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LINEAR RESPONSE OF A PLANAR FGM BEAM WITH NON-LINEAR VARIATION OF THE MECHANICAL PROPERTIES

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Abstract. This contribution aims at proposing an effective First order Shear Deformation Theory (FSDT) capable to tackle the non-trivial effects that a continuous variation of the mechanical properties induces on stresses, displacement, and stiffness distributions within a planar beam made of Functionally Graded Material (FGM). In greater detail, the beam model assumes the Timoshenko beam kinematics and it results naturally expressed by six Ordinary Differential Equations (ODEs) considering both cross-section displacements and internal forces as unknowns. Furthermore, exploiting a recently proposed analysis tool, the paper provides also effective tools for the accurate reconstruction of cross-section stress distributions (with special emphasis on shear stresses) and the beam stiffness estimation. A simple numerical example demonstrates that the proposed beam model can catch with good accuracy the main effects induced by variations of the mechanical properties, allowing for a simple and effective modeling of a large class of structures and opening the doors to a new family of enhanced beam models.

1 INTRODUCTION

Functionally Graded Material (FGM) beams and plates are more and more used in several engineering fields since they allow for a optimal distribution of strength and stiffness within the structure leading, as an example, to the maximal strength exploitation and significant cost saving. Furthermore, this trend benefits also from new additive-layer technologies that allow for complex object manufacture without significant increase of production costs. Conversely, also the presence of defects (e.g., a local deviation of fiber direction) could be seen as a smooth variation of the mechanical or geometrical properties of the structural element. In both cases, an effective modeling is mandatory for an accurate description of the structural element response.

According to a consolidated practice in engineering field, the most natural choice for the modeling of such structures is the usage of 1D (beam) or 2D (plate) Partial Differential Equations

(PDEs). Unfortunately, standard models are developed for prismatic bodies with constant mechanical properties along the axis/mid-plane and turn out to be too coarse and inadequate for enhanced engineering applications. As an example, a non-homogeneous and non-symmetric distribution of the mechanical properties within the beam body induces significant transversal displacements in a beam subjected to axial load. Since standard beams model separately the axial and the shear bending problem, they are not able to catch the so far described effect and, therefore, ad-hoc models must be developed.

Furthermore, even limiting to FGM beams where the mechanical properties vary only along the thickness, the shear stress distribution is significantly different from both homogeneous and layered beams [1, 2] and accurate evaluations of both the shear correction factor and the shear stiffness are required. Therefore, extending the attention to FGM beams where the mechanical properties vary both along the thickness and the axis, the shear stress distribution can vary significantly from one cross-section to the other [3, 4]. Consequently, also the shear correction factor varies along the beam axis leading the shear stiffness to have a non-trivial distribution along the beam axis [4].

Last but not least, a recent paper has highlighted that any variation of the geometry and/or mechanical properties along the beam axis deeply influences the cross-section shear stress distribution that, furthermore, turns out to depend on all the internal forces [5]. As a consequence, also the shear deformation depends on all internal forces and, for symmetry of the constitutive relations, both axial elongation and curvature depend on shear internal force [6, 5]. In other words, according to the notation introduced by [6], the beam's constitutive relations are represented by a full matrix whereas, for standard prismatic beams, they are represented by a diagonal matrix.

This paper aims at proposing a First order Shear Deformation Theory (FSDT) capable to tackle the non-trivial effects that a continuous variation of the mechanical properties induces on stresses, displacement and stiffness distributions within a planar FGM beam. In greater detail, the beam model assumes the Timoshenko beam kinematics and it results naturally expressed by six linear Ordinary Differential Equations (ODEs) with non-constant coefficients where both cross-section displacements and internal forces are the problem unknowns. Furthermore, exploiting a recently proposed modeling strategy [5], the paper provides also an effective tool for the beam stiffness estimation based on an accurate cross-section stress analysis.

The paper is structured as follows. Section 2 resumes the beam model's ODEs proposed in [5] and specializes the stress representation to FGM beams with general variation of mechanical properties. Section 3 discusses a simple numerical example that shows the main capabilities of the proposed beam model. Section 4 reports final remarks and delineate future research developments.

2 1D PLANAR BEAM MODEL

The model consists of 5 main steps detailed in the following subsections: (1) the beam's mechanical properties evaluation (2) the compatibility equations definition, (3)the equilibrium equations definition, (4) the stress representation, and (5) the simplified constitutive relations definition. The herein presented beam model ODEs was derived according to the procedure detailed in [6, 5]. For this reason, details on derivation path are not given in this section and readers may refer to the cited literature for further details.

2.1 Beam's mechanical properties and loads

The object of our study is a planar rectangular beam with non-homogeneous and non-linear distribution of the mechanical properties both within the thickness and along the beam axis. Furthermore, we assume that it behaves under the hypothesis of small displacements and plane stress state.

The horizontal stiffness and the first order of stiffness are defined as

$$A^* (x) = b \int_{h_1}^{h_{n+1}} E (x, y)\, dy; \quad S^* (x) = b \int_{h_1}^{h_{n+1}} E (x, y)\, y dy \tag{1}$$

Consequently, the beam centerline reads

$$c (x) = \frac{S^* (x)}{A^* (x)} \tag{2}$$

Thereafter, the bending stiffness reads

$$I^* (x) = b \int_{h_1}^{h_{n+1}} E (y) \left(y - c (x)\right)^2 dy \tag{3}$$

Finally, we define the horizontal, vertical, and bending resulting loads acting on the cross-section as $q (x)$, $p (x)$, and $m (x)$, respectively.

2.2 Compatibility equations

We assume the kinematics usually adopted for prismatic Timoshenko beam models. Therefore, the 2D displacement field $\boldsymbol{s} (x, y)$ is approximated as

$$\boldsymbol{s} (x, y) \approx \left\{ \begin{array}{c} u (x) + (y - c (x))\, \varphi (x) \\ v (x) \end{array} \right\} \tag{4}$$

where $\varphi (x)$, $u (x)$, and $v (x)$ indicate the rotation, the horizontal, and the vertical displacements of the cross-section, respectively.

Furthermore, indicating the horizontal strain, the curvature, and the shear strain as $\varepsilon_0 (x)$, $\chi (x)$, and $\gamma (x)$, respectively, the beam compatibility is expressed through the following ODEs

$$\varepsilon_0 (x) = u' (x) - c' (x)\, \varphi (x) \tag{5}$$

$$\chi (x) = -\varphi' (x) \tag{6}$$

$$\gamma (x) = v' (x) + \varphi (x) \tag{7}$$

2.3 Equilibrium equations

Indicating the bending moment, the horizontal, and the vertical internal forces as, $M (x)$, $H (x)$, and $V (x)$, respectively, the equilibrium ODEs read

$$H' (x) = -q (x) \tag{8}$$

$$M' (x) - H (x) \cdot c' (x) + V (x) = -m (x) \tag{9}$$

$$V' (x) = -p (x) \tag{10}$$

2.4 Stress representation

Introducing the horizontal-stress distribution functions $d_\sigma^H(x,y)$ and $d_\sigma^M(x,y)$ defined as

$$d_\sigma^H(x,y) = \frac{E(x,y)}{A^*(x)}; \quad d_\sigma^M(x,y) = \frac{E(x,y)}{I^*(x)}(c(x) - y) \tag{11}$$

the horizontal stress distribution can be defined as follows

$$\sigma_x(x,y) = d_\sigma^H(x,y)\,H(x) + d_\sigma^M(x,y)\,M(x) \tag{12}$$

In order to recover the shear stress distribution within the cross-section we resort to a procedure similar to the one proposed initially by Jourawski [7], used among others by [8], and nowadays adopted in most standard literature [9]. After few simplifications and integration with respect to the y variable, the horizontal equilibrium of a slice of infinitesimal length dx of the FGM beam reads

$$\tau(x,y) = -\int_{h_1}^{y} \sigma_{x,x}(x,t)\,dt \tag{13}$$

where the notation $(\,\cdot\,)_{,x}$ indicates partial derivatives with respect to the x variable. Inserting the horizontal stresses definition (12) into Equation (13), calculating the derivative of σ_x, recalling the beam equilibrium equations (8) and (9), and neglecting the contributions of loads and beam eccentricity (i.e., assuming $q(x) = m(x) = c'(x) = 0$) yield the following expression

$$\tau(x,y) = -\int_{h_1}^{y} d_\sigma^H{}_{,x}(x,t)\,H(x)\,dt - \int_{h_1}^{y} d_\sigma^M{}_{,x}(x,t)\,M(x)\,dt - \int_{h_1}^{y} d_\sigma^M(x,t)\,V(x)\,dt \tag{14}$$

It is worth highlighting once more that Equation (14) leads the shear stress distribution to depend on all the internal forces.

Aiming at providing an expression of shear stress distribution similar to (12), we define the shear stress distributions induced by a unitary vertical internal force $V(x)$ as

$$d_\tau^V(x,y) = -\int_{h_1}^{y} d_\sigma^M(x,t)\;dt \tag{15}$$

Consistently, we define the shear stress distributions induced by a unitary bending moment $M(x)$ as

$$d_\tau^M(x,y) = -\int_{h_1}^{y} d_\sigma^M{}_{,x}(x,t)\,dt \tag{16}$$

Finally, we define the shear stress distributions induced by a unitary horizontal internal force $H(x)$ as

$$d_\tau^H(x,y) = \tilde{d}_\tau^H(x,y) - D_\tau^H(x)\,d_\tau^V(x,y) \tag{17}$$

where

$$\tilde{d}_\tau^H(x,y) = -\int_{h_1}^{y} d_\sigma^H{}_{,x}(x,t)\,dt \tag{18}$$

and its resulting area reads

$$D_\tau^H(x) = \int_{h_1}^{h_{n+1}} \tilde{d}_\tau^H(x,y)\,dy \tag{19}$$

According to all so far introduced definitions, the shear stress distribution can be defined as follows

$$\tau(x,y) = d_\tau^H(x,y)\,H(x) + d_\tau^M(x,y)\,M(x) + d_\tau^V(x,y)\,V(x) \tag{20}$$

2.5 Simplified constitutive relations

To complete the Timoshenko-like beam model we introduce some simplified constitutive relations that define the generalized strains as a function of the internal forces.

Therefore, we consider the stress potential, defined as follows

$$\Psi^* (x, y) = \frac{1}{2} \left(\frac{\sigma_x^2 (x, y)}{E (x, y)} + \frac{\tau^2 (x, y)}{G (x, y)} \right) \tag{21}$$

Substituting the stress recovery relations (12) and (20) in Equation (21), the strains $\varepsilon_0 (x)$, $\chi (x)$, and $\gamma (x)$ result as the derivatives of the stress potential with respect to the corresponding internal forces $H (x)$, $M (x)$, and $V (x)$, respectively

$$\begin{aligned} \varepsilon_0 (x) =& b \int_{h_1}^{h_{n+1}} \frac{\partial \Psi^* (x, y)}{\partial H (x)} dy = \\ & \varepsilon_H (x) H (x) + \varepsilon_M (x) M (x) + \varepsilon_V (x) V (x) \end{aligned} \tag{22}$$

$$\begin{aligned} \chi (x) =& b \int_{h_1}^{h_{n+1}} \frac{\partial \Psi^* (x, y)}{\partial M (x)} dy = \\ & \chi_H (x) H (x) + \chi_M (x) M (x) + \chi_V (x) V (x) \end{aligned} \tag{23}$$

$$\begin{aligned} \gamma (x) =& b \int_{h_1}^{h_{n+1}} \frac{\partial \Psi^* (x, y)}{\partial V (x)} dy = \\ & \gamma_H (x) H (x) + \gamma_M (x) M (x) + \gamma_V (x) V (x) \end{aligned} \tag{24}$$

where

$$\varepsilon_H (x) = b \int_{h_1}^{h_{n+1}} \left(\frac{\left(d_\sigma^H (x,y) \right)^2}{E(x,y)} + \frac{\left(d_\tau^H (x,y) \right)^2}{G(x,y)} \right) dy \tag{25}$$

$$\varepsilon_M (x) = \chi_H (x) = b \int_{h_1}^{h_{n+1}} \frac{d_\sigma^H (x,y) d_\sigma^M (x,y)}{E(x,y)} dy + b \int_{h_1}^{h_{n+1}} \frac{d_\tau^H (x,y) d_\tau^M (x,y)}{G(x,y)} dy \tag{26}$$

$$\varepsilon_V (x) = \gamma_H (x) = b \int_{h_1}^{h_{n+1}} \frac{d_\tau^H (x,y) d_\tau^M (x,y)}{G(x,y)} dy \tag{27}$$

$$\chi_M (x) = b \int_{h_1}^{h_{n+1}} \left(\frac{\left(d_\sigma^M (x,y) \right)^2}{E(x,y)} + \frac{\left(d_\tau^M (x,y) \right)^2}{G(x,y)} \right) dy \tag{28}$$

$$\chi_V (x) = \gamma_M (x) = b \int_{h_1}^{h_{n+1}} \frac{d_\tau^M (x,y) d_\tau^V (x,y)}{G(x,y)} dy \tag{29}$$

$$\gamma_V (x) = b \int_{h_1}^{h_{n+1}} \frac{\left(d_\tau^V (x,y) \right)^2}{G(x,y)} dy \tag{30}$$

2.6 Remarks on beam model's ODEs

In is worth noticing what follows.

- Despite the beam body is rectangular (as usual for planar prismatic beams), the continuous variation of the mechanical properties leads the position of the cross-section centroid to change along the beams axis. As a consequence, the centerline definition (2) is non-trivial and leads the modeling of FGM beams to be closer to non-prismatic than to prismatic beams.

- Despite the strong analogy with prismatic beam coefficients, Definitions (1) and (3) are not sufficient to define the stiffness of the non-prismatic beam, as illustrated in Section 2.5.
- Definitions (16) (17) leads

$$\int_{h_1}^{h_{n+1}} d_\tau^H(x,y)\,dy = \int_{h_1}^{h_{n+1}} d_\tau^M(x,y)\,dy = 0 \tag{31}$$

  As a consequence, only the shear-stress distribution functions $d_\tau^V(x,y)$ depends on the vertical force $V(x)$, leading to a simpler stress representation.
- Equation (22) highlights that curvature and shear strains depend on both bending moment and vertical internal force through a non-trivial relation, substantially different from the one that governs the prismatic beam. Furthermore, Equation (22) also highlights that horizontal and bending stiffnesses depend on both the Young's E and the shear G moduli.
- Following the notation adopted by Gimena et al. [10] the beam model's ODEs (5), (8), and (22) can be expressed as

$$\begin{Bmatrix} H'(x) \\ V'(x) \\ M'(x) \\ \varphi'(x) \\ v'(x) \\ u'(x) \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & \mathbf{0} & \\ -c' & 1 & 0 & & & \\ \hline \chi_H & \chi_V & \chi_M & 0 & 0 & 0 \\ \gamma_H & \gamma_V & \gamma_M & -1 & 0 & 0 \\ \varepsilon_H & \varepsilon_V & \varepsilon_M & c' & 0 & 0 \end{array}\right] \begin{Bmatrix} H(x) \\ V(x) \\ M(x) \\ \varphi(x) \\ v(x) \\ u(x) \end{Bmatrix} - \begin{Bmatrix} q(x) \\ p(x) \\ m(x) \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{32}$$

  The resulting ODEs have the same structure as the ones obtained by Balduzzi et al. [6], but differ due to a more complex definitions of both the centerline $c(x)$ and the constitutive relations. For further comments on the resulting ODEs, readers may refer to Balduzzi et al. [6, 5].

3 NUMERICAL RESULTS

Let us consider the layered FGM beam depicted in Figure 1 where $L = 500\,\text{mm}$, $h_1 = 0\,\text{mm}$, $h_2 = 4\,\text{mm}$, $h_3 = 96\,\text{mm}$, $h_4 = 100\,\text{mm}$, $M = 1\,\text{Nm}$, $E_{min} = 10\,000\,\text{MPa}$, $E_{max} = 500\,000\,\text{MPa}$, and $E_{core} = 5\,000\,\text{MPa}$. Specifically, in bottom and top layers the Young's modulus varies linearly between E_{max} and E_{min} and vice-versa whereas the core has constant mechanical properties. Finally, we assume that the Poisson's coefficient is constant within all the layers and $\nu = 0.25$.

In order to provide a reference solution, the whole beam body has been modeled with 2D Finite Element (FE), through the commercial software ABAQUS Sim [11]. Specifically, we approximate the bottom and top layers with a sequence of 21 equal overlapping wedges with piecewise constant mechanical properties. Furthermore, bottom and top layers are discretized with a un-structured mesh of triangles with characteristic element size of 1mm whereas the core is modeled with a structured mesh of squared elements with the edge size equal to 1mm. The resulting mesh turns out to be made of 56 560 elements.

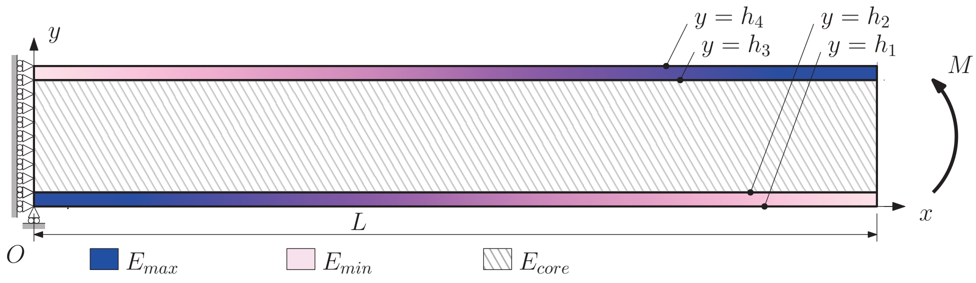


Figure 1: Multilayered FGM beam: geometry and mechanical properties definition.

3.1 Stress distribution

We focus on two cross-sections A_1 and A_2, located at $x = 150\,\text{mm}$ and $x = 250\,\text{mm}$. Figures 2 and 3 depict the cross-section distribution of axial and shear stresses.

It is worth noticing that the approximation of the FGM layers adopted in reference solution introduce some spurious oscillations in both bottom and top layers, nevertheless the proposed model and the reference solution are in good agreement in describing the stresses within these layers. In greater detail, the proposed model predicts the axial stress with errors smaller than 5% whereas the maximal error in shear stress estimation is near 15% (see Figure 3(a), interlayer surfaces).

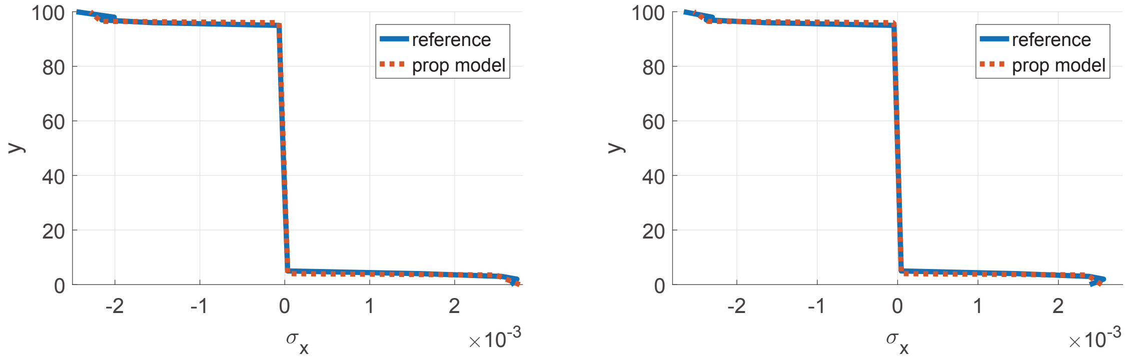
It is worth recalling that the reference solution indicates that the shear stress is absolutely not-negligible since within the beam's core, the ratio between maximal shear and axial stresses is around 1/3. Conversely, the beam results to be under a simple bending load since the material weight is neglected and the initial cross-section is constrained as depicted in Figure 1. Therefore, according to both standard [12, 9] and advanced [3, 4] planar beam models, only axial stresses should resist to the applied load and shear stress should vanish at least in the region far from initial and final cross-sections. Roughly speaking, the presence of so big shear stresses within the considered beam body can be justified by the fact that axial stress migrates according to the stiffness variation along the beam axis and non-vanishing shear stress appears in order to guarantee local equilibrium (see Equation (13)).

As a consequence, it is evident that the prismatic homogeneous beam models existing in literature provide misleading information whereas the herein proposed one has the capability to provide a reasonable description of both axial and shear stresses, resulting therefore significantly more accurate.

3.2 Displacement distribution

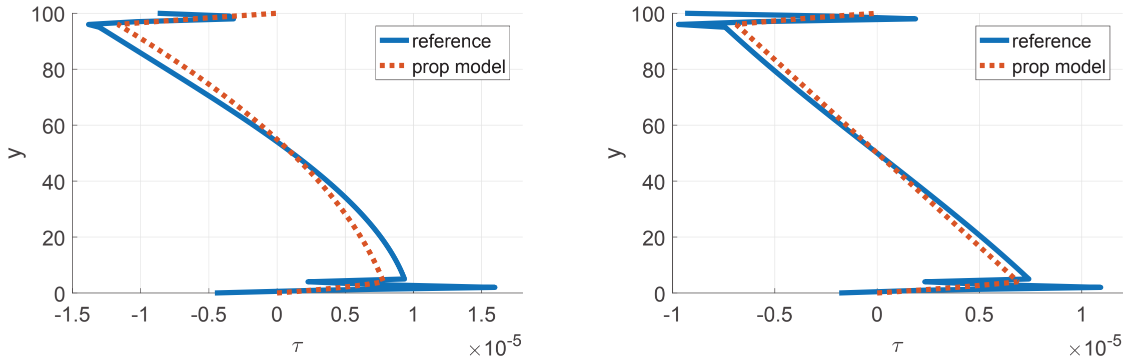
Figure 4 depicts the mean vales of the displacements evaluated along the beam axis. It is worth noticing that the proposed beam model estimates the maximal absolute-value of both horizontal and vertical displacements (occurring at $x = 250\,\text{mm}$ and $x = 500\,\text{mm}$, respectively) with an error of 5%.

Once more, both standard and advanced beam models tackle separately the axial and the shear-bending problems. Therefore, for the considered load, they predict vanishing horizontal displacements. Conversely, the proposed beam model has the capability to effectively tackle the coupling between axial and shear-bending problems, providing more accurate displacement's



(a) A_1, $x = 150$ (b) A_2, $x = 250$

Figure 2: Axial-stress cross-section distributions



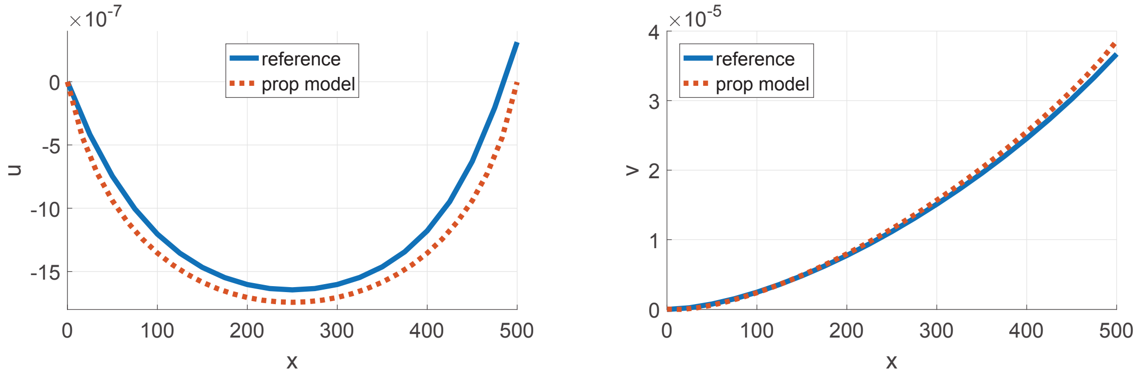
(a) A_1, $x = 150$ (b) A_2, $x = 250$

Figure 3: Shear-stress cross-section distributions

estimations.

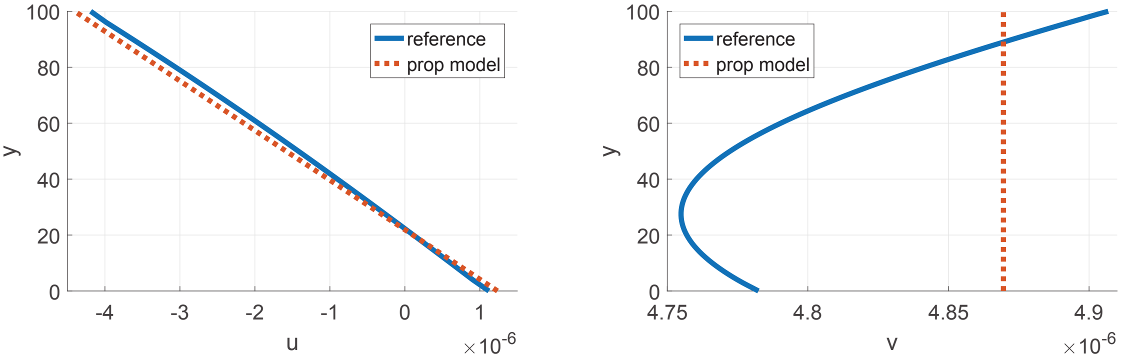
This simple observation confirms that the prismatic homogeneous beam models provide misleading information whereas the herein proposed one has the capability to provide a reasonable description of displacements, resulting therefore significantly more accurate. Furthermore, the results so far illustrated highlights that, aiming at tackling effectively the variation of mechanical properties, prismatic homogeneous beam ODEs must be enriched with suitable additional terms and not only with varying coefficients.

Finally, Figure 5 depicts the cross-section distribution of both horizontal and vertical displacements evaluated at $x = 150\,\text{mm}$. It is evident that the distribution of vertical displacements (Figure 5(b)) is highly non-linear whereas the proposed model only has the capability to estimate the vertical-displacement's mean value. Since the Timoshenko kinematics does not allow to catch higher order effects (clearly not negligible in the considered example), it is reasonable to suppose that the model's fundamental hypothesis represents the most important error source that could be easily eliminated considering more refined kinematics.



(a) $u\,(x)$ (b) $v\,(x)$

Figure 4: Mean value of cross-section displacements



(a) $u\,(150, y)$ (b) $v\,(150, y)$

Figure 5: Distribution of cross-section displacements

4 FINAL REMARKS

This paper proposed a simple FSDT for a planar FGM beams with non-linear variation of the mechanical properties. The numerical results highlight that it has the capability to tackle effectively (i) the variations of the cross-section centroid position, (ii) the non trivial stress distribution, (iii) the complex beam constitutive relations, and (iv) the nontrivial distribution of stiffness and displacements. The main model limitations are strictly related to the hypotheses behind beam model. In particular, it neglects all the boundary effects and the higher order effects.

Further developments of the present work will include the generalization to plate models and the consideration of higher order kinematics.

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