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DISSERTATION

Stochastic Filtering in Pricing and Credit Risk Management

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Univ.-Prof. Dipl.-Math. Dr. rer. nat. habil. Uwe Schmock

Institut für Wirtschaftsmathematik (E 105)
Forschungsgruppe Finanz- und Versicherungsmathematik

sowie unter der Betreuung von

Univ.-Prof. Dipl.-Math. Dr. rer. pol. habil. Rüdiger Frey

Department Finance, Accounting and Statistics
Wirtschaftsuniversität Wien

eingereicht an der Technischen Universität Wien
Fakultät für Mathematik und Geoinformation

von

Dipl.-Math. Lars Rösler

Matrikelnummer 1227843
Kohlweg 30, D-04347 Leipzig

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Kurzfassung der Dissertation

Im Fokus der Dissertation steht die Anwendung und Implementierung von Modellen mit unvollständiger Information im Bereich der dynamischen Kreditrisikomodellierung. "Unvollständige" Information bezeichnet hier den Umstand, dass Faktoren, welche das Modell maßgeblich beeinflussen, nicht vollständig beobachtbar sind. In dieser Dissertation stellen wir jeweils ein Modell mit unvollständiger Information aus der Klasse der Intensitätsmodelle und der strukturellen Kreditrisikomodelle vor und diskutieren jeweils eine Anwendung.

Im ersten Teil der Dissertation wird auf ein in [44] vorgestelltes multivariates Intensitäts-Modell zurückgegriffen. Bei diesem werden die Ausfallintensitäten als Funktion einer nicht direkt beobachtbaren Markovkette modelliert. Insbesondere führt die Annahme der Nicht-Beobachtbarkeit zum Vorhandensein von Ansteckungseffekten. Im Setup von [44] untersuchen wir Kontrahentenausfallrisiken bei Credit Default Swaps, welche over-the-counter gehandelt werden. Darüber hinaus werden auch Absicherungsstrategien (Collateralization) gegen Kontrahentenausfallrisiken untersucht. Insbesondere gehen wir auf den Einfluss der Ansteckungseffekte auf die Kontrahentenrisiken und die Effektivität der Collateralization-Strategien ein.

Der zweite Teil nutzt das univariate strukturelle Kreditrisikomodell aus [41] und wir stellen dieses mitsamt der wichtigsten Resultate in Kapitel 3 vor. Im Gegensatz zu klassischen Arbeiten wird dort angenommen, dass der Wert der Assets V nicht direkt beobachtbar ist; Marktteilnehmer und auch die Firma selbst haben nur eingeschränkte Informationen über V. Im Kapitel 4 gehen wir näher auf die numerischen Implementierung des Modells ein und beschreiben Algorithmen zur Simulation von Pfaden der wichtigsten beteiligten Prozesse. Insbesondere führen wir eine Simulationsstudie durch, bei der wir diese in Punkto Stabilität und Genauigkeit vergleichen. Letztlich nutzen wir in Kapitel 5 das strukturelle Kreditrisikomodell für die Bewertung von Contingent Capital Notes. Contingent Capital Notes sind Unternehmensanleihen, welche mit einem Umwandlungsmechanismus ausgestattet sind, welcher dafür sorgen soll, dass das Eigenkapital der Bank gestärkt wird, falls sie in finanziellen Nöten ist. Wir bewerten unterschiedliche Typen von Contingent Capital Notes und diskutieren die Effektivität verschiedener Umwandlungsmechanismen in Hinblick auf die Stärkung des Eigenkapitals. Insbesondere gehen wir auf die Konsequenzen der unvollständigen Information über V auf die Effektivität der Umwandlungsmechanismen ein.

Abstract

The focus of the thesis is on the application and implementation of models with incomplete information in the context of dynamic credit risk modelling. 'Incomplete information' means that factors which influence the model significantly are not fully observable by the financial market participants. We present a reduced-form-model and a structural credit risk model with incomplete information and discuss for each an application.

In the first part of the thesis we use the multivariate reduced-form model from [44] as vehicle for our analysis. Here, the default intensities are modelled as a function of a finite state Markov chain, which is not fully observable. As a consequence of this, the model framework can incorporate contagion effects. In this set-up we investigate counterparty credit risk for OTC-credit default swaps. Moreover, the effectiveness of different collateralization strategies is discussed in this framework. In particular, we discuss the impact of the contagion effects on counterparty credit risk and on the effectiveness of collateralization strategy and derive a collateralization strategy which explicitly takes contagion effects into account.

The second part uses the univariate structural credit risk model from [41], which we will discuss in Chapter 3. In contrast to classical approaches, in [41] it is assumed that the value of the assets V is not directly observable; market participants only observe noisy observations of the asset process. In Chapter 4 we discuss the numerical aspects, including methods for the simulation of trajectories of the most important processes. In a numerical simulation study we also compare different simulation algorithm in terms of stability and convergence speed. Finally, in Chapter 5 we use the model for the pricing of Contingent Capital Notes. Contingent Capital Notes are corporate bonds which are equipped with a conversion feature. The conversion is designed with the aim at strengthening the equity capital of the bank if it enters into financial distress. We will price different types of Contingent Capital Notes and discuss the effectiveness of the conversion mechanisms. In particular the impact of the assumption of incomplete information on the asset value is discussed.

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Contents

Chapter 1

Introduction

This thesis is devoted to the application and implementation of (dynamic) credit risk models with incomplete information. The term incomplete information addresses situations in which the market participants only partially observe quantities which are relevant for the pricing of securities and contingent claims. In the following we will give a short introduction to dynamic credit risk models and explain where incomplete information effects can arise. For a detailed exposition of incomplete information modelling in credit risk see also the survey [39] and the references within. The two main classes of dynamic credit risk models in mathematical finance are reduced form models and structural credit risk models. Note that the first part of the thesis is based on a reduced form model and the second part on a structural credit risk model.

We begin with the description of the class of reduced form models. For a more detailed exposition on this topic we refer to the Chapter 10 of [63]. The term reduced-form model refers to models where the default mechanism is not modelled explicitly, but only the default times respectively their distribution is specified. Within the class of reduced form models the most common way of specifying the default times $\tau_1, \ldots, \tau_n$ consists of specifying their default intensities $\lambda_1, \ldots, \lambda_n$. These are non-negative processes with the following property: For $i = 1, \ldots, n$ let N_i denote the survival indicator process, that is $N_{i,t} = 1_{\{\tau_i \leq t\}}$ for $t \in [0, \infty)$. Hence, it indicates whether the company i has defaulted yet or not. λ_i is called default intensity of the default time τ_i if the process

$$N_{i,t} - \int_0^{t \wedge \tau_i} \lambda_{i,s} \, ds, t \in [0, \infty)$$

is a martingale. In general not every default time exhibits an intensity, however for some classes of default times the existence of an intensity has been established, for example doubly stochastic default times (see Definition 10.10 of [63]). Moreover, for this class it is possible to construct the default time from their intensity by using exponential thresholds (see Lemma 10.11 in [63]). Frequently, one assumes that the intensities $\lambda_1, \ldots, \lambda_n$ are functions of a stochastic process X, that is there exist functions $f_1, \ldots, f_n$ such that $\lambda_{i,t} = f_i(X_t)$. Since the intensities determine the distribution of the default times including their dependence structure, in this case the process X determines the prices of defaultable securities (see Section 10.5 of [63]); the pricing leads to the computation of conditional expectations of the following form:

$$\mathbb{E}^{\mathbb{Q}}\Big(f(X_t)\Big|\mathcal{F}_t^{\mathbb{M}}\Big), t \in (0, \infty) \tag{1.1}$$

where $\mathbb{F}^{\mathbb{M}} = (\mathcal{F}_t^{\mathbb{M}})_{t\in[0,\infty)}$ denotes the investor information filtration and $\mathbb{Q}$ the risk-neutral measure. Incomplete information effects arise if the process X is not perfectly observable by market participants, that is X is not adapted to the information filtration $\mathbb{F}^{\mathbb{M}}$ ($\mathbb{F}^X \not\subset \mathbb{F}^{\mathbb{M}}$). In this case the computation of (1.1) for different functions f is called stochastic filtering. Herefore, it suffices to find the conditional distribution of X given the available information $\mathcal{F}_t^{\mathbb{M}}$. Unfortunately, this is only possible in a few cases and hence quite often one has to rely on numerical methods. Intensity models respectively reduced-form credit risk models with incomplete information have been considered previously by [27, 34, 38, 72]. In the first part of the thesis we will use set-up from [38] and assume that X is a finite state Markov chain with state space $S^X = \{1, \dots, k\}$. This model framework has several advantages and also improves the hypothetical model where X is observable in some points. Usually this hypothetical model is referred to as complete information model in order to stress out the difference to the original model. Firstly, computations can be mostly done in the context of the complete information model. For this well-established pricing formulas and techniques (see [63]) can be used. The benefits are that the model reflects *spread risk* (random fluctuations of credit spreads between defaults) and *default contagion*. Default contagion refers to the fact that the default of a firm leads to a sudden increase in the credit spread of surviving firms. A prime example for contagion effects is the rise in credit spreads after the default of Lehman Brothers in 2008. The contagion effects in this model are generated by the updating of the conditional distribution of the unobservable factor X in reaction to the incoming default observation.

In Chapter 2 we use the framework described above to study counterparty credit risk for over-the-counter (OTC) Credit Default Swaps (CDS). Counterparty credit risk refers to the risk to each party of a bilateral contract that the counterparty will not live up to its contractual obligations. In this case the surviving party usually will replace the contract and suffers a loss, because the proceeds from the recovery payment are not enough to cover the costs of entering into the new contract. Note that the presence of contagion effects has a significant effect on the size of the price of the new CDS contract and hence also on the replacement costs. Therefore, contagion effects should be taken into account when studying counterparty credit risk. One way to mitigate the involved counterparty credit risk consists of collateralization. Collateralization refers to the practice of posting securities (the so-called collateral) that serve as a pledge for the collateral taker. These securities are liquidated if one of the contracting parties defaults, and the proceeds can be used to cover to the replacement costs of the contract. In order to ensure that the proceeds are sufficient one has to take into account contagion effects. We will discuss different collateralization strategies and also suggest one which takes into account collateralization effects.

The second large class of dynamic credit risk models are structural credit risk models. Structural credit risk models like [9] or [59] are widely used in the analysis of defaultable corporate securities. In these models a firm defaults if a random process V representing the firm's asset value hits some barrier K that is often interpreted as the value of the firm's liabilities. In contrast to reduced form models, where the default is modelled completely exogenously, firm value models offer an intuitive economic interpretation of the default event. Moreover, if the asset value process V is modelled as a diffusion process, then the corresponding default time does not exhibit a default intensity. Additionally, this leads to unrealistically low short-term credit spreads. Another difficulty which arises frequently in the application of structural credit risk models consists of problems in the precise assessment of the asset value for investors in secondary markets. To put it differently, investors in secondary markets only have incomplete information on the asset value V. For example

[35] proposed a model where the market obtains at discrete time points t_n a noisy accounting information of the form $Z_n = \log(V_{t_n}) + \epsilon_n$. In [41] the noisy asset information is modelled by a continuous time process of the form

$$Z_t = \int_0^t a(V_s)\, ds + W_t \text{ for } t \in [0, \infty).$$

for some Brownian motion W independent of V. Interestingly, [35] and [41] establish a relationship between intensity and structural credit risk models by showing the existence of a default intensity of the default time τ in their framework. Moreover, structural models with incomplete information were considered in [23, 26, 41, 58, 64]. Note that [42] represents an updated version of [41] where a slightly different model is considered. The second part of thesis is based on [41] and is organized as follows. In Chapter 3 we present the set-up from [41] and give all the results which are important for the following next chapters. In particular, we show that the pricing of derivative securities naturally leads to stochastic filtering problems of the form

$$\mathbb{E}^{\mathbb{Q}}\left(f(V_t)\middle|\mathcal{F}_t^{\mathbb{M}}\right), t \in (0, \infty).$$

Moreover, a SPDE for the densities of the conditional distributions of V_t given the available market information $\mathcal{F}_t^{\mathbb{M}}$ will be shown.

In Chapter 4 the numerical implementation of the model is discussed. Different algorithms for the simulation of trajectories of all important process are discussed. This includes an Galerkin approximation of the SPDE to obtain a finite dimensional SDE System. Moreover, we study different algorithms for the simulation of this SDE system. We cover the well-known Euler-Maruyama and Milstein method, but also some more advanced methods like the splitting-up method introduced in [48] or the matrix exponential method from [69]. In a simulation study we compare them against each other in terms of stability of the algorithm and convergence speed.

Finally, in Chapter 5 the pricing of Contingent Capital Notes is considered. Contingent capital notes, also known as CoCos, are corporate bonds which are equipped with an conversion mechanism which aims at strengthening the equity capital of the issuer when he enters into financial distress. The conversion trigger is linked to economic measures of the financial strength of the company, for example the share price or some capital adequacy ratio and the conversion takes place as soon as the measure indicates that bank is in financial distress. The modelling and pricing of CoCos has proven to be an interesting and challenging task, because of the various possible definitions of the conversion. The structural credit risk model from [41] is very suitable for these tasks, because all of the different features can be embedded into the model. Moreover, the usage of incomplete information for the pricing offers very special possibilities. This includes the possibility that the used economic measure of the financial strength is subject to observation noise and hence may not correctly reflect the current financial condition of the company. As a result the conversion may be activated too late, such that the company already defaulted. This distinguishes our approach from many other pricing approaches (for example [17]), which implicitly assume that the underlying economic measure always reflects the current condition of the bank correctly. In Chapter 5 we also include a numerical case study of the effectiveness of different conversion mechanisms.

Part I

An Intensity Model with Incomplete Information

Chapter 2

Contagion Effects and Collateralized Credit Value Adjustments for Credit Default Swaps

Apart from small modifications, the following chapter is taken from from [37], which was published in the International Journal of Theoretical and Applied Finance.

2.1 Introduction

The distress of many financial firms in recent years has made counterparty risk management for over-the-counter (OTC) derivatives such as credit default swaps (CDS) an issue of high concern. Crucial tasks in this context are the computation of *credit value adjustments*, which account for the possibility that one of the contracting parties defaults before the maturity of the OTC contract, and the mitigation of counterparty risk by *collateralization.* Collateralization refers to the practice of posting securities (the so-called collateral) that serve as a pledge for the collateral taker. These securities are liquidated if one of the contracting parties defaults before maturity, and the proceeds are used to cover the replacement cost of the contract. In order to ensure that the funds generated in this way are sufficient, the collateral position needs to be adjusted dynamically in reaction to changes in the value of the underlying derivative security. The price dynamics of the collateral thus play a crucial role for the performance of a given collateralization strategy.

In the present paper we study the impact of different price dynamics on the size of value adjustments and on the performance of collateralization strategies for CDSs. We are particularly interested in the influence of contagion. Contagion effects - the fact that the default of a firm leads to a sudden increase in the credit spread of surviving firms - are frequently observed in financial markets; a prime example are the events that surrounded the default of Lehman Brothers in 2008. To see that contagion might be relevant for the performance of collateralization strategies consider the scenario where the protection seller defaults during the runtime of the CDS. In such a case contagion might lead to a substantial increase in the credit spread of the reference entity (the firm on which the CDS is written) and hence in turn to a much higher replacement value for the CDS. In standard collateralization strategies this is taken into account at most in a very crude way, and the amount of collateral posted before the default might be insufficient for replacing the CDS. In our view this issue merits a detailed analysis in the context of dynamic portfolio credit risk models.

We use the reduced-form credit risk model proposed by [44] as vehicle for our analysis. In that model the default times of the reference entity, the protection seller and the protection buyer are conditionally independent given some finite state Markov chain X that models the economic environment. We consider two versions of the model which differ with respect to the amount of information that is available for investors. In the full-information model it is assumed that X is observable so that there are no contagion effects. In the incomplete-information version of the model on the other hand investors observe X in additive Gaussian noise as well as the default history. In that case there is default contagion that is caused by the updating of the conditional distribution of X at the time of default events. An advantage of the set-up of [44] for our purposes is the fact that the the joint distribution of the default times is the same in the two versions of the model. Hence differences in the size of value adjustments or in the performance of collateralization strategies can be attributed purely to the different dynamics of credit spreads (contagion or no contagion) in the two model variants.

In order to compute value adjustments and to measure the performance of collateralization strategies we use the bilateral collateralized credit value adjustment (BCCVA) proposed by [14]. This credit value adjustment accounts for the form of collateralization strategies and for the credit quality of the contracting parties. Our analysis reveals that the impact of contagion on the size of the BCCVA depends strongly on the relative credit quality of the three parties involved and is hard to predict up front. Results on the performance of different collateralization strategies are more clear-cut: we show that while standard market-value based collateralization strategies provide a good protection against losses due to counterparty risk in the full-information setup, they have problems to deal with the contagious jump in credit spreads at a default of the protection seller. Motivated by these findings, we go on and develop improved collateralization strategies that perform well in the presence of contagion. For our analysis we need to compute the BCCVA in both model variants. Using Markov chain theory we derive explicit formulas for the BCCVA under full information; in the incomplete-information setup we rely on simulation arguments.

There is by now a large literature on counterparty risk for CDSs. Existing contributions focus mostly on the computation of value adjustments (with and without collateralization) in various credit risk models. Counterparty credit risk and valuation adjustments for uncollateralized CDS are studied by [3, 10, 16, 54, 60], among others. Counterparty credit risk for collateralized CDS is discussed among others in [5] and [14]. However, none of these contributions covers the issues discussed in this paper in full. [5] analyze the impact of collateralization on counterparty risk in CDS contracts using the Markov copula model which does not exhibit contagion effects. [14] is closest to our contribution: these authors study the impact of contagion on credit value adjustments and on the effectiveness of market-value based collateralization strategies in a Gaussian copula model with stochastic credit spreads. In that model default or event correlation and contagion effects are both driven by the choice of the correlation parameter of the copula. Consequently, it is not possible to disentangle the impact of event correlation and of default contagion on credit value adjustments and on the performance of collateralization strategies. This might be an advantage of our setup. Moreover, [14] do not address the issue of designing collateralization strategies that take default contagion into account.

The remainder of the paper is organized in the following way. In Section 2.2 we discuss the BCCVA of [14]. In Section 2.3 we introduce the credit risk model of [44] that provides the framework for the analysis of the present paper. Section 2.4 is devoted to the computation of the BCCVA in both model variants. In Section 2.5 we discuss different collateralization strategies, and in Section 2.6 we present results from numerical experiments.

2.2 Bilateral Collateralized Credit Value Adjustment (BCCVA)

In this section we discuss the bilateral collateralized credit value adjustment (BCCVA) proposed in [14].

We begin with some notation. Throughout the entire paper we work on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ that fulfills the usual hypotheses. $\mathbb{Q}$ denotes the risk-neutral measure used for pricing, and all expectations are taken with respect to $\mathbb{Q}$. $\mathbb{F}$ is a generic filtration that models the information available to the market participants; we will specify $\mathbb{F}$ when we introduce the credit risk model for our analysis in Section 2.3. We assume throughout that the short rate $r(u)$ is deterministic and we denote the discount factor from time t to time s by $D(t,s) = e^{-\int_t^s r(u)\,du}$. The following parties are involved in the CDS contract: the protection buyer, labeled B; the reference identity, labeled R; the protection seller, labeled S. The default times of these entities are denoted by τ_B, τ_R and τ_S. We introduce the survival indicators $H_t^B := 1_{\{\tau_B>t\}}$, $H_t^R := 1_{\{\tau_R>t\}}$ and $H_t^S := 1_{\{\tau_S>t\}}$ and we put $H := (H^B, H^R, H^S)$. Defaults are observable by assumption so that H is $\mathbb{F}$ adapted and τ_B, τ_R and τ_S are $\mathbb{F}$ stopping times. The first default time is denoted by τ, that is $\tau := \tau_B \wedge \tau_R \wedge \tau_S$. The random variable ξ with values in the set $\{B, R, S\}$ represents the identity of the firm defaulting at τ. Furthermore Rec_B, Rec_R, Rec_S denote the recovery rate and LGD_B, LGD_R, LGD_S the loss given default of of B, R and S, respectively. We assume that recovery rates are constants.

All valuations and cash flows are defined from the perspective of the protection buyer. Therefore positive numbers indicate that a cash flow is received by the protection buyer and negative numbers indicate that a cash flow is received by the protection seller.

Payments of a risk-free CDS. In our context a CDS without counterparty risk, which we call (counterparty-) risk-free CDS, is a CDS where neither the protection seller nor the protection buyer are subject to default risk. For simplicity, we assume that the premium payments are paid continuously. Therefore the sum of all payments in a risk-free CDS from time t to time s discounted to t, is given by:

$$\Pi(t,s) := 1_{\{t<\tau_R\le s\}}\,\mathrm{LGD}_R\, D(t,\tau_R) - \int_t^s S_R D(t,u) 1_{\{\tau_R>u\}}\,du, \tag{2.1}$$

where S_R represents the spread of the CDS. In addition we define the time t price of a risk-free CDS with maturity date $T > t$ as the risk-neutral expectation of $\Pi(t,T)$, that is

$$P_t \ := \ \mathbb{E}^{\mathbb{Q}}(\Pi(t,T)|\mathcal{F}_t).$$

Risky CDS and collateralization. In a CDS with counterparty risk, called *risky CDS* below, the protection buyer or the protection seller might default before the maturity of the CDS. Collateralization is a way to limit the potential loss for the surviving party. To keep things simple we assume that the collateral is posted in form of cash and that the collateral earns the risk-free rate of interest $r(s)$. Many collateralization arrangements are in fact of this form, and the additional valuation adjustments that need to be made if the interest rate paid on the collateral differs from the risk-free rate (see for instance [56]) are not central to the issues studied in this paper. Details of the collateralization procedure are stipulated in the credit support annex (CSA) of the contract. Roughly speaking the procedure works as follows. At $t_0 = 0$ a collateral account is opened. Let C_t denote the cash balance in the account at time t. Here $C_t > 0$ means that S has posted the collateral and that B is the collateral taker, whereas $C_t < 0$ means that B has posted the collateral and that S is the collateral taker. The collateral position is updated at discrete time points $t_1, \dots, t_N \le T$, for instance daily. At t_1 the collateral taker pays interest on the collateral and the cash balance C_{t_1} is adjusted in reaction to changes in the price of the underlying CDS over $(t_0, t_1]$. This procedure continues up to the maturity of the CDS or until the first default occurs. If $\tau > T$ or if $\tau < T$ and $\xi_1 = R$, the collateral account is closed at the "natural end" of the contract so that $C_t \equiv 0$ for $t \ge \tau \wedge T$. If there is an early default of B or S, that is $\tau \le T$ and $\xi_1 \in \{B, S\}$, the collateral is used to reduce the loss of the collateral taker and any remaining collateral is returned; details are specified below.

An issue arising in this context is *re-hypothecation*. The collateral taker has unrestricted access to the posted collateral and he may in particular pledge the funds as collateral in other OTC derivative transaction. Hence a part of the collateral is lost at a default of the collateral taker. We denote by Rec'_B and Rec'_S the recovery rate for the return of collateral and by LGD'_B and LGD'_S the corresponding loss given default (assumed constant). Usually the return of collateral is favored to the settlement of other claims in bankruptcy procedures, so that $\text{Rec}_B \le \text{Rec}'_B$ and $\text{Rec}_S \le \text{Rec}'_S$. Contracts without re-hypothecation are characterized by $\text{Rec}'_B = \text{Rec}'_S = 1$.

We describe the cash balance in the collateral account by some $\mathbb{F}$-adapted semimartingale $C = (C_t)_{0 \le t \le T}$ with RCLL paths, the so-called *collateralization strategy.* For simplicity we assume that interest on the collateral is paid continuously. Since we have assumed that the collateral earns the risk-free rate $r(s)$, from the perspective of the protection buyer collateralization leads to a cumulative cash flow stream given by $C_t - \int_0^t r(s) C_s \, ds$, $t \le T$. The discounted value of that cash-flow stream at $t = 0$ equals

$$C_0 + \int_0^T D(0,s)\, dC_s - \int_0^T D(0,s) r(s) C_s \, ds = D(0,T) C_T,$$

where the second equality follows by applying partial integration to $D(0,t)C_t$. Now $C_T = 0$ on $\{\tau > T\}$ and on $\{\tau \le T\} \cap \{\xi = R\}$. Hence scenarios where neither S nor B default before the end of the underlying CDS can be ignored in the computation of value adjustments for counterparty risk, and it suffices to consider the collateral payments for the case where there is an early default of R or S, that is for $\tau \le T$ and $\xi \in \{B, S\}$.

Payments at an early default. In order to complete the description of the cash flow stream of a risky CDS we need to specify the payments at an early default of B or S. In that case the surviving party is allowed to charge a *close-out amount* from the defaulting one. According to the ISDA Master Agreement the close-out amount is defined as reasonable estimate of the funds needed to close the position. In this paper we assume that the close-out amount is given by P_τ, the value of the risk-free CDS at the first default time. Note that this choice means that the credit quality of the surviving party is completely neglected in the computation of the close-out amount, which is in line with current market practice. However, there are alternative suggestions in the literature; see for instance [12].

We continue with the description of the payments at an early default. To shorten the exposition we concentrate on the payments in the case where the protection seller defaults first. Note that no additional collateral is posted after the first default. Hence we assume that the amount of collateral available during the bankruptcy process is given by $C_{\tau-}$, the amount of collateral that has been posted immediately prior to τ. This distinction matters if the close-out amount P_t jumps at $t = \tau$, for instance due to contagion effects.

In describing the payments at τ we have to consider four scenarios that differ with respect to the sign of P_τ and of $C_{\tau-}$.

1. Suppose that $P_\tau > 0$ and that the protection buyer is the collateral taker, that is $C_{\tau-} > 0$. The collateral is used to reduce the loss of the protection buyer. If $C_{\tau-}$ is smaller than P_τ, the protection buyer claims the difference $P_\tau - C_{\tau-}$ from S. However, B will receive only a recovery payment of size $\mathrm{Rec}_S(P_\tau - C_{\tau-})$ in that case. If $C_{\tau-}$ exceeds P_τ, the excess collateral is returned to the protection seller. With the notation $X^+ := \max(X, 0)$ and $X^- := -\min(X, 0)$,[1] in this scenario the overall payment at τ is given by $\mathrm{Rec}_S(P_\tau - C_{\tau-})^+ - (P_\tau - C_{\tau-})^-$.

2. Suppose next that $P_\tau > 0$ and $C_{\tau-} < 0$, so that the protection seller is the collateral taker. In this situation B is entitled to the repayment of the collateral and to the close-out amount P_τ. However, only a fraction of P_τ and, due to re-hypothecation, of $C_{\tau-}$ will be paid to B. Hence in this scenario the overall payment at τ is given by $\mathrm{Rec}_S\, P_\tau - \mathrm{Rec}'_S\, C_{\tau-}$.

3. Suppose now that $P_\tau < 0$ and that the protection buyer is the collateral taker, that is $C_{\tau-} > 0$. In that case B pays S the close-out amount P_τ and he returns the collateral. Hence from the viewpoint of B, in this scenario the overall payment at τ equals $P_\tau - C_{\tau-}$.

4. Suppose that $P_\tau < 0$ and that B posted some collateral so that $C_{\tau-} < 0$. If $-C_{\tau-} \le -P_\tau$ S keeps the collateral and he moreover receives the difference $-(P_\tau - C_\tau)$. Otherwise the excess collateral has to be returned to B, and there might be losses due to re-hypothecation. Hence in this scenario the overall payment at τ equals $\mathrm{Rec}'_S(P_\tau - C_{\tau-})^+ - (P_\tau - C_{\tau-})^-$.

The payments that arise if the protection buyer defaults first, that is if $\xi = B$, can be described in an analogous manner.

The BCCVA. Given a collateralization strategy C, the *bilateral collateralized credit value adjustment* (BCCVA) is defined as difference of the discounted cash-flow stream of the risk-free and the risky CDS. Following [14], we denote the latter cash-flow stream by $\Pi^D(t, T, C)$, where D stands for 'defaultable'. We thus have

$$\mathrm{BCCVA}(t, T, C) := \mathbb{E}^{\mathbb{Q}}(\Pi(t, T)|\mathcal{F}_t) - \mathbb{E}^{\mathbb{Q}}\big(\Pi^D(t, T, C)\big|\mathcal{F}_t\big). \tag{2.2}$$

Using the above description of the payments at an early default it is straightforward to give an explicit formula for $\Pi^D(t, T, C)$. However, in this paper we use an expression for the BCCVA that does not involve Π^D explicitly (see Proposition 2.3 below) so that we omit the formula and refer to [14] instead.

[1] Note that the convention $X^- := \min(X, 0)$ is used in [14].

By definition the BCCVA measures the difference in value of the cash-flows of a risk-free CDS and a risky CDS. Note that the BCCVA takes the default risk of of S *and* of B into account. The BCCVA thus leads to symmetrical prices in the sense that the adjustment computed from the point of view of the protection buyer equals (with the opposite sign) the adjustment computed from the point of view of the protection seller.

In the sequel we work with the following representation of the BCCVA that is established in [14].

Proposition 2.3. *The* BCCVA *can be decomposed as follows*

$$\mathrm{BCCVA}(t,T,C) = \mathrm{CCVA}(t,T,C) - \mathrm{CDVA}(t,T,C), \tag{2.4}$$

where the collateralized credit value adjustment (CCVA) *and the* collateralized debt value adjustment (CDVA) *are given by:*

$$\begin{aligned}
\mathrm{CCVA}(t,T,C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=S\}}D(t,\tau) \quad (\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ \\
&\qquad\qquad + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+)|\mathcal{F}_t\big),\\
\mathrm{CDVA}(t,T,C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=B\}}D(t,\tau) \quad (\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- \\
&\qquad\qquad + \mathrm{LGD}'_B(P_\tau^+ - C_{\tau-}^+)^-)|\mathcal{F}_t\big).
\end{aligned}$$

Comments. 1. The CCVA reflects the possible loss for B due to an early default of S, whereas the CDVA reflects the loss of S due to an early default of B. Consider for instance the case where $\xi = S$. If $P_\tau > 0$, there are two reasons why B might incur a loss: first, the collateral posted by S might be insufficient to cover the close-out amount of the CDS, which leads to a loss of size $\mathrm{LGD}_S(P_\tau - C_{\tau-}^+)^+$; if $C_{\tau-} < 0$ there is moreover a loss due to re-hypothecation given by $\mathrm{LGD}'_S C_{\tau-}^-$. If $P_\tau < 0$, B incurs a loss of size $\mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)$ (the loss of the excess collateral caused by re-hypothecation). The overall discounted loss incurred by B is thus given by

$$\begin{aligned}
&1_{\{\tau<T\}}1_{\{\xi=S\}}D(t,\tau)\big\{1_{\{P_\tau>0\}}\big(\mathrm{LGD}_S(P_\tau - C_{\tau-}^+)^+ + \mathrm{LGD}'_S C_{\tau-}^-\big)\\
&\qquad\qquad + 1_{\{P_\tau<0\}}\,\mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)\big\}\\
&= 1_{\{\tau<T\}}1_{\{\xi=S\}}D(t,\tau)\big\{\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+\big\},
\end{aligned}$$

which corresponds to the argument of the CCVA-formula above. In a similar way the CDVA can be interpreted as loss of S on $\{\xi = B\}$.

2. Without collateralization, that is if $C_t \equiv 0$, the value adjustments take the form of options on the risk-free CDS price P with strike price $K = 0$ and random maturity date τ. In that case the terms in (2.4) are labelled BCVA (bilateral credit value adjustment), CVA and DVA.

3. Markets often use a simplified value adjustment formula which implicitly assumes that the survival indicators H^B, H^S and the counterparty-risk free CDS price P are independent stochastic processes, an assumption that is known in the counterparty risk literature as *no wrong-way risk.* For $C_t \equiv 0$ the simplified bilateral credit value adjustment at $t = 0$ is given by

$$\begin{aligned}
\mathrm{BCVA}^{\mathrm{indep}} &= \mathrm{LGD}_S \int_0^T \bar{F}_B(s)D(0,s)\mathbb{E}(P_s^+)f_S(s)\,ds\\
&\quad - \mathrm{LGD}_B \int_0^T \bar{F}_S(s)D(0,s)\mathbb{E}(P_s^-)f_B(s)\,ds.
\end{aligned}\tag{2.5}$$

Here $\bar{F}_S(s) = \mathbb{Q}(\tau_S > s)$ and $f_S(s) = -\bar{F}'(s)$ represent the survival function and the density of τ_S and $\bar{F}_B$ and f_B represent the survival function and the density of τ_B. A derivation of (2.5) is given in [51]. The independence assumptions underlying the derivation of (2.5) are clearly unrealistic - just think of the case where B, S and R are financial institutions. In Section 2.6 we therefore study the relation between the "correct" value adjustment (2.4) and the simplified adjustment (2.5). It will turn out that formula (2.5) underestimates the correct value adjustment by a sizeable amount.

2.3 The Model

Next we give the mathematical description of the model framework that is used in the remainder of this paper. We consider a reduced-form model where τ_R, τ_B and τ_S are conditionally independent, doubly-stochastic random times whose default intensity is driven by a finite-state Markov chain $X = (X_t)_{t\geq 0}$ with state space $S^X = \{1, 2, \dots, K\}$, generator matrix $W = (w_{ij})_{1\leq i,j\leq K}$ and initial distribution described by the probability vector π_0 with $\pi_0^k = \mathbb{Q}(X_0 = k)$. Denote by $\mathcal{F}_t^X := \sigma(X_s : s \leq t)$ the filtration generated by X. We assume that for all time points $t_B, t_R, t_S > 0$ one has

$$\mathbb{Q}(\tau_R > t_R, \tau_B > t_B, \tau_S > t_S \mid \mathcal{F}_\infty^X) = \prod_{i\in\{B,R,S\}} \exp\Big(- \int_0^{t_i} \lambda_i(X_s)\, ds\Big), \tag{2.6}$$

where $\lambda_i : S^X \to \mathbb{R}^+$, $i \in \{B, R, S\}$, are deterministic functions. This definition implies that the default times are independent given the realization of the background process X. In our simulation study we consider the case where $\lambda_B(\cdot)$, $\lambda_R(\cdot)$ and $\lambda_S(\cdot)$ are increasing in x. In that case X can be viewed as an abstract representation of the state of the economy, 1 being the best state (low default probability of all firms) and K the worst state (high default probability of all firms).

For technical reasons we moreover assume that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ supports a d-dimensional standard Brownian motion W which is independent of X and of the survival indicator process H; W is used to model investor information under imperfect observation of X (see below). In the sequel we will consider two variants of the model that differ with respect to the assumptions made on investor information.

The full-information case. Here it is assumed that X is *observable* for investors and we take $\mathbb{F} = \mathbb{F}^O$ with

$$\mathbb{F}^O = \mathbb{F}^H \vee \mathbb{F}^X \vee \mathbb{F}^W,$$

where $\mathbb{F}^H$ is the filtration generated by the survival indicators. (The inclusion of $\mathbb{F}^W$ is purely technical and has no impact on the prices of credit derivatives under full information.) It is well-known that for time points $t_R, t_S, t_B > t$ the conditional survival function of (τ_R, τ_S, τ_B) given $\mathcal{F}_t^O$ satisfies

$$\mathbb{Q}(\tau_R > t_R, \tau_S > t_S, \tau_B > t_B \mid \mathcal{F}_t^O) = \prod_{i\in\{R,S,B\}} H_t^i\, \mathbb{E}\Big(\exp\Big(- \int_t^{t_i} \lambda_i(X_s)\, ds\Big) \mid X_t\Big). \tag{2.7}$$

Moreover, the process $\lambda_i(X_t)$, $i \in \{R.S, B\}$, is the $\mathbb{F}^O$ default intensity of τ_i, and the pair process (X, H) is Markov. A derivation of these results can be found in Chapter 10 of [63], among others. Formula (2.7) implies in particular that prior to the default of R the price of the risk-free CDS is a function of t and X_t,

$$P_t^O = \mathbb{E}^{\mathbb{Q}}\big(\Pi(t,T)\big|\mathcal{F}_t^O\big) = H_t^R p^O(t,X_t). \tag{2.8}$$

An explicit formula for the function $p^O(t,k)$ is given in Corollary 2.13 below.

The incomplete-information case. This variant of the model has been studied in detail in [44]. In that paper it is assumed that X is *unobservable* and that investors are confined to a noisy signal of X of the form

$$Z_t \;:=\; \int_0^t a(X_s)\,ds + W_t,$$

where $a : S^X \to \mathbb{R}^d$ is a deterministic function. Hence in this model variant we put $\mathbb{F} = \mathbb{F}^U$ ('unobservable') with

$$\mathbb{F}^U = \mathbb{F}^H \vee \mathbb{F}^Z.$$

Note that $\mathbb{F}^U \subseteq \mathbb{F}^O$ by definition.

Under incomplete information the risk-free CDS-price is given by $P_t^U := \mathbb{E}^{\mathbb{Q}}\big(\Pi(t,T)\big|\mathcal{F}_t^U\big)$. P_t^U can be computed by projecting the full-information price $H_t^B p^O(t,X_t)$ (see (2.8)) on $\mathbb{F}^O$. We get, as $\mathbb{F}^U \subseteq \mathbb{F}^O$,

$$P_t^U = \mathbb{E}^{\mathbb{Q}}\big(\Pi(t,T)\big|\mathcal{F}_t^U\big) = \mathbb{E}^{\mathbb{Q}}\Big(\mathbb{E}^{\mathbb{Q}}\big(\Pi(t,T)\big|\mathcal{F}_t^O\big)\Big|\mathcal{F}_t^U\Big) = H_t^R\mathbb{E}^{\mathbb{Q}}\big(p^O(t,X_t)\big|F_t^U\big). \tag{2.9}$$

Define the conditional probabilities

$$\pi_t^k := \mathbb{Q}(X_t = k \mid \mathcal{F}_t^U),\ 1 \le k \le K, \text{ and let } \pi_t := (\pi_t^1, \dots, \pi_t^K)^\top. \tag{2.10}$$

With this notation (2.9) can be written more succinctly as

$$P_t^U = H_t^R \sum_{k \in S^X} \pi_t^k p^O(t,k). \tag{2.11}$$

Comments. Note that relation (2.11) involves conditional probabilities with respect to the pricing measure $\mathbb{Q}$.

In Proposition 2.16 below we will show that the $\mathbb{Q}$-dynamics of π_t can be described by a K-dimensional SDE system. From this system we may in particular derive an explicit representation for the contagion effects under incomplete information.

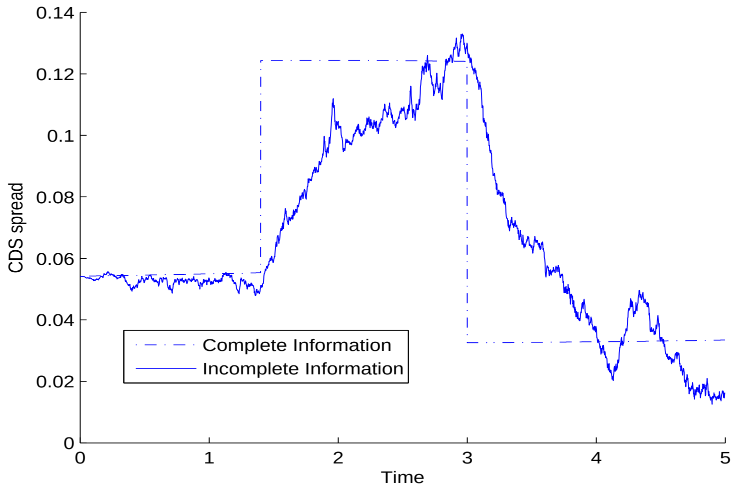
In the practical application of the model the process Z is considered as abstract source of information and the current value of π is calibrated from observed prices of traded credit derivatives; see Section 2.6.1 below.

In both model variants the unconditional joint survival function of τ_R, τ_B and τ_S is given by

$$\mathbb{Q}(\tau_R > t_R, \tau_B > t_B, \tau_S > t_S) = \mathbb{E}\Big(\prod_{i \in \{B,R,S\}} \exp\Big(-\int_0^{t_i} \lambda_i(X_s)\,ds\Big)\Big),$$

so that the distributions of (τ_B, τ_R, τ_S) coincides in both versions of the model. Therefore any differences in the BCCVA or in the performance of collateralization strategies can be attributed to the different dynamics of CDS spreads. For illustrative purposes we plot typical trajectories of CDS spreads in both model variants in Figure 2.1.

Figure 2.1: Trajectories of the fair CDS spread in the complete and incomplete information model.



2.4 Computation of the BCCVA

2.4.1 The case of the full-information model

In order to evaluate the BCCVA-formula ((2.4)) we need to determine the joint distribution of τ, ξ and X_τ. This is done in Proposition 2.12 below. The proof of this result relies on the observation that the distribution of the triple (τ, ξ, X_τ) can be expressed as first entry time of the processes (X, H^R) and (X, H) into specific sets. Since in our setting these processes form a finite-state Markov chain one can use Markov-chain theory to derive their distribution. In order to give precise results, we need to specify the generator matrix of these Markov chains.

For this we assume that the states are ordered in the *inverse lexicographic order*. According to this order a vector $(x_1, \ldots, x_n)$ is smaller than $(y_1, \ldots, y_n)$ if $x_n < y_n$ or if there is some $k < n$ with $x_{l+1} = y_{l+1}$ for $l \in \{k, \ldots, n-1\}$ and with $x_k < y_k$. For example, in the case $K = 2$ the states of the process (X, H^R) are ordered in the following way:

$$(1,0) < (2,0) < (1,1) < (2,1).$$

The transition rate $q_{y,z}$ of (X, H^R) from a state $y = (y_1, y_2)$ to the state $z = (z_1, z_2)$ is given by:

$$q_{y,z} = \begin{cases} w_{y_1\, z_1} & \text{if } y_1 \neq z_1 \text{ and } y_2 = z_2, \\ \lambda_R(y_1) & \text{if } y_1 = z_1, y_2 = 0 \text{ and } z_2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the generator of the process (X, H^R) can be represented by the matrix

$$Q := \begin{pmatrix} W - \Lambda_R & \Lambda_R \\ 0 & W \end{pmatrix}.$$

Here $\Lambda_R = \operatorname{diag}(\lambda_R(1), \ldots, \lambda_R(K))$ denotes a diagonal matrix with entries on the main diagonal given by the elements of the vector λ_R. The transition rates and the generator of (X, H) can be determined by analogous considerations.

Proposition 2.12. *Let $t < s$ and $k \in S^X$. Then the following statements hold:*

(a) The distribution of τ_i with $i \in \{B, R, S\}$ satisfies

$$\mathbb{Q}\big(\tau_i \le s \big| X_t = k, H_t^i = 0\big) \;\; = \;\; 1 - e_k^\top e^{Q_i(s-t)} \boldsymbol{1}_K.$$

Here $Q_i := W - \Lambda_i$ where $\Lambda_i = \mathrm{diag}(\lambda_i(1), \ldots, \lambda_i(K))$, $\boldsymbol{1}_K = (1, \ldots, 1)^\top$ is a column vector of dimension K and e_k denotes the kth unit vector in $\mathbb{R}^K$.

(b) The distribution of the first-to-default time τ can be computed as:

$$\mathbb{Q}(\tau \le s | X_t = k, H_t = 0) \;\; = \;\; 1 - e_k^\top e^{Q_{(1)}(s-t)} \boldsymbol{1}_K,$$

where we defined $Q_{(1)} := W - \sum_{j \in \{B,R,S\}} \Lambda_j$.

(c) The probability that obliger $i \in \{B, R, S\}$ defaults first and before time s is:

$$\mathbb{Q}(\tau_i \le s, \xi = i | X_t = k, H_t = 0) \;\; = \;\; e_k^\top Q_{(1)}^{-1} \Big(e^{Q_{(1)}(s-t)} - I\Big) \Lambda_i \boldsymbol{1}_K.$$

Here $Q_{(1)}^{-1}$ is the inverse of $Q_{(1)}$.

(d) The probability that obliger $i \in \{B, R, S\}$ defaults first and that at default the Markov chain is in the state l equals

$$\mathbb{Q}(X_\tau = l, \tau_i \le s, \xi = i | X_t = k, H_t = 0) \;\; = \;\; e_k^\top Q_{(1)}^{-1} \Big(e^{Q_{(1)}(s-t)} - I\Big) \Lambda_i e_l.$$

The proof of this result can be found in 2.7. The first two claims are well-known and have been derived among others by [50], see also [52]. However we include their proof for the convenience of the reader. Statements c) and d) on the other hand have to the best of our knowledge not appeared previously in the literature.

Using Proposition 2.12 the following well-known formula for the price of a risk-free CDS can be deduced.

Corollary 2.13 (Risk-free CDS price under full-information)**.** *The price P_t^O of a risk-free CDS with generic swap spread S on R given that $X_t = k$ and $\tau_R > t$ is equal to $1_{\{\tau_R > t\}} p^O(t, k)$, where the function $p^O : [0, T] \times S^X \to \mathbb{R}$ is given by*

$$p^O(t, k) = \Big(-\mathrm{LGD}_R\, e_k^\top Q_R - S e_k^\top\Big) \int_t^T D(t, s) e^{(Q_R(s-t)} \, ds \boldsymbol{1}_K.$$

Here $Q_R = W - \mathrm{diag}(\lambda_R(1), \ldots, \lambda_R(K))$, see Proposition 2.12 (a). Moreover, the price of a CDS at $t = 0$ is

$$P_0^O = \Big(-\mathrm{LGD}_R\, \pi_0^\top Q_R - S \pi_0^\top\Big) \int_t^T D(t, s) e^{(Q_R(s-t)} \, ds \boldsymbol{1}_K.$$

Below we will see that for a suitable function $g \colon [0, T] \times S^X \to \mathbb{R}$, collateralization strategies of the form $C_t = g(t, X_t)$ are optimal in the full-information model. For a generic strategy of this form, Theorem 2.12(d) gives the following semi-closed formula for the BCCVA.

Corollary 2.14 (BCCVA formula under full information)**.** *Assume that for $t < \tau$ the collateralization strategy is of the form $C_t = g(t, X_t)$. Then, given $X_t = j$, the* CCVA *and* CDVA *are given by:*

$$\begin{aligned}\mathrm{CCVA}_t = &\sum_{k\in\{1,\dots,K\}} \int_t^T D(t,s)\Big(\mathrm{LGD}_S\big(p^O(s,k)^+ - g(s,k)^+\big)^+ \\ &+ \mathrm{LGD}'_S\big(g(s,k)^- - p^O(s,k)^-\big)^+\Big) f^S_{j,k}(s)\,ds, \\ \mathrm{CDVA}_t = &\sum_{k\in\{1,\dots,K\}} \int_t^T D(t,s)\Big(\mathrm{LGD}_B\big(g(s,k)^- - p(s,k)^-\big)^-, \\ &+ \mathrm{LGD}'_B\big(p(s,k)^+ - g(s,k)^+\big)^-\Big) f^B_{j,k}(s)\,ds.\end{aligned}$$

Here the functions $f^i_{j,k}$, $i \in \{B, S\}$, are given by

$$f^i_{j,k}(s) \;:=\; \frac{d}{ds}\mathbb{Q}(\tau \le s, \xi = i, X_\tau = k \mid X_t = j, H_t = 0) = 1_{\{\tau_i > t\}} e_j^\top e^{Q_{(1)}(s-t)}\Lambda_i e_k.$$

2.4.2 The BCCVA in the incomplete-information model

In this section we discuss the computation of the BCCVA under incomplete information. We begin with a formula for the risk-free CDS price. By combining (2.11) and Corollary 2.13 we obtain

Corollary 2.15 (risk-free CDS prices under incomplete information)**.** *Given that $\{\tau_R > t\}$ the value P^U_t of a risk-free CDS at time t equals*

$$P^U_t \;=\; p^U(t,\pi_t) := \Big(-\mathrm{LGD}_R\,\pi_t^\top Q_R - S\pi_t^\top\Big)(Q_R - rI)^{-1}\Big(e^{(Q_R - rI)(T-t)} - I\Big)\mathbf{1}_K.$$

Note that for $t = 0$ one has $P^O_0 = P^U_0$; this equality reflects of course the fact that the unconditional distributions of the default times coincide in the two model variants.

Under incomplete information the BCCVA is essentially the value of a portfolio of options on the price P^U of the risk-free CDS. Since P^U_t is a function of π_t, in order to compute the BCCVA one thus needs the form of the dynamics of the process π, and we now recall the relevant results from [44]. We begin with some notation. We denote the $\mathbb{Q}$-optional projection of a process $G = (G_t)_{t\in[0,T]}$ with respect to $\mathbb{F}^U$ by $\widehat{G}$, that is $\widehat{G}_t = \mathbb{E}^{\mathbb{Q}}\big(G_t \big| \mathcal{F}^U_t\big)$. In particular,

$$\begin{aligned}(\widehat{\lambda}_i)_t &= \mathbb{E}^{\mathbb{Q}}\big(\lambda_i(X_t)\big|\mathcal{F}^U_t\big) = \sum_{k=1}^K \lambda_i(k)\pi^k_t, \quad i \in \{B, R, S\}, \\ \widehat{a}_t &= \mathbb{E}^{\mathbb{Q}}\big(a(X_t)\big|\mathcal{F}^U_t\big) = \sum_{k=1}^K a(k)\pi^k_t.\end{aligned}$$

Using the Levy-characterization of Brownian motion it is easily seen that

$$\mu_t = (\mu^1_t, \dots, \mu^d_t) \text{ with } \mu^i_t = Z^i_t - \int_0^t (\widehat{a}_i)_s\,ds$$

is a $\mathbb{F}^U$-Brownian motion. In the literature on stochastic filtering such as [1], $(\mu_t)_{0\le t\le T}$ is known as *innovations process.* Moreover, it is well-known that $\widehat{\lambda}_i$ is the $\mathbb{F}^U$ default intensity of firm i, that is for $i \in \{B, R, S\}$,

$$M_t^i := 1_{\{\tau_i \le t\}} - \int_0^t H_s^i \ (\widehat{\lambda}_i)_s \, ds$$

is an $\mathbb{F}^U$ martingale, see for instance Chapter 2 of [11].

Proposition 2.16 (Kushner-Stratonovich-equation)**.** *The process π is the unique solution of the K-dimensional SDE system*

$$d\pi_t^k = \sum_{i=1}^K w_{ik}\pi_t^i dt + \sum_{j\in\{R,B,S\}} \Big(\gamma_j^k(\pi_{t-})\Big)^\top dM_t^j + \Big(\alpha^k(\pi_t)\Big)^\top d\mu_t \,,\ k = 1, \ldots, K,$$

where

$$\begin{aligned} \gamma_j^k(\pi_t) &= \pi_t^k \left(\frac{\lambda_j(k)}{\sum_{i=1}^K \lambda_j(i)\pi_t^i} - 1 \right) \textit{ for } 1 \le j \le K \textit{ and} \\ \alpha^k(\pi_t) &= \pi_t^k \left(a(k) - \sum_{j=1}^K \pi_t^j a(j) \right). \end{aligned}$$

The proposition shows that the process π exhibits jump-diffusion dynamics. In particular, π jumps at default times and the jump height of π_t^k at the default of firm j is equal to $\gamma_j^k(\pi_{\tau_j-})$. Furthermore using the proposition we can compute the size of the information-induced contagion effects: the jump in the $\mathbb{F}^U$-default intensity of firm i at the default of firm j, $j \neq i$ equals

$$(\widehat{\lambda}_i)_{\tau_j} - (\widehat{\lambda}_i)_{\tau_j-} = \sum_{k=1}^K \lambda_i(k)\pi_{\tau_j-}^k \left(\frac{\lambda_j(k)}{\sum_{l=1}^K \lambda_j(l)\pi_{\tau_j-}^l} - 1 \right) = \frac{\operatorname{cov}^{\pi_{\tau_j-}}(\lambda_i, \lambda_j)}{E^{\pi_{\tau_j-}}(\lambda_j)}. \tag{2.17}$$

An inspection of the formula (2.17) shows the following.

- Contagion effects are inversely proportional to the instantaneous default risk of the defaulting entity (firm j): a default of an entity with a better credit quality comes as a bigger surprise and the market impact is larger.
- Contagion effects are proportional to the covariance of the default intensities $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ under the 'a-priori distribution' π_{τ_j-}. In particular, contagion effects are relatively high if the firms have similar characteristics in the sense that the functions $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ are (almost) linearly dependent.

Proposition 2.16 indicates a method to simulate a trajectory of π. The following general approach is suggested in [44].

(1) Generate a trajectory of the Markov chain X.

(2) Generate for the trajectory of X constructed in (i) a trajectory of the default indicator H and the noisy information Z.

(3) Solve the system of SDEs numerically, for instance via a Euler-Maruyama type method.

We close this section with a theoretical result on the relationship between the un-collateralized value adjustments in the two versions of the model.

Proposition 2.18. *Assume that the CDS contract is un-collateralized, i.e.* $C_t \equiv 0$. *Then the following relationships hold:*

$$\mathrm{CVA}_0^O \geq \mathrm{CVA}_0^U \text{ and } \mathrm{DVA}_0^O \geq \mathrm{DVA}_0^U .$$

Proof. We begin with the CVA. Using the definition of the CVA, Jensen's inequality and the relation $P_\tau^U = \mathbb{E}(P_\tau^O \mid \mathcal{F}_\tau^U)$, we get

$$\begin{aligned}
\mathrm{CVA}_0^O &= \mathrm{LGD}_S\, \mathbb{E}^{\mathbb{Q}}\Big(1_{\{t<\tau\leq T\}} 1_{\{\xi=S\}} \big(P_\tau^O\big)^+\Big) \\
&= \mathrm{LGD}_S\, \mathbb{E}^{\mathbb{Q}}\Big(1_{\{t<\tau\leq T\}} 1_{\{\xi=S\}} \mathbb{E}^{\mathbb{Q}}\Big(\big(P_\tau^O\big)^+ \Big| \mathcal{F}_\tau^U\Big)\Big) \\
&\geq \mathrm{LGD}_S\, \mathbb{E}^{\mathbb{Q}}\bigg(1_{\{t<\tau\leq T\}} 1_{\{\xi=S\}} \Big(\mathbb{E}^{\mathbb{Q}}\big(P_\tau^O | \mathcal{F}_\tau^U\big)\Big)^+\bigg) \\
&= \mathrm{LGD}_S\, \mathbb{E}^{\mathbb{Q}}\big(1_{\{t<\tau\leq T\}} 1_{\{\xi=S\}} (P_\tau^U)^+\big),
\end{aligned}$$

and the last line is obviously equal to CVA_0^U. A similar reasoning applies to the DVA. □

The overall relation of the BCVA in the two model variants is in general unclear, since the BCVA is the difference of the CVA and DVA. If B is of a much higher credit quality than S, the DVA is almost zero and we have the relation $\mathrm{BCVA}^O \geq \mathrm{BCVA}^U$. Similarly, if S is of a much higher credit quality than B, one has $\mathrm{BCVA}^O \leq \mathrm{BCVA}^U$.

The intuition underlying (the proof of) the result is as follows: First, the CVA is the price of an option on the risk-free CDS price with exercise price $K = 0$. Moreover, since $P_\tau^U = \mathbb{E}(P_\tau^O \mid \mathcal{F}_\tau^U)$ the variance of P_τ^U is smaller than the variance of P_τ^O. Since the price of an option increases with increasing variance of the distribution of the underlying asset value, we get that $\mathrm{CVA}^O \geq \mathrm{CVA}^U$.

2.5 Collateralization strategies

Standard collateralization strategies. We consider among others the following collateralization strategies. *No collateralization* corresponds to the strategy $C_t \equiv 0$. The *threshold-collateralization* strategy with initial margin γ and thresholds M_1, $M_2 \geq 0$, labeled C^{γ,M_1,M_2}, is given by

$$C_t^{\gamma,M_1,M_2} := \gamma + (P_t - M_1)1_{\{P_t > M_1\}} + (P_t + M_2)1_{\{P_t < -M_2\}} \; \forall t \in [0, T \wedge \tau).$$

This strategy is used if B and S want to protect themselves against severe losses, while accepting the possibility of small losses in order to simplify the collateralization process. At the beginning of the contract an initial payment of collateral of size γ takes place, which is a crude device to account for contagion effects. Additional collateral is only posted if the exposure of one entity exceeds some threshold (M_1 in case of B and M_2 in case of S). Threshold collateralization is quite popular in practice, see [51]. However, the choice of γ in practice is often based on rules of thumb (compare [29] for Repos), possibly reducing the effectiveness of this strategy. For $\gamma = M_1 = M_2 = 0$ we obtain the special case of *market-value collateralization* C^{market} with $C_t^{\text{market}} = P_t$.

Improved collateralization strategies. In the following we study collateralization strategies that attempt to reduce the overall counterparty-risk exposure of the contracting parties. We use the CCVA to measure the exposure to counterparty risk of B and the CDVA to measure the exposure of S. B and S have obviously conflicting interests: B prefers a collateralization strategy where S posts a large amount of collateral and B posts no collateral and vice versa for B. In order to balance these conflicting interests we consider an $\mathbb{F}$ adapted collateralization strategy C to be optimal if it minimizes the following functional

$$m(C) := \mathrm{CCVA}_0 + \mathrm{CDVA}_0 \tag{2.19}$$
$$= \mathbb{E}^{\mathbb{Q}}\big(1_{\{\tau<T\}}1_{\{\xi=S\}}D(0,\tau)(\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+)\big) \tag{2.20}$$
$$+\, \mathbb{E}^{\mathbb{Q}}\big(1_{\{\tau<T\}}1_{\{\xi=B\}}D(0,\tau)(\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- + \mathrm{LGD}'_B(P_\tau^+ - C_{\tau-}^+)^-)\big). \tag{2.21}$$

In the full-information case we let $\mathbb{F} = \mathbb{F}^O$ and $P_\tau = P_\tau^O$; in the incomplete-information case we let $\mathbb{F} = \mathbb{F}^U$ and $P_\tau = P_\tau^U$.

The analysis of the *full-information model* is straightforward. In that case the market value $(P_t^O)_{t\geq 0}$ is continuous at τ_B respectively at τ_S. Therefore counterparty risk can be eliminated completely by choosing the market-value strategy $C_t^{\mathrm{market}} = P_t^O = p^O(t, X_t)$, $t < \tau$, that is $m(C^{\mathrm{market}}) = 0$. Note that this result holds in all credit risk models where the risk-free CDS price does not jump at τ_S or τ_R, that is for $\Delta P_{\tau_S} = \Delta P_{\tau_R} = 0$, and thus in particular in all models with conditionally independent defaults.

Optimal strategies under incomplete information. The situation is more involved in the incomplete-information model. In that case the jump of π at τ leads to a jump in the market value P_t^U of the CDS at $t = \tau$ and the collateral position cannot be adjusted at that point. Hence for the market value strategy $C_t^{\mathrm{market}} = P_t^U = p^U(t, \pi_t)$, $t < \tau$ holds that $m(C^{\mathrm{market}}) > 0$.

We therefore need to work a bit more in order to find an optimal strategy under incomplete information. As a first step we simplify the functional m by conditioning on $\mathcal{F}_{\tau-}$. It is well-known that τ is $\mathcal{F}_{\tau-}$ measurable and that for any predictable process L the random variable L_τ is $\mathcal{F}_{\tau-}$ measurable; see [70], Sec III.2. Moreover, for $j \in \{R, B, S\}$ it holds that

$$\mathbb{Q}(\xi = j|\mathcal{F}_{\tau-}) = \frac{(\widehat{\lambda}_j)_{\tau-}}{\sum_{i\in\{B,R,S\}}(\widehat{\lambda}_i)_{\tau-}} =: d_j(\pi_{\tau-}). \tag{2.22}$$

We begin with the CCVA component of m. By conditioning on $\mathcal{F}_{\tau-}$ we get that (2.20) equals

$$\mathbb{E}^{\mathbb{Q}}\Big(1_{\{\tau\leq T\}}D(t,\tau)\mathbb{E}^{\mathbb{Q}}\big(1_{\{\xi=S\}}\big(\,\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ + \mathrm{LGD}\,\prime_S(C_{\tau-}^- - P_\tau^-)^+\big)\big|\mathcal{F}_{\tau-}\big)\Big) \tag{2.23}$$

In the sequel we use the notation

$$x_S := x_S(\tau, \pi_{\tau-}) = p^U\Big(\tau, \pi_{\tau-} + \mathrm{diag}\left(\gamma_S^1(\pi_{\tau-}), \ldots, \gamma_S^K(\pi_{\tau-})\right)\Big) \tag{2.24}$$

to denote the price of the CDS immediately after the default of S; similarly, $x_B := x_B(\tau, \pi_{\tau-})$ denotes the price of the CDS immediately after the default of B. Now note that $1_{\{\xi=S\}}P_\tau^+ = x_S^+$. Hence, using (2.22), the inner conditional expectation in (2.23) is given by $d_S\big(\,\mathrm{LGD}_S(x_S^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - x_S^-)^+\big)$, and (2.23) equals

$$\mathbb{E}^{\mathbb{Q}}\big(1_{\{\tau\leq T\}}D(t,\tau)d_S\big(\mathrm{LGD}_S(x_S^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - x_S^-)^+\big)\big).$$

Similarly we get that (2.21), the CDVA component of m, is equal to

$$\mathbb{E}^{\mathbb{Q}}\big(1_{\{\tau\le T\}}D(t,\tau)d_B\big(\mathrm{LGD}_B(C^-_{\tau-}-x^-_B)^- + \mathrm{LGD}'_B(x^+_B - C^+_{\tau-})^-\big)\big).$$

Define now the 'infinitesimal loss function' by

$$\begin{aligned} l(t,\pi,c) = {} & d_S(\pi)\big((\mathrm{LGD}_S(x_S(t,\pi)^+ - c^+)^+) + \mathrm{LGD}'_S(c^- - x_S(t,\pi)^-)^+\big) \\ & + d_B(\pi)\big((\mathrm{LGD}_B(c^- - x_B(t,\pi)^-)^- + \mathrm{LGD}'_B(x_B(t,\pi)^+ - c^+)^-\big). \end{aligned}$$

The above computations show that $m(C)$ can be written in the form

$$m(C) = \mathbb{E}^{\mathbb{Q}}(D(t,\tau)l(\tau,\pi_{\tau-},C_{\tau-})).$$

Now suppose that we find an $\mathbb{F}^U$-adapted RCLL-process C^* such that a.s.

$$C^*_t(\omega) \in \operatorname{argmin}\{l(t,\pi_t(\omega),c)\colon c\in\mathbb{R}\}.$$

Then C^* is an optimal collateralization strategy - a minimizer of $m(\cdot)$ - in the incomplete-information model. This leads to the following proposition.

Proposition 2.25. *Denote by $x_S = x_S(t,\pi_t)$ and $x_B = x_B(t,\pi_t)$ the risk-free CDS price at time t given $\tau = t, \xi = S$ respectively $\tau = t, \xi = B$ (see (2.24)) and let*

$$d_S = d_S(\pi_t) = \frac{(\widehat{\lambda_S})_t}{\sum_{i\in\{B,R,S\}}(\widehat{\lambda_i})_t} \quad \text{and } d_B = d_B(\pi_t) = \frac{(\widehat{\lambda_B})_t}{\sum_{i\in\{B,R,S\}}(\widehat{\lambda_i})_t}.$$

Then an $\mathbb{F}^U$-adapted RCLL process C^* is an optimal collateralization strategy under incomplete information if and only if the following relations hold $\mathbb{Q}$-a.s. for $t<\tau$:

$$C^*_t = \begin{cases} x_S & \text{if } 0\le x^B\le x^S, \mathrm{LGD}'_B\, d_B < \mathrm{LGD}_S\, d_S,\\ x_B & \text{if } 0\le x^B\le x^S, \mathrm{LGD}'_B\, d_B > \mathrm{LGD}_S\, d_S,\\ x_S & \text{if } x_B\le x_S\le 0, \mathrm{LGD}_B\, d_B < \mathrm{LGD}'_S\, d_S,\\ x_B & \text{if } x_B\le x_S\le 0, \mathrm{LGD}_B\, d_B > \mathrm{LGD}'_S\, d_S,\\ \operatorname{argmin}\{l(\tau,\pi_t,c)\colon c = x_B, 0, x_S\} & \text{if } x_B<0<x_S. \end{cases}$$

$$C^*_t \in \begin{cases} [x_B,x_S] & \text{if } 0\le x_B\le x_S, \mathrm{LGD}'_B\, d_B = \mathrm{LGD}_S\, d_S,\\ [x_B,x_S] & \text{if } x_B\le x_S\le 0, \mathrm{LGD}_B\, d_B = \mathrm{LGD}'_S\, d_S,\\ [x_S,x_B] & \text{if } x_S\le x_B. \end{cases}$$

In particular for any such strategy it holds that $C^*_t\in[\min\{x^S,x^B\},\max\{x^S,x^B\}]$ $\forall t$ and that $l(t,\pi,C^*_t)=0$ for $x^S\le x^B$.

Proof. The proof relies on the preceding arguments. In order to find an optimal strategy we have to find the minimizers of the function $c\mapsto l(\tau,\pi,c)$. This is a piecewise linear function, which converges to ∞ for $c\to\pm\infty$ and fixed t,π. Therefore a minimum exists; it can be found by a case-by-case analysis. Consider for instance the case $0<x_B<x_S$. In that case l takes the form

$$l(t,\pi,c) = (\mathrm{LGD}_S(x_S-c^+)^+ + \mathrm{LGD}'_S\, c^-)d_S + (\mathrm{LGD}'_B(x_B-c^+)^-)d_B,$$

and l is decreasing in the interval $(-\infty, x_B]$ and increasing in $[x_S, \infty)$. Therefore the optimal c lies in $[x_B, x_S]$. For $c \in [x_B, x_S]$, l is given by:

$$l(\tau, \pi_{\tau-}, c) = c(\mathrm{LGD}'_B\, d_B - \mathrm{LGD}_S\, d_S) + \mathrm{LGD}_S\, x_S d_S - \mathrm{LGD}'_B\, x_B d_B.$$

Therefore the result follows. The other cases can be handled in a similar way.

□

We will see below that this optimal collateralization strategy reduces counterparty risk by a large amount compared to standard market-value collateralization. However, if $\mathbb{Q}\big(x^B(t,\pi_t) > x^S(t,\pi_t)\big) > 0$ there remains some risk, that is $m(C^*) > 0$. This remaining risk is due to the fact that in an inhomogeneous portfolio the size of the contagion effects at τ depends on the identity of the defaulting firm which cannot be predicted upfront given the information contained in $\mathcal{F}_{\tau-}$.

Model-independent strategies. The optimal strategy derived in Proposition 2.25 depends on d_B, d_S, and, most importantly, on the market value x_S and x_B of the risk-free CDS after the default of S or B and hence on the size of contagion effects. While these quantities can be computed within a specific reduced-form credit risk model with contagion such as the model of [44] or the model considered by [12], they do depend on the structure of the model and on the parameter values used. It is therefore of interest to develop a 'model-independent' version of C^*.

For this one needs to estimate d_B, d_S, x_B and x_S in a 'model-independent way'. Given the well-known rule of thumb that the CDS spread of a firm is roughly equal to the product of its default intensity and loss given default, in view of the definition of d_B and d_S in (2.22) it is natural to estimate d_B and d_S by

$$\widehat{d_S} = \frac{\frac{S_S}{\mathrm{LGD}_S}}{\frac{S_B}{\mathrm{LGD}_B} + \frac{S_S}{\mathrm{LGD}_S} + \frac{S_R}{\mathrm{LGD}_R}} \quad \text{and} \quad \widehat{d_B} = \frac{\frac{S_S}{\mathrm{LGD}_S}}{\frac{S_B}{\mathrm{LGD}_B} + \frac{S_B}{\mathrm{LGD}_B} + \frac{S_R}{\mathrm{LGD}_R}}, \tag{2.26}$$

where S_B, S_R and S_S represent the fair CDS spread for B, R and S observed in the market. Estimating x_S and x_B is less straightforward. Here one could use ad-hoc assumptions, based on the analysis of historical contagious events. Alternatively we propose to use our results on contagion effects in the [44]-model. Fix some $t \in [0, T]$. First we use the approximations

$$x_B \approx p^{\mathrm{const}}\Big((\widehat{\lambda_R})\mid_{t=\tau_B}\Big) \text{ and } x_S \approx p^{\mathrm{const}}\Big((\widehat{\lambda_R})\mid_{t=\tau_S}\Big) \tag{2.27}$$

where $P^{\mathrm{const}}(\lambda)$ denotes the price of the risk-free CDS on R in a model with constant intensity λ. Now $(\widehat{\lambda_R})_t \approx S_R/\,\mathrm{LGD}_R$. Hence we get from (2.17)

$$(\widehat{\lambda_R})\mid_{t=\tau_B} \approx \frac{S_R}{\mathrm{LGD}_R} + (\Delta\widehat{\lambda_R})\mid_{t=\tau_B} = \frac{S_R}{\mathrm{LGD}_R} + \frac{\mathrm{cov}^{\pi_t}(\lambda_R, \lambda_B)}{(\widehat{\lambda_R})_t}.$$

Now we suggest to proxy $\mathrm{cov}^{\pi_t}(\lambda_R, \lambda_B)$ by $(\mathrm{LGD}_R\,\mathrm{LGD}_B)^{-1}\widehat{\mathrm{cov}}_t(S_R, S_B)$ where $\widehat{\mathrm{cov}}_t(S_R, S_B)$ is an estimate of the time series covariance of the observed CDS spreads at time t obtained for instance by some exponentially weighted historical average. Plugging this into (2.27) gives the estimators

$$\begin{aligned} (\widehat{x_B})_t &= p^{\mathrm{const}}\left(\frac{S_R}{\mathrm{LGD}_R} + \frac{\widehat{\mathrm{cov}}_t(S_R, S_B)}{\mathrm{LGD}_R\,\mathrm{LGD}_B}\right) \text{ and similarly} \\ (\widehat{x_S})_t &= p^{\mathrm{const}}\left(\frac{S_R}{\mathrm{LGD}_R} + \frac{\widehat{\mathrm{cov}}_t(S_R, S_S)}{\mathrm{LGD}_R\,\mathrm{LGD}_S}\right). \end{aligned} \tag{2.28}$$

Note that the proposed estimators for d_B, d_S, x_B and x_S can be computed directly from a time series of observed CDS spreads without making reference to a particular model. In the next section we compare the performance of the model-independent refined strategy with the performance of market-value collateralization on the one hand and with the optimal strategy on the other.

2.6 Numerical Experiments

In this section we discuss results from a number of numerical experiments.

2.6.1 Setup and Calibration

We considered a Markov chain X with $K = 8$ states. It was assumed that X exhibits next-neighbor dynamics (X_t jumps only to $X_t \pm 1$), so that only the values on the main diagonal and on the first off-diagonal of the generator matrix W differ from zero. During the simulation analysis we set the entries on the off-diagonal equal to 0.25, meaning that the Markov chain jumps on average every second year. We have put the short-rate equal to $r = 0.015$. Throughout the study it was assumed that $\mathrm{Rec}_B = \mathrm{Rec}_S = \mathrm{Rec}_R = 0.5$ and $\mathrm{Rec}'_B = \mathrm{Rec}'_S = 0.75$. [2] We calibrated the model to the risk-free CDS spreads and default correlations for R, B and S given in Table 2.1. We considered five different scenarios, labeled *Base; Base 2; Symmetric; Risky protection buyer; Risky protection seller.* These scenarios differ mainly with respect to the relative riskiness of the firms involved in the CDS contract; their choice serves to illustrate the impact of the relative riskiness of the different firms on credit value adjustments. The fair CDS spreads (in basis points) and default correlations (in percentage points) corresponding to these scenarios can be found in Table 2.1.

Table 2.1: Risk scenarios: CDS-spreads in base points, default correlations in percentage points.

Name of scenario	B	R	S	ρ_{BR}	ρ_{BS}	ρ_{RS}
Base	50	1000	500	2.0	1.5	5.0
Base2	500	1000	50	5.0	1.5	2.0
Symmetric	500	1000	500	5.0	3.0	5.0
Risky PB	1000	500	50	5.0	2.0	1.5
Risky PS	50	500	1000	1.5	2.0	5.0

Table 2.2: Results of model calibration for the base scenario

state	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
π_0	0.0810	0.0000	0.2831	0.0548	0.0000	0.0000	0.0000	0.5811
λ_B	0.0000	0.0010	0.0027	0.0040	0.0050	0.0059	0.0091	0.0195
λ_R	0.0031	0.0669	0.1187	0.1482	0.1687	0.1855	0.2393	0.3668
λ_S	0.0007	0.0245	0.0482	0.0627	0.0732	0.0818	0.1108	0.1840

[2]Following a suggestion of the referee who was rightly concerned with the robustness of our findings we ran our simulations also with different forms of the generator matrix W. This led to qualitatively similar results. We do not report these results here as we do not want to overload the paper with numbers.

In this context model model calibration amounts to finding the initial distribution of the Markov chain π_0 and the parameters $\lambda_B, \lambda_R, \lambda_S$. For calibration purposes we used a modification of the algorithm presented in [44]; since the focus of this paper is not on model calibration we omit the details. All in all the calibration procedure performed well, with very small errors for CDS spreads (absolute errors are less than 0.5 bp) and acceptable results for default correlations (relative errors are around 3%). The calibrated values of π_0 and of λ_B, λ_R and λ_S can be found at the end of the chapter, Table 2.2. Note in particular that the calibrated functions $\lambda_B(\cdot)$, $\lambda_S(\cdot)$ and $\lambda_R(\cdot)$ are increasing in x. In the incomplete-information model we moreover need to choose the parameters $a(1), \ldots, a(K)$. We took $a = c * b$, where $b = [-1.75, -1.25, -0.75, -0.25, 0.25, 0.75, 1.25, 1.75]$ and where $c \geq 0$ If not mentioned otherwise, c was taken equal to one.

2.6.2 Results for the un-collateralized case

The main findings regarding the qualitative behavior of the CVA and the DVA in the un-collateralized case can be summarized as follows.

a) The size of the credit value adjustments depends largely on the relative riskiness of the firms. In particular, the CVA is comparatively high if the *first-to-default probability* $\mathbb{Q}(\tau \leq T, \xi = S)$ is relatively large; similarly, the DVA is comparatively high if $\mathbb{Q}(\tau \leq T, \xi = B)$ is relatively large. This can be seen by comparing the size of the value adjustments for the Base and Base2 scenarios or the RiskyPB and the RiskyPS scenarios in Table 2.3: as shown in Table 2.4, $\mathbb{Q}(\tau \leq T, \xi = S)$ is relatively large in the Base and the RiskyPS scenarios, leading to a high CVA; similarly, $\mathbb{Q}(\tau \leq T, \xi = B)$ is relatively large in the Base2 and the RiskyPB scenarios, leading to a high DVA. Note that the first-to-default probabilities are identical in both versions of the model. They are largely driven by the (relative) riskiness of the three firms as given by the risk-free CDS spread in the three scenarios.

Table 2.3: Value adjustments in different scenarios for the complete-information model (left) and for the incomplete information model (right), both for the uncollateralized case.

	full information			incomplete information		
scenario	CVA	DVA	BCVA	CVA	DVA	BCVA
Base	94	1	92	83	1	82
Base2	10	26	-16	9	15	-6
Symmetric	74	5	68	72	4	68
RiskyPB	6	45	-39	6	27	-21
RiskyPS	115	1	114	97	1	96

Table 2.4: The first-to-default probabilities for different scenarios

scenario	B	R	S
Base	0.0293	0.4238	0.2463
Base2	0.2463	0.4238	0.0293
Symmetric	0.1851	0.3972	0.1851
RiskyPB	0.4238	0.2463	0.0293
RiskyPS	0.0293	0.2463	0.4263

b) We have $\mathrm{CVA}^U < \mathrm{CVA}^O$ and $\mathrm{DVA}^U < \mathrm{CVA}^O$, as predicted by Proposition 2.18. The differences between the model variants decreases with decreasing observation noise, that is for higher values of the parameter c in the definition of the function a, as can be seen by inspection of Table 2.5.

Table 2.5: Un-collateralized value adjustments under incomplete information for different values of the parameter c (low values of c correspond to a high observation noise) in the base scenario

noise parameter	CVA	DVA	BCVA
$c = 0$	68	0	68
$c = 1$	83	1	82
$c = 2$	89	1	88
$c = 5$	92	1	90

c) The conditional default probability of the reference entity given an early default of the protection seller is much higher than the unconditional default probability of R (so-called wrong-way risk). In the full-information case this can be seen from Table 2.6 which gives the distribution of X_τ for the case $\xi = B$ and $\xi = S$. Clearly, X_t tends to be in a higher state (compare the high probabilities for x_8) at a default. Hence we expect that the simplified value adjustments given in (2.5) underestimate the true value adjustments by a sizeable amount. This is indeed true, see the numbers reported in the last row of Table 2.7.

Table 2.6: Distribution of X at τ in the base scenario for $\xi = B$ and $\xi = S$.

state	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$\xi = B$	0.0001	0.0144	0.0740	0.0500	0.0208	0.0221	0.0982	0.7203
$\xi = S$	0.0011	0.0309	0.1188	0.0713	0.0277	0.0279	0.1074	0.6149

2.6.3 Results for the case with collateralization

We go on with the analysis of various collateralization strategies. Since collateralization is only relevant on paths where $\tau < T$ and where $\xi \in \{B, S\}$, we illustrate the performance of collateralization strategies by plotting the conditional distribution function of the random variables

$$L_B(C) := 1_{\{\xi=S\}}\big(\text{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ + \text{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+\big)$$
$$L_S(C) := D(t,\tau)1_{\{\xi=B\}}\big(\text{LGD}_B(C_{\tau-}^- - P_\tau^-)^- + \text{LGD}'_B(P_\tau^+)^- - C_{\tau-}^+\big),$$

given that $\{\tau \le T, \xi \in \{B, S\}\}$. Note that for a given collateralization strategy C, $L_B(C)$ gives discounted loss to B that arises from an early default of S, whereas $L_S(C)$ gives the discounted loss to S that arises from an early default of B. We analyzed strategies of the following type:

- Threshold-collateralization with initial margin γ and thresholds $M_1 = M_2 := M$, denoted $C^{\gamma,M}$;
- Market collateralization $C^{\text{market}} = C^{0,0}$;
- The strategy C^* derived in Proposition 2.25 and the "model-independent optimal strategy" based on the estimators (2.26) and (2.28) for the incomplete-information model.

Our findings can be summarized as follows:

Table 2.7: Value adjustments in the complete-information model (left) and in the incomplete-information model (right) with threshold-collateralization and market value collateralization ($M_1 = M_2 = 0$) for $\gamma = 0$ in the Base scenario. In the last row we report the value adjustment corresponding to the simplified value adjustment formula (2.5).

	full information			incomplete information		
threshold	CCVA	CDVA	BCCVA	CCVA	CDVA	BCCVA
$M_1 = M_2 = 0$	0	0	0	35	0	35
$M_1 = M_2 = 0.02$	16	0	15	45	0	45
$M_1 = M_2 = 0.05$	38	1	37	60	0	60
no collateralization with						
(i) correct formula	93	1	92	83	1	82
(ii) simplified formula	68	6	62	54	4	49

a) Threshold collateralization with $\gamma = 0$ is very effective in the complete-information model. For a threshold $M > 0$ counterparty risk is largely reduced as can be seen from Table 2.7. Counterparty credit risk even vanishes completely for $M = 0$. Moreover, losses are bounded when threshold-collateralization is used. This can be seen from Figure 2.2 which displays the empirical cdf of L_B given $\tau \leq T$ and $\xi \in \{B, S\}$ in the complete information model for different scenarios.

b) Under incomplete information the performance of threshold collateralization with $\gamma = 0$ and threshold M is not fully satisfactory. The main reason is the fact that because of the contagion effects threshold collateralization systematically underestimates the market value of the CDS at τ which leads to losses for the protection buyer in case that $\xi = S$. As a consequence we observe high values for the CCVA in scenarios such as the Base scenario where $\mathbb{Q}(\tau \leq T, \xi = S)$ is comparatively high, compare Table 2.7. The losses of the protection seller on the other hand are always smaller than the threshold M. This behavior can be seen from Figures 2.3 and 2.4 where the conditional cdf of L_B and L_S is plotted in various scenarios.

A nonzero initial margin γ can improve the performance of threshold collateralization in scenarios where the credit quality of B is much better than the credit quality of S as in the Base scenario. In that case $\mathbb{Q}(\xi = S \mid \tau \leq T, \xi \in \{B, S\})$ is close to one and one essentially knows that $\xi = S$ in case of an early default. Consequently it is possible to hedge a large part of the contagion effects by choosing a positive initial margin γ. This can be seen from Figure 2.5 where $m(C^{\gamma,M})$ is plotted in the Base scenario for various values of γ and M. In a symmetric scenario where B and S have similar credit quality on the other hand, the identity of the first defaulting firm cannot be predicted and choosing a nonzero initial margin does not help much to improve the effectiveness of threshold collateralization, as can be seen from Figure 2.6. This is clear intuitively: a large initial margin $\gamma > 0$ will lead to a loss for S in case that $\xi = B$ because of re-hypothecation; on the other hand for $\gamma \leq 0$ there will be a loss for B in case that $\xi = S$ because of contagion effects, and neither of the two cases can be ruled out a-priori because B and S have similar credit quality.

c) The optimal strategy C^* from Proposition 2.25 on the other hand performs well under incomplete information and reduces counterparty risk substantially as can be seen from Table 2.8 where various credit value adjustments and the value of $m(C)$ are given. The strategy is particularly effective in scenarios where the credit quality of B is higher than the credit quality of S so that $x_S \leq x_B$ such as the Base scenario and the Risky-PS scenario. On the other hand C^* does not fully eliminate counterparty risk in scenarios where the credit

Figure 2.2: Empirical cdf of L_B for different threshold-collateralization strategies with $\gamma = 0$ in the Base scenario in the complete-information model given $\tau \le T$ and $\xi \in \{B, S\}$. Note that without collateralization the probability that L_B is large is quite high since in the base scenario $\mathbb{Q}(\xi = S \mid \tau \le T, \xi \in \{B, S\}) = 0.245/(0.245 + 0.029) \approx 1$ (see Table 2.4). We can see that threshold collateralization reduces counterparty credit risk very effectively in that case.

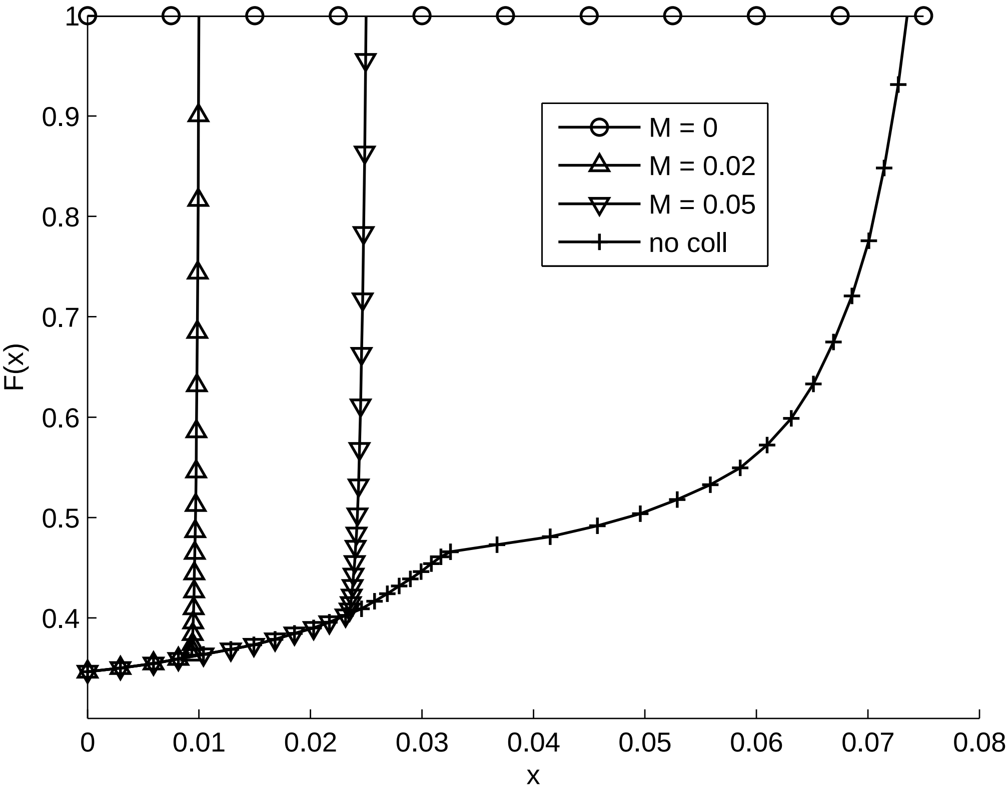


Table 2.8: Performance of different collateralizion strategies in the incomplete-information model as measured by $m(C) = \text{CCVA} + \text{CDVA}$. Note that $m(C^*)$ is small in all scenarios and that $m(C^*) = 0$ in the Base- and RiskyPS scenarios where $x_S \le x_B$. Moreover, the strategy based on the model-independent estimators (2.26) and (2.28) performs much better than market-value collateralization in all scenarios where there is a non-negligible probability that S defaults first.

scenario	C^*	C^{market}	C^* based on (2.26) and (2.28)
Base	2	36	24
Base2	5	7	7
Symmetric	2	32	22
RiskyPB	8	8	8
RiskyPS	0	41	27

Figure 2.3: Empirical cdf of L_B for different threshold-collateralization strategies with $\gamma = 0$ in the Base scenario in the incomplete-information model given $\tau \leq T$ and $\xi \in \{B, S\}$. In that case threshold collateralization with $\gamma = 0$ is not very effective: even for $M = 0$ there is roughly a 20% probability that L_B exceeds 300bp.

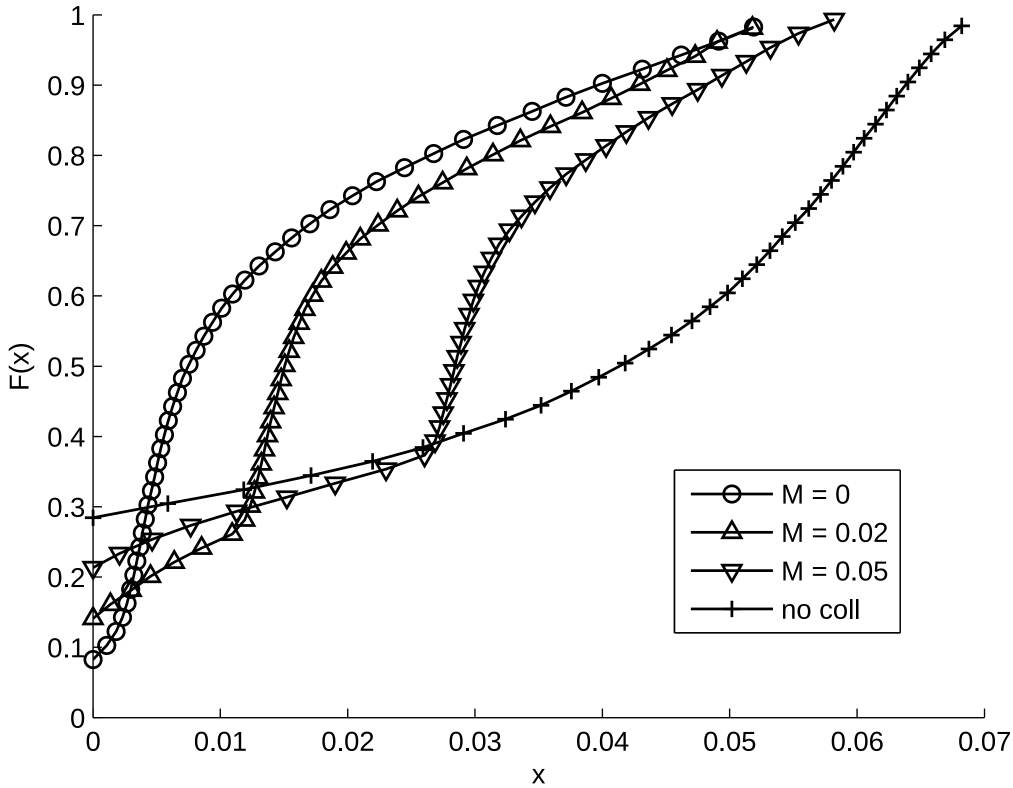
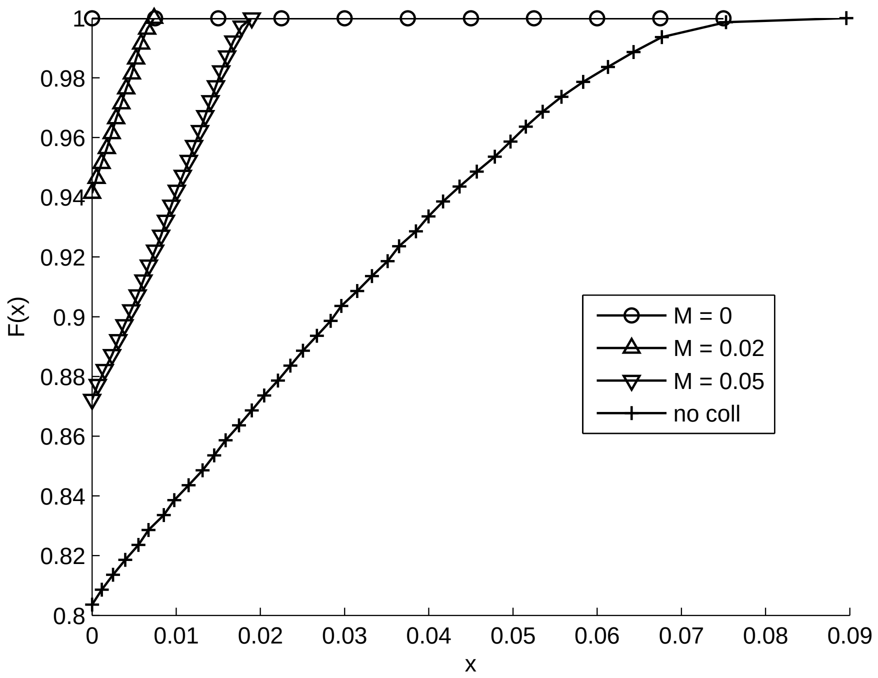


Figure 2.4: Empirical cdf of L_S using threshold-collateralization for the Base2 scenario in the incomplete-information model given $\tau \leq T$ and $\xi \in \{B, S\}$. In this scenario $\mathbb{Q}(\xi = B \mid \tau \leq T, \xi \in \{B, S\})$ is close to one so that threshold collateralization is quite effective even under incomplete information.



quality of S is worse than the credit quality of B such as the Base2 and the Risky-PB scenario, as is evident from Table 2.8. However, even in these scenarios the probability that some party suffers a large loss is fairly small. Of course, the superior performance of the refined collateralization strategy is related to the fact that within our model the quantities x_B and x_S can be computed exactly. We therefore compared the performance of C^* to the performance of the "model-independent optimal strategy" based on (2.26) and (2.28) on the one hand and to the performance of market-value collateralization on the other (see Table 2.8. Of course the model-independent version of our strategy performs worse than C^*. However, in scenarios where there is a non-negligible probability that the protection seller defaults first it performs significantly better than market-value collateralization, This shows that refined collateralization strategies that account for contagion effects have the potential to reduce counterparty credit risk significantly.

2.7 Proof of Proposition 2.12

By symmetry, it suffices to consider the case $i = B$.
a) The default time τ_B is the time, at which the Markov chain (X, H^B) first enters the absorbing set $A = \{(1,1), \dots, (K,1)\}$ and leaves the set $A^c := \{(1,0), \dots, (K,0)\}$. Hence we get:

$$\begin{aligned} \mathbb{Q}\big(\tau_B > s \big| X_t = k, H_t^B = 0\big) &= \mathbb{Q}\big((X_s, H_s^B) \in A^c \big| X_t = k, H_t^B = 0\big) \\ &= 1_{\{\tau_B > t\}} (e_k^\top, 0) e^{Q(s-t)} (\mathbf{1}_K^\top, 0)^\top \end{aligned}$$

Here Q denotes the generator of (X, H^B). Q^n is of the form:

$$Q^n = \begin{pmatrix} W - \Lambda_B & \Lambda_B \\ 0 & W \end{pmatrix}^n = \begin{pmatrix} (W - \Lambda_B)^n & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} Q_B^n & * \\ 0 & * \end{pmatrix}.$$

Therefore the entries in the upper left part of the matrix exponential $e^{Q(s-t)}$ are given by $e^{Q_B(t-s)}$ and we can conclude:

$$\mathbb{Q}\big(\tau_B > s \big| X_t = k, H_t^B = 0\big) = 1_{\{\tau_B > t\}} e_k^\top e^{Q_B(s-t)} \mathbf{1}_K.$$

b) The default times are conditionally independent doubly stochastic random times, and hence the first-to-default time exhibits an intensity which is given by the sum of the individual intensities (see [63], Lemma 17.6), and the result follows from a).

c) We consider the Markov chain $\Psi_t = (X, H^B, H^R, H^S)_{\tau \wedge t}$ (the chain stopped at the first default time.) Ignoring the states where more than one company defaults (and which can therefore never be reached by Ψ), the infinitesimal generator of Ψ is given by:

$$\bar{Q} = \begin{pmatrix} W - \sum_{j \in \{B,R,S\}} \Lambda_j & \Lambda_B & \Lambda_R & \Lambda_S \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The protection buyer B defaults first and before time s if and only if the stopped Markov chain Ψ is in the set $\tilde{A} := \{(1,1,0,0), \dots, (K,1,0,0)\}$ at time s. Therefore:

$$\begin{aligned} \mathbb{Q}(\tau \le s, \xi = B | X_t = k, H_t = (0,0,0)) &= \mathbb{Q}\Big(\Psi_s \in \tilde{A} \Big| \Psi_t = (k,0,0,0)\Big) \\ &= 1_{\{\tau > t\}} (e_k^\top, 0) e^{\bar{Q}(s-t)} (0, \mathbf{1}_K^\top, 0, 0)^\top. \end{aligned}$$

Figure 2.5: Graph of $m(C^{\gamma,M})$ (sum of CCVA and CDVA) under incomplete information for the threshold strategy $C^{\gamma,M}$ for varying values of the initial margin γ and the threshold M in the Base scenario. The function $m(C^{\gamma,M})$ is minimal for $M = 0$ and a positive initial threshold $\gamma^* \approx 0.12$ leading to an optimal value $m(C^{\gamma^*,0}) = 3\text{bp}$, so that counterparty risk can in effect be mitigated by a proper choice of the initial margin.

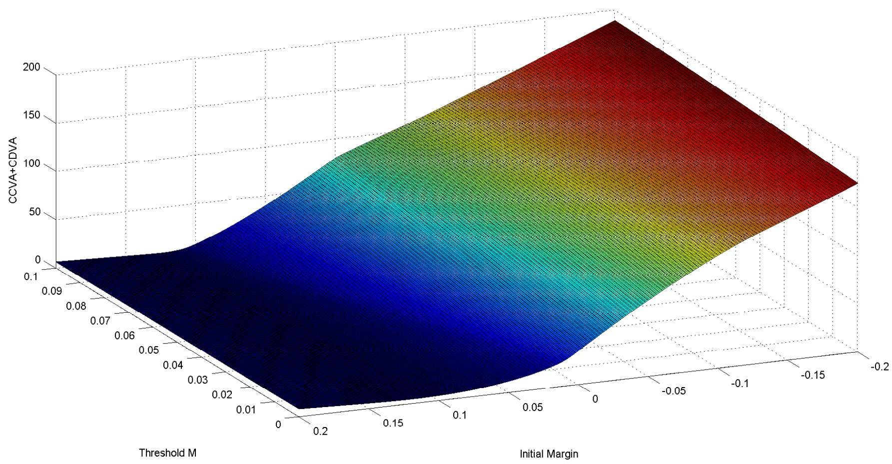
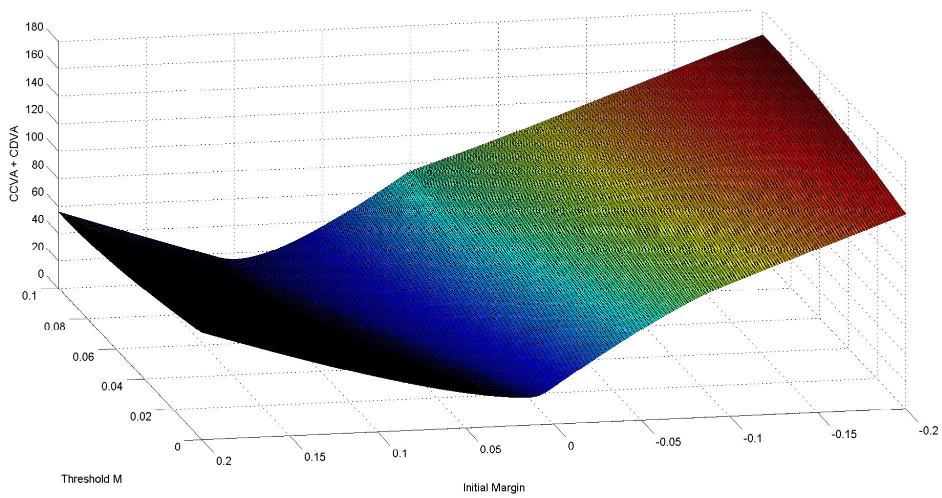


Figure 2.6: Graph of $m(C^{\gamma,M})$ (sum of CCVA and CDVA) under incomplete information for the threshold strategy $C^{\gamma,M}$ for varying values of the initial margin γ and the threshold M in the symmetric scenario. The function $m(C^{\gamma,M})$ is minimal for $M = 0$ and a small initial threshold $\gamma^* \approx 0.01$. Note that $m(C^{\gamma^*,0}) = 15\text{bp}$ whereas for the optimal strategy from Proposition 2.25 one has $m(C^*) = 0$.



So we have to compute the entries of a submatrix of the matrix exponential $e^{\bar{Q}(s-t)}$. Since the n-th power of the matrix $\bar{Q}(s-t)$ is given by $(n > 0)$:

$$(\bar{Q}(s-t))^n = (s-t)^n \begin{pmatrix} Q_{(1)}^n & Q_{(1)}^{n-1}\Lambda_B & Q_{(1)}^{n-1}\Lambda_R & Q_{(1)}^{n-1}\Lambda_S \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the relevant submatrix is given by

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{Q_B^{n-1}}{n!}(s-t)^n \Lambda_B &= Q_{(1)}^{-1}\left(\sum_{n=0}^{\infty} \frac{Q_{(1)}^{n-1}}{n!}(s-t)^n - I\right)\Lambda_B \\ &= Q_{(1)}^{-1}\left(e^{Q_{(1)}(s-t)} - I\right)\Lambda_B, \end{aligned}$$

and the claim follows.

d) The result follows from similar considerations as in c).

Part II

A Structural Credit Risk Model with Incomplete Information

Chapter 3

A Structural Credit Risk Model with Incomplete Information

3.1 Introduction

The second part of the thesis deals with a structural credit risk model which was considered in [41]. In the framework of [41] a default of the company occurs as soon as the asset value process V of the company drops below a critical threshold K, which can be interpreted as the value of the liabilities of the company. In contrast to many classical structural credit risk models like [9] or [59] the asset value process will not be perfectly observable to market participants. In this chapter we will present the model framework from [41] in detail and discuss the pricing of derivative securities. We will show that the pricing of derivative securities naturally leads to stochastic filtering problems with respect to the asset value process and show how to solve the arising filter problems. At this, the material covered in this chapter is mostly taken from [41]. However, only results which are relevant for the sampling of trajectories of processes (see Chapter 4) and the pricing of derivative securities will be presented. In particular, we refer to [41] for proofs and for results on the dynamics of securities. Note that there exists an updated version of [41], where a slightly different model is considered, see [42].

This chapter is structured in the following way: in Section 3.2 we introduce the model framework; the pricing of basic corporate securities is discussed in Section 3.3; Section 3.4 is concerned with the solution of the stochastic filtering problem; Section 3.5 is devoted to derivative pricing. Note that the calibration of the model will de discussed later in Chapter 5.

3.2 Model framework

We work on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t\in[0,\infty)}, \mathbb{P})$ and we assume that all processes introduced below are $\mathbb{G}$-adapted. We consider a company with asset value process $V = (V_t)_{t\in[0,\infty)}$. The company is subject to default risk and the default time is given by

$$\tau = \{t \in [0, \infty) : V_t \leq K\} \text{ for some } K > 0. \tag{3.1}$$

In practice the default barrier might represent debt covenants as in [9] or, in the case of financial institutions, solvency capital requirements imposed by regulators. It is well known that absence of arbitrage implies the existence of a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that for any traded security the corresponding discounted gains from trade are $\mathbb{Q}$-martingales. Since we are mainly interested in pricing, it is thus sufficient to specify the $\mathbb{Q}$-dynamics of all economic variables introduced.

Assumption 3.2. *(Dividends and asset value process)*

(i) The risk free rate of interest is constant and equal to $r \geq 0$.

(ii) The company is about to pay dividends of size $(d_n)_{n\in\mathbb{N}}$ at time points $(T_n)_{n\in\mathbb{N}}$. We assume that $(T_n)_{n\in\mathbb{N}}$ are the jump times of a Poisson process with intensity $\lambda^D > 0$. The size d_n of the n-th dividend is given by:

$$d_n = \delta_n V_{T_n} \tag{3.3}$$

for an i.i.d. sequence of noise variables $(\delta_n)_{n\in\mathbb{N}}$, independent of V, taking values in $(0,1)$, with density function f_δ and mean $\bar{\delta} = \mathbb{E}^{\mathbb{Q}}(\delta_1)$. We denote the cumulative dividend process by $D_t = \sum_{n:T_n\leq t} d_n$. The conditional distribution of d_n given the history of the asset value process is thus of the form

$$\varphi(y, V_{T_n-})dy \text{ where } \varphi(y,v) = v^{-1} f_\delta(y/v).$$

We assume that for all $y \in \mathbb{R}^+$ the map $v \mapsto \varphi(y,v)$ is bounded and twice continuously differentiable on $[K,\infty)$.

(iii) The asset value process V solves the following SDE

$$dV_t = (r - \lambda^D\bar{\delta})V_t dt + \sigma_V V_t dB_t, V_0 = V \tag{3.4}$$

for a constant $\sigma_V > 0$ and a standard $\mathbb{Q}$-Brownian motion B. Moreover, V has Lebesgue density $\pi_0(v)$ for a continuously differentiable function $\pi_0 : [K,\infty) \to \mathbb{R}^+$ with $\pi_0(K) = 0$.

For notational convenience the denote the random measure associated with the marked point process $(T_n, d_n)_{n\in\mathbb{N}}$ by $\mu^D(dy,dt)$. The $\mathbb{G}$-compensator of μ^D is given by $\gamma^D(dy,dt) = \varphi(y,V_t)dy\lambda dt$.

Remarks 3.5. *The insertion of the term $\lambda^D\bar{\delta}V_t dt$ in the drift of the geometric Brownian motion means, that money is continuously extracted from the company, which is used to finance the dividend payments. Since $\bar{\delta} = \mathbb{E}^{\mathbb{Q}}(\delta_1)$ and λ^D is the average number of dividend payments per year, the extracted amount of money is on average sufficient to cover the dividend payments.*

Assumption 3.6. *(Investor information) The following pieces of information are used by the market in the pricing of corporate securities.*

(i) Default information: The market observes the default state $N_t = 1_{\{\tau\leq t\}}$ of the firm. We denote the default history by $\mathbb{F}^N = (\mathcal{F}_t^N)_{t\in[0,\infty)}$.

(ii) Dividend information: Information about dividend payments or equivalently the cumulative dividend process D with $D_t := \sum_{n:T_n\leq t} d_n$ is available to the market; the corresponding filtration is denoted by $\mathbb{F}^D = (\mathcal{F}_t^D)_{t\in[0,\infty)}$. Note that the dividends carry information on V_t as the distribution of the dividend size depends on the asset value at the dividend date.

(iii) Noisy asset observation: The market observes functions of V in additive Gaussian noise. Formally, this bit of information is modelled via some process Z of the form

$$Z_t = \int_0^t a(V_s)\, ds + W_t. \tag{3.7}$$

Here, W is an l-dimensional standard $\mathbb{G}$-Brownian motion independent of B, and we assume that a is a smooth function from $\mathbb{R}^+$ to $\mathbb{R}^l$ with $a(K) = 0$[1]. Finally, $\mathbb{F}^Z = (\mathcal{F}_t^Z)_{t\in[0,\infty)}$ represents the filtration generated by Z. We view the process Z as an abstract representation of all economic information on V that is used by the market in addition to the publicly observed dividend payments.

Summarizing, the information set of the market at time t is given by the σ-field $\mathcal{F}_t^M = \mathcal{F}_t^N \vee \mathcal{F}_t^Z \vee \mathcal{F}_t^D$; the corresponding filtration is denoted by $\mathbb{F}^{\mathbb{M}}$. Note that $\mathbb{F}^{\mathbb{M}} \subset \mathbb{G}$ and that V is not adapted to $\mathbb{F}^{\mathbb{M}}$. There are many possibilities for the form of the function a. A frequent choice is $a(v) = c(\log(v) - \log(K))$; Here the parameter $c \in [0, \infty)$ models the information contained in Z; for c large the asset value can be observed with high precision whereas for c close to zero the process Z conveys almost no information.

3.3 Pricing of Basic Corporate Securities and Nonlinear Filtering

For many reasons the availability of tractable pricing methods for basic corporate securities is important. For example, the calibration method we will suggest in Section 5.5 relies on tractable methods for the pricing of CDS and equity. Moreover, the cash flows of some complex types of securities (like contingent convertible notes from Chapter 5) will depend on basic corporate security. Thus, Monte-Carlo methods which are frequently used for the pricing of these derivatives also require tractable pricing methods for basic securities. In the following we will discuss the pricing of basic corporate securities whose associated cash flow stream depends on future dividend payments and on the occurrence of default and is thus $\mathbb{F}^N \vee \mathbb{F}^D$-adapted. In particular, we will see that the pricing of these leads to a nonlinear filtering problem in a straightforward way. The ex-dividend price of a generic security with $\mathbb{F}^{\mathbb{M}}$-adapted cash flow stream $(H_t)_{t\in[0,T]}$ and maturity date T is given by

$$p_t^H := \mathbb{E}^{\mathbb{Q}}\left(\int_t^T e^{-r(s-t)}\, dH_s \middle| \mathcal{F}_t^{\mathbb{M}} \right), t \in [0, T]. \tag{3.8}$$

Note that p_t^H is defined as the conditional expectation[2] with respect to the σ-field $\mathcal{F}_t^{\mathbb{M}}$ that describes the information available to the market at time t. In the sequel we mostly consider the pre-default value of the security given by $1_{\{\tau>t\}} p_t^H$ (pricing for $\tau \le t$ is largely related to the modelling of recovery rates which is of no concern to us here). Using iterated conditional expectations we get that

$$1_{\{\tau>t\}} p_t^H = \mathbb{E}^{\mathbb{Q}}\left(\mathbb{E}^{\mathbb{Q}}\left(1_{\{\tau>t\}} \int_t^T e^{-r(s-t)}\, dH_s \middle| \mathcal{G}_t \right) \middle| \mathcal{F}_t^{\mathbb{M}} \right).$$

[1]The assumption $a(K) = 0$ is no real restriction as the function a can be replaced by $a - a(K)$ without altering the information content of $\mathbb{F}^M$.

[2]The prices of contingent convertible notes in Section 5.4 will also be defined via formula (3.8).

By the Markov property of V, for typical basic corporate securities with $\mathbb{F}^N \vee \mathbb{F}^D$-adapted cash flow stream the inner conditional expectation can be expressed as a function of time of the current asset value, that is

$$\mathbb{E}^{\mathbb{Q}}\left(1_{\{\tau>t\}} \int_t^T e^{-r(s-t)\,dH_s} \middle| \mathcal{G}_t\right) = 1_{\{\tau>t\}} h(t, V_t). \tag{3.9}$$

The function h will be called *full-information value* of the claim. Below we compute this function for several important examples. We thus get from (3.9) that

$$1_{\{\tau>t\}} p_t^H = 1_{\{\tau>t\}} \mathbb{E}^{\mathbb{Q}}\left(h(t, V_t) \middle| \mathcal{F}_t^{\mathbb{M}}\right).$$

Since V is not $\mathbb{F}^{\mathbb{M}}$ adapted, the valuation of this conditional expectation is a nonlinear filtering problem that is discussed is Section 3.4. Note that martingale pricing generally leads to nonlinear filtering problems under the martingale measure $\mathbb{Q}$ rather than the physical measure $\mathbb{P}$. In the remainder of this section we explain how the full information value h can be computed for debt-related securities such as CDSs and for the equity of the firm.

3.3.1 Full-information value of debt securities

It is well-known that the valuation of debt securities of the firm can be reduced to the pricing of two building blocks, namely a so-called *survival claim* and a so-called *payment-at-default* claim. A survival claim pays one unit of account at the maturity date T provided that $\tau > T$. A payment-at-default claim with maturity T pays one unit directly at τ, provided that $\tau \leq T$. The corresponding full information values will be denoted by h^{surv} respectively h^{def}. Since V is modelled as a geometric Brownian motion, $\log V_t$ satisfies $\log V_t = \mu t + \sigma B_t$ with $\mu := r - \lambda^D \bar{\delta} - \frac{1}{2}\sigma^2$. Hence h^{surv} and h^{def} can be computed using results for the first passage time of Brownian motion with drift. Denote by

$$f(t; b) := \frac{|b|}{\sqrt{2\pi t^3}\sigma} \exp\left(-\frac{(b-\mu t)^2}{2\sigma^2 t}\right)$$

the density function of the first passage time of the process $\sigma B_t + \mu t$ to the level b, see for instance [57], Section 3.5.C. Then we have for $v > K$:

$$h^{\text{surv}}(t, v) = e^{-r(T-t)}\left(1 - \int_0^{T-t} f(s, \log\left(\frac{K}{v}\right) ds\right) \text{ and } h^{\text{def}}(t, v) = \int_0^{T-t} e^{-rs} f(s, \log\left(\frac{K}{v}\right) ds.$$

3.3.2 Full-information value of equity

In our setup the value of the firm's equity is given by the expected discounted value of future dividend payments up to the default time τ, that is

$$1_{\{\tau>t\}} h^{eq}(t, v) = \mathbb{E}^{\mathbb{Q}}\left(\int_t^\infty 1_{\{\tau>s\}} e^{-r(s-t)}\, dD_s \middle| V_t = v\right).$$

In the case of Poissonian dividends [43] show that the pre-default value $1_{\{\tau>t\}} h^{eq}(t, V_t)$ does not explicitly depend on the time t. Thus, we have

$$h^{eq}(v) = \mathbb{E}^{\mathbb{Q}}\left(\int_0^\infty 1_{\{\tau>s\}} e^{-rs}\, dD_s \middle| V_0 = v\right),$$

where we dropped the time t in the notation of h^{eq}. Moreover, [43] show that $h^{eq} = v$ if it holds that $K = 0$ and that for $K > 0$ it holds that

$$h_{eq}(v) = v - \left(\frac{v}{K}\right)^{\alpha^*}, \tag{3.10}$$

where α^* is the negative root of the equation $(r - \lambda^D \bar{\delta})\alpha + 0.5\,\sigma^2\alpha(\alpha - 1) - r = 0$.

Remarks 3.11. *The value of the equity can also be considered as the market capitalization of the bank, which is the product of the number of shares of the company and the share price. Given this relationship, we sometimes call h^{eq} the share price of the company and exploit this relationship on several occasions.*

3.4 Stochastic Filtering of the Asset Value

The last section showed that the pricing of basic corporate securities naturally leads to filtering problems of the form

$$1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\Big(g(V_t)\Big|\mathcal{F}_t^{\mathbb{M}}\Big), t \in [0, T]. \tag{3.12}$$

In the following we will present results from [41] on the solution of the filtering problem (3.12). We will derive an unnormalized density of the conditional distribution of V_t given $\mathcal{F}_t^{\mathbb{M}}$, so that expressions like (3.12) can be essentially computed as integrals with respect to the conditional distribution. In the derivation of the unnormalized density the inclusion of the default information $\mathbb{F}^N$ in the investor information creates problems, because the default time τ does not exhibit an intensity under the full information and standard filtering theory for point processes as in [11] can not be applied. This difficulty is addressed by the using the following result:

Proposition 3.13. *(Proposition 4.1 from [41]) Denote by $V^\tau = (V_{t\wedge\tau})_{t\in[0,\infty)}$ the asset value process stopped at the default boundary, by $\widetilde{Z} = \int_0^t a(V_s^\tau)\,ds + W_t$ the noisy asset information corresponding to the signal process V^τ and by $\widetilde{D}_t := \sum_{n:T_n\leq t}\delta_n V_{T_n}^\tau$ the cumulative dividend process corresponding to V^τ. Then, we have for $g \in L^\infty([K,\infty))$*

$$1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\big(g(V_t^\tau)\big|\mathcal{F}_t^M\big) = 1_{\{\tau>t\}}\frac{\mathbb{E}^{\mathbb{Q}}\Big(g(V_t^\tau)1_{\{V_t^\tau>K\}}\Big|\mathcal{F}_t^{\widetilde{Z}} \vee \mathcal{F}_t^{\widetilde{D}}\Big)}{\mathbb{Q}\Big(V_t^\tau > K\Big|\mathcal{F}_t^{\widetilde{Z}} \vee \mathcal{F}_t^{\widetilde{D}}\Big)}. \tag{3.14}$$

With the notation $f(v) := g(v)1_{\{v>K\}}$ Proposition 3.13 shows that in order to evaluate the right side of (3.12) one has to compute for generic $f \in L^\infty([K,\infty))$ conditional expectations of the form

$$\mathbb{E}^{\mathbb{Q}}\Big(f(V_t^\tau)\Big|\mathcal{F}_t^{\widetilde{Z}} \vee \mathcal{F}_t^{\widetilde{D}}\Big). \tag{3.15}$$

This is a stochastic filtering problem with signal process given by V^τ and with standard diffusion and point process information. [41] apply results from [66] to solve (3.15). The framework from [66] only applies to the filtering of diffusions stopped at the first exit time of some bounded domain, hence they introduce the stopping time $\sigma_N = \inf\{t \in [0,\infty) : V_t \geq N\}$ for some large N and the process $V^N := (V_{t\wedge\sigma_N})_{t\in[0,\infty)}$. Applying Proposition

3.13 to the process V^N leads to a filtering problem with signal process $X := (V^N)^\tau$. More precisely, one has to compute conditional expectations of the form

$$\mathbb{E}^{\mathbb{Q}}\big(f(X_t)\big|\mathcal{F}_t^Z \vee \mathcal{F}_t^D\big), \tag{3.16}$$

where, with a slight abuse of notation, $Z_t = \int_0^t a(X_s)\,ds + W_t$ and $D_t = \sum_{n:T_n \le t} \delta_n X_{T_n}$. Note that $\tau \wedge \sigma_N$ is the first exit time of V of the domain (K, N). Hence, the state space of X is given by $S^X := [K, N]$ and the analysis of [66] applies to the problem (3.16). In [41] it is shown that replacing V by V^N does not affect the financial implications of the analysis, provided that N is sufficiently large. In fact they establish the convergence of the filtering problem (3.16) to

$$\mathbb{E}^{\mathbb{Q}}\Big(f(V_t^\tau)\Big|\mathcal{F}_t^{\widetilde{Z}} \vee \mathcal{F}_t^{\widetilde{D}}\Big).$$

3.4.1 Reference Filtering Approach

In [41] the reference filtering approach is used to find the solution to the filtering problem (3.16). Under this approach one considers the model under an equivalent measure $\mathbb{Q}^*$ such that Z, D and X are independent and reverts to the original dynamics via a change of measure. Recall that we denote the dividend dates by T_n, $n \in \mathbb{N}$, that d_n denotes the dividend paid at T_n and that the conditional density of d_n given $X_{T_n} = x$ is denoted by $\varphi(y, x)$. It will be convenient to model the processes X, Z and D on a product space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q}^*)$. Denote by $(\Omega_2, \mathcal{G}^2, \mathbb{G}^2, \mathbb{Q}_2^*)$ some filtered probability space that supports an l-dimensional Wiener process $Z = (Z_t(\omega_2))_{t\in[0,T]}$. Moreover, fix some strictly positive reference density $\varphi^*(y)$ and suppose the space $(\Omega_2, \mathcal{G}^2, \mathbb{G}^2, \mathbb{Q}_2^*)$ supports a random measure $\mu^D(dy, dt)$ with compensating measure equal to $\gamma^{D,*} := \varphi^*(y)dy\lambda^D dt$ and that μ^D is independent of the Brownian motion Z. Given some probability space $(\Omega_1, \mathcal{G}^1, \mathbb{G}^1, \mathbb{Q}_1^*)$ supporting the process X we let $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$, $\mathbb{G} = \mathbb{G}_1 \otimes \mathbb{G}_2$ and $\mathbb{Q}^* = \mathbb{Q}_1 \otimes \mathbb{Q}_2$. Note that this construction implies that under $\mathbb{Q}^*$, Z is an l-dimensional Brownian motion independent of X. In order to revert to the original model dynamics we introduce the density martingale $L_t = L_t^1 L_t^2$ where L^1 is given by

$$L_t^1 = \exp\bigg(\int_0^t a(X_s)^\top\,dZ_s - \frac{1}{2}\int_0^t |a(X_s)|^2\,ds\bigg)$$

and

$$L_t^2 = 1 + \int_0^t \int_{\mathbb{R}^+} L_{s-}^2 \bigg(\frac{\varphi(y, X_s)}{\varphi^*(y)} - 1\bigg)(\mu^D - \gamma^{D,*})(dy, ds).$$

Since a is a continuous function and the state space S^X is bounded, L is indeed a martingale. Since $\varphi(\cdot, x)$ and φ^* are densities we obtain that

$$\int_{\mathbb{R}^+} \bigg(\frac{\varphi(y, x)}{\varphi^*(y)} - 1\bigg)\,dy = \int_{\mathbb{R}^+} (\varphi(y, x) - \varphi^*(y))\,dy = 0. \tag{3.17}$$

Hence, we get that

$$L_t^2 = 1 + \int_0^t \int_{\mathbb{R}^+} L_{s-}^2 \bigg(\frac{\varphi(y, X_s)}{\varphi^*(y)} - 1\bigg)\mu^D(dy, ds) = \prod_{T_n \le T} \frac{\varphi(d_n, X_{T_n})}{\varphi^*(d_n)}. \tag{3.18}$$

Lemma 3.19. *It holds that* $\mathbb{E}^{\mathbb{Q}^*}(L_T) = 1$. *Define the measure* $\mathbb{Q}$ *by* $(d\mathbb{Q}/d\mathbb{Q}^*)|_{\mathcal{G}_T} = L_T$. *Then under* $\mathbb{Q}$ *the random measure* μ^D *has* $\mathbb{G}$*-compensator* $\gamma^D(dy,dt) = \varphi(y, X_t)\, dy\lambda^D\, dt$. *Moreover, the triple* (X, Z, D) *with* $D_t = \int_0^t y\mu^D(dy, ds)$ *has the joint law postulated in Assumption 3.2.*

The filtering problem (3.16) with respect to $\mathbb{Q}$ can be transferred to the filtering problem with respect to $\mathbb{Q}^*$ by using the abstract Bayes formula. One has for $f \in L^\infty\big(S^X\big)$ that

$$\mathbb{E}^{\mathbb{Q}}\big(f(X_t)\big|\mathcal{F}_t^Z\big) = \frac{\mathbb{E}^{\mathbb{Q}^*}\big(f(X_t)L_t\big|\mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}{\mathbb{E}^{\mathbb{Q}^*}\big(L_t\big|\mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}. \tag{3.20}$$

We concentrate on the numerator. Using the product structure of the underlying probability space we get

$$\mathbb{E}^{\mathbb{Q}^*}\big(f(X_t)L_t\big|\mathcal{F}_t^Z \vee \mathcal{F}_t^D\big) = \mathbb{E}^{\mathbb{Q}_1}(f(X_t)L_t(\cdot,\omega_2)) =: \Sigma_t f(\omega). \tag{3.21}$$

Remark 3.22. *Neglect for the moment the dividend information in the definition of* Σ_t, *that is for* $f \in L\big(S^X\big)$ *we define*

$$\Sigma_t f(\omega) = \mathbb{E}^{\mathbb{Q}^*}\big(f(X_t)L_t\big|\mathcal{F}_t^Z\big). \tag{3.23}$$

Denote by $(T_t)_{t\in[0,\infty)}$ *the transition semigroup of the Markov process* X, *that is for* $x \in S^X$, $T_t f(x) = E_x(f(X_t))$. *Then the following equation holds*

$$\Sigma_t f = \Sigma_0(T_t f) + \sum_{i=1}^{l} \int_0^t \Sigma_s(a_i T_{t-s} f)\, dZ_s^i. \tag{3.24}$$

Equation (3.24) *can be viewed as mild form of the classical Zakai equation. Herefore, note that if* f *is in the domain of the generator of* $\mathcal{L}_X$, *then* (3.24) *is equivalent to the so-called (classical) Zakai-Equation*

$$\Sigma_t f = \Sigma_0(f) + \int_0^t \Sigma_s(\mathcal{L}_X f)\, ds + \sum_{i=1}^{l} \int_0^t \Sigma_s(a_i f)\, dZ_s^i. \tag{3.25}$$

3.4.2 SPDE for the Filter Density

Σ_t defined by (3.21) will be essentially described in terms of an integral with respect to a filter density u. This filter density will be given as the unique solution of an SPDE. Herefore, we introduce some necessary notation. First, we introduce the Sobolev spaces

$$H^k(S^X) = \{f \in L^2\big(S^X\big) : \frac{d^\alpha f}{dx^\alpha} \in L^2\big(S^X\big) \text{ for } \alpha \in \{1,\dots,k\}\}, \tag{3.26}$$

where the derivatives are assumed to exist in the weak sense. Moreover, let $H^1(S^X) = \{f \in H_0^1(S^X) : f = 0 \text{ on } \partial S^X\}$. For an introduction to Sobolev spaces we refer to [36]. The scalar product in $L^2\big(S^X\big)$ is denoted by $(\cdot,\cdot)_{S^X}$.

Consider for $f \in H^2(S^X)$ the differential operator $\mathcal{L}^*$ with

$$\mathcal{L}^* f = \frac{1}{2}\frac{d^2}{dx^2}(\sigma^2 x^2 f)(x) - \frac{d}{dx}((r - \lambda^D\bar{\delta})xf)(x), \tag{3.27}$$

hence $\mathcal{L}^*$ is adjoint to $\mathcal{L}_V$: one has $(f, \mathcal{L}_V g)_{S^X} = (\mathcal{L}^*, g)_{S^X}$ for $f, g \in H^2(S^X) \cap H_0^1(S^X)$. We define an extension of $-\mathcal{L}^*$ to the entire space $H_0^1(S^X)$. For this, we denote by $H_0^1(S^X)'$ the dual space of $H_0^1(S^X)$ and by $\langle \cdot, \cdot \rangle$ the duality pairing between $H_0^1(S^X)'$ and $H_0^1(S^X)$. Then, we define a bounded operator $\mathcal{A}^*$ from $H_0^1(S^X)$ to $H_0^1(S^X)'$ by

$$\langle \mathcal{A}^* f, g \rangle = \frac{1}{2}\left(\sigma^2 x^2 \frac{df}{dx}, \frac{dg}{dx}\right)_{S^X} + \left((\sigma^2 - r + \bar{\delta}\lambda^D) x f, \frac{dg}{dx}\right)_{S^X}.$$

By using partial integration one can show that $\langle \mathcal{A}^* f, g \rangle = -(\mathcal{L}^* f, g)_{S^X}$, so that $\mathcal{A}^*$ is indeed an extension.

In the following we introduce the filter density $u : \Omega \times [0,T] \to H_0^1(S^X)$. From time to time, we will consider u as a mapping from $\Omega \times [0,T] \times S^X$ to $\mathbb{R}_0^+$. In this case, we will drop the notation of the argument $\omega \in \Omega$ of u. In [41] the filter density u is introduced as the solution to the following SPDE:

$$du(t) = -\mathcal{A}^* u(t) dt + a^T u(t) dZ_t + \int_{\mathbb{R}^+} u(t-)\left(\frac{\varphi(y,\cdot)}{\varphi^*(y)} - 1\right)(\mu^D - \gamma^{D,*})(dy, dt), \tag{3.28}$$

with initial condition $u(0) = \pi_0$. Note that this equation is to be understood as an equation in the space $H_0^1(S^X)'$, that is for every $v \in H_0^1(S^X)$ one has the relation

$$\begin{aligned}(u(t), v)_{S^X} &= (u(0), v)_{S^X} - \int_0^t \langle \mathcal{A}^* u(s), v \rangle \, ds + \left(a^T u(s), v\right)_{S^X} dZ_s \\ &\quad + \int_0^t \int_{\mathbb{R}^+} \left(u(s-)\frac{\varphi(y,\cdot)}{\varphi^*(y)} - 1, v\right)_{S^X} (\mu^D - \gamma^{D,*})(dy, ds).\end{aligned} \tag{3.29}$$

Note that (3.17) shows that the integral wrt $\gamma^{D,*}$ can be dropped in (3.28). Therefore, the dynamics of u between two dividend dates, that is on (T_{n-1}, T_n) are

$$du(t) = -\mathcal{A}^* u(t) dt + a^T u(t) dZ_t. \tag{3.30}$$

with initial value $u(T_{n-1})$. At $t = T_n$ one has:

$$u(T_n, x) = u(T_n-, x)\frac{\varphi(d_n, x)}{\varphi^*(d_n)}. \tag{3.31}$$

Therefore, existence and uniqueness of a solution of the SPDE (3.28) can be established by verifying the existence and uniqueness of the following SPDE

$$du(t) = -\mathcal{A}^* u(t) dt + a^T u(t) dZ_t, \; u(0) = \pi_0. \tag{3.32}$$

Theorem 3.33. *(Theorem 4.4 from [41])*
There is a unique $\mathbb{F}^Z$-adapted solution $u \in L^2(\Omega \times [0,T], \mathbb{Q}^* \times dt; H_0^1(S^X))$ of equation (3.32). Moreover, $u(t) \in H^2(S^X)$ a.s. the trajectories belong to $C([0,T], H_0^1(S^X))$ and $u(t,\cdot) \geq 0$ $\mathbb{Q}^*$ a.s.

Finally we give a representation of Σ_t, which shows that u essentially describes Σ_t:

Proposition 3.34. *(Proposition 4.8 from [41])*
Denote by $u(t)$ the solution of the SPDE (3.28) *and define*

$$\nu_K(t) = \frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(s, K) ds + \int_0^t \int_{\mathbb{R}^+} \nu_K(s-)\left(\frac{\varphi(y,K)}{\varphi^*(y)} - 1\right)(\mu^D - \gamma^{D,*})(dy, ds)$$

$$\begin{aligned}\nu_N(t) &= -\int_0^t \frac{1}{2}\sigma^2 N^2 \frac{du}{dx}(s, N) ds + \int_0^t a^T(N) \nu_N(s) \, dZ_s \\ &\quad + \int_0^t \int_{\mathbb{R}^+} \nu_N(s-)\left(\frac{\varphi(y,N)}{\varphi^*(y)} - 1\right)(\mu^D - \gamma^{D,*})(dy, dt).\end{aligned}$$

Then, it holds that $\Sigma_t f = (u(t), f)_{S^X} + \nu_K(t)f(K) + \nu_N(t)f(N)$.

The last proposition shows that the measure Σ_t has a Lebesgue density on the interior of S^X and a point mass on the boundary points K and N. Provided N is large, ν_N is largely irrelevant for practical purposes.

Remark 3.35. *For the filtering results it does not matter that the* d_n *are dividend payments, so that our analysis applies also to other types of noisy asset information arriving discretely in time. In particular, this applies to the Core Tier One Ratio which we will introduce in Chapter 5.*

Finally we return to the filtering problem with respect to the market filtration $\mathbb{F}^{\mathbb{M}}$.

Corollary 3.36. *One has for* $f \in L^\infty(S^X)$*:*

$$1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\Big(f(X_t)\Big|\mathcal{F}_t^{\mathbb{M}}\Big) = 1_{\{\tau>t\}}((\pi(t,\cdot), f)_{S^X} + \pi_N(t)f(N)),$$

with $\pi(t,x) = u(t,x)/C(t)$ *and* $\pi_N(t,x) = \nu_N(t,x)/C(t)$. *Here,* $C(t)$ *is given by* $C(t) = (u(t), 1)_{S^X} + \nu_N(t)$.

Remark 3.37. *For practical purposes the* ν_N*-term that corresponds to the conditional probability of reaching the upper boundary of* S^X *prior to the horizon date can be dropped. With this simplification we get for* $t \in [0, \tau)$*:*

$$\mathbb{E}^{\mathbb{Q}}\big(f(V_t)\big|\mathcal{F}_t^M\big) \approx (\tilde\pi(t,\cdot), f) \text{ with } \tilde\pi(t,x) = 1_{(K,N)}(x)\frac{u(t,x)}{(u(t),1)_{S^X}}. \tag{3.38}$$

We close this section with an interesting result concerning the classification of the outlined structural credit risk model:

Theorem 3.39. *(Theorem 5.1 from [41]) The* $\mathbb{F}^{\mathbb{M}}$*-compensator of* N_t *is given by the process* $(\Lambda_{t\wedge\tau})_{t\in[0,\infty)}$ *where* $\Lambda_t = \int_0^t \lambda_s ds$ *and where the default intensity* λ_t *is given by*

$$\lambda_t = \frac{1}{2}\sigma^2K^2\frac{\partial\pi}{\partial x}(t,K). \tag{3.40}$$

3.5 Derivative Pricing

In this section we discuss the pricing of derivative securities in our set-up. The pricing of basic corporate securities with $\mathbb{F}^N \vee \mathbb{F}^D$ adapted payoff stream such as equity and debt has been discussed in Section 3.3.1. Here, we will discuss the pricing of more complex derivatives. In principle, the pricing of many derivatives can be done by using Monte-Carlo methods. Therefore, we provide an algorithm for the simulation of trajectories of all the relevant processes under the measure $\mathbb{Q}$ in Chapter 4. Note that the pricing of the contingent convertible notes from Chapter 5 has been done in this way. In the following we will additionally present some general results on the structure of prices of *options on basic securities*, that is securities whose pay-off depends on the price of traded basic securities. Examples for such products include equity options, bond options and contingent convertible notes.

We begin with a general result on the pricing of a survival claim with pay-off $1_{\{\tau>T\}}H$ for some $\mathcal{F}_T^Z \vee \mathcal{F}_T^D$ measurable random variable H. The result shows that the pricing of this claim can be reduced to the problem of computing a conditional expectation with respect to the reference measure $\mathbb{Q}^*$ and the background filtration $\mathcal{F}_t^Z \vee \mathcal{F}_t^D$.

Proposition 3.41. *Consider some integrable, $\mathcal{F}_T^Z \vee \mathcal{F}_T^D$ measurable random variable H. Then it holds for $t \le T$ that*

$$\mathbb{E}^{\mathbb{Q}}\Big(1_{\{\tau>T\}} H \Big| \mathcal{F}_t^{\mathbb{M}}\Big) = 1_{\{\tau>t\}} \frac{\mathbb{E}^{\mathbb{Q}^*}\big(H((u(T),1)_{S^X} + \nu_N(T)) \big| \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}{(u(t),1)_{S^X} + \nu_N(t)}.$$

Next we specialize this general result to options on traded basic corporate securities. From now on we ignore the point mass $\nu_N(t)$ at the upper boundary of S^X. Consider for concreteness an option on the stock price of the firm with maturity T and pay-off $H = g(S_T)$. We get for the price of this claim that

$$p_t^H = \mathbb{E}^{\mathbb{Q}^*}\Big(e^{-r(T-t)} 1_{\{\tau>T\}} g(S_T) \Big| \mathcal{F}_t^{\mathbb{M}}\Big) + e^{-r(T-t)} g(0) \mathbb{Q}(\tau \le T).$$

The computation of the default probability $\mathbb{Q}(\tau \le T)$ has been discussed in detail in Section 3.3.1, so that we concentrate on the first term. Using Proposition 3.41. and the fact that $S_T = (u(T), h^{eq})_{S^X} / (u(T), 1)_{S^X}$ we get that this term equals

$$1_{\{\tau>t\}} \frac{1}{(u(t),1)_{S^X}} \mathbb{E}^{\mathbb{Q}}\bigg(g\bigg(\frac{(u(T),h^{eq})_{S^X}}{(u(T),1)_{S^X}}\bigg)(u(T),1)_{S^X} \bigg| \mathcal{F}_t^Z \vee \mathcal{F}_t^D\bigg).$$

Now standard results on the Markov property of solutions of SPDEs such as Theorem 9.30 of [67] imply that under $\mathbb{Q}^*$ the solution $u(t)$ of the SPDE (3.28) is a Markov process. Hence

$$\frac{1}{(u(t),1)_{S^X}} \mathbb{E}^{\mathbb{Q}}\bigg(g\bigg(\frac{(u(T),h^{eq})_{S^X}}{(u(T),1)_{S^X}}\bigg)(u(T),1)_{S^X} \bigg| \mathcal{F}_t^Z \vee \mathcal{F}_t^D\bigg) = \widetilde{C}(t, u(t))$$

for some function $\widetilde{C}(t, u(t))$ of time and the current value of the unnormalized filter density. Moreover, since the SPDE (3.28) is linear $\widetilde{C}$ is homogeneous of degree zero in u. Hence we may without loss of generality replace $u(t)$ by the current filter density $\pi(t) = u(t)/(u(t),1)_{S^X}$, and we get $\mathbb{E}^{\mathbb{Q}}\big(e^{-r(T-t)} 1_{\{\tau>T\}} g(S_T) \big| \mathcal{F}_t^{\mathbb{M}}\big) = 1_{\{\tau>t\}} C(t, \pi(t))$ where

$$C(t,\pi) = \mathbb{E}^{\mathbb{Q}^*}\bigg(g\bigg(\frac{(u(T),h^{eq})_{S^X}}{(u(T),1)_{S^X}}\bigg)(u(T),1)_{S^X} \bigg| u(t) = \pi(t)\bigg). \tag{3.42}$$

The actual computation of C is best done using Monte Carlo methods, using a numerical method to solve the SPDE (3.30). The Galerkin approximation which will be described in Chapter 4 is particularly well suited for this purpose. Note that (3.42) is an expectation with respect to the reference measure $\mathbb{Q}^*$ and not $\mathbb{Q}$. Hence one needs to sample from the SPDE (3.28) under $\mathbb{Q}^*$, that is the driving process Z is a Brownian motion and the random measure μ^D has compensator $\gamma^{D,*}$.

Chapter 4

Numerical Implementation of the Structural Credit Risk Model

In this chapter we discuss several algorithms related to the numerical implementation of the structural credit risk model from Chapter 3. Remember that the prices of *basic corporate securities* can be represented in terms of their full-information value and the conditional distribution π. Securities which belong to the class of *options on basic corporate securities* can be priced by using Monte Carlo methods. Hence, one needs a good algorithm for the simulation of trajectories of V, D, π and S. For this purpose, we propose the following algorithm from [41]:

Algorithm 4.1. *(Sampling trajectories for a given initial filter density π_0)*

1. *Generate a random variable $V_0 \sim \pi_0$, a trajectory $(V_s)_{0\leq s\leq T}$ of the asset value process with inital value $V_0 = V$ and the associated trajectory $(N_s)_{0\leq s\leq T}$ of the default indicator process.*

2. *Generate realizations $(D_s)_{0\leq s\leq T}$ and $(Z_s)_{0\leq s\leq T}$ of the cumulative dividend process and of the noisy asset information, using the trajectory of the asset value process generated in Step 1 as input.*

3. *Compute for the observation generated in Step 2 a trajectory $u(s)_{0\leq s\leq T}$ of the unnormalized filter density with initial value $u(0) = \pi_0$ using a Galerkin approximation, which we will describe in Section 4.1. Return $\pi(t) = (1 - N_t)(u(t)/(u(t), 1)_{S^X})$ and $S_t = (1 - N_t)(\pi(t), h_{eq})_{S^X}$ for all $t \in [0, T]$.*

In the following, we consider the algorithm in more detail. Since the simulation of trajectories of V and Z in the Steps 1 and 2 can be done efficiently by using Euler-Maruyama methods, we only consider the sampling from π_0 in Section 4.2. A finite-dimensional approximation of the SPDE (3.30) for u is discussed in Section 4.1. This approximation leads to a system of SDEs. In Section 4.3 we study different methods for solving this SDE system. We will numerically test the different methods against a Kalman Filter in a simplified setting. The simplified setting as well as the Kalman Filter will be described in Section 4.4. Afterwards, the results of the numerical case study are presented in Section 4.5. The complete algorithm was implemented in Matlab and the corresponding code is available on request.

4.1 Finite-Dimensional Approximation of the Zakai Equation

Since u is the unique solution of the SPDE (3.3.1), it is a infinite dimensional object and we have to approximate it by a finite dimensional object. The following method was presented in [41]. At first we explain the method without dividend payments. Consider m linearly independent basis functions $e_1, \ldots, e_m \in H_0^1(S^X) \cap H^2(S^X)$ generating the subspace $\mathcal{H}^m \subset H_0^1(S^X)$ and denote by $pr^m : H_0^1(S^X) \to \mathcal{H}^m$ the projection on this subspace wrt $(\cdot,\cdot)_{S^X}$. Hence, for $\pi \in H_0^1(S^X)$ its projection is of the form $pr^m(\pi) = \sum_{i=1}^m \psi_i e_i$. Note that the coefficients $(\psi_1, \ldots, \psi_m)^\top$ of the projection are given by

$$(\psi_1, \ldots, \psi_m)^\top = A^{-1}((\pi_0, e_1)_{S^X}, \ldots, (\pi_0, e_m)_{S^X})^\top,$$

where the matrix A is defined by $A_{ij} = (e_i, e_j)_{S^X}$ for $i, j = 1, \ldots, m$. In the Galerkin method the solution $\widetilde{u}$ of the equation

$$d\tilde{u}(t) = pr^m \circ \mathcal{L}^* \circ pr^m \tilde{u}(t)dt + pr^m\big(a^T pr^m \tilde{u}(t)\big)dZ_t,\ \tilde{u}(0) = pr^m \pi_0. \tag{4.2}$$

is used as an approximation of u. Since projections are self-adjoint, we get that for $v \in H_0^1(S^X)$

$$d(\tilde{u}(t), v)_{S^X} = (\mathcal{L}^* \circ pr^m \tilde{u}(t), pr^m v)_{S^X} dt + \Big(a^\top pr^m \tilde{u}(t), pr^m v\Big) dZ_t. \tag{4.3}$$

Hence $d(\tilde{u}, v) = 0$ if v belongs to $(\mathcal{H}^m)^\perp$. Since moreover $\tilde{u}(0) = pr^m \pi_0 \in \mathcal{H}^m$ we conclude that $\tilde{u}(t) \in \mathcal{H}^{(m)}$ for all t. Therefore, $\widetilde{u}$ is of the form $\tilde{u}(t) = \sum_{i=1}^m \psi_i(t) e_i$. In the following we derive a system of SDEs for the m dimensional coefficient process $\Psi^m = (\psi_1, \ldots, \psi_m)^\top$. Using $v = e_j$ in (4.3) we get for $j \in \{1, \ldots, m\}$

$$d(\tilde{u}(t), e_j)_{S^X} = \sum_{i=1}^m \psi_i(t)(\mathcal{L}^* e_i, e_j)_{S^X} dt + \sum_{k=1}^l \sum_{i=1}^m (a_k e_i, e_j)_{S^X} \psi_i(t) dZ_t^k. \tag{4.4}$$

On the other hand,

$$d(\tilde{u}(t), e_j)_{S^X} = \sum_{i=1}^m (e_i, e_j)_{S^X}. \tag{4.5}$$

Equating (4.4) and (4.5) gives the following system of SDEs for Ψ^m

$$d\Psi_t^m = A^{-1} B^\top \Psi_t^m dt + \sum_{k=1}^l A^{-1} C^k \Psi_t^m dZ_t^k, \tag{4.6}$$

where the matrices B and $C^1, \ldots, C^l$ are defined by $B_{ij} = (\mathcal{L}^* e_i, e_j)_{S^X}$ and $C_{ij}^k = (a_k e_i, e_j)_{S^X}$. The initial condition of (4.6) is given by

$$\Psi_0^{(m)} = A^{-1}((\pi_0, e_1), \ldots, (\pi_0, e_m))^\top. \tag{4.7}$$

Conditions for convergence $\widetilde{u} \to u$ can be found in [46]: The Galerkin approximation for the filter density converges for $m \to \infty$ if and only if the Galerkin approximation for the deterministic forward PDE $\frac{du}{dt}(t) = \mathcal{L}^* u(t)$ converges.

In the case with dividend information the Galerkin method is applied successively on each interval $[T_{n-1}, T_n)$, $n = 1, 2, \ldots$. Let $\widetilde{u}^n$ denote the approximating density over the interval $[T_{n-1}, T_n)$. In line with relation (3.31), the initial condition for the interval (T_n, T_{n+1}) is then given by

$$\widetilde{u}(T_n) = pr^m\left(\widetilde{u}^n(T_n-,.)\frac{\varphi(d_n,.)}{\varphi^*(d_n)}\right),$$

that is by projecting the updated density $\tilde{u}^n(T_n-,.)\frac{\varphi(d_n,.)}{\varphi^*(d_n)}$ onto $\mathcal{H}^m$. If one is only interested in finding π, one only needs to compute $\tilde{u}^n$ up to proportionality. Since

$$\widetilde{u}(T_n) = pr^m\left(\widetilde{u}^n(T_n-,.)\frac{\varphi(d_n,.)}{\varphi^*(d_n)}\right) \propto pr^m(\widetilde{u}^n(T_n-,.)\varphi(d_n,.)). \tag{4.8}$$

it suffices to compute the right hand side of (4.8) in this case. Note that this shows that the conditional density π does not depend on the specification of φ^*.

4.2 Simulation of the Initial Density of V

In the first step of the algorithm 4.1 we need to sample from π_0, the distribution of V_0. In some cases the distribution of V_0 is well known and good sampling algorithms are available. For example, this holds true if the distribution of V_0 is a displaced lognormal distribution, which means that $(V_0 - K) \sim LogN(a, b)$. On the other hand the initial density π_0 is often chosen as a convex combination of the basis functions, precisely $\pi_0 = \sum_{i=1}^m \alpha_i e_i$ with coefficients $\alpha_1, \ldots, \alpha_m$. Suppose that the basis functions are densities, which can be attained by normalizing the original basis functions if necessary. The following steps produce a sample from π_0:

1. Generate a sample from the random variable $L : \Omega \to \{1, \ldots, m\}$ with probability density $\alpha_1, \ldots, \alpha_m$ wrt the uniform measure on $\{1, \ldots, m\}$.

2. Given $L = i$ generate a sample from the distribution corresponding to the basis function i. This sample is used as a sample of V_0.

4.3 Simulation of the Galerkin Approximation $\widetilde{\Psi}^m$ of the Unnormalized Density

In the general the system of SDEs (4.6) of the Galerkin approximation Ψ^m has no explicit solution. Therefore, we have to apply numerical methods which will give us an approximation $\widetilde{\Psi}^m$ of the Galerkin approximation Ψ^m. At this, the numerical methods provide us with an approximation at a finite set of time points $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$. We will use an equidistant time grid with an interval length $\Delta = T/n$ such that $t_i = i\Delta$.

All of the algorithms which will be presented in the following define $\widetilde{\Psi}^m_{t_0}$ via the initial condition (4.7) and compute the approximations $\left(\widetilde{\Psi}^m_t\right)_{t=t_1,\ldots,t_n}$ iteratively. That is, for each algorithm there exists a function f such that

$$\widetilde{\Psi}^m_{t_{i+1}} = \widetilde{\Psi}^m_{t_i} + f(\Delta, \widetilde{\Psi}^m_{t_i}, \Delta Z_{t_i}) \text{ for } i = 1, \ldots, n-1, \tag{4.9}$$

where ΔZ_{t_i} denotes the vector of differences of the noisy asset information:

$$\Delta Z_{t_i} := (Z^1_{t_{i+1}} - Z^1_{t_i}, \dots, Z^l_{t_{i+1}} - Z^l_{t_i})^T.$$

When calculating $\widetilde{\Psi}^m_{t_{i+1}}$ from $\widetilde{\Psi}^m_{t_i}$, the SDE system (4.6) restricted to the interval $[t_i, t_{i+1}]$ is used as the starting point for the derivation of an approximation. Remember that this SDE system is equivalent to

$$\Psi^m_{t_{i+1}} = \Psi^m_{t_i} + \int_{t_i}^{t_{i+1}} A^{-1} B^T \Psi^m_s \, ds + \int_{t_i}^{t_{i+1}} \sum_{k=1}^{l} A^{-1} C^k \Psi^m_s \, dZ^k_s. \tag{4.10}$$

In the following we will discuss the Euler-Maruyama method, the Milstein method, the Splitting up method and the Matrix-exponential method.

4.3.1 Euler-Maruyama Method

The Euler-Maruyama scheme is probably the most used method for the simulation of SDEs. It has been extensively studied and we refer to [68] for a thoroughly discussion of its properties. The basic idea of this algorithm consists of approximating the integrals in (4.10):

$$\int_{t_i}^{t_{i+1}} A^{-1} B^T \Psi^m_s \, ds \approx A^{-1} B^T \Psi^m_{t_i} \Delta$$
$$\int_{t_i}^{t_{i+1}} A^{-1} C^k \Psi^m_s \, dZ^k_s \approx A^{-1} C^k \Psi^m_{t_i} \Delta Z^k_{t_i} \text{ for } k = 1, \dots, l,$$

Thus, for $i = 0, \dots, n-1$ we can compute $\widetilde{\Psi}^m_{t_i}$ by the equation

$$\widetilde{\Psi}^m_{t_{i+1}} = \widetilde{\Psi}^m_{t_i} + A^{-1} B^T \widetilde{\Psi}^m_{t_i} \Delta + \sum_{k=1}^{l} A^{-1} C^k \widetilde{\Psi}_{t_i} \Delta Z^k_{t_i}. \tag{4.11}$$

4.3.2 Milstein Method

Apart from the Euler-Maruyama scheme the Milstein scheme is one of the most used algorithms for the simulation of SDEs. Again [68] can be consulted for much more information than we will give here. Compared to the Euler method one uses higher order terms to approximate the stochastic integral and takes into account the quadratic variation of Z. For $l = 1$ we have:

$$\int_{t_i}^{t_{i+1}} A^{-1} C^1 \Psi^m_s \, dZ^1_s \approx \sum_{i=0}^{n-1} A^{-1} C^1 \Psi^m_{t_i} (\Delta Z_{t_i}) + \frac{1}{2} (A^{-1} C^1)^2 \Psi^m_{t_i} ((\Delta Z_{t_i})^2 - \Delta). \tag{4.12}$$

The approximation of the Lebesgue-Integral is not changed:

$$\int_{t_i}^{t_{i+1}} A^{-1} B^T \Psi^m_s \, ds \approx A^{-1} B^T \Psi^m_{t_i} \Delta \tag{4.13}$$

So for $i = 0, \dots, n-1$ we set

$$\widetilde{\Psi}^m_{t_{i+1}} := \widetilde{\Psi}^m_{t_i} + A^{-1} B^T \widetilde{\Psi}^m_{t_i} \Delta + A^{-1} C^1 \widetilde{\Psi}^m_{t_i} \Delta Z_{t_i} + \frac{1}{2} \Big(A^{-1} C^k \Big)^2 \widetilde{\Psi}^m_{t_i} \big((\Delta Z_{t_i})^2 - \Delta \big). \tag{4.14}$$

For $l > 1$ the scheme becomes more complicated and also more time consuming, because double Wiener-Integrals arise. These can't be computed in closed form and have to be approximated as well. This is the main reason why we only consider the case $l = 1$. We refer to [68] for the case $l > 1$.

4.3.3 Splitting-Up Method

The splitting-up method was introduced in [48]. An application of this method in the context of filtering can also be found in [40]. The idea of the algorithm is to split the original SDE into a deterministic part Ψ^{Det} and a stochastic part Ψ^{Sto} and to solve each of them separately. Suppose we already have $\widetilde{\Psi}^m_{t_i}$ and want to find $\widetilde{\Psi}^m_{t_{i+1}}$. We define the deterministic part Ψ^{Det} on the interval $[t_i, t_{i+1}]$ as the solution of the following SDE

$$d\Psi^{Det}_t := A^{-1}B^T\Psi^{Det}_t dt,\ \Psi^{Det}_{t_i} = \widetilde{\Psi}^m_{t_i} \tag{4.15}$$

and the stochastic part Ψ^{Sto} by the solution of

$$d\Psi^{Sto}_t = \sum_{k=1}^{l} A^{-1}C^k\Psi^{Sto}_t dZ^k_t,\ \Psi^{Sto}_{t_i} = \Psi^{Det}_{t_{i+1}}. \tag{4.16}$$

Note that the solutions of the SDEs (4.15) and (4.16) can be represented in terms of matrix exponentials:

$$\Psi^{Det}_{t_{i+1}} = \operatorname{expm}\big(A^{-1}B^T\Delta\big)\widetilde{\Psi}^m_{t_i}, \tag{4.17}$$

$$\Psi^{Sto}_{t_{i+1}} = \operatorname{expm}\left(\sum_{k=1}^{l} A^{-1}C^k(\Delta Z^k_{t_i}) - \frac{1}{2}\sum_{k=1}^{l}(A^{-1}C^k)^2\Delta\right)\Psi^{Det}_{t_{i+1}}. \tag{4.18}$$

Finally we define $\widetilde{\Psi}^m_{t_{i+1}} := \Psi^{Sto}_{t_{i+1}}$. The computation of matrix exponentials in (4.17) and (4.18) can be computationally intensive. Thus, it is reasonable to avoid repeated computations of these matrix exponentials. Provided $A^{-1}B^T$ is diagonalizable it is possible to compute its matrix exponential in the following way.

Precompute the eigenvalue decomposition of $A^{-1}B^T$. Let D be a corresponding diagonal matrix of eigenvalues $\lambda_1, \dots, \lambda_n$ and V be an invertible matrix such that $A^{-1}B^T = VDV^{-1}$. Then (4.17) becomes

$$\Psi^{Det}_{t_{i+1}} = V\operatorname{expm}(D\Delta)V^{-1}\widetilde{\Psi}^m_{t_i}.$$

Note that the matrix exponential $\operatorname{expm}(D\Delta)$ is again a diagonal matrix with entries on the main diagonal given by $e^{\lambda_1\Delta}, \dots, e^{\lambda_n\Delta}$. The matrix exponential in (4.18) factorizes if the matrices $A^{-1}C^1, \dots, A^{-1}C^l$ commute. If the matrices furthermore are diagonalizable, then the we can proceed similarly as for (4.17).

4.3.4 Matrix-exponential Method

The matrix-exponential method was firstly studied in [69] and is motivated by the following result. Let B be a n-dimensional Brownian motion and let $M, M_1, \dots, M_l$ be $n\times n$ matrices. If $M, M_1, \dots, M_l$ commute, then the solution of the following SDE

$$dX_t = MX_t dt + \sum_{i=1}^{l} M_i X_t dB^i_t$$

is given in terms of a matrix exponential:

$$X_t = \operatorname{expm}\left(\int_0^t \left(M - \sum_{i=1}^{l} \frac{1}{2} M_i^2\right) ds + \sum_{i=1}^{l} M_i B_t^i\right) X_0$$
$$= \operatorname{expm}\left(\left(M - \sum_{i=1}^{l} \frac{1}{2} M_i^2\right) t + \sum_{i=1}^{l} M_i B_t^i\right) X_0. \tag{4.19}$$

The matrix-exponential method applies the relationship (4.19) on the SDE (4.6) for each interval $[t_i, t_{i+1}]$ although the involved matrices do *not* commute. Hence, for $i = 0, \ldots, n-1$ we define $\widetilde{\Psi}^m_{t_{i+1}}$ by

$$\widetilde{\Psi}^m_{t_{i+1}} = \operatorname{expm}\left(\left(A^{-1}B^T - \frac{1}{2}\sum_{k=1}^{l}(A^{-1}C^k)^2\right)\Delta + \sum_{k=1}^{l} A^{-1}C^k \Delta Z^k_{t_i}\right)\widetilde{\Psi}^m_{t_i}. \tag{4.20}$$

If the matrices $A^{-1}B^T, A^{-1}C^1, \ldots, A^{-1}C^l$ commute, then (4.20) becomes

$$\widetilde{\Psi}^m_{t_{i+1}} = \operatorname{expm}\left(\sum_{k=1}^{l} A^{-1}C^k(\Delta Z^k_{t_i}) - \frac{1}{2}\sum_{k=1}^{l}(A^{-1}C^k)^2\Delta\right)\operatorname{expm}\big(A^{-1}B^T\Delta\big)\widetilde{\Psi}^m_{t_i}. \tag{4.21}$$

Therefore, splitting-up method and matrix exponential method give the same results provided commutativity holds true.

It is also possible to derive an upper estimate for the error of the splitting of the matrix-exponential in our setting. Let $\|\cdot\|_2$ denote the matrix norm corresponding to the euclidean norm and let L_1 and L_2 be two $m \times m$ matrices. Then, from Theorem 2.1 from [73] we have

$$\|\operatorname{expm}(L_1 + L_2) - \operatorname{expm}(L_1)\operatorname{expm}(L_2)\|_2 \leq \frac{1}{2}\|[L_1, L_2]\|_2 e^{\|L_1\|_2 + \|L_2\|_2}, \tag{4.22}$$

where $[L_1, L_2] := L_1 L_2 - L_2 L_1$ is the commutator of the matrices L_1 and L_2. Note that $[L_1, L_2] = 0$ if L_1 and L_2 commute.

We apply (4.22) to the matrices $L_1 = \sqrt{\Delta} M_1$ and $L_2 = \Delta M_2$, where

$$M_1 = \sum_{k=1}^{l} A^{-1}C^k \frac{(\Delta Z^k_{t_i})}{\sqrt{\Delta}} - \frac{1}{2}\sum_{k=1}^{l}(A^{-1}C^k)^2\sqrt{\Delta}, \tag{4.23}$$
$$M_2 = A^{-1}B^T. \tag{4.24}$$

and obtain

$$\|\operatorname{expm}(L_1 + L_2) - \operatorname{expm}(L_1)\operatorname{expm}(L_2)\|_2 \leq \frac{1}{2}\Delta^{3/2}\|[M_1, M_2]\|_2 e^{\|L_1\|_2 + \|L_2\|_2}. \tag{4.25}$$

Note that $\frac{(\Delta Z^k_{t_i})}{\sqrt{\Delta}} \sim N(0,1)$ under the measure $\mathbb{Q}^*$ and hence[1] $\|[M_1, M_2]\|_2 = \mathcal{O}(1)$ for $\Delta \to 0$. Moreover, $\|L_1\|_2 = \mathcal{O}(1)$ and $\|L_2\|_2 = \mathcal{O}(1)$ for $\Delta \to 0$. Therefore, the order of convergence of the error (4.25) for $\Delta \to 0$ is $\Delta^{3/2}$ and splitting-Up method and matrix-exponential method will give very similar results provided Δ is small.

[1] $f = \mathcal{O}(g)$ means that $\limsup_{x \to 0}\left|\frac{f(x)}{g(x)}\right| < \infty$

4.4 Benchmark - Kalman Filter

The simulation algorithms from Section 4.3 will be tested in a more simple situation, where we assume that

i) no dividends are paid, i.e. $\mu = 0$,

ii) a is a one-dimensional function with $a(x) = c\log(x)$,

iii) no default takes place, i.e. we consider the limiting case $K = 0$,

iv) $V_0 \sim LogN(a, b)$.

Under these assumptions the original filtering problem (3.12) consists of the computation of

$$\mathbb{E}^{\mathbb{Q}}\big(f(V_t)\big|\mathcal{F}_t^Z\big) \tag{4.26}$$

for functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f(V_t)$ is integrable. Here, V and Z are the processes defined in (3.4) and (3.7), so they solve the SDEs

$$dV_t = rV_t dt + \sigma V_t dB_t,\ V_0 \sim LogN(a, b), \tag{4.27}$$
$$dZ_t = a(V_t)dt + dW_t,\ Z_0 = 0. \tag{4.28}$$

In order to test the simulation algorithms from Section 4.3 we solve the SPDE for u associated to the filtering problem (4.26) numerically and compare the results with the correct solution. The correct solution can be obtained from the following filtering problem:

$$\mathbb{E}^{\mathbb{Q}}\Big(f(\widetilde{V}_t)\Big|\mathcal{F}_t^{\widetilde{Z}}\Big), \tag{4.29}$$

where $\widetilde{V}$ and $\widetilde{Z}$ are defined by

$$\begin{aligned} d\widetilde{V}_t &= \sigma dB_t,\ \widetilde{V}_t = \log(V_0), \\ d\widetilde{Z}_t &= c\widetilde{V}_t dt + dW_t,\ \widetilde{Z}_0 = 0. \end{aligned}$$

Note that the conditional density of $\widetilde{V}_t$ given $\mathcal{F}_t^{\widetilde{Z}}$, which provides a solution to (4.29), can be obtained by using a Kalman filter, see for example [65]. In the following we show how to recover the solution of (4.26) from the Kalman filter. Note that

$$d\widetilde{Z}_t = c\log(V_0)dt + c\sigma B_t dt + dW_t.$$

On the other hand, by plugging the solution of (4.27) into (4.28) we have

$$\begin{aligned} dZ(t) &= c\bigg(\log(V_0) + \Big(r - \frac{1}{2}\sigma^2\Big)t + \sigma dB_t\bigg)dt + dW_t \\ &= c\log(V_0)dt + c\Big(r - \frac{1}{2}\sigma^2\Big)tdt + c\sigma B_t dt + dW_t. \end{aligned}$$

Hence, Z and $\widetilde{Z}$ only differ by a deterministic part and replacing Z by $\widetilde{Z}$ does not change the filtering problem. Therefore, for $v \geq 0$ we get

$$\begin{aligned}\mathbb{Q}\big(V_t \leq v \big| \mathcal{F}_t^Z\big) &= \mathbb{Q}\Big(V_t \leq v \Big| \mathcal{F}_t^{\widetilde{Z}}\Big) = \mathbb{Q}\Big(V_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t} \leq v \Big| \mathcal{F}_t^{\widetilde{Z}}\Big) \\ &= \mathbb{Q}\bigg(\log(V_0) + \sigma B_t \leq -(r - \frac{1}{2}\sigma^2)t + \log(v) \bigg| \mathcal{F}_t^{\widetilde{Z}}\bigg) \\ &= \mathbb{Q}\bigg(\widetilde{V}_t \leq -\Big(r - \frac{1}{2}\sigma^2\Big)t + \log(v) \bigg| \mathcal{F}_t^{\widetilde{Z}}\bigg). \end{aligned} \tag{4.30}$$

By differentiating (4.30) we obtain a relationship between the conditional densities of V_t and $\widetilde{V}_t$

$$f_{V_t|Z_t}(v) = f_{\widetilde{V}_t|\widetilde{Z}_t}\bigg(-\Big(r - \frac{1}{2}\sigma^2\Big)t + \log(v)\bigg)\frac{1}{v}. \tag{4.31}$$

With the conditional density (4.31) the conditional expectation (4.26) can be computed easily.

4.5 Numerical Case Study

Finally, we present a numerical case study, where we tested the different algorithms from Section 4.3 against the Kalman filter from Section 4.4. At this, we work within the simplified framework from Section 4.4. Moreover, we choose $l = 1$, $r = 0.02$, $\sigma = 0.2$ and $T = 10$ and investigate the effectivity of the algorithms for $\Delta \in \{0.001, 0.005\}$ and $c \in \{1, 2\}$.

4.5.1 Choice of the Basis Functions

For the Galerkin approximation we use 32 basis functions. The basis functions are plotted in (4.1). The considerably hugh number of basis functions makes sure that the class of functions $f : [K, \infty) \to \mathbb{R}$ which can be approximated well, is big enough. However, it is possible to reduce the number of basis functions by using more sophisticated methods like adapted schemes (see [40] fur further details).

In order to ensure that u can be approximated well in scenarios where $\widehat{V}$ is close to the default boundary K, we define the first basis function e_1 as a displaced density of a gamma distribution with location and scale parameter 2 and 0.6, i.e

$$e_1(x) := f_{\Gamma(2;\,0.6)}(x - K).$$

All other basis functions are truncated densities of displaced normal distributions, i.e. for $i = 2, \ldots, 32$ we define e_i by

$$e_i(x) := 1_{\{K<x\}} C_i^{-1} f_{N(\mu_i,\sigma_i^2)}(x - K),$$

where $f_{N(\mu_i,\sigma_i^2)}$ represents the density of a normal distribution with mean μ_i and variance σ_i^2. $C_i := \int_K^\infty f_{N(\mu_i,\sigma_i^2)}(x - K)\,dx$ denotes a normalization constant to obtain a density. We choose $\mu_i := K - 1.5 + 3.0\,i$ for $i = 2, \ldots, 32$ and $\sigma_i := 1.8$ for all i. Due to the truncation the density are not differentiable at K. However, the choice of the means μ_i and variances σ_i^2 implies that $C_i \approx 1$ for $i = 1, \ldots, n$. Hence, for practical purposes the densities $(e_i)_{i=1}^{32}$ can be considered as $H^2(S^X)$ functions.

In order to study the influence of the choice of the basis functions we will use a second set of basis functions:

- e_1 is again Gamma distributed with location and scale parameters 2 and 0.6,
- e_i is truncated normal for $i = 2, .., 17$ with $\mu_i := K + 4.5 + 6.0(i - 2)$ and $\sigma_i := 2 + \frac{1}{5}\sqrt{2(i-2)}$.

4.5.2 Results of the Numerical Case Study

Section 4.6 contains various graphics and results. The Figures 4.3(a) - 4.3(e) show $\widetilde{\Psi}_T^m$ for the simulation algorithms from 4.3. The Figures 4.4(a) - 4.7(d) show the paths of V and $\widehat{V}$ different simulation of algorithms of $\widetilde{\Psi}^m$. Since Δ was chosen relatively small, the matrix exponential method and the splitting-up method nearly produced the same results, what was already predicted by the error estimate (4.25). It turned out that the matrix-exponential and the splitting-up method performed better than the Euler-Maruyama and the Milstein method in terms of stability. While the matrix-exponential method and the split-up method never showed numerical instabilities, Euler-Maruyama and the Milstein method tend to produce these instabilities when Δ and c are rather large (for example if $\Delta = 0.005$ and $c = 2$). To this regard, eye-catching are the Pictures 4.3(d) and 4.6(a), which show an instability the Euler-Maruyama method. During our numerical analysis the Milstein scheme also showed these instabilities. The results from [40] support our findings. However, it remains unclear how to choose Δ precisely to guarantee the stability of the Euler and the Milstein method apriori. This is one of the main arguments for the use of the advanced methods.

Additionally, our numerical analysis shows that matrix-exponential respectively the splitting-up method provide lower differences between the density from the Galerkin approximation and the Kalman filter. While there are not much differences in the approximation errors between the different algorithms for $\Delta = 0.001$ (see 4.3(a) - 4.3(b)) and all methods provide good results, substantial differences arise for $\Delta = 0.005$ (see 4.3(c) and 4.3(d)). Here, the matrix-exponential-method respectively the splitting-up method performs better than Milstein-method. The good performance of the matrix-exponential-method compared to Euler-Maruyama schemes is also found in [69]. This furthermore underlines the usefulness of the matrix-exponential-method and the splitting-up method.

From the point of speed Euler-Maruyama, the Milstein and the splitting-up method are fine. By far the slowest algorithm is the matrix-exponential method. In case of the Euler-Maruyama and the Milstein method this is mainly due to their simplicity. In case of the splitting-up method we precomputed the eigenvalue-decomposition of the matrices $A^{-1}B^T$ and $A^{-1}C^1$ and used the method in 4.3.3 to compute matrix exponentials in terms of the eigenvalue decomposition.

Finally, we discuss the influence of the choice of basis functions. 4.7 gives the Galerkin approximations for the second set of basis functions. It shows the superior performance of the matrixexponential and the splitting-up method compared to the other methods. However, in 4.7 the filter densities 'fluctuate' around the Kalman filter. Basically this is due to the smaller number of basis functions. Hence, depending on the desired level of approximation one has to choose the number of basis functions.

All in all this shows that due to their stability the matrix-exponential method and the splitting-up method are better suited than ordinary schemes like Euler-Maruyama or Milstein schemes. In our case the splitting-up method outperforms the matrix-exponential method in case of computation time.

4.6 Pictures

Figure 4.1: Basis Functions

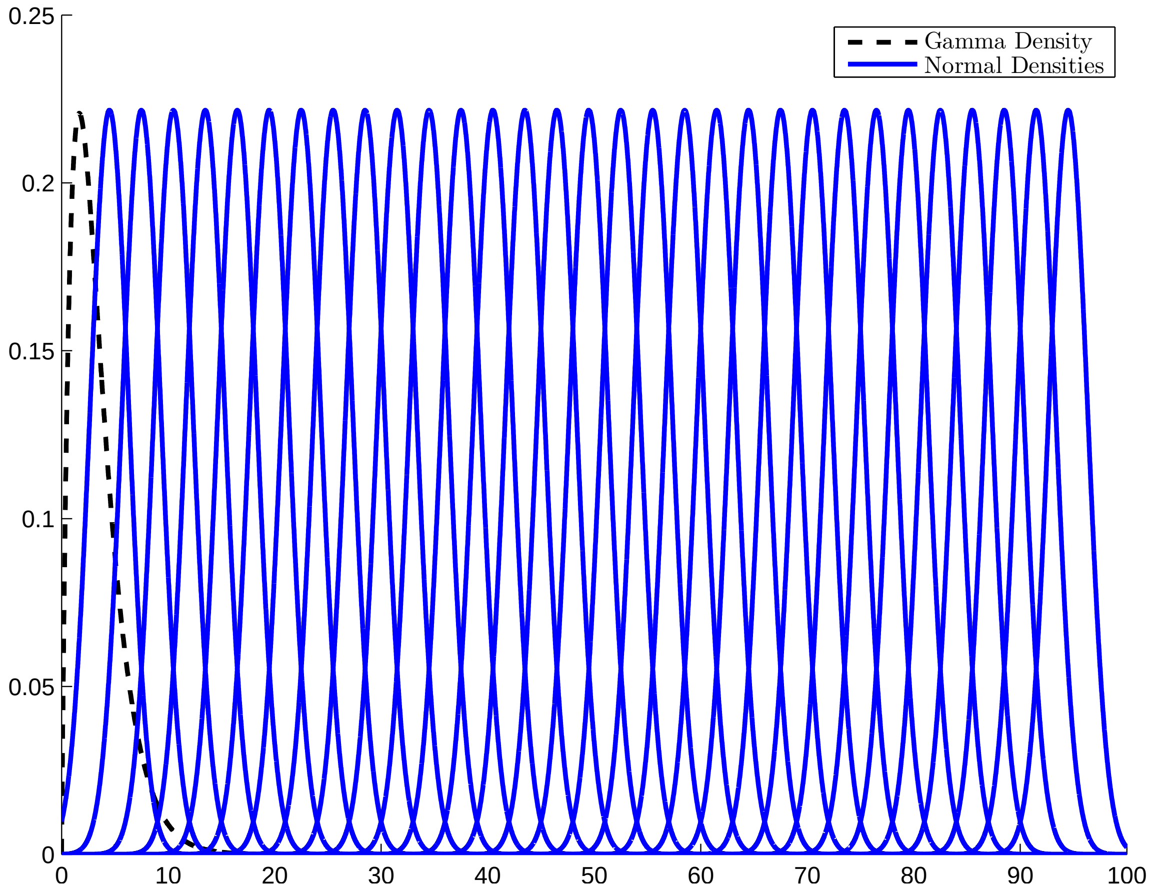
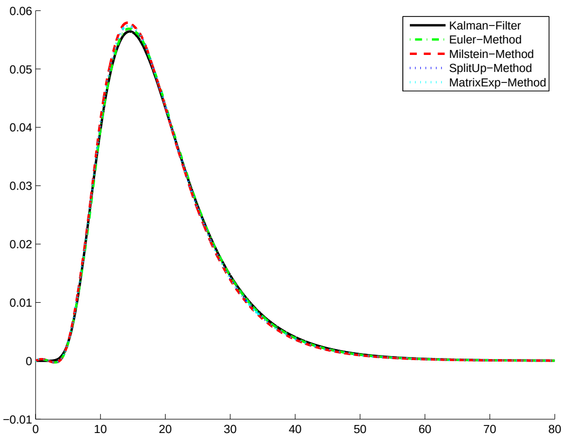
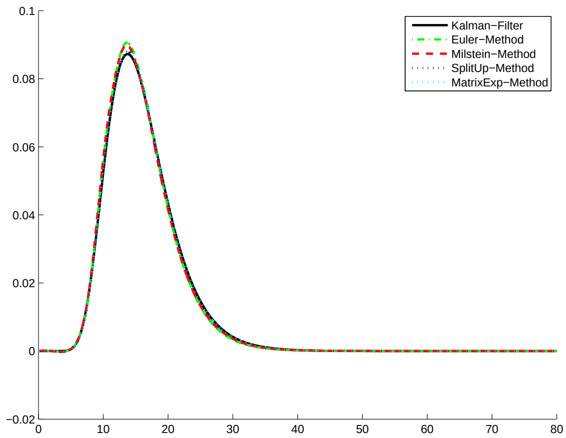


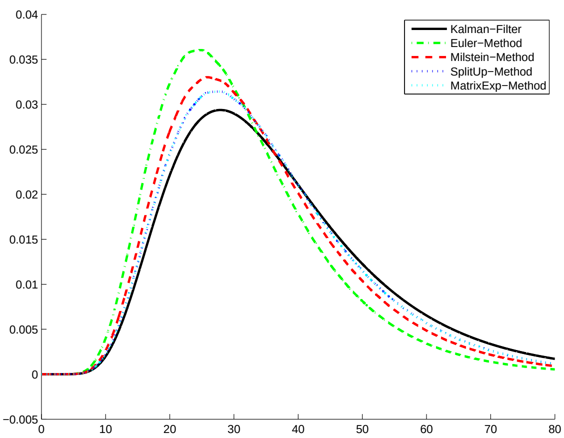
Figure 4.2: Conditional Density of π_T for different values of Δ and c



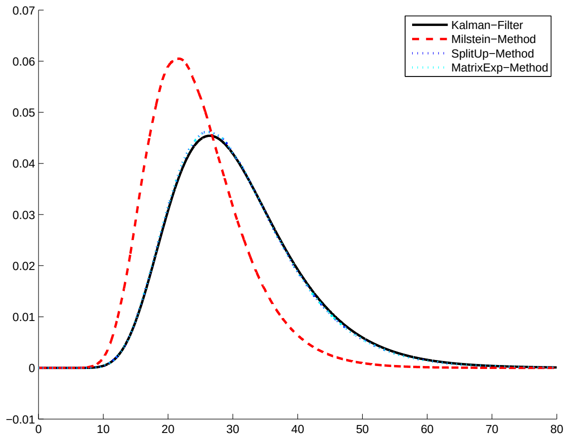
(a) $\Delta = 0.001$ and $c = 1$



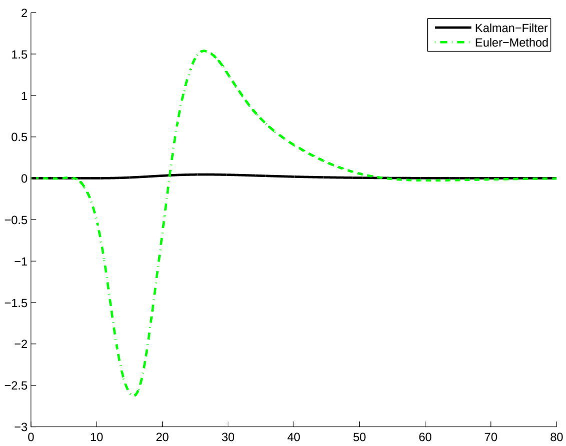
(b) $\Delta = 0.001$ and $c = 2$



(c) $\Delta = 0.005$ and $c = 1$

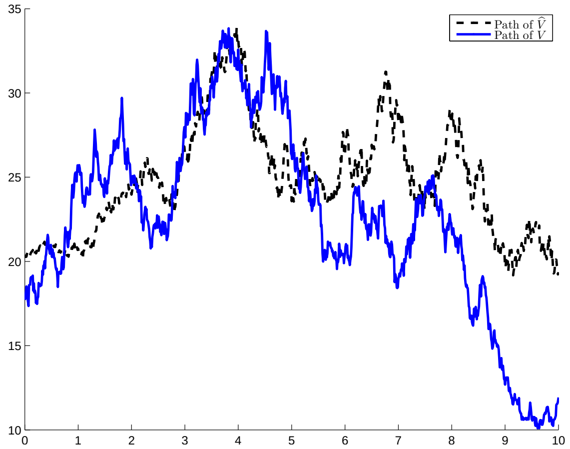


(d) $\Delta = 0.005$ and $c = 2$

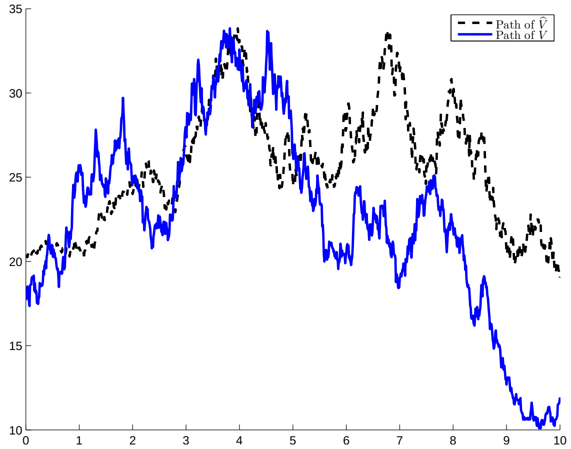


(e) $\Delta = 0.005$ and $c = 2$

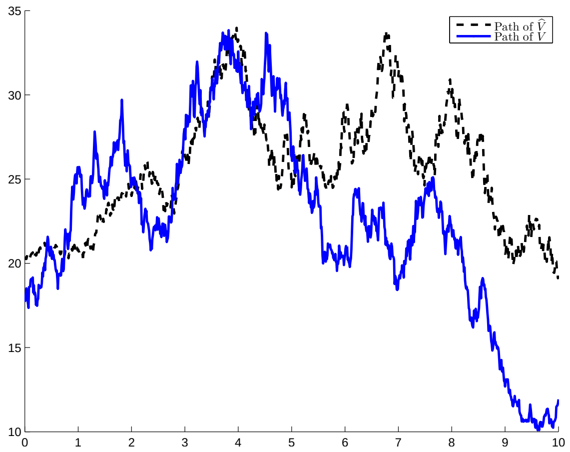
Figure 4.3: Comparison of differenent simulation techniques for $\Delta = 0.001$ and $c = 1$



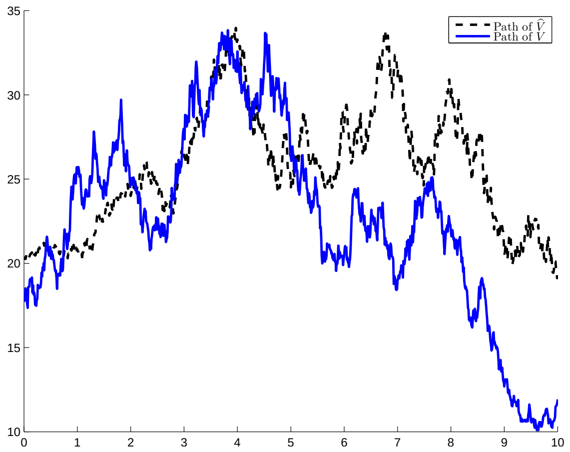
(a) Euler



(b) Milstein

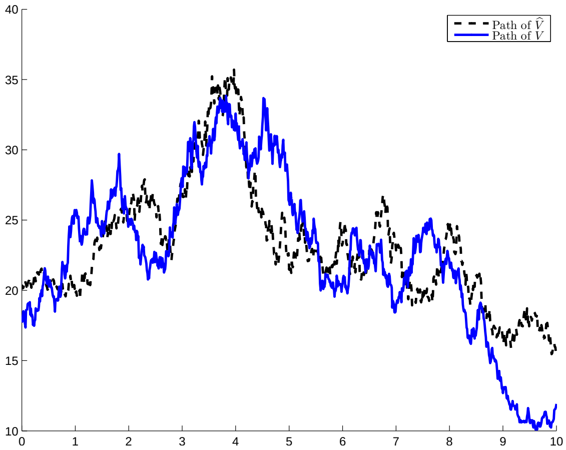


(c) Splitting Up

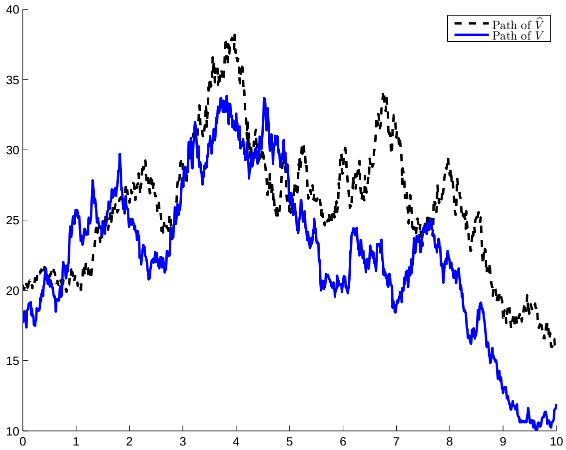


(d) Matrixexponential

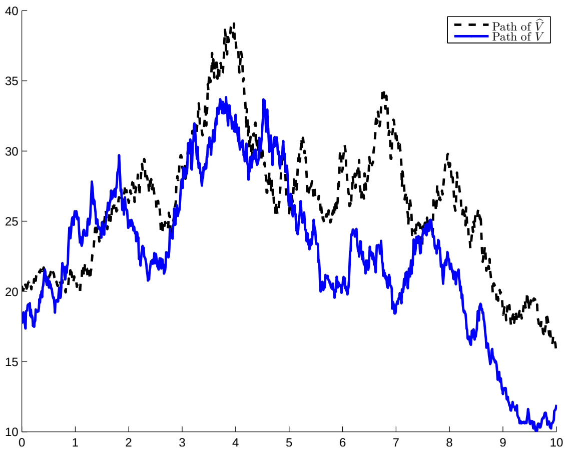
Figure 4.4: Comparison of differenent simulation techniques for $\Delta = 0.001$ and $c = 2$



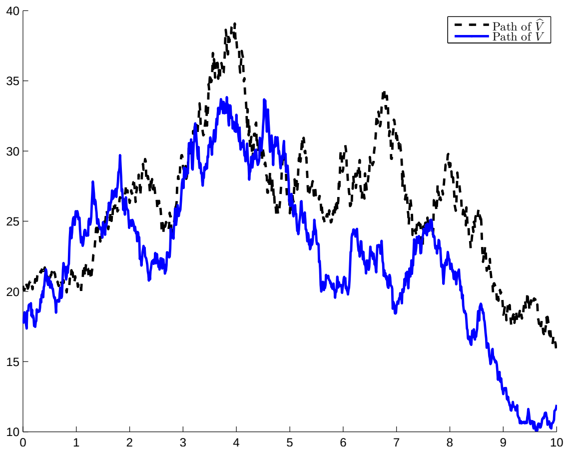
(a) Euler



(b) Milstein

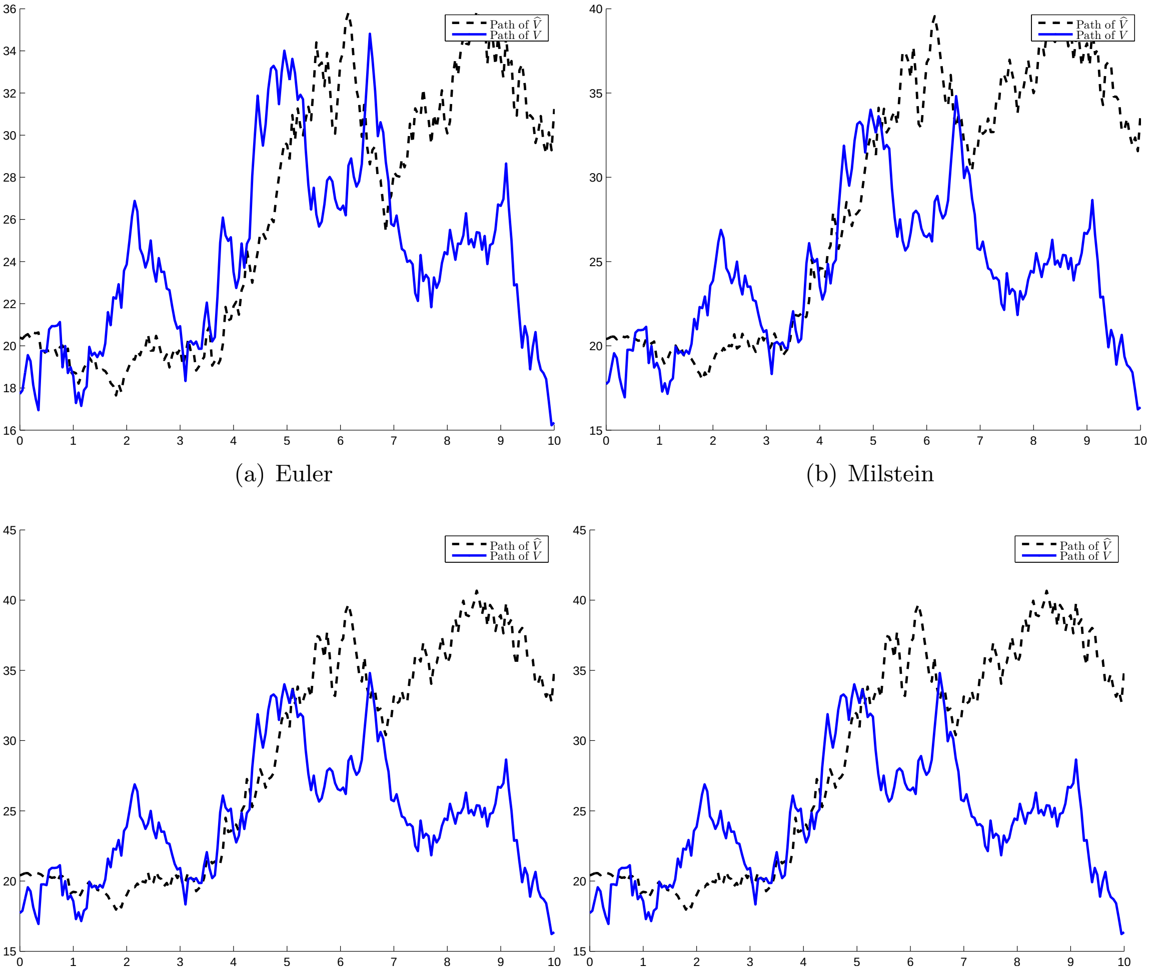


(c) Splitting Up



(d) Matrixexponential

Figure 4.5: Comparison of different simulation techniques for $\Delta = 0.005$ and $c = 1$



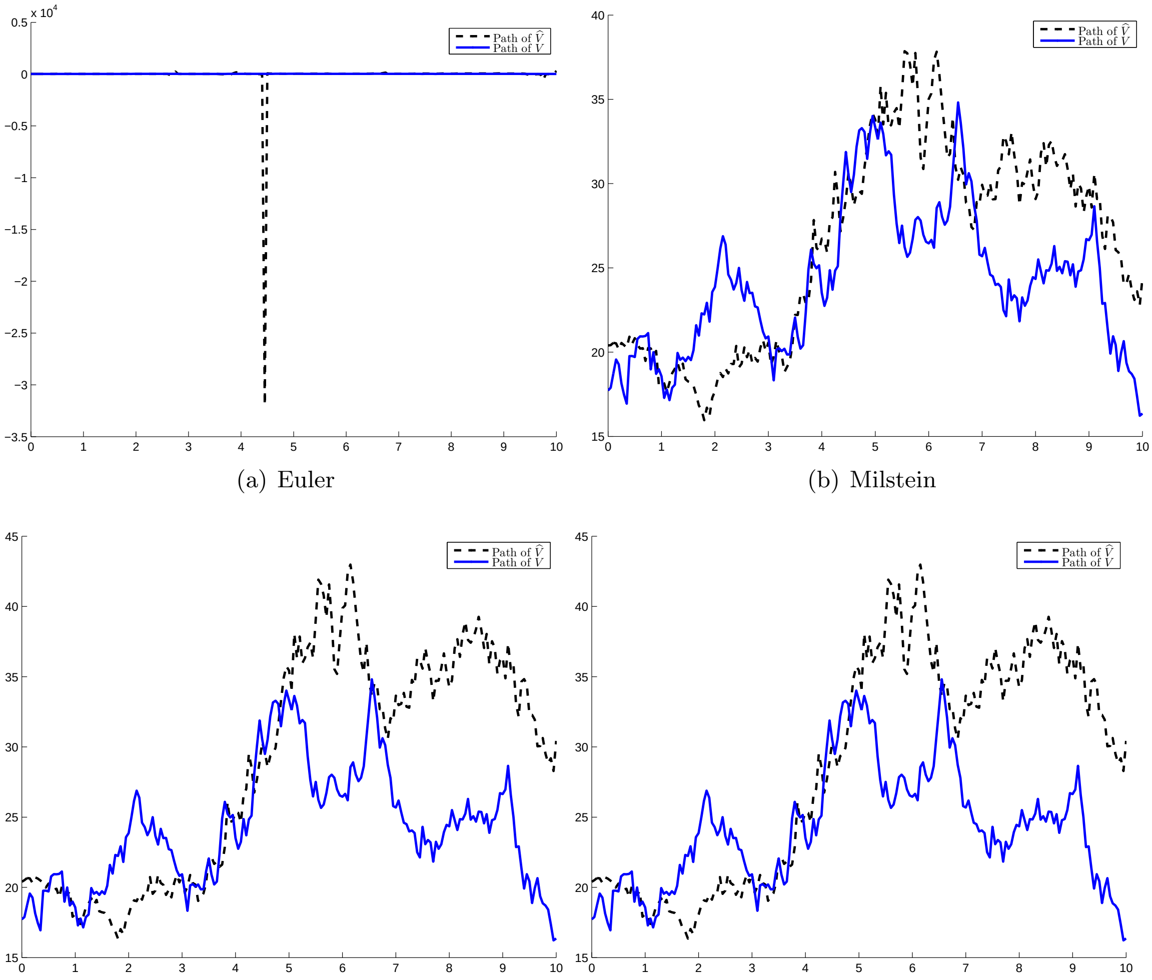
(a) Euler

(b) Milstein

(c) Splitting Up

(d) Matrixexponential

Figure 4.6: Comparison of different simulation techniques for $\Delta = 0.005$ and $c = 2$



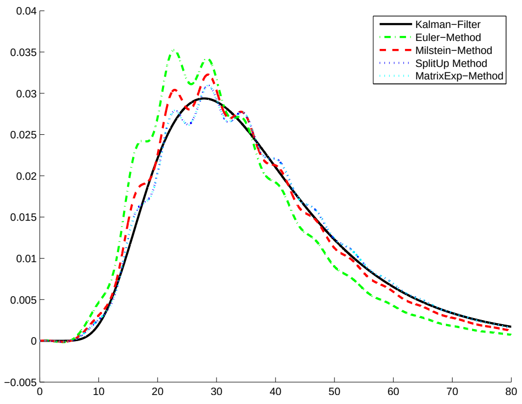
(a) Euler

(b) Milstein

(c) Splitting Up

(d) Matrixexponential

Figure 4.7: Impact of the choice of the basis functions



Chapter 5

Pricing of Contingent Convertibles Notes

5.1 Introduction

The financial crises highlighted the too-big-too-fail problem for huge banks. The consequences of a default of a large bank were considered as so severe for the financial system that most of the governments decided to rescue endangered banks instead of letting them default. At this, debt holders have not been involved in the rescue programs, e.g. by waiving the coupon payments, because their participation would have meant a default. So the problem arose that debt holders profited from the financial aid without participating in the rescue programs. The issuance of contingent capital notes by banks is often suggested to weaken this problem. Contingent capital notes, also known as CoCos, are corporate bonds which are equipped with a conversion mechanism. The conversion mechanism is designed with the intention to strengthen the equity capital of the issuer when he enters into financial distress. In order to achieve this, the face value of the CoCo can be written down or the CoCo can be converted to an amount of equity. The conversion gets triggered when predetermined conditions are fulfilled[1]. Hence, depending on the specified conditions one obtains different types of conversion triggers. The following types of conversion triggers are usually considered:

- Accounting trigger: These triggers are based on capital adequacy ratios, e.g. the Core Tier One Capital Ratio (short CT1R). Roughly speaking, the CT1R is the ratio of the equity of the bank and its risk weighted assets. The conversion takes place as soon as the ratio drops below a given threshold.

- Market trigger: The bond is converted as soon as a quantity observable at a financial market falls below a given threshold. Herefore, usually the share price of the issuing company is used.

- Regulatory trigger: A regulatory authority like a central bank or a federal supervisory authority decides when the conversion takes place.

- Combinations of the triggers mentioned above can be used.

[1] Therefore, convertible bonds and CoCos have to be distinguished.

For an overview of recently issued CoCos and their triggers we refer to [47] and [28]. On the basis of the list in [47] it can be seen that combinations of accounting and regulatory triggers have prevailed. The main reason for the sparse use of market triggers lies in the fear that the market quantity may not reflect the state of the bank correctly. In addition, there is concern that the trigger is intentionally activated or that the activation is prevented by market manipulations. On the other side, accounting triggers have the disadvantage that they depend on complex accounting numbers like the CT1R, so that it is difficult to accurately estimate the probabilities of conversion. Even more difficult is the determination of the influence of a regulatory trigger. Therefore, most of the pricing approaches and also the approach used here do not take into account regulatory trigger. On the other hand, market triggers are much easier to handle from a valuation perspective.

Note that the pricing of CoCos has generated a lot of interest in the last years. For an introduction and a discussion of some general approaches see [75]. Among others, the equity approach for the pricing of CoCos is presented. In this approach the CoCo is priced as an equity derivative, which requires that the cash flow stream of the CoCo is adapted to the filtration generated by the share price of the company. This implies that the CoCo is equipped with a market trigger. Moreover, papers using the equity approach usually assume implicitly that the conversion *always* takes place before the default of the company. The equity approach was used in [28, 62, 76]. Structural credit risk models are used for instance in [17, 21, 49]. These papers use accounting trigger based on capital adequacy ratios. [17] and [21] explicitly model the CT1R. However, they assume that the modelled capital adequacy ratios can be observed continuously. This is in strong contrast to reality, in which the capital adequacy ratios like the CT1R are published at most quarterly.

Another approach based on classical techniques from credit risk modelling can be found in [25]. [25] use an intensity framework and model the conversion time respectively the default time as the first respectively the second jump time of a Cox process. Worth mentioning is also [61], where theory from conic finance is used. In this chapter we want to study the pricing of CoCos with different features in the structural credit risk from Chapter 3. In particular, the model distinguishes itself from other credit risk models which have been used for the pricing of CoCos in two major points. Firstly, it incorporates incomplete information effects: The asset value V is not perfectly observable, but only noisy observations are available. Secondly, our model differs from other pricing approaches, because we allow for the possibility of a default before the conversion. As a consequence CoCos are subject to credit risk in our model and the equity approach can not be used. In contrast to many other approaches our model allows for the pricing of various types of CoCos: It is capable of covering accounting and market triggers on the one hand and write-down and conversion-to-shares features on the other hand. In contrast to [17] and [21] we present an accounting trigger, which is based on discrete-time observations of the CT1R and discuss the consequences of assuming discrete-time observations.

This chapter is structured in the following way. The cash flows of a CoCo are described in detail in Section 5.2. Section 5.3 introduces different trigger definitions. Section 5.4 goes on by discussing the pricing and Section 5.5 by presenting the calibration of the model. Finally, the chapter closes with a pricing and sensitivity study of a CoCo issued by Deutsche Bank in 2014.

5.2 Cash Flows in a Contingent Capital Bond

We denote the cumulative sum of the cash flows from time t up to its maturity T by Π_t. Since the cash flows depend on the trigger and the conversion type we will make use of superscripts to distinguish them. Common to all CoCos considered here is the payment of coupons $c_1, \dots, c_l$, due at the coupon dates $t_1, \dots, t_l$ and the notional N at the maturity T, assuming a default or conversion did not occur. Since CoCos are subordinated bonds, we assume that neither coupons nor the notional are paid after a default, regardless of whether a conversion took place or not. Moreover, we assume that a conversion of the CoCo is not possible *after* a default. Hence, in this case we set the conversion time θ equal to ∞. If the trigger is activated *before* the default, then a conversion of the CoCo takes place. Two types of conversion mechanisms are mainly used: a write-down conversion and a conversion of the bond to an amount of shares. If a write-down conversion is used, then at the trigger event the nominal of the CoCo is written down, possibly even written-off, and only a fraction $\omega \in [0,1)$ of the originally intended coupons and nominal will be paid. Hence, ω can be considered as some type of recovery rate and accordingly $1-\omega$ as a loss given conversion. The cumulative cash flows of a CoCo with write-down feature are denoted by Π^W and for $t \in [0,\infty)$ we define:

$$\Pi_t^W := \sum_{i=1}^{l} 1_{\{t_i \leq t\}} 1_{\{t_i < \tau\}} c_i \big(1_{\{t_i<\theta\}} + \omega\, 1_{\{\theta < t_i\}}\big) + 1_{\{T \leq t\}} 1_{\{T<\tau\}} N \big(1_{\{T<\theta\}} + \omega\, 1_{\{\theta<T\}}\big). \tag{5.1}$$

The second mechanism is the conversion of the bond to shares. The number of shares C_r which are obtained per bond is called conversion ratio and is usually defined in terms of the conversion price C_p by the following relationship

$$C_r = \frac{N}{C_p}.$$

C_p can be interpreted as a price, which is paid by the holder of the bond per share. An interpretation of this conversion as a recovery payment can be found in [75]. [75] present a rule of thumb, where they neglect coupon payments as well as discounting effects. They approximate the loss L to owners of the CoCo due to the conversion by the difference of the nominal and the value of the received shares:

$$L \approx N - C_r S_\theta = N - \frac{N}{C_p} S_\theta = N\left(1 - \frac{S_\theta}{C_p}\right).$$

Hence, S_θ / C_p can be considered as a recovery rate and $1 - S_\theta/C_p$ as a loss given conversion. Therefore, a low conversion price is advantageous for the bond owner, whereas a high conversion price is advantageous for the original share holders. The conversion price C_p should only depend on information which is available up to time θ, which means that C_p has to be a $\mathcal{F}_\theta^{\mathbb{M}}$-measurable random variable. Hence, the choice of $C_p = S_0$ or $C_p = S_\theta$ comes into consideration. As the share price at θ is rather low, CoCo owners would prefer the second choice. It should be noted that the choice $C_p = S_\theta$ implies that the recovery rate is equal to one respectively that the loss is equal to zero. Finally, we define the cumulative cash flows Π^S of a CoCo with a conversion to shares for $t \in [0,\infty)$ by

$$\Pi_t^S = \sum_{i=1}^{l} c_i 1_{\{t_i \leq t\}} 1_{\{t_i < \min(\theta,\tau)\}} + 1_{\{T \leq t\}} N 1_{\{T < \min(\theta,\tau)\}} + 1_{\{\theta < t\}} 1_{\{\theta<\tau\}} C_r S_\theta. \tag{5.2}$$

5.3 Trigger Mechanisms

In this section we give different definitions of the conversion time θ of the CoCo. We discuss an accounting trigger and a market trigger. If θ is given by an accounting trigger, then we use the notion θ_{acc}. If θ is given by a market trigger, then we use the notion θ_{ma}.

5.3.1 The Accounting Trigger

The accounting trigger we consider is based on the Core Tier One Ratio, shortly CT1R, which is used as the core measure of a bank's financial strength from a regulator's point of view. It is defined as the quotient of the core tier one capital and the risk-weighted assets of the bank. The CT1R is usually announced quarterly and hence, we introduce the set $M := \{0.25, 0.5, 0.75, 1, ...\}$. For $t \in M$ let CT1R denote the CT1R announced at time t[2]. For the modelling of the core tier one ratio we use various approximations. At first we approximate the core tier one capital by the equity of the company, which is given by $V_t - K$ in our model. Moreover, the risk-weighted assets are modelled by a multiple βV_t of the asset value with a constant $\beta > 0$. We account for these approximations by defining the CT1R in terms of a regression model for its logarithm. Accordingly, we use the logarithm of the ratio of the equity and the risk weighted assets as the explanatory variable. Working with logarithmic values ensures that the *CT1R* is always positive. Moreover, we assume that the CT1R is zero after a default of the bank. Finally, for $t < \tau$ we obtain that

$$\log(CT1R_t) := \tilde{c} + d \log\left(\frac{V_t - K}{\beta V_t}\right) + \kappa_t, \tag{5.3}$$

with constants $\tilde{c} \in \mathbb{R}$, $d > 0$ and a sequence of i.i.d. noise variables $(\kappa_t)_{t \in M}$ with mean 0 and variance σ_κ^2. The restriction $d > 0$ is made to ensure that the $CT1R$ is indeed a measure of the financial strength of the bank in our model. By definition of the default time lower asset values mean increasing default probabilities. It is therefore reasonable to make sure that the CT1R is an increasing function in V. To see that the restriction $d > 0$ implies this, note that Equation (5.3) can be simplified by using the variable $c := \tilde{c} - d \log(\beta)$:

$$\log(CT1R_t) = c + d \log\left(1 - \frac{K}{V_t}\right) + \kappa_t, t < \tau. \tag{5.4}$$

From this formulation, it is also apparent that the $CT1R$ depends only on the ratio of V and K, which will play a role in the calibration of the underlying model. The parameters c and d in the Definition (5.4) are usually estimated from balance sheet data. For this purpose, least-squares respectively restricted least-squares methods are appropriate. In Section 5.6.1 we estimate the parameters c and d from balance sheet data from the Deutsche Bank.

Definition of the Accounting Trigger based on the CT1R.
As the CT1R is a measure for the financial strength of the bank, it is reasonable to assume that the conversion gets triggered as soon as the CT1R drops below a threshold $K_{acc} > 0$ and introduce a preliminary version of the conversion time:

$$\widetilde{\theta}_{acc} := \inf\{t \in M : CT1R_t \leq K_{acc}\}. \tag{5.5}$$

[2] Note that most of the publications dealing with the pricing of CoCos with accounting trigger assume that the CT1R is continuously observable.

It is possible that a default of the bank occurs before the CT1R drops below the threshold K_{acc}. For instance due to a strong decrease of the asset value between two observation times of the CT1R. Therefore, we modify $\widetilde{\theta}_{acc}$ to ensure that a conversion never takes place after a default.

$$\theta_{acc} := \begin{cases} \tilde{\theta}_{acc} & \text{if } \widetilde{\theta}_{acc} < \tau, \\ \infty & \text{if } \widetilde{\theta}_{acc} \geq \tau. \end{cases} \tag{5.6}$$

If the trigger would have been based on continuous-time observations of the CT1R, then a default without an activation of the trigger would not be possible. To see this, note that the following relationship holds

$$CT1R_t \leq K_{acc} \Leftrightarrow V_t \leq \left(1 - \exp\left(\frac{\log(K_{acc}) - c - \kappa_t}{d}\right)\right)^{-1} K,$$

which implies that

$$V_t \leq K \Rightarrow CT1R_t < K_{acc}.$$

5.3.2 The Market Trigger

The second class of trigger which we consider are market trigger. Basically, the market trigger we will consider is defined as the first time the incomplete information stock price S drops below a threshold $K_{ma} > 0$. Hence, we introduce the stopping time $\widetilde{\theta}_{ma}$:

$$\widetilde{\theta}_{ma} := \inf\{t \in [0, \infty) : S_t \leq K_{ma}\}.$$

In contrast to the full information function h_{eq} there is no 1-to-1 relationship between the stock price and the asset value in the incomplete information model. Therefore it is possible that the bank defaults before the trigger gets activated. and introduce the market trigger θ_{ma} as a modification of $\widetilde{\theta}_{ma}$, which ensures that a conversion never takes place after a default:

$$\theta_{ma} := \begin{cases} \widetilde{\theta}_{ma} & \text{if } \widetilde{\theta}_{ma} < \tau, \\ \infty & \text{if } \widetilde{\theta}_{ma} \geq \tau. \end{cases} \tag{5.7}$$

5.3.3 Comparison of Accounting and Market Trigger

In general it is difficult to compare the accounting trigger θ_{acc} and the market trigger θ_{ma}, because of the presence of the noise variables κ_t and the assumption that the $CT1R$ is not continuously observable. Therefore, we consider a slightly different definition of the original accounting trigger (5.6), which we denote by θ_{ass}. Here, we substitute the set M by $\mathbb{R}_0^+$ and assume that the noise variables κ_t are equal to zero:

$$\theta_{ass} := \inf\left\{t \in [0, \infty) : \exp\left(c + d \log\left(\frac{V_t - K}{V_t}\right)\right) \leq K_{acc}\right\}. \tag{5.8}$$

Rearranging (5.8) leads to

$$\theta_{ass} = \inf\left\{t \in [0, \infty) : V_t \leq \left(1 - \exp\left(\frac{\log(K_{acc}) - c}{d}\right)\right)^{-1} K\right\}. \tag{5.9}$$

Therefore, θ_{ass} may be referred to as an asset trigger with threshold given by $K_{ass} := \left(1 - \exp\left(\frac{\log(K_{acc})-c}{d}\right)\right)^{-1} K$. Moreover, Equation (5.9) shows that a default can not happen before the asset trigger has been activated. The following proposition gives a relationship between an asset trigger and a market trigger with threshold $K_{ma} := h^{eq}(K_{ass})$. Here, h^{eq} denotes the full-information equity function from Section 3.3.2.

Proposition 5.10. *Let $T > 0$ denote the maturity of the CoCo. On the set $\{\theta_{ma} \leq T\}$ the market trigger is conservative in the sense that the average asset value at conversion is higher than K_{ass}, precisely we have*

$$\mathbb{E}^{\mathbb{Q}}(V_{\theta_{ma}}|\theta_{ma} \leq T) > K_{ass}. \tag{5.11}$$

Proof. Essentially the proof is based on the concavity of $h_{eq}(\cdot)$ and on Jensen's inequality. We now give the details. At first note that $\theta_{ma} \leq T$ implies that the conversion was triggered before the default. As $S_{\theta_{ma}} = K_{ma}$ on $\{\theta_{ma} \leq T\}$, we have

$$K_{ma} = \frac{\mathbb{E}^{\mathbb{Q}}\big(S_{\theta_{ma}} 1_{\{\theta_{ma} \leq T\}}\big)}{\mathbb{Q}(\theta_{ma} \leq T)} \overset{(*)}{=} \frac{\mathbb{E}^{\mathbb{Q}}\big(h_{eq}(V_{\theta_{ma}}) 1_{\{\theta_{ma} \leq T\}}\big)}{\mathbb{Q}(\theta_{ma} \leq T)} = \mathbb{E}^{\mathbb{Q}}(h_{eq}(V_{\theta_{ma}})|\theta_{ma} \leq T),$$

where the equality $(*)$ follows from the defining equation of optional projections. Since $h_{eq}(\cdot)$ is strictly concave on $[K, \infty)$ and the rv $V_{\theta_{ma}}$ takes values in that interval given that $\theta_{ma} \leq T$, Jensen's inequality gives

$$\mathbb{E}^{\mathbb{Q}}(h_{eq}(V_{\theta_{ma}})|\theta_{ma} \leq T) < h_{eq}(\mathbb{E}^{\mathbb{Q}}(V_{\theta_{ma}}|\theta_{ma} \leq T)).$$

Since $h_{eq}(\cdot)$ is strictly increasing on $[K, \infty)$ we get

$$K_{ass} = h_{eq}^{-1}(K_{ma}) < \mathbb{E}^{\mathbb{Q}}(V_{\theta_{ma}}|\theta_{ma} \leq T).$$

□

5.4 Pricing of CoCos

We define the price p_t of a CoCo at time t as the expected value of all future discounted payments, namely

$$p_t := \mathbb{E}^{\mathbb{Q}}\left(\int_t^T e^{-r(s-t)}\, d\Pi_s \middle| \mathcal{F}_t^{\mathbb{M}}\right), \tag{5.12}$$

where we assume that for all $t \in [0, \infty)$ the expectation value of $\int_t^T e^{-r(s-t)}\, d\Pi_s$ exists. Here, Π denotes one of the streams of cash flows considered in Section 5.2. The CoCo may be equipped with the accounting trigger (5.6) or with the market trigger (5.7). The conversion can take place in form of a write-down or a conversion to shares.

Remark 5.13. *From a modelling perspective, it would be reasonable to account for the conversion in the capital structure of the bank. For example, one could replace the default threshold K by $K - 1_{\{\theta<t\}}\omega N$ in case of the write-down conversion. However, the debt K of a typical bank is much larger than the nominal N of the CoCo and therefore, the replacement of K will not have a strong impact on the pricing of CoCo. Therefore, for simplicity we neglect the effects of the write-down conversion. For a conversion to shares, we assume that the number of new shares issued through the conversion to shares is relatively small, so that it won't have an impact on the pricing.*

By directly applying (5.12) to the cash flow stream of a CoCo with a write-down feature given by (5.1), we obtain for p_t^W:

$$\begin{aligned} p_t^W &= \sum_{i=1}^{l} 1_{\{t<t_i\}} e^{-r(t_i-t)} c_i \Big(\mathbb{Q}\Big(t_i < \min(\theta,\tau) \Big| \mathcal{F}_t^{\mathbb{M}} \Big) + \omega \mathbb{Q}\Big(\theta < t_i < \tau \Big| \mathcal{F}_t^{\mathbb{M}} \Big) \Big) \\ &\quad + e^{-r(T-t)} N \Big(\mathbb{Q}\Big(T < \min(\theta,\tau) \Big| \mathcal{F}_t^{\mathbb{M}} \Big) + \omega \mathbb{Q}\Big(\theta < T < \tau \Big| \mathcal{F}_t^{\mathbb{M}} \Big) \Big). \end{aligned} \tag{5.14}$$

On the other hand, the price of a CoCo with a conversion to shares with cash flows given by formula (5.2), leads to:

$$\begin{aligned} p_t^S &= \sum_{i=1}^{l} 1_{\{t<t_i\}} e^{-r(t_i-t)} c_i \mathbb{Q}\Big(t_i < \min(\theta,\tau) \Big| \mathcal{F}_t^{\mathbb{M}} \Big) + e^{-r(T-t)} N \mathbb{Q}\Big(T < \min(\theta,\tau) \Big| \mathcal{F}_t^{\mathbb{M}} \Big) \\ &\quad + \mathbb{E}^{\mathbb{Q}}\Big(1_{\{t<\theta<T\wedge\tau\}} e^{-r(\theta-t)} C_r S_\theta \Big| \mathcal{F}_t^{\mathbb{M}} \Big). \end{aligned} \tag{5.15}$$

Apart from the last term in Equation (5.15), the price of a CoCo can be expressed in terms of conditional default and conversion probabilities. Unfortunately, we were not capable of deducing tractable formulas for these conditional conversion probabilities. Hence, a direct evaluation of p^W or p^S seems not to be possible. Hence, it is reasonable to use the pricing approaches for *basic corporate securities* and *options on basic corporate securities* from Chapter 3. For the pricing of the considered CoCos we will rely on Monte-Carlo pricing. Herefore, we use the simulation algorithm 4.1 and the splitting-up method from (4.3.3). Before we discuss the pricing of CoCos in more detail, we discuss some modifications of the model framework, which are necessary if an accounting trigger is used.

5.4.1 Modification of the Filtering Problem due to an Accounting Trigger

When using an accounting trigger of the form (5.4) one has to acknowledge the fact that the CT1R contains additional information about the asset value. Hence, one has to incorporate the filtration $\mathcal{F}^{CT1R}$ which is given by

$$\mathcal{F}_t^{CT1R} := \sigma(CT1R_s : s \in [0,t], s \in M)$$

into the market filtration by setting $\mathbb{F}^{\mathbb{M}} = \mathbb{F}^Z \vee \mathbb{F}^N \vee \mathbb{F}^D \vee \mathbb{F}^{CT1R}$. The incorporation of the CT1R requires an updating of the conditional density π. Because $\mathcal{F}^{CT1R}$ consists of discrete observations at fixed time points M, one can use a Bayesian updating procedure, analogous to the updating due to dividend payments given by formula (3.31). At this, note that for $x > 0$ and $v > K$ the conditional density of $CT1R_t$ given $V_t = v$ is given by

$$\begin{aligned} \mathbb{Q}(CT1R \le x | V_t = v) &= \mathbb{Q}\Big(\kappa_t \le \log(x) - c - d \log\Big(\frac{v-K}{v} \Big) \Big) \\ &= \mathbb{Q}\Big(\kappa_t \le \log(x) - c - d \log\Big(1 - \frac{K}{v} \Big) \Big). \end{aligned} \tag{5.16}$$

Therefore, if κ_t exhibits the density f_{κ_t}, differentiating (5.16) shows that the conditional density of $CT1R_t$ given $V_t = v$ is given by

$$f_{CT1R_t|V_t=v}(x) = \frac{1}{x} f_{\kappa_t}\Big(\log(x) - c - d \log\Big(1 - \frac{K}{v} \Big) \Big).$$

5.4.2 Classification of CoCos

In Chapter 3 the pricing of two classes of securities was discussed, *basic corporate securities* and *options on basic corporate securities*. In the following we want to classify the different types of CoCos into these two classes. Remember that basic corporate securities are securities whose associated cash flow stream H depends on future dividend payments and on the occurrence of a default. Finding the price p_t^H of typical corporate securities of this class is equivalent to solving the filtering problem

$$1_{\{\tau>t\}} p_t^H = 1_{\{\tau>t\}} \mathbb{E}^{\mathbb{Q}}\Big(h(t, V_t)\Big|\mathcal{F}_t^{\mathbb{M}}\Big), \tag{5.17}$$

where the full information value h fulfills

$$1_{\{\tau>t\}} h(t, V_t) = \mathbb{E}^{\mathbb{Q}}\bigg(1_{\{\tau>t\}} \int_t^T e^{-r(s-t)}\, dH_s \bigg| \mathcal{G}_t\bigg).$$

Hence, crucial for the validity of Equation (5.17) was that the pre-default value in the full information model is a function of the current time t and the current asset value V_t. If we consider a CoCo with an accounting trigger and write-down-feature, that is $H = \Pi^{W,acc}$, then its cash flows $\Pi^{W,acc}$ depend apart from the occurrence of a default also on the CT1Rs via the conversion time θ_{acc}. Since the CT1R at time t does not depend on past realisations of the noise variables $(\kappa_s)_{s<t}$, conditional conversion probabilities wrt. $\mathcal{G}$ are functions of time and the asset value only. Therefore, the pre-conversion price $p_t^{W,acc}$ of a CoCo with an accounting trigger and a write-down-feature can be represented by

$$1_{\{\theta_{acc}>t\}} 1_{\{\tau>t\}} p_t^{W,acc} = 1_{\{\theta_{acc}>t\}} 1_{\{\tau>t\}} \mathbb{E}^{\mathbb{Q}}\Big(h_{W,acc}(t, V_t)\Big|\mathcal{F}_t^{\mathbb{M}}\Big),$$

where the full information value function $h_{W,acc}$ fulfils

$$1_{\{\theta_{acc}>t\}} 1_{\{\tau>t\}} h_{W,acc}(t, V_t) = \mathbb{E}^{\mathbb{Q}}\bigg(1_{\{\tau>t\}} 1_{\{\theta_{acc}>t\}} \int_t^T e^{-r(s-t)}\, d\Pi_s^{W,acc} \bigg| \mathcal{G}_t\bigg). \tag{5.18}$$

However, we don't have a closed-formula for $h_{W,acc}$ such as in the case of the share price h_{eq}. Therefore, we will use Monte Carlo pricing to find $p_t^{W,acc}$. Since the cash flows $\Pi^{W,acc}$ only depend on the default time and the conversion time, there is no need to simulate trajectories of the filter density π_t or S. If the CoCo has been equipped with the market trigger or a conversion to shares is used, then the cash flows depend on the stock price S. Since the stock price S depends on the conditional distribution π, these CoCos can not be considered as *basic corporate securities*, but as *options on basic corporate securities*. Remember that the cash flows of *options on basic corporate securities* depend on traded assets. As suggested at the end of Chapter 3 we use the simulation algorithm 4.1 to find the price of the CoCo.

5.5 Calibration

The price of the CoCo depends on various parameters. This raises the inevitable question of model calibration. In the following, we propose a calibration method which fits the model parameters K, λ, σ_V and also the initial density π_0 of V_0. We assume that all other parameters have already been chosen before the start of calibration procedure. Note that the calibration method can be used for all types of CoCos. For calibration, we assume that the share price of the bank and the fair CDS spreads of l liquidly traded CDS are available. Since we work with the fair spreads of the CDS, we will assume that their corresponding market prices are equal to zero.

For the model calibration we suggest the following two-stage procedure. The first step consists of fitting V_0, K, λ and σ_V under the full information model. Note that we assume in this step that V_0 is constant. We will account for the randomness later in the second step. We find those values of the parameters such that model prices from the full information model match market prices. At this, a calibration of V_0 and K is necessary to make sure that the share price is matched. In order to see this, note that the following inequality holds

$$h_{eq}(V_0) = V_0 - K\left(\frac{V_0}{K}\right)^{\alpha} \geq V_0 - K. \tag{5.19}$$

For most banks, the market capitalization is relatively small compared to the book equity value $V_0 - K$. Hence, if we choose V_0 and K according to balance sheet data, for example as the total value of assets and the debt of the bank, then inequality (5.19) is violated.

5.5.1 Calibration of the full information model

The calibration of the full information model is divided into two parts. At first, we choose initial values V^{init} and K^{init} for V_0 and K and use λ and σ to fit the CDS prices. Here, it is possible to choose the total assets and the debt of the bank as initial values. If there is just one liquidly traded CDS available, then one can restrict the problem to fitting σ_V and choose λ at the beginning. In this case, a perfect fit can be obtained. Then, we use the obtained values of λ and σ and fit V_0 and K so that the share price S_0 is matched. Since in this case we have a single nonlinear equation with two unknowns, we add the equation $V_0 = (V^{init}/K^{init})K$. This ensures that the ratios V^{init}/K^{init} and V_0/K coincide and hence, the default probabilities and CDS prices which were obtained in the previous step do not change. Note that this choice also implies that the probabilities that the accounting trigger gets activated will not be influenced by the stock price. We obtain the following two equations

$$S_0 = V_0 - K\left(\frac{V_0}{K}\right)^{\alpha} \text{ and } V_0 = \frac{V^{init}}{K^{init}}K. \tag{5.20}$$

The unique solution of (5.20) is given by

$$V_0 = \frac{S_0}{1 - \left(\frac{V^{init}}{K^{init}}\right)^{\alpha-1}} \text{ and } K = \frac{S_0}{\frac{V^{init}}{K^{init}} - \left(\frac{V^{init}}{K^{init}}\right)^{\alpha}}. \tag{5.21}$$

In summary, we obtain the following algorithm:

1) Choose initial values for V^{init} and K^{init} of V_0 and K.

2) Obtain values for λ and σ by solving the following optimization problem:

$$(\lambda, \sigma_V) = \text{argmin}_{\widetilde{\lambda},\widetilde{\sigma_V}} \sum_{j=1}^{l} \Big(h_j(V^{init}, K^{init}, \widetilde{\lambda}, \widetilde{\sigma_V})\Big)^2. \tag{5.22}$$

Here, $h_1, \dots, h_l$ denote the full information price functions of the CDS, which are given as the difference of the corresponding default and the premium legs[3]. Moreover, the parameter values which entered into the computation of the full information price where given in brackets.

3) Set V_0 and K as

$$V_0 = \frac{S_0}{1 - \left(\frac{V^{init}}{K^{init}}\right)^{\alpha-1}} \text{ and } K = \frac{S_0}{\frac{V^{init}}{K^{init}} - \left(\frac{V^{init}}{K^{init}}\right)^{\alpha}}.$$

5.5.2 Calibration of the incomplete information model

At first, based on V_0 and K we choose appropriate density functions $(e_i)_{i=1,\dots,m} \in H_0^2([K,\infty))$, which will serve as a basis for the Galerkin approximation, which was described in Section 4.1. We restrict the set of possible initial densities π_0 to those which can be expressed as convex combinations of the basis function, e.g. there is a non-negative vector $\psi \in \mathbb{R}^n$ with $\sum_{i=1}^m \psi_i = 1$ and $\pi_0 = \sum_{i=1}^m \psi_i e_i$. A crucial observation in this context is that the prices of the securities are linear functions of π_0. For example, the price of the first CDS is given by

$$(\pi_0, h_1) = \sum_{i=1}^m \psi_i (e_i, h_1).$$

Recall that $(\cdot,\cdot)_{S^X}$ denotes the $L^2(S^X)$ scalar product. Therefore $(e_i, h_k)_{S^X}$ is the price of security k given that e_i is used as initial density for V_0. The following equations ensure that observed market prices3 and model prices match

$$\begin{aligned} &\sum_{i=1}^m \psi_i (e_i, h_0) = S_0 \\ &\sum_{i=1}^m \psi_i (e_i, h_k) = 0 \text{ for } k \in \{1, 2, \dots, l\}, \end{aligned} \tag{5.23}$$

where $h_0 = h_{eq}$ is the equity price function from equation (3.10). Moreover, we add the following equality to ensure that $\int_K^\infty v\pi_0(dv) = V_0$:

$$\sum_{i=1}^m \psi_i (e_i, id) = V_0. \tag{5.24}$$

Here, $id(v) = v$ denotes the identity function. Note that the system of equations (5.23) together with (5.24) typically does not have a unique solution, because $m < l$. A unique solution can be obtained if a suitable regularization procedure is applied. For instance, one can try to find the solution ψ of (5.23) and (5.24) whose associated density π_0 minimizes $\int_K^\infty (\pi_0^{''})^2 \, dv$. Note that

$$\begin{aligned} \int_K^\infty (\pi_0^{''}(v))^2 \, dv &= \int_K^\infty \left(\sum_{i=1}^m \psi_i e_i^{''}(v) \right)^2 dv = \sum_{i=1}^m \sum_{j=1}^m \int_K^\infty \psi_i \psi_j e_i^{''}(v) e_j^{''}(v) \, dv \\ &= \sum_{i=1}^m \sum_{j=1}^m \psi_i \psi_j \left(e_i^{''}(v), e_j^{''}(v) \right) = \psi^T C \psi, \end{aligned}$$

[3]Recall that we assume that the CDSs are traded at the fair spread and hence its price is zero

where C denotes an $m \times m$ matrix with $C_{i,j} = \left(e_i'', e_j''\right)$. Hence, one obtains a quadratic optimization problem for ψ.

5.6 Pricing of the CoCo of the Deutsche Bank

This section consists of a study of the CoCo with ISIN DE000DB7XHP3, which was issued by Deutsche Bank in May 2014. This CoCo is equipped with an accounting trigger, which is activated if the CT1R of the Deutsche Bank falls below 5.125 percent. At the trigger event the notional of the CoCo will be written down. Since the dynamics of the share price do not influence the pricing of this CoCo, we additionally consider a CoCo which is equipped a conversion to shares. Since both types of CoCos are priced within the same model, we only have to calibrate the model once.

CoCo of the Deutsche Bank

Table 5.1 gives some relevant information about the CoCo issued by the Deutsche Bank. For more detailed information consult the term sheet [2]. On closer examination of the term sheet, it is found that more features are included in the contract. However, in order to reduce the complexity of the pricing we neglect these features. The specification we use can be found in the 'Features'-column of the Table 5.1. Moreover, we set ω equal to zero

Property	Term Sheet	Features used in Valuation
Maturity	no Maturity	30.04.2022
Call Dates for DB	annually, from 30. April 2022 on	no call dates
Coupon Rate	up to the first call date 6 % annually, afterwards 4.698 + 5-year swap-rate	6 % annually
Trigger Activation	as soon as $CT1R \leq 5.125\%$	as soon as $CT1R \leq 5.125\%$
Write-Down amount	the minimum amount so that $CT1R >= 5.125\%$	complete write-down

Table 5.1: Term Sheet Data and Features used in Valuation

and $N = 1$. Under this assumption the cash flow stream given by (5.1) simplifies to

$$\Pi_t^{W,acc} = \sum_{i=1}^{l} 1_{\{t \leq t_i\}} 1_{\{t_i < \min(\theta_{acc},\tau)\}} c_i + 1_{\{t \leq T\}} 1_{\{T < \min(\theta_{acc},\tau)\}}. \tag{5.25}$$

Hence, its price at $t = 0$ is given by

$$p_0^{W,acc} = \sum_{i=1}^{l} e^{-rt_i} c_i \mathbb{Q}(t_i < \min(\theta_{acc}, \tau)) + e^{-rT} \mathbb{Q}(T < \min(\theta_{acc}, \tau)). \tag{5.26}$$

In Section 5.6.2 we discuss the results of the pricing of this CoCo from the 1st of October 2014 to the 17th of December 2014. Moreover, we carry out a sensitivity analysis based on the results of the 1st of December.

Modified CoCo - Conversion to Shares

We also consider a CoCo with an accounting trigger which can convert to shares. Herefore, we take over all the specifications from the 'Features'-column of the Table 5.1 except the conversion specifications. We replace the write-down feature by a conversion to shares with conversion price $C_p = 30.207$, which is the price of a share of the Deutsche Bank at the issuance of the CoCo. Hence, its cash flow stream is given by

$$\Pi_t^{S,acc} = \sum_{i=1}^{l} c_i 1_{\{t_i \leq t\}} 1_{\{t_i < \min(\theta_{acc},\tau)\}} + 1_{\{T \leq t\}} 1_{\{T < \min(\theta_{acc},\tau)\}} + 1_{\{\theta_{acc} < t\}} 1_{\{\theta_{acc} < \tau\}} \frac{S_{\theta_{acc}}}{C_p}.$$

and its price at $t = 0$ is given by

$$\begin{aligned} p_0^{S,acc} &= \sum_{i=1}^{l} 1_{\{t<t_i\}} e^{-rt_i} c_i \mathbb{Q}(t_i < \min(\theta_{acc},\tau)) + e^{-rT} \mathbb{Q}(T < \min(\theta_{acc},\tau)) \\ &+ \mathbb{E}^{\mathbb{Q}} \left(1_{\{\theta_{acc} < \min(T,\tau)\}} e^{-r\theta_{acc}} \frac{S_{\theta_{acc}}}{C_p} \right). \end{aligned}$$

In Section 5.6.2 we also discuss the results of the sensitivity analysis for the modified CoCo based on the data of the Deutsche Bank from the 1st of December.

5.6.1 Calibration

At first, we estimate the parameters c and d in the definition of the $CT1R$ from balance sheet data by using a least-squares estimation. Table 5.2 gives balance sheet data, which was used for the regression. The table contains information about the value of the assets (in trillion), the debt (in trillion) and the CT1R of Deutsche Bank from the forth quarter 2013 to the third quarter 2014. Due to the change of the regulatory framework from Basel II to Basel III the definition of the CT1R drastically changed and all CT1Rs which were computed before the forth quarter of 2013 can not be compared with the following ones. Hence, we only take into account balance sheet data starting from forth quarter of 2013. Afterwards, we calibrate the model for each valuation day. For this, we use 7-year OIS rate

Quarter	4Q2013	1Q 2014	2Q 2014	3Q 2014
V	1.611	1.637	1.665	1.709
$V - K$	0.0547	0.0558	0.0647	0.0664
K	1.5563	1.5812	1.6003	1.6426
$(V - K)/V$	0.0351	0.0353	0.0404	0.0404
$log((V - K)/V)$	-3.3827	-3.3788	-3.2478	-3.2480
CT1R	0.097	0.095	0.115	0.115
$log(CT1R)$	-2.3330	-2.3539	-2.1628	-2.1628

Table 5.2: Regression Data: $c = 1.3563$, $d = 2.2420$

for the interest rate r, the stock price of the Deutsche Bank and the 5-year CDS spread. From the stock price of the Deutsche Bank we compute the total market capitalization of the Deutsche Bank by multiplying it with the number of outstanding shares. Figure 5.1 shows graphs of the input data. Moreover, Table 5.3 gives the parameters which have been chosen at the beginning of the calibration. The results of the calibration for the 1st December 2014 can be found in Table 5.4. Moreover, the Figures 5.3(a), 5.3(b) and 5.3(c) show the graphs of V_0, K and σ_V.

LGD	Distribution of κ	μ	λ	Dividends
0.5	$\kappa_1 \sim N(0, 0.01)$	$6.5 \cdot 10^{-4}$	1	$\delta_1 \sim N(0.00065, 0.0002)$

Table 5.3: Parameter

V	K	σ_V	π_l for $l \notin \{7, 8\}$	π_7	π_8
0.0473	0.0455	0.0121	0	0.86	0.14

Table 5.4: Results of the Calibration of the 1st December

5.6.2 Pricing and Sensitivity Analysis

In the following we discuss the results of the pricing of the CoCo issued by Deutsche Bank and its modified version. Moreover, we present some results on the market trigger from Section 5.3.2. Several figures can be found in Section 5.7. Among others, there are

- graphs of the stock price, the interest rate or the CDS spread (Figures 5.1) as the main input data for the calibration,
- the results of the calibration (Figures 5.2),
- figures related to model and market prices of the CoCo of the Deutsche Bank (Figures 5.3),
- figures presenting the dependence of the prices and other different key quantities (trigger probabilities) on different parameters (Figures 5.4 to 5.10) and
- figures related to the market trigger from Section 5.3.2.

For the Monte Carlo pricing of the CoCo of the Deutsche Bank we used 100000 trajectories, for the modified one and the results for the market trigger 5000. Hence, figures related to modified CoCo and the market trigger may not look like as smooth as the other ones. Note that the underlying model has been recalibrated for each day from the 1st of October to the 17th of December. In the following we will present the most interest findings.

Calibration and CoCo of the Deutsche Bank

Noticeable features of the calibration and the pricing of the CoCo of the Deutsche Bank include the following points.

- The parallel assignment of V_0 and K: This is caused by the relation $V_0 = V^{init}/K^{init}K$.
- A decrease of the volatility σ_V over the whole observation period: Since the interest rate r decreased over the considered time period, the volatility σ_V has to decrease to obtain little fluctuating CDS prices.
- Small differences of model and market prices: However, a perfect fit should not be expected, because the price of the CoCo was not involved in the calibration.
- The market movements mainly driven by the conversion probability, compare Figure 5.4(d).

Sensitivity Analysis of the Pricing of the CoCo of the Deutsche Bank

The Figures 5.4 to 5.7 present the results of the sensitivity analysis of the CoCo of the Deutsche Bank for the 1st December. This includes the analysis of the impact of changing r, S_0, the CDS spread, σ_V, K_{acc} and the standard deviation of κ. It is important to keep in mind that the model has been recalibrated after the interest rate, the stock price or the CDS spread has been changed. On the other hand, the model has not been recalibrated if the volatility of V, the threshold of the trigger or the standard deviation of κ has been changed. In the following we list the most interesting observations of the sensitivity analysis of the CoCo of the Deutsche Bank:

- Low sensitivity with respect to the initial share price: This CoCo can be considered as a fixed income derivative.

- With a small increase in interest rates (up to 0.02) the price of the CoCo increases, because the trigger probability decreases. For large increases of the short rate r the effects of discounting outweigh this effect.

- Quite different influence of the interest rate r on the conversion probabilities for different periods of time (see Figure 5.8(b)): While the probabilities for short periods of time decreased, the probabilities for longer periods of time increased. This is mainly caused by the recalibration of the model. If the interest rate is increased c.p., then default probabilities and trigger probabilities decrease. To compensate for this, σ_V is increased during the calibration of the model to account for the CDS spread. But this increase of σ_V effects the trigger probabilities differently. The larger the period of time the stronger is the effect of the increase of σ_V. For short periods the direct effect of increasing r is stronger than the indirect effect of increasing σ_V. On the other hand, for longer periods the indirect effect dominates.

Sensitivity Analysis of the Modified CoCo

We also discuss the dependence of the price of the modified CoCo on the different parameters, see 5.5. Note that this CoCo has been equipped with an accounting trigger and a conversion to shares. Therefore, the trigger probabilities related to this CoCo and the CoCo of the Deutsche Bank are the same. The difference between the considered CoCos is the conversion; for the modified CoCo the distribution of the share price matters due to the conversion payment $C_p S_\theta$. It can be seen in Figure 5.11(c) that increasing the current stock price increases the stock price at conversion. Hence, if the stock prices increases ceteris paribus, the price of the CoCo increases, see Figure 5.6(c).

Effectiveness of Different Trigger

Of particular interest is the identification of scenarios where the used trigger is not effective, e.g. where the conversion does not take place before the default. For this, we study the dependence of the ratio $\mathbb{Q}(\tau < \min(\theta, T))/\mathbb{Q}(\theta < T)$ and $\mathbb{Q}(\tau < \min(\theta, T))$ on different parameters, which is presented in At this, we not only consider an accounting trigger, but also an market trigger. the Figures 5.9. Additionally, the Figures 5.8 show the dependence of $\mathbb{Q}(\tau < \min(\theta_{acc}, T))$ on different parameters. We find that:

- The effectiveness of the accounting trigger is influenced the most by σ_V.
  $\mathbb{Q}(\tau < \min(\theta_{acc}, T))$ and $\mathbb{Q}(\tau < \min(\theta_{acc}, T))/\mathbb{Q}(\theta_{acc} < T)$ increases strongly with σ_V.

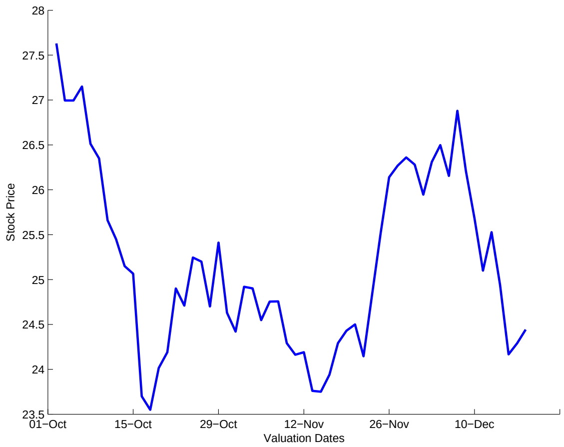
- Also the interest rate r and the CDS spread influence the effectiveness positively. Here, σ_V acts as a transmission channel for the change of the interest rate and the CDS spread.
- On the other hand, the share price, the threshold K_{acc} and the variance of κ have no significant influence on the effectiveness of the trigger.

Finally, Figures 5.11 and 5.12 give some results for the market trigger from Section 5.3.2. It appears that:

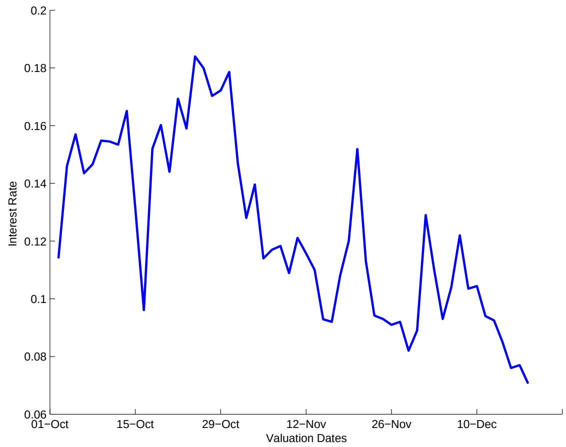
- The numerical results confirm the conservativeness of the market trigger from Proposition 5.10.
- The effectiveness of the market trigger θ_{ma} increases with the threshold K_{ma} and with increasing information content in the noisy observation process Z: The larger parameter c in the function $a(x) := c\log(x)$ is, the higher the effectiveness.

5.7 Pictures

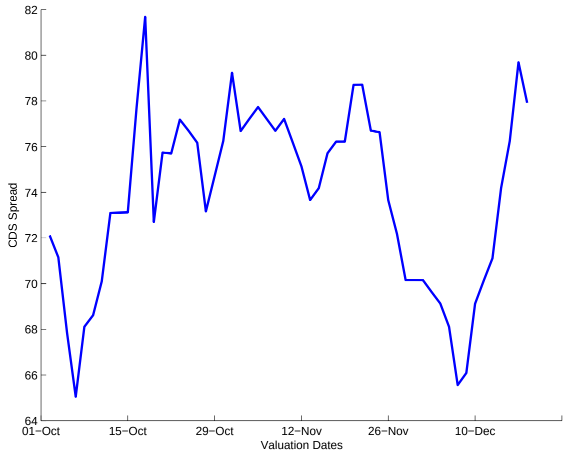
Figure 5.1: Input Data from 1st October to 17th December



(a) Share Price

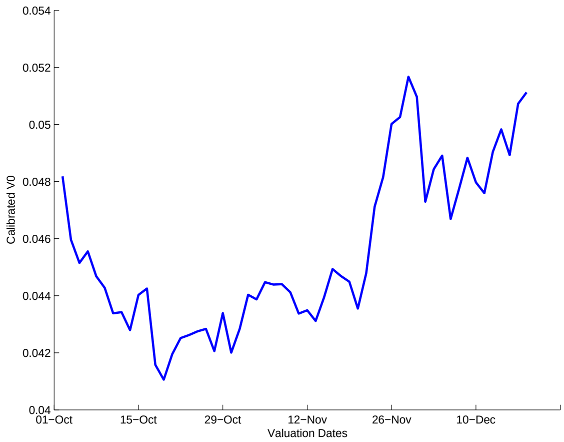


(b) 7-year OIS Rate in %

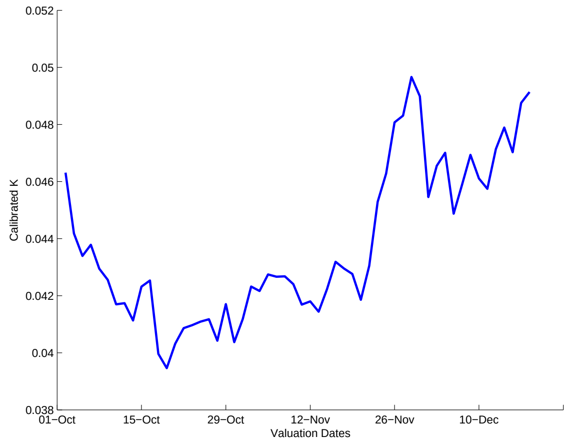


(c) 5-year CDS Spread in %

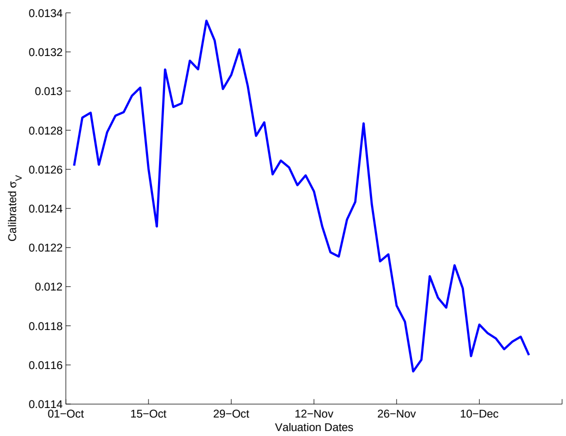
Figure 5.2: Calibration Results from 1rd October to 17th December



(a) Calibrated V_0

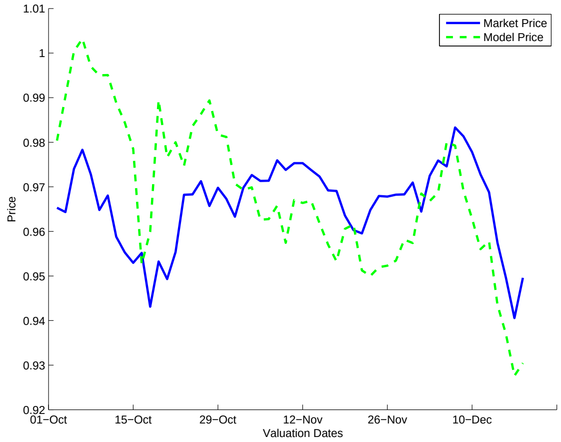


(b) Calibrated K

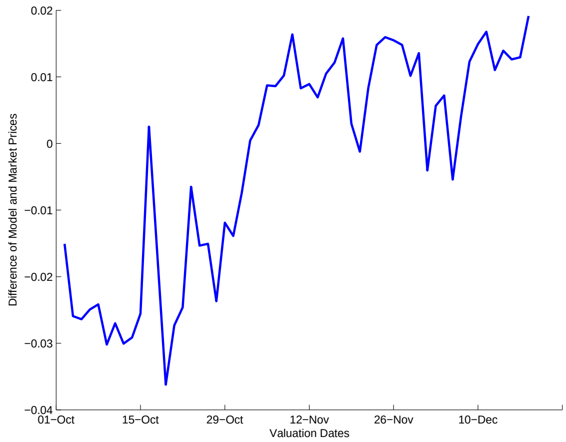


(c) Calibrated σ_V

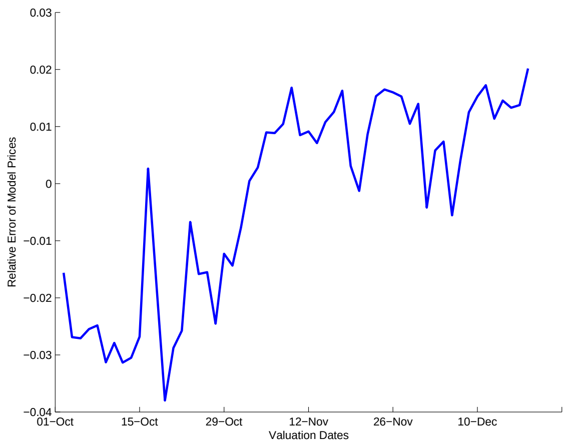
Figure 5.3: Prices and Trigger Probabilities from 1st October to 17th December



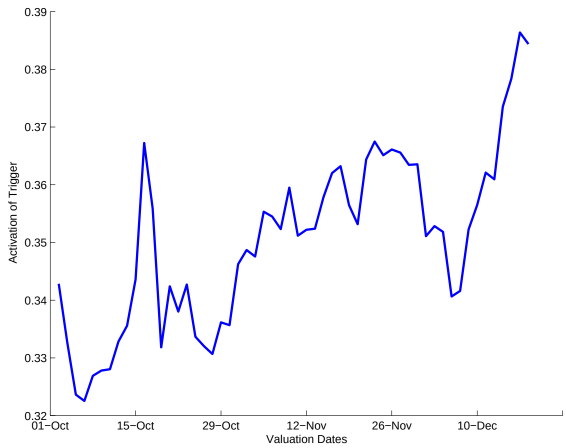
(a) Model and Market Prices



(b) Absolute Differences of Model and Market Prices

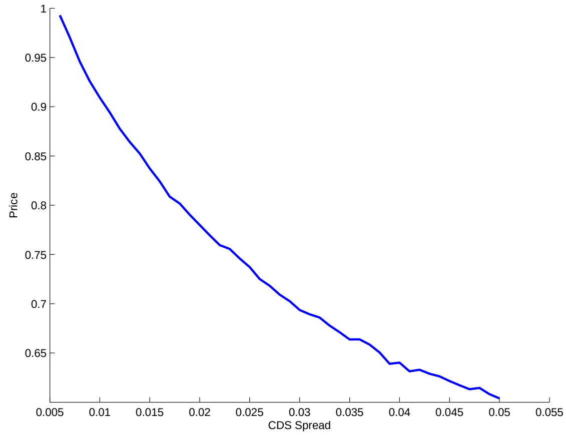


(c) Relative Differences of Model and Market Prices

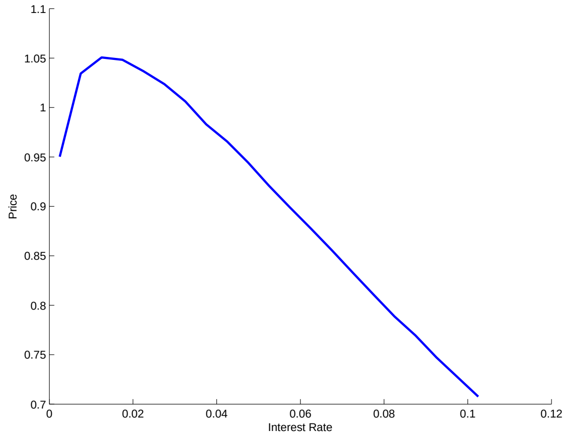


(d) Probability that Trigger is activated until T

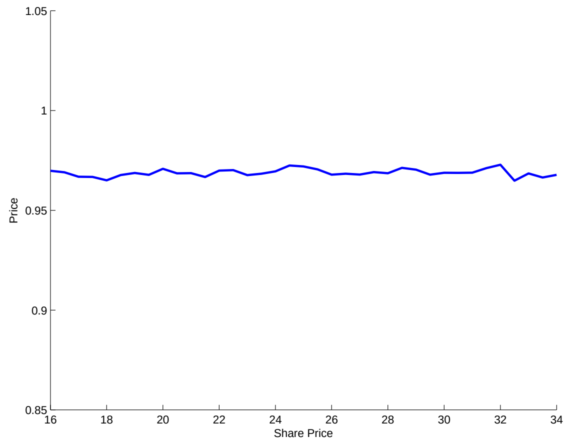
Figure 5.4: Dependence of the Price of the CoCo on different Parameters



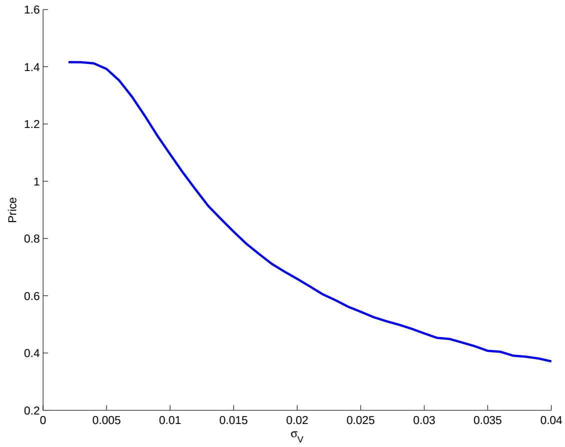
(a) CDS Spread in %



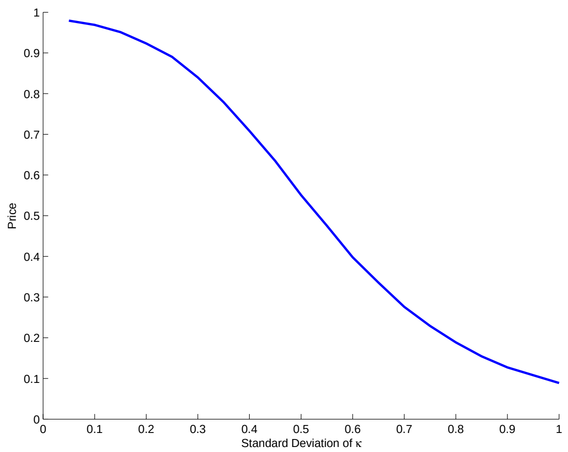
(b) Interest Rate in %



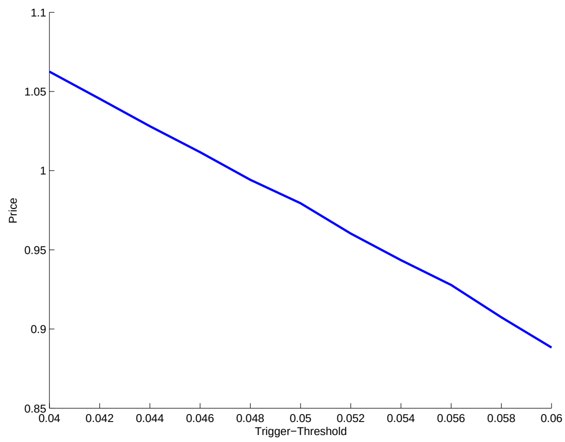
(c) Share Price



(d) σ_V

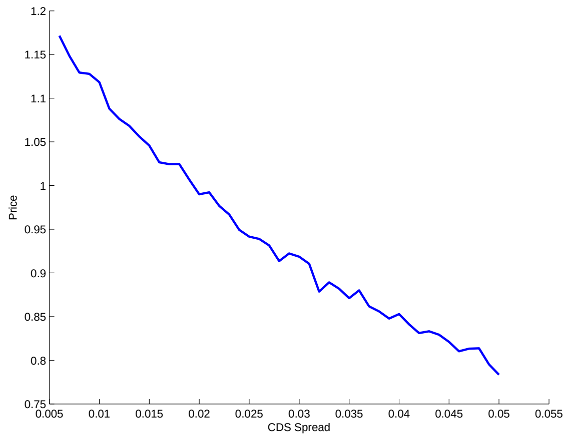


(e) Standard Deviation of κ

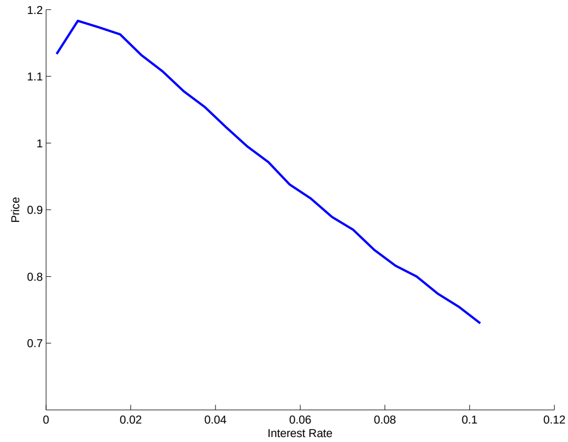


(f) Threshold of the Trigger

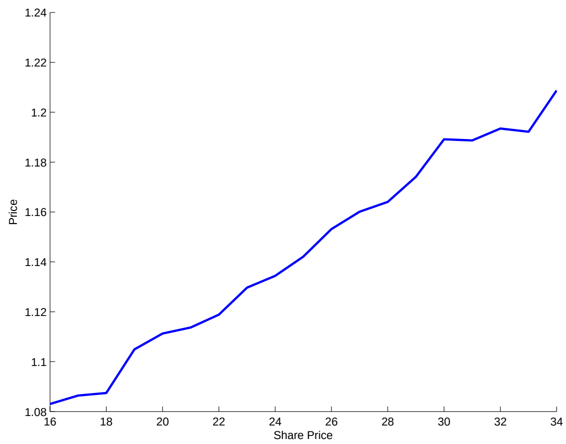
Figure 5.5: Dependence of the Price of the CoCo with Conversion to Shares on different Parameters



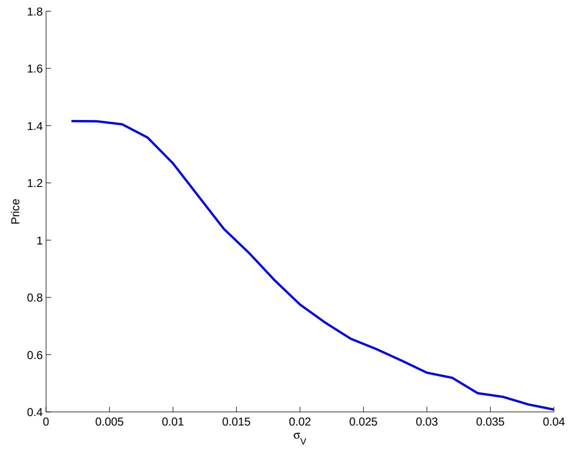
(a) CDS Spread in %



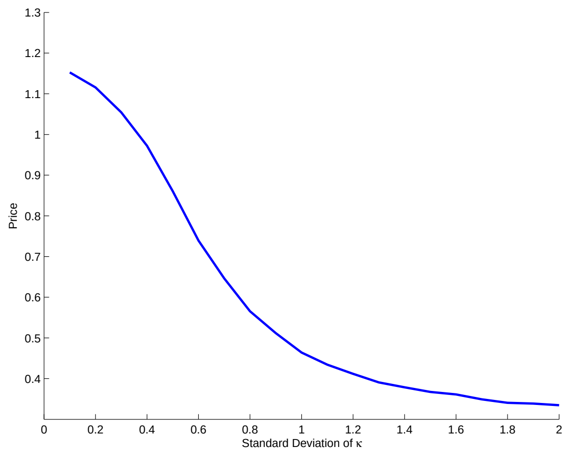
(b) Interest Rate in %



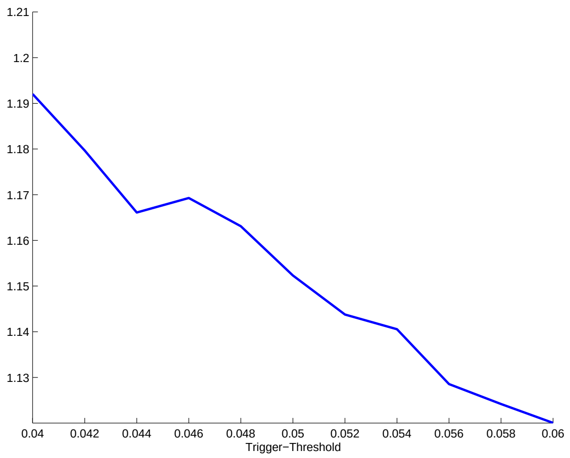
(c) Share Price



(d) σ_V



(e) Standard Deviation of κ



(f) Threshold of the Trigger

Figure 5.6: Dependence of the Default Probability on different Parameters

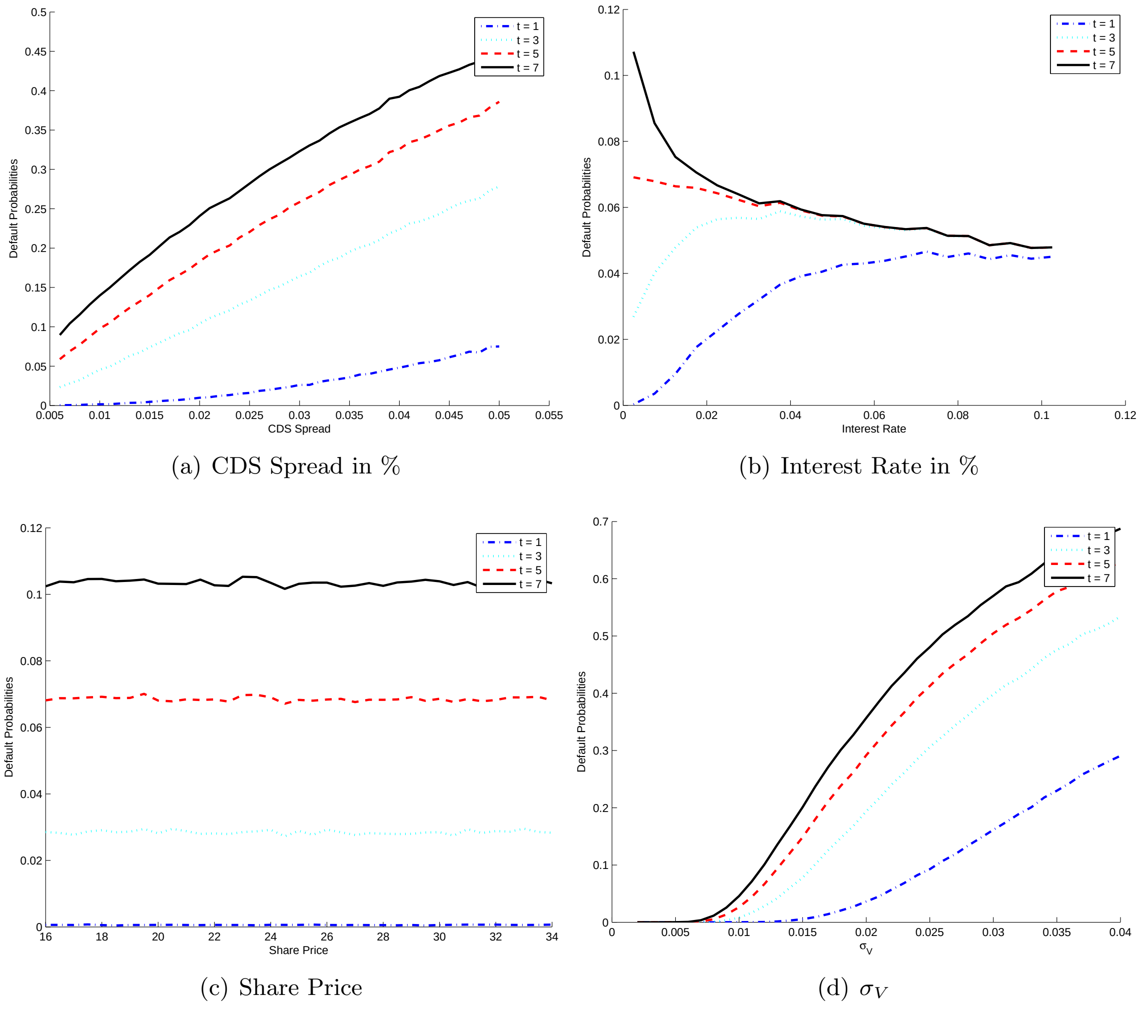
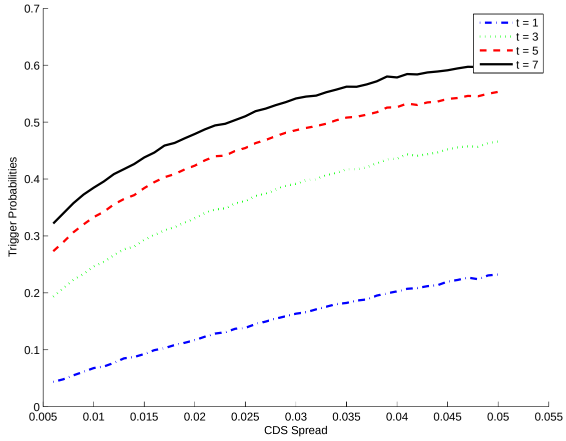
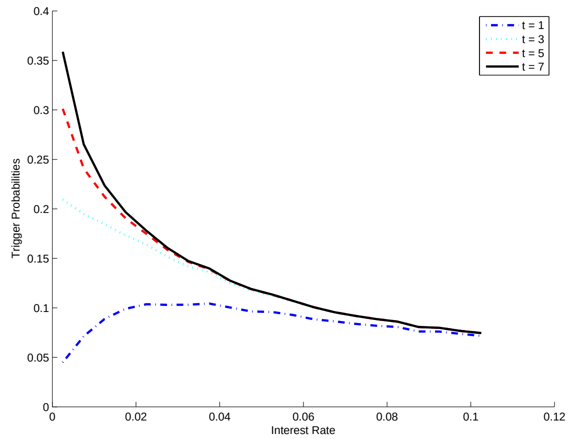


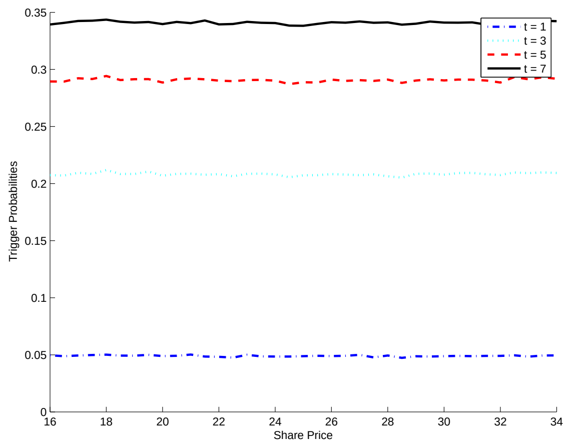
Figure 5.7: Dependence of the Probability $\mathbb{Q}(\theta_{acc} < t)$ on different Parameters



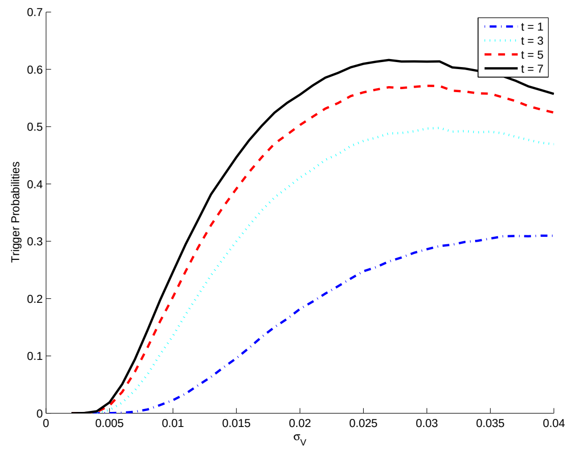
(a) CDS Spread in %



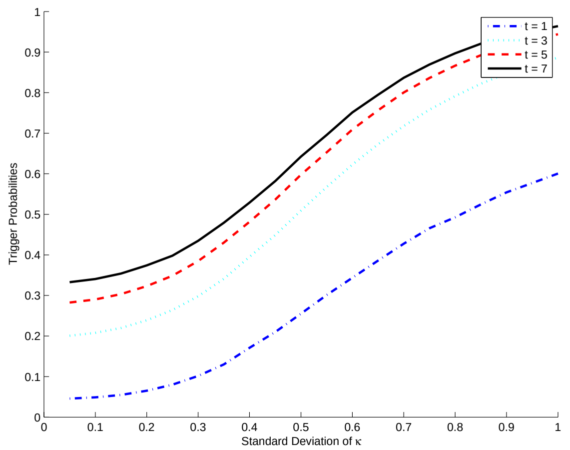
(b) Interest Rate in %



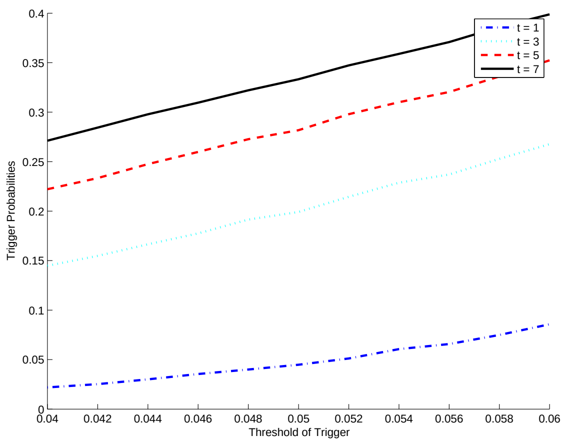
(c) Share Price



(d) σ_V

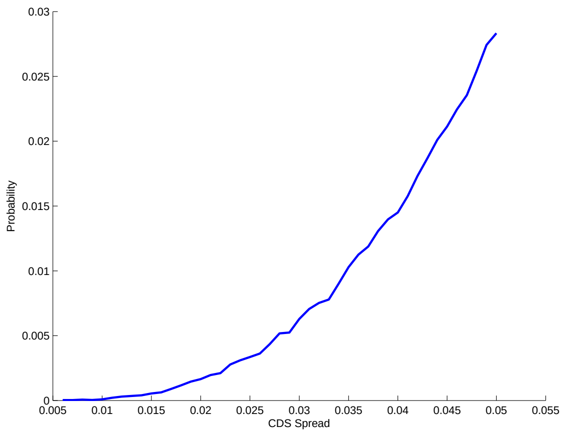


(e) Standard Deviation of κ

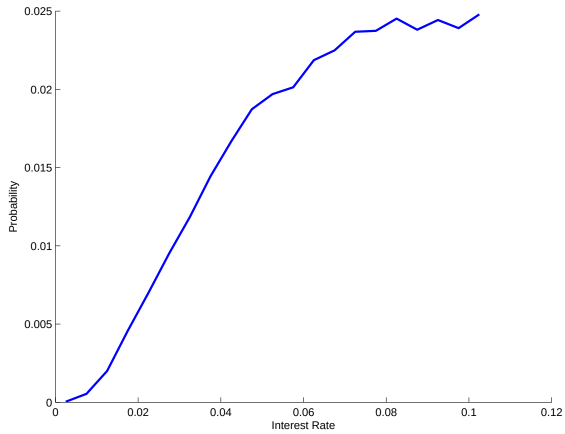


(f) Threshold of the Trigger

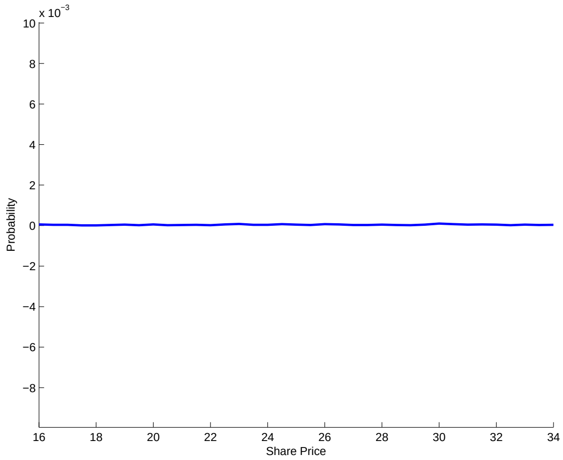
Figure 5.8: Dependence of the Probability $\mathbb{Q}(\tau < \min(\theta_{acc}, T))$ on different parameters



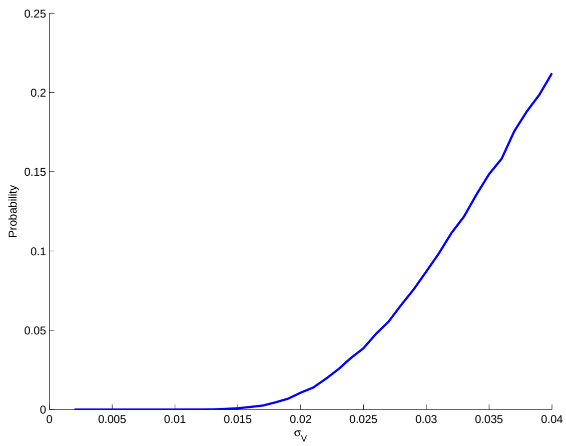
(a) CDS Spread in bp



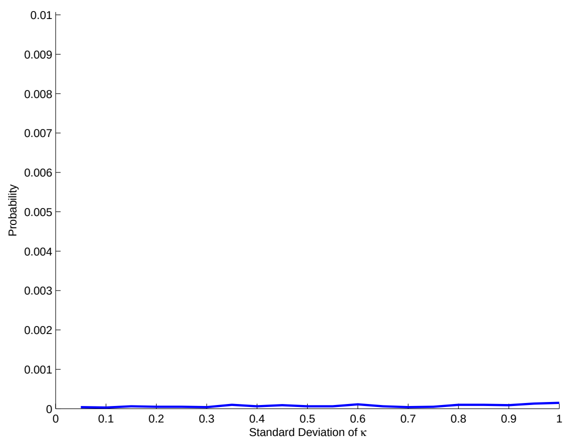
(b) Interest Rate in %



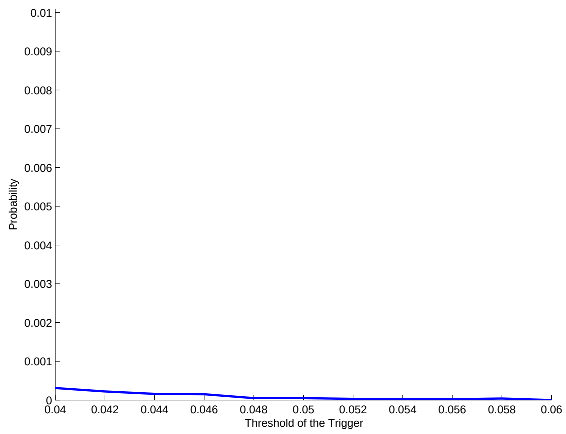
(c) Share Price



(d) σ_V

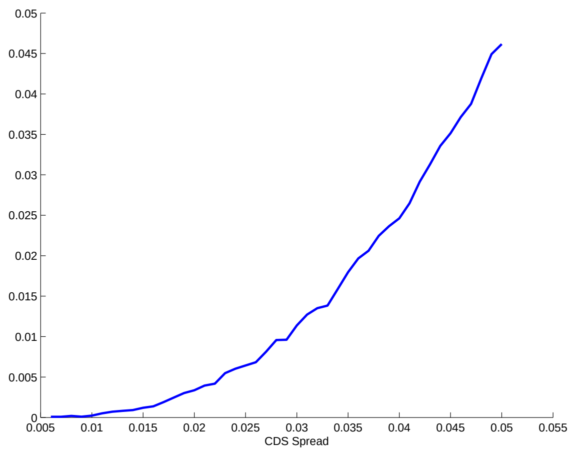


(e) Standard Deviation of κ

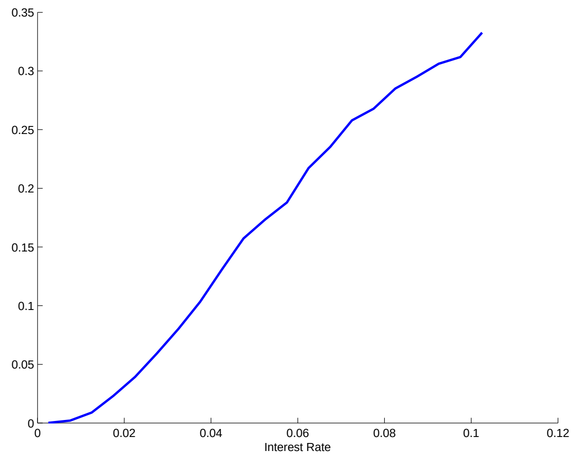


(f) Threshold of the Trigger

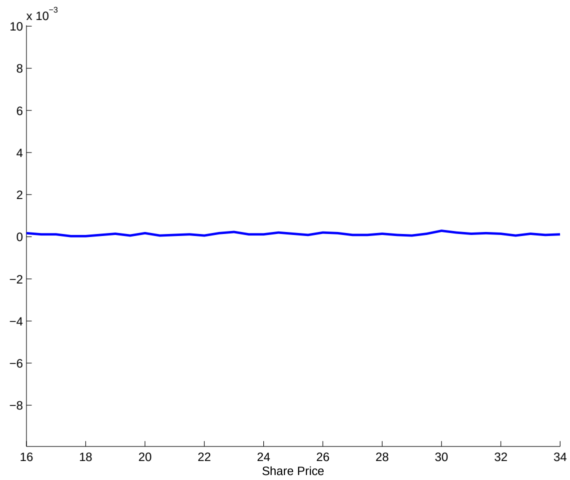
Figure 5.9: Dependence of the Ratio $\mathbb{Q}(\tau < \min(\theta_{acc}, T))/\mathbb{Q}(\theta_{acc} < T)$ on different parameters



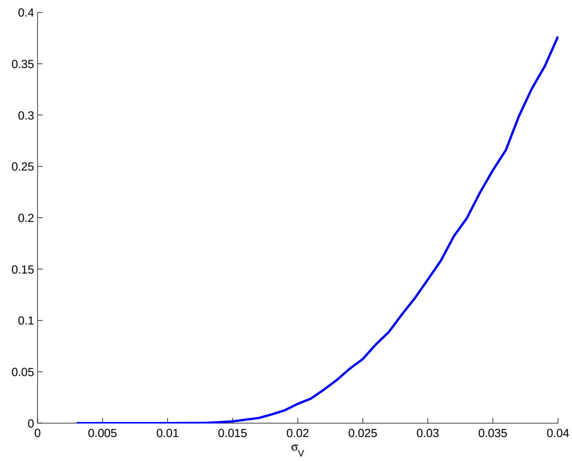
(a) CDS Spread in %



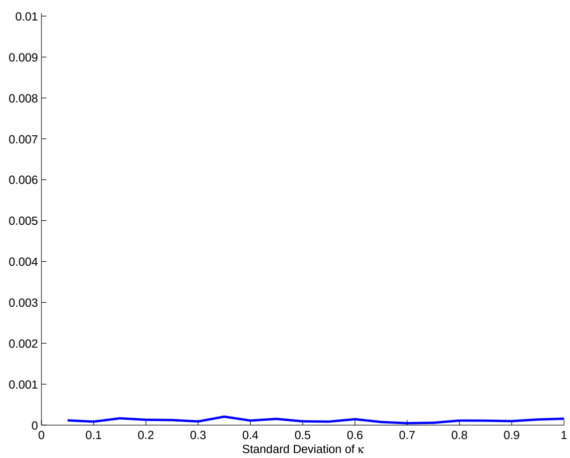
(b) Interest Rate in %



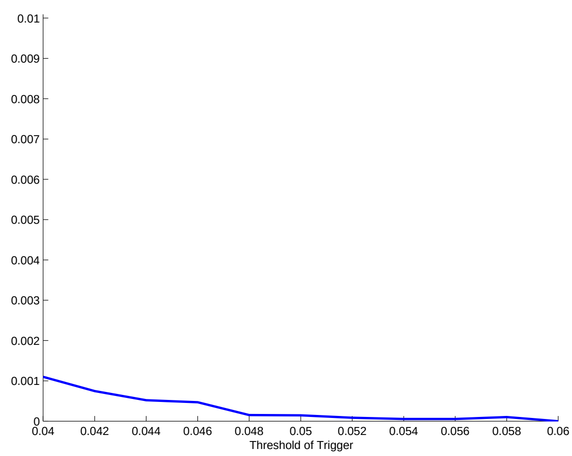
(c) Share Price



(d) σ_V

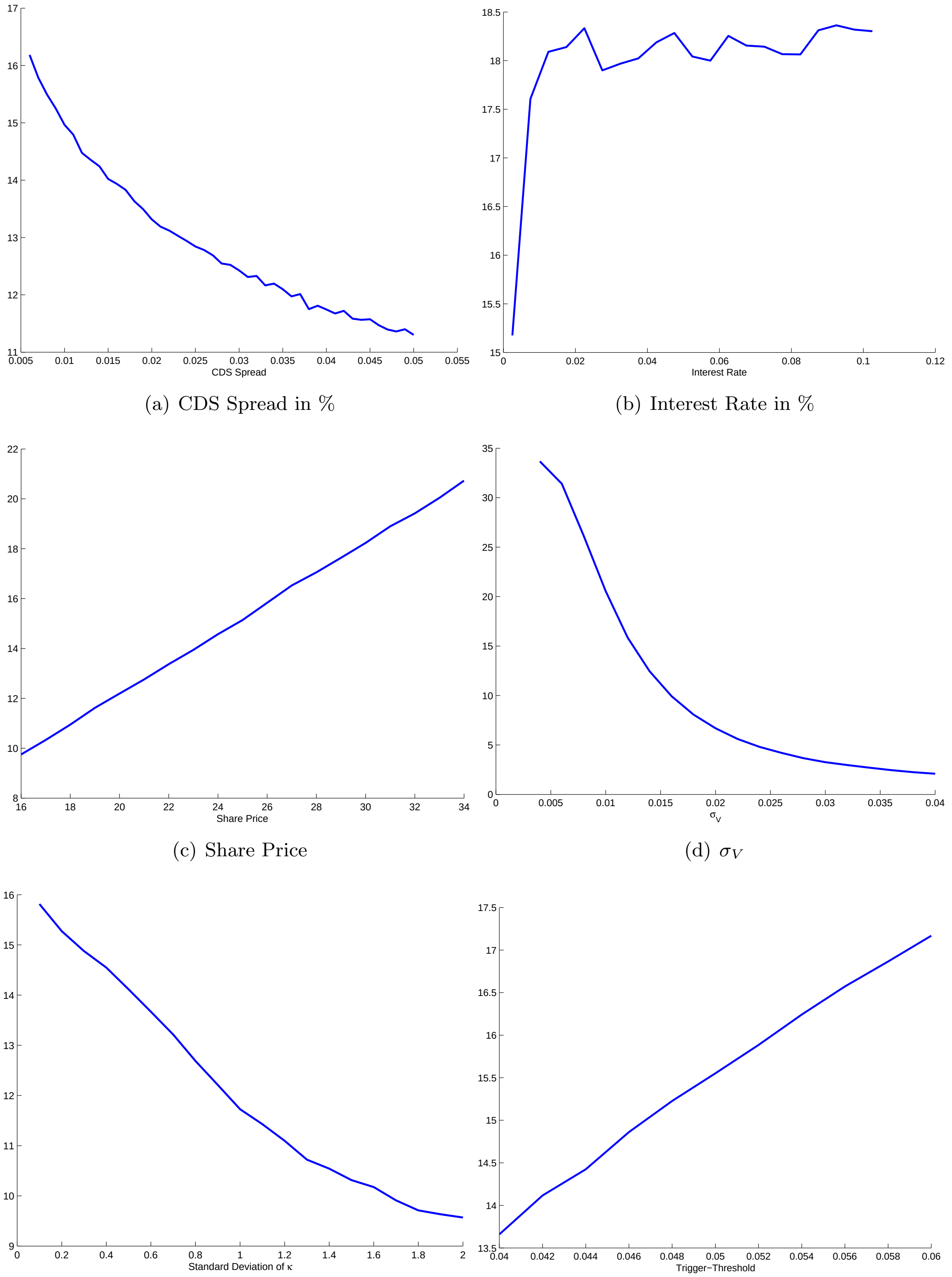


(e) Standard Deviation of κ



(f) Threshold of the Trigger

Figure 5.10: Dependence of $\mathbb{E}^{\mathbb{Q}}(S_\theta)$ on different Parameters



(a) CDS Spread in %

(b) Interest Rate in %

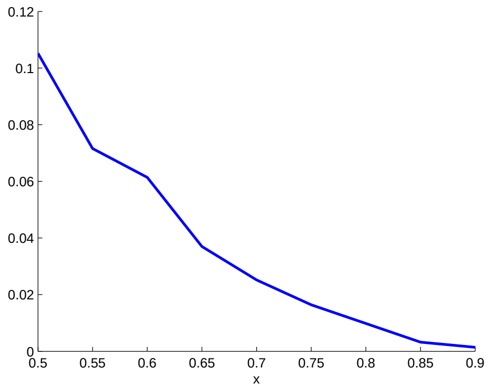
(c) Share Price

(d) σ_V

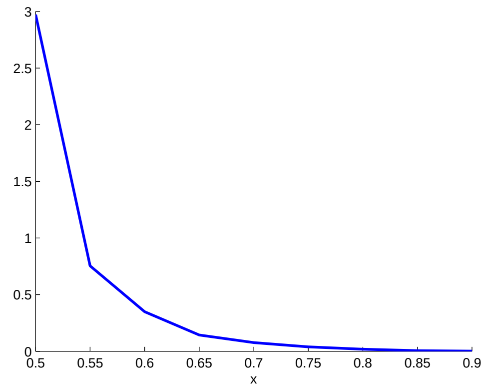
(e) Standard Deviation of κ

(f) Threshold of the Trigger

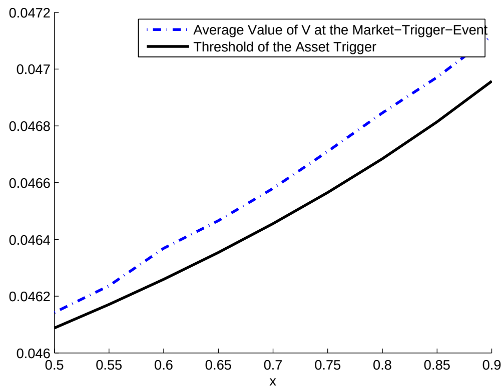
Figure 5.11: Results for the Market Trigger: Conversion threshold is given by $\theta_{ma} = xS_0$



(a) $\mathbb{Q}(\tau < \min(\theta_{ma}, T))$

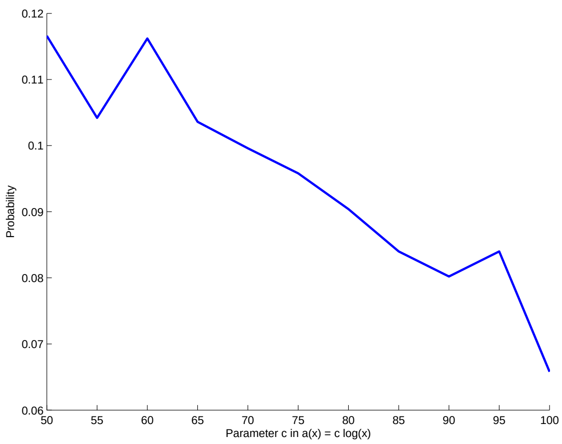


(b) $\mathbb{Q}(\tau < \min(\theta_{ma}, T))/\mathbb{Q}(\theta_{ma} < T)$

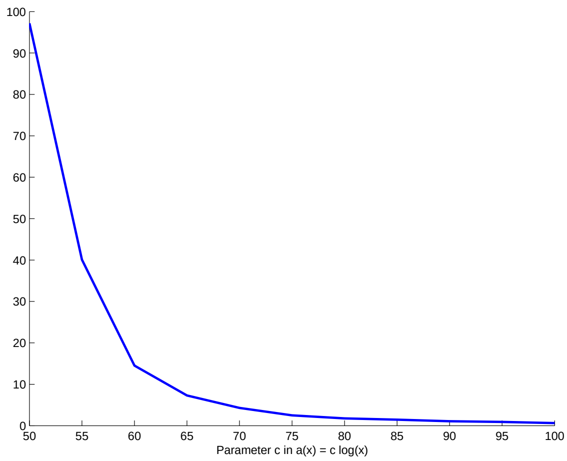


(c) Conservativeness of Market Trigger

Figure 5.12: Results for the Market Trigger: Dependence on the Parameter c in $a(x) = c\log(x)$



(a) $\mathbb{Q}(\tau < \min(\theta_{ma}, T))$



(b) $\mathbb{Q}(\tau < \min(\theta_{ma}, T))/\mathbb{Q}(\theta_{ma} < T)$

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Lars Rösler

Curriculum Vitae

Education

2012-2015	PhD student at the Vienna University of Economics and Business
2010-2012	PhD student at the Max-Planck-Institut for Mathematics in the Sciences and at the University of Leipzig
2009	Diploma degree in mathematics from Otto-von-Guericke-Universität Magdeburg
2004-2009	Study of mathematics (Diplom) at the Otto-von-Guericke-Universität Magdeburg
2004-2009	General qualification for university entrance (Allgemeine Hochschulreife)

Working Experience

Since 2015	Employed by Optimax Energy GmbH
2013-2015	Research associate at WU Vienna
2011-2013	Research associate at the University of Leipzig
2010-2011	Stipendiary of the International Max-Planck-Research-School (IMPRS) in Leipzig
2008-2009	Internship at the consulting corporate Finbridge GmbH

Publications

2015	„Corporate Security Prices in Structural Credit Risk Models with Incomplete Information“ submitted to Mathematical Finance (jointly with Prof. Rüdiger Frey (WU Vienna))
2014	„Contagion Effects and Collateralized Credit Value Adjustments for Credit Default Swaps“ published in International Journal of Theorethical and Applied Finance (jointly with Prof. Rüdiger Frey (WU Vienna))