $s \in \mathbb{N}$ with $s \geq \frac{d+2}{2}$.

Remark 4.2.9. The points $i) - iii)$ are prerequisites of theorem 4.1.26 and corollary 4.1.27.

We now prove the next theorem via a 'bootstrapping'- argument:

Theorem 4.2.10. *Let $f \in L^\infty(0,T; W^{s,\infty}(\mathbb{R}^d, \mathbb{R}^d))$ and $u_0 \in H^s(\mathbb{R}^d)$. Then, under the above assumptions, there exists a weak solution $u \in C(0,T; H^s(\mathbb{R}^d)) \cap L^2(0,T; H^{s+1}(\mathbb{R}^d))$ of (4.1).*

Proof. Let $u \in W^{1,2}(0,T; H^1, L^2)$ be the weak solution of (4.1), whose existence is assured by corollary 4.1.27 and its uniqueness by theorem 4.1.5. Consider now the following linear initial value problem in v:

$$\partial_t v - \operatorname{div}(\sigma \nabla v) + \sum_{i,j} \frac{\partial}{\partial x_i} \sigma_{i,j} \frac{\partial}{\partial x_j} v = \sum_{i,j} \frac{\partial}{\partial x_i} \sigma_{i,j} \frac{\partial}{\partial x_j} u + \operatorname{div}\left(u \cdot f(x,t)\right), \quad (x,t) \in \mathbb{R}^d \times [0,T], \tag{4.63}$$

$$v \mid_{t=0} = u_0, \quad x \in \mathbb{R}^d. \tag{4.64}$$

Obviously u is also a weak solution to (4.63)-(4.64). For the next argument assume σ and v are smooth functions. Then

$$\operatorname{div}(\sigma \nabla v) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{d} \sigma_{i,j} \frac{\partial}{\partial x_j} v \right) = \sum_{i,j} \sigma_{i,j} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} v + \sum_{i,j} \frac{\partial}{\partial x_i} \sigma_{i,j} \frac{\partial}{\partial x_j} v. \tag{4.65}$$

Hence (4.63) can also be written as

$$\partial_t v - \sum_{i,j} \sigma_{i,j} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} v = \sum_{i,j} \frac{\partial}{\partial x_i} \sigma_{i,j} \frac{\partial}{\partial x_j} u + \operatorname{div}\left(u \cdot f(x,t)\right), \quad (x,t) \in \mathbb{R}^d \times [0,T]. \tag{4.66}$$

Under our assumptions, the right hand side of (4.66) is an an element of $L^2(0,T; L^2)$. Choosing $l = 1$, theorem D.2.1 guarantees the existence of $v \in C(0,T; H^1) \cap L^2(0,T; H^2)$ with $\partial_t v \in L^2(0,T; L^2)$, which solves (4.64)+(4.66). As such it also is a solution of (4.63)+(4.64). Because the weak solution of (4.63)+(4.64) is unique, see theorem (D.1.1), $u = v$ almost everywhere.
Therefore the right-hand side of (4.66) is even an element of $L^2(0,T; H^1)$. Which in turn, with theorem D.2.1 and the same arguments as above, leads to $u \in C(0,T; H^2) \cap L^2(0,T; H^3)$ and as such to a even more regular right hand side of (4.66). This argumentation can now be repeated until $u \in C(0,T; H^s) \cap L^2(0,T; H^{s+1})$. □

5. Existence for the Intermediate System (PDE I)

This chapter focuses on proving the existence of a weak solution u_η of the intermediate system (PDE I) for fixed η. This non-linear problem is dealt with by an application of Banach's fix-point theorem. The later parts of this chapter focus on discussing the extension to a global solution and the uniqueness of u_η.

5.1. The Non-Local Diffusion System

Let $s, n \in \mathbb{N}$ with $s > \frac{d}{2} + 2$ and $u^0 \in H^s(\mathbb{R}^d, \mathbb{R}^n)$. We turn to the System (PDE I) of non-linear equations:

$$\partial_t u_{\eta,i} - \operatorname{div}(\sigma(x,t)_i \nabla u_{\eta,i}) = \operatorname{div}\left(\sum_{j=1}^{n} u_{\eta,i} \nabla V_{ij}^{\eta} * u_{\eta,j} + \sum_{l=1}^{d} u_{\eta,i} \frac{\partial(\sigma_i)_{\cdot,l}}{\partial x_l}\right) \quad in\ \mathbb{R}^d \times [0,\infty),$$
$$u_{\eta,i}\,|_{t=0} = u_i^0, \quad in\ \mathbb{R}^d,\ i \in \{1,2,\dots,n\}.$$

Here σ_i satisfies the conditions for either theorem 4.2.4 or theorem 4.2.10.[1] Both these theorems demand σ_i to be symmetric and uniform coercive; see section 1.1 for the definitions. Furthermore, we assume $\nabla \sigma_i$ to be an element of $L^2(0,T;H^s) \cap L^\infty(0,T;W^{s,\infty})$.
To simplify the notation, we set $g_{ij} := \frac{1}{n}\sum_{l=1}^{d} \frac{\partial}{\partial x_l}(\sigma_i)_{\cdot,l}$. Notice that g_{ij} is therefore an element of $L^2(0,T;H^s) \cap L^\infty(0,T;W^{s,\infty})$.

Remark 5.1.1. If $v_j \in L^\infty(0,T;H^s)$, one can show that $f_{ij} := \nabla V_{ij}^\eta * v_j \in L^\infty(0,T;W^{s+1,\infty}) \cap L^\infty(0,T;H^s)$:
Let $t \in (0,T)$ be arbitrary but fixed and $(\phi_k)_{k\in\mathbb{N}} \in D(\mathbb{R}^d, \mathbb{R}^n)$ be any sequence such that $\lim_{k\to\infty} \phi_k = v_j(t)$ in $H^1(\mathbb{R}^d;\mathbb{R}^n)$. Then we get

$$\frac{\partial}{\partial x_i} f_{ij}(x,t) = (\frac{\partial}{\partial x_i}\nabla V_{ij}^\eta) * v_j(t)[x] = \int_{\mathbb{R}^n} (\frac{\partial}{\partial x_i}\nabla V_{ij}^\eta)(x-y) v_j(y)dy$$

$$= \lim_{k\to\infty} \int_{\mathbb{R}^n} (\frac{\partial}{\partial x_i}\nabla V_{ij}^\eta)(x-y)\phi_k(y)dy$$

$$= \lim_{k\to\infty} \int_{\mathbb{R}^n} (-1)\frac{\partial}{\partial y_i}(\nabla V_{ij}^\eta(x-y))\phi_k(y)dy = \lim_{k\to\infty} \int_{\mathbb{R}^n} \nabla V_{ij}^\eta(x-y)\frac{\partial}{\partial y_i}\phi_k(y)dy$$

[1] Both these theorems deal with the existence of a weak solution $u \in C(0,T;H^s) \cap L^2(0,T;H^{s+1})$ for the corresponding linear problem.

$$= \int_{\mathbb{R}^n} \nabla V_{ij}^{\eta}(x-y)\frac{\partial}{\partial y_i} v_j(y)dy = \nabla V_{ij}^{\eta} * \frac{\partial v_j}{\partial x_i}(x).$$

The above argument can now iteratively be used to show $f_j(t) \in C^s$. Because $\nabla V_{ij} \in C_0^1$ one obtains even $f_j(t) \in C^{s+1}$. For the L^∞ bound we calculate for an arbitrary derivative D^α of order $|\alpha| \leq s$ and $\frac{\partial}{\partial x_i}$, the estimate

$$\left|\frac{\partial}{\partial x_i} D^\alpha(\nabla V_{ij}^{\eta} * v_j(t))[x]\right| \leq \int_{\mathbb{R}^n} \left|(\frac{\partial}{\partial x_i}\nabla V_{ij}^{\eta})(x-y)(D^\alpha v_j)(y)\right|dy \leq \left\|\frac{\partial}{\partial x_i}\nabla V_{ij}^{\eta}\right\|_{L^2} \|D^\alpha v_j\|_{L^2}$$

$$\leq \left\|\frac{\partial}{\partial x_i}\nabla V_{ij}^{\eta}\right\|_{L^2} \|v_j\|_{H^s}.$$

As $v_j \in L^\infty(0,T;H^s)$ and $V_{ij}^\eta \in C_b^2$, the last expression is uniformly bounded in $[0,T]$. Hence $f_{ij} \in L^\infty(0,T;W^{s+1,\infty})$. For $f_{ij} \in L^\infty(0,T;H^s)$ we apply Young's convolution inequality (see A.7.1).

5.1.1. Local existence and uniqueness

We are interested in finding a unique solution $u_\eta \in L^\infty(0;T,H^s) \cap L^2(0,T;H^{s+1})$ for system (PDE I) for fixed η. We will proceed as in [5]. Hence the next theorem deals with the local existence and is proven by an application of the Banach fixed-point theorem (see [5, Lemma 4]). The proof of uniqueness afterwards is based on the proof of [5, Lemma 6].

Remark 5.1.2. The higher regularity is necessary because u_η is used in the next chapter to approximate the solution u of (PDE II). This leads to u being an element of $L^\infty(0;T,H^s) \cap L^2(0,T;H^{s+1})$ as well. Due to the embedding $H^s \hookrightarrow C^2$, u then satisfies the Lipschitz condition (2.4), which is a necessary prerequisite for theorem 2.1.2, which deals with the existence regarding the stochastic system (SDE II).

Theorem 5.1.3 (Local existence)**.** *Let $s \in \mathbb{N}$ and $u^0 = (u_1^0, \ldots, u_n^0)$, where $u^0 \in H^s(\mathbb{R}^d, \mathbb{R}^n)$. If $s > \frac{d+2}{2}$, then there exists $u_\eta \in C([0,T^*], H^s(\mathbb{R}^d, \mathbb{R}^n)) \cap L^2(0,T^*;H^{s+1}(\mathbb{R}^d,\mathbb{R}^n))$, so that u_η is a weak solution of (PDE I). The corresponding time $T^* > 0$ depends on the initial value u^0 in such a way, that T^* is bounded from below for small values of $\|u^0\|_{H^s}$ and it does not depend on η. Additionally, there exists a constant $C > 0$ such that*

$$\|u_\eta(t)\|_{H^s}^2 + \epsilon\|\nabla u_\eta\|_{L^2(0,t;H^s)}^2 \leq \|u^0\|_{H^s}^2 + C\|u^0\|_{H^s}^2(\exp(Ct)-1),\ t \in [0,T^*], \tag{5.1}$$

$$\|u_\eta(t)\|_{H^s}^2 \leq 1 + \|u^0\|_{H^s}^2,\ t \in [0,T^*]. \tag{5.2}$$

Here C does not depend on η and ϵ is defined as $\epsilon = \min_{1\leq i\leq j} \epsilon_i$. In the case of $u^0 \geq 0$ almost everywhere, $u_\eta(t)$ is non-negative almost everywhere for all $t \in [0,T^*)$

Proof. Let $T^* > 0$ be arbitrary but fixed. Consider the space

$$\mathcal{X} := \{v \in L^\infty(0,T;H^s(\mathbb{R}^d,\mathbb{R}^n)) : \sup_{0<t<T^*} \|v(\cdot,t)\|_{H^s}^2 \leq M := 1 + \|u^0\|_{H^s}^2\}$$

which endowed with the norm $\|u-v\|_{\mathcal{X}} := \sup_{0<t<T^*} \|(u-v)(t)\|_{L^2}$, with $u, v \in \mathcal{X}$, can be shown to be a Banach- space.

Because the linearity of $\mathcal{X}$ is obvious, we only prove the completeness of $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$. Let therefore $v_n \in \mathcal{X},\ n \in \mathbb{N}$ be a arbitrary Cauchy- sequence regarding $\|\cdot\|_{\mathcal{X}}$. Hence for any choice of $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$\epsilon \geq \|v_n - v_m\|_{\mathcal{X}} = \sup_{0<t<T^*} \|(v_n - v_m)(t)\|_{L^2} \geq \|(v_n - v_m)(s)\|_{L^2}$$

for all $n, m \geq n_\epsilon$ and $s \in (0, T^*)$. Therefore $v_n(s)$ is in particular a Cauchy sequence in $L^2(\mathbb{R}^d, \mathbb{R}^n)$ and has as such a limit $v(s) \in L^2(\mathbb{R}^d, \mathbb{R}^n)$. Due to the uniform boundedness of $v_n(s),\ n \in \mathbb{N}$ regarding the H^s- norm, there exists a subsequence $v_{n_k}(s)$ which is weakly convergent in H^s. Because of the uniqueness of limits and the lower semicontinuity of the norm, $v(s) \in H^s(\mathbb{R}^d, \mathbb{R}^n)$ and $\|v\|_{H^s}^2 \leq M$. It is now straightforward to verify that $v(\cdot) \in \mathcal{X}$ and $\|v_n - v\|_{\mathcal{X}} \to 0$ for $n \to \infty$. As $v_n,\ n \in \mathbb{N}$ was an arbitrary Cauchy sequence, $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is therefore complete and hence a Banach space.

We introduce the operator $S : \mathcal{X} \to \mathcal{X}$, where $S(v) = u$ is the unique weak solution of the linear-advection problem

$$\partial_t u_i = \operatorname{div}(\sigma_i \nabla u_i) + \operatorname{div}\left(\sum_{j=1}^n u_i(\nabla V_{ij}^\eta * v_j + g_{ij})\right) \quad in\ \mathbb{R}^d \times [0, \infty), \tag{5.3}$$

$$u_i \mid_{t=0} = u_i^0, \quad in\ \mathbb{R}^d,\ i \in \{1, 2, \ldots, n\}.$$

Every fixed-point of S is a weak solution of our non-local diffusion system (PDE I). We show existence of such a point by applying Banach's fixed-point theorem. This involves the following steps:

1)Show $S : \mathcal{X} \to \mathcal{X}$ is well defined.

As $f_{ij} := \nabla V_{ij}^\eta * v_j + g_{ij} \in L^\infty(0, T; W^{s+1,\infty})$ (see remark 5.1.1) we can apply theorem 4.2.4 or theorem 4.2.10, i.e. $S(v) : \mathcal{X} :\to C(0, T; H^s)$ is well defined. Furthermore, theorem 4.1.3 guarantees the non-negativity of $S(v)$ as long as the initial condition $u_i^0 \geq 0$ for all $1 \leq i \leq n$. To show $S(v) \in \mathcal{X}$, we need to prove $\sup_{0<t<T^*} \|S(v)[\cdot, t]\|_{H^s}^2 \leq M$. Due to the fact that $S(v)_i = u_i$ is a weak solution of (4.1), we can make use of theorem 4.1.6, i.e.

$$\frac{d}{dt}\|u_i\|_{H^s}^2 + \epsilon\|\nabla u_i\|_{H^s}^2 \leq C\left(1 + \left(\left\|\sum_{j=1}^n f_{ij}\right\|_{L^\infty} + c\left\|D^s \sum_{j=1}^n f_{ij}\right\|_{L^2}\right)^2\right)\|u_i\|_{H^s}^2 \tag{5.4}$$

where $c, C > 0$ are constants depending only on $s, d, \frac{1}{\epsilon}$ and on $\|\sigma_i\|_{L^\infty(0,\infty;W^{s,\infty})}$. We sum the equations (5.4) from $i = 1, 2, \ldots, n$ and use the triangle inequality:

$$\frac{d}{dt}\|u\|_{H^s}^2 + \epsilon\|\nabla u\|_{H^s}^2 \leq C\|u\|_{H^s}^2 \sum_{i=1}^n \left(1 + \left(\sum_{j=1}^n \|f_{ij}\|_{L^\infty} + c\|D^s f_{ij}\|_{L^2}\right)^2\right).$$

The remaining terms which depend on f_{ij} can be further estimated by using Young's convolution inequality (see A.7.1) and Hölder's inequality:

$$\left\|D^s \nabla V_{ij}^\eta * v_j\right\|_{L^2} = \left\|\nabla V_{ij}^\eta * D^s v_j\right\|_{L^2} \leq \left\|\nabla V_{ij}^\eta\right\|_{L^1} \|D^s v\|_{L^2} \leq \left\|\nabla V_{ij}^\eta\right\|_{L^1} \|v\|_{H^s}$$

$$\left\|\nabla V_{ij}^\eta * v_j\right\|_{L^\infty} \leq \left\|\nabla V_{ij}^\eta\right\|_{L^1} \|v\|_{L^\infty} \leq c_{H^s \hookrightarrow L^\infty} \left\|\nabla V_{ij}^\eta\right\|_{L^1} \|v\|_{H^s}$$

where $c_{H^s \hookrightarrow L^\infty}$ is the operator norm of the continuous embedding $\iota : H^s \to L^\infty$. We thus have

$$\|D^s f_{ij}\|_{L^2} \leq \left\|\nabla V_{ij}^\eta\right\|_{L^1} \|v\|_{H^s} + \|g_{ij}\|_{L^\infty(0,T;H^s)},$$

$$\|f_{ij}\|_{L^\infty} \leq c_{H^s \hookrightarrow L^\infty} \left\|\nabla V_{ij}^\eta\right\|_{L^1} \|v\|_{H^s} + \|g_{ij}\|_{L^\infty(0,T;L^\infty)}.$$

Recalling $v \in \mathcal{X}$ and putting all the above results together, we get

$$\frac{d}{dt}\|u\|_{H^s}^2 + \epsilon \|\nabla u\|_{H^s}^2 \leq \tilde{C}\|u\|_{H^s}^2 (1 + \|v\|_{H^s}^2) \leq \tilde{C}(M+1)\|u\|_{H^s}^2 \tag{5.5}$$

or rather

$$\frac{d}{dt}\|u\|_{H^s}^2 \leq \tilde{C}(M+1)\|u\|_{H^s}^2 \tag{5.6}$$

where the constant $\tilde{C} > 0$ does not depend on T^*. Applying Gronwall's lemma to (5.6) leads to

$$\|u(t)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 e^{\tilde{C}(M+1)t} \leq (M-1)e^{\tilde{C}(M+1)T^*}, \quad t \in [0, T^*]. \tag{5.7}$$

We choose T^* small enough so that $(M-1)e^{\tilde{C}(M+1)T^*} \leq M$. Because (5.7) implies $\|u\|_{H^s}^2 \leq M$, $S(v) = u \in \mathcal{X}$. Notice that because $\left\|V_{ij}^\eta\right\|_{L^1} = \|V_{ij}\|_{L^1}$, T^* does not depend on η. Using the inequality (5.7) in estimate (5.5), integrating the result will lead to estimate (5.1).

2)Show $S : \mathcal{X} \to \mathcal{X}$ is a contraction.

Let $v, w \in \mathcal{X}$ and $\phi \in H^1$. Then $S(v)_i$ and $S(w)_i$ satisfy the following equations:

$$\langle \partial_t S(v)_i, \phi \rangle_{H^{-1}} + \langle \sigma_i \nabla S(v)_i, \nabla \phi \rangle_{L^2} = -\langle \sum_{j=1}^n S(v)_i (\nabla V_{ij}^\eta * v_j + g_{ij}), \nabla \phi \rangle_{L^2}, \tag{5.8}$$

$$\langle \partial_t S(w)_i, \phi \rangle_{H^{-1}} + \langle \sigma_i \nabla S(w)_i, \nabla \phi \rangle_{L^2} = -\langle \sum_{j=1}^n S(w)_i (\nabla V_{ij}^\eta * w_j + g_{ij}), \nabla \phi \rangle_{L^2}. \tag{5.9}$$

We choose $\phi = S(v)_i - S(w)_i$ and take the difference of (5.8) and (5.9):

$$\langle \partial_t (S(v)_i - S(w)_i), S(v)_i - S(w)_i \rangle_{H^{-1}} + \langle \sigma_i \nabla (S(v)_i - S(w)_i), \nabla (S(v)_i - S(w)_i) \rangle_{L^2}$$

$$= -\langle \sum_{j=1}^{n} S(v)_i \nabla V_{ij}^{\eta} * v_j, \nabla(S(v)_i - S(w)_i) \rangle_{L^2} - \langle \sum_{j=1}^{n} S(w)_i \nabla V_{ij}^{\eta} * w_j, \nabla(S(v)_i - S(w)_i) \rangle_{L^2}$$

$$= -\langle \sum_{j=1}^{n} S(v)_i \nabla V_{ij}^{\eta} * v_j, \nabla(S(v)_i - S(w)_i) \rangle_{L^2} \pm \langle \sum_{j=1}^{n} S(v)_i \nabla V_{ij}^{\eta} * w_j, \nabla(S(v)_i - S(w)_i) \rangle_{L^2}$$

$$+ \langle \sum_{j=1}^{n} S(w)_i \nabla V_{ij}^{\eta} * w_j, \nabla(S(v)_i - S(w)_i) \rangle_{L^2} - \langle \sum_{j=1}^{n} (S(v)_i - S(w)_i) g_{ij}, \nabla(S(v)_i - S(w)_i) \rangle_{L^2}$$

$$= -\langle \sum_{j=1}^{n} S(v)_i \nabla V_{ij}^{\eta} * (v_j - w_j), \nabla(S(v)_i - S(w)_i) \rangle_{L^2}$$

$$+ \langle \sum_{j=1}^{n} (S(w)_i - S(v)_i) \nabla V_{ij}^{\eta} * w_j, \nabla(S(v)_i - S(w)_i) \rangle_{L^2}$$

$$- \langle \sum_{j=1}^{n} (S(v)_i - S(w)_i) g_{ij}, \nabla(S(v)_i - S(w)_i) \rangle_{L^2}.$$

We use the coercivity of σ_i, the triangle inequality and the Cauchy- Schwarz inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\|S(v)_i - S(w)_i\|_{L^2}^2 + \epsilon_i \|\nabla(S(v)_i - S(w)_i)\|_{L^2}^2$$

$$\leq \sum_{j=1}^{n} \left\| S(v)_i \nabla V_{ij}^{\eta} * (v_j - w_j) \right\|_{L^2} \|\nabla(S(v)_i - S(w)_i)\|_{L^2}$$

$$+ \sum_{j=1}^{n} \left\| (S(w)_i - S(v)_i) \nabla V_{ij}^{\eta} * (w_j) \right\|_{L^2} \|\nabla(S(v)_i - S(w)_i)\|_{L^2}$$

$$+ \sum_{j=1}^{n} \|(S(v)_i - S(w)_i) g_{ij}\|_{L^2} \|\nabla(S(v)_i - S(w)_i)\|_{L^2}.$$

With Young's inequality for products ($\delta > 0$), we separate the terms on the right-hand side

$$\frac{1}{2}\frac{d}{dt}\|S(v)_i - S(w)_i\|_{L^2}^2 + \epsilon_i \|\nabla(S(v)_i - S(w)_i)\|_{L^2}^2$$

$$\leq \sum_{j=1}^{n} \frac{1}{2\delta} \left\| S(v)_i \nabla V_{ij}^{\eta} * (v_j - w_j) \right\|_{L^2}^2 + \frac{\delta}{2} \|\nabla(S(v)_i - S(w)_i)\|_{L^2}^2$$

$$+\sum_{j=1}^{n}\frac{1}{2\delta}\left\|(S(w)_i-S(v)_i)\nabla V_{ij}^{\eta}*(w_j)\right\|_{L^2}^2+\frac{\delta}{2}\|\nabla(S(v)_i-S(w)_i)\|_{L^2}^2$$

$$+\sum_{j=1}^{n}\frac{1}{2\delta}\|(S(v)_i-S(w)_i)g_{ij}\|_{L^2}^2+\frac{\delta}{2}\|\nabla(S(v)_i-S(w)_i)\|_{L^2}^2.$$

If we choose $\delta=\frac{2\epsilon_i}{3}$, some of the resulting terms cancel each other out:

$$\frac{1}{2}\frac{d}{dt}\|S(v)_i-S(w)_i\|_{L^2}^2$$

$$\leq\sum_{j=1}^{n}C(\frac{1}{\epsilon_i})\left\|S(v)_i\nabla V_{ij}^{\eta}*(v_j-w_j)\right\|_{L^2}^2+\sum_{j=1}^{n}C(\frac{1}{\epsilon_i})\left\|(S(w)_i-S(v)_i)\nabla V_{ij}^{\eta}*(w_j)\right\|_{L^2}^2$$

$$+\sum_{j=1}^{n}C(\frac{1}{\epsilon_i})\|(S(v)_i-S(w)_i)g_{ij}\|_{L^2}^2.$$

Summing from $i=1,2,\ldots,n$, and making some rough estimates of the terms on the right hand side, yields

$$\frac{d}{dt}\|S(v)-S(w)\|_{L^2}^2\leq C(\frac{1}{\epsilon})\times$$

$$\times\sum_{i,j=1}^{n}\left(\|S(v)_i\|_{L^\infty}^2\left\|\nabla V_{ij}^{\eta}*(v_j-w_j)\right\|_{L^2}^2+\|S(w)_i-S(v)_i\|_{L^2}^2\left(\left\|\nabla V_{ij}^{\eta}*(w_j)\right\|_{L^\infty}^2+\|g_{ij}\|_{L^\infty}^2\right)\right) \tag{5.10}$$

where $\epsilon=\min_{i\in\{1,2,\ldots,n\}}\epsilon_i$. We apply Young's convolution inequality to (5.10) and we recall that $H^s\hookrightarrow L^\infty$:

$$\frac{d}{dt}\|S(v)-S(w)\|_{L^2}^2\leq C(\frac{1}{\epsilon},g)\times$$

$$\times\sum_{i,j=1}^{n}\left(\underbrace{\|S(v)_i\|_{H^s}^2}_{\leq M^2}\left\|\nabla V_{ij}^{\eta}\right\|_{L^1}^2\|(v_j-w_j)\|_{L^2}^2+\|S(w)_i-S(v)_i\|_{L^2}^2\left(1+\left\|\nabla V_{ij}^{\eta}\right\|_{L^1}^2\underbrace{\|(w_j)\|_{H^s}^2}_{\leq M^2}\right)\right).$$

All in all we have

$$\frac{d}{dt}\|S(v)-S(w)\|_{L^2}^2\leq\underbrace{nC(\frac{1}{\epsilon},g)\left(1+M^2\max_{1\leq i,j\leq n}\left\|\nabla V_{ij}^{\eta}\right\|_{L^1}^2\right)}_{:=C(M)}\left(\|v-w\|_{L^2}^2+\|S(w)-S(v)\|_{L^2}^2\right).$$

Notice here as well that because $\left\|V_{ij}^{\eta}\right\|_{L^1}=\|V_{ij}\|_{L^1}$, $C(M)$ does not depend on η. Due to the fact that $S(v)$ and $S(w)$ satisfy the same initial condition, i.e. $S(v)\mid_{t=0}\equiv S(w)\mid_{t=0}\equiv u^0$, integrating from 0 to $t<T^*$ results in

$$\|S(v)(t)-S(w)(t)\|_{L^2}^2\leq C(M)t\sup_{0<s<T^*}\|v(s)-w(s)\|_{L^2}^2+\int_0^t C(M)\|S(w)(s)-S(v)(s)\|_{L^2}^2ds.$$

Here we can apply the version of Gronwall's lemma for Borel measures, see C.3, which leads to:

$$\|S(v)(t) - S(w)(t)\|_{L^2}^2 \leq C(M)t \sup_{0<s<T^*} \|v(s) - w(s)\|_{L^2}^2 e^{C(M)t}$$

and hence to

$$\sup_{0<s<T^*} \|S(v)(s) - S(w)(s)\|_{L^2}^2 \leq C(M)T^* \sup_{0<s<T^*} \|v(s) - w(s)\|_{L^2}^2 e^{C(M)T^*}.$$

In terms of the $\mathcal{X}$-norm, the last inequality can also be written as

$$\|S(v) - S(w)\|_{\mathcal{X}} \leq C(M)T^* e^{C(M)T^*} \|v - w\|_{\mathcal{X}}.$$

We choose T^* small enough so that $C(M)T^* e^{C(M)T^*} < 1$. Then $S : \mathcal{X} \to \mathcal{X}$ is a contraction and we can apply Banach's fixed-point theorem to complete the proof. □

For initial value conditions u^0, with sufficiently small norms $\|u^0\|_{H^s}$, we can show uniqueness for the local solution u_η of (PDE I):

Lemma 5.1.4 (Uniqueness)**.** *Let there exist a constant $0 < \gamma < \epsilon$, such that*

$$\epsilon - c_{H^s \hookrightarrow L^\infty} \sum_{i,j=1}^{n} \left(\sqrt{1 + \|u_i^0\|_{H^s}^2} \right) \left\| V_{ij}^\eta \right\|_{L^1} \geq \gamma \tag{5.11}$$

where $c_{H^s \hookrightarrow L^\infty}$ is the operator norm of the embedding $\iota : H^s \hookrightarrow L^\infty$ and $\epsilon = \min_{1 \leq i \leq n} \epsilon_i$. Then the non-local solution u_η, characterized by theorem 5.1.3, is the unique solution of (PDE I) on $[0, T]$ for every $T \leq T^*$ in the space $L^\infty(0, T; H^s(\mathbb{R}^d, \mathbb{R}^n)) \cap L^2(0, T; H^{s+1}(\mathbb{R}^d, \mathbb{R}^n))$.

Proof. We proceed as in [5, Lemma 6]. Let $\phi \in L^2(0, T; H^1(\mathbb{R}^d, \mathbb{R}^n))$ be arbitrary but fixed. Furthermore let $u = u_\eta$ and let v be another solution of (PDE I), fulfilling the condition $v(0) = u^0$. Then u_i and v_i satisfy the equations

$$\langle \partial_t u_i, \phi_i \rangle_{H^{-1}} + \langle \sigma_i \nabla u_i, \nabla \phi_i \rangle_{L^2} = -\langle \sum_{j=1}^{n} u_i (V_{ij}^\eta * \nabla u_j + g_{ij}), \nabla \phi_i \rangle_{L^2},$$

$$\langle \partial_t v_i, \phi_i \rangle_{H^{-1}} + \langle \sigma_i \nabla v_i, \nabla \phi_i \rangle_{L^2} = -\langle \sum_{j=1}^{n} v_i (V_{ij}^\eta * \nabla v_j + g_{ij}), \nabla \phi_i \rangle_{L^2}.$$

We choose $\phi = u - v$, use the coercivity of σ_i and take the difference of the equations satisfied by u_i and v_i:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u_i - v_i\|_{L^2}^2 + \epsilon_i \|\nabla(u_i - v_i)\|_{L^2}^2 &\leq \left| \sum_{j=1}^{n} \langle u_i (V_{ij}^\eta * \nabla u_j + g_{ij}) - v_i (V_{ij}^\eta * \nabla v_j + g_{ij}), \nabla(u_i - v_i) \rangle_{L^2} \right| \\
&\leq \left| \sum_{j=1}^{n} \langle u_i V_{ij}^\eta * \nabla u_j - v_i V_{ij}^\eta * \nabla v_j, \nabla(u_i - v_i) \rangle_{L^2} \right| \\
&\quad + \left| \sum_{j=1}^{n} \langle (u_i - v_i) g_{ij}, \nabla(u_i - v_i) \rangle_{L^2} \right| \\
&=: I_1 + I_2.
\end{aligned}$$

The first term I_1 on the right hand side can be further estimated using the Cauchy- Schwarz inequality and Young's convolution inequality:

$$I_1 = \left| \sum_{j=1}^{n} \langle u_i V_{ij}^{\eta} * \nabla u_j - v_i V_{ij}^{\eta} * \nabla v_j, \nabla(u_i - v_i) \rangle_{L^2} \right|$$

$$= \left| \sum_{j=1}^{n} \langle u_i V_{ij}^{\eta} * \nabla u_j \pm u_i V_{ij}^{\eta} * \nabla v_j - v_i V_{ij}^{\eta} * \nabla v_j, \nabla(u_i - v_i) \rangle_{L^2} \right|$$

$$= \left| \sum_{j=1}^{n} \langle u_i V_{ij}^{\eta} * \nabla(u_j - v_j) - (v_i - u_i) V_{ij}^{\eta} * \nabla v_j, \nabla(u_i - v_i) \rangle_{L^2} \right|$$

$$\leq \sum_{j=1}^{n} \left(\left\| u_i V_{ij}^{\eta} * \nabla(u_j - v_j) \right\|_{L^2} + \left\| (v_i - u_i) V_{ij}^{\eta} * \nabla v_j \right\|_{L^2} \right) \|\nabla(u_i - v_i)\|_{L^2}$$

$$\leq \sum_{j=1}^{n} \left(\|u_i\|_{L^\infty} \left\| V_{ij}^{\eta} * \nabla(u_j - v_j) \right\|_{L^2} + \left\| V_{ij}^{\eta} * \nabla v_j \right\|_{L^\infty} \|v_i - u_i\|_{L^2} \right) \|\nabla(u_i - v_i)\|_{L^2}.$$

$$\leq \sum_{j=1}^{n} \left(\|u_i\|_{L^\infty} \left\| V_{ij}^{\eta} \right\|_{L^1} \|\nabla(u_j - v_j)\|_{L^2} + \left\| V_{ij}^{\eta} \right\|_{L^1} \|\nabla v_j\|_{L^\infty} \|v_i - u_i\|_{L^2} \right) \|\nabla(u_i - v_i)\|_{L^2}$$

$$\leq c_{H^s \hookrightarrow L^\infty} \sum_{j=1}^{n} \left\| V_{ij}^{\eta} \right\|_{L^1} (\|u\|_{H^s} \|\nabla(u - v)\|_{L^2} + \|\nabla v\|_{H^s} \|v - u\|_{L^2}) \|\nabla(u - v)\|_{L^2}.$$

To the remaining term I_2 we apply the Cauchy-Schwarz inequality:

$$I_2 = \left| \sum_{j=1}^{n} \langle (u_i - v_i) g_{ij}, \nabla(u_i - v_i) \rangle_{L^2} \right| \leq \sum_{j=1}^{n} \|g_{ij}\|_{L^\infty} \|u_i - v_i\|_{L^2} \|\nabla(u_i - v_i)\|_{L^2}.$$

Summing over the index i, where $1 \leq i \leq n$, yields therefore

$$\frac{1}{2} \frac{d}{dt} \|u - v\|_{L^2}^2 + \epsilon \|\nabla(u - v)\|_{L^2}^2$$

$$\leq c_{H^s \hookrightarrow L^\infty} \sum_{i,j=1}^{n} \left\| V_{ij}^{\eta} \right\|_{L^1} \left(\|u\|_{H^s} \|\nabla(u - v)\|_{L^2} + \left(\|\nabla v\|_{H^s} + \|g_{ij}\|_{L^\infty} \right) \|v - u\|_{L^2} \right) \|\nabla(u - v)\|_{L^2}$$

$$= c_{H^s \hookrightarrow L^\infty} \|\nabla(u - v)\|_{L^2}^2 \sum_{i,j=1}^{n} \left\| V_{ij}^{\eta} \right\|_{L^1} \|u\|_{H^s}$$

$$+ c_{H^s \hookrightarrow L^\infty} \left(\|\nabla v\|_{H^s} + \max_{1 \leq i,j \leq n} \|g_{ij}\|_{L^\infty} \right) \|v - u\|_{L^2} \|\nabla(u - v)\|_{L^2} \sum_{i,j=1}^{n} \left\| V_{ij}^{\eta} \right\|_{L^1}$$

where $\epsilon = \min_{1\le i\le n} \epsilon_i$. We set $\max_{1\le i,j\le n} \|g_{ij}\|_{L^\infty} := G$. Due to theorem 5.1.3, $\|u(t)\|_{H^s} \le \sqrt{1+\|u^0_\eta\|^2_{H^s}}$ for all $t\in[0,T^*]$. In particular, by assumption 5.11, the following inequality holds true:

$$\frac{\epsilon-\gamma}{c_{H^s\hookrightarrow L^\infty}} \ge \sum_{i,j=1}^n \left\|V^\eta_{ij}\right\|_{L^1}\sqrt{1+\|u^0\|^2_{H^s}} \ge \sum_{i,j=1}^n \left\|V^\eta_{ij}\right\|_{L^1}\|u(t)\|_{H^s}.$$

This, together with an application of Young's inequality for products, results in

$$\frac{1}{2}\frac{d}{dt}\|u-v\|^2_{L^2} + \epsilon\|\nabla(u-v)\|^2_{L^2}$$

$$\le (\epsilon-\gamma)\|\nabla(u-v)\|^2_{L^2} + \delta\|\nabla(u-v)\|^2_{L^2} + C(\delta,s,d)\left(\|\nabla v\|_{H^s}+G\right)^2\|v-u\|^2_{L^2}\left(\sum_{i,j=1}^n\left\|V^\eta_{ij}\right\|_{L^1}\right)^2$$

where $\delta>0$ is arbitrary. We choose δ such that $\delta-\gamma\le 0$. Then we have

$$\frac{1}{2}\frac{d}{dt}\|u-v\|^2_{L^2} + \epsilon\|\nabla(u-v)\|^2_{L^2} \le \epsilon\|\nabla(u-v)\|^2_{L^2} + C(\delta,s,d,\eta)\left(\|\nabla v\|_{H^s}+G\right)^2\|v-u\|^2_{L^2}$$

and thus

$$\frac{1}{2}\frac{d}{dt}\|u-v\|^2_{L^2} \le C(\delta,s,d,\eta)\left(\|\nabla v\|_{H^s}+G\right)^2\|v-u\|^2_{L^2}.$$

As $v\in L^2(0,T;H^{s+1})$ and $g_{ij}\in L^\infty(0,T;W^{s,\infty})$, the right-hand side of the last inequality is integrable and we can apply Gronwall's inequality, see C.3. We therefore get

$$\|(u-v)(t)\|^2_{L^2}\le 0$$

for $t\in[0,T]$. Hence $u(t)=v(t)$ almost everywhere for $t\in[0,T]$. □

5.1.2. Global solution

Let $0<T<T^*$ where T^* is as in theorem 5.1.3. To extend a local solution whose existence is guaranteed by theorem 5.1.3 to a global one, we need to make sure that no blow-up occurs. In particular if for a solution u_η of (PDE I), its norm $\|u_\eta(t)\|_{H^s}$ is monotonically non- increasing on $[0,T]$, then $\left\|u^0_\eta\right\|_{H^s} \ge \|u_\eta(T)\|_{H^s}$ and we can use theorem 5.1.3 to extend u_η to $[0,2T]$ by using $u_\eta(T)$ as new initial value condition. We will show that under the right circumstances this argument can be repeated indefinitely, which leads to a global solution.

Let $s>\frac{d+2}{2}$, $\epsilon=\min_{1\le i\le n}\epsilon_i$ and $0<\gamma<\epsilon$ for some $\gamma\in\mathbb{R}$. Going forward we assume that the initial condition $u^0\in H^s$ satisfies

$$\epsilon - C\sum_{i=1}^n\sum_{1\le|\gamma|\le s}\|D^\gamma\sigma_i\|_{L^\infty(0,T;L^\infty)} - K\sum_{i,j=1}^n\left\|u^0_i\right\|_{H^s}\|V_{ij}\|_{L^1}\ge\gamma \tag{5.12}$$

for a specific constants $C>0$ and $K\ge c_{H^s\hookrightarrow L^\infty}$, which does only depend on s and d.[2] The constant $c_{H^s\hookrightarrow L^\infty}$ denotes the operator norm of the embedding $\iota: H^s\hookrightarrow L^\infty$.

[2] C and K will be determined in the proof of lemma 5.1.7.

Remark 5.1.5. Recall that $\|V_{ij}\|_{L^1} = \left\|V_{ij}^\eta\right\|_{L^1}$.

Remark 5.1.6. Assumption (6.11) from theorem 5.1.4, can be directly concluded from (5.12).

Lemma 5.1.7. *Let $s \in \mathbb{N}$, with $s > \frac{d+2}{2}$ and $u^0 \in H^s(\mathbb{R}^d, \mathbb{R}^n)$. Furthermore let assumption 5.12 hold and $g_{ij} \equiv 0$ for $1 \le i, j \le n$. Then for a solution $u_\eta \in C(0,T; H^s(\mathbb{R}^d, \mathbb{R}^n)) \cap L^2(0,T; H^{s+1}(\mathbb{R}^d, \mathbb{R}^n))$ of (PDE I) the following estimate holds:*

$$\|u_\eta(t)\|_{H^s}^2 + 2\gamma \|\nabla u_\eta\|_{L^2(0,t;H^s)}^2 \le \left\|u^0\right\|_{H^s}^2, \ t \in [0,T]. \tag{5.13}$$

Furthermore, $\|u_{\eta,i}(t)\|_{H^s}$ is monotonically decreasing for all $1 \le i \le n$.

Proof. Let $u = u_\eta$. As u_i is a weak solution of 4.1, with $f = \sum_{j=1}^n \nabla V_{ij}^\eta * u_j$, it fulfils equation[3] (4.9):

$$\langle \partial_t D^\alpha u_i, D^\alpha u_i \rangle_{H^{-1}} + \int_{\mathbb{R}^d} D^\alpha \nabla u_i^T \sigma_i D^\alpha \nabla u_i dx = -\int_{\mathbb{R}^d} \left(\sum_{|\beta| \le |\alpha|-1, |\gamma| \le |\alpha|} c_{\beta,\gamma} D^\gamma \sigma_i D^\beta \nabla u_i \right) \cdot \nabla D^\alpha u_i dx$$

$$- \int_{\mathbb{R}^d} D^\alpha \left(\sum_{j=1}^n u_i \nabla V_{ij}^\eta * u_j \right) \cdot \nabla D^\alpha u_i dx,$$

where $c_{\beta,\gamma} \in \{0,1\}$, $|\alpha| \le s$ and we chose $\psi = D^\alpha u_i$. Applying the Cauchy-Schwarz inequality, the triangle inequality and using the coercivity of σ_i leads to

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha u_i\|_{L^2}^2 + \epsilon_i \|D^\alpha \nabla u_i\|_{L^2}^2 \le \left\| \sum_{|\beta| \le |\alpha|-1, 1 \le |\gamma| \le |\alpha|} c_{\beta,\gamma} D^\gamma \sigma_i D^\beta \nabla u_i \right\|_{L^2} \|D^\alpha \nabla u_i\|_{L^2}$$

$$+ \sum_{j=1}^n \left\| D^\alpha \left(u_i \nabla V_{ij}^\eta * u_j \right) \right\|_{L^2} \|\nabla D^\alpha u_i\|_{L^2},$$

and due to estimate 4.10, to

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha u_i\|_{L^2}^2 + \epsilon_i \|D^\alpha \nabla u_i\|_{L^2}^2 \le \sum_{|\beta| \le |\alpha|-1, 1 \le |\gamma| \le |\alpha|} \|D^\gamma \sigma_i\|_{L^\infty} \left\| D^\beta \nabla u_i \right\|_{L^2} \|D^\alpha \nabla u_i\|_{L^2}$$

$$+ \sum_{j=1}^n \left\| D^\alpha (u_i \underbrace{\nabla V_{ij}^\eta * u_j}_{= V_{ij}^\eta * \nabla u_j}) \right\|_{L^2} \|\nabla D^\alpha u_i\|_{L^2}.$$

[3] See the proof of theorem 4.1.6

We apply the Moser-type calculus inequality and Young's convolution inequality to the term on the right-hand side:

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha u_i\|_{L^2}^2 + \epsilon_i\|D^\alpha \nabla u_i\|_{L^2}^2 \le \sum_{|\beta|\le|\alpha|-1, 1\le|\gamma|\le|\alpha|} \|D^\gamma \sigma_i\|_{L^\infty}\|\nabla u_i\|_{H^s}\|D^\alpha \nabla u_i\|_{L^2}$$

$$+K\sum_{j=1}^n \left(\|D^s u_i\|_{L^2}\left\|V_{ij}^\eta * \nabla u_j\right\|_{L^\infty} + \|u_i\|_{L^\infty}\left\|V_{ij}^\eta * D^s\nabla u_j\right\|_{L^2}\right)\|\nabla D^\alpha u_i\|_{L^2}$$

$$\le \sum_{|\beta|\le|\alpha|-1, 1\le|\gamma|\le|\alpha|} \|D^\gamma \sigma_i\|_{L^\infty}\|\nabla u_i\|_{H^s}\|D^\alpha \nabla u_i\|_{L^2}$$

$$+K\sum_{j=1}^n \left(\|D^s u_i\|_{L^2}\left\|V_{ij}^\eta\right\|_{L^1}\|\nabla u_j\|_{L^\infty} + \|u_i\|_{L^\infty}\left\|V_{ij}^\eta\right\|_{L^1}\|D^s\nabla u_j\|_{L^2}\right)\|D^\alpha \nabla u_i\|_{L^2}$$

$$\le C(|\alpha|)\sum_{1\le|\gamma|\le|\alpha|} \|D^\gamma \sigma_i\|_{L^\infty}\|\nabla u_i\|_{H^s}\|D^\alpha \nabla u_i\|_{L^2}$$

$$+K(s,d)\sum_{j=1}^n \left(\|D^s u_i\|_{L^2}\left\|V_{ij}^\eta\right\|_{L^1}\|\nabla u_j\|_{H^s} + \|u_i\|_{H^s}\left\|V_{ij}^\eta\right\|_{L^1}\|D^s\nabla u_j\|_{L^2}\right)\|D^\alpha \nabla u_i\|_{L^2}.$$

Summing over all α, with $|\alpha|\le s$, yields

$$\frac{1}{2}\frac{d}{dt}\|u_i\|_{H^s}^2 + \epsilon_i\|\nabla u_i\|_{H^s}^2$$

$$\le \|\nabla u_i\|_{H^s}^2\left(\tilde{C}(s)\sum_{1\le|\gamma|\le s}\|D^\gamma\sigma_i\|_{L^\infty(0,T;L^\infty)} + \tilde{K}(s,d)\|u_i\|_{H^s}\sum_{j=1}^n\left\|V_{ij}^\eta\right\|_{L^1}\right)$$

and after rearranging some terms we get

$$\frac{1}{2}\frac{d}{dt}\|u_i\|_{H^s}^2 \le -\|\nabla u_i\|_{H^s}^2\left(\epsilon_i - \tilde{C}(s)\sum_{1\le|\gamma|\le s}\|D^\gamma\sigma_i\|_{L^\infty(0,T;L^\infty)} - \tilde{K}(s,d)\|u_i\|_{H^s}\sum_{j=1}^n\left\|V_{ij}^\eta\right\|_{L^1}\right). \tag{5.14}$$

Under assumption (5.12) we have

$$\frac{a}{b} := \frac{\epsilon_i - \tilde{C}(s)\sum_{1\le|\gamma|\le s}\|D^\gamma\sigma_i\|_{L^\infty(0,T;L^\infty)}}{\tilde{K}(s,d)\sum_{j=1}^n\left\|V_{ij}^\eta\right\|_{L^1}} > \frac{\epsilon_i - \tilde{C}(s)\sum_{1\le|\gamma|\le s}\|D^\gamma\sigma_i\|_{L^\infty(0,T;L^\infty)} - \gamma}{\tilde{K}(s,d)\sum_{j=1}^n\left\|V_{ij}^\eta\right\|_{L^1}} \ge \|u_i^0\|_{H^s}.$$

In particular $\left\|u_i^0\right\|_{H^s}^2 \leq \left(\frac{a}{b}\right)^2$ holds true. We choose a, b as above $g = -\|\nabla u_i\|_{H^s}^2$ and $f = \|u_i\|_{H^s}^2$ in the Gronwall-type inequality (see E.0.2). This implies $\|u_i\|_{H^s} \leq \frac{a}{b}$ and hence together with (5.14):

$$\frac{d}{dt}\|u_i\|_{H^s}^2 \leq 0$$

almost everywhere. We conclude $\|u_i(t)\|_{H^s} \leq \left\|u_i^0\right\|_{H^s}$ for all $t \in [0,T]$, and therefore

$$\frac{1}{2}\frac{d}{dt}\|u_i\|_{H^s}^2 \leq -\|\nabla u_i\|_{H^s}^2 \left(\epsilon_i - \tilde{C}(s) \sum_{1\leq|\gamma|\leq s} \|D^\gamma \sigma_i\|_{L^\infty(0,T;L^\infty)} - \tilde{K}(s,d)\|u_i^0\|_{H^s} \sum_{j=1}^{n} \left\|V_{ij}^\eta\right\|_{L^1}\right)$$

$$\leq -\|\nabla u_i\|_{H^s}^2 \gamma.$$

We sum over all $1 \leq i \leq n$ and integrate both sides:

$$\|u(t)\|_{H^s}^2 - \left\|u^0\right\|_{H^s}^2 \leq -2\gamma\|\nabla u\|_{L^2(0,t,H^s)}^2.$$

This completes the proof. □

Corollary 5.1.8. *Let the assumption of theorem 5.1.3 and lemma 5.1.7 hold. Furthermore, let all the assumptions made for σ_i hold for $T = \infty$. Then there exists a global unique solution $u_\eta \in C([0,\infty); H^s(\mathbb{R}^d,\mathbb{R}^n))$ with $\nabla u_\eta \in L^2(0,\infty; H^s(\mathbb{R}^d,\mathbb{R}^n))$ of the intermediate system (PDE I).*

Proof. Due to theorem 5.1.3, there exists $T > 0$ and $u_\eta \in C([0,T], H^s) \cap L^2(0,T,H^{s+1})$ such that u_η is a solution of (PDE I). The term $\|u_\eta\|_{H^s}$ is monotonically decreasing (see lemma 5.1.7). In particular $\|u_\eta(T)\|_{H^s} \leq \left\|u_\eta^0\right\|_{H^s}$. As the time T does depend on the initial value in such a way that it is increasing for smaller values of $\left\|u_\eta^0\right\|_{H^s}$, we can therefore find a solution of (PDE I) on the interval $[T, 2T]$ with the help of theorem 5.1.3. Hence we can extend u_η to the interval $[0, 2T]$. As $\|u_\eta(T)\|_{H^s} \leq \left\|u_\eta^0\right\|_{H^s}$, condition 5.12 is satisfied and we can again use lemma 5.1.7 and theorem 5.1.3 to extend to $[0, 3T]$. This argumentation can be repeated until we arrive at $u_\eta \in C([0,\infty); H^s) \cap L^2(0,t;H^{s+1})$ for all $t > 0$. As in particular $u_\eta \in C(0,t;H^s) \cap L^2(0,t;H^{s+1})$, $t > 0$, estimate (5.13) holds:

$$\|u_\eta(t)\|_{H^s}^2 + 2\gamma\|\nabla u_\eta\|_{L^2(0,t;H^s)}^2 \leq \left\|u_\eta^0\right\|_{H^s}^2, \ t > 0.$$

Using the monotone convergence theorem and performing the limit $t \to \infty$, shows $\nabla u \in L^2(0,\infty;H^s)$. □

6. Existence for the Cross-Diffusion System (PDE II)

In this chapter the existence of a solution u of the cross-diffusion system (PDE II) is established. This is accomplished by approximating u with the solutions of the intermediate systems u_η, which can be shown to converge towards a solution of (PDE II) for $\eta \to 0$. The latter part of this chapter concentrates on estimates of the difference $u - u_\eta$.

6.1. Existence and Uniqueness

Let the assumptions of the previous chapter hold. We focus on the cross- diffusion system (PDE II):

$$\partial_t u_i - \operatorname{div}\left(\sigma(x,t)_i \nabla u_i\right) = \operatorname{div}\left(\sum_{j=1}^n u_i \left(a_{ij}\nabla u_j + g_{ij}\right)\right) \quad \text{in } \mathbb{R}^d \times [0,T],$$

$$u_{\eta,i}\,|_{t=0} = u_i^0, \quad \text{in } \mathbb{R}^d,\ i \in \{1,2,\dots,n\}$$

where $a_{ij} := \int_{\mathbb{R}^d} V_{ij}(|x|)dx = \int_{\mathbb{R}^d} V_{ij}^\eta(|x|)dx$, for $1 \le i,j \le n$. Furthermore, recall that $g_{ij} = \frac{1}{n}\sum_{l=1}^d \frac{\partial}{\partial x_l}(\sigma_i)_{\cdot,l}$.

6.1.1. Local existence

We will see that the system (PDE II) can be seen as the problem obtained by performing the limit $\eta \to 0$ in (PDE I).

We proceed as in the proof of [5, Lemma 7]:

Lemma 6.1.1. *Let $u_{\eta_n}, n \in \mathbb{N}$, be a series of solutions of problem (PDE I) with initial condition $u^0 \in H^s(\mathbb{R}^d, \mathbb{R}^n)$ and $\lim_{n\to\infty} \eta_n = 0$. Furthermore, let u_{η_n} be uniformly bounded in $L^2(0,T;H^{s+1}(\mathbb{R}^d,\mathbb{R}^n))$ for some $T > 0$. Then there exists $u \in L^2(0,T;H^{s+1}(\mathbb{R}^d,\mathbb{R}^n))$ and a subseries $u_{\eta_{n_k}}$, such that $u_{\eta_n} \rightharpoonup u$ weakly in $L^2(0,T;H^{s+1}(\mathbb{R}^d,\mathbb{R}^n))$ and $V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} \rightharpoonup a_{ij}\nabla u_j$ weakly in $L^2(0,T;L^2(\mathbb{R}^d,\mathbb{R}^n))$.*

Proof. Due to the uniform boundedness of the sequence u_{η_n} in $L^2(0,T;H^{s+1})$, there exists a subsequence, which we will again denote with u_{η_n}, which converges weakly in $L^2(0,T;H^{s+1})$ towards some element u.

We want to show $V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} \rightharpoonup a_{ij}\nabla u_j$ weakly in $L^2(0,T;L^2)(\mathbb{R}^d,\mathbb{R}^n))$. Let therefore $\phi \in L^2(0,T;L^2(\mathbb{R}^d,\mathbb{R}^n))$ be arbitrary but fixed. Then we have

$$\left|\int_0^T \langle V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} - a_{ij}\nabla u_j, \phi\rangle_{L^2} dt\right| = \left|\int_0^T \int_{\mathbb{R}^d} \left(V_{ij}^{\eta_n} * \nabla u_{\eta_n,j}(t)[x] - a_{ij}\nabla u_j(x,t)\right)\cdot \phi(x,t)dxdt\right|$$

$$= \left| \int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} V_{ij}^{\eta_n}(x-y) \nabla u_{\eta_n,j}(y,t) dy - a_{ij} \nabla u_j(x,t) \right) \cdot \phi(x,t) dx dt \right|$$

$$= \left| \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_{ij}^{\eta_n}(x-y)\phi(x,t)dx \cdot \nabla u_{\eta_n,j}(y,t) dy dt - \int_0^T \int_{\mathbb{R}^d} a_{ij} \nabla u_j(x,t) \cdot \phi(x,t) dx dt \right| \tag{6.1}$$

where we used Fubini's theorem to exchange the integrals. By assumption $V_{ij}^{\eta}(x) = V_{ij}\left(\frac{|x|}{\eta}\right)$. Thus V_{ij}^{η} is only depended on the absolute value of x and $V_{ij}^{\eta}(x-y) = V_{ij}^{\eta}(y-x)$ for all $x, y \in \mathbb{R}^d$. The term $V_{ij}^{\eta_n} * \phi$ converges strongly in $L^2(0,T;L^2)$ towards $a_{ij}\phi$ (see theorem 9.10 in [8]). As u_{η_n} converges weakly in $L^2(0,T;H^{s+1})$, the sequence ∇u_{η_n} converges weakly in $L^2(0,T;L^2)$ towards ∇u_j. Due to the boundedness of $\|u_{\eta_n}\|_{L^2(0,T;L^2)}$, we can therefore pass to the limit $\eta_n \to 0$:

$$(6.1) \leq \left| \int_0^T \int_{\mathbb{R}^d} \underbrace{\int_{\mathbb{R}^d} V_{ij}^{\eta_n}(y-x)\phi(x,t)dx}_{=V_{ij}^{\eta} * \phi(y)} \cdot \nabla u_{\eta_n,j}(y,t) dy dt - \int_0^T \int_{\mathbb{R}^d} a_{ij}\phi(y,t) \cdot \nabla u_{\eta_n,j}(y,t) dy dt \right|$$

$$+ \left| \int_0^T \int_{\mathbb{R}^d} a_{ij}\phi(y,t) \cdot \nabla u_{\eta_n,j}(y,t) dy dt - \int_0^T \int_{\mathbb{R}^d} a_{ij}\phi(y,t) \cdot \nabla u_j(y,t) dy dt \right|$$

$$\leq \left\| V_{ij}^{\eta} * \phi - a_{ij}\phi \right\|_{L^2(0,T;L^2)} \|u_{\eta_n,j}\|_{L^2(0,T;L^2)} + \left| \int_0^T \int_{\mathbb{R}^d} a_{ij}\phi(y,t) \cdot \left(\nabla u_{\eta_n,j}(y,t) - \nabla u_j(y,t) \right) dy dt \right|$$

$$\to 0.$$

□

Lemma 6.1.2. *Let the assumptions of lemma 6.1.1 hold. Then there exists a solution $u \in L^\infty(0,T;H^s(\mathbb{R}^d,\mathbb{R}^n))$, with $\nabla u \in L^2(0,T;H^s(\mathbb{R}^d,\mathbb{R}^{n\times d}))$, to problem (PDE II). If additionally the solutions u_η of (PDE I) are uniformly bounded in $L^\infty(0,T;H^s(\mathbb{R}^d,\mathbb{R}^n))$ for small η, then u fulfils*

$$\sup_{0<t<T} \|u(t)\|_{H^s} \leq \sup_{0<\eta} \|u_\eta\|_{L^\infty(0,T;H^s)}. \tag{6.2}$$

Furthermore, if $g_{ij} \equiv 0$ for all $1 \leq i,j \leq n$, then, under assumption (5.12), u satisfies the estimates

$$2\gamma \|\nabla u\|^2_{L^2(0,T;H^s)} \leq \left\|u^0\right\|^2_{H^s}, \tag{6.3}$$

$$\sup_{0<t<T} \|u(t)\|_{H^s} \leq \left\|u^0\right\|_{H^s}. \tag{6.4}$$

Proof. Let $T > 0$ be arbitrary but fixed and let u and u_{η_n} be as in lemma 6.1.1 . Then $u_{\eta_n,i}$ satisfies the equation

$$\int_0^T \langle \partial_t u_{\eta_n,i}, \phi \rangle_{H^{-1}} dt + \int_0^T \langle \sigma_i \nabla u_{\eta_n,i}, \nabla \phi \rangle_{L^2} dt = - \int_0^T \langle \sum_{j=1}^n u_{\eta_n,i} \left(V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} + g_{ij} \right), \nabla \phi \rangle_{L^2} dt, \tag{6.5}$$

for all $\phi \in L^2(0,T;H^1)$. We therefore have

$$\left| \int_0^T \langle \partial_t u_{\eta_n,i}, \phi \rangle_{H^{-1}} dt \right| \leq \| \sigma_i \nabla u_{\eta_n,i} \|_{L^2(0,T;L^2)} \| \nabla \phi \|_{L^2(0,T;L^2)}$$

$$+ \sum_{j=1}^n \left\| u_{\eta_n,i} V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} \right\|_{L^2(0,T;L^2)} \| \nabla \phi \|_{L^2(0,T;L^2)}$$

$$+ \sum_{i,j=1}^n \| u_{\eta_n,i} g_{ij} \|_{L^2(0,T;L^2)} \| \nabla \phi \|_{L^2(0,T;L^2)}.$$

We use estimate (5.13) and the embedding $H^s \to L^\infty$ together with Young's convolution inequality, to conclude $u_{\eta_n,i} V_{ij}^{\eta_n} * \nabla u_{\eta_n,j}$, $n \in \mathbb{N}$, bounded in $L^2(0,T;L^2)$:

$$\left\| u_{\eta_n,i} V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} \right\|_{L^2} \leq \| u_{\eta_n,i} \|_{L^\infty} \left\| V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} \right\|_{L^2(0,T;L^2)}$$

$$\leq c_{H^s \hookrightarrow L^\infty} \| u_{\eta_n,i} \|_{H^s} \left\| V_{ij}^{\eta_n} \right\|_{L^1} \| \nabla u_{\eta_n,j} \|_{L^2} \leq C. \tag{6.6}$$

In similar a fashion we get

$$\| u_{\eta_n,i} g_{ij} \|_{L^2(0,T;L^2)} \leq \| u_{\eta_n,i} \|_{L^\infty} \| g_{ij} \|_{L^2(0,T;L^2)} \leq c_{H^s \hookrightarrow L^\infty} \| u_{\eta_n,i} \|_{H^s} \| g_{ij} \|_{L^2(0,T;L^2)} \leq C.$$

Due to the uniform boundedness of $u_{\eta_n,i}$ in $L^2(0,T;H^{s+1})$ we thus have

$$\left| \int_0^T \langle \partial_t u_{\eta_n,i}, \phi \rangle_{H^{-1}} dt \right| \leq \tilde{C} \| \nabla \phi \|_{L^2(0,T;L^2)}$$

where $\tilde{C} > 0$ does not depend on n. This implies $\| \partial_t u_{\eta_n,i} \|_{L^2(0,T;H^{-1})} \leq \tilde{C}$ and hence the series $\partial_t u_{\eta_n,i}$ converges weakly in $L^2(0,T;H^{-1})$ towards $\partial_t u_i$, because for $\phi \in D(\mathbb{R}^d \times (0,T))$

$$\lim_{\eta \to 0} \int_0^T \langle \partial_t u_{\eta_n,i}, \phi \rangle_{H^{-1}} dt = \lim_{\eta \to 0} - \int_0^T \langle u_{\eta_n,i}, \partial_t \phi \rangle_{L^2} dt = - \int_0^T \langle u_i, \partial_t \phi \rangle_{L^2} dt = \int_0^T \langle \partial_t u_i, \phi \rangle_{H^{-1}} dt.$$

Let $R > 0$ be arbitrary but fixed and $B_R(0) := \{ x \in \mathbb{R}^d : \ |x| \leq R \}$. Since $B_R(0)$ is a compact subset of $\mathbb{R}^d$, the embedding $H^1(B_R(0)) \hookrightarrow L^2(B_R(0))$ is compact (see the theorem of

Rellich- Kondrachov A.2.11). The sequence ∇u_{η_n} is uniformly bounded in $W^{1,2}(0,T;H^1,L^2)$. The Aubin lemma (see A.3.21) assures therefore the existence of a subsequence, which we again denote with u_{η_n}, which converges (strongly) in $L^2(0,T;L^2(B_R(0))$. Because of the uniqueness of limits, $\lim_{\eta_n \to 0} u_{\eta_n} = u$ in $L^2(0,T;L^2(B_R(0))$, as u_{η_n} converges weakly towards u in $L^2(0,T;H^{s+1})$. Due to the uniform estimate 6.6, we can again restrict ourselves to the subsequence for which additionally $u_{\eta_n,i}V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} \rightharpoonup v$ weakly in $L^2(0,T;L^2)$, where v is some element of $L^2(0,T;L^2)$. Summing up the (sub-)sequence u_{η_n} has the following properties:

1) $u_{\eta_n} \rightharpoonup u$ weakly in $W^{1,2}(0,T;H^1;L^2)$.

2) $u_{\eta_n} \to u$ in $L^2(0,T;L^2(B_R(0)))$.

3) $V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} \rightharpoonup a_{ij}\nabla u_j$ weakly in $L^2(0,T;L^2)$.

4) $u_{\eta_n,i}V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} \rightharpoonup v$ weakly in $L^2(0,T;L^2)$.

5) $u_{\eta_n,i}g_{ij} \rightharpoonup \tilde{v}$ weakly in $L^2(0,T;L^2)$.

6) $u_{\eta_n} \rightharpoonup u$ weakly in $L^2(0,T;H^{s+1})$.

From the points 2) and 3) we conclude $u_{\eta_n,i}V_{ij}^{\eta_n} * \nabla u_{\eta_n,j} \rightharpoonup u_i a_{ij} \nabla u_j$ weakly in $L^1(0,T;L^1(B_R(0)))$. Due to point 4) and the uniqueness of limits, we get $u_i a_{ij}\nabla u_j = v$. Using the same argumentation, we also see $\tilde{v} = u_i g_{ij}$. From here we can perform the limit $\eta_n \to 0$ in 6.5 for $\phi \in \mathcal{D}(B_R(0) \times [0,T])$:

$$\int_0^T \langle \partial_t u_i, \phi \rangle_{H^{-1}} dt + \int_0^T \langle \sigma_i \nabla u_i, \nabla \phi \rangle_{L^2} dt = -\int_0^T \langle \sum_{j=1}^n u_i \left(a_{ij}\nabla u_j + g_{ij} \right), \nabla \phi \rangle_{L^2} dt. \tag{6.7}$$

For every test-function $\phi \in \mathcal{D}(\mathbb{R}^d \times [0,T])$ there exists a $r > 0$ such that $\phi \in \mathcal{D}(B_r(0) \times [0,T])$. Because R was arbitrary, equation (6.7) holds therefore also for $\phi \in \mathcal{D}(\mathbb{R}^d \times [0,T])$. But the set of test-functions $\mathcal{D}(\mathbb{R}^d \times [0,T])$ is dense in $L^2(0,T;H^1)$ and consequently equation (6.7) holds true for $\phi \in L^2(0,T;H^1)$ as well. Hence u satisfies the weak formulation of (PDE II). The estimate 6.3 follows from the lower semi- continuity of the norm and estimate 5.13.

Notice that $u \in C([0,T^*];H^s)$ due to the embedding $W^{1,2}(0,T;H^{s+1},H^s) \hookrightarrow C([0,T^*];H^s)$ and $H^{-1} \subset H^{-s-1}$ (see proposition A.3.19). Let $\phi(x,t) = \psi(x)\zeta(t)$ with $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $\zeta \in \mathcal{D}([0,T))$. The initial value condition $u \mid_{t=0} = u^0$ can be confirmed by looking at the term

$$\int_0^T \langle \partial_t u_{\eta_n,i}, \phi \rangle_{H^{-1}} dt = -\int_0^T \langle u_{\eta_n,i}, \partial_t \phi \rangle_{L^2} dt - \langle u_i^0, \phi(0) \rangle_{L^2}$$

and performing the limit $\eta_n \to 0$. Then we have

$$-\int_0^T \langle u_i, \partial_t \psi \rangle_{L^2} \zeta dt - \langle u_i(0), \psi \rangle_{L^2} \zeta(0) = \int_0^T \langle \partial_t u_i, \psi \rangle_{H^{-1}} \zeta dt = -\int_0^T \langle u_i, \partial_t \psi \rangle_{L^2} \zeta dt - \langle u_i^0, \psi \rangle_{L^2} \zeta(0).$$

We choose ζ such that $\zeta(0) \neq 0$. This implies

$$\langle u_i(0), \psi \rangle_{L^2} = \langle u_i^0, \psi \rangle_{L^2}$$

for all $\psi \in \mathcal{D}(\mathbb{R}^d)$. Hence we can apply the fundamental lemma of calculus of variations:

$$u_i(0) = u_i^0,$$

almost everywhere. Additionally we want to show that $\|u(t)\|_{H^s} \leq \sup_{0<\eta} \|u_\eta\|_{L^\infty(0,T;H^s)}$. Let therefore $\zeta \in \mathcal{D}(0,T]$, with $\zeta(T) > 0$:

$$\sum_{\alpha \leq s} (-1)^{|\alpha|} \int_0^T \langle \partial_t u_{\eta_n,i}, D^{2\alpha}\psi \rangle_{H^{-1}} \zeta dt$$

$$= \sum_{|\alpha| \leq s} \left((-1)^{|\alpha|+1} \int_0^T \langle u_{\eta_n,i}, \partial_t D^{2\alpha}\psi \rangle_{L^2} \zeta dt + \langle D^\alpha u_{\eta_n,i}(T), D^\alpha \psi \rangle_{L^2} \zeta(T) \right)$$

$$= \sum_{|\alpha| \leq s} (-1)^{|\alpha|+1} \int_0^T \langle u_{\eta_n,i}, \partial_t D^{2\alpha}\psi \rangle_{L^2} \zeta dt + \langle u_{\eta_n,i}(T), \psi \rangle_{H^s} \zeta(T)$$

$$\leq \sum_{|\alpha| \leq s} (-1)^{|\alpha|+1} \int_0^T \langle u_{\eta_n,i}, \partial_t D^{2\alpha}\psi \rangle_{L^2} \zeta dt + \underbrace{\|u_{i,\eta_n}(T)\|_{H^s}}_{\leq \sup_{0<\eta} \|u_\eta\|_{L^\infty(0,T;H^s)}} \|\psi\|_{H^s} \zeta(T)$$

By passing over to the limit $\eta_n \to 0$, we get

$$\sum_{|\alpha| \leq s} (-1)^{|\alpha|} \int_0^T \langle \partial_t u_i, D^{2\alpha}\psi \rangle_{H^{-1}} \zeta dt = \sum_{|\alpha| \leq s} (-1)^{|\alpha|+1} \int_0^T \langle u_i, \partial_t D^{2\alpha}\psi \rangle_{L^2} \zeta dt + \langle u_i(T), \psi \rangle_{H^s} \zeta(T)$$

$$\leq \sum_{|\alpha| \leq s} (-1)^{|\alpha|+1} \int_0^T \langle u_i, \partial_t D^{\alpha}\psi \rangle_{H^s} \zeta dt + \sup_{0<\eta} \|u_\eta\|_{L^\infty(0,T;H^s)} \|\psi\|_{H^s} \zeta(T).$$

This holds for all $\psi \in \mathcal{D}(\mathbb{R}^d)$ and in particular for $-\psi$. Hence

$$|\langle u_i(T), \psi \rangle_{H^s}| \leq \sup_{0<\eta} \|u_\eta\|_{L^\infty(0,T;H^s)} \|\psi\|_{H^s}.$$

This implies $\|u_i(T)\|_{H^s} \leq \sup_{0<\eta} \|u_\eta\|_{L^\infty(0,T;H^s)}$. Due to the fact that all the required convergences also hold true on $[0,t]$ for $0 < t < T$, we can argue as above to get $\|u_i(t)\|_{H^s} \leq \sup_{0<\eta} \|u_\eta\|_{L^\infty(0,T;H^s)}$. The estimates (6.3) and (6.4) follow now from (5.13). □

From the previous chapter we know of the local existence of solutions $u_\eta \in L^2(0,T^*(\eta);H^{s+1})$ to the intermediate problem. Here $T^*(\eta)$ depends on η via the initial value u_η^0 (see theorem 5.1.3). Thus, if these solutions satisfy the same initial value condition $u_\eta |_{t=0} = u^0$, then $T = T(\eta)$. This leads to an uniform bound in $L^2(0,T^*;H^{s+1})$ and we can apply lemma 6.1.2. We therefore have proven:

Corollary 6.1.3 (Local existence). *Let $u^0 \in H^s(\mathbb{R}^d, \mathbb{R}^n)$ and $s > \frac{d}{2} + 2$. Then there exists $0 < T^* \le T$ such that there is a series of solutions $u_{\eta_n} \in L^2(0, T^*; H^{s+1}(\mathbb{R}^d, \mathbb{R}^n))$ of (PDE I) and a solution $u \in L^2(0, T^*; H^{s+1}(\mathbb{R}^d, \mathbb{R}^n))$ of problem (PDE II) with the same initial value u^0. The sequence u_{η_n} is bounded in $L^\infty(0, T^*; H^s(\mathbb{R}^d, \mathbb{R}^n)) \cap L^2(0, T^*; H^{s+1}(\mathbb{R}^d, \mathbb{R}^n))$ and the following estimate holds:*

$$\sup_{0<t<T^*} \|u(t)\|_{H^s} \le \sup_{0<\eta} \|u_\eta\|_{L^\infty(0,T^*;H^s)}. \tag{6.8}$$

Furthermore, if $g_{ij} \equiv 0$ for all $1 \le i, j \le n$ then, under assumption (5.12), $T = T^*$ and u satisfies the additional estimates

$$2\gamma \|\nabla u\|^2_{L^2(0,T;H^s)} \le \|u^0\|^2_{H^s}, \tag{6.9}$$

$$\sup_{0<t<T} \|u(t)\|_{H^s} \le \|u^0\|_{H^s}. \tag{6.10}$$

6.1.2. Uniqueness and global existence

As of now, we only have proven the existence of a solution to (PDE II) until a specific finite time T. In theory, two solutions (u_1, u_2), corresponding to two, possibly different times (T_1, T_2), do not have to coincide on the 'shared' time-interval $[0, \min\{T_1, T_2\}]$. We will therefore show uniqueness on such intervals, which will then imply $u_1 = u_2$ on $[0, \min\{T_1, T_2\}]$. Under the assumptions of corollary 5.1.8, this will furthermore lead to a global solution, as the times T_1, T_2 can be chosen arbitrarily large in this case.

Lemma 6.1.4 (Uniqueness). *Let u be the weak solution of (PDE II) characterized by corollary 6.1.3 and let the constant $0 < \gamma < \epsilon$ be such that*

$$\epsilon - c_{H^s \hookrightarrow L^\infty} \sum_{i,j=1}^n \left(\sqrt{1 + \|u^0\|^2_{H^s}} \right) \|V_{ij}\|_{L^1} \ge \gamma \tag{6.11}$$

where $c_{H^s \hookrightarrow L^\infty}$ is the operator norm of the embedding $\iota : H^s \hookrightarrow L^\infty$ and $\epsilon = \min_{1\le i\le n} \epsilon_i$.

Then u is the unique solution of (PDE II) on $[0, T']$ in the space $L^\infty(0, T'; H^s(\mathbb{R}^d, \mathbb{R}^n)) \cap L^2(0, T'; H^{s+1}(\mathbb{R}^d, \mathbb{R}^n))$ for every $T' \le T^*$.

Proof. Let $v \in L^\infty(0, T; H^s) \cap L^2(0, T; H^{s+1})$ be another weak solution of problem (PDE II). Then u_i and v_i satisfy the following equations for every $\phi \in L^2(0, T; H^1)$ and $1 \le i \le n$:

$$\langle \partial_t u_i, \phi \rangle_{H^{-1}} + \langle \sigma_i \nabla u_i, \nabla \phi \rangle_{L^2} = -\langle \sum_{j=1}^n u_i(a_{ij} \nabla u_j + g_{ij}), \nabla \phi \rangle_{L^2},$$

$$\langle \partial_t v_i, \phi \rangle_{H^{-1}} + \langle \sigma_i \nabla v_i, \nabla \phi \rangle_{L^2} = -\langle \sum_{j=1}^n v_i(a_{ij} \nabla v_j + g_{ij}), \nabla \phi \rangle_{L^2}.$$

We choose $\phi = u_i - v_i$, take the difference of the resulting two equations and sum over all $1 \le i \le n$. This yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u - v\|^2_{L^2} + \sum_{i=1}^n \langle \sigma_i \nabla(u_i - v_i), \nabla(u_i - v_i) \rangle_{L^2} = &- \sum_{i,j=1}^n \langle u_i(a_{ij} \nabla u_j + g_{ij}), \nabla(u_i - v_i) \rangle_{L^2} \\ &+ \sum_{i,j=1}^n \langle v_i(a_{ij} \nabla v_j + g_{ij}), \nabla(u_i - v_i) \rangle_{L^2} \end{aligned}$$

where we recalled that $\langle \partial_t u_i - v_i, u_i - v_i\rangle_{H^{-1}} = \frac{1}{2}\frac{d}{dt}\|u_i - v_i\|_{L^2}^2$. Using the coercivity of σ_i and expanding the left hand side leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 + \epsilon\|\nabla(u-v)\|_{L^2}^2\\
\le& -\sum_{i,j=1}^n \langle u_i a_{ij}\nabla u_j, \nabla(u_i-v_i)\rangle_{L^2} \pm \sum_{i,j=1}^n \langle u_i a_{ij}\nabla v_j, \nabla(u_i-v_i)\rangle_{L^2} + \sum_{i,j=1}^n \langle v_i a_{ij}\nabla v_j, \nabla(u_i-v_i)\rangle_{L^2}\\
&-\sum_{i,j=1}^n \langle (u_i-v_i)g_{ij}, \nabla(u_i-v_i)\rangle_{L^2}\\
=& -\sum_{i,j=1}^n \langle u_i a_{ij}\nabla (u_j-v_j), \nabla(u_i-v_i)\rangle_{L^2} + \sum_{i,j=1}^n \langle (v_i-u_i) a_{ij}\nabla v_j, \nabla(u_i-v_i)\rangle_{L^2}\\
&-\sum_{i,j=1}^n \langle (u_i-v_i)g_{ij}, \nabla(u_i-v_i)\rangle_{L^2}
\end{aligned}
\tag{6.12}
$$

where again $\epsilon = \min_{1\le i\le n} \epsilon_i$. We use the Cauchy-Schwarz inequality in (6.12) and Young's inequality for products to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 + \epsilon\|\nabla(u-v)\|_{L^2}^2\\
\le& \sum_{i,j=1}^n |a_{ij}|\|u_i\|_{L^\infty}\|\nabla(u_i-v_i)\|_{L^2}^2 + n\delta\|\nabla(u-v)\|_{L^2}^2 + \frac{1}{2\delta}\sum_{i,j=1}^n \left(|a_{ij}|\|\nabla v_j\|_{L^\infty}\|u_i-v_i\|_{L^2}\right)^2\\
&+\frac{1}{2\delta}\sum_{i,j=1}^n \left(\|g_{ij}\|_{L^\infty}\|u_i-v_i\|_{L^2}\right)^2\\
\le& c_{H^s\hookrightarrow L^\infty}\sum_{i,j=1}^n \|V_{ij}\|_{L^1}\|u_i\|_{H^s}\|\nabla(u_i-v_i)\|_{L^2}^2 + n\delta\|\nabla(u-v)\|_{L^2}^2 + \frac{1}{2\delta}\sum_{i,j=1}^n \left(\|g_{ij}\|_{L^\infty}\|u_i-v_i\|_{L^2}\right)^2\\
&+\frac{1}{2\delta}\sum_{i,j=1}^n \left(c_{H^s\hookrightarrow L^\infty}|a_{ij}|\|\nabla v_j\|_{H^s}\|(u_i-v_i)\|_{L^2}\right)^2.
\end{aligned}
$$

Here $\delta > 0$ is arbitrary but fixed and $c_{H^s\hookrightarrow L^\infty}$ is the constant resulting from the continuous embedding $H^s \hookrightarrow L^\infty$. Due to the estimates in corollary 6.1.3 and theorem 5.1.3, where we recall that $u_\eta, \eta > 0$, are *unique* we have

$$
\|u(t)\|_{H^s} \le \sup_{0<\eta}\|u_\eta\|_{L^\infty(0,T;H^s)} \le \sqrt{1+\|u^0\|_{H^s}^2}, \quad \text{almost everywhere.}
$$

Thus, we can make use of the assumption (6.11):

$$
c_{H^s\hookrightarrow L^\infty}\sum_{i,j=1}^n \|V_{ij}\|_{L^1}\|u_i\|_{H^s} \le c_{H^s\hookrightarrow L^\infty}\sum_{i,j=1}^n \left(\sqrt{1+\|u^0\|_{H^s}^2}\right)\|V_{ij}\|_{L^1} \le \epsilon - \gamma.
$$

This leads to

$$
\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 + \epsilon\|\nabla(u-v)\|_{L^2}^2
$$

$$\leq (\epsilon - \gamma + n\delta)\|\nabla(u-v)\|_{L^2}^2 + C(s,d,n,\delta)\left(\|\nabla v\|_{H^s} + \|g_{ij}\|_{L^\infty}\right)^2 \|u-v\|_{L^2}^2.$$

We choose $\delta \leq \frac{\gamma}{n}$. Then the second term on the left-hand side absorbs the first term on the right-hand side and we are left with

$$\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 \leq C(s,d,n,\gamma)\left(\|\nabla v\|_{H^s} + \|g_{ij}\|_{L^\infty}\right)^2 \|u-v\|_{L^2}^2.$$

Notice that $C(s,d,n,\gamma)\left(\|\nabla v\|_{H^s} + \|g_{ij}\|_{L^\infty}\right)^2$ is a integrable function due to $v \in L^2(0,T;H^{s+1})$ and $g_{ij} \in L^\infty(0,T;W^{s,\infty})$. We can therefore apply Gronwall's lemma to get

$$\|u(t) - v(t)\|_{L^2}^2 \leq 0, \quad t \in [0,T]$$

because u and v fulfil the same initial value condition. Thus $u = v$. □

Corollary 6.1.5 (Global existence)**.** *Let the assumption of corollary 6.1.3 hold true. Furthermore, let all the assumptions made for σ_i in section 1.1 hold for $T = \infty$. If condition 5.12 is fulfilled and $g_{ij} \equiv 0$ for all $1 \leq i,j \leq n$, then there exists a global unique solution $u \in L(0,\infty;H^s(\mathbb{R}^d,\mathbb{R}^n))$, with $\nabla u \in L^2(0,\infty;H^s(\mathbb{R}^d,\mathbb{R}^{n\times d}))$, of the cross diffusion system (PDE II).*

Proof. This is a direct conclusion from corollary 6.1.3 and lemma 6.1.4. □

6.2. Error-Estimate between u and u_η

We have already seen in the proof of lemma 6.1.2 (local existence) that, under the right conditions, there exists a sequence of solutions of the non-local diffusion problem (PDE I), which converges weakly to the solution u of the cross diffusion system (PDE II) for $\eta \to 0$. But one can even show general convergence of solutions of (PDE I) towards u in a strong sense as initially proposed in [5, Proposition 2]:

Theorem 6.2.1 (Error-estimate)**.** *Let $u_{\eta_n} \in L^2(0,T^*;H^{s+1}(\mathbb{R}^d,\mathbb{R}^n))$ be a series of solutions of (PDE I) and $u \in L^2(0,T^*;H^{s+1}(\mathbb{R}^d,\mathbb{R}^n))$ be the solution of problem (PDE II) fulfilling the same initial value condition. Furthermore, let*

$$\epsilon - c_{H^s \hookrightarrow L^\infty} \sum_{i,j=1}^{n} \left(\sqrt{1 + \|u^0\|_{H^s}^2}\right) \|V_{ij}\|_{L^1} \geq \gamma \tag{6.13}$$

for some $\gamma \in (0,\epsilon)$. Then the following error- estimate holds:

$$\|u - u_\eta\|_{L^\infty(0,t;L^2)} + \|\nabla(u-u_\eta)\|_{L^2(0,t;L^2)} \leq C(t)\eta, \quad t \in [0,T^*] \tag{6.14}$$

where $C(t)$ is a continuous function of t and T^* is as in theorem 5.1.3.

Proof. We proceed as in the proof of lemma 8 in [5].

u_i and $u_{\eta,i}$ fulfil the following equations for every $\phi \in L^2(0,T^*;H^1)$ and $1 \leq i \leq n$:

$$\langle \partial_t u_i, \phi \rangle_{H^{-1}} + \langle \sigma_i \nabla u_i, \nabla \phi \rangle_{L^2} = -\langle \sum_{j=1}^{n} u_i \left(a_{ij}\nabla u_j + g_{ij}\right), \nabla \phi \rangle_{L^2} \tag{6.15}$$

$$\langle \partial_t u_{\eta,i}, \phi \rangle_{H^{-1}} + \langle \sigma_i \nabla u_{\eta,i}, \nabla \phi \rangle_{L^2} = -\langle \sum_{j=1}^{n} u_{\eta,i} \left(V_{ij}^{\eta} * \nabla u_{\eta,j} + g_{ij} \right), \nabla \phi \rangle_{L^2}. \tag{6.16}$$

By choosing $\phi = u_i - u_{\eta,i}$ and taking the difference of (6.15) and (6.16), we obtain

$$\langle \partial_t u_i - u_{\eta,i}, u_i - u_{\eta,i} \rangle_{H^{-1}} + \langle \sigma_i \nabla (u_i - u_{\eta,i}), \nabla (u_i - u_{\eta,i}) \rangle_{L^2}$$

$$= -\langle \sum_{j=1}^{n} u_i \left(a_{ij} \nabla u_j + g_{ij} \right), \nabla (u_i - u_{\eta,i}) \rangle_{L^2} + \langle \sum_{j=1}^{n} u_{\eta,i} \left(V_{ij}^{\eta} * \nabla u_{\eta,j} + g_{ij} \right), \nabla (u_i - u_{\eta,i}) \rangle_{L^2}.$$

We sum from $i = 1, 2, \dots, n$, expand the term on the right-hand side and use the Cauchy-Schwarz inequality, as well as the coercivity of $\sigma_i (\epsilon = \min_{1 \le i \le n} \epsilon_i)$. Recall that $\langle \partial_t u_i - u_{\eta,i}, u_i - u_{\eta,i} \rangle_{H^{-1}} = \frac{1}{2} \frac{d}{dt} \| u_i - u_{\eta,i} \|_{L^2}^2$:

$$\frac{1}{2} \frac{d}{dt} \| u - u_\eta \|_{L^2}^2 + \epsilon \| \nabla (u - u_\eta) \|_{L^2}^2$$

$$\le -\langle \sum_{i,j=1}^{n} u_i a_{ij} \nabla u_j, \nabla (u_i - u_{\eta,i}) \rangle_{L^2} \pm \langle \sum_{i,j=1}^{n} u_i a_{ij} \nabla u_{\eta,j}, \nabla (u_i - u_{\eta,i}) \rangle_{L^2}$$

$$\pm \langle \sum_{i,j=1}^{n} u_i V_{ij}^{\eta} * \nabla u_{\eta,j}, \nabla (u_i - u_{\eta,i}) \rangle_{L^2} + \langle \sum_{i,j=1}^{n} u_{\eta,i} V_{ij}^{\eta} * \nabla u_{\eta,j}, \nabla (u_i - u_{\eta,i}) \rangle_{L^2}$$

$$-\langle \sum_{i,j=1}^{n} (u_i - u_{i,\eta}) \, g_{ij}, \nabla (u_i - u_{\eta,i}) \rangle_{L^2}$$

$$= -\sum_{i,j=1}^{n} \langle u_i a_{ij} \nabla (u_j - u_{\eta,j}), \nabla (u_i - u_{\eta,i}) \rangle_{L^2}$$

$$+ \sum_{i,j=1}^{n} \langle u_i (V_{ij}^{\eta} * \nabla u_{\eta,j} - a_{ij} \nabla u_{\eta,j}), \nabla (u_i - u_{\eta,i}) \rangle_{L^2}$$

$$+ \sum_{i,j=1}^{n} \langle (u_{\eta,i} - u_i) V_{ij}^{\eta} * \nabla u_{\eta,j}, \nabla (u_i - u_{\eta,i}) \rangle_{L^2}$$

$$- \sum_{i,j=1}^{n} \langle (u_i - u_{i,\eta}) \, g_{ij}, \nabla (u_i - u_{\eta,i}) \rangle_{L^2}$$

$$\le \sum_{i,j=1}^{n} \| u_i \|_{L^\infty} |a_{ij}| \| \nabla (u_j - u_{\eta,j}) \|_{L^2} \| \nabla (u_i - u_{\eta,i}) \|_{L^2} \tag{6.17}$$

$$+\sum_{i,j=1}^{n}\|u_i\|_{L^\infty}\left\|(V_{ij}^{\eta}*\nabla u_{\eta,j}-a_{ij}\nabla u_{\eta,j})\right\|_{L^2}\|\nabla(u_i-u_{\eta,i})\|_{L^2} \tag{6.18}$$

$$+\sum_{i,j=1}^{n}\|u_{\eta,i}-u_i\|_{L^2}\left\|V_{ij}^{\eta}*\nabla u_{\eta,j}\right\|_{L^\infty}\|\nabla(u_i-u_{\eta,i})\|_{L^2} \tag{6.19}$$

$$+\sum_{i,j=1}^{n}\|u_i-u_{i,\eta}\|_{L^2}\|g_{ij}\|_{L^\infty}\|\nabla(u_i-u_{\eta,i})\|_{L^2}. \tag{6.20}$$

The term (6.17) can be further estimated by making use of the fact that H^s is continuously embedded in L^∞. We recall the inequality-chain

$$\|u(t)\|_{H^s}\leq\sup_{0<\eta}\|u_\eta\|_{L^\infty(0,T;H^s)}\leq\sqrt{1+\|u^0\|_{H^s}^2},\quad \text{almost everywhere, as show in}$$

corollary 6.1.3 and theorem 5.1.3. Then, together with assumption (6.13), we get:

$$(6.17)\leq\|\nabla(u-u_\eta)\|_{L^2}^2c_{H^s\hookrightarrow L^\infty}\sum_{i,j=1}^{n}\|u_i\|_{H^s}|a_{ij}|$$

$$\leq\|\nabla(u-u_\eta)\|_{L^2}^2c_{H^s\hookrightarrow L^\infty}\sum_{i,j=1}^{n}\|u_i\|_{H^s}\|V_{ij}\|_{L^1}$$

$$\leq(\epsilon-\gamma)\|\nabla(u-u_\eta)\|_{L^2}^2,$$

where we recall $a_{ij}=\int_{\mathbb{R}^d}V_{ij}(x)dx=\int_{\mathbb{R}^d}V_{ij}^{\eta}(x)dx$.

To estimate the term (6.18), we first want to show $\left\|(V_{ij}^{\eta}*\nabla u_{\eta,j}-a_{ij}\nabla u_{\eta,j})\right\|_{L^2}\leq C\eta$ for some constant C, by proving $\left|\langle(V_{ij}^{\eta}*\nabla u_{\eta,j}-a_{ij}\nabla u_{\eta,j}),v\rangle_{L^2}\right|\leq C\eta\|v\|_{L^2}$ for all $v\in L^2(\mathbb{R}^d,\mathbb{R}^n)$. Let therefore $v\in L^2(\mathbb{R}^d,\mathbb{R}^n)$ be arbitrary but fixed. Then we have

$$\left|\langle(V_{ij}^{\eta}*\nabla u_{\eta,j}-a_{ij}\nabla u_{\eta,j}),v\rangle_{L^2}\right|=\left|\int_{\mathbb{R}^d}(V_{ij}^{\eta}*\nabla u_{\eta,j}-a_{ij}\nabla u_{\eta,j})\cdot v dx\right|$$

$$=\left|\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}V_{ij}^{\eta}(y)\nabla u_{\eta,j}(x-y)dy-\int_{\mathbb{R}}V_{ij}^{\eta}(y)\nabla u_{\eta,j}(x)dy\right)\cdot v(x)dx\right|.$$

$$=\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}V_{ij}^{\eta}(y)\left(\nabla u_{\eta,j}(x-y)-\nabla u_{\eta,j}(x)\right)dy\cdot v(x)dx\right|$$

$$= \left| \int_{\mathbb{R}^d} \int_{B_\eta(0)} V_{ij}^\eta(y) \left(\nabla u_{\eta,j}(x-y) - \nabla u_{\eta,j}(x)\right) \cdot v(x) dy dx \right|$$

where in the last step we used supp $V_{ij} \subset B_\eta(0) := \{x \in \mathbb{R}^d : \ |x| \leq \eta\}$. Because $s > \frac{d}{2} + 1$, $H^{s+1}(\Omega)$ is continuously embedded in $C_b^2(\Omega)$ for every bounded and open subset of $\mathbb{R}^d$. In particular $u_\eta \in C^2(\mathbb{R}^d, \mathbb{R}^n)$. Hence we can apply the mean value theorem:

$$= \left| \int_{\mathbb{R}^d} \int_{B_\eta(0)} V_{ij}^\eta(y) \left(\nabla u_{\eta,j}(x-y) - \nabla u_{\eta,j}(x)\right) \cdot v(x) dy dx \right|$$

$$= \left| \int_{\mathbb{R}^d} \int_{B_\eta(0)} V_{ij}^\eta(y) \sum_{l=1}^{d} \left(\frac{\partial u_{\eta,j}}{\partial x_l}(x-y) - \frac{\partial u_{\eta,j}}{\partial x_l}(x) \right) v_l(x) dy dx \right|$$

$$= \left| \int_{\mathbb{R}^d} \int_{B_\eta(0)} V_{ij}^\eta(y) \sum_{l=1}^{d} \left(\int_0^1 \nabla \frac{\partial u_{\eta,j}}{\partial x_l}(x - y\zeta) d\zeta \right) \cdot (-y) v_l(x) dy dx \right|$$

$$= \left| \int_{\mathbb{R}^d} \int_{B_\eta(0)} V_{ij}^\eta(y) \sum_{k,l=1}^{d} \left(\int_0^1 \frac{\partial^2 u_{\eta,j}}{\partial x_k \partial x_l}(x - y\zeta) d\zeta \right) y_k v_l(x) dy dx \right|$$

$$= \left| \int_0^1 \int_{\mathbb{R}^d} \int_{B_\eta(0)} V_{ij}^\eta(y) y^T D^2 u_{\eta,j}(x - y\zeta) v(x) dy dx d\zeta \right|$$

$$\leq \int_0^1 \int_{B_\eta(0)} \left| V_{ij}^\eta(y) \right| \int_{\mathbb{R}^d} |y| \left| D^2 u_{\eta,j}(x - y\zeta) \right| |v(x)| dx dy d\zeta$$

$$\leq \eta \int_0^1 \int_{B_\eta(0)} \left| V_{ij}^\eta(y) \right| \left\| D^2 u_{\eta,j}(\cdot - y\zeta) \right\|_{L^2} \|v\|_{L^2} dy d\zeta$$

$$\leq \eta \int_0^1 \left\| V_{ij}^\eta \right\|_{L^1} \left\| D^2 u_{\eta,j} \right\|_{L^2} \|v\|_{L^2} d\zeta = \eta \|V_{ij}\|_{L^1} \left\| D^2 u_{\eta,j} \right\|_{L^2} \|v\|_{L^2}.$$

Due to $u \in L^\infty(0, T^*; H^s)$,

$$\left| \langle (V_{ij}^\eta * \nabla u_{\eta,j} - a_{ij} \nabla u_{\eta,j}), v \rangle_{L^2} \right| \leq \eta \|V_{ij}\|_{L^1} \left\| D^2 u_{\eta,j} \right\|_{L^2} \|v\|_{L^2} \leq \eta C \|v\|_{L^2}$$

for some constant C. As v was arbitrary, we get

$$\left\| V_{ij}^{\eta} * \nabla u_{\eta,j} - a_{ij} \nabla u_{\eta,j} \right\|_{L^2} \leq \eta \|V_{ij}\|_{L^1} \left\| D^2 u_{\eta,j} \right\|_{L^2} \leq \eta C. \tag{6.21}$$

This, together with an application of Young's inequality for products, leads to

$$(6.18) \leq \frac{n\delta}{2} \|\nabla(u - u_\eta)\|^2 + \frac{\|u\|_{H^s}^2 c_{H^s \hookrightarrow L^\infty}^2}{2\delta} \sum_{i,j=1}^{n} \left\| V_{ij}^{\eta} * \nabla u_{\eta,j} - a_{ij} \nabla u_{\eta,j} \right\|_{L^2}^2$$

$$\leq \frac{n\delta}{2} \|\nabla(u - u_\eta)\|^2 + C(\delta) \|u_0\|_{H^s}^2 \eta^2$$

where $\delta > 0$ is arbitrary. In the case of the term (6.19) we apply Young's inequality for products. Furthermore we make use of the embedding $H^s \hookrightarrow L^\infty$:

$$(6.19) \leq \sum_{i,j=1}^{n} \|u_{\eta,i} - u_i\|_{L^2} \left\| V_{ij}^{\eta} \right\|_{L^1} \|\nabla u_{\eta,j}\|_{L^\infty} \|\nabla(u_i - u_{\eta,i})\|_{L^2}$$

$$\leq \frac{n\delta}{2} \|\nabla(u - u_\eta)\|_{L^2}^2 + \frac{c_{H^s \hookrightarrow L^\infty}^2}{2\delta} \sum_{i,j=1}^{n} \|u_{\eta,i} - u_i\|_{L^2}^2 \left\| V_{ij}^{\eta} \right\|_{L^1}^2 \|\nabla u_{\eta,j}\|_{H^s}^2$$

$$\leq \frac{n\delta}{2} \|\nabla(u - u_\eta)\|_{L^2}^2 + C(\delta) \|\nabla u_\eta\|_{H^s}^2 \|u_\eta - u\|_{L^2}^2.$$

The remaining term 6.20 undergoes a similar treatment:

$$(6.20) \leq \sum_{i,j=1}^{n} \|u_i - u_{i,\eta}\|_{L^2} c_{H^s \hookrightarrow L^\infty} \|g_{ij}\|_{H^s} \|\nabla(u_i - u_{\eta,i})\|_{L^2}$$

$$\leq \frac{n\delta}{2} \|\nabla(u - u_\eta)\|_{L^2}^2 + \frac{n c_{H^s \hookrightarrow L^\infty}^2 \max_{1 \leq i,j \leq n} \|g_{ij}\|_{L^\infty(0,T;H^s)}^2}{2\delta} \|u - u_\eta\|_{L^2}^2.$$

All in all, we have

$$\frac{1}{2} \frac{d}{dt} \|u - u_\eta\|_{L^2}^2 + \epsilon \|\nabla(u - u_\eta)\|_{L^2}^2$$

$$\leq (\epsilon - \gamma + \frac{3n\delta}{2}) \|\nabla(u - u_\eta)\|_{L^2}^2 + C(\delta) \|u_0\|_{H^s}^2 \eta^2 + C(\delta) \left(\|\nabla u_\eta\|_{H^s}^2 + 1 \right) \|u_\eta - u\|_{L^2}^2.$$

We choose $\delta = \frac{\gamma}{3n}$. Then some terms on the right hand side get absorbed by the corresponding term on the left hand side and we end up with

$$\frac{1}{2} \frac{d}{dt} \|u - u_\eta\|_{L^2}^2 + \frac{\gamma}{2} \|\nabla(u - u_\eta)\|_{L^2}^2 \leq C\eta^2 + C_1 \left(\|\nabla u_\eta\|_{H^s}^2 + 1 \right) \|u_\eta - u\|_{L^2}^2$$

or alternatively in integral form:

$$\|(u-u_\eta)(t)\|_{L^2}^2 + \frac{\gamma}{2}\|\nabla(u-u_\eta)\|_{L^2(0,t,L^2)}^2 \leq C\eta^2 t + C_1 \int_0^t \left(\|\nabla u_\eta(r)\|_{H^s}^2 + 1\right) \|(u_\eta - u)(r)\|_{L^2}^2 dr \tag{6.22}$$

where we recall that u and u_δ satisfy the same initial value condition. We apply Gronwall's inequality (see C.3)

$$\|(u-u_\eta)(t)\|_{L^2}^2 \leq C\eta^2 t + C_1\eta^2 \int_0^t r \exp(\int_r^t \|\nabla u_\eta(\zeta)\|_{H^s}^2 d\zeta)\|\nabla u_\eta(r)\|_{H^s}^2 dr.$$

Because $\|\nabla u_\eta\|_{L^2(0,T^*,H^s)}$ is by assumption uniformly bounded, we have

$$\|(u-u_\eta)(t)\|_{L^2}^2 \leq C\eta^2 t + C_1 t^2\eta^2. \tag{6.23}$$

Using (6.23) in (6.22) and recalling the uniform boundedness of u_η again finishes the proof. □

7. Intermediate Results

In this chapter the results of the previous chapters are summarized and put into context. Furthermore, an idea is given how to show that the occurring probability density functions, corresponding to (SDE I) and (SDE II), are the solutions of the intermediate system (PDE I) and the cross-diffusion system (PDE II).

7.1. Recapitulation

So far, we showed the existence of weak solutions for two cases:

i) $\sigma_i \in L^\infty(0,T;L^\infty)$ where σ_i does not depend on the space variable $x \in \mathbb{R}^d$, that is to say $\sigma_i(x,t) = \sigma_i(t)$ for all $t \in [0,T]$.

ii) $\sigma_i \in C(0,T;H^s) \cap L^\infty(0,T;W^{s+1,\infty})$ and $\partial_t \sigma_i \in C(0,T;H^{s-2}) \cap L^2(0,T;H^{s-1})$ where s is a natural number with $s > \frac{d}{2} + 2$.

In both cases σ_i has to be symmetric, uniform coercive and satisfy the following Hölder-condition (see theorem 4.2.4, theorem 4.2.8 and remark 4.1.20):
There exist constants $\beta, C > 0$ such that $\frac{2}{3} < \beta \leq 1$ and

$$\left|y^T \sigma_i(x,t)y - y^T \sigma_i(x,s)y\right| \leq C|t-s|^\beta |y|^2 \tag{7.1}$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0,T]$.
Additionally, for the existence of solutions to the stochastic systems (SDE 0)-(SDE II), $\sqrt{\sigma_i}$ has to satisfy

$$\begin{aligned} |\sqrt{\sigma_i}(x,t) - \sqrt{\sigma_i}(y,t)| &\leq C|x-y|, \\ |\sqrt{\sigma_i}(x,t)| &\leq C(1+|x|), \end{aligned} \tag{7.2}$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0,T]$ for some $C > 0$. The same condition needs to hold for the first derivative of the solution u of (PDE II) as well. But they are automatically satisfied for ∇u on the interval $[0,T^*]$ because $H^s \subset C^2$ for $s > \frac{d}{2} + 2$, and the second derivative is uniformly bounded due to $u \in C(0,T^*;H^s)$ and $H^s \hookrightarrow L^\infty$. Therefore an application of the mean value theorem yields the Lipschitz condition. The linear growth condition is satisfied due to the boundedness of ∇u which is, due to Sobolev embeddings, an element of $L^\infty(0,T^*;C_b^1)$. We therefore have:

Corollary 7.1.1. *Let σ_i, $1 \leq i \leq n$, satisfy either of the two cases i) or ii) above and let it fulfil the conditions (7.1) and (7.2). Then there exists $0 < T^* \leq T$ such that there are unique stochastic processes which solve the systems (SDE 0)-(SDE II) on the time interval $[0,T^*)$. Additionally, the solutions of (SDE I) and (SDE II) have probability density functions. If the assumptions on σ_i, $1 \leq i \leq n$, are fulfilled for $T = \infty$ and $g_{ij} = \frac{1}{n}\sum_{l=1}^d \frac{\partial}{\partial x_l}(\sigma_i)_{\cdot,l} \equiv 0$ then, under condition 6.11, $T = T^* = \infty$.*

7.2. Identification of the Probability-Density Functions

In this section we want to give an idea on how to prove that the density functions of the processes $\widehat{X}_i^k$ and $\bar{X}_{\eta,i}^k$ solve the differential equations corresponding to (PDE I) and (PDE II). Because we do not want to loose ourselves in details, not all of the following arguments are fully worked out.

Let i and k, with $1 \leq i \leq n$ and $1 \leq k \leq N_i$, be arbitrary but fixed and $s > \frac{d}{2} + 2,\ s \in \mathbb{N}$. Furthermore, let $\tilde{u}_i$ be the probability density function of the process $\widehat{X}_i^k$, which solves the stochastic differential equation

$$\begin{aligned} d\widehat{X}_i^k(t) &= -\sum_{j=1}^{n} a_{ij}\nabla u_j(\widehat{X}_i^k(t))dt + \sqrt{2}\sqrt{\sigma_i}(\widehat{X}_i^k(t),t)dW_i^k(t) \\ \widehat{X}_i^k(0) &= \xi_i^k, \end{aligned}$$

where u is the solution of (PDE II). Then $\tilde{u}_i$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(x,t)\tilde{u}_i(x,t)dx &= \int_{\mathbb{R}^d} \phi(x,0)\tilde{u}_i(x,0)dx \\ &+ \int_0^t \int_{\mathbb{R}^d} \frac{\partial \phi}{\partial s}(x,s)\tilde{u}_i(x,s)dxds - \int_0^t \int_{\mathbb{R}^d} \sum_{j=1}^{n} a_{ij}\tilde{u}_i(x,s)\nabla u_j(x,s)\cdot\nabla\phi(x,s)dxds \\ &+ \int_0^t \int_{\mathbb{R}^d} \sum_{k,l=1}^{d} \frac{\partial \phi}{\partial x_k \partial x_l}(x,s)\sigma_i(x,s)_{kl}\tilde{u}_i(x,s)dxds \end{aligned} \tag{7.3}$$

for all $\phi \in C_b^2(\mathbb{R}^d \times [0,T^*])$ and $t \in [0,T^*]$ (see lemma 3.2.1). Because this is a weak formulation of (PDE II), 7.3 holds for u_i as well. Let $\bar{u}_i := \tilde{u}_i - u_i$. Then we have

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(x,t)\bar{u}_i(x,t)dx &= \int_0^t \int_{\mathbb{R}^d} \frac{\partial \phi}{\partial s}(x,s)\bar{u}_i(x,s)dxds - \int_0^t \int_{\mathbb{R}^d} \sum_{j=1}^{n} a_{ij}\bar{u}_i(x,s)\nabla u_j(x,s)\cdot\nabla\phi(x,s)dxds \\ &+ \int_0^t \int_{\mathbb{R}^d} \sum_{k,l=1}^{d} \frac{\partial \phi}{\partial x_k \partial x_l}(x,s)\sigma_i(x,s)_{kl}\bar{u}_i(x,s)dxds \end{aligned}$$

where we recall that $u_i\,|_{t=0}$ is the density function of ξ_i^k and therefore $u_i(x,0) = \tilde{u}_i(x,0)$ almost everywhere. The above calculation can also be written as

$$\int_{\mathbb{R}^d} \phi(x,t)\bar{u}_i(x,t)dx$$

$$= \int_0^t \int_{\mathbb{R}^d} \left(\frac{\partial \phi}{\partial s}(x,s) - \sum_{j=1}^{n} a_{ij}\nabla u_j(x,s)\cdot\nabla\phi(x,s) + \sum_{k,l=1}^{d} \frac{\partial \phi}{\partial x_k \partial x_l}(x,s)\sigma_i(x,s)_{kl} \right) \bar{u}_i(x,s)dxds.$$

We now assume that for every $\psi \in \mathcal{D}(\mathbb{R}^d)$ we can find a smooth solution ϕ of the (backwards) evolution equation

$$\frac{\partial \phi}{\partial s}(x,s) - \sum_{j=1}^{n} a_{ij} \nabla u_j(x,s) \cdot \nabla \phi(x,s) + \sum_{k,l=1}^{d} \frac{\partial \phi}{\partial x_k \partial x_l}(x,s) \sigma_i(x,s)_{kl} = 0, \tag{7.4}$$

for (x,t) in $\mathbb{R}^d \times [0,T^*]$, with $\phi\,|_{s=t} = \psi$. Under this assumption, we can apply the fundamental lemma of calculus of variations. This would yield

$$\bar{u}_i(x,t) = 0$$

almost everywhere for all $t \in [0,T^*]$ and thus, $\tilde{u}_i = u_i$ almost everywhere. Our next step is therefore to prove the existence of a solution to the problem

$$\frac{\partial \phi}{\partial s}(x,s) - \sum_{k,l=1}^{d} \frac{\partial \phi}{\partial x_k \partial x_l}(x,s) \sigma_i(x,t-s)_{kl} = -\sum_{j=1}^{n} a_{ij} \nabla u_j(x,t-s) \cdot \nabla \phi(x,s) \tag{7.5}$$

with $\phi\,|_{s=0} = \psi$, which is equivalent to (7.4). We rewrite (7.5) to get

$$\frac{\partial \phi}{\partial s} - \operatorname{div}(\sigma_i \nabla \phi) = -\sum_{j=1}^{n} a_{ij} \nabla u_j \cdot \nabla \phi - \sum_{k=1}^{d} \left(\frac{\partial \sigma}{\partial x_k} \nabla \phi \right)_k. \tag{7.6}$$

For (7.6) we can find a weak solution $\phi \in L^\infty(0,T;H^s) \cap L^2(0,T;H^{s+1})$ with the same techniques we used for (PDE II), via the modified intermediate system

$$\frac{\partial \phi}{\partial s} - \operatorname{div}(\sigma_i \nabla \phi) = -\sum_{j=1}^{n} \nabla (V_{ij}^\eta * u_j) \cdot \nabla \phi - \sum_{k=1}^{d} \left(\frac{\partial \sigma}{\partial x_k} \nabla \phi \right)_k, \tag{7.7}$$

as demonstrated in chapter 6 (see in particular theorem 6.1.2 and lemma 6.1.1). For the equation (7.7) we can forgo as in chapter 4 due to the linearity and the similar multiplicative structure of the reaction term on the right-hand side. Let $\Psi \in \mathcal{D}(\mathbb{R}^d \times [0,T^*])$. Then ϕ satisfies

$$\begin{aligned}
\int_0^t \langle \frac{\partial \phi}{\partial s}, \Psi \rangle_{H^{-1}} ds &= -\int_0^t \langle \nabla \phi^T \sigma_i, \nabla \Psi \rangle_{L^2} ds - \sum_{j=1}^{n} \int_0^t \langle a_{ij} \nabla u_j \cdot \nabla \phi, \Psi \rangle_{L^2} ds \\
&\quad - \int_0^t \langle \sum_{k=1}^{d} \left(\frac{\partial \sigma}{\partial x_k} \nabla \phi \right)_k, \Psi \rangle_{L^2} ds \\
&= \int_0^t \langle \operatorname{div}\left(\nabla \phi^T \sigma_i \right), \Psi \rangle_{L^2} ds - \sum_{j=1}^{n} \int_0^t \langle a_{ij} \nabla u_j \cdot \nabla \phi, \Psi \rangle_{L^2} ds \\
&\quad - \int_0^t \langle \sum_{k=1}^{d} \left(\frac{\partial \sigma}{\partial x_k} \nabla \phi \right)_k, \Psi \rangle_{L^2} ds.
\end{aligned}$$

$\int_0^t \langle \frac{\partial \phi}{\partial s}, \Psi \rangle_{H^{-1}} ds$ can therefore be extended to a continuous functional for $\Psi \in L^2(0,t;L^2)$. Thus, $\frac{\partial \phi}{\partial s} \in (L^2(0,t;L^2))' = L^2(0,t;L^2)$. Similarly one can show $\frac{\partial \phi}{\partial s} \in L^2(0,t;H^{s-1})$ by testing with $D^\alpha \Psi$ with $|\alpha| \leq s-1$. Because we need differentiability regarding the time-variable to make use of the identity (7.3), we approximate ϕ with differentiable functions. Let θ therefore be an element of $C_c^\infty(\mathbb{R}^d \times (0,t))$ with

i) $\theta \geq 0$,

ii) $\operatorname{supp} \theta \subset B_1(0)$,

iii) $\int\limits_{-\infty}^{\infty} \int\limits_{\mathbb{R}^d} \theta(x,t) dx dt = 1$.

We define[1] $\theta_r(x,t) := \frac{1}{r^{d+1}} \theta(\frac{x}{r}, \frac{t}{r})$. Notice that $\operatorname{supp} \theta_r \subset B_r(0)$. In the case of $f \in L^p$, the family $f * \theta_r \in L^p$ converges towards f regarding the L^p-norm for $r \to 0$ (see [31, Theorem 19.15]). Let $\tilde{\phi}$ be defined via

$$\tilde{\phi} := \begin{cases} \phi(t) & ,s > t \\ \phi(s) & ,0 \leq s \leq t \\ \phi(0) & ,s < 0 \end{cases} .$$

Let $0 < \delta << 1$. Then $g_{\delta,r} := \theta_r * \left(\chi_{[-\delta,t+\delta]} \tilde{\phi} \right)$ converges towards $\chi_{[-\delta,t+\delta]} \tilde{\phi}$ in $L^2(\mathbb{R}^d \times (-\delta, t+\delta))$ for $r \to 0$. Because we can identify $L^2(a,b;L^2(\Omega))$ with $L^2(\Omega \times (a,b))$, we get convergence regarding the $L^2(-\delta, t+\delta; L^2)$-norm as well. For $\delta \to 0$ we get additionally convergence of $\chi_{[-\delta,t+\delta]} \tilde{\phi}$ towards ϕ in $L^2(0,t,L^2)$, due to the monotone convergence theorem. Therefore there exists a sub-series of $g_{\delta,r}$ which converges towards ϕ in $L^2(0,t,L^2)$. This convergence can be extended to $L^2(0,t,H^{s+1})$ because $\phi \in L^2(0,t;H^{s+1})$ and

$$D^\alpha \left(\theta_r * \left(\chi_{[-\delta,t+\delta]} \tilde{\phi} \right) \right) = \theta_r * \left(D^\alpha \chi_{[-\delta,t+\delta]} \tilde{\phi} \right) = \theta_r * D^\alpha \left(\chi_{[-\delta,t+\delta]} \tilde{\phi} \right) = \theta_r * \left(\chi_{[-\delta,t+\delta]} D^\alpha \tilde{\phi} \right)$$

where $|\alpha| \leq s+1$. For the time derivative in particular we get

$$\begin{aligned} \frac{\partial}{\partial s} \left(\theta_r * \left(\chi_{[-\delta,t+\delta]} \tilde{\phi} \right) (x,s) \right) &= \frac{\partial \theta_r}{\partial s} * \left(\chi_{[-\delta,t+\delta]} \tilde{\phi} \right) (x,s) \\ &= \int\limits_{-\infty}^{\infty} \int\limits_{\mathbb{R}^d} \chi_{[-\delta,t+\delta]} \tilde{\phi}(y,\tau) \frac{\partial \theta_r}{\partial s}(x-y, s-\tau) dy d\tau \\ &= \int\limits_{-\delta}^{t+\delta} \int\limits_{\mathbb{R}^d} \tilde{\phi}(y,\tau) \frac{\partial \theta_r}{\partial s}(x-y, s-\tau) dy d\tau. \end{aligned}$$

This yields us

$$\frac{\partial}{\partial s} \left(\theta_r * \left(\chi_{[-\delta,t+\delta]} \tilde{\phi} \right) (x,s) \right)$$

[1] In literature, the family $\theta_r, r > 0$ is often called a mollifier.

$$= \int_{-\delta}^{0} \langle \tilde{\phi}(\tau), \frac{\partial \theta_r}{\partial s}(x - \cdot, s - \tau) \rangle_{L^2} d\tau + \int_{0}^{t} \langle \phi(\tau), \frac{\partial \theta_r}{\partial s}(x - \cdot, s - \tau) \rangle_{L^2} d\tau$$
$$+ \int_{t}^{t+\delta} \langle \tilde{\phi}(\tau), \frac{\partial \theta_r}{\partial s}(x - \cdot, s - \tau) \rangle_{L^2} d\tau$$
$$= - \langle \phi(0), \theta_r(x - \cdot, s - \tau) \rangle_{L^2} |_{\tau=-\delta}^{0} + \int_{0}^{t} \langle \frac{\partial \phi}{\partial \tau}(\tau), \theta_r(x - \cdot, s - \tau) \rangle_{H^{-1}} d\tau$$
$$- \langle \phi(\tau), \theta_r(x - \cdot, s - \tau) \rangle_{L^2} |_{\tau=0}^{t} - \langle \phi(t), \theta_r(x - \cdot, s - \tau) \rangle_{L^2} |_{\tau=t}^{t+\delta}$$
$$= \langle \phi(0), \theta_r(x - \cdot, s + \delta) \rangle_{L^2} - \langle \phi(t + \delta), \theta_r(x - \cdot, s - \delta - t) \rangle_{L^2} + \int_{0}^{t} \langle \frac{\partial \phi}{\partial \tau}(\tau), \theta_r(x - \cdot, s - \tau) \rangle_{L^2} d\tau$$
$$= \langle \phi(0), \theta_r(x - \cdot, s + \delta) \rangle_{L^2} - \langle \phi(t + \delta), \theta_r(x - \cdot, s - \delta - t) \rangle_{L^2} + \theta_r * \left(\chi_{[0,t]} \frac{\partial \phi}{\partial s} \right) (x, s).$$

In the case of $r \leq \delta$ and $s \in [0, t]$, we especially get

$$\frac{\partial}{\partial s} \left(\theta_r * \left(\chi_{[-\delta, t+\delta]} \tilde{\phi} \right) \right) (x, s) = \theta_r * \left(\chi_{[0,t]} \frac{\partial \phi}{\partial s} \right) (x, s).$$

Therefore $\frac{\partial}{\partial s} \left(\theta_r * \left(\chi_{[-\delta,t+\delta]} \tilde{\phi} \right) \right)$ converges in $L^2(0, t; L^2)$ towards $\frac{\partial \phi}{\partial s}$.

With the same argumentation as above we can extend this convergence to $L^2(0, t; H^{s-1})$ and due to $s > \frac{d}{2} + 2$, all the considered convergences hold also in $L^2(0, t; L^\infty)$ as $H^{s-1} \hookrightarrow L^\infty$. This is necessary as we only have $\tilde{u}_i(s) \in L^\infty(0, T^*; L^1)$. Furthermore we get convergence in $C([0, t], H^s)$ due to the continuous embedding $W^{1,2}(0, t; H^{s+1}; H^s) \hookrightarrow C([0, t], H^s)$. As differentiable function, once in time and at least twice in space, $g_{\delta,r}(x, t - s)$ can be used in (7.3). The limit[2] $r, \delta \to 0$ leads then to

$$\int_{\mathbb{R}^d} \psi \bar{u}_i(x, t) dx$$
$$= \int_{0}^{t} \int_{\mathbb{R}^d} \left(-\frac{\partial \phi}{\partial s}(x, t - s) - \sum_{j=1}^{n} a_{ij} \nabla u_j(x, s) \cdot \nabla \phi(x, t - s) + \sum_{k,l=1}^{d} \frac{\partial \phi}{\partial x_k \partial x_l}(x, t - s) \sigma_i(x, s)_{kl} \right) \times$$
$$\times \bar{u}_i(x, s) dx ds = 0. \tag{7.8}$$

As ψ was arbitrary, $u_i(\cdot, t) = \tilde{u}_i(\cdot, t)$ almost everywhere. The same argumentation can be applied to the probability density function of the solution $\bar{X}_{\eta,i}^k$ to (SDE I) and the weak solution of (PDE I). We therefore have:

Corollary 7.2.1. *Let the assumptions of corollary 7.1.1 hold. Then the probability density functions of the processes $\bar{X}_{\eta,i}^k$ and $\widehat{X}_i^k$, characterized by (SDE I) and (SDE II), form the unique weak solutions of (PDE I) and (PDE II).*

[2] We use the sub-series of (δ, r) for which the above convergences hold.

8. Error-Estimate

This chapter focuses on estimating the mean absolute error made when approximating the solution $X_{\eta,i}^{N_i,k}$ of (SDE 0) with the solution $\widehat{X}_i^k$ of (SDE II).

8.1. Estimate for the Approximation-Error

The stochastic differential equation (SDE III) originates as the formal limit of (SDE I) when $\eta \to 0$ and $N_i >> 1$ (see section 1.1). For that reason one might expect that the solution $\widehat{X}_i^k$ of (SDE III) can be used to approximate the solution $X_{\eta,i}^{k,N_i}$ of (SDE I) in a certain sense. To make this intuition rigorous we will prove the following error-estimate. A proof of this estimate for constant diffusion $\sigma > 0$ can be found in [5](see [5, Theorem 3]).

Theorem 8.1.1. *Let the assumptions of theorem 6.2.1 and lemma 2.1.1 hold. Furthermore let $s > \frac{d}{2} + 2$, $0 < T^{'} \leq T^*$, where T^* is as in corollary 6.1.3 with $T^{'} \neq \infty$ and let $0 < \nu < 1$ be sufficiently small. If for fixed $\eta > 0$ and $N := \min\limits_{1 \leq i \leq n} N_i$ the inequality $\nu \log N \geq \eta^{-2d-4}$ is satisfied, then*

$$\sup_{1 \leq k \leq N} \mathbb{E}\left(\sum_{i=1}^{n} \sup_{0<s<t} \left| X_{\eta,i}^{k,N} - \widehat{X}_i^k \right| \right) \leq C_{T'}(t)\eta, \quad t \in (0, T^{'}] \tag{8.1}$$

where $C_{T'}(t) \geq 0$ is a continuous function of t which depends additionally on $T^{'}, n, \left\|D^2 V_{ij}\right\|_{L^\infty}$, $\|D\sigma_i\|_{L^\infty(0,T;L^\infty)}$ and u^0.

To show the result above, we will proceed as in the proof of theorem 3 of [5]. The left hand side of (8.1) can be bounded above by

$$\sup_{1 \leq k \leq N} \mathbb{E}\left(\sum_{i=1}^{n} \sup_{0<s<t} \left| X_{\eta,i}^{k,N} - \widehat{X}_i^{k,N} \right| \right)$$

$$\leq \sup_{1 \leq k \leq N} \mathbb{E}\left(\sum_{i=1}^{n} \sup_{0<s<t} \left| X_{\eta,i}^{k,N} - \bar{X}_{\eta,i}^k \right| \right) + \sup_{1 \leq k \leq N} \mathbb{E}\left(\sum_{i=1}^{n} \sup_{0<s<t} \left| \bar{X}_{\eta,i}^k - \widehat{X}_i^k \right| \right) \tag{8.2}$$

which can easily be seen by applying the triangle inequality. Here $\bar{X}_{\eta,i}^{k,N}$ is the solution of the intermediate stochastic system (SDE II). We will derive error estimates for the two terms on the right hand side of (8.2) to conclude theorem 8.1.

Lemma 8.1.2. *Let the assumptions of theorem 8.1.1 hold. Then*

$$\sup_{1 \leq k \leq N} \mathbb{E}\left(\sum_{i=1}^{n} \sup_{0<s<t} \left| X_{\eta,i}^{k,N}(s) - \bar{X}_{\eta,i}^k(s) \right| \right) \leq C(t) N^{(C(t)\nu - 1)/2}, \quad t \in (0, T^{'}]$$

where $C(t) \geq 0$ is a continuous function on $[0, T]$, which depends additionally on n, $\left\|D^2 V_{ij}\right\|_{L^\infty}$, $\|D\sigma_i\|_{L^\infty(0,T;L^\infty)}$ and u_0.

Proof. We proceed as in the proof of lemma 13 in [5]. Let $1 \leq k \leq N$. We define the stochastic processes S_t^k via

$$S_t^k = \sum_{i=1}^n \sup_{0<s<t} \left| X_{\eta,i}^{k,N}(s) - \bar{X}_{\eta,i}^k(s) \right|^2.$$

Subtracting the equation (SDE II) from (SDE I) results in

$$\begin{aligned} X_{\eta,i}^{k,N}(s) - \bar{X}_{\eta,i}^k(s) &= -\int_0^s \sum_{j=1}^n \frac{1}{N_j} \sum_{l=1}^{N_j} \left(\nabla V_{ij}^\eta (X_{\eta,i}^{k,N_i}(r) - X_{\eta,j}^{l,N_j}(r)) - \nabla V_{ij}^\eta * u_{\eta,j}(\bar{X}_{\eta,i}^k(r), r) \right) dr \\ &+ \sqrt{2} \int_0^s \sqrt{\sigma_i}(X_{\eta,i}^{k,N_i}(r), r) - \sqrt{\sigma_i}(\bar{X}_{\eta,i}^k(r), r) dW_i^k(r) \\ &=: I_1(s) + I_2(s) \end{aligned}$$

which in turn implies

$$\begin{aligned} \mathbb{E}\left[S_t^k\right] = \mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} \left| X_{\eta,i}^{k,N_j}(s) - \bar{X}_{\eta,i}^k(s) \right|^2\right] &= \mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} |I_1(s) + I_2(s)|^2\right] \\ &\leq 2\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} |I_1(s)|^2\right] + 2\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} |I_2(s)|^2\right] \end{aligned}$$

where we used the triangle inequality and the fact that $2ab \leq a^2 + b^2$. We take a closer look at the first of the two remaining terms:

$$\begin{aligned} &\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} |I_1(s)|^2\right] \\ =&\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} \left| \int_0^s \sum_{j=1}^n \frac{1}{N_j} \sum_{l=1}^{N_j} \left(\nabla V_{ij}^\eta (X_{\eta,i}^{k,N_i}(r) - X_{\eta,j}^{l,N_j}(r)) - \nabla V_{ij}^\eta * u_{\eta,j}(\bar{X}_{\eta,i}^k(r), r) \right) dr \right|^2\right] \\ \leq&\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} \left(\int_0^s \sum_{j=1}^n \frac{1}{N_j} \left| \sum_{l=1}^{N_j} \nabla V_{ij}^\eta (X_{\eta,i}^{k,N_i}(r) - X_{\eta,j}^{l,N_j}(r)) - \nabla V_{ij}^\eta * u_{\eta,j}(\bar{X}_{\eta,i}^k(r), r) \right| dr \right)^2\right] \\ \leq&\mathbb{E}\left[\sum_{i=1}^n \left(\int_0^t \sum_{j=1}^n \frac{1}{N_j} \left| \sum_{l=1}^{N_j} \nabla V_{ij}^\eta (X_{\eta,i}^{k,N_i}(r) - X_{\eta,j}^{l,N_j}(r)) - \nabla V_{ij}^\eta * u_{\eta,j}(\bar{X}_{\eta,i}^k(r), r) \right| dr \right)^2\right]. \end{aligned}$$

Applying the Cauchy-Schwarz inequality regarding t and j, we get

$$\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} |I_1(s)|^2\right]$$

$$\leq \mathbb{E}\left[\sum_{i=1}^n \int_0^t \sum_{j=1}^n \frac{tn}{N_j^2} \left| \sum_{l=1}^{N_j} \nabla V_{ij}^\eta (X_{\eta,i}^{k,N_i}(r) - X_{\eta,j}^{l,N_j}(r)) - \nabla V_{ij}^\eta * u_{\eta,j}(\bar{X}_{\eta,i}^k(r), r) \right|^2 dr \right]$$

$$\leq \mathbb{E}\left[\sum_{i=1}^n \int_0^t \sum_{j=1}^n \frac{3tn}{N_j^2} \left| \sum_{l=1}^{N_j} \nabla V_{ij}^\eta (X_{\eta,i}^{k,N_i}(r) - X_{\eta,j}^{l,N_j}(r)) - \nabla V_{ij}^\eta (X_{\eta,i}^{k,N_i}(r) - \bar{X}_{\eta,j}^l(r)) \right|^2 dr \right]$$

$$+ \mathbb{E}\left[\sum_{i=1}^n \int_0^t \sum_{j=1}^n \frac{3tn}{N_j} \left| \sum_{l=1}^{N_j^2} \nabla V_{ij}^\eta (X_{\eta,i}^{k,N_i}(r) - \bar{X}_{\eta,j}^l(r))) - \nabla V_{ij}^\eta (\bar{X}_{\eta,i}^{k,N_i}(r) - \bar{X}_{\eta,j}^l(r)) \right|^2 dr \right]$$

$$+ \mathbb{E}\left[\sum_{i=1}^n \int_0^t \sum_{j=1}^n \frac{3tn}{N_j^2} \left| \sum_{l=1}^{N_j} \nabla V_{ij}^\eta (\bar{X}_{\eta,i}^k(r) - \bar{X}_{\eta,j}^l(r))) - (\nabla V_{ij}^\eta * u_{\eta,j})(\bar{X}_{\eta,i}^k(r), r) \right|^2 dr \right]$$

$$=: I_1^1 + I_1^2 + I_1^3$$

where in the last step we expanded the innermost term, applied the triangle inequality and used the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. Let $x, y \in \mathbb{R}^d$. Because $V_{ij} \in C_b^2(\mathbb{R}^d)$, the mean value theorem implies

$$\left| \nabla V_{ij}^\eta(x) - \nabla V_{ij}^\eta(y) \right| = \left| \int_0^1 D^2 V_{ij}^\eta (x + (x-y)t) dt (x-y) \right| \leq \left\| D^2 V_{ij}^\eta \right\|_{L^\infty} |x-y|.$$

Hence the Lipschitz-constant L_{ij}^η of ∇V_{ij}^η is bounded by $\left\| D^2 V_{ij}^\eta \right\|_{L^\infty}$. Due to $V_{ij}^\eta(x) = \frac{1}{\eta^d} V_{ij}(\frac{x}{\eta})$, this results in $L_{ij}^\eta \leq \frac{1}{\eta^{d+2}} \left\| D^2 V_{ij} \right\|_{L^\infty}$. Using Cauchy-Schwarz inequality regarding l, we therefore obtain the following estimates for the terms I_1^1 and I_1^2:

$$\begin{aligned}
I_1^1 \leq & \mathbb{E}\left[\sum_{i=1}^n \int_0^t \sum_{j=1}^n \frac{3tn}{N_j} \sum_{l=1}^{N_j} (L_{ij}^\eta)^2 \left| X_{\eta,j}^{l,N_j}(r) - \bar{X}_{\eta,j}^l(r) \right|^2 dr \right] \\
\leq & 3tn \sum_{i,j=1}^n (L_{ij}^\eta)^2 \int_0^t \frac{1}{N_j} \sum_{l=1}^{N_j} \mathbb{E}\left[\left| X_{\eta,j}^{l,N_j}(r) - \bar{X}_{\eta,j}^l(r) \right|^2 \right] dr \\
\leq & 3tn \sum_{i,j=1}^n (L_{ij}^\eta)^2 \int_0^t \frac{1}{N_j} \sum_{l=1}^{N_j} \mathbb{E}\left[\sup_{0<r<t} \left| X_{\eta,j}^{l,N_j}(r) - \bar{X}_{\eta,j}^l(r) \right|^2 \right] dr \\
= & 3tn \sum_{i,j=1}^n (L_{ij}^\eta)^2 \int_0^t \mathbb{E}\left[\sup_{0<r<t} \left| X_{\eta,j}^{l,N_j}(r) - \bar{X}_{\eta,j}^l(r) \right|^2 \right] dr
\end{aligned}$$

$$\leq \frac{3tn}{\eta^{2d+4}} \max_{1\leq i,j\leq n} \left\| D^2 V_{ij} \right\|_{L^\infty}^2 \sum_{i,j=1}^n \int_0^t \mathbb{E}\left[\sup_{0<\zeta<r} \left| X_{\eta,j}^{l,N_j}(\zeta) - \bar{X}_{\eta,j}^l(\zeta) \right|^2 \right] dr$$

$$\leq \frac{3tn^2}{\eta^{2d+4}} \max_{1\leq i,j\leq n} \left\| D^2 V_{ij} \right\|_{L^\infty}^2 \int_0^t \mathbb{E}\left[S_r^l \right] dr = \frac{3tn^2}{\eta^{2d+4}} \max_{1\leq i,j\leq n} \left\| D^2 V_{ij} \right\|_{L^\infty}^2 \int_0^t \mathbb{E}\left[S_s^k \right] ds.$$

Here the last equality holds true because S_t^k and S_t^l are identically distributed for all $1 \leq k, l \leq n$. Repeating these arguments for the term I_1^2 yields the same estimate:

$$I_1^2 \leq \frac{3tn^2}{\eta^{2d+4}} \max_{1\leq i,j\leq n} \left\| D^2 V_{ij} \right\|_{L^\infty}^2 \int_0^t \mathbb{E}\left[S_s^k \right] ds.$$

To assess the term I_1^3, we establish some uniform estimates for ∇V_{ij}^η. We recall that $V_{ij}^\eta(x) = \frac{1}{\eta^d} V_{ij}(\frac{x}{\eta}) \in C_b^2(\mathbb{R}^d)$. Thus,

$$\left| \nabla V_{ij}^\eta \right| \leq \max_{1\leq i,j\leq n} \|\nabla V_{ij}\|_{L^\infty} \frac{1}{\eta^{d+1}} \leq \frac{C}{\eta^{d+1}} \tag{8.3}$$

and

$$\left\| \nabla V_{ij}^\eta \right\|_{L^2}^2 = \frac{1}{\eta^{2d+2}} \int_{\mathbb{R}^d} \left| \nabla V_{ij}(\frac{x}{\eta}) \right|^2 dx = \frac{1}{\eta^{2d+2}} \int_{\mathbb{R}^d} |\nabla V_{ij}(x)|^2 \eta^d dx = \frac{1}{\eta^2} \max_{1\leq i,j\leq n} \|\nabla V_{ij}\|_{L^2}^2 \leq \frac{C^2}{\eta^2}. \tag{8.4}$$

Simplifying notation, we define the stochastic process $Y_{i,j}^{k,l}$ via

$$Y_{i,j}^{k,l} := \nabla V_{ij}^\eta(\bar{X}_{\eta,i}^k(r) - \bar{X}_{\eta,j}^l(r)) - (\nabla V_{ij}^\eta * u_{\eta,j})(\bar{X}_{\eta,i}(r), r).$$

The term I_1^3 can then be written as

$$\begin{aligned}
I_1^3 =& \mathbb{E}\left[\sum_{i=1}^n \int_0^t \sum_{j=1}^n \frac{3tn}{N_j^2} \left| \sum_{l=1}^{N_j} \nabla V_{ij}^\eta(\bar{X}_{\eta,i}^k(r) - \bar{X}_{\eta,j}^l(r))) - (\nabla V_{ij}^\eta * u_{\eta,j})(\bar{X}_{\eta,i}^k(r), r) \right|^2 dr \right] \\
=& \sum_{i,j=1}^n \frac{3tn}{N_j^2} \int_0^t \mathbb{E}\left[\left| \sum_{l=1}^{N_j} Y_{i,j}^{k,l}(r) \right|^2 \right] dr \\
=& \sum_{i,j=1}^n \frac{3tn}{N_j^2} \int_0^t \mathbb{E}\left[\sum_{l=1}^{N_j} Y_{i,j}^{k,l}(r) \cdot \sum_{m=1}^{N_j} Y_{i,j}^{k,m}(r) \right] dr \\
=& \sum_{i,j=1}^n \frac{3tn}{N_j^2} \int_0^t \sum_{l=1}^{N_j} \mathbb{E}\left[\left| Y_{i,j}^{k,l}(r) \right|^2 \right] dr + \sum_{i,j=1}^n \frac{3tn}{N_j^2} \int_0^t \sum_{l\neq m} \mathbb{E}\left[Y_{i,j}^{k,l}(r) \cdot Y_{i,j}^{k,m}(r) \right] dr
\end{aligned}$$

Here we used the relation $x \cdot x = \langle x, x \rangle_2 = |x|^2$, where $x \in \mathbb{R}^d$. We will show that the random variables $Y_{i,j}^{k,l}(r), Y_{i,j}^{k,m}(r)$ for $l \neq m$, are uncorrelated. Then the second term, on the right-hand side of the last equation, will vanish and we can use 8.3 and 8.4 to estimate the remaining term. Let therefore $1 \leq l, m \leq n$ and $l \neq m$. Then we have

$$\begin{aligned}
&\mathbb{E}\left[Y_{i,j}^{k,l}(r) \cdot Y_{i,j}^{k,m}(r) \right] \\
=& \mathbb{E}\left[\left(\nabla V_{ij}^\eta(\bar{X}_{\eta,i}^k(r) - \bar{X}_{\eta,j}^l(r))) - \nabla V_{ij}^\eta * u_{\eta,j})(\bar{X}_{\eta,i}^k(r), r) \right) \right. \\
&\left. \cdot \left(\nabla V_{ij}^\eta(\bar{X}_{\eta,i}^k(r) - \bar{X}_{\eta,j}^m(r))) - \nabla V_{ij}^\eta * u_{\eta,j})(\bar{X}_{\eta,i}^k(r), r) \right) \right].
\end{aligned}$$

The stochastic process $\bar{X}^k_{\eta,i}(r)$ is adapted to the filtration $\mathcal{F}^k_i$, which is the filtration initiated by W^k_i and ξ^k_i. As the Brownian motions W^k_i and the initial values ξ^k_i are all independent so are the filtrations F^k_i and thus the stochastic processes $\bar{X}^k_{\eta,i}(r)$. Hence, in the case $i \neq j$, or $i = j$ where simultaneously $k \neq m$ and $k \neq l$, the joint distribution of the stochastic vector $(\bar{X}^k_{\eta,i}(r), \bar{X}^l_{\eta,j}(r), \bar{X}^m_{\eta,j}(r))$ is given by the product of their individual probability density's. In these instances we get

$$\begin{aligned}
\mathbb{E}\left[Y^{k,l}_{i,j}(r)\cdot Y^{k,m}_{i,j}(r)\right] =& \mathbb{E}\left[\nabla V^\eta_{ij}(\bar{X}^k_{\eta,i}(r)-\bar{X}^l_{\eta,j}(r)))\cdot\nabla V^\eta_{ij}(\bar{X}^k_{\eta,i}(r)-\bar{X}^m_{\eta,j}(r)))\right]\\
&+\mathbb{E}\left[(\nabla V^\eta_{ij}*u_{\eta,j})(\bar{X}^k_{\eta,i}(r),r)\cdot(\nabla V^\eta_{ij}*u_{\eta,j})(\bar{X}^k_{\eta,i}(r),r)\right]\\
&-\mathbb{E}\left[\nabla V^\eta_{ij}(\bar{X}^k_{\eta,i}(r)-\bar{X}^l_{\eta,j}(r)))\cdot(\nabla V^\eta_{ij}*u_{\eta,j})(\bar{X}^k_{\eta,i}(r),r)\right]\\
&-\mathbb{E}\left[\nabla V^\eta_{ij}(\bar{X}^k_{\eta,i}(r)-\bar{X}^m_{\eta,j}(r)))\cdot(\nabla V^\eta_{ij}*u_{\eta,j})(\bar{X}^k_{\eta,i}(r),r)\right]\\
=&\mathbb{E}\left[\nabla V^\eta_{ij}(\bar{X}^k_{\eta,i}(r)-\bar{X}^l_{\eta,j}(r)))\cdot\nabla V^\eta_{ij}(\bar{X}^k_{\eta,i}(r)-\bar{X}^m_{\eta,j}(r)))\right]\\
&+\mathbb{E}\left[(\nabla V^\eta_{ij}*u_{\eta,j})^2(\bar{X}^k_{\eta,i}(r),r)\right]\\
&-2\mathbb{E}\left[\nabla V^\eta_{ij}(\bar{X}^k_{\eta,i}(r)-\bar{X}^l_{\eta,j}(r)))\cdot(\nabla V^\eta_{ij}*u_{\eta,j})(\bar{X}^k_{\eta,i}(r),r)\right].
\end{aligned}$$

We use the absolute continuity of the stochastic processes $\bar{X}^k_{\eta,i}(r)$ regarding the Lebesgue-measure to obtain:

$$\begin{aligned}
\mathbb{E}\left[Y^{k,l}_{i,j}(r)\cdot Y^{k,m}_{i,j}(r)\right] =& \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\nabla V^\eta_{ij}(x-y)\cdot\nabla V^\eta_{ij}(x-z)u_{\eta,i}(x)u_{\eta,j}(y)u_{\eta,j}(z)dxdydz\\
&+\int\limits_{\mathbb{R}^d}(\nabla V^\eta_{ij}*u_{\eta,j})^2(x)u(x)_{\eta,i}dx\\
&-2\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\nabla V^\eta_{ij}(x-y)\cdot(\nabla V^\eta_{ij}*u_{\eta,j})(x)u_{\eta,i}(x)u_{\eta,j}(y)dxdy.
\end{aligned}\tag{8.5}$$

Furthermore, we have

$$\begin{aligned}
&=\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\nabla V^\eta_{ij}(x-y)\cdot\nabla V^\eta_{ij}(x-z)u_{\eta,i}(x)u_{\eta,j}(y)u_{\eta,j}(z)dxdydz\\
&=\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\nabla V^\eta_{ij}(x-y)u_{\eta,j}(y)dy\cdot\nabla V^\eta_{ij}(x-z)u_{\eta,i}(x)u_{\eta,j}(z)dxdz\\
&=\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}(\nabla V^\eta_{ij}*u_{\eta,j})(x)\cdot\nabla V^\eta_{ij}(x-z)u_{\eta,i}(x)u_{\eta,j}(z)dxdz\\
&=\int\limits_{\mathbb{R}^d}(\nabla V^\eta_{ij}*u_{\eta,j})(x)\cdot\int\limits_{\mathbb{R}^d}\nabla V^\eta_{ij}(x-z)u_{\eta,j}(z)dzu_{\eta,i}(x)dx\\
&=\int\limits_{\mathbb{R}^d}(\nabla V^\eta_{ij}*u_{\eta,j})(x)\cdot(\nabla V^\eta_{ij}*u_{\eta,j})(x)u_{\eta,i}(x)dx\\
&=\int\limits_{\mathbb{R}^d}(\nabla V^\eta_{ij}*u_{\eta,j})^2(x)u(x_{\eta,i})dx
\end{aligned}$$

and thus, together with (8.5),

$$\mathbb{E}\left[Y_{i,j}^{k,l}(r)\cdot Y_{i,j}^{k,m}(r)\right]=0. \tag{8.6}$$

The remaining case $i=j$ with $k=l$ or $k=m$, can be treated similarly. Therefore (8.6) holds true for every combination of (i,j,m,l) where $m\neq l$ and we obtain

$$\begin{aligned}
I_1^3&=\sum_{i,j=1}^{n}\frac{3tn}{N_j^2}\int_0^t\sum_{l=1}^{N_j}\mathbb{E}\left[\left|Y_{i,j}^{k,l}(r)\right|^2\right]dr\\
&=\sum_{i,j=1}^{n}\frac{3tn}{N_j^2}\int_0^t\sum_{l=1}^{N_j}\mathbb{E}\left[\left|\nabla V_{ij}^{\eta}(\bar{X}_{\eta,i}^{k}(r)-\bar{X}_{\eta,j}^{l}(r))-(\nabla V_{ij}^{\eta}*u_{\eta,j})(\bar{X}_{\eta,i}(r),r)\right|^2\right]dr\\
&\leq\sum_{i,j=1}^{n}\frac{6tn}{N_j^2}\int_0^t\sum_{l=1}^{N_j}\mathbb{E}\left[\left|\nabla V_{ij}^{\eta}(\bar{X}_{\eta,i}^{k}(r)-\bar{X}_{\eta,j}^{l}(r))\right|^2+\left|(\nabla V_{ij}^{\eta}*u_{\eta,j})(\bar{X}_{\eta,i}(r),r)\right|^2\right]dr.
\end{aligned}$$

We use the estimates (8.3) and (8.4) to conclude:

$$\begin{aligned}
I_1^3&\leq\sum_{i,j=1}^{n}\frac{6tn}{N_j^2}\int_0^t\sum_{l=1}^{N_j}\mathbb{E}\left[\frac{C^2}{\eta^{2d+2}}+\left\|\nabla V_{ij}^{\eta}\right\|_{L^2}^2\|u_{\eta,j}\|_{L^2}^2\right]dr\\
&\leq\sum_{i,j=1}^{n}\frac{6tn}{N_j^2}\int_0^t\sum_{l=1}^{N_j}\frac{C^2}{\eta^{2d+2}}+\frac{C^2}{\eta^2}\|u_{\eta,j}\|_{L^\infty(0,T;L^2)}^2dr\\
&\leq\frac{6n^2C^2t^2}{\eta^{2d+2}}\sum_{j=1}^{n}\frac{1}{N_j}\leq\frac{6n^3C^2t^2}{N\eta^{2d+2}}
\end{aligned}$$

for small $\eta>0$ and $N=\min_{1\leq j\leq n}N_j$. All in all we have

$$\mathbb{E}\left[\sum_{i=1}^{n}\sup_{0<s<t}|I_1(s)|^2\right]\leq\frac{6tn^2}{\eta^{2d+4}}\max_{1\leq i,j\leq n}\left\|D^2V_{ij}\right\|_{L^\infty}^2\int_0^t\mathbb{E}\left[S_s^k\right]ds+\frac{6n^3Ct^2}{N\eta^{2d+2}}. \tag{8.7}$$

For the remaining term $\mathbb{E}\left[\sum_{i=1}^{n}\sup_{0<s<t}|I_2(s)|^2\right]$ we recall from stochastic analysis that the projection on the $j-th$ variable

$$I_{2,j}(s)=\sqrt{2}\int_0^s(\sqrt{\sigma_i})_{j,\cdot}(X_{\eta,i}^{k,N_i}(r),r)-(\sqrt{\sigma_i})_{j,\cdot}(\bar{X}_{\eta,i}^{k}(r),r)dW_i^k(r)$$

is a martingale with continuous paths, because I_2 is defined as the Ito-integral of some square-integrable stochastic process. In particular, $|I_{2,j}(s)|$ is a positive sub-martingale. This is a direct conclusion from the convexity and positivity of $|\cdot|$ in combination with monotony of the expectation. Using Doob's L^p-inequality, we get

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} |I_2(s)|^2\right] =&\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} \sum_{j=1}^d |I_{2,j}(s)|^2\right] \\
\leq&\mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^d \sup_{0<s<t} |I_{2,j}(s)|^2\right] = \sum_{i=1}^n \sum_{j=1}^d \mathbb{E}\left[\left(\sup_{0<s<t} |I_{2,j}(s)|\right)^2\right] \\
\leq&4\sum_{i=1}^n \sum_{j=1}^d \mathbb{E}\left[|I_{2,j}(t)|^2\right] \\
=&8\sum_{i=1}^n \sum_{j=1}^d \mathbb{E}\left[\left|\sum_{l=1}^d \int_0^t (\sqrt{\sigma_i})_{j,l}(X_{\eta,i}^{k,N_i}(r),r) - (\sqrt{\sigma_i})_{j,l}(\bar{X}_{\eta,i}^k(r),r) dW_{i,l}^k\right|^2\right] \\
\leq&8d\sum_{i=1}^n \sum_{j=1}^d \mathbb{E}\left[\sum_{l=1}^d \left|\int_0^t (\sqrt{\sigma_i})_{j,l}(X_{\eta,i}^{k,N_i}(r),r) - (\sqrt{\sigma_i})_{j,l}(\bar{X}_{\eta,i}^k(r),r) dW_{i,l}^k\right|^2\right].
\end{aligned} \tag{8.8}$$

Let $G_{j,l}^i(r)$ be the Lipschitz-constant of $(\sqrt{\sigma_i})_{j,l}(r)$. Using the Ito-isometry yields

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t} |I_2(s)|^2\right] \leq&8d\sum_{i=1}^n \sum_{j=1}^d \sum_{l=1}^d \int_0^t \mathbb{E}\left[\left|(\sqrt{\sigma_i})_{j,l}(X_{\eta,i}^{k,N_i}(r),r) - (\sqrt{\sigma_i})_{j,l}(\bar{X}_{\eta,i}^k(r),r)\right|^2\right] dr \\
\leq&8d\sum_{i=1}^n \sum_{j=1}^d \sum_{l=1}^d \int_0^t \mathbb{E}\left[(G_{j,l}^i)^2(r)\left|X_{\eta,i}^{k,N_i}(r) - \bar{X}_{\eta,i}^k(r)\right|^2\right] dr \\
\leq&8d\sum_{i=1}^n \sum_{j=1}^d \sum_{l=1}^d \int_0^t \mathbb{E}\left[\|D(\sqrt{\sigma_i})_{jl}\|^2_{L^\infty(0,T;L^\infty)} \sup_{0<s<r}\left|X_{\eta,i}^{k,N_i}(r) - \bar{X}_{\eta,i}^k(r)\right|^2\right] dr \\
\leq&8d^3 \max_{1\leq j,l\leq d} \|D(\sqrt{\sigma_i})_{jl}\|^2_{L^\infty(0,T;L^\infty)} \int_0^t \mathbb{E}\left[S_r^k\right] dr.
\end{aligned} \tag{8.9}$$

Combining the estimates above and (8.7), we have shown that

$$\mathbb{E}\left[S_t^k\right] \leq \left(\frac{6tn^2}{\eta^{2d+4}} \max_{1\leq i,j\leq n} \left\|D^2 V_{ij}\right\|^2_{L^\infty} + 8d^3 \max_{1\leq m,l\leq d} \|D(\sqrt{\sigma_i})_{ml}\|^2_{L^\infty(0,T;L^\infty)}\right) \int_0^t \mathbb{E}\left[S_s^k\right] ds + \frac{6n^3Ct^2}{N\eta^{2d+2}}$$

and thus

$$\begin{aligned}
&\sup_{1\leq k\leq N} \mathbb{E}\left[S_t^k\right] \\
\leq&\left(\frac{6tn^2}{\eta^{2d+4}} \max_{1\leq i,j\leq n} \left\|D^2 V_{ij}\right\|^2_{L^\infty} + 8d^3 \max_{1\leq m,l\leq d} \|D(\sqrt{\sigma_i})_{ml}\|^2_{L^\infty(0,T;L^\infty)}\right) \int_0^t \mathbb{E}\left[\sup_{1\leq k\leq N} S_s^k\right] ds + \frac{6n^3Ct^2}{N\eta^{2d+2}} \\
\leq&\left(\frac{C_0 t}{\eta^{2d+4}} + C_1\right) \int_0^t \mathbb{E}\left[\sup_{1\leq k\leq N_i} S_s^k\right] ds + \frac{C_2 t^2}{N\eta^{2d+2}}.
\end{aligned}$$

We apply Gronwall's inequality and choose N and ν such that $\nu \log(N) \geq \frac{1}{\eta^{4d+2}}$. In particular the inequality chain

$$N^\nu = \exp(\nu \log(N)) \geq \nu \log(N) \geq \frac{1}{\eta^{2d+4}} \geq \frac{1}{\eta^{2d+2}}$$

holds true. We therefore obtain for $0 < \eta < 1$:

$$\begin{aligned}
\sup_{1\leq k\leq N} \mathbb{E}\left[S_t^k\right] \leq & \frac{C_2 t^2}{N\eta^{2d+2}} \exp\left(\frac{C_0 t^2}{\eta^{2d+4}} + C_1 t\right) \\
\leq & \frac{C_2 t^2}{N\eta^{2d+2}} \exp\left(\frac{C_0 t^2 + C_1 t}{\eta^{2d+4}}\right) \\
\leq & C_2 t^2 N^{\nu-1} \exp((C_0 t^2 + C_1 t)\nu \log(N)) \\
= & C_2 t^2 N^{\nu-1} N^{(C_0 t^2 + C_1 t)\nu)}.
\end{aligned}$$

We set $C(t) := \max\{C_0, C_2\} t^2 + C_1 t + 1$:

$$\sup_{1\leq k\leq N} \mathbb{E}\left[S_t^k\right] \leq C(t) N^{C(t)\nu - 1}.$$

An application of the Cauchy-Schwarz inequality finishes the proof:

$$\begin{aligned}
\sup_{1\leq k\leq N} \mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t}\left|X_{\eta,i}^{k,N}(s) - \bar{X}_{\eta,i}^k(s)\right|\right] \leq & \sqrt{n} \sup_{1\leq k\leq N} \sqrt{\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t}\left|X_{\eta,i}^{k,N_j}(s) - \bar{X}_{\eta,i}^k(s)\right|^2\right]} \sqrt{\mathbb{E}\left[|1|^2\right]} \\
\leq & \sqrt{n} C(t) N^{(C(t)\nu - 1)/2}.
\end{aligned}$$

□

We now show an estimate for the remaining term in (8.2), regarding the difference $\bar{X}_{\eta,i}^k(s) - \widehat{X}_i^k(s)$:

Lemma 8.1.3. *Let the assumptions of theorem 8.1.1 hold. Then*

$$\sup_{1\leq k\leq N} \mathbb{E}\left(\sum_{i=1}^n \sup_{0<s<t}\left|\bar{X}_{\eta,i}^k(s) - \widehat{X}_i^k(s)\right|\right) \leq C(t)\eta, \quad t \in (0, T^{'}]$$

where $C(t) \geq 0$ is a continuous function of t, which depends additionally on n, $\left\|D^2 V_{ij}\right\|_{L^\infty}$, $\|D\sigma_i\|_{L^\infty(0,T;L^\infty)}$ and u_0.

Proof. We proceed as in the proof of lemma 14 in [5].

As in the proof of lemma 8.1.2, we define

$$S_t^k := \sum_{i=1}^n \sup_{0\leq s\leq t}\left|\bar{X}_{\eta,i}^k(s) - \widehat{X}_i^k(s)\right|^2.$$

We take the difference of the two equations (SDE II) and (SDE III) to obtain

$$\bar{X}^k_{\eta,i}(s) - \widehat{X}^k_i(s) = -\int_0^s \sum_{j=1}^n \nabla V^\eta_{ij} * u_{\eta,j}(\bar{X}^k_{\eta,i}(r),r) - a_{ij}\nabla u_j(\widehat{X}^k_i(r),r)dr$$

$$+ \sqrt{2}\int_0^s \sqrt{\sigma_i}(\bar{X}^k_{\eta,i}(r),r) - \sqrt{\sigma_i}(\widehat{X}^k_i(r),r)dW^k_i(r)$$

$$=: I_1(s) + I_2(s)$$

which implies

$$\mathbb{E}\left[S^k_t\right] = \mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t}\left|\bar{X}^k_{\eta,i}(s) - \widehat{X}^k_i(s)\right|^2\right] = \mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t}|I_1(s)+I_2(s)|^2\right]$$

$$\leq 2\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t}|I_1(s)|^2\right] + 2\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t}|I_2(s)|^2\right].$$

For the second term on the right-hand side, one can prove the estimate

$$\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t}|I_2(s)|^2\right] \leq 8d^3 \max_{1\leq j,l\leq d}\|D(\sqrt{\sigma_i})_{jl}\|^2_{L^\infty(0,T;L^\infty)}\int_0^t \mathbb{E}\left[S^k_s\right]ds \tag{8.10}$$

with the same arguments as in the proof of lemma 8.1.2 (see also (8.8) and (8.9)). Otherwise, for the first term on the right-hand side we get:

$$\mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t}|I_1(s)|^2\right]$$

$$\leq \mathbb{E}\left[\sum_{i=1}^n \sup_{0<s<t}\left(\int_0^s \sum_{j=1}^n\left|(\nabla V^\eta_{ij} * u_{\eta,j})(\bar{X}^k_{\eta,i}(r),r) - a_{ij}\nabla u_j(\widehat{X}^k_i(r),r)\right|dr\right)^2\right]$$

$$\leq \mathbb{E}\left[\sum_{i=1}^n \left(\int_0^t \sum_{j=1}^n\left|(\nabla V^\eta_{ij} * u_{\eta,j})(\bar{X}^k_{\eta,i}(r),r) - a_{ij}\nabla u_j(\widehat{X}^k_i(r),r)\right|dr\right)^2\right]$$

$$\leq 3\mathbb{E}\left[\sum_{i=1}^n \left(\int_0^t \sum_{j=1}^n\left|(\nabla V^\eta_{ij} * u_j)(\widehat{X}^k_i(r),r) - a_{ij}\nabla u_j(\widehat{X}^k_i(r),r)\right|dr\right)^2\right]$$

$$+ 3\mathbb{E}\left[\sum_{i=1}^n \left(\int_0^t \sum_{j=1}^n\left|(\nabla V^\eta_{ij} * u_j)(\widehat{X}^k_i(r),r) - (\nabla V^\eta_{ij} * u_{\eta,j})(\widehat{X}^k_i(r),r)\right|dr\right)^2\right]$$

$$+ 3\mathbb{E}\left[\sum_{i=1}^n \left(\int_0^t \sum_{j=1}^n\left|(\nabla V^\eta_{ij} * u_{\eta,j})(\widehat{X}^k_i(r),r) - (\nabla V^\eta_{ij} * u_{\eta,j})(\bar{X}^k_{\eta,i}(r),r)\right|dr\right)^2\right]$$

$$=: I^1_1 + I^2_1 + I^3_1$$

where we expanded the right hand side used the triangle inequality and applied the inequality $(a+b+c)^2 \leq 3a^2+3b^2+3c^2$.
We recall the estimates (6.14) and (6.21),

$$\|u-u_\eta\|_{L^\infty(0,t;L^2)} + \|\nabla(u-u_\eta)\|_{L^2(0,t;L^2)} \leq C(t)\eta, \quad t\in[0,T^*],$$

$$\left\|V_{ij}^\eta * \nabla u_j - a_{ij}\nabla u_j\right\|_{L^2} \leq \|V_{ij}\|_{L^1}\left\|D^2u_j\right\|_{L^2}\eta, \quad t\in[0,T^*],$$

and thus conclude for the terms I_1^1 and I_1^2:

$$\begin{aligned}
I_1^1 \leq & 3\sum_{i=1}^n\left(\sum_{j=1}^n\int_0^t\int_{\mathbb{R}^d}\left|(\nabla V_{ij}^\eta * u_j)(x,r) - a_{ij}\nabla u_j(x,r)\right||u_i(x,r)|dxdr\right)^2\\
\leq & 3\sum_{i=1}^n\left(\sum_{j=1}^n\left\|(\nabla V_{ij}^\eta * u_j) - a_{ij}\nabla u_j\right\|_{L^2(0,t;L^2)}\|u_i\|_{L^\infty(0,t;L^\infty)}\right)^2\\
\leq & 3n^2c_{H^s\hookrightarrow L^\infty}^2\|u\|_{L^\infty(0,T;H^s)}^2\sum_{j=1}^n\left\|(\nabla V_{ij}^\eta * u_j) - a_{ij}\nabla u_j\right\|_{L^2(0,T;L^2)}^2\\
\leq & C\eta^2,
\end{aligned}$$

$$\begin{aligned}
I_1^2 \leq & 3\sum_{i=1}^n\left(\sum_{j=1}^n\int_0^t\int_{\mathbb{R}^d}\left|(\nabla V_{ij}^\eta * u_j)(x,r) - (\nabla V_{ij}^\eta * u_{\eta,j})(x,r)\right||u_i(x,r)|drdx\right)^2\\
\leq & 3\sum_{i=1}^n\left(\sum_{j=1}^n\left\|V_{ij}^\eta * \nabla(u_j-u_{\eta,j})\right\|_{L^2(0,t;L^2)}\|u_i\|_{L^2(0,t,L^2)}\right)^2\\
\leq & 3n^2\|u\|_{L^2(0,T,L^2)}^2\sum_{j=1}^n\left\|V_{ij}^\eta * \nabla(u_j-u_{\eta,j})\right\|_{L^2(0,T;L^2)}^2\\
\leq & C\sum_{j=1}^n\left\|V_{ij}^\eta\right\|_{L^1}^2\|\nabla(u_j-u_{\eta,j})\|_{L^2(0,T;L^2)}^2\\
\leq & C(t)\eta^2
\end{aligned}$$

where in the penultimate step we used Young's convolution inequality. To asses the last term I_1^3, we recall that $\nabla(V_{ij}^\eta * u_{\eta,j})$ is Lipschitz and its Lipschitz-constant L_{ij}^η is bounded by

$$L_{ij}^\eta \leq \left\|D^2(V_{ij}^\eta * u_{\eta,j})\right\|_{L^\infty} = \left\|V_{ij}^\eta * D^2u_{\eta,j}\right\|_{L^\infty} \leq \left\|V_{ij}^\eta\right\|_{L^1}\left\|D^2u_{\eta,j}\right\|_{L^\infty}$$

where the last inequality is an conclusion of Young's convolution inequality. Due to the embedding $H^s\hookrightarrow W^{2,\infty}$, we have $D^2u_{\eta,j}\in L^\infty(0,T;\infty)$. Applying the Cauchy-Schwarz inequality, we therefore get

$$
\begin{aligned}
I_1^3 \leq & 3\mathbb{E}\left[\sum_{i=1}^{n} tn \int_0^t \sum_{j=1}^{n} \left|(\nabla V_{ij}^{\eta} * u_{\eta,j})(\widehat{X}_i^k(r), r) - (\nabla V_{ij}^{\eta} * u_{\eta,j})(\bar{X}_{\eta,i}^k(r), r)\right|^2 dr\right] \\
\leq & 3tn \sum_{i,j=1}^{n} \int_0^t \mathbb{E}\left[(L_{ij}^{\eta})^2 \left|\widehat{X}_i^k(r) - \bar{X}_{\eta,i}^k(r)\right|^2\right] dr \\
\leq & 3tn^2 \max_{1\leq i,j\leq n} \left\|V_{ij}^{\eta}\right\|_{L^1}^2 \left\|D^2 u_{\eta}\right\|_{L^\infty(0,T;L^\infty)}^2 \int_0^t \mathbb{E}\left[\sum_{i=1}^{n} \sup_{0<s<r} \left|\widehat{X}_i^k(s) - \bar{X}_{\eta,i}^k(s)\right|^2\right] dr \\
\leq & tC \int_0^t \mathbb{E}\left[S_s^k\right] ds.
\end{aligned}
\tag{8.11}
$$

Adding all the estimates above together, we have

$$
\mathbb{E}\left[S_t^k\right] \leq C\eta^2 + C(t+1) \int_0^t \mathbb{E}\left[S_s^k\right] ds
$$

for some $C > 0$. Gronwall's inequality implies

$$
\mathbb{E}\left[S_t^k\right] \leq C\eta^2 \exp(C(t+1)t) \leq C(t)\eta^2,
$$

where $C(t)$ is a continuous function of t on $[0, T']$. Applying the Cauchy-Schwarz inequality finishes the proof:

$$
\begin{aligned}
\sup_{1\leq k\leq N} \mathbb{E}\left[\sum_{i=1}^{n} \sup_{0<s<t} \left|\bar{X}_{\eta,i}^k(s) - \widehat{X}_i^k(s)\right|\right] \leq & \sqrt{n} \sup_{1\leq k\leq N} \sqrt{\mathbb{E}\left[\sum_{i=1}^{n} \sup_{0<s<t} \left|\bar{X}_{\eta,i}^k(s) - \widehat{X}_i^k(s)\right|^2\right]} \sqrt{\mathbb{E}\left[|1|^2\right]} \\
\leq & \sqrt{n} \sup_{1\leq k\leq N} \sqrt{\mathbb{E}\left[S_t^k\right]} \leq C(t)\eta.
\end{aligned}
$$

□

Proof of theorem 8.1.

Due to lemma 8.1.2 and lemma 8.1.3, we have

$$
\begin{aligned}
& \sup_{1\leq k\leq N} \mathbb{E}\left(\sum_{i=1}^{n} \sup_{0<s<t} \left|X_{\eta,i}^{k,N} - \widehat{X}_i^{k,N}\right|\right) \\
\leq & \sup_{1\leq k\leq N} \mathbb{E}\left(\sum_{i=1}^{n} \sup_{0<s<t} \left|X_{\eta,i}^{k,N} - \bar{X}_{\eta,i}^k\right|\right) + \sup_{1\leq k\leq N} \mathbb{E}\left(\sum_{i=1}^{n} \sup_{0<s<t} \left|\bar{X}_{\eta,i}^k - \widehat{X}_i^k\right|\right) \\
\leq & C(t)N^{(C(t)\nu-1)/2} + \tilde{C}(t)\eta.
\end{aligned}
$$

Because $C(t)$ is continuous on the compact set $[0, T']$, it is bounded. We can therefore find $\nu > 0$ such that $C(t)\nu - 1 \leq c < 0$ for some constant c and $t \in [0, T']$. Recall that N depends on ν by means of $\nu \log(N) \geq \frac{1}{\eta^{2d+4}}$. We therefore get

$$N^{(C(t)\nu-1)/2} = \exp(\frac{1}{2}(C(t)\nu - 1)\log(N)) \leq \exp((C(t)\nu - 1)\frac{1}{2\eta^{2d+4}}) \leq \frac{2\eta^{2d+4}}{1 - C(t)\nu}$$

where, in the last step, we used the inequality $\exp(-x) \leq \frac{1}{x},\ x \geq 0$. Altogether, we now have for $0 < \eta < 1$:

$$\sup_{1\leq k\leq N} \mathbb{E}\left(\sum_{i=1}^{n} \sup_{0<s<t}\left|X_{\eta,i}^{k,N} - \widehat{X}_i^{k,N}\right|\right) \leq \frac{2\eta^{2d+4}}{1 - C(t)\nu} + \tilde{C}(t)\eta \leq \widehat{C}(t)\eta.$$

□

A. Auxiliary Results in Functional Analysis

A.1. Notation

For a Hilbert-space H we denote its inner product by $\langle\cdot,\cdot\rangle_H$. In the case of a Banach-space X and its (topological) dual X', we use the same brackets to indicate

$$\langle y, x\rangle_{X'} = y(x), \quad x \in X, y \in X'.$$

We denote the space of all bounded linear function of a Banach-space X with $\mathbb{B}[X]$. Let $m \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$. For a subset Ω of $\mathbb{R}^n$ we use the following notations for important function-spaces:

- $C^m(\Omega)$, the set of all continuous functions with continuous derivatives, up to order m.
- $C_b^m(\Omega)$, the set of all bounded functions $f \in C^m(\Omega)$ with bounded derivatives.
- $C_c^m(\Omega)$, the set of all functions $f \in C^m(\Omega)$ with compact support.
- $C_0^m(\Omega)$, the set of all functions $f \in C^m(\Omega)$ with $f(x) \to 0$ for $|x| \to \infty$.
- $\mathcal{D}(\Omega) \cong C_c^\infty(\Omega)$, the set of test-functions.

A.2. Sobolev Spaces

Definition A.2.1. Let $k, n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be an open domain and $1 \le p \le \infty$. Then we denote by $W^{k,p}(\Omega)$ the space of all measurable functions $u : \Omega \to \mathbb{R}$ where all derivatives $D^\alpha u$ with order $|\alpha| \le k$ exist in the weak sense and belong to $L^p(\Omega)$. For $W^{k,2}(\Omega)$ we also write $H^k(\Omega)$. The norm on $W^{k,p}(\Omega)$ for $p < \infty$ is given by

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p dx\right)^{1/p} = \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p}^p\right)^{1/p}.$$

For $p = \infty$ we set

$$\|u\|_{W^{k,\infty}} = \max_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty}.$$

Remark A.2.2. We define the space $W^{k,p}(\Omega, \mathbb{R}^m)$ as the set of all $\mathbb{R}^m$-valued functions u with $u_i \in W^{k,p}(\Omega)$ for all $1 \le i \le n$.

Remark A.2.3. Let $k \in \mathbb{N}$. The Sobolev spaces $H^k(\Omega)$ endowed with the scalar product

$$\langle u, v\rangle_{H^k} := \sum_{|\alpha|\le k} \int_\Omega D^\alpha u D^\alpha v dx = \sum_{|\alpha|\le k} \langle D^\alpha u D^\alpha, v\rangle_{L^2(\Omega)}, \quad u, v \in H^k(\Omega),$$

is a Hilbert space.

Remark A.2.4. The space $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$, see [19, Section 5.3.2 and 5.3.3]. For $\Omega = \mathbb{R}^n$ the space $C_c^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$, see [20, Chapter 8].

Definition A.2.5. Let $k \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be an open domain and $1 \leq p \leq \infty$. Then we denote by $H^{-k}(\Omega)$ the topological dual of $H^k(\Omega)$.

Lemma A.2.6 (Product-rule for Sobolev-spaces)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open domain and $1 \leq p \leq \infty$. Additionally let $g \in W^{1,\infty}(\Omega)$ and $u \in W^{1,p}(\Omega)$. Then $gu \in W^{1,p}(\Omega)$ and $\nabla(gu) = (\nabla g)\, u + g\,(\nabla u)$.*

Proof. Let $\phi \in C_c^\infty(\Omega)$ and $\psi_k \in C^\infty(\Omega)$, where $k \in \mathbb{N}$ and $\psi_k \to u$ in $W^{1,p}$ for $k \to \infty$. Then we have for $1 \leq i \leq n$,

$$\int\limits_\Omega \frac{\partial g}{\partial x_i} \psi_k \phi dx = - \int\limits_\Omega g \left(\frac{\partial \psi_k}{\partial x_i} \phi + \frac{\partial \phi}{\partial x_i} \psi_k \right) dx.$$

Taking the limit $k \to \infty$ therefore yields

$$\int\limits_\Omega \left(\frac{\partial u}{\partial x_i} g + \frac{\partial g}{\partial x_i} u \right) \phi dx = - \int\limits_\Omega g u \frac{\partial \phi}{\partial x_i} dx.$$

Because ϕ was arbitrary, we get $ug \in W^{1,p}$ and

$$\frac{\partial u}{\partial x_i} g + \frac{\partial g}{\partial x_i} u = \frac{\partial (ug)}{\partial x_i},$$

for all $1 \leq i \leq n$. □

Theorem A.2.7 (Stampacchias Theorem; Lemma 1.1. in [7])**.** *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a bounded, open domain with Lipschitz-boundary. Furthermore, let $1 < p < \infty$ and $F : \mathbb{R} \to \mathbb{R}$ be a Lipschitz-function with $F(0) = 0$. Then:*

i) $F \circ u \in W^{1,p}(\Omega)$ for $u \in W^{1,p}(\Omega)$.

ii) $\frac{\partial}{\partial x_i} F \circ u = F' \circ \frac{\partial}{\partial x_i} u$ almost everywhere for $1 \leq i \leq n$, if F has only a finite number of discontinuities.

Remark A.2.8. Choose $F = \chi_{\{x>0\}}$ in theorem A.2.7, where $\chi_{\{x>0\}}$ is the characteristic function regarding the set $\mathbb{R}^+$. Then $u^+ = \chi_{x>0}(u) \in W^{1,p}(\Omega)$, for $u \in W^{1,p}(\Omega)$ and $\nabla u^+ = \chi_{x>0}(u) \nabla u$ almost everywhere.

Theorem A.2.9 (Nirenberg-Gagliardo inequality; Section 1.6 in [3])**.** *Let $\Omega \subset \mathbb{R}^n$ a open domain with C^1- boundary or $\Omega = \mathbb{R}^n$ and $p \geq q, r \geq 1$. Additionally let $0 \leq \theta \leq 1$ and*

$$k - \frac{n}{p} \leq \theta(m - \frac{n}{q}) - n(1 - \theta)\frac{1}{r},$$

for a fixed $k, m \in \mathbb{N}$. Then there exists a constant $C > 0$ such that for $u \in W^{m,q}(\Omega) \cap L^r(\Omega)$,

$$\|u\|_{W^{k,p}} \leq C \|u\|_{W^{m,q}}^\theta \|u\|_{L^r}^{1-\theta}.$$

Theorem A.2.10 (Sobolev- imbedding theorem; Theorem 4.12 in [21])**.** *Let $m, n \in \mathbb{N}$, $1 \leq p < \infty$ and let them satisfy $mp > n$ or, if m=n, p=1. Then, for $m \geq 1$ and $j \in \mathbb{N} \cup \{0\}$, the following embeddings are continuous:*

i) $W^{m+j,p}(\mathbb{R}^n) \hookrightarrow C_b^j(\mathbb{R}^n)$

ii) $W^{m+j,p}(\mathbb{R}^n) \hookrightarrow W^{j,q}(\mathbb{R}^n)$, *for* $p \leq q \leq \infty$.

Here, $C_b^j \subset C^j$ *is the set of all functions, where all derivatives up to order* j *are bounded. In particular, the functions itself are bounded.*

Theorem A.2.11 (Rellich-Kondrachov; Theorem 2.5.17 in [23])**.** *Let* $m, n \in \mathbb{N}$, $1 \leq p < \infty$ *and let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with Lipschitz-boundary. Then, for* $m \geq 1$ *and* $j \in \mathbb{N} \cup \{0\}$, *the embedding* $W^{m+j,p}(\Omega) \hookrightarrow W^{j,q}(\Omega)$ *is compact in the following cases:*

i) $mp < n$ *and* $1 \leq q \leq \frac{np}{n-mp}$.

ii) $mp = n$ *and* $1 \leq q < \infty$.

iii) $mp > n$ *and* $1 \leq q \leq \infty$.

Lemma A.2.12 (Moser-type calculus inequality; Proposition 2.1. in [16])**.** *Let* $s, n \in \mathbb{R}^n$ *with* $s > \frac{n}{2} + 1$ *and let* $\alpha \in \mathbb{N}^n$ *a multi-index satisfying*$|\alpha| \leq s$. *Then there exists a constant* $C > 0$ *such that, for* $f, g \in H^s(\mathbb{R}^d)$,

$$\|D^\alpha(fg)\|_{L^2} \leq C(\|f\|_{L^\infty}\|D^s g\|_{L^2} + \|g\|_{L^\infty}\|D^s f\|_{L^2}).$$

Here, C *does only depend on* s *and* n.

A.3. Integrability and Differentiability in Banach-Spaces

Let B be a Banach-space over $\mathbb{R}$ and $0 < T \leq \infty$. We denote the Lebesgue measure on $[0, T]$ with λ.

Remark A.3.1*.* The results in this chapter are all formulated for the interval $[0, T]$. This interval can be replaced by any other subinterval of $[0, \infty)$ and any of the following statements still holds true.

A.3.1. Bochner-Integral

Definition A.3.2 (Simple function)**.** Let $n \in \mathbb{N}$, $x_i \in B$ and let A_i be a measurable subset of $[0, T]$, for all $1 \leq i \leq n$. Then a B-valued function f of the form

$$f(t) = \sum_{i=1}^{n} x_i \chi_{A_i}(t),$$

is called simple.

Definition A.3.3 (Bochner integral for simple functions)**.** Let f be a simple function according to definition A.3.2. Then we set

$$\int_0^T f(t)dt = \sum_{i=1}^{n} x_i \lambda(A_i). \tag{A.1}$$

(A.1) is called the Bochner-integral of f.

Definition A.3.4 (Bochner measurable)**.** $f : [0, T] \to B$ is called Bochner-measurable, if there exists a sequence of simple functions $s_n, n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \|f(t) - s_n(t)\|_B = 0,$$

almost everywhere.

Definition A.3.5 (Bochner-integrable)**.** A Bochner-measurable function $f : [0, T] \to$ is called Bochner-integrable, if there exists a sequence of simple functions $s_n, n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \int_0^T \|f(t) - s_n(t)\|_B dt = 0.$$

The Bochner-integral of f is thereby defined via

$$\int_0^T f(t)dt := \lim_{n\to\infty} \int_0^T s_n(t)dt.$$

Lemma A.3.6 (Lemma 1.7 in [25])**.** *Let $f : [0, T] \to B$ be Bochner-measurable. Then the function $t \to \|f(t)\|_B$ is Lebesgue-measurable.*

Theorem A.3.7 (Bochner criterion; Theorem 1.12 in [25])**.** *A Bochner- measurable function f is Bochner-integrable if and only if the function $\|f(t)\|_B$ is Lebesgue integrable.*

Lemma A.3.8 (Corollary 1.14 in [25])**.** *Let $f : [0, T] \to B$ be Bochner-integrable. Then it holds that*

$$\left\| \int_0^T f(t)dt \right\|_B \leq \int_0^T \|f(t)\|_B dt.$$

Furthermore, let $g \in B'$. Then the following identity holds true:

$$\langle g, \int_0^T f(t)dt \rangle_{(B',B)} = \int_0^T \langle g, f(t) \rangle_{(B',B)} dt,$$

where $\langle g, h \rangle_{(B',B)} = g(h)$ for $h \in B$.

Theorem A.3.9 (Fundamental theorem of calculus)**.** *Let $f : [0, T] \to B$ be a Bochner integrable function. Then $F(t) := \int_0^t f(s)ds$ is almost everywhere differentiable and $\frac{\partial F}{\partial t} = f$ almost everywhere.*

Proof. Let $0 < h < T - t$ and $0 < t < T]$. The statement follows from

$$\|F(t+h) - F(t)\|_B = \left\| \frac{1}{h} \int_t^{t+h} f(s)ds - f(t) \right\|_B = \left\| \frac{1}{h} \int_t^{t+h} f(s) - f(t)ds \right\|_B$$

$$\leq \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|_B ds$$

and the fundamental theorem of calculus for the Lebesgue-integral. □

A.3.2. L^p-Spaces regarding Banach-Spaces

Definition A.3.10 (L^p-Space)**.** Let $1 \leq p < \infty$. We define the space $L^p(0,T;B)$ as all Bochner measurable functions $f : [0,T] \to B$ such that

$$\int_0^T \|f(t)\|_B^p dt < \infty.$$

The space $L^\infty(0,T;B)$ is the space of all Bochner-measurable functions f, such that

$$\|f(t)\|_B \leq M_f,$$

almost everywhere, where $M_f > 0$ depends only on f.

Theorem A.3.11 (Completeness; Theorem 1.22 in [25])**.** *Let B be a complete Banach space. Then, for $1 < p < \infty$, the space $L^p(0,T;B)$ endowed with the norm*

$$\|f\|_{L^p(0,T;B)} := \int_0^T \|f\|_B^p dt$$

and the space $L^\infty(0,T;B)$ with the norm

$$\|f\|_{L^\infty(0,T;B)} := \operatorname*{ess\,sup}_{0<t<T} \|f(t)\|_B,$$

are complete Banach spaces.

Remark A.3.12. Analogous to the standard L^p-spaces, the elements of $L^p(0,T;B)$ are equivalence-classes. Two Bochner-measurable functions f, g are in the same equivalence-class of $L^p(0,T;B)$ if $\|f-g\|_{L^p(0,T;B)} = 0$.

Remark A.3.13. Let H be a Hilber space. Then $L^2(0,T;H)$ endowed with the inner product

$$\langle u, v\rangle_{L^2(0,T;H)} := \int_0^T \langle u(t), v(t)\rangle_H dt, \quad u, v \in L^2(0,T;H),$$

is a Hilbert space.

Lemma A.3.14 (Lemma 1.23 in [25])**.** *The set of simple functions is dense in $L^p(0,T;B)$ for $1 \leq p \leq \infty$.*

Lemma A.3.15 (Proposition 2.15 in [26])**.** *The set $C_c^\infty((0,T),B)$ is dense in $L^p(0,T;B)$ for $1 \leq p < \infty$.*

Proposition A.3.16 (Dual-space; Proposition 23.7 and exercise 23.12d in [18])**.** *Let B be a reflexive and separable Banach-space and $1 \leq p < \infty$, $1 < q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the following identification of the dual space of $L^p(0,T;B)$ holds:*

$$(L^p(0,T;B))' = L^q(0,T;B').$$

Definition A.3.17 (Evolutionary triple)**.** Let V be a reflexive, separable Banach-space and H a separable Hilbert space. If V can be continuously and densely embedded in H, then the triple $V \hookrightarrow H \hookrightarrow V'$ is called an evolutionary triple.

Definition A.3.18 ($W^{1,p}(0,T;V,H)$)**.** The space $W^{1,p}(0,T;V,H)$ is the set of all $u \in L^p(0,T;V)$ such that $u_t \in L^q(0,T;V')$, where $\frac{1}{p}+\frac{1}{q}=1$ and $1<p<\infty$.

Proposition A.3.19 (Proposition 1.2 in [28])**.** *Let (V,H,V') be a evolutionary triple and $1 < p < \infty$. Then,*

i) the space $W^{1,p}(0,T;V,H)$ endowed with the norm

$$\|u\|_{W^{1,p}(0,T;V,H)} = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^q(0,T;V')}, \quad u \in W^{1,p}(0,T;V,H),$$

is a Banach space.

ii) $W^{1,p}(0,T;V,H)$ can be continuously embedded in $C([0,T];H)$.

iii) the mapping $t \to \|u(t)\|_H$ is absolute continuous for every $u \in W^{1,p}(0,T;V,H)$. Furthermore it holds that

$$\frac{d}{dt}\|u\|_H^2 = 2\langle u_t(t), u(t)\rangle_{V'},$$

almost everywhere.

Proposition A.3.20 (Theorem 8.1.9 in [27])**.** *Let (V,H,V') be a evolutionary triple and $1 < p < \infty$. Then the space $C^1([0,T],V)$ is dense in $W^{1,p}(0,T;V,H)$.*

Theorem A.3.21 (Lemma of Aubin; Proposition 1.3 in [28])**.** *Let (V,H,V') be a evolutionary triple and $1<p<\infty$. Furthermore, let the embedding $V \hookrightarrow H$ be compact. Then the embedding $W^{1,p}(0,T;V,H) \hookrightarrow L^p(0,T;H)$ is compact as well.*

Proposition A.3.22. *Let $\Omega \subset \mathbb{R}^n$ be an bounded, open domain with Lipschitz boundary, or $\Omega = \mathbb{R}^n$ and let $u \in W^{1,2}(0,T;H^1(\Omega),L^2(\Omega))$. Then the mapping $t \to \|u^+\|^2_{L^2(\Omega)}$ is absolute continuous and*

$$\frac{d}{dt}\|u^+\|^2_{L^2(\Omega)} = 2\langle u_t, u^+\rangle_{H^{-1}}.$$

Proof. Let $g \in C_b^1(\mathbb{R})$ with $g(0)=0$. Furthermore, let $u \in C^1([0,T];H^1)$. By Stampacchias theorem, $g(u) \in H^1$, see theorem A.2.7. Let $t \in (0,T)$ be arbitrary but fixed. The sequence $D_t^{\frac{1}{n}}u := \frac{u(t)-u(t-\frac{1}{n})}{\frac{1}{n}}$ converges for $n \to \infty$ towards u_t in H^1 and therefore in particular in L^2. Theorem C.2 now implies the existence of a subsequence $n_k =: \frac{1}{h_k}$, such that $D_t^{h_k}u$ converges almost everywhere point-wise towards $u_t(t)$. We show that $\frac{g(u(t))-g(u(t-h_k))}{h_k}$ converges in L^2 towards $g'(u(t))u_t(t)$. We have

$$\frac{g(u(t))-g(u(t-h_k))}{h_k} - g'(u(t))u_t(t) = g'(\zeta)\frac{u(t)-u(t-h_k)}{h_k} - g'(u(t))u_t(t)$$

$$= \left(g'(\zeta)-g'(u(t))\right)\frac{u(t)-u(t-h_k)}{h_k} + g'(u(t))\left(\frac{u(t)-u(t-h_k)}{h_k} - u_t(t)\right), \tag{A.2}$$

for some $\zeta \in (u(t-h_k), u(t))$. Because $D_t^{h_k}u(x,t) \to u_t(t)$ for almost all $x \in \Omega$ and therefore also $u(t-h_k) \to u(t)$, (A.2) converges almost everywhere towards 0. Furthermore, (A.2) has the (square)- integrable majorant

$$C\frac{u(t)-u(t-h_k)}{h_k} + C\left(\frac{u(t)-u(t-h_k)}{h_k} - u_t(t)\right),$$

where we notice that, as L^2- convergent series $\frac{u(t)-u(t-h_k)}{h_k}$ is bounded in L^2. We apply the dominated converge theorem and get

$$\frac{g(u(t)) - g(u(t-h_k))}{h_k} \to g'(u(t))u_t(t)$$

in L^2 for $k \to \infty$. This implies $\frac{d}{dt}\langle u(t), g(u(t))\rangle_{L^2} = \langle u_t(t), g(u(t))\rangle_{L^2} + \langle u(t), g'(u(t))u_t(t)\rangle_{L^2}$ as one can assure oneself by writing

$$\frac{d}{dt}\langle u(t), g(u(t))\rangle_{L^2} = \lim_{h_k \to 0} \int_\Omega \frac{u(t)g(u(t)) - u(t-h_k)g(u(t-h_k))}{h_k} dx$$

$$= \lim_{h_k \to 0} \int_\Omega g(u(t)) \frac{u(t) - u(t-h_k)}{h_k} dx + \int_\Omega u(t-h_k) \frac{g(u(t)) - g(u(t-h_k))}{h_k} dx.$$

We hence have

$$\langle u(t), g(u(t))\rangle_{L^2} = \langle u(0), g(u(0))\rangle_{L^2} + \int_0^t \langle u_s(s), g(u(s))\rangle_{L^2} + \langle u(s), g'(u(s))u_s(s)\rangle_{L^2} ds$$

We set now $g_\epsilon(x) = \chi_{\mathbb{R}^+}(x)\left(\sqrt{x^2+\epsilon^2} - \epsilon\right)$. A straightforward calculations shows that $g_\epsilon \in C_b^1(\mathbb{R})$ and $g_\epsilon(0) = 0$. Additionally $g_\epsilon(x) \to x^+$ for $\epsilon \to 0$ and all $x \in \mathbb{R}$. Due to

$$|g'(x)| \leq \left| \frac{2x}{\sqrt{x^2+\epsilon^2}} \right| \leq 2$$

$\|g'\|_{L^\infty}$ is uniformly bounded, $g'(x) \to \chi_{\mathbb{R}^+}(x)$ for all $x > 0$ and we get

$$\lim_{\epsilon \to 0}\langle u(t), g_\epsilon(u(t))\rangle_{L^2} = \lim_{\epsilon \to 0}\langle u(0), g_\epsilon(u(0))\rangle_{L^2} + \int_0^t \langle u_s(s), g_\epsilon(u(s))\rangle_{L^2} + \langle u(s), g_\epsilon'(u(s))u_s(s)\rangle_{L^2} ds$$

$$= \langle u(s), u^+(s)\rangle_{L^2} = \langle u(0), u^+(0)\rangle_{L^2} + \int_0^t \langle u_s(s), u^+(s)\rangle_{L^2} + \langle u^+(s), u_s(s)\rangle_{L^2} ds$$

$$= \langle u(0), u^+(0)\rangle_{L^2} + 2\int_0^t \langle u_s(s), u^+(s)\rangle_{L^2} ds$$

$$= \langle u(0), u^+(0)\rangle_{L^2} + 2\int_0^t \langle u_s(s), u^+(s)\rangle_{H^{-1}} ds$$

We therefore have shown the statement for $u \in C^1(0,T;H^1)$. Because this space is dense in $W^{1,2}(0,T;H^1,L^2)$, a simple denseness-argument finishes the proof.

□

A.4. Weak Compactness and Dense Embeddings for Banach Spaces

Lemma A.4.1 (Exercise 18.6 in [18]). *Let X, Y be Banach-spaces, where X lies dense in Y. Furthermore, let the embedding $X \hookrightarrow Y$ be continuous. Then the embedding $Y' \hookrightarrow X'$ is continuous, where X', Y' denote the respective topological dual. For reflexive Y, Y' is even dense in X'.*

Remark A.4.2. The space L^p is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$. The same holds true for the space $W^{m,p}$, $m \in \mathbb{N}$, see [21, Chapter 2].

Theorem A.4.3 (Eberlein-Šmuljan, Theorem 21.D in [18]). *Let $u_k, k \in \mathbb{N}$ be a bounded sequence in a reflexive Banach-space X. Then there exists a subsequence u_{k_n} such that u_{k_n} converges weakly in X.*

Remark A.4.4. Due to the theorem of Banach-Steinhaus, $\|u\|_X \leq \liminf_{n\to\infty} \|u_{k_n}\|_X$, where u is the weak limit of u_{k_n}.

A.5. Operator-Theory

Let H, H_1, H_2 be Hilbert-spaces and B_1, B_2 be Banach spaces.

Definition A.5.1 (Linear operator). A linear function $A : D(A) \subset B_1 \to B_2$ is called a linear operator, if its domain of definition $D(A)$ is a linear subspace of B_1.

Definition A.5.2. A linear Operator $A : D(A) \subset B_1 \to B_2$ is called densely defined, if $D(A)$ is dense in B_1.

Definition A.5.3 (Closed linear operator). A linear Operator $A : D(A) \subset B_1 \to B_2$ is called closed if its graph $\mathrm{gr}(A) := \{(x, Ax) : \ x \in D(A)\}$ is closed reagrding the sum-norm of B_1 and B_2.

Definition A.5.4. We call a linear operator $A : D(A) \subset H \to H$ accretive if $\mathrm{Re}\ \langle Au, u\rangle_H \geq 0$ for all $u \in D(A)$. In this case $-A$ is also called dissipative.

Definition A.5.5. If there is no proper accretive extension (dissipative extension), we call a accretive (dissipative) linear operator maximal accretive (maximal dissipative).

Definition A.5.6 (Adjoint operator). Let $A : D(A) \subset H_1 \to H_2$ be a linear, densely defined Operator. Let Ω be the set of all $y \in H_2$ such that the linear functional

$$x \to \langle A(x), y\rangle_{H_2}, \quad x \in D(A)$$

is bounded. Because A is densely defined, for fixed $y \in \Omega$ there exists $\tilde{y}$ such that

$$\langle A(x), y\rangle_{H_2} = \langle x, \tilde{y}\rangle_{H_1}$$

for all $x \in D(A)$. The linear operator $A^* : D(A^*) = \Omega \to H_1 : y \to \tilde{y}$ is called the adjoint operator of A.

Definition A.5.7. A linear operator $A : D(A) \subset H \to H$ is called symmetric if

$$\langle A(x), y\rangle_H = \langle x, A(y)\rangle_H$$

for all $x, y \in D(A)$.

Definition A.5.8. A linear, densely defined operator $A : D(A) \subset H \to H$ is called self-adjoint if $A = A^*$.

Definition A.5.9. A non- negative, symmetric sesquilinearform $f : D(f) \times D(f) \subseteq H \times H \to \mathbb{R}$ is called *closed* if the existence of a subsequence $\{u_n\}_{n\in\mathbb{N}} \subseteq D(f)$, which satisfies $f(u_n - u_m, u_n - u_m) \to_{\mathbb{R}} 0$ and $u_n \to_X u$, for $n, m \to \infty$, implies $\lim_{n\to\infty} f(u_n, f_{u_n}) = f(u, u)$ and in particular $u \in D(f)$.

Definition A.5.10. A sesquilinearform $f : D(f) \times D(f) \subseteq H \times H \to \mathbb{C}$ is called regular if:

i) $D(f)$ is dense in H.

ii) Re f is a closed, symmetric, non-negative sesquilinearform.

iii) There exists a constant $C \geq 0$ such that $|\text{Im } f(u, u)| \leq C\text{Re } f(u, u)$ for all $u \in D(f)$. C is also called the index of f.

Theorem A.5.11 (Theorem 2.1 in [2]). *Let $f : D(f) \times D(f) \subseteq H \times H \to \mathbb{C}$ be a regular sesquilinearform. Then there exists a unique maximal accretive, closed Operator $A : D(A) \subset H \to H$ such that $D(A) \subset D(f)$ and*

$$f(u, v) = \langle Au, v\rangle$$

for $u \in D(A)$ and $v \in D(f)$.

Theorem A.5.12 (Theorem 4.2 in [2]). *Let $a(t), t \in I \subset \mathbb{R}$ be a family of regular sesquilinearforms with the respective maximale accretive operators $A(t), t \in I$ (see theorem A.5.11) on a Hilbert-space H. Let $D(a(t)) = D$ be independent of t and let there exist constants $C > 0$ and $0 < \beta$ such that*

$$|a(t)[u, u] - a(s)[u, u]| \leq C|t - s|^{\beta}|Re\ a(t)[u, u]|$$

for all $u \in D$ and $s, t \in I$. Additionally let 0 be in the resolvent set of $A(t)$ for every $t \in I$. Then for $0 \leq \alpha < \frac{1}{2}$

$$\left\|A(t)^{\alpha}A(s)^{-\alpha} - Id_H\right\|_{\mathbb{B}(\mathbb{H})} \leq C_{\alpha}|t - s|^{\beta}, \quad s, t \in I$$

where $C_\alpha \geq 0$ depends only on α. In particular the operator $A(t)^{\alpha}A(s)^{-\alpha}$ is bounded.

Proposition A.5.13 (Theorem 5.1.5 in [22]). *Let $A : D(A) \subset H \to H$ be a linear, densely defined Operator. Then its adjoint operator A^* is closed and we have*

$$ker(A^*) = ran(A)^{\perp},$$

where $ker(A) = \{x \in H : \ A(x) = 0\}$ and $ran(A) = A(D(A))$.

Proposition A.5.14 (Proposition 5.1.7 in [22]). *Let $A : D(A) \subset H \to H$ be a linear, densely defined, closed Operator. If additionally, A is injective and $ran(A)$ is dense, then A^* and A^{-1} are densely defined and closed operators. Furthermore, they satisfy*

$$(A^*)^{-1} = (A^{-1})^*.$$

Theorem A.5.15 (Spectral theorem for self-adjoint operators; Theorem 10.4 in [24]). *Let $A : D(A) \subset H \to H$ be a self-adjoint operator. Then there exists a unique projection-valued measure E on the spectrum $\sigma(A)$ of A, such that*

$$\int_{\sigma(A)} \lambda dE(\lambda) = A.$$

Theorem A.5.16 (Theorem of Hille-Yoshida; page 363 in [32]). *Let $A : D(A) \subset H \to H$ be a linear, closed and densely defined operator. If $(0, \infty) \subset \rho(A) = \sigma(A)^c$ and*

$$\|(A - \lambda Id_H)^n\|_{\mathbb{B}(H)} \leq \frac{1}{\lambda^n}$$

for every $n \in \mathbb{N}$ and $\lambda > 0$, then A is the infinitesimal generator of a contraction semi-group.

Proposition A.5.17 (Theorem 1.1.3 in [4]). *If an operator $A : D(A) \subset H \to H$ is the generator of a contraction semi-group, then A is densely defined and maximal dissipative.*

A.6. Fundamental Lemma of Calculus of Variations

Lemma A.6.1 (Fundamental lemma of calculus of variations, Lemma 3.31 in [21]). *Let $\Omega \subset \mathbb{R}^n$ be a open set and $g \in L^1_{loc}(\Omega)$. If*

$$\int_{\Omega} \phi g dx = 0,$$

for all $\phi \in C_c^\infty(\Omega)$, then $g = 0$ almost everywhere.

A.7. Young's Convolution Inequality

Lemma A.7.1 (Young's convolution inequality; Formula (7) on page 107 in [17]). *Let $n \in \mathbb{N}$ and let $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. Then, for $f \in L^p$ and $g \in L^q$,*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

B. Auxiliary Results in Stochastic Analysis

B.1. Ito Processes and Stochastic Differential Equations

Definition B.1.1 (Probability space)**.** Let Ω be an arbitrary set, $\mathcal{F}$ a σ-algebra regarding Ω and P a probability measure on $(\Omega, \mathcal{F})$. Then the triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Definition B.1.2 (Stochastic process)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family $(X_t, t \geq 0)$, of random variables $X_t : \Omega \to \mathbb{R}$ defined on $(\Omega, \mathcal{F}, P)$, is called a stochastic process.

Definition B.1.3 (Filtration)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family of σ-algebras $(\mathcal{F}_t, t \geq 0)$, with $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$, $s \leq t$, is called a filtration.

Definition B.1.4 (Adapted process)**.** Let $(\mathcal{F}_t, t \geq 0)$, be a filtration. A stochastic process $(X_t, t \geq 0)$, is called adapted to the filtration $(\mathcal{F}_t, t \geq 0)$, if X_t is $\mathcal{F}_t$- measurable.

Definition B.1.5 (Simple process)**.** Let $m \in \mathbb{N}$ and $0 \leq i \leq m$. Additionally let X_i be stochastic processes defined on a probability space $(\Omega, \mathcal{F}, P)$ and $t_i \in \mathbb{R}$ with $0 = t_0 \leq t_1 \leq \cdots \leq t_{m-1} \leq t_m$. Then stochastic processes of the form

$$S(t) = \sum_{i=0}^{m-1} \chi_{[t_i, t_{i+1})}(t) X_i$$

are called simple processes, where χ_A is the characteristic function of the set A.

Definition B.1.6 (Wiener process)**.** A stochastic process $(W_t, t \geq 0)$ defined on a probability space $(\Omega, \mathcal{F}, P)$, is called a Wiener process, or Brownian Motion, regarding the filtration $(F_t, t \geq 0)$, if it is adapted to $(F_t, t \geq 0)$ and satisfies

i) $W_0 = 0$,

ii) $W(\omega)_t$ is P- almost surely a continuous function in t,

iii) The increments $(W_t - W_s)$,$0 \leq s < t$, of W are independent from $\mathcal{F}_s$,

iv) $(W_t - W_s)$ is $N(0, t-s)$-distributed for all $0 \leq s < t$. That is to say $(W_t - W_s)$ is normally distributed.

Definition B.1.7 (Ito integral for simple processes)**.** Let $(W_t, t \geq 0)$, be a Wiener process regarding the filtration $(\mathcal{F}_t, t \geq 0)$ and S a $(\mathcal{F}_t, t \geq 0)$-adapted, simple process. Then the random variable

$$\phi(S) := \sum_{i=0}^{m-1} X_i (W_{t_{i+1}} - W_{t_i}),$$

is called the Ito integral of S.

Lemma B.1.8 (Chapter 2 section 1 in [13])**.** *Let $\mathcal{B}$ the Borel σ- algebra of $[0,\infty)$ and let $(W_t, t \geq 0)$ be a $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$- Wiener process on the probability space $(\Omega, \mathcal{F}, P)$. Additionally let $(X_t, t \geq 0)$, be a $\mathbb{F}$-adapted, stochastic process, which satisfies*

i) $(t, \omega) \to X_t(\omega)$ *is* $\mathcal{B} \times \mathcal{F}$- *measurable,*

ii) $\int_0^\infty \mathbb{E}\left[|X_t|^2\right] dt < \infty.$

Then there exists a series of square integrable, simple processes $S_n, n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \int_0^\infty \mathbb{E}\left[|X_t - S_n(t)|^2\right] dt = 0.$$

Furthermore $\phi(S_n)$ converges towards some element Ψ in $L^2(\Omega, \mathcal{F}, P)$, where $\phi(S_n)$ is the Ito integral regarding $(W_t, t \geq 0)$.

Definition B.1.9 (Ito integral)**.** Let the assumptions of lemma B.1.8 hold and let X_t and Ψ be as in lemma B.1.8. Then Ψ is called the stochastic integral regarding the $\mathbb{F}$-Brownian motion $(W_t, t \geq 0)$, of the process $(X_t, t \geq 0)$, . We write

$$\Psi = \int_0^\infty X_t dW(t).$$

Theorem B.1.10 (Ito isometry; Chapter 2 section 1 in [13])**.** *Let the assumptions of lemma B.1.8 hold and let X_t be as in lemma B.1.8. Then*

$$\mathbb{E}\left[\left|\int_0^\infty X_t dW(t)\right|^2\right] = \int_0^\infty \mathbb{E}\left[|X_t|^2\right] dt.$$

Lemma B.1.11 (Chapter 2 section 1 in [13])**.** *Let the assumptions of lemma B.1.8 hold and let X_t be as in lemma B.1.8. Define $\phi_t := \int_0^t X_s dW(s) := \int_0^\infty \chi_{[0,t]} X_s dW(s)$. Then ϕ_t is a martingale regarding the filtration $\mathbb{F}$ with continuous paths a.e. and*

$$\mathbb{E}\left[\int_0^t X_s dW(s)\right] = 0,$$

for all $t \geq 0$.

Definition B.1.12 (Multidimensional Wiener process)**.** Let $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$, be a filtration on the probability space $(\Omega, \mathcal{F}, P)$. The (multidimensional) stochastic process $W_t = (W_1(t), W_2(t), \dots, W_n(t))$ is called a n-dimensional Wiener process, or Brownian motion, if $W_i(t), 1 \leq i \leq n$, are independent, one-dimensional Wiener processes regarding $\mathbb{F}$ and the increments $W_t - W_s$ are $N(0, (t-s)Id)$-distributed. Here, $N(0, Id)$ is the standard normal distribution, where Id is the unit-matrix in $\mathbb{R}^{n\times n}$.

Definition B.1.13 (Correlated Wiener process)**.** The process $(W_t, t \geq 0)$, with $W_t = (W_1(t), W_2(t), \dots, W_n(t))$ is called a vector of correlated Wiener processes if $\tilde{W}_t$ is a n-dimensional Wiener process and there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that $W_t = B\tilde{W}_t$. Here, W_t is $N(0, \Sigma)$ distributed, where $\Sigma = BB^T$.

Definition B.1.14 (Ito process)**.** Let $0 < T$ and $(W_t, t \geq 0)$ a n-dimensional, vector of uncorrelated $\mathbb{F}$-Wiener processes. Additionally let $a(t) \in \mathbb{R}^n$ and $b(t) \in \mathbb{R}^{n\times n}$ be stochastic, $\mathbb{F}$-adapted processes with

i) $\int_0^T |a_i(t)| dt < \infty$,

ii) $\int_0^T |b_{ij}(t)|^2 dt < \infty$,

P-almost surely for all $1 \leq i, j \leq n$. Then a stochastic process $(X_t, t \geq 0)$ of the form

$$X_{i,t} = X_{i,0} + \int_0^t a_{i,s} ds + \sum_{j=1}^n \int_0^t b_{ij,s} dW_j(s), \quad t \in [0,T], \quad 1 \leq i, j \leq n \tag{B.1}$$

is called an Ito process.

Remark B.1.15. To denote the relation (B.1), we also write

$$dX_{i,t} = a_i dt + \sum_{j=1}^n b_{ij} dW_j, \quad t \in [0,T], \quad 1 \leq i, j \leq n.$$

Theorem B.1.16 (Multidimensional Ito formula regarding a vector of uncorrelated Wiener processes; [14])**.** *Let $(X_t, t \geq 0)$ a Ito process according to definition B.1.14 and let the corresponding Wiener process W_t be a vector of uncorrelated Wiener processes. Additionally let $f : \mathbb{R}^n \times [0,T) \to \mathbb{R}, (x,t) \to f(x,t)$, be twice continuously differentiable in x and once in t. Then $(f(X_t,t), t \geq 0)$ is a Ito process and satisfies*

$$df(X_t,t) = \left(\frac{\partial f}{\partial t}(X_t,t) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_t,t) a_i(t) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(X_t,t)(bb^T)_{ij}(t) \right) dt$$
$$+ \sum_{i,j=1}^n \frac{\partial f}{\partial x_i}(X_t,t) b_{ij}(t) dW_j,$$

for $t \in [0,T]$.

Theorem B.1.17 (Uniqueness and existence for stochastic differential equations; Theorem 5.2.1 in [10])**.** *Let $T < 0$ and $a : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $b : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$ be measurable functions which satisfy*

$$|a(x,t)| + |b(x,t)| \leq C(1 + |x|), \quad x \in \mathbb{R}^n, t \in [0,T],$$

$$|a(x,t) - a(y,t)| + |b(x,t) - b(y,t)| \leq C|x - y|, \quad x, y \in \mathbb{R}^n, t \in [0,T],$$

for some constant $C > 0$. Let Z be a random variable which is independent from the Wiener process $(W_t, t \geq 0)$ and satisfies

$$\mathbb{E}\left[|Z|^2\right] < \infty.$$

Then the stochastic differential equation

$$dX_t = a(X_t, t)dt + b(X_t, t)dW(t), \quad t \in [0, T],$$
$$X \mid_{t=0} = Z,$$

has a unique t-continuous solution X_t which is adapted to the filtration F_t generated by Z and $(W_t, 0 \leq s \leq t)$. Furthermore

$$\int_0^T \mathbb{E}\left[|X_t|^2\right] dt < \infty.$$

Remark B.1.18. $|b| = \sqrt{\sum_{i,j=1} |b_{ij}|^2}$ for $b \in \mathbb{R}^{n \times m}$.

B.2. Martingale Inequality's

Theorem B.2.1. *(Doob's L^p- inequality; Theorem II.1.7 in [15]) Let $M_t, t \geq 0$ be martingale or a positive submartingale with right- continuous trajectories $t \to M(\omega)_t$ almost everywhere. Then for all $p > 1$, $T \geq 0$ and $\lambda > 0$, the following inequality holds true:*

$$P\left[\sup_{0 \leq t \leq T} |M_t|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|M_T|^p\right].$$

C. Auxiliary Results in Measurement Theory

C.1. Dominated Convergence Theorem

Theorem C.1.1 (Dominated convergence theorem; Theorem 9.33 in [30])**.** *Let* $(\Omega, \mathcal{F}, \nu)$ *be a measure space with the underlying set* Ω*, the* σ*-algebra* $\mathcal{F}$ *and the measure* ν *and let* f_n *be a sequence of* $\mathcal{F}$*-measurable functions which converges* ν*-almost everywhere. Furthermore let there be a* $\mathcal{F}$*-measurable, integrable function* g *with* $|f_n| \leq g$*. Then the function* $f := \lim_{n\to\infty} f_n$ *is integrable and*

$$\lim_{n\to\infty} \int_\Omega |f_n - f| d\nu = 0, \qquad \lim_{n\to\infty} \int_\Omega f_n d\nu = \int_\Omega f_n d\nu.$$

C.2. Reversal of the Dominated Convergence Theorem

Theorem C.2.1 (Reversal of the dominated convergence theorem; Theorem 4.9 in [20])**.** *Let* $(\Omega, \mathcal{F}, \nu)$ *be a measure space with the underlying set* Ω*, the* σ*-algebra* $\mathcal{F}$ *and the measure* ν*. Let* $1 \leq p \leq \infty$ *and* $f_n, f \in L^p(\Omega, \mathcal{F}, \nu), n \in \mathbb{N}$ *with* $f_n \to f$ *in* L^p*. Then there exists* $g \in L^p$ *and a sub-series* f_{n_k} *such that*

i) $|f_{n_k}(\omega)| \leq g(\omega)$ *for* ν*-almost every* ω*.*

ii) $f_{n_k}(\omega) \to f(\omega)$ *for* ν*-almost every* ω*.*

C.3. Gronwall's Inequality

Theorem C.3.1 (Gronwall's inequality for Borel measures; Theorem 5.1 in the appendix of [29])**.** *Let* ν *be a Borel measure on* $[0, \infty)$*,* $\epsilon > 0$ *and let* f *be a Borel measurable function which is bounded on bounded subsets of* $[0, \infty)$*. If* f *satisfies*

$$f(t) \leq \epsilon + \int_0^t f(s) d\nu(s), \qquad t \geq 0,$$

then

$$f(t) \leq \epsilon e^{\nu([0,t))}.$$

D. Auxiliary Results regarding Linear PDE's

D.1. A Useful Uniqueness Result

Theorem D.1.1. *Let $n \in \mathbb{N}$, $f \in L^2(0,T;L^2(\mathbb{R}^n,\mathbb{R}))$, $u_0 \in L^2(\mathbb{R}^n,\mathbb{R})$ and $B \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^{n\times n}))$. Furthermore $B(x,t)$ is symmetric and uniformly coercive for $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$. Then the weak solution $u \in W^{1,2}(0,T;H^1;L^2)$ of the following linear initial value problem*

$$\partial_t u - div(B\nabla u) + \sum_i (\frac{\partial B}{\partial x_i}\nabla u)_i = f, \quad in\ \mathbb{R}^n \times [0,T), \tag{D.1}$$

$$u\mid_{t=0} = u_0, \quad in\ \mathbb{R}^n \tag{D.2}$$

is unique.

Remark D.1.2.

$$\operatorname{div}(B\nabla u) - \sum_{i=1}^n (\frac{\partial B}{\partial x_i}\nabla u)_i = \sum_{i,j=1}^n B\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u.$$

Proof. Assume that $v, u \in W^{1,2}(0,T;H^1,L^2)$ are both weak solutions of (D.1)-(D.2). Due to the linearity of (D.1) they satisfy

$$\langle \partial_t(u-v),\phi\rangle_{H^{-1}} + \langle B\nabla(u-v),\nabla\phi\rangle_{L^2} + \langle \sum_i (\frac{\partial B}{\partial x_i}\nabla u - v)_i, \phi\rangle_{L^2} = 0$$

for arbitrary $\phi \in H^1$. We choose $\phi = u - v$ and use the uniform coercivity of B:

$$-\langle \sum_i (\frac{\partial B}{\partial x_i}\nabla u - v)_i, u-v\rangle_{L^2} = \langle \partial_t(u-v), u-v\rangle_{H^{-1}} + \langle B\nabla(u-v),\nabla(u-v)\rangle_{L^2}$$

$$\geq \underbrace{\langle \partial_t(u-v), u-v\rangle_{H^{-1}}}_{=\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2} + \epsilon\|\nabla(u-v)\|_{L^2}^2.$$

Let $\delta > 0$. We use Youngs inequality for products on the term on the left hand side.

$$\left|\langle \sum_i (\frac{\partial B}{\partial x_i}\nabla u - v)_i, u-v\rangle_{L^2}\right| = \left|\int_{\mathbb{R}^n}\left(\sum_i (\frac{\partial B}{\partial x_i}\nabla u - v)_i\right)(u-v)dx\right|$$

$$\leq \frac{\delta}{2}\left\|\sum_{i,j}(\frac{\partial B}{\partial x_i}\nabla u - v)_i\right\|_{L^2}^2 + \frac{1}{2\delta}\|u-v\|_{L^2}^2 \leq \frac{n\delta}{2}\|B\|_{W^{1,\infty}}^2\|\nabla(u-v)\|_{L^2}^2 + \frac{1}{2\delta}\|u-v\|_{L^2}^2.$$

All in all we have:

$$\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 + \epsilon\|\nabla(u-v)\|_{L^2}^2 \leq \frac{n\delta}{2}\|B\|_{W^{1,\infty}}^2\|\nabla(u-v)\|_{L^2}^2 + \frac{1}{2\delta}\|u-v\|_{L^2}^2.$$

If we choose δ such that $\frac{n\delta}{2}\|B\|_{L^\infty(0,T;W^{1,\infty})}^2 \leq \epsilon$ the first term on the right-hand side of the last inequality can almost everywhere be absorbed by the corresponding term on the left-hand side. This leaves us with:

$$\frac{d}{dt}\|u-v\|_{L^2}^2 \leq \frac{1}{\delta}\|u-v\|_{L^2}^2$$

almost everywhere. We apply Gronwalls lemma:

$$\|u-v\|_{L^2}^2 \leq 0,$$

for all $t \in [0,T]$. This implies that $\|u-v\|_{L^2} \equiv 0$ on $[0,T]$. Hence $u(t) = v(t)$ almost everywhere for $t \in [0,T]$. □

D.2. Existence for Linear Parabolic Systems

The following paragraph and the theorem contained therein, was contextually abstracted from [6]:

Let $n, m \in \mathbb{N}$. We focus upon the following initial value problem:

$$A(x,t)\frac{\partial v}{\partial t} - \sum_{i,j=1}^{n} B^{i,j}(x,t)\frac{\partial^2 v}{\partial x_i \partial x_j} = f(x,t), \quad in\ \mathbb{R}^n \times \mathbb{R}^+, \tag{D.3}$$

$$v\,|_{t=0} = v_0, \quad in\ \mathbb{R}^n. \tag{D.4}$$

Prerequisites :

i) $A \in \mathbb{R}^{m\times m}$ is positive definite and symmetric.

ii) $B^{i,j} \in \mathbb{R}^{m\times m}$ satisfy $B^{i,j} = B^{j,i}$, with $i,j \in \{1,2,\ldots,n\}$. $\sum_{i,j=1}^{n} B^{i,j} w_i w_j$ is symmetric and positive for all $w \in \mathbb{R}^n$ with $|w| = 1$.

iii) $v, f, v_0 \in \mathbb{R}^m$.

iv) $l, s \in \mathbb{N}$, with $1 \leq l \leq s$ and $s > \frac{n}{2} + 1$.

v) $A, B^{i,j} \in C(0,T;H^s(\mathbb{R}^n))$, $\frac{\partial A}{\partial t}$, $\frac{\partial B^{i,j}}{\partial t} \in C(0,T;H^{s-2}) \cap L^2(0,T;H^{s-1})$, $f \in L^2(0,T;H^{l-1})$, $v_0 \in H^l(\mathbb{R}^n)$.

Theorem D.2.1 (Uniqueness and Existence)**.** *Let the above assumptions hold true. Then problem (D.3) with initial value condition (D.4) has a unique solution* $v \in C(0,T;H^l) \cap L^2(0,T;H^{l+1})$ *with* $\partial_t v \in L^2(0,T;H^{l-1})$.

E. Useful Inequalities

Lemma E.0.1 (Young's inequality with ϵ; Special case of Young's inequality for products)**.** *Let $a, b \in \mathbb{R}$ and $\epsilon > 0$. Then $ab \leq \frac{a^2}{2\epsilon} + \frac{b^2\epsilon}{2}$.*

Proof. Set $x := \frac{a}{\sqrt{\epsilon}}$ and $y := b\sqrt{\epsilon}$. Then the statement follows from the elementary inequality $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$. □

Lemma E.0.2 (Gronwall-type inequality; Lemma 17 in [5])**.** *Let $T > 0$, $g \in C([0,T])$ with $g : [0,T] \to [0,\infty)$ and let $f : [0,T] \to [0,\infty)$ be absolute continuous. Furthermore let $a, b > 0$ and*

$$f'(t) \leq -g(t)(a - b\sqrt{f(t)}), \tag{E.1}$$

for $t \in (0,T]$, where $0 < f(0) < \left(\frac{a}{b}\right)^2$. Then $f(t) > \left(\frac{a}{b}\right)^2$ for all $t \in [0,T]$.

Proof. We proceed as in the proof of lemma 17 in [5]. We assume that there exists $t' \in [0,T]$ such that $f(t') \geq \left(\frac{a}{b}\right)^2$. Due to the absolute continuity of f, there exists some $t'' \in [0,T]$ such that $f(t'') > 0$ and $f(t'') < \left(\frac{a}{b}\right)^2$. But this is a contradiction to (E.1). □

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