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Vienna University of Technology

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Canonical Systems

Institute of
Analysis and Scientific Computing
Technische Universität Wien

Supervisor
Assoz. Prof. Dr. Harald WORACEK

Author
Johannes KAISER

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1 Introduction

In this section we introduce an initial value problem and show that it always has a unique solution and that this solution is an entire function of z . Further we establish some basic properties including a rough growth estimate.

We consider an initial value problem of the form

$$\begin{cases} y'(x) = -zA(x)y(x), & x \in [a, b], \\ y(a) = y_0, \end{cases} \quad (1.1)$$

where $z \in \mathbb{C}$, $y_0 \in \mathbb{C}^2$ and $A \in L^1([a, b], \mathbb{C}^{2 \times 2})$.

We call $y : [a, b] \rightarrow \mathbb{C}^2$ a *solution* of (1.1), if y is defined on $[a, b]$, takes values in \mathbb{C}^2 , is componentwise absolutely continuous, its derivative fulfills (1.1) for almost all $x \in [a, b]$ and $y(a) = y_0$.

We will mainly focus on *canonical systems*, which are initial value problems of the form (1.1), with $A(x) = JH(x)$, where $H \in L^1([a, b], \mathbb{R}^{2 \times 2})$ is real and positive semidefinite, and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Remark 1.1. Note that in some books (1.1) is given without the "–" on the right hand side. This results in minor changes to some theorems.

This section will follow lecture notes of Professor Michael Kaltenböck from the Technische Universität Wien.

1.1 Existence and Uniqueness of Solutions

Theorem 1.2 (Existence and Uniqueness). *Let $z \in \mathbb{C}$, $A \in L^1([a, b], \mathbb{C}^{2 \times 2})$ and $y_0 \in \mathbb{C}^2$, then there exists a unique function $y : [a, b] \rightarrow \mathbb{C}^2$, such that y is a solution of (1.1).*

Proof. Let $\mathcal{B}^M([a, b], \mathbb{C}^2)$ be the space of all bounded and Borel measurable functions on $[a, b]$ with values in \mathbb{C}^2 equipped with $\|f\|_\infty := \sup_{x \in [a, b]} \|f(x)\|_2$. This is a closed subspace of the Banach space $\mathcal{B}([a, b], \mathbb{C}^2)$ of all bounded functions on $[a, b]$ with values in \mathbb{C}^2 , because uniform convergence preserves measurability. Hence $\mathcal{B}^M([a, b], \mathbb{C}^2)$ is a Banach space as well. We define

$$\Lambda(x) := \int_a^x \|A(t)\| dt, \quad x \in [a, b],$$

where $\|\cdot\|$ denotes the operator norm on \mathbb{C}^2 . Let $G(x) := \exp(-|z|\Lambda(x))$, then

$$\|f\|_G := \sup_{x \in [a, b]} \|f(x)\|_2 G(x)$$

gives a norm equivalent to $\|\cdot\|_\infty$ on $\mathcal{B}^M([a, b], \mathbb{C}^2)$. Now let $T : \mathcal{B}^M([a, b], \mathbb{C}^2) \rightarrow \mathcal{B}^M([a, b], \mathbb{C}^2)$ be defined by

$$T(f)(x) := y_0 - \int_a^x zA(t)f(t)dt.$$

As an integral of an L^1 -function $T(f)$ is absolutely continuous, and T indeed maps into $\mathcal{B}^M([a, b], \mathbb{C}^2)$. Further for $f, g \in \mathcal{B}^M([a, b], \mathbb{C}^2)$ we obtain

$$\begin{aligned} G(x) \|T(f)(x) - T(g)(x)\|_2 &= G(x) \left\| \int_a^x z A(t) (g(t) - f(t)) dt \right\|_2 \leq \\ G(x) \int_a^x |z| \|A(t)\| \exp(|z|\Lambda(t)) \exp(-|z|\Lambda(t)) \|f(t) - g(t)\|_2 dt &\leq \\ G(x) \|f - g\|_G \int_a^x |z| \|A(t)\| \exp(|z|\Lambda(t)) dt &= G(x) \|f - g\|_G \int_a^x \frac{d}{dt} \left(\exp(|z|\Lambda(t)) \right) dt \leq \\ \|f - g\|_G \exp(-|z|\Lambda(x)) \left(\exp(|z|\Lambda(x)) - 1 \right). & \end{aligned}$$

Taking the supremum over $x \in [a, b]$, we get

$$\|T(f) - T(g)\|_G \leq \left(1 - \exp(-|z|\Lambda(b))\right) \cdot \|f - g\|_G.$$

By the Banach Fixed Point Theorem there exists a unique $y \in \mathcal{B}^M([a, b], \mathbb{C}^2)$, satisfying $y = Ty$. In particular y is absolutely continuous and obviously $y(a) = y_0$ holds as well. Taking the almost everywhere defined derivative, we obtain that y satisfies (1.1).

If, on the other hand, an absolutely continuous u satisfies $u(a) = y_0$ and the differential equation (1.1) almost everywhere, we get $u \in \mathcal{B}^M([a, b], \mathbb{C}^2)$ and by the fundamental theorem of calculus, we obtain $Tu = u$. Hence $y = u$. \square

In the next theorem we show that the solution depends holomorphically on z .

Theorem 1.3. *For each $y_0 \in \mathbb{C}^2$, there exists a unique function $y : [a, b] \times \mathbb{C} \rightarrow \mathbb{C}^2$ such that*

- (i) *for each $z \in \mathbb{C}$ the function $y(\cdot, z)$ is the solution of (1.1),*
- (ii) *y is continuous on $[a, b] \times \mathbb{C}$,*
- (iii) *for each $x \in [a, b]$ the function $y(x, \cdot)$ is holomorphic.*

Remark 1.4. Note that Theorem 1.3 is a slight modification of Theorem 1.2 and its proof, to obtain holomorphy in the second argument.

To prove this theorem, we need the following lemma.

Lemma 1.5. *Let us now regard $G(x, z) := \exp(-k|z|\Lambda(x))$, for some fixed $k > 1$, as a function on $[a, b] \times \mathbb{C} \rightarrow (0, \infty)$. For a function $f : [a, b] \times \mathbb{C} \rightarrow \mathbb{C}^2$ we define*

$$\|f\|_G := \sup_{z \in \mathbb{C}} \sup_{x \in [a, b]} \|f(x, z)\|_2 \cdot G(x, z).$$

Let X denote the vector space of all $f : [a, b] \times \mathbb{C} \rightarrow \mathbb{C}^2$, such that

- (i) $\|f\|_G < \infty$,
- (ii) for each $z \in \mathbb{C}$ the function $x \mapsto (x, z)$ is continuous.
- (iii) for each $x \in [a, b]$ the function $z \mapsto f(x, z)$ is holomorphic.

Then X equipped with $\|\cdot\|_G$ is a Banach space.

Proof. First it is a well known fact that the space $\mathcal{B}([a, b] \times \mathbb{C}, \mathbb{C}^2)$ of all bounded functions on $[a, b] \times \mathbb{C}$ with values in \mathbb{C}^2 equipped with $\|\cdot\|_\infty$ is a Banach space. Hence the space $\mathcal{B}_G([a, b] \times \mathbb{C}, \mathbb{C}^2)$ of all functions f on $[a, b] \times \mathbb{C}$ with values in \mathbb{C}^2 such that $\|f\|_G < \infty$, equipped with $\|\cdot\|_G$ is a Banach space as well, because it is isometrically isomorphic to $\mathcal{B}([a, b] \times \mathbb{C}, \mathbb{C}^2)$ via $f \mapsto f \cdot G$.

Obviously X is a linear subspace of $\mathcal{B}_G([a, b] \times \mathbb{C}, \mathbb{C}^2)$. Further it is closed: Assume $f_n \in X$ and $f_n \rightarrow f$ in $\mathcal{B}_G([a, b] \times \mathbb{C}, \mathbb{C}^2)$. Then for every $x \in [a, b]$ and for every compact $K \subseteq \mathbb{C}$, the uniform convergence of $f_n|_{[a, b] \times K}$ to $f|_{[a, b] \times K}$, as well as the uniform convergence of $z \mapsto f_n(x, z)$ to $z \mapsto f(x, z)$, for $z \in K$ follows. Hence $f|_{[a, b] \times K}$ and therefore f , are continuous and $z \mapsto f(x, z)$, $z \in \mathbb{C}$ is holomorphic for every x . Hence $f \in X$. \square

Proof of Theorem 1.3. Let X and $G(x, z)$ be defined as in Lemma 1.5, and let $T : X \rightarrow X$ be defined by

$$T(f)(x, z) := y_0 - \int_a^x zA(t)f(t, z)dt.$$

First we want to show that $T(f)$ actually lies in X . Let $(x_1, z_1), (x_2, z_2) \in [a, b] \times \mathbb{C}$ and assume without loss of generality that $x_1 \leq x_2$. Then

$$\begin{aligned} \|T(f)(x_1, z_1) - T(f)(x_2, z_2)\|_2 &\leq \|T(f)(x_1, z_1) - T(f)(x_2, z_1)\|_2 + \|T(f)(x_2, z_1) - T(f)(x_2, z_2)\|_2 \leq \\ &\left\| \int_{x_1}^{x_2} z_1 A(t) f(t, z_1) dt \right\|_2 + \left\| \int_a^{x_2} A(t) (z_2 f(t, z_2) - z_1 f(t, z_1)) dt \right\|_2 \leq \\ &\int_{x_1}^{x_2} \|A(t)\| \|z_1 f(t, z_1)\|_2 dt + \int_a^{x_2} \|A(t)\| \|z_1 f(t, z_1) - z_2 f(t, z_2)\|_2 dt. \end{aligned} \quad (1.2)$$

If we fix (x_1, z_1) , then we get, for $|z_1 - z_2| \leq 1$ and $\rho := |z_1| + 1$,

$$\|z_1 f(t, z_1) - z_2 f(t, z_2)\|_2 \leq 2(|z_1| + 1) \sup_{x \in [a, b], |z| \leq \rho} \|f(x, z)\|_2 \leq 2(|z_1| + 1) \frac{\|f\|_G}{G(b, \rho)}.$$

Hence, by the Dominated Convergence Theorem, the second integral in (1.2) converges to zero for $(x_2, z_2) \rightarrow (x_1, z_1)$. Because $\|z_1 f(t, z_1)\|_2 \leq \rho \frac{\|f\|_G}{G(r, \rho)}$, the same holds for the first integral. Hence $T(f)$ is continuous on $[a, b] \times \mathbb{C}$.

Let us regard $H : z \mapsto T(f)(x, z) = y_0 - \int_a^x zA(t)f(z, t)dt$ as a parameter integral, with holomorphic integrand $z \mapsto zA(t)f(z, t)$ for every t and integrable integrand $t \mapsto zA(t)f(z, t)$ for every z . We have the estimate

$$\|zA(t)f(z, t)\|_2 \leq \|A(t)\| \max_{(\tau, \zeta) \in [a, b] \times K} |\zeta| \|f(\tau, \zeta)\|_2, \quad K \subseteq \mathbb{C} \text{ compact}, \quad z \in K, \quad t \in [a, b],$$

where the right hand side is integrable, and conclude that H is holomorphic. Further we obtain

$$\begin{aligned} \|T(f)(x, z)\|_2 &\leq \|y_0\|_2 + \int_a^x |z| \|A(t)\| \exp(k|z|\Lambda(t)) \exp(-k|z|\Lambda(t)) \|f(t, z)\|_2 dt \leq \\ &\|y_0\|_2 + \frac{1}{k} \left(\exp(k|z|\Lambda(x)) - 1 \right) \|f\|_G. \end{aligned}$$

Multiplying with $G(x, z)$ and taking the supremum over $(x, z) \in [a, b] \times \mathbb{C}$, we obtain $\|T(f)\|_G \leq \|y_0\|_G + \frac{1}{k} \|f\|_G < \infty$. Hence we get $T(f) \in X$.

Finally T is a strict contraction: For $f, g \in X$ we have

$$\|T(f)(x, z) - T(g)(x, z)\|_2 G(x, z) \leq \exp(-k|z|\Lambda(x)) \int_a^x |z| \|A(t)\| \|f(t, z) - g(t, z)\|_2 dt \leq$$

$$\begin{aligned} & \exp(-k|z|\Lambda(x)) \int_a^x |z| \|A(t)\| \exp(k|z|\Lambda(t)) \|f - g\|_G dt = \\ & \frac{1}{k} \exp(-k|z|\Lambda(x)) (\exp(k|z|\Lambda(x)) - 1) \|f - g\|_G, \end{aligned}$$

and hence $\|T(f) - T(g)\|_G \leq \frac{1}{k} \|f - g\|_G$. By the Banach Fixed Point theorem, there exists a unique $f \in X$, which satisfies $Tf = f$. In particular $x \mapsto f(x, z)$ is absolutely continuous. Differentiating almost everywhere, shows that $x \mapsto f(x, z)$ satisfies the differential equation (1.1) and obviously $f(a, z) = y_0$ holds as well. \square

1.2 Properties of the Fundamental Solution

In the proof of Theorem 1.3 we saw that the solution $y(x, z)$ lies in the space X . Thus, for each $k > 1$, it satisfies an estimate of the form ($x \in [a, b]$)

$$\|y(x, z)\|_2 \leq C_k \exp(k|z|\Lambda(x)), \quad t \in \mathbb{C},$$

with some constant $C_k > 0$. The Gronwall Lemma says that such an estimate holds for $k = 1$ as well.

Lemma 1.6 (Gronwall). *Let $y(x, z)$ be a solution of (1.1). Then*

$$\|y(x, z)\|_2 \leq \|y_0\|_2 \exp\left(|z| \int_a^x \|A(t)\| dt\right), \quad (x, z) \in [a, b] \times \mathbb{C}. \quad (1.3)$$

Proof. Obviously it holds that

$$y(x, z) = y_0 - z \int_a^x A(s)y(s, z) ds, \quad (1.4)$$

which implies

$$\|y(x, z)\|_2 \leq \|y_0\|_2 + \left(|z| \int_a^x \|A(t)\| \|y(t, z)\|_2 dt\right).$$

For $a \leq s \leq x \leq b$ we further evaluate

$$\frac{d}{ds} \ln \left(\|y_0\|_2 + |z| \int_a^s \|A(t)\| \|y(t, z)\|_2 dt \right) = \frac{|z| \|A(s)\| \|y(s, z)\|_2}{\|y_0\|_2 + |z| \int_a^s \|A(t)\| \|y(t, z)\|_2 dt}.$$

The right hand side is bounded from above by $|z| \|A(s)\|$, hence integrating from a to x with respect to s yields

$$\ln \left(\|y_0\|_2 + |z| \int_a^x \|A(t)\| \|y(t, z)\|_2 dt \right) - \ln (\|y_0\|_2) \leq \int_a^x |z| \|A(s)\| ds.$$

Applying the exponential function on this inequality completes the proof. \square

Definition 1.7. Let $A \in L^1([a, b], \mathbb{C}^{2 \times 2})$. Let y_1 and y_2 be the solutions of (1.1) with initial value $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. Then the matrix function $M : [a, b] \times \mathbb{C} \rightarrow \mathbb{C}^{2 \times 2}$ defined as

$$M(x, z) := (y_1(x, z) \mid y_2(x, z)),$$

is called *fundamental solution* of (1.1). It is the solution of the matrix-valued initial value problem

$$\begin{cases} \frac{d}{dx} Y(x, z) = -zA(x)Y(x, z), & x \in [a, b], \\ Y(a, z) = I, \end{cases} \quad (1.5)$$

Its value at the right endpoint is called the *monodromy matrix* of (1.1).

Remark 1.8. Note that calling M a fundamental solution is not very precise, since the dimensions for the initial value do not match.

Proposition 1.9. *The fundamental solution $M(x, z)$ of (1.5) has the following properties.*

(i) *Let $a \leq y \leq x \leq b$, and let $M_{y,x}$ be the fundamental solution for $A|_{[y,x]}$. Then*

$$M(c, z) = M_{y,x}(c, z)M(y, z), \quad c \in [y, x], \quad z \in \mathbb{C}.$$

(ii) *M is continuous on $[a, b] \times \mathbb{C} \rightarrow \mathbb{C}^{2 \times 2}$, absolutely continuous in the first argument and holomorphic in the second argument.*

(iii) *If A is real a.e., then the entries $M_{ij}(x, z)$ of $M(x, z)$ satisfy*

$$\overline{M_{ij}(x, \bar{z})} = M_{ij}(x, z), \quad x \in [a, b], \quad z \in \mathbb{C}. \quad (1.6)$$

(iv) *Let $Q \in \text{GL}(2, \mathbb{C})$ and $a \leq y \leq x \leq b$. Then*

$$\|QM(x, z)\| \leq \|QM(y, z)\| \exp\left(|z| \int_y^x \|QA(t)Q^{-1}\| dt\right). \quad (1.7)$$

Further, the fundamental solution is estimated as

$$\|M(x, z)\| \leq \exp\left(|z| \int_a^x \|A(t)\| dt\right), \quad (x, z) \in [a, b] \times \mathbb{C}. \quad (1.8)$$

(v) *The fundamental solution satisfies $\det M(a, z) = 1$ and the differential equation*

$$\frac{d}{dx}(\det M(x, z)) = z \operatorname{tr} A(x) \det M(x, z). \quad (1.9)$$

Hence,

$$\det M(x, z) = \exp\left(z \int_a^x \operatorname{tr} A(t) dt\right). \quad (1.10)$$

Proof. Let $a \leq y \leq x \leq b$. We have $M_{y,x}(y, z) = I$ by definition and hence

$$M(y, z) = M_{y,x}(y, z)M(y, z).$$

Further $M(c, z)$ and $M_{y,x}(c, z)$ both satisfy the differential equation (1.5) for $c \in [x, y]$. Since $M(y, z)$ is a constant factor on the right, also $M_{y,x}(c, z)M(y, z)$ satisfies the differential equation (1.5). By uniqueness of solutions, we obtain (i).

By Theorem 1.3 $y_1, y_2 : [a, b] \times \mathbb{C} \rightarrow \mathbb{C}^2$ are continuous, absolutely continuous in the first argument and holomorphic in the second. Hence $M : [a, b] \times \mathbb{C} \rightarrow \mathbb{C}^{2 \times 2}$ satisfies the same properties and therefore it satisfies (ii) as well.

Since

$$\frac{d}{dx}(\overline{M(x, z)}) = \overline{\frac{d}{dx}M(x, z)} = \overline{-zA(x)M(x, z)}$$

almost everywhere on $[a, b]$, the matrix $\overline{M(t, \bar{z})}$ is the solution of

$$\begin{cases} \frac{d}{dx}Y(x, z) = -z\overline{A(x)}Y(x, z), & x \in [a, b], \\ Y(a, z) = I. \end{cases}$$

In particular, if $A(x) \in \mathbb{R}^{2 \times 2}$ for $x \in [a, b]$, uniqueness of solutions implies

$$\overline{M(x, \bar{z})} = M(x, z), \quad (x, z) \in [a, b] \times \mathbb{C}.$$

For each $a \leq y \leq x \leq b$ a fundamental solution satisfies the integral equation

$$M(x, z) = M(y, z) - z \int_y^x A(t)M(t, z)dt. \quad (1.11)$$

Now let $Q \in \text{GL}(2, \mathbb{C})$. Then we can rewrite this equation as follows

$$QM(x, z) = QM(y, z) - z \int_y^x (QA(t)Q^{-1})QM(t, z)dt.$$

From now on we follow the lines of the proof of Lemma 1.6: The upper equality implies

$$\|QM(x, z)\|_2 \leq \|QM(y, z)\|_2 + \left(|z| \int_a^x \|QA(t)Q^{-1}\| \|QM(t, z)\|_2 dt \right).$$

For $a \leq s \leq x \leq b$ we further evaluate

$$\begin{aligned} \frac{d}{ds} \ln \left(\|QM(y, z)\|_2 + |z| \int_a^s \|QA(t)Q^{-1}\| \|QM(t, z)\|_2 dt \right) = \\ \frac{|z| \|QA(s)Q^{-1}\| \|QM(s, z)\|_2}{\|QM(y, z)\|_2 + |z| \int_a^s \|QA(t)Q^{-1}\| \|QM(t, z)\|_2 dt}. \end{aligned}$$

The right hand side is bounded from above by $|z| \|QA(s)Q^{-1}\|$, hence integrating from a to x with respect to s yields

$$\begin{aligned} \ln \left(\|QM(y, z)\|_2 + |z| \int_a^x \|QA(t)Q^{-1}\| \|QM(t, z)\|_2 dt \right) - \ln (\|QM(y, z)\|_2) \leq \\ \leq \int_a^x |z| \|QA(s)Q^{-1}\| ds. \end{aligned}$$

Applying the exponential function on this inequality yields assertion (v).

For the last property we start by evaluating the left hand side, (omitting the arguments of our fundamental solution to simplify notation)

$$\begin{aligned} \det M' &= (M_{11}M_{22} - M_{12}M_{21})' = M'_{11}M_{22} - M'_{12}M_{21} + M'_{22}M_{11} - M'_{21}M_{12} = \\ &= z \left(A_{11}(x)M_{11}M_{22} + A_{12}(x)M_{21}M_{22} - A_{11}(x)M_{12}M_{21} - A_{12}(x)M_{22}M_{21} + \right. \\ &\quad \left. + A_{21}(x)M_{12}M_{11} + A_{22}(x)M_{22}M_{11} - A_{21}(x)M_{11}M_{12} - A_{22}(x)M_{21}M_{12} \right) = \\ &= z A_{11}(x)M_{11}M_{22} - z A_{11}(x)M_{12}M_{21} + z A_{22}(x)M_{22}M_{11} - z A_{22}(x)M_{21}M_{12} = \\ &= z \text{tr } A(t) \det M \end{aligned}$$

and we obtain the desired equality. Plugging the right hand side of (1.10) in (1.9) completes the proof. \square

The next result is easy to prove, but a very important fact.

Theorem 1.10. *Let $J \in \mathbb{C}^{2 \times 2}$, with $J^* = -J$ and assume that JA is hermitian a.e., then*

$$M(x, w)^* JM(x, z) - J = (\bar{w} - z) \int_a^x M(t, w)^* JA(t)M(t, z)dt \quad z, w \in \mathbb{C}. \quad (1.12)$$

Proof. We evaluate

$$\begin{aligned} \frac{d}{dx} [M(x, w)^* JM(x, z)] &= \left[\frac{d}{dx} M(x, w) \right]^* JM(x, z) + M(x, w)^* J \left[\frac{d}{dx} M(x, z) \right] = \\ &= \left[-J \frac{d}{dx} M(x, w) \right]^* M(x, z) + M(x, w)^* J \left[\frac{d}{dx} M(x, z) \right]. \end{aligned}$$

Because JA is hermitian the right hand side further equals

$$\bar{w}M(x, w)^* JA(x)M(x, z) - zM(x, w)^* JA(x)M(x, z) = (\bar{w} - z)M(x, w)^* JA(x)M(x, z),$$

and the equality follows by integrating with respect to x . \square

Corollary 1.11. *Let $J \in \mathbb{C}^{2 \times 2}$, with $J^* = -J$ and assume that $-JA$ is hermitian and positive semidefinite a.e., then*

$$\frac{M(x, z)^* JM(x, z) - J}{i} \geq 0, \quad x \in [a, b], \quad \text{Im}(z) > 0. \quad (1.13)$$

Proof. We can put $w = z$ in (1.12), because if $-JA$ is hermitian, JA is hermitian as well, and we obtain

$$M(x, z)^* JM(x, z) - J = -2i \text{Im}(z) \int_a^x M(t, z)^* JA(t)M(t, z) dt.$$

After we divide by i , the right hand side is greater or equal to 0, by positive semidefiniteness of $-JA$. \square

Corollary 1.12. *Let $J \in \mathbb{C}^{2 \times 2}$, with $J^* = -J$ and assume that $-JA$ is hermitian and positive semidefinite a.e.. Then the entry $M_{11}(x, z)$ has no nonreal zeroes.*

Proof. Evaluating $M(x, z)^* JM(x, z) - J$, yields (omitting the arguments (x, z) from the matrix entries)

$$\begin{pmatrix} M_{11}\bar{M}_{21} - M_{21}\bar{M}_{11} & M_{12}\bar{M}_{21} - \bar{M}_{11}M_{22} + 1 \\ M_{11}\bar{M}_{22} - M_{21}\bar{M}_{12} - 1 & M_{12}\bar{M}_{22} - M_{22}\bar{M}_{12} \end{pmatrix}.$$

Since, by Corollary 1.11, this is positive semidefinite for $\text{Im}(z) > 0$, after dividing by i , we conclude that

$$\frac{M_{11}\bar{M}_{21} - M_{21}\bar{M}_{11}}{i} \geq 0.$$

Note that, due to which, for $M_{21}(x, z) \neq 0$, is equivalent to

$$\frac{\frac{M_{11}}{M_{21}} - \overline{\left(\frac{M_{11}}{M_{21}}\right)}}{i} \geq 0.$$

Hence $\text{Im}\left(\frac{M_{11}}{M_{21}}\right) \geq 0$. Now, if $\text{Im}\left(\frac{M_{11}}{M_{21}}(x, z_0)\right) = 0$, for a $z_0 \in \mathbb{C}^+$, by the maximum principle we obtain that $\text{Im}\left(\frac{M_{11}}{M_{21}}(x, z)\right) \equiv 0$, for $z \in \mathbb{C}^+$ and hence $\text{Im}(M_{11}(x, z)) \equiv 0$ for $z \in \mathbb{C}$ holds as well. Therefore $\text{Re}(M_{11}(x, z))$ has to be constant as well, and hence has to be equal to the initial value, 1. \square

Corollary 1.13. *Let $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and assume that JA is real and hermitian a.e.. Then $\det M(x, z) = 1$ for $x \in [a, b], z \in \mathbb{C}$.*

Proof. We have that

$$JA(x) = \begin{pmatrix} -A_{21}(x) & -A_{22}(x) \\ A_{11}(x) & A_{12}(x) \end{pmatrix}$$

is real and hermitian, or in other words, $A_{11}(x) = -A_{22}(x)$. Therefore $\text{tr } A(x) = 0$ and hence (1.10) yields the assertion. \square

The results above motivate the following definition.

Definition 1.14. A (2×2) -matrix-valued function $W(z)$ belongs to the class \mathcal{M}_0 if its entries $w_{ij}(z)$ are real entire functions, $W(0) = I$, $\det W(z) = 1$ and if

$$H_W(w, z) := \frac{W^*(w)JW(z) - J}{\bar{w} - z}$$

is a positive semidefinite kernel.

The following lemma shows that \mathcal{M}_0 is closed with respect to products.

Lemma 1.15. *Let $W_1, W_2 \in \mathcal{M}_0$. Then,*

$$H_{W_1W_2} = H_{W_2} + W_2^*(w)H_{W_1}(w, z)W_2(z)$$

Proof. We compute

$$\begin{aligned} H_{W_1W_2} &= \frac{W_2^*(w)W_1^*(w)JW_1(z)W_2(z) - W_2^*(w)JW_2(z) + W_2^*(w)JW_2(z) - J}{\bar{w} - z} = \\ &= W_2^*(w) \frac{W_1^*(w)JW_1(z) - J}{\bar{w} - z} W_2(z) + \frac{W_2^*(w)JW_2(z) - J}{\bar{w} - z} = \\ &= W_2^*(w)H_{W_1}(w, z)W_2(z) + H_{W_2}(w, z), \end{aligned}$$

and have shown the assertion. \square

2 Exponential Type

We recall the notion of exponential type, and present a first result connected with the initial value problem (1.5). Namely we present Theorem 2.4, which provides a much better estimate for the exponential type of the solutions of (1.5) than the Gronwall Lemma (see Lemma 1.6). In fact we show that for canonical systems this upper estimate coincides with the type (Corollary 2.9).

Definition 2.1. Let X be a Banach space, and $f : \mathbb{C} \rightarrow X$. We say f is of *finite exponential type*, if there exist $\alpha \in [0, \infty)$ and $C_\alpha \geq 0$ such that $\|f(z)\| \leq C_\alpha \exp(\alpha|z|)$ for all $z \in \mathbb{C}$.

We call $\tau(f) := \inf\{\alpha > 0 : \exists C_\alpha > 0 : \|f(z)\| \leq C_\alpha \exp(\alpha|z|)\}$ the *exponential type* of f .

Remark 2.2. Assuming that f and g are elements of a Banach algebra, we immediately obtain the following two properties:

$$(i) \quad \tau(fg) \leq \tau(f) + \tau(g),$$

$$(ii) \quad \tau(f + g) \leq \max\{\tau(f), \tau(g)\}.$$

Lemma 2.3. Let $X = \mathbb{C}^{2 \times 2}$, and $A = (A_{i,j})_{i,j \in \{1,2\}} : \mathbb{C} \rightarrow X$. Then

$$\tau(A) = \max_{i,j \in \{1,2\}} \{\tau(A_{ij})\}.$$

Proof. Since all norms are equivalent on $\mathbb{C}^{2 \times 2}$ we can use the maximum entry norm to compute $\tau(A)$. \square

Theorem 2.4. Let $A \in L^1([a, b], \mathbb{C}^{2 \times 2})$, and let $M(x, z)$ be the fundamental solution of (1.5). Set

$$\phi(x) := \inf_{Q \in \text{GL}(2, \mathbb{C})} \|QA(x)Q^{-1}\|, \quad \Phi(x) := \int_a^x \phi(t) dt, \quad x \in [a, b].$$

Then $\tau(M(x, z)) \leq \Phi(x)$ for $x \in [a, b]$.

Proof. First we note that the assertion is trivial if we consider $x = a$. Next we want to verify that $x \mapsto \tau(M(x, z))$ is absolutely continuous. To this end let $a \leq y < x \leq b$. By Proposition 1.9 (i), we have

$$M(x, z) = M_{y,x}(x, z)M(y, z),$$

which is equivalent to $M(y, z) = M_{y,x}(x, z)^{-1}M(x, z)$. Hence, by Remark 2.2 and Lemma 2.3, we obtain

$$\begin{aligned} \tau(M(x, z)) &\leq \tau(M_{y,x}(x, z)) + \tau(M(y, z)), \\ \tau(M(y, z)) &\leq \tau(M_{y,x}(x, z)^{-1}) + \tau(M(x, z)). \end{aligned}$$

Combining these two inequalities yields

$$|\tau(M(x, z)) - \tau(M(y, z))| \leq \max\{\tau(M_{y,x}(x, z)), \tau(M_{y,x}(x, z)^{-1})\}. \quad (2.1)$$

The first term in the maximum can be estimated from above by $\int_y^x \|A(t)\| dt$, due to the Gronwall Lemma. To estimate the second term, let $\text{cof } M_{y,x}(x, z)$ denote the cofactor matrix of $M_{y,x}(x, z)$. By Remark 2.2 and Lemma 2.3, $\tau(\text{cof } M_{y,x}(x, z)) = \tau(M_{y,x}(x, z))$. Cramers rule says

$$M_{y,x}(x, z)^{-1} = \frac{1}{\det M_{y,x}(x, z)} (\text{cof } M_{y,x}(x, z))^T,$$

and applying Proposition 1.9 (v), we obtain

$$\begin{aligned}\tau(M_{y,x}(x,z)^{-1}) &\leq \tau\left(\frac{1}{\det M_{y,x}(x,z)}\right) + \tau((\operatorname{cof} M_{y,x}(x,z))^T) \leq \\ &\leq \left|\int_y^x \operatorname{tr} A(t) dt\right| + \tau(M_{y,x}(x,z)) \leq \int_y^x (|\operatorname{tr} A(t)| + \|A(t)\|) dt.\end{aligned}$$

Hence we can estimate the left hand side of (2.1) by $\int_y^x (|\operatorname{tr} A(t)| + \|A(t)\|) dt$, and conclude that $x \mapsto \tau(M(x,z))$ is absolutely continuous.

Let $Q \in \operatorname{GL}(2, \mathbb{C})$ and let again $a \leq y < x \leq b$. By (1.7) the fundamental solution satisfies

$$\|QM(x,z)\| \leq \|QM(y,z)\| \exp\left(|z| \int_y^x \|QA(t)Q^{-1}\| dt\right).$$

Since multiplying $M(x,z)$ by $Q \in \operatorname{GL}(2, \mathbb{C})$ does not change the exponential type, i.e., $\tau(QM(x,z)) = \tau(M(x,z))$, we get

$$\tau(M(x,z)) - \tau(M(y,z)) \leq \int_y^x \|QA(t)Q^{-1}\| dt.$$

This is equivalent to

$$\frac{\tau(M(x,z)) - \tau(M(y,z))}{x-y} \leq \frac{\int_y^x \|QA(t)Q^{-1}\| dt}{x-y}.$$

For almost every x both sides have a limit when $y \rightarrow x$, and we obtain

$$\tau(M(x,z))' \leq \|QA(x)Q^{-1}\|, \quad x \in [a,b] \text{ a.e..}$$

Taking the infimum over $Q \in \operatorname{GL}(2, \mathbb{C})$, and integrating yields our assertion. \square

Lemma 2.5. *Let $A \in \mathbb{C}^{2 \times 2}$ with $\operatorname{tr} A = 0$ and let $\|\cdot\|$ denote the spectral norm. Then*

$$\inf_{Q \in \operatorname{GL}(2, \mathbb{C})} \|QAQ^{-1}\| = \sqrt{|\det A|}.$$

Proof. Taking an appropriate basis of $\mathbb{C}^{2 \times 2}$ and composing it to a matrix $C \in \operatorname{GL}(2, \mathbb{C})$, we obtain that either

$$CAC^{-1} = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}, \text{ or } CAC^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

By assumption and because $\operatorname{tr}(CAC^{-1}) = \operatorname{tr}(CC^{-1}A) = \operatorname{tr}(A)$, we have $a = 0$ in the first case and $b = -a$ in the second.

In the first case we get $\sqrt{|\det A|} = 0$. For $r > 0$ and $B = \operatorname{diag}(\frac{1}{r}, r)C$, we have $BAB^{-1} = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}$ and hence $\inf_{Q \in \operatorname{GL}(2, \mathbb{C})} \|QAQ^{-1}\| = 0$.

In the second case we have $\sqrt{|\det A|} = |a| = \|CAC^{-1}\|$ and hence

$$\inf_{Q \in \operatorname{GL}(2, \mathbb{C})} \|QAQ^{-1}\| \leq a.$$

For arbitrary $Q \in \operatorname{GL}(2, \mathbb{C})$, the norm $\|QAQ^{-1}\|$ is greater or equal to the maximum of the absolute values of the eigenvalues of QAQ^{-1} , and hence $\inf_{Q \in \operatorname{GL}(2, \mathbb{C})} \|QAQ^{-1}\| \geq |a|$. \square

Corollary 2.6. *If $\text{tr } A(x) = 0$ a.e., then*

$$\tau(M(x, z)) \leq \int_a^x \sqrt{\det A(t)} dt. \quad (2.2)$$

Proof. This result follows immediately from Lemma 2.5 and Theorem 2.4. \square

While the estimate (2.2) of the type from above holds under an assumption on the trace of A , the reverse inequality will follow from a definiteness assumption.

Theorem 2.7. *Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and assume that $-JA$ is a.e. positive semidefinite and real. Then*

$$\tau(M(x, z)) \geq \int_a^x \sqrt{\det A(t)} dt. \quad (2.3)$$

To prove this result we need the following general lemma.

Lemma 2.8. *Let $M(x, z)$ be the fundamental solution corresponding to $A \in L^1([a, b], \mathbb{C}^{2 \times 2})$, and let $\Psi : [a, b] \rightarrow \mathbb{R}$ be some absolutely continuous function with $\Psi(a) = 0$. Then*

$$M_\Psi(x, z) := \exp(iz\Psi(x))M(x, z)$$

is the fundamental solution of the system

$$\begin{cases} \frac{d}{dx}Y(x, z) = -z(A(x) - \Psi'(x)iI)Y(x, z), & x \in [a, b], \\ Y(a, z) = I. \end{cases}$$

Proof. First we note that $M_\Psi(a, z) = M(a, z) = I$. Further we obtain

$$\begin{aligned} \frac{d}{dx}M_\Psi(x, z) &= iz\Psi'(x) \exp(iz\Psi(x))M(x, z) + \exp(iz\Psi(x)) \frac{d}{dx}M(x, z) = \\ &= iz\Psi'(x)M_\Psi(x, z) - \exp(iz\Psi(x))zA(x)M(x, z) \\ &= -z(A(x) - i\Psi'(x)I)M_\Psi(x, z), \end{aligned}$$

and by uniqueness of solutions the proof is complete. \square

Proof of Theorem 2.7. Let $-JA \geq 0$ and write $-JA(x) = \begin{pmatrix} h_1(x) & h_3(x) \\ h_3(x) & h_2(x) \end{pmatrix}$, where $h_i(x)$, $i \in \{1, 2, 3\}$, are real and $h_1, h_2 \geq 0$. Let $\Psi(x) := \int_a^x \sqrt{\det A(t)} dt$, then

$$-J(A(x) - \Psi'(x)iI) = \begin{pmatrix} h_1(x) & h_3(x) - i\sqrt{\det A(x)} \\ h_3(x) + i\sqrt{\det A(x)} & h_2(x) \end{pmatrix}.$$

We have

$$\det \left(-J(A(x) - \Psi'(x)iI) \right) = h_1(x)h_2(x) - h_3^2(x) - \det A(x) = 0,$$

and hence $-J(A(x) - \Psi'(x)iI) \geq 0$. For $y > 0$, this implies by Corollary 1.11

$$\det \left(\frac{M_\Psi(x, iy)^* J M_\Psi(x, iy) - J}{i} \right) \geq 0. \quad (2.4)$$

Now assume that $\tau(M(x, z)) < \int_a^x \sqrt{\det A(t)} dt = \Psi(x)$. Choose $\epsilon > 0$ such that $\tau(M(x, z)) + \epsilon < \Psi(x)$ and $C_\epsilon > 0$ such that

$$|M_{ij}(x, z)| \leq C_\epsilon \exp((\tau(M(x, z)) + \epsilon)|z|), \quad z \in \mathbb{C}.$$

Then

$$\lim_{y \rightarrow \infty} M_{\Psi}(x, iy) = \lim_{y \rightarrow \infty} \exp(-y\Psi(x))M(x, iy) = 0.$$

and hence

$$\lim_{y \rightarrow \infty} \frac{M_{\Psi}(x, iy)^* J M_{\Psi}(x, iy) - J}{i} = iJ.$$

Since $\det iJ = -1$, that contradicts (2.4). □

Putting together Theorem 2.7 and Corollary 2.6 for a canonical system, equality holds in (2.2).

Corollary 2.9. *If $\operatorname{tr} A(x) = 0$ a.e. and $-JA$ is positive semidefinite and real a.e., then*

$$\tau(M(x, z)) = \int_a^x \sqrt{\det A(t)}.$$

Proof. This follows immediately from Corollary 2.6 and Theorem 2.7. □

3 Growth Functions and λ -type

The question arises to which extent Theorem 2.4, or the method leading to its proof, can be used to also capture different growth behaviour, for example with respect to an order smaller than 1.

This chapter is of a preparatory nature. We introduce the notion of growth functions and give some auxiliary results.

Throughout this chapter the order of an entire function will often be of importance or interest. Therefore we start with its definition.

Definition 3.1. Let f be an entire function. Then, we define the *order* of f as

$$\text{ord } f := \limsup_{z \rightarrow \infty} \frac{\log \log (|f(z)|)}{\log (|z|)} \in [0, \infty].$$

3.1 Proximate Orders and Growth Functions

This section follows the section of the same name of [5], but a similar chapter can be found in [6] as well

Definition 3.2. We call a function $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a *growth function*, if it satisfies the following properties

- (i) The limit $\gamma := \lim_{r \rightarrow \infty} \frac{\log(\lambda(r))}{\log(r)}$ exists and is a finite nonnegative number,
- (ii) λ is differentiable for all sufficiently large values of r and $\lim_{r \rightarrow \infty} \frac{r\lambda'(r)}{\lambda(r)} = \gamma$,
- (iii) $\log(r) = o(\lambda(r))$.

Let either $\alpha > 0$ and $\beta_i \in \mathbb{R}$ for $i \in \{1, \dots, n\}$, or $\alpha = 0$, $\beta_1 = 1$, $\beta_j \in \mathbb{R}$ and for $j < i \leq n$. Then an examples for growth functions are given by functions of the form

$$\lambda(r) = r^\alpha (\log(r))^{\beta_1} \cdot \dots \cdot (\log_n(r))^{\beta_n},$$

where $\log_i(r)$ is defined as the i times iterated logarithm,

The following definition gives us information about the relative growth of f compared to λ .

Definition 3.3. Let f be an entire function and λ a growth function. The λ -*type* of f is given by

$$\sigma_f^\lambda := \limsup_{z \rightarrow \infty} \frac{\log^+ (|f(z)|)}{\lambda(|z|)} \in [0, \infty]. \quad (3.1)$$

Further we introduce the following notion, which is closely related to growth functions. We will elaborate their connection in Lemma 3.6.

Definition 3.4. A *proximate order* $\rho(r)$ for the order $\rho \geq 0$, is a function $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

- (i) $\lim_{r \rightarrow \infty} \rho(r) = \rho$,
- (ii) $\lim_{r \rightarrow \infty} \rho'(r)r \log r = 0$.

For the entire function $f(z)$ we set

$$\sigma_f = \limsup_{r \rightarrow \infty} \frac{\log(|f(r)|)}{r^{\rho(r)}}$$

and call σ_f the *type of f with respect to ρ* . If $\sigma_f \in (0, \infty)$, $\rho(r)$ is called a proximate order of the function $f(z)$.

Remark 3.5. We note that for any positive continuous increasing function f of finite order, there exists a proximate order with respect to which f is of finite and nonzero type. For a proof see [5], Appendix II.

Clearly the proximate order and its corresponding type of a given function are not uniquely determined. For example, if we add $\log(c)/\log(r)$ to the proximate order, then we obtain a new proximate order for the function, and now the type has been divided by c .

Lemma 3.6. *The following assertions hold:*

- (i) *Let $\rho(r)$ be a proximate order with respect to the order $\rho > 0$. Then $\lambda(r) := r^{\rho(r)}$ is a growth function.*
- (ii) *Let $\rho(r)$ be a proximate order with respect to the order $\rho = 0$ and $\log(r) = o(r^{\rho(r)})$. Then $\lambda(r) := r^{\rho(r)}$ is a growth function.*
- (iii) *Conversely, if $\lambda(r)$ is a growth function, then $\rho(r) := \frac{\log(\lambda(r))}{\log(r)}$ is a proximate order with respect to the order γ , where $\gamma := \lim_{r \rightarrow \infty} \frac{\log(\lambda(r))}{\log(r)}$ as indicated in Definition 3.2.*

Proof. First we prove (i). To this end let $\rho(r)$ be a proximate order, with respect to the order ρ and $\lambda(r) := r^{\rho(r)}$. Then property (i) of Definition 3.2 follows, because we get $\log(\lambda(r)) = \rho(r)\log(r)$ and apply Definition 3.4 (i). To validate (ii), we evaluate

$$\lambda'(r) = r^{\rho(r)} \left(\log(r^{\rho'(r)}) + \frac{\rho(r)}{r} \right).$$

Multiplying this by $\frac{r}{\lambda(r)}$ and applying Definition 3.4 (i) and (ii) yields the desired assertion. Further, Definition 3.2 (iii) is obvious, because $\rho(x) > 0$ by definition, and we obtain assertion (i).

If we consider the situation in assertion (ii), we observe that Definition 3.2 (i) and (ii) follow analogously. The last point of this definition is fulfilled by the assumption in (ii).

To prove (iii) let λ be a growth function. For $\rho(r) := \frac{\log(\lambda(r))}{\log(r)}$ Definition 3.4 (i) is satisfied, by Definition 3.2 (i), and we observe that $\rho = \gamma$. To verify Definition 3.4 (ii), we evaluate

$$\rho'(r) = \frac{\frac{\lambda'(r)\log(r)}{\lambda(r)} - \frac{\log(\lambda(r))}{r}}{\log^2(r)},$$

and hence we obtain $\lim_{r \rightarrow \infty} \rho'(r)r \log r = 0$, by applying Definition 3.2 (i) and (ii). \square

Proposition 3.7. *If $\rho(r)$ is a proximate order with $\rho > 0$, then $r^{\rho(r)}$ is strictly increasing for r sufficiently large.*

If $\rho(r)$ is a proximate order with $\rho = 0$, then $r^{\rho(r)}$ is increasing for r sufficiently large, if

$$\lim_{r \rightarrow \infty} \frac{r\rho'(r)\log(r)}{\rho(r)} = 0. \quad (3.2)$$

Proof. We have $\frac{d}{dr} (r^{\rho(r)}) = \rho(r)r^{\rho(r)-1} + r^{\rho(r)}\rho'(r)\log r$.

First let $\rho > 0$, then, by Definition 3.4 (i) and (ii), we get $\rho(r) > \rho/2$ and $|\rho'(r)r\log r| < \rho/4$ for all r sufficiently large. For such r , we conclude

$$r^{\rho(r)-1} (\rho(r) + r\rho'(r)\log r) > r^{\rho(r)-1}\rho/4,$$

and because $r^{\rho(r)-1} > 0$ we obtain our assertion.

Let $\rho = 0$, then $\frac{r\rho'(r)\log r}{\rho(r)} \geq -\frac{1}{2}$ for r sufficiently large, by (3.2). Hence

$$\frac{d}{dr} (r^{\rho(r)}) = r^{\rho(r)-1}\rho(r) \left(1 + \frac{r\rho'(r)\log r}{\rho(r)} \right) > 0,$$

for r large enough, which completes the proof. \square

Remark 3.8. Since in the study of asymptotic properties of entire functions we are only interested in their properties for r sufficiently large, we can always change $\rho(r)$ on a bounded set, without affecting the asymptotic properties we study. Thus for $\rho > 0$, we can always assume that $r^{\rho(r)}$ is strictly increasing for $r > 0$.

3.2 Slowly varying functions

Definition 3.9. Let $\Psi : [X, \infty) \rightarrow \mathbb{R}^+$ be a measurable function. We say that Ψ is *slowly varying*, if

$$\lim_{x \rightarrow \infty} \frac{\Psi(kx)}{\Psi(x)} \rightarrow 1, \quad k > 0. \quad (3.3)$$

Remark 3.10. Note that $\lim_{x \rightarrow \infty} \frac{\Psi(kx)}{\Psi(x)} = 1$ if and only if $\lim_{x \rightarrow \infty} \frac{\Psi(\frac{1}{k}x)}{\Psi(x)} = 1$. Hence, in order to have Ψ slowly varying, it is enough to check (3.3) on any set $M \subseteq (0, \infty)$ with $M \cup M^1 = (0, \infty)$.

The following theorem provides a useful result, in the context of slowly varying functions. For the sake of completeness, we present a proof taken from [3], section 1.2..

Theorem 3.11. [*Uniform Convergence*] Let I be a compact interval in $(0, \infty)$ and Ψ be slowly varying, then

$$\lim_{x \rightarrow \infty} \frac{\Psi(kx)}{\Psi(x)} \rightarrow 1, \quad \text{uniformly for } k \in I. \quad (3.4)$$

Proof. Write $h(x) = \log(\Psi(e^x))$. Then our assumption is equivalent to

$$\lim_{x \rightarrow \infty} (h(x+u) - h(x)) = 0, \quad (3.5)$$

for all $u \in \mathbb{R}$ and we have to prove uniform convergence for $u \in I$, where I is any compact interval in \mathbb{R} . It suffices to prove uniform convergence on $[0, A]$, for by translation this gives us uniform convergence on every compact interval.

Now choose $\epsilon \in (0, A)$. For $x > 0$ let

$$\begin{aligned} I_x &:= [x, x + 2A], \\ E_x &:= \{t \in I_x : |h(t) - h(x)| \geq \epsilon/2\}, \\ E_x^* &:= \{t \in [0, 2A] : |h(x+t) - h(x)| \geq \epsilon/2\} = E_x - x. \end{aligned}$$

Then E_x, E_x^* are measurable, as Ψ is, and $|E_x| = |E_x^*|$, where $|\cdot|$ denotes the Lebesgue measure. By (3.5) the indicator function of E_x^* tends pointwise to 0 as $x \rightarrow \infty$. So by dominated convergence its integral $|E_x^*|$ tends to 0. Thus $|E_x| \leq \epsilon/2$ for x large enough, say, for $x \geq x_\epsilon$.

Now, for $c \in [0, A]$, $I_{x+c} \cap I_x = [x+c, x+2A]$ has length $2A - c \geq A$, while for $x \geq x_\epsilon$

$$|E_{x+c} \cup E_x| \leq |E_{x+c}| + |E_x| < \epsilon < A.$$

So for $c \in [0, A]$ and $x \geq x_\epsilon$

$$(I_{x+c} \cap I_x) \setminus (E_{x+c} \cup E_x)$$

has positive measure, and therefore is non-empty. Let t be a point of this set, then

$$\begin{aligned} |h(t) - h(x)| &< \epsilon/2, \\ |h(t) - h(x+c)| &< \epsilon/2. \end{aligned}$$

So, by the triangular inequality, for all $c \in [0, A]$ and $x \geq x_\epsilon$,

$$|h(x+c) - h(x)| < \epsilon,$$

proving the desired uniformity on $[0, A]$, and hence the theorem. \square

The following theorem is very important in the study of proximate orders.

Theorem 3.12. *If $\rho(r)$ is a proximate order, then $\Psi(r) = r^{\rho(r)-\rho}$ is a slowly varying function.*

Proof. By definition of Ψ , we obtain

$$\begin{aligned} \log \left(\frac{\Psi(kr)}{\Psi(r)} \right) &= \log \left((kr)^{\rho(kr)-\rho} \right) - \log \left(r^{\rho(r)-\rho} \right) = \\ &= (\rho(kr) - \rho) (\log(k) + \log(r)) - (\rho(r) - \rho) \log(r) = \\ &= (\rho(kr) - \rho) \log(k) + (\rho(kr) - \rho(r)) \log(r). \end{aligned} \quad (3.6)$$

Let $\epsilon > 0$ and choose r_0 such that

$$|\rho'(x)x \log(x)| \leq \epsilon, \quad x \geq r_0 \quad (3.7)$$

$$\frac{\log(r)}{\log(r) + \log(k)} \leq 2 \quad r \geq r_0 \quad (3.8)$$

$$|\rho(x) - \rho| \leq \epsilon \quad x \geq r_0. \quad (3.9)$$

Now assume $k \in (0, 1)$. Then, by the Mean Value Theorem, $|\rho(r) - \rho(rk)| = |\rho'(\xi)|(r - rk)|$ for some $\xi \in (rk, r)$. Hence, for $r \geq \frac{r_0}{k}$, we have

$$\begin{aligned} \log(r)|\rho(rk) - \rho(r)| &\leq \frac{|\rho'(\xi)\xi \log(\xi)|(r - rk) \log(r)}{\xi \log(\xi)} \leq \\ &\leq \frac{|\rho'(\xi)\xi \log(\xi)|r(1-k) \log(r)}{kr \log(kr)} \leq \\ &\leq \epsilon \frac{1-k}{k} 2, \end{aligned}$$

by (3.7) and (3.8). Further by (3.9)

$$|(\rho(rk) - \rho) \log(k)| \leq \epsilon |\log(k)|$$

and hence, by (3.6), $\left| \log \left(\frac{\Psi(kr)}{\Psi(r)} \right) \right| < \epsilon \left(2 \frac{1-k}{k} + |\log(k)| \right)$.

By Remark 3.10, we conclude that $\left| \log \left(\frac{\Psi(kr)}{\Psi(r)} \right) \right| < \epsilon$ for all $r > r_0$ and $k \in (0, \infty)$. \square

Corollary 3.13. *Let $\epsilon > 0$. Then, for r sufficiently large*

$$(1 - \epsilon)k^\rho r^{\rho(r)} < (kr)^{\rho(kr)} < (1 + \epsilon)k^\rho r^{\rho(r)}, \quad r > r_0(\epsilon).$$

Proof. By Theorem 3.12, we obtain

$$(1 - \epsilon) \leq \frac{(kr)^{\rho(kr)} r^\rho}{(kr)^{\rho r^{\rho(r)}}} \leq (1 + \epsilon),$$

for r sufficiently large, and the assertion follows immediately. \square

Further we will need the following result, and prove it following the lines of [3], section 1.3.

Theorem 3.14. *Let I be a compact interval. Then the following two assertions hold.*

(i) *If the function Ψ is slowly varying, it may be written in the form*

$$\Psi(x) = c(x) \exp\left(\int_a^x \frac{\epsilon(u)}{u} du\right), \quad x \geq a, \quad (3.10)$$

for some $a > 0$, where $c(x)$ and $\epsilon(x)$ are measurable functions with $c(x) \rightarrow c \in (0, \infty)$ and $\epsilon(x) \rightarrow 0$ for $x \rightarrow \infty$, respectively.

(ii) *If the function Ψ can be written in the form given above, it satisfies (3.4).*

Remark 3.15. Note that this will not yield an alternative proof of the Uniform Convergence Theorem 3.11, since we apply it to show assertion (i).

Remark 3.16. Before we start the proof we make some observations.

(i) Since c , Ψ , ϵ may be altered on finite intervals, the value of a is unimportant, and one may also take c bounded. We may rewrite (3.10) as

$$\Psi(x) = \exp\left(c_1(x) + \int_a^x \frac{\epsilon(u)}{u} du\right), \quad (3.11)$$

where $c_1(x)$, $\epsilon(x)$ are bounded and measurable, $\lim_{x \rightarrow \infty} c_1(x) = d \in \mathbb{R}$, $\lim_{x \rightarrow \infty} \epsilon(x) = 0$.

(ii) We will write $h(x) := \log(\Psi(e^x))$. Thus we can prove Theorem 3.14 by showing that h satisfies $\lim_{x \rightarrow \infty} (h(x+y) - h(x)) = 0$, $y \in \mathbb{R}$ if and only if h may be written as

$$h(x) = d(x) + \int_b^x g(y) dy, \quad (3.12)$$

with $x \geq b = \log(a)$, $d(x) = c_1(e^x)$, $g(x) = \epsilon(e^x)$, where $\lim_{x \rightarrow \infty} d(x) \rightarrow d \in \mathbb{R}$, $\lim_{x \rightarrow \infty} g(x) \rightarrow 0$. This follows by applying the definition of h and by substitution in the integral.

Lemma 3.17. *If Ψ is positive, measurable, defined on some interval $[A, \infty)$ and*

$$\lim_{x \rightarrow \infty} \frac{\Psi(kx)}{\Psi(x)} \rightarrow 1,$$

for every $k > 0$, then Ψ is bounded on all finite intervals far enough to the right. If $h(x) := \log(\Psi(e^x))$, h is also bounded on finite intervals far enough to the right.

Proof. By the Uniform Convergence Theorem 3.11 and Remark 3.16 (ii), we get for x_0 sufficiently large,

$$|h(x+u) - h(x)| < 1, \quad x \geq x_0$$

for $u \in [0, 1]$. Hence, if we consider $x = x_0$ and write $y := x_0 + u$ by the triangular inequality, $|h(y)| \leq 1 + |h(x_0)|$ for $y \in [x_0, x_0 + 1]$. By induction we get $|h(y)| \leq h(x_0) + n$ for $y \in [x_0, x_0 + n]$, for $n = 1, 2, \dots$, and we get the conclusion for h . Further this implies the conclusion for $\Psi(x) = \exp(h(\log(x)))$. \square

Proof of Theorem 3.14. Assume that (3.10) holds and let $k \in I$. Then we get

$$\frac{\Psi(kx)}{\Psi(x)} = \frac{c(kx)}{c(x)} \exp\left(\int_x^{kx} \frac{\epsilon(u)}{u} du\right).$$

Choose $\epsilon > 0$. Then, for all sufficiently large x , the right hand side lies between

$$(1 \pm \epsilon) \exp(\pm \epsilon |\log(k)|),$$

from which (3.4) follows.

By Lemma 3.17, h is integrable on finite intervals far enough to the right, by being bounded and measurable on them. For X large enough, we may therefore write

$$h(x) = \int_x^{x+1} (h(x) - h(t)) dt + \int_X^x (h(t+1) - h(t)) dt + \int_X^{X+1} h(t) dt, \quad x \geq X.$$

The last term on the right is a constant, say c . If $g(x) := h(x+1) - h(x)$, then $g(x) \rightarrow 0$ as $x \rightarrow \infty$ by Remark 3.16 (ii). The first term on the right is $\int_0^1 (h(x) - h(x+u)) du$, which tends to 0 as $x \rightarrow \infty$, by the Uniform Convergence Theorem 3.11 and again Remark 3.16 (ii). By putting $d(x) := c + \int_0^1 (h(x) - h(x+u)) du$ we obtain (3.12) \square

3.3 Indicator functions with relation to slowly varying functions

In this section we investigate more closely the growth of functions of order 0. The main result is the theorem below. There we denote, for an entire function f ,

$$m_f(r) := \inf_{|z|=r} |f(z)| \quad \text{and} \quad M_f(r) := \sup_{|z|=r} |f(z)|.$$

Theorem 3.18. *Let f be an entire function and Ψ be a positive slowly varying function. Further assume*

$$\alpha := \limsup_{r \rightarrow \infty} \frac{\log(M_f(r))}{\Psi(r)} < \infty. \quad (3.13)$$

Then also

$$\limsup_{r \rightarrow \infty} \frac{\log(m_f(r))}{\Psi(r)} = \alpha.$$

Remark 3.19. This theorem applies to all functions of zero order. For if f has order zero, and $\rho(r)$ is a proximate order for f , then $\Psi(r) := r^{\rho(r)}$ is slowly varying and (3.13) holds with some $\alpha > 0$. It does not remain true for functions of larger order. For example consider $f(z) := e^z$. Then $M_f(r) = e^r$, while $m_f(r) = e^{-r}$.

A proof of this (and some stronger) facts, without appealing to slowly varying functions, can be found in [4], chapter 3.

The proof of Theorem 3.25 will closely follow the lines of [2] section 6, but first we present a corollary of it.

Corollary 3.20. *Assume that (3.13) holds. Then the indicator of f w.r.t Ψ , i.e. the function*

$$h_{f,\Psi}(\theta) := \limsup_{r \rightarrow \infty} \frac{\log(|f(re^{i\theta})|)}{\Psi(r)}, \quad \theta \in [0, 2\pi), \quad (3.14)$$

is constant equal to α .

Proof of Corollary 3.20. Clearly for every $\theta \in [0, 2\pi]$

$$m_f(r) \leq \left| f(r \exp(i\theta)) \right| \leq M_f(r).$$

Thus our corollary follows from Theorem 3.18. \square

We need a few preliminary observations, before we start with the proof of Theorem 3.25. First, let us observe that validity of (3.13) for some slowly varying function implies that f is of zero order.

Lemma 3.21. *Let Ψ be a slowly varying function. Then for every $\delta > 0$ we have $\Psi(r) = o(r^\delta)$, $r \rightarrow \infty$.*

Proof. Applying Theorem 3.14, we obtain

$$\Psi(r) \leq c(r) \exp \left(\int_a^{r_0} \frac{\max_{u \in [a, r_0]} \epsilon(u)}{s} ds + \int_{r_0}^r \frac{\epsilon(s)}{s} ds \right)$$

For r_0 large enough we get $\epsilon(u) \leq \epsilon'$ for $u \geq r_0$ and hence the second integral is bounded by $\epsilon' \log(r)$. Since, we can choose $\epsilon(x)$ to be bounded, the first one is a constant K , and hence we obtain

$$\Psi(r) \leq c(r) \exp(K + \epsilon' \log(r)) = c(r) e^K r^{\epsilon'}.$$

If we take $\epsilon' \leq \delta$ we obtain

$$\Psi(r) = o(r^\delta), \text{ as } r \rightarrow \infty,$$

for $\delta > 0$, since $\lim_{r \rightarrow \infty} c(r) = c$, by Theorem 3.14. \square

Remark 3.22. Combining Lemma 3.21 with (3.13) shows that f has zero order, because since $\Psi(r) = o(r^\delta)$ holds, we obtain $\limsup_{r \rightarrow \infty} \frac{\log(M_f(r))}{r^\delta} = 0$ for all $\delta > 0$.

Next we want to present two well known results, which we will need during this section. Their proofs can be found for example in [6].

Theorem 3.23 (Jensen). *Let $f(z)$ be holomorphic in a disc of radius R with center at the origin and $f(0) \neq 0$. Then*

$$\int_0^R \frac{n(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log(|f(R \exp(i\theta))|) d\theta - \log(|f(0)|),$$

where $n(t)$ is the number of zeroes of f in the disk $|z| \leq t$.

Theorem 3.24 (Hadamard). *An entire function $f(z)$ of order ρ can be represented in the form*

$$f(z) = z^m \exp(p(z)) \prod_0^\infty G\left(\frac{z}{z_\nu}, q\right),$$

where z_ν are the zeroes of f , $p(z)$ is a polynomial whose degree does not exceed ρ , m is the multiplicity of the zero at the origin, $q \leq \rho$ and

$$G\left(\frac{z}{z_\nu}, q\right) := \left(1 - \frac{z}{z_\nu}\right) \exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^q}{q}\right), \quad G\left(\frac{z}{z_\nu}, 0\right) := 1 - \frac{z}{z_\nu}.$$

The basic connection between $M_f(r)$, $m_f(r)$, and $\Psi(r)$ is given by the following theorem, which provides a very important step in the proof of Theorem 3.25.

Theorem 3.25. *Let Ψ be a slowly varying function, f entire, and assume that (3.13) holds. Then*

$$\int_r^{2r} \log \left(\frac{M_f(t)}{m_f(t)} \right) \frac{1}{t} dt = o(\Psi(r)), \quad (3.15)$$

for $r \rightarrow \infty$.

Remark 3.26. We start the proof with a few preliminary observations.

(i) We can assume w.l.o.g. that $f(0) = 1$. This is because passing from $f(z)$ to $f(z)/(cz^p)$, where p is a nonnegative integer and c a non-zero constant, does not affect the relations (3.13), (3.15).

(ii) By Lemma 3.21 f is of order zero, and hence Hadamard's Theorem 3.24 gives

$$f(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{z_\nu} \right),$$

where z_ν are the zeroes of f . Hence, for $|z| = r$,

$$\prod_{\nu=1}^{\infty} \left| 1 - \frac{r}{|z_\nu|} \right| \leq |f(z)| \leq \prod_{\nu=1}^{\infty} \left| 1 + \frac{r}{|z_\nu|} \right|.$$

Therefore, we get

$$\log \left(\frac{M_f(r)}{m_f(r)} \right) \leq \sum_{\nu=1}^{\infty} \log \left(\frac{r + |z_\nu|}{|r - |z_\nu||} \right). \quad 0 < r < \infty. \quad (3.16)$$

To prove Theorem 3.25 we need an estimate of the sum in (3.16), which we will obtain by means of the following lemmata.

Lemma 3.27. *Assume that f satisfies (3.13), $f(0) = 1$, and denote by $n(r)$ the number of zeroes of f in the disc $\{z \in \mathbb{C} : |z| \leq r\}$, then*

$$n(r) = o(\Psi(r)), \quad r \rightarrow \infty.$$

Proof. Let $K > 1$. From Jensen's Theorem 3.23, (3.13), and the fact that Ψ is slowly varying, we obtain $r_0(K) > 0$ such that

$$\int_0^{Kr} \frac{n(t)}{t} dt \leq \log(M_f(Kr)) \leq (\alpha + 1)\Psi(Kr) \leq 2(\alpha + 1)\Psi(r), \quad r > r_0(K).$$

Hence

$$n(r) \log(K) = n(r) \int_r^{Kr} \frac{1}{t} dt \leq \int_r^{Kr} \frac{n(t)}{t} dt \leq 2(\alpha + 1)\Psi(r), \quad r > r_0(K).$$

Since K can be chosen arbitrarily large, this yields our assertion. \square

Lemma 3.28. *For $s > 0$ we have*

$$\int_0^\infty \log \left(\left| \frac{1 + \frac{s}{t}}{1 - \frac{s}{t}} \right| \right) \frac{1}{t} dt = \frac{\pi}{2}.$$

Proof. We first put $s/t = x$ so that our integral becomes

$$\int_0^\infty \log \left(\left| \frac{1+x}{1-x} \right| \right) \frac{1}{x} dx = \int_0^1 \log \left(\frac{1+x}{1-x} \right) \frac{1}{x} dx + \int_0^1 \log \left(\frac{1+y}{1-y} \right) \frac{1}{y} dy,$$

where we have put $x = 1/y$, when $x > 1$. Further

$$\int_0^1 \log \left(\frac{1+x}{1-x} \right) \frac{1}{x} dx = \int_0^1 \left(-\sum_{k=0}^{\infty} \frac{(-x)^{k+1}}{k+1} + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \right) \frac{1}{x} dx = 2 \sum_{m=1}^{\infty} \int_0^1 \frac{x^{2m-2}}{2m-1} dx,$$

where we used the power series representation of the logarithm. Evaluating the integral in the last sum yields

$$2 \sum_{m=1}^{\infty} \int_0^1 \frac{x^{2m-2}}{2m-1} dx = 2 \sum_{m=1}^{\infty} \left(\frac{1}{2m-1} \right)^2 = \frac{\pi}{4}$$

and this proves our lemma. \square

Lemma 3.29. *We have*

$$\sum_{|z_\nu| > 2r} \log \left(\frac{1 + \frac{r}{|z_\nu|}}{1 - \frac{r}{|z_\nu|}} \right) = o(\Psi(r)). \quad (3.17)$$

Proof. First we note

$$\log \left(\frac{1 + \frac{r}{t}}{1 - \frac{r}{t}} \right) = \log \left(1 + \frac{2r}{(1 - \frac{r}{t})t} \right) \leq \frac{2r}{(1 - \frac{r}{t})t} \leq \frac{4r}{t}, \quad t \geq 2r.$$

Further, by Lemma 3.27 and Lemma 3.21 we get $n(t) = o(\Psi(t)) = o(t^\delta)$ for arbitrary $\delta > 0$. Choosing $\delta < 1$, we conclude

$$\lim_{t \rightarrow \infty} n(t) \log \left(\frac{1 + \frac{r}{t}}{1 - \frac{r}{t}} \right) = 0.$$

Applying this, we can write the sum from the left hand side of (3.17) as

$$\begin{aligned} \int_{2r}^\infty \log \left(\frac{1 + \frac{r}{t}}{1 - \frac{r}{t}} \right) dn(t) &= \left[n(t) \log \left(\frac{1 + \frac{r}{t}}{1 - \frac{r}{t}} \right) \right]_{2r}^\infty + 2r \int_{2r}^\infty \frac{n(t)}{t^2 - r^2} dt = \\ &= -n(2r) \log(3) + o \left(r \int_{2r}^\infty \frac{\Psi(t)}{t^2} dt \right), \end{aligned} \quad (3.18)$$

for r sufficiently large, noting that $\frac{t^2}{t^2 - r^2} \leq 4/3$ for $t \in [2r, \infty)$. Now consider $t \in [2^p r, 2^{p+1} r]$ for $p \in \mathbb{N}$. Then, for r large enough

$$\Psi(t) < (1 + \epsilon) \Psi(r 2^p) < 2^{1/2} \Psi(2r 2^{p-1}) < 2 \Psi(2r 2^{p-2}) < \dots < 2^{p/2} \Psi(2r),$$

because Ψ is slowly varying. Hence, for r sufficiently large, we obtain

$$\int_{r 2^p}^{r 2^{p+1}} \frac{\Psi(t)}{t^2} dt < \int_{r 2^p}^{r 2^{p+1}} \frac{\Psi(2r) 2^{p/2}}{t^2} dt = \frac{\Psi(2r) 2^{p/2}}{r 2^{p+1}} = \frac{\Psi(2r)}{2r} 2^{-p/2}.$$

We get

$$r \int_{2r}^\infty \frac{\Psi(t)}{t^2} dt < \frac{\Psi(2r)}{2} \sum_{p=1}^{\infty} 2^{-p/2} = O(\Psi(r)).$$

Thus, (3.18) and Lemma 3.27 yield the assertion. \square

Proof of Theorem 3.25. Let $r < t < 2r$. Then, by Lemma 3.29,

$$\sum_{|z_\nu| > 4r} \log \left(\frac{1 + \frac{t}{|z_\nu|}}{1 - \frac{t}{|z_\nu|}} \right) \leq \sum_{|z_\nu| > 4r} \log \left(\frac{1 + \frac{2r}{|z_\nu|}}{1 - \frac{2r}{|z_\nu|}} \right) = o(\Psi(r)),$$

and hence

$$\sum_{|z_\nu| > 4r} \log \left(\frac{|z_\nu| + t}{|z_\nu| - t} \right) = o(\Psi(r)).$$

Thus

$$\int_r^{2r} \sum_{|z_\nu| > 4r} \log \left(\frac{|z_\nu| + t}{|z_\nu| - t} \right) \frac{1}{t} dt = o(\Psi(r)) \int_r^{2r} \frac{1}{t} dt = o(\Psi(r)). \quad (3.19)$$

Further, by Lemma 3.28

$$\int_r^{2r} \sum_{|z_\nu| \leq 4r} \log \left(\left| \frac{|z_\nu| + t}{|z_\nu| - t} \right| \right) \frac{1}{t} dt \leq \sum_{|z_\nu| \leq 4r} \int_0^\infty \log \left(\left| \frac{1+t}{1-t} \right| \right) \frac{1}{t} dt \leq \frac{\pi^2}{2} n(4r) = o(\Psi(r)). \quad (3.20)$$

Putting together (3.16), (3.19) and (3.20) yields our assertion. \square

Proof of Theorem 3.18. Let $\epsilon > 0$ and choose a large r , such that

$$\log(M_f(r)) > (\alpha - \epsilon)\Psi(r).$$

Let $k \in [1, 2]$. Then, because Ψ is slowly varying, we get $\Psi(kr) \leq (1 + \epsilon_1)\Psi(r)$ and hence

$$(\alpha - \epsilon)\Psi(r) \geq \frac{\alpha - \epsilon}{1 + \epsilon_1} \Psi(t) \geq (\alpha - 2\epsilon)\Psi(t),$$

with $r \leq t \leq 2r$, for r sufficiently large. Further, since $\log(M_f(r))$ increases with r we obtain that if r is sufficiently large

$$\log(M_f(t)) > (\alpha - 2\epsilon)\Psi(t), \quad r \leq t \leq 2r. \quad (3.21)$$

Again, because Ψ is slowly varying, we get $\Psi(kr) \geq (1 - \epsilon_2)\Psi(r)$, and hence

$$-\epsilon\Psi(r) \geq -\frac{\epsilon}{1 - \epsilon_2} \Psi(t) \geq -2\epsilon\Psi(t), \quad r \leq t \leq 2r$$

Applying this, and by the mean value theorem for integrals, it follows from Theorem 3.25 that we can choose t , such that $r \leq t \leq 2r$ and

$$\log(m_f(t)) > \log(M_f(t)) - \epsilon\Psi(r) \geq \log(M_f(t)) - 2\epsilon\Psi(t),$$

if r is sufficiently large.

On combining this with (3.21) we obtain

$$\log(m_f(t)) > (\alpha - 4\epsilon)\Psi(t).$$

Since ϵ is arbitrarily small we obtain

$$\limsup_{t \rightarrow \infty} \frac{\log(m_f(t))}{\Psi(t)} \geq \alpha.$$

Since $m_f(r) \leq M_f(r)$ we have from (3.13)

$$\limsup_{t \rightarrow \infty} \frac{\log(m_f(t))}{\Psi(t)} \leq \alpha,$$

which proves the corollary. \square

3.4 A Formula for Type

The next theorem will yield a formula for the type of an entire function with respect to a proximate order $\rho(r)$. Since by Proposition 3.12, $r^{\rho(r)}$ is an increasing function for $r > 0$, if $\rho > 0$ the equation $t = r^{\rho(r)}$ admits a unique solution for $t > 0$. We will denote this solution by $r = \phi(t)$. Therefore $\phi(t)$ is just the inverse function of $r^{\rho(r)}$. Of course $\phi(t)$ depends on $\rho(r)$, but we will not denote this dependence. The proof we present is based on the proofs of this theorem given in [5], [6].

Theorem 3.30. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$, $z \in \mathbb{C}$, be the Taylor series expansion of the entire function $f(z)$ of finite order $\rho > 0$ and of proximate order $\rho(r)$. Then the type σ_f with respect to the proximate order $\rho(r)$ is given by*

$$\frac{1}{\rho} \log(\sigma_f) = \limsup_{n \rightarrow \infty} \left(\frac{1}{n} \log(|c_n|) + \log(\phi(n)) \right) - \frac{1}{\rho} - \frac{\log(\rho)}{\rho}, \quad \rho > 0, \quad (3.22)$$

where $\phi(t)$ is the inverse function of $r^{\rho(r)}$.

Proof. We divide the proof in several steps.

Step 1: We show that $\lim_{s \rightarrow \infty} \frac{\phi(ks)}{\phi(s)} = k^{\frac{1}{\rho}}$, $k \in (0, \infty)$. Set $t = r^{\rho(r)}$. Since, by definition, $\log(t(r)) = \rho(r) \log(r)$, we get

$$\frac{d(\log(t))}{d(\log(r))} = \rho'(r)r \log(r) + \rho(r),$$

which, by Definition 3.4, tends to ρ when r tends to infinity. Furthermore

$$\frac{d(\log(t(r)))}{d(\log(r))} = \frac{\frac{d}{dt} \log(t(r))}{\frac{d}{dt} \log(r)} = \frac{\frac{d}{dt} \log(t(r))}{\frac{d}{dt} \log(\phi(t(r)))}.$$

So combining these two equalities yields, for r sufficiently large,

$$\left(\frac{1}{\rho} - \epsilon \right) \frac{d}{dt} \log(t(r)) < \frac{d}{dt} \log(\phi(t(r))) < \left(\frac{1}{\rho} + \epsilon \right) \frac{d}{dt} \log(t(r)).$$

If we integrate from s to ks , we obtain

$$\left(\frac{1}{\rho} - \epsilon \right) \log(k) < \log\left(\frac{\phi(ks)}{\phi(s)}\right) < \left(\frac{1}{\rho} + \epsilon \right) \log(k),$$

and hence taking exponentials yields

$$\lim_{s \rightarrow \infty} \frac{\phi(ks)}{\phi(s)} = k^{\frac{1}{\rho}}. \quad (3.23)$$

Step 2: By the Cauchy Integral Formula we have the following estimate for the coefficients of the power series

$$|c_n| \leq \frac{\sup_{|z| \leq r} |f(z)|}{r^n}.$$

Since $f(z)$ is of finite order ρ , there exists r_0 such that for $r \geq r_0$, $k \geq \rho$ and $\tilde{\sigma} > \sigma_f^{r^k}$

$$\sup_{|z| \leq r} |f(z)| \leq e^{\tilde{\sigma} r^k}$$

and hence

$$|c_n| \leq e^{\tilde{\sigma}r^k} r^{-n}, \quad r \geq r_0. \quad (3.24)$$

Considering the function on the right hand side as a function of r , with $r \in \mathbb{R}^+$, we want to determine its smallest value. Therefore, we evaluate

$$\frac{d}{dr} \left[e^{\tilde{\sigma}r^k} r^{-n} \right] = e^{\tilde{\sigma}r^k} \tilde{\sigma} k r^{k-1} r^{-n} - n e^{\tilde{\sigma}r^k} r^{-n-1} = e^{\tilde{\sigma}r^k} r^{-n-1} (\tilde{\sigma} k r^k - n),$$

where the right hand side equals zero for

$$r = \left(\frac{n}{\tilde{\sigma}k} \right)^{1/k}.$$

To verify that this is indeed the smallest value we regard

$$\begin{aligned} & \frac{d^2}{dr^2} \left[e^{\tilde{\sigma}r^k} r^{-n} \right] = \\ & = e^{\tilde{\sigma}r^k} \tilde{\sigma} k r^{k-1} r^{-n-1} (\tilde{\sigma} k r^k - n) - (n+1) e^{\tilde{\sigma}r^k} r^{-n-2} (\tilde{\sigma} k r^k - n) + e^{\tilde{\sigma}r^k} r^{-n-1} \tilde{\sigma} k^2 r^{k-1}. \end{aligned}$$

Plugging in the value we obtained for r , we notice that $(\tilde{\sigma} k r^k - n) = 0$ and that only the last term, which is obviously positive, remains and we are done. Therefore, we obtain

$$|c_n| \leq \left(\frac{e \tilde{\sigma} k}{n} \right)^{n/k}, \quad (3.25)$$

for $\left(\frac{n}{\tilde{\sigma}k} \right)^{1/k} \geq r_0$ and $k \geq \rho$ and $\tilde{\sigma} > \sigma_f^{r^k}$.

Conversely, assume that (3.25) holds for all $n \geq n(k, \tilde{\sigma})$ and let us estimate $\sup_{|z| \leq r} |f(z)|$. If we choose r such that $m_r = 2^k e \tilde{\sigma} k r^k \geq n(k, \tilde{\sigma})$, we have by (3.25)

$$|c_n z^n| \leq 2^{-n}$$

for $n \geq m_r = 2^k e \tilde{\sigma} k r^k$, and therefore

$$|f(z)| \leq \sum_{n=0}^{m_r} |c_n| r^n + 2^{m_r}.$$

Introducing the notation

$$\mu(r) := \max_n |c_n| r^n$$

we have

$$\sup_{|z| \leq r} f(z) \leq (1 + 2^k e \tilde{\sigma} k r^k) \mu(r) + 2^{m_r} \quad (3.26)$$

Step 3: In this step we show that " \geq " holds in (3.22), if $\sigma_f < \infty$. By (3.24) we obtain

$$\log(|c_n|) < \tilde{\sigma} r^{\rho(r)} - n \log(r),$$

for $\tilde{\sigma} > \sigma_f$ and r large enough. If r_n is the solution of the equation $n = \tilde{\sigma} \rho r^{\rho(r)}$, then for n sufficiently large we have

$$\log(|c_n|) < \frac{n}{\rho} - n \log \left(\phi \left(\frac{n}{\rho \tilde{\sigma}} \right) \right).$$

Dividing by n and adding $\log(\phi(n))$ to this yields

$$\log\left(\phi(n)|c_n|^{1/n}\right) < \frac{1}{\rho} + \log\left(\frac{\phi(n)}{\phi\left(\frac{n}{\rho\tilde{\sigma}}\right)}\right),$$

and hence, taking the exponential again, $\limsup_{n \rightarrow \infty} (\phi(n)|c_n|^{1/n}) \leq (\tilde{\sigma}\rho e)^{1/\rho}$ by (3.23). This is true for all $\tilde{\sigma} > \sigma_f$. Passing to the limit we have

$$\limsup_{n \rightarrow \infty} (\phi(n)|c_n|^{1/n}) \leq (\tilde{\sigma}\rho e)^{1/\rho},$$

and since $\tilde{\sigma} > \sigma_f$ was arbitrary

$$\limsup_{n \rightarrow \infty} (\phi(n)|c_n|^{1/n}) \leq (\sigma_f \rho e)^{1/\rho}, \quad (3.27)$$

Step 4: This step completes the proof by showing that " \leq " holds in (3.22) if the right hand side is finite. Let $\hat{\sigma}$ be defined by the formula $\limsup_{n \rightarrow \infty} (\phi(n)|c_n|^{1/n}) = (\hat{\sigma}\rho e)^{\frac{1}{\rho}}$. We shall show that $\hat{\sigma} < \sigma_f$ leads to a contradiction. Suppose $\hat{\sigma} < \sigma_f$ and choose σ' such that $\hat{\sigma} < \sigma' < \sigma_f$.

$$|c_n| < \left(\frac{(\sigma'\rho e)^{1/\rho}}{\phi(n)}\right)^n \leq \left(\frac{e^{1/\rho}}{\phi\left(\frac{n}{\sigma'\rho}\right)}\right)^n,$$

by (3.23). Thus, there exists n_0 such that for $n \geq n_0$,

$$|c_n|r^n \leq \left(\frac{e^{1/\rho}}{\phi\left(\frac{n}{\sigma'\rho}\right)}\right)^n r^n.$$

and hence

$$\mu(r) < \max_{n > n_0} \left(\frac{e^{1/\rho}}{\phi\left(\frac{n}{\sigma'\rho}\right)}\right)^n r^n.$$

This maximum is obtained at $\sigma\rho r^{\rho(r)}$, according to [5], [6]. Therefore, let n be the biggest integer smaller than $\sigma\rho r^{\rho(r)}$. Then, by (3.23), we get $\phi\left(\frac{n}{\sigma'\rho}\right) \geq \phi\left(\frac{\sigma'\rho r^{\rho(r)}}{\sigma'\rho}\right)(1 - \epsilon_1)$. Hence, choosing ϵ_2 such that $1 + \epsilon_2 = \frac{1}{1 - \epsilon_1}$, we get, by definition of ϕ

$$\mu(r) \leq \left(\frac{e^{1/\rho}}{\phi(r^{\rho(r)})}\right)^n \cdot ((1 + \epsilon_2)r)^n \leq (e^{1/\rho}(1 + \epsilon_2))^n,$$

which is strictly increasing in n , and therefore

$$(e^{1/\rho}(1 + \epsilon_2))^n \leq ((1 + \epsilon_2)e^{1/\rho})^{\sigma'\rho r^{\rho(r)}},$$

for r sufficiently large. Hence from (3.26), we see that for all $\rho_1 > \rho$, $\sigma' > 0$

$$\sup_{|z| \leq r} |f(z)| \leq (1 + 2^{\rho_1} e \sigma_f \rho_1 r^{\rho_1}) e^{\sigma' r^{\rho(r)}} (1 + \epsilon_2)^{\sigma' \rho r^{\rho(r)}} + 1.$$

Further, since we can choose our ϵ_1 accordingly, we can get $(1 + \epsilon_2) \leq e^{\epsilon_3/\rho}$, for an arbitrary $\epsilon_3 > 0$. Therefore, and by definition of σ_f , this leads to $\sigma_f \leq \sigma'(1 + \epsilon_3)$, where ϵ_3 is arbitrarily small, and hence a contradiction. Thus the inequality sign in (3.27) cannot occur and the theorem is proved. \square

3.5 λ -type of Matrices of the Class \mathcal{M}_0

Definition 3.31. The Nevanlinna Class \mathcal{N}_0 is defined as the set of all functions q , that are analytic in $\mathbb{C} \setminus \mathbb{R}$, satisfy $q = q^\# := \overline{q(\bar{z})}$ and have nonnegative imaginary part in \mathbb{C}^+ .

In the following we want to investigate the connection between the Nevanlinna Class and matrices of the class \mathcal{M}_0 , as given in Definition 1.14. The most important result will be Corollary 3.36. We start with the following theorem, which is taken from [1]:

Theorem 3.32. *Let A, B be entire functions such that $\frac{B}{A} \in \mathcal{N}_0$ and let λ be a growth function. Then $\sigma_A^\lambda = \sigma_B^\lambda$.*

Proof. We start by showing that $\frac{B}{A} \in \mathcal{N}_0$ implies $-\frac{A}{B} \in \mathcal{N}_0$. Clearly $\frac{B}{A} = \left(\frac{B}{A}\right)^\#$ implies $-\frac{A}{B} = -\left(\frac{A}{B}\right)^\#$. Further, if we write $\frac{B}{A}(z) = v(z) + w(z)i$, inverting yields $-\frac{A}{B}(z) = -\frac{v(z) - w(z)i}{(v(z) + w(z)i)^2}$ and hence the nonnegativity of the imaginary part translates as well. Further, if $\text{Im} \frac{B}{A}(z) = 0$ for some $z \in \mathbb{C}^+$, by the maximum principle this would hold for all $z \in \mathbb{C}^+$. Hence the real part would be constant as well and the theorem follows immediately. Therefore we only need to consider $\frac{B}{A}$ without zeroes.

If both σ_A^λ and σ_B^λ are both equal to ∞ we are done. Therefore, let $\sigma_A^\lambda < \infty$. We show that this implies $\sigma_A^\lambda \geq \sigma_B^\lambda$, which will complete the proof. Since we have $\frac{B}{A} \in \mathcal{N}_0$, it has by the Herglotz Theorem (see Appendix), an integral representation of the form

$$\frac{B(z)}{A(z)} = az + b + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\mu(t), \quad z \in \mathbb{C}^+,$$

where $a \geq 0$, $b \in \mathbb{R}$ and μ is a positive Borel measure on \mathbb{R} such that $\int_{\mathbb{R}} (t^2+1)^{-1} d\mu(t) < \infty$. In the present case in fact μ is a discrete measure with point masses at the zeroes of A . Hence

$$\begin{aligned} \left| \frac{B(z)}{A(z)} \right| &= \left| az + b + \int_{\mathbb{R}} \frac{tz+1}{(t^2+1)(t-z)} d\mu(t) \right| \leq \\ &\leq \left| az + b + z \int_{\mathbb{R}} \frac{1}{t^2+1} d\mu(t) \right| + \left| (z^2+1) \int_{\mathbb{R}} \frac{1}{(t^2+1)(t-z)} d\mu(t) \right|, \quad z \in \mathbb{C}^+. \end{aligned}$$

Therefore,

$$\log (|B(z)|) \leq \log (|A(z)|) + C_1 \log (|z| + 2) + C_2 \log^+ \frac{1}{|\text{Im}(z)|} + C_3$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$. In particular

$$\log (|B(z)|) \leq \log (|A(z)|) + C_1 \log (|z| + 2), \quad |\text{Im}(z)| \geq 1. \quad (3.28)$$

Now let $\text{Im}(z) < 1$. Then by subharmonicity of $\log |B|$,

$$\begin{aligned} \log (|B|) &\leq \frac{1}{2\pi} \int_0^{2\pi} \log (|B(z + \exp(i\phi))|) d\phi \leq \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log (|A(z + \exp(i\phi))|) d\phi + \\ &+ C_4 \log (|z| + 2) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|\text{Im}(z + \exp(i\phi))|} d\phi. \end{aligned}$$

Clearly

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|\operatorname{Im}(z + \exp(i\phi))|} d\phi \leq C_5$$

for all z with $|\operatorname{Im}(z)| < 1$. Consequently

$$\log(|B(z)|) \leq \frac{1}{2\pi} \int_0^{2\pi} \log(|A(z + \exp(i\phi))|) d\phi + C_4 \log(|z| + 2) + C_5, \quad \operatorname{Im}(z) < 1. \quad (3.29)$$

Since $\sigma_A^\lambda < \infty$ and $\log(r) = o(\lambda(r))$, from (3.29) and (3.28) it follows that $\sigma_B^\lambda < \infty$ and $\sigma_B^\lambda \leq \sigma_A^\lambda$. \square

From now on let M denote the fundamental solution of a system (1.5), hence satisfying the properties shown in Proposition 1.9.

Lemma 3.33. *Let $M(x, z) \in \mathcal{M}_0$. Then $\frac{M_{11}(x, z)}{M_{12}(x, z)}$ and $\frac{M_{22}(x, z)}{M_{21}(x, z)} \in \mathcal{N}_0$.*

Proof. We evaluate

$$\operatorname{diag} \left(\frac{M^*(x, w)JM(x, z) - J}{\bar{w} - z} \right) = \left(\frac{-M_{11}(x, z)\overline{M_{12}(x, w)} + M_{12}(x, z)\overline{M_{11}(x, w)}}{\bar{w} - z}, \frac{-M_{21}(x, z)\overline{M_{22}(x, w)} + M_{22}(x, z)\overline{M_{21}(x, w)}}{\bar{w} - z} \right).$$

If we consider the first entry, it is equal to $\frac{1}{M_{12}(x, \bar{w})M_{12}(x, z)} \frac{-\frac{M_{11}(x, z) + M_{11}(x, \bar{w})}{M_{12}(x, z) + M_{12}(x, \bar{w})}}{\bar{w} - z}$. Putting $z = w$ yields

$$\frac{1}{M_{12}(x, z)M_{12}(x, \bar{z})} \frac{\operatorname{Im} \frac{M_{11}(x, z)}{M_{12}(x, z)}}{\operatorname{Im} z}$$

Now we consider the vector $a = \begin{pmatrix} -M_{12}(x, z)i \\ 0 \end{pmatrix}$ and since $M \in \mathcal{M}_0$ we have $\bar{a}^T H_M(z, z)a \geq 0$.

Therefore $\operatorname{Im} \frac{M_{11}(x, z)}{M_{12}(x, z)} \geq 0$ for $z \in \mathbb{C}^+$. The other two properties are immediate conclusions from the properties of M .

If we consider the second entry of the diagonal matrix, we analogously obtain that $\operatorname{Im} \frac{M_{21}(x, z)}{M_{22}(x, z)} \geq 0$. \square

Lemma 3.34. *Let $V := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $M \in \mathcal{M}_0$, if and only if $VM^{-1}V \in \mathcal{M}_0$.*

Proof. Note that $V = V^{-1} = V^*$ and $VJV = -J = J^{-1}$. Now the following computation shows that the equation

$$\begin{aligned} H_{VM^{-1}V}(w, z) &= \frac{(VM^{-1}V)^*(w)J(VM^{-1}V)(z) - J}{\bar{w} - z} = \\ &= (M^{-1}(w)V)^* \frac{-J - M^*(w)VJVM(z)}{\bar{w} - z} M^{-1}(z)V \\ &= (M^{-1}(w)V)^* \frac{M^*(w)JM(z) - J}{\bar{w} - z} M^{-1}(z)V \\ &= (M^{-1}(w)V)^* H_M(w, z)M^{-1}(z)V \end{aligned}$$

holds and the assertion follows immediately. \square

Lemma 3.35. *If $M \in \mathcal{M}_0$ and $\tau(z) \in \mathcal{N}_0$, then*

$$\frac{M_{12}(z)\tau(z) - M_{11}(z)}{M_{22}(z)\tau(z) - M_{21}(z)} \in \mathcal{N}_0.$$

Proof. Define

$$(M \star \tau)(z) := \frac{M_{12}(z)\tau(z) - M_{11}(z)}{M_{22}(z)\tau(z) - M_{21}(z)}.$$

Assume that

$$\begin{aligned} (M_{22}(\bar{w})\tau(\bar{w}) - M_{21}(\bar{w})) \frac{(M \star \tau)(\bar{w}) - (M \star \tau)(z)}{\bar{w} - z} (M_{22}(z)\tau(z) - M_{21}(z)) = \\ \begin{pmatrix} -\tau(\bar{w}) \\ 1 \end{pmatrix}^T \frac{(VM^{-1}V)^*(w)J(VM^{-1}V)(z) - J \begin{pmatrix} -\tau(z) \\ 1 \end{pmatrix}}{\bar{w} - z} \begin{pmatrix} -\tau(\bar{w}) \\ 1 \end{pmatrix} + \frac{\tau(\bar{w}) - \tau(z)}{\bar{w} - z}. \end{aligned} \quad (3.30)$$

Since, due to the previous lemma, in our case the right hand side is greater or equal to 0, the same holds for the left hand side, and we conclude that the lemma holds.

What remains, is to verify (3.30), which will follow by a few elementary computations. First we note that

$$\begin{pmatrix} -\tau(\bar{w}) \\ 1 \end{pmatrix}^T J \begin{pmatrix} -\tau(z) \\ 1 \end{pmatrix} = \tau(\bar{w}) - \tau(z).$$

Applying this and multiplying by $(\bar{w} - z)$ in (3.30) and furthermore evaluating the left hand side of it, yields

$$\begin{aligned} (M_{12}(\bar{w})\tau(\bar{w}) - M_{11}(\bar{w}))(M_{22}(z)\tau(z) - M_{21}(z)) - \\ -(M_{12}(z)\tau(z) - M_{11}(z))(M_{22}(\bar{w})\tau(\bar{w}) - M_{21}(\bar{w})) = \\ = \begin{pmatrix} -\tau(\bar{w}) \\ 1 \end{pmatrix}^T (VM^{-1}V)^*(w)J(VM^{-1}V)(z) \begin{pmatrix} -\tau(z) \\ 1 \end{pmatrix}. \end{aligned} \quad (3.31)$$

Further, applying Proposition 1.9 (iii) and $VM^{-1}V(z) = \begin{pmatrix} M_{22}(z) & M_{21}(z) \\ M_{12}(z) & M_{11}(z) \end{pmatrix}$, we rewrite the right hand side of (3.31) to

$$\begin{aligned} \begin{pmatrix} -\tau(\bar{w}) \\ 1 \end{pmatrix}^T (VM^{-1}V)^*(w)J(VM^{-1}V)(z) \begin{pmatrix} -\tau(z) \\ 1 \end{pmatrix} = \\ \begin{pmatrix} -M_{22}(\bar{w})\tau(\bar{w}) + M_{21}(\bar{w}) \\ -M_{12}(\bar{w})\tau(\bar{w}) + M_{11}(\bar{w}) \end{pmatrix}^T \begin{pmatrix} -M_{12}(z) & -M_{11}(z) \\ M_{22}(z) & M_{21}(z) \end{pmatrix} \begin{pmatrix} -\tau(z) \\ 1 \end{pmatrix} = \\ = \tau(z)\tau(\bar{w})(-M_{22}(\bar{w})M_{12}(z) + M_{12}(\bar{w})M_{22}(z)) + \tau(z)(M_{21}(\bar{w})M_{12}(z) - M_{11}(\bar{w})M_{22}(z)) + \\ + \tau(\bar{w})(M_{22}(\bar{w})M_{11}(z) - M_{12}(\bar{w})M_{21}(z)) - M_{21}(\bar{w})M_{11}(z) + M_{11}(\bar{w})M_{21}(z), \end{aligned}$$

which equals the left hand side of (3.31), and we are done. \square

Corollary 3.36. *If $M \in \mathcal{M}_0$, all the entries of M have the same λ -type.*

Proof. By putting $\tau = 0$ in the previous lemma and applying Theorem 3.32, we get

$$\sigma_{M_{11}}^\lambda = \sigma_{M_{21}}^\lambda.$$

By Lemma 3.33 and again applying Theorem 3.32 we obtain the equalities

$$\sigma_{M_{12}}^\lambda = \sigma_{M_{11}}^\lambda, \quad \sigma_{M_{21}}^\lambda = \sigma_{M_{22}}^\lambda,$$

and we obtain the corollary by combining the equalities above. \square

Definition 3.37. Due to this corollary we can now define the λ -type of $M \in \mathcal{M}_0$ via

$$\sigma_M^\lambda := \sigma_{M_{i,j}}^\lambda, \quad i, j \in \{1, 2\}.$$

Corollary 3.38. If $M \in \mathcal{M}_0$, all the entries of M have the same order.

Proof. If all entries of M have infinite order, we are trivially done. Therefore, assume that the order of M_{ij} , $i, j \in \{1, 2\}$ is finite. Then we can write

$$\text{ord } M_{ij} = \inf \left\{ \alpha > 0 : \sigma_{M_{ij}}^{r^\alpha} < \infty \right\}.$$

By the previous Corollary 3.36 we get that the r^α -types of the other entries are finite as well. Hence their order is smaller or equal to α . W.l.o.g., assume that M_{kl} , $k, l \in \{1, 2\}$, has order $\beta < \alpha$, then repeating the argument above M_{ij} has to have order smaller or equal to β , which is a contradiction to the assumption that it has order α . Therefore we conclude that every entry has order α , and we are done. \square

Lemma 3.39. Let $c \in [a, b]$, M be the fundamental solution of a system (1.5) and let $M_{a,c}$, $M_{c,b}$ be the fundamental solutions of (1.5), restricted to the respective intervals. Then, if $M_{a,c}$, $M_{c,b} \in \mathcal{M}_0$, the following hold:

(i)

$$\sigma_M^\lambda \leq \sigma_{M_{a,c}}^\lambda + \sigma_{M_{c,b}}^\lambda.$$

(ii) If either

$$\begin{aligned} \limsup_{z \rightarrow \infty} \frac{\log^+(|M_{ac}(z)|)}{\lambda(|z|)} &= \lim_{z \rightarrow \infty} \frac{\log^+(|M_{ac}(z)|)}{\lambda(|z|)}, \text{ or} \\ \limsup_{z \rightarrow \infty} \frac{\log^+(|M_{cb}(z)|)}{\lambda(|z|)} &= \lim_{z \rightarrow \infty} \frac{\log^+(|M_{cb}(z)|)}{\lambda(|z|)}, \end{aligned}$$

then

$$\sigma_M^\lambda \geq \sigma_{M_{a,c}}^\lambda + \sigma_{M_{c,b}}^\lambda.$$

Proof. First we note that by Lemma 1.15 $M \in \mathcal{M}_0$, hence σ_M^λ is actually defined. Further we obtain that $M(x, z) = M_{c,b}(x, z)M_{a,c}(c, z)$ holds, with the same argument we used in Proposition 1.9 (i).

Now we write

$$M_{c,b}(b, z) := \begin{pmatrix} M_{11}(z) & M_{12}(z) \\ M_{21}(z) & M_{22}(z) \end{pmatrix}, \quad M_{a,c}(c, z) := \begin{pmatrix} \tilde{M}_{11}(z) & \tilde{M}_{12}(z) \\ \tilde{M}_{21}(z) & \tilde{M}_{22}(z) \end{pmatrix},$$

and obtain

$$M(b, z) = \begin{pmatrix} \tilde{M}_{11}(z)M_{11}(z) + \tilde{M}_{12}(z)M_{21}(z) & \tilde{M}_{11}(z)M_{12}(z) + \tilde{M}_{12}(z)M_{22}(z) \\ \tilde{M}_{21}(z)M_{11}(z) + \tilde{M}_{22}(z)M_{21}(z) & \tilde{M}_{21}(z)M_{12}(z) + \tilde{M}_{22}(z)M_{22}(z) \end{pmatrix}.$$

Now we rewrite the first entry of this matrix to

$$\tilde{M}_{11}(z)M_{11}(z) + \tilde{M}_{12}(z)M_{21}(z) = \tilde{M}_{11}(z)M_{21}(z) \left(\frac{M_{11}(z)}{M_{21}(z)} + \frac{\tilde{M}_{12}(z)}{\tilde{M}_{11}(z)} \right).$$

By Lemma 3.33 and by Lemma 3.35, we obtain $\frac{M_{11}(z)}{M_{21}(z)}, \frac{\tilde{M}_{12}(z)}{\tilde{M}_{11}(z)} \in \mathcal{N}_0$, and hence the last term is an element of \mathcal{N}_0 as well. Let us write $h(z) := \frac{M_{11}(z)}{M_{21}(z)} + \frac{\tilde{M}_{12}(z)}{\tilde{M}_{11}(z)}$. Then, evaluation of the λ -type of the functions in the upper equation, therefore yields

$$\begin{aligned} \sigma_M^\lambda &= \limsup_{z \rightarrow \infty} \frac{\log^+ (|\tilde{M}_{11}(z)M_{11}(z) + \tilde{M}_{12}(z)M_{21}(z)|)}{\lambda(|z|)} = \\ &= \limsup_{z \rightarrow \infty} \left(\frac{\log^+ (|\tilde{M}_{11}(z)|)}{\lambda(|z|)} + \frac{\log^+ (|M_{21}(z)|)}{\lambda(|z|)} + \frac{\log^+ (|h(z)|)}{\lambda(|z|)} \right). \end{aligned}$$

We note that $h(z) \in \mathcal{N}_0$ implies $\frac{1}{\operatorname{Im}(z)} \leq h(z) \leq \operatorname{Im}(z)$ and hence that $\sigma_h^\lambda = 0$ which is not just its limes superior, but its limit. Therefore the last term does not affect the result on the right hand side.

Now we obtain assertion (i), because of the well known identity

$$\limsup_{z \rightarrow \infty} (a(z) + b(z)) \leq \limsup_{z \rightarrow \infty} a(z) + \limsup_{z \rightarrow \infty} (b(z)). \quad (3.32)$$

Further, we obtain assertion (ii), because, by assumption, one of the remaining limes superior has to be a limit, which gives us equality in (3.32). Therefore we get

$$\sigma_M^\lambda = \limsup_{z \rightarrow \infty} \left(\frac{\log^+ (|\tilde{M}_{11}(z)|)}{\lambda(|z|)} \right) + \limsup_{z \rightarrow \infty} \left(\frac{\log^+ (|M_{21}(z)|)}{\lambda(|z|)} \right) = \sigma_{\tilde{M}}^\lambda + \sigma_M^\lambda.$$

□

4 Canonical Systems

In this section we extend Theorems 1 and 2 of [7]. From now on we will consider our differential equation (1.5) for a specific A . Therefore, let H be a positive summable function on $[a, b]$ with values in 2×2 matrices of reals, such that $H(t)$ is symmetric and positive semidefinite almost everywhere and let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We put $A = JH(t)$ and define a *canonical system* $(H, [a, b])$ as the matrix differential equation of the form

$$J \frac{dY}{dx} = zHY, \quad z \in \mathbb{C}, \quad x \in [a, b]. \quad (4.1)$$

Note that by writing $\|\cdot\|$ we refer to the matrix spectral norm. The matrix H is referred to as *Hamiltonian* and we assume, without loss of generality, that $\text{tr}(H) = 1$ almost everywhere.

Definition 4.1. A Hamiltonian H is of *finite rank*, if there exist numbers x_j , $a = x_0 < x_1 < \dots < x_n = b$, and a finite sequence of numbers, $(\phi_j)_{j=0}^{n-1}$, $\phi_j \in [0, 2\pi)$, such that

$$H(x) = \begin{pmatrix} \sin^2(\phi_j) & \sin(\phi_j) \cos(\phi_j) \\ \sin(\phi_j) \cos(\phi_j) & \cos^2(\phi_j) \end{pmatrix}, \quad x \in (x_j, x_{j+1}), \quad j = 0, \dots, n-1.$$

The sequences $(x_j)_{j=0}^{n-1}$, $(\phi_j)_{j=0}^{n-1}$ are called parameters and n is called rank of the Hamiltonian. The intervals (x_j, x_{j+1}) are called *constancy intervals*.

Definition 4.2. Let $\xi_\phi := (\cos \phi, \sin \phi)^T$, for some $\phi \in [0, \pi)$. The open interval $I_\phi \subseteq [a, b]$ is called *H-indivisible*, if the relation

$$\xi_\phi^T JH = 0, \quad \text{almost everywhere on } I_\phi$$

holds. In particular $\det H = 0$ almost everywhere on I_ϕ . An *H-indivisible interval* is called maximal if it is not a proper subset of an *H-indivisible interval*.

4.1 Main Growth Theorem of Canonical Systems

Before we formulate the first important theorem regarding the growth of canonical systems, we present a preparatory result, which we will use to prove the theorem.

Lemma 4.3. *Let $M(x, z)$ be a fundamental solution of (1.5), let x_j , $a = x_0 < x_1 < \dots < x_n = b$ and $Q_j \in \text{GL}(2, \mathbb{C})$ for $0 \leq j \leq n$. Then*

$$\|M(b, z)\| \leq \exp \left(|z| \sum_{j=1}^n \int_{x_{j-1}}^{x_j} \|Q_j^{-1} A(t) Q_j\| dt \right) \|Q_n^{-1}\| \prod_{j=0}^{n-1} \|Q_{j+1} Q_j^{-1}\| \|Q_0\|.$$

Proof. Applying (1.7), we obtain

$$\|QM(x_j, z)\| \leq \|QM(x_{j-1}, z)\| \exp \left(|z| \int_{x_{j-1}}^{x_j} \|QA(t)Q^{-1}\| dt \right),$$

for arbitrary $Q \in \text{GL}(2, \mathbb{C})$ and $0 \leq j \leq n$. Hence, we get

$$\begin{aligned} \|Q_{j+1}M(x_{j+1}, z)\| &\leq \|Q_{j+1}Q_j^{-1}\| \|Q_jM(x_{j+1}, z)\| \leq \\ &\leq \|Q_{j+1}Q_j^{-1}\| \|Q_jM(x_j, z)\| \exp \left(|z| \int_{x_j}^{x_{j+1}} \|Q_j A(t) Q_j^{-1}\| dt \right). \end{aligned}$$

We recall that $M(x_n, z) = M(b, z)$ and $M(x_0, z) = M(a, z) = I$ and we apply the upper inequality inductively for $\|Q_n M(x_n, z)\|$ and get

$$\|Q_n M(b, z)\| \leq \|Q_0\| \prod_{j=0}^{n-1} \|Q_{j+1} Q_j^{-1}\| \exp \left(|z| \int_{x_j}^{x_{j+1}} \|Q_j A(t) Q_j^{-1}\| dt \right).$$

Multiplying this inequality with $\|Q_n^{-1}\|$ yields the desired result. \square

For $\phi \in [0, 2\pi)$, we define $P(\phi)$ as the following matrix of rank one

$$P(\phi) := \begin{pmatrix} \sin^2(\phi) & \sin(\phi) \cos(\phi) \\ \sin(\phi) \cos(\phi) & \cos^2(\phi) \end{pmatrix}.$$

Theorem 4.4. *Let $(H, [a, b])$ be a canonical system and let $0 \leq d \leq 1$. Suppose that there exists $C \geq 0$, such that for each R large enough, there exists a Hamiltonian $H(R)$ of a finite rank $n(R)$, defined on (a, b) and a sequence of numbers (depending on R) $(a_j(R))_{j=0}^{n(R)-1}$, $0 < a_j(R) \leq 1$, for which the following conditions are satisfied:*

$$(i) \sum_{j=0}^{n-1} \frac{1}{a_j^2(R)} \int_{x_j}^{x_{j+1}} \|H(t) - H_R(t)\| dt \leq C_1 \frac{\lambda(R)}{R},$$

$$(ii) \sum_{j=0}^{n-1} a_j^2(R) (x_{j+1} - x_j) \leq C_2 \frac{\lambda(R)}{R},$$

$$(iii) \log a_0(R) + \log a_{n(R)-1}^{-1}(R) + \sum_{j=0}^{n-1} \log \left(\min(a_j(R) a_{j+1}^{-1}(R), a_{j+1}(R) a_j^{-1}(R)) \right) \leq C_3 \lambda(R),$$

Then the monodromy matrix has finite λ -type.

Proof. Let $(H, [a, b])$ be a canonical system. For an arbitrary finite set of numbers x_j , $0 \leq j \leq n$, such that $a = x_0 < x_1 < \dots < x_n = b$ and arbitrary invertible matrices Q_j , $0 \leq j \leq n$, we obtain, by Lemma 4.3

$$\|M(b, z)\| \leq \|Q_0\| \|Q_n^{-1}\| \prod_{j=0}^{n-1} \|Q_{j+1} Q_j^{-1}\| \exp \left(|z| \int_{x_j}^{x_{j+1}} \|Q_j JH(t) Q_j^{-1}\| dt \right)$$

Taking the logarithm we obtain

$$\log \|M(b, z)\| \leq |z| \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \|Q_j JH(t) Q_j^{-1}\| dt + \sum_{j=0}^{n-1} \log \|Q_{j+1} Q_j^{-1}\| + \log \|Q_0\| + \log \|Q_n^{-1}\| \quad (4.2)$$

A summand in the first sum can be estimated, applying the triangle inequality, the following way:

$$\begin{aligned} \int_{x_j}^{x_{j+1}} \|Q_j JH(t) Q_j^{-1}\| dt &\leq \int_{x_j}^{x_{j+1}} \left(\|Q_j J(H(t) - P(\phi_j)) Q_j^{-1}\| + \|Q_j JP(\phi_j) Q_j^{-1}\| \right) dt \leq \\ &(x_{j+1} - x_j) \|Q_j JP(\phi_j) Q_j^{-1}\| + \int_{x_j}^{x_{j+1}} \|Q_j J(H(t) - P(\phi_j)) Q_j^{-1}\| dt \leq \\ &(x_{j+1} - x_j) \|Q_j JP(\phi_j) Q_j^{-1}\| + \|Q_j\| \|Q_j^{-1}\| \int_{x_j}^{x_{j+1}} \|H(t) - P(\phi_j)\| dt. \end{aligned} \quad (4.3)$$

The next step is to choose appropriate matrices Q_j . We want to choose them such that equality in Lemma 2.5 is nearly fulfilled. Since $\det P(\phi_j) = 0$, we take

$$Q_j = \text{diag} \left(a_j^{-1}(R), a_j(R) \right) U_j,$$

where $a_j(R) \in (0, 1]$ and U_j is a unitary transform reducing $JP(\phi_j)$ into its Jordan form,

$$U_j JP(\phi_j) U_j^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

More precisely we get $U_j = e^{-\phi_j J} = \begin{pmatrix} \cos \phi_j & \sin \phi_j \\ -\sin \phi_j & \cos \phi_j \end{pmatrix}$ and hence

$$Q_j = \begin{pmatrix} a_j^{-1}(R) \cos \phi_j & a_j^{-1}(R) \sin \phi_j \\ -a_j(R) \sin \phi_j & a_j(R) \cos \phi_j \end{pmatrix} \text{ and } Q_j^{-1} = \begin{pmatrix} a_j(R) \cos \phi_j & -a_j^{-1}(R) \sin \phi_j \\ a_j(R) \sin \phi_j & a_j^{-1}(R) \cos \phi_j \end{pmatrix}.$$

We can compute the spectral norms $\|Q_j\| = \|Q_j^{-1}\| = a_j^{-1}(R)$ and $\|Q_j JP_j Q_j\| = a_j^2(R)$, and hence continue to estimate the right hand side of (4.3) from above by

$$\frac{1}{a_j^2(R)} \int_{x_j}^{x_{j+1}} \|H(t) - P(\phi_j)\| dt + a_j^2(R)(x_{j+1} - x_j). \quad (4.4)$$

For $j \leq n-1$, computing the spectral norm, we get

$$\|Q_{j+1} Q_j^{-1}\| = \min \left(a_j(R) a_{j+1}^{-1}(R), a_{j+1}(R) a_j^{-1}(R) \right). \quad (4.5)$$

Hence

$$\log \|Q_{j+1} Q_j^{-1}\| \leq \left| \log \left(\min \{ a_j(R) a_{j+1}^{-1}(R), a_{j+1}(R) a_j^{-1}(R) \} \right) \right|.$$

Plugging this and (4.4) into (4.2), taking into account that $\|Q_0\| = a_0^{-1}(R)$, $\|Q_n^{-1}\| = a_{n-1}^{-1}(R)$ and applying (i) - (iii), we obtain

$$\log \|M(L, z)\| \leq (C_1 + C_2) \frac{\lambda(R)}{R} |z| + C_3 \lambda(R) = (C_1 + C_2 + C_3) \lambda(R),$$

which concludes the proof. \square

4.2 An Application of Theorem 4.4

If we consider the case that $a_j(R)$ is independent of j , we get a corollary as a consequence of the previous theorem.

Corollary 4.5. *Assume $\lambda(R) = R^d$, with $\frac{1}{2} \leq d < 1$ and that $a_j(R)$ is independent of j . For any $\epsilon > 0$ let a finite rank Hamiltonian H^ϵ defined on (a, b) exist such that*

$$\|H - H^\epsilon\|_{L^1(a,b)} \leq \epsilon \quad (4.6)$$

Then the order of the system $(H, [a, b])$ is not greater than d .

Proof. We note that if $a_j(R)$ is independent of j condition (ii) of Theorem 4.4 reduces to $a_j^2(R) = O(R^{d-1})$ and the left hand sides in condition (i) is monotonely decreasing in $a_j(R)$. The left hand side of condition (iii) of Theorem 4.4 reduces to $\log(a_j(R)) + \log(a_j(R)^{-1}) = \log(a_j(R) a_j(R)^{-1}) = 0$ and hence is trivially fulfilled. Hence, the system $(H, [a, b])$ obeys the conditions of Theorem 4.4 with $H_R = H^{\epsilon(R)}$, $\epsilon(R) = R^{2(d-1)}$, $a_j(R) = R^{\frac{(d-1)}{2}}$, because (4.6) implies (i), and (ii) is immediate. \square

4.3 Improvements on Theorem 4.4

Now we will try to find a way to get a better estimate than in the theorem above.

Definition 4.6. Let $A : [a, b] \rightarrow \mathbb{C}^{2 \times 2}$ and let $(x_j)_{j=0}^n$ be a partition of $[a, b]$ such that $a = x_0 < x_1 < \dots < x_n = b$. Then let $Q := \langle (x_j)_{j=0}^n, (Q_j)_{j=0}^n \rangle$, where $Q_j \in \text{GL}(2, \mathbb{C})$, for $j = 0, \dots, n$. We will call $\text{est}(Q) := \|Q_0^{-1}\| \|Q_n\| \prod_{j=0}^n \|Q_j^{-1} Q_{j-1}\|$ the *estimator*, for our problem.

The estimate from Lemma 4.3 provides us with a rough upper bound for the growth of our fundamental solutions. Our next goal will be to choose $(Q_j)_{j=1}^n$ such that $\text{est}(Q)$ becomes as small as possible. To this end we note the following simple properties of $\text{est}(Q)$.

Lemma 4.7. Let $Q = \langle (x_j)_{j=0}^n, (Q_j)_{j=0}^n \rangle$. Then the following statements hold true:

- (i) Let $Q' = \langle (x'_j)_{j=0}^n, (Q'_j)_{j=0}^n \rangle$ be such that $x'_j = x_j$ and $Q'_j = \alpha_j Q_j$, for $\alpha_j \neq 0$. Then $\text{est}(Q) = \text{est}(Q')$.
- (ii) Let $Q' = \langle (x'_j)_{j=0}^n, (Q'_j)_{j=0}^n \rangle$ be such that $x'_j = x_j$ and $Q'_j = Q_j U_j$, where U_j are unitary matrices for $j \in \{1, \dots, n\}$. Then $\text{est}(Q) = \text{est}(Q')$.
- (iii) Let $Q' = \langle (x'_i)_{i=0}^M, (Q'_i)_{i=0}^M \rangle$ be such that $\{x_j : j = 0, \dots, N\} \subseteq \{x'_i : i = 1, \dots, M\}$ and $Q'_i = Q_j$, if $x'_i \in (x_{j-1}, x_j]$ and $i = \{1, \dots, M\}$. Then $\text{est}(Q) = \text{est}(Q')$.

Proof. (i) trivially holds.

(ii) obviously holds as well, because $\|U_j^{-1}\| = \|U_j\| = 1$, for $j = 1, \dots, n$, by unitarity of U_j .

Since $Q'_i = Q_{i-1}$, whenever $x'_{i-1} \neq x_j$ for $j = 1, \dots, n$, we observe that in that case $\|Q'_i{}^{-1} Q'_{i-1}\| = 1$ and hence we obtain (iii). \square

Corollary 4.8. Let Q be defined with $Q_j = \exp(\phi_j J) \begin{pmatrix} a_j^{-1} & 0 \\ 0 & a_j \end{pmatrix}$ and Q' such that $Q'_j = \exp((\phi_j + \frac{\pi}{2})J) \begin{pmatrix} 0 & a_j^2 \\ 1 & 0 \end{pmatrix}$, then $\text{est}(Q) = \text{est}(Q')$.

Proof. We observe

$$Q'_j = \exp\left(\left(\phi_j + \frac{\pi}{2}\right)J\right) \begin{pmatrix} 0 & a_j^2 \\ 1 & 0 \end{pmatrix} = \exp(\phi_j J) \begin{pmatrix} -1 & 0 \\ 0 & a_j^2 \end{pmatrix} = a_j Q_j \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and conclude our proof by Lemma 4.7 (i). \square

Remark 4.9. Recall that we have proven Theorem 4.4 using matrices of the form $\exp(\phi_j J) \begin{pmatrix} a_j^{-1} & 0 \\ 0 & a_j \end{pmatrix}$. With the corollary above we see that the estimator stays the same if

we use matrices of the form $Q(\phi, a_j) := Q'_j = \exp(\phi_j J) \begin{pmatrix} 0 & a_j \\ 1 & 0 \end{pmatrix}$.

Further $\|Q'_j\| = \|Q_j\|$, $\|(Q'_j)^{-1}\| = \|Q_j^{-1}\|$ and $\|Q'_j J P(\phi_j) (Q'_j)^{-1}\| = \|Q_j J P(\phi_j) (Q_j)^{-1}\|$ hold, because multiplication by unitary matrices does not change the spectral norm. Hence we obtain the same estimate for the order for these matrices.

This is of special interest, because in some applications $Q_j(\phi, a)$ has actually led to better results than the matrices presented in that proof. Because of this we will now turn our attention to minimizing the estimator of a partition with matrices $Q_j(\phi, a)$. To this end we want to find a more precise estimate to replace conditions (iii) and (iv) from Theorem 4.4. The following lemma shall serve this purpose.

Lemma 4.10. *Let $0 < v, w \leq 1$ and $Q(\phi, v) := \exp(\phi J) \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix}$ and write $C := Q(\phi, v)^{-1}Q(\psi, w)$. Then*

$$\|C\| \leq \sqrt{|\lambda(v, w)|},$$

where

$$\lambda(v, w) := \max \left\{ \frac{1}{2} \left(\operatorname{tr}(C^*C) + \sqrt{\operatorname{tr}(C^*C)^2 - 4 \det(C)^2} \right), \frac{1}{2} \left(\operatorname{tr}(C^*C) - \sqrt{\operatorname{tr}(C^*C)^2 - 4 \det(C)^2} \right) \right\}.$$

Proof. We evaluate

$$\begin{aligned} C &= \left(\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \cdot \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix} \cdot \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} -\sin \phi & \cos \phi \\ v^{-1} \cos \phi & v^{-1} \sin \phi \end{pmatrix} \cdot \begin{pmatrix} -\sin \psi & w \cos \psi \\ \cos \psi & w \sin \psi \end{pmatrix} = \\ &= \begin{pmatrix} \sin \phi \sin \psi + \cos \phi \cos \psi & -w \sin \phi \cos \psi + w \cos \phi \sin \psi \\ -v^{-1} \cos \phi \sin \psi + v^{-1} \sin \phi \cos \psi & wv^{-1} \cos \phi \cos \psi + wv^{-1} \sin \phi \sin \psi \end{pmatrix} = \\ &= \begin{pmatrix} \cos \psi - \phi & w \sin(\psi - \phi) \\ -v^{-1} \sin(\psi - \phi) & wv^{-1} \cos(\psi - \phi) \end{pmatrix}. \end{aligned}$$

Now we want to compute the spectral norm of C . To this end we consider

$$C^*C = \cos^2(\psi - \phi) \begin{pmatrix} 1 & 0 \\ 0 & w^2v^{-2} \end{pmatrix} + \sin^2(\psi - \phi) \begin{pmatrix} v^{-2} & 0 \\ 0 & w^2 \end{pmatrix} + D,$$

where

$$D := \sin(\psi - \phi) \cos(\psi - \phi) \begin{pmatrix} 0 & w - wv^{-2} \\ w - wv^{-2} & 0 \end{pmatrix}$$

denotes the appropriate offdiagonal matrix such that the equation is fulfilled. Hence, to compute the eigenvalues of this matrix, we have to solve the following quadratic equation

$$\lambda^2 - \operatorname{tr}(C^*C)\lambda + \det(C)^2 = 0.$$

Therefore, the eigenvalues of C^*C are

$$\begin{aligned} \lambda_1(v, w) &= \frac{1}{2} \left(\operatorname{tr}(C^*C) + \sqrt{\operatorname{tr}(C^*C)^2 - 4 \det(C)^2} \right), \\ \lambda_2(v, w) &= \frac{1}{2} \left(\operatorname{tr}(C^*C) - \sqrt{\operatorname{tr}(C^*C)^2 - 4 \det(C)^2} \right), \end{aligned}$$

and we are done. \square

Corollary 4.11. *Let $(H, [a, b])$ be a canonical system and let $0 \leq d \leq 1$. Suppose that there exists a $C \geq 0$, such that for each R large enough, there exists a Hamiltonian $H(R)$ of a finite rank, $n(R)$, defined on (a, b) and a sequence of numbers (depending on R) $(a_j(R))_{j=0}^{n(R)-1}$, $0 < a_j(R) \leq 1$, for which the following conditions are satisfied:*

$$\begin{aligned} (i) \quad & \sum_{j=0}^{n-1} \frac{1}{a_j^2(R)} \int_{x_j}^{x_{j+1}} \|H(t) - H_R(t)\| dt \leq C_1 \frac{\lambda(R)}{R}, \\ (ii) \quad & \sum_{j=0}^{n-1} a_j^2(R) (x_{j+1} - x_j) \leq C_2 \frac{\lambda(R)}{R}, \end{aligned}$$

$$(iii) \log a_0^{-1}(R) + \log a_{n(R)-1}^{-1}(R) + \sum_{j=0}^{n-1} \log (\lambda(a_j(R), a_{j+1}(R))) \leq C_3 \lambda(R).$$

Then the entries of the monodromy matrix have finite λ -type.

Proof. The proof of this corollary follows the same lines as the proof of Theorem 4.4, except that we replace the estimate (4.5) by the estimate we get from Lemma 4.10. \square

4.4 Diagonal Hamiltonians

An important class of canonical systems is constituted by systems with diagonal matrices $H(x)$, which we call *diagonal Hamiltonians*. We will now apply Theorem 4.4 to diagonal Hamiltonians to find their order.

In the context of the order problem we are only interested in Hamiltonians with determinant equal to 0. Hence, a diagonal Hamiltonian may take only two values, $H_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $H_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Our next result says that in this case the upper bound implied by Theorem 4.4 coincides with the actual order. The formulation is as follows. Define $X_1 := \{x \in (a, b) : H(x) = H_1\}$, $X_2 := \{x \in (a, b) : H(x) = H_2\}$, and let $|\cdot|$ denote the Lebesgue measure.

Theorem 4.12. *Suppose that for almost every $x \in [a, b]$ either $H(x) = H_1$ or $H(x) = H_2$. Then the order of the system $(H, [a, b])$ coincides with the infimum of $d \in (0, 1]$, for which there exists $C > 0$ such that for n large enough there exists a covering of the interval (a, b) by at most n intervals ω_j , such that*

$$\sum_{j=1}^n \sqrt{|\omega_j \cap X_1| |\omega_j \cap X_2|} \leq C n^{1-d^{-1}} \quad (4.7)$$

We shall split the proof into several parts. The first part will be the proof of the following proposition, which we will use to show that the order of the system is not greater than the infimum.

Proposition 4.13. *Under the assumptions of Theorem 4.12 the order of the system $(H, [a, b])$ is less or equal to the infimum of $d \in (0, 1]$, for which there exists a positive C such that for each R large enough there exists a covering of the interval (a, b) by $n(R) \leq CR^d$ intervals $\omega_j(R)$, such that*

$$\sum_{j=1}^{n(R)} \sqrt{|\omega_j(R) \cap X_1| |\omega_j(R) \cap X_2|} \leq CR^{d-1} \quad (4.8)$$

Proof. Let d be any number larger than the infimum mentioned in the statement. Then we find $\epsilon > 0$, $C > 0$, and intervals $\omega_j(R)$, such that $n(R) \leq CR^{d-\epsilon}$ and (4.8) holds with the exponent $d - \epsilon - 1$ on the right side. Since $R^{d-\epsilon} \leq \frac{R^d}{\log(R)}$ for R large, we may assume that we have $C > 0$, $\omega_j(R)$, with $n(R) \leq \frac{CR^d}{\log(R)}$, such that (4.8) holds with exponent $d - 1$.

The stated inequality will be established if we show that the order is not greater than d . Without loss of generality, one can assume that the intervals $\omega_j(R)$ are mutually disjoint. Define

$$H_R(x) = \begin{cases} H_1, & x \in \omega_j(R), \quad |\omega_j(R) \cap X_1| \geq \frac{|\omega_j(R)|}{2}, \\ H_2, & \text{otherwise.} \end{cases}$$

With this choice of H_R ,

$$\int_{\omega_j(R)} \|H(t) - H_R(t)\| dt \leq 2 \min\{|\omega_j(R) \cap X_1|, |\omega_j(R) \cap X_2|\};$$

hence the left hand side of condition (i) of Theorem 4.4 takes the form

$$\sum_{j=0}^{n(R)} \frac{2}{a_j(R)^2} \min\{|\omega_j(R) \cap X_1|, |\omega_j(R) \cap X_2|\}. \quad (4.9)$$

To further estimate this, we show that

$$\min\{|\omega_j(R) \cap X_1|, |\omega_j(R) \cap X_2|\} \asymp |\omega_j(R) \cap X_1| |\omega_j(R) \cap X_2| / |\omega_j(R)|$$

holds. Therefore we observe that $\min\{|\omega_j \cap X_1|, |\omega_j(R) \cap X_2|\}$, is obviously bounded from below by $|\omega_j(R) \cap X_1| |\omega_j(R) \cap X_2| / |\omega_j(R)|$. To find a $C > 0$ such that $\min\{|\omega_j(R) \cap X_1|, |\omega_j(R) \cap X_2|\} \leq C |\omega_j(R) \cap X_1| |\omega_j(R) \cap X_2| / |\omega_j(R)|$ holds we can assume without loss of generality that the minimum is taken by $|\omega_j(R) \cap X_1|$. Then it follows that $|\omega_j(R) \cap X_2| / |\omega_j(R)| \geq 1/2$ and therefore we can choose $C = 2$ and the inequality holds. Hence (4.9) is equivalent to

$$\sum_{j=0}^{n(R)} \frac{2}{a_j^2(R) |\omega_j(R)|} |\omega_j(R) \cap X_1| |\omega_j(R) \cap X_2|. \quad (4.10)$$

The left hand side of condition (ii) obviously has the form

$$\sum_{j=0}^{n(R)} a_j(R)^2 |\omega_j(R)|. \quad (4.11)$$

Condition (iii) in the situation under consideration, is estimated as follows

$$\begin{aligned} \log a_0^{-1}(R) + \log a_{n(R)-1}^{-1}(R) + \sum_{j=0}^{n(R)-1} \log \left(\min(a_j(R) a_{j+1}^{-1}(R), a_{j+1}(R) a_j^{-1}(R)) \right) &\leq \\ &\leq \log a_0^{-1}(R) + \log a_{n(R)-1}^{-1}(R) + \sum_{j=0}^{n(R)-1} \log \left(a_j^{-1}(R) \right) \leq 2 \sum_{j=0}^{n(R)} \log(1 + a_j(R)^{-2}), \end{aligned} \quad (4.12)$$

because $a_j(R) < 1$ for $j \in \{0, \dots, n\}$ and by monotonicity of the logarithm. Now we define numbers $a_j(R)$ to further estimate the three previous terms.

First we consider $|\omega_j(R)| \leq \frac{2}{R}$. We write $\mathcal{N} := \{j : |\omega_j(R)| \leq \frac{2}{R}\}$ and set $a_j(R) = 1$ for $j \in \mathcal{N}$. The parts of sums in (4.10), (4.11), (4.12) over $j \in \mathcal{N}$ are estimated from above by $4n(R)R^{-1}$, $2n(R)R^{-1}$ and $n(R)$, respectively, hence all of them satisfy the bounds required for Theorem 4.4. For $|\omega_j(R)| > \frac{2}{R}$ we optimize the choice of $a_j(R)$ over the summands in (4.10), (4.11) and (4.12), by taking

$$a_j(R)^2 = \max \left\{ \frac{1}{R |\omega_j(R)|}, \frac{\sqrt{|\omega_j(R) \cap X_1| |\omega_j(R) \cap X_2|}}{|\omega_j(R)|} \right\}.$$

Note that with this choice we can estimate (4.10) and (4.12) from above by taking either value for $a_j(R)$, because we regard its inverse. With this choice the sums over $j \notin \mathcal{N}$ in (4.10), (4.11) and (4.12) are estimated from above by

$$2 \sum_{j=0}^{n(R)} \sqrt{|\omega_j(R) \cap X_1| |\omega_j(R) \cap X_2|},$$

$$2R^{-1}n(R) + 2 \sum_{j=0}^{n(R)} \sqrt{|\omega_j(R) \cap X_1| |\omega_j(R) \cap X_2|},$$

$$\sum_{j=0}^{n(R)} \log(1 + R|\omega_j(R)|) \leq cn(R) \log R, \quad c \in [0, \infty).$$

Due to condition (4.8) and our choice of $n(R)$ it follows that the order is not greater than d by Theorem 4.4. \square

The next step will be the following proposition. To prove it we show a few general properties of diagonal Hamiltonians, some of which we will then apply. To simplify notation, we set

$$\rho_1(c, d) := \int_c^d h_1(t) dt, \quad \rho_2(c, d) := \int_c^d h_2(t) dt.$$

Proposition 4.14. *If H is a diagonal Hamiltonian, it holds*

$$\text{ord } M(x, \cdot) \geq \limsup_{\tau \rightarrow \infty} \frac{\log \left(\int_a^x \frac{\mathbb{1}_{X_2}(t)}{\rho_2(r(\tau, t), t)} dt \right)}{\log(\tau)}, \quad x \in [a, b], \quad (4.13)$$

where $\mathbb{1}_{X_2}$ denotes the indicator function on X_2 and $r(\tau, x)$ is a strictly increasing function we will specify later on.

To this end we start with a few preliminary observations. We write $H(t) = \begin{pmatrix} h_1(t) & 0 \\ 0 & h_2(t) \end{pmatrix}$, i.e. $h_1(x) = \mathbb{1}_{X_1}$, $h_2(x) = \mathbb{1}_{X_2}$. By definition of X_1 and X_2 , we always get that either $h_1(t) = 1$ and $h_2(t) = 0$ or vice versa. Therefore our canonical system, writing out the components, has the form

$$\begin{aligned} \frac{d}{dx} M_{21}(x, z) &= -zh_1(x)M_{11}(x), \\ \frac{d}{dx} M_{22}(x, z) &= -zh_1(x)M_{12}(x), \\ \frac{d}{dx} M_{11}(x, z) &= zh_2(x)M_{21}(x), \\ \frac{d}{dx} M_{12}(x, z) &= zh_2(x)M_{22}(x). \end{aligned} \quad (4.14)$$

Written in integral form the whole system has the form

$$JM(x, z) - J = z \int_a^x H(t)M(t, z) dt.$$

Again writing out the components, we obtain

$$\begin{aligned} M_{21}(x, z) &= -z \int_a^x h_1(t)M_{11}(t) dt, \\ M_{22}(x, z) &= 1 - z \int_a^x h_1(t)M_{12}(t) dt, \\ M_{11}(x, z) &= 1 + z \int_a^x h_2(t)M_{21}(t) dt, \\ M_{12}(x, z) &= z \int_a^x h_2(t)M_{22}(t) dt. \end{aligned} \quad (4.15)$$

If we combine the first and third and the second and fourth equations respectively we get four second order differential equations for $M_{ij}(x, z)$, $i, j \in \{1, 2\}$:

$$M_{11}(x, z) = 1 - z^2 \int_a^x h_2(t) \int_a^t h_1(s) M_{11}(s, z) ds dt, \quad (4.16)$$

$$\begin{aligned} M_{21}(x, z) &= -z \int_a^x h_1(t) dt - z^2 \int_a^x h_1(t) \int_a^t h_2(s) M_{21}(s, z) ds dt, \\ M_{12}(x, z) &= z \int_a^x h_2(t) dt - z^2 \int_a^x h_2(t) \int_a^t h_1(s) M_{12}(s, z) ds dt, \\ M_{22}(x, z) &= 1 - z^2 \int_a^x h_1(t) \int_a^t h_2(s) M_{22}(s, z) ds dt. \end{aligned} \quad (4.17)$$

Then, since $\mathbb{1}_{[a,t]}(s) = \mathbb{1}_{[s,x]}(t)$, if $s, t \in [a, x]$, and by changing the order of integration, we get

$$\begin{aligned} \int_a^x h_2(t) \int_a^t h_1(s) M_{11}(s, z) ds dt &= \int_a^x h_2(t) \int_a^x \mathbb{1}_{[a,t]}(s) h_1(s) M_{11}(s, z) ds dt = \\ &= \int_a^x h_2(t) \int_a^x \mathbb{1}_{[s,x]}(t) h_1(s) M_{11}(s, z) ds dt = \\ &= \int_a^x \rho_2(s, x) h_1(s) M_{11}(s, z) ds. \end{aligned}$$

Applying this (and similar computations for the other three equations) to (4.16), we obtain

$$\begin{aligned} M_{11}(x, z) &= 1 - z^2 \int_a^x \rho_2(s, x) h_1(s) M_{11}(s, z) ds, \\ M_{21}(x, z) &= -z \rho_1(a, x) - z^2 \int_a^x \rho_1(s, x) h_2(s) M_{21}(s, z) ds, \\ M_{12}(x, z) &= z \rho_2(a, x) - z^2 \int_a^x \rho_2(s, x) h_1(s) M_{12}(s, z) ds, \\ M_{22}(x, z) &= 1 - z^2 \int_a^x \rho_1(s, x) h_2(s) M_{22}(s, z) ds. \end{aligned} \quad (4.18)$$

Soon we will focus on M_{11} to estimate its order, and hence, by Corollary 3.38, the order of M , but first we make the following general observation:

Lemma 4.15. *The functions $M_{11}(x, z)$ and $M_{22}(x, z)$ are even, while $M_{21}(x, z)$ and $M_{12}(x, z)$ are odd.*

Proof. Set $V := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and consider $\tilde{M}(x, z) = VM(x, z)V$. Using that $JV = -VJ$ and that $VH(x) = H(x)V$, if $H(x)$ is diagonal, we obtain

$$J \frac{d}{dx} \tilde{M}(x, z) = JV \frac{d}{dx} M(x, z) V = -V \left(J \frac{d}{dx} M(x, z) \right) V = -zH(x) \tilde{M}(x, z).$$

Further $\tilde{M}(a, z) = I$ and we conclude that $M(x, z) = \tilde{M}(x, -z)$ i.e.

$$\begin{pmatrix} M_{11}(x, z) & -M_{12}(x, z) \\ -M_{21}(x, z) & M_{22}(x, z) \end{pmatrix} = \begin{pmatrix} M_{11}(x, -z) & M_{12}(x, -z) \\ M_{21}(x, -z) & M_{22}(x, -z) \end{pmatrix}.$$

□

Lemma 4.16. For $\tau > 0$, set $\xi_\tau(x) := M_{11}(x, i\tau)$. Then ξ_τ is real, positive nondecreasing and $\xi_\tau(a) = 1$.

Proof. Using Lemma 4.15 and Proposition 1.9 (iii) we obtain

$$\xi_\tau(x) = M_{11}(x, i\tau) = M_{11}(x, -i\tau) = \overline{M_{11}(x, i\tau)} = \overline{\xi_\tau(x)},$$

hence $\xi_\tau(x)$ is real.

Fix $x \in [a, b]$. Since $M_{11}(x, \cdot)$, by Corollary 1.12, has no nonreal zeroes and $M_{11}(x, 0) = 1$, continuity of $\tau \mapsto \xi_\tau(x)$ yields that $\xi_\tau(x) > 0$ for all $\tau > 0$. Now fix $\tau > 0$. Then (4.18) becomes

$$\xi_\tau(x) = 1 + \tau^2 \int_a^x h_2(t) \int_a^t h_1(s) \xi_\tau(s) ds dt.$$

Since the integrand is a nonnegative function, $\xi_\tau(x)$ is nondecreasing. \square

Lemma 4.17. Let $\tau > 0$, $x \in [a, b]$, and let $r : [a, b] \mapsto [a, b]$ such that $r(t) \leq t$. Then

$$\log(\xi_\tau(x)) \geq \int_a^x \frac{\tau^2 \rho_1(r(t), t) h_2(t)}{1 + \tau^2 \rho_1(r(t), t) \rho_2(r(t), t)} dt. \quad (4.19)$$

Proof. We shall estimate $\frac{d}{dx}(\log(\xi_\tau(x)))$ in two ways.

First, using (4.16), and that ξ_τ is nondecreasing, we get

$$\xi_\tau(x)' = \tau^2 h_2(x) \int_a^x h_1(s) \xi_\tau(s) ds \geq \tau^2 h_2(x) \int_{r(x)}^x h_1(s) \xi_\tau(s) ds \geq \tau^2 h_2(x) \xi_\tau(r(x)) \rho_1(r(x), x).$$

Dividing by $\xi_\tau(x)$ yields

$$\tau^2 h_2(x) \rho_1(r(x), x) \frac{\xi_\tau(r(x))}{\xi_\tau(x)} \leq \frac{\xi_\tau'(x)}{\xi_\tau(x)}. \quad (4.20)$$

Second, using (4.18) and that ρ_2 is nonincreasing in its first argument, we obtain

$$\begin{aligned} h_2(x)(\xi_\tau(x) - \xi_\tau(r(x))) &= h_2(x) \tau^2 \int_{r(x)}^x \rho_2(s, x) h_1(s) \xi_\tau(s) ds \leq \\ &\leq h_2(x) \tau^2 \rho_2(r(x), x) \int_{r(x)}^x h_1(s) \xi_\tau(s) ds \leq \\ &\leq \tau^2 \rho_2(r(x), x) h_2(x) \int_a^x h_1(s) \xi_\tau(s) ds = \rho_2(r(x), x) \xi_\tau'(x). \end{aligned}$$

Multiplying this by $\frac{\tau^2 \rho_1(r(x), x)}{\xi_\tau(x)}$, we get

$$\tau^2 \rho_1(r(x), x) h_2(x) \left(1 - \frac{\xi_\tau(r(x))}{\xi_\tau(x)}\right) \leq \tau^2 \rho_1(r(x), x) \rho_2(r(x), x) \frac{\xi_\tau'(r(x))}{\xi_\tau(x)}. \quad (4.21)$$

Summing up (4.20) and (4.21) gives us

$$\frac{\tau^2 \rho_1(r(x), x) h_2(x)}{1 + \tau^2 \rho_1(r(x), x) \rho_2(r(x), x)} \leq \frac{\xi_\tau'(r(x))}{\xi_\tau(x)}.$$

Since $\xi_\tau(a) = 1$, integrating over $[a, x]$ yields the assertion. \square

To simplify (4.19), we make a particular choice of $r(x)$ depending on τ .

Lemma 4.18. *Let $c, \alpha > 0$ and assume that $H(x)$ starts with two indivisible intervals*

$$H(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad x \in [a - 2\alpha, a - \alpha], \quad H(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad x \in [a - \alpha, a].$$

Then there exists a unique function $r_c : [\sqrt{c}, \infty) \times [a, b] \rightarrow [a, b]$, such that

(i) $r_c(\tau, x) \leq x, \quad x \in [a, b],$

(ii) $\tau^2 \rho_1(r_c(\tau, x), x) \rho_2(r_c(\tau, x), x) = c\alpha^2, \quad \tau \in [\sqrt{c}, \infty), \quad x \in [a, b].$

Further this function has the following properties:

(iii) $r_c(\sqrt{c}, a) = a - 2\alpha,$

(iv) *The functions $\tau \mapsto r_c(\tau, x)$ and $x \mapsto r_c(\tau, x)$ are strictly increasing and*

(v) $x - r_c(\tau, x) \geq \frac{\sqrt{c}\alpha}{\tau}.$

Proof. First we want to show that there exists a unique function $r_c(\tau, x)$ that fulfills conditions (i), (ii). Therefore, fix some $x \in [a, b]$. Writing out (ii) yields

$$\tau^2 \left(\int_{r_c(\tau, x)}^x h_1(s) ds \right) \left(\int_{r_c(\tau, x)}^x h_2(s) ds \right) = c\alpha^2.$$

By definition of h_1, h_2 , this is equivalent to

$$\left(\int_{r_c(\tau, x)}^x \mathbb{1}_{X_1}(s) ds \right) \left(\int_{r_c(\tau, x)}^x \mathbb{1}_{X_2}(s) ds \right) = \frac{c\alpha^2}{\tau^2},$$

and further, if $|\cdot|$ denotes the Lebesgue measure, we obtain that $r_c(\tau, x)$ has to satisfy

$$|(r_c(\tau, x), x) \cap X_1| |(r_c(\tau, x), x) \cap X_2| = \frac{c\alpha^2}{\tau^2}. \quad (4.22)$$

Now we need to show that there exists a value for $r_c(\tau, x)$ such that this is fulfilled. Therefore, we consider $\tau \geq \sqrt{c}$ and hence $\frac{c}{\tau^2} \leq 1$. Further, by assumption, we have $|(a - 2\alpha, x) \cap X_i| \geq \alpha$, for $x \in [a, b], i \in \{1, 2\}$. To sum up we have

$$|(a - 2\alpha, x) \cap X_1| |(a - 2\alpha, x) \cap X_2| \geq \alpha^2 \geq \frac{c\alpha^2}{\tau^2}, \quad x \in [a, b].$$

On the other hand obviously $\lim_{t \rightarrow x} |(t, x) \cap X_i| = 0 < \frac{c\alpha^2}{\tau^2}, x \in [a, b], i \in \{1, 2\}$ and since $|(\cdot, x) \cap X_1| |(\cdot, x) \cap X_2|$ maps continuously and strictly decreasing from $[a - 2\alpha, x]$ to \mathbb{R} we get the existence of a unique $r_c(\tau, x) \in [a - 2\alpha, x]$ such that (4.22) and assertion (i) are fulfilled.

To prove that (iii) is satisfied, we plug $r_c(\sqrt{c}, a)$ into (4.22):

$$|(r_c(\sqrt{c}, a), a) \cap X_1| |(r_c(\sqrt{c}, a), a) \cap X_2| = \alpha^2. \quad (4.23)$$

Due to the assumption we know the form of X_1 and X_2 on $[a - 2\alpha, a]$, and hence can evaluate

$$\begin{aligned} |(r_c(\sqrt{c}, a), a) \cap X_1| &= \begin{cases} \alpha & a - 2\alpha \leq r_c(\sqrt{c}, a) \leq a - \alpha, \\ a - r_c(\sqrt{c}, a) & a - \alpha \leq r_c(\sqrt{c}, a) \leq a, \end{cases} \\ |(r_c(\sqrt{c}, a), a) \cap X_2| &= \begin{cases} a + \alpha - r_c(\sqrt{c}, a) & a - 2\alpha \leq r_c(\sqrt{c}, a) \leq a - \alpha, \\ 0 & a - \alpha \leq r_c(\sqrt{c}, a) \leq a. \end{cases} \end{aligned}$$

Therefore, (4.23) is satisfied, if and only if $r_c(\sqrt{c}, a) = a - 2\alpha$.

Now we want to show the strict monotonicity of $x \mapsto r_c(\tau, x)$. To this end let $\tau \in [\sqrt{c}, \infty)$ be fixed, $x_1 < x_2$ and assume $r_c(\tau, x_2) \leq r_c(\tau, x_1)$. Then we get the contradiction

$$\begin{aligned} \frac{c\alpha^2}{\tau^2} &= |(r_c(\tau, x_1), x_1) \cap X_1| |(r_c(\tau, x_1), x_1) \cap X_2| < \\ &< |(r_c(\tau, x_2), x_2) \cap X_1| |(r_c(\tau, x_2), x_2) \cap X_2| = \frac{c\alpha^2}{\tau^2}. \end{aligned}$$

For the strict monotonicity of $\tau \mapsto r_c(\tau, x)$ let $x \in [a, b]$ be fixed. Now, because the right hand side in (4.22) strictly decreases with a bigger τ , to decrease the left hand side as well we have to strictly increase $r_c(\tau, x)$.

To see that (v) holds, by (4.22) we obtain

$$|(r_c(\tau, x), x)| |(r_c(\tau, x), x)| \geq |(r_c(\tau, x), x) \cap X_1| |(r_c(\tau, x), x) \cap X_2| = \frac{c\alpha^2}{\tau^2},$$

and hence $\frac{\sqrt{c}\alpha}{\tau} \leq x - r_c(\tau, x)$. □

Corollary 4.19. *Assume that H is given as indicated in the previous lemma, let $c > 0$ and $x \in [a, b]$. Then*

$$\log(\xi_\tau(x)) \geq \frac{c\alpha^2}{c\alpha^2 + 1} \int_a^x \frac{h_2(t)}{\rho_2(r_c(\tau, t), t)} dt, \quad \tau \in [\sqrt{c}, \infty), \quad x \in [a, b]. \quad (4.24)$$

Proof. By (4.19) and due to Lemma 4.18 (ii), we have

$$\begin{aligned} \log(\xi_\tau(x)) &\geq \int_a^x \frac{\tau^2 \rho_1(r_c(\tau, t), t) h_2(t)}{1 + \tau^2 \rho_1(r_c(\tau, t), t) \rho_2(r_c(\tau, t), t)} dt = \\ &= \int_a^x \frac{\tau^2 \rho_1(r_c(\tau, t), t) \rho_2(r_c(\tau, t), t) h_2(t)}{\rho_2(r_c(\tau, t), t) + \tau^2 \rho_1(r_c(\tau, t), t) \rho_2(r_c(\tau, t), t)} dt = \\ &= \frac{c\alpha^2}{c\alpha^2 + 1} \int_a^x \frac{h_2(t)}{\rho_2(r_c(\tau, t), t)} dt. \end{aligned}$$

□

Proof of Proposition 4.14. First let $z \in \mathbb{R}$ and define N as the set of all zeroes of $M_{11}(x, \cdot)$, which are always real by Corollary 1.12. Then, since M_{11} is even, we can write

$$M_{11}(z) = \prod_{z_j \in N} \left(1 - \frac{z}{z_j}\right) = \prod_{\substack{z_j \in N, \\ z_j > 0}} \left(1 - \frac{z}{z_j}\right) \left(1 - \frac{z}{-z_j}\right).$$

Using this we can bound

$$|M_{11}(z)| \leq \prod_{\substack{z_j \in N, \\ z_j > 0}} \left(1 + \frac{|z|^2}{z_j^2}\right) = \prod_{\substack{z_j \in N, \\ z_j > 0}} \left(1 + \frac{|iz|^2}{z_j^2}\right) = M_{11}(i|z|).$$

Therefore, and because by Corollary 3.36 the orders of all components are the same, we get $\text{ord } M(x, \cdot) = \text{ord } \xi(x)$. We fix some $\alpha > 0$ and we have

$$\text{ord } M(x, \cdot) = \limsup_{\tau \rightarrow \infty} \frac{\log(\xi_\tau(x))}{\log(\tau)} \geq \limsup_{\tau \rightarrow \infty} \frac{\frac{c\alpha^2}{c\alpha^2 + 1} \int_a^x \frac{h_2(t)}{\rho_2(r_c(\tau, t), t)} dt}{\log(\tau)}.$$

We obtain the assertion, since for fixed α and $\epsilon > 0$ we can always find c large enough such that

$$\frac{c\alpha^2}{c\alpha^2 + 1} > 1 - \epsilon,$$

and regard $r_c(\tau, t)$ for such a c . □

Proof of Theorem 4.12. To prove Theorem 4.12 let us assume that for any d

$$\int_a^b \frac{h_2(x)}{\rho_2(r(R, x), x)} dx = O(R^d), \quad R \rightarrow \infty, \quad (4.25)$$

holds. Then, after taking the logarithm and dividing by $\log(R)$, Proposition 4.14 yields

$$\text{ord } M(x, \cdot) \geq \frac{\log \left(\int_a^b \frac{h_2(x)}{\rho_2(r(R, x), x)} dx \right)}{\log(R)} = d. \quad (4.26)$$

We can attach two intervals, as indicated by Lemma 4.18, and the order of the fundamental solution of the canonical system $(H, [a - 2\alpha, b])$ remains the same. Now we regard this system and we want to show that for every d , such that (4.25) is satisfied, the interval $(a - 2\alpha, b)$ can be covered by $n(R) \leq O(R^d \log(R))$ intervals, $\omega_j(R)$, such that

$$\sum_{j=0}^{n(R)} \sqrt{|\omega_j(R) \cap X_1| |\omega_j(R) \cap X_2|} \leq O(R^{d-1}) \quad (4.27)$$

holds.

For each R large enough define a monotone decreasing sequence $(x_j(R))_{j=1}^{n(R)}$, as follows: $x_1(R) := b$, $x_{j+1}(R) = r_c(R, x_j)$, if $j \geq 1$ and $x_j(R) \geq a$. If $x_{j-1}(R) \geq a$, $x_j(R) < a$, then $x_{j+1}(R) := a - 2\alpha$ and the sequence terminates. First we show that the sequence is finite. To this end let $\alpha := 1$. Then, by the Lemma 4.18 (i) and (v), we get

$$c = R^2 \rho_1(r_c(R, x), x) \rho_2(r_c(R, x), x) \leq R^2 |x - r_c(R, x)|^2.$$

This implies that $x_j(R) - x_{j+1}(R) \geq \sqrt{c}/R^{-1}$, so the sequence has at most $O(R)$ members. Define $\omega_j(R) := [x_{j+1}(R), x_j(R)]$. By construction $[a - 2\alpha, b] \subseteq \bigcup_j \omega_j(R)$. To simplify notation we will write x_j instead of $x_j(R)$ from now on.

We claim that $\omega_j(R)$ is the required covering. First we have to show that $n(R)$, the number of intervals in the covering, is $O(R^d \log(R))$. To this end, recall that for $x \in [x_{j+2}, x_j]$, it holds that $\rho_2(r_c(R, x), x) = \rho_2(x_{j+1}, x_j) \leq \rho_2(x_{j+2}, x_j)$. Defining $s_j := \rho_2(x_{j+1}, x_j)$ we can estimate the left hand side of (4.25) from below by

$$\sum_{j=1}^{n(R)-3} \int_{x_{j+1}}^{x_j} \frac{h_2(x)}{\rho_2(r(\tau, x), x)} dx \geq \sum_{j=1}^{n(R)-3} \int_{x_{j+1}}^{x_j} \frac{h_2(x)}{\rho_2(x_{j+2}, x_j)} = \sum_{j=1}^{n(R)-3} \frac{s_j}{s_j + s_{j+1}}. \quad (4.28)$$

Let $\mathcal{G}_1 := \{j : \frac{s_{j+1}}{s_j} \leq 2\}$, $\mathcal{G}_2 := \{j : \frac{s_{j+1}}{s_j} > 2\}$ and let $n(\mathcal{G}_1)$, $n(\mathcal{G}_2)$ be the respective numbers of elements. When $j \in \mathcal{G}_1$ the summand in (4.28) is bounded below, because in this case $\frac{s_j}{s_j + s_{j+1}} \geq \frac{1}{3}$. Hence $n(\mathcal{G}_1) \leq O(R^d)$ by (4.25). To estimate $n(\mathcal{G}_2)$, notice that $s_j \geq \frac{c}{(b - (a - 2\alpha))R^2}$, because $c = R^2 |\omega_j(R) \cap X_1| s_j \leq R^2 (b - (a - 2\alpha)) s_j$. It follows that if k is the length of a discrete interval of the set \mathcal{G}_2 and m is the right end of it, then

$$x_{m+k} - x_{m+k-1} \geq s_{m+k-1} \geq 2s_{m+k-2} \geq \dots \geq 2^{k-1} s_m \geq \frac{2^{k-1} c}{(b - (a - 2\alpha))R^2}.$$

On the other hand $x_m - x_{m+1} \leq b - (a - 2\alpha)$ trivially, hence $k \leq C + 2 \log_2 R$. Hence we have estimated the length of one interval of \mathcal{G}_1 . The number of those intervals can not exceed $O(R^d)$, because there has to be an interval of \mathcal{G}_1 between two intervals of $n(\mathcal{G}_2)$. In conclusion we get $n(\mathcal{G}_2) \leq O(R^d \log R)$ and therefore, $n(R) = n(\mathcal{G}_1) + n(\mathcal{G}_2) \leq O(R^d \log R)$ as required. Now we can always estimate $O(R^d \log(R)) \leq O(R^{d+\epsilon})$, for $\epsilon > 0$. Hence, the same holds true for the infimum of such ϵ , and we see that we can limit the number of intervals by $O(R^d)$.

Further, we note that by the definition of $r_c(R, x)$, the summand in (4.27) is in $O(R^{-1})$, and hence (4.27) holds. Therefore, by Proposition 4.13 the fundamental solution $M(x, z)$ of the canonical system $(H, [a - 2\alpha, b])$ satisfies $\text{ord } M(x, \cdot) \leq d$. Hence, we also get that the fundamental solution $M(x, z)$ of the canonical system $(H, [a, b])$ satisfies $\text{ord } M(x, \cdot) \leq d$. This completes the proof of Theorem 4.12, by putting $n := R^d$ in (4.7). \square

5 Appendix A - The Herglotz Integral Representation

Theorem 5.1 (Herglotz). *Let h be analytic in \mathbb{C}^+ . Then we have $\operatorname{Im} h(w) \geq 0$, $w \in \mathbb{C}^+$ if and only if h has a representation of the form*

$$h(w) = aw + b + \int_{\mathbb{R}} \left(\frac{1}{t-w} - \frac{t}{t^2+1} \right) d\nu(t), \quad w \in \mathbb{C}^+,$$

where $a \geq 0$, $b \in \mathbb{R}$ and where ν is a positive Borel measure on \mathbb{R} such that

$$\int_{\mathbb{R}} (t^2 + 1)^{-1} d\nu(t) < \infty.$$

Proof. We divide the proof in three steps:

Step 1: Let f be analytic on some domain containing the closed unit disc and $z \in \mathbb{D}$ and let $\operatorname{Re} f(0) \geq 0$. Then, by the Cauchy Integral Theorem and the Cauchy Integral Formula, we obtain

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \left(\frac{f(\zeta)}{\zeta-z} - \frac{f(\zeta)}{\zeta-\frac{1}{\bar{z}}} \right) d\zeta. \quad (5.1)$$

Further the following equality holds:

$$\frac{1}{\zeta} \operatorname{Re} \left(\frac{\zeta+z}{\zeta-z} \right) = \frac{1}{\zeta-z} - \frac{1}{\zeta-\frac{1}{\bar{z}}}. \quad (5.2)$$

To verify this we consider the left hand side and, using that fact that $|\zeta| = 1$, obtain

$$\begin{aligned} \frac{1}{\zeta} \operatorname{Re} \left(\frac{\zeta+z}{\zeta-z} \right) &= \frac{1}{2\zeta} \left(\frac{\zeta+z}{\zeta-z} + \frac{\bar{\zeta}+\bar{z}}{\bar{\zeta}-\bar{z}} \right) = \frac{1}{2\zeta} \left(\frac{\zeta+z}{\zeta-z} + \frac{1+\zeta\bar{z}}{1-\zeta\bar{z}} \right) = \\ &= \frac{\zeta+z-\zeta^2\bar{z}-\zeta|z|^2+\zeta+\zeta^2\bar{z}-z-\zeta|z|^2}{2\zeta(\zeta-z)(1-\zeta\bar{z})} = \frac{2\zeta(1-|z|^2)}{2\zeta(\zeta-z)(1-\zeta\bar{z})} = \\ &= \frac{1-|z|^2}{(\zeta-z)(1-\zeta\bar{z})}. \end{aligned}$$

The right hand side satisfies

$$\frac{1}{\zeta-z} - \frac{1}{\zeta-\frac{1}{\bar{z}}} = \frac{1}{\zeta-z} + \frac{\bar{z}}{1-\zeta\bar{z}} = \frac{1-\zeta\bar{z}+\zeta\bar{z}-z\bar{z}}{(1-\zeta\bar{z})(\zeta-z)} = \frac{1-|z|^2}{(1-\zeta\bar{z})(\zeta-z)}.$$

and hence we obtain (5.2). Inserting (5.2) in (5.1) and with the substitution $\zeta = e^{i\theta}$, we get

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \operatorname{Re} \left(\frac{\zeta+z}{\zeta-z} \right) f(z) \frac{1}{\zeta} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left(\frac{e^{i\theta}+z}{e^{i\theta}-z} \right) f(e^{i\theta}) d\theta.$$

Now let

$$k(z) := \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \operatorname{Re} f(e^{i\theta}) d\theta,$$

which is holomorphic on the unit disc. We have $\operatorname{Re} k(z) = \operatorname{Re} f(z)$, and hence $k(z) = f(z) + ci$ for some $c \in \mathbb{R}$. Since, by the mean value property of holomorphic functions $k(0) = \operatorname{Re} f(0)$ we obtain $c = -\operatorname{Im} f(0)$, which implies

$$f(z) = i \operatorname{Im} f(0) + \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \operatorname{Re} f(e^{i\theta}) d\theta. \quad (5.3)$$

Step 2: Let f be analytic in the open unit disc and satisfy $\operatorname{Re} f(z) \geq 0$, $z \in \mathbb{D}$. Define $\phi : \overline{[0, 2\pi)} \rightarrow \mathbb{T}$ as

$$\phi(\theta) = e^{i\theta}. \quad (5.4)$$

For $r \in [0, 1)$ consider the positive measure η_r on $[0, 2\pi)$, which shall be defined such that

$$d\eta_r = \frac{1}{2\pi} \operatorname{Re} f(re^{i\theta}) d\theta$$

and further the image measure $\mu_r := \eta_r \circ \phi^{-1}$. Applying (5.3) with the function $z \mapsto f(rz)$, we obtain

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu_r(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \operatorname{Re} f(re^{i\theta}) d\theta = f(rz) - i \operatorname{Im} f(0).$$

For $z = 0$, this yields

$$\|\mu_r\| = \int_{\mathbb{T}} d\mu_r = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(e^{i\theta}) d\theta = f(0) - i \operatorname{Im} f(0) = \operatorname{Re} f(0).$$

Having a uniform bound for $\|\mu_r\|$ we can apply the Banach-Alaoglu Theorem, which gives a positive measure μ such that for every net μ_{r_i} with $r_i \rightarrow 1$, there exists a subnet $\mu_{r_{i_n}}$ which satisfies $\lim_{r_{i_n} \rightarrow 1} \mu_{r_{i_n}}(z) \xrightarrow{\omega^*} \mu(z)$. Passing to the limit in (5.3), yields

$$f(z) = i \operatorname{Im} f(0) + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta), z \in \mathbb{D}. \quad (5.5)$$

Step 3: Now we have to perform a fractional linear transform from the unit disc to the upper halfplane. Consider the fractional linear transformation $\beta : \mathbb{C}^+ \rightarrow \mathbb{D}$, defined by

$$\beta(\omega) = \frac{\omega - i}{\omega + i}, \quad \omega \in \mathbb{C}^+.$$

Let g be analytic in $\mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re} g(0) \geq 0$ and $f(z) := (g \circ \beta^{-1})(z)$. Now we want to evaluate $g(\omega) = (g \circ \beta^{-1})(\beta(\omega))$. Obviously $\beta^{-1}(1) = \infty$, and we separate the right hand side of (5.5) into two parts

$$\begin{aligned} f(z) &= i \operatorname{Im} f(0) + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) = i \operatorname{Im} f(0) + \int_{\{1\}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) + \int_{\mathbb{T} \setminus \{1\}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) = \\ &= i \operatorname{Im} f(0) + \mu(\{1\}) \frac{1+z}{1-z} + \int_{\mathbb{R}} \frac{\beta(t) + z}{\beta(t) - z} d(\mu \circ \beta^{-1})(t). \end{aligned}$$

Now we can evaluate

$$g(\omega) = (g \circ \beta^{-1})(\beta(\omega)) = i \operatorname{Im} f(0) + \mu(\{1\}) \frac{1 + \beta(\omega)}{1 - \beta(\omega)} + \int_{\mathbb{R}} \frac{\beta(t) + \beta(\omega)}{\beta(t) - \beta(\omega)} d(\mu \circ \beta^{-1})(t) = \quad (5.6)$$

$$= i \operatorname{Im} f(0) + \mu(\{1\}) \frac{1 + \frac{\omega - i}{\omega + i}}{1 - \frac{\omega - i}{\omega + i}} + \int_{\mathbb{R}} \left(\frac{\frac{t-i}{t+i} + \frac{\omega-i}{\omega+i}}{\frac{t-i}{t+i} - \frac{\omega-i}{\omega+i}} \right) d(\mu \circ \beta^{-1})(t) = \quad (5.7)$$

$$= i \operatorname{Im} f(0) + \mu(\{1\}) \frac{\omega}{i} + \int_{\mathbb{R}} \frac{1 + t\omega}{i(t - \omega)} d(\mu \circ \beta^{-1})(t). \quad (5.8)$$

Now we consider

$$\nu(\omega) := (1 + t^2) (\mu \circ \beta^{-1}) (\omega).$$

Then, since $|\int_{\mathbb{T}} d\mu| < \infty$, it holds that $\int_{\mathbb{R}} d(\mu \circ \beta^{-1}) < \infty$ and because of this we obtain

$$\int_{\mathbb{R}} (1 + t^2)^{-1} d\nu(t) < \infty.$$

Finally, there holds

$$\left(\frac{1}{t - \omega} - \frac{t}{1 + t^2} \right) (1 + t^2) = \frac{1 + t\omega}{t - \omega} d(\mu \circ \beta^{-1}),$$

which yields the assertion, if we apply it to (5.6) and consider $h(w) := ig(w)$. □

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