Diese Dissertation haben begutachtet:

DISSERTATION

Analysis of kinetic and diffusive multi-species systems

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Es ist nicht das Wissen, sondern das Lernen, nicht das Besitzen, sondern das Erwerben, nicht das Dasein, sondern das Hinkommen, was den größten Genuß gewährt.

It is not the knowledge, but the act of learning, not the possession of but the act of getting there, which grants the greatest enjoyment.

Carl Friedrich Gauss (1777–1855)

Kurzfassung

Ziel der vorliegenden Arbeit ist das mathematische Studium der Analysis diverser kinetischer und diffusiver Mehrkomponentensysteme, die in der Physik, der Biologie und der Chemie ihre Anwendung finden, und gewisse Kreuzeffekte zwischen den Spezies aufweisen. Die Arbeit ist in zwei Hauptteile gegliedert, und zwar in einen kinetischen Teil, der in Kapitel 4 und Kapitel 5 behandelt wird, und in einen diffusiven Teil, der in Kapitel 6 und Kapitel 7 ausgearbeitet wird.

Der erste Teil ist physikalischen Mehrkomponentensystemen kinetischer Gleichungen gewidmet, und untersucht das Langzeitverhalten von Gasgemischen, welche aus monoatomischen, nichtreaktiven Gaspartikeln mehrerer Spezies bestehen. Unser Ziel ist es, unter physikalisch möglichst plausiblen Voraussetzungen mit expliziten Methoden zu zeigen, dass diese Gemische für hinreichend reguläre Anfangsdaten mit exponentieller Abklingrate zum globalen Gleichgewichtszustand konvergieren. Mathematisch verwenden wir hierfür die mehrkomponentige Boltzmann-Gleichung auf dem dreidimensionalen Torus, welche aus einem degeneriert-dissipativen Kollisionsoperator und einem konservativen Transportoperator besteht. Im Gegensatz zur Theorie der klassischen einkomponentigen Boltzmann-Gleichung treten hier während des Kollisionsvorgangs zusätzliche Kreuzeffekten auf, die dadurch entstehen, dass im Gasgemisch auch Teilchen unterschiedlicher Spezies miteinander kollidieren können. Dieses Phänomen stellt somit die Hauptschwierigkeit unseres Problems dar. Unsere Studie führen wir in zwei Teilen durch.

In Kapitel 4 arbeiten wir mit der linearisierten Gleichung und unter der vereinfachenden Voraussetzung, dass alle Teilchen die gleiche Masse besitzen. Es gelingt uns, eine bisher nur für den einkomponentige Boltzmannschen Kollisionsoperator gezeigte degenerierte Koerzivitätsungleichung auf den Mehrkomponentenfall zu verallgemeinern. Diese Ungleichung wird in der englischsprachigen Fachliteratur neben dem Namen "coercivity estimate" auch unter dem Namen "spectral-gap estimate" angeführt, da sie nicht nur das degeneriert dissipative Verhalten des Kollisionsoperators beweist, sondern weiters auch eine für das Abklingverhalten wichtige Eigenschaft des Spektrums des Operators zeigt, nämlich dass er einen "spectral gap" besitzt. Mit Hilfe der Hypokoerzivitätsmethode von C. Mouhot und L. Neumann [113] zusammen mit der von uns verallgemeinerten Koerzivitätsungleichung zeigen wir schlussendlich exponentielle Konvergenz zum globalen Gleichgewicht für Lösungen der mehrkomponentigen linearisierten Boltzmann-Gleichung.

In Kapitel 5 verfeinern wir unsere Studie in mehrfacher Hinsicht. Einerseits schwächen wir die physikalischen Voraussetzungen an unser Modell insofern ab, dass wir nun auch Gasgemische zulassen, die aus Spezies mit unterschiedlichen Massen bestehen. Andererseits betrachten wir nicht mehr nur die um das globale Gleichgewicht herum linearisierte Gleichung, sondern erlauben kleine Störungen. Dies ist mathematisch äquivalent dazu, die volle nichtlineare mehrkomponentige Boltzmann-Gleichung nahe des globalen Gleichgewichts zu

betrachten. Der Schwerpunkt unserer Untersuchungen liegt nun darin, in aus physikalischer Sicht bestmöglich passenden Funktionenräumen eine rigorose Cauchytheorie (Existenz, Eindeutigkeit, exponenzielle Konvergenz zum Gleichgewicht) für diese Gleichung zu beweisen, indem wir neueste Techniken für das mehrkomponentige Problem verallgemeinern. Dabei zeigen wir auch eine neue Carleman-Darstellung und eine Art Povzner Ungleichung für Gasgemische mit unterschiedlichen Massen. Ausgangspunkt ist wieder die linearisierte Theorie, für die sich heraustellt, dass sich auch im Falle ungleicher Massen eine Koerzivitätsungleichung herleiten lässt. Nach Kombination mehrerer (zum Teil hypokoerziver) Methoden, wobei die Stärke und Neuartigkeit dieser Kombination darin liegt, dass sie ohne die sonst übliche Verwendung von Sobolevräumen höherer Ordnung auskommt, erhalten wir schließlich Existenz, Eindeutigkeit, und exponentielle Konvergenz zum globalen Gleichgewicht der Lösung im Raum $L^1_v L^\infty_x(m)$, wobei $m \sim (1 + |v|^k)$ ein polynomielles Gewicht der Ordnung $k > k_0$, und k_0 ein physikalischer Schwellenwert ist. Im Spezialfall von Spezies mit gleichen Massen und unter der "hard-spheres" Voraussetzung liefert dies den Schwellenwert $k_0 = 2$, und somit (zumindest fast) den physikalisch "optimalen" Raum aller Dichtefunktionen mit endlicher Masse und Energie.

Der zweite Teil dieser Arbeit ist diffusiven Mehrkomponentensystemen gewidmet, die in der Populationsdynamik der Biologie (Kapitel 6) und in der Reaktionskinetik der Chemie (Kapitel 7) eine wichtige Rolle spielen.

In Kapitel 6 untersuchen wir das nach Shigesada, Kawasaki und Teramoto [129] benannte SKT-Modell der Populationsdynamik, welches ein Kreuzdiffusionsmodell ist, und das Phänomen der räumlichen Abkapselung von miteinander sich im Konkurrenzkampf (z.B. um eine Futterquelle) befindenden Spezies beschreibt. Unser Ziel ist es, globale Existenz schwacher Lösungen des SKT Models zu beweisen. Neu an unseren Ergebnissen ist, dass wir (unter algebraischen Bedingungen an die Diffusionskoeffizienten) Existenz von Lösungen für eine beliebige Anzahl an Spezies zeigen können. Bisherige Ergebnisse beinhalteten (bis auf Spezialfälle in [51]) nur den Fall zweier Populationen. Dabei verwenden und erweitern wir eine von A. Jüngel [94] entwickelte Entropiemethode, die uns die für den Existenzbeweis wichtigen Gradientenabschätzungen liefert. Die Bedingungen an die Diffusionskoeffizienten, die wir erhalten, stellen sicher, dass die Entropie monoton fallend ist, und lassen sich salopp folgendermaßen charakterisieren: entweder wir benötigen eine aus der Theorie von Markovketten her bekannte Bedingung namens "detailed balance", oder die Kreuzdiffusionseffekte müssen schwach im Vergleich zu den Selbstdiffusionseffekten sein. Mehrere Gegenbeispiele illustrieren, dass die Entropie (anfangs) monoton steigt statt zu fallen, falls keine dieser beiden Bedingungen erfüllt ist. Schlussendlich leiten wir eine graphentheoretische Bedingung her, unter der "detailed balance" erfüllt ist.

In Kapitel 7 beweisen wir den rigorosen Limes von verallgemeinerten Reaktions- Diffusionssystemen aus der Reaktionskinetik zu neuartigen Kreuzdiffusionssystemen mit Reaktionstermen, indem wir annehmen, dass einige der chemischen Reaktionen im Vergleich zu allen anderen extrem schnell ablaufen. Dieser Limes besitzt nun die interessante Eigenschaft, dass er Entropie erhaltend ist, und somit Entropien zu neuartigen Klassen von Kreuzdiffusionssystemen mit Reaktionstermen generieren kann. Für den rigorosen Konvergenzbeweis verwenden wir neben Entropieabschätzungen auch die von M. Pierre und D. Schmitt entwickelte Dualitätsmethode [121].

Abstract

The objective of this thesis is the analysis of kinetic and diffusive multi-species systems with certain cross effects between the species, which are very important in many applications in physics, biology and chemistry.

In the first part of this thesis, we study the multi-species Boltzmann equation for hard potentials or Maxwellian molecules under Grad's angular cut-off condition on the torus, which describes the evolution of a dilute gaseous mixture. First, we work on the linearized level with same molar masses, where we prove a multi-species spectral-gap estimate of the collision operator, which leads to exponential trend to global equilibrium using the hypocoercive properties of the linearized Boltzmann equation. Next, we study the full Cauchy theory of the nonlinear multi-species Boltzmann equation close to global equilibrium for different molar masses in $L^1_v L^\infty_x(m)$, where $m \sim (1 + |v|^k)$ is a polynomial weight of order $k > k_0$, recovering the optimal physical threshold of finite energy $k_0 = 2$ in the particular case of a multi-species hard spheres mixture with same molar masses.

The second part of this thesis is devotet to cross-diffusion systems. First, we prove global existence of weak solutions for a generalized SKT cross-diffusion population dynamics model with an arbitrary number of species under detailed balance or weak cross-diffusion condition using entropy methods. Finally, we study a rigorous fast-reaction limit from reaction-diffusion systems to cross-diffusion systems using entropy estimates and additional duality estimates. Since the reaction-diffusion system exhibits an entropy structure, performing the fast-reaction limit leads to a limiting entropy of the limiting cross-diffusion system. In this way, we obtain new entropies for new classes of cross-diffusion systems.

Contents

Part I.

Introduction

1. General introduction

This thesis is devoted to the analysis of kinetic and diffusive multi-species systems in physics, biology and chemistry, where the interactions between the different species play an important role. On both the kinetic and the macroscopic level of description, we will study how these interactions create certain cross effects between the species, which are able to describe cross-diffusion phenomena.

1.1. Overview

Kinetic theory [32, 33, 34, 138] is a part of non-equilibrium statistical physics, and started in the 19th century with the founding works of J. Clerk Maxwell [107, 108] and L. Boltzmann [59] on the evolution of a dilute gas, described by the Boltzmann equation. The basic idea is to use a statistical description to model the evolution of a system consisting of many particles, like a gas, plasma or galaxy, by assuming that it can be regarded as a continuum, instead of describing the trajectory of each particle individually by Newton's laws of classical mechanics. However, the Boltzmann equation is not a "first principle" of physics, but represents a *mesoscopic level*, since it can be formally derived from the *microscopic level* of molecular dynamics of a gas (see [33][34]), and it was rigorously derived from molecular gas dynamics in the Boltzmann-Grad limit (see for short times Landford's theorem [102], or more recently [70, 125]).

Cross diffusion in multi-species systems represents the effect when the gradient of the concentration of one of the species induces a flux on the other species. Cross-diffusion equations are *macroscopic* models, and they can be derived from kinetic equations in the diffusive limit [21, 91, 18]. Cross diffusion was first conjectured by Onsager and Fuoss [117] in 1932, and experimentally justified by Gosting and Dunlop [65] in 1955. One of the most prominent experiments was performed by Duncan and Toor [64], showing that cross effects between the species can completely change the diffusive fluxes. Nowadays, the field of cross-diffusion in multi-species systems is a very active research area, see [95] for a mathematical viewpoint, and [136] and the references therein for many experimental results on cross-diffusion phenomena.

1.2. The main results

The results presented in this thesis can be summarized as follows. On the kinetic level, assuming hard potentials or Maxwellian molecules under Grad's cutoff assumption, we prove an explicit spectral-gap estimate for the multi-species linearized Boltzmann collision operator for same molar masses, and show exponential convergence to global equilibrium for the inhomogeneous linearized equation using hypocoercive techniques. Afterwards, we study the Cauchy theory for the perturbed multi-species Boltzmann equation with different molar masses in physically relevant function spaces, without using higher order Sobolev regularity. The asymmetry due to the different molar masses leads to new multi-species results, including a spectral-gap estimate for different molar masses, a multi-species Carleman representation, and a new multi-species Povzner-type inequality.

On the macroscopic level, we investigate two different issues. First, we study the existence of global weak solutions of a generalized Shigesada-Kawasaki-Teramoto type cross-diffusion system for an arbitrary number of species under a detailed balance of weak cross-diffusion condition, by using entropy methods. This has been proven (up to some very specific cases) only for two species so far. Finally, by starting already from a macroscopic equation of reaction-diffusion type, we perform a rigorous fast-reaction limit in order to obtain more complicated cross-diffusion systems with known entropy structure in the limit. This is done with the help of entropy estimates and additional duality estimates.

1.3. Presentation of the mathematical problems

We start our observations by considering a distribution function $F_i(t,x,v)$ for $1 \le i \le n$, which describes the probability density of the i-th species of a gaseous mixture of $n \ge 2$ different species with same molar masses on the phase space $(x,v) \in \mathbb{T}^3 \times \mathbb{R}^3$. The quantity $F_i(t,x,v)\,dxdv$ measures the number of particles of the species i in the mixture at time t in an elementary volume of the phase space of the size $dxdv$ centered at the point (x,v). The particles are assumed to be mono-atomic and non-reactive, moreover, we restrict our observations to classical mechanics, ignoring any kind of relativistic or quantum effects. Then, the evolution of F_i can be described by the multi-species Boltzmann equation

$$\partial_t F_i + v \cdot \nabla_x F_i = Q_i(F), \quad 1 \le i \le n, \quad t > 0, \tag{1.3.1}$$

with initial conditions and periodic boundary conditions. The term $v \cdot \nabla_x F_i$ models the transport of the particles, whereas $Q(F) = (Q_1(F), \dots, Q_n(F))$ is the quadratic multi-species Boltzmann collision operator, local in time t and in position x, with

$$Q_i(F)(v) = \sum_{j=1}^{n} Q_{ij}(F_i, F_j)(v), \quad 1 \le i \le n.$$

It models the collisions between particles of the same $(i = j)$ or of different $(i \ne j)$ species, where

$$Q_{ij}(F_i,F_j)(v) = \sum_{j=1}^{n} \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij}(|v - v_*|, \cos(\theta))(F_i(v')F_j(v'^*) - F_i(v)F_j(v^*))dv^* d\sigma.$$

The most important property of the Boltzmann equation is the H-theorem, first proved by L. Boltzmann for a single-species gas, and later for reactive multi-species kinetic models in [122, 23, 52]. It states that the entropy is nonincreasing in time along the flow of (1.3.1) towards its global equilibrium called global Maxwellian $M(v) = (M_1(v), \dots, M_n(v))$, which is a Gaussian function. This coincides with the Second Law of Thermodynamics, which says that the physical entropy (which is the negative mathematical entropy) of an isolated system is nondecreasing in time. Besides the existence of a unique global equilibrium, there exist also local equilibria, which are in equilibrium with respect to the velocity v, but not with respect to the position x. It turns out that the entropy stops decaying whenever it reaches a local equilibrium, and only the interplay between the conservative transport part and the dissipative collision part of the kinetic equation leads to convergence to global equilibrium. This property of the Boltzmann equation and also of several other kinetic models is called hypocoercivity [139]. It was first investigated for the single-species kinetic Fokker-Planck equation in [54, 86], and then for the confined nonlinear single-species Boltzmann equation assuming regularity in [55]. Hypocoercivity of a general class of linear single-species kinetic models was investigated in [113] and in [58]. In both papers, a linear energy method is established, which combines the coercivity of the collision operator in the velocity space (spectral-gap estimate) together with transport effects in order to overcome the lack of coercivity on the set of local equilibria. The abstract strategy presented in [113] includes the linearized Boltzmann equation for hard spheres on the torus, yielding exponential convergence in H^1. The work [58] studies hypocoercivity for a large class of single-species linear kinetic models on the whole space with confinement potential in L^2, but for only one-dimensional collision kernels. Recently, an abstract space-extension technique was established in [78], which is capable of enlarging the functional space of the explicit convergence result of a semigroup, by using a high-order quantitative factorization argument on the resolvent of the semigroup. In this way, a first constructive proof of exponential convergence for the full nonlinear Boltzmann equation for hard spheres was proved in $L^1_v L^\infty_x(1 + |v|^k)$ for $k > 2$.

In **Chapter 4**, we show that also the multi-species linearized Boltzmann equation

$$\partial_t f_i + v \cdot \nabla_x f_i = L_i(f), \quad 1 \le i \le n, \quad t > 0,$$

$$L_i(f)(v) = \sum_{j=1}^{n} L_{ij}(f_i, f_j)(v),$$

with

$$L_{ij}(f_i, f_j) = M_i^{-1/2} \big(Q_{ij}(M_i, M_j^{1/2} f_j) + Q_{ij}(M_i^{1/2} f_i, M_j) \big)$$

exhibits a hypocoercive property by generalizing the abstract hypocoercivity procedure of C. Mouhot and L. Neumann [113] to the multi-species case. The core idea in [113] is to construct a modified entropy by adding an additional mixed term of the form (2.1.15) to the H^1-norm with an appropriate weight, whose dissipation functional is coercive. This yields exponential convergence to global equilibrium with an explicitly computable exponential convergence rate in $H^1(\mathbb{T}^3 \times \mathbb{R}^3)$. The crucial part for our multi-species problem will be to show that an explicit spectral-gap estimate like in [111] holds true also in the multi-species case, by carefully exploiting the multi-species conservation laws. These conservation laws turn out to be significantly different from the single-species ones due to certain cross effects between the species, namely, that the mass of each species is conserved individually, but only the sum of the momenta and the sum of the energies is conserved, which can be derived from the multi-species H-theorem [52]. Thus, the idea is to study the differences of the momenta and the differences of the energies, in order to handle the cross effects arising due to the collisions between different species, see section 2.1.5, and in particular the estimate on the differences of the energies and differences of the momenta presented in (2.1.11) for an explicitly computable constant $C > 0$:

$$-(\Pi^m(f), L^b(\Pi^m(f)))_{L^2_v} \geq C \sum_{i,j=1}^{n} \big(|u_i - u_j|^2 + |e_i - e_j|^2\big),$$

where we decomposed the linearized operator $L = L^m + L^b$ into a mono-species part L^m and a bi-species part L^b, and thus $\Pi^m(f)$ denotes the orthogonal projection of f onto the kernel of L^m in L^2, and u_i are the momenta and e_i are the energies of f, defined in (4.4.4). In this way, we obtain the explicit multi-species spectral-gap estimate

$$-(f, L(f))_{L^2_v} \geq \lambda \|f - \Pi^L(f)\|^2_{L^2_v(\nu)},$$

where $\lambda > 0$ is an explicitly computable constant, $L^2(\nu)$ is a weighted (vector-valued) L^2-space weighted by the collision frequency ν, and $\Pi^L(f)$ denotes the orthogonal projection of $f = (f_1, \dots, f_n)$ onto the null space of the linearized multi-species collision operator $L = (L_1, \dots, L_n)$ with respect to the L^2 scalar product. This part of the thesis is based on a research collaboration with A. Jüngel (TU Wien), C. Mouhot (University of Cambridge) and Nicola Zamponi (TU Wien), and was published under the title *Hypocoercivity for a linearized multispecies Boltzmann system* in the journal *SIAM Journal on Mathematical Analysis* [46].

In **Chapter 5**, we extend the previous results, by establishing a full Cauchy theory in $L^1_v L^\infty_x(m)$ for the nonlinear multi-species Boltzmann equation with different molar masses on the torus, where $m \sim (1 + |v|^k)$ is a polynomial weight of order $k \geq k_0$ with an explicit threshold k_0, under the assumption that f_i is close to global equilibrium, which is equivalent of solving the perturbed multi-species Boltzmann equation. In the particular case of a multi-species hard spheres mixture with same molar masses, this result recovers the optimal physical threshold of finite energy $k_0 = 2$, which was obtained for the single-species equation in [78]. The strategy of our proof is to combine and adapt several very recent methods, combined with new hypocoercivity estimates, in order to develop a new constructive approach that allows to deal with polynomial weights without requiring any spatial Sobolev regularity. Moreover, dealing with different masses induces a loss of symmetry in the Boltzmann operator, which prevents the direct adaptation of standard mono-species methods. Thus, our results include a refinement of the multi-species spectral-gap estimate [46] (see Theorem 5.6), a multi-species Carleman representation (section 5.4.1), which takes into account the concentric spheres arising due to the different masses already pointed out by Pettersson in [119, p. 364], and a new multi-species Povzner-type inequality (see Proposition 5.22). The different molar masses do not disturb too much the procedure for proving the multi-species spectral-gap estimate which we developed in [46] and presented in Chapter 4. The Carleman representation for different molar masses has the interesting feature of producing admissible Carleman sets of the form of a hyperplane that passes through $V_E(v, v')$ defined in (5.4.7) and is orthogonal to $v - v'$ (similar to the single-species case), or of a sphere defined in (5.4.9), depending on which change of variables we perform. For the Povzner-type inequality, we used a method similar to [9, Lemma 1 and Corollary 3], and our main idea is to consider kinetic energies $m_i\,|v'|^2$ and $m_j\,|v'_*|^2$ in order to exhibit the problematic term arising from $m_i - m_j$, which can be non-zero in the case of different masses.

Before, global existence of L^1-renormalized solutions was proved in [123] for reactive multi-species Boltzmann systems with initial data only satisfying the natural condition that total mass, energy and entropy are finite. In the recent work [27], additional L^∞ stability estimates for the multi-species Boltzmann equation are derived. Other methods for proving the compactness of the compact part of the multi-species linearized collision operator can be found in [19, 22], and for previous results on a multi-species Povzner-type inequality for a coupling of two Boltzmann-like equations modeling the evolution of dust particles in a rarefied atmosphere, we refer to [36]. The work presented in this thesis is based on a research collaboration with M. Briant (Paris 6), which will appear under the title *The Boltzmann equation for a multi-species mixture close to global equilibrium* in the journal *Archive for Rational Mechanics and Analysis* [28].

Starting from a diffusive scaling of the multi-species Boltzmann equation (1.3.1) under the assumption of volume-filling and with $\varepsilon > 0$ representing the mean-free path

$$\varepsilon\,\partial_t f_i^\varepsilon + v \cdot \nabla_x f_i^\varepsilon = \frac{1}{\varepsilon}\,Q_i^\varepsilon(f^\varepsilon), \quad 1 \le i \le n,$$

a class of cross-diffusion systems called Maxwell-Stefan equations can be derived in the diffusive limit $\varepsilon \to 0$, see [21, 91, 18]. Existence of solutions for this system was investigated *e.g.* in [72, 12, 20, 96], including a global existence result of weak solutions under quite general assumptions in [96] by using entropy methods. A general strategy, which generalizes the approach in [96] of proving global existence of weak solutions for strongly coupled cross-diffusion systems of the form

$$\partial_t u - \operatorname{div}(A(u)\nabla_x u) = f(u), \quad u = (u_1, \dots, u_n), \tag{1.3.2}$$

where the diffusion matrix $A(u)$ is neither symmetric nor positive definite and $f(u)$ represents the reaction part, was established in [94] (see also [95]) by the Boundedness-by-entropy method. The core assumption for this method is the existence of an entropy, which is a Lyapunov functional along the flow of the cross-diffusion system, such that this system can be rewritten in (formal) gradient-flow form. In particular, we assume that equation (1.3.2) can be written as

$$\partial_t u - \operatorname{div}\left(B\nabla\frac{\delta\mathcal{H}}{\delta u}\right) = f(u), \tag{1.3.3}$$

where B is a positive semi-definite matrix and $\frac{\delta\mathcal{H}}{\delta u}$ is the variational derivative of the entropy functional $\mathcal{H}[u] = \int_\Omega h(u)dx$ with an entropy density $h : (0,\infty)^n \to [0,\infty)$. If we identify

$\frac{\delta \mathcal{H}}{\delta u}$ by its Riesz representative $h'(u)$ (the derivative of h) and introduce the entropy variable $w = h'(u)$, we obtain the equation

$$\partial_t u - \operatorname{div}\left(B(w)\nabla w\right) = f(u). \tag{1.3.4}$$

The simple calculation $\nabla w = \nabla h'(u) = h''(u)\nabla u$ yields $B(w)h''(u) = A(u)$, which leads to

$$B(w) = A(u)\left(h''(u)\right)^{-1}. \tag{1.3.5}$$

Thus we can also study equation (1.3.4) and by supposing that $h' : (0,\infty)^n \to \mathbb{R}^n$ is invertible, transform back from the w- variable to the u- variable.

The important consequence of the gradient-flow formulation is that by just assuming the matrix B to be positive-semidefinite, we can show (omitting the reaction term) that $\mathcal{H}\left[u(t)\right]$ is a Lyapunov functional along the solutions $u(t)$ of the reaction diffusion system (1.3.2), *i.e.*

$$\frac{d}{dt}\mathcal{H}\left[u(t)\right] = \frac{d}{dt}\int_\Omega h(u)dx = -\int_\Omega \nabla w : B(w)\nabla w dx = -\int_\Omega \nabla u : h''(u)A(u)\nabla u dx \le 0.$$

The resulting entropy-dissipation inequality provides the necessary a priori estimates for the gradients in order to apply a nonlinear Aubin-Lions lemma.

In **Chapter 6**, we extend the Boundedness-by-entropy method [94, 95] to a population dynamics cross-diffusion system, which was first investigated by Shigesada, Kawasaki and Teramoto in 1979 in order to study the segregation between two competing species due to repulsive effects [129]. First global existence results for this so called SKT model in the case of two species by using entropy methods were shown in [37, 38]. Later, these techniques were refined for generalized SKT models by combining entropy estimates with duality estimates [50, 51]. In [94], global existence of weak solutions for a generalized SKT model was shown in the case of only two species. We extend this result to an arbitrary number of species. The model has the form (1.3.2), with diffusion matrix

$$A_{ij}(u) = \delta_{ij}p_i(u) + u_i\frac{\partial p_i}{\partial u_j}(u), \quad p_i(u) = a_{i0} + \sum_{k=1}^n a_{ik}u_k^s, \quad a_{ij} \ge 0, \quad s > 0,$$

and Lotka-Volterra competition term

$$f_i(u) = u_i\left(b_{i0} - \sum_{j=1}^n b_{ij}u_j^\sigma\right), \quad 0 \le \sigma < 2s - 1 + 2/d,$$

on a d-dimensional open bounded domain $\Omega \subseteq \mathbb{R}^d$. For $s = 1, n = 2$ and $\sigma = 1$, this yields the classical SKT model of Shigesada, Kawasaki and Teramoto [129]. The existence proof is carried out under a detailed balance or weak cross-diffusion condition. The detailed balance condition is related to the symmetry of the mobility matrix, mirroring Onsager's principle in thermodynamics. Under detailed balance (and without reaction term), the entropy is non-increasing in time, but counter-examples show that the entropy may increase initially if the detailed balance condition does not hold. The crucial gradient estimate in the linear case $s = 1$ is derived from the following entropy-dissipation inequality:

$$\frac{d}{dt}\mathcal{H}[u] + 4\int_\Omega \sum_{i=1}^n \pi_i a_{i0} |\nabla \sqrt{u_i}|^2 dx + 2\int_\Omega \sum_{i=1}^n \pi_i a_{ii} |\nabla u_i|^2 dx \le 0,$$

where $\pi_i > 0$ are positive constants inside the entropy functional

$$\mathcal{H}[u] = \sum_{i=1}^n \int_\Omega \pi_i (u_i \log u_i - u_i + 1)\, dx,$$

which satisfy the detailed balance condition

$$\pi_i a_{ij} = \pi_j a_{ji}, \quad 1 \le i, j \le n.$$

These results are based on a research collaboration with X. Chen (Beijing University of Posts and Telecommunications) and A. Jüngel (TU Wien), which is submitted under the title *Global existence analysis of cross-diffusion population systems for multiple species* [39].

In **Chapter 7**, we study the rigorous derivation of a class of (reaction-) cross diffusion systems in the fast-reaction limit, by already starting from a macroscopic equation of reaction-diffusion type. We consider two concrete examples and use a strategy similar to [16]. In the first model, we study the fast-reaction limit $\varepsilon \to 0$ of a system of three reaction-diffusion equations of the form

$$\begin{cases} \partial_t u_1^\varepsilon - \Delta_x f_1(u_1^\varepsilon) = -\frac{1}{\varepsilon}\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)\, q_3(u_3^\varepsilon)\Big), \\ \partial_t u_2^\varepsilon - \Delta_x f_2(u_2^\varepsilon) = +\frac{1}{\varepsilon}\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)\, q_3(u_3^\varepsilon)\Big), \\ \partial_t u_3^\varepsilon - \Delta_x f_3(u_3^\varepsilon) = +\frac{1}{\varepsilon}\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)\, q_3(u_3^\varepsilon)\Big), \end{cases} \tag{1.3.6}$$

for $\varepsilon > 0$ on a bounded domain $\Omega \subseteq \mathbb{R}^d$ together with no-flux boundary conditions and nonnegative initial conditions. The functions $u_i^\varepsilon := u_i^\varepsilon(t,x) \ge 0$ for $i = 1,2,3$ represent the concentrations of the i-th reacting species, and $f_i(u_i^\varepsilon)$ and $q_i(u_i^\varepsilon)$ are general smooth functions with some restrictions on their growth. The advantage of this rather general system is, that its structure is still simple enough that an entropy naturally appears as a generalization of the physical entropy of systems arising from reversible reaction chemistry. Then, by using entropy estimates and duality estimates, we show that this system converges rigorously in the fast-reaction limit $\varepsilon \to 0$ towards

$$q_1(u_1) - q_2(u_2) q_3(u_3) = 0, \tag{1.3.7}$$

$$\partial_t\,(u_1 + u_2) - \Delta_x\,(f_1(u_1) + f_2(u_2)) = 0, \tag{1.3.8}$$

$$\partial_t\,(u_1 + u_3) - \Delta_x\,(f_1(u_1) + f_3(u_3)) = 0. \tag{1.3.9}$$

By using the algebraic condition (1.3.7) and performing the change of variables $w_2 := u_1 + u_2$ and $w_3 := u_1 + u_3$ in (1.3.8) and (1.3.9), this limiting system turns out to be a system of two cross-diffusion equations for which an entropy exists: it is the limiting entropy of the well-known physical entropy of the system (1.3.6) for $\varepsilon \to 0$. In this way, we obtain a new large class of cross-diffusion systems with known entropy structure. Our second model is one example of a more complex situation, in which a system of four equations of reaction-diffusion type converges in the fast-reaction limit towards a "hybrid" system of reaction-cross diffusion consisting of three equations, still presenting an entropy structure. In both rigorous convergence proofs, entropy estimates as well as duality estimates play an important role. Entropy estimates are obtained from the entropy-dissipation inequality of the Lyapunov functional of these systems:

$$\sup_{t\in[0,T]} H[u^\varepsilon(t)] + \int_0^T D_\varepsilon[u^\varepsilon(s)]ds \leq H[u^{in}],$$

where $H[u^\varepsilon(t)]$ is the entropy (Lyapunov function) along the flow of the system, and $D_\varepsilon[u^\varepsilon(t)] = -\frac{d}{dt}H[u^\varepsilon(t)]$ is the entropy dissipation. The use of duality estimates for reaction-diffusion systems goes back to M. Pierre and D. Schmitt [121] and has been refined in many variants since then, see *e.g* [49, 29]. Recently, they have been introduced also for cross-diffusion systems, see for instance [50, 51]. We present two of the variants explicitly in Lemma 7.5 and Lemma 7.21, which can be found in [49] and [29] respectively. The core idea of these duality estimates is to gain regularity for a system of the type

$$\partial_t u - \Delta_x(Mu) = 0, \quad M = M(t,x) \geq 0, \quad u = u(t,x) \geq 0, \quad u = (u_1, \dots, u_n),$$

with no-flux boundary conditions and regular initial conditions, roughly speaking, by multiplying the system by a positive solution of its adjoint problem backward in time, *i.e.* its dual problem. This part of the thesis is based on a research collaboration with L. Desvillettes (Paris Diderot) and A. Jüngel (TU Wien) under the title *Cross-diffusion systems and fast-reaction limit*, and will be submitted soon.

1.4. Outline of the thesis

In **Chapter 1**, we introduce the mathematical problems investigated in this thesis, discuss how they fit into the context of the existing literature, and briefly sketch some ideas of the proofs. A more detailed introduction into the models presented in this thesis can be found in **Chapter 2** and **Chapter 3**, which are based on my own works [46, 28] and [39] respectively. We refer to these two chapters for more details on the state of the art and the methods used for the proofs. In **Chapter 4, 5, 6, 7** we present the detailed proofs. The proofs of the first three chapters (Chapter 4, 5, 6) can be found in [46, 28, 39].

2. Introduction to the kinetic multi-species models

2.1. The linearized Boltzmann system with same molar masses

In Chaper 4, our goal is to prove an explicit spectral-gap estimate of the linearized Boltzmann operator for gas mixtures in the case of hard and Maxwellian potentials as well as the exponential decay of solutions to a *multi-species* Boltzmann system. Spectral-gap estimates and the large-time behavior of the *mono-species* Boltzmann equations were intensively studied in the literature, but are unknown for *multi-species* systems. Our aim is to extend the spectral-gap analysis to the case of the linearized *multi-species* Boltzmann system modeling an ideal gas mixture. This is achieved by generalizing the coercivity method of [113], including quantitative estimates on the spectral gap for the multi-species collision operator. A crucial step of our analysis is the observation that the multi-species version of the H-theorem implies conservation of mass for each species but conservation of momentum and energy only for the sum of all species. As a consequence, we need to study carefully the “cross-effects” of the collisions, i.e., how collisions between different species act on distribution functions which are elements of the nullspace of the mono-species collision operator. The crucial step is to relate these “cross-effects” to the differences of momentum and energy.

2.1.1. The model for same molar masses

The evolution of a dilute ideal gas composed of $n \geq 2$ different species of chemically non-interacting mono-atomic particles (see [52] for chemically reacting gases) with the same particle mass (see Chapter 5 for different molar masses) can be modeled by the following system of Boltzmann equations, stated on the three-dimensional torus $\mathbb{T}^3$,

$$\partial_t F_i + v \cdot \nabla_x F_i = Q_i(F), \quad t > 0, \quad F_i(x, v, 0) = F_{I,i}(x, v), \quad (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3, \tag{2.1.1}$$

where $1 \leq i \leq n$. The vector $F = (F_1, \ldots, F_n)$ is the distribution function of the system, with F_i describing the ith species. The variables are the position $x \in \mathbb{T}^3$, the velocity $v \in \mathbb{R}^3$, and the time $t \geq 0$. The right-hand side of the kinetic equation in (2.1.1) is the ith component of the nonlinear collision operator, defined by

$$Q_i(F) = \sum_{j=1}^{n} Q_{ij}(F_i, F_j), \quad 1 \leq i \leq n,$$

where Q_{ij} models interactions between particles of the same ($i = j$) or of different species ($i \neq j$),

$$Q_{ij}(F_i, F_j)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij}(|v - v^*|, \cos\vartheta)(F_i' F_j'^* - F_i F_j^*) dv^* d\sigma,$$

with the abbreviations $F_i' = F_i(v')$, $F_i^* = F_i(v^*)$, $F_i'^* = F_i(v'^*)$, the three-dimensional unit sphere $\mathbb{S}^2$, and

$$v' = \frac{v+v^*}{2} + \frac{|v-v^*|}{2}\sigma, \quad v'^* = \frac{v+v^*}{2} - \frac{|v-v^*|}{2}\sigma \tag{2.1.2}$$

are the pre-collisional velocities depending on the post-collisional velocities (v, v^*). These expressions follow from the fact that we assume the collisions to be elastic, i.e., the momentum and kinetic energy are conserved on the microscopic level:

$$v' + v'^* = v + v^*, \quad \frac{1}{2}|v'|^2 + \frac{1}{2}|v'^*|^2 = \frac{1}{2}|v|^2 + \frac{1}{2}|v^*|^2. \tag{2.1.3}$$

The collision kernels B_{ij} are nonnegative functions of the modulus $|v - v^*|$ and the cosine of the deviation angle $\vartheta \in [0, \pi]$, defined by $\cos\vartheta = \sigma \cdot (v - v^*)/|v - v^*|$.

Let us first recall the main properties of the nonlinear operator Q_i. Using the techniques from [34, pp. 36-42], it is not difficult to see that $Q := (Q_1, \ldots, Q_n)$ conserves the mass of each species but only total momentum and energy, i.e.

$$\int_{\mathbb{R}^3} \sum_{i,j=1}^n Q_{ij}(F_i, F_j)\psi_i(v)dv = 0$$

if and only if $\psi(v) \in \operatorname{span}\{e^{(1)}, \ldots, e^{(n)}, v_1\mathbf{1}, v_2\mathbf{1}, v_3\mathbf{1}, |v|^2\mathbf{1}\}$, where $e^{(i)}$ is the ith unit vector in $\mathbb{R}^n$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. It is shown in [52] that Q satisfies a multi-species version of the H-theorem which implies that any local equilibrium, i.e. any function F being the maximum of the Boltzmann entropy, has the form of a local Maxwellian $M_{\text{loc}} = (M_{\text{loc},1}, \ldots, M_{\text{loc},n})$ with

$$F_i(x, v, t) = M_{\text{loc},i}(x, v, t) = \frac{\rho_{\text{loc},i}(x, t)}{(2\pi\theta_{\text{loc}}(x, t))^{3/2}} \exp\left(-\frac{|v - u_{\text{loc}}(x, t)|^2}{2\theta_{\text{loc}}(x, t)}\right),$$

where, introducing the total local density $\rho_{\text{loc}} = \sum_{i=1}^n \rho_{\text{loc},i}$,

$$\rho_{\text{loc},i} = \int_{\mathbb{R}^3} F_i dv, \quad u_{\text{loc}} = \frac{1}{\rho_{\text{loc}}} \sum_{i=1}^n \int_{\mathbb{R}^3} F_i v dv, \quad \theta_{\text{loc}} = \frac{1}{3\rho_{\text{loc}}} \sum_{i=1}^n \int_{\mathbb{R}^3} F_i |v - u|^2 dv$$

are the (local) masses of the species, the total momentum and total energy, respectively.

On the other hand, the global equilibrium, which is the unique stationary solution F to (2.1.1), is given by $M = (M_1, \ldots, M_n)$ with

$$F_i(x, v) = M_i(v) = \frac{\rho_{\infty,i}}{(2\pi\theta_\infty)^{3/2}} \exp\left(-\frac{|v - u_\infty|^2}{2\theta_\infty}\right),$$

where now, setting $\rho_\infty = \sum_{i=1}^n \rho_{\infty,i}$,

$$\rho_{\infty,i} = \int_{\mathbb{T}^3\times\mathbb{R}^3} F_i dxdv, \quad u_\infty = \frac{1}{\rho_\infty} \int_{\mathbb{T}^3\times\mathbb{R}^3} F_i v dxdv, \quad \theta_\infty = \frac{1}{3\rho_\infty} \int_{\mathbb{T}^3\times\mathbb{R}^3} F_i |v - u|^2 dxdv$$

do not depend on (x, t). By translating and scaling the coordinate system, we may assume that $u_\infty = 0$ and $\theta_\infty = 1$ such that the global equilibrium becomes

$$M_i(v) = \frac{\rho_{\infty,i}}{(2\pi)^{3/2}} e^{-|v|^2/2}, \quad 1 \le i \le n. \tag{2.1.4}$$

2.1.2. Linearized Boltzmann collision operator

We assume that the distribution function F_i is close to the global equilibrium such that we can write $F_i = M_i + M_i^{1/2} f_i$ for some small perturbation f_i, where M_i is given by (2.1.4). Then, dropping the small nonlinear remaining term, f_i satisfies the linearized equation

$$\partial_t f_i + v \cdot \nabla_x f_i = L_i(f), \quad t > 0, \quad f_i(x, v, 0) = f_{I,i}(x, v), \quad (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3, \qquad (2.1.5)$$

for $1 \le i \le n$, where $f = (f_1, \dots, f_n)$ and the ith component of the linearized collision operator $L = (L_1, \dots, L_n)$ is given by

$$L_i(f) = \sum_{j=1}^{n} L_{ij}(f_i, f_j), \quad 1 \le i \le n,$$

with

$$\begin{aligned} L_{ij}(f_i, f_j) &= M_i^{-1/2}\big(Q_{ij}(M_i, M_j^{1/2} f_j) + Q_{ij}(M_i^{1/2} f_i, f_j)\big) \\ &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} M_i^{1/2} M_j^*(h_i' + h_j'^* - h_i - h_j^*) dv^* d\sigma, \quad h_i := M_i^{-1/2} f_i. \end{aligned} \qquad (2.1.6)$$

Here, we have used $M_i'^* M_j' = M_i^* M_j$ for any i, j, which follows from (2.1.3). Notice that we have chosen the linearization considered in, e.g., [113, 135], which allows to work in a pure $L^2_{x,v}$ framework without weight function. Another linearization is given by $F_i = M_i + g_i$, which we use in Chapter 5, making it necessary to work in a weighted L^2 space, where the weight function is a Maxwellian of the form $1/M_i$. Note that this choice gives the same results as with the linearization (2.1.6) since both linearizations correspond to the same space of solutions. It turned out that in Chapter 4 the computations are easier using (2.1.6), whereas in Chapter 5 they are easier using the other linearization.

The linearized Boltzmann system satisfies an H-theorem with the linearized entropy $H(f) = \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^3} f_i^2 dv$,

$$-\frac{dH}{dt} = -\sum_{i=1}^{n} \int_{\mathbb{R}^3} f_i L_i(f) dv =: -(f, L(f))_{L^2_v} \ge 0,$$

where $(\cdot, \cdot)_{L^2_v}$ is the scalar product on $L^2_v := L^2(\mathbb{R}^3; \mathbb{R}^n)$. We will prove in Lemma 4.4 that $(f, L(f))_{L^2_v} = 0$ if and only if $M_i^{-1/2} f_i$ lies in $\operatorname{span}\{e^{(1)}, \dots, e^{(n)}, v_1\mathbf{1}, v_2\mathbf{1}, v_3\mathbf{1}, |v|^2\mathbf{1}\}$, which is the null space $\operatorname{Ker}(L)$ of the linear operator L.

Our main goal is to show that, under suitable assumptions on the collision kernels, there exists a constant $\lambda > 0$, which can be computed explicitly, such that for all suitable functions f, $-(f, L(f))_{L^2_v} \ge \lambda \|f - \Pi^L(f)\|^2_{\mathcal{H}}$, where Π^L is the projection onto $\operatorname{Ker}(L)$ and $\mathcal{H}$ is a subset of L^2_v (see Theorem 2.3 for the precise statement). This spectral-gap estimate, together with hypocoercivity techniques, allows us to conclude that exponential decay of the solutions $f(t)$ towards the global equilibrium holds (see Theorem 2.4).

2.1.3. State of the art

The study of the linearized mono-species collision operator, in the spatially homogeneous and hard-potential case, goes back to Hilbert [87]. For this operator, Carleman [30] proved the existence of a spectral gap. The results were extended by Grad [74] for hard potentials with cut-off. Baranger and Mouhot [4] derived constructive estimates in the hard-sphere case. For Maxwell molecules, Fourier transform methods were employed in [35] to achieve explicit spectral properties. A spectral-gap estimate for the linearized Boltzmann operator, consisting of the sum of the linearized collision operator and the transport operator, was first shown by Ukai [134]. Improved estimates (in smaller spaces of Sobolev type), still for hard potentials, were established in [113]. In [114], spectral-gap estimates for moderately soft potentials (without angular cut-off) were proved, improving and extending previous results by Pao [118]. Hypoelliptic estimates for the linearized operator without cut-off can be found in [1] and references therein. A spectral analysis with relaxed tail decay and regularity conditions on the solutions was performed recently in an abstract framework [78], which we will use in Chapter 5. Dolbeault et al. [58] derived exponential decay rates in weighted L^2 spaces, which improves previous Sobolev estimates. For further references, we refer to [114, Section 1.5].

Spectral properties of the linearized Boltzmann operator were already investigated by Grad [75]. Based on these results, Schechter [128] located the essential spectrum of the classical collision operator in L^2. The spectrum of the Boltzmann operator for hard spheres was also analyzed in L^p for $p \neq 2$; see [100]. We refer to the recent work [63] for further results in L^p for $1 \leq p \leq \infty$ and more references. A detailed analysis of the resolvent and spectrum of the linearized Boltzmann operator can be found in [135, Section 2.2]. A complete analysis for the essential and discrete spectra for the linearized collision operator with hard potentials was performed in [112].

Before we state our main results, let us introduce the general assumptions.

2.1.4. Assumptions on the collision kernels

We impose the following assumptions on the collision kernels B_{ij} arising in (2.1.6). For a discussion of these assumptions, see section 4.1.2.

(A1) The collision kernels satisfy

$$B_{ij}(|v-v^*|,\cos\vartheta) = B_{ji}(|v-v^*|,\cos\vartheta) \quad \text{for } 1 \leq i,j \leq n.$$

(A2) The collision kernels decompose in the kinetic part $\Phi_{ij} \geq 0$ and the angular part $b_{ij} \geq 0$ according to

$$B_{ij}(|v-v^*|,\cos\vartheta) = \Phi_{ij}(|v-v^*|)b_{ij}(\cos\vartheta), \quad 1 \leq i,j \leq n.$$

(A3) For the kinetic part, there exist constants C_1, $C_2 > 0$, $\gamma \in [0,1]$, and $\delta \in (0,1)$ such that for all $1 \leq i,j \leq n$ and $r > 0$,

$$C_1 r^\gamma \leq \Phi_{ij}(r) \leq C_2(r + r^{-\delta}).$$

(A4) For the angular part, there exist constants C_3, $C_4 > 0$ such that for all $1 \leq i,j \leq n$ and $\vartheta \in [0,\pi]$,

$$0 < b_{ij}(\cos\vartheta) \leq C_3 |\sin\vartheta|\,|\cos\vartheta|, \quad b'_{ij}(\cos\vartheta) \leq C_4.$$

Furthermore,

$$C^b := \min_{1\leq i\leq n} \inf_{\sigma_1,\sigma_2\in\mathbb{S}^2} \int_{\mathbb{S}^2} \min\big\{b_{ii}(\sigma_1\cdot\sigma_3), b_{ii}(\sigma_2\cdot\sigma_3)\big\} d\sigma_3 > 0.$$

(A5) For all $1 \leq i,j \leq n$, b_{ij} is even in $[-1,1]$ and the mapping $v \mapsto \Phi'_{ij}(|v|)$ on $\mathbb{R}^3$ is locally integrable on $\mathbb{R}^3$ and bounded as $|v| \to \infty$.

(A6) There exists $\beta > 0$ such that for all $1 \leq i,j \leq n$, $s > 0$, and $\sigma \in [-1,1]$, we have $B_{ij}(s,\sigma) \leq \beta B_{ii}(s,\sigma)$.

Following [113], since the functions b_{ij} are integrable, we define

$$\ell^b := \min_{1,\leq i,j\leq n} \int_0^\pi b_{ij}(\cos\theta)\sin\theta d\theta > 0. \tag{2.1.7}$$

2.1.5. Main results for same molar masses

We present the main results and discuss them briefly. The first result is a geometric property of the essential spectrum of the linearized collision operator L and the linearized Boltzmann operator $G = L - T$, where T denotes the transport part $T = v \cdot \nabla_x$.

Theorem 2.1 (Essential spectrum of L and $L - T$)**.** *Let the collision kernels B_{ij} satisfy assumptions (A1)-(A4) and set*

$$J = \cup_{i=1}^n \mathrm{Im}(\nu_i) \subset [\nu_0, \infty),$$

where $\nu_0 = \min_{i=1,\dots,n} \sup_{v\in\mathbb{R}^3} \nu_i(v) > 0$, where ν_i is the i-th component of the collision frequency ν_i, given by

$$\nu_i(v) = \sum_{j=1}^n \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij}(|v^* - v|, \cos\vartheta) M_j^* dv^* d\sigma, \quad i = 1,\dots,n. \tag{2.1.8}$$

Then

$$\sigma_{ess}(L) = -J, \quad \sigma_{ess}(L - T) = \{\lambda \in \mathbb{C} : \Re(\lambda) \in -J\},$$

where $\sigma_{ess}(L)$ denotes the essential spectrum of the multi-species operator L.

Remark 2.2. We observe that if $\lim_{|v|\to\infty} \nu_i(v) = \infty$ for $i = 1,\dots,n$ then

$$\sigma_{\mathrm{ess}}(L) = (-\infty, -\nu_0], \quad \sigma_{\mathrm{ess}}(L - T) = \{\lambda \in \mathbb{C} : \Re(\lambda) \leq -\nu_0\}.$$

Indeed, under the assumption $\nu_i(v) \to \infty$ as $|v| \to \infty$, the continuity of ν_i, and the Weierstraß theorem show that $J = [\nu_0, \infty)$. Thus, the essential spectrum of the linearized *multi-species* collision operator is very similar to the *mono-species* operator, where ν_0 corresponds to the infimum in $\mathbb{R}^3$ of the single collision frequency; see [111, Section 3] and [112, Prop. 3.1]. □

The proof of Theorem 2.1 is based on perturbation theory [97, Chap. IV] and is similar to the proof for the mono-species collision operator [135], but we show new explicit spectral-gap estimates related to the particular structure of the kernel in the multi-species case. More precisely, we write $L = K - \Lambda$, where it turns out that $K = L + \Lambda$ is compact on L^2_v (see section 4.3 for details). Weyl's theorem [84, Theorem S] states that the essential spectrum of $L = K - \Lambda$ coincides with that of $-\Lambda$. Thus it remains to show that $\sigma_{\text{ess}}(\Lambda) = J$. This is done by using Weyl's singular sequences, which allow for a sufficient and necessary condition for $\lambda \in \mathbb{C}$ being an element of the essential spectrum of the selfadjoint operator Λ.

The proof of the second statement in Theorem 2.1 is more involved since K is not compact on $L^2_{x,v}$ and hence, Weyl's theorem cannot be applied directly. The idea is to employ an extended Weyl theorem, which states that the essential spectrum is conserved under a *relatively compact* perturbation [97, Section IV.5.6, Theorem 5.35]. Indeed, if K is relatively compact with respect to $\Lambda + T$ then $\sigma_{\text{ess}}(L - T) = \sigma_{\text{ess}}(K - (\Lambda + T)) = -\sigma_{\text{ess}}(\Lambda + T)$, and it remains to compute the essential spectrum of $\Lambda + T$.

The next theorem concerns an explicit spectral-gap estimate, which is our main result here.

Theorem 2.3 (Explicit spectral-gap estimate). *Let the collision kernels B_{ij} satisfy assumptions (A1)-(A4). Then there exists a constant $\lambda > 0$ such that*

$$-(f, L(f))_{L^2_v} \geq \lambda \|f - \Pi^L(f)\|^2_{\mathcal{H}} \quad \text{for all } f \in \mathcal{D}, \tag{2.1.9}$$

where Π^L is the projection onto the null space $\text{Ker}(L)$*, $\mathcal{D}$ denotes the domain of L, and $\mathcal{H}$ is a weighted $L^2(\nu)$ space weighted by the collision frequency ν, see* (4.1.6) *for the precise definitions. If additionally hypothesis (A6) holds, the constant λ can be computed explicitly:*

$$\lambda = \frac{\eta D^b}{8C_k}, \quad \eta = \min\left\{1, \frac{4C^m C_k}{16C_k + D^b}\right\},$$

where C^m, D^b, and C_k are defined in (4.4.8)*,* (4.4.13)*, and* (4.4.15)*, respectively.*

We present two proofs of this theorem. The first proof is non-constructive and relies on an abstract functional theoretical argument, based on the decomposition $L = K - \Lambda$ and Weyl's perturbation theorem. This abstract spectral-gap estimate is proved in Lemma 4.9. The second proof provides a constructive spectral-gap estimate, generalizing the result in [111] (also see [110, Theorem 6.1]) from the mono-species to the multi-species case. For this, we split the operator $L = L^m + L^b$ in the mono-species part $L^m = (L^m_1, \dots, L^m_n)$ and the bi-species part $L^b = (L^b_1, \dots, L^b_n)$,

$$L^m_i(f_i) = L_{ii}(f_i, f_i), \quad L^b_i(f) = \sum_{j \neq i} L_{ij}(f_i, f_j).$$

The proof consists of four main steps.

Step 1: Coercivity of the mono-species operator L^m. The bi-species part of L satisfies $-(f, L^b(f))_{L^2_v} \geq 0$ for all $f \in \mathcal{D}$. Furthermore, the results of [110, Theorem 6.1] show that for the mono-species part,

$$-(f, L^m(f))_{L^2_v} \geq C^m \|f - \Pi^m(f)\|^2_{\mathcal{H}} \quad \text{for } f \in \text{Dom}(L^m), \tag{2.1.10}$$

where the constant $C^m > 0$ can be computed explicitly and Π^m is the projection onto $\mathrm{Ker}(L^m)$ (see Lemma 4.10). Inequality (2.1.10) may be interpreted as a coercivity estimate for L^m on $\mathrm{Ker}(L^m)^\perp$. It is related to the "microscopic coercivity" in [58, Section 1.3] for the mono-species setting. Hence, we obtain the "naive" spectral-gap estimate

$$-(f, L(f))_{L^2_v} \geq C^m \|f - \Pi^m(f)\|^2_{\mathcal{H}} \quad \text{for } f \in \mathcal{D}.$$

This estimate is not sharp enough for the multi-species case since we need an inequality for all $f - \Pi^L(f) \in \mathrm{Ker}(L)^\perp$ and not only for $f - \Pi^m(f) \in \mathrm{Ker}(L^m)^\perp \subset \mathrm{Ker}(L)^\perp$. By projecting onto $\mathrm{Ker}(L^m)^\perp$ only, we neglect the "cross-effects" coming from the bi-species part of the collision operator. Thus, we need a better estimate for $-(f, L(f))_{L^2_v}$, which is achieved as follows.

Step 2: Absorption of the orthogonal parts. The contribution $f^\perp := f - \Pi^m(f)$ in $-(f, L^b(f))_{L^2_v}$ can be absorbed by the $\mathcal{H}$ norm of $f^\perp$ (see Lemma 4.11), giving for a certain $\eta > 0$,

$$-(f, L(f))_{L^2_v} \geq (C^m - 4\eta)\|f^\perp\|^2_{\mathcal{H}} - \frac{\eta}{2}(\Pi^m(f), L^b(\Pi^m(f)))_{L^2_v}.$$

Step 3: Coercivity of the bi-species operator L^b. The projection $\Pi^m(f)$ depends on the velocities u_i and energies e_i of the ith species, and thus, the cross terms can be bounded by the differences of momentum and differences of energies,

$$-(\Pi^m(f), L^b(\Pi^m(f)))_{L^2_v} \geq C \sum_{i,j=1}^{n} \big(|u_i - u_j|^2 + |e_i - e_j|^2\big), \tag{2.1.11}$$

for some constant $C > 0$. This is the key step of the proof. The inequality may be considered as a coercivity estimate for the bi-species operator. A key observation is that the differences of momenta and energies converge to zero as f approaches the global equilibrium. Whereas (2.1.10) acts on $\mathrm{Ker}(L^m)^\perp$, (2.1.11) gives an estimate on the orthogonal complement $\mathrm{Ker}(L^m)$.

Step 4: Lower bound for the differences of momenta and energy. The last step consists in estimating the differences $|u_i - u_j|$ and $|e_i - e_j|$ from below by the error made by projecting onto $\mathrm{Ker}(L^m)^\perp$ instead of $\mathrm{Ker}(L)^\perp$:

$$\sum_{i,j=1}^{n} \big(|u_i - u_j|^2 + |e_i - e_j|^2\big) \geq C\|f - \Pi^L(f)\|^2_{\mathcal{H}} - 2C\|f - \Pi^m(f)\|^2_{\mathcal{H}}.$$

Putting together the above inequalities, Theorem 2.3 follows; we refer to section 4.4 for details.

As a consequence of the spectral-gap estimate, we are able to prove the exponential decay of the solution $f(t)$ to (2.1.5) to the global equilibrium with an explicit decay rate.

Theorem 2.4 (Convergence to equilibrium)**.** *Let the collision kernels B_{ij} satisfy assumptions (A1)-(A5) and let $f_I \in H^1_{x,v}$. Then the linearized Boltzmann operator $G = L - T$ with $T = v \cdot \nabla_x$ generates a strongly continuous semigroup $S_G(t)$ on $H^1_{x,v}$, which satisfies*

$$\|S_G(t)(I - \Pi^G)\|_{H^1_{x,v}} \leq C\mathrm{e}^{-\tau t}, \quad t \geq 0, \tag{2.1.12}$$

for some constants C, $\tau > 0$. In particular, the solution $f(t) = S_G(t)f_I$ to (2.1.5) satisfies

$$\|f(t) - f_\infty\|_{H^1_{x,v}} \le C\mathrm{e}^{-\tau t}\|f_I - f_\infty\|_{H^1_{x,v}}, \quad t \ge 0, \tag{2.1.13}$$

where $f_\infty := \Pi^G(f_I)$ is the global equilibrium of (2.1.5). Moreover, under the additional assumption (A6) and lower bound in (A4), the constants C and τ depend only on the constants appearing in hypotheses (M1)-(M3) in section 4.5 and in particular on λ defined in Theorem 2.3.

The idea of the proof is to employ the hypocoercivity of the linearized Boltzmann operator $L - T$, using the interplay between the degenerate-dissipative properties of L and the conservative properties of T. The aim is to find a functional $\widetilde{H}[f]$ which is equivalent to the square of the norm of a Banach space (here, $H^1_{x,v}$),

$$\kappa_1\|f\|^2_{H^1_{x,v}} \le \widetilde{H}[f] \le \kappa_2\|f\|^2_{H^1_{x,v}} \quad \text{for } f \in H^1_{x,v},$$

leading to

$$\frac{d}{dt}\widetilde{H}[f(t)] \le -\kappa\|f(t)\|^2_{H^1_{x,v}}, \quad t > 0, \tag{2.1.14}$$

where κ_1, κ_2, $\kappa > 0$ and $f(t) = S_G(t)f_I$. These two estimates yield exponential convergence of $f(t)$ in $H^1_{x,v}$. It turns out that the obvious choice $\widetilde{H}[f] = c_1\|f\|_{L^2_{x,v}} + c_2\|\nabla_x f\|_{L^2_{x,v}} + c_3\|\nabla_v f\|_{L^2_{x,v}}$ does not lead to a closed estimate. The key idea, inspired from [139] and worked out in [113], is to add the "mixed term" of the form

$$c_4(\nabla_x f, \nabla_v f)_{L^2_{x,v}} \tag{2.1.15}$$

to the definition of $\widetilde{H}[f]$. Then

$$\frac{d}{dt}(\nabla_x f, \nabla_v f)_{L^2_{x,v}} = -\|\nabla_x f\|^2_{L^2_{x,v}} + 2(\nabla_x L(f), \nabla_v f)_{L^2_{x,v}},$$

and the last term can be estimated in terms of expressions arising from the time derivative of the other norms in $\widetilde{H}[f(t)]$. Thus, choosing $c_i > 0$ in a suitable way, one may conclude that (2.1.14) holds.

In [113], the calculation of (2.1.14) is reduced to the validity of certain abstract conditions on the operators K and Λ (see section 4.5). These conditions state that Λ is coercive in a certain sense, K has a regularizing effect, and $L = K - \Lambda$ has a local spectral gap. The last condition is proved in Theorem 2.3, while the other conditions follow from direct calculations, since the operators K and Λ are given explicitly. As a consequence, the proof of Theorem 2.4 essentially consists in verifying the abstract conditions stated in [113]. In contrast to the estimate of Theorem 2.3, where the multi-species character plays a role in the spectral-gap estimate, there are no "cross-effects" here and the same modified functional $\widetilde{H}[f]$ as above, including the mixed term, can be used. However, the decay rate τ changes, since the constant in hypothesis (M3) (see section 4.5) differs in the mono- and multi-species case and τ depends also on that constant.

2.1.6. Comments and outlook

First, it seems to be not trivial to extend the results to the whole-space case. The problem is that one loses the compactness in the x-space. One possibility is to assume some confinement potential which, under some appropriate weighted Poincaré inequality, can yield compactness of the resolvent and hence a spectral gap. For instance, Duan [62] used non-constructive techniques to prove decay rates for the mono-species linearized Boltzmann equation. Still in the mono-species case, with one-dimensional collisional invariants and using constructive methods, the decay is investigated by, e.g., Hérau and Nier [86] and Villani [139], working in the space $H^1_{x,v}$, and by Hérau [85] and Dolbeault, Mouhot, Schmeiser [58], working in the space $L^2_{x,v}$. The tasks in the multi-species case are first to extend the non-constructive methods, which probably does not contain new difficulties, and second to devise a constructive method, which is more involved and work in progress; see [57].

Second, the convergence result based on hypocoercivity requires some regularity on the initial data, namely $f_I \in H^1_{x,v}$. The extension of the exponential decay to initial data from $L^2_{x,v}$ is done in Chapter 5, where a Cauchy theory for the full nonlinear multi-species Boltzmann equation in a perturbative framework is presented. In [58], exponential convergence was proved for solutions to scalar linear kinetic equations in the whole space with a confining potential, where just L^2 regularity for the initial data was needed, but only for one-dimensional collision kernels which is not the case for the Boltzmann equation.

Third, the technique of proving a multi-species explicit spectral-gap estimate seems to be quite robust also to other multi-species models, since it has been recently succesfully adapted to the case of a multi-species Landau equation with soft potentials [77].

2.2. The Boltzmann system with different molar masses close to equilibrium

In Chapter 5 our goal will be to extend the previous results obtained in Chapter 4 to the more general case of a gaseous mixture with different molar masses. Furthermore, our goal will be not only to study the trend to global equilibrium, but to perform a full Cauchy theory for this equation, including also existence, uniqueness, positivity besides exponential trend to equilibrium in appropriate functional spaces.

The physically most relevant space for such a Cauchy theory for this nonlinear equation close to equilibrium is the space of density functions that only have finite mass and energy, which are the first and second moments in the velocity variable. We will obtain this result in the space $L^1_v L^\infty_x(1 + |v|^k)$ for any $k > k_0$, where k_0 is an explicit threshold depending heavily on the differences of the masses, recovering the physically optimal threshold $k_0 = 2$ when all the masses of the mixture are the same and the particles are approximated to be hard spheres.

Moreover, by combining very recent strategies in combination with new hypocoercivity estimates, we develop a new constructive approach that deals with polynomial weights without the need of any spatial Sobolev regularity. This is new even in the mono-species case even though the final result we obtain has recently been proved for the mono-species hard sphere model [78]) (which we therefore also extend to more general hard and Maxwellian potential kernels.).

Also, as a by-product, we prove explicitly that the linear operator $\mathbf{L} - v \cdot \nabla_x$ generates a strongly continuous semigroup with exponential decay both in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ and in $L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$; such constructive and direct results on the torus are new to our knowledge, even for the single-species Boltzmann equation.

At last, we derive new estimates in order to deal with different masses and the multi-species cross-interaction operators, and we also extend recent mono-species estimates to more general collision kernels. Note that the asymmetry of the elastic collisions requires to derive a new description of Carleman's representation of the Boltzmann operator as well as new Povzner-type inequalities suitable for this lack of symmetry.

2.2.1. The model for different molar masses

Since we will use a slightly different notation in Chapter 5 compared to Chapter 4, let us briefly introduce the model. For a detailed presentation of the properties of this model, we refer to subsection 5.1.1 and 5.1.2.

We are now interested in the evolution of a dilute gas on the torus $\mathbb{T}^3$ composed of N different species of chemically non-reacting mono-atomic particles with different molar masses, which can be modeled by the following system of Boltzmann equations, stated on $\mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3$,

$$\forall\, 1 \leq i \leq N, \quad \partial_t F_i(t,x,v) + v \cdot \nabla_x F_i(t,x,v) = Q_i(\mathbf{F})(t,x,v) \tag{2.2.1}$$

with initial data

$$\forall\, 1 \leq i \leq N,\ \forall (x,v) \in \mathbb{T}^3 \times \mathbb{R}^3, \quad F_i(0,x,v) = F_{0,i}(x,v).$$

The distribution function of the system is given by the vector $\mathbf{F} = (F_1, \dots, F_N)$, where F_i describes the i^{th} species at time t, position x and velocity v.

The Boltzmann operator $\mathbf{Q}(\mathbf{F}) = (Q_1(\mathbf{F}), \dots, Q_N(\mathbf{F}))$ is given for all i by

$$Q_i(\mathbf{F}) = \sum_{j=1}^N Q_{ij}(F_i, F_j),$$

where Q_{ij} describes interactions between particles of either the same ($i = j$) or of different ($i \neq j$) species and are local in time and space.

$$Q_{ij}(F_i, F_j)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij}\left(|v - v_*|, \cos\theta\right) \left[F_i' F_j'^{*} - F_i F_j^*\right] dv_* d\sigma,$$

where we used the shorthands $F_i' = F_i(v')$, $F_i = F_i(v)$, $F_j'^{*} = F_j(v_*')$ and $F_j^* = F_j(v_*)$. Due to the fact that now we allow also different molar masses for each species, the microscopic collision rules have the following form:

$$\begin{cases} v' &= \dfrac{1}{m_i + m_j}\left(m_i v + m_j v_* + m_j |v - v_*| \sigma\right) \\ v_*' &= \dfrac{1}{m_i + m_j}\left(m_i v + m_j v_* - m_i |v - v_*| \sigma\right) \end{cases}, \text{ and } \quad \cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle.$$

which comes from the conservation of momentum and kinetic energy on the microscopic level due to elastic collisions:

$$\begin{aligned} m_i v + m_j v_* &= m_i v' + m_j v'_*, \\ \frac{1}{2} m_i |v|^2 + \frac{1}{2} m_j |v_*|^2 &= \frac{1}{2} m_i |v'|^2 + \frac{1}{2} m_j |v'_*|^2 . \end{aligned} \tag{2.2.2}$$

We want to mention the lack of symmetry between v' and v'_* compared to v for the transformation $(v, v_*) \mapsto (v_*, v)$ due to different masses, which creates additional technical difficulties. By translating and rescaling the coordinate system, we can always assume that the only global equilibrium is the normalized Maxwellian

$$\boldsymbol{\mu} = (\mu_i)_{1 \leq i \leq N} \quad \text{with} \quad \mu_i(v) = c_{\infty,i} \left(\frac{m_i}{2\pi}\right)^{3/2} e^{-m_i \frac{|v|^2}{2}}. \tag{2.2.3}$$

We mention that we have the following macroscopic conservation laws: conservation of the total number density $c_{\infty,i}$ of each species, of the total momentum of the gas $\rho_\infty u_\infty$ and its total energy $3\rho_\infty \theta_\infty / 2$:

$$\begin{aligned} \forall t \geq 0, \quad c_{\infty,i} &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_i(t,x,v)\, dxdv \quad (1 \leq i \leq N) \\ u_\infty &= \frac{1}{\rho_\infty} \sum_{i=1}^N \int_{\mathbb{T}^3 \times \mathbb{R}^3} m_i v F_i(t,x,v)\, dxdv \\ \theta_\infty &= \frac{1}{3\rho_\infty} \sum_{i=1}^N \int_{\mathbb{T}^3 \times \mathbb{R}^3} m_i |v - u_\infty|^2 F_i(t,x,v)\, dxdv, \end{aligned} \tag{2.2.4}$$

where $\rho_\infty = \sum_{i=1}^N m_i c_{\infty,i}$ is the global density of the gas. For more details on the modified conservation rules and other differences due to these different molar masses, we refer to subsection 5.1.2.

The aim is to construct a Cauchy theory for the multi-species Boltzmann equation (2.2.1) around the global equilibrium $\boldsymbol{\mu}$. In other terms we study the existence, uniqueness and exponential decay of solutions of the form $F_i(t,x,v) = \mu_i(v) + f_i(t,x,v)$ for all i.

Under this perturbative regime, the Cauchy problem amounts to solving the perturbed multi-species Boltzmann system of equations

$$\partial_t \mathbf{f} + v \cdot \nabla_x \mathbf{f} = \mathbf{L}(\mathbf{f}) + \mathbf{Q}(\mathbf{f}), \tag{2.2.5}$$

or equivalently in the non-vectorial form

$$\forall\, 1 \leq i \leq N, \quad \partial_t f_i + v \cdot \nabla_x f_i = L_i(\mathbf{f}) + Q_i(\mathbf{f}),$$

where $\mathbf{f} = (f_1, \dots, f_N)$ and the operator $\mathbf{L} = (L_1, \dots, L_N)$ is the linear Boltzmann operator given for all $1 \leq i \leq N$ by

$$L_i(\mathbf{f}) = \sum_{j=1}^N L_{ij}(f_i, f_j),$$

with

$$L_{ij}(f_i, f_j) = Q_{ij}(\mu_i, f_j) + Q_{ij}(f_i, \mu_j).$$

As mentioned previously, we prove the existence, uniqueness, positivity and exponential trend to equilibrium for the full nonlinear multi-species Boltzmann equation (2.2.1) in $L^1_v L^\infty_x\left((1+|v|)^k\right)$ with the explicit threshold $k > k_0$ defined in Lemma 5.20, when the initial data $\mathbf{F_0}$ is close enough to the global equilibrium $\boldsymbol{\mu}$. This is equivalent to solving the perturbed equation (2.2.5) for small $\mathbf{f_0}$.

2.2.2. State of the art for the Cauchy problem

The perturbed single-species Boltzmann equation around a global Maxwellian has been extensively studied over the past fifty years (see [135] for an exhaustive review). Starting with Grad [76], the Cauchy problem has been tackled in $L^2_v H^s_x\left(\mu^{-1/2}\right)$ spaces [134], in $H^s_{x,v}\left(\mu^{-1/2}(1+|v|)^k\right)$ [81][143] was then extended to $H^s_{x,v}\left(\mu^{-1/2}\right)$ where an exponential trend to equilibrium has also been obtained [113][82]. Recently, [78] proved existence and uniqueness for single-species Boltzmann equation in more the general spaces $\left(W^{\alpha,1}_v \cap W^{\alpha,q}_v\right) W^{\beta,p}_x\left((1+|v|)^k\right)$ for $\alpha \leq \beta$ and β and k large enough with explicit thresholds. The latter paper thus includes $L^1_v L^\infty_x\left((1+|v|)^k\right)$. All the results presented above hold in the case of the torus for hard and Maxwellian potentials. We refer the reader interested in the Cauchy problem to the review [135].

All the works mentioned above involve to working in spaces with derivatives in the space variable x (we shall discuss some of the reasons later) with exponential weight. The recent breakthrough [78] gets rid of both the Sobolev regularity and the exponential weight but uses a new extension method which still requires to have a well-established linear theory in $H^s_{x,v}\left(\mu^{-1/2}\right)$.

2.2.3. Assumptions on the collision kernels

We will use the following assumptions on the collision kernels B_{ij}.

(H1) The following symmetry holds

$$B_{ij}(|v-v_*|, \cos\theta) = B_{ji}(|v-v_*|, \cos\theta) \quad \text{for } 1 \leq i,j \leq N.$$

(H2) The collision kernels decompose into the product

$$B_{ij}(|v-v_*|, \cos\theta) = \Phi_{ij}(|v-v_*|)b_{ij}(\cos\theta), \quad 1 \leq i,j \leq N,$$

where the functions $\Phi_{ij} \geq 0$ are called kinetic part and $b_{ij} \geq 0$ angular part.

(H3) The kinetic part has the form of hard or Maxwellian ($\gamma = 0$) potentials, *i.e.*

$$\Phi_{ij}(|v-v_*|) = C^\Phi_{ij}|v-v_*|^\gamma, \quad C^\Phi_{ij} > 0, \;\; \gamma \in [0,1], \quad \forall\, 1 \leq i,j \leq N.$$

(H4) For the angular part, we assume a strong form of Grad's angular cutoff (first introduced in [74]), that is: there exist constants $C_{b1}, C_{b2} > 0$ such that for all $1 \leq i,j \leq N$ and $\theta \in [0,\pi]$,

$$0 < b_{ij}(\cos\theta) \leq C_{b1}|\sin\theta|\,|\cos\theta|, \quad b'_{ij}(\cos\theta) \leq C_{b2}.$$

Furthermore,

$$C^b := \min_{1\leq i\leq N} \inf_{\sigma_1,\sigma_2\in\mathbb{S}^2} \int_{\mathbb{S}^2} \min\left\{b_{ii}(\sigma_1\cdot\sigma_3), b_{ii}(\sigma_2\cdot\sigma_3)\right\} d\sigma_3 > 0.$$

Moreover, we shall use the (standard) shorthand notations

$$b_{ij}^\infty = \|b_{ij}\|_{L^\infty_{[-1,1]}} \quad \text{and} \quad l_{b_{ij}} = \|b\circ\cos\|_{L^1_{\mathbb{S}^2}}. \tag{2.2.6}$$

and

$$\langle v\rangle = \sqrt{1+|v|^2}. \tag{2.2.7}$$

2.2.4. Main results for different molar masses close to equilibrium

As already explained, the ultimate goal in this chapter will be to perform a full perturbative Cauchy theory for the multi-species Boltzmann equation (2.2.1). Along the way, we shall also prove the following important results about the linear perturbed operator $\mathbf{L} - v\cdot\nabla_x$.

Theorem 2.5. *Let the collision kernels B_{ij} satisfy assumptions $(H1)-(H4)$. Then the following holds.*

(i) The operator $\mathbf{L}$ is a closed self-adjoint operator in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$ and there exists $\lambda_L > 0$ such that

$$\forall \mathbf{f} \in L^2_v\left(\boldsymbol{\mu}^{-1/2}\right), \quad \langle \mathbf{f}, \mathbf{L}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \leq -\lambda_L \|\mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})\|^2_{L^2_v\left(\langle v\rangle^{\gamma/2}\boldsymbol{\mu}^{-1/2}\right)};$$

(ii) Let $E = L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ or $E = L^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)$ with $\beta > 3/2$. The linear perturbed operator $\mathbf{G} = \mathbf{L} - v\cdot\nabla_x$ generates a strongly continuous semigroup $S_{\mathbf{G}}(t)$ on E and there exist C_E, $\lambda_E > 0$ such that

$$\forall t \geq 0, \quad \|S_{\mathbf{G}}(t)\left(Id - \Pi_{\mathbf{G}}\right)\|_E \leq C_E e^{-\lambda_E t},$$

where $\pi_{\mathbf{L}}$ is the orthogonal projection onto Ker$(\mathbf{L})$ in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$ and $\Pi_{\mathbf{G}}$ is the orthogonal projection onto Ker$(\mathbf{G})$ in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$.
The constants λ_L, C_E and λ_E are explicit and depend on N, E, the different masses m_i and the collision kernels.

Spectral gap for the linear operator in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$. It has been known for long that the single-species linear Boltzmann operator L is a self-adjoint non positive linear operator in the space $L^2_v\left(\mu^{-1/2}\right)$. Moreover it has a spectral gap λ_0. This has been proved in [30][74][75] with non constructive methods for hard potential with cutoff and in [6][7] in the Maxwellian case. These results were made constructive in [4][111] for more general collision operators. One can easily extend this spectral gap to Sobolev spaces $H^s_v\left(\mu^{-1/2}\right)$ (see for instance [78] Section 4.1).

In Chapter 4 based on [46], we proved the existence of an explicit spectral gap for the operator $\mathbf{L}$ for multi-species mixtures where all the masses are the same ($m_i = m_j$). Our constructive spectral gap estimate in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$ proved in this chapter for different molar masses closely follows the methods presented in Chapter 4 that consist in proving that the cross-interactions between different species do not perturb too much the spectral gap that is known to exist for the diagonal operator L_{ii} (single-species operators). We emphasize here that not only we adapt the methods of [46] to fit the different masses framework but we also derive estimates on the collision frequencies that allow us to get rid of their strong requirement on the collision kernels: $B_{ij} \leq \beta B_{ii}$ for all i, j. The latter assumption is indeed physically irrelevant in our framework.

$L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ theory for the full perturbed linear operator. The next step is to prove that the existence of a spectral gap for $\mathbf{L}$ in the sole velocity variable can be transposed to $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ when one adds the skew-symmetric transport operator $-v\cdot\nabla_x$. In other words, we prove that $\mathbf{G} = \mathbf{L} - v\cdot\nabla_x$ generates a strongly continuous semigroup in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ with exponential decay.

One thus wants to derive an exponential decay for solutions to the linear perturbed Boltzmann equation

$$\partial_t \mathbf{f} + v\cdot\nabla_x \mathbf{f} = L\left(\mathbf{f}\right).$$

A straightforward use of the spectral gap λ_L of $\mathbf{L}$ shows for such a solution

$$\frac{d}{dt}\left\|\mathbf{f}\right\|^2_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)} \leq -2\lambda_L \left\|\mathbf{f} - \pi_{\mathbf{L}}\left(\mathbf{f}\right)\right\|^2_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)},$$

where $\pi_{\mathbf{L}}$ stands for the orthogonal projection in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$ onto the kernel of the operator $\mathbf{L}$. This inequality exhibits the hypocoercivity of $\mathbf{L}$. Roughly speaking, the exponential decay in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ would follow for solutions $\mathbf{f}$ if the microscopic part $\pi^{\perp}_{\mathbf{L}}(\mathbf{f}) = \mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})$ controls the fluid part which has the following form

$$\forall 1 \leq i \leq N, \quad \pi_{\mathbf{L}}(\mathbf{f})_i(t,x,v) = \left[a_i(t,x) + b(t,x)\cdot v + c(t,x)\frac{|v|^2 - 3m_i^{-1}}{2}\right] m_i\mu_i(v),$$

where $a_i(t,x), c(t,x) \in \mathbb{R}$ and $b(t,x) \in \mathbb{R}^3$ are the coordinates of an orthogonal basis.

The standard strategies in the case of the single-species Boltzmann equation are based on higher Sobolev regularity either from hypocoercivity methods [113] or elliptic regularity of the coefficients a, b and c [80][82]. Roughly speaking one has [80][82]

$$\Delta\pi_L(f) \sim \partial^2\pi^{\perp}_L f + \text{higher order terms}, \tag{2.2.8}$$

which can be combined with elliptic estimates to control the fluid part by the microscopic part in Sobolev spaces H^s. Our main contribution to avoid involving high regularity is based on an adaptation of the recent work [66] (dealing with the single-species Boltzmann equation with diffusive boundary conditions). The key idea consists in integrating against test functions that contains a weak version of the elliptic regularity of $\mathbf{a}(t,x)$, $b(t,x)$ and $c(t,x)$. Basically, the elliptic regularity of $\pi_{\mathbf{L}}(\mathbf{f})$ will be recovered thanks to the transport part applied to these test functions while on the other side $\mathbf{L}$ encodes the control by $\pi_{\mathbf{L}}^{\perp}(\mathbf{f})$.

It has to be emphasized that thanks to boundary conditions, [66] only needed the conservation of mass whereas in our case this "weak version" of estimates (2.2.8) strongly relies on all the conservation laws. The choice of test functions thus has to take into account the delicate interaction between each species and the total mixture we already pointed out. This leads to intricate technicalities since for each species we need to deal with different reference rates of decay m_i. Finally, our proof also involves elliptic regularity in negative Sobolev spaces to deal with $\partial_t \mathbf{a}$, $\partial_t b$ and $\partial_t c$.

$L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$ theory for the full nonlinear equation. Thanks to the first two steps we have a satisfactory L^2 semigroup theory for the full linear operator. Unfortunately, as it is already the case for the single-species Boltzmann equation (see [33][34] or [138] for instance), the underlying $L^2_{x,v}$-norm is not an algebraic norm for the nonlinear operator $\mathbf{Q}$ whereas the $L^\infty_{x,v}$-norm is.

The key idea of proving a semigroup property in L^∞ is thanks to an L^2-L^∞ theory "à la Guo" [83], where the L^∞-norm will be controlled by the L^2-norm along the characteristics. As we shall see, each component L_i can be decomposed into $L_i = K_i - \nu_i$ where $\nu_i(f) = \nu_i(v)f_i$ is a multiplicative operator. If we denote by $\mathbf{S_G}(t)$ the semigroup generated by $\mathbf{G} = \mathbf{L} - v\cdot\nabla_x$, we have the following implicit Duhamel representation of its i^{th} component along the characteristics

$$S_{\mathbf{G}}(t)_i = e^{-\nu_i(v)t} + \int_0^t e^{-\nu_i(v)(t-s)} K_i\left[\mathbf{S_G}(s)\right]\, ds.$$

Following the idea of Vidav [137] and later used in [83], an iteration of the above should yield a certain compactness property. Hiding here all the cross-interactions, we end up with

$$\begin{aligned}\mathbf{S_G}(t) =& e^{-\boldsymbol{\nu}(v)t} + \int_0^t e^{-\boldsymbol{\nu}(v)(t-s)}\mathbf{K}e^{-\boldsymbol{\nu}(v)s}\, ds\\ &+ \int_0^t\int_0^s e^{-\boldsymbol{\nu}(v)(t-s)}\mathbf{K}e^{-\boldsymbol{\nu}(v)(s-s_1)}\mathbf{K}\left[\mathbf{S_G}(s_1)\right]\, ds_1 ds.\end{aligned}$$

We shall prove that $\mathbf{K}$ is compact and is a kernel operator. The first two terms will be easily estimated and the last term will be roughly of the form

$$\int_0^t\int_0^s\int_{v_1,v_2 \text{ bounded}} \left|\mathbf{S_G}(s_1, x-(t-s)v-(s-s_1)v_1, v_2\right|\, dv_2 dv_1 ds_1 ds.$$

The double integration implies that v_1 and v_2 are independent and we can thus perform a change of variables which changes the integral in v_1 into an integral over $\mathbb{T}^3$ that we can bound thanks to the previous L^2 theory. For integrability reasons, this third step actually proves that $\mathbf{G}$ generates a strongly continuous semigroup with exponential decay in $L^\infty\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$ for $\beta > 3/2$.

Our work provides two key contributions to prove the latter result. First, to prove the desired pointwise estimate for the kernel of $\mathbf{K}$, we need to give a new representation of the operator in terms of the parameters (v', v'_*) instead of (v_*, σ). In the single-species case, such a representation is the well-known Carleman representation [30] and requires integration onto the so-called Carleman hyperplanes $\langle v' - v, v'_* - v\rangle = 0$. However, when particles have different masses, the lack of symmetry between v' and v'_* compared to v obliges us to derive new Carleman admissible sets (some become spheres). Second, the decay of the exponential weight differs from one species to the other. To obtain estimates that are similar to the case of single-species we exhibit the property that $\mathbf{K}$ mixes the exponential rate of decay among the cross-interaction between species. This enables us to close the L^∞ estimate for the first two terms of the iterated Duhamel representation.

We now state the results we obtain for the full nonlinear equation.

Theorem 2.6. *Let the collision kernels B_{ij} satisfy assumptions $(H1)-(H4)$ and let $E = L^1_v L^\infty_x\left(\langle v\rangle^k\right)$ with $k > k_0$, where k_0 is the minimal integer such that*

$$C_k = \frac{2}{k+2}\,\frac{1-\left[\max_{i,j}\frac{|m_i-m_j|}{m_i+m_j}\right]^{\frac{k+2}{2}}+\left[1-\left(\max_{i,j}\frac{|m_i-m_j|}{m_i+m_j}\right)\right]^{\frac{k+2}{2}}}{1-\max_{i,j}\frac{|m_i-m_j|}{m_i+m_j}}\max_{i,j}\frac{4\pi b^\infty_{ij}}{l_{b_{ij}}} < 1. \tag{2.2.9}$$

where $l_{b_{ij}}$ and b^∞_{ij} are angular kernel constants (2.2.6).
Then there exist η_E, C_E and $\lambda_E > 0$ such that for any $\mathbf{F_0} = \boldsymbol{\mu} + \mathbf{f_0} \geq 0$ satisfying the conservation of mass, momentum and energy (2.2.4) *with $u_\infty = 0$ and $\theta_\infty = 1$, if*

$$\|\mathbf{F_0} - \boldsymbol{\mu}\| \leq \eta_E$$

then there exists a unique solution $\mathbf{F} = \boldsymbol{\mu} + \mathbf{f}$ in E to the multi-species Boltzmann equation (2.2.1) *with initial data $\mathbf{f_0}$. Moreover, $\mathbf{F}$ is non-negative, satisfies the conservation laws and*

$$\forall t \geq 0, \quad \|\mathbf{F} - \boldsymbol{\mu}\|_E \leq C_E e^{-\lambda_E t}\|\mathbf{F_0} - \boldsymbol{\mu}\|_E\,.$$

The constants are explicit and only depend on N, k, the different masses m_i and the collision kernels.

The extension to polynomial weights and $L^1_v L^\infty_x$ space is done by developing an analytic and nonlinear version of the recent work [78], also recently adapted in a nonlinear setting [26]. The main strategy is to find a decomposition of the full linear operator $\mathbf{G}$ into $\mathbf{G_1} + \mathbf{A}$. We shall prove that $\mathbf{G_1}$ acts like a small perturbation of the operator $\mathbf{G}_{\boldsymbol{\nu}} = -v\cdot\nabla_x - \boldsymbol{\nu}(v)$ and is thus hypodissipative, and that $\mathbf{A}$ has a regularizing effect. The regularizing property of the operator $\mathbf{A}$ allows us to decompose the perturbative equation (2.2.5) into a system of differential equations

$$\begin{aligned} \partial_t \mathbf{f_1} &= \mathbf{G_1}(\mathbf{f_1}) + \mathbf{Q}(\mathbf{f_1}+\mathbf{f_2}, \mathbf{f_1}+\mathbf{f_2}) \\ \partial_t \mathbf{f_2} + v \cdot \nabla_x \mathbf{f_2} &= \mathbf{L}(\mathbf{f_2}) + \mathbf{A}(\mathbf{f_1}) \end{aligned}$$

The first equation is solved in $L^\infty_{x,v}(m)$ or $L^1_v L^\infty_x(m)$ with the initial data $\mathbf{f_0}$ thanks to the hypodissipativity of $\mathbf{G_1}$. The regularity of $\mathbf{A}(\mathbf{f_1})$ allows us to use Step 3 and thus solve the second equation with null initial data in $L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$. First, the existence of a solution to the system having exponential decay is obtained thanks to an iterative scheme combined with new estimates on the multi-species operators $\mathbf{G_1}$ and $\mathbf{A}$. Then uniqueness follows a new stability estimate in an equivalent norm (proposed in [78]), that fits the dissipativity of the semigroup generated by $\mathbf{G}$. Finally, positivity of the unique solution comes from a different iterative scheme.

In the case of the single-species Boltzmann equation, the less regular weight $m(v)$ one can achieve with this method is determined by the hypodissipative property of $\mathbf{G_1}$ and gives $m = \langle v\rangle^k$ with $k > 2$, which is indeed obtained also in the multi-species framework of same masses in the case of hard spheres. In the general case of different masses, the threshold k_0 is more intricate (see Theorem 2.6), since it also depends on the different masses m_i.

2.2.5. Comments and outlook

We make a few comments about the theorem above.

(1) As already mentioned, $\boldsymbol{\mu}$ can be replaced by any global equilibrium $\mathbf{M}(c_{i,\infty}, u_\infty, \theta_\infty)$.

(2) The natural weight for this theory is the one associated to the conservation of individual masses and total energy: $(1 + m_i^{k/2}|v|^k)_{1\leq i\leq N}$, which is equivalent to $\langle v\rangle^k$ and we keep the latter weight to work without vector-valued masses outside Subsection 5.5.1.

(3) The uniqueness has to be understood in a perturbative regime, that is among the solutions that can be written under the form $\mathbf{F} = \boldsymbol{\mu} + \mathbf{f}$. We do not give a global uniqueness in $L^1_v L^\infty_x\left(\langle v\rangle^k\right)$ (as proved in [78] for the single-species Boltzmann equation).

(4) As a by-product of the proof of uniqueness, we prove that the spectral-gap estimate of Theorem 2.5 also holds for $E = L^1_v L^\infty_x\left(\langle v\rangle^k\right)$ with $k > k_0$.

(5) In the case of identical masses and hard sphere collision kernels ($b = 1$) we recover $C_k = 4/(k+2)$ and thus $k_0 = 2$ which has recently been obtained in the mono-species case [78].

(6) We want to mention, that a stability analysis of the global equilibrium in L^∞ settings has been recently obtained in [27].

3. Introduction to the diffusive multi-species models

3.1. Cross-diffusion population systems for multiple species

Shigesada, Kawasaki, and Teramoto suggested in their seminal paper [129] a diffusive Lotka-Volterra system for two competing species, which is able to describe the segregation of the population and to show pattern formation when time increases. Starting from an on-lattice random-walk model, this system was extended to an arbitrary number of species in [144, Appendix]. While the existence analysis of global weak solutions to the two-species model is well understood by now [37, 38], only very few results for the n-species model under very restrictive conditions exist (see the discussion below). Here, we provide for the first time a global existence analysis for an arbitrary number of population species using the entropy method of [94], and we reveal an astonishing relation between the monotonicity of the entropy and the detailed balance condition of an associated Markov chain.

3.1.1. The model

We consider the reaction-cross-diffusion equations

$$\partial_t u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right) = f_i(u) \quad \text{in } \Omega,\ t>0, \quad i=1,\ldots,n, \tag{3.1.1}$$

with no-flux boundary and initial conditions

$$\sum_{j=1}^n A_{ij}(u)\nabla u_j\cdot\nu = 0 \quad \text{on } \partial\Omega,\ t>0, \quad u_i(\cdot,0)=u_i^0 \quad \text{in } \Omega. \tag{3.1.2}$$

Here, u_i models the density of the ith species, $u=(u_1,\ldots,u_n)$, $\Omega\subset\mathbb{R}^d$ ($d\geq 1$) is a bounded domain with Lipschitz boundary, and ν is the exterior unit normal vector to $\partial\Omega$. The diffusion coefficients are given by

$$A_{ij}(u) = \delta_{ij}p_i(u) + u_i\frac{\partial p_i}{\partial u_j}(u), \quad p_i(u) = a_{i0} + \sum_{k=1}^n a_{ik}u_k^s, \quad i,j=1,\ldots,n, \tag{3.1.3}$$

where a_{i0}, $a_{ij}\geq 0$ and $s>0$. The functions p_i are the transition rates of the underlying random-walk model [95, 144]. The source terms f_i are of Lotka-Volterra type,

$$f_i(u) = u_i\left(b_{i0} - \sum_{j=1}^n b_{ij}u_j\right), \quad i=1,\ldots,n, \tag{3.1.4}$$

and we suppose that b_{i0}, $b_{ij} \geq 0$ (competition case). Note that (3.1.1) can be written more compactly as

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u), \quad f(u) = (f_1(u), \ldots, f_n(u)).$$

3.1.2. State of the art

From a mathematical viewpoint, the analysis of (3.1.1)-(3.1.2) is highly nontrivial since the diffusion matrix $A(u)$ is neither symmetric nor generally positive definite. Although the maximum principle may be applied to prove the nonnegativity of the densities, it is generally not possible to show upper bounds. Moreover, there is no general regularity theory for diffusion systems, which makes the analysis very delicate. Equations (3.1.1) can be written in the form

$$\partial_t u_i - \Delta(u_i p_i(u)) = f_i(u), \tag{3.1.5}$$

which allows for the proof of an L^{2+s} estimate by the duality method [48, 120], but we will not exploit this method here, only in Chapter 7.

The case of $n = 2$ species and linear transition rates $s = 1$ corresponds to the original population model of Shigesada, Kawasaki, and Teramota [129],

$$\begin{aligned} \partial_t u_1 - \Delta\big(u_1(a_{10} + a_{11}u_1 + a_{12}u_2)\big) &= f_1(u), \\ \partial_t u_2 - \Delta\big(u_2(a_{20} + a_{21}u_1 + a_{22}u_2)\big) &= f_2(u). \end{aligned} \tag{3.1.6}$$

The numbers a_{i0} are the diffusion coefficients, a_{ii} are the self-diffusion coefficients, and a_{ij} for $i \neq j$ are called the cross-diffusion coefficients. This model attracted a lot of attention in the mathematical literature. The first global existence result is due to Kim [99] who studied the equations in one space dimension, neglected self-diffusion, and assumed equal coefficients ($a_{ij} = 1$). His result was extended to higher space dimensions in [56]. Most of the papers made restrictive structural assumptions, for instance supposing that the diffusion matrix is triangular ($a_{21} = 0$), since this allows for the maximum principle in the second equation [2, 103, 106]. Another restriction is to suppose that the cross-diffusion coefficients are small, since in this situation the diffusion matrix becomes positive definite [56, 141].

Significant progress was made by Amann [2] who showed that a priori estimates in the $W^{1,p}$ norm with $p > d$ are sufficient for the solutions to general quasilinear parabolic systems to exist globally in time, and he applied his result to the triangular case. The first global existence result without any restriction on the diffusion coefficients (except positivity) was achieved in [69] in one space dimension and in [37, 38] in several space dimensions. The results were extended to the whole space in [60]. The existence of global classical solutions was proved in, e.g., [104], under suitable conditions on the coefficients.

Nonlinear transition rates, but still for two species, were analyzed by Desvillettes and co-workers, assuming sublinear ($0 < s < 1$) [50] or superlinear rates ($s > 1$) and the weak cross-diffusion condition $((s-1)/(s+1))^2 a_{12}a_{21} \leq a_{11}a_{22}$ [51]. Similar results, but under a slightly stronger weak cross-diffusion hypothesis, were proved in [94].

As already mentioned, there are very few results for more than two species. The existence of positive stationary solutions and the stability of the constant equilibrium was investigated in [3, 127]. The existence of global weak solutions in one space dimension assuming a positive definite diffusion matrix was proved in [140], based on Amann's results. Using an entropy approach, the global existence of solutions was shown in [51] for three species under the condition $0 < s < 1/\sqrt{3}$ (which guarantees that $\det(A(u)) > 0$). To our knowledge, a global existence theorem under more general conditions seems to be not available in the literature. Here, we prove such a result and relate a structural condition on the coefficients a_{ij} with Onsager's principle of thermodynamics.

3.1.3. Key ideas

Before we state the main results, let us explain our strategy. The idea is to find a priori estimates by employing a Lyapunov functional approach with

$$\mathcal{H}[u] = \int_\Omega h(u)dx = \int_\Omega \sum_{i=1}^n \pi_i h_s(u_i)dx, \tag{3.1.7}$$

where $\pi_i > 0$ are some numbers and

$$h_s(z) = \begin{cases} z(\log z - 1) + 1 & \text{for } s = 1, \\ \dfrac{z^s - sz}{s-1} + 1 & \text{for } s \neq 1. \end{cases} \tag{3.1.8}$$

Because of the connection of our method to nonequilibrium thermodynamics [95, Section 4.3], we refer to $\mathcal{H}[u]$ as an entropy and to $h(u)$ as an entropy density. Introducing the so-called entropy variable $w = (w_1, \ldots, w_n)$ (called chemical potential in thermodynamics) by

$$w_i = \frac{\partial h}{\partial u_i}(u) = \begin{cases} \pi_i \log u_i & \text{for } s = 1, \\ \dfrac{s\pi_i}{s-1}(u_i^{s-1} - 1) & \text{for } s \neq 1, \end{cases}$$

equations (3.1.1) can be written as

$$\partial_t u(w) - \operatorname{div}(B(w)\nabla w) = f(u(w)), \quad B(w) = A(u)H(u)^{-1}, \tag{3.1.9}$$

where $u(w) := (h')^{-1}(w)$ is the inverse transformation and $H(u) = h''(u)$ is the Hessian of the entropy density. We claim that if $f = 0$ and $B(w)$ or, equivalently, $H(u)A(u)$ is positive semi-definite (we say that an arbitrary matrix $M \in \mathbb{R}^{n\times n}$ is positive (semi-) definite if $z^\top Mz > (\geq)\ 0$ for all $z \in \mathbb{R}^n$, $z \neq 0$.), $\mathcal{H}[u]$ is a Lyapunov functional along solutions to (3.1.1). Indeed, a (formal) computation shows that

$$\frac{d}{dt}\mathcal{H}[u] = -\int_\Omega \nabla w : B(w)\nabla w dx \leq 0,$$

which implies that $t \mapsto \mathcal{H}[u(t)]$ is nonincreasing. The entropy method provides more than just the monotonicity of $\mathcal{H}[u]$. If, for instance, $z^\top H(u)A(u)z \geq \sum_{i=1}^n c_i u_i^{\alpha-2} z_i^2$ for some constants $\alpha > 0$, $c_i > 0$, it follows that

$$\frac{d}{dt}\mathcal{H}[u] + \frac{4}{\alpha^2}\int_\Omega \sum_{i=1}^n c_i |\nabla u_i^{\alpha/2}|^2 dx \leq 0,$$

which yields gradient estimates for $u_i^{\alpha/2}$. This strategy was employed in many papers on cross-diffusion systems; see, e.g., [37, 38, 50, 60, 69, 94, 144]. In this work, we introduce two new ideas which we explain for the case $s = 1$ ($s \neq 1$ is studied below).

It is known that the entropy (3.1.7) with $\pi_i = 1$ is a Lyapunov functional for the two-species model (3.1.6) with $f_1 = f_2 = 0$. This property is generally not satisfied for the corresponding n-species system. Our *first idea* is to introduce the numbers $\pi = (\pi_1, \dots, \pi_n)$ in the entropy (3.1.7). It turns out that (3.1.7) is a Lyapunov functional and $H(u)A(u)$ is symmetric and positive definite if

$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for all } i, j = 1, \dots, n. \tag{3.1.10}$$

More precisely, this property is *equivalent* to the symmetry of $H(u)A(u)$ (see Proposition 6.19). We recognize (3.1.10) as the detailed balance condition for the Markov chain associated to (a_{ij}). The equivalence of the symmetry and the detailed balance condition is new but not surprising. In fact, the latter condition means that π is a reversible measure, and time-reversibility of a thermodynamic system is equivalent to the symmetry of the so-called Onsager matrix $B(w)$, so symmetry and reversibility are related both from a mathematical and physical viewpoint. We detail these relations in Section 6.5.1. In Section 6.2.1, we derive a refined estimate for $H(u)A(u)$ leading to

$$\frac{d}{dt}\mathcal{H}[u] + 4\int_\Omega \sum_{i=1}^n \pi_i a_{i0} |\nabla \sqrt{u_i}|^2 dx + 2\int_\Omega \sum_{i=1}^n \pi_i a_{ii} |\nabla u_i|^2 dx \le 0, \tag{3.1.11}$$

and thus giving an H^1 estimate for $\sqrt{u_i}$ (if $a_{i0} > 0$) and u_i (if $a_{ii} > 0$). This is the key estimate for the global existence result. (Below we also take into account the reaction terms (3.1.4).)

One may ask whether the detailed balance condition is necessary for the monotonicity of the entropy. It is not. We show that if self-diffusion dominates cross-diffusion in the sense

$$\eta_0 := \min_{i=1,\dots,n} \left(a_{ii} - \frac{s}{2(s+1)} \sum_{j=1}^n \left(\sqrt{a_{ij}} - \sqrt{a_{ji}} \right)^2 \right) > 0, \tag{3.1.12}$$

and detailed balance may be not satisfied, then the estimate leading to (3.1.11) still holds (with different constants), and global existence follows. (Throughout this work, we set $\pi_i = 1$ when detailed balance does not hold.) However, if conditions (3.1.10) or (3.1.12) are both not satisfied, there exist coefficients a_{ij} and initial data u^0 such that $t \mapsto \mathcal{H}[u(t)]$ is increasing on $[0, t_0]$ for some $t_0 > 0$; see Section 6.5.3. Numerical experiments (not shown) indicate that after the initial increase, the entropy decays and, in fact, it stays bounded for all time. We conjecture that the entropy is bounded for all time for all nonnegative coefficients and nonnegative initial data and that global existence of weak solutions holds for any (positive) coefficients a_{ij}.

Our results can be extended to nonlinear transition rates of type (3.1.3). One may choose more general terms $a_{ij}u_j^{s_j}$ with different exponents s_j but the results are easier to formulate if all exponents are equal. Coefficients with exponents $s \neq 1$ were also considered in [50, 51, 94] but in the two-species case only. We generalize these results to the multi-species case for any $n \geq 2$. The entropy method has to be adapted since the inverse of

$h_s'(z) = (s/(s-1))(z^{s-1}-1)$ cannot be defined on $\mathbb{R}$ and thus, $u(w) = (h')^{-1}(w)$ is not defined for all $w \in \mathbb{R}^n$. This issue can be overcome by regularization as in [50, 94]. In fact, we introduce

$$h_\varepsilon(u) = h(u) + \varepsilon \sum_{i=1}^n \big(u_i(\log u_i - 1) + 1\big).$$

Then $h_\varepsilon' : (0,\infty)^n \to \mathbb{R}^n$ can be inverted and $(h_\varepsilon')^{-1} : \mathbb{R}^n \to (0,\infty)^n$ is defined on $\mathbb{R}^n$. As a consequence, $u_i = (h_\varepsilon')^{-1}(w)_i$ is positive for any $w \in \mathbb{R}^n$ and even strongly positive if w varies in a compact subset of $\mathbb{R}^n$.

Unfortunately, the product $H_\varepsilon(u)A(u)$, where $H_\varepsilon(u) = h_\varepsilon''(u)$, is generally not positive definite and we need to approximate $A(u)$. In contrast to the approximations suggested in [50, 94], we employ a *non-diagonal* matrix; see (6.2.5) below. More specifically, we introduce $A_\varepsilon(u) = A(u) + \varepsilon A^0(u) + \varepsilon^\eta A^1(u)$ with non-diagonal $A^0(u)$, diagonal $A^1(u)$, and $\eta \leq 1/2$ such that

$$z^\top H_\varepsilon(u) A_\varepsilon(u) z \geq z^\top H(u)A(u)z \quad \text{for all } z \in \mathbb{R}^n.$$

The choice of the non-diagonal approximation satisfying this inequality is nontrivial, and this construction is our *second idea.*

3.1.4. Main results

First, we show that global existence of weak solutions holds for linear transition rates ($s = 1$). In the following, we set $Q_T = \Omega \times (0,T)$.

Theorem 3.1 (Global existence for linear transition rates)**.** *Let $T > 0$, $s = 1$ and $u^0 = (u_1^0, \ldots, u_n^0)$ be such that $u_i^0 \geq 0$ for $i = 1, \ldots, n$ and $\int_\Omega h(u^0)dx < \infty$. Let either detailed balance and $a_{ii} > 0$ for $i = 1, \ldots, n$; or* (3.1.12) *hold. Then there exists a weak solution $u = (u_1, \ldots, u_n)$ to* (3.1.1)-(3.1.2) *satisfying $u_i \geq 0$ in Ω, $t > 0$, and*

$$\begin{aligned} &u_i \in L^2(0,T;H^1(\Omega)), \quad u_i \in L^\infty(0,T;L^1(\Omega)), \\ &u_i \in L^{2+2/d}(Q_T), \quad \partial_t u_i \in L^{q'}(0,T;W^{1,q}(\Omega)'), \quad i = 1, \ldots, n, \end{aligned}$$

where $q = 2(d+1)$ and $q' = (2d+2)/(2d+1)$. The solution u solves (3.1.1) *in the weak sense*

$$\int_0^T \langle \partial_t u, \phi \rangle dt + \int_0^T \int_\Omega \nabla\phi : A(u)\nabla u dx dt = \int_0^T \int_\Omega f(u) \cdot \phi dx dt \qquad (3.1.13)$$

for all test functions $\phi \in L^q(0,T;W^{1,q}(\Omega))$, and the initial condition in (3.1.2) *is satisfied in the sense of $W^{1,q}(\Omega)'$.*

The theorem can be generalized to the case of vanishing self-diffusion, i.e. $a_{ii} = 0$ if detailed balance, $a_{i0} > 0$, and $b_{ii} > 0$ hold; see Remark 6.12.

Our second result is concerned with nonlinear transition rates ($s \neq 1$). The entropy inequality yields the regularity $u_i \in L^{2s+2/d}(Q_T)$ which may not include L^2 for "small" exponents $s < 1$ and large dimensions d. For this reason, we need to suppose, in the sublinear case, the lower bound $s > 1 - 2/d$ and a weaker growth of the Lotka-Volterra terms:

$$f_i(u) = u_i\bigg(b_{i0} - \sum_{j=1}^n b_{ij}u_j^\sigma\bigg), \quad i = 1, \ldots, n, \quad 0 \le \sigma < 2s - 1 + 2/d. \tag{3.1.14}$$

The superlinear case ($s > 1$) is somehow easier than the sublinear one since the entropy inequality gives the higher regularity $u_i \in L^p(Q_T)$ with $p > 2$. On the other hand, we need a weak cross-diffusion constraint. More precisely, if detailed balance holds, we require that

$$\eta_1 := \min_{i=1,\ldots,n}\bigg(a_{ii} - \frac{s-1}{s+1}\sum_{j=1,\, j\neq i}^n a_{ij}\bigg) > 0, \tag{3.1.15}$$

and if detailed balance does not hold, we suppose that

$$\eta_2 := \min_{i=1,\ldots,n}\bigg(a_{ii} - \frac{1}{2(s+1)}\sum_{j=1,\, j\neq i}\big(s(a_{ij} + a_{ji}) - 2\sqrt{a_{ij}a_{ji}}\big)\bigg) > 0. \tag{3.1.16}$$

For $m \ge 2$ and $1 \le q \le \infty$ we introduce the space

$$W_\nu^{m,q}(\Omega) = \{\phi \in W^{m,q}(\Omega) : \nabla\phi \cdot \nu = 0 \text{ on } \partial\Omega\}. \tag{3.1.17}$$

Theorem 3.2 (Global existence for nonlinear transition rates)**.** *Assume that $T > 0$, $s > \max\{0, 1 - 2/d\}$, and let the initial data u^0 be such that $u_i^0 \ge 0$ for $i = 1, \ldots, n$ and $\int_\Omega h(u^0)dx < \infty$. If $s < 1$, we suppose that* (3.1.14) *and either detailed balance and $a_{ii} > 0$ for $i = 1, \ldots, n$; or* (3.1.12) *hold. If $s > 1$, we suppose that* (3.1.4) *and either detailed balance and* (3.1.15) *or* (3.1.16) *hold. Then there exist a number $2 \le q < \infty$ and a weak solution $u = (u_1, \ldots, u_n)$ to* (3.1.1)-(3.1.2) *satisfying $u_i \ge 0$ in Ω, $t > 0$, and*

$$u_i^s \in L^2(0,T; H^1(\Omega)), \quad u_i \in L^\infty(0,T; L^{\max\{1,s\}}(\Omega)),$$
$$u_i \in L^{p(s)}(Q_T), \quad \partial_t u_i \in L^{q'}(0,T; W_\nu^{m,q}(\Omega)'), \quad i = 1, \ldots, n,$$

where $p(s) = 2s + (2/d)\max\{1, s\}$, $1/q + 1/q' = 1$, and $m > \max\{1, d/2\}$. The solution u solves (6.1.1) *in the "very weak" sense*

$$\int_0^T \langle\partial_t u, \phi\rangle dt - \int_0^T\int_\Omega \sum_{i=1}^n u_i p_i(u)\Delta\phi_i dxdt = \int_0^T\int_\Omega f(u)\cdot\phi dxdt \tag{3.1.18}$$

for all $\phi = (\phi_1, \ldots, \phi_n) \in L^q(0,T; W_\nu^{m,q}(\Omega))$, and the initial condition holds in the sense of $W_\nu^{m,q}(\Omega)'$.

In the superlinear case, it can be shown that the solution satisfies (3.1.1) in the weak sense (3.1.13); see Remark 6.16. Moreover, for any $s > \max\{0, 1 - 2/d\}$, it is sufficient to consider test functions from $L^\beta(0,T; W_\nu^{2,\beta}(\Omega))$ with $1/\beta + 1/p(s) = 1$, and the initial condition holds in the sense of $W_\nu^{2,\beta}(\Omega)'$. We can generalize the theorem to the case of vanishing self-diffusion if either $s > \max\{1, d/2\}$; or $0 < s < 1$, $d = 1$, and $\sigma < s + 1$ hold; see Remark 6.17.

The lower bound $s > 1-2/d$ can be avoided if the regularity $u_i \in L^{2+s}(Q_T)$ holds, which is expected to follow from the duality method [48, 120]. Unfortunately, this method is not compatible with our approximation scheme (see (6.3.1) below). This issue can possibly be overcome by employing the scheme proposed in [51] which is specialized to diffusion systems like (3.1.5). In this work, however, we prefer to employ scheme (6.3.1).

3.2. From reaction diffusion to cross diffusion in the fast-reaction limit

The motivation of this work can be summarized in the following question: For which nonnegative functions $p_2(w), p_3(w) \geq 0$ can we find an entropy for the cross-diffusion system

$$\begin{cases} \partial_t w_2 - \Delta_x \left(p_2(w) w_2\right) = 0, \\ \partial_t w_3 - \Delta_x \left(p_3(w) w_3\right) = 0, \\ \qquad n(x) \cdot \nabla_x w_i = 0 \quad \text{on} \quad \partial\Omega, \quad i = 2,3 \\ \qquad w_i(0,\cdot) = w_{i0}(\cdot) \quad \text{in} \quad \Omega, \quad i = 2,3, \end{cases} \tag{3.2.1}$$

where $w(t,x) = (w_2(t,x), w_3(t,x))$ with x on a bounded domain $\Omega \subseteq \mathbb{R}^N$? This means that we are looking for a Lyapunov functional $H[w_2(t), w_3(t)]$ which is decreasing along the flow of the system (3.2.1)

$$\frac{d}{dt} H[w_2(t), w_3(t)] \leq 0 \quad \forall t \geq 0.$$

The idea will be to start from a reaction-diffusion model with known entropy and by performing a fast-reaction limit to obtain the entropy for the cross-diffusion system (3.2.1).

3.2.1. The strategy of our work

We present the motivation of our work on a toy problem, which was also studied in [16]. Given the (non-standard) cross-diffusion system

$$\begin{cases} \partial_t w_2 - \Delta_x \left(p_2(w) w_2\right) = 0, \quad p_2(w) = \dfrac{d_1 \psi_3(w) + d_2}{\psi_3(w) + 1}, \quad d_i > 0, \\ \partial_t w_3 - \Delta_x \left(p_3(w) w_3\right) = 0, \quad p_3(w) = \dfrac{d_1 \psi_2(w) + d_3}{\psi_2(w) + 1}, \quad d_i > 0, \end{cases} \tag{3.2.2}$$

with

$$\psi_2(w) = \frac{\sqrt{(w_3 - w_2 + 1)^2 + 4w_2}}{2} - \frac{w_3 - w_2 + 1}{2}, \quad \psi_3(w) = \frac{w_3}{1 + \psi_2(w)}, \tag{3.2.3}$$

the question is whether we can find an entropy for this system. Indeed we can, namely we will show that

$$H[w(t)] = \int_\Omega \big(\psi_2(w)\psi_3(w)(\log \psi_2(w)\psi_3(w) - 1) + 1\big)\, dx +$$

$$+\int_\Omega \big(\psi_2(w)(\log\psi_2(w)-1)+1\big)\,dx+\int_\Omega \big(\psi_3(w)(\log\psi_3(w)-1)+1\big)\,dx \tag{3.2.4}$$

is an entropy for (3.2.2), (3.2.3). Our strategy will be to start from a simpler reaction-diffusion system with known entropy and to perform the fast reaction limit.

Let us start from a reaction-diffusion system for species A_i, $i=1,\dots,3$, of the form

$$\begin{cases} \partial_t u_1^\varepsilon - d_1\,\Delta_x u_1^\varepsilon = \dfrac{1}{\varepsilon}\,(u_2^\varepsilon\,u_3^\varepsilon - u_1^\varepsilon) \\ \partial_t u_2^\varepsilon - d_2\,\Delta_x u_2^\varepsilon = -\dfrac{1}{\varepsilon}\,(u_2^\varepsilon\,u_3^\varepsilon - u_1^\varepsilon) \\ \partial_t u_3^\varepsilon - d_3\,\Delta_x u_3^\varepsilon = -\dfrac{1}{\varepsilon}\,(u_2^\varepsilon\,u_3^\varepsilon - u_1^\varepsilon), \end{cases} \tag{3.2.5}$$

with nonnegative initial conditions and no-flux boundary conditions, which describes the reversible chemical reaction

$$A_1 \quad \rightleftharpoons \quad A_2 + A_3$$

with fast reaction rate $1/\varepsilon > 0$, where the functions $u_i^\varepsilon := u_i^\varepsilon(t,x) \geq 0$ represent the concentrations of species A_i with diffusivities $d_i > 0$ on a bounded smooth open domain $\Omega \subseteq \mathbb{R}^N$. Since this system is coming from reversible reaction chemistry, it is not surprising that it has an entropy of the form

$$H[u^\varepsilon(t)] = \sum_{i=1}^{3}\int_\Omega \big(u_i^\varepsilon(\log u_i^\varepsilon - 1)+1\big)\,dx. \tag{3.2.6}$$

Performing the fast reaction limit $\varepsilon \to 0$ leads to the system

$$\begin{cases} u_2\,u_3 - u_1 = 0, \\ \partial_t(u_1+u_2) - \Delta_x(d_1\,u_1 + d_2\,u_2) = 0, \\ \partial_t(u_1+u_3) - \Delta_x(d_1\,u_1 + d_3\,u_3) = 0, \end{cases} \tag{3.2.7}$$

which can be rewritten by using the algebraic condition $u_1 = u_2u_3$ in the following way:

$$\begin{cases} \partial_t(u_2(1+u_3)) + \Delta_x\left[\dfrac{d_1u_3+d_2}{u_3+1}(u_2(1+u_3))\right] = 0, \\ \partial_t(u_3(1+u_2)) + \Delta_x\left[\dfrac{d_1u_2+d_3}{u_2+1}(u_3(1+u_2))\right] = 0. \end{cases}$$

Inverting the following change of variables

$$w_2 := u_1 + u_2 \;\text{ and }\; w_3 := u_1 + u_3 \;\text{ with }\; w = (w_2, w_3), \tag{3.2.8}$$

yields the relation

$$u_2 = \psi_2(w_2,w_3), \quad u_3 = \psi_3(w_2,w_3), \tag{3.2.9}$$

from which we obtain the desired cross-diffusion system (3.2.2)

$$\begin{cases} \partial_t w_2 - \Delta_x \left[\dfrac{d_1 \psi_3(w) + d_2}{\psi_3(w) + 1} w_2 \right] = 0, \\ \partial_t w_3 - \Delta_x \left[\dfrac{d_1 \psi_2(w) + d_3}{\psi_2(w) + 1} w_3 \right] = 0. \end{cases}$$

The key step is the inversion in (3.2.9), where $\psi = (\psi_2, \psi_3) := \varphi^{-1}$ is the inverse of the homeomorphism

$$\varphi : \begin{cases} \mathbb{R}_+^2 \to \mathbb{R}_+^2 \\ (u_2, u_3) \mapsto (\underbrace{u_2(1+u_3)}_{=:w_2}, \underbrace{u_3(1+u_2)}_{:=w_3}). \end{cases}$$

In this toy problem, this leads to the quadratic equation $u_2^2 + (w_3 - w_2 + 1)u_2 - w_2 = 0$, which has the unique nonnegative solution

$$u_2 = \psi_2(w) = \frac{\sqrt{(w_3 - w_2 + 1)^2 + 4w_2}}{2} - \frac{w_3 - w_2 + 1}{2}, \quad u_3 = \psi_3(w) = \frac{w_3}{1 + u_2}.$$

By performing the same limit $\varepsilon \to 0$ and the same variable transformation (3.2.8),(3.2.9) again, but now for the entropy (3.2.6), we obtain the limiting entropy (3.2.4).

3.2.2. Main results

Our goal will be to use the above sketched strategy on two generalized starting systems, namely (3.2.10) and (3.2.15), in order to obtain new entropies for new large classes of cross-diffusion systems. We briefly introduce both models considered and present the main results.

Model 1

The first model is obtained by starting from the following reaction-diffusion system:

$$\begin{cases} \partial_t u_1^\varepsilon - \Delta_x f_1(u_1^\varepsilon) = -\dfrac{1}{\varepsilon}\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)\, q_3(u_3^\varepsilon)\Big), \\ \partial_t u_2^\varepsilon - \Delta_x f_2(u_2^\varepsilon) = +\dfrac{1}{\varepsilon}\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)\, q_3(u_3^\varepsilon)\Big), \\ \partial_t u_3^\varepsilon - \Delta_x f_3(u_3^\varepsilon) = +\dfrac{1}{\varepsilon}\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)\, q_3(u_3^\varepsilon)\Big), \end{cases} \tag{3.2.10}$$

where the entropy of this system reads

$$H[u^\varepsilon(t)] = \sum_{i=1}^{3} \int_\Omega h_i(u_i^\varepsilon)\, dx, \text{ with } h_i(u_i^\varepsilon) = \int_c^{u_i^\varepsilon} \log(q_i(z))\, dz. \tag{3.2.11}$$

The limiting cross-diffusion system for $\varepsilon \to 0$ has the form

$$\begin{cases} q_1(u_1) - q_2(u_2)q_3(u_3) = 0, \\ \partial_t\,(u_1 + u_2) - \Delta_x\,(f_1(u_1) + f_2(u_2)) = 0, \\ \partial_t\,(u_1 + u_3) - \Delta_x\,(f_1(u_1) + f_3(u_3)) = 0, \end{cases} \tag{3.2.12}$$

which can be transformed by inverting the following homeomorphism

$$\varphi : \begin{cases} \mathbb{R}_+^2 \to \mathbb{R}_+^2, \\ (u_2, u_3) \mapsto (w_2, w_3), \end{cases} \tag{3.2.13}$$

where $w_2 := u_2 + q_1^{-1}\,(q_2(u_2)q_3(u_3))$ and $w_3 := u_3 + q_1^{-1}\,(q_2(u_2)q_3(u_3))$, and its inverse

$$\psi := \varphi^{-1}, \quad \psi(w) = \big(\psi_2(w), \psi_3(w)\big). \tag{3.2.14}$$

This leads to the following cross-diffusion system

$$\begin{cases} \partial_t w_2 - \Delta_x \left[\left(\dfrac{f_1(q_1^{-1}\,(q_2(\psi_2(w))q_3(\psi_3(w)))) + f_2(\psi_2(w))}{w_2} \right) w_2 \right] = 0, \\ \partial_t w_3 - \Delta_x \left[\left(\dfrac{f_1(q_1^{-1}\,(q_2(\psi_2(w))q_3(\psi_3(w)))) + f_3(\psi_3(w))}{w_3} \right) w_3 \right] = 0. \end{cases}$$

The entropy can be expressed in the new variables $w = (w_2, w_3)$ in the following way

$$\begin{aligned} H[w(t)] = &\int_\Omega \int_c^{q_1^{-1}(q_2(\psi_2(w))q_3(\psi_3(w)))} \log(q_1(z))\,dzdx \\ &+ \int_\Omega \int_c^{\psi_2(w)} \log(q_2(z))\,dzdx + \int_\Omega \int_c^{\psi_3(w)} \log(q_3(z))\,dzdx. \end{aligned}$$

The assumptions for the rigorous fast-reaction limit are the following:

(B1) There exists a constant $C_1 > 0$, such that (for large x) it holds that

$$0 \le f_i(x) \le C_1\, x^2 \log(x) \quad \text{for all} \;\; x \ge 0, \quad i = 1, 2, 3.$$

(B2) There exist $\alpha > 0$ and $C_2 > 0$, such that

$$q_i(x) \ge C_2\, x^\alpha \quad \text{for all} \;\; x \ge 0, \quad i = 1, 2, 3.$$

(B3) For some $s > 1$ and $1 \le p, p' \le \infty$ with $1/p + 1/p' = 1$, there exist positive constants $C_3, C_4, C_5 > 0$ such that (for large x)

$$\begin{aligned} q_1^s(x) &\le C_3\, x f_1(x) \quad \text{for all} \;\; x \ge 0, \\ q_2^{sp}(x) &\le C_4\, x f_2(x) \quad \text{for all} \;\; x \ge 0, \\ q_2^{sp'}(x) &\le C_5\, x f_3(x) \quad \text{for all} \;\; x \ge 0. \end{aligned}$$

(B4) The functions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $q_i : \mathbb{R}_+^* \to \mathbb{R}_+$ are in C^1 with

$$f_i'(x) > 0 \quad \text{and} \quad q_i'(x) > 0 \quad \text{for all} \quad x \geq 0, \qquad i = 1, 2, 3.$$

(B5) There exists a constant $C_6 > 0$, such that

$$q_i(x) \leq C_6\, x \left(f_i f_i' q_i'\right)(x) \quad \text{for all} \quad x \geq 0, \qquad i = 1, 2, 3.$$

(B6) For $a : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ with $a(u_2, u_3) = (a_2(u_2, u_3), a_3(u_2, u_3))$ it holds that

$$a_i : \begin{cases} \mathbb{R}_+^2 \to \mathbb{R}_+ \\ (u_2, u_3) \mapsto u_i + q_1^{-1}\Big(q_2(u_2) q_3(q_3)\Big)/u_i \end{cases}$$

are continuous for $i = 2, 3$.

The main theorem for this model reads as follows:

Theorem 3.3. *Let assumptions* (*B*1)-(*B*6) *for* $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $q_i : \mathbb{R}_+ \to \mathbb{R}_+$ *hold. Let* $\Omega \subseteq \mathbb{R}^N$ *be a bounded regular open set of* $\mathbb{R}^N$, *and let for any* $\varepsilon > 0$, $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$ *denote a weak solution of the reaction-diffusion system* (3.2.10) *with initial data* $u_i^\varepsilon(0, x) = u_i^{in}(x) \in L^\infty$ *for all* $x \in \Omega$, $i = 1, 2, 3$.

Then the following holds: If $\varepsilon \to 0$, *there exists a subsequence of* $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$ *(which we still denote by* $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$*), which converges to* u_1, u_2, u_3 *in* $L^1_{loc}([0, \infty); L^1(\Omega))$.

Moreover, this limit is a weak solution of the cross-diffusion system (3.2.12) *belonging to* $L^1_{loc}([0, \infty); L^1(\Omega))$.

Model 2

The second model is a 'hybrid" reaction-cross diffusion system in four species, which is derived from the follwing reaction-diffusion system

$$\begin{cases} \partial_t u_1^\varepsilon - d_1\, \Delta_x u_1^\varepsilon = \dfrac{1}{\varepsilon}\,(u_2^\varepsilon\, u_3^\varepsilon - u_1^\varepsilon) + (u_4^\varepsilon - u_1^\varepsilon\, u_3^\varepsilon) \\ \partial_t u_2^\varepsilon - d_2\, \Delta_x u_2^\varepsilon = -\dfrac{1}{\varepsilon}\,(u_2^\varepsilon\, u_3^\varepsilon - u_1^\varepsilon) \\ \partial_t u_3^\varepsilon - d_3\, \Delta_x u_3^\varepsilon = -\dfrac{1}{\varepsilon}\,(u_2^\varepsilon\, u_3^\varepsilon - u_1^\varepsilon) + (u_4^\varepsilon - u_1^\varepsilon\, u_3^\varepsilon) \\ \partial_t u_4^\varepsilon - d_4\, \Delta_x u_4^\varepsilon = -(u_4^\varepsilon - u_1^\varepsilon\, u_3^\varepsilon) \end{cases} \tag{3.2.15}$$

with the entropy

$$H[u^\varepsilon(t)] = \sum_{i=1}^{4} \int_\Omega \left(u_i^\varepsilon(\log u_i^\varepsilon - 1) + 1\right)\, dx. \tag{3.2.16}$$

The limiting system for (w_2, w_3, u_4) with $w_2 = u_2(1+u_3), w_3 = u_3(1+u_2)$ reads

$$\begin{cases} \partial_t w_2 - \Delta_x \left(\dfrac{d_1\psi_3(w) + d_2}{\psi_3(w) + 1}\, w_2 \right) = \big(u_4 - \psi_2^2\psi_3(w)\big), \\ \partial_t w_3 - \Delta_x \left(\dfrac{d_1\psi_2(w) + d_3}{\psi_2(w) + 1}\, w_3 \right) = 2\big(u_4 - \psi_2^2\psi_3(w)\big), \\ \qquad\qquad \partial_t u_4 - d_4\Delta_x u_4 = -\big(u_4 - \psi_2^2\psi_3(w)\big). \end{cases} \tag{3.2.17}$$

and has the limiting entropy

$$H[w_2(t),_3(t), u_4(t)] = \int_\Omega \big(\psi_2(w)\psi_3(w)\big(\log(\psi_2(w)\psi_3(w)) - 1\big) + 1\big)\, dx$$
$$+ \sum_{i=2,3} \int_\Omega \big(\psi_i(w)\big(\log(\psi_i(w)) - 1\big) + 1\big)\, dx + \int_\Omega \big(u_4\big(\log(u_4) - 1\big) + 1\big)\, dx.$$

The main theorem concerning this model states the following.

Theorem 3.4. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded regular open set of $\mathbb{R}^N$, and let for any $\varepsilon > 0$, $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, u_4^\varepsilon$ denote a weak solution of the reaction-diffusion system* (3.2.15) *with initial data $u^{in}\log(u^{in}) \in L^2(\Omega)$.*

Then the following holds: If $\varepsilon \to 0$, there exists a subsequence of $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, u_4^\varepsilon$ (which we still denote by $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, u_4^\varepsilon$), which converges to u_1, u_2, u_3, u_4 in $L^1_{loc}([0,\infty); L^1(\Omega))$.

Moreover, this limit is a weak solution of the reaction-cross-diffusion system (3.2.17) *belonging to $L^2_{loc}([0,\infty); L^2(\Omega))$.*

The proof of this theorem is based on entropy and duality estimates.

3.2.3. State of the art

Our work generalizes the strategy initiated in [16, 79, 17] to perform a fast-reaction limit from reaction-diffusion systems coming from reversible reaction chemistry to cross-diffusion systems. The strategy presented on a toy problem in subsection 3.2.1 was already investigated in [79], and also generalized for more general starting systems including a model with fast and slow reactions similar to our second model (3.2.15). However, the results concerning the rigorous fast-rection limit of our first model (3.2.10) are (up to our knowledge) completely new, only existence of solutions for fixed $\varepsilon > 0$ has been already studied in [17, Theorem 3, p.21] for models including our first model. Moreover, also our results about how the limiting entropy looks like for the limiting cross-diffusion system are completely new and cannot be found in any of these works, neither the robustness analysis for the entropy performed in subsection 7.2.5.

Fast-reaction limits from reaction-diffusion systems have been well-known in the engineering literature for a long time, but the rigorous mathematical treatment started only very recently. For the presentation of the underlying mass action law of the reaction-diffusion model, see [72]. In [89], a fast reaction limit from a reaction-diffusion system modeling the chemical reaction $A + B \rightharpoonup C$ to a free boundary value problem was investigated. In [11], limits of ODE systems were studied with several fast-reaction and additional slow-reaction processes. This was extended in [13] to the case of a simple fast reaction of the type $A \rightleftharpoons B$ from a reaction diffusion to a nonlinear diffusion equation using invariant sets. In [10], limits of reaction-diffusion systems with equal diffusion rates were investigated using also invariant sets. In [5] and [14], Quasi-Steady-State Approximations (QSSA) of the form $A + B \rightleftharpoons C \rightleftharpoons D + E$ with highly reactive intermediate C were investigated. For fast-reaction limits in irreversible chemistry, see [15] for a limit to a free boundary value problem, and for QSSA in irreversible chemistry, see [90] and [43]. We also want to mention the recent work [45], which focusses mainly on the modeling aspects of singular limits from reaction-diffusion equations. Moreover, we want to mention the works [92, 93, 88, 44, 53, 115] for a passage from reaction-diffusion equations to SKT-type cross-diffusion models of population dynamics. We also want to mention, that a rigorous fast-reaction limit starting from a kinetic system to a reaction-diffusion system was investigated in [31, 116].

Part II.

The kinetic models

4. The linearized multi-species Boltzmann system

4.1. The model

In this chapter, we prove a new coercivity estimate on the spectral gap of the linearized Boltzmann collision operator for multiple species under assumptions on the collision kernels including hard and Maxwellian potentials under Grad's angular cut-off condition. We present two proofs: a non-constructive one, based on the decomposition of the collision operator into a compact and a coercive part, and a constructive one, which exploits the "cross-effects" coming from collisions between different species and which yields explicit constants. Furthermore, the essential spectra of the linearized collision operator and the linearized Boltzmann operator are calculated. Based on the spectral-gap estimate, the exponential convergence towards global equilibrium with explicit rate is shown for solutions to the linearized multi-species Boltzmann system on the torus. The convergence is achieved by hypocoercive effects between degenerately dissipative collision term and conservative transport term, proved by using the hypocoercivity method of Mouhot and Neumann.

For a presentation of the model, see subsection 2.1.1 and subsection 2.1.2. Here, we just repeat the main definitions. We are interested in the linearized multi-species Boltzmann system

$$\partial_t f_i + v\cdot\nabla_x f_i = L_i(f), \quad t>0, \quad f_i(x,v,0) = f_{I,i}(x,v), \quad (x,v)\in\mathbb{T}^3\times\mathbb{R}^3, \tag{4.1.1}$$

for $1\le i\le n$, where $f=(f_1,\dots,f_n)$ and the ith component of the linearized collision operator $L=(L_1,\dots,L_n)$ is given by

$$L_i(f) = \sum_{j=1}^n L_{ij}(f_i,f_j), \quad 1\le i\le n,$$

with

$$\begin{aligned} L_{ij}(f_i,f_j) &= M_i^{-1/2}\big(Q_{ij}(M_i, M_j^{1/2}f_j) + Q_{ij}(M_i^{1/2}f_i, f_j)\big)\\ &= \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij}M_i^{1/2}M_j^*(h_i' + h_j'^* - h_i - h_j^*)dv^*d\sigma, \quad h_i := M_i^{-1/2}f_i. \end{aligned} \tag{4.1.2}$$

The global equilibrium has the form

$$M_i(v) = \frac{\rho_{\infty,i}}{(2\pi)^{3/2}}e^{-|v|^2/2}, \quad 1\le i\le n, \tag{4.1.3}$$

and the collision frequency reads

$$\nu_i(v) = \sum_{j=1}^{n} \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij}(|v^* - v|, \cos\vartheta) M_j^* dv^* d\sigma, \quad i = 1, \dots, n. \tag{4.1.4}$$

4.1.1. Assumptions on the collision kernels

We repeat the assumptions on the collision kernels B_{ij} arising in (4.1.2), already introduced in subsection 2.1.4, and discuss them briefly.

(A1) The collision kernels satisfy

$$B_{ij}(|v - v^*|, \cos\vartheta) = B_{ji}(|v - v^*|, \cos\vartheta) \quad \text{for } 1 \le i, j \le n.$$

(A2) The collision kernels decompose in the kinetic part $\Phi_{ij} \ge 0$ and the angular part $b_{ij} \ge 0$ according to

$$B_{ij}(|v - v^*|, \cos\vartheta) = \Phi_{ij}(|v - v^*|) b_{ij}(\cos\vartheta), \quad 1 \le i, j \le n.$$

(A3) For the kinetic part, there exist constants C_1, $C_2 > 0$, $\gamma \in [0, 1]$, and $\delta \in (0, 1)$ such that for all $1 \le i, j \le n$ and $r > 0$,

$$C_1 r^\gamma \le \Phi_{ij}(r) \le C_2(r + r^{-\delta}).$$

(A4) For the angular part, there exist constants C_3, $C_4 > 0$ such that for all $1 \le i, j \le n$ and $\vartheta \in [0, \pi]$,

$$0 < b_{ij}(\cos\vartheta) \le C_3 |\sin\vartheta|\,|\cos\vartheta|, \quad b'_{ij}(\cos\vartheta) \le C_4.$$

Furthermore,

$$C^b := \min_{1\le i\le n} \inf_{\sigma_1,\sigma_2\in\mathbb{S}^2} \int_{\mathbb{S}^2} \min\big\{b_{ii}(\sigma_1\cdot\sigma_3), b_{ii}(\sigma_2\cdot\sigma_3)\big\} d\sigma_3 > 0.$$

(A5) For all $1 \le i, j \le n$, b_{ij} is even in $[-1, 1]$ and the mapping $v \mapsto \Phi'_{ij}(|v|)$ on $\mathbb{R}^3$ is locally integrable on $\mathbb{R}^3$ and bounded as $|v| \to \infty$.

(A6) There exists $\beta > 0$ such that for all $1 \le i, j \le n$, $s > 0$, and $\sigma \in [-1, 1]$, we have $B_{ij}(s, \sigma) \le \beta B_{ii}(s, \sigma)$.

Following [113], since the functions b_{ij} are integrable, we define

$$\ell^b := \min_{1,\le i,j\le n} \int_0^\pi b_{ij}(\cos\theta) \sin\theta d\theta > 0. \tag{4.1.5}$$

4.1.2. Discussion of the assumptions

The first hypothesis (A1) means that the collisions are micro-reversible. Assumption (A2) is satisfied, for instance, for collision kernels derived from interaction potentials behaving like inverse-power laws. The lower bound in hypothesis (A3) includes power-law functions $\Phi_{ij}(r) = r^\gamma$ with $\gamma > 0$ (hard potential) and $\gamma = 0$ (Maxwellian molecules). The assumption $\gamma \geq 0$ is crucial since the linearized collision operator in the mono-species case for soft potentials ($\gamma < 0$) with angular cut-off has no spectral gap [4]; however, degenerate spectral-gap estimates are possible [73, 110]. The upper bound in (A3) means that the kinetic part is of restricted growth for both small and large values of $|v - v^*|$. In hypothesis (A4), the upper bound for b_{ij} implies Grad's cut-off assumption. The positivity of C^b in Assumption (A4) is used in the constructive proof of the multi-species spectral-gap estimate (Theorem 4.2) via the *mono-species* spectral-gap estimate which depends on C^b; see also the proofs of Theorem 1.1 in [4] and Theorem 6.1 in [110]. The positivity of C^b is satisfied for the main physical case of a collision kernel satisfying Grad's cut-off, i.e. for hard spheres with $B_{ij}(|v - v^*|, \cos\vartheta) = |v - v^*|$. Conditions (A1)-(A4) are also imposed in [4, 110, 111] for the linearized mono-species Boltzmann operator.

Assumption (A5) imposes technical conditions needed to verify the abstract hypotheses in [113]. More precisely, the evenness of b_{ij} is employed to show hypothesis (M2) (see section 4.5) and the properties on Φ'_{ij} are used to verify (4.5.2) in hypothesis (M1). The conditions on Φ'_{ij} are satisfied for hard and Maxwellian power-law potentials $\Phi_{ij}(r) = r^\gamma$ with exponent $\gamma \in [0, 1]$, for instance. Finally, condition (A6) states that the ratio of the off-diagonal and diagonal collision kernels can be bounded uniformly from above by a constant $\beta > 0$. This hypothesis will be needed for the explicit computation of the constants in Theorems 4.2 and 4.3. More precisely, (A6) allows us to estimate the mono-species part of the collision operator using the computation of [110]; see Lemma 4.10.

4.1.3. Notation and definitions

We call $\mathrm{Dom}(F)$ the domain of an operator F and $\mathrm{Im}(f)$ the image of a mapping f. We introduce the spaces $L^2_v = L^2(\mathbb{R}^3; \mathbb{R}^n)$, $L^2_{x,v} = L^2(\mathbb{T}^3 \times \mathbb{R}^3; \mathbb{R}^n)$, $H^1_{x,v} = H^1(\mathbb{T}^3 \times \mathbb{R}^3; \mathbb{R}^n)$, and

$$\begin{aligned} \mathcal{H} &= \Big\{ f \in L^2_v : \|f\|^2_{\mathcal{H}} = \sum_{i=1}^n \int_{\mathbb{R}^3} f_i^2 \nu_i dv < \infty \Big\}, \\ \mathcal{D} &= \Big\{ f \in L^2_v : \|f\|^2_{\mathcal{H}} = \sum_{i=1}^n \int_{\mathbb{R}^3} f_i^2 \nu_i^2 dv < \infty \Big\}. \end{aligned} \tag{4.1.6}$$

Here, ν_i is the collision frequency, defined in (4.1.4). For Maxwellian modelcules $\Phi_{ij}(r) =$ const., the collision frequency is constant but for strictly hard potentials $\Phi_{ij}(r) = r^\gamma$ with $0 < \gamma \leq 1$, ν_i is unbounded. In fact, it satisfies $\nu_0(1 + |v|)^\gamma \leq \nu_i(v) \leq \nu_1(1 + |v|)^\gamma$ for some constants $\nu_1 \geq \nu_0 > 0$ [113, p. 991]. In the physically most relevant case of hard spheres ($\gamma = 1$, $b_{ij} = 1$), the collision frequency can be computed explicitly, see formula (2.13) in [34, Section 7.2]. For more properties of the collision frequencies, we refer to [32, Section III.3]. If the collision frequencies are bounded, $\mathcal{H} = L^2_v$. Generally, ν_i is unbounded and so, $\mathcal{H}$ is a proper subset of L^2_v. The norm on L^2_v (and similarly for the other spaces) is defined by

$$\|f\|_{L^2_v}^2 = \sum_{i=1}^{n} \int_{\mathbb{R}^3} f_i^2 dv \quad \text{for } f = (f_1, \dots, f_n) \in L^2_v.$$

We distinguish the following linear operators. We define the operator $\Lambda = (\Lambda_1, \dots, \Lambda_n)$: $\mathrm{Dom}(\Lambda) \to L^2_v$ by

$$\Lambda_i(f) = \nu_i f_i, \quad i = 1, \dots, n,$$

where $\mathrm{Dom}(\Lambda) = \{f \in L^2_v : \Lambda f \in L^2_v\} = \mathcal{D}$. It is closed, densely defined, selfadjoint and, by Lemma 4.6 below, coercive. The linearized collision operator $L : \mathrm{Dom}(L) \to L^2_v$, defined in (4.1.2), can be written as $L = K - \Lambda$, where $K := L + \Lambda$, or, more explicitly,

$$K_i(f) = \sum_{j=1}^{n} \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} M_i^{1/2} M_j^* (h_i' + h_j'^* - h_j^*) dv^* d\sigma, \quad i = 1, \dots, n.$$

It was shown in [19] that K is a compact operator in L^2_v. Thus, $\mathrm{Dom}(L) = \mathrm{Dom}(\Lambda) = \mathcal{D}$ and L is closed and densely defined. Furthermore, L is nonpositive and selfadjoint on L^2_v. We define the transport operator

$$T = v \cdot \nabla_x : \mathrm{Dom}(T) \to L^2_v,$$

where $\mathrm{Dom}(T) = \{f \in L^2_{x,v} : v \cdot \nabla_x f \in L^2_{x,v}\}$. Finally, we consider the linearized Boltzmann operator

$$G = L - T : \mathrm{Dom}(G) \to L^2_v,$$

which is unbounded, closed, and densely defined with $\mathrm{Dom}(G) = \mathrm{Dom}(L) \cap \mathrm{Dom}(T)$.

We denote by $\mathrm{Ker}(A)$ and $\mathrm{Ran}(A)$ the kernel and range of a linear operator A, respectively. Its resolvent set is denoted by $\rho(A)$ and its spectrum by $\sigma(A) = \mathbb{C} \backslash \rho(A)$. For a linear unbounded operator A with $\sigma(A) \subset (-\infty, 0]$, we say that A has a spectral gap when the distance between 0 and $\sigma(A) \backslash \{0\}$ is positive. Finally, the essential spectrum of A is defined as the set of all complex numbers $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not Fredholm, where I is the identity operator. We refer to section 4.3 for details regarding this definition.

4.1.4. Main results

In this subsection, we list the main results of this part of the thesis. For a discussion of these results, see section 2.1.5.

Theorem 4.1 (Essential spectrum of L and $L - T$)**.** *Let the collision kernels B_{ij} satisfy assumptions (A1)-(A4) and set*

$$J = \cup_{i=1}^{n} \mathrm{Im}(\nu_i) \subset [\nu_0, \infty),$$

where $\nu_0 = \min_{i=1,\dots,n} \sup_{v \in \mathbb{R}^3} \nu_i(v) > 0$ (see Lemma 4.6). Then

$$\sigma_{ess}(L) = -J, \quad \sigma_{ess}(L - T) = \{\lambda \in \mathbb{C} : \Re(\lambda) \in -J\}.$$

The next theorem is the main result.

Theorem 4.2 (Explicit spectral-gap estimate)**.** *Let the collision kernels B_{ij} satisfy assumptions (A1)-(A4). Then there exists a constant $\lambda > 0$ such that*

$$-\left(f, L(f)\right)_{L^2_v} \geq \lambda \| f - \Pi^L(f) \|^2_{\mathcal{H}} \quad \text{for all } f \in \mathcal{D}, \tag{4.1.7}$$

where Π^L is the projection onto the null space Ker(L)*. If additionally hypothesis (A6) holds, the constant λ can be computed explicitly:*

$$\lambda = \frac{\eta D^b}{8C_k}, \quad \eta = \min\left\{1, \frac{4C^m C_k}{16C_k + D^b}\right\},$$

where C^m, D^b, and C_k are defined in (4.4.8), (4.4.13), *and* (4.4.15), *respectively.*

Note that the constant C^m depends on the mono-species spectral-gap constant C^b via (4.4.8) below.

As a consequence of the spectral-gap estimate, we are able to prove the exponential decay of the solution $f(t)$ to (4.1.1) to the global equilibrium with an explicit decay rate.

Theorem 4.3 (Convergence to equilibrium)**.** *Let the collision kernels B_{ij} satisfy assumptions (A1)-(A5) and let $f_I \in H^1_{x,v}$. Then the linearized Boltzmann operator $G = L - T$ generates a strongly continuous semigroup $S_G(t)$ on $H^1_{x,v}$, which satisfies*

$$\| S_G(t)(I - \Pi^G) \|_{H^1_{x,v}} \leq C\mathrm{e}^{-\tau t}, \quad t \geq 0, \tag{4.1.8}$$

for some constants C, $\tau > 0$. In particular, the solution $f(t) = S_G(t) f_I$ to (4.1.1) *satisfies*

$$\| f(t) - f_\infty \|_{H^1_{x,v}} \leq C\mathrm{e}^{-\tau t} \| f_I - f_\infty \|_{H^1_{x,v}}, \quad t \geq 0, \tag{4.1.9}$$

where $f_\infty := \Pi^G(f_I)$ is the global equilibrium of (4.1.1)*. Moreover, under the additional assumption (A6) and lower bound in (A4), the constants C and τ depend only on the constants appearing in hypotheses (M1)-(M3) in section 4.5 and in particular on λ defined in Theorem 4.2.*

4.2. Properties of the kinetic model

We show some properties of the linearized collision operator (4.1.2) and the collision frequencies (4.1.4). Let assumptions (A1)-(A4) hold. First we prove an H-theorem for (4.1.2).

Lemma 4.4 (H-theorem for the linearized collision operator)**.** *It holds that $(f, L(f))_{L^2_v} \leq 0$ for all $f \in \mathcal{D}$ and $(f, L(f))_{L^2_v} = 0$ if and only if $f \in$ Ker(L), where*

$$\begin{aligned} \mathrm{Ker}(L) = \big\{ f \in L^2_v : \exists \alpha_1, \ldots, \alpha_n, e \in \mathbb{R},\ u \in \mathbb{R}^3,\ \forall 1 \leq i \leq n, \\ f_i = M_i^{1/2}(\alpha_i + u \cdot v + e|v|^2) \big\}, \end{aligned}$$

and M_i is given by (4.1.3).

The proof is similar to the mono-species case except that the elements of the null space of L depend on the total mean velocity u and total energy e instead of the individual velocities and energies. Therefore, we give a complete proof. We note that an H-theorem for the nonlinear Boltzmann operator for a mixture of reactive gases was proved in [52].

Proof. By the change of variables $(v, v^*) \mapsto (v^*, v)$ and $(v, v^*) \mapsto (v', v'^*)$ and the symmetry of B_{ij} (assumption (A1)), we can write for $f \in L^2_v$,

$$(f, L(f))_{L^2_v} = -\frac{1}{4}\sum_{i,j=1}^{n}\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}M_iM_j^*(h_i' + h_j'^* - h_i - h_j^*)^2 dv^* dv d\sigma,$$

where we recall that $h_i = M_i^{-1/2}f_i$. This shows that $(f, L(f))_{L^2_v} \le 0$ for all $f \in \mathcal{D}$. Moreover, $(f, L(f))_{L^2_v} = 0$ if and only if

$$h_i' + h_j'^* - h_i - h_j^* = 0 \quad \text{for all } (v, v^*) \in \mathbb{R}^3\times\mathbb{R}^3,\ 1 \le i, j \le n. \tag{4.2.1}$$

It is shown in [34, pp. 36-42] that (4.2.1) for $i = j$ implies that h_i has the form $h_i(v) = \alpha_i + u_i \cdot v + e_i|v|^2$ for suitable constants α_i, $e_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^3$. Inserting this expression into (4.2.1) leads to

$$u_i \cdot (v' - v) + u_j \cdot (v'^* - v^*) + e_i(|v'|^2 - |v|^2) + e_j(|v'^*|^2 - |v^*|^2) = 0 \tag{4.2.2}$$

for $1 \le i, j \le n$. We consider the particular type of collisions with $v' = v^*$, $v'^* = v$, and $|v| = |v'|$. For such collisions, $\sigma = (v^* - v)/|v^* - v|$. Then the above equation becomes

$$(u_i - u_j)\cdot(v' - v) = 0, \quad 1 \le i, j \le n.$$

By rotating the velocities v, v' in all possible ways, we deduce that $(u_i - u_j)\cdot w = 0$ for all $w \in \mathbb{R}^3$ and thus, $u_i = u_j$ for all $1 \le i, j \le n$. We set $u := u_1$. This fact, together with the conservation of momentum $v' - v + v'^* - v^* = 0$, implies that (4.2.2) becomes

$$e_i(|v'|^2 - |v|^2) + e_j(|v'^*|^2 - |v^*|^2) = 0, \quad 1 \le i, j \le n.$$

Taking into account the conservation of energy $|v'|^2 - |v|^2 + |v'^*|^2 - |v^*|^2 = 0$, we infer that $(e_i - e_j)(|v'|^2 - |v|^2) = 0$ and consequently, $e_i = e_j$ for all $1 \le i, j \le n$. Set $e := e_1$. We have shown that $(f, L(f))_{L^2_v} = 0$ if and only if there exist $\alpha_1, \ldots, \alpha_n$, $e \in \mathbb{R}$ and $u \in \mathbb{R}^3$ such that for $1 \le i \le n$, $f_i(v) = M_i^{1/2}(\alpha_i + u\cdot v + e|v|^2)$. These functions clearly belong to $\mathrm{Ker}(L)$, which finishes the proof. □

The next result is concerned with the stationary solutions of (4.1.1).

Lemma 4.5. *The global equilibrium $f_\infty = (f_{\infty,1}, \ldots, f_{\infty,n})$ of (4.1.1), i.e. the unique stationary solution, is given by*

$$f_{\infty,i}(v) = M_i^{1/2}(\alpha_i + u\cdot v + e|v|^2), \quad 1 \le i \le n,$$

where α_i, $e \in \mathbb{R}$ and $u \in \mathbb{R}^3$ are uniquely determined by the global conservation laws of mass, momentum, and energy, i.e. by the equations

$$\int_{\mathbb{R}^3} M_i^{1/2}(f_{\infty,i} - f_{I,i})\psi(v)dv = 0, \quad 1 \le i \le n,$$

for $\psi(v) = 1, v_1, v_2, v_3, |v|^2$, where $f_{I,i}$ are the initial data.

Proof. First, we claim that $\mathrm{Ker}(G) = \mathrm{Ker}(L) \cap \mathrm{Ker}(T)$, where $G = L - T$ and $T = v \cdot \nabla_x$ are considered on $\mathbb{T}^3 \times \mathbb{R}^3$. The inclusion $\mathrm{Ker}(L) \cap \mathrm{Ker}(T) \subset \mathrm{Ker}(G)$ being trivial, let $f \in \mathrm{Ker}(G)$. Then, using the skew-symmetry of T,

$$0 = (f, G(f))_{L^2_{x,v}} = (f, L(f))_{L^2_{x,v}} - (f, T(f))_{L^2_{x,v}} = (f, L(f))_{L^2_{x,v}}.$$

Lemma 4.4 shows that $f \in \mathrm{Ker}(L)$. But this implies that $T(f) = L(f) - G(f) = 0$ and hence $f \in \mathrm{Ker}(T)$. This shows the claim. Let f_∞ be a stationary solution. Then $f_\infty \in \mathrm{Ker}(G)$ and by our claim, $f \in \mathrm{Ker}(L) \cap \mathrm{Ker}(T)$. Since $\mathrm{Ker}(T) = \{f \in L^2_{x,v} : \nabla_x f = 0\}$ [25, Lemma B.2], f_∞ does not depend on x. Because of $f_\infty \in \mathrm{Ker}(L)$, Lemma 4.4 shows the result. □

Finally, we prove that the collision frequencies (4.1.4) are strictly positive with bounded derivative.

Lemma 4.6. *Let Assumptions (A2)-(A4) hold. The collision frequencies* (4.1.4) *satisfy*

$$\min_{1\le i\le n} \inf_{v\in\mathbb{R}^3} \nu_i(v) \ge \nu_0 := 2^{3\gamma/2} \frac{C_1 \ell^b \rho_\infty}{\sqrt{\pi}} \Gamma\left(\frac{\gamma+3}{2}\right) > 0, \tag{4.2.3}$$

where $C_1 > 0$ is given by assumption (A3), $\ell^b > 0$ is defined in (4.1.5)*, $\rho_\infty := \sum_{j=1}^n \rho_{j,\infty}$ (see* (2.1.4)*), and Γ is the Gamma function. Furthermore, if additionally (A5) holds, then $\nabla_v \nu_i \in L^\infty_v(\mathbb{R}^3)$, implying that $|\nu_i(v)| \le C_\nu(1 + |v|)$ for some $C_\nu > 0$ and for all $v \in \mathbb{R}^3$, $i = 1, \ldots, n$.*

Proof. The decomposition of B_{ij}, according to assumption (A2), implies that

$$\nu_i(v) = \sum_{j=1}^n \int_{\mathbb{R}^3} \Phi_{ij}(|v - v^*|) M_j^* dv^* \int_{\mathbb{S}^2} b_{ij}(\cos\vartheta) d\sigma.$$

The integral

$$c_{ij} := \int_{\mathbb{S}^2} b_{ij}(\cos\vartheta) d\sigma = 2\pi \int_0^\pi b_{ij}(\cos\vartheta) \sin\vartheta d\vartheta$$

does not depend on v or v^*. We conclude from (A2)-(A4) that

$$\begin{aligned} \nu_i(v) &= (2\pi)^{-3/2} \sum_{j=1}^n c_{ij} \rho_{\infty,j} \int_{\mathbb{R}^3} \Phi_{ij}(|v - v^*|) \mathrm{e}^{-|v^*|^2/2} dv^* \\ &\ge \frac{C_1 \ell^b \rho_\infty}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} |v - v^*|^\gamma \mathrm{e}^{-|v^*|^2/2} dv^*. \end{aligned} \tag{4.2.4}$$

Observe that the function

$$G(v) := \int_{\mathbb{R}^3} |v - v^*|^\gamma \mathrm{e}^{-|v^*|^2/2} dv^*$$

is uniformly positive since the transformation $v^* \mapsto -v^*$ and the elementary inequality

$$|v - v^*|^\gamma + |v + v^*|^\gamma \ge |(v - v^*) + (v + v^*)|^\gamma = 2^\gamma |v^*|^\gamma$$

for $\gamma \in [0,1]$ lead to

$$G(v) = \frac{1}{2}\int_{\mathbb{R}^3} (|v-v^*|^\gamma + |v+v^*|^\gamma)\mathrm{e}^{-|v^*|^2/2}dv^* \geq 2^{\gamma-1}\int_{\mathbb{R}^3} |v^*|^\gamma \mathrm{e}^{-|v^*|^2/2}dv^* = 2^{\gamma-1}G(0).$$

Actually, using spherical coordinates and the change of unknowns $s = r^2/2$,

$$G(0) = 4\pi \int_0^\infty r^{\gamma+2}\mathrm{e}^{-r^2/2}dr = 2^{(\gamma+5)/2}\pi \int_0^\infty s^{(\gamma+1)/2}\mathrm{e}^{-s}ds = 2^{(\gamma+5)/2}\pi\Gamma\left(\frac{\gamma+3}{2}\right).$$

Inserting the above estimate on $G(v)$ into (4.2.4) shows (4.2.3).

It remains to prove that $\nabla_v \nu_i \in L_v^\infty(\mathbb{R}^3)$. To this end, we compute

$$\begin{aligned} |\nabla \nu_i(v)| &= (2\pi)^{-3/2}\left|\sum_{j=1}^n c_{ij}\rho_{\infty,j}\int_{\mathbb{R}^3} \Phi'_{ij}(|v-v^*|)\cdot\frac{v-v^*}{|v-v^*|}\mathrm{e}^{-|v^*|^2/2}dv^*\right| \\ &\leq (2\pi)^{-3/2}\sum_{j=1}^n c_{ij}\rho_{\infty,j}\int_{\mathbb{R}^3} |\Phi'_{ij}(|v-v^*|)|\mathrm{e}^{-|v^*|^2/2}dv^*. \end{aligned} \tag{4.2.5}$$

For given $R > 0$, we decompose

$$|\Phi'_{ij}(|v|)| = |\Phi'_{ij}(|v|)|\chi_{\{|v|<R\}}(v) + |\Phi'_{ij}(|v|)|\chi_{\{|v|\geq R\}}(v).$$

Assumption (A5) means that there exists $R > 0$ such that

$$|\Phi'_{ij}(|\cdot|)|\chi_{\{|\cdot|<R\}} \in L_v^1(\mathbb{R}^3) \quad \text{and} \quad |\Phi'_{ij}(|\cdot|)|\chi_{\{|\cdot|\geq R\}} \in L_v^\infty(\mathbb{R}^3).$$

Thus, the right-hand side of (4.2.5) is bounded since it can be written as the sum of two terms, each of which is the convolution of an L^1 and an L^∞ function. This shows that $\nabla_v \nu_i \in L_v^\infty(\mathbb{R}^3)$. ☐

Remark 4.7. We observe that ν_i is generally not bounded since the kinetic part $\Phi_{ij}(r)$ may grow like r as $r \to \infty$. It is possible to show that ν_i is bounded if Φ_{ij} is bounded. The unboundedness of ν_i implies that the spaces L_v^2 and $\mathcal{H}$ are not isomorphic. ☐

4.3. Geometric properties of the spectrum

In this section, we prove Theorem 4.1 and give a non-constructive proof of the spectral-gap estimate (4.1.7) in Theorem 4.2 by using arguments from functional analysis.

First, we study the essential spectrum of L and $L - T$. There exist several definitions of the essential spectrum of a linear operator. Given a linear, closed and densely defined operator $A : \mathrm{Dom}(A) \subset X \to X$ on a Banach space X, we define

$$\sigma_{\mathrm{ess}}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}.$$

We recall that a linear, closed, and densely defined operator A is Fredholm if its range $\mathrm{Ran}(A)$ is closed and both its kernel and cokernel are finite-dimensional. For other definitions of the essential spectrum, we refer to [84]. The essential spectrum is closed and conserved under compact perturbations, i.e., the bounded operators A and B have the same essential spectrum if $A - B$ is compact (Weyl's theorem; see [84, Theorem S].

If X is a Hilbert space and A is selfadjoint, it holds $\sigma_{\mathrm{ess}}(A) \subset \mathbb{R}$ and for given $\lambda \in \mathbb{R}$, we have $\lambda \in \sigma_{\mathrm{ess}}(A)$ if and only if $A - \lambda I$ is not closed or the kernel of $A - \lambda I$ is infinite dimensional. (This follows from the fact that $\mathrm{Ran}(A - \lambda I)^\perp = \mathrm{Ker}(A - \lambda I)$ for closed, selfadjoint operators A [97, Chap. V.3.1].) Moreover, Weyl's criterion holds [133, Lemma 5.17]: $\lambda \in \sigma_{\mathrm{ess}}(A)$ if and only if $A - \lambda I$ admits a *singular sequence*, i.e. a sequence $(f_k) \subset \mathrm{Dom}(A)$ such that (i) $\|f_k\|_X = 1$ for all $k \in \mathbb{N}$; (ii) $\|(A - \lambda I) f_k\|_X \to 0$ as $k \to \infty$; and (iii) (f_k) has no convergent subsequences in X.

We decompose L as $L = K - \Lambda$, where $K = (K_1, \ldots, K_n)$, $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$, and

$$K_i(f) = \sum_{j=1}^n \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} M_i^{1/2} M_j^* (h_i' + h_j'^* - h_j^*) dv^* d\sigma,$$
$$\Lambda_i(f) = \nu_i f_i, \quad 1 \le i \le n, \tag{4.3.1}$$

and the collision frequencies ν_i are defined in (4.1.4). We recall from Lemma 4.6 that they satisfy $\nu_i(v) \ge \nu_0 > 0$ and $|\nu_i(v)| \le C_\nu(1 + |v|)$ for all $i = 1, \ldots, n$ and $v \in \mathbb{R}^3$.

Proof of Theorem 4.1. Since K is compact on $X = L_v^2$ [19, Prop. 2], it follows that $\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(-\Lambda) = -\sigma_{\mathrm{ess}}(\Lambda)$. Thus, we will first study the essential spectrum of Λ. The proof is divided into several steps. Recall that $J = \cup_{i=1}^n \mathrm{Im}(\nu_i) \subset [\nu_0, \infty)$.

Step 1: $\overline{J} \subset \sigma_{ess}(\Lambda)$. Let $\lambda \in J$. There exists $j \in \{1, \ldots, n\}$ and $\widehat{v} \in \mathbb{R}^3$ such that $\lambda = \nu_j(\widehat{v})$. We define the sequence $(f_k) \subset \mathcal{D}$ by

$$f_{k,i}(v) = (2\pi\sigma_k)^{-3/4} \exp\left(-\frac{|v - \widehat{v}|^2}{4\sigma_k}\right) \quad \text{if } i = j, \quad f_{k,i}(v) = 0 \quad \text{if } i \ne j,$$

where $\sigma_k = 1/k$, $k \in \mathbb{N}$. Clearly, condition (i) for the singular sequence is satisfied. Furthermore,

$$\begin{aligned}\|(\Lambda - \lambda I) f_k\|_{L_v^2}^2 &= \sum_{i=1}^n \int_{\mathbb{R}^3} (\nu_i(v) - \lambda)^2 f_{k,i}(v)^2 dv \\ &= (2\pi\sigma_k)^{-3/2} \int_{\mathbb{R}^3} (\nu_i(v) - \nu_j(\widehat{v}))^2 \exp\left(-\frac{|v - \widehat{v}|^2}{2\sigma_k}\right) dv.\end{aligned}$$

The limit of a sequence of Gaussians with variance tending to zero converges to the delta distribution $\delta_{\widehat{v}}$ (in the sense of distributions), which means that

$$(2\pi\sigma_k)^{-3/2} \int_{\mathbb{R}^3} u(v) \exp\left(-\frac{|v - \widehat{v}|^2}{2\sigma_k}\right) dv \to u(\widehat{v}) \quad \text{as } k \to \infty$$

for all functions $u \in C^0(\mathbb{R}^3)$ with polynomial growth at infinity. Since $|\nu_i(v)| \le C_\nu(1 + |v|)$, this condition is satisfied and we conclude that $\|(\Lambda - \lambda I) f_k\|_{L_v^2} \to 0$ as $k \to \infty$, showing that condition (ii) holds.

Let us assume by contradiction that condition (iii) does not hold. Then there exists a subsequence (f_{k_ℓ}) of (f_k) that converges in L_v^2 to some function $f \in L_v^2$. As a consequence,

$|f_{k_\ell}|^2 \to |f|^2$ in L^1_v as $\ell \to \infty$. In particular, $f \in L^2_v$. However, the distributional limit $|f_{k_\ell}|^2 \to \delta_{\widehat{v}}$ and the uniqueness of the limit imply that $\delta_{\widehat{v}} = |f|^2 \in L^1_v$, which is absurd. Thus, condition (iii) holds, and we infer that $\lambda \in \sigma_{\text{ess}}(\Lambda)$. Then, since $\sigma_{\text{ess}}(\Lambda)$ is closed, $\overline{J} \subset \sigma_{\text{ess}}(\Lambda)$.

Step 2: $\sigma_{ess}(\Lambda) \subset \overline{J}$. Let $\lambda \in \mathbb{R}\backslash\overline{J}$. Then there exists a constant $c > 0$ such that for all $v \in \mathbb{R}^3$ and $i = 1, \dots, n$, $|\nu_i(v) - \lambda| \geq c$. If $(f_k) \subset \mathcal{D}$ with $\|f_k\|_{L^2_v} = 1$ for all $k \in \mathbb{N}$, we have

$$\|(\Lambda - \lambda I)f_k\|^2_{L^2_v} = \sum_{i=1}^n \int_{\mathbb{R}^3} (\nu_i(v) - \lambda)^2 f_{k,i}(v)^2 dv \geq c^2 \sum_{i=1}^n \int_{\mathbb{R}^3} f_{k,i}(v)^2 dv = c^2 > 0$$

for all $k \in \mathbb{N}$. Thus, condition (ii) cannot hold which implies that $\lambda \notin \sigma_{\text{ess}}(\Lambda)$.

Steps 1 and 2 imply that $\sigma_{\text{ess}}(\Lambda) = \overline{J}$.

Step 3: $\{\lambda \in \mathbb{C} : \Re(\lambda) \in J\} \subset \sigma_{ess}(\Lambda + T)$. Let $\lambda \in \mathbb{C}$ be such that $\Re(\lambda) \in J$. It follows from Step 1 that $\Re(\lambda) \in \sigma_{\text{ess}}(\Lambda)$. Since Λ is selfadjoint on the Hilbert space L^2_v, $\Lambda - \Re(\lambda)I$ is not closed or the kernel of $\Lambda - \Re(\lambda)I$ is infinite dimensional. As the operator $\Lambda - \Re(\lambda)I$ is closed, its kernel must be infinite dimensional. Therefore, there exists a sequence $(f_k) \subset L^2_v$ such that $\Lambda(f_k) - \Re(\lambda)f_k = 0$ and $(f_k, f_\ell)_{L^2_v} = \delta_{k\ell}$ for $k, \ell \in \mathbb{N}$. Let us define $\phi(x, v) = \exp(\mathrm{i}\Im(\lambda)x \cdot v/|v|^2)$ and $g_k = \phi f_k \in L^2_{x,v}$. Since $|\phi| = 1$, we have

$$(g_k, g_\ell)_{L^2_{x,v}} = (f_k, f_\ell)_{L^2_v} = \delta_{k\ell} \quad \text{for } k, \ell \in \mathbb{N}. \tag{4.3.2}$$

Furthermore, $\phi \in \text{Dom}(T)$ and $T(\phi) = \mathrm{i}\Im(\lambda)\phi$ for $v \neq 0$, and thus,

$$(\Lambda + T - \lambda I)g_k = \phi(\Lambda - \Re(\lambda)I)f_k + f_k(T - \mathrm{i}\Im(\lambda)I)\phi = 0,$$

which shows that $g_k \in \text{Ker}(\Lambda + T - \lambda I)$ for $k \in \mathbb{N}$. This fact, together with relation (4.3.2), implies that $\text{Ker}(\Lambda + T - \lambda I)$ is infinite dimensional. As a consequence, $\Lambda + T - \lambda I$ is not Fredholm and $\lambda \in \sigma_{\text{ess}}(\Lambda + T)$, which proves the claim.

Step 4: $\{\lambda \in \mathbb{C} : \Re(\lambda) \notin \overline{J}\} \subset \rho(\Lambda + T)$. Clearly, this gives

$$\sigma_{\text{ess}}(\Lambda + T) \subset \sigma(\Lambda + T) \subset \{\lambda \in \mathbb{C} : \Re(\lambda) \in \overline{J}\}.$$

Let $\lambda \in \mathbb{C}$ be such that $\Re(\lambda) \in \mathbb{R}\backslash\overline{J}$. We show first that $\text{Ker}(\Lambda + T - \lambda I) = \{0\}$. We assume by contradiction that there exists $f \in \text{Dom}(\Lambda + T)$ satisfying $\|f\|_{L^2_{x,v}} > 0$ and $(\Lambda + T - \lambda I)f = 0$. In particular, there is an index $\ell \in \{1, \dots, n\}$ such that $\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} f_\ell^2 dxdv > 0$. Then, multiplying $\nu_\ell f_\ell + T(f_\ell) = \lambda f_\ell$ by $\overline{f}_\ell$ (the complex conjugate of f_ℓ) and integrating in $\mathbb{T}^3 \times \mathbb{R}^3$, we obtain

$$\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \nu_\ell |f_\ell|^2 dxdv + \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \overline{f}_\ell v \cdot \nabla_x f_\ell dxdv = \lambda \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} |f_\ell|^2 dxdv. \tag{4.3.3}$$

By the divergence theorem, the real part of the second integral vanishes,

$$\begin{aligned} 2\Re \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \overline{f}_\ell v \cdot \nabla_x f_\ell dxdv &= \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \big(\overline{f}_\ell v \cdot \nabla_x f_\ell + f_\ell v \cdot \nabla_x \overline{f}_\ell\big) dxdv \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} v \cdot \nabla_x |f_\ell|^2 dxdv = 0. \end{aligned} \tag{4.3.4}$$

Then, taking the real part of (4.3.3), we infer that

$$\Re(\lambda) = \frac{\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \nu_\ell |f_\ell|^2 dxdv}{\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} |f_\ell|^2 dxdv}.$$

Consequently, $\inf_{\mathbb{R}^3} \nu_\ell \le \Re(\lambda) \le \sup_{\mathbb{R}^3} \nu_\ell$ and, thanks to the continuity of ν_ℓ, $\Re(\lambda) \in \overline{\mathrm{Im}(\nu_\ell)} \subset \overline{J}$, which is a contradiction. Thus, $\mathrm{Ker}(\Lambda + T - \lambda I) = \{0\}$. Similarly, we can show that $\mathrm{Ker}((\Lambda + T - \lambda I)^*) = \mathrm{Ker}(\Lambda + T^* - \overline{\lambda} I) = \{0\}$ as well.

The operator $L - T$ is closed [135, Theorems 2.2.1 and 2.2.2]. Thus, the boundedness of the compact operator K and the stability of closedness under bounded perturbations [97, Chap. III, Problem 5.6] imply that $\Lambda + T = K - (L - T)$ is closed (and also densely defined). Hence, $\mathrm{Ran}(\Lambda + T - \lambda I)^\perp = \mathrm{Ker}((\Lambda + T - \lambda I)^*) = \{0\}$, meaning that $\Lambda + T - \lambda I$ is invertible. If $f \in L^2_{x,v}$ is given, there exists $u \in \mathrm{Dom}(\Lambda + T)$ such that $(\Lambda + T - \lambda I)u = f$, which translates into

$$(\nu_j - \Re(\lambda))u_j + (T - \mathrm{i}\Im(\lambda))u_j = f, \quad j = 1, \dots, n. \tag{4.3.5}$$

We point out that, since ν_j is continuous, $\mathrm{Im}(\nu_j)$ is an interval (or a point, in case that ν_j is constant). This fact and the assumption $\Re(\lambda) \notin \overline{J}$ imply that either $\nu_j - \Re(\lambda) > 0$ in $\mathbb{R}^3$ or $\nu_j - \Re(\lambda) < 0$ in $\mathbb{R}^3$. This means that the sign s_j of $\nu_j - \Re(\lambda)$ is constant in $\mathbb{R}^3$, for $j = 1, \dots, n$. By multiplying (4.3.5) by $s_j \overline{u}_j$, integrating over $\mathbb{T}^3 \times \mathbb{R}^3$, taking the real part, and summing over $j = 1, \dots, n$, we find that

$$\sum_{j=1}^n \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} |\nu_j - \Re(\lambda)||u_j|^2 dxdv = \frac{1}{2}\sum_{j=1}^n s_j \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \big(\overline{u}_j f_j + u_j \overline{f}_j\big) dxdv.$$

The (real part of the) second term in (4.3.5) vanishes after integration; see (4.3.4). Since $\lambda \in \mathbb{R}\backslash\overline{J}$, by definition of J, there exists $c_\lambda > 0$ such that $|\nu_j - \Re(\lambda)| \ge c_\lambda$ in $\mathbb{R}^3$ for all $j = 1, \dots, n$. Then the Cauchy-Schwarz inequality shows that $\|u\|_{L^2_{x,v}} \le c_\lambda^{-1} \|f\|_{L^2_{x,v}}$. This means that $(\Lambda + T - \lambda I)^{-1}$ is bounded, so $\lambda \in \rho(\Lambda + T)$.

Steps 3 and 4 show that $\sigma_{\mathrm{ess}}(\Lambda + T) = \{\lambda \in \mathbb{C} : \Re(\lambda) \in \overline{J}\}$.

Step 5: $\sigma_{ess}(-\Lambda - T) = \sigma_{ess}(K - \Lambda - T)$. The operator K is compact on L^2_v but not on $L^2_{x,v}$, so the claim does not follow from the original form of Weyl's theorem. Instead we will employ the fact that the essential spectrum is conserved under a relatively compact perturbation [97, Section IV.5.6, Theorem 5.35]. More precisely, we prove that K is relatively compact with respect to $\Lambda + T$, i.e., $G_z := K(\Lambda + T - zI)^{-1}$ is compact on $L^2_{x,v}$ for some $z \in \mathbb{C}$ with $\Re(z) \in \mathbb{R}\backslash\overline{J}$. (Notice that by Step 4, $z \in \rho(\Lambda + T)$.) Then

$$\sigma_{\mathrm{ess}}(K - \Lambda - T) = \sigma_{\mathrm{ess}}(-\Lambda - T) = \{\lambda \in \mathbb{C} : \Re(\lambda) \in -J\}. \tag{4.3.6}$$

The second identity is a consequence of Steps 3 and 4.

To prove the compactness of G_z, we introduce the space $W := \ell^2(\mathbb{Z}^3; L^2_v)$ of sequences $f = (f_m) \subset L^2_v$ with the canonical norm $\|f\|_W = (\sum_{m\in\mathbb{Z}^3} \|f_m\|^2_{L^2_v})^{1/2}$. Clearly, W is a Hilbert space with the scalar product $(f, g)_W = \sum_{m\in\mathbb{Z}^3}(f_m, g_m)_{L^2_v}$. Furthermore, we introduce the Fourier mapping $F : L^2_{x,v}(\mathbb{T}^3 \times \mathbb{R}^3) \to W$ by

$$F(f) = (\widehat{f}_m), \quad \widehat{f}_m(v) = \int_{\mathbb{T}^3} e^{-2\pi\mathrm{i}m\cdot x} f(x, v)dx \quad \text{for } m \in \mathbb{Z}^3,\ v \in \mathbb{R}^3.$$

This mapping is bounded, invertible, and has a bounded inverse. We wish to show that $\widehat{G}_z = FG_zF^{-1} : W \to W$ is compact. Then also $G_z = F^{-1}\widehat{G}_zF$ is compact as a composition of a compact and two bounded operators. This idea is due to Ukai; see e.g. [135, Section 2.2.1].

Since K and Λ do not depend on x, it holds that $\widehat{G}_z = K(\Lambda+\widehat{T}-z)^{-1}$, where $\widehat{T} = 2\pi\mathrm{i}v\cdot m$. Let $(f^{(k)}) = (f_m^{(k)}) \subset W$ be a bounded sequence in W, i.e., there exists $c_0 > 0$ such that for all $k \in \mathbb{N}$,

$$\|f^{(k)}\|_W^2 = \sum_{m\in\mathbb{Z}^3} \|f_m^{(k)}\|_{L_v^2}^2 \le c_0. \tag{4.3.7}$$

As $\Re(z) \in \mathbb{R}\backslash\overline{J}$, there is a constant $c_z > 0$ such that for all $i = 1,\dots,n$ and $v \in \mathbb{R}^3$, $|\nu_i(v) + 2\pi\mathrm{i}v\cdot m - z| \ge c_z$. Thus,

$$\begin{aligned}\|(\Lambda+\widehat{T}-z)^{-1}f_m^{(k)}\|_{L_v^2}^2 &= \sum_{i=1}^n \int_{\mathbb{R}^3} \big|(\nu_i(v)+2\pi\mathrm{i}v\cdot m - z)^{-1}f_{m,i}^{(k)}\big|^2 dv \\ &\le c_z^{-2}\sum_{i=1}^n\int_{\mathbb{R}^3}|f_{m,i}^{(k)}|^2dv = c_z^{-2}\|f_m^{(k)}\|_{L_v^2}^2.\end{aligned}$$

Summing these inequalities over $m \in \mathbb{Z}^3$, we infer that

$$\|(\Lambda+\widehat{T}-z)^{-1}f^{(k)}\|_W^2 \le c_z^{-2}\|f^{(k)}\|_W^2 \le c_0c_z^{-2}.$$

Consequently, the sequence $g^{(k)} := (\Lambda+\widehat{T}-z)^{-1}f^{(k)}$ is bounded in W. In particular, for any $s \in \mathbb{Z}^3$, $\|g_s^{(k)}\|_{L_v^2}^2 \le \sum_{m\in\mathbb{Z}^3}\|g_m^{(k)}\|_{L_v^2}^2 = \|g^{(k)}\|_W^2 \le c_0c_z^{-2}$. Hence, for any $s \in \mathbb{Z}^3$, the sequence $(g_s^{(k)}) \subset L_v^2$ is bounded in L_v^2. Since $K : L_v^2 \to L_v^2$ is compact and $\mathbb{Z}^3$ is countable, we may apply Cantor's diagonal argument to find a subsequence $(g^{(k_\ell)})$ of $(g^{(k)})$ such that $(K(g_m^{(k_\ell)}))$ is convergent in L_v^2 as $\ell \to \infty$, for all $m \in \mathbb{Z}^3$.

We will show that $(\widehat{G}_z(f^{(k_\ell)}))$ is a Cauchy sequence in W. To this end, let ℓ, s, $N \in \mathbb{N}$. We write

$$\begin{aligned}\|\widehat{G}_z(f^{(k_\ell)}) - \widehat{G}_z(f^{(k_s)})\|_W^2 &= \sum_{m\in\mathbb{Z}^3}\|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2 \\ &= \sum_{|m|\le N}\|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2 + \sum_{|m|>N}\|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2,\end{aligned} \tag{4.3.8}$$

where $|m| = \sum_{i=1}^3|m_i|$ for all $m \in \mathbb{Z}^3$. First, we consider the second sum on the right-hand side. Denote by $\|\cdot\|_{\mathscr{L}(L_v^2)}$ the norm in the space of linear bounded operators on L_v^2. By the definition of $g_m^{(k)}$, we obtain

$$\begin{aligned}\sum_{|m|>N}\|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2 &= \sum_{|m|>N}\|K(\Lambda+2\pi\mathrm{i}v\cdot m - z)^{-1}(f_m^{(k_\ell)} - f_m^{(k_s)})\|_{L_v^2}^2 \\ &\le 2\sum_{|m|>N}\|K(\Lambda+2\pi\mathrm{i}v\cdot m-z)^{-1}\|_{\mathscr{L}(L_v^2)}^2\big(\|f_m^{(k_\ell)}\|_{L_v^2}^2 + \|f_m^{(k_s)}\|_{L_v^2}^2\big).\end{aligned} \tag{4.3.9}$$

For the operator norm, we employ Prop. 2.2.6 in [135], which can be applied since $\Re(z) \in \mathbb{R}\backslash\overline{J}$:

$$\|K(\Lambda + 2\pi \mathrm{i} v \cdot m - z)^{-1}\|^2_{\mathscr{L}(L^2_v)} \le c_1(1+|m|)^{-\alpha} \quad \text{for all } m \in \mathbb{Z}^3$$

for some suitable constant $c_1 > 0$ (depending on z) and a suitable exponent $\alpha \in (0,1)$ (actually, $\alpha = 4/13$). Let $0 < \beta < 2\alpha/3$. By Hölder's inequality and (4.3.7), we estimate

$$\begin{aligned}\sum_{|m|>N} \|K(\Lambda + 2\pi \mathrm{i} v \cdot m - z)^{-1}\|^2_{\mathscr{L}(L^2_v)} \|f_m^{(k)}\|^2_{L^2_v} &\le c_0^{\beta/2} c_1 \sum_{|m|>N} (1+|m|)^{-\alpha} \|f_m^{(k)}\|^{2-\beta}_{L^2_v} \\ &\le c_0^{\beta/2} c_1 \bigg(\sum_{|m|>N} (1+|m|)^{-2\alpha/\beta} \bigg)^{\beta/2} \bigg(\sum_{|m|>N} \|f_m^{(k)}\|^2_{L^2_v} \bigg)^{1-\beta/2} \\ &\le c_0 c_1 \bigg(\sum_{|m|>N} (1+|m|)^{-2\alpha/\beta} \bigg)^{\beta/2}.\end{aligned}$$

Using this estimate in (4.3.9), it follows that

$$\sup_{\ell, s \in \mathbb{N}} \sum_{|m|>N} \|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|^2_{L^2_v} \le 2c_0 c_1 \bigg(\sum_{|m|>N} (1+|m|)^{-2\alpha/\beta} \bigg)^{\beta/2}.$$

The choice of β implies that $2\alpha/\beta > 3$ and hence, the sum over $|m| > N$ is finite. In particular, $\sum_{|m|>N} (1+|m|)^{-2\alpha/\beta} \to 0$ as $N \to \infty$. As a consequence, for given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\sup_{\ell, s \in \mathbb{N}} \sum_{|m|>N_\varepsilon} \|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|^2_{L^2_v} < \frac{\varepsilon}{2}.$$

Finally, since $(K(g_m^{(k_\ell)}))$ is convergent in L^2_v for all $m \in \mathbb{Z}^3$, there is a number $\eta = \eta(\varepsilon) > 0$ such that for all ℓ, $s > \eta$,

$$\sum_{|m|\le N_\varepsilon} \|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|^2_{L^2_v} < \frac{\varepsilon}{2}.$$

Thus, choosing $N = N_\varepsilon$ in (4.3.8), we deduce that $(\widehat{G}_z(f^{(k_s)}))$ is a Cauchy sequence in the Hilbert space W and consequently, it is convergent. This shows that $\widehat{G}_z : W \to W$ is a compact operator and (4.3.6) holds. This finishes the proof of Theorem 4.1. □

Next, we show the spectral-gap estimate for the linearized collision operator $L = K - \Lambda$, i.e. the first statement of Theorem 4.2. Since K is compact on L^2_v, it remains to prove that $\Lambda : \mathcal{D} \subset L^2_v \to L^2_v$ is coercive.

Lemma 4.8. *Let (A1)-(A4) hold. Then the embedding $\mathcal{H} \hookrightarrow L^2_v$ is continuous and $\Lambda : \mathcal{D} \to L^2_v$, defined in (4.3.1), is a linear unbounded operator with the property*

$$(f, \Lambda(f))_{L^2_v} = \|f\|^2_{\mathcal{H}} \ge C\|f\|^2_{L^2_v} \quad \text{for } f \in \mathcal{H} \tag{4.3.10}$$

for some $C > 0$. Moreover, Λ can be extended by density to a linear bounded operator $\Lambda : \mathcal{H} \to \mathcal{H}'$, where $\mathcal{H}'$ is the dual of $\mathcal{H}$ with respect to the L^2_v scalar product. In particular, the mapping $\mathcal{H} \to \mathbb{R}$, $f \mapsto \langle \Lambda(f), f\rangle$ is continuous, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{H}'$ and $\mathcal{H}$.

Proof. The strict positivity of ν_i in $\mathbb{R}^3$ (see Lemma 4.6) implies that the embedding $\mathcal{H} \hookrightarrow L^2_v$ is continuous. Then the definitions of Λ_i and $\mathcal{H}$ show that for all $f \in \mathcal{H}$, (4.3.10) holds. For given $f \in \mathcal{H}$, the element $\Lambda(f) = (f_1\nu_1, \ldots, f_n\nu_n)$ can be identified with the linear bounded operator $\mathcal{H} \to \mathbb{R}$, $g \mapsto \sum_{i=1}^n \int_{\mathbb{R}^3} g_i f_i \nu_i dv$ and consequently, $\Lambda(f) \in \mathcal{H}'$. It is immediate to see that $\|\Lambda(f)\|_{\mathcal{H}'} = \|f\|_{\mathcal{H}}$, so that $\Lambda : \mathcal{H} \to \mathcal{H}'$ is isometric and thus bounded. Moreover, it follows that $\mathcal{H} \to \mathbb{R}$, $f \mapsto \langle \Lambda(f), f\rangle$, is continuous. □

The following result provides a spectral gap for general operators which decompose into a compact and a coercive part.

Lemma 4.9. *Let $\mathcal{H}_0$ and $\mathcal{H}$ be Hilbert spaces such that $\mathcal{H} \hookrightarrow \mathcal{H}_0$ continuously and let $L : \mathcal{H} \to \mathcal{H}'$ be a linear bounded operator such that $L = K - \Lambda$ with linear bounded operators $\Lambda : \mathcal{H} \to \mathcal{H}'$ and $K : \mathcal{H}_0 \to \mathcal{H}_0$. Furthermore, assume that*

(i) for all $f \in \mathcal{H}$, $\langle L(f), f\rangle \le 0$ with equality holding if and only if $f \in \operatorname{Ker}(L)$;

(ii) the operator $K : \mathcal{H}_0 \to \mathcal{H}_0$ is compact;

(iii) there exists $C_0 > 0$ such that for all $f \in \mathcal{H}$, $\langle \Lambda(f), f\rangle \ge C_0\|f\|^2_{\mathcal{H}}$.

Then there exists a constant $C_1 > 0$ such that

$$-\langle L(f), f\rangle \ge C_1\|f\|^2_{\mathcal{H}} \quad \textit{for all } f \in \mathcal{H} \cap \operatorname{Ker}(L)^\perp.$$

Proof. We argue by contradiction. Let $(f_n) \subset \mathcal{H} \cap \operatorname{Ker}(L)^\perp$ be a sequence such that $\|f_n\|_{\mathcal{H}} = 1$ for $n \ge 1$ but $\langle L(f_n), f_n\rangle \to 0$ as $n \to \infty$. Since (f_n) is bounded in the Hilbert space $\mathcal{H}$, there exists a subsequence, which is not relabeled, such that $f_n \rightharpoonup f$ weakly in $\mathcal{H}$. Because of the continuous embedding $\mathcal{H} \hookrightarrow \mathcal{H}_0$, also $f_n \rightharpoonup f$ weakly in $\mathcal{H}_0$. Since $f_n \in \operatorname{Ker}(L)^\perp$ and $\operatorname{Ker}(L)^\perp$ is weakly closed by Mazur's lemma, $f \in \operatorname{Ker}(L)^\perp$. As the operator $K : \mathcal{H}_0 \to \mathcal{H}_0$ is compact, by hypothesis (ii), the weak convergence of (f_n) in $\mathcal{H}_0$ implies that $K(f_n) \to K(f)$ strongly in $\mathcal{H}_0$. Hence, $(f_n, K(f_n))_{\mathcal{H}_0} \to (f, K(f))_{\mathcal{H}_0}$. Since $\Lambda : \mathcal{H} \to \mathcal{H}'$ is bounded, the mapping $P : \mathcal{H} \to \mathbb{R}$, $f \mapsto \langle \Lambda(f), f\rangle$, is continuous. The linearity of Λ and property (iii) imply that P is also convex. Thus, P is weakly lower semicontinuous [24, Corollary 3.9]. Therefore,

$$-\langle L(f), f\rangle = \langle \Lambda(f), f\rangle - (K(f), f)_{\mathcal{H}_0} \le \liminf_{n\to\infty} \big(\langle \Lambda(f_n), f_n\rangle - (K(f_n), f_n)_{\mathcal{H}_0}\big) = 0,$$

because $\langle L(f_n), f_n\rangle \to 0$ as $n \to \infty$ by assumption. We infer from hypothesis (i) that $f \in \operatorname{Ker}(L)$. But also $f \in \operatorname{Ker}(L)^\perp$, so $f = 0$. Then, by hypothesis (iii),

$$0 < C_0 = C_0\|f_n\|^2_{\mathcal{H}} \le \langle \Lambda(f_n), f_n\rangle = (K(f_n), f_n)_{\mathcal{H}_0} - \langle L(f_n), f_n\rangle \to 0,$$

which is a contradiction. □

Let $\mathcal{H}_0 = L^2_v$. By [19, Prop. 2], assumption (ii) of Lemma 4.9 holds. Furthermore, Lemma 4.8 shows that (iii) holds true. Assumption (i) is a consequence of Lemma 4.4. Let $f \in \mathcal{D} \subset \mathcal{H}$ und set $\widetilde{f} = f - \Pi^L(f) \in \operatorname{Ker}(L)^\perp$. Then

$$-\langle L(f), f\rangle = \langle L(\widetilde{f}), \widetilde{f}\rangle \ge C\|\widetilde{f}\|^2_{\mathcal{H}} = C\|f - \Pi^L(f)\|^2_{\mathcal{H}},$$

since $L(f) \in L^2_v$ and $\langle L(f), f\rangle = (f, L(f))_{L^2_v}$ for $f \in \mathcal{D}$. This proves the first statement in Theorem 4.2.

4.4. Explicit spectral-gap estimate

We present a second proof of the spectral-gap estimate (4.1.7) with explicit constants. The idea is to decompose the collision operator L into a mono-species and a multi-species part and to exploit the fact that the conservation properties of L are different from those of the mono-species part L^m. Let assumptions (A1)-(A4) hold.

4.4.1. Decomposition

We decompose $L = L^m + L^b$, where $L^m = (L_1^m, \dots, L_n^m)$, $L^b = (L_1^b, \dots, L_n^b)$, and

$$L_i^m(f_i) = L_{ii}(f_i, f_i), \quad L_i^b(f) = \sum_{j \neq i} L_{ij}(f_i, f_j). \tag{4.4.1}$$

Denoting by Π^m the orthogonal projection onto $\mathrm{Ker}(L^m)$ (with respect to the scalar product in L_v^2), we can decompose f according to

$$f = f^{\parallel} + f^{\perp}, \quad \text{where } f^{\parallel} := \Pi^m(f), \quad f^{\perp} := f - f^{\parallel}. \tag{4.4.2}$$

Lemma 4.4 shows that

$$f \in \mathrm{Ker}(L) \text{ if and only if } f_i = M_i^{1/2}(\alpha_i + u \cdot v + e|v|^2) \text{ for } \alpha_i, e \in \mathbb{R},\ u \in \mathbb{R}^3, \tag{4.4.3}$$

$$f \in \mathrm{Ker}(L^m) \text{ if and only if } f_i = M_i^{1/2}(\alpha_i + u_i \cdot v + e_i|v|^2) \text{ for } \alpha_i, e_i \in \mathbb{R},\ u_i \in \mathbb{R}^3, \tag{4.4.4}$$

and $f^{\parallel}$ has clearly the form (4.4.4).

For later use, we define the following bilinear forms

$$-(f, L^m(f))_{L_v^2} = \frac{1}{4}\sum_{i=1}^n \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ii}\Delta_i[h_i]^2 M_i M_i^* dv dv^* d\sigma, \tag{4.4.5}$$

$$-(f, L^b(f))_{L_v^2} = \frac{1}{4}\sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i, h_j]^2 M_i M_j^* dv dv^* d\sigma, \tag{4.4.6}$$

where $h_i = M_i^{-1/2} f_i$ and

$$\Delta_i[h_i] := h_i' + h_i'^* - h_i - h_i^*, \quad A_{ij}[h_i, h_j] := h_i' + h_j'^* - h_i - h_j^*.$$

4.4.2. Spectral-gap estimate for the mono-species part

Our starting point is the fact that the mono-species collision operator L^m has an explicitly computable spectral gap. A spectral-gap estimate for the linearized collision operator with $n = 1$ was proved in [110, Theorem 6.1, Remark 1]:

$$\frac{1}{4}\int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ii}\Delta_i[h_i]^2 M_i M_i^* dv dv^* d\sigma \geq \frac{\lambda_m}{\rho_{\infty,i}} \int_{\mathbb{R}^3} (f_i - \Pi^m(f_i))^2 \nu_{ii} dv, \tag{4.4.7}$$

where $\lambda_m = \lambda_m(\gamma, C_1, C^b) > 0$, only depending on γ, C_1, and C^b (see (A3)-(A4)), can be computed explicitly,

$$\nu_{ii}(v) := \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ii}(|v-v^*|,\cos\vartheta)M_i^* dv^* d\sigma,$$

and $i \in \{1,\dots,n\}$ is fixed. This yields the following estimate for L^m, where we recall that the space $\mathcal{H}$ is defined in (4.1.6).

Lemma 4.10. *With L^m defined in* (4.4.1)*, we have*

$$-(f, L^m(f))_{L^2_v} \geq C^m \|f - \Pi^m(f)\|^2_{\mathcal{H}} \quad \text{for all } f \in \mathrm{Dom}(L^m),$$

where

$$C^m = \frac{\lambda^m(\gamma, C_1, C^b)}{\beta\rho_\infty}, \tag{4.4.8}$$

and $\lambda^m = \lambda^m(\gamma, C_1, C^b)$ is given in (4.4.7).

Proof. We sum (4.4.7) over $i = 1,\dots,n$ and employ (4.4.5) to obtain

$$-(f, L^m(f))_{L^2_v} \geq \lambda^m \sum_{i=1}^n \int_{\mathbb{R}^3} (f_i - \Pi^m(f))^2 \frac{\nu_{ii}}{\rho_{\infty,i}} dv. \tag{4.4.9}$$

It remains to estimate ν_{ii} in terms of ν_i, defined in (4.1.4). The definition of M_i implies that $M_j = (\rho_{\infty,j}/\rho_{\infty,i})M_i$. This fact, as well as definition (4.1.4) of ν_i, the lower bound (4.2.3), and assumption (A6) give

$$\nu_i = \sum_{j=1}^n \frac{\rho_{\infty,j}}{\rho_{\infty,i}} \int_{\mathbb{R}^3} B_{ij} M_i^* dv^* d\sigma \leq \beta \sum_{j=1}^n \frac{\rho_{\infty,j}}{\rho_{\infty,i}} \int_{\mathbb{R}^3} B_{ii} M_i^* dv^* d\sigma = \frac{\beta\rho_\infty}{\rho_{\infty,i}} \nu_{ii}.$$

We conclude that $\nu_{ii}/\rho_{\infty,i} \geq \nu_i/(\beta\rho_\infty)$, and inserting this bound into (4.4.9) yields the result. □

Lemma (4.10) and the inequality $-(f, L^b(f))_{L_2^v} \geq 0$ immediately show that

$$-(f, L(f))_{L^2_v} \geq C^m \|f - \Pi^m(f)\|^2_{\mathcal{H}} \quad \text{for all } f \in \mathcal{D}.$$

However, we need the projection onto $\mathrm{Ker}(L)^\perp$ instead of $\mathrm{Ker}(L^m)^\perp$, which is contained in $\mathrm{Ker}(L)^\perp$. Therefore, we will exploit the part $-(f, L^b(f))_{L^2_v}$ to derive a sharper estimate.

4.4.3. Absorption of the orthogonal parts

We prove that the contribution $f^\perp$ (introduced in (4.4.2)) in the term $-(f, L^b(f))_{L^2_v} = -(f^\| + f^\perp, L^b(f^\| + f^\perp))_{L^2_v}$ can be absorbed by the $\mathcal{H}$ norm of $f^\perp$.

Lemma 4.11. *Let $\eta = \min\{1, C^m/8\}$, where $C^m > 0$ is given in Lemma 4.10. Then, for all $f \in \mathcal{D}$,*

$$-(f, L(f))_{L^2_v} \geq (C^m - 4\eta)\|f - f^\|\|^2_{\mathcal{H}} - \frac{\eta}{2}(f^\|, L^b(f^\|))_{L^2_v},$$

where $f^\| = \Pi^m(f)$ is the projection onto $\mathrm{Ker}(L^m)^\perp$.

Proof. By Lemma 4.10, we find that

$$-\left(f, L(f)\right)_{L^2_v} \geq C^m \|f - f^{\|}\|^2_{\mathcal{H}} - (f, L^b(f))_{L^2_v} \geq C^m \|f - f^{\|}\|^2_{\mathcal{H}} - \eta (f, L^b(f))_{L^2_v}, \quad (4.4.10)$$

since $-(1-\eta)(f, L^b(f))_{L^2_v} \geq 0$ for $\eta \in (0,1]$. We estimate first the expression $A_{ij}[h_i, h_j]$ in definition (4.4.6), writing $h_i^{\|} = M_i^{-1/2} f_i^{\|}$ and $h_i^{\perp} = M_i^{-1/2} f_i^{\perp}$,

$$\begin{aligned} A_{ij}[h_i, h_j]^2 &= \left(A_{ij}[h_i^{\|}, h_j^{\|}] + A_{ij}[h_i^{\perp}, h_j^{\perp}]\right)^2 \\ &= A_{ij}[h_i^{\|}, h_j^{\|}]^2 + A_{ij}[h_i^{\perp}, h_j^{\perp}]^2 + 2A_{ij}[h_i^{\|}, h_j^{\|}] A_{ij}[h_i^{\perp}, h_j^{\perp}] \\ &\geq \frac{1}{2} A_{ij}[h_i^{\|}, h_j^{\|}]^2 - A_{ij}[h_i^{\perp}, h_j^{\perp}]^2. \end{aligned}$$

Inserting this estimate into (4.4.6) and (4.4.10) gives

$$\begin{aligned} -(f, L(f))_{L^2_v} \geq C^m \|f^{\perp}\|^2_{\mathcal{H}} &+ \frac{\eta}{8} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i^{\|}, h_j^{\|}]^2 M_i M_j^* dv dv^* d\sigma \\ &- \frac{\eta}{4} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i^{\perp}, h_j^{\perp}]^2 M_i M_j^* dv dv^* d\sigma. \quad (4.4.11) \end{aligned}$$

We claim that the last term on the right-hand side can be estimated from below by $\|f^{\perp}\|^2_{\mathcal{H}}$, up to a small factor. For this, we employ the invariance properties of B_{ij} and the identity $M_i M_j^* = M_i' M_j'^*$:

$$\begin{aligned} &\int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i^{\perp}, h_j^{\perp}]^2 M_i M_j^* dv dv^* d\sigma \\ &\quad \leq 4 \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} \left(((h_i^{\perp})')^2 + ((h_j^{\perp})'^*)^2 + (h_i^{\perp})^2 + ((h_j^{\perp})^*)^2 \right) M_i M_j^* dv dv^* d\sigma \\ &\quad \leq 16 \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (h_i^{\perp})^2 M_i M_j^* dv dv^* d\sigma = 16 \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (f_i^{\perp})^2 M_j^* dv dv^* d\sigma. \end{aligned}$$

Thus, the last term on the right-hand side of (4.4.11) can be estimated as

$$\begin{aligned} &-\frac{\eta}{4} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i^{\perp}, h_j^{\perp}]^2 M_i M_j^* dv dv^* d\sigma \\ &\quad \geq -4\eta \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (f_i^{\perp})^2 M_j^* dv dv^* d\sigma \geq -4\eta \sum_{i=1}^n \int_{\mathbb{R}^3} (f_i^{\perp})^2 \nu_i dv = -4\eta \|f^{\perp}\|^2_{\mathcal{H}}, \end{aligned}$$

taking into account definition (4.1.4) of ν_i. We infer from (4.4.11) that

$$-(f, L(f))_{L^2_v} \geq (C^m - 4\eta) \|f - f^{\|}\|^2_{\mathcal{H}} + \frac{\eta}{8} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i^{\|}, h_j^{\|}]^2 M_i M_j^* dv dv^* d\sigma,$$

and definition (4.4.6) yields the conclusion. □

4.4.4. Estimate for the remaining part

It remains to estimate the term $-(f^{\|}, L^b(f^{\|}))_{L^2_v}$.

Lemma 4.12. *For* $f^{\|} \in \mathrm{Ker}(L^m)$*, i.e.* $f_i^{\|} = M_i^{1/2}(\alpha_i + u_i \cdot v + e_i|v|^2)$ *for some* α_i*,* $e_i \in \mathbb{R}$ *and* $u_i \in \mathbb{R}^3$*, we have*

$$-(f^{\|}, L^b(f^{\|}))_{L^2_v} \geq \frac{D^b}{4} \sum_{i,j=1}^{n} \big(|u_i - u_j|^2 + (e_i - e_j)^2\big),$$

where $D^b > 0$ *is defined in* (4.4.13).

Proof. Thanks to the momentum and energy conservation, we obtain differences of the momenta and energies, which will be crucial in the following:

$$u_i \cdot v' + u_j \cdot v'^* - u_i \cdot v - u_j \cdot v^* = (u_i - u_j) \cdot (v' - v),$$
$$e_i|v'|^2 + e_j|v'^*|^2 - e_i|v|^2 - e_j|v^*|^2 = (e_i - e_j)(|v'|^2 - |v|^2).$$

Using these identities in $A_{ij}[h_i^{\|}, h_j^{\|}]$, where $h_i^{\|} = \alpha_i + u_i \cdot v + e_i|v|^2$, we find that

$$\begin{aligned}
-(f^{\|}, L^b(f^{\|}))_{L^2_v} &= \frac{1}{4}\sum_{i=1}^{n}\sum_{j\neq i}\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij} A_{ij}[h_i^{\|}, h_j^{\|}]^2 M_i M_j^* dv dv^* d\sigma \\
&= \frac{1}{4}\sum_{i=1}^{n}\sum_{j\neq i}\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}\big((u_i - u_j)\cdot(v' - v) + (e_i - e_j)(|v'|^2 - |v|^2)\big)^2 M_i M_j^* dv dv^* d\sigma.
\end{aligned}$$

Using the symmetry of B_{ij} (thanks to assumption (A1)) and of $M_i M_j^*$ with respect to v, the function $G(v, v^*, \sigma) = B_{ij}(u_i - u_j)\cdot(v' - v)(|v'|^2 - |v|^2)$ is odd with respect to (v, v^*, σ) and thus, the mixed term of the square in the above integral vanishes. Therefore, we obtain

$$\begin{aligned}
-(f^{\|}, L^b(f^{\|}))_{L^2_v} = \frac{1}{4}\sum_{i=1}^{n}\sum_{j\neq i}\int_{\mathbb{R}^6\times\mathbb{S}^2} &B_{ij}\big(|(u_i - u_j)\cdot(v' - v)|^2 + (e_i - e_j)^2(|v'|^2 - |v|^2)^2\big) \\
&\times M_i M_j^* dv dv^* d\sigma. \qquad (4.4.12)
\end{aligned}$$

Now, we claim that

$$\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}((u_i - u_j)\cdot(v' - v))^2 M_i M_j^* dv dv^* d\sigma = \frac{|u_i - u_j|^2}{3}\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}|v - v'|^2 M_i M_j^* dv dv^* d\sigma.$$

To prove this identity, we write $u_{i,k}$ and v_k for the kth component of the vectors u_i and v, respectively. The transformation $(v_k, v_k^*, \sigma_k) \mapsto -(v_k, v_k^*, \sigma_k)$ for fixed k leaves B_{ij}, M_i, and M_j^* unchanged but $v_k' \mapsto -v_k'$ such that

$$\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij} v_k' v_\ell M_i M_j^* dv dv^* d\sigma = 0 \quad \text{for } \ell \neq k.$$

Furthermore,

$$\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij} v_k v_\ell M_i M_j^* dv dv^* d\sigma = 0 \quad \text{for } \ell \neq k,$$

since the integrand is odd. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^6\times\mathbb{S}^2} & B_{ij}((u_i - u_j)\cdot(v'-v))^2 M_i M_j^* dv dv^* d\sigma \\ &= \sum_{k,\ell=1}^{3} (u_{i,k}-u_{j,k})(u_{i,\ell}-u_{j,\ell}) \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}(v_k'-v_k)(v_\ell'-v_\ell) M_i M_j^* dv dv^* d\sigma \\ &= \sum_{k=1}^{3} (u_{i,k}-u_{j,k})^2 \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}(v_k - v_k')^2 M_i M_j^* dv dv^* d\sigma.\end{aligned}$$

In fact, we can see that the integral is independent of k, and we infer that

$$\begin{aligned}\int_{\mathbb{R}^6\times\mathbb{S}^2} & B_{ij}((u_i - u_j)\cdot(v'-v))^2 M_i M_j^* dv dv^* d\sigma \\ &= \frac{1}{3}\sum_{k=1}^{3} (u_{i,k}-u_{j,k})^2 \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}|v-v'|^2 M_i M_j^* dv dv^* d\sigma,\end{aligned}$$

from which the claim follows.

Hence, (4.4.12) can be estimated as

$$-(f^{\parallel}, L^b(f^{\parallel}))_{L^2_v} \geq D^b \sum_{i,j=1}^{3} \big(|u_i - u_j|^2 + (e_i - e_j)^2\big),$$

where

$$D^b = \min_{1\leq i,j\leq n} \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij} \min\left\{\frac{1}{3}|v-v'|^2, (|v'|^2 - |v|^2)^2\right\} M_i M_j^* dv dv^* d\sigma. \tag{4.4.13}$$

It remains to show that $D^b > 0$. The integrand of (4.4.13) vanishes if and only if $|v'| = |v|$. However, the set

$$X = \{(v, v^*, \sigma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 : |v'| = |v|\}$$

is closed since it is the pre-image of $\{0\}$ of the continuous function $F(v, v^*, \sigma) = |v'|^2 - |v|^2$, i.e. $X = F^{-1}(\{0\})$, recalling that v' depends on (v, v^*, σ) through (2.1.2). Since $X \neq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$, its complement X^c is open and nonempty and thus has positive Lebesgue measure. Since the integrand in (4.4.13) is positive on X^c, we infer that $D^b > 0$. This finishes the proof. □

4.4.5. Estimate for the momentum and energy differences

The last step is to derive lower bounds for the differences $\sum_{i,j}(|u_i - u_j|^2 + (e_i - e_j)^2)$. First, we recall some moment identities:

$$\int_{\mathbb{R}^3} M_i dv = \rho_i, \quad \int_{\mathbb{R}^3} M_i v_j v_k dv = \rho_i \delta_{jk}, \quad \int_{\mathbb{R}^3} M_i |v|^4 dv = 15\rho_i \tag{4.4.14}$$

for all $1 \leq i \leq n$ and $1 \leq j, k \leq 3$.

Lemma 4.13. *Let $f \in L^2_v$ with $f_i^{\|} = M_i^{1/2}(\alpha_i + u_i \cdot v + e_i|v|^2)$ for $1 \le i \le n$. Then*

$$\int_{\mathbb{R}^3} M_i^{1/2} f_i dv = \rho_i(\alpha_i + 3e_i), \quad \int_{\mathbb{R}^3} M_i^{1/2} f_i v dv = \rho_i u_i, \quad \int_{\mathbb{R}^3} M_i^{1/2} f_i |v|^2 dv = \rho_i(3\alpha_i + 15e_i).$$

Proof. Decomposing $f = f^{\|} + f^{\perp}$, where $f^{\|} = \Pi^m(f)$ and $f^{\perp} = f - \Pi^m(f)$, we infer from $M_i^{1/2} \in \mathrm{Ker}(L^m)$ (see (4.4.4)) that $(M_i^{1/2}, f_i^{\perp})_{L^2_v} = 0$ and hence, by (4.4.4) again,

$$(M_i^{1/2}, f_i^{\|})_{L^2_v} = \int_{\mathbb{R}^3} M_i(\alpha_i + u_i \cdot v + e_i|v|^2) dv = \rho_i(\alpha_i + 3e_i).$$

The other identities can be shown in a similar way. □

Lemma 4.14. *For all $f \in \mathcal{D}$, we have*

$$\sum_{i,j=1}^{n} \big(|u_i - u_j|^2 + (e_i - e_j)^2\big) \ge \frac{1}{C_k}\big(\|f - \Pi^L(f)\|^2_{\mathcal{H}} - 2\|f - \Pi^m(f)\|^2_{\mathcal{H}}\big),$$

where u_i, e_i are the coefficients of the ith component of $\Pi^m(f)$ in (4.4.4), *Π^L is the projection on* $\mathrm{Ker}(L)$, *$C_k > 0$ is given by*

$$C_k = 60 n \rho_\infty \max_{1 \le k,\ell \le 5n} \left| \sum_{i=1}^{n} \int_{\mathbb{R}^3} \psi_k \psi_\ell \nu_i dv \right|, \tag{4.4.15}$$

and (ψ_k) is an arbitrary orthonormal basis of $\mathrm{Ker}(L^m)$ *in L^2_v.*

Proof. We again decompose $f = f^{\|} + f^{\perp}$ with $f^{\|} = \Pi^m(f)$ and $f^{\perp} = f - f^{\|}$. Then

$$\|f - \Pi^L(f)\|^2_{\mathcal{H}} \le 2\big(\|f^{\perp}\|^2_{\mathcal{H}} + \|f^{\|} - \Pi^L(f)\|^2_{\mathcal{H}}\big). \tag{4.4.16}$$

We estimate first the difference $g := f^{\|} - \Pi^L(f) = \Pi^m(f) - \Pi^L(f) \in \mathrm{Ker}(L^m)$ (note that $\mathrm{Ker}(L) \subset \mathrm{Ker}(L^m)$). Let (ψ_k) be an arbitrary orthonormal basis of $\mathrm{Ker}(L^m)$ in L^2_v. Because of (4.4.4) and $\nabla_v \nu_i \in L^\infty(\mathbb{R}^3)$, we have $\psi_k \in \mathcal{H}$. Then, by Young's inequality, we find that

$$\begin{aligned}
\|g\|^2_{\mathcal{H}} &= \sum_{i=1}^{n} \int_{\mathbb{R}^3} \left| \sum_{k=1}^{5n} (g, \psi_k)_{L^2_v} \psi_k \right|^2 \nu_i(v) dv = \sum_{k,\ell=1}^{5n} (g,\psi_k)_{L^2_v} (g,\psi_\ell)_{L^2_v} \sum_{i=1}^{n} \int_{\mathbb{R}^3} \psi_k \psi_\ell \nu_i(v) dv \\
&= \sum_{k,\ell=1}^{5n} (g,\psi_k)_{L^2_v} (g,\psi_\ell)_{L^2_v} (\psi_k, \psi_\ell)_{\mathcal{H}} \\
&\le \frac{1}{2} \max_{1 \le k,\ell \le 5n} |(\psi_k,\psi_\ell)_{\mathcal{H}}| \sum_{k,\ell=1}^{5n} \big((g,\psi_k)^2_{L^2_v} + (g,\psi_\ell)^2_{L^2_v}\big) \\
&= 5n \max_{1 \le k,\ell \le 5n} |(\psi_k,\psi_\ell)_{\mathcal{H}}| \sum_{k=1}^{5n} (g,\psi_k)^2_{L^2_v} = 5n \max_{1 \le k,\ell \le 5n} |(\psi_k,\psi_\ell)_{\mathcal{H}}| \, \|g\|^2_{L^2_v}.
\end{aligned}$$

Thus, we infer from (4.4.16) that

$$\|f - \Pi^L(f)\|_{\mathcal{H}}^2 \le 2\|f^\perp\|_{\mathcal{H}}^2 + 10n \max_{1\le k,\ell\le 5n} |(\psi_k,\psi_\ell)_{\mathcal{H}}| \, \|f^\parallel - \Pi^L(f)\|_{L^2_v}^2.$$

Because of $\mathrm{Ker}(L) \subset \mathrm{Ker}(L^m)$, we have $\Pi^m \Pi^L = \Pi^L$ and

$$\begin{aligned}\|f^\parallel - \Pi^L(f)\|_{L^2_v}^2 &= \|f^\parallel\|_{L^2_v}^2 - 2(\Pi^m(f), \Pi^L(f))_{L^2_v} + \|\Pi^L(f)\|_{L^2_v}^2 \\ &= \|f^\parallel\|_{L^2_v}^2 - 2(f, \Pi^L(f))_{L^2_v} + \|\Pi^L(f)\|_{L^2_v}^2 = \|f^\parallel\|_{L^2_v}^2 - \|\Pi^L(f)\|_{L^2_v}^2.\end{aligned}$$

Consequently, setting $k_0 = 10n \max_{1\le k,\ell\le n} |(\psi_k,\psi_\ell)_{\mathcal{H}}|$,

$$\|f - \Pi^L(f)\|_{\mathcal{H}}^2 \le 2\|f^\perp\|_{\mathcal{H}}^2 + k_0\big(\|f^\parallel\|_{L^2_v}^2 - \|\Pi^L(f)\|_{L^2_v}^2\big). \tag{4.4.17}$$

Next, we compute the L^2_v norms of $f^\parallel$ and $\Pi^L(f)$. Moment identities (4.4.14) show that

$$\begin{aligned}\|f^\parallel\|_{L^2_v}^2 &= \sum_{i=1}^n \int_{\mathbb{R}^3} M_i(\alpha_i + u_i\cdot v + e_i|v|^2)^2 dv \\ &= \sum_{i=1}^n \int_{\mathbb{R}^3} M_i(\alpha_i^2 + (u_i\cdot v)^2 + e_i^2|v|^4 + 2\alpha_i e_i |v|^2) dv \\ &= \sum_{i=1}^n \rho_{\infty,i}(\alpha_i^2 + |u_i|^2 + 15e_i^2 + 6\alpha_i e_i).\end{aligned}$$

For the computation of the L^2_v norm of $\Pi^L(f)$, we choose the following orthonormal basis $(\phi_j) = (\phi_{j,i})_{i=1,\dots,n}$ of $\mathrm{Ker}(L)$ in L^2_v:

$$\phi_{j,i} = \rho_\infty^{-1/2} M_j^{1/2}\delta_{ij}, \quad \phi_{n+k,i} = \rho_\infty^{-1/2} M_i^{1/2} v_k, \quad \phi_{n+4,i} = (6\rho_\infty)^{-1/2} M_i^{1/2}(|v|^2 - 3),$$

where $1 \le j \le n$ and $1 \le k \le 3$. Then, using the moment identities of Lemma 4.13,

$$\|\Pi^L(f)\|_{L^2_v}^2 = \sum_{j=1}^{n+4} (f,\phi_j)_{L^2_v}^2 = \sum_{i=1}^n \rho_{\infty,i}(\alpha_i + 3e_i)^2 + \rho_\infty \left|\sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} u_i\right|^2 + 6\rho_\infty\left(\sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} e_i\right)^2.$$

Inserting the above identities for $\|f^\parallel\|_{L^2_v}^2$ and $\|\Pi^L(f)\|_{L^2_v}^2$ into (4.4.17), we conclude that

$$\begin{aligned}\|f - \Pi^L(f)\|_{\mathcal{H}}^2 \le 2\|f - \Pi^m(f)\|_{\mathcal{H}}^2 &+ k_0\rho_\infty \left(\sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty}|u_i|^2 - \left|\sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} u_i\right|^2\right) \\ &+ 6k_0\rho_\infty\left(\sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} e_i^2 - \left(\sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} e_i\right)^2\right).\end{aligned}$$

Then, if the inequalities

$$\sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty}|u_i|^2 - \left|\sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} u_i\right|^2 \le \sum_{i,j=1}^n |u_i - u_j|^2, \tag{4.4.18}$$

$$\sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} e_i^2 - \left(\sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} e_i \right)^2 \leq \sum_{i,j=1}^n (e_i - e_j)^2 \tag{4.4.19}$$

hold, the lemma follows with $C_k = 6k_0\rho_\infty$.

It remains to prove (4.4.18) and (4.4.19). To this end, we define the following scalar product on $\mathbb{R}^{3n}$:

$$(u,v)_\rho = \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} u_i \cdot v_i, \quad u = (u_1, \ldots, u_n),\, v = (v_1, \ldots, v_n) \in \mathbb{R}^{3n},$$

where $u_i \cdot v_i$ denotes the usual scalar product in $\mathbb{R}^3$. The corresponding norm is $\|u\|_\rho = (u,u)_\rho^{1/2}$. Then $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{3n}$ satisfies $\|\mathbf{1}\|_\rho = 1$. The elementary identity

$$\|u\|_\rho^2 - (u, \mathbf{1})_\rho^2 = \|u - (u,\mathbf{1})_\rho \mathbf{1}\|_\rho^2$$

can be equivalently written as

$$I := \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} |u_i|^2 - \left| \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} u_i \right|^2 = \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} \left| u_i - \sum_{j=1}^n \frac{\rho_{\infty,j}}{\rho_\infty} u_j \right|^2.$$

Then, using $\sum_{j=1}^n \rho_{\infty,j} = \rho_\infty$,

$$\begin{aligned}
I &= \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} \left| \left(1 - \frac{\rho_{\infty,i}}{\rho_\infty}\right) u_i - \sum_{j\neq i} \frac{\rho_{\infty,j}}{\rho_\infty} u_j \right|^2 = \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} \left| \sum_{j\neq i} \frac{\rho_{\infty,j}}{\rho_\infty} (u_i - u_j) \right|^2 \\
&= \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} \left(\sum_{k\neq i} \frac{\rho_{\infty,k}}{\rho_\infty} \right)^2 \left| \frac{\sum_{j\neq i} (\rho_{\infty,j}/\rho_\infty)(u_i - u_j)}{\sum_{k\neq i} \rho_{\infty,k}/\rho_\infty} \right|^2 \\
&= \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} \left(\sum_{k\neq i} \frac{\rho_{\infty,k}}{\rho_\infty} \right)^2 \left| \sum_{j\neq i} \lambda_j (u_i - u_j) \right|^2,
\end{aligned}$$

where $\lambda_j = (\rho_{\infty,j}/\rho_\infty)(\sum_{k\neq i}(\rho_{\infty,k}/\rho_\infty))^{-1}$. Since $\sum_{j\neq i} \lambda_j = 1$, we may apply Jensen's inequality to this convex combination, leading to

$$\begin{aligned}
I &\leq \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} \left(\sum_{k\neq i} \frac{\rho_{\infty,k}}{\rho_\infty} \right)^2 \sum_{j\neq i} \lambda_j |u_i - u_j|^2 \\
&= \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_\infty} \left(1 - \frac{\rho_{\infty,i}}{\rho_\infty}\right) \sum_{j\neq i} \frac{\rho_{\infty,j}}{\rho_\infty} |u_i - u_j|^2 \leq \sum_{i,j=1}^n |u_i - u_j|^2,
\end{aligned}$$

since $\rho_{\infty,j} \leq \rho_\infty$. This ends the proof. □

Now, we are able to prove Theorem 4.2.

Proof of Theorem 4.2. By Lemmas 4.11, 4.12, and 4.14, we obtain

$$\begin{aligned}-(f, L(f))_{L^2_v} &\geq (C^m - 4\eta)\|f - f^{\|}\|^2_{\mathcal{H}} + \frac{\eta D^b}{8}\sum_{i,j=1}^n \left(|u_i - u_j|^2 + (e_i - e_j)^2\right) \\ &\geq \left(C^m - 4\eta - \frac{\eta D^b}{4C_k}\right)\|f - f^{\|}\|^2_{\mathcal{H}} + \frac{\eta D^b}{8C_k}\|f - \Pi^L(f)\|^2_{\mathcal{H}}.\end{aligned}$$

The first term on the right-hand side is nonnegative if we choose $\eta = \min\{1, 4C^m C_k/(16C_k + D^b)\}$, and estimate (4.1.7) follows with $\lambda = \eta D^b/(8C_k)$. □

4.5. Convergence to equilibrium

In this section, we prove Theorem 4.3. The idea of the proof is to adapt the hypocoercivity method of [113] to the multi-species setting. To this end, we need to verify the structural assumptions (H1)-(H3) in the paper of C. Mouhot and L. Neumann [113, Theorem 1.1], which we call (M1)-(M3) here to avoid confusing with other assumptions in this thesis. The setting is as follows.

Let L be a closed, densely defined, and self-adjoint operator on $\mathrm{Dom}(L) \subset L^2_v$ such that $L = K - \Lambda$ and the operators K and Λ satisfy the following assumptions:

(M1) The operator Λ is coercive in the following sense: There exist a norm $\|\cdot\|_{\mathcal{H}}$ on $\mathcal{H} \subset L^2_v$ and positive constants $\overline{\nu}_i$ $(0 \leq i \leq 4)$ such that for all $f \in \mathrm{Dom}(L) \subset \mathcal{H}$,

$$\overline{\nu}_0\|f\|^2_{L^2_v} \leq \overline{\nu}_1\|f\|^2_{\mathcal{H}} \leq (f, \Lambda(f))_{L^2_v} \leq \overline{\nu}_2\|f\|^2_{\mathcal{H}}, \tag{4.5.1}$$

$$(\nabla_v f, \nabla_v \Lambda(f))_{L^2_v} \geq \overline{\nu}_3\|\nabla_v f\|^2_{\mathcal{H}} - \overline{\nu}_4\|f\|^2_{L^2_v}. \tag{4.5.2}$$

Moreover, there exists a constant $C_L > 0$ such that for all f, $g \in \mathrm{Dom}(L)$,

$$(L(f), g)_{L^2_v} \leq C_L\|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}}. \tag{4.5.3}$$

(M2) The operator K has a regularizing effect in the following sense: For all $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for all $f \in H^1_v$,

$$(\nabla_v f, \nabla_v K(f))_{L^2_v} \leq \varepsilon\|\nabla_v f\|^2_{L^2_v} + C(\varepsilon)\|f\|^2_{L^2_v}.$$

(M3) The operator L has a finite-dimensional kernel and the following local spectral-gap assumption holds: There exists $\lambda > 0$ such that for all $f \in \mathrm{Dom}(L)$,

$$-(f, L(f))_{L^2_v} \geq \lambda\|f - \Pi^L(f)\|^2_{\mathcal{H}},$$

where Π^L is the projection on $\mathrm{Ker}(L)$.

Assumption (M3) is a consequence of Theorem 4.2. Next, we verify assumption (M1). Using Lemma 4.8 and the continuous embedding $\mathcal{H} \hookrightarrow L^2_v$, we see that (4.5.1) holds. For the proof of (4.5.2), we employ Young's inequality:

$$(\nabla_v f, \nabla_v \Lambda(f))_{L^2_v} = \sum_{i=1}^n \int_{\mathbb{R}^3} \nabla_v f_i \cdot \nabla_v(\nu_i f_i)dv$$

$$
\begin{aligned}
&= \sum_{i=1}^n \left(\int_{\mathbb{R}^3} f_i \nabla_v f_i \cdot \nabla_v \nu_i dv + \int_{\mathbb{R}^3} |\nabla_v f_i|^2 \nu_i dv \right) \\
&\geq \frac{1}{2} \sum_{i=1}^n \left(- \int_{\mathbb{R}^3} \frac{|\nabla_v \nu_i|^2}{\nu_i} f_i^2 dv + \int_{\mathbb{R}^3} |\nabla_v f_i|^2 \nu_i dv \right) \\
&\geq \overline{\nu}_3 \|\nabla_v f\|_{\mathcal{H}}^2 - \overline{\nu}_4 \|f\|_{L_v^2}^2,
\end{aligned}
$$

where $\overline{\nu}_3 = 1/2$ and $\overline{\nu}_4 = \max_{1\leq i\leq n} \sup_{v\in\mathbb{R}^3} |\nabla_v \nu_i|^2/(2\nu_i)$. Note that $\overline{\nu}_4$ is finite since $\nabla_v \nu_i$ is bounded and ν_i is strictly positive (see Lemma 4.6). Finally, inequality (4.5.3) follows from the decomposition $L = K - \Lambda$, the compactness and hence continuity of K, the explicit expression for Λ, and the Cauchy-Schwarz inequality applied to $(L(f), g)_{L_v^2}$.

It remains to verify assumption (M2). Let $N := \rho_{\infty,i}^{-1/2} M_i^{1/2} = (2\pi)^{-3/4} \exp(-|v|^2/4)$. We decompose $K = K^{(1)} - K^{(2)}$, where $K^{(j)} = (K_1^{(j)}, \ldots, K_n^{(j)})$ and

$$
\begin{aligned}
K_i^{(1)} &= \sum_{j=1}^n \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij} M_i^{1/2} M_j^* \left(\frac{f_i'}{(M_i')^{1/2}} + \frac{f_j'^*}{(M_j'^*)^{1/2}} \right) dv^* d\sigma, \\
K_i^{(2)} &= \sum_{j=1}^n \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij} (M_i M_j^*)^{1/2} f_j^* dv^* d\sigma
\end{aligned}
$$

for $1 \leq i \leq n$. Because of $M_k' M_k'^* = M_k M_k^*$ for all k, we find that

$$
\begin{aligned}
K_i^{(1)}(f) &= \sum_{j=1}^n \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij} M_i^{1/2} M_j^* \left(\frac{(M_i'^*)^{1/2} f_i'}{(M_i' M_i'^*)^{1/2}} + \frac{(M_j')^{1/2} f_j'^*}{(M_j'^* M_j')^{1/2}} \right) dv^* d\sigma \\
&= \sum_{j=1}^n \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij} M_i^{1/2} M_j' \left(\frac{(M_i'^*)^{1/2} f_i'}{(M_i M_i^*)^{1/2}} + \frac{(M_j')^{1/2} f_j'^*}{(M_j^* M_j)^{1/2}} \right) dv^* d\sigma \\
&= \sum_{j=1}^n \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij} \big(\rho_{\infty,j}^{1/2} N'^* f_i' + \rho_{\infty,i}^{1/2} N' f_j'^* \big) \rho_{\infty,j}^{1/2} N^* dv^* d\sigma.
\end{aligned}
$$

The transformation $\sigma \mapsto -\sigma$ leaves v and v^* unchanged and exchanges v' and v'^*. Assumption (A5) (b_{ij} is an even function) ensures that B_{ij} is unchanged under this transformation. Therefore,

$$
\int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij} f_i' N'^* N^* dv^* d\sigma = \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij} f_i'^* N' N^* dv^* d\sigma,
$$

and we can write $K^{(1)}$ as

$$
K_i^{(1)}(f) = \frac{1}{2} \sum_{j=1}^n \rho_{\infty,j}^{1/2} \big(\rho_{\infty,j}^{1/2} K_{ij}^{(1)}(f_i) + \rho_{\infty,i}^{1/2} K_{ij}^{(1)}(f_j) \big), \tag{4.5.4}
$$

where

$$
K_{ij}^{(1)}(f_k) = \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij} (N'^* f_k' + N' f_k'^*) N^* dv^* d\sigma, \quad 1 \leq i, j, k \leq n.
$$

Note that $K_{ij}^{(1)} = K_{ji}^{(1)}$. In a similar way, we can decompose the operator $K^{(2)}$:

$$K_i^{(2)}(f) = \sum_{j=1}^n (\rho_{\infty,i}\rho_{\infty,j})^{1/2} N \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij} f_j^* N^* dv^* d\sigma = \sum_{j=1}^n (\rho_{\infty,i}\rho_{\infty,j})^{1/2} K_{ij}^{(2)}(f_j),$$

where

$$K_{ij}^{(2)}(f_j) = N \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij} N^* f_j^* dv^* d\sigma.$$

Next, we estimate the derivatives of $K_{ij}^{(\ell)}$. It is shown in [113, Eqs. (5.15)-(5.18)] that for all $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for any $f \in H_v^1$, $1 \leq i,j,k \leq n$, and $\ell = 1,2$,

$$\|\nabla_v K_{ij}^{(\ell)}(f_k)\|_{L_v^2}^2 \leq \varepsilon \|\nabla_v f_k\|_{L_v^2}^2 + C(\varepsilon)\|f_k\|_{L_v^2}^2. \tag{4.5.5}$$

Then we infer from (4.5.4) that

$$\begin{aligned}
\|\nabla_v K^{(1)}(f)\|_{L_v^2}^2 &= \sum_{i=1}^n \left\| \frac{1}{2}\sum_{j=1}^n \rho_{\infty,j}^{1/2}\big(\rho_{\infty,j}^{1/2}\nabla_v K_{ij}^{(1)}(f_i) + \rho_{\infty,i}^{1/2}\nabla_v K_{ij}^{(1)}(f_j)\big)\right\|_{L_v^2}^2 \\
&\leq \frac{n}{4}\sum_{i,j=1}^n \left\|\rho_{\infty,j}^{1/2}\big(\rho_{\infty,j}^{1/2}\nabla_v K_{ij}^{(1)}(f_i) + \rho_{\infty,i}^{1/2}\nabla_v K_{ij}^{(1)}(f_j)\big)\right\|_{L_v^2}^2 \\
&\leq n(\max_{1\leq i\leq n}\rho_{\infty,i})^2 \sum_{i,j=1}^n \|\nabla_v K_{ij}^{(1)}(f_i)\|_{L_v^2}^2.
\end{aligned}$$

Thus, by (4.5.5), it follows that for $\ell = 1$,

$$\|\nabla_v K^{(\ell)}(f)\|_{L_v^2}^2 \leq n^2(\max_{1\leq i\leq n}\rho_{\infty,i})^2 \sum_{i=1}^n \big(\varepsilon\|\nabla_v f_i\|_{L_v^2}^2 + C(\varepsilon)\|f_i\|_{L_v^2}^2\big).$$

A similar computation shows that this estimate also holds for $\ell = 2$. We infer that

$$\|\nabla K(f)\|_{L_v^2}^2 \leq 4n^2(\max_{1\leq i\leq n}\rho_{\infty,i})^2 \sum_{i=1}^n \big(\varepsilon\|\nabla_v f_i\|_{L_v^2}^2 + C(\varepsilon)\|f_i\|_{L_v^2}^2\big).$$

This proves assumption (M2) since $\varepsilon > 0$ is arbitrary.

Proof of Theorem 4.3. We have verified that (M1)-(M3) are satisfied. Then, using exactly the same arguments as in the proof of Theorem 1.1 in [113], but now for the multi-species case, we conclude the exponential decay (4.1.8) of the semigroup $S_G(t)$, which is the first property of the theorem.

It remains to show that the decay estimate (4.1.9) follows from (4.1.8). For this, we write the initial value f_I as $f_I = \Pi^G(f_I) + (I - \Pi^G)(f_I)$, where Π^G is the projection onto $\mathrm{Ker}(G)$ in $L_{x,v}^2$. Then the solution to (2.1.5) is given by

$$f(t) = S_G(t)f_I = S_G(t)\Pi^G(f_I) + S_G(t)(I - \Pi^G)(f_I), \quad t \geq 0.$$

We have already shown that

$$\|S_G(t)(I - \Pi^G)g\|_{H^1_{x,v}} \leq C\mathrm{e}^{-\tau t}\|g\|_{H^1_{x,v}} \quad \text{for all } g \in H^1_{x,v},\ t > 0.$$

In particular, the choice $g = (I - \Pi^G)(f_I)$ and the property $(I - \Pi^G)^2 = I - \Pi^G$ lead to

$$\|S_G(t)(I - \Pi^G)(f_I)\|_{H^1_{x,v}} \leq C\mathrm{e}^{-\tau t}\|(I - \Pi^G)(f_I)\|_{H^1_{x,v}}.$$

It remains to prove that $f_\infty = \Pi^G(f_I) = S_G(t)\Pi^G(f_I)$ is the global equilibrium. Since $G\Pi^G(f_I) = 0$ and $\Pi^G(f_I)$ does not depend on time, the constant-in-time function $g = \Pi^G(f_I)$ is the unique solution to the Cauchy problem

$$\partial_t g = Gg, \quad t > 0, \quad g(0) = \Pi^G(f_I).$$

This shows that $\Pi^G(f_I) = S_G(t)g(0) = S_G(t)\Pi^G(f_I)$ and finishes the proof. □

5. The nonlinear multi-species Boltzmann system close to global equilibrium

In this chapter we use a setting with different molar masses, and a different linearization compared to Chapter 4. Since we also slightly change the notation, let us briefly recap the model used in this chapter. A more detailed introduction for this model can be found in Section 2.2.1.

5.1. The model

5.1.1. Different molar masses

We want to study the evolution of a dilute gas on the torus $\mathbb{T}^3$ composed of N different species of chemically non-reacting mono-atomic particles with different molar masses, modeled by the system of Boltzmann equations on $\mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3$,

$$\forall\, 1 \leq i \leq N, \quad \partial_t F_i(t,x,v) + v \cdot \nabla_x F_i(t,x,v) = Q_i(\mathbf{F})(t,x,v) \tag{5.1.1}$$

with initial data

$$\forall\, 1 \leq i \leq N,\ \forall (x,v) \in \mathbb{T}^3 \times \mathbb{R}^3, \quad F_i(0,x,v) = F_{0,i}(x,v).$$

The distribution function of the system is given by the vector $\mathbf{F} = (F_1, \dots, F_N)$, with F_i describing the i^{th} species at time t, position x and velocity v.

The Boltzmann operator $\mathbf{Q}(\mathbf{F}) = (Q_1(\mathbf{F}), \dots, Q_N(\mathbf{F}))$ has the form

$$Q_i(\mathbf{F}) = \sum_{j=1}^{N} Q_{ij}(F_i, F_j),$$

where Q_{ij} models interactions between particles of either the same ($i = j$) or of different ($i \neq j$) species and are local in time and space.

$$Q_{ij}(F_i, F_j)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij}\left(|v - v_*|, \cos\theta\right) \left[F_i' F_j'^{*} - F_i F_j^{*}\right] dv_* d\sigma,$$

with the shorthands $F_i' = F_i(v')$, $F_i = F_i(v)$, $F_j'^{*} = F_j(v_*')$ and $F_j^{*} = F_j(v_*)$. Since now we allow also different molar masses for each species, the microscopic collision rules read as follows:

$$\begin{cases} v' &= \dfrac{1}{m_i+m_j}\left(m_i v + m_j v_* + m_j|v-v_*|\sigma\right) \\ v'_* &= \dfrac{1}{m_i+m_j}\left(m_i v + m_j v_* - m_i|v-v_*|\sigma\right) \end{cases}, \text{ and } \quad \cos\theta = \left\langle \frac{v-v_*}{|v-v_*|}, \sigma \right\rangle.$$

Note that these expressions imply that we deal with gases where only binary elastic collisions occur (the mass m_i of all molecules of species i remains the same, since there is no reaction). Indeed, v' and v'_* are the velocities of two molecules of species i and j before collision giving post-collisional velocities v and v_* respectively, with conservation of momentum and kinetic energy:

$$\begin{gathered} m_i v + m_j v_* = m_i v' + m_j v'_*, \\ \frac{1}{2}m_i\left|v\right|^2 + \frac{1}{2}m_j\left|v_*\right|^2 = \frac{1}{2}m_i\left|v'\right|^2 + \frac{1}{2}m_j\left|v'_*\right|^2. \end{gathered} \tag{5.1.2}$$

5.1.2. The perturbative regime and its motivation

Using the standard changes of variables $(v,v_*) \mapsto (v',v'_*)$ and $(v,v_*) \mapsto (v_*,v)$ (note the lack of symmetry between v' and v'_* compared to v for the second transformation due to different masses) together with the symmetries of the collision operators (see [33][34][138] among others and [52][46] and in particular [21] for multi-species specifically), we recover the following weak forms:

$$\int_{\mathbb{R}^3} Q_{ij}(F_i,F_j)(v)\psi_i(v)\,dv = \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} B_{ij}(|v-v_*|,\cos(\theta))F_iF_j^*\left(\psi'_i-\psi_i\right)\,d\sigma dv dv_*$$

and

$$\begin{gathered} \int_{\mathbb{R}^3} Q_{ij}(F_i,F_j)(v)\psi_i(v)\,dv + \int_{\mathbb{R}^3} Q_{ji}(F_j,F_i)(v)\psi_j(v)\,dv = \\ -\frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} B_{ij}(|v-v_*|,\cos(\theta))\left(F'_iF'^*_j - F_iF_j^*\right)\left(\psi'_i+\psi'^*_j-\psi_i-\psi_j^*\right)\,d\sigma dv dv_*. \end{gathered} \tag{5.1.3}$$

Thus

$$\sum_{i,j=1}^{N}\int_{\mathbb{R}^3} Q_{ij}(F_i,F_j)(v)\psi_i(v)\,dv = 0 \tag{5.1.4}$$

if and only if $\psi(v)$ belongs to Span $\left\{\mathbf{e_1},\dots,\mathbf{e_N}, v_1\mathbf{m}, v_2\mathbf{m}, v_3\mathbf{m}, |v|^2\mathbf{m}\right\}$, where $\mathbf{e_k}$ stands for the k^{th} unit vector in $\mathbb{R}^N$ and $\mathbf{m} = (m_1,\dots,m_N)$. The fact that we need to sum over i has interesting consequences and implies a fundamental difference compared with the single-species Boltzmann equation. In particular it implies conservation of the total number density $c_{\infty,i}$ of each species, of the total momentum of the gas $\rho_\infty u_\infty$ and its total energy $3\rho_\infty\theta_\infty/2$:

$$\begin{aligned}
\forall t \geq 0, \quad c_{\infty,i} &= \int_{\mathbb{T}^3\times\mathbb{R}^3} F_i(t,x,v)\,dxdv \quad (1 \leq i \leq N)\\
u_\infty &= \frac{1}{\rho_\infty}\sum_{i=1}^N \int_{\mathbb{T}^3\times\mathbb{R}^3} m_i v F_i(t,x,v)\,dxdv\\
\theta_\infty &= \frac{1}{3\rho_\infty}\sum_{i=1}^N \int_{\mathbb{T}^3\times\mathbb{R}^3} m_i\left|v-u_\infty\right|^2 F_i(t,x,v)\,dxdv,
\end{aligned} \tag{5.1.5}$$

where $\rho_\infty = \sum_{i=1}^N m_i c_{\infty,i}$ is the global density of the gas. Note that this already shows intricate interactions between each species and the total mixture itself.

The operator $\mathbf{Q} = (Q_1,\dots,Q_N)$ also satisfies a multi-species version of the classical H-theorem [52] which implies that any local equilibrium, i.e. any function $\mathbf{F} = (F_1,\dots,F_N)$ being the maximum of the Boltzmann entropy, has the form of a local Maxwellian, that is

$$\forall\, 1 \leq i \leq N,\ F_i(t,x,v) = c_{\mathrm{loc},i}(t,x)\left(\frac{m_i}{2\pi k_B\theta_{\mathrm{loc}}(t,x)}\right)^{3/2}\exp\left[-m_i\frac{|v-u_{\mathrm{loc}}(t,x)|^2}{2k_B\theta_{\mathrm{loc}}(t,x)}\right].$$

Here k_B is the Boltzmann constant and, denoting the total local mass density by $\rho_{\mathrm{loc}} = \sum_{i=1}^N m_i c_{\mathrm{loc},i}$, we used the following local definitions

$$\forall\, 1 \leq i \leq N, \quad c_{\mathrm{loc},i}(t,x) = \int_{\mathbb{R}^3} F_i(t,x,v)\,dv,$$

$$u_{\mathrm{loc}}(t,x) = \frac{1}{\rho_{\mathrm{loc}}}\sum_{i=1}^N\int_{\mathbb{R}^3} m_i v F_i\,dv, \quad \theta_{\mathrm{loc}}(t,x) = \frac{1}{3\rho_{\mathrm{loc}}}\sum_{i=1}^N\int_{\mathbb{R}^3} m_i\left|v-u_{\mathrm{loc}}\right|^2 F_i\,dv.$$

On the torus, this multi-species H-theorem also implies that the global equilibrium, i.e. a stationary solution $\mathbf{F}$ to (5.1.1), associated to the initial data $\mathbf{F_0}(x,v) = (F_{0,1},\dots,F_{0,N})$ is uniquely given by the global Maxwellian

$$\forall\, 1 \leq i \leq N, \quad F_i(t,x,v) = F_i(v) = c_{\infty,i}\left(\frac{m_i}{2\pi k_B\theta_\infty}\right)^{3/2}\exp\left[-m_i\frac{|v-u_\infty|^2}{2k_B\theta_\infty}\right].$$

By translating and rescaling the coordinate system we can always assume that $u_\infty = 0$ and $k_B\theta_\infty = 1$ so that the only global equilibrium is the normalized Maxwellian

$$\boldsymbol{\mu} = (\mu_i)_{1\leq i\leq N} \quad \text{with} \quad \mu_i(v) = c_{\infty,i}\left(\frac{m_i}{2\pi}\right)^{3/2} e^{-m_i\frac{|v|^2}{2}}. \tag{5.1.6}$$

The aim is to construct a Cauchy theory for the multi-species Boltzmann equation (5.1.1) around the global equilibrium $\boldsymbol{\mu}$. In other terms we study the existence, uniqueness and exponential decay of solutions of the form $F_i(t,x,v) = \mu_i(v) + f_i(t,x,v)$ for all i.

Under this perturbative regime, the Cauchy problem amounts to solving the perturbed multi-species Boltzmann system of equations

$$\partial_t \mathbf{f} + v \cdot \nabla_x \mathbf{f} = \mathbf{L}(\mathbf{f}) + \mathbf{Q}(\mathbf{f}), \tag{5.1.7}$$

or equivalently in the non-vectorial form

$$\forall\, 1 \leq i \leq N, \quad \partial_t f_i + v \cdot \nabla_x f_i = L_i(\mathbf{f}) + Q_i(\mathbf{f}),$$

where $\mathbf{f} = (f_1, \dots, f_N)$ and the operator $\mathbf{L} = (L_1, \dots, L_N)$ is the linear Boltzmann operator given for all $1 \leq i \leq N$ by

$$L_i(\mathbf{f}) = \sum_{j=1}^{N} L_{ij}(f_i, f_j),$$

with

$$L_{ij}(f_i, f_j) = Q_{ij}(\mu_i, f_j) + Q_{ij}(f_i, \mu_j).$$

Since we are looking for solutions $\mathbf{F}$ preserving individual mass, total momentum and total energy (5.1.5) we have the equivalent perturbed conservation laws for $\mathbf{f} = \mathbf{F} - \boldsymbol{\mu}$ which are given by

$$\begin{aligned} \forall t \geq 0, \quad 0 &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_i(t,x,v)\, dxdv \quad (1 \leq i \leq N) \\ 0 &= \sum_{i=1}^{N} \int_{\mathbb{T}^3 \times \mathbb{R}^3} m_i v f_i(t,x,v)\, dxdv \\ 0 &= \sum_{i=1}^{N} \int_{\mathbb{T}^3 \times \mathbb{R}^3} m_i \left|v\right|^2 f_i(t,x,v)\, dxdv. \end{aligned} \tag{5.1.8}$$

5.1.3. Notations and assumptions on the collision kernel

First, to avoid any confusion, vectors and vector-valued operators in $\mathbb{R}^N$ will be denoted by a bold symbol, whereas their components by the same indexed symbol. For instance, $\mathbf{W}$ represents the vector or vector-valued operator $(W_1, \dots, W_N)$.

We define the Euclidian scalar product in $\mathbb{R}^N$ weighted by a vector $\mathbf{W}$ by

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathbf{W}} = \sum_{i=1}^{N} f_i g_i W_i.$$

In the case $\mathbf{W} = \mathbf{1} = (1, \dots, 1)$ we may omit the index $\mathbf{1}$.

Function spaces. We define the following shorthand notation

$$\langle v \rangle = \sqrt{1 + \left|v\right|^2}.$$

The convention we choose is to index the space by the name of the concerned variable, so we have for p in $[1,+\infty]$

$$L^p_{[0,T]} = L^p([0,T]), \quad L^p_t = L^p\left(\mathbb{R}^+\right), \quad L^p_x = L^p\left(\mathbb{T}^3\right), \quad L^p_v = L^p\left(\mathbb{R}^3\right).$$

For $\mathbf{W} = (W_1,\dots,W_N) : \mathbb{R}^3 \longrightarrow \mathbb{R}^+$ a strictly positive measurable function in v, we will use the following vector-valued weighted Lebesgue spaces defined by their norms

$$\begin{aligned}
\|f\|_{L^2_v(\mathbf{W})} &= \left(\sum_{i=1}^N \|f_i\|^2_{L^2_v(W_i)}\right)^{1/2}, &\quad \|f_i\|_{L^2_v(W_i)} &= \|f_i W_i(v)\|_{L^2_v},\\
\|f\|_{L^2_{x,v}(\mathbf{W})} &= \left(\sum_{i=1}^N \|f_i\|^2_{L^2_{x,v}(W_i)}\right)^{1/2}, &\quad \|f_i\|_{L^2_{x,v}(W_i)} &= \left\|\|f_i\|_{L^2_x} W_i(v)\right\|_{L^2_v},\\
\|f\|_{L^\infty_{x,v}(\mathbf{W})} &= \sum_{i=1}^N \|f_i\|_{L^\infty_{x,v}(W_i)}, &\quad \|f_i\|_{L^\infty_{x,v}(W_i)} &= \sup_{(x,v)\in\mathbb{T}^3\times\mathbb{R}^3} \left(|f_i(x,v)|\, W_i(v)\right),\\
\|f\|_{L^1_vL^\infty_x(\mathbf{W})} &= \sum_{i=1}^N \|f_i\|_{L^1_vL^\infty_x(W_i)}, &\quad \|f_i\|_{L^1_vL^\infty_x(W_i)} &= \left\|\sup_{x\in\mathbb{T}^3} |f_i(x,v)|\, W_i(v)\right\|_{L^1_v}.
\end{aligned}$$

Note that $L^2_v(\mathbf{W})$ and $L^2_{x,v}(\mathbf{W})$ are Hilbert spaces with respect to the scalar products

$$\begin{aligned}
\langle \mathbf{f}, \mathbf{g}\rangle_{L^2_v(\mathbf{W})} &= \sum_{i=1}^N \langle f_i, g_i\rangle_{L^2_v(W_i)} = \sum_{i=1}^N \int_{\mathbb{R}^3} f_i g_i W_i^2\, dv,\\
\langle \mathbf{f}, \mathbf{g}\rangle_{L^2_{x,v}(\mathbf{W})} &= \sum_{i=1}^N \langle f_i, g_i\rangle_{L^2_{x,v}(W_i)} = \sum_{i=1}^N \int_{\mathbb{T}^3\times\mathbb{R}^3} f_i g_i W_i^2\, dx dv.
\end{aligned}$$

Assumptions on the collision kernel.
We will use the following assumptions on the collision kernels B_{ij}.

(H1) The following symmetry holds

$$B_{ij}(|v-v_*|, \cos\theta) = B_{ji}(|v-v_*|, \cos\theta) \quad \text{for } 1 \leq i,j \leq N.$$

(H2) The collision kernels decompose into the product

$$B_{ij}(|v-v_*|, \cos\theta) = \Phi_{ij}(|v-v_*|) b_{ij}(\cos\theta), \quad 1 \leq i,j \leq N,$$

where the functions $\Phi_{ij} \geq 0$ are called kinetic part and $b_{ij} \geq 0$ angular part. This is a common assumption as it is technically more convenient and also covers a wide range of physical applications.

(H3) The kinetic part has the form of hard or Maxwellian ($\gamma = 0$) potentials, *i.e.*

$$\Phi_{ij}(|v-v_*|) = C^\Phi_{ij}|v-v_*|^\gamma, \quad C^\Phi_{ij} > 0, \ \ \gamma \in [0,1], \quad \forall\, 1 \leq i,j \leq N.$$

(H4) For the angular part, we assume a strong form of Grad's angular cutoff (first introduced in [74]), that is: there exist constants $C_{b1}, C_{b2} > 0$ such that for all $1 \leq i, j \leq N$ and $\theta \in [0, \pi]$,

$$0 < b_{ij}(\cos\theta) \leq C_{b1} |\sin\theta| \, |\cos\theta|, \quad b'_{ij}(\cos\theta) \leq C_{b2}.$$

Furthermore,

$$C^b := \min_{1 \leq i \leq N} \inf_{\sigma_1, \sigma_2 \in \mathbb{S}^2} \int_{\mathbb{S}^2} \min \left\{ b_{ii}(\sigma_1 \cdot \sigma_3), b_{ii}(\sigma_2 \cdot \sigma_3) \right\} d\sigma_3 > 0.$$

We emphasize here that the important cases of Maxwellian molecules ($\gamma = 0$ and $b = 1$) and of hard spheres ($\gamma = b = 1$) are included in our study. We shall use the standard shorthand notations

$$b_{ij}^{\infty} = \|b_{ij}\|_{L^{\infty}_{[-1,1]}} \quad \text{and} \quad l_{b_{ij}} = \|b \circ \cos\|_{L^1_{\mathbb{S}^2}}. \tag{5.1.9}$$

5.1.4. Main results

For the convenience of the reader, we summarize here the main results of this chapter. For a discussion of these results, we refer to Subsection 2.2.4.

Theorem 5.1. *Let the collision kernels B_{ij} satisfy assumptions $(H1) - (H4)$. Then the following holds.*

(i) The operator $\mathbf{L}$ is a closed self-adjoint operator in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$ and there exists $\lambda_L > 0$ such that

$$\forall \mathbf{f} \in L^2_v\left(\boldsymbol{\mu}^{-1/2}\right), \quad \langle \mathbf{f}, \mathbf{L}(\mathbf{f}) \rangle_{L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)} \leq -\lambda_L \left\|\mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_v\left(\langle v \rangle^{\gamma/2} \boldsymbol{\mu}^{-1/2}\right)};$$

(ii) Let $E = L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ or $E = L^{\infty}_{x,v}\left(\langle v \rangle^{\beta} \boldsymbol{\mu}^{-1/2}\right)$ with $\beta > 3/2$. The linear perturbed operator $\mathbf{G} = \mathbf{L} - v \cdot \nabla_x$ generates a strongly continuous semigroup $S_{\mathbf{G}}(t)$ on E and there exist C_E, $\lambda_E > 0$ such that

$$\forall t \geq 0, \quad \left\|S_{\mathbf{G}}(t) \left(Id - \Pi_{\mathbf{G}}\right)\right\|_E \leq C_E e^{-\lambda_E t},$$

where $\pi_{\mathbf{L}}$ is the orthogonal projection onto $Ker(\mathbf{L})$ in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$ and $\Pi_{\mathbf{G}}$ is the orthogonal projection onto $Ker(\mathbf{G})$ in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$.
The constants λ_L, C_E and λ_E are explicit and depend on N, E, the different masses m_i and the collision kernels.

Remark 5.2. *This Theorem is split into three parts, and handled separately in Theorem 5.6, Theorem 5.11 and Theorem 5.17.*

The next result concerns the nonlinear equation close to global equilibrium.

Theorem 5.3. *Let the collision kernels B_{ij} satisfy assumptions $(H1)-(H4)$ and let $E = L^1_v L^\infty_x \left(\langle v\rangle^k\right)$ with $k > k_0$, where k_0 is the minimal integer such that*

$$C_k = \frac{2}{k+2}\frac{1-\left[\max\limits_{i,j}\frac{|m_i-m_j|}{m_i+m_j}\right]^{\frac{k+2}{2}}+\left[1-\left(\max\limits_{i,j}\frac{|m_i-m_j|}{m_i+m_j}\right)\right]^{\frac{k+2}{2}}}{1-\max\limits_{i,j}\frac{|m_i-m_j|}{m_i+m_j}}\max_{i,j}\frac{4\pi b^\infty_{ij}}{l_{b_{ij}}} < 1. \quad (5.1.10)$$

where $l_{b_{ij}}$ and b^∞_{ij} are angular kernel constants (5.1.9).
Then there exist η_E, C_E and $\lambda_E > 0$ such that for any $\mathbf{F_0} = \boldsymbol{\mu} + \mathbf{f_0} \geq 0$ satisfying the conservation of mass, momentum and energy (5.1.5) with $u_\infty = 0$ and $\theta_\infty = 1$, if

$$\|\mathbf{F_0} - \boldsymbol{\mu}\| \leq \eta_E$$

then there exists a unique solution $\mathbf{F} = \boldsymbol{\mu} + \mathbf{f}$ in E to the multi-species Boltzmann equation (5.1.1) with initial data $\mathbf{f_0}$. Moreover, $\mathbf{F}$ is non-negative, satisfies the conservation laws and

$$\forall t \geq 0, \quad \|\mathbf{F} - \boldsymbol{\mu}\|_E \leq C_E e^{-\lambda_E t}\|\mathbf{F_0} - \boldsymbol{\mu}\|_E\,.$$

The constants are explicit and only depend on N, k, the different masses m_i and the collision kernels.

The proof of this theorem is pressented in Section 5.5.

5.1.5. Organisation of this chapter

Section 5.2 deals with the multi-species spectral gap of $\mathbf{L}$ in the case of different molar masses. The semigroup property in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ is treated in Section 5.3. This property is then passed on to $L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$ in Section 5.4. At last, we work out the Cauchy problem for the full nonlinear equation in Section 5.5.

5.2. Spectral gap with different masses

5.2.1. First properties of the linear multi-species Boltzmann operator

We start by describing some properties of the linear multi-species Boltzmann operator $\mathbf{L} = (L_i)_{1\leq i\leq N}$. First recall

$$L_i(\mathbf{f}) = \sum_{j=1}^N L_{ij}(f_i, f_j), \quad 1 \leq i \leq N,$$

with

$$\begin{aligned} L_{ij}(f_i,f_j) &= Q_{ij}(\mu_i, f_j) + Q_{ij}(f_i,\mu_j) \\ &= \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij}(|v-v_*|,\cos(\theta))\left(\mu'^*_j f'_i + \mu'_i f'^*_j - \mu^*_j f_i - \mu_i f^*_j\right)\, dv_* d\sigma, \end{aligned}$$

where we have used $\mu_i^{\prime *}\mu_j^{\prime} = \mu_i^*\mu_j$ for any i, j, which follows from the laws of elastic collisions (5.1.2).

In Chapter 4 (see also [46]), we proved an explicite multi-species spectral gap in the case of species having same mass ($m_i = m_j$). Most of the proofs are directly applicable in the case of different masses, and we therefore refer to this chapter for detailed proofs.

$\mathbf{L}$ is a self-adjoint operator in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$ with $\langle \mathbf{f}, \mathbf{L}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} = 0$ if and only if $\mathbf{f}$ belongs to Ker($\mathbf{L}$).

$$\mathrm{Ker}\,(\mathbf{L}) = \mathrm{Span}\left\{\boldsymbol{\phi}_1(v), \dots, \boldsymbol{\phi}_{N+4}(v)\right\},$$

where $(\boldsymbol{\phi}_i)_{1\leq i\leq N+4}$ is an orthonormal basis of Ker$\,(\mathbf{L})$ in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$. More precisely, if we denote $\pi_{\mathbf{L}}$ the orthogonal projection onto Ker$\,(\mathbf{L})$ in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$:

$$\pi_{\mathbf{L}}(\mathbf{f}) = \sum_{k=1}^{N+4}\left(\int_{\mathbb{R}^3}\langle \mathbf{f}(v), \boldsymbol{\phi}_k(v)\rangle_{\boldsymbol{\mu}^{-1/2}}\,dv\right)\boldsymbol{\phi}_k(v),$$

and

$$\mathbf{e_k} = (\delta_{ik})_{1\leq i\leq N},$$

we can write

$$\begin{cases} \boldsymbol{\phi}_k(v) = \dfrac{1}{\sqrt{c_{\infty,k}}}\,\mu_k\mathbf{e_k}, \quad 1\leq k\leq N \\ \boldsymbol{\phi}_k(v) = \dfrac{v_{k-N}}{\left(\sum\limits_{i=1}^N m_i c_{\infty,i}\right)^{1/2}}\,(m_i\mu_i)_{1\leq i\leq N}, \quad N+1\leq k\leq N+3. \\ \\ \boldsymbol{\phi}_{N+4}(v) = \dfrac{1}{\left(\sum\limits_{i=1}^N c_{\infty,i}\right)^{1/2}}\left(\dfrac{|v|^2 - 3m_i^{-1}}{\sqrt{6}}m_i\mu_i\right)_{1\leq i\leq N}. \end{cases} \tag{5.2.1}$$

Finally, we denote $\pi_{\mathbf{L}}^{\perp} = \mathrm{Id} - \pi_{\mathbf{L}}$. The projection $\pi_{\mathbf{L}}(\mathbf{f}(t,x,\cdot))(v)$ of $\mathbf{f}(t,x,v)$ onto the kernel of $\mathbf{L}$ is called its fluid part whereas $\pi_{\mathbf{L}}^{\perp}(\mathbf{f})$ is its microscopic part.

$\mathbf{L}$ can be written under the following form

$$\mathbf{L} = -\boldsymbol{\nu}(v) + \mathbf{K}, \tag{5.2.2}$$

where $\boldsymbol{\nu} = (\nu_i)_{1\leq i\leq N}$ is a multiplicative operator called the collision frequency

$$\nu_i(v) = \sum_{j=1}^N \nu_{ij}(v), \tag{5.2.3}$$

with

$$\nu_{ij}(v) = C_{ij}^{\Phi}\int_{\mathbb{R}^3\times\mathbb{S}^2} b_{ij}(\cos\theta)\,|v - v_*|^{\gamma}\,\mu_j(v_*)\,d\sigma dv_*.$$

Each of the ν_{ij} could be seen as the collision frequency $\nu(v)$ of a single-species Boltzmann kernel with kernel B_{ij}. It is well-known (for instance [33][34][138][78]) that under our assumptions: $\nu(v) \sim (1+|v|^\gamma) \sim \langle v\rangle^\gamma$. This means that for all i, j there exist $\nu_{ij}^{(0)}$, $\nu_{ij}^{(1)} > 0$ (they are explicit, see the references above) such that

$$\forall v \in \mathbb{R}^3, \quad \nu_{ij}^{(0)}\left(1+|v|^\gamma\right) \leq \nu_{ij}(v) \leq \nu_{ij}^{(1)}\left(1+|v|^\gamma\right),$$

Every constant being strictly positive, the following lemma follows straightforwardly.

Lemma 5.4. *There exists a constant* $\beta > 0$*, and for all* i *in* $\{1,\dots,N\}$ *there exist* $\nu_i^{(0)}$, $\nu_i^{(1)} > 0$ *such that*

$$\forall v \in \mathbb{R}^3, \quad \nu_i^{(0)}\left(1+|v|^\gamma\right) \leq \nu_i(v) \leq \nu_i^{(1)}\left(1+|v|^\gamma\right). \tag{5.2.4}$$

Thus, we get the following relation between the collision frequencies

$$\forall v \in \mathbb{R}^3, \quad \nu_i(v) \leq \beta\nu_{ii}(v). \tag{5.2.5}$$

Remark 5.5. *Estimate* (5.2.5) *is a crucial step in the proof of Lemma 5.7. In Chapter 4 (see also [46]), the additional assumption* $B_{ij} \leq CB_{ii}$ *for a constant* $C > 0$ *has been used in order to get* (5.2.5)*. We want to point out that despite of even having different masses to handle, we manage to get rid of this assumption. The prize we have to pay is a slightly more restrictive assumption on the collision kernel* B *in assumption* $(H3)$*.*

Next we decompose the operator $\mathbf{L}$ into its mono-species part $\mathbf{L^m} = (L_i^m)_{1\leq i\leq N}$ and its bi-species part $\mathbf{L^b} = (L_i^b)_{1\leq i\leq N}$ according to

$$\mathbf{L} = \mathbf{L^m} + \mathbf{L^b}, \quad L_i^m(f_i) = L_{ii}(f_i,f_i), \quad L_i^b(f) = \sum_{j\neq i} L_{ij}(f_i,f_j). \tag{5.2.6}$$

Thus $\mathbf{f}$ can be written as

$$\mathbf{f} = \pi_{\mathbf{L^m}}(\mathbf{f}) + \pi_{\mathbf{L^m}}^{\perp}(\mathbf{f}), \tag{5.2.7}$$

where $\pi_{\mathbf{L^m}}$ is the orthogonal projection on $\mathrm{Ker}(\mathbf{L^m})$ with respect to $L_v^2\left(\boldsymbol{\mu}^{-1/2}\right)$, and

$$\pi_{\mathbf{L^m}}^{\perp} := \left(1 - \pi_{\mathbf{L^m}}\right).$$

By employing the standard change of variables, the Dirichlet forms of $\mathbf{L^m}$ and $\mathbf{L^b}$ have the form

$$\langle \mathbf{f}, \mathbf{L^m}(\mathbf{f})\rangle_{L_v^2\left(\boldsymbol{\mu}^{-1/2}\right)} = -\frac{1}{4}\sum_{i=1}^N \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ii}\mu_i\mu_i^*\left(A_{ii}\left[f_i\mu_i^{-1}, f_i\mu_i^{-1}\right]\right)^2, \tag{5.2.8}$$

$$\left\langle \mathbf{f}, \mathbf{L^b}(\mathbf{f})\right\rangle_{L_v^2\left(\boldsymbol{\mu}^{-1/2}\right)} = -\frac{1}{4}\sum_{i=1}^N\sum_{j\neq i} \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}\mu_i\mu_j^*\left(A_{ij}\left[f_i\mu_i^{-1}, f_j\mu_j^{-1}\right]\right)^2, \tag{5.2.9}$$

with the shorthands

$$A_{ij}\left[f_i\mu_i^{-1}, f_j\mu_j^{-1}\right] := \left(f_i\mu_i^{-1}\right)' + \left(f_j\mu_j^{-1}\right)'^{*} - \left(f_i\mu_i^{-1}\right) - \left(f_j\mu_j^{-1}\right)^{*}. \tag{5.2.10}$$

Since $\mathbf{L^m}$ describes a multi-species operator when all the cross-interactions are null,

$$\pi_{\mathbf{L^m}}(\mathbf{f})_i = m_i\mu_i(v)(a_i(t,x) + u_i(t,x)\cdot v + e_i(t,x)|v|^2), \quad 1 \leq i \leq N, \tag{5.2.11}$$

where $a_i \in \mathbb{R}, u_i \in \mathbb{R}^3$ and $e_i \in \mathbb{R}$ are the coordinates of $\pi_{\mathbf{L^m}}(\mathbf{f})$ with respect to a $5N$-dimensional basis, while

$$\pi_{\mathbf{L}}(\mathbf{f})_i = m_i\mu_i(v)(a_i(t,x) + u(t,x)\cdot v + e(t,x)|v|^2) \quad 1 \leq i \leq N, \tag{5.2.12}$$

where $a_i \in \mathbb{R}, u \in \mathbb{R}^3$ and $e \in \mathbb{R}$ are the coordinates of $\pi_{\mathbf{L}}(\mathbf{f})$ with respect to an $(N+4)$-dimensional basis.

Finally, since

$$\int_{\mathbb{R}^3} \mu_i\, dv = c_i, \quad \int_{\mathbb{R}^3} \mu_i|v|^2\, dv = 3c_i m_i^{-1}, \quad \int_{\mathbb{R}^3} \mu_i|v|^4\, dv = 15c_i m_i^{-2}, \tag{5.2.13}$$

the following moment identities hold for a_i, u_i, e_i defined in (5.2.11)

$$\begin{aligned} &\int_{\mathbb{R}^3} f_i\, dv = c_i(m_i a_i + 3e_i), \\ &\int_{\mathbb{R}^3} f_i v\, dv = c_i u_i, \\ &\int_{\mathbb{R}^3} f_i|v|^2\, dv = c_i(3a_i + 15e_i m_i^{-1}). \end{aligned} \tag{5.2.14}$$

5.2.2. Explicit spectral gap

This subsection is devoted to the proof of the following constructive spectral gap estimate for the multi-species linear operator $\mathbf{L}$ with different masses.

Theorem 5.6. *Let the collision kernels B_{ij} satisfy assumptions (H1)-(H4). Then there exists an explicit constant $\lambda_L > 0$ such that*

$$\langle \mathbf{f}, \mathbf{L}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \leq -\lambda_L \|\mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})\|^2_{L^2_v(\langle v\rangle^{\gamma/2}\boldsymbol{\mu}^{-1/2})} \quad \forall \mathbf{f} \in Dom(\mathbf{L}),$$

where λ_L depends on the properties of the collision kernel, the number of species N and the different masses.

The next two lemmas are crucial for the proof of Theorem 5.6, generalizing the strategy of [46] presented in Chapter 4 to the case of different masses. The key idea is to decompose $\mathbf{L}$ into $\mathbf{L} = \mathbf{L^m} + \mathbf{L^b}$ (see (5.2.6)), and to derive separately a spectral-gap estimate for the mono-species part $\mathbf{L^m}$ on its domain $\mathrm{Dom}(\mathbf{L^m})$ (see Lemma 5.7), and a spectral-gap type estimate for the bi-species part $\mathbf{L^b}$ on $\mathrm{Ker}(\mathbf{L^m})$ (see Lemma 5.8) measured in terms of the following functional

$$\mathcal{E} : \mathrm{Ker}(\mathbf{L^m}) \to \mathbb{R}^+, \quad \mathcal{E}(\mathbf{f}) := \sum_{i,j=1}^{N} \left(\left| u_i^{(f)} - u_j^{(f)} \right|^2 + \left(e_i^{(f)} - e_j^{(f)} \right)^2 \right),$$

where for a fixed $\mathbf{f} \in \mathrm{Ker}(\mathbf{L^m})$, $u_i^{(f)}$ and $e_i^{(f)}$ describe the coordinates of the i^{th} component of $\mathbf{f}$ with respect to the basis defined in (5.2.11). To lighten computations, we introduce the following Hilbert space $\mathcal{H} := L^2_v\left(\boldsymbol{\nu}^{1/2}\boldsymbol{\mu}^{-1/2}\right)$, which is equivalent to $L^2_v\left(\langle v\rangle^{\gamma/2}\boldsymbol{\mu}^{-1/2}\right)$:

$$\mathcal{H} = \left\{ f \in L^2_v(\boldsymbol{\mu}^{\mathbf{-1/2}}) : \|\mathbf{f}\|^2_{\mathcal{H}} = \sum_{i=1}^{N} \int_{\mathbb{R}^3} f_i^2 \nu_i \mu_i^{-1}\, dv < \infty \right\}. \tag{5.2.15}$$

Lemma 5.7. *For all* $\mathbf{f}$ *in Dom*$(\mathbf{L^m})$ *there exists an explicit constant* $C_1 > 0$*, such that*

$$\langle \mathbf{f}, \mathbf{L^m}(\mathbf{f}) \rangle_{L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)} \leq -C_1 \|\mathbf{f} - \pi_{\mathbf{L^m}}(\mathbf{f})\|^2_{L^2_v\left(\langle v\rangle^{\gamma/2}\boldsymbol{\mu}^{-1/2}\right)},$$

where C_1 *depends on the properties of the collision kernel, the number of species* N *and the different masses.*

Proof. By [111, Theorem 1.1, Remark 1] together with the shorthand introduced in (5.2.10),

$$\frac{1}{4} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ii} \left(A_{ii} \left[f_i \mu_i^{-1}, f_i \mu_i^{-1} \right] \right)^2 \mu_i \mu_i^* \, dv dv_* d\sigma \geq \lambda_m c_{\infty,i} \int_{\mathbb{R}^3} (f_i - \pi_{\mathbf{L^m}}(\mathbf{f})_i)^2 \nu_{ii} \mu_i^{-1}\, dv,$$

where $\lambda_m > 0$ depends on the properties of the collision kernel, the number of species N and the different masses. Summing this estimate over $i = 1, \dots, N$ and employing (5.2.9) yields

$$- \langle \mathbf{f}, \mathbf{L^m}(\mathbf{f}) \rangle_{L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)} \geq \lambda^m \sum_{i=1}^{N} c_{\infty,i} \int_{\mathbb{R}^3} (f_i - \pi_{\mathbf{L^m}}(f_i))^2 \frac{\nu_{ii}}{\mu_i}\, dv. \tag{5.2.16}$$

Now we can estimate ν_{ii} in terms of ν_i by using (5.2.5), and plugging this bound into (5.2.16) together with the fact that $\mathcal{H}$ is equivalent to $L^2_v\left(\langle v\rangle^{\gamma/2}\boldsymbol{\mu}^{-1/2}\right)$ finishes the proof. $\square$

Lemma 5.8. *For all* $\mathbf{f}$ *in Ker*$(\mathbf{L^m}) \cap$ *Dom*$(\mathbf{L^b})$ *there exists an explicit* $C_2 > 0$ *such that*

$$\left\langle \mathbf{f}, \mathbf{L^b}(\mathbf{f}) \right\rangle_{L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)} \leq -C_2\, \mathcal{E}(\mathbf{f}),$$

with the functional $\mathcal{E}$ defined by

$$\mathcal{E}:\ Ker(\mathbf{L^m}) \to \mathbb{R}^+, \quad \mathcal{E}(\mathbf{f}) := \sum_{i,j=1}^{N} \left(\left| u_i^{(f)} - u_j^{(f)} \right|^2 + \left(e_i^{(f)} - e_j^{(f)} \right)^2 \right), \tag{5.2.17}$$

where for fixed $\mathbf{f} \in Ker(\mathbf{L^m})$ it holds that $u_i^{(f)}$, $e_i^{(f)}$ describe the coordinates of the i^{th} component of $\mathbf{f}$ with respect to the basis defined in (5.2.11)*, and $C_2 > 0$ is defined in* (5.2.19)*.*

Remark 5.9. *Note that for $\mathbf{f}$ in $Ker(\mathbf{L^m})$ it holds that*

$$\mathcal{E}(\mathbf{f}) = 0 \quad \Leftrightarrow \quad \mathbf{f} \in Ker(\mathbf{L^b}),$$

since $Ker(\mathbf{L}) = Ker(\mathbf{L^m}) \cap Ker(\mathbf{L^b})$. This fact together with a multi-species version of the H-theorem show that the left-hand side of the estimate in Lemma 5.8 is null if and only if the right-hand side is null.

Proof. Let $\mathbf{f} \in \mathrm{Ker}(\mathbf{L^m}) \cap \mathrm{Dom}(\mathbf{L^b})$. Writing $\mathbf{f}$ in the form (5.2.11) and applying the microscopic conservation laws (5.1.2) yields

$$A_{ij}[f_i\mu_i^{-1}, f_j\mu_j^{-1}] = m_i(u_i - u_j)\cdot(v' - v) + m_i(e_i - e_j)(|v'|^2 - |v|^2),$$

and thus

$$\begin{aligned} &-\langle \mathbf{f}, \mathbf{L^b}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \\ &\quad = \frac{1}{4}\sum_{\substack{i,j=1\\ j\neq i}}^{N} m_i^2 \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}\left[(u_i - u_j)\cdot(v'-v) + (e_i - e_j)(|v'|^2 - |v|^2)\right]^2 \mu_i\mu_j^*. \end{aligned}$$

Using the symmetry of B_{ij} and of $\mu_i\mu_j^*$ together with the oddity of the function $G(v, v_*, \sigma) = B_{ij}(u_i - u_j)\cdot(v'-v)(|v'|^2 - |v|^2)$ with respect to (v, v_*, σ) yields that the mixed term in the square of the integral above vanishes. Thus we obtain

$$\begin{aligned} -\langle \mathbf{f}, \mathbf{L^b}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} &= \frac{1}{4}\sum_{\substack{i,j=1\\ j\neq i}}^{N} m_i^2 \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij} \\ &\times \left(|(u_i - u_j)\cdot(v'-v)|^2 + (e_i - e_j)^2(|v'|^2 - |v|^2)^2\right)\mu_i\mu_j^*\, dv dv_* d\sigma. \end{aligned} \tag{5.2.18}$$

We claim that the following holds

$$\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}((u_i - u_j)\cdot(v'-v))^2\mu_i\mu_j^*\, dv dv_* d\sigma = \frac{|u_i - u_j|^2}{3}\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}|v - v'|^2\mu_i\mu_j^*\, dv dv_* d\sigma.$$

To prove this identity, we write $u_{i,k}$ and v_k for the kth component of the vectors u_i and v, respectively. The change of variables $(v_k, v_k^*, \sigma_k) \mapsto -(v_k, v_k^*, \sigma_k)$ for fixed k leaves B_{ij}, μ_i, and μ_j^* unchanged but $v_k' \mapsto -v_k'$, such that

$$\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij} v_k' v_\ell \mu_i\mu_j^*\, dv dv_* d\sigma = 0 \quad \text{for } \ell \neq k.$$

Moreover,

$$\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij} v_k v_\ell \mu_i \mu_j^* \, dv dv_* d\sigma = 0 \quad \text{for } \ell \neq k,$$

since the integrand is odd. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}((u_i - u_j)\cdot(v'-v))^2 \mu_i\mu_j^* \, dv dv_* d\sigma \\
&\qquad = \sum_{k,\ell=1}^{3} (u_{i,k} - u_{j,k})(u_{i,\ell} - u_{j,\ell}) \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}(v_k' - v_k)(v_\ell' - v_\ell)\mu_i\mu_j^* \, dv dv_* d\sigma \\
&\qquad = \sum_{k=1}^{3} (u_{i,k} - u_{j,k})^2 \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}(v_k - v_k')^2 \mu_i\mu_j^* \, dv dv_* d\sigma.
\end{aligned}$$

Since the integral is independent of k, we get

$$\begin{aligned}
&\int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}((u_i - u_j)\cdot(v'-v))^2 \mu_i\mu_j^* \, dv dv_* d\sigma \\
&\qquad = \frac{1}{3}\sum_{k=1}^{3} (u_{i,k} - u_{j,k})^2 \int_{\mathbb{R}^6\times\mathbb{S}^2} B_{ij}|v - v'|^2 \mu_i\mu_j^* \, dv dv_* d\sigma,
\end{aligned}$$

which proves the claim.

This implies that for all $\mathbf{f}$ in $\mathrm{Ker}(\mathbf{L}^{\mathbf{m}}) \cap \mathrm{Dom}(\mathbf{L}^{\mathbf{b}})$ it holds that

$$\langle \mathbf{f}, \mathbf{L}^{\mathbf{b}}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \leq -C_2\, \mathcal{E}(\mathbf{f}),$$

where $\mathcal{E}(\cdot)$ is defined in (5.2.17) and

$$C_2 = \frac{1}{4}\min_{1\leq i,j\leq n} \int_{\mathbb{R}^6\times\mathbb{S}^2} m_i^2 B_{ij} \min\left\{\frac{1}{3}|v-v'|^2, (|v'|^2 - |v|^2)^2\right\} \mu_i\mu_j^* \, dv dv_* d\sigma. \tag{5.2.19}$$

The last part is to prove that $C_2 > 0$. For this we note that the integrand of (5.2.19) vanishes if and only if $|v'| = |v|$. However, the set

$$X = \{(v, v_*, \sigma) \in \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2 : |v'| = |v|\}$$

is closed since it is the pre-image of $\{0\}$ of the function $H(v, v_*, \sigma) = |v'|^2 - |v|^2$ which is continuous. Now X^c is open and nonempty and thus has positive Lebesgue measure, and since the integrand in (5.2.19) is positive on X^c, we get that $C_2 > 0$, which finishes the proof. □

Proof of Theorem 5.6. The proof will be performed in 4 steps, using the strategy worked out in Chapter 4 in order to get an explicit spectral-gap estimate, but now for different molar masses. To lighten notation, we will use the following shorthands for $\mathbf{f} \in \mathrm{Dom}(\mathbf{L})$:

$$\mathbf{f}^{\parallel} = \pi_{\mathbf{L}^{\mathbf{m}}}(\mathbf{f}), \quad \mathbf{f}^{\perp} = \mathbf{f} - \mathbf{f}^{\parallel}, \quad h_i^{\parallel} = \mu_i^{-1} f_i^{\parallel}, \quad h_i^{\perp} = \mu_i^{-1} h_i^{\perp}. \tag{5.2.20}$$

Step 1 : **Absorption of the orthogonal part.**
The nonnegativity of $-\langle \mathbf{f}, \mathbf{L^b}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \geq 0$ and Lemma 5.7 imply that

$$-\langle(\mathbf{f}, \mathbf{L}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \geq C_1 \|\mathbf{f} - \mathbf{f}^{\|}\|^2_{\mathcal{H}} - \eta \langle \mathbf{f}, \mathbf{L^b}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})}, \tag{5.2.21}$$

where $\eta \in (0,1]$ and $C_1 > 0$ was defined in Lemma 5.7. Now it holds that

$$A_{ij}[h_i, h_j]^2 \geq \frac{1}{2} A_{ij}[h_i^{\|}, h_j^{\|}]^2 - A_{ij}[h_i^{\perp}, h_j^{\perp}]^2,$$

and plugging this into (5.2.9) and (5.2.21) implies

$$\begin{aligned} -\langle \mathbf{f}, \mathbf{L}(\mathbf{f}))\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \geq C_1 \|f^{\perp}\|^2_{\mathcal{H}} + \frac{\eta}{8} \sum_{i=1}^N \sum_{j\neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i^{\|}, h_j^{\|}]^2 \mu_i \mu_j^* \, dv dv_* d\sigma \\ - \frac{\eta}{4} \sum_{i=1}^N \sum_{j\neq i}^N \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i^{\perp}, h_j^{\perp}]^2 \mu_i \mu_j^* \, dv dv_* d\sigma. \end{aligned} \tag{5.2.22}$$

Now we prove that (up to a small factor) the last term on the right-hand side can be estimated from below by $\|\mathbf{f}^{\perp}\|^2_{\mathcal{H}}$. For this we perform the standard change of variables $(v, v_*) \to (v_*, v)$ together with $i \leftrightarrow j$ and $(v, v_*) \to (v', v'_*)$, and by using the identity $\mu_i \mu_j^* = \mu'_i \mu'^*_j$ we obtain

$$\begin{aligned} &\sum_{i=1}^N \sum_{j\neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i^{\perp}, h_j^{\perp}]^2 \mu_i \mu_j^* \, dv dv_* d\sigma \\ &\quad \leq 4 \sum_{i=1}^N \sum_{j\neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} \big(((h_i^{\perp})')^2 + ((h_j^{\perp})'^*)^2 + (h_i^{\perp})^2 + ((h_j^{\perp})^*)^2 \big) \mu_i \mu_j^* \, dv dv_* d\sigma \\ &\quad \leq 16 \sum_{i=1}^N \sum_{j\neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (h_i^{\perp})^2 \mu_i \mu_j^* \, dv dv_* d\sigma. \end{aligned}$$

Taking into account the definition (5.2.3) of ν_i, we get for the last term on the right-hand side of (5.2.22)

$$\begin{aligned} &-\frac{\eta}{4} \sum_{i=1}^N \sum_{j\neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i^{\perp}, h_j^{\perp}]^2 \mu_i \mu_j^* \, dv dv_* d\sigma \\ &\qquad \geq -4\eta \sum_{i,j=1}^N \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (f_i^{\perp})^2 \mu_j^* \mu_i^{-1} \, dv dv_* d\sigma \\ &\qquad \geq -4\eta \sum_{i=1}^N \int_{\mathbb{R}^3} (f_i^{\perp})^2 \nu_i \mu_i^{-1} \, dv = -4\eta \|f^{\perp}\|^2_{\mathcal{H}}. \end{aligned}$$

Finally (5.2.22) yields

$$-\langle \mathbf{f}, \mathbf{L}(\mathbf{f}) \rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \geq (C_1 - 4\eta) \left\| \mathbf{f} - \mathbf{f}^{\parallel} \right\|^2_{\mathcal{H}} + \frac{\eta}{8} \sum_{i=1}^{N} \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij} \left[h_i^{\parallel}, h_j^{\parallel} \right]^2 \mu_i \mu_j^* \, dv dv_* d\sigma.$$

Thus

$$\langle \mathbf{f}, \mathbf{L}(\mathbf{f}) \rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \leq -(C_1 - 4\eta) \left\| \mathbf{f} - \mathbf{f}^{\parallel} \right\|^2_{\mathcal{H}} + \frac{\eta}{2} \langle \mathbf{f}^{\parallel}, \mathbf{L^b}(\mathbf{f}^{\parallel}) \rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})}, \tag{5.2.23}$$

where $0 < \eta \leq \min\{1, C_1/8\}$.

Step 2 : **Estimate for for the remaining part.** Due to Lemma 5.8 there exists an explicit $C_2 > 0$ such that

$$\left\langle \mathbf{f}^{\parallel}, \mathbf{L^b}(\mathbf{f}^{\parallel}) \right\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \leq -C_2 \, \mathcal{E}\left(\mathbf{f}^{\parallel} \right).$$

Step 3 : **Estimate for the momentum and energy differences.**
We need to find a relation between $\mathcal{E}(\mathbf{f}^{\parallel})$, $\left\| \mathbf{f} - \mathbf{f}^{\parallel} \right\|$ and $\|\mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})\|$ respectively. To this end, we decompose $\mathbf{f} = \mathbf{f}^{\parallel} + \mathbf{f}^{\perp}$ recalling that $\mathbf{f}^{\parallel} = \pi_{\mathbf{L^m}}(\mathbf{f})$ and $\mathbf{f}^{\perp} = \mathbf{f} - \mathbf{f}^{\parallel}$. Using an arbitrary orthonormal basis $(\boldsymbol{\psi}_k)_{1 \leq k \leq 5N}$ of $\mathrm{Ker}(\mathbf{L}^m)$ in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$, we first show that

$$\|\mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})\|^2_{\mathcal{H}} \leq 2\|\mathbf{f}^{\perp}\|^2_{\mathcal{H}} + k_0 \left(\|\mathbf{f}^{\parallel}\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})} - \|\pi_{\mathbf{L}}(\mathbf{f})\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})} \right), \tag{5.2.24}$$

where $k_0 = 10N \max_{1 \leq k, \ell \leq 5N} |\langle \boldsymbol{\psi_k}, \boldsymbol{\psi_\ell} \rangle_{\mathcal{H}}|$.
To this end, we start with

$$\|\mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})\|^2_{\mathcal{H}} \leq 2\big(\|\mathbf{f}^{\perp}\|^2_{\mathcal{H}} + \|\mathbf{f}^{\parallel} - \pi_{\mathbf{L}}(\mathbf{f})\|^2_{\mathcal{H}}\big). \tag{5.2.25}$$

Denoting the last term by $\mathbf{g} := \mathbf{f}^{\parallel} - \pi_{\mathbf{L}}(\mathbf{f}) \in \mathrm{Ker}(\mathbf{L}^m)$ (note that $\mathrm{Ker}(\mathbf{L}) \subset \mathrm{Ker}(\mathbf{L^m})$) and using Young's inequality implies

$$\begin{aligned}
\|\mathbf{g}\|^2_{\mathcal{H}} &= \sum_{i=1}^{N} \int_{\mathbb{R}^3} \left| \sum_{k=1}^{5N} \langle \mathbf{g}, \boldsymbol{\psi_k} \rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \psi_{k,i} \right|^2 \nu_i(v) \, dv \\
&= \sum_{k,\ell=1}^{5N} \langle \mathbf{g}, \boldsymbol{\psi}_k \rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \langle \mathbf{g}, \boldsymbol{\psi}_\ell \rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \langle \boldsymbol{\psi}_k, \boldsymbol{\psi}_\ell \rangle_{\mathcal{H}} \\
&\leq \frac{1}{2} \max_{1 \leq k, \ell \leq 5N} |\langle \boldsymbol{\psi_k}, \boldsymbol{\psi}_\ell \rangle_{\mathcal{H}}| \sum_{k,\ell=1}^{5N} \left(\langle \mathbf{g}, \boldsymbol{\psi_k} \rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} + \langle \mathbf{g}, \boldsymbol{\psi}_\ell \rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \right) \\
&= 5N \max_{1 \leq k, \ell \leq 5N} |\langle \boldsymbol{\psi_k}, \boldsymbol{\psi_\ell} \rangle_{\mathcal{H}}| \, \|\mathbf{g}\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})}.
\end{aligned}$$

Thus, (5.2.25) implies

$$\|\mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})\|^2_{\mathcal{H}} \leq 2\|\mathbf{f}^{\perp}\|^2_{\mathcal{H}} + 10N \max_{1 \leq k, \ell \leq 5N} |\langle \boldsymbol{\psi_k}, \boldsymbol{\psi_\ell} \rangle_{\mathcal{H}}| \, \|\mathbf{f}^{\parallel} - \pi_{\mathbf{L}}(\mathbf{f})\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})}.$$

Now $\mathrm{Ker}(\mathbf{L}) \subset \mathrm{Ker}(\mathbf{L}^{\mathbf{m}})$ implies $\pi_{\mathbf{L}^{\mathbf{m}}}\pi_{\mathbf{L}} = \pi_{\mathbf{L}}$, thus

$$\|\mathbf{f}^{\parallel} - \pi_{\mathbf{L}}(\mathbf{f})\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})} = \|\mathbf{f}^{\parallel}\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})} - \|\pi_{\mathbf{L}}(\mathbf{f})\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})},$$

which indeed yields (5.2.24).

Now the moment identities (5.2.13) and (5.2.14) yield

$$\|\mathbf{f}^{\parallel}\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})} = \sum_{i=1}^N c_{\infty,i}(m_i^2 a_i^2 + m_i|u_i|^2 + 15e_i^2 + 6m_i a_i e_i),$$

and

$$\begin{aligned}\|\pi_{\mathbf{L}}(\mathbf{f})\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})} &= \sum_{j=1}^{N+4} \langle \mathbf{f}, \boldsymbol{\phi_j}\rangle^2_{L^2_v(\boldsymbol{\mu}^{-1/2})} \\ &= \sum_{i=1}^N c_{\infty,i}(m_i a_i + 3e_i)^2 + \rho_\infty \left|\sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty} u_i\right|^2 + 6c_\infty\left(\sum_{i=1}^N \frac{c_{\infty,i}}{c_\infty} e_i\right)^2,\end{aligned}$$

where $\left(\boldsymbol{\phi}_j\right)_{1\leq j\leq N+4}$ is the orthonormal basis of $\mathrm{Ker}(\mathbf{L})$ in $L^2_v(\boldsymbol{\mu}^{-1/2})$ introduced in (5.2.1).

Inserting these expressions into (5.2.24), we conclude that

$$\begin{aligned}\|\mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})\|^2_{\mathcal{H}} \leq 2\|\mathbf{f} - \mathbf{f}^{\parallel}\|^2_{\mathcal{H}} &+ k_0\rho_\infty\left(\sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}|u_i|^2 - \left|\sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty} u_i\right|^2\right) \\ &+ 6k_0 c_\infty\left(\sum_{i=1}^N \frac{c_{\infty,i}}{c_\infty} e_i^2 - \left(\sum_{i=1}^N \frac{c_{\infty,i}}{c_\infty} e_i\right)^2\right).\end{aligned}$$

The next step is to prove that the following estimates hold:

$$I_1 := \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}|u_i|^2 - \left|\sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty} u_i\right|^2 \leq \sum_{i,j=1}^N |u_i - u_j|^2, \tag{5.2.26}$$

$$I_2 := \sum_{i=1}^N \frac{c_{\infty,i}}{c_\infty} e_i^2 - \left(\sum_{i=1}^N \frac{c_{\infty,i}}{c_\infty} e_i\right)^2 \leq \sum_{i,j=1}^N (e_i - e_j)^2. \tag{5.2.27}$$

Note that we only need to prove the estimate for I_1, since the arguments for I_2 are exactly the same. In order to handle the expression I_1, we define for $\mathbf{u} = (u_i)_{1\leq i\leq N}$ and $\mathbf{v} = (v_i)_{1\leq i\leq N} \in \mathbb{R}^{3N}$ the following scalar product on $\mathbb{R}^{3N}$ with corresponding norm

$$\langle \mathbf{u}, \mathbf{v}\rangle_\rho = \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty} u_i \cdot v_i, \quad \|\mathbf{u}\|_\rho = \langle \mathbf{u}, \mathbf{u}\rangle_\rho^{1/2},$$

where $u_i \cdot v_i$ denotes the standard Euclidean scalar product in $\mathbb{R}^3$. Note that the vector $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{3N}$ satisfies $\|\mathbf{1}\|_\rho = 1$. Now we use the following elementary identity

$$\|\mathbf{u}\|_\rho^2 - \langle \mathbf{u}, \mathbf{1}\rangle_\rho^2 = \|\mathbf{u} - \langle \mathbf{u}, \mathbf{1}\rangle_\rho \mathbf{1}\|_\rho^2,$$

which can be written as

$$I_1 = \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}|u_i|^2 - \left|\sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty} u_i\right|^2 = \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}\left|u_i - \sum_{j=1}^N \frac{\rho_{\infty,j}}{\rho_\infty} u_j\right|^2.$$

By using the fact that $\sum_{j=1}^N \rho_{\infty,j} = \rho_\infty$, we get

$$I_1 = \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}\left|\left(1 - \frac{\rho_{\infty,i}}{\rho_\infty}\right) u_i - \sum_{j\neq i} \frac{\rho_{\infty,j}}{\rho_\infty} u_j\right|^2 = \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}\left|\sum_{j\neq i} \frac{\rho_{\infty,j}}{\rho_\infty}(u_i - u_j)\right|^2.$$

Inserting the additional factor $(\sum_{j\neq i} \rho_{\infty,k}/\rho_\infty)^2$ leads to a convex combination of λ_j such that $\sum_{j\neq i} \lambda_j = 1$:

$$\begin{aligned} I_1 &= \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}\left(\sum_{k\neq i}\frac{\rho_{\infty,k}}{\rho_\infty}\right)^2 \left|\frac{\sum_{j\neq i}(\rho_{\infty,j}/\rho_\infty)(u_i - u_j)}{\sum_{k\neq i}\rho_{\infty,k}/\rho_\infty}\right|^2 \\ &= \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}\left(\sum_{k\neq i}\frac{\rho_{\infty,k}}{\rho_\infty}\right)^2 \left|\sum_{j\neq i}\lambda_j(u_i - u_j)\right|^2, \end{aligned}$$

where $\lambda_j = (\rho_{\infty,j}/\rho_\infty)(\sum_{k\neq i}(\rho_{\infty,k}/\rho_\infty))^{-1}$. Thus we can apply Jensen's inequality to this convex combination and obtain

$$I_1 = \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}\left(\sum_{k\neq i}\frac{\rho_{\infty,k}}{\rho_\infty}\right)^2 \left|\sum_{j\neq i}\lambda_j(u_i - u_j)\right|^2 \leq \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}\left(\sum_{k\neq i}\frac{\rho_{\infty,k}}{\rho_\infty}\right)^2 \sum_{j\neq i}\lambda_j |u_i - u_j|^2.$$

Finally, we can estimate the right-hand side easily by using the definition of the λ_j and that $\rho_{\infty,j} \leq \rho_\infty$ to obtain

$$I_1 \leq \sum_{i=1}^N \frac{\rho_{\infty,i}}{\rho_\infty}\left(1 - \frac{\rho_{\infty,i}}{\rho_\infty}\right)\sum_{j\neq i}\frac{\rho_{\infty,j}}{\rho_\infty}|u_i - u_j|^2 \leq \sum_{i,j=1}^N |u_i - u_j|^2.$$

For I_2 in (5.2.27) exactly the same calculations hold. This implies that

$$-\mathcal{E}(\mathbf{f}^{\|}) \leq -C_3\big(\|\mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})\|_{\mathcal{H}}^2 - 2\|\mathbf{f} - \mathbf{f}^{\|}\|_{\mathcal{H}}^2\big), \tag{5.2.28}$$

where $C_3 = 1/C_k > 0$, with

$$C_k = 10N \max_{1\leq k,\ell\leq 5N}\left|\sum_{i=1}^N \int_{\mathbb{R}^3} \psi_{k,i}\psi_{\ell,i}\nu_i \, dv\right| \max\{\rho_\infty, 6c_\infty\},$$

recalling that $(\psi_k)_{1\leq k\leq 5N}$ is an arbitrary orthonormal basis of Ker$(\mathbf{L}^{\mathbf{m}})$ in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$.

Step 4: End of the proof.
Putting together (5.2.23), Lemma 5.8, and (5.2.28) yields

$$\begin{aligned}\langle \mathbf{f}, \mathbf{L}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} &\leq -(C_1-4\eta)\|\mathbf{f}-\mathbf{f}^{\|}\|^2_{\mathcal{H}} - C_2/2\,\mathcal{E}(\mathbf{f}^{\|})\\ &\leq -\left(C_1-4\eta-C_2C_3\eta\right)\|\mathbf{f}-\mathbf{f}^{\|}\|^2_{\mathcal{H}} - (C_2C_3\eta)/2\|\mathbf{f}-\pi_{\mathbf{L}}(\mathbf{f})\,\|^2_{\mathcal{H}}.\end{aligned}$$

The first term on the right-hand side is nonnegative if we choose

$$0<\eta\leq \min\left\{1, C_1/(4+C_2C_3)\right\},$$

and the desired spectral-gap estimate follows with $\lambda_L = (C_2C_3C_4\eta)/2$, where the additional constant $C_4>0$ takes care of the fact that $\mathcal{H}$ is equivalent to $L^2_v\left(\langle v\rangle^{\gamma/2}\boldsymbol{\mu}^{-1/2}\right)$. □

Remark 5.10. *(1) We obtain the following relation between the spectral-gap constant λ derived for same masses $m_i = m_j$ for $1\leq i,j\leq N$ in Theorem 4.2 and our new constant λ_L for different masses in Theorem 5.6 :*

$$\lambda_L = \lambda \min_{1\leq i\leq N} m_i^2 \frac{6\rho_\infty}{\max\{\rho_\infty, 6c_\infty\}},$$

where $\rho_\infty = \sum_{i=1}^N m_i c_{\infty,i}$ and $c_\infty = \sum_{i=1}^N c_{\infty,i}$. Thus, increasing the difference between the masses m_i makes the the spectral-gap constant λ_L smaller, while in the special case of identical masses the two spectral-gap constants λ and λ_L are equal.

(2) Furthermore, the spectral-gap result of Theorem 5.6 only holds for a finite number of species $1\leq N<\infty$, since for $N\to\infty$ we get that $\lambda_L\to\infty$. It remains an open problem whether or not it is possible to extend the result of Theorem 5.6 to the limit $N\to\infty$.

5.3. L^2-theory for the linear part with Maxwellian weight

This section is devoted to the study of the linear perturbed operator $\mathbf{G} = \mathbf{L} - v\cdot\nabla_x$ in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$, which is the natural space for $\mathbf{L}$. We shall show that $\mathbf{G}$ generates a strongly continuous semigroup on this space.

Theorem 5.11. *We assume that assumptions $(H1)-(H4)$ hold for the collision kernel. Then the linear perturbed operator $\mathbf{G} = \mathbf{L} - v\cdot\nabla_x$ generates a strongly continuous semigroup $S_{\mathbf{G}}(t)$ on $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ which satisfies*

$$\forall t\geq 0,\quad \|S_{\mathbf{G}}(t)\,(Id-\Pi_{\mathbf{G}})\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} \leq C_G e^{-\lambda_G t},$$

where $\Pi_{\mathbf{G}}$ is the orthogonal projection onto Ker$(\mathbf{G})$ in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$.
The constants C_G, $\lambda_G>0$ are explicit and depend on N, the different masses m_i and the collision kernels.

Let us first make an important remark about $\Pi_{\mathbf{G}}$. Note that $\mathbf{G}(\mathbf{f}) = 0$ means

$$\forall i \in \{1, \dots, N\}, \forall (x,v) \in \mathbb{T}^3 \times \mathbb{R}^3, \quad v \cdot \nabla_x f_i(x,v) = L_i(\mathbf{f}(x,\cdot))(v)$$

Multiplying by $\mu_i^{-1}(v) f_i(x,v)$ and integrating over $\mathbb{T}^3 \times \mathbb{R}^3$ implies

$$0 = \int_{\mathbb{T}^3} \langle L_i(\mathbf{f}(x,\cdot)), f_i(x,\cdot)\rangle_{L^2_v\left(\mu_i^{-1/2}\right)} \, dx$$

and therefore by summing over i in $\{1, \dots, N\}$

$$0 = \int_{\mathbb{T}^3} \langle \mathbf{L}(\mathbf{f}(x,\cdot)), \mathbf{f}(x,\cdot)\rangle_{L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)} \, dx.$$

The integrand is nonpositive thanks to the spectral gap of $\mathbf{L}$ and hence

$$\forall x \in \mathbb{T}^3, \, \forall v \in \mathbb{R}^3, \quad \mathbf{f}(x,v) = \pi_{\mathbf{L}}(\mathbf{f}(x,\cdot))(v)$$

and therefore $\mathbf{L}(\mathbf{f}(x,\cdot)) = 0$. The latter further implies that $v \cdot \nabla_x \mathbf{f}(x,v) = 0$ which in turn implies that $\mathbf{f}$ does not depend on x [25, Lemma B.2].

We can thus define the projection in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ onto the kernel of $\mathbf{G}$

$$\Pi_{\mathbf{G}}(\mathbf{f}) = \sum_{k=1}^{N+4} \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} \langle \mathbf{f}(x,v), \boldsymbol{\phi}_k(v)\rangle_{\boldsymbol{\mu}^{-1/2}} \, dxdv \right) \boldsymbol{\phi}_k(v), \tag{5.3.1}$$

where the $\boldsymbol{\phi}_k$ were defined in (5.2.1). Again we define $\Pi_{\mathbf{G}}^{\perp} = \mathrm{Id} - \Pi_{\mathbf{G}}$. Note that $\Pi_{\mathbf{G}}^{\perp}(\mathbf{f}) = 0$ amounts to saying that $\mathbf{f}$ satifies the multi-species perturbed conservation laws (5.1.8), *i.e.* null individual mass, sum of momentum and sum of energy.

In Subsection 5.3.1, we show the key lemma of the proof that is the *a priori* control of the fluid part of $S_{\mathbf{G}}(t)$ by its orthogonal part, thus recovering some coercivity for $\mathbf{G}$ in the set of solutions to the linear perturbed equation. Subsection 5.3.2 is dedicated to the proof of Theorem 5.11.

5.3.1. A priori control of the fluid part by the microscopic part

As seen in the previous section, the operator $\mathbf{L}$ is only coercive on the orthogonal part. The key argument is to show that we recover some coercivity for solutions to the differential equation. Namely, that for these specific functions, the microscopic part controls the fluid part. This is the purpose of the next lemma

Lemma 5.12. *Let* $\mathbf{f_0}(x,v)$ *and* $\mathbf{g}(t,x,v)$ *be in* $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ *such that* $\Pi_{\mathbf{G}}(\mathbf{f_0}) = \Pi_{\mathbf{G}}(\mathbf{g}) = 0$. *Suppose that* $\mathbf{f}(t,x,v)$ *in* $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ *is solution to the equation*

$$\partial_t \mathbf{f} = \mathbf{L}(\mathbf{f}) - v \cdot \nabla_x \mathbf{f} + \mathbf{g} \tag{5.3.2}$$

with initial value $\mathbf{f_0}$ *and satisfying the multi-species conservation laws. Then there exist an explicit* $C_\perp > 0$ *and a function* $N_{\mathbf{f}}(t)$ *such that for all* $t \geq 0$

(i) $|N_{\mathbf{f}}(t)| \leq C_\perp \left\|\mathbf{f}(t)\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}$;

(ii)

$$\int_0^t \left\|\pi_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\, ds \leq N_{\mathbf{f}}(t) - N_{\mathbf{f}}(0) + C_\perp \int_0^t \left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\, ds$$
$$+ C_\perp \int_0^t \left\|\mathbf{g}\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\, ds.$$

The constant $C_\perp$ is independent of $\mathbf{f}$ *and* $\mathbf{g}$.

The methods of the proof are a technical adaptation of the method proposed in [66] in the case of bounded domain with diffusive boundary conditions. The description of $\mathrm{Ker}(\mathbf{L})$ associated with the global equilibrium $\boldsymbol{\mu}$ is given by orthogonal functions in L^2_v but that are not of norm one. Unlike [66] where only mass conservation holds but boundary conditions overcome the lack of conservation laws, we strongly need the conservation of mass, momentum and energy.

Proof of Lemma 5.12. We recall (5.2.1) the definition of $\pi_{\mathbf{L}}(\mathbf{f}) = (\pi_i(\mathbf{f}))_{1\leq i\leq N}$ and we define $(a_i(t,x))_{1\leq i\leq N}$, $b(t,x)$ and $c(t,x)$ to be the coordinates of $\pi_{\mathbf{L}}(\mathbf{f})$:

$$\forall\, 1\leq i\leq N, \quad \pi_i(\mathbf{f})(t,x,v) = \left[a_i(t,x) + b(t,x)\cdot v + c(t,x)\frac{|v^2| - 3m_i^{-1}}{2}\right] m_i\mu_i(v). \quad (5.3.3)$$

Note that we are working with an orthogonal but not orthonormal basis of $\mathrm{Ker}(\mathbf{L})$ in $L^2_{x,v}(\boldsymbol{\mu}^{-\mathbf{1/2}})$ in order to lighten computations. We will denote by ρ_i the mass of $m_i\mu_i$.

The key idea of the proof is to choose suitable test functions $\boldsymbol{\psi} = (\psi_i)_{1\leq i\leq N}$ in $H^1_{x,v}$ that will catch the elliptic regularity of a_i, b and c and estimate them.

For a test function $\boldsymbol{\psi} = \boldsymbol{\psi}(t,x,v)$ integrated against the differential equation (5.3.2) we have by Green's formula on each coordinate

$$\int_0^t \frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} \langle \boldsymbol{\psi}, \mathbf{f}\rangle_{\mathbf{1}}\, dxdvds = \int_{\mathbb{T}^3\times\mathbb{R}^3} \langle \boldsymbol{\psi}(t), \mathbf{f}(t)\rangle_{\mathbf{1}}\, dxdv - \int_{\mathbb{T}^3\times\mathbb{R}^3} \langle \boldsymbol{\psi}_0, \mathbf{f_0}\rangle_{\mathbf{1}}\, dxdv$$
$$= \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \langle \mathbf{f}, \partial_t\boldsymbol{\psi}\rangle_{\mathbf{1}}\, dxdvds + \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \langle \mathbf{L}\left(\mathbf{f}\right), \boldsymbol{\psi}\rangle_{\mathbf{1}}\, dxdvds$$
$$+ \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} f_i v\cdot\nabla_x\psi_i\, dxdvds + \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \langle \boldsymbol{\psi}, \mathbf{g}\rangle_{\mathbf{1}}\, dxdvds.$$

We decompose $\mathbf{f} = \pi_{\mathbf{L}}(\mathbf{f}) + \pi^\perp_{\mathbf{L}}(\mathbf{f})$ in the term involving $v\cdot\nabla_x$ and use the fact that $\mathbf{L}\left(\mathbf{f}\right) = \mathbf{L}[\pi^\perp_{\mathbf{L}}(\mathbf{f})]$ to obtain the weak formulation

$$-\sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_i(\mathbf{f}) v\cdot\nabla_x\psi_i\, dxdvds = \Psi_1(t) + \Psi_2(t) + \Psi_3(t) + \Psi_4(t) + \Psi_5(t) \quad (5.3.4)$$

with the following definitions

$$\begin{aligned}
\Psi_1(t) &= \int_{\mathbb{T}^3\times\mathbb{R}^3} \langle \boldsymbol{\psi}_0, \mathbf{f_0}\rangle_{\mathbf{1}}\, dxdv - \int_{\mathbb{T}^3\times\mathbb{R}^3} \langle \boldsymbol{\psi}(t), \mathbf{f}(t)\rangle_{\mathbf{1}}\, dxdv, &(5.3.5)\\
\Psi_2(t) &= \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_{\mathbf{L}}^{\perp}(\mathbf{f})_i v\cdot\nabla_x \psi_i\, dxdvds, &(5.3.6)\\
\Psi_3(t) &= \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \mathbf{L}\left(\pi_{\mathbf{L}}^{\perp}(\mathbf{f})\right)_i \psi_i\, dxdvds, &(5.3.7)\\
\Psi_4(t) &= \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} f_i\partial_s\psi_i\, dxdvds, &(5.3.8)\\
\Psi_5(t) &= \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \langle \boldsymbol{\psi}, \mathbf{g}\rangle_{\mathbf{1}}\, dxdvds. &(5.3.9)
\end{aligned}$$

For each of the functions $\mathbf{a} = (a_i)_{1\leq i\leq N}$, b and c, we construct a $\boldsymbol{\psi}$ such that the left-hand side of (5.3.4) is exactly the L^2_x-norm of the function and the rest of the proof is estimating the four different terms $\Psi_i(t)$. Note that $\Psi_1(t)$ is already under the desired form

$$\Psi_1(t) = N_{\mathbf{f}}(t) - N_{\mathbf{f}}(0) \tag{5.3.10}$$

with $|N_{\mathbf{f}}(s)| \leq C\,\|\mathbf{f}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}$ if $\psi_i(x,v)\mu_i^{1/2}(v)$ is in $L^2_{x,v}$ for all i and their norm is controlled by the one of $\mathbf{f}$ (which will be the case in our next choices).

Remark 5.13. *The linear perturbed equation* (5.3.2) *and the conservation laws are invariant under standard time mollification. We therefore consider for simplicity in the rest of the proof that all functions are smooth in the variable t. Exactly the same estimates can be derived for more general functions and the method would obviously be to study time mollified equation and then take the limit in the smoothing parameter.*

For clarity, every positive constant will be denoted by C_k.

Estimate for $\mathbf{a} = (a_i)_{1\leq i\leq N}$. By assumption $\mathbf{f}$ preserves the mass which is equivalent to

$$0 = \int_{\mathbb{T}^3\times\mathbb{R}^3} \mathbf{f}(t,x,v)\, dxdv = \int_{\mathbb{T}^3}\left(\int_{\mathbb{R}^3} \langle \mathbf{f}(t,x,v), \boldsymbol{\mu}\rangle_{\boldsymbol{\mu}^{-1/2}}\, dv\right) dx = \int_{\mathbb{T}^3} \mathbf{a}(t,x)\, dx,$$

where we used the fact that $\boldsymbol{\mu}\in \mathrm{Ker}(\mathbf{G})$, $\mathbf{f_0}\in\mathrm{Ker}(\mathbf{G})^{\perp}$ and the orthogonality of the basis defined in (5.3.3). Define a test function $\boldsymbol{\psi}_{\mathbf{a}} = (\psi_i)_{1\leq i\leq N}$ by

$$\psi_i(t,x,v) = \left(|v|^2 - \alpha_i\right) v\cdot\nabla_x\phi_i(t,x)$$

where

$$-\Delta_x\phi_i(t,x) = a_i(t,x)$$

and $\alpha_i > 0$ is chosen such that for all $1 \leq k \leq 3$

$$\int_{\mathbb{R}^3} \left(|v|^2 - \alpha_i\right) \frac{|v|^2 - 3m_i^{-1}}{2} v_k^2 \mu_i(v)\, dv = 0.$$

The integral over $\mathbb{T}^3$ of $a_i(t,\cdot)$ is null and therefore standard elliptic estimate [67] yields:

$$\forall t \geq 0, \quad \|\phi_i(t)\|_{H^2_x} \leq C_0 \|a_i(t)\|_{L^2_x}. \tag{5.3.11}$$

The latter estimate provides both the control of $\Psi_1 = N^{(a)}_{\mathbf{f}}(t) - N^{(a)}_{\mathbf{f}}(0)$, as discussed before, and the control of (5.3.9), using Cauchy-Schwarz and Young's inequality,

$$\begin{aligned} |\Psi_5(t)| &\leq C \sum_{i=1}^N \int_0^t \|\sqrt{\rho_i}\phi_i\|_{L^2_x} \|g_i\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)}\, ds \\ &\leq \frac{C_1}{4} \int_0^t \|\mathbf{a}\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds + C_5 \int_0^t \|\mathbf{g}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\, ds, \end{aligned} \tag{5.3.12}$$

where $C_1 > 0$ is given in (5.3.13) below and where we defined $\boldsymbol{\rho} = (\rho_i)_{1\leq i\leq N}$ the vector of the masses associated to $(m_i\mu_i)_{1\leq i\leq N}$.

Firstly, we compute the term on the left-hand side of (5.3.4).

$$\begin{aligned} &-\sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_i(\mathbf{f}) v\cdot\nabla_x\psi_i\, dxdvds \\ &= -\sum_{i=1}^N \sum_{1\leq j,k\leq 3} \int_0^t \int_{\mathbb{T}^3} a_i(s,x) \left(\int_{\mathbb{R}^3} \left(|v|^2-\alpha_i\right) v_j v_k m_i \mu_i(v)\, dv\right) \partial_{x_j}\partial_{x_k}\phi_i\, dxds \\ &\quad -\sum_{i=1}^N \sum_{1\leq j,k\leq 3} \int_0^t \int_{\mathbb{T}^3} b(s,x)\cdot \left(\int_{\mathbb{R}^3} v\left(|v|^2-\alpha_i\right) v_j v_k m_i \mu_i(v)\, dv\right) \partial_{x_j}\partial_{x_k}\phi_i\, dxds \\ &\quad -\sum_{i=1}^N \sum_{1\leq j,k\leq 3} \int_0^t \int_{\mathbb{T}^3} c(s,x) \left(\int_{\mathbb{R}^3} \left(|v|^2-\alpha_i\right) \frac{|v|^2-3m_i^{-1}}{2} v_j v_k m_i \mu_i dv\right) \partial_{x_j}\partial_{x_k}\phi_i. \end{aligned}$$

The second term is null as well as the first and last ones when $j \neq k$ thanks to the oddity in v. In the last term when $j = k$ we recover our choice of α_i which makes the last term being null too. It remains the first term when $k = j$. In this case, the integral in v gives a constant C_1 independent of i times ρ_i. Direct computations give $\alpha_i = 10/m_i$ and $C_1 > 0$. It follows

$$\begin{aligned} -\sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_i(\mathbf{f}) v\cdot\nabla_x\psi_i\, dxdvds &= -C_1 \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3} a_i(s,x)\rho_i\Delta_x\phi_i(s,x)\, dxds \\ &= C_1 \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3} a_i^2 \rho_i\, ds \end{aligned}$$

$$= \quad C_1 \int_0^t \|\mathbf{a}(s)\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds. \tag{5.3.13}$$

We recall $\mathbf{L} = -\boldsymbol{\nu}(v) + \mathbf{K}$ where $\mathbf{K}$ is a bounded operator in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$. Moreover, the H^2_x-norm of $\phi_i(t,x)$ is bounded by the L^2_x-norm of $a_i(t,x)$. Multiplying by $\mu_i^{1/2}(v)\mu_i(v)^{-1/2}$ inside the i^{th} integral of Ψ_2 (5.3.6) and of Ψ_3 (5.3.7) a mere Cauchy-Schwarz inequality yields

$$\begin{aligned}
\forall k \in \{2,3\}, \quad |\Psi_k(t)| &\leq C \sum_{i=1}^N \int_0^t \|\sqrt{\rho_i} a_i\|_{L^2_x} \left\|\pi_i^\perp(\mathbf{f})\right\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)} ds \\
&\leq \frac{C_1}{4} \int_0^t \|\mathbf{a}\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds + C_2 \int_0^t \left\|\pi_{\mathbf{L}}^\perp(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\, ds.
\end{aligned} \tag{5.3.14}$$

We used Young's inequality for the last inequality, with C_1 defined in (5.3.13).

It remains to estimate the term with time derivatives (5.3.8). It reads

$$\begin{aligned}
\Psi_4(t) &= \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} f_i \left(|v|^2 - \alpha_i\right) v\cdot[\partial_t \nabla_x \phi_i]\, dxdvds \\
&= \sum_{i=1}^N \sum_{k=1}^3 \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_i(\mathbf{f}) \left(|v|^2 - \alpha_i\right) v_k \partial_t \partial_{x_k}\phi_i\, dxdvds \\
&\quad + \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_i^\perp(\mathbf{f}) \left(|v|^2 - \alpha_i\right) v\cdot[\partial_t \nabla_x \phi_i]\, dxdvds
\end{aligned}$$

Using oddity properties for the first integral on the right-hand side and then Cauchy-Schwarz with the following bound

$$\int_{\mathbb{R}^3} \left(|v|^2 - \alpha_i\right)^2 |v|^2 \mu_i(v)\, dv = C\rho_i < +\infty$$

we get

$$|\Psi_4(t)| \leq C \sum_{i=1}^N \int_0^t \left[\sum_{k=1}^3 \|\rho_i b_k\|_{L^2_x} + \left\|\pi_i^\perp(\mathbf{f})\right\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)}\right] \|\partial_t \nabla_x \phi_i\|_{L^2_x}\, ds. \tag{5.3.15}$$

Estimating $\|\partial_t \nabla_x \phi_a\|_{L^2_x}$ will come from elliptic estimates in negative Sobolev spaces. We use the decomposition of the weak formulation (5.3.4) between t and $t+\varepsilon$ (instead of between 0 and t) with $\boldsymbol{\psi}(t,x,v) = \phi(x)\mathbf{e}_i \in H^1_x$, where $\mathbf{e}_i = (\delta_{ji})_{1\leq j\leq N}$. We furthermore require that $\phi(x)$ has a null integral over $\mathbb{T}^3$. $\boldsymbol{\psi}$ only depends on x and therefore $\Psi_4(t) = 0$. Moreover, multiplying by $\mu_i(v)\mu_i^{-1}(v)$ in the i^{th} integral of Ψ_3 yields

$$\Psi_3(t) = \int_t^{t+\varepsilon} \int_{\mathbb{T}^3} \langle \mathbf{L}(\mathbf{f}), \mu_i \mathbf{e}_i\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \phi(x)\, dxdvds = 0,$$

by definition of Ker($\mathbf{L}$).

From the weak formulation (5.3.4) it therefore remains

$$\begin{aligned}
&\int_{\mathbb{T}^3\times\mathbb{R}^3} \phi(x)\langle \mathbf{e}_i, \mathbf{f}(t+\varepsilon)\rangle_{\mathbf{1}}\, dxdv - \int_{\mathbb{T}^3\times\mathbb{R}^3} \phi(x)\langle \mathbf{e}_i, \mathbf{f}(t)\rangle_{\mathbf{1}}\, dxdv \\
&\quad = \int_t^{t+\varepsilon}\int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_i(\mathbf{f})v\cdot\nabla_x\phi(x)\, dxdvds + \int_t^{t+\varepsilon}\int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_i^\perp(\mathbf{f})v\cdot\nabla_x\phi(x)\, dxdvds \\
&\qquad + \int_t^{t+\varepsilon}\int_{\Omega\times\mathbb{R}^3} g_i(s,x,v)\phi(x)\, dxdvds
\end{aligned}$$

which is equal to

$$\begin{aligned}
\int_{\mathbb{T}^3} \rho_i\left[a_i(t+\varepsilon)-a_i(t)\right]\phi(x)\, dx \quad = \quad & C\int_t^{t+\varepsilon}\int_{\mathbb{T}^3} \rho_i b(s,x)\cdot\nabla_x\phi(x)\, dxds \\
& + \int_t^{t+\varepsilon}\int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_i^\perp(\mathbf{f})\mu_i(v)^{-1/2}\mu_i(v)^{1/2}v\cdot\nabla_x\phi(x) \\
& + \int_t^{t+\varepsilon}\int_{\Omega\times\mathbb{R}^3} g_i(s,x,v)\phi(x)\, dxdvds,
\end{aligned}$$

where C does not depend on i.

Dividing by $\rho_i\varepsilon$ and taking the limit as ε goes to 0 yields, after a mere Cauchy-Schwarz inequality on the right-hand side

$$\begin{aligned}
\left|\int_{\mathbb{T}^3} \partial_t a_i(s,x)\phi(x)\, dx\right| \quad \leq \quad & C\left[\|b(t,x)\|_{L^2_x} + \left\|\pi_i^\perp(\mathbf{f})\right\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)}\right]\|\nabla_x\phi(x)\|_{L^2_x} \\
& + C\,\|g_i\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)}\|\phi\|_{L^2_x} \\
\leq \quad & C\left[\|b(t,x)\|_{L^2_x} + \left\|\pi_i^\perp(\mathbf{f})\right\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)} + \|g_i\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)}\right] \\
& \times\|\nabla_x\phi(x)\|_{L^2_x}.
\end{aligned}$$

We used Poincaré inequality since $\phi(x)$ has a null integral over $\mathbb{T}^d$. The latter inequality is true for all ϕ in H^1_x with a null integral and therefore implies for all $t\geq 0$

$$\|\partial_t a_i(t,x)\|_{(\mathcal{H}^1_x)^*} \leq C\left[\|b(t,x)\|_{L^2_x} + \left\|\pi_i^\perp(\mathbf{f})\right\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)} + \|g_i\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)}\right] \tag{5.3.16}$$

where $\left(\mathcal{H}^1_x\right)^*$ is the dual of the set of functions in H^1_x with null integral.

Thanks to the conservation of mass we have that $\partial_t a_i(t,x)$ have a zero integral on the torus and we can construct $\Phi_i(t,x)$ such that

$$-\Delta_x\Phi_i(t,x) = \partial_t a_i(t,x)$$

and by standard elliptic estimate [67]:

$$\|\Phi_i\|_{\mathcal{H}^1_x} \leq \|\partial_t a_i\|_{(\mathcal{H}^1_x)^*} \leq C\left[\|b(t,x)\|_{L^2_x} + \left\|\pi_i^\perp(\mathbf{f})\right\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)} + \|g_i\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)}\right],$$

where we used (5.3.16). Combining this estimate with

$$\|\partial_t \nabla_x \phi_i\|_{L^2_x} = \left\|\nabla_x \Delta^{-1}\partial_t a_i\right\|_{L^2_x} \leq \left\|\Delta^{-1}\partial_t a_i\right\|_{H^1_x} = \|\Phi_i\|_{H^1_x}$$

we can further control Ψ_4 in (5.3.15) using $\rho_i = \sqrt{\rho_i}\sqrt{\rho_i}$

$$|\Psi_4(t)| \leq C_5 \int_0^t \left(\sum_{i=1}^N \|\sqrt{\rho_i} b\|^2_{L^2_x} + \left\|\pi_i^\perp(\mathbf{f})\right\|^2_{L^2_{x,v}\left(\mu_i^{-1/2}\right)} + \|g_i\|^2_{L^2_{x,v}\left(\mu_i^{-1/2}\right)}\right) ds. \tag{5.3.17}$$

We now plug (5.3.13), (5.3.10), (5.3.14), (5.3.17) and (5.3.12) into (5.3.4)

$$\begin{aligned}\int_0^t \|\mathbf{a}\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds \leq & N_{\mathbf{f}}^{(a)}(t) - N_{\mathbf{f}}^{(a)}(0) + C_{a,b}\int_0^t \|b\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds \\ & + C_a \int_0^t \left[\left\|\pi_{\mathbf{L}}^\perp(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right] ds.\end{aligned} \tag{5.3.18}$$

Estimate for b. The choice of function to integrate against to deal with the b term is more involved technically. We emphasize that $b(t,x)$ is a vector $(b_1(t,x), b_2(t,x), b_3(t,x))$, thus we used the obvious short-hand notation

$$\|b\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} = \sum_{i=1}^N \sum_{k=1}^3 \|\sqrt{\rho_i} b_k\|^2_{L^2_x}.$$

Fix J in $\{1,2,3\}$ and the conservation of momentum implies that for all $t \geq 0$

$$\int_{\mathbb{T}^3} b_J(t,x)\, dx = 0.$$

Define $\boldsymbol{\psi}_{b_J}(t,x,v) = (\psi_{iJ}(t,x,v))_{1\leq i \leq N}$ with

$$\psi_{iJ}(t,x,v) = \sum_{j=1}^3 \varphi_{ij}^{(J)}(t,x,v),$$

with

$$\varphi_{ij}^{(J)}(t,x,v) = \begin{cases} |v|^2 v_j v_J \partial_{x_j}\phi_J(t,x) - \dfrac{7}{2m_i}\left(v_j^2 - m_i^{-1}\right)\partial_{x_J}\phi_J(t,x), & \text{if } j \neq J \\ \dfrac{7}{2m_i}\left(v_J^2 - m_i^{-1}\right)\partial_{x_J}\phi_J(t,x), & \text{if } j = J. \end{cases}$$

where

$$-\Delta_x \phi_J(t,x) = b_J(t,x).$$

Since it will be important, we emphasize here that for all $j \neq k$

$$\int_{\mathbb{R}^3} \left(v_j^2 - m_i^{-1}\right)\mu_i(v)\, dv = 0 \quad \text{and} \quad \int_{\mathbb{R}^3}\left(v_j^2 - m_i^{-1}\right) v_k^2 \mu_i(v)\, dv = 0. \tag{5.3.19}$$

The null integral of b_J implies by standard elliptic estimate [67]

$$\forall t \geq 0, \quad \|\phi_J(t)\|_{H^2_x} \leq C_0 \|b_J(t)\|_{L^2_x}. \tag{5.3.20}$$

Again, this estimate provides the control of $\Psi_1(t) = N^{(J)}_{\mathbf{f}}(t) - N^{(J)}_{\mathbf{f}}(0)$ and of $\Psi_5(t)$ as in (5.3.12):

$$|\Psi_5(t)| \leq \frac{C_1}{4}\int_0^t \|b_J\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds + C_5 \int_0^t \|\mathbf{g}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\, ds, \tag{5.3.21}$$

where $C_1 > 0$ is given in (5.3.22) below.

We start by the left-hand side of (5.3.4). By oddity, there is neither contribution from any of the $a_i(s,x)$ nor from $c(s,x)$. Hence, for all i in $\{1,\dots,N\}$

$$\begin{aligned}
&-\int_0^t \int_{\Omega\times\mathbb{R}^3} \pi_i(\mathbf{f}) v\cdot\nabla_x \psi_{iJ}\, dxdvds \\
&= -\sum_{1\leq k,l\leq 3} \sum_{\substack{j=1\\ j\neq J}}^3 \int_0^t\int_\Omega b_l(s,x)\left(\int_{\mathbb{R}^3} \left|v^2\right| v_l v_k v_j v_J m_i \mu_i(v)\, dv\right) \partial_{x_k}\partial_{x_j}\phi_J(s,x)\, dxds \\
&\quad + \frac{7}{2m_i} \sum_{1\leq k,l\leq 3} \sum_{\substack{j=1\\ j\neq J}}^3 \int_0^t\int_\Omega b_l(s,x)\left(\int_{\mathbb{R}^3} \left(v_j^2 - m_i^{-1}\right) v_l v_k m_i \mu_i dv\right) \partial_{x_k}\partial_{x_J}\phi_J\, dxds \\
&\quad - \frac{7}{2m_i} \sum_{1\leq k,l\leq 3} \int_0^t\int_\Omega b_l(s,x)\left(\int_{\mathbb{R}^3} \left(v_J^2 - m_i^{-1}\right) v_l v_k m_i \mu_i(v)\, dv\right) \partial_{x_k}\partial_{x_J}\phi_J\, dxds.
\end{aligned}$$

The last two integrals on $\mathbb{R}^3$ are zero if $l \neq k$. Moreover, when $l = k$ and $l \neq J$ it is also zero by (5.3.19). We compute directly for $l = J$

$$\int_{\mathbb{R}^3} \left(v_J^2 - m_i^{-1}\right) v_J^2 m_i \mu_i(v)\, dv = \frac{2}{m_i^2}\rho_i.$$

The first term is composed by integrals in v of the form

$$\int_{\mathbb{R}^3} |v|^2 v_k v_j v_l v_J \mu_i(v)\, dv$$

which is always null unless two indices are equals to the other two. Therefore if $j = l$ then $k = J$ and if $j \neq l$ we only have two options: $k = j$ and $l = J$ or $k = l$ and $j = J$. Hence, for all i in $\{1,\dots,N\}$

$$\begin{aligned}
&-\int_0^t \int_{\Omega\times\mathbb{R}^3} \pi_i(\mathbf{f}) v\cdot\nabla_x \psi_J\, dxdvds \\
&= -\sum_{\substack{j=1\\ j\neq J}}^3 \int_0^t\int_\Omega b_J(s,x)\partial_{x_j x_j}\phi_J \left(\int_{\mathbb{R}^3} |v|^2 v_j^2 v_J^2 m_i\mu_i(v)\, dv\right) dxds
\end{aligned}$$

$$-\sum_{\substack{j=1\\j\neq J}}^{3}\int_0^t\int_\Omega b_j(s,x)\partial_{x_jx_J}\phi_J\left(\int_{\mathbb{R}^3}|v|^2\,v_j^2v_J^2m_i\mu_i(v)\,dv\right)dxds$$
$$+\frac{7}{m_i^3}\sum_{\substack{j=1\\j\neq J}}^{3}\int_0^t\int_\Omega \rho_i b_j(s,x)\partial_{x_jx_J}\phi_J\,dxds-\frac{7}{m_i^3}\int_0^t\int_\Omega\rho_i b_J(s,x)\partial_{x_J}\partial_{x_J}\phi_J(s,x)\,dxds.$$

To conlude we compute for $j\neq J$

$$\int_{\mathbb{R}^3}\left|v^2\right|v_j^2v_J^2m_i\mu_i(v)\,dv=\frac{7}{m_i^3}\rho_i$$

and it thus only remains the following equality for all i in $\{1,\dots,N\}$.

$$\begin{aligned}-\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_i(\mathbf{f})v\cdot\nabla_x\psi_J\,dxdvds &= -\frac{7}{m_i^3}\int_0^t\int_\Omega\rho_i b_J(s,x)\Delta_x\phi_J(s,x)\,dxds\\ &= \frac{7}{m_i^3}\int_0^t\|\sqrt{\rho_i}b_J\|^2_{L^2_x}\,ds.\end{aligned}$$

Summing over i yields

$$-\sum_{i=1}^{N}\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_j(\mathbf{f})v\cdot\nabla_x\psi_J=\frac{7}{m_i^3}\int_0^t\|b_J\|_{L^2_x(\boldsymbol{\rho}^{1/2})}\,dxdvds. \tag{5.3.22}$$

We recall $\boldsymbol{\rho}=(\rho_i)_{1\leq i\leq N}$.

Then the terms Ψ_2 and Ψ_3 are dealt with as in (5.3.14)

$$\forall k\in\{2,3\}\,,\quad |\Psi_k(t)|\leq\frac{7}{4}\int_0^t\|b_J\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\,ds+C_2\int_0^t\left\|\pi_{\mathbf{L}}^{\perp}(\mathbf{f})\right\|^2_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)}\,ds. \tag{5.3.23}$$

It remains to estimate Ψ_4 which involves time derivative (5.3.8):

$$\begin{aligned}\Psi_4(t) &= \sum_{i=1}^{N}\sum_{j=1}^{3}\int_0^t\int_{\Omega\times\mathbb{R}^3}f_i\partial_t\varphi_{ij}^{(J)}(s,x,v)\,dxdvds\\ &= \sum_{i=1}^{N}\sum_{j=1}^{3}\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_i^{\perp}(\mathbf{f})\partial_t\varphi_{ij}^{(J)}(s,x,v)\,dxdvds\\ &\quad+\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq J}}^{3}\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_i(\mathbf{f})\,|v|^2\,v_jv_J\partial_{x_j}\phi_J\,dxdvds\\ &\quad+\sum_{i=1}^{N}\sum_{j=1}^{3}\pm\frac{7}{2m_i}\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_i(\mathbf{f})\left(v_j^2-m_j^{-1}\right)\partial_{x_J}\phi_J\,dxdvds.\end{aligned}$$

By oddity arguments, only terms in $a_i(s,x)$ and $c(s,x)$ can contribute to the last two terms on the right-hand side. However, $j \neq J$ implies that the second term is zero as well as the contribution of $a_i(s,x)$ in the third term thanks to (5.3.19). Finally, a Cauchy-Schwarz inequality on both integrals yields as in (5.3.15)

$$|\Psi_4(t)| \leq C \sum_{i=1}^{N} \int_0^t \left[\|\rho_i c\|_{L^2_x} + \left\|\pi_i^\perp(\mathbf{f})\right\|_{L^2_{x,v}\left(\mu_i^{-1/2}\right)} \right] \|\partial_t \nabla_x \phi_J\|_{L^2_x} \, ds. \tag{5.3.24}$$

To estimate $\|\partial_t \nabla_x \phi_J\|_{L^2_x}$ we follow the idea developed for $\mathbf{a}(s,x)$ about negative Sobolev regularity. We apply the weak formulation (5.3.4) to a specific function between t and $t+\varepsilon$. The test function is $\boldsymbol{\psi}(x,v) = \phi(x) v_J \mathbf{m}$ with ϕ in H^1_x with a zero integral over $\mathbb{T}^3$. Note that ψ does not depend on t so $\Psi_4 = 0$ and multiplying by $\mu_i(v)\mu_i^{-1}(v)$ in the i^{th} integral of Ψ_3 yields

$$\Psi_3(t) = \int_0^t \int_{\mathbb{T}^3} \langle \mathbf{L}(\mathbf{f}), v_J (m_i \mu_i)_{1 \leq i \leq N} \rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})} \partial_{x_k} \phi(x) \, dx dv ds = 0,$$

by definition of $\mathrm{Ker}(\mathbf{L})$.

It remains

$$\begin{aligned} C \sum_{i=1}^{N} \int_\Omega \rho_i \left[b_J(t+\varepsilon) - b_J(t) \right] \phi(x) \, dx \\ = \sum_{i=1}^{N} \int_t^{t+\varepsilon} \int_{\Omega \times \mathbb{R}^3} \pi_i(\mathbf{f}) v_J v \cdot \nabla_x \phi(x) \, dx dv ds \\ + \sum_{i=1}^{N} \int_t^{t+\varepsilon} \int_{\Omega \times \mathbb{R}^3} \pi_i^\perp(\mathbf{f}) v_J v \cdot \nabla_x \phi(x) \, dx dv ds \\ + \sum_{i=1}^{N} \int_t^{t+\varepsilon} \int_{\Omega \times \mathbb{R}^3} g_i v_J \phi(x) \, dx dv ds. \end{aligned}$$

As for $a_i(t,x)$ we divide by ε and take the limit as ε goes to 0. By oddity, the first integral on the right-hand side only gives terms with $a_i(s,x)$ and $c(s,x)$. The other two integrals are dealt with by a Cauchy-Schwarz inequality and Poincaré. This yields

$$\begin{aligned} &\left| \int_\Omega \partial_t b_J(t,x) \phi(x) \, dx \right| \\ &\leq C \left[\|\mathbf{a}\|_{L^2_x(\boldsymbol{\rho}^{1/2})} + \|c\|_{L^2_x(\boldsymbol{\rho}^{1/2})} + \left\|\pi_{\mathbf{L}}^\perp(\mathbf{f})\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} \right] \|\nabla_x \phi\|_{L^2_x}. \end{aligned} \tag{5.3.25}$$

The latter is true for all $\phi(x)$ in H^1_x with a null integral over $\mathbb{T}^3$. We thus fix t and apply the inequality above to

$$-\Delta_x \phi(t,x) = \partial_t b_J(t,x)$$

which has a zero integral thanks to the conservation of momentum and obtain

$$\|\partial_t \nabla_x \phi_J\|^2_{L^2_x} = \left\|\nabla_x \Delta^{-1} \partial_t b_J\right\|^2_{L^2_x} = \int_\Omega \left(\nabla_x \Delta^{-1} \partial_t b_J\right) \nabla_x \phi(x)\, dx.$$

We integrate by parts

$$\|\partial_t \nabla_x \phi_J\|^2_{L^2_x} = \int_\Omega \partial_t b_J(t,x) \phi(x)\, dx.$$

At last, we use (5.3.25)

$$\begin{aligned}
&\|\partial_t \nabla_x \phi_J\|^2_{L^2_x} \\
&\leq C \left[\|\mathbf{a}\|_{L^2_x(\boldsymbol{\rho}^{1/2})} + \|c\|_{L^2_x(\boldsymbol{\rho}^{1/2})} + \left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right] \|\nabla_x \phi\|_{L^2_x} \\
&= C \left[\|\mathbf{a}\|_{L^2_x(\boldsymbol{\rho}^{1/2})} + \|c\|_{L^2_x(\boldsymbol{\rho}^{1/2})} + \left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right] \left\|\nabla_x \Delta_x^{-1} \partial_t b_J\right\|_{L^2_x} \\
&= C \left[\|\mathbf{a}\|_{L^2_x(\boldsymbol{\rho}^{1/2})} + \|c\|_{L^2_x(\boldsymbol{\rho}^{1/2})} + \left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right] \|\partial_t \nabla_x \phi_J\|_{L^2_x}.
\end{aligned}$$

Combining this estimate with (5.3.24) and using Young's inequality with any $\varepsilon_b > 0$

$$\begin{aligned}
|\Psi_4(t)| \leq &\varepsilon_b \int_0^t \|\mathbf{a}\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds \\
&+ C_5(\varepsilon_b) \int_0^t \left[\|c\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} + \left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right] ds.
\end{aligned} \tag{5.3.26}$$

We now gather (5.3.22), (5.3.10), (5.3.23), (5.3.26) and (5.3.21)

$$\begin{aligned}
\int_0^t \|b_J\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds \leq &N^{(J)}_{\mathbf{f}}(t) - N^{(J)}_{\mathbf{f}}(0) + \varepsilon_b \int_0^t \|a\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds + C_{J,c}(\varepsilon_b) \int_0^t \|c\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} \\
&+ C_J(\varepsilon_b) \int_0^t \left[\|\mathbf{g}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right] ds.
\end{aligned}$$

Finally, summing over all J in $\{1,2,3\}$

$$\begin{aligned}
\int_0^t \|b\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds \leq &N^{(b)}_{\mathbf{f}}(t) - N^{(b)}_{\mathbf{f}}(0) + \varepsilon_b \int_0^t \|\mathbf{a}\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} + C_{b,c} \int_0^t \|c\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} \\
&+ C_b \int_0^t \left[\left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_{x,v}} + \|\mathbf{g}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right] ds,
\end{aligned} \tag{5.3.27}$$

with $C_{b,c}$ and C_b depending on ε_b.

Estimate for c. The contribution of $c(t,x)$ is really similar to the one of $\mathbf{a}(t,x)$. Since $\mathbf{f}$ preserves mass and energy the following holds

$$\int_{\mathbb{T}^3} c(t,x)\, dx = 0.$$

Define the test function $\boldsymbol{\psi} = (\psi_{ic}(t,x,v))_{1\leq i\leq N}$ with

$$\psi_{ic}(t,x,v) = \left(|v|^2 - \alpha_{ic}\right) v \cdot \nabla_x \phi_c(t,x)$$

where

$$-\Delta_x \phi_c(t,x) = c(t,x)$$

and $\alpha_{ic} > 0$ is chosen such that for all $1 \leq k \leq 3$

$$\int_{\mathbb{R}^3} \left(|v|^2 - \alpha_{ic}\right) v_k^2\, \mu_i(v)\, dv = 0.$$

Again, the null integral of c and standard elliptic estimate [67] show

$$\forall t \geq 0, \quad \|\phi_c(t)\|_{H^2_x} \leq C_0 \|c(t)\|_{L^2_x}. \tag{5.3.28}$$

Again, this estimate provides the control of $\Psi_1 = N_{\mathbf{f}}^{(c)}(t) - N_{\mathbf{f}}^{(c)}(0)$ and of $\Psi_5(t)$ as in (5.3.12):

$$|\Psi_5(t)| \leq \frac{C_1}{4}\int_0^t \|c\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds + C_5 \int_0^t \|\mathbf{g}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\, ds, \tag{5.3.29}$$

where $C_1 > 0$ is given in (5.3.30) below.

We start by the left-hand side of (5.3.4).

$$\begin{aligned}
&-\sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_i(\mathbf{f}) v \cdot \nabla_x \psi_c\, dxdvds \\
&= -\sum_{i=1}^N \sum_{1\leq j,k\leq 3} \int_0^t \int_{\mathbb{T}^3} a_i(s,x) \left(\int_{\mathbb{R}^3} \left(|v|^2 - \alpha_{ic}\right) v_j v_k m_i \mu_i(v)\, dv\right) \partial_{x_j}\partial_{x_k}\phi_c\, dxds \\
&\quad -\sum_{i=1}^N \sum_{1\leq j,k\leq 3} \int_0^t \int_{\mathbb{T}^3} b(s,x) \cdot \left(\int_{\mathbb{R}^3} v \left(|v|^2 - \alpha_{ic}\right) v_j v_k m_i \mu_i(v)\, dv\right) \partial_{x_j}\partial_{x_k}\phi_c\, dxds \\
&\quad -\sum_{i=1}^N \sum_{1\leq j,k\leq 3} \int_0^t \int_{\mathbb{T}^3} c(s,x) \left(\int_{\mathbb{R}^3} \left(|v|^2 - \alpha_{ic}\right) \frac{|v|^2 - 3m_i^{-1}}{2} v_j v_k m_i \mu_i dv\right) \partial_{x_j}\partial_{x_k}\phi_c.
\end{aligned}$$

By oddity, the second integral vanishes, as well as all the others if $j \neq k$. Our choice of α_{ic} makes the first integral vanish even for $j = k$. It only remains the last integral with terms $j = k$ and therefore the definition of $\Delta_x \phi_c(t,x)$ gives

$$\begin{aligned}
-\sum_{i=1}^N \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} \pi_i(\mathbf{f}) v \cdot \nabla_x \psi_c\, dxdvds &= C_1 \int_0^t \sum_{i=1}^N \int_{\mathbb{T}^3} \rho_i c(s,x)^2\, dxds \\
&= C_1 \int_0^t \|c(s)\|_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds.
\end{aligned} \tag{5.3.30}$$

Again, direct computations give $\alpha_{ic} = 5/m_i$ and $C_1 > 0$.

Then the terms Ψ_2 and Ψ_3 are dealt with as in (5.3.14)

$$\forall k \in \{2,3\}, \quad |\Psi_k(t)| \leq \frac{C_1}{4}\int_0^t \|c\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\,ds + C_2\int_0^t \left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\,ds. \quad (5.3.31)$$

As for $\mathbf{a}(t,x)$ the estimate on Ψ_4 (5.3.8) will follow from elliptic regularity in negative Sobolev spaces. With exactly the same computations as for (5.3.15) we have

$$|\Psi_4(t)| \leq C\int_0^t \left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} \|\partial_t\nabla_x\phi_c\|_{L^2_x}\,ds. \quad (5.3.32)$$

Note that the contribution of $\pi_{\mathbf{L}}$ was null by oddity for the $\mathbf{a}(t,x)$ and $c(t,x)$ terms and also for the $b(t,x)$ terms thanks to our choice of α_{ic}.

To estimate $\|\partial_t\nabla_x\phi_c\|_{L^2_x}$ we use the decomposition of the weak formulation (5.3.4) between t and $t+\varepsilon$ (instead of between 0 and t) with

$$\boldsymbol{\psi}(t,x,v) = \Big(m_i(|v|^2 - 3m_i^{-1})\phi(x)\Big)_{1\leq i\leq N}$$

where ϕ belongs to H^1_x and has a zero integral on the torus. ψ does not depend on t and therefore $\Psi_4(t) = 0$. Moreover, multiplying by $\mu_i(v)\mu_i^{-1}(v)$ in the i^{th} integral of Ψ_3 yields

$$\Psi_3(t) = \int_0^t\int_{\mathbb{T}^3} \langle \mathbf{L}(\mathbf{f}), \left(\frac{|v|^2 - 3m_i^{-1}}{2}m_i\mu_i\right)_{1\leq i\leq N}\rangle_{L^2_v(\boldsymbol{\mu}^{-1/2})}\partial_{x_k}\phi(x)\,dxdvds = 0,$$

by definition of Ker($\mathbf{L}$).

From the weak formulation (5.3.4) it therefore remains

$$\begin{aligned} C\int_{\mathbb{T}^3}[c(t+\varepsilon)-c(t)]\,\phi(x)\,dx \;=\; & \sum_{i=1}^N\int_t^{t+\varepsilon}\int_{\mathbb{T}^3\times\mathbb{R}^3}\pi_i(\mathbf{f})\frac{m_i|v|^2-3}{2}v\cdot\nabla_x\phi(x) \\ + & \sum_{i=1}^N\int_t^{t+\varepsilon}\int_{\mathbb{T}^3\times\mathbb{R}^3}\pi_i^\perp(\mathbf{f})\frac{m_i|v|^2-3}{2}v\cdot\nabla_x\phi(x) \\ + & \sum_{i=1}^N\int_t^{t+\varepsilon}\int_{\mathbb{T}^3\times\mathbb{R}^3}g_i(s,x,v)\frac{m_i|v|^2-3}{2}\phi(x). \end{aligned}$$

As for $\mathbf{a}(t,x)$ we divide by ε and take the limit as ε goes to 0. By oddity, the first integral on the right-hand side only gives terms with $\rho_i b(s,x)$. The last two terms are dealt with by multiplying by $\mu_i(v)^{-1/2}\mu_i(v)^{1/2}$ inside each integral and applying a Cauchy-Schwarz inequality. Note that again we also apply Poincaré inequality. This yields

$$\begin{aligned} &\left|\int_{\mathbb{T}^3}\partial_t c(t,x)\phi(x)\,dx\right| \\ &\qquad \leq C\left[\|b\|_{L^2_x(\boldsymbol{\rho}^{1/2})} + \left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right]\|\nabla_x\phi\|_{L^2_x}. \end{aligned}$$

That estimate holds for all $\phi(x)$ in H^1_x with null integral over $\mathbb{T}^3$. We copy the arguments made for $\mathbf{a}(t,x)$ or $b_J(t,x)$ and construct

$$-\Delta_x \Phi_c(t,x) = \partial_t c(t,x)$$

and obtain by elliptic estimates

$$\begin{aligned} \|\partial_t \nabla_x \phi_c\|_{L^2_x} &= \left\|\nabla_x \Delta^{-1}\partial_t c\right\|_{L^2_x} \leq \left\|\Delta^{-1}\partial_t c\right\|_{H^1_x} = \|\Phi_c\|_{H^1_x} \\ &\leq C \left\|\partial_t c(t,x)\right\|_{(H^1_x)^*} \\ &\leq C \left[\|b\|_{L^2_x(\boldsymbol{\rho}^{1/2})} + \left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right]. \end{aligned}$$

Combining this estimate with (5.3.32) and using Young's inequality with any $\varepsilon_c > 0$

$$|\Psi_4(t)| \leq \varepsilon_c \int_0^t \|b\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds + C_5(\varepsilon_c) \int_0^t \left[\left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right] ds. \tag{5.3.33}$$

We now gather (5.3.30), (5.3.10), (5.3.31), (5.3.33) and (5.3.29) into (5.3.4):

$$\begin{aligned} \int_0^t \|c\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds \leq & N^{(c)}_{\mathbf{f}}(t) - N^{(c)}_{\mathbf{f}}(0) + \varepsilon_c \int_0^t \|b\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\, ds \\ & + C_c(\varepsilon_c) \int_0^t \left[\left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right] ds. \end{aligned} \tag{5.3.34}$$

Conclusion of the proof. We gather together the estimates we derived for $\mathbf{a}$, b and c. We compute the linear combination (5.3.18) + $\alpha \times$ (5.3.27) + $\beta \times$ (5.3.34). For all $\varepsilon_b > 0$ and $\varepsilon_c > 0$ this implies

$$\begin{aligned} &\int_0^t \left[\|\mathbf{a}\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} + \alpha \|b\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} + \beta \|c\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\right] ds \\ &\leq N_{\mathbf{f}}(t) - N_{\mathbf{f}}(0) + C_\perp \int_0^t \left[\left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \|\mathbf{g}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right] ds \\ &\quad + \int_0^t \left[\alpha\varepsilon_b \|\mathbf{a}\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} + (C_{a,b} + \beta\varepsilon_c) \|b\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} + \alpha C_{b,c}(\varepsilon_b) \|c\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}\right] ds. \end{aligned}$$

We first choose $\alpha > C_{a,b}$, then ε_b such that $\alpha\varepsilon_b < 1$ and then $\beta > \alpha C_{b,c}(\varepsilon_b)$. Finally, we fix ε_c small enough such that $C_{a,b} + \beta\varepsilon_c < \alpha$. With such choices we can absorb the last term on the right-hand side by the left-hand side. This concludes the proof of Lemma 5.12 since

$$\|\pi_{\mathbf{L}}(\mathbf{f})\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} = \|\mathbf{a}\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} + \|b\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})} + \|c\|^2_{L^2_x(\boldsymbol{\rho}^{1/2})}.$$

□

5.3.2. Generation of a semigroup

We now have the tools to develop the hypocoercivity of $\mathbf{G}$ into a semigroup property.

Proof of Theorem 5.11. Let $\mathbf{f_0}$ be in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ and consider the following equation

$$\partial_t \mathbf{f} = \mathbf{L}\left(\mathbf{f}\right) - v\cdot\nabla_x \mathbf{f} \tag{5.3.35}$$

with initial data $\mathbf{f_0}$.

Since the transport part $-v\cdot\nabla_x$ is skew-symmetric in $L^2_{x,v}\left(\mu_i^{-1/2}\right)$ (mere integration by part) and $\mathbf{L}$ is self-adjoint, Ker($\mathbf{G}$) and $(\mathrm{Ker}(\mathbf{G}))^\perp$ are stable under (5.3.35). We therefore consider only the case $\mathbf{f_0}$ in $(\mathrm{Ker}(\mathbf{G}))^\perp$ and the associated solution stays in $(\mathrm{Ker}(\mathbf{G}))^\perp$ for all t.

Moreover, $\mathbf{L}$ has a spectral gap λ_L and so by Theorem 5.6, if $\mathbf{f} = (f_i)_{1\leq i\leq N}$ is a solution to (5.3.35) we have the following

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{f}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} &= \int_{\mathbb{T}^3\times\mathbb{R}^3}\langle\partial_t\mathbf{f},\mathbf{f}\rangle_{\boldsymbol{\mu}^{-1/2}}\,dxdv \\
&= -\sum_{i=1}^N\int_{\mathbb{T}^3\times\mathbb{R}^3} v\cdot\nabla_x\left(f_i(t,x,v)^2\right)\mu_i^{-1}(v)\,dxdv \\
&\quad + \int_{\mathbb{T}^3}\langle\mathbf{L}(\mathbf{f})(t,x,\cdot),\mathbf{f}(t,x,\cdot)\rangle_{L^2_v(\mu^{-1/2})}\,dx \\
&\leq -\lambda_L\left\|\pi^\perp_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}.
\end{aligned} \tag{5.3.36}$$

We remind that $\pi^\perp_{\mathbf{L}} = \mathrm{Id} - \pi_{\mathbf{L}}$ where $\pi_{\mathbf{L}}$ is the orthogonal projection (5.2.1) onto Ker(L) in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$. The norm is thus decreasing under the flow and it therefore follows that $\mathbf{G}$ generates a strongly continuous semigroup on $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$, we refer the reader to [98] (general theory) or [134][135] (for the special case of single species Boltzmann equation).

Let $\mathbf{f} = S_{\mathbf{G}}(t)\mathbf{f_0}$ and define $\widetilde{\mathbf{f}}(t,x,v) = e^{\lambda t}\mathbf{f}(t,x,v)$ for $\lambda > 0$ to be defined later. $\widetilde{\mathbf{f}}$ satisfies the conservation laws and is solution in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ to the following equation

$$\partial_t\widetilde{\mathbf{f}} = \mathbf{G}(\widetilde{\mathbf{f}}) + \lambda\widetilde{\mathbf{f}}.$$

As for (5.3.36) we obtain the following estimate

$$\left\|\widetilde{\mathbf{f}}\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} \leq \|\mathbf{f_0}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} - 2\lambda_L\int_0^t\left\|\pi^\perp_{\mathbf{L}}(\widetilde{\mathbf{f}})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + 2\lambda\int_0^t\left\|\widetilde{\mathbf{f}}\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}. \tag{5.3.37}$$

Along with the latter estimate, we have the following control given by Lemma 5.12 with $\mathbf{g} = \lambda\widetilde{\mathbf{f}}$

$$\begin{aligned}
\int_0^t\left\|\pi_{\mathbf{L}}(\widetilde{\mathbf{f}})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\,ds \leq & N_{\widetilde{\mathbf{f}}}(t) - N_{\widetilde{\mathbf{f}}}(0) + C_\perp\int_0^t\left\|\pi^\perp_{\mathbf{L}}(\widetilde{\mathbf{f}})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\,ds \\
& + C_\perp\lambda^2\int_0^t\left\|\widetilde{\mathbf{f}}\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\,ds
\end{aligned} \tag{5.3.38}$$

where $C_\perp > 0$ is independent of $\mathbf{f}$ and $\left|N_{\widetilde{\mathbf{f}}}(s)\right| \leq C \left\|\widetilde{\mathbf{f}}(s)\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}$, then $\varepsilon \times$ (5.3.38) + (5.3.37) yields

$$\left[\left\|\widetilde{\mathbf{f}}\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} - \varepsilon N_{\widetilde{\mathbf{f}}}(t)\right] + C_\varepsilon \int_0^t \left(\left\|\pi_{\mathbf{L}}(\widetilde{\mathbf{f}})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \left\|\pi_{\mathbf{L}}^\perp(\widetilde{\mathbf{f}})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\right)$$
$$\leq \|\mathbf{f_0}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} - \varepsilon N_{\widetilde{f}}(0) + \left(2\lambda + \varepsilon C_\perp \lambda^2\right) \int_0^t \left\|\widetilde{\mathbf{f}}\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} ds.$$

where $C_\varepsilon = \min\{2\lambda_L - \varepsilon C_\perp, \varepsilon\}$. By the control on $\left|N_{\widetilde{\mathbf{f}}}(s)\right|$ and the fact that

$$\left\|\pi_{\mathbf{L}}(\widetilde{\mathbf{f}})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} + \left\|\pi_{\mathbf{L}}^\perp(\widetilde{\mathbf{f}})\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} = \left\|\widetilde{\mathbf{f}}\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}$$

we can choose ε small enough such that $C_\varepsilon > 0$ and then λ small enough such that $\left(2\lambda + \varepsilon C_\perp \lambda^2\right) < C_\varepsilon$. Such choices imply that $\left\|\widetilde{\mathbf{f}}\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}$ is uniformly bounded in time by $C\|\mathbf{f_0}\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}$.

By definition of $\widetilde{\mathbf{f}}$, this shows an exponential decay for $\mathbf{f}$ and concludes the proof of Theorem 5.11.

□

5.4. L^∞-theory for the linear part with Maxwellian weight

As explained in the introduction, the L^2 setting is not algebraic for the nonlinear operator $\mathbf{Q}$. We therefore need to work in an L^∞ framework. We first give a pointwise control on the linear operator $\mathbf{K}$ in Subsection 5.4.1 and then we prove that the linear part of the perturbed equation (5.1.7) generates a strongly continuous semigroup in $L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$ in Subsection 5.4.2.

5.4.1. Pointwise estimate on the compact part

We recall that $\mathbf{L}$ can be written under the following form

$$\mathbf{L} = -\boldsymbol{\nu}(v) + \mathbf{K},$$

where $\boldsymbol{\nu} = (\nu_i)_{1\leq i\leq N}$ is a multiplicative operator satisfying (5.2.4):

$$\forall v \in \mathbb{R}^3, \quad \nu_i^{(0)}(1+|v|^\gamma) \leq \nu_i(v) \leq \nu_i^{(1)}(1+|v|^\gamma),$$

with $\nu_i^{(0)}$, $\nu_i^{(1)} > 0$.

In the case of single-species Boltzmann equation, the operator $\mathbf{K}$ can be written as a kernel operator ([75] or [34] Section 7.2) and we give here a similar property where the different exponential decay rates, due to the different masses, are explicitely taken into account. These explicit bounds will be strongly needed for the L^∞ theory.

Lemma 5.14. *Let* $\mathbf{f}$ *be in* $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$. *Then for all* i *in* $\{1,\dots,N\}$ *there exists* $\mathbf{k}^{(i)}$ *such that*

$$K_i(\mathbf{f})(v) = \int_{\mathbb{R}^3} \langle \mathbf{k}^{(i)}(v,v_*), \mathbf{f}(v_*)\rangle \, dv_*.$$

Moreover there exist m, $C_K > 0$ *such that for all* i *in* $\{1,\dots,N\}$ *and for all* $1 \leq j \leq N$

$$\left|k_j^{(i)}(v,v_*)\right| \leq C_K \sqrt{\frac{\mu_i(v)}{\mu_j(v_*)}} \left[|v-v_*|^\gamma + |v-v_*|^{\gamma-2}\right] e^{-m|v-v_*|^2 - m\frac{\left||v|^2-|v_*|^2\right|^2}{|v-v_*|^2}}. \tag{5.4.1}$$

The constants m *and* C_K *are explicit and depend only on* $(m_i)_{1\leq i\leq N}$ *and the collision kernel* B.

Proof of Lemma 5.14. By definition, $\mathbf{K} = (K_i)_{1\leq i\leq N}$ with

$$K_i(\mathbf{f})(v) = \sum_{j=1}^N \int_{\mathbb{S}^2\times\mathbb{R}^3} B_{ij}\left(|v-v_*|, \cos\theta\right) \left[\mu_j'^{*} f_i' + \mu_i' f_j'^{*} - \mu_i f_j^*\right] \, d\sigma dv_*. \tag{5.4.2}$$

We used the identity $\mu_i(v)\mu_j(v_*) = \mu_i(v')\mu_j(v'_*)$ that is a consequence of the conservation of energy during an elastic collision.

Step 1: A kernel form. The third term in the integral is already in the desired form. The first two terms require a new representation of the collision kernel where the integrand parameters will be v' and v'_* instead of v_* and σ. Such a representation has been obtained in the case of a single-species Boltzmann equation and is called the Carleman representation [30]. We derive below the Carleman representation associated with the multi-species Boltzmann operator. We follow the methods used in [30][34] Section 7.2 and [71]. However, the existence of different masses generates an asymmetry between v' and v'_* as we shall see.

The laws of elastic collisions gives

$$v' = V + \frac{m_j}{m_i+m_j}|v-v_*|\,\sigma \quad \text{and} \quad v'_* = V - \frac{m_i}{m_i+m_j}|v-v_*|\,\sigma$$

where V is the center of mass of the particles i and j:

$$V = \frac{m_i}{m_i+m_j}v + \frac{m_j}{m_i+m_j}v_*.$$

We can also express

$$v = V + \frac{m_j}{m_i+m_j}\left(v-v_*\right) \quad \text{and} \quad v_* = V - \frac{m_i}{m_i+m_j}\left(v-v_*\right).$$

Note that

$$|v-v_*| = \left|v'-v'_*\right|, \tag{5.4.3}$$

$$\begin{aligned} \left|v - v'\right| &\leq \frac{2m_j}{m_i + m_j}\left|v' - v'_*\right|, && (5.4.4) \\ \left|v - v'_*\right| &\leq \left|v' - v'_*\right|. && (5.4.5) \end{aligned}$$

The points v, v_*, v' and v'_* therefore belong to the plane defined by V and $\mathrm{Span}(\sigma, v - v_*)$. We have the following geometric configuration, which gives a perfect circle in the case of equal masses.

Geometrically, $m_j^{-1}(v-V)$, $m_j^{-1}(v'-V)$, $m_i^{-1}(v_*-V)$ and $m_i^{-1}(v'_*-V)$ are on the same circle of diameter $\left|m_i^{-1}(v'^* - V) - m_j(v'-V)\right| = \frac{2}{m_i+m_j}|v - v_*|$. Therefore,

$$\left\langle \frac{1}{m_i}(v'_* - V) - \frac{1}{m_j}(v - V), \frac{1}{m_j}(v - V) - \frac{1}{m_j}(v' - V)\right\rangle = 0.$$

Using the laws of elasticity to see that

$$V = \frac{m_i}{m_i + m_j}v' + \frac{m_j}{m_i + m_j}v'_*$$

we end up with the following orthogonal property (that is also easily checked by direct computations)

$$\left\langle v'_* - \left(\frac{m_i + m_j}{2m_j}v - \frac{m_i - m_j}{2m_j}v'\right), v - v'\right\rangle = 0. \qquad (5.4.6)$$

We can now apply the change of variables $(v_*, \sigma) \mapsto (v', v'_*)$, where v' evolves in $\mathbb{R}^3$ and v'_* in $E^{ij}_{vv'}$. $E^{ij}_{vv'}$ is the hyperplane that passes through

$$V_E(v, v') = \frac{m_i + m_j}{2m_j}v - \frac{m_i - m_j}{2m_j}v' \qquad (5.4.7)$$

and is orthogonal to $v - v'$; we denote $dE(v'_*)$ the Lebesgue measure on it. Note that $v_* = V(v', v'_*)$ is now a function of v' and v'_*:

$$V(v', v'_*) = v'_* + m_i m_j^{-1} v' - m_i m_j^{-1} v.$$

Up to the translation and dilatation (generating a constant $C_{ij} > 0$ only depending on m_i and m_j) from v to the origin of $E^{ij}_{vv'}$, this change of variables works as derived in [71]. Our operator thus reads

$$\begin{aligned} &\int_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) f' g'^* \, dv_* d\sigma \\ &= C_{ij} \int_{\mathbb{R}^3} \frac{1}{|v - v'|}\left(\int_{E^{ij}_{vv'}} \frac{B\left(v - V(v', v'_*), \frac{v'_* - v'}{|v'_* - v'|}\right)}{|v'_* - v'|} g'^* \, dE(v'_*)\right) f' \, dv'. \end{aligned} \qquad (5.4.8)$$

We can also give a Carleman representation where we first integrate against v'_*. In the case $m_i = m_j$ the orthogonal property (5.4.6) is entirely symmetric in v' and v'_* and we reach the same representation (5.4.8) with the role of v' and v'_* swapped. This is the classical case of a single-species Boltzmann operator.

In the case $m_i \neq m_j$, (5.4.6) is equivalent to

$$\left|v'\right|^2 - 2\left\langle v', \frac{m_i}{m_i - m_j}v - \frac{m_j}{m_i - m_j}v'_*\right\rangle = \left\langle v, \frac{2m_j}{m_i - m_j}v'_* - \frac{m_i + m_j}{m_i - m_j}v\right\rangle$$

which is itself equivalent to

$$\left|v' + \left(\frac{m_j}{m_i - m_j}v'_* - \frac{m_i}{m_i - m_j}v\right)\right|^2 = \left|\frac{m_j}{m_i - m_j}v'_* - \frac{m_j}{m_i - m_j}v\right|^2. \tag{5.4.9}$$

The same change of variables as before but $(v_*, \sigma) \mapsto (v'_*, v')$ instead of $(v_*, \sigma) \mapsto (v', v'_*)$ thus yields

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{S}^2} & B(v - v_*, \sigma) f' g'^{*} \, dv_* d\sigma \\ &= C_{ij} \int_{\mathbb{R}^3} \frac{1}{|v - v'_*|} \left(\int_{\widetilde{E}^{ij}_{vv'_*}} \frac{B\left(v - V(v', v'_*), \frac{v'_* - v'}{|v'_* - v'|}\right)}{|v'_* - v'|} f' \, dE(v') \right) g'^{*} \, dv'_*,\end{aligned} \tag{5.4.10}$$

where $\widetilde{E}^{ij}_{vv'_*}$ stands for $E^{ij}_{vv'_*}$ if $m_i = m_j$ or for the sphere defined by (5.4.9); and dE is the Lebesgue measure on it.

We therefore conclude gathering (5.4.2), (5.4.8) and (5.4.10) with a relabelling of the integrated variables,

$$\begin{aligned}K_i(\mathbf{f})(v) = &\sum_{j=1}^N C_{ij} \int_{\mathbb{R}^3} \left(\frac{1}{|v - v_*|} \int_{\widetilde{E}^{ij}_{vv_*}} \frac{B_{ij}\left(v - V(u, v_*), \frac{v_* - u}{|u - v_*|}\right)}{|u - v_*|} \mu_i(u) \, dE(u) \right) f_j^* \, dv_* \\ &+ \sum_{j=1}^N C_{ji} \int_{\mathbb{R}^3} \left(\frac{1}{|v - v_*|} \int_{E^{ij}_{vv_*}} \frac{B_{ij}\left(v - V(v_*, u), \frac{u - v_*}{|u - v_*|}\right)}{|u - v_*|} \mu_j(u) \, dE(u) \right) f_i^* \, dv_* \\ &- \sum_{j=1}^N \int_{\mathbb{R}^3} B_{ij}\left(|v - v_*|, \cos\theta\right) \mu_i(v) f_j^* \, dv_*.\end{aligned} \tag{5.4.11}$$

This concludes the fact that K_i is a kernel operator.

Step 2: Pointwise estimate. It remains to show the pointwise estimate (5.4.1). The assumptions on B_{ij} imply that

$$\left| B_{ij}\left(v - V(v_*, u), \frac{u - v_*}{|u - v_*|}\right)\right| \leq C \left|v - V(v_*, u)\right|^\gamma,$$

where C denotes any positive constant independent of v and v_*. We shall bound each of the three terms in (5.4.11) separately.

From elastic collision laws (5.4.3), for u in $E^{ij}_{vv_*}$ one has $|v - V(v_*, u)| = |u - v_*|$, and hence

$$\left| \int_{E^{ij}_{vv_*}} \frac{B_{ij}\left(v - V(v_*, u), \frac{u-v_*}{|u-v_*|}\right)}{|u - v_*|} \mu_j(u)\, dE(u) \right| \leq C \int_{E^{ij}_{vv_*}} \frac{1}{|u - v_*|^{1-\gamma}} e^{-m_j \frac{|u|^2}{2}}\, dE(u).$$

We can further bound, since (5.4.4) is valid on $E^{ij}_{vv_*}$,

$$|u - v_*| \geq \frac{m_i + m_j}{2m_j} |v - v_*|\,,$$

and get

$$\left| \int_{E^{ij}_{vv_*}} \frac{B_{ij}\left(v - V(v_*, u), \frac{u-v_*}{|u-v_*|}\right)}{|u - v_*|} \mu_j(u)\, dE(u) \right| \leq \frac{C}{|v - v_*|^{1-\gamma}} \int_{E_{vv_*}} e^{-m_j \frac{|u|^2}{2}}\, dE(u).$$

To estimate the integral over $E^{ij}_{vv_*}$ we make the change of variables

$$u = V_E(v, v_*) + w$$

with $V_E(v, v_*)$ the origin (5.4.7) of $E^{ij}_{vv_*}$ and w in $(\mathrm{Span}(v - v_*))^\perp$. Using $\langle v, w\rangle = \langle v_*, w\rangle$ we compute

$$\begin{aligned} |u|^2 = |V_E(v, v_*) + w|^2 &= \left| w + \frac{1}{2}(v + v_*) + \frac{m_i}{2m_j}(v - v_*) \right|^2 \\ &= \left| w + \frac{1}{2}(v + v_*) \right|^2 + \frac{m_i^2}{4m_j^2} |v - v_*|^2 + \frac{m_i}{2m_j}\left(|v|^2 - |v_*|^2\right). \end{aligned}$$

Now we decompose $v + v_* = V^\perp + V^\parallel$ where $V^\parallel$ is the projection onto $\mathrm{Span}(v - v_*)$ and $V^\perp$ is the orthogonal part. This implies

$$|u|^2 = \left| w + \frac{1}{2}V^\perp \right|^2 + \frac{1}{4}\left| V^\parallel \right|^2 + \frac{m_i^2}{4m_j^2} |v - v_*|^2 + \frac{m_i}{2m_j}\left(|v|^2 - |v_*|^2\right).$$

By definition,

$$\left| V^\parallel \right|^2 = \frac{\langle v + v_*, v - v_*\rangle^2}{|v - v_*|^2} = \frac{\left| |v|^2 - |v_*|^2 \right|^2}{|v - v_*|^2}$$

and therefore the following holds

$$\begin{aligned} &\left| \frac{1}{|v - v_*|} \int_{E^{ij}_{vv_*}} \frac{B_{ij}\left(v - V(v_*, u), \frac{u-v_*}{|u-v_*|}\right)}{|u - v_*|} \mu_j(u)\, dE(u) \right| \\ &\quad \leq \frac{C}{|v - v_*|^{2-\gamma}} e^{-\frac{m_i^2}{8m_j}|v-v_*|^2 - \frac{m_j}{8}\frac{\left||v|^2 - |v_*|^2\right|^2}{|v-v_*|^2}} \sqrt{\frac{\mu_i(v)}{\mu_i(v_*)}} \left[\int_{(v-v_*)^\perp} e^{-\frac{m_j}{2}\left|w + \frac{1}{2}V^\perp\right|^2}\, dE(w) \right]. \end{aligned} \tag{5.4.12}$$

The space $(v-v_*)^\perp$ is invariant by translation of vector $-2^{-1}V^\perp$ and the exponential term inside the integral only depends on the norm and therefore the integral term is a constant not depending on v or v_*.

We now turn to the term involving $\widetilde{E}^{ij}_{vv'}$ which is a bit more technical. In the case $m_i = m_j$ then $\widetilde{E}^{ij}_{vv'} = E^{ij}_{vv'}$. We therefore have the bound (5.4.12) to which we use $\mu_i(v)\mu_i^{-1}(v) = C_{ij}\mu_j(v)\mu_i^{-1}(v)$ since $m_i = m_j$.

Assume now that $m_i \neq m_j$. As for $E^{ij}_{vv_*}$, the elastic collision properties (5.4.3) and (5.4.5) give for all v_* in $\mathbb{R}^3$ and u in $\widetilde{E}^{ij}_{vv_*}$

$$\left|\int_{\widetilde{E}^{ij}_{vv_*}} \frac{B_{ij}\left(v - V(u,v_*), \frac{v_*-u}{|u-v_*|}\right)}{|u-v_*|}\mu_i(u)\,dE(u)\right| \leq \frac{C}{|v-v_*|^{1-\gamma}}\int_{\widetilde{E}^{ij}_{vv_*}} e^{-m_i\frac{|u|^2}{2}}\,dE(u).$$

Since $\widetilde{E}^{ij}_{vv_*}$ is the sphere of radius

$$R_{vv_*} = \frac{m_j}{|m_i - m_j|}\,|v-v_*|$$

and centered at

$$O_{vv_*} = \frac{m_i}{m_i - m_j}v - \frac{m_j}{m_i - m_j}v_*.$$

We make a change of variables to end up on $\mathbb{S}^2$:

$$\begin{aligned}&\left|\frac{1}{|v-v_*|}\int_{\widetilde{E}^{ij}_{vv_*}} \frac{B_{ij}\left(v - V(u,v_*), \frac{v_*-u}{|u-v_*|}\right)}{|u-v_*|}\mu_i(u)\,dE(u)\right| \\ &\qquad\leq C\,|v-v_*|^\gamma \int_{\mathbb{S}^2} e^{-\frac{m_i}{2}|R_{vv_*}u + O_{vv_*}|^2}\,d\sigma(u).\end{aligned} \tag{5.4.13}$$

Decomposing the norm inside the integral and using Cauchy-Schwarz inequality yields

$$\begin{aligned}-\frac{m_i}{2}\,|R_{vv_*}u + O_{vv_*}|^2 \leq &- \frac{m_i m_j^2}{2(m_i-m_j)^2}\,|v-v_*|^2 - \frac{m_i}{2(m_i-m_j)^2}\,|m_i v - m_j v_*|^2 \\ &+ \frac{m_i m_j}{(m_i - m_j)^2}\,|v-v_*|\,|m_i v - m_j v_*|\end{aligned} \tag{5.4.14}$$

The idea is to express everything in terms of $|v-v_*|$ and $\frac{|v|^2-|v_*|^2}{|v-v_*|}$. We recall that we defined $v+v_* = V^\perp + V^\parallel$ with $V^\perp$ orthogonal to $\mathrm{Span}(v-v_*)$ and $V^\parallel = \frac{\langle v+v_*, v-v_*\rangle}{|v-v_*|}(v-v_*)$. We first use the identity

$$|v-v_*|\,|m_i v - m_j v_*| = \frac{1}{4}\left[|(1+m_i)v - (1+m_j)v_*|^2 - |(1-m_i)v - (1-m_j)v_*|^2\right]$$

and then the following equality that holds for all a and b,

$$|av - bv_*|^2 \quad = \quad \left|\frac{a-b}{2}(v+v_*) + \frac{a+b}{2}(v-v_*)\right|^2$$

$$= \frac{(a-b)^2}{4}\left|V^\perp\right|^2 + \frac{(a-b)^2}{4}\frac{\left||v|^2-|v_*|^2\right|^2}{|v-v_*|^2} + \frac{(a-b)(a+b)}{2}\left(|v|^2-|v_*|^2\right) + \frac{(a+b)^2}{4}|v-v_*|^2. \tag{5.4.15}$$

Direct computations from (5.4.14) then yield

$$-\frac{m_i}{2}\left|R_{vv_*}u+O_{vv_*}\right|^2 \leq -\frac{m_i}{8}\left|V^\perp\right|^2 - \frac{m_i}{8}\frac{\left||v|^2-|v_*|^2\right|^2}{|v-v_*|^2} - \frac{m_i}{4}\left(|v|^2-|v_*|^2\right) - \frac{m_i}{8}|v-v_*|^2.$$

Taking $(a,b)=(1,0)$ and $(a,b)=(0,1)$ in (5.4.15) we have

$$\frac{m_i}{4}|v|^2 - \frac{m_j}{4}|v_*|^2 = \frac{m_i-m_j}{16}\left|V^\perp\right|^2 + \frac{m_i-m_j}{16}\frac{\left||v|^2-|v_*|^2\right|^2}{|v-v_*|^2} + \frac{m_i+m_j}{8}\left(|v|^2-|v_*|^2\right) + \frac{m_i-m_j}{16}|v-v_*|^2.$$

At last we obtain

$$-\frac{m_i}{2}\left|R_{vv_*}u+O_{vv_*}\right|^2 \leq -\frac{m_i}{4}|v|^2 + \frac{m_j}{4}|v_*|^2 - \frac{m_i+m_j}{16}\left|V^\perp\right|^2 + U\left(\frac{|v|^2-|v_*|}{|v-v_*|}, |v-v_*|\right) \tag{5.4.16}$$

where $U(x,y)$ is a quadratic form defined by

$$U(x,v) = -\frac{m_i+m_j}{16}x^2 - \frac{m_i-m_j}{8}xy - \frac{m_i+m_j}{16}y^2.$$

The latter quadratic form is associated with the symmetric matrix

$$\begin{pmatrix} -\frac{m_i+m_j}{16} & -\frac{m_i-m_j}{16} \\ -\frac{m_i-m_j}{16} & -\frac{m_i+m_j}{16} \end{pmatrix}$$

which has a negative trace and determinant $m_im_j/64 > 0$. It therefore is a negative definite symmetric matrix and thus, denoting by $-\lambda(m_i,m_j) < 0$ its largest eigenvalue we have

$$\forall (x,x)\in\mathbb{R}^2, \quad U(x,v) \leq -\lambda(m_i,m_j)\left[x^2+y^2\right].$$

Plugging the latter into (5.4.16) and going back to the integral of interest (5.4.13) we get

$$\left|\frac{1}{|v-v_*|}\int_{\widetilde{E}^{ij}_{vv_*}} \frac{B_{ij}\left(v-V(u,v_*),\frac{v_*-u}{|u-v_*|}\right)}{|u-v_*|}\mu_i(u)\,dE(u)\right| \leq C\,|v-v_*|^\gamma\, e^{-\lambda(m_i,m_j)|v-v_*|^2-\lambda(m_i,m_j)\frac{\left||v|^2-|v_*|^2\right|^2}{|v-v_*|^2}}\sqrt{\frac{\mu_i(v)}{\mu_j(v_*)}}. \tag{5.4.17}$$

To conclude we turn to the last integral term in (5.4.11) which is easily bounded by

$$\begin{aligned} |B_{ij}(|v-v_*|, \cos\theta)\, \mu_i(v)| &\leq C\,|v-v_*|^\gamma \, \mu_i(v) \\ &\leq C\,|v-v_*|^\gamma \, e^{-\frac{1}{4}\left(m_i|v|^2+m_j|v_*|^2\right)}\sqrt{\frac{\mu_i(v)}{\mu_j(v_*)}}. \end{aligned}$$

Using Cauchy-Schwartz

$$\begin{aligned} |v-v_*|^2 + \frac{\left||v|^2-|v_*|^2\right|^2}{|v-v_*|^2} &= |v-v_*|^2 + \frac{|\langle v-v_*, v+v_*\rangle|^2}{|v-v_*|^2} \\ &\leq |v-v_*|^2 + |v+v_*|^2 = 2\left(|v|^2+|v_*|^2\right), \end{aligned}$$

this implies

$$|B_{ij}(|v-v_*|, \cos\theta)\, \mu_i(v)| \leq C\,|v-v_*|^\gamma \, e^{-\frac{m_{ij}}{8}|v-v_*|^2-\frac{m_{ij}}{8}\frac{\left||v|^2-|v_*|^2\right|^2}{|v-v_*|^2}}\sqrt{\frac{\mu_i(v)}{\mu_j(v_*)}}, \qquad (5.4.18)$$

where $m_{ij} = \min\{m_i, m_j\}$.

Gathering (5.4.11)-(5.4.12)-(5.4.17)-(5.4.18) gives the desired estimate on $k_j^{(i)}$. □

The pointwise estimate on $k_j^{(i)}$ can be transferred into a decay of the L^1_v-norm with a relatively important weight. This has been proved in [83, Lemma 7] for the right-hand side of (5.4.1) with $m = 1/8$. The case of general m is identical and leads to

Lemma 5.15. *Let $\beta > 0$ and θ in $[0, 1/(32m))$. There exists $C_{\theta,\beta} > 0$ and $\varepsilon_{\theta,\beta} > 0$ such that for all i, j in $\{1,\dots,N\}$ and all ε in $[0,\varepsilon_{\theta,\beta})$,*

$$\int_{\mathbb{R}^3} \left|k_j^{(i)}(v,v_*)\right| e^{\varepsilon m|v-v_*|^2+\varepsilon m\frac{\left||v|^2-|v_*|^2\right|^2}{|v-v_*|^2}} \frac{\langle v\rangle^\beta e^{\theta|v|^2}\mu_i(v)^{-1/2}}{\langle v_*\rangle^\beta e^{\theta|v_*|^2}\mu_j(v_*)^{-1/2}}\, dv_* \leq \frac{C_{\beta,\theta}}{1+|v|}.$$

From Lemma 5.14 and 5.15 we conclude that $\mathbf{K}$ is a bounded operator on $L^\infty_v\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$.

5.4.2. Semigroup generated by the linear part

Following ideas developed in [83] in the case of bounded domains, the L^2 theory could be used to construct a L^∞ one by using the flow of characteristics to transfer pointwise estimates at $x - vt$ into integral in the space variable. Such a method is the core of the L^∞ theory thanks to the following lemma.

Lemma 5.16. *Let $\beta > 3/2$ and let $(H1)-(H4)$ hold for the collision kernel. Assume that there exist $T_0 > 0$ and λ, $C_{T_0} > 0$ such that for all $\mathbf{f}(t,x,v)$ in $L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$ solution to*

$$\partial_t \mathbf{f} + v\cdot\nabla_x \mathbf{f} = \mathbf{L}(\mathbf{f}) \tag{5.4.19}$$

with initial data $\mathbf{f_0}$, the following holds for all t in $[0,T_0]$

$$\|\mathbf{f}(t)\|_{L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)} \leq e^{\lambda(T_0-2t)}\|\mathbf{f_0}\|_{L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)} + C_{T_0}\int_0^t \|\mathbf{f}(s)\|_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)}\,ds.$$

Then for all $0 < \widetilde{\lambda} < \min\{\lambda, \lambda_G\}$, defined in Theorem 5.11, there exists $C = C\left(\beta, \widetilde{\lambda}\right) > 0$ such that for all $\mathbf{f}$ solution to (5.4.19) in $L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$ satisfying $\Pi_{\mathbf{G}}(\mathbf{f}) = 0$,

$$\forall t \geq 0, \quad \|\mathbf{f}(t)\|_{L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)} \leq Ce^{-\widetilde{\lambda}t}\|\mathbf{f_0}\|_{L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)}.$$

Proof of Lemma 5.16. To shorten the computations we use the following notation $\mathbf{w}_\beta(v) = \langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}$.

Let $\mathbf{f}$ be a solution to (5.4.19) in $L^\infty_{x,v}(\mathbf{w}_\beta)$ associated with the initial data $\mathbf{f_0}$. Taking n in $\mathbb{N}$ we can apply the assumption of the lemma to $\widetilde{\mathbf{f}}(t,x,v) = \mathbf{f}(t+nT_0,x,v)$. This yields, with a change of variables $t \mapsto t - nT_0$,

$$\|\mathbf{f}((n+1)T_0)\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \leq e^{-\lambda T_0}\|\mathbf{f}(nT_0)\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} + C_{T_0}\int_{nT_0}^{(n+1)T_0}\|\mathbf{f}(s)\|_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)}\,ds.$$

We can iterate the process for $\mathbf{f}(nT_0)$ as long as $n \neq 0$. We thus obtain

$$\begin{aligned}\|\mathbf{f}((n+1)T_0)\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \leq& e^{-(n+1)\lambda T_0}\|\mathbf{f_0}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \\ &+ C_{T_0}\sum_{k=0}^n e^{-k\lambda T_0}\int_{(n-k)T_0}^{(n+1-k)T_0}\|\mathbf{f}(s)\|_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)}\,ds.\end{aligned} \tag{5.4.20}$$

We see that multiplying and dividing by $\langle v\rangle^\beta$ gives

$$\|\mathbf{f}\|^2_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)} = \sum_{i=1}^N \int_{\mathbb{T}^3\times\mathbb{R}^3} f_i^2\mu_i^{-1}\,dxdv \leq \left|\mathbb{T}^3\right| \left(\int_{\mathbb{R}^3}\frac{dv}{\left(1+|v|^2\right)^\beta}\right)\|\mathbf{f}\|^2_{L^\infty_{x,v}(\mathbf{w}_\beta)}.$$

Since $\beta > 3/2$, the integral is finite and $\mathbf{f}$ also belongs to $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$. By Theorem 5.11 it follows that $\mathbf{f}(t) = S_{\mathbf{G}}(t)(\mathbf{f_0})$ and thus if $\Pi_{\mathbf{G}}(\mathbf{f}) = 0$ we have the following exponential decay

$$\forall t \geq 0, \quad \|\mathbf{f}(t)\|_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)} \leq C_G e^{-\lambda_G t}\|\mathbf{f_0}\|_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)} \leq C_{G,\beta}e^{-\lambda_G t}\|\mathbf{f_0}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}.$$

Plugging the latter into (5.4.20) and taking $0 < \widetilde{\lambda} < \lambda_1 \leq \min\{\lambda, \lambda_G\}$

$$\begin{aligned}
&\|\mathbf{f}((n+1)T_0)\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \\
&\quad \leq \left[e^{-(n+1)\lambda T_0} + C_{\beta,G}\left(\sum_{k=0}^{n} e^{-k\lambda_1 T_0}\int_{(n-k)T_0}^{(n+1-k)T_0} e^{-\lambda_1 s}\,ds\right)\right]\|\mathbf{f_0}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \\
&\quad \leq \left[e^{-(n+1)\lambda T_0} + \frac{C_{\beta,G}e^{\lambda_1 T_0}}{\lambda_1}(n+1)e^{-(n+1)\lambda_1 T_0}\right]\|\mathbf{f_0}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \\
&\quad \leq C_{T_0,\widetilde{\lambda}}e^{-(n+1)\widetilde{\lambda}T_0}\|\mathbf{f_0}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)},
\end{aligned}$$

where we used $(n+1)e^{-(n+1)\lambda_1 T_0} \leq Ce^{-(n+1)\widetilde{\lambda}T_0}$.

At last, for $t \geq 0$ there exists n in $\mathbb{N}$ such that $nT_0 \leq t \leq (n+1)T_0$. Using the inequality satisfied by $\sup_{0\leq t\leq T_0}\|\mathbf{f}(t-nT_0,x,v)\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}$, same computations as above gives

$$\begin{aligned}
\|\mathbf{f}(t)\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} &\leq C\|\mathbf{f}((n+1)T_0)\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \leq Ce^{-(n+1)\widetilde{\lambda}T_0}\|\mathbf{f_0}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \\
&\leq Ce^{-\widetilde{\lambda}t}\|\mathbf{f_0}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)},
\end{aligned}$$

where C is any positive constants depending on T_0. This concludes the proof. □

We now state the theorem about the linear perturbed equation.

Theorem 5.17. *Let $\beta > 3/2$ and let assumptions $(H1)-(H4)$ hold for the collision kernel. The linear perturbed operator $\mathbf{G} = \mathbf{L} - v\cdot\nabla_x$ generates a semigroup $S_\mathbf{G}(t)$ on $L^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)$. Moreover, there exists λ_∞ and $C_\infty > 0$ such that*

$$\forall t\geq 0, \quad \|S_\mathbf{G}(t)\,(Id-\Pi_\mathbf{G})\|_{L^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)} \leq C_\infty e^{-\lambda_\infty t},$$

where $\Pi_\mathbf{G}$ is the orthogonal projection onto Ker$(\mathbf{G})$ in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$.
The constants C_∞ and λ_∞ are explicit and depend on β, N and the collision kernel.

Proof of Theorem 5.17. As before, we use the shorthand notations $\mathbf{w}_\beta = \langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}$ and $w_{\beta i} = \langle v\rangle^\beta\mu_i^{-1/2}$.

Let $\mathbf{f_0}$ be in $L^\infty_{x,v}(\mathbf{w}_\beta)$ with $\beta > 3/2$. If $\mathbf{f}$ is solution of (5.4.19):

$$\partial_t\mathbf{f} = \mathbf{G}(\mathbf{f})$$

in $L^\infty_{x,v}(\mathbf{w}_\beta)$ with initial data $\mathbf{f_0}$ then because $\beta > 3/2$ we have that $\mathbf{f}$ belongs to $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ and $\mathbf{f}(t) = S_\mathbf{G}(t)\mathbf{f_0}$ in this space. This implies first that $\mathbf{f}$ has to be unique and second that Ker$(\mathbf{G})$ and $(\text{Ker}(\mathbf{G}))^\perp$ are stable under the flow of the equation (5.4.19). It suffices to consider $\mathbf{f_0}$ such that $\Pi_{\mathbf{G}}(\mathbf{f_0}) = 0$ and to prove existence and exponential decay of solutions to (5.4.19) in $L^\infty_{x,v}(\mathbf{w}_\beta)$ with initial data $\mathbf{f_0}$.

We recall that $\boldsymbol{\nu}(v) = (\nu_i(v))_{1\leq i\leq N}$ is a multiplicative operator and so the existence of solutions to equation (5.4.19) is equivalent to the existence of a fixed point to its Duhamel's form along the characteristics of the free transport equation. These characteristic trajectories are straight lines of constant speed. We thus need to have existence and exponential decay of a fixed point $\mathbf{f} = (f_i)_{1\leq i\leq N}$ to the following problem for all i in $\{1,\dots,N\}$:

$$f_i(t,x,v) = e^{-\nu_i(v)t} f_{0i}(x-vt,v) + \int_0^t e^{-\nu_i(v)(t-s)} K_i\left(\mathbf{f}(s,x-(t-s)v,\cdot)\right)(v)\,ds.$$

Thanks to Lemma 5.14, each operator K_i is a kernel operator and we thus have for all i in $\{1,\dots,N\}$,

$$\begin{aligned} f_i(t,x,v) =& e^{-\nu_i(v)t} f_{0i}(x-vt,v) \\ &+ \sum_{j=1}^N \int_0^t \int_{\mathbb{R}^3} e^{-\nu_i(v)(t-s)} k_j^{(i)}(v,v_*) f_j(s,x-(t-s)v,v_*)\,dv_*ds. \end{aligned}$$

Iterating this Duhamel's form we end up with the following formulation

$$f_i(t,x,v) = D_1^{(i)}(\mathbf{f_0})(t,x,v) + D_2^{(i)}(\mathbf{f_0})(t,x,v) + D_3^{(i)}(\mathbf{f})(t,x,v) \tag{5.4.21}$$

where we define

$$D_1^{(i)}(\mathbf{f_0}) = e^{-\nu_i(v)t} f_{0i}(x-vt,v), \tag{5.4.22}$$

$$\begin{aligned} D_2^{(i)}(\mathbf{f_0}) =& \sum_{j=1}^N \int_0^t \int_{\mathbb{R}^3} e^{-\nu_i(v)(t-s)} e^{-\nu_j(v_*)s} k_j^{(i)}(v,v_*) \\ &\times f_{0j}(x-(t-s)v-sv_*,v_*)\,dv_*ds, \end{aligned} \tag{5.4.23}$$

$$\begin{aligned} D_3^{(i)}(\mathbf{f}) =& \sum_{j=1}^N \sum_{l=1}^N \int_0^t \int_0^s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_i(v)(t-s)} e^{-\nu_j(v_*)(s-s_1)} k_j^{(i)}(v,v_*) k_l^{(j)}(v_*,v_{**}) \\ &\times f_l(s_1,x-(t-s)v-(s-s_1)v_*,v_{**})\,dv_{**}dv_*ds_1ds. \end{aligned} \tag{5.4.24}$$

Thanks to this Duhamel's formulation, the existence of a fixed point to (5.4.21) in $L^\infty_t L^\infty_{x,v}(\mathbf{w}_\beta)$ follows from a contraction argument. The computations required to prove such a contraction property follow exactly the ones leading to the exponential decay of the latter fixed point. We therefore solely prove that if $\mathbf{f}$ satisfies (5.4.21) then $\mathbf{f}$ decreases exponentially in $L^\infty_{x,v}(\mathbf{w}_\beta)$.

We shall bound each of the terms (5.4.22), (5.4.23) and (5.4.24) separately. From (5.2.4), for all i there exists $\nu_0^{(i)} = \min_{v\in\mathbb{R}^3}\{\nu_i(v)\} > 0$. We define by $\nu_0 > 0$ the minimum of the $\nu_0^{(i)}$ and every positive constant independent of i and $\mathbf{f}$ will be denoted by C_k.

The first term (5.4.22) is straightforwardly bounded.

$$\left\| D_1^{(i)}(\mathbf{f_0})(t) \right\|_{L^\infty_{x,v}(w_{\beta i})} \leq e^{-\nu_0 t} \left\| f_{0i} \right\|_{L^\infty_{x,v}(w_{\beta i})}. \tag{5.4.25}$$

In the second term (5.4.23) we multiply and divide inside the v_* integral by $w_{\beta j}(v_*)$ and take the supremum $\mathbb{T}^3 \times \mathbb{R}^3$.

$$\begin{aligned}\left|w_{\beta i}(v) D_2^{(i)}(\mathbf{f_0})(t)\right| \leq & Cte^{-\nu_0 t} \\ & \times \sum_{j=1}^N \left(\int_{\mathbb{R}^3} \left|k_j^{(i)}(v,v_*)\right| \frac{\langle v\rangle^\beta \mu_i^{-1/2}}{\langle v_*\rangle^\beta \mu_{j*}^{-1/2}}\, dv_*\right) \|f_{0j}\|_{L^\infty_{x,v}(w_{\beta j})}.\end{aligned}$$

Applying Lemma 5.15 with $\theta = \varepsilon = 0$, the integral term is bounded uniformly in i, j and v. Hence

$$\left\|D_2^{(i)}(\mathbf{f_0})(t)\right\|_{L^\infty_{x,v}(w_{\beta i})} \leq C_2 te^{-\nu_0 t} \|\mathbf{f_0}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}. \tag{5.4.26}$$

The third and last term (5.4.24) is more involved analytically and requires to consider the cases $|v| \geq R$ and $|v| \leq R$, for R to be chosen later, separately.

Step 1: $|\mathbf{v}| \geq \mathbf{R}$. We multiply and divide by $w_{\beta l}(v_{**})$ inside the v_* integral of (5.4.24) and take the supremum in space and velocity for f_l. The exponential factor can be bounded by

$$e^{-\nu_i(v)(t-s)-\nu_j(v_*)(s-s_1)} \leq e^{-\frac{\nu_0}{2}t} e^{-\frac{\nu_0}{2}(t-s)} e^{-\frac{\nu_0}{2}(t-s_1)} e^{\frac{\nu_0}{2}s}.$$

Hence, for all t, x and v,

$$\begin{aligned}&\left|w_{\beta i}(v) D_3^{(i)}(\mathbf{f})(t,x,v)\right| \\ &\quad \leq e^{-\frac{\nu_0}{2}t} \sum_{1\leq j,l\leq N} \int_0^t \int_0^s e^{-\frac{\nu_0}{2}(t-s_1)} \left(e^{\frac{\nu_0}{2}s} \|f_l\|_{L^\infty_{x,v}(w_{\beta l})}\right) \\ &\quad\quad \times \left[\int_{\mathbb{R}^3} \left|k_j^{(i)}(v,v_*)\right| \frac{w_{\beta i}(v)}{w_{\beta i}(v_*)} \left(\int_{\mathbb{R}^3} \left|k_l^{(j)}(v_*,v_{**})\right| \frac{w_{\beta i}(v_*)}{w_{\beta l}(v_{**})}\, dv_{**}\right) dv_*\right] ds_1 ds.\end{aligned} \tag{5.4.27}$$

We use Lemma 5.15 twice to bound the term inside bracket independently of j, l and v by

$$\frac{C_\beta^2}{1+|v|} \leq \frac{C_\beta^2}{1+R}.$$

We conclude

$$\sup_{|v|\geq R} \left|w_{\beta i}(v) D_3^{(i)}(\mathbf{f})(t,x,v)\right| \leq \frac{C_3}{1+R} e^{-\frac{\nu_0}{2}t} \sup_{0\leq s\leq t} \left[e^{\frac{\nu_0}{2}} \|\mathbf{f}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}\right]. \tag{5.4.28}$$

Step 2: $|\mathbf{v}| \leq \mathbf{R}$. In order for the change of variables $y = x - (t-s)v - (s-s_1)v_*$ in the v_* integral to be well-defined we need $s - s_1$ bounded from below. Moreover, in order to make the L^2-norm appearing we would need to have $k_j^{(i)}(v,v_*)$ uniformly bounded which is not the case. We therefore need to approximate it uniformly by compactly supported functions, which is possible on compact domains. We take $\eta > 0$ and divide (5.4.24) into four parts

$$\begin{aligned} D_3^{(i)}(\mathbf{f}) = & \int_0^t \int_{s-\eta}^s \int_{\mathbb{R}^3\times\mathbb{R}^3} d_3^{(i)} + \int_0^t \int_0^{s-\eta} \int_{|v_*|\geq 2R} \int_{\mathbb{R}^3} d_3^{(i)} \\ & + \int_0^t \int_0^{s-\eta} \int_{|v_*|\leq 2R} \int_{|v_{**}|\geq 3R} d_3^{(i)} + \int_0^t \int_0^{s-\eta} \int_{|v_*|\leq 2R} \int_{|v_{**}|\leq 3R} d_3^{(i)}, \end{aligned} \tag{5.4.29}$$

where, using

$$e^{-\nu_i(v)(t-s)-\nu_j(v_*)(s-s_1)} \leq e^{-\nu_0(t-s_1)},$$

we have the following bound

$$\begin{aligned} d_3^{(i)} \leq e^{-\nu_0(t-s_1)} & \sum_{1\leq j,l\leq N} k_j^{(i)}(v,v_*)k_l^{(j)}(v_*,v_{**}) \\ & \times |f_l(s_1, x-(t-s)v-(s-s_1)v_*, v_{**})| \; dv_{**}dv_*ds_1ds. \end{aligned}$$

The first integral in (5.4.29) is dealt with by using Lemma 5.15 twice, as for (5.4.27). We get

$$\begin{aligned} \left| w_{\beta i} \int_0^t \int_{s-\eta}^s \int_{\mathbb{R}^3\times\mathbb{R}^3} d_3^{(i)} \right| & \leq Ce^{-\frac{\nu_0}{2}t} \left(\int_0^t \int_{s-\eta}^s e^{-\frac{\nu_0}{2}(t-s_1)} \, ds_1 ds \right) \sup_{0\leq s\leq t} \left[e^{\frac{\nu_0}{2}} \|\mathbf{f}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \right] \\ & \leq \eta C e^{-\frac{\nu_0}{2}t} \sup_{0\leq s\leq t} \left[e^{\frac{\nu_0}{2}} \|\mathbf{f}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \right]. \end{aligned} \tag{5.4.30}$$

For the second and third terms in (5.4.29) we remark that for $|v| \leq R$ we always have either $|v - v_*| \geq R$ or $|v_* - v_{**}| \geq R$ in the domain of integration and therefore we have for any $\varepsilon > 0$ either one of the following inequalities

$$\begin{aligned} \left| k_j^{(i)}(v,v_*) \right| & \leq e^{-m\varepsilon R^2} \left| k_j^{(i)}(v,v_*) e^{m\varepsilon|v-v_*|^2} \right| \\ \left| k_l^{(j)}(v_*,v_{**}) \right| & \leq e^{-m\varepsilon R^2} \left| k_l^{(j)}(v_*,v_{**}) e^{m\varepsilon|v_*-v_{**}|^2} \right|. \end{aligned}$$

Now we take ε small enough to apply Lemma 5.15 as before but with the first inequality above for $|v_*| \geq 2R$ or the second inequality above for $|v_*| \leq 2R$ and $|v_{**}| \geq 3R$. Exactly the same computations as (5.4.27) before yields

$$\left| w_{\beta i} \int_0^t \int_0^{s-\eta} \int_{|v_*|\geq 2R} \int_{\mathbb{R}^3} d_3^{(i)} \right| \leq Ce^{-m\varepsilon R^2} e^{-\frac{\nu_0}{2}t} \sup_{0\leq s\leq t} \left[e^{\frac{\nu_0}{2}} \|\mathbf{f}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \right] \tag{5.4.31}$$

$$\left| w_{\beta i} \int_0^t \int_0^{s-\eta} \int_{|v_*|\geq 2R} \int_{|v_{**}|\geq 3} d_3^{(i)} \right| \leq Ce^{-m\varepsilon R^2} e^{-\frac{\nu_0}{2}t} \sup_{0\leq s\leq t} \left[e^{\frac{\nu_0}{2}} \|\mathbf{f}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)} \right] \tag{5.4.32}$$

At last, the last term in (5.4.29) deals with a set included in the compact support

$$\Omega = \left\{ (v, v_*, v_{**}) \in \mathbb{R}^3, \quad |v| \leq 3R, \; |v_*| \leq 2R, \; |v_{**}| \leq 3R \right\}.$$

As discussed earlier, Lemma 5.14 shows that $k_j^{(i)}(v,v_*)$ has a possible blow-up in $|v-v_*|^\gamma$. However, since Ω is compact we can approximate $k_j^{(i)}(v,v_*)$, for all i and j, by a smooth and compactly supported function $k_{R,j}^{(i)}(v,v_*)$ in the following uniform sense

$$\sup_{|v|\leq 3R}\int_{|v_*|\leq 3R}\left|k_j^{(i)}(v,v_*)-k_{R,j}^{(i)}(v,v_*)\right|\frac{w_{\beta i(v)}}{w_{\beta i}(v_*)}\,dv_*\leq\frac{1}{R}. \tag{5.4.33}$$

Thanks to the following equality

$$\begin{aligned}k_j^{(i)}(v,v_*)k_l^{(j)}(v_*,v_{**}) &= \left(k_j^{(i)}(v,v_*)-k_{R,j}^{(i)}(v,v_*)\right)k_l^{(j)}(v_*,v_{**})\\ &\quad+\left(k_l^{(j)}(v_*,v_{**})-k_{R,l}^{(j)}(v_*,v_{**})\right)k_{R,j}^{(i)}(v,v_*)\\ &\quad+k_{R,j}^{(i)}(v,v_*)k_{R,l}^{(j)}(v_*,v_{**})\end{aligned}$$

the last term in (5.4.29) is bounded by

$$\begin{aligned}&\left|w_{\beta i}\int_0^t\int_0^{s-\eta}\int_{|v_*|\leq 2R}\int_{|v_{**}|\leq 3R}d_3^{(i)}\right|\\ &\leq\frac{C}{R}e^{-\frac{\nu_0}{2}t}\sup_{0\leq s\leq t}\left[e^{\frac{\nu_0}{2}}\|\mathbf{f}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}\right]\sup_{1\leq i,j,l\leq N}\left(\sup_{|v_*|\leq 2R}\int_{|v_{**}|\leq 3R}\left|k_l^{(j)}(v_*,v_{**})\right|\frac{w_{\beta i}(v_*)}{w_{\beta l}(v_{**})}\,dv_{**}\right)\\ &\quad+\frac{C}{R}e^{-\frac{\nu_0}{2}t}\sup_{0\leq s\leq t}\left[e^{\frac{\nu_0}{2}}\|\mathbf{f}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}\right]\sup_{1\leq i,j\leq N}\left(\sup_{|v|\leq R}\int_{|v_*|\leq 2R}\left|k_{R,j}^{(i)}(v,v_*)\right|\frac{w_{\beta i(v)}}{w_{\beta i}(v_*)}\,dv_*\right)\\ &\quad+\sum_{1\leq j,l\leq N}\int_0^t\int_0^{s-\eta}e^{-\nu_0(t-s_1)}\int_{\substack{|v_*|\leq 2R\\|v_{**}|\leq 3R}}\left|k_{R,j}^{(i)}(v,v_*)k_{R,l}^{(j)}(v_*,v_{**})\right||f_l(s_1,y(v_*),v_{**})|\end{aligned}$$

where we made the usual controls (5.4.27) and used (5.4.33). We also defined $y(v_*)=x-(t-s)v-(s-s_1)v_*$. The first two terms are dealt with using Lemma 5.15 while we can bound $k_{R,j}^{(i)}k_{R,l}^{(j)}$ by a constant C_R depending only on R (note that all constants only depending on R will be denoted by C_R). This yields

$$\begin{aligned}&\left|w_{\beta i}\int_0^t\int_0^{s-\eta}\int_{|v_*|\leq 2R}\int_{|v_{**}|\leq 3R}d_3^{(i)}\right|\\ &\quad\leq\frac{C}{R}e^{-\frac{\nu_0}{2}t}\sup_{0\leq s\leq t}\left[e^{\frac{\nu_0}{2}}\|\mathbf{f}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}\right]+C_R\sum_{l=1}^N\int_0^t\int_0^{s-\eta}\int_{\substack{|v_*|\leq 2R\\|v_{**}|\leq 3R}}|f_l(s_1,y(v_*),v_{**})|\,.\end{aligned}$$

We first integrate over v_*. We make the change of variables $y=y(v_*)$ which has a jacobian $|s-s_1|^{-3}\leq\eta^{-3}$. Since we are on the periodic box, y has to be understood as the class of equivalence of $y(v_*)$ and is therefore not one-to-one. However, v_* being bounded by $2R$ we cover $\mathbb{T}^3$ only finitely many times (depending on R). Hence,

$$\begin{aligned}&\left|w_{\beta i}\int_0^t\int_0^{s-\eta}\int_{|v_*|\leq 2R}\int_{|v_{**}|\leq 3R}d_3^{(i)}\right|\\ &\quad\leq\frac{C}{R}e^{-\frac{\nu_0}{2}t}\sup_{0\leq s\leq t}\left[e^{\frac{\nu_0}{2}}\|\mathbf{f}\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}\right]+\frac{C_R}{\eta^3}\sum_{l=1}^N\int_0^t\int_0^{s-\eta}\int_{\mathbb{T}^3\times\mathbb{R}^3}|f_l(s_1,y,v_{**})|\,.\end{aligned}$$

Finally, a Cauchy-Schwarz inequality against $\mu_l^{-1/2}(v_{**})/\mu_l^{-1/2}(v_{**})$ yields the following estimate

$$\begin{aligned}&\left|w_{\beta i}\int_0^t\int_0^{s-\eta}\int_{|v_*|\leq 2R}\int_{|v_{**}|\leq 3R} d_3^{(i)}\right|\\&\leq \frac{C}{R}e^{-\frac{\nu_0}{2}t}\sup_{0\leq s\leq t}\left[e^{\frac{\nu_0}{2}}\left\|\mathbf{f}\right\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}\right]+\frac{C_R}{\eta^3}t\int_0^t\left\|\mathbf{f}(s_1)\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\,ds_1.\end{aligned}\tag{5.4.34}$$

Plugging (5.4.30), (5.4.31), (5.4.32) and (5.4.34) into (5.4.29) gives the final estimate

$$\begin{aligned}\sup_{|v|\leq R}\left|w_{\beta i}(v)D_3^{(i)}(\mathbf{f})\right|\leq & C_4e^{-\frac{\nu_0}{2}t}\left(\eta+e^{-m\varepsilon R^2}+\frac{1}{R}\right)\sup_{0\leq s\leq t}\left[e^{\frac{\nu_0}{2}}\left\|\mathbf{f}\right\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}\right]\\&+C_{R,\eta}t\int_0^t\left\|\mathbf{f}(s)\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\,ds.\end{aligned}\tag{5.4.35}$$

We can now conclude the proof by gathering (5.4.21), (5.4.25), (5.4.26), (5.4.28) and (5.4.35). We get that for all i in $\{1,\dots,N\}$

$$\begin{aligned}e^{\frac{\nu_0}{2}t}\left\|f_i(t)\right\|_{L^\infty_{x,v}(w_{\beta i})}\leq &(1+C_2t)\,e^{-\frac{\nu_0}{2}t}\left\|\mathbf{f_0}\right\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}+C_{R,\eta}t\int_0^t\left\|\mathbf{f}(s)\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\,ds\\&+C_5\left(\eta+e^{-m\varepsilon R^2}+\frac{1}{R}\right)\sup_{0\leq s\leq t}\left[e^{\frac{\nu_0}{2}}\left\|\mathbf{f}\right\|_{L^\infty_{x,v}(\mathbf{w}_\beta)}\right].\end{aligned}\tag{5.4.36}$$

We remind the reader that C_2 and C_5 are independent of η, R and t; moreover $\varepsilon>0$ is fixed. We choose R large enough and η small enough such that

$$C_5\left(\eta+e^{-m\varepsilon R^2}+\frac{1}{R}\right)\leq\frac{1}{2}$$

and $T_0>0$ such that

$$2(1+C_2T_0)e^{-\nu_0T_0}=e^{-\frac{\nu_0}{2}T_0}.$$

Such choices with (5.4.36) yields that for all t in $[0,T_0]$,

$$\left\|\mathbf{f}(t)\right\|_{L^\infty_{x,v}(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2})}\leq e^{\frac{\nu_0}{2}(T_0-2t)}\left\|\mathbf{f_0}\right\|_{L^\infty_{x,v}(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2})}+CT_0\int_0^t\left\|\mathbf{f}(s)\right\|_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})}\,ds.$$

Lemma 5.16 then concludes the proof of Theorem 5.17. □

5.5. The full nonlinear equation in a perturbative regime

This section is devoted to the proof of Theorem 5.3. We divide our study in three steps. Subsection 5.5.1 deals with the existence of a solution with exponential decay to the perturbed multi-species Boltzmann equation that reads

$$\partial_t \mathbf{f} + v \cdot \nabla_x \mathbf{f} = \mathbf{L}(\mathbf{f}) + \mathbf{Q}(\mathbf{f}). \tag{5.5.1}$$

Then Subsection 5.5.2 proves the uniqueness of such solutions and, at last, Subsection 5.5.3 shows the positivity of the latter.

5.5.1. Existence of a solution that decays exponentially

We refer to the definition of $\Pi_{\mathbf{G}}$ (5.3.1) and recall that $\Pi_{\mathbf{G}}(\mathbf{f}) = 0$ is a convenient way to say that $\mathbf{f}$ satisfies the conservation laws (5.1.5) with $\theta_\infty = 1$ and $u_\infty = 0$.

This subsection is dedicated to the proof of the following proposition.

Proposition 5.18. *Let assumptions* $(H1) - (H4)$ *hold for the collision kernel, and let* $k > k_0$*, where* k_0 *is the smallest integer such that* $C_{k_0} < 1$ *where* C_k *was given by* (5.1.10)*. There exists* η_k*,* C_k *and* $\lambda_k > 0$ *such that for any* $\mathbf{f_0}$ *in* $L^1_v L^\infty_x \left(\langle v \rangle^k\right)$ *satisfying* $\Pi_{\mathbf{G}}(\mathbf{f_0}) = 0$*, if*

$$\|\mathbf{f_0}\| \leq \eta_k$$

then there exists $\mathbf{f}$ *in* $L^1_v L^\infty_x \left(\langle v \rangle^k\right)$ *with* $\Pi_{\mathbf{G}}(\mathbf{f}) = 0$ *solution to* (5.5.1) *with initial data* $\mathbf{f_0}$ *such that*

$$\forall t \geq 0, \quad \|\mathbf{f}\|_{L^1_v L^\infty_x(\langle v \rangle^k)} \leq C_k e^{-\lambda_k t} \|\mathbf{f_0}\|_{L^1_v L^\infty_x(\langle v \rangle^k)}.$$

The constants are explicit and only depend on N*,* k *and the collision kernels.*

Decomposition of the perturbed equation and toolbox

As explained in the introduction, the main strategy is to find a decomposition of the perturbed Boltzmann equation (5.5.1) into a system of differential equations where we could make use of the L^∞ semigroup theory developed in Section 5.4. More precisely, one would like to solve a somewhat simpler equation in $L^1_v L^\infty_x \left(\langle v \rangle^k\right)$ and that the remainder part has regularising properties and could thus be handled in the more regular space $L^\infty_{x,v}\left(\langle v \rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$. Then the exponential decay of $S_{\mathbf{G}}(t)$ in the more regular space could be carried up to the bigger space.

Remark that

$$L^\infty_{x,v}\left(\langle v \rangle^\beta \boldsymbol{\mu}^{-1/2}\right) \subset L^1_v L^\infty_x \left(\langle v \rangle^k\right).$$

We propose here a decomposition of the mutli-species linear operator $\mathbf{G} = \mathbf{L} - v \cdot \nabla_x$ that follows the idea used in [78] for the single-species Boltzmann operator. We define for

$\delta \in (0,1)$ to be chosen later the truncation function $\Theta(v, v^*, \sigma) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ bounded by one on the set

$$\left\{|v| \leq \delta^{-1} \quad \text{and} \quad 2\delta \leq |v-v^*| \leq \delta^{-1} \quad \text{and} \quad |\cos\theta| \leq 1-2\delta\right\},$$

and its support included in the set

$$\left\{|v| \leq 2\delta^{-1} \quad \text{and} \quad \delta \leq |v-v^*| \leq 2\delta^{-1} \quad \text{and} \quad |\cos\theta| \leq 1-\delta\right\}.$$

Thus we can define the splitting

$$\mathbf{G} = \mathbf{L} - v\cdot\nabla_x = \mathbf{A}^{(\delta)} + \mathbf{B}^{(\delta)} - \boldsymbol{\nu} - v\cdot\nabla_x,$$

with the operators $\mathbf{A}^{(\delta)} = \left(A_i^{(\delta)}\right)_{1\leq i\leq N}$ and $\mathbf{B}^{(\delta)} = \left(B_i^{(\delta)}\right)_{1\leq i\leq N}$ defined as

$$A_i^{(\delta)}(\mathbf{f}(v)) = \sum_{j=1}^N C_{ij}^\Phi \int_{\mathbb{R}^3\times\mathbb{S}^2} \Theta_\delta \left(\mu_j'^* f_i' + \mu_i' f_j'^* - \mu_i f_j^*\right) b_{ij}(\cos\theta)|v-v^*|^\gamma d\sigma dv^*,$$
$$B_i^{(\delta)}(\mathbf{f}(v)) = \sum_{j=1}^N C_{ij}^\Phi \int_{\mathbb{R}^3\times\mathbb{S}^2} (1-\Theta_\delta) \left(\mu_j'^* f_i' + \mu_i' f_j'^* - \mu_i f_j^*\right) b_{ij}(\cos\theta)|v-v^*|^\gamma d\sigma dv^*.$$

Our goal is to show that $\mathbf{A}^{(\delta)}$ has some regularizing effects and that $\mathbf{G_1} := \mathbf{B}^{(\delta)} - \boldsymbol{\nu} - v\cdot\nabla_x$ acts like a small perturbation of $\boldsymbol{G_\nu} := -\boldsymbol{\nu} - v\cdot\nabla_x$ and is thus hypodissipative.

Lemma 5.19. *For any k in $\mathbb{N}$, $\beta > 0$ and δ in $(0,1)$, there exists $C_A > 0$ such that for all $\mathbf{f}$ in $L^1_v L^\infty_x\left(\langle v\rangle^k\right)$*

$$\left\|\mathbf{A}^{(\delta)}\left(\mathbf{f}\right)\right\|_{L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)} \leq C_A \|\mathbf{f}\|_{L^1_v L^\infty_x\left(\langle v\rangle^k\right)}.$$

The constant C_A is constructive and only depends on k, β, δ, N and the collision kernels.

Proof of Lemma 5.19. As we proved it in Lemma 5.14, the operator $\mathbf{A}^{(\delta)}$ can be written as a kernel operator thanks to Carleman representation:

$$\forall\, i \in \{1,\dots,N\}, \quad A_i^{(\delta)}(\mathbf{f})(x,v) = \int_{\mathbb{R}^3} \langle \mathbf{k}_{\mathbf{A}}^{(\mathbf{i}),(\delta)}(v,v_*), \mathbf{f}(x,v_*)\rangle\, dv_*.$$

Moreover, by definition of $\mathbf{A}^{(\delta)}$, its kernels $\mathbf{k}_{\mathbf{A}}^{(\mathbf{i}),(\delta)}$ are of compact support which implies the desired estimate. $\square$

Thanks to the regularizing property above of the operator $\mathbf{A}^{(\delta)}$ we are looking for solutions to the perturbed Boltzmann equation

$$\partial_t \mathbf{f} = \mathbf{G}\left(\mathbf{f}\right) + \mathbf{Q}(\mathbf{f})$$

in the form of $\mathbf{f} = \mathbf{f_1} + \mathbf{f_2}$ with $\mathbf{f_1}$ in $L^1_v L^\infty_x \left(\langle v\rangle^k\right)$ and $\mathbf{f_2}$ in $L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$ and $(\mathbf{f_1},\mathbf{f_2})$ satisfying the following system of equation

$$\partial_t \mathbf{f_1} = \mathbf{G_1}^{(\delta)}(\mathbf{f_1}) + \mathbf{Q}(\mathbf{f_1}+\mathbf{f_2}) \quad \text{and} \quad \mathbf{f_1}(0,x,v) = \mathbf{f_0}(x,v), \tag{5.5.2}$$

$$\partial_t \mathbf{f_2} = \mathbf{G}(\mathbf{f_2}) + \mathbf{A}^{(\delta)}(\mathbf{f_1}) \quad \text{and} \quad \mathbf{f_2}(0,x,v) = 0. \tag{5.5.3}$$

The equation in the smaller space (5.5.3) will be treated thanks to the semigroup generated by $\mathbf{G}$ in $L^\infty\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$ whilst we expect an exponential decay for solutions in the larger space (5.5.2). Indeed, $\mathbf{B}^{(\delta)}$ can be controlled by the multiplicative operator $\boldsymbol{\nu}(v)$ thanks to the following lemma.

Lemma 5.20. *Define*

$$\overline{\mathbf{w_k}} = \left(1 + m_i^{k/2}\,|v|^k\right)_{1\leq i\leq N} \quad \text{and} \quad \overline{\mathbf{w_k}}\boldsymbol{\nu} = \left((1 + m_i^{k/2}\,|v|^k)\nu_i(v)\right)_{1\leq i\leq N}.$$

There exists k_0 in $\mathbb{N}$ such that for any $k \geq k_0$ and δ in $(0,1)$ there exists $C_B(k,\delta) > 0$ such that for all $\mathbf{f}$ in $L^1_v L^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})$,

$$\left\|\mathbf{B}^{(\delta)}(\mathbf{f})\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}})} \leq C_B(k,\delta)\,\|\mathbf{f}\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})}.$$

Moreover we have the following formula

$$C_B(k,\delta) = C_k + \varepsilon_k(\delta)$$

where $\varepsilon_k(\delta)$ is an explicit function that tends to 0 as δ tends to 0 and C_k is defined by (5.1.10) and k_0 is the minimal integer such that $C_{k_0} < 1$.

We make an important remark.

Remark 5.21. *We emphasize here that for $k > k_0$ we have that $\lim_{\delta\to 0} C_B(k,\delta) = C_k < 1$. Until the end we fix $\delta_k > 0$ such that $C_B(k,\delta_k) < 1$. For convenience we will drop the exponent and use the following notations: $\mathbf{B} = \mathbf{B}^{(\delta_k)}$, $\mathbf{A} = \mathbf{A}^{(\delta_k)}$, $\mathbf{G_1} = \mathbf{G_1}^{(\delta_k)}$ and finally $C_B = C_B(k,\delta_k)$. The equivalent of this result for the mono-species Boltzmann equation can be found in [78, Lemma 4.4] for $k > 2$ which is recovered here when $m_i = m_j$ (note that our Lemma deals with more general collision kernels).*

We also notice here that the weighted norm $\overline{\mathbf{w_k}}$ required for this sharp lemma is equivalent to $\langle v\rangle^k$.

The proof of Lemma 5.20 relies on a Povzner-type inequality. Such inequalities are now common in the mono-species Boltzmann literature (for both elastic and inelastic collisions) [124][109][8][9][78] and state that the integral on $\mathbb{S}^2$ of $\left[|v'|^k + |v'_*|^k\right]$ can be controlled strictly by the integral on $\mathbb{S}^2$ of $C_k\left[|v|^k + |v_*|^k\right]$ with $C_k = 4/(k+2) < 1$ (for hard sphere collision kernels) and a remainder term of lower order when $k > 2$. As we shall see, the asymmetry brought by the difference of masses generates a larger constant C_k that can still be less than 1 if k is large enough.

The method proposed here to prove such a Povzner inequality is inspired by [9, Lemma 1 and Corollary 3]. The main idea is to consider kinetic energies $m_i\,|v'|^2$ and $m_j\,|v'_*|^2$ to exhibit the problematic term arising from $m_i - m_j$ which can be non-zero. We state our result, which covers the mono-species case when $m_i = m_j$.

Proposition 5.22 (Povzner-type inequality)**.** *Let i and j in $\{1,\dots,N\}$. Then for all $k > 2$,*

$$\int_{\mathbb{S}^2}\left[m_i^{k/2}\left|v'\right|^k + m_j^{k/2}\left|v'_*\right|^k\right]\,d\sigma \leq \frac{l_{b_{ij}}}{b_{ij}^\infty}C_k\left[m_i\,|v|^2 + m_j\,|v_*|^2\right]^{k/2}$$

where C_k was defined by (5.1.10) *and $l_{b_{ij}}$, b_{ij}^∞ by* (5.1.9)*.*

Proof of Proposition 5.22. By definition of v' and v'_* we can expand $|v'|^2$ and $|v'_*|^2$ as follows

$$\begin{aligned} m_i\left|v'\right|^2 &= E\frac{1 + a_{ij} + b_{ij}\langle e,\sigma\rangle}{2} \\ m_i\left|v'_*\right|^2 &= E\frac{1 - a_{ij} - b_{ij}\langle e,\sigma\rangle}{2} \end{aligned}$$

where we denoted by e the direction of the vector $m_i v + m_j v_*$ and we defined

$$\begin{aligned} E &= m_i\,|v|^2 + m_j\,|v_*|^2\,, \\ a_{ij} &= \frac{1}{E}\frac{m_i - m_j}{m_i + m_j}\left[m_i\frac{m_i - m_j}{m_i + m_j}\,|v|^2 + m_j\frac{m_j - m_i}{m_i + m_j}\,|v_*|^2 + 4\frac{m_i m_j}{m_i + m_j}\langle v, v_*\rangle\right], \\ b_{ij} &= \frac{1}{E}\frac{4m_i m_j}{(m_i + m_j)^2}\,|v - v_*|\,|m_i v + m_j v_*|\,. \end{aligned} \tag{5.5.4}$$

We will drop the dependencies on v and v_* The first important property to notice is that for all σ on $\mathbb{S}^2$, $m_i\,|v'|^2$ and $m_j\,|v'_*|^2$ are positive and this implies

$$|a_{ij}| \leq 1, \quad |a_{ij} + b_{ij}| \leq 1 \quad \text{and} \quad |a_{ij} - b_{ij}| \leq 1. \tag{5.5.5}$$

Plugging these equalities inside the integral yields

$$\begin{aligned} &\int_{\mathbb{S}^2}\left[m_i^{k/2}\left|v'\right|^k + m_j^{k/2}\left|v'_*\right|^k\right]\,d\sigma \\ &\quad= E^{k/2}\int_{\mathbb{S}^2}\left[\left(\frac{1 + a_{ij} + b_{ij}\langle e,\sigma\rangle}{2}\right)^{k/2} + \left(\frac{1 - a_{ij} - b_{ij}\langle e,\sigma\rangle}{2}\right)^{k/2}\right]\,d\sigma \\ &\quad= 2\pi E^{k/2}\int_{-1}^{1}\left[\left(\frac{1 + a_{ij} + b_{ij}z}{2}\right)^{k/2} + \left(\frac{1 - a_{ij} - b_{ij}z}{2}\right)^{k/2}\right]\,dz \end{aligned}$$

$$= \frac{8\pi}{k+2} E^{k/2} \left[F_{k/2}(|a_{ij}|, b_{ij}) + F_{k/2}(-|a_{ij}|, b_{ij}) \right] \tag{5.5.6}$$

where

$$F_p(a,b) = \frac{\left(\frac{1+a+b}{2}\right)^{p+1} - \left(\frac{1+a-b}{2}\right)^{p+1}}{b}.$$

When $|a| \leq 1$, a mere study of the function $F_p(a,\cdot)$ shows that the latter function is increasing on $[0, 1+a]$ if $p \geq 0$. Therefore, since $|a_{ij}| \leq 1$, for $k \geq 2$ we can bound $F_{k/2}(|a_{ij}|, b_{ij})$ and $F_{k/2}(-|a_{ij}|, b_{ij})$ with their value at an upper bound on b_{ij}. Using (5.5.5) we see that $0 \leq b_{ij} \leq 1 - |a_{ij}|$. Bounding into (5.5.6), this gives

$$\int_{\mathbb{S}^2} \left[m_i^{k/2} |v'|^k + m_j^{k/2} |v'_*|^k \right] d\sigma \leq \frac{8\pi}{k+2} E^{k/2} \frac{1 - |a_{ij}|^{k/2+1} + (1-|a_{ij}|)^{k/2+1}}{1-|a_{ij}|}. \tag{5.5.7}$$

To conclude the proof it suffices to see that the function

$$a \mapsto \frac{1 - |a|^{k/2+1} + (1-|a|)^{k/2+1}}{1-|a|}$$

is increasing on $[0,1]$. Proposition 5.22 will follow if $|a_{ij}| \leq |m_i - m_j| / (m_i + m_j)$.

Going back to the definition of a_{ij} and decomposing v_* as $v_* = \lambda v + v^\perp$ with $v^\perp$ orthogonal to v we see that

$$|a_{ij}| = \frac{1}{E} \frac{|m_i - m_j|}{m_i + m_j} \left| \left(\frac{m_i^2 + m_i m_j (4\lambda - \lambda^2 - 1) + \lambda^2 m_j^2}{m_i + m_j} \right) |v|^2 + m_j \frac{m_i - m_j}{m_i + m_j} \left| v^\perp \right|^2 \right|.$$

But then, direct computations show first

$$\left| m_j \frac{m_i - m_j}{m_i + m_j} \right| \leq m_j$$

and second

$$\begin{aligned} &\left| m_i^2 + m_i m_j (4\lambda - \lambda^2 - 1) + \lambda^2 m_j^2 \right|^2 - (m_i + m_j)^2 (m_i + \lambda^2 m_j)^2 \\ &\quad = -4 m_i m_j (1-\lambda)^2 (m_i + \lambda m_j)^2 \leq 0. \end{aligned}$$

Hence

$$|a_{ij}| \leq \frac{|m_i - m_j|}{m_i + m_j} \frac{(m_i + \lambda^2 m_j)|v|^2 + m_j \left| v^\perp \right|^2}{E}$$

which terminates the proof of the proposition. □

Now we can prove the estimate on $\mathbf{B}^{(\delta)}$.

Proof of Lemma 5.20. We use the definition $\overline{\mathbf{w_k}} = (1 + m_i^{k/2}|v|^k)_{1\leq i\leq N}$. Moreover, as we will drastically bound $\mathbf{B}^{(\delta)}(\mathbf{f})$ by the absolute value inside the integral in v, it is enough to show Lemma 5.20 only for $f = f(v)$.

With the multi-species Povzner inequality (Proposition 5.22), the proof follows closely the proof of [78, Lemma 4.4] with appropriate characteristic functions that fit the invariance of the elastic collisions (5.1.2).

First we bound the truncation function from above by cutting the integral in the following way

$$
\begin{aligned}
&\left\|\mathbf{B}^{(\delta)}(\mathbf{f})\right\|_{L^1_v(\overline{\mathbf{w_k}})} \\
&\leq \sum_{i,j=1}^N C_{ij}^\Phi \int_{\mathbb{R}^6\times\mathbb{S}^2} (1-\Theta_\delta)\left[\mu_j^{'*}|f_i'| + \mu_i'|f_j{'^*}| + \mu_i|f_j^*|\right] b_{ij}(\cos\theta)|v-v_*|^\gamma \overline{w_k}_i dv_* d\sigma \\
&\leq \sum_{i,j=1}^N C_{ij}^\Phi \int_{\{|\cos\theta|\in[1-\delta,1]\}} b_{ij}(\cos\theta)|v-v_*|^\gamma \mu_j^*|f_i|(\overline{w_k}_i' + \overline{w_k}_j'^* + \overline{w_k}_j^*) dv dv_* d\sigma \\
&\quad + \sum_{i,j=1}^N C_{ij}^\Phi \int_{|v-v_*|\leq\delta} b_{ij}(\cos\theta)|v-v_*|^\gamma \mu_j^*|f_i|(\overline{w_k}_i' + \overline{w_k}_j'^* + \overline{w_k}_j^*) dv dv_* d\sigma \\
&\quad + \sum_{i,j=1}^N C_{ij}^\Phi \int_{\{|v|\geq\delta^{-1} \text{ or } |v-v_*|\geq\delta^{-1}\}} \left[\mu_j^{'*}|f_i'| + \mu_i'|f_j^{'*}| + \mu_i|f_j^*|\right] b_{ij}(\cos\theta)|v-v_*|^\gamma \overline{w_k}_i.
\end{aligned}
$$

Note that we used the change of variables $(v, v_*, \sigma) \to (v', v'^*, v - v_*/|v-v_*|)$ for $\mu_j^{'*} f_i'$. Then for $\mu_i' f_j^{'*}$ we used first $(v, v_*, \sigma) \to (v_*, v, -\sigma)$ which sends $(v_{ij}', v_{ij}^{'*})$ to $(v_{ji}^{'*}, v_{ji}')$ and then relabelling i and j we come back to the first term $\mu_j^{'*} f_i'$.

Defining the characteristic function χ_A on the set

$$
A = \left\{\sqrt{m_i|v|^2 + m_j|v_*|^2} \geq \min\left\{\sqrt{m_i}, \sqrt{m_j}\right\}\delta^{-1} \quad \text{or} \quad |v-v_*| \geq \delta^{-1}\right\}
$$

we can bound $b(\cos\theta)$ by its supremum b_∞ and use the equivalence between ν_i and $1+|v|^\gamma$ to get

$$
\begin{aligned}
&\left\|\mathbf{B}^{(\delta)}(\mathbf{f})\right\|_{L^1_v(\overline{\mathbf{w_k}})} \\
&\leq \delta C(k)\|\mathbf{f}\|_{L^1_v(\overline{\mathbf{w_k}}\boldsymbol{\nu})} \\
&\quad + \sum_{i,j=1}^N C_{ij}^\Phi \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \chi_A\left[\mu_j'^*|f_i'| + \mu_i'|f_j'^*| + \mu_i|f_j^*|\right] b_{ij}(\cos\theta)|v-v_*|^\gamma \overline{w_k}_i \, dv dv_* d\sigma
\end{aligned}
\tag{5.5.8}
$$

where $C(k)$ will denote any positive constant independent on δ and $\mathbf{f}$.

We shall deal with the second term on the right-hand side of (5.5.8) thanks to the Povzner inequality. Indeed, the set A is invariant by the changes of variables already mentioned (remember that when changing v to v_* we also change i and j) and therefore

$$
\begin{aligned}
&\sum_{i,j=1}^{N} C_{ij}^{\Phi} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \chi_A \left[\mu_j'^{*}|f_i'| + \mu_i'|f_j'^{*}| + \mu_i|f_j^{*}|\right] b_{ij}(\cos\theta)|v-v_*|^{\gamma}\overline{w_{k}}_i \, dv dv_* d\sigma \\
&\quad = \sum_{i,j=1}^{N} C_{ij}^{\Phi} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \chi_A b_{ij}(\cos\theta)|v-v_*|^{\gamma}\mu_j^{*}|f_i|(\overline{w_k}_i^{*\prime} + \overline{w_k}_i' + \overline{w_k}_i^{*}) \, dv dv_* d\sigma \\
&\quad \leq \sum_{i,j=1}^{N} C_{ij}^{\Phi} b_{ij}^{\infty} \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_A |v-v_*|^{\gamma}\mu_j^{*}|f_i| \left(\int_{\mathbb{S}^2}\left[\overline{w_k}_i' + \overline{w_k}_j'^{*} - \overline{w_k}_j^{*} - \overline{w_k}_i\right] d\sigma\right) dv dv_* \\
&\qquad + 8\pi \sum_{i,j=1}^{N} C_{ij}^{\Phi} b_{ij}^{\infty} \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_A |v-v_*|^{\gamma}\mu_j^{*}|f_i|\overline{w_k}_i^{*} \, dv dv_* \\
&\qquad + 4\pi \sum_{i,j=1}^{N} C_{ij}^{\Phi} b_{ij}^{\infty} \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_A |v-v_*|^{\gamma}\mu_j^{*}|f_i|\overline{w_k}_i \, dv dv_*
\end{aligned}
\tag{5.5.9}
$$

We can use Proposition 5.22 for the first term on the right-hand side of the inequality. Indeed,

$$
\begin{aligned}
&\int_{\mathbb{S}^2}\left[\overline{w_k}_j' + \overline{w_k}_j'^{*} - \overline{w_k}_j^{*} - \overline{w_k}_i\right] \, d\sigma \\
&\qquad \leq \frac{l_{b_{ij}}}{b_{ij}^{\infty}} C_k \left(m_i\,|v|^2 + m_j\,|v_*|^2\right)^{k/2} - 4\pi m_i^{k/2}\,|v|^k - 4\pi m_j^{k/2}\,|v_*|^k \\
&\qquad \leq 2^{k/2}\frac{l_{b_{ij}}}{b_{ij}^{\infty}} C_k \left[(m_i\,|v|^2)^{k/2-1/2}(m_j\,|v_*|^2)^{1/2} + (m_i\,|v|^2)^{1/2}(m_j\,|v_*|^2)^{k/2-1/2}\right] \\
&\qquad\quad - 4\pi\left(1 - \frac{l_{b_{ij}}}{4\pi b_{ij}^{\infty}} C_k\right)\left[m_i^{k/2}\,|v|^k + m_j^{k/2}\,|v_*|^k\right]
\end{aligned}
$$

For $k \geq k_0$ we have that $C_k < 1$, hence $\frac{l_{b_{ij}}}{4\pi b_{ij}^{\infty}} C_k < 1$. We can thus plug this back into

(5.5.9) we find, recalling that $\overline{w_k}_i = 1 + m_i^{k/2}\left|v\right|^k$

$$
\begin{aligned}
\sum_{i,j=1}^N C_{ij}^\Phi \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \chi_A \left[\mu_j'^*|f_i'| + \mu_i'|f_j'^*| + \mu_i|f_j^*|\right] b_{ij}(\cos\theta)|v-v_*|^\gamma \overline{w_k}_i \, dvdv_*d\sigma \\
\leq C(k)\sum_{i,j=1}^N \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_A|v-v_*|^\gamma \mu_j^*|f_i| \left[\left|v\right|^{k-1}\left|v_*\right| + \left|v\right|\left|v_*\right|^{k-1}\right]\, dvdv_* \\
+ 12\pi \sum_{i,j=1}^N C_{ij}^\Phi b_{ij}^\infty \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_A|v-v_*|^\gamma \mu_j^*|f_i|\, dvdv_* + \\
+ 8\pi \sum_{i,j=1}^N C_{ij}^\Phi b_{ij}^\infty \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_A|v-v_*|^\gamma \mu_j^*|f_i| m_j^{k/2}\left|v_*\right|^k\, dvdv_* \\
+ C_k \sum_{i,j=1}^N C_{ij}^\Phi l_{b_{ij}} \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_A|v-v_*|^\gamma \mu_j^*|f_i| m_i^{k/2}\left|v\right|^k\, dvdv_*
\end{aligned}
$$

From here we can use that

$$
\chi_A(v,v_*) \leq 2\max_{i,j}\{m_i,m_j\}\,\delta(m_i\left|v\right|^2 + m_j\left|v^*\right|^2)
$$

and the fact that $\gamma+1 < k_0 \leq k$ to bound the first, second and third term on the right-hand side by $\delta C(k)\left\|\mathbf{f}\right\|_{L^1_v(\overline{\mathbf{w_k}})}$. And finally, we exactly have the definition of $\nu_i(v)$ in the last term on the right-hand side. This gives

$$
\begin{aligned}
\sum_{i,j=1}^N C_{ij}^\Phi \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \chi_A \left[\mu_j'^*|f_i'| + \mu_i'|f_j'^*| + \mu_i|f_j^*|\right] b_{ij}(\cos\theta)|v-v_*|^\gamma \overline{w_k}_i \, dvdv_*d\sigma \\
\leq C_k \sum_{i=1}^N \left\|f_i\right\|_{L^1_v(\overline{w_k}_i\nu_i)} + \delta C(k)\left\|\mathbf{f}\right\|_{L^1_v(\overline{\mathbf{w_k}})}.
\end{aligned}
\tag{5.5.10}
$$

Combining (5.5.8) and (5.5.10) yields the desired estimate. □

We conclude this subsection with a control on the nonlinear term.

Lemma 5.23. *Define* $\widetilde{\mathbf{Q}}(\mathbf{f},\mathbf{g})$ *by*

$$
\forall\, 1 \leq i \leq N, \quad \widetilde{Q}_i(\mathbf{f},\mathbf{g}) = \frac{1}{2}\sum_{j=1}^N \left(Q_{ij}(f_i,g_j) + Q_{ij}(g_i,f_j)\right). \tag{5.5.11}
$$

Then for all $\mathbf{f}$, $\mathbf{g}$ *such that* $\widetilde{\mathbf{Q}}(\mathbf{f},\mathbf{g})$ *is well-defined, the latter belongs to* $[Ker(L)]^\perp$*:*

$$
\pi_{\mathbf{L}}\left(\widetilde{\mathbf{Q}}(\mathbf{f},\mathbf{g})\right) = 0.
$$

Moreover, there exists $C_Q > 0$ such that for all i in $\{1, \dots, N\}$ and every $\mathbf{f}$ and $\mathbf{g}$,

$$\left\|\widetilde{Q}_i(\mathbf{f}, \mathbf{g})\right\|_{L^1_v L^\infty_x(\langle v\rangle^k)} \leq C_Q \Big[\|f_i\|_{L^1_v L^\infty_x(\langle v\rangle^k)} \|\mathbf{g}\|_{L^1_v L^\infty_x(\langle v\rangle^k \boldsymbol{\nu})} + \|f_i\|_{L^1_v L^\infty_x(\nu_i\langle v\rangle^k)} \|\mathbf{g}\|_{L^1_v L^\infty_x(\langle v\rangle^k)} \Big],$$

The constant C_Q is explicit and depends only on k, N and the kernel of the collision operator.

Proof of Lemma 5.23. The orthogonality property is well-known for the single-species case [25, Appendix A.2],[21] and follows in the same way as (5.1.3), (5.1.4). The estimate also follows standard computations from the mono-species case, we adapt them to the case of multi-species for the sake of completeness. Since we are dealing with hard potential kernels, we can decompose the bilinear operator $Q_{ij}(f_i, g_j)$, for any i, j in $\{1, \dots, N\}$, as

$$Q_{ij}(f_i, g_j) = \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij}\left(|v - v_*|, \cos\theta\right) f_i' g_j'^{*}\, dv_* d\sigma - \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij}\left(|v - v_*|, \cos\theta\right) f_i g_j^{*}\, dv_* d\sigma.$$

By Minkowski integral inequality we have for all q in $[1, \infty)$,

$$\int_{\mathbb{R}^3} \langle v\rangle^k \left[\int_{\mathbb{T}^3} |Q_{ij}(f_i, g_j)|^q\, dx\right]^{1/q} dv \leq \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} \langle v\rangle^k \left[\int_{\mathbb{T}^3} \left|B_{ij} f_i' g_j'^{*}\right|^q dx\right]^{1/q} d\sigma dv_* dv + \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} \langle v\rangle^k \left[\int_{\mathbb{T}^3} \left|B_{ij} f_i g_j^{*}\right|^q dx\right]^{1/q} d\sigma dv_* dv.$$

Since the function $(v, v_*) \mapsto (v', v_*')$ is its own inverse and does not change the value of $B_{ij}\left(|v - v_*|, \cos\theta\right)$, we make the latter change of variables in the first integral and we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \langle v\rangle^k \left[\int_{\mathbb{T}^3} |Q_{ij}(f_i, g_j)|^q\, dx\right]^{1/q} dv \\
&\quad\leq \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} \left(\langle v\rangle^k + \langle v'\rangle^k\right) \left[\int_{\mathbb{T}^3} \left|B_{ij} f_i g_j^{*}\right|^q dx\right]^{1/q} d\sigma dv_* dv \\
&\quad\leq C_{ij} \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} \langle v\rangle^k \langle v_*\rangle^k |v - v_*|^\gamma \left[\int_{\mathbb{T}^3} \left|f_i g_j^{*}\right|^q dx\right]^{1/q} d\sigma dv_* dv.
\end{aligned}$$

The constant $C_{ij} > 0$ will stand for any constant depending only on m_i, m_j, the integral over the sphere of b_{ij} and C^Φ_{ij} (see assumptions on the kernel B_{ij}). Finally we use the fact that $|v - v_*|^\gamma \leq \langle v\rangle^\gamma + \langle v^*\rangle^\gamma$.

$$\begin{aligned}
&\int_{\mathbb{R}^3} \langle v\rangle^k \left[\int_{\mathbb{T}^3} |Q_{ij}(f_i, g_j)|^q\, dx\right]^{1/q} dv \\
&\quad\leq C_{ij} \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} \left(\langle v\rangle^{k+\gamma} \langle v_*\rangle^k + \langle v\rangle^k \langle v_*\rangle^{k+\gamma}\right) \left[\int_{\mathbb{T}^3} \left|f_i g_j^{*}\right|^q dx\right]^{1/q} d\sigma dv_* dv.
\end{aligned}$$

We take the limit as q tends to infinity and conclude

$$\|Q_{ij}(f_i,g_j)\|_{L^1_vL^\infty_x(\langle v\rangle^k)} \leq C_{ij}\Big[\,\|f_i\|_{L^1_vL^\infty_x(\langle v\rangle^k)}\|g_j\|_{L^1_vL^\infty_x(\langle v\rangle^{k+\gamma})} + \|f_i\|_{L^1_vL^\infty_x(\langle v\rangle^{k+\gamma})}\|g_j\|_{L^1_vL^\infty_x(\langle v\rangle^k)}\Big].$$

We remind (5.2.4) which states that $\nu_i(v)\sim\langle v\rangle^\gamma$ and the lemma follows after summing over j, C_Q being the maximum of all the C_{ij}. □

Study of the equations in $L^1_vL^\infty_x\left(\langle v\rangle^k\right)$

We start with the well-posedness of the system (5.5.2) in $L^1_vL^\infty_x\left(\langle v\rangle^k\right)$.

Proposition 5.24. *Let $k > k_0$. Let $\mathbf{f_0}$ be in $L^1_vL^\infty_x\left(\langle v\rangle^k\right)$ and $\mathbf{g}$ in $L^\infty_tL^1_vL^\infty_x\left(\boldsymbol{\nu}\langle v\rangle^k\right)$. There exist $\eta_1,\lambda_1>0$ such that if*

$$\|\mathbf{f_0}\|_{L^1_vL^\infty_x(\langle v\rangle^k)}\leq\eta_1 \quad \text{and} \quad \exists C,\,\lambda>0 \quad \|\mathbf{g}(t)\|_{L^1_vL^\infty_x(\boldsymbol{\nu}\langle v\rangle^k)}\leq C\,\|\mathbf{f_0}\|_{L^1_vL^\infty_x(\langle v\rangle^k)}\,e^{-\lambda t}$$

then there exists a function $\mathbf{f_1}$ in $L^\infty_tL^1_vL^\infty_x\left(\langle v\rangle^k\right)$ such that

$$\partial_t\mathbf{f_1}=\mathbf{G_1}\left(\mathbf{f_1}\right)+\mathbf{Q}\left(\mathbf{f_1}+\mathbf{g}\right) \quad \text{and} \quad \mathbf{f_1}(0,x,v)=\mathbf{f_0}(x,v).$$

Moreover, any solution $\mathbf{f_1}$ satisfies

$$\forall t\geq 0,\quad \|\mathbf{f_1}(t)\|_{L^1_vL^\infty_x(\langle v\rangle^k)}\leq C_1e^{-\lambda_1 t}\,\|\mathbf{f_0}\|_{L^1_vL^\infty_x(\langle v\rangle^k)}\,.$$

The constants C_1, δ_1, η_1 and λ_1 are independent of $\mathbf{f_0}$ and $\mathbf{g}$ and depends on N, k and the collision kernel.

Proof of Proposition 5.24. We start by showing the exponential decay and then prove existence. As a matter of fact, we saw in Lemma 5.20 that the natural weight to estimate $\mathbf{B}$ is $\overline{\mathbf{w_k}}=1+\mathbf{m}^{k/2}\,|v|^k$ which is equivalent to $\langle v\rangle^k$. We will therefore rather work in $L^1_vL^\infty_x\left(\overline{\mathbf{w_k}}\right)$ which just modifies the definition for C_1, δ_1 and η_1.

Step 1: *a priori* exponential decay. Suppose that $\mathbf{f_1}$ is a solution to the differential equation in $L^1_vL^\infty_x\left(\overline{\mathbf{w_k}}\right)$ with initial data $\mathbf{f_0}$.

We recall that for q in $[1,\infty)$,

$$\|\mathbf{f_1}\|_{L^1_vL^q_x(\overline{\mathbf{w_k}})}=\sum_{i=1}^N\int_{\mathbb{R}^3}\left(1+m_i^{k/2}\,|v|^k\right)\left(\int_{\mathbb{T}^3}|f_{1i}|^q\,dx\right)^{1/q}\,dv.$$

Therefore we can compute for all i in $\{1,\dots,N\}$

$$\frac{d}{dt}\|f_{1i}\|_{L^1_vL^q_x\left(1+m_i^{k/2}|v|^k\right)}$$
$$= \int_{\mathbb{R}^3}\left(1+m_i^{k/2}|v|^k\right)\|f_{1i}\|^{1-q}_{L^q_x}\left(\int_{\mathbb{T}^3}\mathrm{sgn}(f_{i1})\,|f_{1i}|^{q-1}\,\partial_t f_{1i}\,dx\right)\,dv.$$

Observing that

$$\partial_t f_{1i} = -v\cdot\nabla_x f_{1i} - \nu_i(v)f_{1i} + B_i\left(\mathbf{f_1}\right) + Q_i\left(\mathbf{f_1}+\mathbf{g}\right),$$

that the transport gives null contribution

$$\int_{\mathbb{T}^3}\mathrm{sgn}(f_{1i})\,|f_{1i}|^{q-1}\,v\cdot\nabla_x f_{1i}\,dx = \frac{1}{q}v\cdot\int_{\mathbb{T}^3}\nabla_x\left(|f_{1i}|^q\right)\,dx = 0,$$

that the multiplicative part gives a negative contribution,

$$-\int_{\mathbb{T}^3}\nu_i(v)f_{1i}\,\mathrm{sgn}(f_{1i})\,|f_{1i}|^{q-1}\,dx \leq -\nu_i(v)\,\|f_{1i}\|^q_{L^q_x}$$

and that by Hölder inequality with q and $q/(q-1)$,

$$\left|\int_{\mathbb{T}^3}\mathrm{sgn}(f_{1i})\,|f_{1i}|^{q-1}\,g_i\,dx\right| \leq \|f_{1i}\|^{q-1}_{L^q_x}\,\|g_i\|_{L^q_x}, \tag{5.5.12}$$

we deduce

$$\frac{d}{dt}\|f_{i1}\|_{L^1_vL^q_x\left(1+m_i^{k/2}|v|^k\right)} \leq -\|f_{1i}\|_{L^1_vL^q_x\left(\nu_i(1+m_i^{k/2}|v|^k)\right)} + \|B_i\left(\mathbf{f_1}\right)\|_{L^1_vL^q_x\left(1+m_i^{k/2}|v|^k\right)}$$
$$+\|Q_i\left(\mathbf{f_1}+\mathbf{g}\right)\|_{L^1_vL^q_x\left(1+m_i^{k/2}|v|^k\right)}.$$

First sum over i in $\{1,\dots,N\}$ and then let q tend to infinity (on the torus the limit is thus the L^∞-norm). This yields for all $t\geq 0$.

$$\begin{aligned}\frac{d}{dt}\|\mathbf{f_1}\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}})} \leq &-\|\mathbf{f_1}\|_{L^1_vL^\infty_x(\boldsymbol{\nu}\overline{\mathbf{w_k}})} + \|\mathbf{B}\left(\mathbf{f_1}\right)\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}})}\\ &+\|\mathbf{Q}\left(\mathbf{f_1}+\mathbf{g}\right)\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}})}.\end{aligned} \tag{5.5.13}$$

We use Lemma 5.20 to control $\mathbf{B}$, recalling that $0<C_B<1$, and the control of $\mathbf{Q}$ given in Lemma 5.23 for $\mathbf{Q}$ (of course, since $\overline{\mathbf{w_k}}\sim\langle v\rangle^k$ the Lemma still holds with a different C_Q). We get that for all $t\geq 0$,

$$\frac{d}{dt}\|\mathbf{f_1}\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}})} \leq -\left[1-C_B-2C_Q\left(\|\mathbf{f_1}\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}})}+2\|\mathbf{g}\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}})}\right)\right]\|\mathbf{f_1}\|_{L^1_vL^\infty_x(\boldsymbol{\nu}\overline{\mathbf{w_k}})}$$
$$+C_Q\,\|\mathbf{g}(t)\|^2_{L^1_vL^\infty_x(\boldsymbol{\nu}\overline{\mathbf{w_k}})}.$$

Since $C_B<1$, if $\|\mathbf{f_1}(0)\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}})}$ is sufficiently small and thanks to the exponential decay of $\|\mathbf{g}(t)\|_{L^\infty_tL^1_vL^\infty_x(\boldsymbol{\nu}\overline{\mathbf{w_k}})}$, a direct application of Grönwall lemma yields the desired exponential decay.

Step 2: existence. Let $\mathbf{f^{(0)}} = 0$ and consider the following iterative scheme

$$\partial_t \mathbf{f^{(n+1)}} + v \cdot \nabla_x \mathbf{f^{(n+1)}} = -\boldsymbol{\nu}(v)\left(\mathbf{f^{(n+1)}}\right) + \mathbf{B}\left(\mathbf{f^{(n)}}\right) + \widetilde{\mathbf{Q}}\left(\mathbf{f^{(n)}} + \mathbf{g}\right)$$

with the initial data $\mathbf{f^{(n+1)}}(0,x,v) = \mathbf{f_0}$.

For each n in $\mathbb{N}$, $\mathbf{f^{(n+1)}}$ is well-defined by induction since we have the explicit Duhamel formula along the characteristics for all i in $\{1,\dots,N\}$

$$f_i^{(n+1)}(t,x,v) = e^{-\nu_i(v)t} f_{0i} + \int_0^t e^{-\nu_i(v)(t-s)}\left[B_i\left(\mathbf{f^{(n)}}\right) + Q_i\left(\mathbf{f^{(n)}} + \mathbf{g}\right)\right](x - sv, v)\, ds.$$

We are about to show that $\left(\mathbf{f^{(n)}}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_t L^1_v L^\infty_x\left(\overline{\mathbf{w_k}}\right)$.

Direct computations on the nonlinear operator gives

$$\begin{aligned}\partial_t\left(\mathbf{f^{(n+1)}} - \mathbf{f^{(n)}}\right) = &- \boldsymbol{\nu}(v)\left(\mathbf{f^{(n+1)}} - \mathbf{f^{(n)}}\right) + \mathbf{B}\left(\mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}\right)\\ &+ \widetilde{\mathbf{Q}}\left(\mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}, \mathbf{f^{(n-1)}} + \mathbf{g}\right) + \widetilde{\mathbf{Q}}\left(\mathbf{f^{(n)}} + \mathbf{g}, \mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}\right),\end{aligned}$$

where we remind that $\widetilde{\mathbf{Q}}$ was defined by (5.5.11) and that $\widetilde{\mathbf{Q}}(\mathbf{a},\mathbf{a}) - \widetilde{\mathbf{Q}}(\mathbf{b},\mathbf{b}) = \widetilde{\mathbf{Q}}(\mathbf{a}-\mathbf{b},\mathbf{b}) + \widetilde{\mathbf{Q}}(\mathbf{a},\mathbf{a}-\mathbf{b})$.

Taking the $L^1_v L^\infty_x\left(\overline{\mathbf{w_k}}\right)$-norm of $\left(\mathbf{f^{(n+1)}} - \mathbf{f^{(n)}}\right)$ and summing over i from 1 to N gives for all $t \geq 0$

$$\begin{aligned}&\left\|\mathbf{f^{(n+1)}}(t) - \mathbf{f^{(n)}}(t)\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}})}\\ &\quad\leq \sum_{i=1}^N \int_0^t ds \int_{\mathbb{R}^3} dv\, e^{-\nu_i(v)(t-s)}\left(1 + m_i^{k/2}|v|^k\right)\left\|\Delta_{ni}\left(\mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}\right)\right\|_{L^\infty_x}.\end{aligned}$$

where we defined

$$\begin{aligned}&\boldsymbol{\Delta}_n\left(\mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}\right)\\ &\quad= \mathbf{B}\left(\mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}\right) + \widetilde{\mathbf{Q}}\left(\mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}, \mathbf{f^{(n-1)}} + \mathbf{g}\right) + \widetilde{\mathbf{Q}}\left(\mathbf{f^{(n)}} + \mathbf{g}, \mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}\right).\end{aligned}$$

As $\nu_i(v) \geq \nu_0$ for all i and v we further get

$$\begin{aligned}\left\|\mathbf{f^{(n+1)}}(t) - \mathbf{f^{(n)}}(t)\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}})} &\leq \int_0^t e^{-\nu_0(t-s)}\left\|\boldsymbol{\Delta}_n\left(\mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}\right)\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}})} ds\\ &\leq \left[C_B + C_Q\left(\left\|\mathbf{f^{(n)}}\right\|_{L^\infty_t L^1_v L^\infty_x(\overline{\mathbf{w_k}})} + \left\|\mathbf{f^{(n-1)}}\right\|_{L^\infty_t L^1_v L^\infty_x(\overline{\mathbf{w_k}})} + 2\left\|\mathbf{g}\right\|_{L^\infty_t L^1_v L^\infty_x(\overline{\mathbf{w_k}})}\right)\right]\\ &\quad\times \int_0^t e^{-\nu_0(t-s)}\left\|\mathbf{f^{(n)}}(s) - \mathbf{f^{(n-1)}}(s)\right\|_{L^1_v L^\infty_x(\boldsymbol{\nu}\overline{\mathbf{w_k}})} ds\\ &\quad + C_Q\left[\int_0^t e^{-\nu_0(t-s)}\left(\left\|\mathbf{f^{(n)}}\right\|_{L^1_v L^\infty_x(\boldsymbol{\nu}\overline{\mathbf{w_k}})} + \left\|\mathbf{f^{(n-1)}}\right\|_{L^1_v L^\infty_x(\boldsymbol{\nu}\overline{\mathbf{w_k}})}\right) ds\right]\\ &\quad\times \sup_{s\in[0,t]}\left\|\mathbf{f^{(n)}}(s) - \mathbf{f^{(n-1)}}(s)\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}})}.\end{aligned} \tag{5.5.14}$$

where, as above, we used Lemma 5.20 and the estimate of Lemma 5.23.

Let us look at the terms inside the time integrals. Thus, we take the $L^1_t L^1_v L^\infty_x(\boldsymbol{\nu}\overline{\mathbf{w_k}})$-norm of $\left(\mathbf{f^{(n+1)}} - \mathbf{f^{(n)}}\right)$ and sum over i.

$$\begin{aligned}
&\int_0^t \left\|\mathbf{f^{(n+1)}}(s) - \mathbf{f^{(n)}}(s)\right\|_{L^1_v L^\infty_x\left(\langle v\rangle^k \boldsymbol{\nu}\right)} ds \\
&\qquad \leq \sum_{i=1}^N \int_0^t \int_0^s \int_{\mathbb{R}^3} e^{-\nu_i(v)(s-s_1)} \nu_i(v) \overline{w}_{ki}(v) \left\|\boldsymbol{\Delta}_n \left(\mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}\right)\right\|_{L^\infty_x} (s_1)\, ds_1 ds.
\end{aligned}$$

We exchange the integration domains in s and s_1, which implies

$$\begin{aligned}
&\int_0^t \left\|\mathbf{f^{(n+1)}}(s) - \mathbf{f^{(n)}}(s)\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})} ds \\
&\leq \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^3} \left(\int_{s_1}^t e^{-\nu_i(v)(s-s_1)} \nu_i(v)\, ds\right) \overline{w}_{ki}(v) \left\|\boldsymbol{\Delta}_n \left(\mathbf{f^{(n)}} - \mathbf{f^{(n-1)}}\right)\right\|_{L^\infty_x} (s_1)\, ds_1.
\end{aligned}$$

Since the integral in s is bounded by 1, we use Lemma 5.20 and Lemma 5.23 again and obtain

$$\begin{aligned}
&\int_0^t \left\|\mathbf{f^{(n+1)}}(s) - \mathbf{f^{(n)}}(s)\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})} ds \\
&\leq \left[C_B + C_Q\left(\left\|\mathbf{f^{(n)}}\right\|_{L^\infty_t L^1_v L^\infty_x(\overline{\mathbf{w_k}})} + \left\|\mathbf{f^{(n-1)}}\right\|_{L^\infty_t L^1_v L^\infty_x(\overline{\mathbf{w_k}})} + 2\left\|\mathbf{g}\right\|_{L^\infty_t L^1_v L^\infty_x(\overline{\mathbf{w_k}})}\right)\right] \\
&\quad \times \int_0^t \left\|\mathbf{f^{(n)}}(s_1) - \mathbf{f^{(n-1)}}(s_1)\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})} ds_1 \\
&\quad + C_Q \left[\int_0^t \left(\left\|\mathbf{f^{(n)}}\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})} + \left\|\mathbf{f^{(n-1)}}\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})}\right) ds_1\right] \\
&\qquad \times \sup_{s\in[0,t]} \left\|\mathbf{f^{(n)}}(s) - \mathbf{f^{(n-1)}}(s)\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}})}
\end{aligned} \tag{5.5.15}$$

We now conclude the proof of existence. Indeed, exact same computations but subtracting $e^{-\boldsymbol{\nu}(v)t}\mathbf{f_0}$ instead of $\mathbf{f^{(n)}}$ lead to (5.5.14) and (5.5.15) with $\mathbf{f^{(n-1)}}$ replaced by 0. Therefore, since $C_B < 1$ it follows that for $\left\|\mathbf{f_0}\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}})}$ and $\left\|\mathbf{g}\right\|_{L^\infty_t L^1_v L^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})}$ sufficiently small we have that there exists $C > 0$ such that for all n in $\mathbb{N}$ and all $t \geq 0$,

$$\left\|\mathbf{f^{(n)}}(t)\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}})} \leq C \left\|\mathbf{f_0}\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}})}$$

and

$$\int_0^t \left\|\mathbf{f^{(n)}}(s)\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})} ds \leq C \int_0^t \left\|\mathbf{f^{(1)}}\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})} ds \leq C \left\|\mathbf{f_0}\right\|_{L^1_v L^\infty_x(\overline{\mathbf{w_k}})}.$$

Therefore, denoting by C any positive constant independent of $\mathbf{f^{(n)}}$ and $\mathbf{g}$, adding (5.5.14) and (5.5.15) yields

$$\begin{aligned}
&\left\|\mathbf{f^{(n+1)}}(t)-\mathbf{f^{(n)}}(t)\right\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}})}+\int_0^t\left\|\mathbf{f^{(n+1)}}(s)-\mathbf{f^{(n)}}(s)\right\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})}\,ds\\
&\quad\leq C\eta_1\sup_{s\in[0,t]}\left\|\mathbf{f^{(n)}}(s)-\mathbf{f^{(n-1)}}(s)\right\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}})}\\
&\quad\quad+\left[C_B+C\eta_1\right]\int_0^t\left\|\mathbf{f^{(n)}}(s)-\mathbf{f^{(n-1)}}(s)\right\|_{L^1_vL^\infty_x(\overline{\mathbf{w_k}}\boldsymbol{\nu})}\,ds.
\end{aligned}$$

Since $C_B<1$, choosing η_1 such that $C_B+C\eta_1<1$ implies that $\left(\mathbf{f}^{(n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_tL^1_vL^\infty_x\left(\overline{\mathbf{w_k}}\right)$. Hence, $\left(\mathbf{f}^{(n)}\right)_{n\in\mathbb{N}}$ converges to a function $\mathbf{f_1}$ in $L^\infty_tL^1_vL^\infty_x\left(\overline{\mathbf{w_k}}\right)$ and since $k>k_0>\gamma$ we can take the limit inside the iterative scheme and $\mathbf{f_1}$ is thus a solution of our differential equation. □

Study of the equations in $L^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)$

We turn to the system (5.5.3) in $L^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)$ with $\beta>3/2$ so that Theorem 5.17 holds.

Proposition 5.25. *Let $k>k_0$, $\beta>3/2$ and let assumptions $(H1)-(H4)$ hold for the collision kernel. Let $\mathbf{g}=\mathbf{g}(t,x,v)$ be in $L^\infty_tL^1_vL^\infty_x\left(\langle v\rangle^k\right)$. Then there exists a unique function $\mathbf{f_2}$ in $L^\infty_tL^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)$ such that*

$$\partial_t\mathbf{f_2}=\mathbf{G}\left(\mathbf{f_2}\right)+\mathbf{A}\left(\mathbf{g}\right)\quad\text{and}\quad\mathbf{f_2}(0,x,v)=0.$$

Moreover, if $\Pi_{\mathbf{G}}\left(\mathbf{f_2}+\mathbf{g}\right)=0$ and if

$$\exists\,\lambda_g,\ \eta_g>0,\ \forall t\geq 0,\ \|\mathbf{g}(t)\|_{L^1_vL^\infty_x\left(\langle v\rangle^k\right)}\leq\eta_g e^{-\lambda_g t},$$

then for any $0<\lambda_2<\min\left\{\lambda_g,\ \lambda_\infty\right\}$, with λ_∞ defined in Theorem 5.17, there exist $C_2>0$ such that

$$\forall t\geq 0,\quad\|\mathbf{f_2}(t)\|_{L^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)}\leq C_2\eta_g e^{-\lambda_2 t}.$$

The constant C_2 only depends on λ_2.

Proof of Proposition 5.25. Thanks to the regularising property of $\mathbf{A}$, Lemma 5.19, $\mathbf{A}\left(\mathbf{g}\right)$ belongs to $L^\infty_tL^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)$. Theorem 5.17 implies that there is indeed a unique $\mathbf{f_2}$ solution to the differential system, given by

$$\mathbf{f_2}=\int_0^t S_{\mathbf{G}}(t-s)\left[\mathbf{A}\left(\mathbf{g}\right)(s)\right]\,ds,$$

where $S_{\mathbf{G}}(t)$ is the semigroup generated by $\mathbf{G}$ in $L^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)$.

Suppose that $\Pi_{\mathbf{G}}(\mathbf{f_2}+\mathbf{g})=0$ and that $\exists \eta_2>0$ such that $\|\mathbf{g}(t)\|_{L^1_vL^\infty_x(\langle v\rangle^k)} \leq \eta_2 e^{-\lambda t}$.

Using the definition of $\Pi_{\mathbf{G}}$ (5.3.1), the projection part of $\mathbf{f_2}$ is straightforwardly bounded for all $t\geq 0$:

$$\begin{aligned}\|\Pi_{\mathbf{G}}(\mathbf{f_2})(t)\|_{L^\infty_{x,v}(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2})} = \|\Pi_{\mathbf{G}}(\mathbf{g})(t)\|_{L^\infty_{x,v}(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2})} &\leq C_{\Pi_G}\|\mathbf{g}\|_{L^1_vL^\infty_x(\langle v\rangle^k)} \\ &\leq C_{\Pi_G}\eta_g\, e^{-\lambda_g t}.\end{aligned} \tag{5.5.16}$$

Applying $\Pi_{\mathbf{G}}^\perp = \mathrm{Id}-\Pi_{\mathbf{G}}$ to the equation satisfied by $\mathbf{f_2}$ we get, thanks to the fact the definition of $\Pi_{\mathbf{G}}$ (5.3.1) which is independent of t,

$$\partial_t\left[\Pi_{\mathbf{G}}^\perp(\mathbf{f_2})\right] = \mathbf{G}\left[\Pi_{\mathbf{G}}^\perp(\mathbf{f_2})\right] + \Pi_{\mathbf{G}}^\perp(\mathbf{A}(\mathbf{g})).$$

This yields

$$\Pi_{\mathbf{G}}^\perp(\mathbf{f_2}) = \int_0^t S_{\mathbf{G}}(t-s)\left[\Pi_{\mathbf{G}}^\perp(\mathbf{A}(\mathbf{g}))(s)\right]\,ds.$$

We now use the exponential decay of $S_{\mathbf{G}}(t)$ on $(\mathrm{Ker}(\mathbf{G}))^\perp$, see Theorem 5.17.

$$\left\|\Pi_{\mathbf{G}}^\perp(\mathbf{f_2})\right\|_{L^\infty_{x,v}(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2})} \leq C_\infty\int_0^t e^{-\lambda_\infty(t-s)}\left\|\Pi_{\mathbf{G}}^\perp(\mathbf{A}(\mathbf{g}))(s)\right\|_{L^\infty_{x,v}(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2})}\,ds.$$

Using the definition of $\Pi_{\mathbf{G}}$ (5.3.1) and then the regularising property of $\mathbf{A}$ Lemma 5.19 we further bound, for a fixed $\lambda_2<\min\{\lambda_\infty,\,\lambda_g\}$,

$$\begin{aligned}\left\|\Pi_{\mathbf{G}}^\perp(\mathbf{f_2})\right\|_{L^\infty_{x,v}(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2})} &\leq C_\infty C_{\Pi_G}C_AC_g\eta_g\int_0^t e^{-\lambda_\infty(t-s)}e^{-\lambda_g s}\,ds \\ &\leq C_GC_\infty C_{\Pi_G}C_AC_g\eta_g\, te^{-\min\{\lambda_g,\lambda_\infty\}t} \\ &\leq C_2(\lambda_2)\eta_g e^{-\lambda_2 t}.\end{aligned} \tag{5.5.17}$$

Gathering (5.5.16) and (5.5.17) yields the desired exponential decay. □

Proof of Proposition 5.18

Take $\mathbf{f_0}$ in $L^1_vL^\infty_x(\langle v\rangle^k)$ such that $\Pi_{\mathbf{G}}(\mathbf{f_0})=0$.

The existence will be proved by an iterative scheme. We start with $\mathbf{f_1^{(0)}}=\mathbf{f_2^{(0)}}=0$ and we approximate the system of equation (5.5.2) – (5.5.3) as follows.

$$\begin{aligned}\partial_t\mathbf{f_1^{(n+1)}} &= \mathbf{G_1}\left(\mathbf{f_1^{(n+1)}}\right) + \mathbf{Q}\left(\mathbf{f_1^{(n+1)}}+\mathbf{f_2^{(n)}}\right) \\ \partial_t\mathbf{f_2^{(n+1)}} &= \mathbf{G}\left(\mathbf{f_2^{(n+1)}}\right) + \mathbf{A}^{(\delta)}\left(\mathbf{f_1^{(n+1)}}\right),\end{aligned}$$

with the following initial data

$$\mathbf{f_1^{(n+1)}}(0,x,v)=\mathbf{f_0}(x,v) \quad\text{and}\quad \mathbf{f_2^{(n+1)}}(0,x,v)=0.$$

Assume that $(1 + C_1C_2)\|\mathbf{f_0}\| \leq \eta_1$, where C_1, η_1 were defined in Proposition 5.24 and C_2 was defined in Proposition 5.25. Thanks to Proposition 5.24 and Proposition 5.25, an induction proves first that $\left(\mathbf{f_1^{(n)}}\right)_{n\in\mathbb{N}}$ and $\left(\mathbf{f_2^{(n)}}\right)_{n\in\mathbb{N}}$ are well-defined sequences and second that for all n in $\mathbb{N}$ and all $t \geq 0$

$$\begin{aligned} \left\|\mathbf{f_1^{(n)}}(t)\right\|_{L^1_vL^\infty_x\left(\langle v\rangle^k\right)} &\leq e^{-\lambda_1 t}\|\mathbf{f_0}\|_{L^1_vL^\infty_x\left(\langle v\rangle^k\right)} &(5.5.18)\\ \left\|\mathbf{f_2^{(n)}}(t)\right\|_{L^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)} &\leq C_1C_2e^{-\lambda_2 t}\|\mathbf{f_0}\|_{L^1_vL^\infty_x\left(\langle v\rangle^k\right)}, &(5.5.19)\end{aligned}$$

with $\lambda_2 < \min\{\lambda_1, \lambda_\infty\}$. Indeed, if we constructed $\mathbf{f_1^{(n)}}$ and $\mathbf{f_2^{(n)}}$ satisfying the exponential decay above then we can construct $\mathbf{f_1^{(n+1)}}$ with Proposition 5.24 and $\mathbf{g} = \mathbf{f_2^{(n)}}$, which has the required exponential decay (5.5.18), and then construct $\mathbf{f_2^{(n+1)}}$ with Proposition 5.25 and $\mathbf{g} = \mathbf{f_1^{(n+1)}}$. Finally, we have the following equality

$$\partial_t\left(\mathbf{f_1^{(n+1)}} + \mathbf{f_2^{(n+1)}}\right) = \mathbf{G}\left(\mathbf{f_1^{(n+1)}} + \mathbf{f_2^{(n+1)}}\right) + \mathbf{Q}\left(\mathbf{f_1^{(n+1)}} + \mathbf{f_2^{(n)}}\right).$$

Thanks to orthogonality property of $\mathbf{Q}$ in Lemma 5.23 and the definition of $\Pi_{\mathbf{G}}$ (5.3.1) we obtain that the projection is constant with time and thus

$$\Pi_{\mathbf{G}}\left(\mathbf{f_1^{(n+1)}} + \mathbf{f_2^{(n+1)}}\right) = \Pi_{\mathbf{G}}(\mathbf{f_0}) = 0.$$

Applying Proposition 5.25 we obtain the exponential decay (5.5.19) for $\mathbf{f_2^{(n+1)}}$.

We recognize exactly the same iterative scheme for $\mathbf{f_1^{n+1}}$ as in the proof of Proposition 5.24 with $\mathbf{g}$ replaced by $\mathbf{f_2^{(n)}}$. Moreover, the uniform bound (5.5.19) allows us to derive the same estimates as in the latter proof independently of $\mathbf{f_2^{(n)}}$. As a conclusion, $\left(\mathbf{f_1^{(n)}}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_tL^1_vL^\infty_x\left(\langle v\rangle^k\right)$ and therefore converges strongly towards a function $\mathbf{f_1}$.

By (5.5.19), the sequence $\left(\mathbf{f_2^{(n)}}\right)_{n\in\mathbb{N}}$ is bounded in $L^\infty_tL^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)$ and is therefore weakly-* compact and therefore converges, up to a subsequence, weakly-* towards $\mathbf{f_2}$ in $L^\infty_tL^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)$.

Since the function inside the collision operator behaves like $|v-v_*|^\gamma$ and that in our weighted spaces $k > k_0 > \gamma$, we can take the weak limit inside the iterative scheme. This implies that $(\mathbf{f_1}, \mathbf{f_2})$ is solution to the system (5.5.2) – (5.5.3) and thus $\mathbf{f} = \mathbf{f_1} + \mathbf{f_2}$ is solution to the perturbed multi-species equation (5.5.1). Moreover, taking the limit inside the exponential decays (5.5.18) and (5.5.19) yields the expected exponential decay for $\mathbf{f}$.

5.5.2. Uniqueness of solutions in the perturbative regime

As said in Remark 2.2.5, we are solely interested in the uniqueness of solutions to the multi-species Boltzmann equation (5.1.1) in the perturbative setting. In other terms, uniqueness of solutions of the form $\mathbf{F} = \boldsymbol{\mu} + \mathbf{f}$ as long as $\mathbf{F_0}$ is close enough to the global equilibrium

$\boldsymbol{\mu}$. This is equivalent to proving the uniqueness of solutions to the perturbed multi-species equation

$$\partial_t \mathbf{f} = \mathbf{G}(\mathbf{f}) + \mathbf{Q}(\mathbf{f}) \tag{5.5.20}$$

for $\mathbf{f_0}$ small.

Proposition 5.26. *Let $k > k_0$ and let assumptions $(H1) - (H4)$ hold for the collision kernel. There exists $\eta_k > 0$ such that for any $\mathbf{f_0}$ in $L^1_v L^\infty_x \left(\langle v \rangle^k\right)$; if $\|\mathbf{f_0}\|_{L^1_v L^\infty_x\left(\langle v \rangle^k\right)} \leq \eta_k$ then there exists at most one solution to the perturbed multi-species equation* (5.5.20)*.*
The constant η_k only depends on k, N and the collision kernels.

The uniqueness will follow from the study of the semigroup generated by $\mathbf{G}$ in a dissipative norm as well as a new *a priori* stability estimate for solutions to (5.5.20) in the latter norm. They are the purpose of the next two lemmas.

Lemma 5.27. *Let $k > k_0$ and let assumptions $(H1) - (H4)$ hold for the collision kernel. The operator $\mathbf{G}$ generates a semigroup in $L^1_v L^\infty_x \left(\langle v \rangle^k\right)$. Moreover, there exist C_k, $\lambda_k > 0$ such that for all $\mathbf{f_0}$ in $L^1_v L^\infty_x \left(\langle v \rangle^k\right)$ with $\Pi_{\mathbf{G}}(\mathbf{f_0}) = 0$*

$$\forall t \geq 0, \quad \|S_{\mathbf{G}}(\mathbf{f})\|_{L^1_v L^\infty_x\left(\langle v \rangle^k\right)} \leq C_k e^{-\lambda_k t} \|\mathbf{f_0}\|_{L^1_v L^\infty_x\left(\langle v \rangle^k\right)}.$$

Proof of Lemma 5.27. From Proposition 5.18 with a collision operator $\mathbf{Q} = 0$ we have the existence of a solution to the equation

$$\partial_t \mathbf{f} = \mathbf{G}\left(\mathbf{f}\right)$$

with initial data $\mathbf{f_0}$ in $L^1_v L^\infty_x \left(\langle v \rangle^k\right)$. Moreover, that solution satisfies $\Pi_{\mathbf{G}}(\mathbf{f}) = 0$ and it decays exponentially fast with rate λ_k.

Let $\mathbf{g}$ be another solution to the linear equation then

$$\partial_t \left(\mathbf{f} - \mathbf{g}\right) = \left[-v \cdot \nabla_x - \boldsymbol{\nu} + \mathbf{B} + \mathbf{A}\right]\left(\mathbf{f} - \mathbf{g}\right).$$

Similar computations as to obtain (5.5.13) yield

$$\begin{aligned} \frac{d}{dt}\|\mathbf{f} - \mathbf{g}\|_{L^1_v L^\infty_x\left(\langle v \rangle^k\right)} \leq &- \|\mathbf{f} - \mathbf{g}\|_{L^1_v L^\infty_x\left(\langle v \rangle^k \boldsymbol{\nu}\right)} + \|\mathbf{A}(\mathbf{f} - \mathbf{g})\|_{L^1_v L^\infty_x\left(\langle v \rangle^k\right)} \\ &+ \|\mathbf{B}(\mathbf{f} - \mathbf{g})\|_{L^1_v L^\infty_x\left(\langle v \rangle^k\right)}. \end{aligned}$$

Using Lemma 5.19 and Lemma 5.20, there exists $0 < C_B < 1$ such that

$$\frac{d}{dt}\|\mathbf{f} - \mathbf{g}\|_{L^1_v L^\infty_x\left(\langle v \rangle^k\right)} \leq -(1 - C_B)\|\mathbf{f} - \mathbf{g}\|_{L^1_v L^\infty_x\left(\langle v \rangle^k \boldsymbol{\nu}\right)} + C_A \|\mathbf{f} - \mathbf{g}\|_{L^1_v L^\infty_x\left(\langle v \rangle^k\right)}. \tag{5.5.21}$$

Since $(1 - C_B) > 0$ we can further bound

$$\frac{d}{dt}\|\mathbf{f} - \mathbf{g}\|_{L^1_v L^\infty_x\left(\langle v \rangle^k\right)} \leq \left[C_A - (1 - C_B)\right]\|\mathbf{f} - \mathbf{g}\|_{L^1_v L^\infty_x\left(\langle v \rangle^k\right)}$$

and a Grönwall lemma therefore yields $\mathbf{f} = \mathbf{g}$ if $\mathbf{g_0} = \mathbf{f_0}$.

We thus obtain existence and uniqueness of solution to the linear equation which means that $\mathbf{G}$ generates a semigroup in $L^1_v L^\infty_x \left(\langle v\rangle^k\right)$. Moreover it has an exponential decay of rate $\lambda_k > 0$ for functions in $(\mathrm{Ker}(\mathbf{G}))^\perp$. □

We now derive a stability estimate in an equivalent norm that catches the dissipativity of the linear operator.

Lemma 5.28. *Let $k > k_0$ and let assumptions $(H1) - (H4)$ hold for the collision kernel. For $\alpha > 0$, we define*

$$\|\mathbf{f}\|_{\alpha,k} = \alpha \|\mathbf{f}\|_{L^1_v L^\infty_x(\langle v\rangle^k)} + \int_0^{+\infty} \|S_{\mathbf{G}}(s)(\mathbf{f})\|_{L^1_v L^\infty_x(\langle v\rangle^k)}\, ds.$$

There exist η, α, C_1, C_2 and $\lambda > 0$ such that $\|\cdot\|_{\alpha,k} \sim \|\cdot\|_{L^1_v L^\infty_x(\langle v\rangle^k)}$ and for all $\mathbf{f_0}$ in $L^1_v L^\infty_x \left(\langle v\rangle^k\right)$ with $\Pi_{\mathbf{G}}(\mathbf{f_0}) = 0$ and such that

$$\|\mathbf{f_0}\|_{L^1_v L^\infty_x(\langle v\rangle^k)} \leq \eta;$$

if $\mathbf{f}$ in $L^1_v L^\infty_x \left(\langle v\rangle^k\right)$ with $\Pi_{\mathbf{G}}(\mathbf{f}) = 0$ is solution to the perturbed equation (5.5.20) *with initial data $\mathbf{f_0}$ then*

$$\frac{d}{dt}\|\mathbf{f}\|_{\alpha,k} \leq -\left(C_1 - C_2 \|\mathbf{f}\|_{\alpha,k}\right)\|\mathbf{f}\|_{\alpha,k,\boldsymbol{\nu}},$$

where the subscript $\boldsymbol{\nu}$ refers to the fact that the weight is multiplied by $\nu_i(v)$ on each coordinate.

Proof of Lemma 5.28. Start with the new norm. Lemma 5.27 proved that for all $\mathbf{f_0}$ such that $\Pi_{\mathbf{G}}(\mathbf{f_0}) = 0$ and all $s \geq 0$,

$$\|S_{\mathbf{G}}(s)(\mathbf{f_0})\|_{L^1_v L^\infty_x(\langle v\rangle^k)} \leq C_k e^{-\lambda_k s}\|\mathbf{f_0}\|_{L^1_v L^\infty_x(\langle v\rangle^k)}$$

and hence

$$\alpha\|\mathbf{f_0}\|_{L^1_v L^\infty_x(\langle v\rangle^k)} \leq \|\mathbf{f_0}\|_{\alpha,k} \leq \left(\alpha + \frac{C_k}{\lambda_k}\right)\|\mathbf{f_0}\|_{L^1_v L^\infty_x(\langle v\rangle^k)}.$$

Suppose that $\mathbf{f}$ is the solution described in Lemma 5.28. Same computations as to obtain (5.5.13) and (5.5.21) yields

$$\frac{d}{dt}\|\mathbf{f}\|_{L^1_v L^\infty_x(\langle v\rangle^k)} \leq -(1 - C_B)\|\mathbf{f}\|_{L^1_v L^\infty_x(\langle v\rangle^k \boldsymbol{\nu})} + C_A\|\mathbf{f}\|_{L^1_v L^\infty_x(\langle v\rangle^k)} + \|\mathbf{Q}(\mathbf{f})\|_{L^1_v L^\infty_x(\langle v\rangle^k)}.$$

To which we can apply Lemma 5.23:

$$\frac{d}{dt}\|\mathbf{f}\|_{L^1_v L^\infty_x(\langle v\rangle^k)} \leq -\left(1 - C_B - C_Q\|\mathbf{f}\|_{L^1_v L^\infty_x(\langle v\rangle^k)}\right)\|\mathbf{f}\|_{L^1_v L^\infty_x(\langle v\rangle^k \boldsymbol{\nu})} + C_A\|\mathbf{f}\|_{L^1_v L^\infty_x(\langle v\rangle^k)}. \tag{5.5.22}$$

We now turn to the second term in the $\|\cdot\|_{\alpha,k}$ norm. For q in $[1,\infty)$ we denote $\Phi_q(\mathbf{F}) = \text{sgn}(\mathbf{F})\,|\mathbf{F}|^{q-1}$, where it has to be understood component by component. We thus have

$$\begin{aligned}\frac{d}{dt}\int_0^{+\infty} & \|S_{\mathbf{G}}(s)\,(\mathbf{f}(t))\|_{L^1_vL^q_x(\langle v\rangle^k)}\,ds \\ &= \int_0^{+\infty}\int_{\mathbb{R}^3}\langle v\rangle^k\,\|S_{\mathbf{G}}(s)(\mathbf{f})\|_{L^q_x}^{1-q}\left(\int_{\mathbb{T}^3}\Phi_q\left(S_{\mathbf{G}}(s)(\mathbf{f})\right)S_{\mathbf{G}}(s)\left[\mathbf{G}(\mathbf{f})\right]\,dx\right)dvds \\ &\quad + \int_0^{+\infty}\int_{\mathbb{R}^3}\langle v\rangle^k\,\|S_{\mathbf{G}}(s)(\mathbf{f})\|_{L^q_x}^{1-q}\left(\int_{\mathbb{T}^3}\Phi_q\left(S_{\mathbf{G}}(s)(\mathbf{f})\right)S_{\mathbf{G}}(s)\left[\mathbf{Q}(\mathbf{f})\right]\,dx\right)dvds\end{aligned}$$

First, by definition of $S_{\mathbf{G}}(s)$ we have that

$$\Phi_q\left(S_{\mathbf{G}}(s)(\mathbf{f}(t))\right)S_{\mathbf{G}}(s)\left[\mathbf{G}(\mathbf{f}(t))\right] = \frac{d}{ds}\left|S_{\mathbf{G}}(s)(\mathbf{f}(t))\right|^q.$$

Second, by Hölder inequality with q and $q/(q-1)$ (see (5.5.12)):

$$\int_{\mathbb{T}^3}\Phi_q\left(S_{\mathbf{G}}(s)(\mathbf{f}(t))\right)S_{\mathbf{G}}(s)\left[\mathbf{Q}(\mathbf{f}(t))\right]\,dx \leq \|S_{\mathbf{G}}(s)(\mathbf{f}(t))\|_{L^q_x}^{q-1}\,\|S_{\mathbf{G}}(s)(\mathbf{Q}(\mathbf{f}(t)))\|_{L^q_x}.$$

We therefore get

$$\begin{aligned}\frac{d}{dt}\int_0^{+\infty}\|S_{\mathbf{G}}(s)\,(\mathbf{f}(t))\|_{L^1_vL^q_x(\langle v\rangle^k)}\,ds \leq & \int_0^{+\infty}\frac{d}{ds}\|S_{\mathbf{G}}(\mathbf{f}(t))\|_{L^1_vL^q_x(\langle v\rangle^k)}\,ds \\ & + \int_0^{+\infty}\|S_{\mathbf{G}}(s)(\mathbf{Q}(\mathbf{f}(t)))\|_{L^1_vL^q_x(\langle v\rangle^k)}\,ds.\end{aligned}$$

We make q tend to infinity. Then we have $\Pi_{\mathbf{G}}(\mathbf{Q}(\mathbf{f}(t))) = 0$ by Lemma 5.23 so we are able to use the exponential decay of $S_{\mathbf{G}}(s)$ Lemma 5.27. This yields

$$\begin{aligned}\frac{d}{dt}\int_0^{+\infty}\|S_{\mathbf{G}}(s)\,(\mathbf{f}(t))\|_{L^1_vL^\infty_x(\langle v\rangle^k)}\,ds \leq & -\|\mathbf{f}(t)\|_{L^1_vL^\infty_x(\langle v\rangle^k)} \\ & + C_k\left(\int_0^{+\infty}e^{-\lambda_k s}\,ds\right)\|\mathbf{Q}(\mathbf{f}(t))\|_{L^1_vL^\infty_x(\langle v\rangle^k)}.\end{aligned}$$

With Lemma 5.23 we control $\mathbf{Q}(\mathbf{f})$:

$$\begin{aligned}\frac{d}{dt}\int_0^{+\infty}\|S_{\mathbf{G}}(s)\,(\mathbf{f}(t))\|_{L^1_vL^\infty_x(\langle v\rangle^k)}\,ds \leq & -\|\mathbf{f}(t)\|_{L^1_vL^\infty_x(\langle v\rangle^k)} \\ & + \frac{C_kC_Q}{\lambda_k}\|\mathbf{f}\|_{L^1_vL^\infty_x(\langle v\rangle^k)}\|\mathbf{f}\|_{L^1_vL^\infty_x(\langle v\rangle^k\boldsymbol{\nu})}.\end{aligned} \tag{5.5.23}$$

To conclude we add $\alpha\times(5.5.22)+(5.5.23)$,

$$\begin{aligned}\frac{d}{dt}\|\mathbf{f}\|_{k,\alpha} \leq & -\left[\alpha(1-C_B) - \left(\alpha C_Q + \frac{C_kC_Q}{\lambda_k}\right)\|\mathbf{f}\|_{L^1_vL^\infty_x(\langle v\rangle^k)}\right]\|\mathbf{f}\|_{L^1_vL^\infty_x(\langle v\rangle^k\boldsymbol{\nu})} \\ & + \left[\alpha C_A - 1\right]\|\mathbf{f}\|_{L^1_vL^\infty_x(\langle v\rangle^k)}.\end{aligned} \tag{5.5.24}$$

Choosing α such that $(\alpha C_A - 1) < 0$ yields the desired estimate. □

We now prove the uniqueness proposition.

Proof of Proposition 5.26. Let $\mathbf{f}$ and $\mathbf{g}$ in $L^1_vL^\infty_x\left(\langle v\rangle^k\right)$, $k > k_0$, be two solutions of the perturbed equation with initial datum $\mathbf{f_0}$.

Thanks to Lemma 5.28, if $\|\mathbf{f_0}\|_{L^1_vL^\infty_x(\langle v\rangle^k)}$ is small enough we can deduce from the differential inequality that for all $t \geq 0$,

$$\frac{d}{dt}\|\mathbf{f}\|_{\alpha,k} \leq -C_k\|\mathbf{f}\|_{\alpha,k,\boldsymbol{\nu}},$$

and the same holds for $\mathbf{g}$ with the same constant $C_k > 0$. We therefore have two estimates on $\mathbf{f}$ and $\mathbf{g}$. Either by integrating from 0 to t:

$$\forall t \geq 0, \quad \int_0^t \|\mathbf{f}(s)\|_{\alpha,k,\boldsymbol{\nu}}\, ds \leq C_k^{-1}\|\mathbf{f_0}\|_{\alpha,k}; \tag{5.5.25}$$

or by Grönwall lemma:

$$\forall t \geq 0, \quad \|\mathbf{f}(t)\|_{\alpha,k} \leq e^{-C_k t}\|\mathbf{f_0}\|_{\alpha,k}. \tag{5.5.26}$$

The same estimate holds for $\mathbf{g}$.

Recalling the definition (5.5.11) of the operator $\widetilde{\mathbf{Q}}$, we find the differential equation satisfied by $\mathbf{f}-\mathbf{g}$:

$$\partial_t(\mathbf{f}-\mathbf{g}) = \mathbf{G}(\mathbf{f}-\mathbf{g}) + \widetilde{\mathbf{Q}}(\mathbf{f}-\mathbf{g},\mathbf{f}) + \widetilde{\mathbf{Q}}(\mathbf{g},\mathbf{f}-\mathbf{g}).$$

Using controls on $\mathbf{B}$ (Lemma 5.20), $\mathbf{A}$ (Lemma 5.19), $\widetilde{\mathbf{Q}}$ (Lemma 5.23) and the semigoup property (Lemma 5.27), exact same computations as for (5.5.24), gives

$$\begin{aligned}\frac{d}{dt}\|\mathbf{f}-\mathbf{g}\|_{k,\alpha} \leq &-\left[\alpha(1-C_B) - \left(\alpha C_Q + \frac{C_kC_Q}{\lambda_k}\right)\left(\|\mathbf{f}\|_{\alpha,k} + \|\mathbf{g}\|_{\alpha,k}\right)\right]\|\mathbf{f}-\mathbf{g}\|_{\alpha,k,\boldsymbol{\nu}} \\ &+ \left[\alpha C_A - 1 + C_Q\left(\|\mathbf{f}\|_{\alpha,k,\boldsymbol{\nu}} + \|\mathbf{g}\|_{\alpha,k,\boldsymbol{\nu}}\right)\right]\|\mathbf{f}-\mathbf{g}\|_{\alpha,k}.\end{aligned}$$

Note that we used the equivalence of the $\|\cdot\|_{\alpha,k}$ norm and our usual norm (see Lemma 5.28).

First, by (5.5.26) and $C_B < 1$, if $\mathbf{f_0}$ is small enough then for all $t \geq 0$,

$$\alpha(1-C_B) - \left(\alpha C_Q + \frac{C_kC_Q}{\lambda_k}\right)\left(\|\mathbf{f}\|_{\alpha,k} + \|\mathbf{g}\|_{\alpha,k}\right) \leq 0.$$

Second we take α small enough so that $(\alpha C_A - 1) < 0$. Hence, integrating the differential inequality from 0 to t:

$$\|\mathbf{f}(t)-\mathbf{g}(t)\|_{k,\alpha} \leq C_Q\left[\int_0^t\left(\|\mathbf{f}(s)\|_{\alpha,k,\boldsymbol{\nu}} + \|\mathbf{g}(\mathbf{s})\|_{\alpha,k,\boldsymbol{\nu}}\right)ds\right]\sup_{s\in[0,t]}\|\mathbf{f}(s)-\mathbf{g}(s)\|_{k,\alpha}.$$

To conclude we use (5.5.25) to obtain

$$\forall t \geq 0, \quad \|\mathbf{f}(t)-\mathbf{g}(t)\|_{k,\alpha} \leq \frac{2C_Q}{C_k}\|\mathbf{f_0}\|_{\alpha,k}\left(\sup_{s\in[0,t]}\|\mathbf{f}(s)-\mathbf{g}(s)\|_{k,\alpha}\right),$$

which implies $\mathbf{f} = \mathbf{g}$ if $\|\mathbf{f_0}\|_{\alpha,k}$ is small enough. □

5.5.3. Positivity of solutions

This last subsection is dedicated to the positivity of the solution to the multi-species Boltzmann equation

$$\partial_t \mathbf{F} + v \cdot \nabla_x \mathbf{F} = \mathbf{Q}(\mathbf{F}) \tag{5.5.27}$$

in the perturbative setting studied above.

Proposition 5.29. *Let $k > k_0$, let assumptions $(H1)-(H4)$ hold for the collision kernel, and let $\mathbf{f_0}$ be in $L^1_v L^\infty_x\left(\langle v\rangle^k\right)$ with $\Pi_{\mathbf{G}}(\mathbf{f_0}) = 0$ and*

$$\|\mathbf{f_0}\|_{L^1_v L^\infty_x\left(\langle v\rangle^k\right)} \leq \eta_k,$$

where $\eta_k > 0$ is chosen such that Proposition 5.18 and Proposition 5.26 hold and denote $\mathbf{f}$ the unique solution of the perturbed multi-species equation associated to $\mathbf{f_0}$.
Suppose that $\mathbf{F_0} = \boldsymbol{\mu} + \mathbf{f_0} \geq 0$ then $\mathbf{F} = \boldsymbol{\mu} + \mathbf{f} \geq 0$.

Proof of Proposition 5.29. Since we are working with the Grad's cutoff assumption we can decompose the nonlinear operator into

$$\mathbf{Q}(\mathbf{F}) = -\mathbf{Q_1}(\mathbf{F}) + \mathbf{Q_2}(\mathbf{F})$$

where

$$\begin{aligned} Q_{1i}(\mathbf{F}) &= \sum_{j=1}^N \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij}\left(|v - v_*|, \cos\theta\right) F_i F_j^* \, dv_* d\sigma \\ Q_{2i}(\mathbf{F}) &= \sum_{j=1}^N \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij}\left(|v - v_*|, \cos\theta\right) F_i' F_j'^* \, dv_* d\sigma. \end{aligned}$$

Following the idea of [83], we construct an interative scheme for the multi-species Boltzmann equation

$$\partial_t \mathbf{F^{(n+1)}} + v \cdot \nabla_x \mathbf{F^{(n+1)}} + \overline{\mathbf{Q}}_\mathbf{1}(\mathbf{F^{(n+1)}}, \mathbf{F^{(n)}}) = \mathbf{Q_2}(\mathbf{F^{(n)}}),$$

with the non-symmetriized bilinear form $\overline{\mathbf{Q}}_\mathbf{1}$ defined as

$$\begin{aligned} \overline{Q}_{1i}(\mathbf{F}, \mathbf{G}) &= \sum_{j=1}^N \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij}\left(|v - v_*|, \cos\theta\right) F_i G_j^* \, dv_* d\sigma \\ \overline{Q}_{2i}(\mathbf{F}, \mathbf{G}) &= \sum_{j=1}^N \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij}\left(|v - v_*|, \cos\theta\right) F_i' G_j'^* \, dv_* d\sigma. \end{aligned}$$

Defining $\mathbf{f^{(n)}} = \mathbf{F^n} - \boldsymbol{\mu}$ we have the following differential iterative scheme

$$\partial_t \mathbf{f^{(n+1)}} + v \cdot \nabla_x \mathbf{f^{(n+1)}} = -\boldsymbol{\nu}(v)\left(\mathbf{f^{(n+1)}}\right) + \mathbf{K}\left(\mathbf{f^{(n)}}\right) + \mathbf{Q_2}\left(\mathbf{f^{(n)}}\right) - \widetilde{\mathbf{Q}}_\mathbf{1}\left(\mathbf{f^{(n+1)}}, \mathbf{f^{(n)}}\right).$$

As before, we can prove that $\left(\mathbf{f^{(n)}}\right)_{n\in\mathbb{N}}$ is well-defined and converges in $L^1_v L^\infty_x \left(\langle v\rangle^k\right)$ towards $\mathbf{f}$, the unique solution of the perturbed multi-species equation and thus the same holds for $\mathbf{F^n}$ converging towards $\mathbf{F}$ the unique perturbed solution of the original multi-species Boltzmann equation.

We prove that $\mathbf{F^n} \geq 0$ by an induction on N.
By definition we see that

$$\widetilde{\mathbf{Q}}_{\mathbf{1}}(\mathbf{F^{(n+1)}}, \mathbf{F^{(n)}}) = q_1(\mathbf{F^{(n)}})\mathbf{F^{(n+1)}},$$

and thus applying the Duhamel formula along the characteristics gives

$$\begin{aligned}
&\mathbf{F^{(n+1)}}(t,x,v) \\
&\quad = \exp\left[-\int_0^t q_1(\mathbf{F^{(n)}})(s, x-(t-s)v, v)\, ds\right] \mathbf{F_0}(x-tv, v) \\
&\qquad + \int_0^t \exp\left[-\int_s^t q_1(\mathbf{F^{(n)}})(s_1, x-(t-s_1)v, v)\, ds_1\right] \mathbf{Q_2}(\mathbf{F^{(n)}})(s, x-(t-s)v, v)\, ds.
\end{aligned}$$

By positivity of $\mathbf{F^{(n)}}$, all the terms on the right-hand side are positive and therefore $\mathbf{F^{n+1}} \geq 0$. Passing to the limit implies that $\mathbf{F} \geq 0$. $\square$

Part III.

The diffusive models

6. Cross-diffusion population systems for multiple species

Our goal in this chapter is the proof of global existence of weak solutions to reaction-cross-diffusion systems for an arbitrary number of competing population species in a bounded domain with homogeneous Neumann boundary conditions. In the case of linear transition rates, the model considered extends the two-species population model of Shigesada, Kawasaki, and Teramoto. The existence proof uses a refined entropy method with the help of a new approximation scheme. Global existence can be proved under a detailed balance or weak cross-diffusion condition. The detailed balance condition is related to the symmetry of the mobility matrix, which mirrors Onsager's principle in thermodynamics. Under detailed balance (and without reaction), we can show that the entropy is nonincreasing in time, but counter-examples illustrate that the entropy can increase initially if detailed balance is not satisfied.

6.1. The model

We briefly recall the model. As already introduced in Section 3.1, we consider the reaction-cross-diffusion equations

$$\partial_t u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right) = f_i(u) \quad \text{in } \Omega,\ t>0, \quad i=1,\ldots,n, \tag{6.1.1}$$

with homogeneous Neumann boundary conditions and initial conditions

$$\sum_{j=1}^n A_{ij}(u)\nabla u_j\cdot\nu = 0 \quad \text{on } \partial\Omega,\ t>0, \quad u_i(\cdot,0)=u_i^0 \quad \text{in } \Omega, \tag{6.1.2}$$

where the variable u_i describes the density of the ith species of $u=(u_1,\ldots,u_n)$, $\Omega\subset\mathbb{R}^d$ ($d\geq 1$) is a bounded domain with Lipschitz boundary, and ν is the exterior unit normal vector to $\partial\Omega$. The diffusion coefficients have the form

$$A_{ij}(u) = \delta_{ij}p_i(u) + u_i\frac{\partial p_i}{\partial u_j}(u), \quad p_i(u) = a_{i0} + \sum_{k=1}^n a_{ik}u_k^s, \quad i,j=1,\ldots,n, \tag{6.1.3}$$

with a_{i0}, $a_{ij}\geq 0$ and $s>0$. The reaction terms f_i are of Lotka-Volterra type,

$$f_i(u) = u_i\left(b_{i0} - \sum_{j=1}^n b_{ij}u_j\right), \quad i=1,\ldots,n, \tag{6.1.4}$$

and we assume that we are in the competition case, thus b_{i0}, $b_{ij} \geq 0$. We notice that equations (6.1.1) can be also written in the following form

$$\partial_t u_i - \Delta(u_i p_i(u)) = f_i(u), \quad 1 \leq i \leq n. \tag{6.1.5}$$

The entropy reads

$$\mathcal{H}[u] = \int_\Omega h(u)dx = \int_\Omega \sum_{i=1}^n \pi_i h_s(u_i)dx, \tag{6.1.6}$$

where $\pi_i > 0$ are some positive numbers and

$$h_s(z) = \begin{cases} z(\log z - 1) + 1 & \text{for } s = 1, \\ \dfrac{z^s - sz}{s-1} + 1 & \text{for } s \neq 1. \end{cases} \tag{6.1.7}$$

If we use the so-called entropy variable $w = (w_1, \dots, w_n)$ by

$$w_i = \frac{\partial h}{\partial u_i}(u) = \begin{cases} \pi_i \log u_i & \text{for } s = 1, \\ \dfrac{s\pi_i}{s-1}(u_i^{s-1} - 1) & \text{for } s \neq 1, \end{cases}$$

we can rewrite equations (6.1.1) as

$$\partial_t u(w) - \operatorname{div}(B(w)\nabla w) = f(u(w)), \quad B(w) = A(u)H(u)^{-1}, \tag{6.1.8}$$

with $u(w) := (h')^{-1}(w)$ describing the inverse transformation and $H(u) = h''(u)$ the Hessian of the entropy density.

Now we introduce two important conditions, a detailed-balance condition and a weak cross-diffusion condition, which are important in the context of deriving the crucial gradient estimate in Section 6.2.1, which has the form

$$\frac{d}{dt}\mathcal{H}[u] + 4\int_\Omega \sum_{i=1}^n \pi_i a_{i0} |\nabla \sqrt{u_i}|^2 dx + 2\int_\Omega \sum_{i=1}^n \pi_i a_{ii} |\nabla u_i|^2 dx \leq 0. \tag{6.1.9}$$

This inequality implies an H^1 estimate for $\sqrt{u_i}$ if $a_{i0} > 0$, and an H^1 estimate for u_i if $a_{ii} > 0$. First, it turns out that (6.1.6) is a Lyapunov functional and the condition $H(u)A(u)$ is symmetric and positive definite if

$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for all } i, j = 1, \dots, n, \tag{6.1.10}$$

which we will call detailed balance condition.

Moreover, we show that if self-diffusion dominates cross-diffusion in the sense

$$\eta_0 := \min_{i=1,\dots,n} \left(a_{ii} - \frac{s}{2(s+1)} \sum_{j=1}^n \left(\sqrt{a_{ij}} - \sqrt{a_{ji}}\right)^2 \right) > 0, \tag{6.1.11}$$

and detailed balance may be not satisfied, then the estimate leading to (6.1.9) still holds true, but with different constants, leading to global existence. We remark that we set $\pi_i = 1$ whenever detailed balance does not hold.

6.1.1. Main results

Theorem 6.1 (Global existence for linear transition rates)**.** *Let $T > 0$, $s = 1$ and $u^0 = (u_1^0, \ldots, u_n^0)$ be such that $u_i^0 \geq 0$ for $i = 1, \ldots, n$ and $\int_\Omega h(u^0)dx < \infty$. Let either detailed balance and $a_{ii} > 0$ for $i = 1, \ldots, n$; or* (6.1.11) *hold. Then there exists a weak solution $u = (u_1, \ldots, u_n)$ to* (6.1.1)-(6.1.2) *satisfying $u_i \geq 0$ in Ω, $t > 0$, and*

$$u_i \in L^2(0, T; H^1(\Omega)), \quad u_i \in L^\infty(0, T; L^1(\Omega)),$$
$$u_i \in L^{2+2/d}(Q_T), \quad \partial_t u_i \in L^{q'}(0, T; W^{1,q}(\Omega)'), \quad i = 1, \ldots, n,$$

where $q = 2(d + 1)$ and $q' = (2d + 2)/(2d + 1)$. The solution u solves (6.1.1) *in the weak sense*

$$\int_0^T \langle \partial_t u, \phi \rangle dt + \int_0^T \int_\Omega \nabla \phi : A(u) \nabla u dx dt = \int_0^T \int_\Omega f(u) \cdot \phi dx dt \tag{6.1.12}$$

for all test functions $\phi \in L^q(0, T; W^{1,q}(\Omega))$, and the initial condition in (6.1.2) *is satisfied in the sense of $W^{1,q}(\Omega)'$.*

For a generalization of this theorem to the case of vanishing self-diffusion, see Remark 6.12.

The second result works for nonlinear transition rates ($s \neq 1$). In the sublinear case (*i.e.* $s < 1$), we face the probem that the entropy inequality only yields the regularity $u_i \in L^{2s+2/d}(Q_T)$. This doe not yield an L^2 estimate for "small" exponents $s < 1$ and large dimensions d. Thus, we need to assume the lower bound $s > 1 - 2/d$ and a weaker growth of the Lotka-Volterra terms:

$$f_i(u) = u_i \bigg(b_{i0} - \sum_{j=1}^n b_{ij} u_j^\sigma \bigg), \quad i = 1, \ldots, n, \quad 0 \leq \sigma < 2s - 1 + 2/d. \tag{6.1.13}$$

The superlinear case (*i.e.* $s > 1$) is in some sense easier than the sublinear one, because the entropy inequality gives the higher regularity $u_i \in L^p(Q_T)$ with $p > 2$. On the other hand, we need a weak cross-diffusion constraint. Thus, under detailed balance, we require that

$$\eta_1 := \min_{i=1,\ldots,n} \bigg(a_{ii} - \frac{s-1}{s+1} \sum_{j=1,\, j \neq i}^n a_{ij} \bigg) > 0, \tag{6.1.14}$$

and if detailed balance is not satisfied, we assume that

$$\eta_2 := \min_{i=1,\ldots,n} \bigg(a_{ii} - \frac{1}{2(s+1)} \sum_{j=1,\, j \neq i} \big(s(a_{ij} + a_{ji}) - 2\sqrt{a_{ij} a_{ji}} \big) \bigg) > 0. \tag{6.1.15}$$

For $m \geq 2$ and $1 \leq q \leq \infty$ we introduce the space

$$W_\nu^{m,q}(\Omega) = \{\phi \in W^{m,q}(\Omega) : \nabla \phi \cdot \nu = 0 \text{ on } \partial\Omega\}. \tag{6.1.16}$$

Theorem 6.2 (Global existence for nonlinear transition rates)**.** *Assume that $T>0$, $s>\max\{0,1-2/d\}$, and let the initial data u^0 be such that $u_i^0\geq 0$ for $i=1,\ldots,n$ and $\int_\Omega h(u^0)dx<\infty$. If $s<1$, we suppose that* (6.1.13) *and either detailed balance and $a_{ii}>0$ for $i=1,\ldots,n$; or* (6.1.11) *hold. If $s>1$, we suppose that* (6.1.4) *and either detailed balance and* (6.1.14) *or* (6.1.15) *hold. Then there exist a number $2\leq q<\infty$ and a weak solution $u=(u_1,\ldots,u_n)$ to* (6.1.1)-(6.1.2) *satisfying $u_i\geq 0$ in Ω, $t>0$, and*

$$u_i^s\in L^2(0,T;H^1(\Omega)),\quad u_i\in L^\infty(0,T;L^{\max\{1,s\}}(\Omega)),$$
$$u_i\in L^{p(s)}(Q_T),\quad \partial_t u_i\in L^{q'}(0,T;W_\nu^{m,q}(\Omega)'),\quad i=1,\ldots,n,$$

where $p(s)=2s+(2/d)\max\{1,s\}$, $1/q+1/q'=1$, and $m>\max\{1,d/2\}$. The solution u solves (6.1.1) *in the "very weak" sense*

$$\int_0^T\langle\partial_t u,\phi\rangle dt-\int_0^T\int_\Omega\sum_{i=1}^n u_ip_i(u)\Delta\phi_i dxdt=\int_0^T\int_\Omega f(u)\cdot\phi dxdt \tag{6.1.17}$$

for all $\phi=(\phi_1,\ldots,\phi_n)\in L^q(0,T;W_\nu^{m,q}(\Omega))$, and the initial condition holds in the sense of $W_\nu^{m,q}(\Omega)'$.

Again, we can generalize the theorem to the case of vanishing self-diffusion if either $s>\max\{1,d/2\}$; or $0<s<1$, $d=1$, and $\sigma<s+1$ hold; which is sketched in Remark 6.17.

The chapter is organized as follows. Section 6.2 is concerned with the positive definiteness of the matrices $H(u)A(u)$ and $H_\varepsilon(u)A_\varepsilon(u)$. The existence theorems are proved in Sections 6.3 and 6.4, respectively. In the final Section 6.5, we detail the connection between the detailed balance condition and the symmetry of $H(u)A(u)$, prove a nonlinear Aubin-Lions compactness lemma needed in the proof of Theorem 6.2, and show that the entropy may be increasing initially for special initial data.

6.2. Positive definiteness of the mobility matrix

We derive sufficient conditions for the positive definiteness of the matrix $H(u)A(u)$. Let $\mathbb{R}_+=(0,\infty)$. Recall that

$$A_{ij}(u)=\delta_{ij}\left(a_{i0}+\sum_{k=1}^n a_{ik}u_k^s\right)+sa_{ij}u_iu_j^{s-1},\quad H_{ij}(u)=\delta_{ij}s\pi_iu_i^{s-2}.$$

The following result is valid for any $s>0$.

Lemma 6.3. *Let $s>0$. Then, for any $z\in\mathbb{R}^n$ and $u\in\mathbb{R}_+^n$,*

$$\begin{aligned}z^\top H(u)A(u)z &\geq s\sum_{i=1}^n\pi_ia_{i0}u_i^{s-2}z_i^2+s(1-s)\sum_{i,j=1,\,i\neq j}^n\pi_ia_{ij}u_j^su_i^{s-2}z_i^2\\ &\quad+s\sum_{i=1}^n\left((s+1)\pi_ia_{ii}-\frac{s}{2}\sum_{j=1}^n\left(\sqrt{\pi_ia_{ij}}-\sqrt{\pi_ja_{ji}}\right)^2\right)u_i^{2(s-1)}z_i^2.\end{aligned} \tag{6.2.1}$$

Proof. The elements of the matrix $H(u)A(u)$ equal

$$\begin{aligned}(H(u)A(u))_{ij} &= \delta_{ij} s\pi_i \bigg(a_{i0} u_i^{s-2} + \sum_{k=1}^n a_{ik} u_k^s u_i^{s-2} \bigg) + s^2 a_{ij} (u_i u_j)^{s-1} \\ &= \delta_{ij} \big(s\pi_i a_{i0} u_i^{s-2} + s(s+1)\pi_i a_{ii} u_i^{2(s-1)} \big) \\ &\quad + \delta_{ij} s\pi_i \sum_{k=1,\, k\neq i}^n a_{ik} u_k^s u_i^{s-2} + (1-\delta_{ij}) s^2 \pi_i a_{ij} (u_i u_j)^{s-1}.\end{aligned}$$

Therefore, for $z \in \mathbb{R}^n$,

$$\begin{aligned}z^\top H(u)A(u)z &= s\sum_{i=1}^n \pi_i a_{i0} u_i^{s-2} z_i^2 + s(s+1)\sum_{i=1}^n \pi_i a_{ii} u_i^{2(s-1)} z_i^2 \\ &\quad + s \sum_{i,j=1,\, i\neq j}^n \pi_i a_{ij} u_j^s u_i^{s-2} z_i^2 + s^2 \sum_{i,j=1,\, i\neq j}^n \pi_i a_{ij} (u_i u_j)^{s-1} z_i z_j \qquad (6.2.2) \\ &=: I_1 + \cdots + I_4.\end{aligned}$$

The sum I_1 is the same as the first term on the right-hand side of (6.2.1), and I_2 equals the first part of the last term on this right-hand side. The remaining terms are written as

$$\begin{aligned}I_3 + I_4 &= s^2 \sum_{i,j=1,\, i\neq j}^n \pi_i a_{ij} u_j^s u_i^{s-2} z_i^2 + s(1-s) \sum_{i,j=1,\, i\neq j}^n \pi_i a_{ij} u_j^s u_i^{s-2} z_i^2 \\ &\quad + s^2 \sum_{i,j=1,\, i\neq j}^n \pi_i a_{ij} (u_i u_j)^{s-1} z_i z_j.\end{aligned}$$

The second term corresponds to the second term on the right-hand side of (6.2.1). Thus, it remains to prove that

$$\begin{aligned}J &:= s^2 \sum_{i,j=1,\, i\neq j}^n \pi_i a_{ij} u_j^s u_i^{s-2} z_i^2 + s^2 \sum_{i,j=1,\, i\neq j}^n \pi_i a_{ij} (u_i u_j)^{s-1} z_i z_j \\ &\geq -\frac{s^2}{2} \sum_{j=1}^n \big(\sqrt{\pi_i a_{ij}} - \sqrt{\pi_j a_{ji}}\big)^2 u_i^{2(s-1)} z_i^2.\end{aligned}$$

For this, we employ twice the inequality $b^2 + c^2 \geq 2bc$:

$$\begin{aligned}J &= s^2 \sum_{i,j=1,\, i<j}^n \pi_i a_{ij} u_j^s u_i^{s-2} z_i^2 + s^2 \sum_{i,j=1,\, i>j}^n \pi_i a_{ij} u_j^s u_i^{s-2} z_i^2 \\ &\quad + s^2 \sum_{i,j=1,\, i<j}^n \pi_i a_{ij} (u_i u_j)^{s-1} z_i z_j + s^2 \sum_{i,j=1,\, i>j}^n \pi_i a_{ij} (u_i u_j)^{s-1} z_i z_j \\ &= s^2 \sum_{i,j=1,\, i<j}^n \Big(\pi_i a_{ij} u_j^s u_i^{s-2} z_i^2 + \pi_j a_{ji} u_i^s u_j^{s-2} z_j^2 + (\pi_i a_{ij} + \pi_j a_{ji})(u_i u_j)^{s-1} z_i z_j \Big)\end{aligned}$$

$$
\begin{aligned}
&\geq s^2 \sum_{i,j=1,\, i<j}^{n} \Big(2\sqrt{\pi_i a_{ij}\pi_j a_{ji}}(u_iu_j)^{s-1}|z_iz_j| - (\pi_i a_{ij} + \pi_j a_{ji})(u_iu_j)^{s-1}|z_iz_j|\Big) \\
&= -s^2 \sum_{i,j=1,\, i<j}^{n} \big(\sqrt{\pi_i a_{ij}} - \sqrt{\pi_j a_{ji}}\big)^2 \big|(u_i^{s-1}z_i)(u_j^{s-1}z_j)\big| \\
&\geq -\frac{s^2}{2} \sum_{i,j=1,\, i<j}^{n} \big(\sqrt{\pi_i a_{ij}} - \sqrt{\pi_j a_{ji}}\big)^2 \big((u_i^{s-1}z_i)^2 + (u_j^{s-1}z_j)^2\big) \\
&= -\frac{s^2}{2} \sum_{i,j=1,\, i\neq j}^{n} \big(\sqrt{\pi_i a_{ij}} - \sqrt{\pi_j a_{ji}}\big)^2 (u_i^{s-1}z_i)^2.
\end{aligned}
$$

This finishes the proof. □

6.2.1. Sublinear and linear transition rates

For $s \leq 1$, Lemma 6.3 provides immediately the positive definiteness of $H(u)A(u)$ if detailed balance (6.1.10) holds. However, we can derive a sharper result.

Lemma 6.4 (Detailed balance)**.** *Let $0 < s \leq 1$ and $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i \neq j$. Then, for all $z \in \mathbb{R}^n$ and $u \in \mathbb{R}^n_+$,*

$$
\begin{aligned}
z^\top H(u)A(u)z \geq{}& s\sum_{i=1}^{n} \pi_i u_i^{s-2}\big(a_{i0} + (s+1)a_{ii}u_i^s\big)z_i^2 \\
&+ \frac{s^2}{2} \sum_{i,j=1,\, i\neq j}^{n} \pi_i a_{ij}(u_iu_j)^{s-1}\left(\sqrt{\frac{u_j}{u_i}}z_i + \sqrt{\frac{u_i}{u_j}}z_j\right)^2. \qquad (6.2.3)
\end{aligned}
$$

Proof. The sum of the terms I_1 and I_2 in (6.2.2) is exactly the first term on the right-hand side of (6.2.3). Using detailed balance, we find that

$$
\begin{aligned}
I_3 + I_4 ={}& \frac{s}{2} \sum_{i,j=1,\, i\neq j}^{n} \pi_i a_{ij}(u_iu_j)^{s-1}\frac{u_j}{u_i}z_i^2 + \frac{s}{2} \sum_{i,j=1,\, i\neq j}^{n} \pi_i a_{ij}(u_ju_i)^{s-1}\frac{u_i}{u_j}z_j^2 \\
&+ s^2 \sum_{i,j=1,\, i\neq j}^{n} \pi_i a_{ij}(u_iu_j)^{s-1}z_iz_j \\
={}& \frac{s^2}{2} \sum_{i,j=1,\, i\neq j}^{n} \pi_i a_{ij}(u_iu_j)^{s-1}\frac{u_j}{u_i}z_i^2 + \frac{s^2}{2} \sum_{i,j=1,\, i\neq j}^{n} \pi_i a_{ij}(u_ju_i)^{s-1}\frac{u_i}{u_j}z_j^2 \\
&+ s^2 \sum_{i,j=1,\, i\neq j}^{n} \pi_i a_{ij}(u_iu_j)^{s-1}z_iz_j + \frac{s}{2}(1-s) \sum_{i,j=1,\, i\neq j}^{n} \pi_i a_{ij}(u_iu_j)^{s-1}\frac{u_j}{u_i}z_i^2 \\
&+ \frac{s}{2}(1-s) \sum_{i,j=1,\, i\neq j}^{n} \pi_i a_{ij}(u_ju_i)^{s-1}\frac{u_i}{u_j}z_j^2.
\end{aligned}
$$

The sum of the first three terms equal the second term on the right-hand side of (6.2.3), and the remaining two terms are nonnegative since $s \leq 1$. □

Remark 6.5. *In the existence proof, we will choose $z_i = \nabla u_i$ (with a slight abuse of notation). Then the first term in* (6.2.3) *gives an estimate for $\nabla u_i^{s/2}$ in L^2 (if $a_{i0} > 0$) and the better bound $\nabla u_i^s \in L^2$ (if $a_{ii} > 0$). If $a_{ii} = 0$, we lose the latter regularity. This loss can be compensated by the last term in* (6.2.3) *giving*

$$(u_i u_j)^{s-1}\left|\sqrt{\frac{u_j}{u_i}}\nabla u_i + \sqrt{\frac{u_i}{u_j}}\nabla u_j\right|^2 = \frac{4}{s^2}|\nabla(u_i u_j)^{s/2}|^2, \quad i \neq j,$$

and consequently a bound for $\nabla(u_i u_j)^{s/2}$ in L^2. This observation is used in Remark 6.12. □

Lemma 6.6 (Non detailed balance)**.** *Let $0 < s \le 1$. If*

$$\eta_0 := \min_{i=1,\ldots,n}\left(a_{ii} - \frac{s}{2(s+1)}\sum_{j=1}^n\left(\sqrt{a_{ij}} - \sqrt{a_{ji}}\right)^2\right) \ge 0,$$

then $H(u)A(u)$ is positive definite. Under the slightly stronger condition $\eta_0 > 0$, it holds for all $z \in \mathbb{R}^n$ and $u \in \mathbb{R}^n_+$ that

$$z^\top H(u)A(u)z \ge s\sum_{i=1}^n a_{i0}u_i^{s-2}z_i^2 + \eta_0 s(s+1)\sum_{i=1}^n u_i^{2(s-1)}z_i^2.$$

The lemma follows from Lemma 6.3 after choosing $\pi_i = 1$ for $i = 1, \ldots, n$. Observe that $\eta_0 > 0$ holds if $a_{ii} > 0$ for all i and (a_{ij}) is symmetric.

It is possible to show the positive definiteness of $H(u)A(u)$ without any restriction on (a_{ij}) (except positivity) if we restrict the choice of the parameter s; see the following lemma.

Lemma 6.7. *Let $a_{ij} + a_{ji} > 0$ for $i, j = 1, \ldots, n$ and $0 < s \le s_0$, where*

$$s_0 := \min_{i,j=1,\ldots,n}\frac{2\sqrt{a_{ij}a_{ji}}}{a_{ij} + a_{ji}} \le 1.$$

Then, for all $z \in \mathbb{R}^n$ and $u \in \mathbb{R}^n_+$,

$$z^\top H(u)A(u)z \ge s\sum_{i=1}^n a_{i0}u_i^{s-2}z_i^2 + s(s+1)\sum_{i=1}^n a_{ii}u_i^{2(s-1)}z_i^2.$$

Proof. We choose $\pi_i = 1$ for $i = 1, \ldots, n$. With the notation of the proof of Lemma 6.3, we only need to show that $I_3 + I_4 \ge 0$. Employing the inequality $b^2 + c^2 \ge 2bc$, we find that

$$\begin{aligned} I_3 + I_4 &= s\sum_{i,j=1,\, i<j}^n\left(a_{ij}u_j^s u_i^{s-2}z_i^2 + a_{ji}u_i^s u_j^{s-2}z_j^2 + s(a_{ij} + a_{ji})(u_i u_j)^{s-1}z_i z_j\right) \\ &\ge s\sum_{i,j=1,\, i<j}^n\left(2\sqrt{a_{ij}a_{ji}}(u_i u_j)^{s-1}|z_i z_j| - s(a_{ij} + a_{ji})(u_i u_j)^{s-1}|z_i z_j|\right) \\ &= s\sum_{i,j=1,\, i<j}^n(a_{ij} + a_{ji})\left(\frac{2\sqrt{a_{ij}a_{ji}}}{a_{ij} + a_{ji}} - s\right)(u_i u_j)^{s-1}|z_i z_j|, \end{aligned}$$

and this expression is nonnegative if $s \le s_0$. □

6.2.2. Superlinear transition rates

Again, we assume first that detailed balance holds.

Lemma 6.8 (Detailed balance)**.** *Let $s > 1$ and $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i \neq j$. If*

$$\eta_1 := \min_{i=1,\ldots,n} \left(a_{ii} - \frac{s-1}{s+1} \sum_{j=1,\, j\neq i}^{n} a_{ij} \right) \geq 0,$$

then $H(u)A(u)$ is positive definite. Furthermore, if $\eta_1 > 0$, then, for all $z \in \mathbb{R}^n$ and $u \in \mathbb{R}^n_+$,

$$z^\top H(u)A(u)z \geq s \sum_{i=1}^{n} \pi_i a_{i0} u_i^{s-2} z_i^2 + \eta_1 s(s+1) \sum_{i=1}^{n} \pi_i u_i^{2(s-1)} z_i^2.$$

Proof. It is sufficient to estimate the sum $I_3 + I_4$, defined in the proof of Lemma 6.3:

$$\begin{aligned}
I_3 + I_4 &= s \sum_{i,j=1,\, i<j}^{n} \Big(\pi_i a_{ij} u_j^s u_i^{s-2} z_i^2 + \pi_j a_{ji} u_i^s u_j^{s-2} z_j^2 + s(\pi_i a_{ij} + \pi_j a_{ji})(u_i u_j)^{s-1} z_i z_j \Big) \\
&\geq s \sum_{i,j=1,\, i<j}^{n} \Big(2\sqrt{\pi_i a_{ij} \pi_j a_{ji}} (u_i u_j)^{s-1} |z_i z_j| - s(\pi_i a_{ij} + \pi_j a_{ji})(u_i u_j)^{s-1} |z_i z_j| \Big) \\
&= -s \sum_{i,j=1,\, i<j}^{n} \Big(s(\pi_i a_{ij} + \pi_j a_{ji}) - 2\sqrt{\pi_i a_{ij} \pi_j a_{ji}} \Big) (u_i u_j)^{s-1} |z_i z_j| \\
&\geq -\frac{s}{2} \sum_{i,j=1,\, i<j}^{n} \Big(s(\pi_i a_{ij} + \pi_j a_{ji}) - 2\sqrt{\pi_i a_{ij} \pi_j a_{ji}} \Big) \big((u_i^{s-1} z_i)^2 + (u_j^{s-1} z_j)^2 \big) \\
&= -\frac{s}{2} \sum_{i,j=1,\, i\neq j}^{n} \Big(s(\pi_i a_{ij} + \pi_j a_{ji}) - 2\sqrt{\pi_i a_{ij} \pi_j a_{ji}} \Big) (u_i^{s-1} z_i)^2.
\end{aligned}$$

This expression simplifies because of the detailed balance condition:

$$I_3 + I_4 \geq -s(s-1) \sum_{i,j=1,\, i\neq j}^{n} \pi_i a_{ij} (u_i^{s-1} z_i)^2,$$

and we end up with

$$z^\top H(u)A(u)z \geq s \sum_{i=1}^{n} \pi_i a_{i0} u_i^{s-2} z_i^2 + s(s+1) \sum_{i=1}^{n} \pi_i \left(a_{ii} - \frac{s-1}{s+1} \sum_{j=1,\, j\neq i}^{n} a_{ij} \right) u_i^{2(s-1)} z_i^2,$$

from which we conclude the result. □

Remark 6.9. *Let $n = 2$. Then the condition $\eta_1 \geq 0$ on the coefficients (a_{ij}) becomes $a_{11} \geq a_{12}(s-1)/(s+1)$ and $a_{22} \geq a_{21}(s-1)/(s+1)$. The product*

$$a_{11} a_{22} \geq \left(\frac{s-1}{s+1} \right)^2 a_{12} a_{21}$$

is the same as the condition imposed in [51, Section 5.1] but weaker than

$$a_{11}a_{22} \geq \left(\frac{s-1}{s}\right)^2 a_{12}a_{21},$$

which was needed in [94, Lemma 11]. Furthermore, under the slightly stronger condition $\eta_1 > 0$*, that is*

$$a_{11}a_{22} > \left(\frac{s-1}{s+1}\right)^2 a_{12}a_{21},$$

our weak solution satisfies the stronger estimate $u_i^s \in L^2(0,T;H^1(\Omega))$ *than that in [51, Section 5.1].* □

Lemma 6.10 (Non detailed balance)**.** *Let* $s > 1$ *and let*

$$\eta_2 := \min_{i=1,\ldots,n}\left(a_{ii} - \frac{1}{2(s+1)}\sum_{j=1,\,j\neq i}\big(s(a_{ij}+a_{ji}) - 2\sqrt{a_{ij}a_{ji}}\big)\right) \geq 0.$$

Then $H(u)A(u)$ *is positive definite. Moreover, if* $\eta_2 > 0$*, then, for all* $z \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^n_+$*,*

$$z^\top H(u)A(u)z \geq s\sum_{i=1}^n a_{i0}u_i^{s-2}z_i^2 + \eta_2 s(s+1)\sum_{i=1}^n u_i^{2(s-1)}z_i^2.$$

Proof. We choose $\pi_i = 1$ for $i = 1,\ldots,n$. Then, as in the previous proof,

$$I_3 + I_4 \geq -\frac{s}{2}\sum_{i,j=1,\,i\neq j}^n \big(s(a_{ij}+a_{ji}) - 2\sqrt{a_{ij}a_{ji}}\big)u_i^{2(s-1)}z_i^2$$

and

$$\begin{aligned}
z^\top H(u)A(u)z &\geq s\sum_{i=1}^n a_{i0}u_i^{s-2}z_i^2 + s(s+1)\sum_{i=1}^n a_{ii}u_i^{2(s-1)}z_i^2 \\
&\quad - \frac{s}{2}\sum_{i,j=1,\,i\neq j}^n \big(s(a_{ij}+a_{ji}) - 2\sqrt{a_{ij}a_{ji}}\big)u_i^{2(s-1)}z_i^2 \\
&= s\sum_{i=1}^n a_{i0}u_i^{s-2}z_i^2 \\
&\quad + s(s+1)\sum_{i=1}^n\left(a_{ii} - \frac{1}{2(s+1)}\sum_{i,j=1,\,i\neq j}^n \big(s(a_{ij}+a_{ji}) - 2\sqrt{a_{ij}a_{ji}}\big)\right)u_i^{2(s-1)}z_i^2.
\end{aligned}$$

By definition of η_2, the result follows. □

6.2.3. Approximate matrices

Our theory requires that the range of the derivative h' equals $\mathbb{R}^n$. Since this is not the case if $s \neq 1$, we need to approximate the entropy density and consequently also the diffusion matrix. The approximate entropy density

$$h_\varepsilon(u) = h(u) + \varepsilon \sum_{i=1}^n \big(u_i(\log u_i - 1) + 1\big) \tag{6.2.4}$$

possesses the property that the range of its derivative is $\mathbb{R}^n$. We set $H(u) = h''(u) = (\delta_{ij} s\pi_i u_i^{s-2})_{i,j=1,\ldots,n}$ for its Hessian and

$$\begin{aligned} H_\varepsilon(u) &= H(u) + \varepsilon H^0(u), \quad H^0_{ij}(u) = \delta_{ij} u_i^{-1}, \\ A_\varepsilon(u) &= A(u) + \varepsilon A^0(u) + \varepsilon^\eta A^1(u), \end{aligned} \tag{6.2.5}$$

where $\eta < 1/2$ and

$$\begin{aligned} A^0_{ij}(u) &= \delta_{ij} \frac{u_i}{\pi_i}\mu_i - (1-\delta_{ij})\frac{u_i}{\pi_i}a_{ji}, \quad A^1_{ij}(u) = \delta_{ij} u_i, \\ \mu_i &:= \frac{\pi_i}{2}\sum_{j=1,\, j\neq i}^n \left(\frac{a_{ji}}{\pi_i} + \frac{a_{ij}}{\pi_j}\right), \quad i = 1,\ldots,n. \end{aligned}$$

The approximation $\varepsilon^\eta A^1(u)$ is needed to achieve bounds for $\varepsilon^{(\eta+1)/2}\nabla u_i$ in L^2, which are necessary for the limit $\varepsilon \to 0$. The off-diagonal terms in $A^0(u)$ are needed to preserve the entropy structure in the sense that $H_\varepsilon(u)A_\varepsilon(u)$ is still positive definite. This is shown in the following lemma.

Lemma 6.11. *Let $s > 0$. Then, for all $z \in \mathbb{R}^n$ and $u \in \mathbb{R}^n_+$,*

$$z^\top H_\varepsilon(u)A_\varepsilon(u)z \geq z^\top H(u)A(u)z + \varepsilon^\eta s \sum_{i=1}^n \pi_i u_i^{s-1} z_i^2 + \varepsilon^{\eta+1}\sum_{i=1}^n z_i^2.$$

Proof. We decompose the product $H_\varepsilon(u)A_\varepsilon(u)$ as

$$\begin{aligned} H_\varepsilon(u)A_\varepsilon(u) = {} & H(u)A(u) + \varepsilon^\eta H_\varepsilon(u)A^1(u) + \varepsilon\big(H^0(u)A(u) + H(u)A^0(u)\big) \\ & + \varepsilon^2 H^0(u)A^0(u). \end{aligned}$$

The ε^2-term becomes

$$\begin{aligned} (H^0(u)A^0(u))_{ij} &= \sum_{k=1}^n \delta_{ik} u_k^{-1}\left(\delta_{kj}\frac{u_k}{\pi_k}\mu_k - (1-\delta_{kj})\frac{u_k}{\pi_k}a_{jk}\right) \\ &= \delta_{ij}\frac{\mu_i}{\pi_i} - (1-\delta_{ij})\frac{a_{ji}}{\pi_i}. \end{aligned}$$

We obtain for $z \in \mathbb{R}^n$:

$$z^\top H^0(u)A^0(u)z = \sum_{i=1}^n \frac{\mu_i}{\pi_i}z_i^2 - \sum_{i,j=1,\, i\neq j}^n \frac{a_{ji}}{\pi_i}z_i z_j$$

$$
\begin{aligned}
&\geq \sum_{i=1}^n \frac{\mu_i}{\pi_i} z_i^2 - \frac{1}{2} \sum_{i,j=1,\, i\neq j}^n \frac{a_{ji}}{\pi_i}(z_i^2 + z_j^2) \\
&= \sum_{i=1}^n \frac{\mu_i}{\pi_i} z_i^2 - \frac{1}{2} \sum_{i=1}^n \bigg(\sum_{j=1,\, j\neq i}^n \frac{a_{ji}}{\pi_i} \bigg) z_i^2 - \frac{1}{2} \sum_{i=1}^n \bigg(\sum_{j=1,\, j\neq i}^n \frac{a_{ij}}{\pi_j} \bigg) z_i^2 \\
&= 0.
\end{aligned}
$$

Next, we consider the ε-terms:

$$
\begin{aligned}
(H^0(u)A(u))_{ij} &= \sum_{k=1}^n \delta_{ik} u_i^{-1} \bigg(\delta_{kj} \bigg(a_{k0} + \sum_{\ell=1}^n a_{k\ell} u_\ell^s + s a_{kk} u_k^s \bigg) + (1-\delta_{kj}) s a_{kj} u_j^{s-1} u_k \bigg) \\
&= \delta_{ij} \bigg(a_{i0} u_i^{-1} + \sum_{\ell=1}^n a_{i\ell} u_\ell^s u_i^{-1} + s a_{ii} u_i^{s-1} \bigg) + (1-\delta_{ij}) s a_{ij} u_j^{s-1}, \\
(H(u)A^0(u))_{ij} &= \sum_{k=1}^n \delta_{ik} s \pi_i u_i^{s-2} \bigg(\delta_{kj} \frac{u_k}{\pi_k} \mu_k - (1-\delta_{kj}) \frac{u_k}{\pi_k} a_{jk} \bigg) \\
&= \delta_{ij} s u_i^{s-1} \mu_i - (1-\delta_{ij}) s a_{ji} u_i^{s-1}.
\end{aligned}
$$

Summing these expressions and neglecting some positive contributions, we find that

$$
\begin{aligned}
z^\top \big(H^0(u)A(u) + H(u)A^0(u) \big) z &\geq \sum_{i=1}^n (a_{i0} u_i^{-1} + s a_{ii} u_i^{s-1}) z_i^2 \\
&\quad + s \sum_{i,j=1}^n (1-\delta_{ij}) a_{ij} u_j^{s-1} z_i z_j - s \sum_{i,j=1}^n (1-\delta_{ij}) a_{ji} u_i^{s-1} z_i z_j \\
&= \sum_{i=1}^n \big(a_{i0} u_i^{-1} + s a_{ii} u_i^{s-1} \big) z_i^2 \geq s \sum_{i=1}^n a_{ii} u_i^{s-1} z_i^2.
\end{aligned}
$$

Here we see how we constructed $A_{ij}^0(u)$: The off-diagonal coefficients are chosen in such a way that the mixed terms in $z_i z_j$ cancel, and the diagonal elements (namely μ_i) are sufficiently large to obtain positive definiteness of $H^0(u)A^0(u)$. Finally, we have $(H_\varepsilon(u)A^1(u))_{ij} = \delta_{ij}(s\pi_i u_i^{s-1} + \varepsilon)$ and

$$
z^\top H_\varepsilon(u) A^1(u) z = \sum_{i=1}^n (s\pi_i u_i^{s-1} + \varepsilon) z_i^2,
$$

which proves the lemma. □

6.3. Linear transition rates: proof of Theorem 6.1

In this section, we prove Theorem 6.1. Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, $\varepsilon > 0$, and $m \in \mathbb{N}$ with $m > d/2$. This ensures that the embedding $H^m(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact. We assume that $u_i^0(x) \in [a,b]$ for $x \in \Omega$, $i = 1, \ldots, n$, where $0 < a < b < \infty$. Then, clearly, $w^0 =$

$h'(u^0) \in L^\infty(\Omega;\mathbb{R}^n)$. For general $u_i^0 \geq 0$, we may first consider $u_\varepsilon^0 = (Q_\varepsilon(u_1^0), \ldots, Q_\varepsilon(u_n^0))$, where $0 < \varepsilon < 1$ and Q_ε is the cut-off function

$$Q_\varepsilon(z) = \begin{cases} \varepsilon & \text{for } 0 \leq z < \varepsilon, \\ z & \text{for } \varepsilon \leq z < \varepsilon^{-1/2}, \\ \varepsilon^{-1/2} & \text{for } z \geq \varepsilon^{-1/2}, \end{cases}$$

and then pass to the limit $\varepsilon \to 0$. We leave the details to the reader.

Step 1: solution of an approximated problem. Given $w^{k-1} \in L^\infty(\Omega;\mathbb{R}^n)$ for $k \in \mathbb{N}$, we wish to find $w^k \in H^m(\Omega;\mathbb{R}^n)$ such that

$$\begin{aligned} &\frac{1}{\tau}\int_\Omega \big(u(w^k) - u(w^{k-1})\big)\cdot \phi dx + \int_\Omega \nabla\phi : B(w^k)\nabla w^k dx \\ &\quad + \varepsilon\int_\Omega \Big(\sum_{|\alpha|=m} D^\alpha w^k \cdot D^\alpha \phi + w^k\cdot\phi\Big)dx = \int_\Omega f(u(w^k))\cdot\phi dx \end{aligned} \tag{6.3.1}$$

for all $\phi \in H^m(\Omega;\mathbb{R}^n)$. Here, $u(w^k) = (h')^{-1}(w^k)$, $B(w^k) = A(u(w^k))H(u(w^k))^{-1}$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \ldots + \alpha_d = m$ is a multiindex, and $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d})$ is a partial derivative of order m. If $k = 1$, we define $w^0 = h'(u^0)$. Equation (6.3.1) is an implicit Euler discretization of (6.1.1) including an H^m regularization term.

We recall that the entropy is given by

$$\mathcal{H}[u] = \int_\Omega h(u)dx = \int_\Omega \sum_{i=1}^n \pi_i h_1(u_i)dx, \quad h_1(u_i) = u_i(\log u_i - 1) + 1.$$

Then the entropy variables equal $w_i = \partial h/\partial u_i = \pi_i \log u_i$. In particular, $h' : \mathbb{R}_+^n \to \mathbb{R}^n$ is invertible on $\mathbb{R}^n$, i.e., Hypothesis (H1) in [94] is satisfied. By Lemmas 6.4 and 6.6, $H(u)A(u)$ is positive definite, i.e., Hypothesis (H2) in [94] holds as well. (At this step, we only need that $H(u)A(u)$ is positive semi-definite.) Furthermore, f_i grows at most linearly which implies that

$$\sum_{i=1}^n f_i(u)\pi_i \log u_i \leq C_f(1 + h(u)),$$

where $C_f > 0$ depends only on (b_{ij}) and π. This means that Hypothesis (H3) in [94] is also satisfied. Thus, we can apply Lemma 5 in [94] giving a weak solution $w^k \in H^m(\Omega;\mathbb{R}^n)$ to (6.3.1) satisfying the discrete entropy inequality

$$\begin{aligned} &(1 - C_f\tau)\int_\Omega h(u(w^k))dx + \tau\int_\Omega \nabla w^k : B(w^k)\nabla w^k dx \\ &\quad + \varepsilon\tau\int_\Omega \Big(\sum_{|\alpha|=m} |D^\alpha w^k|^2 + |w^k|^2\Big)dx \leq \int_\Omega h(u(w^{k-1}))dx + C_f\tau\text{meas}(\Omega). \end{aligned} \tag{6.3.2}$$

Step 2: uniform estimates. We set $u^k = u(w^k)$ and introduce the piecewise in time constant functions $w^{(\tau)}(x,t) = w^k(x)$ and $u^{(\tau)}(x,t) = u^k(x)$ for $x \in \Omega$, $t \in ((k-1)\tau, k\tau]$.

At time $t=0$, we set $w^{(\tau)}(\cdot,0)=h'(u^0)=w^0$ and $u^{(\tau)}(\cdot,0)=u^0$. Let $u^{(\tau)}=(u_1^{(\tau)},\ldots,u_n^{(\tau)})$. We define the backward shift operator $(\sigma_\tau u^{(\tau)})(x,t)=u(w^{k-1}(x))$ for $x\in\Omega$, $t\in((k-1)\tau,k\tau]$. Then $u^{(\tau)}$ solves

$$\frac{1}{\tau}\int_0^T\int_\Omega(u^{(\tau)}-\sigma_\tau u^{(\tau)})\cdot\phi dxdt+\int_0^T\int_\Omega\nabla\phi:B(w^{(\tau)})\nabla w^{(\tau)}dxdt$$
$$+\varepsilon\int_0^T\int_\Omega\Big(\sum_{|\alpha|=m}D^\alpha w^{(\tau)}\cdot D^\alpha\phi+w^{(\tau)}\cdot\phi\Big)dxdt=\int_0^T\int_\Omega f(u^{(\tau)})\cdot\phi dxdt \tag{6.3.3}$$

for piecewise constant functions $\phi:(0,T)\to H^m(\Omega;\mathbb{R}^n)$. By a density argument, this equation also holds for all $\phi\in L^2(0,T;H^m(\Omega;\mathbb{R}^n))$ [126, Prop. 1.36].

By Lemmas 6.4 and 6.6, we have

$$\nabla w^k:B(w^k)\nabla w^k=\nabla u^k:H(u^k)A(u^k)\nabla u^k\geq 2\eta_0\sum_{i=1}^n|\nabla u_i^k|^2,$$

where $\eta_0=\min_{i=1,\ldots,n}\pi_i a_{ii}>0$ if detailed balance holds, and $\eta_0>0$ is given by (6.1.11) otherwise. By the generalized Poincaré inequality [132, Chapter 2, Section 1.4], it holds that

$$\int_\Omega\Big(\sum_{|\alpha|=m}|D^\alpha w^k|^2+|w^k|^2\Big)dx\geq C_P\|w^k\|^2_{H^m(\Omega)},$$

where $C_P>0$ is the Poincaré constant. Then the discrete entropy inequality (6.3.2) gives

$$(1-C_f\tau)\int_\Omega h(u^k)dx+2\eta_0\tau\int_\Omega|\nabla u^k|^2dx+\varepsilon C_P\tau\|w^k\|^2_{H^m(\Omega)}$$
$$\leq\int_\Omega h(u^{k-1})dx+C_f\tau\text{meas}(\Omega).$$

Summing these inequalities over $k=1,\ldots,j$, it follows that

$$(1-C_f\tau)\int_\Omega h(u^j)dx+2\eta_0\tau\sum_{j=1}^k\int_\Omega|\nabla u^k|^2dx+\varepsilon C_P\tau\sum_{j=1}^k\|w^k\|^2_{H^m(\Omega)}$$
$$\leq\int_\Omega h(u^0)dx+C_f\tau\sum_{k=1}^{j-1}\int_\Omega h(u^k)dx+C_fT\text{meas}(\Omega).$$

By the discrete Gronwall inequality [42], if $\tau<1/C_f$,

$$\int_\Omega h(u^j)dx+\tau\sum_{j=1}^k\int_\Omega|\nabla u^k|^2dx+\varepsilon\tau\sum_{j=1}^k\|w^k\|^2_{H^m(\Omega)}\leq C,$$

where here and in the following, $C>0$ denotes a generic constant independent of τ and ε. Then, observing that the entropy density dominates the L^1 norm and consequently, $u^{(\tau)}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega;\mathbb{R}^n))$, we obtain

$$\|u^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}+\|u^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}+\varepsilon^{1/2}\|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))}\leq C. \tag{6.3.4}$$

We wish to derive more a priori estimates. Set $Q_T = \Omega\times(0,T)$. The Gagliardo-Nirenberg inequality with $p = 2 + 2/d$ and $\theta = 2d(p-1)/(dp+2p) \in [0,1]$ (such that $\theta p = 2$) yields for $i = 1,\dots,n$,

$$\begin{aligned}\|u_i^{(\tau)}\|_{L^p(Q_T)}^p &= \int_0^T \|u_i^{(\tau)}\|_{L^p(\Omega)}^p dt \le C\int_0^T \|u_i^{(\tau)}\|_{H^1(\Omega)}^{\theta p}\|u_i^{(\tau)}\|_{L^1(\Omega)}^{(1-\theta)p} dt \\ &\le C\|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}^{(1-\theta)p}\|u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}^{\theta p} \le C. \end{aligned} \tag{6.3.5}$$

In order to apply a compactness result, we need a uniform estimate for the discrete time derivative of $u^{(\tau)}$. Let $q = 2(d+1)$ and $\phi \in L^q(0,T;W^{m,q}(\Omega;\mathbb{R}^n))$. Then $1/p+1/q+1/2 = 1$ and, by Hölder's inequality,

$$\begin{aligned}\frac{1}{\tau}\left|\int_0^T\int_\Omega (u^{(\tau)} - \sigma_\tau u^{(\tau)})\cdot\phi dxdt\right| \le \sum_{i,j=1}^n &\|A_{ij}(u^{(\tau)})\|_{L^p(Q_T)}\|\nabla u_j^{(\tau)}\|_{L^2(Q_T)}\|\nabla\phi_i\|_{L^q(Q_T)} \\ &+ \varepsilon\|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))}\|\phi\|_{L^2(0,T;H^m(\Omega))} \\ &+ \|f(u^{(\tau)})\|_{L^{q'}(Q_T)}\|\phi\|_{L^q(Q_T)},\end{aligned}$$

where $q' = (2d+2)/(2d+1)$. Estimate (6.3.5) and the linear growth of $A_{ij}(u^{(\tau)})$ with respect to $u^{(\tau)}$ show that the first term on the right-hand side is bounded. The second term is bounded because of (6.3.4). Finally, $|f_i(u^{(\tau)})|$ is growing at most like $(u_i^{(\tau)})^2$ such that

$$\|f(u^{(\tau)})\|_{L^{q'}(Q_T)} \le C\big(1 + \|u^{(\tau)}\|_{L^{2q'}(Q_T)}^2\big) \le C,$$

since $2q' \le p$. We conclude that

$$\tau^{-1}\|u^{(\tau)} - \sigma_\tau u^{(\tau)}\|_{L^{q'}(0,T;W^{m,q}(\Omega)')} \le C. \tag{6.3.6}$$

Step 3: the limit $(\varepsilon,\tau)\to 0$. In view of (6.3.4) and (6.3.6), we can apply the Aubin-Lions lemma in the version of [61], which yields the existence of a subsequence, which is not relabeled, such that, as $(\tau,\varepsilon)\to 0$,

$$u^{(\tau)} \to u \quad \text{strongly in } L^2(Q_T) \text{ and a.e.,} \tag{6.3.7}$$

$$u^{(\tau)} \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{6.3.8}$$

$$\varepsilon w^{(\tau)} \to 0 \quad \text{strongly in } L^2(0,T;H^m(\Omega)), \tag{6.3.9}$$

$$\tau^{-1}(u^{(\tau)} - \sigma_\tau u^{(\tau)}) \rightharpoonup \partial_t u \quad \text{weakly in } L^{q'}(0,T;W^{m,q}(\Omega)'), \tag{6.3.10}$$

where $u = (u_1,\dots,u_n)$. In view of the a.e. convergence (6.3.7) and the uniform bound (6.3.5), we have

$$u^{(\tau)} \to u \quad \text{strongly in } L^\gamma(Q_T) \text{ for all } \gamma < 2 + 2/d. \tag{6.3.11}$$

Then, together with (6.3.8),

$$u_i^{(\tau)}\nabla u_j^{(\tau)} \rightharpoonup u_i\nabla u_j \quad \text{weakly in } L^1(Q_T).$$

We deduce from the $L^{q'}(Q_T)$ bound for $A(u^{(\tau)})\nabla u^{(\tau)}$ that

$$B(w^{(\tau)})\nabla w^{(\tau)} = A(u^{(\tau)})\nabla u^{(\tau)} \rightharpoonup A(u)\nabla u \quad \text{weakly in } L^{q'}(Q_T).$$

Furthermore, taking into account (6.3.11) and the uniform bound for $f_i(u^{(\tau)})$ in $L^{q'}(Q_T)$,

$$f_i(u^{(\tau)}) \rightharpoonup f_i(u) \quad \text{weakly in } L^{q'}(Q_T).$$

Then (6.3.9) and (6.3.10) allow us to perform the limit $(\varepsilon, \tau) \to 0$ in (6.3.3) with $\phi \in L^q(0,T;\, W^{m,q}(\Omega))$, which directly yields (6.1.12). Since $\partial_t u = \operatorname{div}(A(u)\nabla u) + f(u) \in L^{q'}(0,T; W^{1,q}(\Omega)')$, a density argument shows that the weak formulation holds for all $\phi \in L^q(0,T; W^{1,q}(\Omega))$. Moreover, $u_i \in W^{1,q'}(0,T; W^{1,q}(\Omega)') \hookrightarrow C^0([0,T]; W^{1,q}(\Omega)')$, which shows that the initial condition is satisfied in $W^{1,q}(\Omega)'$. This ends the proof.

Remark 6.12 (Detailed balance and vanishing self-diffusion)**.** *In the case of detailed balance, we may allow for vanishing self-diffusion. If $a_{ii} = 0$ but $a_{i0} > 0$, Lemma 6.4 implies that only $\nabla(u_i^{(\tau)})^{1/2}$ is bounded in $L^2(Q_T)$. This situation was considered in [38] for the two-species case, and we sketch the generalization to the n-species case.*

Applying the Gagliardo-Nirenberg inequality similarly as in Step 2 of the previous proof, we conclude that $(u_i^{(\tau)})^{1/2} \in L^{\widetilde{p}}(Q_T)$ with $\widetilde{p} = 2 + 4/d$. Then

$$\|\nabla u_i^{(\tau)}\|_{L^{\widetilde{q}}(Q_T)} = 2\|(u_i^{(\tau)})^{1/2}\|_{L^{\widetilde{p}}(Q_T)}\,\|\nabla(u_i^{(\tau)})^{1/2}\|_{L^2(Q_T)} \le C, \quad \widetilde{q} = \frac{d+2}{d+1},$$

and thus, $(u_i^{(\tau)})$ is bounded in $L^{\widetilde{q}}(0,T; W^{1,\widetilde{q}}(\Omega))$ instead of $L^2(0,T; H^1(\Omega))$. This loss of regularity is problematic for the estimate of the discrete time derivative of $u_i^{(\tau)}$. In order to compensate this, we need the last sum in (6.2.3). Indeed, Remark 6.5 shows that for any $i \ne j$, $(u_i^{(\tau)}u_j^{(\tau)})^{1/2}$ is bounded in $L^2(0,T; H^1(\Omega))$. Moreover, $(u_i^{(\tau)}u_j^{(\tau)})^{1/2}$ is bounded in $L^\infty(0,T; L^1(\Omega))$. We infer from the Gagliardo-Nirenberg inequality that $(u_i^{(\tau)}u_j^{(\tau)})^{1/2}$ is bounded in $L^p(Q_T)$ with $p = 2 + 2/d$.

Next we exploit the structure of the equations,

$$\sum_{j=1}^n A_{ij}(u^{(\tau)})\nabla u_j^{(\tau)} = \nabla(u_i^{(\tau)}p_i(u^{(\tau)})), \quad p_i(u^{(\tau)}) = a_{i0} + \sum_{j=1}^n a_{ij}u_j^{(\tau)}.$$

Thus, to show that $A_{ij}(u^{(\tau)})\nabla u_j^{(\tau)}$ is bounded, we only need to verify that $\nabla(u_i^{(\tau)}u_j^{(\tau)})$ is bounded:

$$\|\nabla(u_i^{(\tau)}u_j^{(\tau)})\|_{L^{q'}(Q_T)} \le 2\|(u_i^{(\tau)}u_j^{(\tau)})^{1/2}\|_{L^p(Q_T)}\|\nabla(u_i^{(\tau)}u_j^{(\tau)})^{1/2}\|_{L^2(Q_T)} \le C,$$

where $q' = (2d{+}2)/(2d{+}1)$. The estimate for the Lotka-Volterra term is more delicate since we have only the regularity $u_i^{(\tau)} \in L^{1+1/d}(Q_T)$. Here, we need to suppose that $b_{ii} > 0$, since this assumption provides an estimate for $(u_i^{(\tau)})^2 \log u_i^{(\tau)}$ in $L^1(Q_T)$. Then the discrete time derivative of $u_i^{(\tau)}$ is bounded in $L^1(0,T; W^{m,q}(\Omega)')$ – but not in $L^{q'}(0,T; W^{m,q}(\Omega)')$. By the Aubin-Lions lemma, there exists a subsequence (not relabeled) such that, as $(\varepsilon, \tau) \to 0$,

$$u_i^{(\tau)} \to u_i \quad \textit{strongly in } L^{q'}(Q_T).$$

The problem now is to show that (a subsequence of) the discrete time derivative of $u_i^{(\tau)}$ converges to $\partial_t u_i$ since $L^1(0,T;W^{m,q}(\Omega)')$ is not reflexive. The idea is to apply a result from [142] which provides a criterium for weak compactness in $L^1(0,T;X)$, where X is a reflexive Banach space. For details, we refer to [38]. □

6.4. Nonlinear transition rates: proof of Theorem 6.2

The strategy of the proof is similar to the proof of Theorem 6.1 but the nonlinear transition rates complicate the proof significantly. As outlined in Section 6.2.3, we approximate the entropy density by (6.2.4) and the diffusion matrix by (6.2.5). Again, we assume without loss of generality that $u_i^0(x) \in [a,b]$ for $x \in \Omega$, $i = 1,\ldots,n$, where $0 < a < b < \infty$.

Step 1: solution of an approximated problem. We employ the transformation $w_i = \partial h_\varepsilon/\partial u_i$ and define $B_\varepsilon(w) = A_\varepsilon(u(w))H_\varepsilon(u(w))^{-1}$. Given $w^{k-1} \in L^\infty(\Omega;\mathbb{R}^n)$, we wish to find $w^k \in H^m(\Omega;\mathbb{R}^n)$ solving

$$\frac{1}{\tau}\int_\Omega \big(u(w^k) - u(w^{k-1})\big)\cdot \phi dx + \int_\Omega \nabla\phi : B_\varepsilon(w^k)\nabla w^k dx \\ + \varepsilon\int_\Omega\bigg(\sum_{|\alpha|=m} D^\alpha w^k \cdot D^\alpha\phi + w^k\cdot\phi\bigg)dx = \int_\Omega f(u(w^k))\cdot\phi dx \tag{6.4.1}$$

for all $\phi \in H^m(\Omega;\mathbb{R}^n)$. If $k = 1$, we define $w^0 = h_\varepsilon'(u^0)$ such that $u(w^0) = u^0$.

The construction of h_ε ensures that Hypothesis (H1) of [94] is satisfied. By Lemma 6.11, Hypothesis (H2) holds as well. Also Hypothesis (H3) holds true since, for some $C_f > 0$,

$$f(u)\cdot w = \sum_{I=1}^n \bigg(b_{i0} - \sum_{j=1}^n b_{ij}u_j^\sigma\bigg)(su_i^s + \varepsilon u_i \log u_i) \le C_f(1 + h_\varepsilon(u)),$$

where $\sigma = 1$ if $s > 1$ and $0 \le \sigma \le \max\{0, 2s-1+2/d\}$ if $s < 1$. We apply Lemma 5 in [94] to deduce the existence of a weak solution $w^k \in H^m(\Omega;\mathbb{R}^n)$ to the above problem, which satisfies the discrete entropy inequality

$$(1 - C_f\tau)\int_\Omega h_\varepsilon(u(w^k))dx + \tau\int_\Omega \nabla w^k : B_\varepsilon(w^k)\nabla w^k dx \\ + \varepsilon\tau\int_\Omega\bigg(\sum_{|\alpha|=m}|D^\alpha w^k|^2 + |w^k|^2\bigg)dx \le \int_\Omega h_\varepsilon(u(w^{k-1}))dx + C_f\tau\mathrm{meas}(\Omega). \tag{6.4.2}$$

Setting $u^k := u(w^k)$ and employing Lemma 6.11, the second integral can be estimated as follows:

$$\int_\Omega \nabla w^k : B_\varepsilon(w^k)\nabla w^k dx = \int_\Omega u^k : H_\varepsilon(u^k)A_\varepsilon(u^k)\nabla u^k dx \\ \ge s(s+1)\int_\Omega \sum_{i=1}^n \min\{a_{ii}\pi_i, \eta_0, \eta_1\pi_i, \eta_2\}(u_i^k)^{2(s-1)}|\nabla u_i^k|^2 dx$$

$$
\begin{aligned}
&+ \varepsilon^\eta s \int_\Omega \sum_{i=1}^n \pi_i (u_i^k)^{s-1} |\nabla u_i^k|^2 dx + \varepsilon^{\eta+1} \int_\Omega \sum_{i=1}^n |\nabla u_i^k|^2 dx \\
&\geq C_s \int_\Omega \sum_{i=1}^n |\nabla (u_i^k)^s|^2 dx \\
&\quad + \frac{4\varepsilon^\eta s}{(s+1)^2} \int_\Omega \sum_{i=1}^n \pi_i |\nabla (u_i^k)^{(s+1)/2}|^2 dx + \varepsilon^{\eta+1} \int_\Omega \sum_{i=1}^n |\nabla u_i^k|^2 dx,
\end{aligned}
\tag{6.4.3}
$$

where $C_s = s^{-1}(s+1)\min\{a_{11}\pi_1, \ldots, a_{nn}\pi_n, \eta_0, \eta_1\pi_1, \ldots, \eta_1\pi_n, \eta_2\}$.

To finish this step, we wish to write the "very weak" formulation for the solution $u^{(\tau)}$, which is defined from u^k as in the previous section. First, we observe that

$$
\begin{aligned}
(B_\varepsilon(w^k)\nabla w^k)_i &= (A_\varepsilon(u^k)\nabla u^k)_i = \varepsilon(A^0(u^k)\nabla u^k)_i + \varepsilon^\eta (A^1(u^k)\nabla u^k)_i + \nabla(u_i^k p_i(u^k)) \\
&= \varepsilon(A^0(u^k)\nabla u^k)_i + \frac{\varepsilon^\eta}{2}\nabla(u_i^k)^2 + \nabla(u_i^k p_i(u^k)).
\end{aligned}
$$

Next, we choose a test function $\phi = (\phi_1, \ldots, \phi_n) \in L^q(0,T;\, W_\nu^{m,q}(\Omega))$, where $m > \max\{1, d/2\}$ and $q \geq 2$ will be determined below. Recall that $W_\nu^{m,q}(\Omega)$ is defined in (6.1.16). Integrating by parts in (6.4.1), $u^{(\tau)}$ solves

$$
\begin{aligned}
&\frac{1}{\tau}\int_0^T \int_\Omega \big(u^{(\tau)} - \sigma_\tau u^{(\tau)}\big) \cdot \phi dx dt - \int_0^T \int_\Omega \sum_{i=1}^n u_i^{(\tau)} p_i(u^{(\tau)}) \Delta \phi_i dx dt \\
&+ \varepsilon \int_0^T \int_\Omega \nabla \phi : A^0(u^{(\tau)}) \nabla u^{(\tau)} dx dt - \frac{\varepsilon^\eta}{2} \int_0^T \int_\Omega \sum_{i=1}^n (u_i^{(\tau)})^2 \Delta \phi_i dx dt \\
&+ \varepsilon \int_0^T \int_\Omega \Big(\sum_{|\alpha|=m} D^\alpha w^{(\tau)} \cdot D^\alpha \phi + w^{(\tau)} \cdot \phi \Big) dx dt = \int_0^T \int_\Omega f(u^{(\tau)}) \cdot \phi dx dt.
\end{aligned}
\tag{6.4.4}
$$

Step 2: uniform estimates. Arguing as in Step 2 of the proof of Theorem 6.1, we obtain from (6.4.2) and (6.4.3) for suffiently small $\tau > 0$ the following uniform estimates.

Lemma 6.13. *It holds for $i = 1, \ldots, n$ that*

$$
\|u_i^{(\tau)}\|_{L^\infty(0,T;L^{\max\{1,s\}}(\Omega))} + \|(u_i^{(\tau)})^s\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{6.4.5}
$$

$$
\varepsilon^{\eta/2} \|(u_i^{(\tau)})^{(s+1)/2}\|_{L^2(0,T;H^1(\Omega))} + \varepsilon^{(\eta+1)/2} \|u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{6.4.6}
$$

$$
\varepsilon^{1/2} \|w_i^{(\tau)}\|_{L^2(0,T;H^m(\Omega))} \leq C. \tag{6.4.7}
$$

Here, we used the fact that $\int_\Omega h_\varepsilon(u^0) dx$ is uniformly bounded and that $s < 1$ implies that $u^{(\tau)} \leq C(1 + h(u^{(\tau)}))$ for some $C > 0$, from which we deduce that $(u_i^{(\tau)})$ is bounded in $L^\infty(0,T;L^1(\Omega))$. We need more a priori estimates.

Lemma 6.14. *Let $s > \max\{0, 1 - 2/d\}$. It holds that*

$$
\|u^{(\tau)}\|_{L^{p(s)}(Q_T)} + \varepsilon^{\eta/r(s)} \|u^{(\tau)}\|_{L^{r(s)}(Q_T)} \leq C, \tag{6.4.8}
$$

where $p(s) = 2s + (2/d)\max\{1, s\}$ and $r(s) = s + 1 + (2/d)\max\{1, s\} > 2$.

Proof. The estimates are consequences of Lemma 6.13 and the Gagliardo-Nirenberg inequality. First, let $s < 1$. We employ the Gagliardo-Nirenberg inequality, with $\theta = ds/(ds+1) \in (0,1)$:

$$\begin{aligned}\|u_i^{(\tau)}\|_{L^{p(s)}(Q_T)}^{p(s)} &= \int_0^T \|(u_i^{(\tau)})^s\|_{L^{p(s)/s}(\Omega)}^{p(s)/s} dt \le C\int_0^T \|(u_i^{(\tau)})^s\|_{H^1(\Omega)}^{\theta p(s)/s}\|(u_i^{(\tau)})^s\|_{L^{1/s}(\Omega)}^{(1-\theta)p(s)/s} dt \\ &\le C\|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}^{(1-\theta)p(s)} \int_0^T \|(u_i^{(\tau)})^s\|_{H^1(\Omega)}^{\theta p(s)/s} dt, \quad i=1,\ldots,n.\end{aligned}$$

It holds that $\theta p(s)/s = 2$. By (6.4.5), $\|u^{(\tau)}\|_{L^{p(s)}(Q_T)} \le C$.

Next, let $s > 1$. Then, with $\theta = d/(d+1) \in (0,1)$,

$$\begin{aligned}\|(u_i^{(\tau)})^s\|_{L^{2+2/d}(Q_T)}^{2+2/d} &\le C\int_0^T \|(u_i^{(\tau)})^s\|_{H^1(\Omega)}^{\theta(2+2/d)}\|(u_i^{(\tau)})^s\|_{L^1(\Omega)}^{(1-\theta)(2+2/d)} dt \\ &\le C\|(u_i^{(\tau)})^s\|_{L^2(0,T;H^1(\Omega)}^2 \|u_i^{(\tau)}\|_{L^\infty(0,T;L^s(\Omega))}^{s(1-\theta)(2+2/d)} \le C,\end{aligned}$$

again taking into account estimate (6.4.5). This shows that $(u^{(\tau)})$ is bounded in $L^{p(s)}(Q_T)$.

Finally, let $\max\{0, 1-2/d\} < s < 1$. Then $r(s) = s+1+2/d$. We apply the Gagliardo-Nirenberg inequality with $\theta = d(s+1)/(2+d(s+1)) \in (0,1)$ such that $\theta \cdot 2r(s)/(s+1) = 2$,

$$\begin{aligned}\varepsilon^\eta \|u_i^{(\tau)}\|_{L^{r(s)}(Q_T)}^{r(s)} &= \varepsilon^\eta \|(u_i^{(\tau)})^{(s+1)/2}\|_{L^{2r(s)/(s+1)}(Q_T)}^{2r(s)/(s+1)} \\ &\le \varepsilon^\eta C\int_0^T \|(u_i^{(\tau)})^{(s+1)/2}\|_{H^1(\Omega)}^{2r(s)\theta/(s+1)} \|(u_i^{(\tau)})^{(s+1)/2}\|_{L^{2/(s+1)}(\Omega)}^{2r(s)(1-\theta)/(s+1)} dt \\ &\le C\varepsilon^\eta \|(u_i^{(\tau)})^{(s+1)/2}\|_{L^2(0,T;H^1(\Omega))}^2 \|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}^{(1-\theta)r(s)} \le C,\end{aligned}$$

using (6.4.5) and (6.4.6). If $s > 1$, we have $r(s) = s+1+2s/d$, and applying the Gagliardo-Nirenberg inequality with $\theta = d(s+1)/(2s+d(s+1)) \in (0,1)$, we obtain in a similar way as above

$$\begin{aligned}\varepsilon^\eta \|u_i^{(\tau)}\|_{L^{r(s)}(Q_T)}^{r(s)} &= \varepsilon^\eta \|(u_i^{(\tau)})^{(s+1)/2}\|_{L^{2r(s)/(s+1)}(Q_T)}^{2r(s)/(s+1)} \\ &\le \varepsilon^\eta C\int_0^T \|(u_i^{(\tau)})^{(s+1)/2}\|_{H^1(\Omega)}^{2r(s)\theta/(s+1)} \|(u_i^{(\tau)})^{(s+1)/2}\|_{L^{2s/(s+1)}(\Omega)}^{2r(s)(1-\theta)/(s+1)} dt \\ &\le C\varepsilon^\eta \|(u_i^{(\tau)})^{(s+1)/2}\|_{L^2(0,T;H^1(\Omega))}^2 \|u_i^{(\tau)}\|_{L^\infty(0,T;L^s(\Omega))}^{(1-\theta)r(s)} \le C.\end{aligned}$$

This shows the lemma. □

Lemma 6.15. *Let $s > \max\{0, 1-2/d\}$ and $m > \max\{1, d/2\}$. Then there exist $2 \le q < \infty$ and $C > 0$ such that*

$$\tau^{-1}\|u^{(\tau)} - \sigma_\tau u^{(\tau)}\|_{L^{q'}(0,T;W^{m,q}(\Omega)')} \le C, \tag{6.4.9}$$

and $1/q + 1/q' = 1$.

Proof. Let $\phi \in L^q(0,T;W_\nu^{m,q}(\Omega))$, where $q \ge 2$ has to be determined. Recall that $W_\nu^{m,q}(\Omega)$ is defined in (6.1.16) and that $m > \max\{1, d/2\}$. Then, by (6.4.4),

$$\tau^{-1}\left|\int_\Omega (u^{(\tau)} - \sigma_\tau u^{(\tau)}) \cdot \phi dx\right| \le \varepsilon\|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))}\|\phi\|_{L^2(0,T;H^m(\Omega))}$$

$$
\begin{aligned}
&+\sum_{i=1}^n \|u_i^{(\tau)}p_i(u^{(\tau)})\|_{L^{q'}(Q_T)}\|\Delta\phi_i\|_{L^q(Q_T)} + \varepsilon\sum_{i,j=1}^n \|A_{ij}^0(u^{(\tau)})\nabla u_j^{(\tau)}\|_{L^{q'}(Q_T)}\|\nabla\phi_j\|_{L^q(Q_T)} \\
&+\frac{\varepsilon^\eta}{2}\sum_{i=1}^n \|(u_i^{(\tau)})^2\|_{L^{q'}(Q_T)}\|\Delta\phi_i\|_{L^q(Q_T)} + \|f(u^{(\tau)})\|_{L^{q'}(Q_T)}\|\phi\|_{L^q(Q_T)} \qquad (6.4.10)\\
&=: I_1+\cdots+I_5,
\end{aligned}
$$

where $1/q+1/q'=1$.

By (6.4.7), I_1 is bounded. We deduce from (6.4.8) that $u_i^{(\tau)}(u_j^{(\tau)})^s$ is uniformly bounded in $L^{p(s)/(s+1)}(Q_T)$, and so does $u_i^{(\tau)}p_i(u^{(\tau)})$. As $s>1-2/d$, we have $q_1:=p(s)/(s+1)>1$. We conclude that I_2 is bounded with $q'\le\min\{2,q_1\}$.

Since $A_{ij}^0(u^{(\tau)})$ depends linearly on $u^{(\tau)}$, it is sufficient to prove that $\varepsilon u_i^{(\tau)}\nabla u_j^{(\tau)}$ is uniformly bounded in some $L^{q_2}(Q_T)$ for all i,j. Let $q_2=2r(s)/(r(s)+2)$, where $r(s)=s+1+2/d$ is defined in Lemma 6.14. As $r(s)>2$, it holds that $q_2>1$. Then, by Hölder's inequality, (6.4.6), and (6.4.8),

$$
\varepsilon^{\eta/r(s)+(\eta+1)/2}\|u_i^{(\tau)}\nabla u_j^{(\tau)}\|_{L^{q_2}(Q_T)} \le \varepsilon^{\eta/r(s)}\|u_i^{(\tau)}\|_{L^{r(s)}(Q_T)}\cdot\varepsilon^{(\eta+1)/2}\|\nabla u_j^{(\tau)}\|_{L^2(Q_T)} \le C.
$$

The property $r(s)>2$ also implies that $\eta/r(s)+(\eta+1)/2<1$. This shows the bound on I_3 with $q'\le\min\{2,q_2\}$.

Set $q_3=r(s)/2>1$. Using the second estimate in (6.4.8) and $1-2/r(s)>0$, we find that

$$
\varepsilon^\eta\|(u_i^{(\tau)})^2\|_{L^{q_3}(Q_T)} = \varepsilon^{(1-2/r(s))\eta}\big(\varepsilon^{\eta/r(s)}\|u_i^{(\tau)}\|_{L^{r(s)}(Q_T)}\big)^2 \le C,
$$

proving that I_4 is bounded with $q'\le\min\{2,q_3\}$.

Finally, in view of (6.1.13), $|f_i(u^{(\tau)})|$ grows at most like $(u_i^{(\tau)})^{1+\sigma}$, where $\sigma=1$ if $s>1$ and $\sigma<2s-1+2/d$ if $s<1$. Therefore, we have $q_4:=p(s)/(1+\sigma)>1$ and

$$
\|f(u^{(\tau)})\|_{L^{q_4}(Q_T)} \le C\big(1+\|u^{(\tau)}\|^{1+\sigma}_{L^{(1+\sigma)q_4}(Q_T)}\big) = C\big(1+\|u^{(\tau)}\|^{1+\sigma}_{L^{p(s)}(Q_T)}\big) \le C.
$$

Hence, I_5 is bounded with $q'\le\min\{2,q_4\}$. We conclude that the lemma follows with $q':=\min\{2,q_1,q_2,q_3,q_4\}>1$ and $q=q'/(q'-1)$. ☐

Step 3: the limit $(\varepsilon,\tau)\to 0$. Estimates (6.4.5) and (6.4.9) allow us to apply the nonlinear Aubin-Lions lemma (Theorem 6.21 if $s\ge 1/2$ or Theorem 6.22 if $s<1/2$) to obtain the existence of a subsequence which is not relabeled such that, as $(\varepsilon,\tau)\to 0$,

$$
u^{(\tau)}\to u \quad \text{strongly in } L^\gamma(Q_T) \quad \text{for all } 1\le\gamma<p(s).
$$

In particular, $u^{(\tau)}\to u$ a.e. in Q_T. By estimates (6.4.5), (6.4.7), and (6.4.9), we have, up to subsequences,

$$
\begin{aligned}
(u_i^{(\tau)})^s &\rightharpoonup u_i^s \quad \text{weakly in } L^2(0,T;H^1(\Omega)),\\
\varepsilon w^{(\tau)} &\to 0 \quad \text{strongly in } L^2(0,T;H^m(\Omega)),\\
\tau^{-1}(u^{(\tau)}-\sigma_\tau u^{(\tau)}) &\rightharpoonup \partial_t u \quad \text{weakly in } L^{q'}(0,T;W^{m,q}(\Omega)').
\end{aligned}
$$

We have shown in the proof of Lemma 6.15 that $(u_i^{(\tau)}p_i(u^{(\tau)}))$ is bounded uniformly in $L^{p(s)/(s+1)}(Q_T)$. Taking into account the a.e. convergence $u_i^{(\tau)}p_i(u^{(\tau)}) \to u_ip_i(u)$ in Q_T, we infer that

$$u_i^{(\tau)}p_i(u^{(\tau)}) \to u_ip_i(u) \quad \text{strongly in } L^1(Q_T).$$

Furthermore, we proved that $(\varepsilon^{\eta/r(s)+(\eta+1)/2}A_{ij}^0(u^{(\tau)})\nabla u_j^{(\tau)})$ is bounded in $L^{q_2}(Q_T)$ with $q_2 = 2r(s)/(r(s)+2)$ such that

$$\begin{aligned}\varepsilon A_{ij}^0(u^{(\tau)})\nabla u_j^{(\tau)} &= \varepsilon^{1-\eta/r(s)-(\eta+1)/2} \cdot \varepsilon^{\eta/r(s)+(\eta+1)/2}A_{ij}^0(u^{(\tau)})\nabla u_j^{(\tau)} \\ &\to 0 \quad \text{strongly in } L^1(Q_T).\end{aligned}$$

Here, we used the fact that $\eta/r(s)+(\eta+1)/2 < 1$ such that $\varepsilon^{1-\eta/r(s)-(\eta+1)/2} \to 0$ as $\varepsilon \to 0$. We know from (6.4.8) that $(\varepsilon^{\eta/r(s)}u_i^{(\tau)})$ is bounded in $L^2(Q_T)$. Consequently,

$$\varepsilon^\eta(u_i^{(\tau)})^2 = \varepsilon^{\eta(1-2/r(s))}\big(\varepsilon^{\eta/r(s)}u_i^{(\tau)}\big)^2 \to 0 \quad \text{strongly in } L^1(Q_T),$$

since $\varepsilon^{\eta(1-2/r(s))} \to 0$ as $\varepsilon \to 0$ because of $r(s) > 2$. Finally, $f_i(u^{(\tau)}) \to f_i(u)$ a.e. and the uniform bound $\|f_i(u^{(\tau)}\|_{L^{q_4}(Q_T)} \le C$ with $q_4 = p(s)/(1+\sigma) > 1$ imply that

$$f_i(u^{(\tau)}) \to f_i(u) \quad \text{strongly in } L^1(Q_T).$$

Then, performing the limit $(\varepsilon,\tau) \to 0$ in (6.4.4) with $\phi \in L^\infty(0,T;W_\nu^{m,\infty}(\Omega))$, it follows that u solves (6.1.17) for such test functions. A density argument shows that, in fact, u solves (6.1.17) for $\phi \in L^q(0,T;W_\nu^{m,q}(\Omega))$, finishing the proof.

Remark 6.16 (Weak formulation)**.** *In the superlinear case $s > 1$, the solution constructed in the previous proof satisfies* (6.1.1) *even in the weak sense* (6.1.12) *with test functions $\phi \in L^q(0,T;W^{1,q}(\Omega))$. In order to see this, it is sufficient to show that*

$$A_{ij}(u^{(\tau)})\nabla u_j^{(\tau)} \rightharpoonup A_{ij}(u)\nabla u_j \quad \textit{weakly in } L^{q'}(Q_T)$$

for some $1 < q' \le 2$. Because of the structure of A_{ij}, we only need to verify that

$$\begin{aligned}u_i^{(\tau)}(u_j^{(\tau)})^{s-1}\nabla u_j^{(\tau)} &\rightharpoonup u_iu_j^{s-1}\nabla u_j \quad \textit{weakly in } L^{q'}(Q_T), \\ (u_i^{(\tau)})^s\nabla u_j^{(\tau)} &\rightharpoonup u_i^s\nabla u_j \quad \textit{weakly in } L^{q'}(Q_T).\end{aligned}$$

Indeed, we have the convergences $u_i^{(\tau)} \to u_i$ strongly in $L^\gamma(Q_T)$ for any $2 < \gamma < p(s)$ and $(u_i^{(\tau)})^s \rightharpoonup u_i^s$ weakly in $L^2(0,T;H^1(\Omega))$ and hence,

$$u_i^{(\tau)}(u_j^{(\tau)})^{s-1}\nabla u_j^{(\tau)} = s^{-1}u_i^{(\tau)}\nabla(u_j^{(\tau)})^s \rightharpoonup s^{-1}u_i\nabla u_j^s = u_iu_j^{s-1}\nabla u_j \quad \textit{weakly in } L^{q'}(Q_T),$$

choosing $q' = 2\gamma/(\gamma+2) > 1$. For the remaining convergence, we need to integrate by parts. It holds for $\phi_i \in L^q(0,T;W_\nu^{2,q}(\Omega))$ that

$$\int_0^T\int_\Omega (u_i^{(\tau)})^s\nabla u_j^{(\tau)} \cdot \nabla\phi_i dxdt$$

$$
\begin{aligned}
&= -\int_0^T \int_\Omega u_j^{(\tau)} \nabla (u_i^{(\tau)})^s \cdot \nabla \phi_i dx dt - \int_0^T \int_\Omega (u_i^{(\tau)})^s u_j^{(\tau)} \Delta \phi_i dx dt \\
&\to -\int_0^T u_j \nabla u_i^s \cdot \nabla \phi_i dx dt - \int_0^T \int_\Omega u_i^s u_j \Delta \phi_i dx dt = \int_0^t \int_\Omega u_i^s \nabla u_j \cdot \nabla \phi_i dx dt.
\end{aligned}
$$

A density argument shows that the weak formulation also holds for $\phi_i \in L^q(0,T;W^{1,q}(\Omega))$. ☐

Remark 6.17 (Vanishing self-diffusion). *Assume that* $a_{i0} > 0$ *and* $a_{ii} = 0$. *The difficulty is that we obtain a uniform bound only for* $\nabla (u_i^{(\tau)})^{s/2}$ *instead for* $\nabla (u_i^{(\tau)})^s$ *in* $L^2(Q_T)$. *In order to compensate this loss of regularity, we need additional assumptions, namely either* $s > \max\{1, d/2\}$ *(superlinear rates); or* $0 < s < 1$, $d = 1$, *and* $\sigma < s + 1$ *(sublinear rates). Under these conditions, the statement of Theorem 6.2 holds true.*

For the proof, we remark that the regularity for $(u_i^{(\tau)})^s$ *in* $L^2(0,T;H^1(\Omega))$ *is employed in the estimate of* $u_i^{(\tau)}$ *in* $L^{p(s)}(Q_T)$. *If only* $(u_i^{(\tau)})^{s/2}$ *is bounded in* $L^2(0,T;H^1(\Omega))$, *the Gagliardo-Nirenberg inequality gives a weaker result: for* $0 < s < 1$ *with* $\theta = ds/(ds+2)$ *and* $\rho = s + 2/d$,

$$
\begin{aligned}
\|u_i^{(\tau)}\|_{L^\rho(Q_T)}^\rho &= \|(u_i^{(\tau)})^{s/2}\|_{L^{2\rho/s}(Q_T)}^{2\rho/s} \le C \int_0^T \|(u_i^{(\tau)})^{s/2}\|_{H^1(\Omega)}^{2\theta\rho/s} \|(u_i^{(\tau)})^{s/2}\|_{L^{2/s}(\Omega)}^{2(1-\theta)\rho/s} dt \\
&\le C \|(u_i^{(\tau)})^{s/2}\|_{L^2(0,T;H^1(\Omega))}^2 \|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}^{(1-\theta)\rho} \le C,
\end{aligned}
$$

since $2\theta\rho/s = 2$; *and for* $s > 1$ *with* $\theta = d/(d+2)$ *and* $\rho = s + 2s/d$,

$$
\|u_i^{(\tau)}\|_{L^\rho(Q_T)}^\rho \le C \|(u_i^{(\tau)})^{s/2}\|_{L^2(0,T;H^1(\Omega))}^2 \|u_i^{(\tau)}\|_{L^\infty(0,T;L^s(\Omega))}^{(1-\theta)\rho} \le C,
$$

since $2\theta\rho/s = 2$. *Consequently,* $(u_i^{(\tau)})$ *is bounded in* $L^\rho(Q_T)$ *with* $\rho = s + (2/d)\max\{1, s\}$.

We claim that this estimate is sufficient to derive a bound for the discrete time derivative. Since the ε*-terms in* (6.4.10) *do not need the estimate for* $u_i^{(\tau)}$ *in* $L^\rho(Q_T)$, *it is sufficient to bound* $u_i^{(\tau)} p_i(u^{(\tau)})$ *and* $(u_i^{(\tau)})^{\sigma+1}$ *in some* $L^{q'}(Q_T)$ *with* $q' > 1$. *This is possible as long as* $\rho > s + 1$ *and* $\rho > \sigma + 1$, *respectively. If* $0 < s < 1$, *these two inequalities are equivalent to* $d = 1$ *and* $\sigma < s - 1 + d/2 = s + 1$. *If* $s > 1$ *(in this case* $\sigma = 1$*), they give the restriction* $s > d/2$, *thus* $s > \max\{1, d/2\}$. *This shows the claim.* ☐

6.5. Additional and auxiliary results

6.5.1. Detailed balance condition

We wish to interpret the detailed balance condition (6.1.10) and to explain how the numbers π_i can be computed from the coefficients (a_{ij}). We assume that the coefficients are normalized in the sense that $a_{ij} \ge 0$ and $\sum_{k=1,\,k\ne j} a_{kj} \le 1$ for all i, j. The idea is to use a probabilistic approach, interpreting the coefficients a_{ij} as the transition rates between two discrete states i and j of the state space $S := \{1, \ldots, n\}$. Then

$$
a_{ij} = \mathrm{P}(X_k = j | X_{k-1} = i)
$$

is the conditional probability for a random variable $X : \mathbb{N} \to S$. This variable represents the Markov chain associated to the stochastic matrix $Q = (Q_{ij})_{i,j} \in \mathbb{R}^{n\times n}$, defined by $Q_{ij} = a_{ij}$ for $i \neq j$ and $Q_{ii} = 1 - \sum_{i=1,\, i\neq j} a_{ij}$ for $i = 1, \dots, n$. A Markov chain is called reversible if there exists a probability distribution $\pi = (\pi_1, \dots, \pi_n)$ on S (called an invariant measure) such that

$$\pi_i a_{ij} = \pi_j a_{ji}, \quad i, j = 1, \dots, n. \tag{6.5.1}$$

The Markov chain can be interpreted as a directed graph, where the states $i \in S$ are the nodes and the edges are labeled by the probabilities a_{ij} going from state i to state j.

The state space S can be partitioned into so-called communicating classes. We write $i \to j$ if there exist $i_0, i_1, \dots, i_{n+1} \in S$ such that $a_{i_0,i_1} a_{i_1,i_2} \cdots a_{i_n,i_{n+1}} > 0$ for $i_0 = i$ and $i_{n+1} = j$. We say that i communicates with j if both $i \to j$ and $j \to i$. A set of states $\sigma \subset S$ is a communicating class if every pair in σ communicates with each other. This defines an equivalence relation, and communicating classes are the equivalence classes.

Consider the following properties:

(A1) For all $i, j \in S$, it holds that either $a_{ij} = a_{ji} = 0$ or $a_{ij} a_{ji} > 0$.

(A2) For any periodic cycle $i_0, i_1, \dots, i_{m+1} = i_0$,

$$\prod_{k=0}^{m} a_{i_k,i_{k+1}} = \prod_{k=0}^{m} a_{i_{k+1},i_k}.$$

The detailed balance condition (6.5.1) implies (A1) and (A2). It is shown in [131] that the converse is true and that the invariant measure π can be constructed explicitly.

Proposition 6.18. *Let (A1)-(A2) hold. Then there exists an invariant measure $\pi = (\pi_1, \dots, \pi_n)$ such that the detailed balance condition* (6.5.1) *is satisfied. Moreover, π can be computed explicitly by choosing an i_0 in each communicating class and defining π_j for i_0 and j belonging in the same class by*

$$\pi_j := \prod_{k=1}^{n-1} \frac{a_{i_k,i_{k+1}}}{a_{i_{k+1},i_k}}$$

depending only on i_0 and j, where $i_1, i_2, \dots, i_n = j$ are such that $a_{i_k,i_{k+1}} > 0$ for $k = 0, \dots, n-1$.

For instance, if $n = 3$, we need to suppose (according to (A2)) that

$$a_{12} a_{23} a_{31} = a_{13} a_{32} a_{21}, \tag{6.5.2}$$

and the invariant measure is given by $\pi = c(1, a_{12}/a_{21}, a_{13}/a_{31})$, where $c = (1 + a_{12}/a_{21} + a_{13}/a_{31})^{-1}$.

The following result relates the detailed balance condition and the symmetry of the matrix $H(u)A(u)$.

Proposition 6.19. *The following three properties are equivalent:*

(i) *Graph-theoretical condition: (A1) and (A2) hold.*

(ii) *Detailed balance condition:* $\pi_i a_{ij} = \pi_j a_{ji}$ *for* $i \neq j$.

(iii) *Symmetry: The matrix* $H(u)A(u)$ *is symmetric.*

Proof. The implication (i) $\Leftarrow$ (ii) is shown in Proposition 6.18. The converse can be proved directly using the detailed balance condition. Finally, the equivalence (ii) $\Leftrightarrow$ (iii) follows from an explicit calculation of $H(u)A(u)$. □

Remark 6.20. *The equivalence of the symmetry of* $H(u)A(u)$ *and the detailed balance condition is related to the Onsager principle of thermodynamics. Indeed, the diffusion matrix* $B = A(u)H(u)^{-1}$ *in*

$$\partial_t u - \operatorname{div}(B\nabla w) = f(u),$$

where $w = h'(u)$ *is the vector of entropy variables, is the Onsager matrix which is symmetric, according to Onsager, if and only if the thermodynamic system is time-reversible. Time-reversibility means that the Markov chain associated to the matrix* (a_{ij}) *is reversible, and the symmetry of* B *is equivalent to the symmetry of* $H(u)A(u)$*. Thus, the equivalence (ii)* $\Leftrightarrow$ *(iii) corresponds to the equivalence of the symmetry of* B *and the time-reversibility. For details on the detailed balance principle in thermodynamics, we refer to [47].* □

6.5.2. Nonlinear Aubin-Lions lemmas

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) be a bounded domain with Lipschitz boundary. Let $(u^{(\tau)})$ be a family of nonnegative functions which are piecewise constant in time with uniform time step size $\tau > 0$. We introduce the time shift operator $(\sigma_\tau u^{(\tau)})(t) = u^{(\tau)}(t-\tau)$ for $t \geq \tau$.

If there exist uniform estimates for the gradient $(\nabla u^{(\tau)})$ and the discrete time derivative $\tau^{-1}(u^{(\tau)} - \sigma_\tau u^{(\tau)})$, then, by the Aubin-Lions theorem and under suitable conditions on the spaces, $(u^{(\tau)})$ is relatively compact in some L^q space. In the case of nonlinear transition rates, we obtain uniform estimates only for $(\nabla(u^{(\tau)})^s)$, where $s > 0$. Then relative compactness follows from a nonlinear version of the Aubin-Lions theorem [40]. We recall a special case of this result.

Theorem 6.21 (Nonlinear Aubin-Lions lemma for $s \geq 1/2$)**.** *Let* $s \geq 1/2$, $m \geq 0$, $1 \leq q < \infty$, *and there exists* $C > 0$ *such that for all* $\tau > 0$,

$$\|(u^{(\tau)})^s\|_{L^2(0,T;W^{1,q}(\Omega))} + \tau^{-1}\|u^{(\tau)} - \sigma_\tau u^{(\tau)}\|_{L^1(\tau,T;H^m(\Omega)')} \leq C.$$

Then there exists a subsequence of $(u^{(\tau)})$*, which is not relabeled, such that, as* $\tau \to 0$,

$$u^{(\tau)} \to u \quad \text{strongly in } L^{2s}(0,T;L^{ps}(\Omega)),$$

where $p \geq \max\{1, 1/s\}$ *is such that the embedding* $W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$ *is compact.*

Theorem 6.21 can be extended to the case $s < 1/2$ if $(u^{(\tau)})$ is additionally bounded in $L^\infty(0,T;L^1(\Omega))$ which generally follows from the entropy inequality. This result is new.

Theorem 6.22 (Nonlinear Aubin-Lions lemma for $s < 1/2$). *Let* $\max\{0, 1/2 - 1/d\} < s < 1/2$, $m \geq 0$, *and there exists* $C > 0$ *such that for all* $\tau > 0$,

$$\|u^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} + \|(u^{(\tau)})^s\|_{L^2(0,T;H^1(\Omega))} + \tau^{-1}\|u^{(\tau)} - \sigma_\tau u^{(\tau)}\|_{L^1(\tau,T;H^m(\Omega)')} \leq C.$$

Then there exists a subsequence of $(u^{(\tau)})$, *which is not relabeled, such that, as* $\tau \to 0$,

$$u^{(\tau)} \to u \quad \textit{strongly in } L^1(0,T;L^1(\Omega)).$$

Proof. The result follows from Theorem 6.21 and the Hölder inequality. Indeed, we have

$$\begin{aligned}\|\nabla(u^{(\tau)})^{1/2}\|_{L^2(0,T;L^{1/(1-s)}(\Omega))} &= (2s)^{-1}\|(u^{(\tau)})^{1/2-s}\nabla(u^{(\tau)})^s\|_{L^2(0,T;L^{1/(1-s)}(\Omega))} \\ &\leq (2s)^{-1}\|(u^{(\tau)})^{(1-2s)/2}\|_{L^\infty(0,T;L^{2/(1-2s)}(\Omega))}\|\nabla(u^{(\tau)})^s\|_{L^2(0,T;L^2(\Omega))} \\ &= \|u^{(\tau)}\|^{(1-2s)/2}_{L^\infty(0,T;L^1(\Omega))}\|\nabla(u^{(\tau)})^s\|_{L^2(0,T;L^2(\Omega))} \leq C.\end{aligned}$$

Therefore, $(u^{(\tau)})^{1/2}$ is uniformly bounded in $L^2(0,T;W^{1,1/(1-s)}(\Omega))$. By Rellich-Kondrachov's theorem, the embedding $W^{1,1/(1-s)}(\Omega) \hookrightarrow L^2(\Omega)$ is compact for $s > 0$ if $d \leq 2$ and $s > 1/2 - 1/d$ if $d \geq 3$. Applying Theorem 6.21 with $s = 1/2$, $q = 1/(1-s)$, and $p = 2$, we infer that $(u^{(\tau)})$ is relatively compact in $L^1(0,T;L^1(\Omega))$. □

6.5.3. Increasing entropies

If detailed balance or a weak cross-diffusion condition hold, we have shown that the entropy is nonincreasing in time along solutions to (6.1.1)-(6.1.2). In this section, we show that the entropy may be increasing for small times if these conditions do not hold. To simplify the presentation, we restrict ourselves to the case $n = 3$ (three species), $s = 1$ (linear transition rates), and $\Omega = (0,1)$.

Lemma 6.23 (Vanishing diffusion coefficients a_{i0}). *Let* $a_{13} = a_{32} = a_{21} = 1$ *and* $a_{ij} = 0$ *else. For any* $\varepsilon > 0$, *there exist initial data* u^0 *such that*

$$\frac{d\mathcal{H}}{dt}[u^0] \geq \frac{1}{\varepsilon}.$$

In particular, if $t \mapsto \mathcal{H}[u(t)]$ *is continuous, there exists* $t_0 > 0$ *such that* $t \mapsto \mathcal{H}[u(t)]$ *is increasing on* $[0, t_0]$.

Proof. Observe that (6.5.2) is not satisfied, and hence detailed balance does not hold. Furthermore, we have

$$H(u)A(u) = \begin{pmatrix} 1/u_1 & 0 & 0 \\ 0 & 1/u_2 & 0 \\ 0 & 0 & 1/u_3 \end{pmatrix}\begin{pmatrix} u_3 & 0 & u_1 \\ u_2 & u_1 & 0 \\ 0 & u_3 & u_2 \end{pmatrix} = \begin{pmatrix} u_3/u_1 & 0 & 1 \\ 1 & u_1/u_2 & 0 \\ 0 & 1 & u_2/u_3 \end{pmatrix}.$$

Let $0 < \varepsilon < 0.5$ and define $u^0 = (u_1^0, u_2^0, u_3^0)$ by $u_1^0(x) = 1$ for $x \in (0,1)$ and

$$u_2^0(x) = \begin{cases} 3 & \text{for } 0 < x < 0.5, \\ 3 - \varepsilon^{-1}(x - 0.5) & \text{for } 0.5 < x < 0.5 + \varepsilon, \\ 2 & \text{for } 0.5 + \varepsilon < x < 1, \end{cases}$$

$$u_3^0(x) = \begin{cases} 9 & \text{for } 0 < x < 0.5, \\ 9 + \varepsilon^{-1}(x - 0.5) & \text{for } 0.5 < x < 0.5 + \varepsilon, \\ 10 & \text{for } 0.5 + \varepsilon < x < 1, \end{cases}$$

Then

$$\begin{aligned} \int_0^1 (\partial_x u^0)^\top H(u^0) A(u^0) \partial_x u^0 dx &= \frac{1}{\varepsilon^2} \int_{0.5}^{0.5+\varepsilon} \left(\frac{1}{u_2^0(x)} - 1 + \frac{u_2^0(x)}{u_3^0(x)} \right) dx \\ &\le \frac{1}{\varepsilon} \left(\frac{1}{2} - 1 + \frac{3}{9} \right) = -\frac{1}{6\varepsilon}, \end{aligned}$$

which implies that $(d\mathcal{H}/dt)[u^0] \ge 1/(6\varepsilon)$. □

One may ask if a similar result as above holds if the diffusion coefficients a_{i0} do not vanish, since they give positive contributions to the entropy production. The next lemma shows that the entropy may be increasing even if $a_{i0} > 0$ is chosen arbitrarily.

Lemma 6.24 (Positive diffusion coefficients a_{i0}). *Let $a_{13} = a_{32} = a_{21} = 1$, $a_{i0} > 0$ for $i = 1, 2, 3$, and $a_{ij} = 0$ else. For any $\varepsilon > 0$, there exist initial data u^0 such that*

$$\frac{d\mathcal{H}}{dt}[u^0] \ge \frac{1}{\varepsilon}.$$

In particular, if $t \mapsto \mathcal{H}[u(t)]$ is continuous, there exists $t_0 > 0$ such that $t \mapsto \mathcal{H}[u(t)]$ is increasing on $[0, t_0]$.

Proof. We choose the initial datum

$$\begin{aligned} u_1^0(x) &= \frac{a_{20}(2a_{20} + a_{30})}{8a_{20} + 4a_{30}}, \\ u_2^0(x) &= \begin{cases} 4a_{20} & \text{for } 0 < x < 0.5, \\ a_{20}(4 - \varepsilon^{-1}(x - 0.5)) & \text{for } 0.5 < x < 0.5 + \varepsilon, \\ 3a_{20} & \text{for } 0.5 + \varepsilon < x < 1, \end{cases} \\ u_3^0(x) &= \begin{cases} 8a_{20} + 4a_{30} & \text{for } 0 < x < 0.5, \\ a_{20}(8 - \varepsilon^{-1}(x - 0.5)) + 4a_{30} & \text{for } 0.5 < x < 0.5 + \varepsilon, \\ 9a_{20} + 4a_{30} & \text{for } 0.5 + \varepsilon < x < 1, \end{cases} \end{aligned}$$

Then

$$\begin{aligned} &\int_0^1 (\partial_x u^0)^\top H(u^0) A(u^0) \partial_x u^0 dx \\ &= \int_{0.5}^{0.5+\varepsilon} \left(\frac{u_1^0}{u_2^0} (\partial_x u_2^0)^2 + \frac{a_{20}}{u_2^0} (\partial_x u_2^0)^2 + \partial_x u_2^0 \partial_x u_3^0 + \frac{u_2^0 + a_{30}}{u_3^0} (\partial_x u_3^0)^2 \right) dx \\ &\le \frac{a_{20}^2}{\varepsilon^2} \int_{0.5}^{0.5+\varepsilon} \left(\frac{2a_{20} + a_{30}}{3(8a_{20} + 4a_{30})} \frac{a_{20}}{3a_{20}} - 1 + \frac{4a_{20} + a_{30}}{8a_{20} + 4a_{30}} \right) dx \\ &\le \frac{a_{20}^2}{\varepsilon^2} \left(\frac{1}{12} - \frac{1}{3} - 1 + \frac{1}{2} \right) = -\frac{a_{20}^2}{12\varepsilon}, \end{aligned}$$

which proves the result. □

7. From reaction diffusion to cross diffusion in the fast-reaction limit

7.1. Motivation

Our goal is to derive new entropies to new classes of cross-diffusion systems by using a fast-reaction limit. For a detailed presentation of our strategy, see Section 3.2.

7.1.1. Outline

The first model consists of three species with one fast reaction and general reaction and diffusion functions q_i and f_i, which we will study in Section 7.2. The second model is a "hybrid" reaction-cross diffusion system in four species with linear diffusivities and quadratic reaction terms, studied in Section 7.3.

7.1.2. Notation

We will use the following notation in this chapter:

$$\mathbb{R}_+ := [0,\infty), \quad \mathbb{R}_+^* := (0,\infty), \quad \Omega_T := \Omega \times (0,T),$$

$$\|u_i\|_{(L\log L)(\Omega_T)} := \int_0^T \int_\Omega u_i \log(u_i)\, dxdt,$$
$$\|u_i\|_{(L^2(\log L)^2)(\Omega_T)} := \int_0^T \int_\Omega (u_i)^2(\log(u_i))^2\, dxdt.$$

7.2. Rigorous limit from three species with one fast reaction

In this section we prove the rigorous limit of a quite general quasilinear reaction-diffusion system consisting of three species with one fast reaction to a limiting cross-diffusion system when the fast-reaction rate $1/\varepsilon$ tends to infinity.

7.2.1. Model for three species with one fast reaction

The model

For the concentrations $u_i^\varepsilon := u_i^\varepsilon(t,x) \geq 0$ of species A_i, $i = 1,2,3,$ we consider the following system of reaction-diffusion equations with fast-reaction term

$$\begin{cases} \partial_t u_1^\varepsilon - \Delta_x f_1(u_1^\varepsilon) = -\dfrac{1}{\varepsilon}\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)\, q_3(u_3^\varepsilon)\Big), \\ \partial_t u_2^\varepsilon - \Delta_x f_2(u_2^\varepsilon) = +\dfrac{1}{\varepsilon}\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)\, q_3(u_3^\varepsilon)\Big), \\ \partial_t u_3^\varepsilon - \Delta_x f_3(u_3^\varepsilon) = +\dfrac{1}{\varepsilon}\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)\, q_3(u_3^\varepsilon)\Big), \end{cases} \tag{7.2.1}$$

together with initial data

$$u_i^\varepsilon(0,x) = u_i^{in}(x) \geq 0, \quad x \in \Omega, \quad i = 1, \dots, 3,$$

and homogeneous Neumann boundary conditions on the boundary $\partial\Omega \in C^2$ of a bounded smooth open subset Ω of $\mathbb{R}^N$

$$n(x) \cdot \nabla_x u_i^\varepsilon(t,x) = 0, \quad x \in \partial\Omega, \quad i = 1, \dots, 3.$$

Limiting cross-diffusion system

Our goal is to study the fast-reaction limit $\varepsilon \to 0$ such that $u_i^\varepsilon \to u_i$, with u_i satisfying the limiting cross-diffusion system:

$$\begin{cases} q_1(u_1) - q_2(u_2) q_3(u_3) = 0, \\ \partial_t (u_1 + u_2) - \Delta_x (f_1(u_1) + f_2(u_2)) = 0, \\ \partial_t (u_1 + u_3) - \Delta_x (f_1(u_1) + f_3(u_3)) = 0. \end{cases} \tag{7.2.2}$$

7.2.2. Existence of solutions

Assumptions for subsection 7.2.2

Our first goal is to prove global existence of weak solutions to system (7.2.1) for fixed $\varepsilon > 0$. To this end, we impose the following set of assumptions for this subsection.

(C1) The functions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ are in $C^2(\mathbb{R}_+)$ with $f_i(0) = 0$, and there exists a constant $C_1 > 0$ such that

$$f_i'(u_i) \geq C_1 > 0 \quad \text{for all} \quad u_i \geq 0, \quad i = 1, 2, 3.$$

Consequently,

$$f_i(u_i) \geq \widetilde{C_1} u_i > 0 \quad \text{for all} \quad u_i \geq 0, \quad i = 1, 2, 3.$$

(C2) The functions $q_i : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous. Moreover, there exists a constant $C_2 > 0$, such that

$$q_i(u_i) \leq C_2 \frac{u_i}{1 + u_i} \quad \text{for all} \quad u_i \geq 0, \quad i = 1, 2, 3.$$

(C3) There exists a constant $\mu > 0$, such that $u_i^{in}(x) \geq \mu > 0$, for all $x \in \Omega$, $i = 1, 2, 3$.

Existence theorem

Theorem 7.1. *Assume that $(C1)$-$(C3)$ are satisfied with $u_i^{in}(x) \in L^\infty(\Omega),\ i = 1, 2, 3.$*

Then for any fixed $\varepsilon > 0$, there exists a global weak solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ to system (7.2.1), satisfying

$$u_i^\varepsilon \geq 0, \quad u_i^\varepsilon \in L^2(0,T;H^1(\Omega)), \quad u_i^\varepsilon \in L^\infty(\Omega_T), \quad \partial_t u_i^\varepsilon \in L^2(0,T;H^{-1}(\Omega)),$$

and u^ε satisfies for all $\phi \in C_0^\infty([0,T) \times \overline{\Omega}; \mathbb{R}^3)$ and for $\beta = (1,-1,-1)$ that

$$\int_0^T \int_\Omega u^\varepsilon \partial_t \phi \, dxdt + \int_0^T \int_\Omega \nabla_x \phi : f'(u^\varepsilon) \nabla_x u^\varepsilon \, dxdt = \tag{7.2.3}$$

$$\frac{(-1)^\beta}{\varepsilon} \int_0^T \int_\Omega \big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon) q_3(u_3^\varepsilon)\big) \phi \, dxdt + \int_\Omega u^{in}(x) \phi(0,x) \, dx. \tag{7.2.4}$$

Proof. Since $\varepsilon > 0$ is fixed, we will omit for simplicity the ε-dependency of u for this proof, and assume without loss of generality that $\varepsilon = 1$ for the right-hand side.

Step 1: solution for an approximated problem. Let $1 \leq k \leq N$ and $T > 0$. We consider the time step $\tau = T/N$. Then, for $u_i^{k-1} \in L^\infty(\Omega)$ we can introduce the following time-discretized scheme in $\tau > 0$:

Given $u^{k-1} = (u_1^{k-1}, u_2^{k-1}, u_3^{k-1}) \in L^\infty(\Omega; \mathbb{R}^3)$ nonnegative and bounded, find the k-th approximated solution $u^k = (u_1^k, u_2^k, u_3^k) \in L^\infty(\Omega; \mathbb{R}^3)$, such that (in the strong sense) it holds

$$\begin{cases} \tau^{-1}(u_i^k - u_i^{k-1}) - \Delta_x f_i(u_i^k) = Q_i(u^k) \quad \text{in } \Omega, \quad 1 \leq i \leq 3, \\ \nabla_x u_i^k(x) \cdot n(x) = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{7.2.5}$$

where $Q_i(u^k) = (-1)^\beta \big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon) q_3(u_3^\varepsilon)\big)$ with $\beta = (1,-1,-1)$ is the i-th component of the reaction term on the right-hand side. If $k = 1$, we set $u_i^{k-1}(x) := u_i^{in}(x)$ for the initial condition.

In order to prove existence of smooth solutions for the scheme, we first show that due to assumptions $(C1) - (C3)$ there exists a constant $C > 0$ uniform in time step $\tau > 0$, such that

$$u_i^k(x) \geq C > 0, \quad \text{for all } \ x \in \Omega, \ \ i = 1, 2, 3.$$

To this end, we assume without loss of generality that $i = 2$, and define $\widetilde{x} \in \overline{\Omega}$ such that $u_2^k(\widetilde{x}) = \inf_{x \in \overline{\Omega}} u_2^k(x)$. We need to distinguish the cases $\widetilde{x} \in \Omega$ or $\widetilde{x} \in \partial\Omega$. If $\widetilde{x} \in \partial\Omega$, we can use a strong Hopf lemma to obtain that $\nabla_x u_2^k \cdot n(x) > 0$, in contradiction to our assumption of homogeneous Neumann boundary conditions. If $\widetilde{x} \in \Omega$, then $\nabla_x u_2^k(\widetilde{x}) = 0$ and $\Delta_x u_2^k(\widetilde{x}) \geq 0$. Thus, by $(C1)$,

$$\Delta_x(f_2(u_2^k(\widetilde{x}))) = f_2''(u_2^k(\widetilde{x})) |\nabla u_2^k(\widetilde{x})|^2 + f_2'(u_2^k(\widetilde{x})) \Delta_x u_2^k(\widetilde{x}) \geq 0.$$

Consequently, by assumption $(C2)$, it holds for all $1 \leq k \leq N$,

$$\frac{1}{\tau}(u_2^k(\widetilde{x}) - u_2^{k-1}(\widetilde{x})) \geq q_1(u_1^k(\widetilde{x})) - q_2(u_2^k(\widetilde{x}))q_3(u_3^k(\widetilde{x})) \geq -C_2C_3u_2^k(\widetilde{x}).$$

This yields

$$u_2^k(\widetilde{x}) \geq (1 + C_2C_3\tau)^{-1}u_2^{k-1}(\widetilde{x}) \geq \ldots \geq (1 + C_2C_3\tau)^{-k}u_2^{in}(\widetilde{x}) \geq (1 + C_2C_3T/N)^{-N}\mu > 0, \tag{7.2.6}$$

where we used assumption $(C3)$ in the last step. Thus,

$$\inf_{x\in\Omega} u_2^k(x) \geq (1 + C_2C_3T/N)^{-N}\mu > 0. \tag{7.2.7}$$

Since for $N \to \infty$ the term $(1 + C_2C_3T/N)^{-N}\mu$ converges to $\exp(-C_2C_3T) > 0$, the lower bound in (7.2.7) is positive uniformly in $\tau = T/N > 0$. Exactly the same arguments are also valid for u_1^k and u_3^k, leading to similar positive lower bounds uniform in $\tau > 0$.

Now, due to this strict positivity of u_i^k uniform in $\tau > 0$ for $1 \leq k \leq N$, assumptions $(H1)$-$(H3)$ in [51, Theorem 2.2] are satisfied. In fact, assumption $(H1)$ is clear, the lower boundedness of $f_i(u_i)/u_i$ follows from assumption $(C1)$, and for the boundedness of $q_1(u_1^k)/u_1^k$ and $q_2(u_2^k)q_3(u_3^k)/u_i^k$, $i = 2, 3$ in $(H2)$ we can use the boundedness of $q_i(u_i^k)$ due to $(C2)$ together with the strict positivity of the u_i^k. For assumption $(H3)$, that is proving that $f : \mathbb{R}_+^3 \to \mathbb{R}_+^3$ is a homeomorphism, we can use [51, Proposition 5.2]. Thus by applying [51, Theorem 2.2], we get the existence of a strong solution $(u^k)_{1\leq k\leq N-1} \in L^\infty(\Omega; \mathbb{R}^3)$. However, we notice that the L^∞-bound provided by this theorem is not uniform with respect to the time step $\tau > 0$.

Step 2: uniform (in τ) a priori estimates. Our strategy of proving that $u_i^k \in L^\infty(\Omega)$ uniformly in τ is similar to the proof of the strict positivity of the u_i^k. Define $\widetilde{x} \in \overline{\Omega}$ such that $u_i^k(\widetilde{x}) = \sup_{x\in\overline{\Omega}} u_i^k(x)$. If $\widetilde{x} \in \partial\Omega$, we can use again a strong Hopf lemma to obtain that $\nabla_x u_2^k \cdot n(x) > 0$, in contradiction to our assumption of homogeneous Neumann boundary conditions. If $\widetilde{x} \in \Omega$, it holds that $\nabla_x u_i^k(\widetilde{x}) = 0$ and $\Delta_x u_i^k(\widetilde{x}) \leq 0$. Moreover, it holds that

$$\Delta_x(f_1(u_1^k(\widetilde{x}))) = f''(u_1^k(\widetilde{x}))|\nabla u_1^k(\widetilde{x})|^2 + f'(u_1^k(\widetilde{x}))\Delta_x u_1^k(\widetilde{x}) \leq 0.$$

Thus, due to assumption $(C2)$, it holds for all $1 \leq k \leq N$ that

$$\frac{1}{\tau}(u_1^k(\widetilde{x}) - u_1^{k-1}(\widetilde{x})) \leq C_2,$$

yielding recursively

$$u_1^k(\widetilde{x}) \leq C_2\tau + u_1^{k-1}(\widetilde{x}) \leq \ldots \leq C_2N\tau + u_1^{in}(\widetilde{x}).$$

Since $\tau = T/N$, this yields the uniform in $\tau > 0$ upper bound

$$\|u_1^k\|_{L^\infty(\Omega)} \leq C_2T + \|u_1^{in}\|_{L^\infty(\Omega)}, \tag{7.2.8}$$

Same arguments are also valid for u_2^k and u_3^k.

Next, we derive a uniform estimate for the gradient $\nabla_x u_i^k(x)$. Multiplying (7.2.5) by $\phi := u_i^k$ and integrating in Ω yields

$$\tau^{-1}\int_\Omega (u_i^k - u_i^{k-1})u_i^k\,dx + \int_\Omega f_i'(u_i^k)|\nabla u_i^k|^2\,dx = \int_\Omega Q_i(u^k)u_i^k\,dx.$$

By using assumption $(C1)$ and $(C2)$, we get

$$\int_\Omega (u_i^k)^2\,dx + C_1\tau\int_\Omega |\nabla u_i^k|^2\,dx \le C_2\tau\int_\Omega u_i^k\,dx + \int_\Omega u_i^{k-1}u_i^k\,dx.$$

Using Young's inequality and summing over $k = 1, \ldots, m$ yields

$$\int_\Omega (u_i^m)^2\,dx + 2C_1\tau\sum_{k=1}^m\int_\Omega |\nabla u_i^k|^2\,dx \le 2C_2\tau\sum_{k=1}^m\int_\Omega u_i^k\,dx + \int_\Omega (u_i^{in})^2\,dx. \tag{7.2.9}$$

This yields the bound

$$\|\nabla_x u_i^k\|_{L^2(\Omega)} \le C_T.$$

Now we define the piecewise constant functions in time $u^{(\tau)}(x,t) := u^k(x)$ for x in Ω and $t \in ((k-1)\tau, k\tau]$, $k = 1, \ldots, N$, and at $t = 0$ set $u^{(\tau)}(\cdot, t) = u^{in}(x)$. Moreover, we define the shift operator $\sigma_\tau u^{(\tau)}(x,t) := u^{k-1}(x)$ for x in Ω and $t \in ((k-1)\tau, k\tau]$, $k = 1, \ldots, N$.

Identifying the sum over $k = 1, \ldots, m$ multiplied by the time step τ as the time integral over $(0,T)$ in (7.2.9), we get that

$$\|\nabla_x u_i^{(\tau)}\|_{L^2(0,T;L^2(\Omega))} \le C_T \quad \text{(uniformly in } \tau).$$

Summarizing, we have shown that

$$\|u_i^{(\tau)}\|_{L^\infty(0,T;L^\infty(\Omega))} + \|u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \le C_T \quad \text{(uniformly in } \tau). \tag{7.2.10}$$

Finally, we need to derive a uniform estimate for the discrete time derivative. To this end, let $\phi \in L^2(0,T;H^1(\Omega;\mathbb{R}^3))$. Then

$$\begin{aligned}\tau^{-1}\left|\int_\tau^T (u^{(\tau)} - \sigma_\tau u^{(\tau)})\phi\,dxdt\right| &\le \|f'(u^{(\tau)})\nabla u^{(\tau)}\|_{L^2(\Omega_T)}\|\nabla\phi\|_{L^2(\Omega_T)} + \widetilde{C}\|\phi\|_{L^1(\Omega_T)}\\ &\le C\|\phi\|_{L^2(0,T;H^1(\Omega))}.\end{aligned}$$

Thus,

$$\tau^{-1}\|u^{(\tau)} - \sigma_\tau u^{(\tau)}\|_{L^2(0,T;H^{-1}(\Omega))} \le C_T. \tag{7.2.11}$$

Step 3: passing to the limit $\tau \to 0$. Due to the uniform estimates (7.2.10) and (7.2.11), we can use the Aubin-Lions lemma [40, Theorem 3] to derive that $u_i^{(\tau)} \to u_i$ strongly in $L^2(\Omega_T)$. Since $u_i^{(\tau)}$ is uniformly bounded in $L^\infty(\Omega_T)$ and f' and Q are continuous, also

$f'(u^{(\tau)})$ and $Q(u^{(\tau)})$ are uniformly bounded in $L^\infty(\Omega_T)$, moreover, we get the convergence $u_i(\tau) \to u_i$ strongly in $L^s(\Omega)$ for all $s < \infty$ and a.e. in Ω_T (up to subsequences). Thus we have

$$\begin{aligned} u^{(\tau)} &\to u \quad \text{strongly in } L^s(0,T;L^s(\Omega)), && (7.2.12)\\ u^{(\tau)} &\rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1(\Omega)), && (7.2.13)\\ f'(u^{(\tau)})\nabla u^{(\tau)} &\rightharpoonup X \quad \text{weakly in } L^2(0,T,L^2(\Omega)), && (7.2.14)\\ \tau^{-1}(u^{(\tau)} - \sigma_\tau u^{(\tau)}) &\rightharpoonup \partial_t u \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)). && (7.2.15) \end{aligned}$$

Due to (7.2.12) and (7.2.13), it holds that

$$f'(u^{(\tau)})\nabla u^{(\tau)} \rightharpoonup f'(u)\nabla u \quad \text{weakly in} \quad L^p(\Omega_T) \quad \text{for all} \quad p < 2,$$

and consequently $X = f'(u)\nabla u$. Similarly as in the proof of [41, pp. 1208-1209], we derive from (7.2.12) and (7.2.11) that

$$\int_0^T \int_\Omega \tau^{-1}(\sigma_\tau u^{(\tau)} - u^{(\tau)} \cdot \phi)\, dxdt \to \int_0^T \int_\Omega u \cdot \partial_t \phi \, dxdt - \int_\Omega u^0 \cdot \phi(0,x)\, dx.$$

This and the convergence results in (7.2.12)-(7.2.15) are sufficient to pass to the limit $\tau \to 0$, yielding (7.2.3).

□

Remark 7.2. *In fact, assuming enough regularity for the data, the solutions are even classical solutions. This can be shown by a bootstrap argument.*

7.2.3. Fast-reaction limit to a cross-diffusion system

Our next goal is to study the rigorous limit of system (7.2.1) to the cross-diffusion system (7.2.2). To this end, we impose the following set of assumptions for this subsection.

Assumptions for subsection 7.2.3

(B1) There exists a constant $C_1 > 0$, such that (for large x) it holds that

$$0 \le f_i(x) \le C_1\, x^2 \log(x) \quad \text{for all} \quad x \ge 0, \quad i = 1,2,3.$$

(B2) There exist $\alpha > 0$ and $C_2 > 0$, such that

$$q_i(x) \ge C_2\, x^\alpha \quad \text{for all} \quad x \ge 0, \quad i = 1,2,3.$$

(B3) For some $s > 1$ and $1 \leq p, p' \leq \infty$ with $1/p + 1/p' = 1$, there exist positive constants $C_3, C_4, C_5 > 0$ such that (for large x)

$$q_1^s(x) \leq C_3\, x f_1(x) \quad \text{for all} \quad x \geq 0,$$
$$q_2^{sp}(x) \leq C_4\, x f_2(x) \quad \text{for all} \quad x \geq 0,$$
$$q_2^{sp'}(x) \leq C_5\, x f_3(x) \quad \text{for all} \quad x \geq 0.$$

(B4) The functions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $q_i : \mathbb{R}_+^* \to \mathbb{R}_+$ are in C^1 with

$$f_i'(x) > 0 \;\text{ and }\; q_i'(x) > 0 \;\text{ for all }\; x \geq 0, \quad i = 1, 2, 3.$$

(B5) There exists a constant $C_6 > 0$, such that

$$q_i(x) \leq C_6\, x \left(f_i f_i' q_i'\right)(x) \quad \text{for all} \;\; x \geq 0, \quad i = 1, 2, 3.$$

(B6) For $a : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ with $a(u_2, u_3) = (a_2(u_2, u_3), a_3(u_2, u_3))$ it holds that

$$a_i : \begin{cases} \mathbb{R}_+^2 \to \mathbb{R}_+ \\ (u_2, u_3) \mapsto u_i + q_1^{-1}\Big(q_2(u_2) q_3(q_3)\Big)/u_i \end{cases}$$

are continuous for $i = 2, 3$.

Fast-reaction limit

Theorem 7.3. *Let assumptions* $(B1) - (B6)$ *hold. Let* $\Omega \subseteq \mathbb{R}^N$ *be a bounded regular open set of* $\mathbb{R}^N$*, and let for any* $\varepsilon > 0$, $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$ *denote a weak solution of the reaction-diffusion system* (7.2.1) *with initial data* $u_i^\varepsilon(0, x) = u_i^{in}(x) \in L^\infty$ *for all* $x \in \Omega$, $i = 1, 2, 3$.

Then the following holds: If $\varepsilon \to 0$*, there exists a subsequence of* $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$ *(which we still denote by* $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$*), which converges to* u_1, u_2, u_3 *in* $L^1_{loc}([0, \infty); L^1(\Omega))$.

Moreover, this limit is a weak solution of the cross-diffusion system (7.2.2) *belonging to* $L^1_{loc}([0, \infty); L^1(\Omega))$.

A priori estimates

Defining the entropy

$$H[u^\varepsilon(t)] = \sum_{i=1}^{3} \int_\Omega h_i(u_i^\varepsilon)\, dx, \text{ with } h_i(u_i^\varepsilon) = \int_c^{u_i^\varepsilon} \log(q_i(z))\, dz, \tag{7.2.16}$$

the entropy dissipation along the flow of system (7.2.1) has the following form

$$
\begin{aligned}
D_\varepsilon[u^\varepsilon(t)] &= -\frac{d}{dt}H[u^\varepsilon(t)] \\
&= -\sum_{i=1}^{3}\int_\Omega \log q_i(u_i^\varepsilon)\,\partial_t u_i^\varepsilon\,dx \\
&= \sum_{i=1}^{3}\int_\Omega \left(\frac{f_i' q_i'}{q_i}\right)(u_i^\varepsilon)\,|\nabla_x u_i^\varepsilon|^2\,dx \\
&\quad + \frac{1}{\varepsilon}\int_\Omega \Big(\log(q_1(u_1^\varepsilon)) - \log(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\Big)\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\Big)\,dx \\
&\geq 0.
\end{aligned}
$$

Integrating this in time leads to the standard entropy-entropy-dissipation inequality, where the right-hand side is uniform in ε:

$$
\sup_{t\in[0,T]} H[u^\varepsilon(t)] + \int_0^T D_\varepsilon[u^\varepsilon(s)]ds \leq H[u^{in}]. \tag{7.2.17}
$$

Lemma 7.4 (Entropy estimates)**.** *The following estimates hold for u_i^ε with $i = 1, 2, 3$:*

$$
\sum_{i=1}^{3}\int_0^T\int_\Omega \left(\frac{f_i' q_i'}{q_i}\right)(u_i^\varepsilon)\,|\nabla_x u_i^\varepsilon|^2\,dxdt \leq C_T, \tag{7.2.18}
$$

$$
\int_\Omega \Big(\log(q_1(u_1^\varepsilon)) - \log(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\Big)\Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\Big)\,dxdt \leq \varepsilon C_T, \tag{7.2.19}
$$

$$
\sup_{t\in[0,T]}\int_\Omega u_i^\varepsilon \log(u_i^\varepsilon)\,dx \leq C_T. \tag{7.2.20}
$$

Proof. Estimates (7.2.18) and (7.2.19) follow from the entropy-entropy-dissipation inequality (7.2.17).
Estimate (7.2.20) can be proved in the following way: Thanks to $(B2)$ we know that

$$
h_i(x) = \int_c^x \log(q_i(z))\,dz \geq \alpha C\int_c^x \log(z)\,dz.
$$

This together with (7.2.17) (and the conservations of $\int_\Omega(u_1^\varepsilon + u_i^\varepsilon)$) yields

$$
\sup_{t\in[0,T]}\sum_{i=1}^{3}\int_\Omega u_i^\varepsilon|\log(u_i^\varepsilon)|\,dx \leq C\sup_{t\in[0,T]} H[u^\varepsilon(t)] \leq C_T.
$$

□

Now we present the following duality lemma from [29], which is a variant of [121] introduced by M. Pierre and D. Schmitt.

Lemma 7.5 (Duality lemma). *If for $u \geq 0$ and $A := A(t,x) \geq 0$ it holds that*

$$\begin{cases} \partial_t u - \Delta_x(Au) \leq 0 \quad in \quad \Omega, \\ n(x) \cdot \nabla_x(Au) = 0 \quad on \quad \partial\Omega, \\ \qquad u(0,x) = u^{in}(x) \in L^\infty(\Omega), \end{cases}$$

then

$$\int_0^T \int_\Omega Au^2 \, dxdt \leq C \left(1 + \int_0^T \int_\Omega A \, dxdt \right),$$

where C depends only on $||u^{in}||_{L^\infty(\Omega)}$.

Proof. We integrate $\partial_t u - \Delta_x(Au) \leq 0$ against a test function $w \geq 0$, which is defined as a positive solution of the dual problem

$$\begin{cases} \partial_t w + A\Delta_x w = -Au, \\ \nabla_x w \cdot n(x) = 0 \ \text{ on } \ \partial\Omega, \\ w(T, \cdot) = 0. \end{cases} \tag{7.2.21}$$

Note that w is indeed nonnegative because u is also assumed to be nonnegative. This leads to

$$\int_0^T \int_\Omega \big(w\partial_t u - w\Delta_x (Au) \big) \, dxdt \leq 0.$$

Integration by parts yields

$$-\int_0^T \int_\Omega u\partial_t w \, dxdt - \int_\Omega (wu)(0) \, dx - \int_0^T \int_\Omega Au\Delta_x w \, dxdt \leq 0$$

and using the dual problem (7.2.21) leads to

$$\int_0^T \int_\Omega Au^2 \, dxdt \leq \int_\Omega (wu)(0) \, dx.$$

Using Hölder's inequality yields

$$\int_0^T \int_\Omega Au^2 \, dxdt \leq \|u(0)\|_{L^\infty(\Omega)} \int_\Omega |w(0)| \, dx. \tag{7.2.22}$$

Next we multiply the dual problem (7.2.21) by $\Delta_x w$ and integrate, which yields

$$-\int_0^T \partial_t \int_\Omega \frac{|\nabla_x w|^2}{2} \, dxdt + \int_0^T \int_\Omega A \left(\Delta_x w \right)^2 \, dxdt = -\int_0^T \int_\Omega Au\Delta_x w \, dxdt.$$

Using Young's inequality leads to

$$\frac{1}{2} \int_\Omega |\nabla_x w(0)|^2 \, dx + \int_0^T \int_\Omega A \left(\Delta_x w \right)^2 \, dxdt \leq \frac{1}{2} \int_0^T \int_\Omega A \left(\Delta_x w \right)^2 \, dxdt + \frac{1}{2} \int_0^T \int_\Omega Au^2 \, dxdt.$$

Thus

$$\int_\Omega |\nabla_x w(0)|^2 \, dx + \int_0^T \int_\Omega A \, (\Delta_x w)^2 \, dxdt \le \int_0^T \int_\Omega Au^2 \, dxdt. \tag{7.2.23}$$

Now we can write

$$w(0) = -\int_0^T \partial_t w \, dt = \int_0^T \big(A\Delta_x w + Au\big) \, dt.$$

Thus

$$\int_\Omega |w(0)| \, dx \le \int_0^T \int_\Omega \big(|A\Delta_x w| + |Au|\big) \, dxdt.$$

Using Young's inequality for both terms on the right-hand side leads to

$$\int_\Omega |w(0)| \, dx \le \frac{\varepsilon}{2} \int_0^T \int_\Omega Au^2 \, dxdt + \frac{\varepsilon}{2} \int_0^T \int_\Omega A(\Delta_x w)^2 \, dxdt + \frac{1}{\varepsilon} \int_0^T \int_\Omega A \, dxdt,$$

and thanks to (7.2.23) we get that

$$\int_\Omega |w(0)| \, dx \le \frac{1}{\varepsilon} \int_0^T \int_\Omega A \, dxdt + \varepsilon \int_0^T \int_\Omega Au^2 \, dxdt. \tag{7.2.24}$$

Now we use (7.2.22) and (7.2.24) to obtain

$$\begin{aligned} \int_0^T \int_\Omega Au^2 \, dxdt &\le \|u(0)\|_{L^\infty(\Omega)} \int_\Omega |w(0)| \, dx \\ &\le \|u(0)\|_{L^\infty(\Omega)} \left(\frac{1}{\varepsilon} \int_0^T \int_\Omega A \, dxdt + \varepsilon \int_0^T \int_\Omega Au^2 \, dxdt \right), \end{aligned}$$

which proves the duality lemma by choosing $\varepsilon > 0$ small enough. □

Lemma 7.6 (Duality estimate)**.** *It holds that*

$$\int_0^T \int_\Omega \Big(f_1(u_1^\varepsilon) + f_i(u_i^\varepsilon)\Big)\big(u_1^\varepsilon + u_i^\varepsilon\big) \, dxdt \le C_T, \quad i = 2, 3.$$

Proof. It holds that

$$\partial_t \left(u_1^\varepsilon + u_i^\varepsilon\right) - \Delta_x \left(\left(\frac{f_1(u_1^\varepsilon) + f_i(u_i^\varepsilon)}{u_1^\varepsilon + u_i^\varepsilon} \big(u_1^\varepsilon + u_i^\varepsilon\big) \right) \right) = 0, \quad i = 2, 3. \tag{7.2.25}$$

Due to (B1) and (7.2.20) we know that

$$\int_0^T \int_\Omega \frac{f_1(u_1^\varepsilon) + f_i(u_i^\varepsilon)}{u_1^\varepsilon + u_i^\varepsilon} \, dxdt \le C \int_0^T \int_\Omega u_1^\varepsilon \log(u_1^\varepsilon) \, dxdt + \int_\Omega u_i^\varepsilon \log(u_i^\varepsilon) \, dxdt \le C_T.$$

Thus we can use Lemma 7.5 for equation (7.2.25) and get the desired estimate. □

Lemma 7.7. *Recalling that* $u_i^{in}(x) \in L^\infty(\Omega)$ *for* $i = 1, 2, 3$*, we get the following estimates:*

$$\begin{aligned} \left\| (q_1(u_1^\varepsilon))^{1/2} - \left(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\right)^{1/2} \right\|_{L^2(\Omega_T)} &\le \sqrt{\varepsilon} C_T, \\ \| q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon) \|_{L^1(\Omega_T)} &\le \sqrt{\varepsilon} C_T. \end{aligned}$$

Proof. The elementary inequality

$$4\left|\sqrt{a} - \sqrt{b}\right|^2 \le (a-b)(\log a - \log b), \quad a, b \ge 0,$$

together with Lemma 7.4 yields

$$\begin{aligned} 4 & \left\| (q_1(u_1^\varepsilon))^{1/2} - \left(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\right)^{1/2} \right\|_{L^2(\Omega_T)}^2 = \\ &= 4 \int_0^T \int_\Omega \left| (q_1(u_1^\varepsilon))^{1/2} - \left(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\right)^{1/2} \right|^2 dxdt \\ &\le \int_0^T \int_\Omega \Big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon) \Big) \big(\log\left(q_1(u_1^\varepsilon)\right) - \log\left(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\right) \big)\, dxdt \\ &\le \varepsilon C_T, \end{aligned}$$

which shows the first estimate. By using the Cauchy-Schwarz inequality, we get that

$$\begin{aligned} & \| q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon) \|_{L^1(\Omega_T)} = \\ &= \int_0^T \int_\Omega \left((q_1(u_1^\varepsilon))^{1/2} - \left(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\right)^{1/2} \right) \left((q_1(u_1^\varepsilon))^{1/2} + \left(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\right)^{1/2} \right) dxdt \\ &\le \left(\int_{Q_T} \left| q_1(u_1^\varepsilon)^{1/2} - \left(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\right)^{1/2} \right|^2 \right)^{1/2} \left(\int_{Q_T} \left| q_1(u_1^\varepsilon)^{1/2} + \left(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\right)^{1/2} \right|^2 \right)^{1/2} \end{aligned}$$

For the first factor we can use the first part of this lemma:

$$\int_0^T \int_\Omega \left| (q_1(u_1^\varepsilon))^{1/2} - \left(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\right)^{1/2} \right|^2 dxdt \le \sqrt{\varepsilon} C_T,$$

and the second factor is bounded due to Young's inequality, $(B3)$ and Lemma 7.6:

$$\begin{aligned} & \int_0^T \int_\Omega \left| (q_1(u_1^\varepsilon))^{1/2} + \left(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\right)^{1/2} \right|^2 dxdt \le \\ & 2 \int_0^T \int_\Omega q_1(u_1^\varepsilon)\, dxdt + 2 \int_0^T \int_\Omega q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\, dxdt \\ & \le C + C \int_0^T \int_\Omega \left(u_1^\varepsilon f_1(u_1^\varepsilon) + u_2^\varepsilon f_2(u_2^\varepsilon) + u_3^\varepsilon f_3(u_3^\varepsilon) \right) dxdt \le C_T. \end{aligned}$$

This finishes the proof of the second estimate. □

Lemma 7.8. *It holds for* $i = 1, 2, 3$ *that*

$$\| \nabla_x u_i^\varepsilon \|_{L^1(\Omega_T)} \le C_T,$$

which shows that u_i^ε *is bounded in* $L^1(0, T; W^{1,1}(\Omega))$.

Proof. It holds by Cauchy-Schwarz that

$$\int_0^T\int_\Omega |\nabla_x u_i^\varepsilon|\,dxdt \le \left(\int_0^T\int_\Omega \left(\frac{f_i'q_i'}{q_i}\right)(u_i^\varepsilon)\,|\nabla_x u_i^\varepsilon|^2\,dxdt\right)^{1/2}\left(\int_0^T\int_\Omega \left(\frac{q_i}{f_i'q_i'}\right)(u_i^\varepsilon)\,dxdt\right)^{1/2}$$

Now it holds that

$$\int_0^T\int_\Omega \left(\frac{f_i'q_i'}{q_i}\right)(u_i^\varepsilon)\,|\nabla_x u_i^\varepsilon|^2\,dxdt \le C_T$$

due to Lemma 7.4, and thanks to $(B5)$ and Lemma 7.6 we get that

$$\int_0^T\int_\Omega \left(\frac{q_i}{f_i'q_i'}\right)(u_i^\varepsilon)\,dxdt \le C\int u_i^\varepsilon f_i(u_i^\varepsilon)\,dxdt \le C_T,$$

which finishes the proof. □

Strong compactness for u_i^ε

The difficulty in proving strong compactness for u_i^ε in system (7.2.1) relies in the fact that we cannot use the Aubin-Lions lemma [130] for u_i^ε directly, due to blow-up in the time dervatives $\partial_t u_i^\varepsilon$ for $\varepsilon \to 0$. However, we are still able to prove strong compactness for the u_i^ε with the help of the following lemmata.

Lemma 7.9. *The sequences* $(u_1^\varepsilon + u_i^\varepsilon)\,,\ i = 2,3$ *are relatively compact in* $L^1(\Omega_T)$.

Proof. For the terms $u_1^\varepsilon + u_i^\varepsilon,\ i = 2,3,$ in system (7.2.1), we see that the right-hand side does not blow-up for $\varepsilon \to 0$, thus we apply the Aubin-Lions lemma to these terms and get strong compactness for them.

The terms $u_1^\varepsilon + u_i^\varepsilon,\ i = 2,3,$ are bounded in $L^1(0,T;W^{1,1}(\Omega))$ due to Lemma 7.8. For the time derivatives $\partial_t(u_1^\varepsilon + u_i^\varepsilon),\ i = 2,3,$ we have that

$$\partial_t(u_1^\varepsilon + u_i^\varepsilon) = \Delta_x\big(f_1(u_1^\varepsilon) + f_i(u_i^\varepsilon)\big).$$

Now it holds by Lemma 7.6, (B1) and Lemma 7.4 that

$$\begin{aligned}\int_0^T\int_\Omega f_i(u_i^\varepsilon)\,dxdt &\le \left(\int_0^T\int_\Omega f_i(u_i^\varepsilon)u_i^\varepsilon\,dxdt\right)^{1/2}\left(\int_0^T\int_\Omega \frac{f_i(u_i^\varepsilon)}{u_i^\varepsilon}\,dxdt\right)^{1/2}\\ &\le C\left(\int_0^T\int_\Omega f_i(u_i^\varepsilon)u_i^\varepsilon\,dxdt\right)^{1/2}\left(\int_0^T\int_\Omega u_i^\varepsilon\log(u_i^\varepsilon)\,dxdt\right)^{1/2}\\ &\le C_T,\end{aligned}$$

which implies that $\partial_t(u_1^\varepsilon + u_i^\varepsilon)$ is bounded in $L^1(0,T;W^{-2,1}(\Omega))$. Thus by the standard Aubin-Lions lemma [130] we get that $(u_1^\varepsilon + u_i^\varepsilon)$ is relatively compact in $L^1(\Omega_T)$. □

Lemma 7.10. *The function*

$$\varphi : \begin{cases} \mathbb{R}_+^2 \to \mathbb{R}_+^2 \\ (u_2, u_3) \mapsto \Big(u_2 + q_1^{-1}\big(q_2(u_2)q_3(u_3)\big), u_3 + q_1^{-1}\big(q_2(u_2)q_3(u_3)\big)\Big) \end{cases}$$

is a homeomorphism on $\mathbb{R}_+^2$.

Proof. The function φ can be written as

$$\varphi : \begin{cases} \mathbb{R}_+^2 \to \mathbb{R}_+^2 \\ (u_2, u_3) \mapsto \big(a_2(u_2, u_3)u_2, a_3(u_2, u_3)u_3\big), \end{cases}$$

where for $i = 2, 3$ we have $a_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ with

$$a_i(u_2, u_3) = 1 + \frac{q_1^{-1}\big(q_2(u_2)q_3(u_3)\big)}{u_i}.$$

By $(B6)$ it holds that the functions a_i are continuous on $\mathbb{R}_+^2$. Clearly they are lower bounded by 1. Moreover, on $\mathbb{R}_+ \times \mathbb{R}_+$ it holds thanks to $(B4)$ that φ is strictly increasing (in the sense that each component is strictly increasing w.r.t. each component), and on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ it holds that φ is C^1. The determinant of the Jacobian of φ is strictly positive on $\mathbb{R}_+^* \times \mathbb{R}_+^*$:

$$\det D(\varphi) = 1 + \left(q_1^{-1}\right)' (q_2(u_2)q_3(u_3)) \cdot \big(q_2'(u_2)q_3(u_3) + q_2(u_2)q_3'(u_3)\big) > 0 \quad \forall u_i > 0.$$

Thus we can apply [51, Proposition 5.1] and get that φ is a homeomorphism on $\mathbb{R}_+^2$. ☐

Lemma 7.11. *If* (u_i^ε), $i = 1, 2, 3$ *are sequences in* $\mathbb{R}_+$ *satisfying for* $\varepsilon \to 0$ *that*

$$\begin{aligned} (u_1^\varepsilon + u_2^\varepsilon) &\to B \quad \text{a.e. in } \Omega_T, \\ (u_1^\varepsilon + u_3^\varepsilon) &\to C \quad \text{a.e. in } \Omega_T, \end{aligned} \tag{7.2.26}$$

and if additionally

$$\big(u_1^\varepsilon - q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\big) \to 0 \quad \text{a.e. in } \Omega_T,$$

then it holds that

$$\begin{aligned} u_1^\varepsilon &\to u_1 \quad \text{a.e. in } \Omega_T, \\ u_2^\varepsilon &\to u_2 \quad \text{a.e. in } \Omega_T, \\ u_3^\varepsilon &\to u_3 \quad \text{a.e. in } \Omega_T, \end{aligned}$$

with

$$u_2 := \psi_2(B, C), \quad u_3 := \psi_3(B, C), \quad u_1 := q_1^{-1}\big(q_2(\psi_2(B, C))q_3(\psi_3(B, C))\big), \tag{7.2.27}$$

where $\psi = (\psi_2, \psi_2) := \varphi^{-1}$, *and* φ *denotes the homeomorphism introduced in Lemma 7.10.*

Proof. Due to the assumptions, it holds that

$$u_2^\varepsilon + q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)) \to B \text{ a.e. in } \Omega_T,$$
$$u_3^\varepsilon + q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)) \to C \text{ a.e. in } \Omega_T.$$

Moreover, since φ is bijective with the inverse $\psi := \varphi^{-1}$, it holds that

$$(u_2^\varepsilon, u_3^\varepsilon) = \psi\big(u_2^\varepsilon + q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)),\ \ u_3^\varepsilon + q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\big) \text{ for all } \varepsilon > 0,$$

and since ψ is continuous, it holds a.e. in Ω_T that

$$\begin{aligned}(u_2, u_3) &= \lim_{\varepsilon\to 0}(u_2^\varepsilon, u_3^\varepsilon)\\ &= \lim_{\varepsilon\to 0}\Big(\psi_2\big(u_2^\varepsilon + q_1^{-1}(q_2q_3), u_3^\varepsilon + q_1^{-1}(q_2q_3)\big), \psi_3\big(u_2^\varepsilon + q_1^{-1}(q_2q_3), u_3^\varepsilon + q_1^{-1}(q_2q_3)\big)\Big)\\ &= \big(\psi_2(B,C), \psi_3(B,C)\big),\end{aligned}$$

with B, C defined in (7.2.26). This shows that $u_2^\varepsilon \to u_2$ a.e. in Ω_T, and $u_3^\varepsilon \to u_3$ a.e. in Ω_T with $u_2 = \psi_2(B,C)$ and $u_3 = \psi_3(B,C)$. Finally, since $u_1^\varepsilon - q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)) \to 0$ a.e. in Ω_T, we get that $u_1^\varepsilon \to q_1^{-1}\big(q_2(\psi_2(B,C))q_3(\psi_3(B,C))\big)$ a.e. in Ω_T, which finishes the proof. □

Theorem 7.12. *For $i = 1, 2, 3$ it holds that*

$$u_i^\varepsilon \to u_i \textit{ strongly in } L^1(\Omega_T) \quad \textit{and a.e..}$$

Proof. Due to Lemma 7.9 we have (for a subsequence) that $u_1^\varepsilon + u_2^\varepsilon \to B$ a.e. in Ω_T and $u_1^\varepsilon + u_3^\varepsilon \to C$ a.e. in Ω_T, and due to Lemma 7.7 it holds (for a subsequence) that $\big(u_1^\varepsilon - q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\big) \to 0$ a.e. in Ω_T. Thus we can apply Lemma 7.11, and get that $u_i^\varepsilon \to u_i$ a.e. in Ω_T.

Thanks to Lemma 7.4 we have for $i = 1, 2, 3$ that

$$\|u_i^\varepsilon\|_{L(\log L)(\Omega_T)} \le C_T,$$

thus by [105, Lemma 1.3] we get that

$$u_i^\varepsilon \to u_i \text{ strongly in } L^1(\Omega_T).$$

□

Uniform integrability

Lemma 7.13. *For $\varepsilon \to 0$ it holds that*

$$q_1(u_1^\varepsilon) \to q_1(u_1) \textit{ strongly in } L^1(\Omega_T),$$
$$q_2(u_2^\varepsilon)q_3(u_3^\varepsilon) \to q_2(u_2)q_3(u_3) \textit{ strongly in } L^1(\Omega_T).$$

Proof. Due to Theorem 7.12 we know that $u_i^\varepsilon \to u_i$ a.e. in Ω_T. Since q_i is continuous, it holds that

$$\begin{aligned} q_1(u_1^\varepsilon) &\to q_1(u_1) \text{ a.e. in } \Omega_T, \\ q_2(u_2^\varepsilon)q_3(u_3^\varepsilon) &\to q_2(u_2)q_3(u_3) \text{ a.e. in } \Omega_T. \end{aligned}$$

Thanks to (B3) and Lemma 7.6 we have for some $s > 1$:

$$\begin{aligned} \|q_1(u_1^\varepsilon)\|_{L^s(\Omega_T)} &\le C_T, \\ \|q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\|_{L^s(\Omega_T)} &\le C_T. \end{aligned}$$

Thus $q_1(u_1^\varepsilon)$ and $q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)$ are uniformly integrable, and applying [105, Lemma 1.3] finishes the proof. □

Lemma 7.14. *It holds that $f_i(u_i^\varepsilon) \to f_i(u_i)$ strongly in $L^1(\Omega_T)$ for $i = 1, 2, 3$.*

Proof. First we prove that $f_i(u_i^\varepsilon)$ is uniformly integrable, where we recall that $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $f_i' > 0$. In the case that that f_i is bounded, this is clearly true. In the case that f_i is unbounded, f_i^{-1} is also unbounded, and we proceed in the following way. Defining the function

$$\Phi_i : \begin{cases} \mathbb{R}_+ \to \mathbb{R}_+ \\ x \mapsto f_i^{-1}(x)x. \end{cases}$$

for $i = 1, 2, 3$, it holds that Φ_i is strictly increasing and that $\Phi_i(x)/x \to \infty$ for $x \to \infty$. Finally, thanks to Lemma 7.6 we have for all $\varepsilon > 0$ and for $i = 1, 2, 3$ that

$$\int_0^T \int_\Omega \Phi_i\left(|f_i(u_i^\varepsilon)|\right) dxdt = \int_0^T \int_\Omega u_i^\varepsilon f_i(u_i^\varepsilon)\, dxdt \le C_T.$$

Thus $\big(f_i(u_i^\varepsilon)\big)$ is uniformly integrable for $i = 1, 2, 3$ also in this case. Since $f_i(u_i^\varepsilon) \to f_i(u_i)$ a.e. in Ω_T, we can apply [105, Lemma 1.3] to get the desired result. □

Passing to the limit

The estimates derived in the last subsections are sufficient to pass to the limit $\varepsilon \to 0$ in the weak formulation of the reaction-diffusion system (7.2.1). Thus we are now able to prove Theorem 7.3.

Proof of Theorem 7.3. The only step which remains to prove is to pass to the limit $\varepsilon \to 0$ for the terms $u_1^\varepsilon + u_i^\varepsilon$, $i = 2, 3$ in the weak formulations of the reaction-diffusion system (7.2.1). First we write down the equation satisfied for $u_1^\varepsilon + u_i^\varepsilon$, $i = 2, 3$:

$$\begin{cases} \partial_t(u_1^\varepsilon + u_i^\varepsilon) - \Delta_x\big(f_1(u_1^\varepsilon) + f_i(u_i^\varepsilon)\big) = 0, \\ \qquad (u_1^\varepsilon + u_i^\varepsilon)(0, x) = \big(u_1^{in} + u_i^{in}\big)(x), \\ \qquad n(x) \cdot \nabla_x(u_1^\varepsilon + u_i^\varepsilon)(t, x) = 0. \end{cases}$$

The weak form can be written in the following way: For any test function $\psi = \psi(t,x)$ in the set

$$\big\{\psi \in C^2([0,\infty)\times\overline{\Omega}) : \psi(t) = 0 \text{ for } t \geq T \text{ and } n(x)\cdot\nabla_x\psi(t,x) = 0\big\},$$

we have that

$$\begin{aligned} 0 = &\int_{\Omega_T} (u_1^\varepsilon + u_i^\varepsilon)(t,x)\partial_t\psi(t,x)\,dxdt + \int_\Omega (u_1^{in} + u_i^{in})(x)\psi(0,x)\,dx \\ &+ \int_{\Omega_T} \big(f_1(u_1^\varepsilon) + f_i(u_i^\varepsilon)\big)\Delta_x\psi(t,x)\,dxdt. \end{aligned}$$

Now, by using that

$$\begin{aligned} &u_i^\varepsilon \to u_i \;\text{ strongly in }\; L^1(\Omega_T), \quad i = 1,2,3, \\ &f_i(u_i^\varepsilon) \to f_i(u_i) \;\text{ strongly in }\; L^1(\Omega_T), \quad i = 1,2,3, \end{aligned} \tag{7.2.28}$$

we can pass to the limit $\varepsilon \to 0$ and get that

$$\begin{aligned} 0 = &\int_{\Omega_T} (u_1 + u_i)(t,x)\partial_t\psi(t,x)\,dxdt + \int_\Omega (u_1^{in} + u_i^{in})(x)\psi(0,x)\,dx \\ &+ \int_{\Omega_T} \big(f_1(u_1) + f_i(u_i)\big)\Delta_x\psi(t,x)\,dxdt. \end{aligned} \tag{7.2.29}$$

Next we observe that thanks to Lemma 7.13 (second inequality) and Lemma 7.7, we get $q_1(u_1) - q_2(u_2)q_3(u_3)$.

This finishes the proof. □

Remark 7.15 (Initial layers)**.** *The last proof shows that the initial data needs to be well-prepared, i.e.*

$$q_2(u_2^{in})q_3(u_3^{in}) = q_1(u_1^{in}) \quad a.e. \text{ in } \Omega.$$

If this condition is not satisfied, there is an initial layer appearing. But due to the fact that

$$\|q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\|_{L^1(\Omega_T)} \leq \sqrt{\varepsilon}C,$$

this initial layer vanishes after $o_{\varepsilon\to 0}(1)-$ time.

7.2.4. Transformed system

First we recall the definition of the homeomorphism $\varphi(u) = (\varphi_2(u), \varphi_3(u))$ for $u = (u_2, u_3)$ defined in Lemma 7.10:

$$\varphi : \begin{cases} \mathbb{R}_+^2 \to \mathbb{R}_+^2, \\ (u_2, u_3) \mapsto (w_2, w_3), \end{cases} \tag{7.2.30}$$

where $w_2 := u_2 + q_1^{-1}\left(q_2(u_2)q_3(u_3)\right)$ and $w_3 := u_3 + q_1^{-1}\left(q_2(u_2)q_3(u_3)\right)$, and its inverse

$$\psi := \varphi^{-1}, \quad \psi(w) = \big(\psi_2(w), \psi_3(w)\big). \tag{7.2.31}$$

Thus we can formally rewrite the limiting system in terms of the new variables (w_2, w_3) in the following way:

$$\begin{cases} \partial_t w_2 - \Delta_x \left[\left(\dfrac{f_1(q_1^{-1}\left(q_2(\psi_2(w))q_3(\psi_3(w))\right)) + f_2(\psi_2(w))}{w_2} \right) w_2 \right] = 0, \\ \partial_t w_3 - \Delta_x \left[\left(\dfrac{f_1(q_1^{-1}\left(q_2(\psi_2(w))q_3(\psi_3(w))\right)) + f_3(\psi_3(w))}{w_3} \right) w_3 \right] = 0. \end{cases}$$

The entropy can be expressed in the new variables $w = (w_2, w_3)$ in the following way

$$\begin{aligned} H[w(t)] = &\int_\Omega \int_c^{q_1^{-1}(q_2(\psi_2(w))q_3(\psi_3(w)))} \log(q_1(z))\, dzdx \\ &+ \int_\Omega \int_c^{\psi_2(w)} \log(q_2(z))\, dzdx + \int_\Omega \int_c^{\psi_3(w)} \log(q_3(z))\, dzdx. \end{aligned}$$

7.2.5. Robustness of the method

Robustness of the method for quadratic diffusivities

Now we study the microscopic model with quadratic diffusivities $f_i(u_i^\varepsilon) = (u_i^\varepsilon)^2$, $i = 1, 2, 3$, where we allow a small pertubation of cross-diffusion terms in $u_1^\varepsilon, u_2^\varepsilon$ for $\delta > 0, \gamma > 0$:

$$\begin{cases} \partial_t u_1^\varepsilon - \Delta_x \Big(u_1^\varepsilon \big(u_1^\varepsilon + \delta u_2^\varepsilon \big) \Big) = -\dfrac{1}{\varepsilon} \left(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon) q_3(u_3^\varepsilon) \right), \\ \partial_t u_2^\varepsilon - \Delta_x \Big(u_2^\varepsilon \big(u_2^\varepsilon + \gamma u_1^\varepsilon \big) \Big) = +\dfrac{1}{\varepsilon} \left(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon) q_3(u_3^\varepsilon) \right), \\ \partial_t u_3^\varepsilon - \Delta_x \big((u_3^\varepsilon)^2 \big) = +\dfrac{1}{\varepsilon} \left(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon) q_3(u_3^\varepsilon) \right). \end{cases} \tag{7.2.32}$$

The following lemma shows that our method of obtaining entropies for the limiting cross-diffusion systems is robust under small perurbation of cross-diffusion terms.

Lemma 7.16. *Let $0 < \delta \leq 4$ and $0 < \gamma \leq 4$. Assume that there exist positive constants $C_1, C_2 > 0$, such that*

$$C_1 \frac{q_2'(u_2)}{q_2(u_2)} \leq \frac{q_1'(u_1)}{q_1(u_1)} \leq C_2 \frac{q_2'(u_2)}{q_2(u_2)}. \tag{7.2.33}$$

Moreover, assume that

$$\delta - \frac{\gamma}{2C_1} \geq 0, \qquad \gamma - \frac{\delta C_2}{2} \geq 0. \tag{7.2.34}$$

Then the entropy is decreasing along the flow of the pertubed system (7.2.32)*, i.e.*

$$\forall \varepsilon > 0 : \quad \frac{d}{dt} H[u^\varepsilon(t)] \leq 0.$$

Proof. It holds that

$$\begin{aligned}
\frac{d}{dt}H[u^\varepsilon(t)] &= \int_\Omega \log\big(q_1(u_1^\varepsilon)\big)\partial_t u_1^\varepsilon\,dx + \int_\Omega \log\big(q_2(u_2^\varepsilon)\big)\partial_t u_2^\varepsilon\,dx + \int_\Omega \log\big(q_3(u_3^\varepsilon)\big)\partial_t u_3^\varepsilon\,dx \\
&= -\frac{1}{\varepsilon}\int_\Omega \big(\log(q_1(u_1^\varepsilon)) - \log(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\big)\big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big)\,dx \\
&\quad - 2\int_\Omega \frac{q_1'(u_1^\varepsilon)}{q_1(u_1^\varepsilon)}u_1^\varepsilon|\nabla_x(u_1^\varepsilon)|^2\,dx - 2\int_\Omega \frac{q_2'(u_2^\varepsilon)}{q_2(u_2^\varepsilon)}u_2^\varepsilon|\nabla_x(u_2^\varepsilon)|^2\,dx \\
&\quad - 2\int_\Omega \frac{q_3'(u_3^\varepsilon)}{q_3(u_3^\varepsilon)}u_3^\varepsilon|\nabla_x(u_3^\varepsilon)|^2\,dx - \delta\int_\Omega \frac{q_1'(u_1^\varepsilon)}{q_1(u_1^\varepsilon)}u_2^\varepsilon|\nabla_x(u_1^\varepsilon)|^2\,dx \\
&\quad - \gamma\int_\Omega \frac{q_2'(u_2^\varepsilon)}{q_2(u_2^\varepsilon)}u_1^\varepsilon|\nabla_x(u_2^\varepsilon)|^2\,dx - \delta\int_\Omega \frac{q_1'(u_1^\varepsilon)}{q_1(u_1^\varepsilon)}u_1^\varepsilon\nabla_x(u_1^\varepsilon)\cdot\nabla_x(u_2^\varepsilon)\,dx \\
&\quad - \gamma\int_\Omega \frac{q_2'(u_2^\varepsilon)}{q_2(u_2^\varepsilon)}u_2^\varepsilon\nabla_x(u_1^\varepsilon)\cdot\nabla_x(u_2^\varepsilon)\,dx.
\end{aligned}$$

Using Young's inequality

$$\int_\Omega \frac{q_i'(u_i^\varepsilon)}{q_i(u_i^\varepsilon)}u_i^\varepsilon\nabla_x(u_i^\varepsilon)\cdot\nabla_x(u_j^\varepsilon)\,dx \le \frac{1}{2}\int_\Omega \frac{q_i'(u_i^\varepsilon)}{q_i(u_i^\varepsilon)}u_i^\varepsilon|\nabla_x(u_i^\varepsilon)|^2\,dx + \frac{1}{2}\int_\Omega \frac{q_i'(u_i^\varepsilon)}{q_i(u_i^\varepsilon)}u_i^\varepsilon|\nabla_x(u_j^\varepsilon)|^2\,dx$$

we get that

$$\begin{aligned}
\frac{d}{dt}H[u^\varepsilon(t)] &\le -\frac{1}{\varepsilon}\int_\Omega \big(\log(q_1(u_1^\varepsilon)) - \log(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\big)\big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big)\,dx \\
&\quad - \left(2-\frac{\delta}{2}\right)\int_\Omega \frac{q_1'(u_1^\varepsilon)}{q_1(u_1^\varepsilon)}u_1^\varepsilon|\nabla_x(u_1^\varepsilon)|^2\,dx - \left(2-\frac{\gamma}{2}\right)\int_\Omega \frac{q_2'(u_2^\varepsilon)}{q_2(u_2^\varepsilon)}u_2^\varepsilon|\nabla_x(u_2^\varepsilon)|^2\,dx \\
&\quad - 2\int_\Omega \frac{q_3'(u_3^\varepsilon)}{q_3(u_3^\varepsilon)}u_3^\varepsilon|\nabla_x(u_3^\varepsilon)|^2\,dx - \left(\delta - \frac{\gamma}{2C_1}\right)\int_\Omega \frac{q_1'(u_1^\varepsilon)}{q_1(u_1^\varepsilon)}u_2^\varepsilon|\nabla_x(u_1^\varepsilon)|^2\,dx \\
&\quad - \left(\gamma - \frac{\delta C_2}{2}\right)\int_\Omega \frac{q_2'(u_2^\varepsilon)}{q_2(u_2^\varepsilon)}u_1^\varepsilon|\nabla_x(u_2^\varepsilon)|^2\,dx \\
&\le 0.
\end{aligned}$$

□

Robustness of the method for cubic diffusivities

Now we study the microscopic model with cubic diffusivities $f_i(u_i^\varepsilon) = (u_i^\varepsilon)^3,\quad i = 1,2,3,$ where we allow a small pertubation of cross-diffusion terms in $u_1^\varepsilon, u_2^\varepsilon$ for $\delta > 0, \gamma > 0$:

$$\begin{cases}
\partial_t u_1^\varepsilon - \Delta_x\Big(u_1^\varepsilon\big((u_1^\varepsilon)^2 + \delta\,(u_2^\varepsilon)^2\big)\Big) = -\frac{1}{\varepsilon}\big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big), \\
\partial_t u_2^\varepsilon - \Delta_x\Big(u_2^\varepsilon\big((u_2^\varepsilon)^2 + \gamma\,(u_1^\varepsilon)^2\big)\Big) = +\frac{1}{\varepsilon}\big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big), \\
\partial_t u_3^\varepsilon - \Delta_x\big((u_3^\varepsilon)^3\big) = +\frac{1}{\varepsilon}\big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big).
\end{cases} \tag{7.2.35}$$

The following lemma shows that our method of obtaining entropies for the limiting cross-diffusion system is robust under small perurbation of cross-diffusion terms also for this model.

Lemma 7.17. *Let $\delta > 0$, $\gamma > 0$ and assume that there exist two positive constants $C_1, C_2 > 0$ such that assumption* (7.2.33) *holds. Moreover, assume that*

$$3 - \delta - \frac{\gamma}{C_1} \geq 0, \qquad\qquad 3 - \gamma - \delta C_2 \geq 0. \tag{7.2.36}$$

Then the entropy is decreasing along the flow of the pertubed system (7.2.35), *i.e.*

$$\forall \varepsilon > 0 : \quad \frac{d}{dt} H[u^\varepsilon(t)] \leq 0.$$

Proof. It holds that

$$\begin{aligned}
\frac{d}{dt} H[u^\varepsilon(t)] &= \int_\Omega \log\big(q_1(u_1^\varepsilon)\big)\partial_t u_1^\varepsilon \, dx + \int_\Omega \log\big(q_2(u_2^\varepsilon)\big)\partial_t u_2^\varepsilon \, dx + \int_\Omega \log\big(q_3(u_3^\varepsilon)\big)\partial_t u_3^\varepsilon \, dx \\
&= -\frac{1}{\varepsilon} \int_\Omega \big(\log(q_1(u_1^\varepsilon)) - \log(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\big)\big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big) \, dx \\
&\quad - 3 \int_\Omega \frac{q_3'(u_3^\varepsilon)}{q_3(u_3^\varepsilon)} (u_3^\varepsilon)^2 \, |\nabla u_3^\varepsilon|^2 \, dx \\
&\quad - 3 \int_\Omega \frac{q_1'(u_1^\varepsilon)}{q_1(u_1^\varepsilon)} (u_1^\varepsilon)^2 \, |\nabla_x (u_1^\varepsilon)|^2 \, dx - 3 \int_\Omega \frac{q_2'(u_2^\varepsilon)}{q_2(u_2^\varepsilon)} (u_2^\varepsilon)^2 \, |\nabla_x (u_2^\varepsilon)|^2 \, dx \\
&\quad - \delta \int_\Omega \frac{q_1'(u_1^\varepsilon)}{q_1(u_1^\varepsilon)} (u_2^\varepsilon)^2 \, |\nabla_x (u_1^\varepsilon)|^2 \, dx \\
&\quad - \gamma \int_\Omega \frac{q_2'(u_2^\varepsilon)}{q_2(u_2^\varepsilon)} (u_1^\varepsilon)^2 \, |\nabla_x (u_2^\varepsilon)|^2 \, dx - 2\delta \int_\Omega \frac{q_1'(u_1^\varepsilon)}{q_1(u_1^\varepsilon)} u_1^\varepsilon u_2^\varepsilon \nabla_x (u_1^\varepsilon) \cdot \nabla_x (u_2^\varepsilon) \, dx \\
&\quad - 2\gamma \int_\Omega \frac{q_2'(u_2^\varepsilon)}{q_2(u_2^\varepsilon)} u_1^\varepsilon u_2^\varepsilon \nabla_x (u_1^\varepsilon) \cdot \nabla_x (u_2^\varepsilon) \, dx.
\end{aligned}$$

Using Young's inequality in the following way

$$\begin{aligned}
&\int_\Omega \frac{q_i'(u_i^\varepsilon)}{q_i(u_i^\varepsilon)} u_i^\varepsilon u_j^\varepsilon \nabla_x (u_i^\varepsilon) \cdot \nabla_x (u_j^\varepsilon) \, dx \\
&\qquad \leq \frac{1}{2} \int_\Omega \frac{q_i'(u_i^\varepsilon)}{q_i(u_i^\varepsilon)} (u_i^\varepsilon)^2 \, |\nabla_x (u_i^\varepsilon)|^2 \, dx + \frac{1}{2} \int_\Omega \frac{q_i'(u_i^\varepsilon)}{q_i(u_i^\varepsilon)} \big(u_j^\varepsilon\big)^2 \, |\nabla_x (u_j^\varepsilon)|^2 \, dx
\end{aligned}$$

leads to

$$\begin{aligned}
\frac{d}{dt} H[u^\varepsilon(t)] &\leq -\frac{1}{\varepsilon} \int_\Omega \big(\log(q_1(u_1^\varepsilon)) - \log(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\big)\big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big) \, dx \\
&\quad - 3 \int_\Omega \frac{q_3'(u_3^\varepsilon)}{q_3(u_3^\varepsilon)} u_3^\varepsilon |\nabla_x (u_3^\varepsilon)|^2 \, dx \\
&\quad - \Big(3 - \delta - \frac{\gamma}{C_1}\Big) \int_\Omega \frac{q_1'(u_1^\varepsilon)}{q_1(u_1^\varepsilon)} (u_1^\varepsilon)^2 \, |\nabla_x (u_1^\varepsilon)|^2 \, dx \\
&\quad - \Big(3 - \gamma - \delta C_2\Big) \int_\Omega \frac{q_2'(u_2^\varepsilon)}{q_2(u_2^\varepsilon)} (u_2^\varepsilon)^2 \, |\nabla_x (u_2^\varepsilon)|^2 \, dx \\
&\quad - \delta \int_\Omega \frac{q_1'(u_1^\varepsilon)}{q_1(u_1^\varepsilon)} (u_2^\varepsilon)^2 \, |\nabla_x (u_1^\varepsilon)|^2 \, dx - \gamma \int_\Omega \frac{q_2'(u_2^\varepsilon)}{q_2(u_2^\varepsilon)} (u_1^\varepsilon)^2 \, |\nabla_x (u_2^\varepsilon)|^2 \, dx \\
&\leq 0.
\end{aligned}$$

□

7.3. Rigorous limit from four species with one fast and one slow reaction

In this section we prove the rigorous limit of a reaction-diffusion system with one fast reaction and one slow reaction to a limiting cross-diffusion system when the fast-reaction rate $1/\varepsilon$ tends to infinity.

7.3.1. Model for four species with one fast and one slow reaction

Chemical system considered:

Fast reaction (rate $1/\varepsilon$):

$$A_1 \quad \rightleftharpoons \quad A_2 + A_3,$$

Slow reaction (rate 1):

$$A_1 + A_3 \quad \rightleftharpoons \quad A_4.$$

Corresponding system of PDEs for the concentrations $u_i^\varepsilon := u_i^\varepsilon(t,x) \geq 0$ of species A_i for $i = 1,2,3,4$ with diffusivities $d_i > 0$:

$$\begin{cases} \partial_t u_1^\varepsilon - d_1\, \Delta_x u_1^\varepsilon = \dfrac{1}{\varepsilon}\,(u_2^\varepsilon\, u_3^\varepsilon - u_1^\varepsilon) + (u_4^\varepsilon - u_1^\varepsilon\, u_3^\varepsilon) \\ \partial_t u_2^\varepsilon - d_2\, \Delta_x u_2^\varepsilon = -\dfrac{1}{\varepsilon}\,(u_2^\varepsilon\, u_3^\varepsilon - u_1^\varepsilon) \\ \partial_t u_3^\varepsilon - d_3\, \Delta_x u_3^\varepsilon = -\dfrac{1}{\varepsilon}\,(u_2^\varepsilon\, u_3^\varepsilon - u_1^\varepsilon) + (u_4^\varepsilon - u_1^\varepsilon\, u_3^\varepsilon) \\ \partial_t u_4^\varepsilon - d_4\, \Delta_x u_4^\varepsilon = -(u_4^\varepsilon - u_1^\varepsilon\, u_3^\varepsilon) \end{cases} \tag{7.3.1}$$

together with initial data

$$u_i^\varepsilon(0,x) = u_i^{in}(x) \geq 0, \quad i = 1,2,3,4, \quad x \in \Omega,$$

and homogeneous Neumann boundary conditions on the boundary $\partial\Omega$ of a bounded smooth open set Ω of $\mathbb{R}^N$

$$n(x) \cdot \nabla_x u_i^\varepsilon(t,x) = 0, \quad i = 1,2,3,4, \quad x \in \partial\Omega.$$

Formal limit: We are interested in the rigorous limit $\varepsilon \to 0$, with u_i satisfying the following reaction-cross-diffusion system:

$$\begin{cases} u_2\, u_3 - u_1 = 0, \\ \partial_t(u_1 + u_2) - \Delta_x(d_1\, u_1 + d_2\, u_2) = u_4 - u_1\, u_3, \\ \partial_t(u_1 + u_3) - \Delta_x(d_1\, u_1 + d_3\, u_3) = 2\,(u_4 - u_1\, u_3). \\ \partial_t u_4 - d_4\, \Delta_x u_4 = -(u_4 - u_1\, u_3). \end{cases} \tag{7.3.2}$$

7.3.2. Existence of solutions

Theorem 7.18. *For any $\varepsilon > 0$, there exists a weak solution $u^\varepsilon \in L^2_{loc}\big([0,\infty); L^2(\Omega;\mathbb{R}^4)\big)$ to system* (7.3.1)*.*

Proof. This follows from [49, Theorem 4.1]. □

7.3.3. Fast-reaction limit to a cross-diffusion system

In this section we will prove the rigorous limit of the reaction-diffusion system (7.3.1) to the limiting reaction-cross-diffusion system (7.3.2). The idea of the proof is, that by duality, we get an $L^2(\log L)^2$ estimate, which directly implies the uniform integrability of the nonlinearities. By compactness properties (Aubin-Lions lemma), we get $L^{4/3}(\Omega_T)$-compactness of (u_i^ε). Thus we get, up to a subsequence, convergence of the right-hand-side in $L^1(\Omega_T)$.

Theorem 7.19. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded regular open set of $\mathbb{R}^N$, and let for any $\varepsilon > 0$, $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, u_4^\varepsilon$ denote a weak solution of the reaction-diffusion system* (7.3.1) *with initial data $u^{in}\log(u^{in}) \in L^2(\Omega)$.*

Then the following holds: If $\varepsilon \to 0$, there exists a subsequence of $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, u_4^\varepsilon$ (which we still denote by $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, u_4^\varepsilon$), which converges to u_1, u_2, u_3, u_4 in $L^1_{loc}([0,\infty); L^1(\Omega))$.

Moreover, this limit is a weak solution of the reaction-cross-diffusion system (7.3.2) *belonging to $L^2_{loc}([0,\infty); L^2(\Omega))$.*

7.3.4. A priori estimates

Using the entropy

$$H[u^\varepsilon(t)] = \sum_{i=1}^{4} \int_\Omega \left(u_i^\varepsilon(\log u_i^\varepsilon - 1) + 1\right)\, dx, \tag{7.3.3}$$

the entropy dissipation

$$D_\varepsilon[u^\varepsilon(t)] = -\frac{d}{dt} H[u^\varepsilon(t)]$$

along the flow of system (7.3.1) has the following form

$$\begin{aligned} D_\varepsilon[u^\varepsilon(t)] = \sum_{i=1}^{4} d_i \int_\Omega \frac{|\nabla_x u_i^\varepsilon|^2}{u_i^\varepsilon} dx &+ \frac{1}{\varepsilon}\int_\Omega (u_2^\varepsilon u_3^\varepsilon - u_1^\varepsilon)(\log(u_2^\varepsilon u_3^\varepsilon) - \log u_1^\varepsilon)\, dx \\ &+ \int_\Omega (u_4^\varepsilon - u_1^\varepsilon u_3^\varepsilon)(\log u_4^\varepsilon - \log(u_1^\varepsilon u_3^\varepsilon))\, dx. \end{aligned}$$

Integrating this in time leads to the standard entropy-entropy-dissipation inequality, where the right-hand side is uniform in ε:

$$\sup_{t\in[0,T]} H[u^\varepsilon(t)] + \int_0^T D_\varepsilon[u^\varepsilon(s)]ds \le H[u^{in}]. \tag{7.3.4}$$

Lemma 7.20 (Entropy estimates). *Recalling that $u^{in}\log(u^{in}) \in L^2(\Omega)$, the following estimates hold for u_i^ε with $i = 1, 2, 3, 4$:*

$$\begin{aligned}
\|u_i^\varepsilon\|_{L^\infty(0,T;L\log L)} + \|(u_i^\varepsilon)^{1/2}\|_{L^2(0,T;H^1(\Omega))} &\le C_T, \\
\int_0^T \int_\Omega (u_4^\varepsilon - u_1^\varepsilon u_3^\varepsilon)(\log u_4^\varepsilon - \log(u_1^\varepsilon u_3^\varepsilon)) &\le C_T, \\
\int_0^T \int_\Omega (u_1^\varepsilon - u_2^\varepsilon u_3^\varepsilon)(\log u_1^\varepsilon - \log(u_2^\varepsilon u_3^\varepsilon)) &\le \varepsilon C_T.
\end{aligned}$$

Proof. This follows directly from the entropy-entropy-dissipation inequality (7.3.4) by using the fact that $4|\nabla_x (u_i^\varepsilon)^{1/2}|^2 = |\nabla_x u_i^\varepsilon|^2/u_i^\varepsilon$. □

The next lemma can be found in [49, Theorem 8.1].

Lemma 7.21 (Duality lemma). *If h is a strong (i.e. $C^2([0,T]\times\overline{\Omega})$) nonnegative solution to the parabolic problem*

$$\begin{cases} \partial_t h - \Delta_x (Ah) \le 0, \\ \nabla_x (Ah)\cdot n(x) = 0 \quad on \quad \partial\Omega, \\ h(0,\cdot) = h^{in}(\cdot) \in L^2(\Omega), \end{cases} \tag{7.3.5}$$

where A is bounded from below and from above by some positive constants $C_1, C_2 > 0$ in the following way

$$0 < C_1 \le A(x,t) \le C_2,$$

then

$$\|h\|_{L^2(\Omega_T)} \le C_T \|h(0)\|_{L^2(\Omega)}.$$

Proof. We integrate $\partial_t h - \Delta_x (Ah) \le 0$ against a test function $w \ge 0$ which is a defined as a positive solution of the dual problem

$$\begin{cases} -(\partial_t w + A\Delta_x w) = \Theta \in C_0^\infty(\Omega_T), \quad \Theta \ge 0, \\ n(x)\cdot\nabla_x w(t,x) = 0 \quad \text{on} \quad \partial\Omega, \\ w(T,\cdot) = 0. \end{cases} \tag{7.3.6}$$

This leads to

$$\int_0^T \int_\Omega \big(w\partial_t h - w\Delta_x(Ah)\big)\, dxdt \le 0.$$

Integrating by parts yields

$$-\int_0^T \int_\Omega h\partial_t w \, dxdt - \int_\Omega (wh)(0)\, dx - \int_0^T \int_\Omega Ah\Delta_x w \, dxdt \le 0,$$

and using the dual problem (7.3.6) leads to

$$\int_0^T \int_\Omega h\Theta \, dxdt \le \int_\Omega (hw)(0)\, dx.$$

Next we multiply the dual problem (7.3.6) by $\Delta_x w$ and integrate, which yields

$$-\frac{1}{2}\int_0^T \partial_t \int_\Omega |\nabla_x w(t)|^2 \, dxdt + \int_0^T \int_\Omega A(\Delta_x w)^2 \, dxdt = -\int_0^T \int_\Omega \Theta \Delta_x w \, dx.$$

Using Young's inequality leads to

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla_x w(0)|^2 \, dx + \int_0^T \int_\Omega A(\Delta_x w)^2 \, dxdt \\ &\quad \le \frac{C_1}{2}\int_0^T \int_\Omega (\Delta_x w)^2 \, dxdt + C(C_1)\int_0^T \int_\Omega \Theta^2 \, dxdt, \end{aligned}$$

where we recall that $A \ge C_1 > 0$. Thus

$$\int_0^T \int_\Omega (\Delta_x w)^2 \, dxdt \le \widetilde{C} \int_0^T \int_\Omega \Theta^2 \, dxdt. \tag{7.3.7}$$

Let us now estimate $w(0)$ in $L^2(\Omega)$. For this we write

$$w(0) = -\int_0^T \partial_t w \, dt.$$

Thus by using the dual problem we get that

$$\int_\Omega (w(0))^2 \, dx = -\int_0^T \int_\Omega (\partial_t w) w(0) \, dxdt = \int_0^T \int_\Omega w(0)\Theta \, dxdt + \int_0^T \int_\Omega w(0) A \Delta_x w \, dxdt.$$

Using Young's inequality, we get that

$$\int_\Omega w(0)^2 \, dx \le C \int_0^T \int_\Omega \Theta^2 \, dxdt.$$

Putting all this estimates together leads to

$$\begin{aligned} \int_0^T \int_\Omega h\Theta \, dxdt &\le \int_\Omega h(0) w(0) \, dx \\ &\le \|w(0)\|_{L^2(\Omega)} \|h(0)\|_{L^2(\Omega)} \\ &\le C\|h(0)\|_{L^2(\Omega)} \int_0^T \int_\Omega \Theta^2 \, dxdt. \end{aligned}$$

Thus by duality we get that

$$\|h\|_{L^2(\Omega_T)} \le C\|h(0)\|_{L^2(\Omega)},$$

□

which finishes the proof.

Lemma 7.22 (Duality estimate)**.** *For $i = 1, 2, 3, 4$ it holds that*

$$\|u_i^\varepsilon\|_{L^2(\log L)^2(\Omega_T)} \le C_T.$$

Proof. Set

$$\begin{aligned} h_\varepsilon &:= \sum_{i=1}^4 (u_i^\varepsilon \,\ln u_i^\varepsilon - u_i^\varepsilon + 1), \\ w_\varepsilon &:= \sum_{i=1}^4 d_i\,(u_i^\varepsilon \,\ln u_i^\varepsilon - u_i^\varepsilon + 1), \\ j_\varepsilon &:= \sum_{i=1}^4 d_i\,\frac{|\nabla_x u_i^\varepsilon|^2}{u_i^\varepsilon} + \frac{1}{\varepsilon}\,(u_2^\varepsilon\,u_3^\varepsilon - u_1^\varepsilon)\left(\ln(u_2^\varepsilon\,u_3^\varepsilon) - \ln u_1^\varepsilon\right) \\ &\qquad\qquad + (u_4^\varepsilon - u_1^\varepsilon\,u_3^\varepsilon)\,(\ln u_4^\varepsilon - \ln(u_1^\varepsilon\,u_3^\varepsilon)), \end{aligned}$$

then system (7.3.1) exhibits the following entropy structure

$$\partial_t h_\varepsilon - \Delta_x w_\varepsilon = -j_\varepsilon.$$

This implies that the entropy density h_ε is a strong ($i.e.$ $C^2([0,T]\times\overline{\Omega})$) nonnegative solution to the parabolic problem

$$\begin{cases} \partial_t h_\varepsilon - \Delta_x\,(A_\varepsilon h_\varepsilon) \le 0, \quad \text{with} \;\; A_\varepsilon = w_\varepsilon/h_\varepsilon, \\ \nabla_x(A_\varepsilon h_\varepsilon)\cdot n = 0 \;\; \text{on} \;\; \partial\Omega, \\ h_\varepsilon(0,\cdot) = h^{in}(\cdot) \in L^2(\Omega), \end{cases} \tag{7.3.8}$$

where $h^{in} > 0$ is independent of ε, and A_ε is bounded from below and from above in the following way

$$0 < \min_{1\le i\le 4} d_i \le A_\varepsilon \le \max_{1\le i\le 4} d_i.$$

Thus we can apply Lemma 7.21 to h_ε and get the desired result.

□

Lemma 7.23. *Recalling that $u^{in}\log\left(u^{in}\right) \in L^2(\Omega)$, we get the following estimates:*

$$\begin{aligned} \|(u_1^\varepsilon)^{1/2} - (u_2^\varepsilon u_3^\varepsilon)^{1/2}\|_{L^2(\Omega_T)} &\le \sqrt{\varepsilon}C_T, \\ \|u_1^\varepsilon - u_2^\varepsilon u_3^\varepsilon\|_{L^1(\Omega_T)} &\le \sqrt{\varepsilon}C_T, \\ \|(u_4^\varepsilon)^{1/2} - (u_1^\varepsilon u_3^\varepsilon)^{1/2}\|_{L^2(\Omega_T)} &\le C_T, \\ \|u_4^\varepsilon - u_1^\varepsilon u_3^\varepsilon\|_{L^1(\Omega_T)} &\le C_T. \end{aligned}$$

Proof. The elementary inequality

$$4\left|\sqrt{a} - \sqrt{b}\right|^2 \le (a-b)(\log a - \log b), \quad a, b \ge 0,$$

together with Lemma 7.22 yields

$$4\|(u_1^\varepsilon)^{1/2} - (u_2^\varepsilon u_3^\varepsilon)^{1/2}\|^2_{L^2(\Omega_T)} \le 4\int_0^T\int_\Omega \left|(u_1^\varepsilon)^{1/2} - (u_2^\varepsilon u_3^\varepsilon)^{1/2}\right|^2\,dxdt$$

$$\leq \int_0^T \int_\Omega (u_1^\varepsilon - u_2^\varepsilon u_3^\varepsilon)(\log(u_1^\varepsilon) - \log(u_2^\varepsilon u_3^\varepsilon))\, dxdt$$
$$\leq \varepsilon C_T,$$

which shows the first estimate. By using the Cauchy-Schwarz inequality, we get that

$$\|u_1^\varepsilon - u_2^\varepsilon u_3^\varepsilon\|_{L^1(\Omega_T)} = \int_0^T \int_\Omega \left((u_1^\varepsilon)^{1/2} - (u_2^\varepsilon u_3^\varepsilon)^{1/2}\right)\left((u_1^\varepsilon)^{1/2} + (u_2^\varepsilon u_3^\varepsilon)^{1/2}\right) dxdt$$
$$\leq \left(\int_0^T \int_\Omega \left((u_1^\varepsilon)^{1/2} - (u_2^\varepsilon u_3^\varepsilon)^{1/2}\right)^2 dxdt\right)^{1/2} \left(\int_0^T \int_\Omega \left((u_1^\varepsilon)^{1/2} + (u_2^\varepsilon u_3^\varepsilon)^{1/2}\right)^2 dxdt\right)^{1/2}.$$

The boundedness of the last factor by Lemma 7.22 and the previous estimate lead to

$$\|u_1^\varepsilon - u_2^\varepsilon u_3^\varepsilon\|_{L^1(\Omega_T)} \leq \sqrt{\varepsilon} C_T.$$

Similarly,

$$4\|(u_4^\varepsilon)^{1/2} - (u_1^\varepsilon u_3^\varepsilon)^{1/2}\|_{L^2(\Omega_T)}^2 \leq 4\int_0^T \int_\Omega \left|(u_4^\varepsilon)^{1/2} - (u_1^\varepsilon u_3^\varepsilon)^{1/2}\right|^2 dxdt$$
$$\leq \int_0^T \int_\Omega (u_4^\varepsilon - u_1^\varepsilon u_3^\varepsilon)(\log(u_4^\varepsilon) - \log(u_1^\varepsilon u_3^\varepsilon))\, dxdt$$
$$\leq C_T,$$

and again by using the Cauchy-Schwarz inequality, Lemma 7.22 and the previous estimate, we get that

$$\|u_4^\varepsilon - u_1^\varepsilon u_3^\varepsilon\|_{L^1(\Omega_T)} \leq C_T.$$

□

Lemma 7.24. *It holds that*

$$\|\nabla_x u_i^\varepsilon\|_{L^{4/3}(\Omega_T)} \leq C_T.$$

Proof. Using the fact that $\nabla_x u_i^\varepsilon = 2\sqrt{u_i^\varepsilon}(\nabla_x \sqrt{u_i^\varepsilon})$ with $\sqrt{u_i^\varepsilon}$ bounded in $L^4(\Omega \times [0,T])$ due to the duality estimate in Lemma 7.22 and $\nabla_x \sqrt{u_i^\varepsilon}$ bounded in $L^2(\Omega \times [0,T])$ due to the entropy estimate in Lemma 7.20, we get by using Hölder's inequality with $p = 3, q = 3/2$ and $1/p + 1/q = 1$, that the product $\nabla_x u_i^\varepsilon$ is bounded in $L^{4/3}(\Omega \times [0,T])$:

$$\int_0^T \int_\Omega |\nabla_x u_i^\varepsilon|^{4/3}\, dxdt = 2^{4/3} \int_0^T \int_\Omega |\sqrt{u_i^\varepsilon}|^{4/3} |\nabla_x \sqrt{u_i^\varepsilon}|^{4/3}\, dxdt$$
$$\leq 2^{4/3} \left(\int_0^T \int_\Omega |\sqrt{u_i^\varepsilon}|^4\, dxdt\right)^{1/3} \left(\int_0^T \int_\Omega |\nabla_x \sqrt{u_i^\varepsilon}|^2\, dxdt\right)^{2/3}$$
$$\leq C_T.$$

□

7.3.5. Strong compactness

The difficulty in proving strong compactness for the species u_i^ε in system (7.3.1) relies in the fact that we cannot use the Aubin-Lions lemma [130] for u_i^ε directly, due to blow-up in the time dervatives $\partial_t u_i^\varepsilon$ for $\varepsilon \to 0$. However, we are still able to prove strong compactness for the u_i^ε with the help of the following lemmata.

Lemma 7.25. *The sequences $(u_1^\varepsilon + u_2^\varepsilon)$ and $(u_1^\varepsilon + u_3^\varepsilon)$ are relatively compact in $L^{4/3}(\Omega_T)$.*

Proof. For the sums $u_1^\varepsilon + u_2^\varepsilon$ and $u_1^\varepsilon + u_3^\varepsilon$ in system (7.3.1), we see that the right-hand side does not blow-up for $\varepsilon \to 0$, thus we apply the Aubin-Lions lemma to these terms and get strong compactness for them. The terms $u_1^\varepsilon + u_2^\varepsilon$ and $u_1^\varepsilon + u_3^\varepsilon$ are bounded in $L^{4/3}(0,T;W^{1,4/3}(\Omega))$ due to Lemma 7.24. For the time derivative $\partial_t(u_1^\varepsilon + u_2^\varepsilon)$, we have that

$$\partial_t(u_1^\varepsilon + u_2^\varepsilon) = d_1\Delta_x u_1^\varepsilon + d_2\Delta_x u_2^\varepsilon + (u_4^\varepsilon - u_1^\varepsilon u_3^\varepsilon),$$

thus $\partial_t(u_1^\varepsilon + u_2^\varepsilon)$ is bounded in

$$L^2(0,T;W^{-2,2}(\Omega)) \cap L^1(0,T;L^1(\Omega)) \subseteq L^1(0,T;W^{-2,1}(\Omega))$$

due to the duality estimate in Lemma 7.22, and the same holds for $\partial_t(u_1^\varepsilon + u_3^\varepsilon)$.

Summarizing, for $f^\varepsilon = u_1^\varepsilon + u_2^\varepsilon$ and $f^\varepsilon = u_1^\varepsilon + u_3^\varepsilon$ respectively, the following estimates hold: (f^ε) is bounded in $L^{4/3}(0,T;W^{1,4/3}(\Omega))$, and $(\partial_t f^\varepsilon)$ is bounded in $L^1(0,T;W^{-2,1}(\Omega))$, and thus by the standard Aubin-Lions lemma [130] we get that (f^ε) is relatively compact in $L^{4/3}(\Omega_T)$. □

Lemma 7.26. *The function*

$$\varphi : \begin{cases} \mathbb{R}_+^2 \to \mathbb{R}_+^2 \\ (x,y) \mapsto \big((1+y)x, (1+x)y\big) \end{cases}$$

is a homeomorphism on $(\mathbb{R}_+)^2$.

Proof. The function φ can be written as

$$\varphi : \begin{cases} \mathbb{R}_+^2 \to \mathbb{R}_+^2 \\ (x,y) \mapsto \big(a(x,y)x, b(x,y)y\big), \end{cases}$$

where $a : \mathbb{R}_+^2 \to \mathbb{R}_+,\ a(x,y) = 1+y$ and $b : \mathbb{R}_+^2 \to \mathbb{R}_+,\ b(x,y) = 1+x$. Thus we see that a and b are continuous and lower bounded by 1. Moreover, on $\mathbb{R}_+ \times \mathbb{R}_+$ it holds that φ is strictly increasing (in the sense that each component is strictly increasing w.r.t. each component), and on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ it holds that φ is $\mathcal{C}^1$. The determinant of the Jacobian of φ is strictly positive on $\mathbb{R}_+^* \times \mathbb{R}_+^*$:

$$\det D(\varphi) = 1 + x + y > 0 \quad \forall x > 0, \forall y > 0.$$

Thus we can apply [51, Proposition 5.1] and get that φ is a homeomorphism on $\mathbb{R}_+^2$. □

Lemma 7.27. *If $(a_n), (b_n)$ and (c_n) with $n \in \mathbb{N}$ are sequences in $\mathbb{R}_+$ satisfying for $n \to \infty$ that*

$$\begin{aligned} (a_n + b_n) &\to B \quad a.e.\ in\ \ \Omega_T, \\ (a_n + c_n) &\to C \quad a.e.\ in\ \ \Omega_T, \end{aligned} \tag{7.3.9}$$

and additionally

$$(a_n - b_n c_n) \to 0 \quad a.e.\ in\ \ \Omega_T,$$

then it holds that

$$\begin{aligned} (a_n) &\to a \quad a.e.\ in\ \ \Omega_T, \\ (b_n) &\to b \quad a.e.\ in\ \ \Omega_T, \\ (c_n) &\to c \quad a.e.\ in\ \ \Omega_T, \end{aligned}$$

with

$$b := \psi_2(B, C), \quad c := \psi_3(B, C), \quad a := bc, \tag{7.3.10}$$

where $\psi = (\psi_2, \psi_3) := \varphi^{-1}$, and φ denotes the homeomorphism introduced in Lemma 7.26

$$\varphi : \begin{cases} \mathbb{R}_+^2 \to \mathbb{R}_+^2 \\ (x, y) \mapsto (x(1 + y), y(1 + x)). \end{cases} \tag{7.3.11}$$

Proof. Due to the assumptions, we can write $a_n = b_n c_n + \varepsilon_n$ with $\varepsilon_n \to 0$ for $n \to \infty$, and get that $b_n c_n + b_n + \varepsilon_n \to B$ and $b_n c_n + c_n + \varepsilon_n \to C$. Thus it holds that

$$\begin{aligned} b_n(1 + c_n) &\to B \quad \text{a.e. in } \ \Omega_T, \\ c_n(1 + b_n) &\to C \quad \text{a.e. in } \ \Omega_T. \end{aligned} \tag{7.3.12}$$

Since φ is bijective with the inverse $\psi := \varphi^{-1}$, it holds that

$$(b_n, c_n) = \psi\left(b_n(1 + c_n), c_n(1 + b_n)\right) \quad \text{for all} \quad n \in \mathbb{N},$$

and since ψ is continuous, it holds a.e. in Ω_T that

$$(b, c) = \lim_{n\to\infty} (b_n, c_n) = \lim_{n\to\infty} \psi\left(b_n(1 + c_n), c_n(1 + b_n)\right) = \big(\psi_2\left(B, C\right), \psi_3\left(B, C\right)\big),$$

with B, C defined in (7.3.9). This shows that $b_n \to b$ a.e. in Ω_T, and $c_n \to c$ a.e. in Ω_T with b and c defined in (7.3.10). Finally, since $a_n - b_n c_n \to 0$ a.e. in Ω_T and $b_n c_n \to bc$ a.e. in Ω_T, we get that $a_n \to bc$ a.e. in Ω_T, which finishes the proof. □

Theorem 7.28. *For $i = 1, 2, 3, 4$ it holds that*

$$u_i^\varepsilon \to u_i \ \textit{strongly in}\ L^2(\Omega_T).$$

Proof. Due to Lemma 7.25 we have that $u_1^\varepsilon + u_2^\varepsilon \to B$ a.e. in Ω_T and $u_1^\varepsilon + u_3^\varepsilon \to C$ a.e. in Ω_T, and due to Lemma 7.23 it holds that $(u_1^\varepsilon - u_2^\varepsilon u_3^\varepsilon) \to 0$ a.e. in Ω_T. Thus we can apply Theorem 7.27, and get that $u_i^\varepsilon \to u_i$ a.e. in Ω_T.

Thus we have for $i = 1, 2, 3, 4$ that

$$\begin{aligned} &u_i^\varepsilon \to u_i \quad \text{a.e. in} \quad \Omega_T, \\ &\|u_i^\varepsilon\|_{L^2(\log L)^2(\Omega_T)} \leq C_T, \end{aligned}$$

thus by [105, Lemma 1.3] we get that

$$u_i^\varepsilon \to u_i \quad \text{strongly in} \quad L^2(\Omega_T),$$

which finishes the proof. □

Corollary 7.29. *For $\varepsilon \to 0$ it holds that $\left(u_i^\varepsilon u_j^\varepsilon\right) \to u_i u_j$ strongly in $L^1(\Omega_T)$ for all $i, j \in \{1, 2, 3, 4\}$.*

Proof. Define $g^\varepsilon := u_i^\varepsilon u_j^\varepsilon$ and $g := u_i u_j$ for i, j in $\{1, 2, 3, 4\}$. Then it holds that $g^\varepsilon \to g$ a.e. in Ω_T, and that $\|g^\varepsilon\|_{L^1(\log L)} \leq C_T$ due to the fact that

$$u_i^\varepsilon u_j^\varepsilon \log(u_i^\varepsilon u_j^\varepsilon) = u_i^\varepsilon \big(u_j^\varepsilon \log(u_j^\varepsilon)\big) + u_j^\varepsilon \big(u_i^\varepsilon \log(u_i^\varepsilon)\big) \quad \text{is bounded in } L^1(\Omega_T).$$

Thus [105, Lemma 1.3] implies that $g^\varepsilon \to g$ strongly in $L^1(\Omega_T)$, and thus $\left(u_i^\varepsilon u_j^\varepsilon\right) \to u_i u_j$ strongly in $L^1(\Omega_T)$ for all $i, j \in \{1, 2, 3, 4\}$. □

7.3.6. Passing to the limit

The estimates derived in the last subsections are sufficient to pass to the limit $\varepsilon \to 0$ in the weak formulation of the reaction-diffusion system (7.3.1). Thus we are now able to prove Theorem 7.19.

Proof of Theorem 7.19. The only step which remains to prove is to pass to the limit $\varepsilon \to 0$ for the terms $u_1^\varepsilon + u_2^\varepsilon$, $u_1^\varepsilon + u_2^\varepsilon$, u_2^ε and u_4^ε in the weak formulations of the reaction-diffusion system (7.3.1). First we write down the equation satisfied for $u_1^\varepsilon + u_2^\varepsilon$:

$$\begin{cases} \partial_t(u_1^\varepsilon + u_2^\varepsilon) - d_1 \,\Delta_x u_1^\varepsilon - d_2 \,\Delta_x u_1^\varepsilon = (u_4^\varepsilon - u_1^\varepsilon \, u_3^\varepsilon), \\ (u_1^\varepsilon + u_2^\varepsilon)(0, x) = u_1^{in} + u_2^{in}, \\ n(x) \cdot \nabla_x (u_1^\varepsilon + u_2^\varepsilon)(t, x) = 0. \end{cases}$$

The weak form can be written in the following way: For any testfunction $\psi = \psi(t, x)$ in the set

$$\big\{\psi \in C^2([0, \infty) \times \overline{\Omega}) : \psi(t) = 0 \text{ for } t \geq T \text{ and } n(x) \cdot \nabla_x \psi(t, x) = 0\big\},$$

we have that

$$\begin{aligned} -\int_{\Omega_T} (u_4^\varepsilon - u_1^\varepsilon u_3^\varepsilon)\psi(t, x)\, dxdt = &\int_{\Omega_T} (u_1^\varepsilon + u_2^\varepsilon)\partial_t \psi(t, x)\, dxdt + \int_\Omega (u_1^{in} + u_2^{in})(x)\partial_t \psi(0, x)\, dx \\ &+ \int_{\Omega_T} (d_1 u_1^\varepsilon + d_2 u_2^\varepsilon)\Delta_x \psi(t, x)\, dxdt. \end{aligned}$$

Now, by using the following results obtained above, namely that

$$\begin{aligned} &u_i^\varepsilon \to u_i \;\text{ strongly in }\; L^2(\Omega_T), \quad i = 1,2,3,4,\\ &u_i^\varepsilon u_j^\varepsilon \to u_i u_j \;\text{ strongly in }\; L^1(\Omega_T), \quad i,j \in \{1,2,3,4\}, \end{aligned} \tag{7.3.13}$$

we can pass to the limit $\varepsilon \to 0$ and get that

$$\begin{aligned} -\int_{\Omega_T} (u_4 - u_1 u_3)\psi(t,x)\,dxdt = &\int_{\Omega_T} (u_1 + u_2)\partial_t\psi(t,x)\,dxdt + \int_\Omega (u_1^{in} + u_2^{in})(x)\partial_t\psi(0,x)\,dx\\ &+ \int_{\Omega_T} (d_1 u_1 + d_2 u_2)\Delta_x\psi(t,x)\,dxdt. \end{aligned}$$

The procedure for $u_1^\varepsilon + u_3^\varepsilon$ is exactly the same, thus

$$\begin{aligned} -2\int_{\Omega_T} (u_4^\varepsilon - u_1^\varepsilon u_3^\varepsilon)\psi(t,x)dxdt = &\int_{\Omega_T} (u_1^\varepsilon + u_3^\varepsilon)\partial_t\psi(t,x)dxdt + \int_\Omega (u_1^{in} + u_3^{in})(x)\partial_t\psi(0,x)dx\\ &+ \int_{\Omega_T} (d_1 u_1^\varepsilon + d_3 u_3^\varepsilon)\Delta_x\psi(t,x)dxdt. \end{aligned}$$

converges for $\varepsilon \to 0$ to the limiting system

$$\begin{aligned} -2\int_{\Omega_T} (u_4 - u_1 u_3)\psi(t,x)dxdt = &\int_{\Omega_T} (u_1 + u_3)\partial_t\psi(t,x)dxdt + \int_\Omega (u_1^{in} + u_3^{in})(x)\partial_t\psi(0,x)dx\\ &+ \int_{\Omega_T} (d_1 u_1 + d_3 u_3)\Delta_x\psi(t,x)dxdt. \end{aligned}$$

For u_4^ε, we obtain by the same arguments that

$$\begin{aligned} \int_{\Omega_T} (u_4^\varepsilon - u_1^\varepsilon u_3^\varepsilon)\psi(t,x)\,dxdt = &\int_{\Omega_T} u_4^\varepsilon\partial_t\psi(t,x)\,dxdt + \int_\Omega u_4^{in}(x)\partial_t\psi(0,x)\,dx\\ &+ \int_{\Omega_T} d_4 u_4^\varepsilon\Delta_x\psi(t,x)\,dxdt. \end{aligned}$$

converges for $\varepsilon \to 0$ to

$$\begin{aligned} \int_{\Omega_T} (u_4 - u_1 u_3)\psi(t,x)\,dxdt = &\int_{\Omega_T} u_4\partial_t\psi(t,x)\,dxdt + \int_\Omega u_4^{in}(x)\partial_t\psi(0,x)\,dx\\ &+ \int_{\Omega_T} d_4 u_4\Delta_x\psi(t,x)\,dxdt. \end{aligned}$$

Using Lemma 7.23 and Theorem 7.28, we get $u_2 u_3 - u_1 = 0$.

This shows that the reaction-diffusion system (7.3.1) converges weakly to the reaction-cross-diffusion system (7.3.2), and since the duality estimate in Lemma 7.22 is uniform in ε, we get that the solution of the limiting cross-diffusion system (7.3.2) is bounded in $L^2_{loc}([0,\infty); L^2(\Omega))$. □

Remark 7.30 (Initial layers)**.** *The last proof shows that the initial data needs to be well-prepared, i.e.*

$$u_2^{in} u_3^{in} = u_1^{in} \quad a.e. \ in \ \ \Omega.$$

If this condition is not satisfied, there is an initial layer appearing. But due to the fact that

$$\|u_2^\varepsilon u_3^\varepsilon - u_1^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \sqrt{\varepsilon} C,$$

this initial layer is vanishing after $o_{\varepsilon \to 0}(1)-$*time.*

7.3.7. Global strong solutions to the reaction-diffusion system

Theorem 7.31. *Let* Ω *be a bounded and regular (*$C^{2+\alpha}$ *or* C^∞*) domain of* $\mathbb{R}^N$ *for* $N \leq 5$ *and* $d_i > 0$*,* $i = 1, \ldots, 4$*. If the nonnegative initial data* u_i^{in}*,* $i = 1, \ldots, 4$ *are smooth (*$C^2(\overline{\Omega})$*) and compatible with the Neumann boundary conditions, then for each fixed* $\varepsilon > 0$ *the solution of* (7.3.1) *is strong and unique.*

Proof. We fix $\varepsilon > 0$ for the entire proof. The idea now is to use the bootstrap argument of [48, Proof of Theorem 3.1]. Due to the duality estimate of Lemma 7.22, we know that u_i^ε is bounded in $L^2([0,T] \times \Omega)$ for $i = 1, \ldots, 4$. Since

$$\partial_t u_2^\varepsilon - d_2 \Delta_x u_2^\varepsilon = \frac{1}{\varepsilon} u_1^\varepsilon - \frac{1}{\varepsilon} u_2^\varepsilon u_3^\varepsilon \leq \frac{1}{\varepsilon} u_1^\varepsilon \text{ is bounded in } L^2([0,T] \times \Omega),$$

we can use the property of the heat kernel in Lemma 7.33 and get that

$$u_2^\varepsilon \in L^{q_0}_{loc}([0,\infty) \times \overline{\Omega}) \text{ for } \frac{1}{q_0} > \frac{N-2}{2(N+2)}.$$

The same argument holds for u_3^ε, namely since

$$\partial_t u_3^\varepsilon - d_3 \Delta_x u_3^\varepsilon \leq \frac{1}{\varepsilon} u_1^\varepsilon + u_4^\varepsilon \text{ is bounded in } L^2([0,T] \times \Omega),$$

we get by the property of the heat kernel in Lemma 7.33 that

$$u_3^\varepsilon \in L^{q_0}_{loc}([0,\infty) \times \overline{\Omega}) \text{ for } \frac{1}{q_0} > \frac{N-2}{2(N+2)}.$$

Thus by Young's inequality with $p = q = 2$ and $1/p + 1/q = 1$, we conclude that

$$u_2^\varepsilon u_3^\varepsilon \in L^{p_0}_{loc}([0,\infty) \times \overline{\Omega}) \text{ for } \frac{1}{p_0} > \frac{N-2}{N+2}.$$

Again by Young's inequality with $p = (q_0+2)/q_0, q = (q_0+2)/2$, where $\frac{1}{q_0} > \frac{N-2}{2(N+2)}$ and $1/p + 1/q = 1$, we conclude that

$$u_1^\varepsilon u_3^\varepsilon \in L^{p_1}_{loc}([0,\infty) \times \overline{\Omega}) \text{ for } \frac{1}{p_1} > \frac{N}{N+2}.$$

Now we consider the reaction-diffusion equation satisfied for u_4^ε:

$$\partial_t u_4^\varepsilon - d_4 \Delta_x u_4^\varepsilon \le \frac{1}{\varepsilon} u_1^\varepsilon u_3^\varepsilon \text{ bounded in } L^{p_1}([0,T]\times\Omega) \text{ for } \frac{1}{p_1} > \frac{N}{N+2},$$

and again by the property of the heat kernel, we get that

$$u_4^\varepsilon \in L^{q_1}_{loc}([0,\infty)\times\overline{\Omega}) \text{ for } \frac{1}{q_1} > \frac{N-2}{N+2},$$

and using the same arguments for u_1^ε leads to

$$u_1^\varepsilon \in L^{q_1}_{loc}([0,\infty)\times\overline{\Omega}) \text{ for } \frac{1}{q_1} > \frac{N-2}{N+2}.$$

For the remaining part of the proof we suppose that $N = 5$, since the case $N < 5$ can be handled with the same arguments and possibly less steps of bootstrapping. For $N = 5$, we have showed so far that

$$u_2^\varepsilon, u_3^\varepsilon \in L^{q_0}_{loc}([0,\infty)\times\overline{\Omega}) \text{ for } \frac{1}{q_0} > \frac{3}{14}, \quad u_1^\varepsilon, u_4^\varepsilon \in L^{q_1}_{loc}([0,\infty)\times\overline{\Omega}) \text{ for } \frac{1}{q_1} > \frac{3}{7}.$$

Since

$$\partial_t u_i^\varepsilon - d_i \Delta_x u_i^\varepsilon \text{ is bounded in } L^{7/3}([0,T]\times\Omega) \text{ for } i = 2,3,$$

we get by the property of the heat kernel that $u_i \in L^{7-\delta}_{loc}([0,\infty),\overline{\Omega})$, for $i = 2,3$, and a small $\delta > 0$. Thus by using Young's inequality with $p = 4, q = 4/3$ and $1/p + 1/q = 1$, we get that

$$u_1^\varepsilon u_3^\varepsilon \in L^{7/4-\delta}([0,T]\times\Omega), \quad u_2^\varepsilon u_3^\varepsilon \in L^{7/4-\delta}([0,T]\times\Omega).$$

Thus

$$\partial_t u_i^\varepsilon - d_i \Delta_x u_i^\varepsilon \text{ is bounded in } L^{7/4-\delta}([0,T]\times\Omega), \text{ for } i = 1,4,$$

and by the property of the heat kernel we get that

$$u_i^\varepsilon \in L^{7/2-\delta}_{loc}([0,\infty)\times\overline{\Omega}), \text{ for } i = 1,4,$$

showing that

$$\partial_t u_i^\varepsilon - d_i \Delta_x u_i^\varepsilon \text{ is bounded in } L^{7/2-\delta}([0,T]\times\Omega), \text{ for } i = 2,3,$$

and by the heat kernel we get that

$$u_i^\varepsilon \in L^{q_2}_{loc}([0,\infty)\times\overline{\Omega}), \quad q_2 \in [1,+\infty), \text{ for } i = 2,3.$$

Using Young's inequality leads to

$$u_1^\varepsilon u_3^\varepsilon \in L^{r_2}([0,T]\times\Omega) \text{ for } \frac{1}{r_2} > \frac{2}{7},$$

and together with the fact that

$$\partial_t u_i^\varepsilon - d_i \Delta_x u_i^\varepsilon \text{ is bounded in } L^{7/2-\delta}([0,T]\times\Omega), \text{ for } i = 1,4$$

and the property of the kernel leads to

$$u_i^\varepsilon \in L^{q_2}_{loc}([0,\infty)\times\overline{\Omega}), \quad q_2 \in [1,+\infty) \text{ for } i = 1,4.$$

Finally, using the property of the heat kernel leads to

$$u_i^\varepsilon \in L^{\infty}_{loc}([0,\infty)\times\overline{\Omega}), \quad i = 1,2,3,4.$$

Smoothness ($C^2([0,\infty]\times\overline{\Omega})$) and uniqueness can be proved by using methods in [101]. □

Remark 7.32. *The previous theorem only provides uniqueness for a given ε, and this does not imply any information about uniqueness of solutions to the limiting cross-diffusion system* (7.3.2)*. However, uniqueness of weak solutions to* (7.3.2) *may be obtained under additional assumptions by using the technique of Gajewski [68].*

7.3.8. Transformed system

Introducing the transformation φ, with $\varphi(u) = (\varphi_2(u), \varphi_3(u))$, $u = (u_2, u_3)$

$$\varphi : \begin{cases} (\mathbb{R}_+)^2 \to (\mathbb{R}_+)^2 \\ (u_2,u_3) \mapsto (w_2,w_3), \quad w = (w_2,w_3), \quad w_2 := u_2(1+u_3), \quad w_3 := u_3(1+u_2), \end{cases} \tag{7.3.14}$$

and its inverse

$$\psi := \varphi^{-1}, \quad \psi(w) = (\psi_2(w), \psi_3(w)), \tag{7.3.15}$$

we can rewrite the limiting system formally in terms of the new variables (w_2, w_3, u_4) as

$$\begin{cases} \partial_t w_2 - \Delta_x \left(\dfrac{d_1\psi_3(w) + d_2}{\psi_3(w)+1}\, w_2 \right) = \big(u_4 - \psi_2^2\psi_3(w)\big), \\ \partial_t w_3 - \Delta_x \left(\dfrac{d_1\psi_2(w) + d_3}{\psi_2(w)+1}\, w_3 \right) = 2\big(u_4 - \psi_2^2\psi_3(w)\big), \\ \qquad\qquad \partial_t u_4 - d_4\Delta_x u_4 = -\big(u_4 - \psi_2^2\psi_3(w)\big). \end{cases}$$

The entropy can be expressed in the new variables $w = (w_2, w_3, u_4)$ in the following way

$$\begin{aligned} H[w_2(t), w_3(t), u_4(t)] = &\int_\Omega \big(\psi_2(w)\psi_3(w)\big(\log(\psi_2(w)\psi_3(w)) - 1\big) + 1\big)\, dx \\ &+ \int_\Omega \big(\psi_2(w)\big(\log(\psi_2(w)) - 1\big) + 1\big)\, dx \\ &+ \int_\Omega \big(\psi_3(w)\big(\log(\psi_3(w)) - 1\big) + 1\big)\, dx \\ &+ \int_\Omega \big(u_4\big(\log(u_4) - 1\big) + 1\big)\, dx. \end{aligned}$$

7.4. Auxiliary result: regularizing effect of the heat kernel

The following result [48, p.5] concerning the regularizing effect of the heat kernel has been used in this chapter.

Lemma 7.33. *If $f \in L^p([0,T] \times \Omega)$, then the solution $u := u(t,x)$ of*

$$\begin{cases} \partial_t u - \Delta_x u = f, \text{ for } t \in [0,T] \text{ and } x \in \Omega, \\ n \cdot \nabla_x u = 0, \text{ for } t \in [0,T] \text{ and } x \in \partial\Omega, \\ u(0,\cdot) \in L^\infty(\Omega), \end{cases}$$

lies in $L^q([0,T] \times \Omega)$ for all $q \in [1,+\infty]$ satisfying

$$\frac{1}{p} + \frac{N}{N+2} - 1 < \frac{1}{q} \quad \Rightarrow \quad \frac{1}{q} > \frac{1}{p} - \frac{2}{N+2}.$$

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