The Viability Kernel Algorithm -
Convergence and Application

Ausgeführt am Institut für
Stochastik und Wirtschaftsmathematik

unter Anleitung von
Univ.Prof. Dipl.-Ing. Dr.techn. Vladimir Veliov

durch

Axel Böhm, BSc.

Maria-Theresien-Straße 30/12, 1010 Wien

December 7, 2015

Axel Böhm
Zusammenfassung

Abstract

This thesis is devoted to the problem of computing viability kernels in the context of viability theory. Furthermore, the convergence order of the viability kernel algorithm is investigated. The main focus is put on the class of differential inclusions with one-sided Lipschitz continuous right hand side. A broad analysis of the stability of the viability kernel with respect to perturbations in the constraint set is done. Several counter examples are presented where linear dependence does not hold as well as rather restrictive but sufficient conditions. Furthermore numerical results in the context of value functions of infinite horizon optimal control problems on top of various other examples are illustrated.
Contents

1 Basic Set-Valued Analysis .................................................. 4
  1.1 Operating with Sets .................................................. 4
  1.2 Set-Valued Maps ..................................................... 6
  1.3 Differential Inclusions .............................................. 7

2 Viability Theory .............................................................. 11
  2.1 Viability Domain ..................................................... 11
  2.2 The Viability Theorem ................................................ 12
  2.3 Viability Kernel ....................................................... 13

3 Approximating the Viability Kernel .................................... 15
  3.1 Constructing Discrete Viability Kernels ................................ 15
  3.2 Viability Kernel Convergence ....................................... 17
  3.3 Using Finite valued Set-Valued Maps .............................. 20

4 Estimates for the Viability Kernel Algorithm ......................... 25
  4.1 Results ........................................................... 26
  4.2 One-Sided Lipschitz Continuous Right Hand Sides ............... 31
  4.3 Counter example ..................................................... 35

5 Computing Value Functions and Hamilton-Jacobi-Equation ........ 42
  5.1 Introduction ....................................................... 42
  5.2 Characterizing the Value Function ................................ 43
  5.3 Example ........................................................... 46

6 Numerical Examples ........................................................ 48
  6.1 A Simple Example .................................................. 48
  6.2 A Circle .......................................................... 51
  6.3 Lotka-Volterra type Predator-Prey-model .......................... 52
  6.4 A Bio-Economic Model .............................................. 53

7 Summary and Outlook ..................................................... 57
Introduction

We want to describe the evolution of a system over time which is subjected to constraints regarding the state. For modeling the dynamics of this system we shall use a differential inclusion:

\[ \dot{x}(t) \in F(t, x(t)). \] (1)

In order to motivate the use of differential inclusions we start by presenting some of their applications. See [11] for the following examples and a more detailed investigation.

Consider the following ordinary differential equation used in mechanics:

\[ \ddot{x}(t) = -s(\dot{x}(t)) + \varphi(t, x(t), \dot{x}(t)), \quad x(0) = x_0, \quad \dot{x}(0) = x'_0. \]

The acceleration on the left hand side is determined by a term \( s(\dot{x}(t)) \), representing dry friction, and a part \( \varphi(t, x(t), \dot{x}(t)) \) that governs viscous friction as well as elastic and external forces. Typically the dry friction term \( s(\cdot) \) is discontinuous at zero. Due to this discontinuity, however, the system may fail to have a solution. Filippov [8, 9] suggested to regularize the problem by replacing it with the differential inclusion

\[ \ddot{x}(t) \in -S(\dot{x}(t)) + \varphi(t, x(t), \dot{x}(t)), \quad x(0) = x_0, \quad \dot{x}(0) = x'_0, \]

where \( S(\cdot) \) is now a set-valued map and \( S(y) \) is the convex hull of the limit values of \( s(\cdot) \) left and right of \( y \). By doing so the existence of unique solutions can be ensured, under mild conditions.

Furthermore, consider the following system

\[ \dot{x}(t) = g(t, x(t)) + B(t)v(t), \quad x(0) = x_0, \] (2)

where \( v(t) \in \mathbb{R}^n \) is an unknown perturbation. Reasons for this could be that \( v \) is inherently random, not exactly known or could be a result of model simplification. Either way, reasonable bounds on the magnitude of \( v \) should be available: \( v(t) \in V(t) \subseteq \mathbb{R}^n \). By considering the differential inclusion (1) with right hand side

\[ F(t, x(t)) = g(t, x(t)) + B(t)V(t), \] (3)

instead we still cannot predict the system at a given moment \( T > 0 \) exactly, but it is possible to bound potential outcomes. This is done by enclosing numerically the reachable set:

\[ \{x(T) : \ x(\cdot) \text{ is a solution of (1)} \}. \]

Thirdly, we consider a control problem:

\[
\begin{cases}
\dot{x}(t) & = f(t, x(t), u(t)) \\
u(t) & \in U,
\end{cases}
\] (4)
where $u(t) \in \mathbb{R}^n$ is the control input, which is subjected to the constraint $u(t) \in U$. We observe that solutions of the control system (4) are solutions of the differential inclusion with right hand side $F(t, x) := \{ f(t, x, u) : u \in U \}$. This enables us to use other discretization methods, which may have advantages over directly discretizing (4) for any possible choice of the control $u(\cdot)$.

With these motivation in mind we continue by stating the main problem of viability theory, which is our field of interest. We consider a differential inclusion

$$\dot{x}(t) \in F(x(t)),$$  \hspace{1cm} (5)

combined with a state constraint

$$x(t) \in K$$

for some set $K \subseteq \mathbb{R}^n$. For a given initial state $x_0$ we want to know if there exists a solution of (5) that starts form $x_0$ and stays in $K$ for all $t \geq 0$. Ultimately, we want to study the set of all of these initial values, which we call the viability kernel. The concept of a viability kernel can be useful tool not only in the study of other mathematical problems but also has many applications in biology and economics as described in [5]. In Section 5 the viability kernel is used to compute the value function of an infinite horizon optimal control problem and in Section 6 an application in the management of marine renewable resources is found.

Outline

In Section 1 we discuss the fundamentals of set-valued analysis. Many central concepts of mathematical analysis, such as limits, continuity, or differentiability, have extensions to the set-valued case. They can, however, be generalized in many different ways. In order to keep the comprehensibility high we focus on the most important definitions and results.

Continuing the preliminaries in Section 2, we present an insight into viability theory with the concept of viability kernel leading the way. Furthermore, the very important connection between viability kernels and viability domains is presented.

Since the computation of an exact viability kernel is rather difficult in most cases, in Section 3 we are going to look at how to approximate them. Following [15] we present an appropriate numerical scheme, known as the "Viability Kernel Algorithm", and the corresponding convergence results.

The convergence results from Section 3 tell us that the approximations, computed by the viability kernel algorithm, converge to the true object for decreasing discretization parameter. This, however, gives no information about the size of the error for a particular approximation using time and space discretization. Therefore error estimates for
a special class of right hand sides are presented in section 4. The content of this part is mostly taken from [14], except for the relaxation of the assumption about stability of the viability kernel. This assumption is sometimes not even fulfilled in rather simple cases as shown by examples, which were found by the author. For this reason a proof of the continuity of the viability kernel with respect to perturbations in the constraint set is given. Furthermore, conditions that ensure said property and the according proof are given. This shows linear convergence of the algorithm for a special class of problems.

In Section 5 we find a first application of the above described algorithm, where it is used to compute the value function of infinite horizon optimal control problems (see [5]). This is done by characterizing the epigraph of the value function as the viability kernel of an auxiliary system.

Section 6 gives numerical results for various examples. These range from very simple problems where the exact viability kernel is known, to applications in the management of marine renewable resources.

Finally, in Section 7, we revisit the content of this work. We highlight the most important parts and point out some of the problems which may need further research.
1 Basic Set-Valued Analysis

In this chapter we are going to discuss the fundamentals of set-valued analysis. We
are start by defining limits of sequences of sets and accompanying we look at distances
between sets. Moreover the notion of a tangent cone to a set is defined and motivated.
Furthermore, set-valued maps and their properties are discussed. Subsequently, we
are ready to generalize differential equations to set-valued right hand sides, so-called
differential inclusions. With our knowledge about set-valued maps we look at the existence
and properties of solutions. The following definitions and results can be found in [4, 1, 3].

1.1 Operating with Sets

Limits of Sets

Definition 1.1.1. Let \((X,d)\) be a metric space and \(K \subseteq X\) be a subset of \(X\). The
distance between a point \(x \in X\) and \(K\) is defined by:

\[
d(x,K) := \inf_{k \in K} d(x,k).
\]

Definition 1.1.2. Let \((X,d)\) be a metric space and \((A_n)_{n \in \mathbb{N}}\) be a sequence of sets such
that \(A_n \subseteq X, \forall n \in \mathbb{N}\). The Kuratowski upper limit of this sequence, \(\limsup_{n \to \infty} A_n\), is
defined as:

\[
\limsup_{n \to \infty} A_n := \left\{ x \in X : \liminf_{n \to \infty} d(x, A_n) = 0 \right\}.
\]

Remark 1.1.3 ([4]). The Kuratowski upper limit can be expressed as the set of all cluster
points:

\[
x \in \limsup_{n \to \infty} A_n \Leftrightarrow \exists (a_{n_k})_{k \in \mathbb{N}} : a_{n_k} \in A_{n_k}, \forall k \in \mathbb{N} \wedge d(a_{n_k}, x) \xrightarrow{k \to \infty} 0.
\]

Definition 1.1.4. Let \((X,d)\) be a metric space and \((A_n)_{n \in \mathbb{N}}\) be a sequence of sets such
that \(A_n \subseteq X, \forall n \in \mathbb{N}\). The Kuratowski lower limit of this sequence, \(\liminf_{n \to \infty} A_n\), is
declared as:

\[
\liminf_{n \to \infty} A_n := \left\{ x \in X : \lim_{n \to \infty} d(x, A_n) = 0 \right\}.
\]

Remark 1.1.5 ([4]). The Kuratowski lower limit is the set of limits of sequences:

\[
x \in \liminf_{n \in \mathbb{N}} A_n \Leftrightarrow \exists (a_n)_{n \in \mathbb{N}} : a_n \in A_n, \forall n \in \mathbb{N} \wedge d(a_n, x) \xrightarrow{k \to \infty} 0.
\]

Lemma 1.1.6 ([4]). The Kuratowski lower limit as well as the the Kuratowski upper
limit are closed sets.

Remark 1.1.7 ([4]). It is easy to see that the Kuratowski lower limit is a subset of the
Kuratowski upper limit:

\[
\liminf_{n \to \infty} A_n \subseteq \limsup_{n \to \infty} A_n.
\]
Distances between Sets

Definition 1.1.8. Let \((X, d)\) be a metric space and \(A, B \subseteq X\) two subsets of \(X\), then the Hausdorff semidistance from \(A\) to \(B\) is defined by

\[\text{dist}(A, B) := \sup_{a \in A} d(a, B) = \sup_{a \in A} \inf_{b \in B} d(a, b) .\]

and is sometimes also called one-sided Hausdorff distance. Furthermore the distance can be measured by the (symmetric) Hausdorff distance:

\[d_H(A, B) := \max \{\text{dist}(A, B), \text{dist}(B, A)\} .\]

Remark 1.1.9. The Hausdorff semidistance of subsets of a normed space could have been defined in the following, equivalent way:

\[\text{dist}(A, B) = \inf \{\epsilon > 0 : A \subseteq B + \epsilon B\} .\]

By \(B\) we denote the ball around zero with radius 1.

Remark 1.1.10. The Hausdorff distance defines a metric on the set of closed and bounded subsets of \(X\) but also makes sense for more general sets.

Tangent Cones

The idea here is to somewhat generalize the concept of tangency and differentiability, which go hand in hand. In classical analysis one wants the derivative to be a linear function and the tangent space to be a linear space. If we do not demand anymore that the differential and the tangent space respectively preserve the addition and ask for homogeneity regarding the scalar multiplication only for positive scalars, we end up with what is called a tangent cone. For more information see [1, 4].

Definition 1.1.11. Let \(K \subseteq \mathbb{R}^n\) be an arbitrary set and \(x \in \bar{K}\) then we can define the so-called contingent cone \(T_K(\cdot)\) at \(x\):

\[T_K(x) := \left\{ v \in \mathbb{R}^n : \liminf_{h \to 0^+} \frac{d(x + hv, K)}{h} = 0 \right\} .\]

Remark 1.1.12. If \(x\) is in the interior of \(K\) then the contingent cone to \(K\) at \(x\) is the hole space:

\[x \in \bar{K} \Rightarrow T_K(x) = \mathbb{R}^n .\]

Remark 1.1.13. The tangent cone is a closed set and, in fact, a cone.

Remark 1.1.14. The tangent cone can be characterized by sequences in the following way:

\[v \in T_K(x) \iff \exists (h_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}} \text{ where } h_n \in \mathbb{R}^+ \text{ and } v_n \in \mathbb{R}^n, \forall n \in \mathbb{N} \text{ such that } \lim_{n \to \infty} h_n = 0, \lim_{n \to \infty} v_n = v \text{ and } x + h_nv_n \in K, \forall n \in \mathbb{N}.\]
1.2 Set-Valued Maps

A "set-valued" map might strike the reader as a paradox in itself since it seems to contradict the characteristic property of a function: to be well-defined. Technically a set-valued map from a space $X$ into another space $Y$ is a function into $\mathcal{P}(Y)$ (the power set of $Y$). In the following we want to avoid writing $F : X \rightarrow \mathcal{P}(Y)$, which not only is more difficult to read but also misses the point$^1$. Instead, we write $F : X \rightrightarrows Y$ in order to emphasize that $F$ is set-valued.

From now on $X$ and $Y$ shall always be finite-dimensional topological spaces, if not specified otherwise, for the rest of this section, and $\Omega \subseteq X$ shall be the domain of $F$ (see the next definition).

Definition 1.2.1. Let $F : X \rightrightarrows Y$ be a set-valued map. The domain and the graph of $F$ are defined as:

$$\text{Dom}(F) := \{ x \in X : F(x) \neq \emptyset \}$$

and

$$\text{Graph}(F) := \{ (x, y) \in X \times Y : y \in F(x) \}.$$ 

respectively. Furthermore we shall call $F$ nontrivial if $\text{Dom}(F) \neq \emptyset$.

Definition 1.2.2. For a set-valued map $F : X \rightrightarrows Y$ we sometimes write $F(Q)$ for a set $Q \subseteq X$ and mean by that:

$$F(Q) = \bigcup_{x \in Q} F(x).$$

Definition 1.2.3. Let $F : X \rightrightarrows Y$ be a set-valued map. Its inverse defines a set-valued map in the following way:

$$F^{-1}() := \begin{cases} y \mapsto \{ x \in \Omega : y \in F(x) \} \\ Y \rightrightarrows X. \end{cases}$$

Proposition 1.2.4. Following the last two definitions, the inverse of a set can be characterized by:

$$F^{-1}(Q) = \{ x \in \Omega : F(x) \cap Q \neq \emptyset \}.$$ 

For the continuity of set-valued maps we will mainly focus on the concepts of lower semicontinuity and upper semicontinuity. For the single valued case, both of these come down to ordinary continuity whereas for set-valued maps they provide different concepts.

Definition 1.2.5. A set-valued map $F : X \rightrightarrows Y$ is called upper semicontinuous at $x \in \Omega$ if for every open set $V \supseteq F(x)$ there exists an open set $U$ in $X$ containing $x$ such that $F(x') \subseteq V$, $\forall x' \in U$. We say that $F$ is upper semicontinuous if it is upper semicontinuous at every point $x \in \Omega$.

$^1$various properties of set-valued maps are not defined via topologies on $\mathcal{P}(Y)$ but depend on the topology on $Y$)
Proposition 1.2.6 ([4]). Let \( F : X \rightarrow Y \) be a set-valued map and let its domain \( \Omega \) be closed. Then \( F \) is upper semicontinuous if and only if \( F^{-1}(Q) \) is closed in \( X \) for every closed set \( Q \subseteq Y \).

Proposition 1.2.7 ([4]). The graph of an upper semicontinuous set-valued map \( F : X \rightarrow Y \) with closed values and a closed domain is closed. The converse is true if \( Y \) is compact.

Now we come to the concept of lower semicontinuity as it is found in [3].

Definition 1.2.8. A set-valued map \( F : X \rightarrow Y \) is called lower semicontinuous at \( x \in \Omega \) if for every open set \( V \subseteq Y \) such that \( V \cap F(x) \neq \emptyset \) there exists an open set \( U \subseteq X \) containing \( x \) such that \( F(x') \cap V \neq \emptyset \), \( \forall x' \in U \). We say that \( F \) is lower semicontinuous if it is lower semicontinuous at every point \( x \in \Omega \).

Equivalently we could have used the following characterization to define lower semicontinuity as it is done in [4].

Proposition 1.2.9 ([4]). \( F \) is lower semicontinuous at \( x \in \Omega \) if and only if for any \( y \in F(x) \) and for any sequence \( (x_n)_{n \in \mathbb{N}} \) converging to \( x \) there exists a sequence of elements \( y_n \in F(x_n) \) converging to \( y \).

Proposition 1.2.10 ([4]). \( F \) is lower semicontinuous if and only if \( F^{-1}(U) \) is open in \( X \) for every open set \( U \subseteq Y \).

Definition 1.2.11. We shall say that \( F \) is continuous at a point \( x \in \Omega \) if \( F \) is both upper semicontinuous and lower semicontinuous at \( x \), and that it is continuous if it is continuous at every point \( x \in \Omega \).

Definition 1.2.12. Let \((X, \|\cdot\|_X)\) and \((Y, \|\cdot\|_Y)\) be normed spaces. We shall say that a set-valued map \( F : X \rightarrow Y \) is Lipschitz if there exists a constant \( L \) such that:

\[
\forall x_1, x_2 \in X, \quad F(x_1) \subseteq F(x_2) + L\|x_1 - x_2\|_X B_Y.
\]

In this case we will also refer to \( F \) as \( L \)-Lipschitz.

Definition 1.2.13 ([11]). Let \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a set-valued map with nonempty values, then \( F \) is called one-sided Lipschitz if for every compact set \( K \subseteq \mathbb{R}^n \) there exists a constant \( \lambda \) such that for all \( x, y \in K \) and for all \( u \in F(x) \) there exists a \( v \in F(y) \) such that:

\[
\langle x - y, u - v \rangle_{\mathbb{R}^n} \leq \lambda|x - y|^2.
\]

1.3 Differential Inclusions

Let us consider a set-valued map \( F : X \rightarrow X \) and the corresponding Ordinary Differential Inclusion (ODI):

\[
\dot{x}(t) \in F(x(t)).
\]
Definition 1.3.1. A solution of problem (6) is any absolutely continuous function \(x(\cdot)\) defined on some interval \([0, T)\) such that \(\dot{x}(t) \in F(x(t))\) holds almost everywhere.

Due to our special interest in solutions defined on the entire positive real line \([0, +\infty)\) we will study properties of the right hand side of (6) that ensure existence of such solutions.

Definition 1.3.2. A set-valued map \(F : X \rightrightarrows X\) is said to be of linear growth if there exists a constant \(c > 0\) such that:

\[
\|F(x)\| := \sup_{y \in F(x)} \|y\| \leq c(\|x\| + 1), \quad \forall x \in \text{Dom}(F).
\]  

(7)

Definition 1.3.3. A set-valued map \(F\) is said to be a Marchaud map if it is upper semicontinuous, nontrivial, has closed convex values and linear growth.

Before stating the main result of this section we introduce some notations. The space \(W^{1,1}([0, \infty); X; e^{-bt}dt)\) denotes the space of Lebesgue measurable functions \(x(\cdot)\) from \([0, +\infty)\) to \(X\), such that \(x(\cdot)\) and its derivative \(\dot{x}(\cdot)\) are integrable with respect to the weighted measure \(e^{-bt}dt\). However, we supply this space not with the usual integral norm, but with the following topology: A sequence \((x_n(\cdot))_{n \in \mathbb{N}}\) converges to \(x(\cdot)\) if and only if

\[
\begin{align*}
x_n(\cdot) & \text{ converges to } x(\cdot) \text{ uniformly on compact sets} \\
\dot{x}_n(\cdot) & \text{ converges to } \dot{x}(\cdot) \text{ weakly in } L^1([0, \infty); X; e^{-bt}dt).
\end{align*}
\]

Definition 1.3.4. The solution map \(S_F(\cdot)\) assigns to each initial value \(x_0\) the set of all solution of (6) starting from \(x_0\).

Proposition 1.3.5. For a finite dimensional vector space \((X, \|\cdot\|)\) and a Marchaud map \(F : X \rightrightarrows X\) we set

\[
c := \sup_{x \in \text{Dom}(F)} \frac{\|F\|}{1 + \|x\|} (\langle +\infty).\]

Then the solution map \(S_F(\cdot)\) is upper semicontinuous with non-empty compact images from its domain to \(C([0, \infty); X)\) equipped with the topology of uniform convergence on compact subsets.

For \(b > c\) the solution map is also upper semicontinuous with compact images into the the space \(W^{1,1}([0, \infty); X; e^{-bt}dt)\). Furthermore, the graph of the restriction of \(S_F\) to a compact subset \(L \subseteq X\) is compact in \(X \times W^{1,1}([0, \infty); X; e^{-bt}dt)\).

Proof. A proof can be found in [1].

The importance of the following result is twofold. It can be not only regarded as a stability theorem but also as a existence result.
Theorem 1.3.6 (Filippov). Let \( I = [a, b] \) be an interval, \( y : I \to \mathbb{R}^n \) an absolutely continuous function, \( \beta > 0 \) a constant and let \( Q \subseteq I \times \mathbb{R}^n \) be defined by \( Q := \{(t, x) : |x - y(t)| \leq \beta\} \). Assume that \( F : Q \to \mathbb{R}^n \) is continuous with closed convex values and satisfies
\[
d_H(F(t, x), F(t, y)) \leq k(t)|x - y|.
\]
Assume moreover that
\[
|y(a) - x_0| = \delta \leq \beta
\]
and
\[
d(y'(t), F(t, y(t))) \leq p(t) \quad \text{a.e.}
\]
where \( p(\cdot) \in L^1(I) \). Set
\[
\xi(t) := \delta \exp \left( \int_a^t k(s) \, ds \right) + \int_a^t \exp \left( \int_s^t k(u) \, du \right) p(s) \, ds
\]
and let \( J = [a, \omega] \) be a nonempty interval such that \( t \in J \) implies \( \xi(t) \leq \beta \). Then there exists a solution \( x(\cdot) : J \to \mathbb{R}^n \) of
\[
x'(t) \in F(t, x(t)), \quad x(a) = x_0,
\]
such that
\[
|x(t) - y(t)| \leq \xi(t)
\]
and
\[
|x'(t) - y'(t)| \leq k(t)\xi(t) + p(t) \quad \text{a.e.}
\]
Proof. See [3] for a proof. \( \square \)

Support Functions

See [3] for the following results and more information.

Definition 1.3.7. Let \((X, \| \cdot \|)\) be a Banach space and \( K \subseteq X \) a nonempty subset. With any linear functional \( l \in X^* \) we associate:
\[
\sigma(l|K) := \sigma_K(l) := \sup_{x \in K} \langle l, x \rangle.
\]
The function \( \sigma_K : X^* \to \mathbb{R} \cup \{+\infty\} \) is called support function (of \( K \)).

Proposition 1.3.8. The support function has the following properties:
\[
\sigma(\alpha l|K) = \alpha \sigma(l|K)
\]
\[
\sigma(l_1 + l_2|K) \leq \sigma(l_1|K) + \sigma(l_2|K).
\]
Therefore \( \sigma(\cdot|K) \) is a convex function.
It is a remarkable fact that nonempty closed convex sets can be characterized by their corresponding support function:

\[ K = \{ y \in Y : \langle l, y \rangle \leq \sigma(l|K), \forall l \in Y^* \}. \]

**Definition 1.3.9.** A set-valued map \( F : X \rightharpoonup Y \) is called upper hemicontinuous at \( x_0 \in X \) if the function

\[
\begin{align*}
X & \to \mathbb{R} \\
 x & \mapsto \sigma(p|F(x))
\end{align*}
\]

is upper semicontinuous at \( x_0 \). \( F \) is called upper hemicontinuous if it is upper hemicontinuous at every \( x \in X \).

**Proposition 1.3.10.** Let \( F \) be a upper semicontinuous set-valued map with compact values from a metric space \((X,d)\) into a Banach space \((Y,\|\cdot\|)\), then the function:

\[
(x,l) \in X \times Y^* \to \sigma(l,F(x))
\]

is upper semicontinuous as well.

**Proposition 1.3.11.** Let \( A,B \subseteq \mathbb{R}^n \) be two convex and compact sets. Then their Hausdorff distance can be expressed in terms of the support-function:

\[
d_H(A,B) = \sup_{l \in \mathbb{R}^n,\|l\|=1} |\sigma(l|A) - \sigma(l|B)|.
\]

**The Convergence Theorem**

**Theorem 1.3.12 (Convergence Theorem).** Let \( F \) be a nontrivial upper hemicontinuous map with closed and convex values from a normed space \((X,d)\) into a Hilbert space \(Y\). Moreover, let \( I \subseteq \mathbb{R} \) be an interval and \((x_n(\cdot))_{n \in \mathbb{N}}\) and \((y_n(\cdot))_{n \in \mathbb{N}}\) be sequences of functions from this interval into \(X\) and \(Y\) respectively, satisfying:

For a.e. \( t \in I \) and for every neighborhood \( U \) of the origin in the product space \( X \times Y \) there exists a number \( N := N(t,U) \in \mathbb{N} \) such that:

\[
(x_n(t),y_n(t)) \in \text{Graph}(F) + U, \quad \forall n > N.
\]

If we assume that

\[
\begin{align*}
x_n(\cdot) & \text{ converges to some } x(\cdot) \text{ almost everywhere} \\
y_n(\cdot) & \text{ converges to some } y(\cdot) \text{ weakly in } L^1(I,Y; a(t) \, dt)
\end{align*}
\]

for a measurable and strictly positive function \( a(\cdot) \). Then

\[
y(t) \in F(x(t)), \quad \text{for almost every } t \in I.
\]

**Proof.** See [1] for a proof.
2 Viability Theory

Consider again set-valued map $F : X \rightrightarrows X$ and the corresponding differential inclusion:

$$\dot{x}(t) \in F(x(t)), \quad (8)$$

but this time together with the constraint $x(t) \in K \subseteq X$. We are looking for initial values $x_0 \in K$ such that there exists a solution of (8) that starts at $x_0$ and stays in $K$ for all $t \in [0, \infty)$. These solutions are called viable and the constraints are sometimes referred to as viability constraints. This terminology tries to emphasize that the dynamic system described by (8) can only exist (or ”live”) inside the set $K$. The following material can be found in [1] as well as [4]. Furthermore we shall always assume, if not stated otherwise, that $X$ is a finite dimensional vector space and that $F$ has nonempty images on $K$, i.e.: $K \subseteq \text{Dom}(F)$.

Remark 2.0.13. By definition, a function $x(\cdot) : [0, +\infty) \to X$ is a solution of (6) if it is absolutely continuous and its derivative, which is known to exist almost everywhere, fulfills $\dot{x}(t) \in F(x(t))$ for almost every $t \in [0, +\infty)$.

2.1 Viability Domain

Let $K \subseteq X$ be a subset of the space $X$. We want to find out what it means for a set-valued map $F$ to point into set $K$. We expect that, if $F$ is ”pointing inwards” at every point of $K$ solutions to $\dot{x}(t) \in F(x(t))$ have the possibility to choose their derivative directed towards $K$. Under which assumptions this turns in fact out to be true is discussed in Section 2.2.

Definition 2.1.1. A set $K \subseteq X$ is called a Viability Domain of (8) if

$$F(x) \cap T_K(x) \neq \emptyset \quad \forall x \in K.$$  

Leaving the realm of continuous time, the concept of ”pointing inwards” and viability becomes much simpler. Consider therefore the discrete time dynamical system (also called discrete differential inclusion)

$$x_{n+1} \in G(x_n) \quad \forall n \in \mathbb{N}, \quad (9)$$

for a set-valued map $G : X \rightrightarrows X$.

Definition 2.1.2. A solution $(x_n)_{n \in \mathbb{N}}$ of the discrete differential inclusion (9) is called an orbit (of $G$).

Definition 2.1.3. A set $D \subseteq X$ is called a discrete Viability Domain of (9) if

$$G(x) \cap D \neq \emptyset \quad \forall x \in D.$$  

What has to be proven in the continuous time case is a simple observation for discrete differential inclusions: For any $x$ in a discrete viability domain $D$, there exists an orbit $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in D$, $\forall n \in \mathbb{N}$ and $x_0 = x$. 

11
2.2 The Viability Theorem

Definition 2.2.1. A function \( x(\cdot) : [0, T] \rightarrow X \) is called viable in \( K \) (or simply viable, if it is clear what set we are referring to) if
\[
x(t) \in K, \forall t \in [0, T].
\]
Furthermore \( K \) is said to enjoy the
- local viability property (for a set-valued map \( F \)) if for every initial state \( x_0 \in K \) there exists a \( T > 0 \) and a viable solution on \([0, T]\) starting from \( x_0 \).
- (global) viability property if in the last definition one can take \( T \) to be \( +\infty \) for all \( x_0 \).

Theorem 2.2.2 (Haddad’s Theorem, [4]). Assume that
\[
\begin{align*}
\begin{cases}
F : X &\rightrightarrows X \text{ is upper semicontinuous} \\
F \text{ has convex and compact images} \\
K \text{ is locally compact}
\end{cases}
\end{align*}
\]
Then \( K \) enjoys the local viability property if and only if \( K \) is a viability domain of \( F \).

Remark 2.2.3. All open or closed subsets of a locally compact space are locally compact as well.

Since open subsets of a finite-dimensional topological vector space are locally compact, Theorem 2.2.2 gives an extension of the well known Peano Theorem to the set-valued case.

Theorem 2.2.4 ([1]). Let \( F : X \rightrightarrows X \) be an upper semicontinuous set-valued map with convex and compact values and let \( D \subseteq \text{Dom}(F) \) be open. Then, for any \( x_0 \in D \), there exists \( T > 0 \) such that the differential inclusion (8) has a solution on the interval \([0, T]\) starting from \( x_0 \).

In the context of viability theory, however, the interesting case is the one where the subset \( K \) is closed. For this situation we can make the statement of Theorem 2.2.2 more precise.

Theorem 2.2.5 (local Viability Theorem, [4]). Let \( F : X \rightrightarrows X \) be an upper semicontinuous set-valued map with convex, compact values and let \( K \subseteq \text{Dom}(F) \) be closed. Then \( K \) has the local viability property if and only if it is a viability domain. In fact for any \( x_0 \in K \) there exists a solution defined on \([0, T]\) for some \( T > 0 \) such that \( T \) is either equal to \(+\infty\) or \( \limsup_{t \rightarrow T^{-}} \| x(t) \| = +\infty \).

If we impose conditions about the growth of \( F \) we can exclude the case where \( \limsup_{t \rightarrow T^{-}} \| x(t) \| = +\infty \).

Theorem 2.2.6 (Viability Theorem, [4]). Let \( F : X \rightrightarrows X \) be upper semicontinuous with convex, compact values and have the linear growth property (7), i.e. let \( F \) be a Marchaud map. If \( K \) is a viability domain then for all \( x_0 \in K \) there exists a viable solution of (6) which is defined on \([0, +\infty)\), starts from \( x_0 \) and belongs to \( W^{1,1}(0, +\infty; X, e^{-b t} \, dt) \).
2.3 Viability Kernel

Discrete-Time Viability Kernel

Definition 2.3.1. Consider a discrete dynamical system (9) and a set $K \subseteq \text{Dom}(G)$. The discrete viability kernel of $K$ is the largest closed discrete viability domain contained in $K$ and is denoted by $\overrightarrow{\text{Viab}}_G(K)$.

Remark 2.3.2. The discrete viability kernel of a set $K$ can be characterized in this more intuitive way:
A point $x$ belongs to the viability kernel if and only if there exists a corresponding sequence $(x_n)_{n \in \mathbb{N}}$ with the following properties

$$
x_{n+1} \in G(x_n) \quad \forall n \in \mathbb{N}
$$

$$
x_n \in K \quad \forall n \in \mathbb{N}
$$

$$
x_0 = x \in K.
$$

One has to check that the so defined set is closed. Proposition 3.1.1 gives a constructive proof of the existence of such an object.

Continuous-Time Viability Kernel

Definition 2.3.3 ([1]). Let $F : X \rightsquigarrow X$ be a set-valued map and $K \subseteq \text{Dom}(F)$. Consider the differential inclusion with right hand side $F$. We call the largest closed subset of $K$ viable under $F$ the viability kernel of $K$ (for $F$) and denote it by $\text{Viab}_F(K)$ or merely $\text{Viab}(K)$. If $\text{Viab}(K)$ is empty we say that $K$ is a repeller.

The next theorem states that such an object does in fact exists.

Theorem 2.3.4 ([1]). Let $F : X \rightsquigarrow X$ be an upper semicontinuous set-valued map with convex, compact values and let $K \subseteq \text{Dom}(F)$ be closed. Then the viability kernel exists (it might be empty) and is equal to the set of all initial values $x_0 \in K$ such that there exists a solution starting at $x_0$ that is viable in $K$.

$$
\text{Viab}_F(K) := \{ x_0 \in K : S_F(x_0) \cap K \neq \emptyset \}
$$

(10)

where $K \subseteq C(0, +\infty; X)$ denotes the set of all continuous functions viable in $K$.

We reproduce the proof in [1] for completeness.

Proof. We have to show that $\text{Viab}_F(K)$ defined by (10) is closed, viable and the largest among all the sets fulfilling the first two properties.

Closedness: Let $x^n_0 \in \text{Viab}_F(K)$ converge to some $x^*_0 \in X$ and let $x_n(\cdot) \in S_F(x^n_0) \cap K$ be a sequence of corresponding solutions. Since $X$ is a finite dimensional space the sequence $(x^n_0)_{n \in \mathbb{N}}$ remains in a compact subset $L \subseteq K$. So $(x^n_0, x_n(\cdot))$ belongs to $\text{Graph}(S_F|_{L})$ which is compact by Theorem 1.3.5. So there exists a subsequence converging to some
$(x_0^*, x^*(\cdot)) \in Graph(S_F|_L)$. Since $S_F(x_0) \cap \mathcal{K}$ is closed $x^*(\cdot)$ is an element of this set. This proofs that $x_0^*$ belongs to $\text{Viab}_F(K)$.

$\text{Viab}_F(K)$ is in fact viable: Let $x_0 \in \text{Viab}_F(K)$ and $x(\cdot)$ the corresponding viable solution. Then $\tau \mapsto y^i(\tau) := x(\tau + t)$ is a viable solution to (6) starting from $x(t)$. Therefore $x(t)$ belongs to $\text{Viab}_F(K)$.

Maximality: Let $L \subseteq K$ be a viable subset of $K$. For all $x_0 \in L$ there exists a solution remaining in $L$ and thus in $K$. Therefore $S_F(x_0) \cap \mathcal{K}$ is nonempty which shows that $L \subseteq \text{Viab}_F(K)$.

Remark 2.3.5. The previous theorem shows that it makes no difference whether we require the viability kernel to be the largest viable subset of $K$ or just the largest subset viable in $K$. Either way, every solution starting from a point in $K \setminus \text{Viab}_F(K)$ has to leave $K$ in finite time and never enters the viability kernel.

Theorem 2.3.6 (Saint-Pierre,[1]). Let $F : X \rightsquigarrow X$ be a Marchaud map and $K \subset X$ be closed with nonempty interior. If the viability kernel $\text{Viab}_F(K)$ of $K$ is contained in the interior of $K$, then its boundary is also a viability domain.

Convex Viability Kernels

The following results can be found in [7].

Proposition 2.3.7. Assume that $K$ is convex. If $\text{Graph}(F)$ is convex and if $F$ is a Marchaud, then $\text{Viab}_F(K)$ is convex.

Remark 2.3.8. In particular, when $F$ is of the form $F(x) = f(x,V)$ and $f$ results from a linear control system, that is $f(x,v) := Ax + Bv$ where $A$ and $B$ are matrices and $V$ is convex and compact, then $\text{Graph}(F)$ is convex.

Convexity of the viability kernel is of interest, since it guarantees that the viability kernel has no 'holes' and therefore makes it easier to find its boundary. Moreover, techniques using support functions can be employed.
3 Approximating the Viability Kernel

The main goal of this section is to discuss the method established in [15] to approximate the viability kernel of a given system. This algorithm, often referred to as ”Viability Kernel Algorithm”, gives a fully discretized system which can be executed on a computer and therefore used for numerical purposes. The difficulty at hand is that not only the continuity in time has to be treated (like for ordinary differential equations), but also, due to the fact that the values of the right hand side of the dynamical system are in general not just single points, the state space has to be discretized. Without further mentioning, we will use the terminology and results from [15], if not stated otherwise.

3.1 Constructing Discrete Viability Kernels

Consider again the discrete time dynamical system
\[ x_{n+1} \in G(x_n) \quad \forall n \in \mathbb{N}, \] (11)
together with the state constraint
\[ x_n \in K \quad \forall n \in \mathbb{N}. \]

We are trying to use the property of the viability kernel being the largest viability domain by starting with the largest subset of \( K \), namely \( K \) itself, and iteratively removing points that prohibit the set of being a viability domain. Consider the following sequence of sets:
\[ K^1 := K, \]
\[ K^{n+1} := \{ x \in K^n : G(x) \cap K^n \neq \emptyset \}. \]

Furthermore we denote
\[ K^\infty = \bigcap_{n=1}^{\infty} K^n. \]

Proposition 3.1.1. Let \( G : \Omega \rightrightarrows X \) be an upper semicontinuous set-valued map with closed values and let \( K \subseteq \Omega \) be compact. Then
\[ K^\infty = \overrightarrow{\text{Viab}}_G(K). \]

Proof. First we show that \( \overrightarrow{\text{Viab}}_G(K) \subseteq K^\infty \) by induction. Obviously \( \overrightarrow{\text{Viab}}_G(K) \subseteq K \). For all \( x \in K \setminus K^1 \) it follows that \( G(x) \cap K = \emptyset \), thus \( x \notin \overrightarrow{\text{Viab}}_G(K) \). Therefore
\[ \overrightarrow{\text{Viab}}_G(K) \subseteq K^1, \]
which implies that
\[
\overrightarrow{\text{Viab}_G(K)} = \overrightarrow{\text{Viab}_G(K^1)}.
\]
Let us assume that
\[
\overrightarrow{\text{Viab}_G(K)} = \overrightarrow{\text{Viab}_G(K^{n-1})}.
\]
Again, for \( x \in K^{n-1} \setminus K^n \), it follows that \( G(x) \cap K^{n-1} = \emptyset \). This implies that
\[
x \not\in \overrightarrow{\text{Viab}_G(K^{n-1})}
\]
and therefore
\[
x \not\in \overrightarrow{\text{Viab}_G(K)}.
\]
This shows that
\[
\overrightarrow{\text{Viab}_G(K)} = \overrightarrow{\text{Viab}_G(K^n)} \subseteq K^n, \ \forall n \in \mathbb{N}.
\]
Therefore
\[
\overrightarrow{\text{Viab}_G(K)} \subseteq K^{\infty}.
\]
Conversely, we are going to show that \( K^{\infty} \) is a viability domain. We start by noting that \( K^n \) is closed for all \( n \in \mathbb{N} \) since \( G(\cdot) \) is upper semicontinuous. Let \( x_0 \) be a point in \( K^{\infty} \). By construction, this implies that \( x_0 \in K^n \ \forall n \in \mathbb{N} \) and \( G(x_0) \cap K^n \neq \emptyset \ \forall n \in \mathbb{N} \). Therefore \( G(x_0) \cap K^n \) forms a decreasing sequence of compact, nonempty sets, which leaves \( K^{\infty} \cap G(x_0) \) nonempty. Since \( \overrightarrow{\text{Viab}_G(K)} \) is the largest viability domain contained in \( K \), we have
\[
K^{\infty} \subseteq \overrightarrow{\text{Viab}_G(K)},
\]
which completes the proof. 

**Definition 3.1.2.** Let \( G : \Omega \rightrightarrows X \) be an upper semicontinuous set-valued map with closed values and let \( r > 0 \). We call the map \( G^r(\cdot) \) defined by
\[
G^r(x) := G(x) + rB \ \forall x \in \Omega
\]
an extension of \( G \) with a closed ball of radius \( r \).

**Remark 3.1.3.** We use the same notation for sets and write \( A^r \) for \( A + rB \), where \( A \subseteq X \).

**Remark 3.1.4.** If \( G \) is an upper semicontinuous map, so is \( G^r \).

Continuing the idea of this section and combining the previous concepts we define:
\[
K^{r,n+1} := \{ x \in K^n : G^r(x) \cap K^{r,n} \neq \emptyset \}
\]
\[
K^{r,\infty} := \bigcap_{n=1}^{\infty} K^{r,n}.
\]
Corollary 3.1.5. Due to Proposition 3.1.1

\[ \overrightarrow{\text{Viab}_{G^r}}(K) = K^{r,\infty}. \]

Since \( G^r(\cdot) \) is an approximation of \( G(\cdot) \) for decreasing \( r \) we want to show that also the corresponding viability kernels \( \overrightarrow{\text{Viab}_{G^r}}(K) \) approach \( \overrightarrow{\text{Viab}_{G}}(K) \).

Proposition 3.1.6. Let \( G : \Omega \rightrightarrows X \) be an upper semicontinuous set-valued map with closed values and let \( K \subseteq \Omega \) be compact. Then:

\[ \overrightarrow{\text{Viab}_{G}}(K) = \bigcap_{r > 0} \overrightarrow{\text{Viab}_{G^r}}(K). \]

Proof. By showing that \( \bigcap_{r > 0} \overrightarrow{\text{Viab}_{G^r}}(K) \) is a closed viability domain we know that it has to be a subset of \( \overrightarrow{\text{Viab}_{G}}(K) \). Let therefore \( x_0 \in \bigcap_{r > 0} \overrightarrow{\text{Viab}_{G^r}}(K) \). It follows from the definition that:

\[ G^r(x_0) \cap K^{r,\infty} \neq \emptyset \]

and since the left hand side forms a sequence of compact sets

\[ \bigcap_{r > 0} (G^r(x_0) \cap K^{r,\infty}) \neq \emptyset. \]

This yields

\[ G(x_0) \cap \bigcap_{r > 0} K^{r,\infty} \neq \emptyset. \]

Bearing in mind that \( \overrightarrow{\text{Viab}_{G^r}}(K) \) is closed for every \( r > 0 \), this completes the proof of the first inclusion.

On the other hand

\[ \overrightarrow{\text{Viab}_{G}}(K) \subseteq \overrightarrow{\text{Viab}_{G^r}}(K) \quad \forall r > 0, \]

which gives

\[ \overrightarrow{\text{Viab}_{G}}(K) \subseteq \bigcap_{r > 0} \overrightarrow{\text{Viab}_{G^r}}(K). \]

\[ \square \]

3.2 Viability Kernel Convergence

Let \( F : \Omega \rightrightarrows X \) be a upper semicontinuous set-valued map with compact and convex values that satisfies the linear growth condition (7) and let \( \Gamma_\rho \) be a family of set-valued maps that satisfies:

\[ \forall \epsilon > 0, \exists \rho_\epsilon > 0, \forall \rho \in (0, \rho_\epsilon] : \text{Graph} \left( \frac{\Gamma_\rho - 1}{\rho} \right) \subseteq \text{Graph}(F) + \epsilon B_{X \times X} \quad (12) \]

Note that for

\[ F_\rho := \frac{\Gamma_\rho - 1}{\rho} \]
the assumption (12) implies the following:
\[ \limsup_{\rho \to 0^+} \text{Graph} (F_\rho) \subseteq \text{Graph}(F). \] (13)

**Theorem 3.2.1.** Let \( F : \Omega \rightrightarrows X \) be a Marchaud map and \( \Gamma_\rho \) be a family of set-valued maps that satisfies (12). Furthermore let \( K_\rho \) be a family of subsets of \( X \) such that \( \limsup_{\rho \to 0^+} K_\rho = K \).

Then \( K^\sharp := \limsup_{\rho \to 0^+} \overrightarrow{\text{Viab}_{\Gamma_\rho}}(K_\rho) \) is viable under \( F \):
\[ \limsup_{\rho \to 0^+} \overrightarrow{\text{Viab}_{\Gamma_\rho}}(K_\rho) \subseteq \overrightarrow{\text{Viab}_F}(K). \]

**Proof.** See [15] for the proof. \( \square \)

**Examples / Applications**

**Corollary 3.2.2** (thickened Euler-Step). Let the assumptions of Theorem 3.2.1 be fulfilled and let \( \Gamma_\rho \) be defined by
\[ \Gamma_\rho := 1 + \rho F + \frac{ML}{2} \rho^2 B \] (14)
for arbitrary numbers \( M \) and \( L \). Then
\[ \limsup_{\rho \to 0^+} \overrightarrow{\text{Viab}_{\Gamma_\rho}}(K_\rho) \subseteq \overrightarrow{\text{Viab}_F}(K). \]

**Proof.** Considering
\[ \text{Graph} \left( \frac{\Gamma_\rho - 1}{\rho} \right) = \text{Graph} \left( F + \frac{ML}{2} \rho B \right) \subseteq \text{Graph}(F) + \frac{ML}{2} \rho B \]
Theorem 3.2.1 proves the claim. \( \square \)

The following proposition was proven by the author.

**Proposition 3.2.3.** Let \( F : X \rightrightarrows X \) be a Lipschitz continuous set-valued map with compact and convex values that satisfies the linear growth condition (7) and the boundedness condition (15). Let \( \Gamma_H \), sometimes called the Heun scheme, be defined by
\[ \Gamma_H(x) := x + \rho^2 \frac{1}{2} [F(x) + F(x + \rho F(x))]. \]

Then
\[ \limsup_{\rho \to 0^+} \overrightarrow{\text{Viab}_{\Gamma_H}}(K_\rho) \subseteq \overrightarrow{\text{Viab}_F}(K). \]
Proof. It holds that
\[
\text{dist} \left( \frac{1}{2} [F(x) + F(x + \rho F(x))], F(x) \right) \leq \\
\text{dist} \left( \frac{1}{2} [F(x) + F(x) + L\|\rho F(x)\|B], F(x) \right) \leq \\
\text{dist} \left( F(x) + \frac{1}{2} ML\rho B, F(x) \right) \leq \frac{1}{2} ML\rho.
\]

Remark 3.2.4. There exists another set-valued version of the Heun scheme, which can be used if the right hand side \( F \) of the differential inclusion is parametrized by \( F(x) := f(x, U) := \{f(x, u) : u \in U\} \) for some function \( f(\cdot) \) and some set \( U \). This version of the Heun scheme (see [11]) is defined by:
\[
\tilde{\Gamma}_H(x) := \left\{ x + \rho \frac{1}{2}[f(x, u) + f(x + \rho f(x, u), u)] : u \in U \right\}.
\]

**The Lipschitz Case**

Only requiring \( F \) to be Marchaud is not sufficient to prove that the discrete viability kernels we constructed in Theorem 3.2.1 do in fact converge to the "true" viability kernel. As we have seen in the last section we were only able to prove the convergence to a subset. By imposing further assumptions, such as Lipschitz continuity, this result can be strengthened.

**Theorem 3.2.5.** Let \( F : \Omega \rightrightarrows X \) be a Lipschitz continuous set-valued map with compact and convex values that satisfies the linear growth condition (7). Let \( K \) be a closed subset of \( X \) and let the following boundedness condition hold:
\[
M := \sup_{x \in K} \sup_{y \in F(x)} \|y\| < +\infty.
\] (15)

Then
\[
\limsup_{\rho \to 0^+} \text{Viab}_{\Gamma_{\rho}}(K) = \text{Viab}_F(K)
\] (16)
holds for \( \Gamma_{\rho} \) defined in (14).

**Proof.** Due to Corollary 3.2.2 the following statement holds true:
\[
\limsup_{\rho \to 0^+} \text{Viab}_{\Gamma_{\rho}}(K) \subseteq \text{Viab}_F(K).
\]
on the other hand we want to use the Lipschitz property to show the opposite inclusion. Let \( x_0 \in K \) and \( x(\cdot) \in S_F(x_0) \). Note
\[
x(t + \rho) - x(t) = \int_t^{t+\rho} \dot{x}(s) ds \in \int_t^{t+\rho} F(x(s)) ds,
\]
where the last integral is to be interpreted in the Aumann sense. Lipschitz continuity of $F$ implies
\[
\int_t^{t+\rho} F(x(s)) \, ds \subseteq \int_t^{t+\rho} F(x(t)) + L\|x(s) - x(t)\|B \, ds \quad \forall t > 0.
\]
Since $F$ is bounded $\|x(s) - x(t)\| \leq (s - t)M$ and
\[
\int_t^{t+\rho} F(x(t)) + L\|x(s) - x(t)\|B \, ds \subseteq \rho F(x(t)) + \frac{\rho^2 LM}{2}B \quad \forall t > 0.
\]
Therefore
\[
x(t + \rho) - x(t) \in \rho F(x(t)) + \frac{\rho^2 LM}{2}B \quad \forall t > 0.
\]
So we just proved that if $x(\cdot) \in \mathcal{S}_F(x_0)$ the sequence $\xi_n := x(\rho n) \quad \forall n \in \mathbb{N}$ is an orbit of the discrete dynamical system
\[
\xi_{n+1} \in \Gamma_{\rho}(\xi_n) \quad \forall n \in \mathbb{N}
\]
with the same initial value and is viable if $x(\cdot)$ is such. Thus
\[
\text{Viab}_F(K) \subseteq \text{Viab}_{\Gamma_{\rho}}(K) \quad (17)
\]
and
\[
\text{Viab}_F(K) \subseteq \limsup_{\rho \to 0^+} \text{Viab}_{\Gamma_{\rho}}(K).
\]

### 3.3 Using Finite valued Set-Valued Maps

On our journey to a fully discretized system, which can actually be treated on a computer, we are still facing the problem of sets being difficult to handle. Therefore we want to reduce them to some kind of finite grids.

For any $h > 0$ we consider a finite subset $X_h$ of $X$ which approximates $X$ in the following sense:
\[
\forall x \in X, \exists x_h \in X_h : \|x - x_h\| \leq \alpha(h),
\]
where $\alpha(h)$ goes to zero for $h$ converging to zero:
\[
\lim_{h \to 0^+} \alpha(h) = 0.
\]
Let $K_h$ be a subset of $X_h$ and let $G_h : X_h \rightrightarrows X_h$ be a set-valued map where $\text{Dom}(G_h) \subseteq K_h$. 

20
Finite Viability Kernels

In the spirit of the algorithm from Section 3.1 we introduce the following notation:

\[ K_h^1 := K_h, \]
\[ K_h^{n+1} := \{ x_h \in K_h^n : G_h(x) \cap K_h^n \neq \emptyset \}, \]
\[ K_h^\infty := \bigcap_{n=1}^{\infty} K^n. \]

This, however, might be confusing, since in application we are not given a set-valued map \( G_h \) which is defined on \( X_h \) but rather have a map \( G \) defined on \( X \) and want to transfer it to \( X_h \). This is done in the following way: \( G_h(x_h) := G(x_h) \cap X_h \). In order for Proposition 3.1.1 to stay true we have to ensure that \( G_h(x_h) \cap X_h \neq \emptyset \) for all \( x_h \in X_h \). Otherwise \( G_h(\cdot) \) might have empty values which was not allowed in the assumptions of the proposition. Now the definition of an extension of a set-valued map comes in handy since by inflating by \( r \), where \( r \geq \alpha(h) \), we can ensure that this will not happen whilst the extension still remains a reasonable approximation as Proposition 3.1.6 tells us.

Before formulating the corresponding results we introduce more notation:

\[ G_h^r(x_h) := G^r(x_h) \cap X_h, \]
\[ K_h^{r,1} := K_h^r, \]
\[ K_h^{r,n+1} := \{ x_h \in K_h^{r,n} : G_h^r(x) \cap K_h^{r,n} \neq \emptyset \}, \]
\[ K_h^{r,\infty} := \bigcap_{n=1}^{\infty} K_h^{r,n}. \]

**Corollary 3.3.1.** Due to Proposition 3.1.1

\[ K_h^{r,\infty} = \overrightarrow{\text{Viab}_{G_h^r}(K_h)}. \]

**Corollary 3.3.2.** Due to Proposition 3.1.6

\[ \bigcap_{r>0} K_h^{r,\infty} = \bigcap_{r>0} \overrightarrow{\text{Viab}_{G_h^r}(K_h)} = \overrightarrow{\text{Viab}}_{G_h}(K_h). \]

**Remark 3.3.3.** In the process of forming \( K_h^{r,\infty} \) i.e. computing the intersection \( \bigcap_{n=1}^{\infty} K_h^{r,n} \) only finitely many \( K_h^{r,n} \) have to be considered. This is due to the fact that \( K_h \) consists only of finitely many points and therefore there exist a natural number \( n_0 \in \mathbb{N} \) such that \( K_h^{r,n} = K_h^{r,n_0} \) for all \( n \geq n_0 \).

Subsequently we want to know in what sense entire orbits of discrete dynamical systems can be approximated by orbits of the corresponding finite system.

**Lemma 3.3.4.** Let \( G : X \to X \) be a Lipschitz continuous set-valued map with constant \( k \) and let \( r \geq k\alpha(h) \).

Assume that

\[ \forall \xi \in G(x) \exists \xi_h \in G(x) \cap X_h \text{ such that } \| \xi - \xi_h \| \leq \frac{r}{k}. \] (18)
For any orbit $\xi = (\xi_n)_{n \in \mathbb{N}}$ of the discrete dynamical system

$$\xi_{n+1} \in G(\xi_n)$$

there exists an orbit $\xi_h = (\xi_{n,h})_{n \in \mathbb{N}}$ of the finite dynamical system

$$\xi_{n+1,h} \in G'_h(\xi_{n,h})$$

such that

$$\|\xi_n - \xi_{n,h}\| \leq \frac{r}{k} \quad \forall n \in \mathbb{N}.$$  

**Corollary 3.3.5.** Let the assumptions of Lemma 3.3.4 be fulfilled. Then the following holds:

$$\overrightarrow{\text{Viab}}_G(K) \subseteq \overrightarrow{\text{Viab}}_{G'_h}(K_{h}^{r}) + \frac{r}{k} \mathcal{B}.$$  

**Final Result**

Before presenting the final result we state two lemmas.

**Lemma 3.3.6.** Let $D \subseteq X$ be closed, and let $D_\rho$ be a decreasing sequence of closed subsets of $X$ such that $\bigcap_{\rho > 0} D_\rho = D$. Then

$$D = \limsup_{\rho,h \to 0^+} (D_\rho + \alpha(h)) \cap X_h.$$  

**Lemma 3.3.7.** Let $F : \Omega \rightrightarrows X$ be an $L$-Lipschitz continuous set-valued map. Then $\Gamma_\rho := 1 + \rho F + \frac{ML^2}{2} \rho^2 \mathcal{B}$ is $k$-Lipschitz continuous with $k = 1 + \rho L$.

**Theorem 3.3.8 ([15]).** Let $F : \Omega \rightrightarrows X$ be an $L$-Lipschitz continuous set-valued map with compact and convex values that satisfies the linear growth condition (7). Let $G_\rho := 1 + \rho F$ and $\Gamma_\rho := 1 + \rho F + \frac{ML^2}{2} \rho^2 \mathcal{B}$. Assume that $\rho$ and $h$ are chosen in such a way that $\alpha(h) \leq \frac{ML^2}{2} \rho^2$. Furthermore note that $k := 1 + \rho L$ and $R := kML\rho^2$. Let $\Gamma^R_\rho : X \rightrightarrows X$ and $\Gamma^R_{\rho,h} : X_h \rightrightarrows X_h$ be defined by:

$$\Gamma^R_\rho(x) := \Gamma_\rho(x) + RB = x + \rho F + \left(\frac{1}{2} + k\right) ML\rho^2 \mathcal{B}$$

$$\Gamma^R_{\rho,h}(x_h) := \Gamma^R_\rho(x_h) \cap X_h.$$  

Then:

$$\text{Viab}_F(K) = \limsup_{\rho,h \to 0^+} \left( \overrightarrow{\text{Viab}}_{\Gamma_\rho}(K) + \alpha(h) \mathcal{B} \right) \cap X_h$$  \hspace{1cm} (19)

and

$$\text{Viab}_F(K) = \limsup_{\rho,h \to 0^+} \overrightarrow{\text{Viab}}_{\Gamma^R_{\rho,h}}(K_{h}^{ML\rho^2}).$$  \hspace{1cm} (20)
Proof. We start by proofing the first claim. From Theorem 3.2.5 we know that:

$$\limsup_{\rho \to 0^+} \overrightarrow{\text{Viab}}_{\Gamma_{\rho}}(K) = \text{Viab}_F(K)$$

and in its proof we have seen that

$$\text{Viab}_F(K) \subseteq \overrightarrow{\text{Viab}}_{\Gamma_{\rho}}(K).$$

Combining this with Lemma 3.3.6 we obtain the desired result.

For proofing the (20) we want to use Corollary 3.3.5 with \(G = \Gamma_{\rho}\) and \(r = R\). But first we have to check that the thickness-like property (18) is fulfilled. i.e. we need to check that

$$\forall \xi \in \Gamma_{\rho}(x) \exists \xi_h \in \Gamma_{\rho}(x) \cap X_h \text{ such that } \|\xi - \xi_h\| \leq \frac{r}{k} = ML\rho^2.$$  

First we note that

$$\forall \xi \in \Gamma_{\rho}(x) \exists \xi' \in x + \rho F(x) \text{ such that } \|\xi - \xi'\| \leq \frac{ML}{2} \rho^2$$

and from the definition of \(\alpha(h)\) it follows that

$$\forall \xi' \in x + \rho F(x) \exists \xi_h \in \Gamma_{\rho}(x) \cap X_h \text{ such that } \|\xi' - \xi_h\| \leq \alpha(h) \leq \frac{ML}{2} \rho^2,$$

which combined shows that the desired assumption does in fact hold true. Therefore Corollary 3.3.5 says that

$$\overrightarrow{\text{Viab}}_{\Gamma_{\rho}}(K) \subseteq \overrightarrow{\text{Viab}}_{\Gamma_{\rho,h}}(K) + ML\rho^2 \mathcal{B}.$$  

In conjunction with overapproximation property (17) we can conclude that

$$\text{Viab}_F(K) \subseteq \overrightarrow{\text{Viab}}_{\Gamma_{\rho,h}}(K) + ML\rho^2 \mathcal{B}.$$  

(21)

Therefore

$$\text{Viab}_F(K) \subseteq \limsup_{\rho,h \to 0^+} \left( \overrightarrow{\text{Viab}}_{\Gamma_{\rho,h}}(K) + ML\rho^2 \mathcal{B} \right),$$

which implies

$$\text{Viab}_F(K) \subseteq \limsup_{\rho,h \to 0^+} \left( \overrightarrow{\text{Viab}}_{\Gamma_{\rho,h}}(K) \right).$$

For the opposite inclusion we are going to use Theorem 3.2.1. Therefore we have to check that:

$$\forall \epsilon > 0, \exists \rho_\epsilon > 0, \forall \rho \in (0, \rho_\epsilon] : \text{Graph} \left( \frac{\Gamma_{\rho,h}}{\rho} - 1 \right) \subseteq \text{Graph}(F) + \epsilon \mathcal{B} \times \mathcal{B}.$$  

(22)
Let us note that
\[
\text{Graph} \left( \frac{\Gamma_{\rho,h} - 1}{\rho} \right) = \text{Graph} \left( \left( F + \left( \frac{1}{2} + k \right) ML \rho B \right) \cap X_h \right) \subseteq \text{Graph} \left( F + \left( \frac{1}{2} + k \right) ML \rho B \right) \subseteq \text{Graph} \left( F + \left( \frac{3}{2} + \rho L \right) ML \rho B \right).
\]

Clearly,
\[
\forall \epsilon > 0, \exists \rho_\epsilon > 0, \forall \rho \in (0, \rho_\epsilon] : \left( \frac{3}{2} + \rho L \right) ML \rho \leq \epsilon,
\]
which shows that (22) is in fact true.

Therefore Theorem 3.2.1 states that:
\[
\limsup_{\rho,h \to 0^+} \overrightarrow{\text{Viab}_{\rho,h}} (K^2_{ML \rho}) \subseteq \text{Viab}_F (K).
\]

This proofs (20).

At this point we want to draw some attention towards (21) which is one of the most remarkable properties of the described algorithm. It states, that our approximation of the viability kernel contains the true one up the enlargement by $ML \rho^2$. This can be translated to
\[
\text{dist} \left( \overrightarrow{\text{Viab}_F (K), \overrightarrow{\text{Viab}_{\rho,h}} (K^2_{ML \rho})} \right) \leq ML \rho^2
\]
This result could have been obtained in an easier way, but it just goes to show that the difficult part about estimating the Hausdorff distance between the true viability kernel and our approximation lies in
\[
\text{dist} \left( \overrightarrow{\text{Viab}_{\rho,h}} (K^2_{ML \rho}), \text{Viab}_F (K) \right) = \inf \left\{ \epsilon > 0 : \overrightarrow{\text{Viab}_{\rho,h}} (K^2_{ML \rho}) \subseteq \text{Viab}_F (K) + \epsilon B \right\}
\]
i.e. figuring out how much we are overestimating the true viability kernel.
4 Estimates for the Viability Kernel Algorithm

In the last section we saw that under some assumptions about the right hand side of a differential inclusion an approximation of the viability kernel, which converges to the true viability kernel, can be computed. However, we do not know about the quality of a particular approximation. We use the results and proofs presented in [14] with slight modifications to show error estimates in the case of differential inclusions with one-sided Lipschitz right hand side. In [14] it is assumed that there exists an \( \epsilon_0 > 0 \) such that

\[
d_H(\text{Viab}_F(K), \text{Viab}_F(K^{\epsilon})) \leq L_V \epsilon, \quad \forall \ 0 \leq \epsilon \leq \epsilon_0.
\] (24)

This assumption is dropped and instead a proof of the continuity at 0 of said distance is presented in order to achieve a slightly more general result. Furthermore, a sufficient condition for (24) is presented.

Definitions and Assumptions

Let \( X \) be a finite-dimensional vector space and let \( K \subseteq X \) be a compact subset. We consider the differential inclusion

\[
\dot{x}(t) \in F(x(t)),
\] (25)

with right hand side \( F : X \rhd X \).

Definition 4.0.9. A sequence \( (x_n)_{n \in \mathbb{N}} \) satisfying \( d_H(x_{n+1}, G(x_n)) \leq d \) for some \( d \geq 0 \) is called a \( d \)-pseudotrajectory of \( G \).

Definition 4.0.10. A set-valued map \( G : X \rhd X \) is said to have the \( (d, \epsilon) \)-shadowing property in \( K \), if for every \( d \)-pseudotrajectory \( (x_n)_{n \in \mathbb{N}} \) viable in \( K \) there exists an orbit \( (p_n)_{n \in \mathbb{N}} \) of \( G \) such that \( |p_n - x_n| \leq \epsilon \) for all \( n \in \mathbb{N} \).

Definition 4.0.11. A set-valued map \( G : X \rhd X \) has the inverse \( (d, \epsilon) \)-shadowing property in \( K \), if for every continuous map \( \Phi : X \rhd X \) with compact and convex values satisfying

\[
d_H(\Phi(x), G(x)) \leq d
\]

for all \( x \) in a neighborhood of \( K \) every orbit \( (p_n)_{n \in \mathbb{N}} \) of \( G \) viable in \( K \) there exists an orbit \( (x_n)_{n \in \mathbb{N}} \) of \( \Phi \) such that \( |x_n - p_n| \leq \epsilon \) for all \( n \in \mathbb{N} \).

Definition 4.0.12. Let \( \omega(\cdot) \) denote the error in the viability kernel with respect to the enlargement of \( K \):

\[
\omega(\epsilon) := d_H(\text{Viab}_F(K), \text{Viab}_F(K^{\epsilon})).
\]

Assumptions in the following section:

1. There exists a \( d_{(s)}^{(s)} > 0 \) and a \( d_{(is)}^{(is)} > 0 \) such that the \( \rho \)-flow \( G_\rho \) has
1a. the \((d, \varphi(d))\)-shadowing property in \(K^{e_0}\) for \(d \in (0, d^{(is)}_{0,\rho}]\) and

1b. the inverse \((d, \psi(d))\)-shadowing property in \(K\) for \(d \in (0, d^{(is)}_{0,\rho}]\), where \(\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+\) are increasing functions with \(\lim_{d \to 0^+} \varphi(d) = \lim_{d \to 0^+} \psi(d) = 0\), which can depend on \(\rho\).

2. \(F\) is \(L\)-Lipschitz on \(K^{e_0}\) with compact convex values.

Assumption 2 implies that:

\[
M := \|F\|_\infty := \sup_{x \in K^{e_0}} \sup_{y \in F(x)} |y| < +\infty.
\]

**Definition 4.0.13.** The reachable set \(\mathcal{R}(0, \cdot, \cdot)\) of a differential inclusion (25) is defined by:

\[
\mathcal{R}(0, \rho, x) := \{x(\rho) : x(\cdot) \text{ is a solution to (25) and } x(0) = x\}.
\]

Let the Euler-step \(\Gamma_\rho(\cdot)\) and the \(\rho\) -flow \(G_\rho(\cdot)\) be defined by

\[
\Gamma_\rho(\cdot) : \begin{cases}
X \sim X \\
x \mapsto x + \rho F(x)
\end{cases}
\]

\[
G_\rho(\cdot) : \begin{cases}
X \sim X \\
x \mapsto \mathcal{R}(0, \rho, x)
\end{cases}
\]

And the fully discretized system:

\[
\Gamma_{\rho,h}(\cdot) : \begin{cases}
X_h \sim X_h \\
x \mapsto (x_h + \rho F(x_h) + k(\rho, h)B) \cap X_h,
\end{cases}
\]

where \(k(\rho, h) := (2 + L\rho)\alpha(h)\).

**4.1 Results**

This first proposition was proven by the author:

**Proposition 4.1.1.** Let Assumption 2 be fulfilled and let \(K \subseteq X\) be compact, then \(\omega(\cdot)\) is continuous at zero:

\[
\lim_{\epsilon \to 0^+} d_H(\text{Viab}_F(K), \text{Viab}_F(K^\epsilon)) = 0.
\]

**Proof.** We start by proving that

\[
\bigcap_{\epsilon > 0} \text{Viab}_F(K^\epsilon) = \text{Viab}_F(K). \tag{26}
\]
First note that
\[ \text{Viab}_F(K) \subseteq \text{Viab}_F(K') \quad \forall, \epsilon > 0 \]
which gives
\[ \text{Viab}_F(K) \subseteq \bigcap_{\epsilon > 0} \text{Viab}_F(K'). \]

On the other hand let \( x_0 \in \bigcap_{\epsilon > 0} \text{Viab}_F(K') \). It follows that
\[ \forall \epsilon > 0, \exists x_\epsilon(\cdot) \in S_F(x_0) \quad \text{s.t.} \quad x_\epsilon([0, \infty)) \subseteq K'. \]

We know from Proposition 1.3.5 that \( S_F(x_0) \) is compact, which implies the existence of a \( x^*(\cdot) \in S_F(x_0) \) and a subnet again denoted by \( x_\epsilon(\cdot) \) such that \( \lim_{\epsilon \to 0^+} \| x_\epsilon(\cdot) - x^*(\cdot) \|_\infty = 0. \)

Assume that there exists a \( t_0 > 0 \) such that \( x^*(t_0) \not\in K \) which means that \( d(x^*(t_0), K) > 0. \) Combining
\[ \lim_{\epsilon \to 0^+} d(x_\epsilon(t_0), K) = 0 \]
and
\[ \lim_{\epsilon \to 0^+} d(x_\epsilon(t_0), x^*(t_0)) = 0, \]
however, contradicts this. This proves the viability of \( x^*(\cdot) \) in \( K \) and therefore finishes the first part of the proof.

Since \( \text{Viab}_F(K') \) is decreasing for \( \epsilon \to 0 \) the distance \( d(x, \text{Viab}_F(K')) \) is increasing for all \( x \in X \). Therefore
\begin{align*}
\liminf_{\epsilon \to 0^+} d(x, \text{Viab}_F(K')) &= 0 \\
\iff \lim_{\epsilon \to 0^+} d(x, \text{Viab}_F(K')) &= 0 \\
\iff d(x, \text{Viab}_F(K')) &= 0 \quad \forall \epsilon > 0 \\
\iff x &\in \text{Viab}_F(K') \quad \forall \epsilon > 0 \\
\iff x &\in \bigcap_{\epsilon > 0} \text{Viab}_F(K').
\end{align*}

This shows that
\[ \limsup_{\epsilon \to 0^+} \text{Viab}_F(K') = \liminf_{\epsilon \to 0^+} \text{Viab}_F(K') = \text{Viab}_F(K). \]

This says that \( \omega(\cdot) \) is continuous at 0, which implies the continuity with respect to the Hausdorff distance, since every \( \text{Viab}_F(K') \) is compact and they are all contained in a bounded set. \( \square \)
Lemma 4.1.2 ([14]). Any solution $x(\cdot)$ of (25) remaining in $K^{\epsilon_0}$ satisfies
\[ |x(t) - x(0)| \leq \int_0^t |\dot{x}(s)| \, ds \leq tM. \] (27)

If $x(0) \in K^\epsilon$ for $0 < \epsilon < \epsilon_0$ this implies that $x(t) \in K^{\epsilon_0}$ for all $t \in [0, \rho]$ as long as $\rho < \frac{\epsilon_0 - \epsilon}{M}$.

Therefore, if $x(0) \in K^\epsilon$ and $x(\rho) \in K^{\epsilon_0}$ we can apply (27) also backwards in time to get
\[ d(x(s), K^\epsilon) \leq \frac{1}{2} M \rho. \]

Lemma 4.1.3 ([14]). It holds that:
\[ \text{Viab}_F(K) \subseteq \overrightarrow{\text{Viab}}_{G_\rho}(K) \subseteq \overrightarrow{\text{Viab}}_{G_\rho}(K^\epsilon) \subseteq \text{Viab}_F(K^{\epsilon + \frac{1}{2} M \rho}) \]
for $0 \leq \epsilon < \epsilon_0$ and $0 < M \rho < \epsilon_0 - \epsilon$.

Proof. Let $x_0 \in \text{Viab}_F(K)$ then there exists a solution to (25) starting from $x_0$ remaining in $K$. This implies that the sequence $(p_n)_{n \in \mathbb{N}_0}$ defined by $p_n = x(\rho n)$ is viable for $G_\rho$ in $K$ and it starts from $x_0$, which proves the first inclusion.

The second inclusion is an easy observation.

To show the last inclusion let $p_0 \in \overrightarrow{\text{Viab}}_{G_\rho}(K^\epsilon)$ and the correspond orbit $(p_n)_{n \in \mathbb{N}_0}$. It follows from the definition of $G_\rho$ that there exists a solution $x(\cdot)$ to (25) such that $x(\rho n) = p_n$, $\forall n \in \mathbb{N}_0$. Furthermore $x(t)$ remains in $K^{\epsilon + \frac{1}{2} M \rho}$ for all times due to Lemma 4.1.2, which completes the proof.

Lemma 4.1.4. For $0 \leq \epsilon < \epsilon_0$ and $0 < M \rho < \epsilon_0 - \epsilon$ it holds that:
\[ \text{dist}(\overrightarrow{\text{Viab}}_{G_\rho}(K^\epsilon), \text{Viab}_F(K)) \leq \omega(\epsilon + \frac{1}{2} M \rho) \]

Proof. The triangle inequality implies:
\[
\text{dist}(\overrightarrow{\text{Viab}}_{G_\rho}(K^\epsilon), \text{Viab}_F(K)) \leq \\
\text{dist}(\overrightarrow{\text{Viab}}_{G_\rho}(K^\epsilon), \text{Viab}_F(K^{\epsilon + \frac{1}{2} M \rho})) + \text{dist}(\text{Viab}_F(K^{\epsilon + \frac{1}{2} M \rho}), \text{Viab}_F(K)) \leq \\
0 + \omega(\epsilon + \frac{1}{2} M \rho).
\]
The last inequality follows from Lemma 4.1.3.

Lemma 4.1.5 ([14]). The approximation error between $\Gamma_\rho(\cdot)$ and $G_\rho(\cdot)$ is
\[ d_H(G_\rho(x), \Gamma_\rho(x)) \leq M \rho (\epsilon L^\rho - 1) \]
for all $0 < \epsilon < \epsilon_0$, $x \in K^\epsilon$ and $\rho > 0$ such that $M \rho e^{L \rho} \leq \epsilon_0 - \epsilon$.
Lemma 4.1.6 ([14]). If $M \rho (e^{L_0} - 1) \leq d^{(s)}_{0,\rho}, \epsilon_1 := \psi (M \rho (e^{L_0} - 1))$ and $M \rho e^{L_0} \leq \epsilon_0$, then:

$$\text{dist} \left( \text{Viab}_F(K), \overrightarrow{\text{Viab}}_{\rho}(K^{\epsilon_1}) \right) \leq \epsilon_1.$$

Lemma 4.1.7 ([14]). If $\epsilon_1 + \alpha(h) \leq \epsilon_0$, then:

$$\text{dist}(\overrightarrow{\text{Viab}}_{\rho}(K^{\epsilon_1}), \overrightarrow{\text{Viab}}_{\rho,h}(K^{\epsilon_1+\alpha(h)})) \leq \alpha(h).$$

Lemma 4.1.8. Let $\epsilon_2 := \varphi(k(\rho, h) + M \rho (e^{L_0} - 1))$. If $M \rho e^{L_0} \leq \epsilon_0 - \epsilon_1 - \alpha(h), M \rho \leq \epsilon_0 - \epsilon_1 - \alpha(h) - \epsilon_2$ and $k(\rho, h) + M \rho (e^{L_0} - 1) \leq d^{(s)}_0$, then:

$$\text{dist}(\overrightarrow{\text{Viab}}_{\rho,h}(K^{\epsilon_1+\alpha(h)}), \text{Viab}_F(K)) \leq \epsilon_2 + \omega \left( \epsilon_1 + \alpha(h) + \epsilon_2 + \frac{1}{2} M \rho \right).$$

Proof. By Lemma 4.1.5,

$$\text{dist}(\Gamma_{\rho,h}(x_h), G_{\rho}(x_h)) \leq \text{dist}(x_h + \rho F(x_h) + k(\rho, h) B, x_h + \rho F(x_h)) + \text{dist}(x_h + \rho F(x_h), G_{\rho}(x_h)) \leq k(\rho, h) + M \rho (e^{L_0} - 1) =: d$$

for every $x_h \in K^{\epsilon_1+\alpha(h)}$. Thus any trajectory $(\xi_n)_{n \in \mathbb{N}}$ of $\Gamma_{\rho,h}$ which is viable in $K^{\epsilon_1+\alpha(h)}$ is a $d$-pseudotrajectory of $G_{\rho}$ such that $|p_n - \xi_n| \leq \epsilon_2$ for all $n \in \mathbb{N}$. Hence $p_0 \in \overrightarrow{\text{Viab}}_{\rho,h}(K^{\epsilon_1+\alpha(h)+\epsilon_2})$, which means that

$$\text{dist}(\overrightarrow{\text{Viab}}_{\rho,h}(K^{\epsilon_1+\alpha(h)}), \text{Viab}_F(K)) \leq \text{dist}(\overrightarrow{\text{Viab}}_{\rho,h}(K^{\epsilon_1+\alpha(h)}), \overrightarrow{\text{Viab}}_{\rho}(K^{\epsilon_1+\alpha(h)+\epsilon_2})) + \text{dist}(\overrightarrow{\text{Viab}}_{\rho}(K^{\epsilon_1+\alpha(h)+\epsilon_2}), \text{Viab}_F(K)) \leq \epsilon_2 + \omega \left( \epsilon_1 + \alpha(h) + \epsilon_2 + \frac{1}{2} M \rho \right)$$

by Lemma 4.1.4. \qed

Theorem 4.1.9 ([14]). If assumptions 1 and 2 and inequalities

$$k(\rho, h) + M \rho (e^{L_0} - 1) \leq d^{(s)}_{0,\rho}, \quad k(\rho, h) + M \rho (e^{L_0} - 1) \leq d^{(s)}_{0,\rho} \quad (28)$$

$$M \rho e^{L_0} + \epsilon_1 + \alpha(h) \leq \epsilon_0, \quad M \rho + \epsilon_1 + \epsilon_2 + \alpha(h) \leq \epsilon_0 \quad (29)$$

are satisfied, then

$$d_H \left( \overrightarrow{\text{Viab}}_{\rho,h}(K^{\epsilon_1+\alpha(h)}), \text{Viab}_F(K) \right) \leq \max \left\{ \epsilon_1 + \alpha(h), \epsilon_2 + \omega \left( \epsilon_1 + \alpha(h) + \epsilon_2 + \frac{1}{2} M \rho \right) \right\},$$

where $\epsilon_1 := \psi (M \rho (e^{L_0} - 1))$ and $\epsilon_2 := \varphi (M \rho (e^{L_0} - 1) + k(\rho, h))$. 

29
Proof. The statement of the theorem follows from Lemma 4.1.6, Lemma 4.1.7 and Lemma 4.1.8.

Remark 4.1.10. The conditions of Theorem 4.1.9 do not look very appealing. In application $\alpha(h)$ is usually expressed in terms of $\rho$ or vice versa:

$$\alpha(h) = c\rho$$

for some $c > 0$. Under moderate assumption the left hand sides of (28) and (29) converge faster to zero in $\rho$ than the shadowing constants.

Estimates using the shadowing property only

It is possible to abandon the inverse shadowing property by inflating the right hand side of the numerical scheme:

$$\tilde{\Gamma}_\rho(\cdot) : \begin{cases} X \leadsto X \\ x \mapsto x + \rho F(x) + M\rho(e^{L\rho} - 1)B \end{cases}$$

$$\tilde{\Gamma}_{\rho,h}(\cdot) : \begin{cases} X_h \leadsto X_h \\ x_h \mapsto (x_h + \rho F(x_h) + (M\rho(e^{L\rho} - 1) + k(\rho, h)) B) \cap X_h. \end{cases}$$

Lemma 4.1.11 ([14]). If $M\rho e^{L\rho} \leq \epsilon_0$, $\alpha(h) \leq \epsilon_0$, then:

$$\text{dist}(\text{Viab}_F(K), \text{Viab}_{\tilde{\Gamma}_{\rho,h}}(K^\alpha_{\rho,h})) \leq \alpha(h).$$

Lemma 4.1.12. If $\epsilon_3 := \varphi(2M\rho(e^{L\rho} - 1) + k(\rho, h)) \leq \epsilon_0 - \alpha(h) - M\rho$, $M\rho e^{L\rho} \leq \epsilon_0 - \alpha(h)$ and $2M\rho(e^{L\rho} - 1) + k(\rho, h) \leq d_{0,\rho}^{(s)}$, then:

$$\text{dist}(\text{Viab}_{\tilde{\Gamma}_{\rho,h}}(K^\alpha_{\rho,h}), \text{Viab}_F(K)) \leq \epsilon_3 + \omega \left( \alpha(h) + \epsilon_3 + \frac{1}{2}M\rho \right).$$

Proof. The proof is analogous to the proof of Lemma 4.1.8.

Theorem 4.1.13. If assumptions (1a), (2) and inequalities

$$M\rho e^{L\rho} + \alpha(h) \leq \epsilon_0,$$

$$2M\rho(e^{L\rho} - 1) + k(\rho, h) \leq d_{0,\rho}^{(s)},$$

$$M\rho + \epsilon_3 + \alpha(h) \leq \epsilon_0$$

are satisfied, then:

$$d_H(\text{Viab}_F(K), \text{Viab}_{\tilde{\Gamma}_{\rho,h}}(K^\alpha_{\rho,h})) \leq \max \left\{ \alpha(h), \epsilon_3 + \omega \left( \alpha(h) + \epsilon_3 + \frac{1}{2}M\rho \right) \right\},$$

where $\epsilon_3 := \varphi(2M\rho(e^{L\rho} - 1) + k(\rho, h))$.

Proof. The statement of the theorem follows from Lemma 4.1.11 and Lemma 4.1.12.
4.2 One-Sided Lipschitz Continuous Right Hand Sides

In the following we want to apply the previous results to differential inclusions with one-sided Lipschitz continuous right hand sides.

**Theorem 4.2.1** ([13]). Let \( F : X \rightarrow X \) be a Lipschitz continuous set-valued map that satisfies the one-sided Lipschitz condition with constant \( \mu < 0 \). Then \( F \) defines a differential inclusion such that the reachable sets at time \( \rho > 0 \) form a contraction:

\[
d_H(G_\rho(x), G_\rho(x')) = d_H(R(\rho, 0, x), R(\rho, 0, x')) \leq e^{\mu \rho} |x - x'| \quad \forall x, x' \in X.
\]

**Theorem 4.2.2** ([14]). Let \( G : K_{\epsilon_0} \rightarrow X \) be a set-valued map with nonempty, compact values such that:

\[
d_H(G(x), G(x')) \leq \lambda |x - x'| \quad \forall x, x' \in K_{\epsilon_0}
\]

for some \( \lambda \in [0, 1) \). Then \( G \) has the Lipschitz shadowing property on every \( K_\epsilon \) with \( 0 \leq \epsilon < \epsilon_0 \): For every \( d \)-pseudotrajectory \( (x_k)_{k \in \mathbb{Z}} \subseteq K_\epsilon \) with \( d \leq d_0 := \frac{1 - \lambda}{2}(\epsilon_0 - \epsilon) \) there exists an orbit \( (p_k)_{k \in \mathbb{Z}} \) of \( G \) such that

\[
|x_k - p_k| \leq \frac{d}{1 - \lambda} \quad \forall k \in \mathbb{Z}.
\]

Let \( F \) is set-valued map as described in Theorem 4.2.1. Then Theorem 4.2.2 shows that the \( \rho \)-flow \( G_\rho \), induced by the differential inclusion with right hand side \( F \), has the \( (d, d_1 - e^{\mu \rho}) \)-shadowing property on \( K_{\epsilon_0} \) whenever \( d < d^{(s)}_{\rho_0} := \frac{1 - e^{\rho \mu}}{2}(\epsilon_0 - \epsilon) \). It follows that \( \varphi(d) = \frac{d}{1 - e^{\rho \mu}} \), and \( d^{(s)}_{\rho_0} \) as well as \( \varphi(\cdot) \), in fact, depend on \( \rho \).

**Theorem 4.2.3.** Let \( F : K_{\epsilon_0} \rightarrow X \) be a \( L \)-Lipschitz continuous set-valued map with convex and compact values that satisfies the one-sided Lipschitz condition with constant \( \mu < 0 \). Then

\[
M \rho e^{\rho \mu} + \alpha(h) \leq \epsilon_0
\]

and

\[
4M\rho e^{\rho \mu} - \frac{1}{1 - e^{\rho \mu}} + (4 + 2L\rho) \frac{\alpha(h)}{1 - e^{\rho \mu}} + 2\alpha(h) + M \rho \leq \epsilon_0
\]

imply

\[
d_H(\text{Viab}_F(K), \overrightarrow{\text{Viab}_{F_{\rho,h}}}(K_{\rho}^{\alpha(h)})) \leq 2M\rho e^{\rho \mu} - \frac{1}{1 - e^{\rho \mu}} + (2 + L\rho) \frac{\alpha(h)}{1 - e^{\rho \mu}} + \omega \left( \alpha(h) + 2M\rho e^{\rho \mu} - \frac{1}{1 - e^{\rho \mu}} + (2 + L\rho) \frac{\alpha(h)}{1 - e^{\rho \mu}} + \frac{1}{2} M \rho \right).
\]

By establishing a relation between \( h \) and \( \rho \), as discussed in Remark 4.1.10, the error can be expressed just in terms of \( \rho \). In this case of \( \alpha(h) := \rho^2 \), the statement reads as follows:

\[
M \rho e^{\rho \mu} + \rho^2 \leq \epsilon_0
\]
and

\[ 4M \rho \frac{e^{L \rho} - 1}{1 - e^{\mu \rho}} + (4 + 2L \rho) \frac{\rho^2}{1 - e^{\mu \rho}} + 2\rho^2 + M \rho \leq \epsilon_0 \]

imply

\[ d_H(\text{Viab}_F(K), \text{Viab}^{\alpha(h)}_{\bar{\rho},h}(K_{h}^{\alpha(h)})) \leq \]

\[ 2M \rho \frac{e^{L \rho} - 1}{1 - e^{\mu \rho}} + (2 + L \rho) \frac{\rho^2}{1 - e^{\mu \rho}} + \omega \left( \rho^2 + 2M \rho \frac{e^{L \rho} - 1}{1 - e^{\mu \rho}} + (2 + L \rho) \frac{\rho^2}{1 - e^{\mu \rho}} + \frac{1}{2}M \rho \right). \]

**Proof.** The statement follows from Theorem 4.1.13. \(\square\)

**Corollary 4.2.4 ([14]).** If we assume additionally that the mapping

\[ \omega(\epsilon) = d_H(\text{Viab}_F(K), \text{Viab}_F(K^\epsilon)) \]

is \( L_V \)-Lipschitz at 0, i.e.

\[ d_H(\text{Viab}_F(K), \text{Viab}_F(K^\epsilon)) \leq L_V \epsilon, \tag{30} \]

we can estimate \( \omega(\epsilon) \leq L_V \epsilon \). The statement of the last theorem then reads as follows:

\[ d_H(\text{Viab}_F(K), \text{Viab}^{\alpha(h)}_{\bar{\rho},h}(K_{h}^{\alpha(h)})) \leq \]

\[ (1 + L_V) \left( 2M \rho \frac{e^{L \rho} - 1}{1 - e^{\mu \rho}} + (2 + L \rho) \frac{\rho^2}{1 - e^{\mu \rho}} + L_V \left( \rho^2 + \frac{1}{2}M \rho \right) \right). \]

Taking into account that \( \lim_{\rho \to 0^+} \frac{e^{L \rho} - 1}{1 - e^{\mu \rho}} = 0 \) and \( \lim_{\rho \to 0^+} \frac{\rho}{1 - e^{\mu \rho}} = -\frac{1}{\mu} \), this proves linear convergence in \( \rho \) of the algorithm.

Generally speaking we can only say that the convergence speed is as good as the stability of the viability kernel with respect to to perturbations of the constraint set:

\[ \epsilon \mapsto d_H(\text{Viab}_F(K), \text{Viab}_F(K^\epsilon)) \]

but not better than linear.

Since the assumption (30) is rather difficult to check, we want to provide conditions by which it is ensured. Before we do so we establish some notation for convenience.

**Definition 4.2.5.** Let \( a \) and \( b \) be elements of \( \mathbb{R}^n \). We denote by

\[ [a, b] := \{ \lambda a + (1 - \lambda)b : \lambda \in [0, 1] \} \]

the line segment between the two points.

**Lemma 4.2.6.** Let \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) be a \( L \)-Lipschitz continuous set-valued map with compact values and a convex graph. Then the convex combination of two solutions of the differential inclusion with right hand side \( F \) is a solution itself.
Proof. Let \( x(\cdot) \) and \( y(\cdot) \) be solutions of (25). It follows that

\[
\frac{d}{dt} (\lambda x(t) + (1 - \lambda)y(t)) = \\
(\lambda \dot{x}(t) + (1 - \lambda)\dot{y}(t)) \in \\
\lambda F(x(t)) + (1 - \lambda)F(y(t)),
\]

which is a subset of

\[ F(\lambda x(t) + (1 - \lambda)y(t)) \]
due to the convexity of \( \text{Graph}(F) \).

Theorem 4.2.7. Let \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) be a \( L \)-Lipschitz continuous set-valued map with convex and compact values that satisfies the one-sided Lipschitz condition with constant \( \mu < 0 \). We assume that the graph of \( F \) is convex and that \( K \) is a convex and compact subset of \( \mathbb{R}^n \). Additionally we assume the existence of a viable solution \( x^*(\cdot) \) of (25) such that

\[
r := \inf_{t > 0} d(x^*(t), \partial K) > 0.
\]

Then there exists \( L_V > 0 \) such that

\[
d_H(\text{Viab}_F(K), \text{Viab}_F(K^\epsilon)) \leq L_V \epsilon
\]
holds for \( 0 \leq \epsilon < r \).

Proof. First denote by \( R := \sup_{x,y \in K} |x - y| \) the diameter of \( K \). Let \( x_0 \in \text{Viab}_F(K^\epsilon) \) and let \( x(\cdot) \) be the according solution starting from \( x_0 \) viable in \( K^\epsilon \). We claim that \( \tilde{x}(\cdot) \), defined by:

\[
\tilde{x}(t) := \frac{\epsilon}{r} x^*(t) + \left( 1 - \frac{\epsilon}{r} \right) x(t)
\]
is a solution to (25), viable in \( K \), and that there exists a \( C > 0 \) independent of \( x_0 \) and \( \epsilon \) such that

\[
|x_0 - \tilde{x}(0)| \leq C \epsilon.
\]

The fact that \( \tilde{x}(\cdot) \) is a solution to the differential inclusion is ensured by Lemma 4.2.6. Furthermore,

\[
|x_0 - \tilde{x}(0)| \leq \frac{\epsilon}{r} |x_0 - x^*(0)| \leq \frac{R + \epsilon}{r} \leq \frac{R + r}{r}
\]
proofs (32). It remains to show that \( \tilde{x}(\cdot) \) is viable in \( K \). We start by mentioning that \( K^\epsilon \) is convex since \( K \) is so. This implies that \( \tilde{x}(\cdot) \) is viable in \( K^\epsilon \) since it is a convex combination of two solutions which are viable in \( K^\epsilon \).
Let $t' \in [0, \infty)$ be arbitrary. If $x(t') \in K$ it is clear that its convex combination with $x^*(t')$ is in $K$. Therefore we assume that $x(t') \notin K$. Because of the convexity of $K$ there exists only a single point of intersection between $[x^*(t'), x(t')]$ and $\partial K$, which we call $\hat{x}$. At this point there exists a supporting hyperplane $H$ of $K$. It is easy to see that:

$$a := d(x(t'), H) \leq d(x(t'), K) \leq \epsilon, \quad (33)$$
$$b := d(x^*(t'), H) \geq d(x(t'), \partial K) = r. \quad (34)$$

Furthermore, let us define following distances:

$$y := |x^*(t') - \hat{x}|,$$
$$z := |x(t') - \hat{x}|.$$

Figure 1 and simple geometry show that:

$$\frac{b}{y} = \frac{a}{z}. \quad (35)$$

We can also see that $\lambda x^*(t') + (1 - \lambda)x(t')$ is in $K$ whenever $\lambda \geq \frac{z}{z+y}$. Furthermore (33),
(34) and (35) imply
\[
\frac{z}{z + y} \leq \frac{z}{y} = \frac{a}{b} \leq \frac{\epsilon}{r},
\]
which shows the viability of \(\hat{x}(\cdot)\) and therefore finishes the proof.

\textit{Remark 4.2.8.} It should be noted that the assumption about the convexity of the graph of \(F\) is a rather restrictive one. For a single-valued map convexity of the graph is equivalent to being linear.

\textit{Remark 4.2.9.} The assumption (31) about the existence of a uniformly internal solution might seem difficult to check, but is for example ensured if there exists a stationary point of \(F\), that is a point \(\bar{x}\) such that \(0 \in F(\bar{x})\), in the interior of \(K\).

### 4.3 Counter example

Unfortunately the assumption about the linear behavior of \(\omega(\cdot)\) made in Corollary 4.2.4 which was suggested in [14] is not always satisfied even for one-sided Lipschitz right hand sides. This is shown by the following examples which were found by the author. Consider:

\[
\begin{align*}
\dot{x}(t) & \in F(x(t)) \\
x(t) & \in K \subseteq \mathbb{R}^2,
\end{align*}
\]

where \(F(x) = -x\) and \(K = \{(\frac{x}{y}) \in \mathbb{R}^2 : x \in [0, 1], 0 \leq y \leq x^2\}\). It is clear that \(F\) is one-sided Lipschitz with negative one-sided Lipschitz constant and due to the simplicity of the system all trajectories just move in a straight line towards the origin. In fact a solution \(x(\cdot)\) starting from \(x_0 \in \mathbb{R}^2\) is unique and has the form: \(x(t) = e^{-t}x_0\). Therefore a point \(x\) is in \(\text{Viab}_F(K)\) if and only if the line segment connecting 0 and \(x_0\) belongs to \(K\). It follows that

\[
\text{Viab}_F(K) = \left\{ \left(\begin{array}{c} x \\ y \end{array}\right) \in \mathbb{R}^2 : x \in [0, 1], y = 0 \right\},
\]

which is illustrated in Figure 2.

In order to show now that the viability kernel converges slower than expected let us define the following set:

\[
\tilde{K}^\epsilon := \left\{ \left(\begin{array}{c} x \\ y \end{array}\right) \in \mathbb{R}^2 : x \in [0, 1], 0 \leq y \leq x^2 + \epsilon \right\},
\]

which fulfills \(K \subseteq \tilde{K}^\epsilon \subseteq K^\epsilon\). Therefore

\[
\text{dist}(\text{Viab}_F(K^\epsilon), \text{Viab}_F(K)) \geq \text{dist}(\text{Viab}_F(\tilde{K}^\epsilon), \text{Viab}_F(K)).
\]

See Figure 3 for an illustration of \(\tilde{K}^\epsilon\).
Figure 2: Illustration of $K$ and its viability kernel through their boundaries

For computing the viability kernel of $\tilde{K}^\epsilon$ it is essential to find the line, starting from the origin that is tangent to the graph of the function

$$\varphi_\epsilon(\cdot) : \begin{cases} x \mapsto x^2 + \epsilon \\ [0,1] \to \mathbb{R}. \end{cases}$$

This line is given by $\{ t \left( \frac{1}{2\sqrt{\epsilon}} \right) : t \in [0,1] \}$, as its slope and function value $\sqrt{\epsilon}$ agree with the ones of $\varphi(\cdot)$ at this point. Therefore the viability kernel of $\tilde{K}^\epsilon$ is given by:

$$\text{Viab}_F(\tilde{K}^\epsilon) = \left\{ \left( \begin{array}{c} x \\ y \end{array} \right) \in \mathbb{R}^2 : 0 \leq x \leq \sqrt{\epsilon}, 0 \leq y \leq x^2 + \epsilon \right\} \cup \left\{ \left( \begin{array}{c} x \\ y \end{array} \right) \in \mathbb{R}^2 : \epsilon \leq x \leq 1, 0 \leq y \leq 2\sqrt{\epsilon}x \right\}.$$

Since $\text{Viab}_F(K) = \{ (\frac{1}{\sqrt{2}}) \in \mathbb{R}^2 : 0 \leq x \leq 1, y = 0 \}$ the Hausdorff distance between the two viability kernels can be estimated by distance between the points $\left( \frac{1}{2\sqrt{\epsilon}} \right)$ and $\left( \frac{1}{0} \right)$:

$$\text{dist}(\text{Viab}_F(\tilde{K}^\epsilon), \text{Viab}_F(K)) \geq 2\sqrt{\epsilon}.$$ 

This shows that

$$\text{dist}(\text{Viab}_F(K^\epsilon), \text{Viab}_F(K)) \geq 2\sqrt{\epsilon},$$

thus proofing that condition (30) cannot be fulfilled.
Counterexample with convex $K$

One might assume that the troubles in the last example were caused by the fact that $K$ was not convex. This, however, is not true as the next example shows. Consider:

$$
\begin{align*}
\dot{x}(t) &= -ax(t) \\
\dot{y}(t) &= -by(t) \\
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &\in K \subseteq \mathbb{R}^2.
\end{align*}
$$

with $a = 2b > 0$ and $K := \{(\frac{y}{x}) \in \mathbb{R}^2 : x \in [0, 1], x^2 \leq y \leq x\}$. Again it is easy to see that $F$ is one-sided Lipschitz with negative one-sided Lipschitz constant, has compact convex values and the solutions can be computed explicitly. Starting from $(\frac{x_0}{y_0}) \in \mathbb{R}^2$ solutions have the form:

$$
\begin{align*}
x(t) &= e^{-at}x_0 \\
y(t) &= e^{-bt}y_0.
\end{align*}
$$

This shows that the viability kernel of $K$ is in fact only $\{0\}$ since all solutions starting from different points leave $K$ near the origin as we can see in Figure 4.

Let us define:

$$
\tilde{K} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x \in [0, 1], x^2 - \epsilon \leq y \leq x \right\}.
$$
For simplicity we are only going to compute a subset of $\text{Viab}_F(\tilde{K}^\varepsilon)$, as shown in Figure 5. Let $\tilde{V} \subseteq \mathbb{R}^2$ be defined by:

$$\tilde{V} := \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 : x \in [0, \sqrt{\varepsilon}] \right\}.$$ 

We can see that $\tilde{V}$ is not just a subset of $\tilde{K}^\varepsilon$ but also of $\text{Viab}_F(\tilde{K}^\varepsilon)$. This gives:

$$\text{dist}(\text{Viab}_F(K'), \text{Viab}_F(K)) \geq \text{dist}(\text{Viab}_F(\tilde{K}^\varepsilon), \text{Viab}_F(K)) \geq \text{dist}(\tilde{V}, \text{Viab}_F(K)) = \text{dist}(\tilde{V}, \{0\}) = \sqrt{\varepsilon}.$$ 

**Remark 4.3.1.** By setting $K$ equal to $\{(\frac{y}{x}) \in \mathbb{R}^2 : x \in [0,1], 0 \leq y \leq x^n\}$ for a $n \in \mathbb{N}$, the last example shows that the convergence can be arbitrarily slow.

**Smooth Boundary**

Looking for assumptions, which would imply (30), we hope that a smooth boundary of the constraint set $K$ will do the trick. The next example, however, shows that even this assumption and the convexity of the graph of the right hand side are not sufficient.
\[
\begin{align*}
\dot{x}(t) &= -x(t) \\
\dot{y}(t) &= -3y(t) \\
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &\in K \subseteq \mathbb{R}^2.
\end{align*}
\]

with and \( K := \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 \leq 1\} = (0_1) + B \). Starting from \((x_0, y_0) \in \mathbb{R}^2\) solutions have the form:

\[
\begin{align*}
x(t) &= e^{-t}x_0 \\
y(t) &= e^{-3t}y_0.
\end{align*}
\]

This means that all the solutions in the upper halfspace, except for the ones on the axes, can be parametrized by:

\[
\{ x \mapsto \theta x_3 : \theta \in (0, \infty) \}.
\]

The boundary of \( K \) can be expressed, locally around the origin, as the function:

\[
x \mapsto 1 - \sqrt{1 - x^2} \approx \frac{1}{2} x^2 + \mathcal{O}(x^4).
\]

This shows that all solutions, not lying on the \( y \)-axis, leave \( K \) at some point close to the origin. Therefore \( \text{Viab}_F(K) = \{(0, y) \in \mathbb{R}^2 : 0 \leq y \leq 2\} \). See Figure 6 for an illustration.
It is easy to see that
\[ \tilde{V} := \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 : x \in [0, \sqrt{\epsilon}] \right\} \subseteq \text{Viab}_F(K^\epsilon) \]
as shown in Figure 7. This implies:
\[ \text{dist}(\text{Viab}_F(K^\epsilon), \text{Viab}_F(K)) \geq \text{dist}(\tilde{V}, \text{Viab}_F(K)) \geq \sqrt{\epsilon}. \]
Figure 7: $K^\epsilon$ and a subset of its viability kernel.
5 Computing Value Functions and Hamilton-Jacobi-Equation

Let us start with a small motivational introduction. Generally speaking there are two ways of solving optimal control problems. First of which is Pontryagins well known minimum/maximum principle. The second method requires us to compute the problems value function, usually by connecting it to some type of Hamilton-Jacobi-Bellman equation, and in the next step extract in a feedback manner the optimal control and trajectory. Either way, solving optimal control problems is out of the scope of this thesis. Nevertheless we want to investigate the computation of the value function because it gives a nice application to the viability kernel algorithm. We will use the terminology and several results from [5] without further mentioning it to see that the value function’s epigraph can be characterized as the viability kernel of an auxiliary system. This approach is of special interest in the case of state constraints as the usual approaches tend to struggle here.

**Definition 5.0.2.** The epigraph of a function \( \phi : X \to Y \) will be denoted by:

\[
\mathcal{E}p(\phi) := \{(x, y) \in X \times Y : y \geq \phi(x)\}.
\]

**Definition 5.0.3 ([2]).** The contingent epiderivative \( D_\uparrow \phi(x_0) \) of \( \phi \) at a point \( x_0 \) is defined by its epigraph:

\[
\mathcal{E}p(D_\uparrow \phi(x_0)) := T_{\mathcal{E}p(\phi)}(x_0, \phi(x_0)).
\]

**Remark 5.0.4.** Following remark 1.1.14 the contingent epiderivative of \( \phi \) at a point \( x_0 \) in the direction \( u \) can be expressed by

\[
D_\uparrow \phi(x_0)u = \liminf_{h \to 0^+, u' \to u} \frac{\phi(x_0 + hu') - \phi(x_0)}{h}.
\]

**Remark 5.0.5 ([2]).** An extended function is lower semicontinuous if and only if its epigraph is closed.

5.1 Introduction

We start by looking at the following generic control system:

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \\
u(t) &\in U(x(t))
\end{align*}
\] (36)

for a single-valued map \( f : X \to X \) and a set-valued map \( U : X \rightrightarrows Z \). The state space \( X \) and the control space \( Z \) are assumed to be finite dimensional vector spaces (one may think of \( \mathbb{R}^n \) and \( \mathbb{R}^m \) respectively). As before, state constraints of the form \( x(t) \in K, t \geq 0 \) are present as well, but are taken into account implicitly by setting \( \text{Dom}(U(\cdot)) := K \).
By $\mathcal{S}(\cdot)$ we denote the solution map which assigns to an initial value $x_0$ all the state control pairs $(x(\cdot), u(\cdot)) \in W^{1,1}(0, \infty; X, e^{rt} dt) \times L^1(0, \infty; Z, e^{rt} dt)$ which solve (36) and fulfill that $x(0) = x_0$.

**Definition 5.1.1.** The pair $(U, f)$ is said to be Marchaud if

1. $\text{Graph}(U(\cdot))$ is closed and and $U(\cdot)$ has convex images.
2. $f(\cdot)$ is continuous and affine with respect to the control.
3. $f(\cdot)$ and $U(\cdot)$ have linear growth.

**Remark 5.1.2.** This terminology is in line with the previous one in the sense that: If (36) is Marchaud then the parametrized set-valued map $F(x) = f(x, U(x))$ is Marchaud as well.

Now we can start looking at the problem of interest:

$$\inf_{(x(\cdot), u(\cdot)) \in \mathcal{S}(x_0)} \int_0^\infty e^{rs} W(x(s), u(s)) \, ds$$

s.t. $\dot{x}(t) = f(x(t), u(t))$

$u(t) \in U(x(t))$

$x(0) = x_0$.

In this section we shall always assume the listed properties to hold true:

(A1) $(U, f)$ is a Marchaud system;

(A2) $(x, u) \mapsto W(x, u)$ is a nonnegative lower semicontinuous function convex with respect to $u$;

(A3) there exists a $c > 0$ such that $W(x, u) \leq c(\|x\| + 1), \forall (x, u) \in \text{Graph}(U(\cdot))$;

(A4) the discount rate is negative: $r < 0$.

## 5.2 Characterizing the Value Function

**Theorem 5.2.1.** Let $V : X \to \mathbb{R}^+ \cup \{+\infty\}$ be a non-negative extended lower semicontinuous function (regarded as a cost function) and suppose that the assumptions (A1)-(A4) are fulfilled. Furthermore, we assume that there exists a constant $c > 0$ such that

$$\inf_{u \in U(x)} D_1V(x)(f(x, u)) \geq -c(\|x\| + 1), \forall x \in \text{Dom}(U).$$

Then the following statements are equivalent:

1. For every initial state $x_0 \in \text{Dom}(V)$ there exists a solution $(x(\cdot), u(\cdot)) \in \mathcal{S}(x_0)$ to the control system (36) such that:

$$\int_0^t e^{rs} W(x(s), u(s)) \, ds \leq V(x_0) - e^{rt}V(x(t)) \quad \forall t \geq 0$$

(37)
(2) \( V(\cdot) \) is a contingent solution to Hamilton- Jacobi inequality:

\[
\inf_{u \in U(x)} \{ D_t V(x)(f(x, u)) + W(x, u) \} + r V(x) \leq 0 \quad \forall x \in \text{Dom}(V). \tag{38}
\]

**Proof.** We begin by defining the set-valued map \( G : X \times \mathbb{R} \rightrightarrows X \times \mathbb{R} \):

\[
G(x, w) := \{(f(x, u), \lambda) : u \in U(x), \lambda \in [-c(\|x\| + 1) - rw, -W(x, u) - rw]\}. \tag{39}
\]

Due to assumptions (A1)-(A3) and by construction \( G \) has nonempty convex and compact values, linear growth and is upper semicontinuous. In addition we are going to show that the epigraph of \( V(\cdot) \) is a viability domain of \( G(\cdot) \).

Let \( x_0 \in \text{Dom}(V) \) be arbitrary. Since \( D_t V(x_0)(f(x_0, \cdot)) + W(x_0, \cdot) \) is a lower semi-continuous function there exists a point \( u^* \) where the minimum of this function over the compact set \( U(x_0) \) is attained. Assuming (38) the following holds true:

\[
D_t V(x_0)(f(x_0, u^*)) + W(x_0, u^*) + r V(x_0) \leq 0,
\]

which is the same as

\[
D_t V(x_0)(f(x_0, u^*)) \leq -W(x_0, u^*) - r V(x_0).
\]

It follows immediately that

\[
(f(x_0, u^*), -W(x_0, u^*) - r V(x_0)) \in \mathcal{E}p(D_t V(x_0)(\cdot)).
\]

Per definitionem this equivalent to:

\[
(f(x_0, u^*), -W(x_0, u^*) - r V(x_0)) \in T_{\mathcal{E}p(V(\cdot))}(x_0, V(x_0)). \tag{40}
\]

Furthermore note that \( (f(x_0, u^*), -W(x_0, u^*) - r V(x_0)) \in G((x_0, V(x_0))) \).

Now we show that also \( G(x_0, w) \cap T_{\mathcal{E}p(V(\cdot))}(x_0, w) \neq \emptyset \) whenever \( w \) is greater than \( V(x_0) \). First denote by \( \epsilon_w := w - V(x_0) \geq 0 \). Then \( (f(x_0, u^*), -W(x_0, u^*) - r V(x_0) - r \epsilon_w) \in T_{\mathcal{E}p(V(\cdot))}(x_0, w) \) because (40) implies the existence of sequences \( (h_n)_{n \in \mathbb{N}} \) converging to 0, \( (u_n)_{n \in \mathbb{N}} \) converging to \( f(x_0, u^*) \) and \( (d_n)_{n \in \mathbb{N}} \) converging to \(-W(x_0) - r V(x_0)\) such that

\[
(x_0 + h_n u_n, V(x_0) + h_n d_n - h_n r \epsilon_w) = (x_0 + h_n u_n, V(x_0) + h_n d_n - h_n r \epsilon_w).
\]

This implies that \( (f(x_0, u^*), -W(x_0, u^*) - r V(x_0) - r \epsilon_w) \) belongs to \( T_{\mathcal{E}p(V(\cdot))}(x_0, w) \). It remains to prove that \( (f(x_0, u^*), -W(x_0, u^*) - r V(x_0) - r \epsilon_w) \in G((x_0, w)) \). For this it must hold true that:

\[
-W(x_0, u^*) - r V(x_0) + rw - r \epsilon_w \in [-(\|x\| + 1), -W(x_0, u^*)],
\]

which is in fact the case since \(-r V(x_0) + rw - r \epsilon_w \) is equal to 0. This shows that \( \mathcal{E}p(V(\cdot)) \) is indeed a viability domain for \( G \).
The Viability Theorem (2.2.6) therefore proofs the existence of a solution \((x(\cdot), w(\cdot))\) to the differential inclusion:

\[
(\dot{x}(\cdot), \dot{w}(\cdot)) \in G(x(t), w(t))
\]

starting from \((x_0, V(x_0))\) that stays in \(\mathcal{E}_p(V(\cdot))\) for all times. From the definition of \(G\) it follows that the solution has the following properties:

\[
\dot{w}(s) + r w(s) \leq -W(x(s), u(s))
\]

and

\[
V(x(s)) \leq w(s), \quad V(x_0) = w(0).
\]

This implies:

\[
V(x(t)) \leq w(t) \leq e^{-rt}V(x_0) - \int_0^t e^{-a(t-s)}W(x(s), u(s)) \, ds.
\]

For the proof of the other implication see [5].

A cost function \(V : X \mapsto \mathbb{R}_+ \cup \{+\infty\}\) being given there is no reason why the monotonicity property (37) in Theorem 5.2.1 should hold true. We can, however, construct a smallest lower semicontinuous solution to the Hamilton-Jacobi inequality (38) larger than or equal to \(V\).

**Theorem 5.2.2.** Let the assumptions (A1)-(A4) be fulfilled and let \(V : X \mapsto \mathbb{R}_+ \cup \{+\infty\}\) be a non-negative lower semicontinuous function. Then there exists a smallest lower semicontinuous function \(V_\alpha(\cdot)\) larger than or equal to \(V(\cdot)\), which solves the Hamilton-Jacobi inequality (38) (could be constant \(+\infty\) ). It thus also enjoys the monotonicity property (37).

Furthermore, the epigraph of \(V_\alpha(\cdot)\) is given by:

\[
\mathcal{E}_p(V_\alpha) = \text{Viab}_G(\mathcal{E}_p(V)),
\]

where \(G\) is the set-valued map defined in (39).

**Remark 5.2.3.** The last theorem can be found in [5] as Theorem 2.1. The statement that the epigraph of \(V_\alpha(\cdot)\) is equal to the viability kernel of \(\mathcal{E}_p(V)\) under \(G\), however, was only mentioned in the proof. Therefore, we modified the statement of the theorem to emphasize this important fact more.

**Definition 5.2.4.** Let us denote by \(\mathcal{S}_t(x_0)\) the set of solutions to the control system (36) starting from \(x_0\) at time \(t\). The value function is defined by:

\[
U(t, x_0) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{S}_t(x_0)} \int_t^\infty e^{rs}W(x(s), u(s)) \, ds.
\]
Before finally establishing the connection between the previous results and the value function, which is the object of our interest, we need a little bit more preparation. First we remind about the definition of the characteristic function which will take the place of the cost function \( V(\cdot) \) in the last theorems.

**Definition 5.2.5.** Let \( M \) be a set and let \( A \) be a subset of \( M \). The function \( \chi_A(\cdot) : M \to \mathbb{R} \cup \{ +\infty \} \) defined by:

\[
\chi_A(x) := \begin{cases} 
0, & x \in A \\
+\infty, & x \notin A 
\end{cases}
\]

is called characteristic function (of the set \( A \)).

**Theorem 5.2.6.** Let the assumptions (A1)-(A4) hold true and let \( K := \text{Dom}(U) \) be closed. The value function and the smallest lower semicontinuous function \( V_\alpha(\cdot) \), satisfying (37), greater or equal to the characteristic function \( \chi_K(\cdot) \) of the set \( K \) are connected by the formula:

\[
U(t, x_0) = e^{rt}V_\alpha(x_0).
\]

Furthermore a state-control pair \( (\hat{x}(\cdot), \hat{u}(\cdot)) \in S(x_0) \) has the monotonicity property (37) for \( V_\alpha(\cdot) \) if and only if it is a solution to the optimal control problem:

\[
\int_0^\infty e^{rs}W(\hat{x}(s), \hat{u}(s)) \, ds = \inf_{(x(\cdot), u(\cdot)) \in S(x_0)} \int_0^\infty e^{rs}W(x(s), u(s)) \, ds.
\]

In this case it also obeys the "optimality principle":

\[
U(t, \hat{x}(t)) = e^{rt}V_\alpha(\hat{x}(t)) = \int_t^\infty e^{rs}W(\hat{x}(s), \hat{u}(s)) \, ds. \quad (41)
\]

Theorem 5.2.6 and Theorem 5.2.2 together show that the value function is fully determined by the viability kernel of the system described in the latter one.

### 5.3 Example

Let us consider the following optimal control problem found in [10]:

\[
\begin{aligned}
\max_{u(\cdot)} & \int_0^\infty e^{-rt}x(t)u(t) \, dt \\
\text{s.t.} \quad & \dot{x}(t) = -x(t) + u(t) \\
& x(0) = \xi \in \mathbb{R} \\
& u(t) \in [-1, 1], \quad t \geq 0.
\end{aligned} \quad (42)
\]

The exact value function of (42) at \( t = 0 \) is given by:

\[
U(0, \xi) = \begin{cases} 
\frac{1-r\xi}{r(\tau+1)}, & \xi < 0 \\
\frac{1}{r(\tau+1)}, & \xi = 0 \\
\frac{r\xi+1}{r(\tau+1)}, & \xi > 0.
\end{cases} \quad (43)
\]
Figure 8 shows a plot of the value function described in (43) and an approximation computed by the viability kernel algorithm using the characterization of the epigraph of the value function as the viability kernel of an auxiliary system as described in Section 5.2.

![Graph showing the value function and an approximation](image)

Figure 8: The value function and an approximation computed by the method described above. (r=1)
6 Numerical Examples

6.1 A Simple Example

The following example, taken from [1], is of special interest, since the analytic viability kernel is known. Consider:

\[
\begin{align*}
\dot{x}(t) &= y(t) \\
\dot{y}(t) &= u(t) \\
u(t) &\in U = [-c, c] \\
K &= [0, 1] \times \mathbb{R}.
\end{align*}
\]

(44)

![Figure 9: The viability kernel of (44) and an approximation. (c = 1)](image)

One can check that the exact viability kernel of (44) is given by:

\[
\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, -\sqrt{2cx} \leq y \leq \sqrt{2c(1-x)} \}.
\]

See Figure 9 for an illustration of the viability kernel of (44) and an approximation computed by the viability kernel algorithm for discretization parameter \( h = 0.00625 \).
Since the exact viability kernel is known we can take a look at the convergence order of the algorithm. However, no clear conclusion can be drawn from Figure 10 and Table 1 other than the impression that convergence order $\frac{1}{2}$ in the discretization parameter $h$ is not completely unrealistic.

![Figure 10: A log-log plot showing the empirical convergence order for the example above. For orientation lines representing linear convergence and order $\frac{1}{2}$ respectively are drawn.](image)

**Sensitivity to the Constraints**

In Section 4 we look at the stability of the viability kernel with respect to perturbations in the constraint set. Now we investigate this topic numerically. For this reason we consider

<table>
<thead>
<tr>
<th>discretization</th>
<th>Hausdorff distance</th>
<th>computation time [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.87321</td>
<td>&lt; 1</td>
</tr>
<tr>
<td>0.2</td>
<td>0.19086</td>
<td>&lt; 1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.28594</td>
<td>&lt; 1</td>
</tr>
<tr>
<td>0.05</td>
<td>0.097853</td>
<td>1</td>
</tr>
<tr>
<td>0.025</td>
<td>0.060938</td>
<td>2</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.036876</td>
<td>3.5</td>
</tr>
<tr>
<td>0.00625</td>
<td>0.042648</td>
<td>15</td>
</tr>
<tr>
<td>0.003125</td>
<td>0.039062</td>
<td>135</td>
</tr>
</tbody>
</table>

Table 1: Hausdorff distance for decreasing discretization parameter.
Table 2: Hausdorff distance for decreasing discretization parameter.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>Hausdorff distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.39706</td>
</tr>
<tr>
<td>0.1</td>
<td>0.24622</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1118</td>
</tr>
<tr>
<td>0.025</td>
<td>0.063738</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.051539</td>
</tr>
<tr>
<td>0.00625</td>
<td>0.03763</td>
</tr>
</tbody>
</table>

again system (44) for which the viability kernel is known. The viability kernel $\text{Viab}_F(K^{\epsilon})$ of the system with relaxed constraints, however, must be approximated numerically. Both viability kernels are illustrated in Figure 11.

![Sensitivity to the Constraints](image)

Figure 11: $\epsilon = 0.4$

Furthermore, the convergence order is investigated in Table 2 and Figure 12. We can see that convergence order $\frac{1}{2}$ in $\epsilon$ seems realistic.

Remark 6.1.1. Note that we surveyed linear convergence in Section 4 for one-sided Lipschitz right hand side.
Consider:

\[
\begin{cases}
\dot{x}(t) = x(t) + u(t) \\
\dot{y}(t) = y(t) + v(t) \\
(u(t), v(t)) \in U = B_{\mathbb{R}^2} \\
K = [-a, a]^2,
\end{cases}
\]  

(45)

for some $a > 1$. Equivalently, system (45) can be represented by:

\[
\begin{cases}
\dot{x}(t) \in F(x(t)) := x(t) + B_{\mathbb{R}^2} \\
K = [-a, a]^2.
\end{cases}
\]

It is therefore easy to see that $0 \in F(x)$ for all $x \in B_{\mathbb{R}^2}$. All solutions starting from a point outside the unit ball, however, grow exponentially. This shows that $\text{Viab}_F(K) = B_{\mathbb{R}^2}$, which is illustrated in Figure 13. This example was chosen because of the more complicated nature of the control constraints as $U$ is two-dimensional and not just a box.
Figure 13: A plot showing the viability kernel and an approximation

6.3 Lotka-Volterra type Predator-Prey-model

Let us consider the following model:

\[
\begin{aligned}
\dot{x}(t) &= \alpha x(t) - \beta x(t)y(t) \\
\dot{y}(t) &= \delta x(t)y(t) - \gamma y(t) - y(t)u(t) \\
u(t) &\in U = [0, 1] \\
K &= [k_1, +\infty] \times [k_2, +\infty]
\end{aligned}
\] (46)

Ignoring the control at first, this represents a classic Lotka-Volterra model. The first component can be interpreted as some kind of prey and the second coordinate as its predator. When introducing a control component to this system one is left with multiple options. In this case we chose to only hunt/fish the predator as seen by \(-y(t)u(t)\) and the fact that no control appears in the dynamics of prey. The constraints consists of lower bounds for both of the players, representing a minimum threshold for the population not to die out.

In Figure 14 an approximation of the viability kernel of system (46), with sample parameters, is displayed.

Figure 15 shows the viability kernel for the predator-prey model with different parameters. The findings are mostly in line with what we would intuitively expect for such a system. An increase in \(\alpha\), which represents the natural growth rate of the prey, allows
for a larger number of predators without conflicting with the viability constraints. On the other hand we observe that a higher natural 'death rate' $\gamma$ among the predators shift the the viability kernel to the right, meaning that more prey is needed for the system to be sustainable. It can be seen that viability is very sensitive with respect to the predators growth rate, as $\delta = 2$ almost leaves no initial states from where a collapse can be prevented. Strongly related to the predators growth, but not necessarily the same, is the loss of prey by being 'eaten' $\beta x(t)y(t)$. Therefore the observations are similar as $\beta = 1.5$ already results in an empty viability kernel.

6.4 A Bio-Economic Model

Through the following model (cf. [6]) we want to analyze the management of a marine renewable resource (e.g. fish). For the population growth a logistic model is used, meaning that without harvesting the biomass $x(\cdot)$ obeys the equation:

$$\dot{x}(t) = rx(t) \left(1 - \frac{x(t)}{l}\right).$$

The parameter $l$ represent the ecosystems maximal capacity with respect to the biomass and $r$ the intrinsic growth rate. Next we are going to introduce fishing which is proportional to the number of fish and the "fishing effort" $e(\cdot)$ and the "fishability coefficient"
\[ \dot{x}(t) = rx(t) \left(1 - \frac{x(t)}{L}\right) - qx(t)e(t). \]

To model some rigidity in the decision making only the derivative of the fishing effort itself is controlled:

\[ \dot{e}(t) = u(t), \quad u(t) \in U = [u^-, u^+]. \]

This can be interpreted as the fact that the number of active boats cannot be changed immediately.

The state constraints are mostly canonical as the biomass of fish should not fall below a certain level \( x_{\text{min}} \) and fishing effort has to be positive and cannot exceed the number of the available resources: \( 0 \leq e(t) \leq e_{\text{max}} \). On the other hand, fishing does have to be sustainable for the fishers as well meaning that their profit should drop too low. In other words, revenue \( pqx(t)e(t) \) minus costs \( ce(t) \) should be bounded from below giving:

\[ R(x(t), e(t)) := pqx(t)e(t) - ce(t) - C \geq 0 \]
where $p$ represents the price of fish and $c$ is the cost per unit of fishing effort.

Summarizing, the model is:

\[
\begin{align*}
\dot{x}(t) &= rx(t) \left(1 - \frac{x(t)}{T}\right) - qe(t)x(t) \\
\dot{e}(t) &= u(t) \\
u(t) &\in U = [u^-, u^+] \\
x_{\text{min}} &\leq x(t) \\
0 &\leq e(t) \leq e_{\text{max}} \\
R(x(t), e(t)) &= pqe(t)x(t) - ce(t) - C \geq 0.
\end{align*}
\]

(47)

Figure 17 shows approximations of the viability kernel of (47) for different values of $r$. It seems more than reasonable that the viability kernel gets bigger as the natural growth rate of the biomass increases. Ultimately, it just gives the possibility to fish more without decreasing the population too much.
Figure 17: An illustration of the viability kernel for different parameters.
7 Summary and Outlook

The first two sections are devoted to work towards the concept of a viability kernel. In order to do so we give an introduction to viability theory and the necessary set-valued analysis. In Section 3 we discuss how to construct a numerical scheme for computing the viability kernel. Then we talk about error estimates in Section 4. Last but not least sections 5 and 6 highlight several applications of the viability kernel and specifics about the numerics.

One of the goals of this work is to investigate the convergence of the viability kernel algorithm. In Section 4 we look at the case of one-sided Lipschitz continuous right hand sides with negative one-sided Lipschitz constant and found out that the convergence speed mainly depends on the stability of the viability kernel with respect to perturbations of the constraint set:

\[ \epsilon \mapsto d_H(\text{Viab}_F(K), \text{Viab}_F(K^\epsilon)). \quad (48) \]

If this expression converges linearly in \( \epsilon \) then so does the algorithm. However, if the convergence is slower, nothing more than this slower convergence speed can be said about the algorithm. In Section 4.3 it can be seen that, even in examples with very nice properties, linear behavior of (48) can sometimes not be ensured. However, conditions which do in fact provide linear dependence are given in Section 4.2. Either way, due to the large computational effort of the viability kernel algorithm, even for not so small discretization parameters, it is not clear how important this assertion is, as it was already asked in [14].

In addition to the rapid increase in computational effort for decreasing discretization parameters, this problem only gets exponentially worse with increasing state dimensions. This makes the viability kernel algorithm already difficult to use for dimensions greater or equal to 3. The curse of dimensionality strikes again.

A somewhat heuristic trick can be used in order to make the computation more efficient: By not discretizing the entire state space but rather refining in an iterative procedure. What we mean by this is the following: From (23) we know that the approximation is somewhat a superset of the viability kernel. Therefore the question at hand is which of the points close to the boundary of the approximation is in fact in the viability kernel. By refining only around the boundary, we can more or less reduce the dimension of the problem by one.

Another point which may need further research is this: How should we find the exact values of \( M := \sup_{x \in K} \sup_{y \in F(x)} \|y\| \) and \( L \), where \( L \) is the Lipschitz constant of the right hand side \( F \). For complicated systems, these constants might be hard to compute. This does not seem like a problem, since for convergence of the viability kernel algorithm only upper bounds for these values are needed. However, plugging larger numbers into the algorithm makes it dramatically slower.
Appendix

iterative scheme VKA

%%% discretize K
K_h = meshINT(K,n,h);
if strcmp(shapeK,'polytope') % K is not just a rectangle
    K_h = K_h(inpolygon(K_h(:,1),K_h(:,2),K_poly(:,1),...
                      K_poly(:,2)),:);
end

if strcmp(shapeK,'curved') % K might be curvy ("g")
    m = size(K_h,1);
    lin_K = NaN(m,1);
    for i = 1:m
        lin_K(i) = g(K_h(i,:)) <= 0;
    end
    K_h = K_h(logical(lin_K),:);
end

%%% apply the viability kernel algorithm
if isempty(K_h)
    error('K might be empty!?')
else
    VIAB_K = discrete_VKA(K_h,h,n,f,Lip_const,M,U,F,gU,g);
end

%%% refine around the viability kernel and repeat VKA
for k = 1:iterations
    % plot
    hd = plot_VKA(VIAB_K,K,n,h,exactVK,only_boundary,plot);
    if lexactVK_given
        Hausdorffdistance(k) = hd;
    end

    % use the approximated viability kernel as new "K"
    K_h = VIAB_K;
\[% increase accuracy\]
\[h = h/2;\]

%%% refine and thicken

%%% compute time step
\[p = \sqrt{h/(2*\sqrt{2}*M*Lip\_const)};\]

if lcompute\_valuefct
    \% use different refinement if computing value function
    K\_h = refine\_for\_epi(K,K\_h,h,n,Lip\_const,M,p);
else
    K\_h = refine(K\_h,h,n,Lip\_const,M,p);
end

%%% cubes have made the set too big
%%% eliminate all the points which are not in K

if n==1
    K\_h = K\_h(all([K\_h(:,1)>K(1,1),K\_h(:,1)<=K(1,2)],2));
elseif n==2
    K\_h = K\_h(all([K\_h(:,1)>=K(1,1),K\_h(:,1)<=K(1,2),K\_h(:,2)>=K(2,1),...
                   K\_h(:,2)<=K(2,2)],2,:));
    sizeK = size(K\_h,1);
    switch shapeK
    case 'polytope' \% K is not just a rectangle but arbitrary polygon
        K\_h = K\_h(inpolygon(K\_h(:,1),K\_h(:,2),K\_poly(:,1),...
                         K\_poly(:,2)),:);
    case 'curved'
        linK = NaN(sizeK,1);
        for l = 1:sizeK
            linK(1) = g(K\_h(1,:)) <= 0;
        end
        K\_h = K\_h(logical(linK),:);
    end
elseif n==3
    K\_h = K\_h(all([K\_h(:,1)>=K(1,1),K\_h(:,1)<=K(1,2),K\_h(:,2)>=K(2,1),...
                  K\_h(:,2)<=K(2,2),K\_h(:,3)>=K(3,1),K\_h(:,3)<=K(3,2)])

59
%% apply the viability kernel algorithm
if isempty(K_h)
    fprintf('K is a repeller! (or an error occurred?)')
    return
else
    if h < refine_boundary
        % refine only the boundary
        VIAB_K = discrete_VKA2(K_h,h,n,f,Lip_const,M,U,F,gU,g,K,lwith_K_boundary);
    else
        VIAB_K = discrete_VKA(K_h,h,n,f,Lip_const,M,U,F,gU,g);
    end
end

% final plot
hd = plot_VKA(VIAB_K,K,n,exactVK,lonly_boundary,plplot);
if lexactVK_given
    Hausdorffdistance(end) = hd;
end
end

discrete VKA

%% init
% to save memory and make coding easier all points in K_h are identified
% with numbers
sizeK = size(K_h,1);
% guess for upper bound
init_length = 4*sizeK;
F_int_K = NaN(init_length,1);
numbering = (1:sizeK)';
jmax = init_length;
act_length = 1;
sum_time = 0;

%% compute \( F(x) \cap K \) for all \( x \) in \( K \)
for j = 1:sizeK
    % allocate in chunks
    if act_length > jmax - 1
        F_int_K = [F_int_K;NaN(init_length,1)]; %#ok<A4ROW>
        jmax = jmax + init_length;
    end

tcheckboxes = [];

if explicit_rhs %only works if \( F(x) \) is a box
    Fx = Gamma(K_h(:,:));
    % rows of "ll" correspond to the points in \( K \)
    % columns show if the coordinates agree with \( F(x) \)
    ll = NaN(sizeK,2*n);
    for jj = 1:n
        ll(:,(2*(jj-1)+1):(2*jj)) = [K_h(:,jj)>Fx(jj,1),
                                    K_h(:,jj)<=Fx(jj,2)];
    end
    ia = all(ll,2);
else
    Fx = Gamma(K_h(:,:),discreteU);

tcheckboxes = tic;

%% new try
nof_cube_F = size(Fx,1);
ia = NaN(sizeK,nof_cube_F);
for k = 1:nof_cube_F
    if n == 2
        % check which points of \( K \) do lie in \( F(x) \)
        ia(:,k) = all([K_h(:,1) >= Fx(k,1),K_h(:,1) <=
                       Fx(k,2),K_h(:,2) >= Fx(k,3),K_h(:,2) <= Fx(k,
                       4),2]);
    else
        error('n>2 is not yet implemented!')
    end
end
ia = any(logical(ia),2);
sum_time = sum_time + toc(tcheckboxes);
\[
\text{end}
\]
\[
\text{tmp} = K_h(ia,:) ;
\]
\[
\text{size_of_int} = \text{size}(tmp,1) ;
\]
\[
F_{\text{int}\_K}(\text{act\_length}+1: \text{act\_length} + \text{size\_of\_int}) = \text{numbering}(ia) ;
\]
\[
\text{act\_length} = \text{act\_length} + 1 + \text{size\_of\_int} ;
\]
\[
\text{F\_int\_K} = F\_int\_K(1: \text{act\_length} ,1) ;
\]
\[
\text{time} = \text{toc}(tfintk) ;
\]
\[
\text{fprintf('ct for F(x) cap K with discretization h = %.4f is: \%0.3f s
' ,h,time)
\]
\[
\text{if } \sim \text{isempty(tcheckboxes)}
\]
\[
\text{fprintf('ct tsearch/checkboxes with h = %.4f is: \%0.3f s
' ,h,sum_time)
\]
\[
\text{end}
\]
\[
\%	ext{ discrete viability kernel algorithm}
\]
\[
tvka = \text{tic} ;
\]
\[
i = 1 ;
\]
\[
\text{old\_bad\_points} = [] ;
\]
\[
\text{new\_bad\_points} = 1 ; \%	ext{ dummy}
\]
\[
\text{while i<40 && \sim \text{isempty(new\_bad\_points)}
\]
\[
\%	ext{ find back to back NaNs}
\]
\[
\text{find_nans1} = \text{isnan}(F\_int\_K) ;
\]
\[
\text{find_nans} = [\text{find\_nans1},[\text{find\_nans1}(2:end);0]] ;
\]
\[
\text{empty} = \text{all}(\text{find\_nans},2) ;
\]
\[
\text{N} = \text{find}(\text{empty}) ;
\]
\[
\text{nbad\_points} = \text{length}(N) ;
\]
\[
\text{bad\_points} = \text{NaN}(\text{nbad\_points},1) ;
\]
\[
\text{for \ k = 1:nbad\_points}
\]
\[
\text{bad\_points}(k) = \text{sum}(\text{find\_nans1}(1:N(k))) ;
\]
\[
\text{end}
\]
\[
\%	ext{ find bad points that have not been found already}
\]
\[
\text{new\_bad\_points} = \text{setdiff}(\text{bad\_points},\text{old\_bad\_points}) ; \%	ext{ distinguish between old and new bad points!}
\]
old_bad_points = union(bad_points, old_bad_points);
F_int.K = F_int.K(~ismember(F_int.K, new_bad_points));
i = i + 1;
end
time = toc(tvka);
fprintf('Elapsed time for VKA with discretization h = %.4f is: %.3f s\n', h, time)
remaining_points = setdiff(numbering, old_bad_points);

remaining_points = setdiff(numbering, old_bad_points);

end

References


