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D I P L O M A R B E I T

Modeling and energy estimates for viscous compressible Korteweg type equations

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Für meine Eltern

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Kurzfassung

Modellierung und Energieabschätzungen für viskose kompressible Korteweg-Gleichungen

Viskose kompressible Korteweg-Gleichungen sind durch das System

$$\begin{aligned} \partial_t \rho + \mathrm{div}(\rho v) &= 0, \\ \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) &= -\nabla p(\rho) + 2\mathrm{div}(\mu(\rho) D(v)) + \nabla(\lambda(\rho)\mathrm{div}\ v) + \mathrm{div}\ K, \end{aligned}$$

gegeben, wobei v die Geschwindigkeit, ρ die Dichte, $p(\rho)$ der Druck und $\mu(\rho)$, $\lambda(\rho)$ die Zähigkeitskoeffizienten sind. Der Tensor $D(v)$ entspricht dem symmetrischen Anteil des Geschwindigkeitsgradienten und der Tensor K ist der Korteweg Tensor,

$$K = \alpha |\nabla \rho|^2 I + \beta \nabla \rho \otimes \nabla \rho - \gamma \Delta \rho I - \delta \nabla^2 \rho,$$

mit Konstanten $\alpha, \beta, \gamma, \delta$. Die Navier-Stokes Gleichungen sind Erhaltungsgleichungen für die Masse, den Impuls und die Energie eines Systems. Wir zeigen, dass sie auch im hydrodynamischen Limes aus kinetischen Systemen, wie dem Boltzmann BGK Model, folgen. Der Korteweg Tensor modelliert die Wirkung von Kapillarkräften. Er folgt aus der Van der Waal'schen Auffassung der Kapillarität und wird von thermodynamischen konstitutiven Gleichungen abgeleitet.
Wir zeigen, dass es, wenn die Zähigkeitskoeffizienten die Relation

$$\lambda(\rho) = 2(\rho \mu'(\rho) - \mu(\rho))$$

erfüllen, eine zur klassischen Energieabschätzung zusätzliche Entropieabschätzung gibt, aus der a priori Beschränkungen für eine effektive Geschwindigkeit, die von dem Gradienten der Dichte abhängt, folgen:

$$\frac{d}{dt} \int_\Omega \rho |v + \nabla \phi(\rho)|^2 < C.$$

Diese zusätzliche Regularität der Dichte erlaubt es uns auch degenerierte Modelle mit Vakuum-Anfangsbedingungen zu behandeln. So kann die Stabilität von einigen Korteweg-Gleichungen, wie etwa Flach-Wasser Gleichungen und Quanten Navier-Stokes Gleichungen, gezeigt werden. Aber die so genannte BD Entropie Abschätzung kann auch auf Systeme ohne Kapillaritätsterm angewendet werden. Die Stabilität von barotropischen kompressiblen Navier-Stokes Gleichungen mit einem gegebenen Druck $p(\rho) = \rho^\gamma$, $\gamma > 1$, wird in zwei und drei Dimensionen gezeigt.

Introduction

A diffusive capillary model of Korteweg type is given by the balance equations for a fluid with velocity v, density ρ, Lamé coefficients $\mu(\rho)$ and $\lambda(\rho)$, and pressure $p(\rho)$,

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{0.1}$$
$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = -\nabla p(\rho) + 2\operatorname{div}(\mu(\rho)D(v)) + \nabla(\lambda(\rho)\operatorname{div} v) + \operatorname{div} K, \tag{0.2}$$
$$\rho(x,0) = \rho_0 \geq 0, \qquad \rho v(x,0) = m_0. \tag{0.3}$$

where $D(v)$ is the symmetric part of the density gradient and K is the Korteweg tensor,

$$K = \alpha|\nabla\rho|^2 I + \beta\nabla\rho \otimes \nabla\rho - \gamma\Delta\rho I - \delta\nabla^2\rho,$$

with some constants $\alpha, \beta, \gamma, \delta$. Applying Van der Waals' model of capillarity, [32], and considering all possible interactions in an infinitesimal volume, Korteweg postulated this capillary tensor in 1901, [23].

Korteweg systems can be used to describe oceanic and atmospheric flow, water waves in rivers and avalanches, but they are also applied in quantum hydrodynamics in order to describe flow in a superconductor. This wide range of applications made it important to study the existence of solutions for these systems. We will discuss the stability of weak solutions, which results from a priori bounds given by an additional energy estimate, if the viscosity coefficients $\mu(\rho)$ and $\lambda(\rho)$ fulfill the relation

$$\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho)). \tag{0.4}$$

In the first chapter, we will motivate the Navier-Stokes equations and formally derive them from a kinetic model, the Boltzmann BGK model, [29]. Afterwards, we will shortly discuss Van der Waals' model of capillarity and derive a general form of the Cauchy stress tensor from thermodynamic constitutive equations, in which the Korteweg tensor is included. For that purpose, we will follow the derivation given by Heida and Málek, [19].

The second chapter is dedicated to energy estimates and the stability of weak solutions of viscous compressible Korteweg systems. In the first section we discuss the derivation of a new entropy estimate, the BD entropy estimate, which was introduced by Didier Bresch and Benoît Desjardins in their paper *Quelques modèles diffusifs de type Korteweg*, [8]. In order to derive this estimate, we assume to have smooth solutions (ρ, v) of system (0.1)-(0.3), with smooth Lamé coefficients $\mu(\rho), \lambda(\rho)$ satisfying (0.4). By multiplying the momentum balance equation (0.2) by v and integrating over Ω we obtain the classical energy estimate for this system. Proceeding similarly with a mass balance equation (0.1) for a regular enough function $\phi(\rho)$ gives an additional energy estimate on an auxiliary velocity $v + \nabla\phi(\rho)$, the BD entropy estimate. In the following, we assume that the two physical energy identities, the classical and the BD entropy estimate, are satisfied by a sequence of approximate weak solutions $(\rho_n, v_n)_{n\in\mathbb{N}}$ of system (0.1)-(0.2) with initial conditions

$$\rho_n(x,0) = \rho_0^n \geq 0, \qquad \rho_n v_n(x,0) = m_0^n.$$

Note that we will introduce different notions of weak solutions depending on the special cases of Korteweg systems which we will discuss. Then the two energy estimates yield a priori bounds on $(\rho_n, v_n)_{n\in\mathbb{N}}$, which enable us to prove the stability of weak solutions (ρ, v) of system (0.1)-(0.3), i.e. we prove $(\rho_n, v_n) \to (\rho, v)$ strongly in the space of distributions. Especially, the bound on the effective velocity resulting from the BD entropy estimate,

$$\frac{d}{dt}\int_\Omega \rho|v + \nabla\phi(\rho)|^2 < \infty,$$

helps to control weak solutions close to vacuum and allows us to deal with vacuum initial conditions. We will discuss the energy estimates and the a priori bounds on the weak solutions arising thereof for a Korteweg system with viscosity coefficients $\mu(\rho) = \rho, \lambda(\rho) = 0$, Korteweg tensor $\kappa\rho\nabla\Delta\rho$, where κ is the capillary coefficient, and a rather general pressure term, [10].
Afterwards, we will look at a special case of the former system, with a quadratic pressure term $\rho^2/2$, the shallow water equations. We use the a priori bounds on the sequence of approximate weak solutions to prove the stability of weak solutions, [7]. Finally, we will consider a different Korteweg tensor than the one treated in the former two cases, $K = \mu(\rho)\nabla^2\phi(\rho)$, where $\mu(\rho)$ and $\lambda(\rho)$ also satisfy a different relation than (0.4), namely,

$$\lambda(\rho) = \rho\mu'(\rho) - \mu(\rho).$$

These systems cover the quantum Navier-Stokes equations. We rewrite the balance equations entirely in terms of the new effective velocity $v + \nabla\phi(\rho)$ and derive a BD entropy estimate for the rewritten system in which the capillary terms are eliminated, [22].

In chapter three, we use the energy estimates derived in chapter two and the a priori bounds on the weak solutions arising thereof to show the stability of weak solutions of the barotropic Navier-Stokes equations with a power pressure law $p(\rho) = \rho^\gamma$, for $\gamma > 1$ and without capillary terms, in dimension two and three, [27]. This shows that the BD entropy estimate is also useful when treating models without Korteweg terms.

Contents

1. Modeling of Navier-Stokes equations of Korteweg type

In this chapter, we recall the basic physical properties which lead to the Navier-Stokes equations with capillary terms. We motivate the Navier-Stokes equations by rewriting the balance equations for a fluid and formally derive them from a Boltzmann-BGK model. Additionally, the capillary terms will be physically motivated in terms of diffusive interfaces and mathematically deduced from balance laws for the entropy and the entropy production.

1.1. Balance equations

In this section, we want to motivate the Navier-Stokes equations by formally rewriting the balance equations for the mass, momentum and energy of a fluid.

Let $\Omega \subset \mathbb{R}^d$ be the domain occupied by a continuous body at time $t = 0$. A motion of the body is a smooth map Φ,

$$\begin{aligned} &\Phi : \Omega \times [0, T] \to \mathbb{R}^d, \qquad T > 0, \\ &\Phi = \Phi(X, t), \end{aligned}$$

where $\Phi_t := \Phi(\cdot, t)$ is invertible for each fixed time $t \in [0, T]$ and

$$\Phi(X, 0) = X.$$

The domain occupied by the body at time t is $\Omega_t := \{\Phi(X, t) | X \in \Omega\}$. The inverse map of $\Phi_t(X)$ is $\Psi_t(x) = \Psi(x, t)$:

$$\Psi_t(\cdot, t) : \Omega_t \to \Omega.$$

We will further assume that $\Phi \in C^2(\mathbb{R}^d \times [0, T]; \mathbb{R}^d)$ is bijective. The coordinates $X \in \Omega$ and $x \in \Omega_t$ represent the Lagrangian and Eulerian coordinates, respectively. The Lagrangian (material) description of a fluid is given when an observer is following a fluid parcel as it moves through space and time. We can always determine the flow properties of this parcel, like the velocity, in the Lagrangian description. The Eulerian description focuses on specific locations in space through which the fluid flows as time passes.

The velocity V in the Lagrangian description is given by:

$$V(X,t) = \partial_t \Phi(X,t).$$

In the Eulerian description, we have to use the inverse map Ψ_t to obtain the initial location X of the particle that is situated at $x \in \Omega_t$. Then we calculate the velocity at time t of this particle in the Lagrangian description. This gives the velocity in Eulerian coordinates:

$$v(x,t) = \partial_t \Phi(X,t)|_{\Psi_t(x)}. \tag{1.1}$$

Consider a scalar function $f(x,t)$ defined in Eulerian coordinates. We denote by $\dot{\overline{f}}$ the material time derivative of f, i.e. the time derivative at fixed X,

$$\begin{aligned}\dot{\overline{f}}(x,t) &= \frac{d}{dt} f(\Phi(X,t),t) = \partial_t f(x,t)|_{x=\Phi(X,t)} + \sum_{i=1}^{d} \partial_t \Phi(X,t)|_{X=\Psi_t(x)} \cdot \partial_{x_i} f(x,t)|_{x=\Phi(X,t)} \\ &= \partial_t f(x,t) + \sum_{i=1}^{d} v_i(x,t) \cdot \partial_{x_i} f(x,t). \end{aligned} \tag{1.2}$$

Similarly, for a vector valued function f, the material time derivative is given by:

$$\dot{\overline{f}}(x,t) = \partial_t f(x,t) + (\nabla f(x,t)) v(x,t). \tag{1.3}$$

The traction, i.e. the force per unit area, is dependent upon the surface unit normal vector n, $t(x,t,n)$. With given density $\rho(x,t) \in C^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R})$ and force per unit mass $f(x,t)$, the total force $F(x,t)$ acting on a subset $B_t \subset \Omega_t$ is given by

$$F(x,t) = \int_{B_t} f(x,t)\rho(x,t) \, \mathrm{d}x + \int_{\partial B_t} t(x,t,n) \, \mathrm{d}s.$$

The linear momentum, $L(x,t)$, given by

$$L(x,t) = \int_{B_t} v(x,t)\rho(x,t) \, \mathrm{d}x,$$

only changes if there is a force acting onto the set B_t, according to Newton's first law. Therefore, its total time derivative must be equal to the total force acting on

B_t:

$$\frac{d}{dt}\int_{B_t} v(x,t)\rho(x,t)\ \mathrm{d}x = \int_{B_t} f(x,t)\rho(x,t)\ \mathrm{d}x + \int_{\partial B_t} t(x,t,n)\ \mathrm{d}s. \tag{1.4}$$

In three dimensions, we can define the angular momentum as:

$$A(x,t) = \frac{d}{dt}\int_{B_t} x \times v(x,t)\rho(x,t)\ \mathrm{d}x.$$

According to the same principle,

$$\frac{d}{dt}\int_{B_t} x \times v(x,t)\rho(x,t)\ \mathrm{d}x = \int_{B_t} x \times f(x,t)\rho(x,t)\ \mathrm{d}x + \int_{\partial B_t} x \times t(x,t)\ \mathrm{d}s. \tag{1.5}$$

The Cauchy-Euler stress principle states that upon any surface that divides the body, the action of one part of the body on the other part is equivalent to the system of distributed forces. It couples on the surface dividing the body and is represented by the Cauchy-Euler stress vector, i.e. the traction $t = t(x,t,n)$, which depends continuously on the surface unit normal vector n. Cauchy's Stress Theorem, [20], states that there exists a Cartesian tensor $T(x,t)$ of order two, called the Cauchy stress tensor, such that $t(x,t,n)$ can be expressed as

$$t = Tn. \tag{1.6}$$

We will use this representation of the stress vector and the following theorems in order to rewrite the balance equations.

Theorem 1.1 (Reynold's Transport Theorem for scalar functions, [12]).
Let the motion $\Phi : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ be bijective, $\Phi \in C^2(\mathbb{R}^d \times \mathbb{R}^+;\mathbb{R}^d)$ and ν be a continuously differentiable scalar valued function $\nu(x,t) \in C^1(\mathbb{R}^d \times \mathbb{R}^+;\mathbb{R})$, then

$$\frac{d}{dt}\int_{B_t} \nu(x,t)\ dx = \int_{B_t} \partial_t \nu(x,t) + \mathrm{div}(\nu v)(x,t)\ dx,$$

with the velocity v defined in (1.1).

Theorem 1.2 (Reynold's Transport Theorem for vector valued functions, [12]).
Let the motion $\Phi : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ be bijective, $\Phi \in C^2(\mathbb{R}^d \times \mathbb{R}^+;\mathbb{R}^d)$ and ν a continuously differentiable vector valued function $\nu(x,t) \in C^1(\mathbb{R}^d \times \mathbb{R}^+;\mathbb{R}^d)$, then

$$\frac{d}{dt}\int_{B_t} \nu(x,t)\ dx = \int_{B_t} \partial_t \nu(x,t) + \operatorname{div}(\nu \otimes v)(x,t)\ dx,$$

with the velocity v defined in (1.1).

Applying Reynold's Theorem (Theorem 1.2) to the vector valued function $\rho(x,t)v(x,t)$ and applying Cauchy's Stress Theorem (equation (1.6)) to the traction vector $t(x,t,n)$, we can rewrite the balance equations (1.4) and (1.5) as:

$$\int_{B_t} \partial_t(\rho v)(x,t) + \operatorname{div}(\rho v \otimes v)(x,t) - \rho(x,t) f(x,t)\ \mathrm{d}x = \int_{\partial B_t} T(x,t)n\ \mathrm{d}s \tag{1.7}$$

$$\int_{B_t} x \times (\partial_t(\rho v)(x,t) + \operatorname{div}(\rho v \otimes v)(x,t) - \rho(x,t) f(x,t))\ \mathrm{d}x = \int_{\partial B_t} x \times T(x,t)n\ \mathrm{d}s.$$

Using Gauss' Theorem (Theorem A.22) and the Localization Theorem (Theorem A.24) on the linear momentum balance equation (1.7) gives

$$\partial_t(\rho v)(x,t) + \operatorname{div}(\rho v \otimes v)(x,t) = \rho(x,t) f(x,t) + \operatorname{div}\ T(x,t), \tag{1.8}$$

where the divergence of a Matrix is defined row-wise. In three dimensions, it follows from the angular momentum balance equation that T is symmetric, i.e. $T^T = T$. In order to see this, we use Gauss' Theorem on $T(x,t)$ and consider the i^{th} component, $i \in \{1,2,3\}$, of the angular momentum balance equation (1.5):

$$\left(\frac{d}{dt}\int_{B_t} x \times v(x,t)\rho(x,t)\ \mathrm{d}x\right)_i = \left(\int_{B_t} x \times f(x,t)\rho(x,t)\ \mathrm{d}x + \int_{B_t} \operatorname{div}(x \times T(x,t))\ \mathrm{d}x\right)_i.$$

We use index notation and Einstein's summing (Definition A.26) convention to rewrite the i^{th} component of the angular momentum balance equation as

$$\frac{d}{dt}\int_{B_t} \epsilon_{ijk} x_j v_k(x,t)\rho(x,t)\ \mathrm{d}x = \int_{B_t} \epsilon_{ijk} x_j f_k(x,t)\rho(x,t)\ \mathrm{d}x + \int_{B_t} \partial_{x_\ell}(\epsilon_{ijk} x_j T_{k\ell}(x,t))\ \mathrm{d}x,$$

with $i,j,k,\ell \in \{1,2,3\}$ and the Epsilon tensor ϵ (Definition A.25). Partially differentiating the last term gives

$$\int_{B_t} \epsilon_{ijk} x_j \left(\frac{d}{dt}(v_k(x,t)\rho(x,t)) - f_k(x,t)\rho(x,t) - \partial_{x_\ell} T_{k\ell}(x,t)) \right) \mathrm{d}x \\ + \int_{B_t} \epsilon_{ijk}\delta_{j\ell} T_{k\ell}(x,t)) \mathrm{d}x = 0,$$

with the Kronecker Delta symbol,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The first integral is equal to zero, by the linear momentum balance equation, where we didn't apply Reynold's Transport Theorem,

$$\frac{d}{dt} v(x,t)\rho(x,t) = \rho(x,t) f(x,t) + \text{div } T(x,t).$$

Thus,

$$\int_{B_t} \epsilon_{i\ell k} T_{k\ell}(x,t) \mathrm{d}x = 0.$$

We use the Localization Theorem (Theorem A.24) to obtain

$$\epsilon_{i\ell k} T_{k\ell}(x,t) = 0.$$

If T is not identiqually equal to zero, this equality implies that two indices of the Epsilon tensor must be equal, see Definition A.25. This proves that $T_{k\ell} = T_{\ell k}$, i.e. T is symmetric.
Therefore, we will assume that the Cauchy stress tensor T is symmetric in every dimension.

The mass balance equation

$$\frac{d}{dt} \int_{B_t} \rho(x,t) \mathrm{d}x = 0,$$

can be equally written as

$$\partial_t \rho(x,t) + \operatorname{div}(\rho(x,t)v(x,t)) = 0, \tag{1.9}$$

through Reynold's Transport Theorem (Theorem 1.1) applied to the scalar function $\rho(x,t)$ and the Localization Theorem (Theorem A.24).

The total energy $E(x,t)$ of a system is the sum of the potential energy $e(x,t)$ and the kinetic energy $\frac{1}{2}|v|^2$, $E = e + \frac{1}{2}|v|^2$. The energy balance equation is given by

$$\frac{d}{dt}\int_{B_t} \rho(x,t)E(x,t)\ \mathrm{d}x =$$
$$\int_{B_t} \rho(x,t)f(x,t)\cdot v(x,t) + \rho(x,t)r(x,t)\ \mathrm{d}x + \int_{\partial B_t} t(x,t,n)\cdot v(x,t) - q(x,t)\cdot n\ \mathrm{d}s,$$

where $q(x,t)$ is the heat flux through the boundary of B_t and $r(x,t)$ is a specific energy production, for example radiation. It can be rewritten as

$$\int_{\partial B_t} \partial_t(\rho(x,t)E(x,t)) + \operatorname{div}(\rho(x,t)E(x,t)\cdot v(x,t))\ \mathrm{d}x =$$
$$\int_{B_t} \rho(x,t)f(x,t)\cdot v(x,t) + \rho(x,t)r(x,t)\ \mathrm{d}x + \int_{\partial B_t} T(x,t)n\cdot v(x,t) - q(x,t)\cdot n\ \mathrm{d}s,$$

for $E \in C^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^d)$, again, by Cauchy's (equation (1.6)) and Reynold's Theorem (Theorem 1.2). Together with

$$T(x,t)n\cdot v(x,t) = T(x,t)v(x,t)\cdot n,$$

Gauss' (Theorem A.22) and the Localization Theorem (Theorem A.24), we obtain,

$$\begin{aligned} &\partial_t(\rho(x,t)E(x,t)) + \operatorname{div}(\rho(x,t)E(x,t)\cdot v(x,t)) \\ &\quad = \operatorname{div}(T(x,t)v(x,t) - q(x,t)) + \rho(x,t)r(x,t) + \rho(x,t)f(x,t)\cdot v(x,t). \end{aligned} \tag{1.10}$$

Therefore, the balance equations, i.e. the Navier-Stokes equations, are given by:

$$\begin{aligned} \partial_t\rho + \operatorname{div}(\rho v) &= 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v\otimes v) &= \operatorname{div} T + \rho f, \quad T^T = T, \\ \partial_t(\rho E) + \operatorname{div}(\rho E\cdot v) &= \operatorname{div}(Tv - q) + \rho r + \rho f\cdot v. \end{aligned} \tag{1.11}$$

Remark 1.3. Using the mass balance equation (1.9) we can rewrite the momentum and energy balance equations, (1.8) and (1.10), in terms of the material time derivate:

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = \rho \overbrace{(\partial_t v + \rho(\nabla v)v)}^{=\dot{v}} + \overbrace{v \cdot (\partial_t \rho + \operatorname{div}(\rho v))}^{=0}.$$

Inserting this equality into the balance equations (1.8) and (1.10) gives

$$\rho\dot{v} = \operatorname{div}\, T + \rho f, \tag{1.12}$$

and equivalently,

$$\rho\dot{E} = \operatorname{div}\, (Tv - q) + \rho r + \rho f \cdot v. \tag{1.13}$$

In the last two sections of this chapter, we want to deduce a general form of the Cauchy stress tensor T similar to the stress tensor proposed by Korteweg in 1901, [23], namely,

$$T = -pI + 2\mu D(v) + \lambda \operatorname{div}\, v\ I + \alpha|\nabla\rho|^2\ I + \beta\nabla\rho\otimes\nabla\rho - \gamma\Delta\rho\ I - \delta\nabla^2\rho, \tag{1.14}$$

where $D(v)$ is the symmetric part of the velocity gradient, $D(v) = \frac{1}{2}\nabla v + \frac{1}{2}\nabla v^T$, p is the pressure, μ and λ are the viscosity coefficients, $\alpha, \beta, \gamma, \delta$ are some constants and I is the identity matrix. The viscous part of the stress tensor,

$$T^{NS} = -pI + 2\mu D(v) + \lambda \operatorname{div}\, v\ I,$$

characterizes a Navier-Stokes model for compressible fluids without taking capillarity into account. It can be motivated in the following way, [25]:
We assume that the Cauchy stress tensor is of the form

$$T^{NS} = -pI + T',$$

where T' is the part of the momentum flux which is not due to direct transfer with the mass of the moving fluid. Internal friction only takes place if the fluid particles move with different velocities, thus the tensor must be a function of the space derivatives of the velocity. Newton has observed that T' is proportional to the first order derivatives of the velocity. Thus T' must be a linear function of terms of the form $\partial_{x_j} v_i$. Furthermore, there cannot be any terms independent of $\partial_{x_j} v_i$, since

T'_{ij} must vanish for $v =$ const. If we assume that the fluid is isotropic, the form of the tensor T' must be

$$T'_{ij} = \mu \partial_{x_j} v_i + \nu \partial_{x_i} v_j + \lambda \delta_{ij} \partial_{x_k} v_k,$$

where we used index notation and Einstein's summing convention (Definition A.26) with the Kronecker delta δ_{ij} and $i, j, k \in \{i, .., d\}$. By symmetry of T' it follows that $\mu = \nu$. Thus, with $D(v) = \frac{1}{2}\left(\partial_{x_i} v_j + \partial_{x_j} v_i\right)$,

$$T' = 2\mu D(v) + \lambda \text{ div } v \ I,$$

and

$$T^{NS} = -pI + 2\mu D(v) + \lambda \text{ div } v \ I.$$

Example 1.4. For instance, if the fluid is incompressible, div $v = 0$, the Navier-Stokes equations read:

$$\begin{aligned} \partial_t \rho + \nabla \rho \cdot v &= 0 \\ \rho \left(\partial_t v + \text{div}(v \otimes v)\right) &= -\nabla p + \mu \Delta v. \end{aligned}$$

1.2. Formal derivation of the isothermal compressible Navier-Stokes equations from a Boltzmann BGK model

In this section the connection between kinetic gas theory and macroscopic fluid dynamics is discussed. In kinetic theory, a monoatomic gas is represented as a cloud of point-like particles and is fully described by its number density f. The phase space is the set of $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, where x is the space variable and v the velocity variable. The infinitesimal volume dx dv centered at (x, v) contains $f(t, x, v)$ dx dv particles at time t. The Boltzmann BGK model is a kinetic model which only takes the global interactions between the particles, which are expected to lead to thermodynamic equilibrium, into account. Despite the fact that binary interactions, which are treated by the Boltzmann equation, are not considered, this model contains most of the basic properties of hydrodynamics. The conservation of mass, momentum and energy equations are fulfilled. Given the microscopic density $f(x, v, t)$ the BGK model is, [29]:

$$\partial_t f + v \cdot \nabla_x f + a \cdot \nabla_v f = \frac{1}{\nu}(M_f - f), \tag{1.15}$$

$$M_f(x,v,t) = \frac{n(t,x)}{(2\pi T(t,x))^{d/2}} \exp\left(-\frac{|v-u(t,x)|^2}{2T(t,x)}\right), \tag{1.16}$$

$$n(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\ \mathrm{d}v,$$

$$nu(x,t) = \int_{\mathbb{R}^d} f(x,v,t)v\ \mathrm{d}v,$$

$$(nu^2 + dnT)(x,t) = \int_{\mathbb{R}^d} f(x,v,t)|v|^2\ \mathrm{d}v,$$

$$f(x,v,0) = f_0(x,v).$$

Here a is a bulk acceleration attributed to some body force, which is independent of both x and v, M_f is a Maxwellian with the same moments as f, i.e. the density at thermodynamic equilibrium, T is the temperature of the gas and ν is the relaxation parameter.

Formal procedures to derive hydrodynamic limits of the Boltzmann BGK equation consist in introducing a small parameter ϵ, the Knudsen number, which represents the ratio of the mean free path of the particles to some characteristic length of the flow. The Von Karman identity,

$$\epsilon = \frac{Ma}{Re},$$

with the Mach and Reynolds numbers Ma, Re, holds, [29].
Chapman and Enskog, [13], individually proclaimed an expansion of f in ϵ,

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + ...,$$

which is assumed to uniformly converge.
We apply the first order expansion to f and will, at least formally, derive the isothermal Navier-Stokes equations.

In the following we will consider a Boltzmann BGK type equation with a bulk acceleration attributed to some body force, which is produced by a potential Φ, e.g. the electric field produced by an electric potential,

$$\partial_t f + v \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_v f = \frac{1}{\nu}(M_f - f). \tag{1.17}$$

We will now assume that ν is of order ϵ. Thus, the first order in the Chapman-Enskog expansion is given by

$$f = M_f + \nu f^{(1)}, \tag{1.18}$$

since $f^{(0)}$ is given by the density at equilibrium, $f^{(0)} = M_f$, [13]. We define the function

$$g := f^{(1)} = \frac{1}{\nu}(f - M_f).$$

Note that the derivation of the hydrodynamic Navier-Stokes equations from the kinetic BGK model only makes sense if ν is of order smaller than one. In cases where the Knudsen Number is bigger or equal to one, the gas cannot be described as a fluid. For the sake of simplicity, we set the temperature T to one. Thus, the Maxwellian M_f is given by:

$$M_f(x, v, t) = \frac{n(t, x)}{(2\pi)^{d/2}} \exp\left(-\frac{|v - u(t, x)|^2}{2}\right).$$

We integrate the Boltzmann BGK equation (1.17) over $\mathbb{R}^d$ and obtain:

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} \partial_t f + v \cdot \nabla_x f \ \mathrm{d}v \\ &= \partial_t n + \operatorname{div}(nu). \end{aligned} \tag{1.19}$$

Multiplying the BGK equation by v and integrating gives:

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} v\partial_t f + v \otimes v \cdot \nabla_x f + v \otimes \nabla_v f \cdot \nabla_x \Phi \ \mathrm{d}v \\ &= \partial_t(nu) + \operatorname{div} \int_{\mathbb{R}^d} v \otimes v f \ \mathrm{d}v - n\nabla_x \Phi. \end{aligned} \tag{1.20}$$

We apply the Chapman-Enskog expansion (1.18) to $v \otimes vf$,

$$\int_{\mathbb{R}^d} v \otimes vf \ \mathrm{d}v = \int_{\mathbb{R}^d} v \otimes vM_f \ \mathrm{d}v + \nu \int_{\mathbb{R}^d} v \otimes vg \ \mathrm{d}v = nI + nu \otimes u + \nu \int_{\mathbb{R}^d} v \otimes vg \ \mathrm{d}v,$$

and insert this equality into the right-hand side of equation (1.20),

$$\partial_t(nu) + \mathrm{div}(nu \otimes u) + \nabla_x n - n\nabla_x \Phi \ + \nu \ \mathrm{div} \int_{\mathbb{R}^d} v \otimes vg \ \mathrm{d}v = 0. \tag{1.21}$$

By explicitly calculating g,

$$\begin{aligned} g &= -\frac{1}{\nu}(M_f - f) = -\partial_t f - v \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_v f \\ &= -\partial_t M_f - v \cdot \nabla_x M_f - \nabla_x \Phi \cdot \nabla_v M_f + O(\nu), \end{aligned}$$

we see that its second moment is given by:

$$\begin{aligned} &\int_{\mathbb{R}^d} v \otimes v \ g \ \mathrm{d}v \\ &\quad = -\partial_t \int_{\mathbb{R}^d} v \otimes v \ M_f \ \mathrm{d}v - \mathrm{div} \int_{\mathbb{R}^d} v \otimes v \otimes v M_f \ \mathrm{d}v - \nabla_x \Phi \cdot \int_{\mathbb{R}^d} \nabla_v M_f (v \otimes v) \ \mathrm{d}v. \end{aligned} \tag{1.22}$$

Furthermore,

$$\begin{aligned} -\partial_t \int_{\mathbb{R}^d} v \otimes v \ M_f \ \mathrm{d}v &= -\partial_t n I - \partial_t (nu \otimes u) \\ &\overset{(1.19)}{=} \mathrm{div}(nu) I - \partial_t(nu) \otimes u - u \otimes \partial_t(nu) + \mathrm{div}(nu) u \otimes u, \\ &\overset{(1.21)}{=} \mathrm{div}(nu) I + (\mathrm{div}(nu \otimes u) + \nabla_x n - n \nabla_x \Phi) \otimes u \\ &\qquad + u \otimes (\mathrm{div}(nu \otimes u) + \nabla_x n - n\nabla_x \Phi) + \mathrm{div}(nu) u \otimes u + O(\nu), \end{aligned}$$

$$\begin{aligned} &-\mathrm{div} \int_{\mathbb{R}^d} v \otimes v \otimes v M_f \ \mathrm{d}v \\ &\qquad = -\mathrm{div}(nu \otimes u \otimes u) - \mathrm{div}(nu) I - \nabla_x n \otimes u - u \otimes \nabla_x n - 2nD(v). \end{aligned}$$

The third integral in (1.22) gives:

$$\begin{aligned} -\nabla_x \Phi \cdot \int_{\mathbb{R}^d} \nabla_v M_f (v \otimes v) \ \mathrm{d}v &= \nabla_x \Phi \cdot \int_{\mathbb{R}^d} M_f \nabla_v (v \otimes v) \ \mathrm{d}v \\ &= n(\nabla_x \Phi \otimes u + u \otimes \nabla_x \Phi). \end{aligned}$$

By summing up these three integrals, we obtain

$$\int_{\mathbb{R}^d} v \otimes vg \, \mathrm{d}v = -2nD(u).$$

And, finally, by inserting this relation into equation (1.21),

$$\partial_t(nu) + \mathrm{div}(nu \otimes u) + \nabla_x n - n\nabla_x \Phi - 2\nu \mathrm{div}(nD(u)) = 0.$$

Together with equation (1.19) we have obtained a compressible Navier-Stokes system with pressure $p(n) = n$ and viscosity $\mu(n) = \nu n$.

1.3. Motivation and derivation of the Korteweg tensor

In this section, we will derive a general form of the Korteweg tensor from thermodynamic constitutive equations. Firstly, we will motivate the dependency of the energy and entropy on the gradient of the density. Furthermore, we will use some thermodynamic constitutive equations to obtain a general form of the Cauchy tensor in which the Korteweg tensor is included. Finally, we will discuss some special forms of the Kortweg tensor, which will be important in the next chapters.

1.3.1. Van der Waals' concept of capillarity

Van der Waals, [32], developed a model of capillarity in which the internal energy e and the entropy η of the fluid do not only depend on the local variables, such as the temperature T or the density ρ, but also on non-local variables, the gradients of ρ. He found that the interface between two phases, e.g. liquid and vapor, must be smooth, that there exists a small transition zone between the two phases with very steep density gradient. This leads to additional terms in the momentum balance equation (1.8). It is worth noting that these additional quantities are applied to the whole three-dimensional domain rather than to the two-dimensional surface only.
In order to take the diffusive interface into account, Van der Waals postulated the thermodynamical description of a liquid-vapor interface via the free energy

$$F = F_0 + \frac{\lambda}{2}|\nabla\rho|^2, \tag{1.23}$$

where F_0 is the volumetric free energy. Jamet, [21], showed that this special form of the energy leads to an additional term in the momentum balance equation (1.8),

$$\mathrm{div}(\lambda\nabla\rho \otimes \nabla\rho),$$

which is usually called the Korteweg tensor in literature.

Korteweg derived a more complete model by considering all possible interactions between the molecules in two neighboured infinitesimal volume elements based on the diffusive interface model of Van der Waals and postulated the capillarity stress tensor K:

$$K = \alpha|\nabla\rho|^2 I + \beta\nabla\rho\otimes\nabla\rho - \gamma\Delta\rho I - \delta\nabla^2\rho, \tag{1.24}$$

with constants $\alpha, \beta, \gamma, \delta$.

1.3.2. Derivation of the Korteweg tensor from thermodynamic constitutive equations

We discuss the derivation of a general form of the Cauchy stress tensor from thermodynamic constitutive equations, in which the Korteweg tensor is included. Heida and Málek, [19], showed that this can be done by maximization of the entropy which, as motivated above, is assumed to be a function of local variables such as the internal energy e and the density ρ, but also of the nonlocal density gradient $\nabla\rho$.

We assume that there exists a specific entropy $\eta(e, \rho, \nabla\rho)$, which is differentiable in all variables and increasing with respect to e. Then we can apply the implicit function theorem, which implies $e = e(\eta, \rho, \nabla\rho)$, [24].

We would like to obtain an equation for the entropy production ξ of the form

$$\xi(J_\alpha, A_\alpha) = \sum_\alpha J_\alpha A_\alpha, \tag{1.25}$$

where J_α and A_α represent thermodynamic fluxes (such as force per unit area or heat flux) and affinities (i.e. the velocity or the temperature gradient), respectively. We assume the constitutive equation

$$\xi(J_\alpha, A_\alpha) \geq 0,$$

then the second law of thermodynamics is automatically fulfilled by the function ξ. As motivated for a compressible Navier-Stokes-Fourier fluid by Heida and Málek, the following particular form of the entropy production is advantageous:

$$\xi(J_\alpha) = \sum_\alpha \frac{1}{\gamma_\alpha}|J_\alpha|^2. \tag{1.26}$$

We will note this in the derivation, too. Maximization of this equation, provided that (1.25) holds, gives

$$J_\alpha = \gamma_\alpha A_\alpha, \tag{1.27}$$

by a maximization method with Lagrange multipliers, [31].

In order to obtain an equation for the entropy production, we start with the material time derivative of the internal energy (see equation (1.2) and (1.3)):

$$\rho\dot{\overline{e}} = \rho\ \partial_\eta e\ \dot{\overline{\eta}} + \rho\ \partial_\rho e\ \dot{\overline{\rho}} + \rho\ \partial_{\nabla\rho} e\ \dot{\overline{\nabla\rho}}. \tag{1.28}$$

Using the mass balance equation (1.9), we calculate

$$\begin{aligned}
\dot{\overline{\nabla\rho}} &= \partial_t \nabla\rho + \nabla^2\rho \cdot v \\
&= \partial_t \nabla\rho + \nabla(\nabla\rho \cdot v) - \nabla v \cdot \nabla\rho \\
&= \nabla(\partial_t\rho + \nabla\rho \cdot v) - \nabla v \cdot \nabla\rho \\
&\overset{(1.9)}{=} -\nabla(\rho \mathrm{div}\ v) - \nabla v \cdot \nabla\rho \\
&= -\nabla\rho \mathrm{div}\ v - \rho\nabla \mathrm{div}\ v - \nabla v \cdot \nabla\rho.
\end{aligned} \tag{1.29}$$

The thermodynamic temperature θ is defined as

$$\theta = \frac{\partial e}{\partial \eta}.$$

We insert the balance equations, written in terms of the material time derivative, (1.12) and (1.13), and equation (1.29) into the formula for the internal energy (1.28). Then we obtain an equation for the material time derivative of the entropy η,

$$\begin{aligned}
\rho\theta\dot{\overline{\eta}} &= \rho\dot{\overline{e}} - \rho\ \partial_\rho e\ \dot{\overline{\rho}} - \rho\ \partial_{\nabla\rho} e\ \dot{\overline{\nabla\rho}} \\
&= \rho\dot{\overline{E}} - \rho v \cdot \dot{\overline{v}} - \rho\ \partial_\rho e\ \dot{\overline{\rho}} - \rho\ \partial_{\nabla\rho} e\ \dot{\overline{\nabla\rho}} \\
&= \mathrm{div}(Tv - q) + \rho r + \rho f v - \rho v \cdot \dot{\overline{v}} + \rho^2\ \partial_\rho e\ \mathrm{div}\ v \\
&\quad - \rho\ \partial_{\nabla\rho} e \cdot (-[\nabla v]\nabla\rho - \mathrm{div}\ v\ \nabla\rho - \rho\nabla \mathrm{div}\ v) \\
&\overset{(1.12)}{=} T : \nabla v - \mathrm{div}\ q + \rho r + \rho^2\ \partial_\rho e\ \mathrm{div}\ v + \rho\,(\partial_{\nabla\rho} e \otimes \nabla\rho) \cdot \nabla v \\
&\quad + (\rho\ \partial_{\nabla\rho} e \cdot \nabla\rho)\,\mathrm{div}\ v + \rho^2\ \partial_{\nabla\rho} e \nabla \mathrm{div}\ v,
\end{aligned} \tag{1.30}$$

which can be rewritten through partial derivation as:

$$\begin{aligned}\rho\theta\dot{\overline{\eta}} = &(T + \rho\ \partial_{\nabla\rho} e \otimes \nabla\rho) : \nabla v - \text{div}\left(q - \rho^2\ \partial_{\nabla\rho} e\ \text{div}\ v\right) + \rho r \\ &+ \left(\rho^2\ \partial_\rho e + \rho\ \partial_{\nabla\rho} e \cdot \nabla\rho - \text{div}\left(\rho^2\ \partial_{\nabla\rho} e\right)\right) \text{div}\ v.\end{aligned} \tag{1.31}$$

We want to reformulate this equation in form of a balance equation for the entropy

$$\rho\dot{\overline{\eta}} + \text{div}\left(\frac{h}{\theta}\right) = \frac{\xi}{\theta}, \tag{1.32}$$

where h/θ is the entropy flux. We emphasize that we do not a priori assume that the heat and entropy flux, q and h, have to coincide. In order to obtain an equation of the form (1.32), we define the derivatoric parts of the tensors T and D denoted as T^d and D^d, respectively, by

$$\begin{aligned} m :=& \frac{1}{3}\ \text{tr}\ T, \qquad T^d := T - mI, \\ D^d :=& D - \frac{1}{3}\text{div}\ v\ I, \end{aligned} \tag{1.33}$$

where tr T is the trace of a matrix T, tr $T = \sum_{i=1}^{d} T_{ii}$. We set

$$\begin{aligned} p &= \rho^2 \partial_\rho e, \\ P &= p + \frac{4}{3}\rho\ \partial_{\nabla\rho} e \cdot \nabla\rho - \text{div}\left(\rho^2 \partial_{\nabla\rho} e\right), \\ \overline{P} &= P + \rho^2 \nabla\theta \cdot \partial_{\nabla\rho} e, \\ h &= q - \rho^2 (\text{div}\ v)\ \partial_{\nabla\rho} e, \\ T^{dis} &= T^d + \rho\left(\partial_{\nabla\rho} e \otimes \nabla\rho - \frac{1}{3}\left(\partial_{\nabla\rho} e \cdot \nabla\rho\right) I\right), \end{aligned} \tag{1.34}$$

and after inserting these definitions into (1.31), we obtain,

$$\rho\theta\dot{\overline{\eta}} = T^{dis} : (\nabla v)^d + (P + m)\text{div}\ v - \text{div}\ h + \rho r, \tag{1.35}$$

Division by θ and partial derivation of (div h)/θ finally gives:

$$\begin{aligned}\rho\dot{\overline{\eta}} + \text{div}\left(\frac{h}{\theta}\right) &= \frac{1}{\theta}\left(T^{dis} : (\nabla v)^d + (P+m)\text{div } v - \left(\frac{h}{\theta}\right)\cdot\nabla\theta + \rho r\right)\\ &= \frac{1}{\theta}\left(T^{dis} : (\nabla v)^d + (P+m+\rho^2\nabla\theta\cdot\partial_{\nabla\rho}e)\text{div } v - \left(\frac{q}{\theta}\right)\cdot\nabla\theta + \rho r\right)\\ &= \frac{1}{\theta}\left(T^{dis} : (\nabla v)^d + (\overline{P}+m)\text{div } v - \left(\frac{q}{\theta}\right)\cdot\nabla\theta + \rho r\right). \end{aligned} \tag{1.36}$$

For the sake of simplicity, we set $r = 0$ in the following. The tensor T^{dis} is traceless, if it is symmetric (which will be assumed in the following), the entropy balance equation reads as:

$$\rho\dot{\overline{\eta}} + \text{div}\left(\frac{h}{\theta}\right) = \frac{1}{\theta}\left(T^{dis} : D^d + (\overline{P}+m)\text{div } v - \left(\frac{q}{\theta}\right)\cdot\nabla\theta + \rho r\right),$$

since $\nabla v : S = D(V) : S$, for every symmetric tensor S. Therefore, ξ is given by:

$$\xi = T^{dis} : D^d + (\overline{P}+m)\text{div } v - q\cdot\frac{\nabla\theta}{\theta}. \tag{1.37}$$

In the following, we consider a compressible Navier-Stokes-Fourier fluids, which are Newtonian-like fluids which follow Fourier's law,

$$q = -k_B\nabla\theta,$$

where k_B is the Boltzmann constant. Doing a similar calculation for a compressible Navier-Stokes-Fourier fluid while assuming that $e = e(\eta,\rho)$ classically does not depend on $\nabla\rho$, we obtain the same equation as (1.30) without the last three terms, which has the exact same structure as equation (1.35). Thus, we obtain the same equation as (1.37), with p instead of $\overline{P}$, T^d instead of T^{dis} and $h = q$:

$$\tilde{\xi} = T^d\cdot D^d + (p+m)\text{div } v - q\cdot\frac{\nabla\theta}{\theta}. \tag{1.38}$$

The constitutive equations for the compressible Navier-Stokes-Fourier fluids are given by:

$$
\begin{aligned}
T &= -pI + 2\mu D^d + \frac{2\mu + 3\lambda}{3} \text{ div } v \, I, \\
q &= -k\nabla\theta, \\
r &= 0.
\end{aligned}
$$

Therefore, T^d and m are given by

$$
\begin{aligned}
T^d &= 2\mu D^d, \\
m + pI &= \frac{2\mu + 3\lambda}{3} \text{div } v, \\
q &= -k\nabla\theta.
\end{aligned}
$$

Inserting m and T^d into equation (1.38) gives a relation for $\tilde{\xi}$ which either depends entirely on thermodynamic affinities (D^d, div v, $\nabla\theta$),

$$
\tilde{\xi} = 2\mu|D^d|^2 + \frac{2\mu + 3\lambda}{3}(\text{div } v)^2 + \kappa|\nabla\theta|^2,
$$

where $\kappa := k_B/\theta$, or on thermodynamic fluxes,

$$
\tilde{\xi} = \frac{1}{2\mu}|T^d|^2 + \frac{3}{2\mu + 3\lambda}(m + p)^2 + \frac{1}{\kappa}|q|^2. \tag{1.39}
$$

Note that the former two equations only makes sense if all coefficients are positive. The latter form of the energy production is advantageous since it gives a better insight into the physical circumstances under which a material does not dissipate any energy, i.e. produces any entropy. It is not dissipating any energy when there are neither volume chances nor isochoric dissipative processes such as shear nor any heat flux generation.

Since the equation for the entropy production of the non-classic entropy $\eta(e, \rho, \nabla\rho)$, (1.37) has exactly the same structure as the one for the classical entropy (1.38), we assume to have an equation with the same structure as equation (1.39) for a Newtonian-like Navier-Stokes-Fourier fluid with non-classical entropy, too,

$$
\xi = \frac{1}{2\mu}|T^{dis}|^2 + \frac{3}{2\mu + 3\lambda}(m + \overline{P})^2 + \frac{1}{\tilde{\kappa}}|q|^2, \tag{1.40}
$$

where $\mu, 2\mu + 3\lambda$ and $\tilde{\kappa}$ are positive. By maximizing equation (1.40) while assuming that (1.37) is true, we obtain the following identities,

$$\begin{aligned} T^{dis} &= 2\mu D^d, \\ m &= \frac{2\mu + 3\lambda}{3} \text{div } v - \overline{P}, \\ q &= -k\nabla\theta, \end{aligned} \tag{1.41}$$

in accordance with equation (1.27). Here k is given by $k := \kappa/\theta$.

Finally, he general form of the Cauchy-stress tensor T is given by

$$\begin{aligned} T &\overset{(1.33)}{=} T^d + mI \overset{(1.34)}{=} T^{dis} - \rho \left(\partial_{\nabla\rho} e \otimes \nabla\rho - \frac{1}{3} \left(\partial_{\nabla\rho} e \cdot \nabla\rho \right) I \right) + mI \\ &\overset{(1.41)}{=} 2\mu D^d - \rho \left(\partial_{\nabla\rho} e \otimes \nabla\rho - \frac{1}{3} \left(\partial_{\nabla\rho} e \cdot \nabla\rho \right) I \right) + \frac{2\mu + 3\lambda}{3} \text{div } v \ I - pI \\ &\qquad - \frac{4}{3} \rho \partial_{\nabla\rho} e \cdot \nabla\rho I + \text{div} \left(\rho^2 \partial_{\nabla\rho} e \right) I - \rho^2 \nabla\theta \cdot \partial_{\nabla\rho} e I \\ &\overset{(1.33)}{=} 2\mu D + \lambda \text{div } v \ I - \rho \partial_{\nabla\rho} e \otimes \nabla\rho + \rho \text{div} \left(\rho \partial_{\nabla\rho} e \right) I - pI - \rho^2 \left(\partial_{\nabla\rho} e \cdot \nabla\theta \right) I. \end{aligned} \tag{1.42}$$

We have derived a general form of the Cauchy stress tensor from thermodynamic constitutive equations. This formula expresses the constitutive equations characterizing a class of Navier-Stokes-Fourier-Korteweg fluids.

1.3.2.1. Examples of the Korteweg stress tensor associated with capillary effects in the isothermal case

The specific form of the stress tensor depends on the specific given internal energy e. In the following we will consider isothermal processes, i.e. $q = \nabla\theta = 0$. We will assume that the internal energy is a sum of the pure volumetric energy $e_0 = e_0(\eta, \rho)$ and the capillary energy $e_c = e_c(\rho, \nabla\rho)$, in the sense of equation (1.23). Then the pressure is also a sum of the classical thermodynamic pressure p_0 and a capillary pressure p_c, which can be interpreted to be due to inhomogeneous density fields

$$\begin{aligned} e &= e_0(\eta, \rho) + e_c(\rho, \nabla\rho), \\ p &= p_0 + p_c, \end{aligned}$$

where

$$p_0 := \rho^2 \partial_\rho e_0, \qquad p_c := \rho^2 \partial_\rho e_c.$$

The non-viscous part of the stress tensor T in (1.42), denoted T^c, is given by

$$\begin{aligned} T^c &= -\rho \partial_{\nabla\rho} e_c \otimes \nabla\rho + \rho \mathrm{div}\left(\rho \partial_{\nabla\rho} e_c\right) I - p_0 I - \rho^2 \partial_\rho e_c I \\ &= K - p_0 I, \end{aligned} \tag{1.43}$$

where K is called the capillary stress tensor. We consider a specific class of capillary energies given by

$$e_c = \frac{1}{2+\alpha} A(\rho) |\nabla\Psi(\rho)|^{2+\alpha} = \frac{1}{2+\alpha} A(\rho) |\Psi'(\rho)|^{2+\alpha} |\nabla\rho|^{2+\alpha}, \tag{1.44}$$

for scalar functions $\Psi(\rho)$ and $A(\rho)$. Some elements of this energy class give rise to Korteweg tensors which have already been discussed in literature, others have not been examined yet.

Example 1.5. For $\alpha = 0, \Psi(\rho) = \rho$ and $A(\rho) = \sigma\rho^{-1}$, with the surface tension coefficient σ, we obtain from equation (1.43):

$$K = -\sigma\nabla\rho \otimes \nabla\rho + \sigma\rho\Delta\rho I + \frac{\sigma}{2} |\nabla\rho|^2 I. \tag{1.45}$$

The first term is the classical capillary tensor, which is often called the Korteweg tensor in literature. This model represents the model derived by Korteweg, (1.24), with $\alpha = \sigma/2, \beta = -\sigma, \gamma = \rho\sigma$ and $\delta = 0$.

Using the general identity

$$\mathrm{div}(\nabla\Psi \otimes \nabla\Psi) = \Delta\Psi\nabla\Psi + \frac{1}{2}\nabla|\nabla\Psi|^2 I, \tag{1.46}$$

for a smooth function Ψ, the divergence of the capillary tensor is given by:

$$\mathrm{div}\ K = \sigma\rho\nabla\Delta\rho. \tag{1.47}$$

This capillary tensor is treated in Section 2.2 and Section 2.3, where we discuss shallow water equations.

Example 1.6. For $\alpha = 0$ and $A(\rho) = \sigma\rho^{-1}$, equation (1.43) gives

$$\begin{aligned} K &= -\sigma\nabla\Psi(\rho) \otimes \nabla\Psi(\rho) + \sigma\rho \operatorname{div} (\Psi'(\rho)\nabla\Psi(\rho))I - \frac{1}{2}\sigma\rho^2\partial_\rho(\rho^{-1}|\Psi'(\rho)|^2)|\nabla\rho|^2 I \\ &= -\sigma\nabla\Psi(\rho) \otimes \nabla\Psi(\rho) + \sigma\rho(\Psi'(\rho)\Delta\Psi(\rho))I + \frac{1}{2}\sigma|\nabla\Psi(\rho)|^2 I. \end{aligned} \tag{1.48}$$

The divergence of this capillary tensor is given by

$$\operatorname{div} K = \sigma\rho\nabla(\Psi'(\rho)\Delta\Psi(\rho)). \tag{1.49}$$

The mathematical properties of capillary tensors of this type will be discussed in Section 2.1.

2. Introduction of the BD entropy and stability results for weak solutions of Korteweg type equations

In this chapter, we will discuss the mathematical properties of viscous fluid equations with capillary terms of Korteweg type. The structure of this chapter is given as follows: Firstly, an additional energy estimate, the BD entropy, is introduced for diffusive capillary models of Korteweg type. Then, a priori bounds on the solutions of some Korteweg systems, like the viscous shallow water, or Saint-Venant, equations and the quantum Navier-Stokes equations, which arise from the BD entropy estimate, are discussed. Finally, we will prove the stability of weak solutions of the barotropic compressible Navier-Stokes equations using the a priori bounds on the solutions.

2.1. Derivation of the classical energy estimate and the BD entropy estimate for smooth solutions of diffusive capillary models of Korteweg type

In this section, we will recall the derivation of the BD entropy by Bresch and Desjardins in [8]. They discovered the entropy estimate in [6], for two-dimensional Saint-Venant equations, and used it to prove the existence of global weak solutions of these shallow water systems in [7]. Later, they derived a general form of the BD entropy in [8].

In the context of chapter one, ρ is the density, v is the velocity, f is the body force and μ, λ are the Lamé coefficients. We will treat stress tensors T, which are dependent on the surface tension coefficient σ and on a scalar function $\Psi(\rho)$.

We assume isothermal flow and consider a bounded domain with periodic boundary conditions, i.e. $\Omega = T^d$ for the space dimension d, and set the body force $f = 0$. A diffusive capillary model of Korteweg type is given by

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \tag{2.1}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = \operatorname{div} T, \tag{2.2}$$

where the Cauchy tensor T is given by the sum of the viscous and capillary tensors:

$$T = 2\mu D(v) + \lambda \text{ div } v\ I - pI - \sigma\nabla\Psi(\rho)\otimes\nabla\Psi(\rho) + \sigma\rho(\Psi'(\rho)\Delta\Psi(\rho))I + \frac{1}{2}\sigma|\nabla\Psi(\rho)|^2 I.$$

In the non-conservative form the momentum equation reads

$$\partial_t(\rho v) + \text{div}(\rho v\otimes v) = 2\text{div}(\mu D(v)) + \nabla(\lambda \text{ div } v) - \nabla p + \sigma\rho\nabla(\Psi'(\rho)\Delta\Psi(\rho)),$$

see Example 1.6.

We will prove, under certain compatibility assumptions between the diffusion and the capillarity, an extra regularity of a velocity which is related to the gradient of the density. This regularity allows us to control the degeneration of solutions in areas close to vacuum.
We will assume the following definitions of the Lamé coefficients $\mu(\rho), \lambda(\rho)$,

$$\mu(\rho) = \mu_1\Psi(\rho), \qquad \lambda(\rho) = 2\mu_1(\rho\Psi'(\rho) - \Psi(\rho)), \tag{2.3}$$

with a constant μ_1 and a scalar valued function

$$\Psi(\rho) \geq 0.$$

We define $\phi(\rho)$ and $\Pi(\rho)$ such that

$$\rho\phi'(\rho) = \Psi'(\rho), \qquad \rho\Pi'(\rho) - \Pi(\rho) = p(\rho), \tag{2.4}$$

i.e, for a given constant density $\overline{\rho} \geq 0$,

$$\phi(\rho) = \int_{\overline{\rho}}^{\rho} \Psi'(s)s^{-1}ds \quad \text{and} \quad \Pi(\rho) = \rho\int_{\overline{\rho}}^{\rho} p(s)s^{-2}ds. \tag{2.5}$$

Remark 2.1. Note that the relation between $\mu(\rho)$ and $\lambda(\rho)$ is of the general form $\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho))$. This relation is, so far, only a mathematical hypothesis.

Remark 2.2. In particular, we have $\rho\Pi''(\rho) = p'(\rho)$. Thus, the internal energy is a convex function of the density in regions with nondecreasing pressure.

Definition 2.3. The matrix inner product is given by

$$A : B = \sum_i \sum_j A_{ij} B_{ij},$$

for two matrices A and B.

Theorem 2.4.
Under suitable smoothness assumptions on $\mu(\rho)$ and $\lambda(\rho)$, the following two energy equalities hold for smooth solutions of system (2.1)-(2.2):

(1) Classical energy identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|v|^2 \ dx + 2\mu_1 \int_\Omega \Psi(\rho)|D(v)|^2 \ dx + 2\mu_1 \int_\Omega (\Psi'(\rho)\rho - \Psi(\rho))|\text{div } v|^2 \ dx$$
$$+ \frac{\sigma}{2}\frac{d}{dt}\int_\Omega |\nabla\Psi(\rho)|^2 \ dx + \frac{d}{dt}\int_\Omega \Pi(\rho) \ dx = 0 \tag{2.6}$$

(2) BD entropy identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|v + 2\mu_1\nabla\phi|^2 \ dx + 2\mu_1 \int_\Omega \Psi(\rho)|D(v)|^2 \ dx + 2\sigma\mu_1 \int_\Omega \Psi'(\rho)|\Delta\Psi(\rho)|^2 \ dx$$
$$+2\mu_1 \int_\Omega p'(\rho)|\nabla\rho|^2\phi'(\rho) \ dx + \frac{\sigma}{2}\frac{d}{dt}\int_\Omega |\nabla\Psi(\rho)|^2 \ dx + \frac{d}{dt}\int_\Omega \Pi(\rho) \ dx$$
$$= 2\mu_1 \int_\Omega \Psi(\rho)\nabla v : \nabla v^T \ dx \tag{2.7}$$

Proof.

(1)
Firstly, we multiply the mass balance equation (2.1) by $|v|^2/2$ and integrate over Ω:

$$\int_\Omega \frac{|v|^2}{2}\partial_t\rho + \frac{|v|^2}{2}\text{div}(\rho v) \ \text{d}x = 0.$$

Then we multiply equation (2.1) by $-\sigma\Psi'(\rho)\Delta\Psi(\rho)$ and integrate over Ω, again:

$$\int_\Omega -\sigma\Psi'(\rho)\Delta\Psi(\rho)\partial_t\rho - \sigma\Psi'(\rho)\Delta\Psi(\rho)\text{div}(\rho v) \ \text{d}x = 0.$$

We integrate by parts in the first term

$$\int_\Omega -\sigma\Psi'(\rho)\Delta\Psi(\rho)\partial_t\rho \ \mathrm{d}x = \int_\Omega -\sigma\partial_t\Psi(\rho)\Delta\Psi(\rho) \ \mathrm{d}x = \int_\Omega \sigma\nabla\Psi\cdot\partial_t\nabla\Psi \ \mathrm{d}x$$
$$= \int_\Omega \frac{\sigma}{2}\frac{d}{dt}|\nabla\Psi|^2 \ \mathrm{d}x,$$

and multiply the momentum balance equation (2.2) by v to obtain

$$\int_\Omega v\partial_t(\rho v) + v\mathrm{div}(\rho v\otimes v) \ \mathrm{d}x$$
$$= \int_\Omega 2v\mathrm{div}(\mu D(v)) + v\nabla(\lambda \ \mathrm{div} \ v) - v\nabla p + v\sigma\rho\nabla(\Psi'(\rho)\Delta\Psi(\rho)) \ \mathrm{d}x.$$

We use the following equalities, which are established by integrating by parts and using the definitions of $\mu(\rho), \lambda(\rho)$ and $\Pi(\rho)$, to sum up the former three equations,

$$-\int_\Omega v\nabla(\lambda\mathrm{div} \ v) \ \mathrm{d}x = \int_\Omega \lambda|\mathrm{div} \ v|^2 \ \mathrm{d}x \overset{(2.3)}{=} \int_\Omega (\rho\Psi'(\rho) - \Psi(\rho))|\mathrm{div} \ v|^2 \ \mathrm{d}x,$$
$$-\int_\Omega 2v\mathrm{div}(\mu(\rho)D(v)) \ \mathrm{d}x = 2\int_\Omega \mu(\rho)\nabla v : D(v) \ \mathrm{d}x \overset{(2.3)}{=} 2\mu_1\int_\Omega \Psi(\rho)|D(v)|^2 \ \mathrm{d}x,$$

because $\nabla v : D(v) = |D(v)|^2$, since $D(v)$ is symmetric and $\nabla v : S = D(v) : S$ for every symmetric tensor S,

$$\int_\Omega v\mathrm{div}(\rho v\otimes v) \ \mathrm{d}x = \int_\Omega \mathrm{div}(\rho v)|v|^2 + \rho v(\nabla v\cdot v) \ \mathrm{d}x$$
$$= \int_\Omega \mathrm{div}(\rho v)|v|^2 - \mathrm{div}(\rho v)\frac{|v|^2}{2} \ \mathrm{d}x, \tag{2.8}$$
$$\int_\Omega v\partial_t(\rho v) \ \mathrm{d}x = \int_\Omega \rho\frac{d}{dt}\frac{|v|^2}{2} + \partial_t\rho|v|^2 \ \mathrm{d}x,$$
$$\int_\Omega -\sigma\Psi'(\rho)\Delta\Psi(\rho)\mathrm{div}(\rho v) \ \mathrm{d}x = \int_\Omega \sigma\rho v\cdot\nabla(\Psi'(\rho)\Delta\Psi(\rho)) \ \mathrm{d}x,$$
$$-\int_\Omega v\nabla p \ \mathrm{d}x = -\int_\Omega p'\nabla\rho\cdot v \ \mathrm{d}x \overset{(2.4)}{=} -\int_\Omega \rho\Pi''\nabla\rho\cdot v \ \mathrm{d}x$$
$$= -\int_\Omega \rho\nabla\Pi'\cdot v \ \mathrm{d}x = \int_\Omega \Pi'\mathrm{div}(\rho\cdot v) \ \mathrm{d}x$$
$$\overset{(2.1)}{=} -\int_\Omega \Pi'\partial_t\rho \ \mathrm{d}x = -\frac{d}{dt}\int_\Omega \Pi(\rho) \ \mathrm{d}x, \tag{2.9}$$

and observe that all other terms in the sum cancel and we obtain the classical energy estimate (2.6).

(2)
The mass balance equation for the function $\phi(\rho)$ yields:

$$\begin{aligned}\partial_t\phi(\rho) &+ v\cdot\nabla\phi(\rho) + \phi'\rho\mathrm{div}\ v\\ &= \phi'\partial_t\rho + v\cdot\phi'\nabla\rho + \phi'\rho\mathrm{div}\ v\\ &= \phi'\partial_t\rho + v\cdot\phi'\nabla\rho - \phi'\nabla\rho\cdot v + \phi'\mathrm{div}(\rho v)\\ &= \phi'\left(\partial_t\rho + \mathrm{div}(\rho v)\right) = 0.\end{aligned}$$

We differentiate this equation with respect to the spatial variables,

$$\partial_t\nabla\phi(\rho) + \nabla\nabla\phi(\rho)\cdot v + \nabla v\cdot\nabla\phi(\rho) + \nabla(\phi'\rho\mathrm{div}\ v) = 0,$$

multiply it by $\rho\nabla\phi(\rho)$ and integrate over Ω. Using the mass balance equation, again, we obtain:

$$\begin{aligned}0 = \int_\Omega &\rho\nabla\phi\cdot\partial_t\nabla\phi + \rho\nabla\phi\cdot\nabla\nabla\phi\cdot v + \rho\nabla\phi\cdot\nabla v\cdot\nabla\phi + \rho\nabla\phi\cdot\nabla(\phi'\rho\mathrm{div}\ v)\ \mathrm{d}x\\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\nabla\phi|^2\ \mathrm{d}x - \int_\Omega \frac{1}{2}|\nabla\phi|^2\partial_t\rho\ \mathrm{d}x + \int_\Omega \frac{1}{2}\nabla\left(\nabla\phi\cdot\nabla\phi\right)\cdot v\rho\ \mathrm{d}x\\ &\quad + \int_\Omega \rho\nabla\phi\cdot\nabla v\cdot\nabla\phi\ \mathrm{d}x + \int_\Omega \nabla\Psi\cdot\nabla(\phi'\rho\mathrm{div}\ v)\ \mathrm{d}x\\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\nabla\phi|^2\ \mathrm{d}x - \underbrace{\left(\int_\Omega \frac{1}{2}|\nabla\phi|^2\partial_t\rho\ \mathrm{d}x + \int_\Omega \frac{1}{2}|\nabla\phi|^2\mathrm{div}(\rho v)\ \mathrm{d}x\right)}_{=0}\\ &\quad + \int_\Omega \rho\nabla\phi\cdot\nabla v\cdot\nabla\phi\ \mathrm{d}x + \int_\Omega \nabla\Psi\cdot\nabla(\phi'\rho\mathrm{div}\ v)\ \mathrm{d}x\\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\nabla\phi|^2\ \mathrm{d}x + \int_\Omega \rho\nabla\phi\cdot\nabla v\cdot\nabla\phi\ \mathrm{d}x + \int_\Omega \nabla\Psi\cdot\nabla(\phi'\rho\mathrm{div}\ v)\ \mathrm{d}x.\end{aligned}$$

We integrate by parts in the second term on the right-hand side and use the relation (2.4), this gives:

$$\begin{aligned}\int_\Omega \rho\nabla\phi(\rho)\cdot\nabla v\cdot\nabla\phi(\rho)\ \mathrm{d}x &= \int_\Omega \frac{\nabla\Psi\cdot\nabla v\cdot\nabla\Psi}{\rho}\ \mathrm{d}x\\ = \int_\Omega \frac{\Psi\nabla\Psi\cdot\nabla v\cdot\nabla\rho}{\rho^2}\ \mathrm{d}x &- \int_\Omega \frac{\Psi\nabla\mathrm{div}\ v\cdot\nabla\Psi}{\rho}\ \mathrm{d}x - \int_\Omega \frac{\Psi\nabla v : \nabla\nabla\Psi}{\rho}\ \mathrm{d}x.\end{aligned}$$

Thus, we have:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\nabla\phi|^2 \,\mathrm{d}x + \int_\Omega \frac{\Psi\nabla\Psi\cdot\nabla v\cdot\nabla\rho}{\rho^2}\,\mathrm{d}x - \int_\Omega \frac{\Psi\nabla\mathrm{div}\ v\cdot\nabla\Psi}{\rho}\,\mathrm{d}x$$
$$- \int_\Omega \frac{\Psi\nabla v\cdot\nabla\nabla\Psi}{\rho}\,\mathrm{d}x + \int_\Omega \nabla\Psi(\rho)\cdot\nabla(\phi'\rho\mathrm{div}\ v)\,\mathrm{d}x = 0. \quad (2.10)$$

By multiplying the momentum balance equation by $\nabla\Psi(\rho)/\rho$, we obtain

$$\frac{\nabla\Psi(\rho)}{\rho}\partial_t(\rho v) + \frac{\nabla\Psi(\rho)}{\rho}\mathrm{div}(\rho v\otimes v)$$
$$- \frac{\nabla\Psi(\rho)}{\rho}2\mathrm{div}(\mu D(v)) - \frac{\nabla\Psi(\rho)}{\rho}\nabla(\lambda\ \mathrm{div}\ v) + \frac{\nabla\Psi(\rho)}{\rho}\nabla p$$
$$- \frac{\nabla\Psi(\rho)}{\rho}\sigma\rho\nabla(\Psi'(\rho)\Delta\Psi(\rho)) = 0, \quad (2.11)$$

and integration by parts, together with the equations (2.3) and (2.8), gives:

$$\int_\Omega (\partial_t v + v\cdot\nabla v)\nabla\Psi + \frac{\nabla\Psi(\rho)}{\rho}v\cdot\overbrace{(\partial_t\rho + \mathrm{div}(\rho v))}^{=0}\,\mathrm{d}x$$
$$+ \int_\Omega 2\mu_1\Psi(\rho)D(v) : \left(\frac{\nabla\nabla\Psi}{\rho} - \frac{\nabla\Psi\otimes\nabla\rho}{\rho^2}\right)\mathrm{d}x$$
$$- \int_\Omega 2\mu_1\frac{\nabla\Psi(\rho)}{\rho}\nabla((\Psi'(\rho)\rho - \Psi(\rho))\mathrm{div}\ v)\,\mathrm{d}x + \int_\Omega p'(\rho)\Psi'(\rho)\frac{|\nabla\rho|^2}{\rho}\,\mathrm{d}x$$
$$+ \int_\Omega \sigma\Psi'(\rho)|\Delta\Psi(\rho)|^2\,\mathrm{d}x = 0. \quad (2.12)$$

When adding equation (2.12) to the $2\mu_1$ multiple of equation (2.10), we observe that,

$$\int_\Omega 2\mu_1\Psi(\rho)D(v) : \left(\frac{\nabla\nabla\Psi}{\rho} - \frac{\nabla\Psi\otimes\nabla\rho}{\rho^2}\right)\mathrm{d}x + 2\mu_1\int_\Omega \frac{\Psi\nabla\Psi\cdot\nabla v\cdot\nabla\rho}{\rho^2}\,\mathrm{d}x$$
$$- 2\mu_1\int_\Omega \frac{\Psi\nabla v : \nabla\nabla\Psi}{\rho}\,\mathrm{d}x = 0,$$

since both, $\nabla\nabla\Psi$ and $\nabla\Psi\otimes\nabla\rho = \Psi'\nabla\rho\otimes\nabla\rho$, are symmetric. We recall that $D(v) : S = \nabla v : S$ for every symmetric tensor S. Thus, the sum is given by:

$$
\begin{aligned}
&\int_\Omega (\partial_t v + v\cdot\nabla v)\,\nabla\Psi \;\mathrm{d}x - 2\mu_1 \int_\Omega \frac{\Psi\nabla \mathrm{div}\; v\cdot\nabla\Psi}{\rho}\;\mathrm{d}x \\
&\quad + \frac{\mu_1}{2}\frac{d}{dt}\int_\Omega 2\rho|\nabla\phi|^2\;\mathrm{d}x - \int_\Omega 2\mu_1\frac{\nabla\Psi(\rho)}{\rho}\nabla((\Psi'(\rho)\rho-\Psi(\rho))\mathrm{div}\; v)\;\mathrm{d}x \qquad (2.13)\\
&\quad + 2\mu_1\int_\Omega \nabla(\Psi'(\rho)\mathrm{div}\; v)\cdot\nabla\Psi\;\mathrm{d}x + \int_\Omega p'(\rho)\Psi'(\rho)\frac{|\nabla\rho|^2}{\rho}\;\mathrm{d}x + \int_\Omega \sigma\Psi'(\rho)|\Delta\Psi(\rho)|^2\;\mathrm{d}x = 0.
\end{aligned}
$$

We consider all terms containing div v and observe that:

$$
\begin{aligned}
&\int_\Omega 2\mu_1\frac{\nabla\Psi(\rho)}{\rho}\cdot\big(-\Psi\nabla\mathrm{div}\; v - \nabla(\Psi'(\rho)\rho\mathrm{div}\; v) + \nabla(\Psi(\rho)\mathrm{div}\; v) + \rho\nabla(\Psi'(\rho)\mathrm{div}\; v)\big)\;\mathrm{d}x\\
&= \int_\Omega 2\mu_1\frac{\nabla\Psi(\rho)}{\rho}\cdot(-\Psi\nabla\mathrm{div}\; v - \nabla(\Psi'(\rho)\mathrm{div}\; v)\rho - \nabla\Psi(\rho)\mathrm{div}\; v + \nabla\Psi(\rho)\mathrm{div}\; v\\
&\qquad + \Psi(\rho)\nabla\mathrm{div}\; v + \rho\nabla(\Psi'(\rho)\mathrm{div}\; v)) = 0.
\end{aligned}
$$

Now we expand the first term in equation (2.13):

$$
\int_\Omega (\partial_t v + v\cdot\nabla v)\,\nabla\Psi\;\mathrm{d}x = \frac{d}{dt}\int_\Omega v\cdot\nabla\Psi\;\mathrm{d}x - \int_\Omega v\cdot\partial_t\nabla\Psi\;\mathrm{d}x + \int_\Omega v\cdot\nabla v\cdot\nabla\Psi\;\mathrm{d}x.
$$

By integrating by parts in the last two terms of the former equation and using the mass balance equation we obtain:

$$
\begin{aligned}
-\int_\Omega v\cdot\partial_t\nabla\Psi\;\mathrm{d}x &= -\int_\Omega \Psi' v\cdot\partial_t\nabla\rho\;\mathrm{d}x = \int_\Omega \Psi' v\cdot\nabla\mathrm{div}(\rho v)\;\mathrm{d}x\\
&= -\int_\Omega \Psi'(\mathrm{div}\; v)\;\mathrm{div}(\rho v)\;\mathrm{d}x
\end{aligned}
$$

and

$$
\int_\Omega v\cdot\nabla v\cdot\nabla\Psi\;\mathrm{d}x = -\int_\Omega \Psi\; v\cdot\nabla\mathrm{div}\; v\;\mathrm{d}x - \int_\Omega \Psi\nabla v : \nabla v^T\;\mathrm{d}x.
$$

By integrating by parts in the term

$$
-\int_\Omega \Psi v\cdot\nabla\mathrm{div}\; v\;\mathrm{d}x = \int_\Omega \nabla\Psi\cdot v\mathrm{div}\; v\;\mathrm{d}x + \int_\Omega \Psi|\mathrm{div}\; v|^2\;\mathrm{d}x,
$$

and adding it to

$$-\int_\Omega \Psi'(\text{div } v)\ \text{div}(\rho v)\ \mathrm{d}x = -\int_\Omega \Psi'\rho|\text{div } v|^2\ \mathrm{d}x - \int_\Omega \Psi' v\cdot\nabla\rho\ \text{div } v\ \mathrm{d}x,$$

we finally obtain

$$\begin{aligned}&\int_\Omega (\partial_t v + v\cdot\nabla v)\,\nabla\Psi\ \mathrm{d}x\\ &\quad= \frac{d}{dt}\int_\Omega v\cdot\nabla\Psi\ \mathrm{d}x - \int_\Omega(\rho\Psi'(\rho)-\Psi(\rho))|\text{div } v|^2\ \mathrm{d}x - \int_\Omega \Psi\nabla v : \nabla v^T\ \mathrm{d}x.\end{aligned}$$

After inserting this equation into equation (2.13), it reads:

$$\begin{aligned}&\frac{d}{dt}\int_\Omega v\cdot\nabla\Psi\ \mathrm{d}x - \int_\Omega(\rho\Psi'(\rho)-\Psi(\rho))|\text{div } v|^2\ \mathrm{d}x - \int_\Omega \Psi\nabla v : \nabla v^T\ \mathrm{d}x \\ &\quad+ \frac{\mu_1}{2}\frac{d}{dt}\int_\Omega 2\rho|\nabla\phi|^2\ \mathrm{d}x + \int_\Omega p'(\rho)\Psi'(\rho)\frac{|\nabla\rho|^2}{\rho}\ \mathrm{d}x + \int_\Omega \sigma\Psi'(\rho)|\Delta\Psi(\rho)|^2\ \mathrm{d}x = 0.\end{aligned} \tag{2.14}$$

If we now multiply equation (2.14) by $2\mu_1$ and add it to the classical energy estimate (2.6) we obtain the additional energy estimate (2.7), since

$$\begin{aligned}2\mu_1\frac{d}{dt}\int_\Omega v\cdot\nabla\Psi\ \mathrm{d}x + \mu_1\frac{d}{dt}\int_\Omega 2\mu_1\rho|\nabla\phi|^2\ \mathrm{d}x + \frac{1}{2}\frac{d}{dt}\int_\Omega \rho|v|^2\ \mathrm{d}x\\ = \frac{1}{2}\frac{d}{dt}\int_\Omega \rho|v+2\mu_1\nabla\phi|^2\ \mathrm{d}x.\end{aligned}$$

□

Remark 2.5. For viscosity terms of the form $\mu\nabla v$ rather than $2\mu D(v)$ we obtain the same result with $\int_\Omega \mu|\nabla v|^2\ \mathrm{d}x$ instead of $\int_\Omega 2\mu|D(v)|^2\ \mathrm{d}x$, as can be easily seen in the proof.

Remark 2.6. The classical free energy of the system is:

$$E_{cl} = \int_\Omega \frac{1}{2}\rho|v|^2 + \Pi(\rho) + \frac{\sigma}{2}|\nabla\Psi(\rho)|^2\ \mathrm{d}x.$$

Thus, the classical energy estimate (2.6) has the form

$$\frac{dE_{cl}}{dt} + \int_\Omega 2\mu_1 \Psi(\rho)|D(v)|^2 \, \mathrm{d}x + 2\mu_1 \int_\Omega (\Psi'(\rho)\rho - \Psi(\rho))|\mathrm{div}\ v|^2 \, \mathrm{d}x = 0,$$
$$\frac{dE_{cl}}{dt} + \int_\Omega 2\mu|D(v)|^2 \, \mathrm{d}x + \lambda|\mathrm{div}\ v|^2 \, \mathrm{d}x = 0.$$

Remark 2.7. We can rewrite the diffusion term containing ∇v as

$$\int_\Omega 2\mu|D(v)|^2 \, \mathrm{d}x - 2\int_\Omega \mu\nabla v : \nabla v^T \, \mathrm{d}x = \int_\Omega 2\mu|A(v)|^2 \, \mathrm{d}x,$$

where $A(v) = \frac{1}{2}\nabla v - \frac{1}{2}\nabla v^T$ is the antisymmetric part of the velocity gradient.

In the following sections we will emphasize the importance of the new entropy identity for Korteweg systems. Assuming that the classical physical energy estimate must not only hold for smooth solutions, but also for weak solutions of Korteweg systems, gives a priori regularity estimates on the weak solutions. These are often not enough to prove the stability of weak solutions for models which include vacuum. But assuming that also the BD entropy equality must be satisfied by the weak solutions gives extra regularity on the density gradient. Therefore, we can prove stability results for weak solutions of some Korteweg systems with vacuum initial conditions.

2.2. Energy estimates and a priori bounds on weak solutions of Korteweg systems

In this section, we will discuss the classical and the BD entropy estimate for some Korteweg systems with viscosity coefficients $\mu(\rho) = \rho, \lambda(\rho) = 0$ and a general barotropic pressure term $p(\rho)$ which includes the case of a power pressure law but also unstable spinodal regions in which $p(\rho)$ may decrease. We will use these physical energy estimates to derive a priori bounds on weak solutions of these Korteweg systems.

We consider a domain $\Omega = T^d$, for d equal to 2 or 3 with periodic boundary conditions. We assume to have a barotropic pressure law $p = p(\rho)$ and let κ be the capillary coefficient. We will consider the following viscous compressible Korteweg system:

$$\partial_t\rho + \mathrm{div}(\rho v) = 0,$$
$$\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \nabla p = 2\mathrm{div}(\mu D(v)) + \nabla(\lambda \mathrm{div}\ v)) + \kappa\rho\nabla\Delta\rho + \rho f,$$

with initial conditions

$$\rho(0,x) = \rho_0 \geq 0, \qquad \rho v(0,x) = \rho_0 v_0 =: m_0. \tag{2.15}$$

In particular, this problem includes vacuum initial conditions, $\rho_0 = 0$. For the sake of simplicity, we will set $f = 0$ and drop the factor 2, i.e. we consider the viscosity coefficients $\mu(\rho) = \frac{1}{2}\nu\rho$, with $\nu \geq 0$ and $\lambda(\rho) = 0$. Then, the mass and momentum balance equations of the Korteweg system are given by:

$$\begin{aligned} &\partial_t \rho + \operatorname{div}(\rho v) = 0, \\ &\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \nabla p - \nu \operatorname{div}(\rho D(v)) = \kappa \rho \nabla \Delta \rho. \end{aligned} \tag{2.16}$$

We assume that p is in $C^1([0,\infty))$ and satisfies

$$p(\rho) \geq 0, \quad p'(\rho) \geq 0, \tag{2.17}$$

and

$$\Sigma(\rho) \leq A\rho^n \Pi(\rho) \qquad \text{for large enough } \rho, \tag{2.18}$$

where A is a positive constant and, $n < \infty$ when $d = 2$, $n < 4$ when $d = 3$, with

$$\Sigma(\rho) = \int_{\overline{\rho}}^{\rho} s p'(s) ds.$$

Remark 2.8. In particular, the pressure given in a shallow water system, $\rho^2/2$, fulfills the conditions above with $n = 1$ and $A \geq 1$, see Section 2.3.

We want to formally derive the form of the energy identities for this special Korteweg system. We have $\Psi(\rho) = \rho$, then obviously, the requirement (2.4), $\rho\phi'(\rho) = \Psi'(\rho)$, is fulfilled by $\phi(\rho) = \log \rho$. The viscosity coefficients also fulfill the requirement for the BD entropy estimate (2.3), since $\mu = 1/2\ \nu\rho$ and $\lambda = 0$. We have

$$\mu(\rho) = \frac{1}{2}\ \nu \Psi(\rho),$$

and, thus, the BD entropy estimate (2.7) for system (2.16) is given by

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho|v+\nu\nabla\log\rho|^2 \,\mathrm{d}x + \Pi(\rho) + \frac{\kappa}{2}|\nabla\rho|^2\right)\,\mathrm{d}x + 4\nu\int_\Omega p'(\rho)|\nabla\sqrt{\rho}|^2\,\mathrm{d}x \tag{2.19}$$
$$+\,\nu\kappa\int_\Omega |\nabla^2\rho|^2\,\mathrm{d}x + \int_\Omega \nu\rho|A(v)|^2\,\mathrm{d}x = 0,$$

with $A(v) = \frac{1}{2}(\nabla v - \nabla v^T)$ and since

$$\nu\int_\Omega \nabla p(\rho)\cdot\nabla\log\rho\,\mathrm{d}x = 4\nu\int_\Omega p'(\rho)|\nabla\sqrt{\rho}|^2\,\mathrm{d}x.$$

The classical energy estimate is given by

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho|v|^2 + \Pi(\rho) + \frac{\kappa}{2}|\nabla\rho|^2\right)\,\mathrm{d}x + \int_\Omega \nu\rho|D(v)|^2\,\mathrm{d}x = 0. \tag{2.20}$$

We define "weak solutions" for system (2.16) which are not contained in the usual notion of weak solution, since we will use test functions which depend upon the solutions. The regularity of ρ (which will be shown to be in $L^2(0,T;H^2(\Omega))$) allows us to consider test functions $\rho\phi$. These are supported on the set of non-vanishing ρ.

Definition 2.9 ("Weak solutions")**.** We say (ρ, v) is a "weak solution" of system (2.16) on $(0,T)\times\Omega$ if and only if

$$\int_\Omega \left(\kappa\frac{|\nabla\rho_0|^2}{2} + \Pi(\rho_0) + \rho_0\frac{|v_0|^2}{2}\right)\,\mathrm{d}x < +\infty, \tag{2.21}$$
$$2\nu^2\int_\Omega |\nabla\sqrt{\rho_0}|^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega \rho_0|\nu\nabla\log\rho_0|^2\,\mathrm{d}x < +\infty, \tag{2.22}$$

and the following three assumptions are satisfied:

$$\begin{aligned}
&\rho \in L^2(0,T;H^2(\Omega)),\\
&\nabla\rho, \nabla\sqrt{\rho} \in L^\infty(0,T;(L^2(\Omega))^d),\\
&\sqrt{\rho}v \in L^\infty(0,T;(L^2(\Omega))^d),\\
&\sqrt{\rho}D(v) \in L^2(0,T;(L^2(\Omega))^{d\times d}),
\end{aligned} \tag{2.23}$$

with $\rho \geq 0$ a.e. and

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho v) = 0 \quad & \text{in } \mathcal{D}'((0,T) \times \Omega), \\ \rho(0,x) = \rho_0 \quad & \text{in } \mathcal{D}'(\Omega), \end{aligned} \tag{2.24}$$

and for all $\phi \in \mathcal{D}([0,T] \times \Omega)$, such that $\phi(T,\cdot) = 0$, it holds:

$$\begin{aligned} &\int_\Omega \rho_0 v_0 \cdot \rho_0 \phi(0,\cdot)\,\mathrm{d}x + \int_0^T \int_\Omega \rho^2 v \cdot \partial_t \phi + \rho v \otimes \rho v : D(\phi)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \rho^2 (v \cdot \phi)\,\operatorname{div} v - \nu \rho D(v) : \rho D(\phi) - \nu \rho D(v) : \phi \otimes \nabla \rho + \Sigma(\rho)\,\operatorname{div}\phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \kappa \rho^2 \Delta \rho\,\operatorname{div}\phi - 2\kappa \rho (\phi \cdot \nabla \rho) \Delta \rho\,\mathrm{d}x\,\mathrm{d}t = 0. \end{aligned}$$

We want to motivate a stability result for these "weak solutions" of the Korteweg system (2.16).

Theorem 2.10 ([10])**.** *Let d be equal to 2 or 3 and let $(\rho_n, v_n)_{n \in \mathbb{N}}$ be sequence of "weak solutions" of system (2.16) satisfying the entropy inequalities (2.19) and (2.20) with initial conditions*

$$\rho_n(0,x) = \rho_0^n \geq 0, \qquad \rho_0 v_0(0,x) = \rho_0^n v_0^n =: m_0^n, \tag{2.25}$$

where ρ_0^n, v_0^n are such that

$$\rho_0^n \geq 0, \quad \rho_0^n \to \rho_0 \text{ in } L^1(\Omega), \quad \rho_0^n v_0^n \to \rho_0 v_0 \text{ in } L^1(\Omega). \tag{2.26}$$

Then $(\rho_n, v_n)_{n \in \mathbb{N}}$ converges strongly to a weak solution of system (2.16) with initial conditions (2.15 satisfying (2.19) and (2.20)).

Bresch, Desjardins and Lin, [10], introduced the different notion of weak solutions, in order to prove the stability of the Korteweg systems. The proof of compactness failed in regions where the limit density ρ vanishes. They were only able to show the weak compactness of $\sqrt{\rho_n} v_n$, which does not imply the compactness of $\sqrt{\rho_n} v_n \otimes \sqrt{\rho_n} v_n$. This problem is overcome in the next Section, by considering a system with additional drag terms $r_0 v$, for which the energy estimates give extra regularity on the velocity v, so the term $\sqrt{\rho_n} v_n$ is well defined on the vacuum set and converges strongly. As shown in the last chapter of this thesis, assuming additional relations on the viscosity coefficients also helps to show the strong compactness of $\sqrt{\rho_n} v_n$. The multiplication by ρ_n provides the strong convergence of the viscosity term $\rho_n v_n \otimes$

$\rho_n v_n : D(\phi)$, since Bresch, Desjardins and Lin have proven that the momentum $\rho_n v_n$ converges strongly in $L^2(0, T; L^2(\Omega))$.

Remark 2.11. Lions, [26], claimed that the strong convergence of the initial data $(\rho_0^n, \rho_0^n v_0^n)$ to (ρ_0, v_0) in $L^1(\Omega)$ is essential to construct a sequence $(\rho_n, v_n)_{n\in\mathbb{N}}$ with the desired properties. The construction of such a sequence is still an open problem in many cases, [5].

Idea of the proof of Theorem 2.10 The proof of the stability relies upon the a prori bounds on the sequence of "weak solutions" $(\rho_n, v_n)_{n\in\mathbb{N}}$ which follow from the classical (2.20) and the BD entropy identity (2.19).

We have assumed that the sequence of "weak solutions" $(\rho_n, v_n)_{n\in\mathbb{N}}$ satisfies the energy identities and that the initial classical energy of the system is bounded, (2.21), therefore, integrating the classical energy equation with respect to the time yields

$$\begin{aligned}\int_\Omega \left(\frac{1}{2}\rho_n|v_n|^2 + \Pi(\rho_n) + \frac{\kappa}{2}|\nabla\rho_n|^2\right) \mathrm{d}x + \int_0^T \int_\Omega \nu\rho_n|D(v_n)|^2 \mathrm{d}x \mathrm{d}t \\ = \int_\Omega \left(\frac{1}{2}\rho_0|v_0|^2 + \Pi(\rho_0) + \frac{\kappa}{2}|\nabla\rho_0|^2\right) \mathrm{d}x \le C, \qquad (2.27)\end{aligned}$$

for some constant $C > 0$. Thus, we have the following bounds on $(\rho_n, v_n)_{n\in\mathbb{N}}$:

$$\begin{aligned}&||\nabla\rho_n||_{L^\infty(0,T;(L^2(\Omega))^d)} \le C,\\ &||\sqrt{\rho_n}v_n||_{L^\infty(0,T;(L^2(\Omega))^d)} \le C,\\ &||\sqrt{\rho_n}D(v_n)||_{L^2(0,T;(L^2(\Omega))^{d\times d})} \le C.\end{aligned}$$

Proceeding similarly with the BD entropy estimate gives additional bounds on the "weak solutions", since we also assumed that the initial BD entropy is bounded, by (2.21), (2.22) and because

$$2\rho_0 v_0 \nabla \log\rho_0 = 2v_0\nabla\rho_0 = v_0\sqrt{\rho_0}\nabla\sqrt{\rho_0} \in L^1(\Omega),$$

by Hölder's inequality (Theorem A.31):

$$\begin{aligned}\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho_n|v_n + \nu\nabla\log\rho_n|^2 \mathrm{d}x + \Pi(\rho_n) + \frac{\kappa}{2}|\nabla\rho_n|^2\right) \mathrm{d}x + 4\nu\int_\Omega p'(\rho_n)|\nabla\sqrt{\rho_n}|^2 \mathrm{d}x \\ + \nu\kappa\int_\Omega |\nabla^2\rho_n|^2 \mathrm{d}x + \int_\Omega \nu\rho_n|A(v_n)|^2 \mathrm{d}x \le C. \qquad (2.28)\end{aligned}$$

Therefore,

$$
\begin{aligned}
&||\rho_n||_{L^2(0,T;H^2(\Omega))} \leq C,\\
&||\nabla\sqrt{\rho_n}||_{L^\infty(0,T;(L^2(\Omega))^d)} \leq C,\\
&||\sqrt{\rho_n}\nabla v_n||_{L^2(0,T;(L^2(\Omega))^{d\times d})} \leq C,
\end{aligned}
$$

The first bound follows from the inequality (2.28) since $\Pi(\rho_n) \in L^\infty(0,T;L^1(\Omega))$, thus, by assumption (2.18), $\rho_n \in L^\infty(0,T;L^2(\Omega))$, and also $\nabla\rho_n, \nabla^2\rho_n \in L^2(0,T;L^2(\Omega))$ by (2.28). The second one is resulting from the inequality (2.28) by

$$2\nabla\sqrt{\rho_n} = \rho_n\nabla\log\rho_n.$$

The third results from the $L^2(0,T;L^2(\Omega))$ bounds on both, the symmetric and the antisymmetric part of $\sqrt{\rho_n}\nabla v_n$.
The additional bounds on $\nabla\rho_n$ allow for compactness arguments and, thus, allow to prove the convergence of the sequence of "weak solutions" in the desired space, [10]. We will not discuss the proof in more detail, but we will discuss a special case of these Korteweg systems with a power pressure law and prove the stability of weak solutions thereof in the next Section. □

2.3. Energy estimates and the stability of weak solutions to the shallow water equations

Shallow water equations model free surface flow where the vertical scale is much smaller than the horizontal one. Gravity as well as the Coriolis force and drag terms coming from friction are taken into account. These models allow to describe water waves in a river or the ocean, atmospheric flow or avalanches.
The two-dimensional shallow water equations can be formally derived from the depth-averaged incompressible Navier-Stokes equations, as shown in [5]. Depending on the chosen bottom boundary conditions, i.e. friction or no-slip boundary conditions, there are mainly two different types of shallow water systems. In this work, we will focus on the system derived with friction boundary conditions.

Let Ω be a two-dimensional space domain, a friction shallow water system is given by

$$\partial_t h + \operatorname{div}(hv) = 0, \tag{2.29}$$

$$\partial_t(hv) + \operatorname{div}(hv \otimes v) = -h\frac{\nabla h}{Fr^2} - \frac{1}{We}\ h\nabla\Delta h - h\frac{fv^{\perp}}{Ro} + \frac{2}{Re}\operatorname{div}(hD(v)) + \frac{2}{Re}\nabla(h\operatorname{div}\ v) + D, \tag{2.30}$$

$$h(x,0) = h_0, \qquad (hv)(x,0) = m_0 \quad \text{in } \Omega, \tag{2.31}$$

where h denotes the height of the free surface, it represents the density in the two-dimensional case. The velocity v is the vertical average of the horizontal velocity component of the fluid. The function f depends on the latitude y and $v^{\perp}$ is given by $v^{\perp} = (-v_y, v_x)$ when $v = (v_x, v_y)$. The term $fv^{\perp}$ describes the Coriolis force. We denote by D some drag terms which are a result of the friction boundary condition at the bottom. The functions h_0 and q_0 are assumed to satisfy

$$h_0 \geq 0 \text{ a.e. on } \Omega \qquad \frac{|m_0|^2}{h_0} = 0 \text{ a.e. on } \{x \ \in \ \Omega \mid h_0(x) = 0\}. \tag{2.32}$$

The dimensionless numbers Fr, We, Ro and Re denote the

- Froude number: a measure for the ratio of the flow inertia to a gravity field, defined as $Fr = \frac{U}{\sqrt{gL}}$, where U is the characteristic flow velocity, L the characteristic length and g is gravity. It is a coefficient of the non-dimensional Navier-Stokes equation alongside with the Reynolds number,

- Weber number: a measure of the importance of the fluid's inertia to its surface tension. It is useful in the description of multiphase flow, thin film flow or bubbles, $We = \frac{\rho U^2 L}{\sigma}$, where ρ is the density, σ is the surface tension and U and L are defined as for the Froude number,

- Rossby number: the ratio of flow inertia to Coriolis force. It is used to describe the Coriolis acceleration arising from planetary rotation and, therefore, models geophysical phenomena in the oceans or the atmosphere. $Ro = \frac{U}{fL}$, where $f = 2\Omega \sin\phi$ is the Coriolis frequency with Ω the angular frequency of the rotation and ϕ the angular latitude,

- Reynolds number: the ratio of momentum forces to viscous forces, $Re = \frac{\rho UL}{\mu}$, with the density ρ and the viscosity coefficient μ. Flow patterns with similar Reynolds number behave similarly, low Reynolds number indicates laminar flow, high Reynolds number turbulent flow,

respectively, [5].

Remark 2.12. This system of shallow water equations is energetically consistent, i.e. the total kinetic energy of the system is conserved. Also the conservation of the kinetic momentum is fulfilled. It leads to the constraint that the stress tensor has to be symmetric. Viscous terms of the form $h\nu\nabla v$ do not conserve the kinetic momentum, [9].

Remark 2.13. The Euler equations of isentropic gas dynamics with a pressure law $p(\rho) = \rho^2/2Fr^2$ coincide with the inviscid shallow water system (2.29) - (2.31) without taking surface and drag terms into account. In two dimensions, the vorticity is a scalar given by:

$$\omega = \partial_x v_y - \partial_y v_x.$$

We will derive an evolution equation for the vorticity which gives a powerful constraint in large-scale motions in the atmosphere.
We take the curl of the momentum equation of the Euler equations divided by h and use the mass balance equation to obtain:

$$\begin{aligned}
&\operatorname{curl}\left(\frac{1}{h}\left(\partial_t(hv) + \operatorname{div}(hv\otimes v) + h\frac{\nabla h}{Fr^2} + h\frac{f(y)v^\perp}{Ro}\right)\right) = \\
&\quad = \operatorname{curl}\left(\partial_t v + v\cdot\nabla v + \frac{f(y)v^\perp}{Ro}\right) = \\
&\quad = \partial_t\omega + v\cdot\nabla\omega + \omega \operatorname{div} v + \frac{f(y)}{Ro}\operatorname{div} v + \frac{v_y\partial_y f(y)}{Ro} = 0. \qquad (2.33)
\end{aligned}$$

In order to eliminate the term containing div v, we multiply the mass balance equation by $\omega + f(y)/Ro$ and subtract it from equation (2.33) multiplied by h,

$$\begin{aligned}
&h\partial_t\omega + hv\cdot\nabla\omega + h\omega \operatorname{div} v + h\frac{f(y)}{Ro}\operatorname{div} v + h\frac{v_y\partial_y f(y)}{Ro} \\
&\qquad - \left(\omega + \frac{f(y)}{Ro}\right)\partial_t h - \left(\omega + \frac{f(y)}{Ro}\right)\operatorname{div}(hv) \\
&= h\left(\partial_t\left(\omega + \frac{f(y)}{Ro}\right) + v\cdot\nabla\left(\omega + \frac{f(y)}{Ro}\right)\right) - \left(\omega + \frac{f(y)}{Ro}\right)(\partial_t h + v\cdot\nabla h) = 0, \qquad (2.34)
\end{aligned}$$

since

$$h\left(\left(\partial_t\frac{f(y)}{Ro}\right) + v\cdot\nabla\left(\frac{f(y)}{Ro}\right)\right) = h\frac{v_y\partial_y f(y)}{Ro}.$$

Therefore, we obtain the following equation for the relative vorticity $\omega_R := \omega + f(y)/Ro$:

$$\partial_t \left(\frac{\omega_R}{h}\right) + v \cdot \nabla \left(\frac{\omega_R}{h}\right) = 0. \tag{2.35}$$

This means that the material time derivative of the potential vorticity $\omega_P := \omega_R/h$ is zero, i.e. the potential vorticity is conserved along the particle trajectories of the flow. In particular, if we multiply (2.35) by ω_R and use the conservation of mass equation, we get the following conservation equality:

$$\frac{d}{dt}\int_\Omega h \left|\frac{\omega_R}{h}\right|^2 \, \mathrm{d}x = 0.$$

If h decreases in time, $\omega + f/Ro$ must decrease also. In particular, if $\omega + f/Ro$ is constant initially, then also h must remain constant, too.

Remark 2.14. For the sake of completeness, we will give here the description of the no-slip shallow-water system.
Let Ω be a periodic box in the two-dimensional space. We consider a thin-film liquid down an inclined plane with slope angle θ. Then the no-slip system is given by:

$$\begin{aligned}
\partial_t h \ + \ \operatorname{div}(hv) &= 0,\\
\partial_t (hv) \ + \ \operatorname{div}\left(\frac{6}{5} hv \otimes v\right) + \nabla\left(\frac{\cos\theta \ h^2}{Re} - \frac{(2\sin\theta)^2}{75} h^5\right) &- \epsilon^2 \frac{1}{We} \ h\nabla\Delta h\\
&= \frac{1}{\epsilon Re}\left(2\sin\theta h - \frac{3v}{h}\right),\\
h(x,0) = h_0, \qquad (hv)(x,0) = m_0 \quad &\text{in } \Omega,
\end{aligned}$$

where ϵ is the aspect ratio of the domain. The functions m_0 and h_0 should fulfill the conditions (2.32).

In their paper [6], Bresch and Desjardins introduced the BD entropy for the first time, for a shallow-water system where $\lambda(h) = 0$ and $\mu(h = h$, in a two-dimensional bounded domain Ω with periodic boundary conditions, i.e. $\Omega = T^2$,

$$\partial_t h + \operatorname{div}(hv) = 0, \tag{2.36}$$

$$\begin{aligned}
\partial_t(hv) + \operatorname{div}(hv \otimes v) =& - h\frac{\nabla h}{Fr^2} + \kappa \ h\nabla\Delta h - h\frac{v^\perp}{Ro} + 2\nu \operatorname{div}(hD(v))\\
&+ h\tilde{f} - r_0 v - r_1 h|v|v,
\end{aligned} \tag{2.37}$$

$$h(x,0) = h_0, \qquad (hv)(x,0) = q_0 \quad \text{in } \Omega, \tag{2.38}$$

with drag terms $-r_0 v, r_0 > 0$ in the laminar case and $-r_1 h|v|v, r_1 \geq 0$ in the turbulent case. In comparison to system (2.29)-(2.30), we have $\kappa = 1/We$, $\nu = 1/Re$ and $f = 1$. The pressure term $h^2/2Fr^2$ is a quadratic function of h.
They have shown that the following classical energy inequality can be associated with system (2.36) - (2.37):

$$\int_\Omega \left(\frac{h^2}{2Fr^2} + h\frac{|v|^2}{2} + \kappa\frac{|\nabla h|^2}{2} \right) \mathrm{d}x + \int_0^T \int_\Omega 2\nu h|D(v)|^2 \mathrm{~d}x \mathrm{~d}t$$
$$+ \int_0^T \int_\Omega r_0|v|^2 \mathrm{~d}x \mathrm{~d}t + \int_0^T \int_\Omega r_1 h|v|^3 \mathrm{~d}x \mathrm{~d}t$$
$$\leq \int_\Omega \left(\frac{h_0^2}{2Fr^2} + h_0\frac{|v_0|^2}{2} + \kappa\frac{|\nabla h_0|^2}{2} \right) \mathrm{d}x + \int_0^T \int_\Omega h\tilde{f} \cdot v \mathrm{~d}x \mathrm{~d}t. \quad (2.39)$$

It is a special case of the classical energy estimate (2.6). The drag terms multiplied by v give the additional terms, and $v^\perp \cdot v$ vanishes.

Remark 2.15. The conditions of Theorem 2.4 are satisfied by system (2.36) - (2.37), where $\Psi(h) = h$ and $\mu_1 = 1$.

Remark 2.16. Note that the shallow-water system (2.29) - (2.31) does not fulfill the conditions of Theorem 2.4, since $\mu(h) = h$ but $\lambda(h) \neq 0$. Thus, the general friction shallow-water equations are not solved yet, except for the one-dimensional case, [5].

Bresch and Desjardins have proven the following special case of the BD entropy identity for system (2.36) - (2.37):

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}h|v + 2\nu\nabla \log h|^2 + \frac{h^2}{2Fr^2} + \frac{\kappa}{2}|\nabla h|^2 \right) \mathrm{d}x - \int_0^T \int_\Omega 2\nu\nabla h\tilde{f} \mathrm{~d}x \mathrm{~d}t \quad (2.40)$$
$$+ 2\nu\int_\Omega \frac{|\nabla h|^2}{Fr^2} \mathrm{~d}x + 2\nu\frac{v^\perp\nabla h}{Ro} \mathrm{~d}x + 2\nu\kappa\int_\Omega |\nabla^2 h|^2 \mathrm{~d}x + 2\nu r_1\int_\Omega |v|v \cdot \nabla h \mathrm{~d}x$$
$$+ 2\nu r_1\int_\Omega h|v|^3 \mathrm{~d}x - 2\nu r_0\int_\Omega |v|^2 \mathrm{~d}x - 2\nu r_0\frac{d}{dt}\int_\Omega \log h \mathrm{~d}x + \int_\Omega 2\nu\rho|A(v)|^2 \mathrm{~d}x = 0.$$

The derivation is exactly the same as the one given in the proof of Theorem 2.4. We have $\phi(\rho) = \log h$ and after the multiplication by $2\nu\nabla h/h$, the additional terms,

$$\int_\Omega h\frac{v^\perp}{Ro} + r_0 v + r_1 h|v|v - h\tilde{f} \mathrm{~d}x,$$

become

$$\int_\Omega 2\nu \frac{v^\perp \nabla h}{Ro}\ \mathrm{d}x + 2\nu r_1 \int_\Omega |v| v \cdot \nabla h\ \mathrm{d}x - 2\nu r_0 \int_\Omega v \cdot \frac{\nabla h}{h}\ \mathrm{d}x - \int_\Omega 2\nu \nabla h \tilde{f}\ \mathrm{d}x.$$

Using the mass balance equation, we see that the second last term can be rewritten as

$$\begin{aligned} r_0 \int_\Omega v \cdot \frac{\nabla h}{h}\ \mathrm{d}x &= -r_0 \int_\Omega h \nabla \left(\frac{1}{h}\right) \cdot v\ \mathrm{d}x = r_0 \int_\Omega \frac{1}{h} \mathrm{div}(hv)\ \mathrm{d}x = -r_0 \int_\Omega \frac{1}{h} \partial_t h\ \mathrm{d}x \\ &= -r_0\ \frac{d}{dt} \int_\Omega \log h\ \mathrm{d}x. \end{aligned}$$

Using the energy estimates (2.39) and (2.40), we are able to prove the stability of weak solutions of shallow water equations with vanishing capillary terms (in the presence of drag terms) in two dimensions, [7].

Definition 2.17 (Weak solutions)**.** We say (h, v) is a weak solution of system (2.36) - (2.37) on $(0, T)$ if (2.38) holds in $\mathcal{D}'(\Omega)$, the energy inequality (2.39) and the equality (2.40) are satisfied a.e. for nonnegative t, system (2.36) - (2.37) holds in $\mathcal{D}'((0,T) \times \Omega)$ and if the following regularity properties are satisfied:

$$\begin{aligned} \nabla\sqrt{h} &\in L^\infty(0,T;(L^2(\Omega))^2), \qquad &\sqrt{h}v \in L^\infty(0,T;(L^2(\Omega))^2), \\ \sqrt{h}\nabla v &\in L^2(0,T;(L^2(\Omega))^4), \qquad &\nabla h \in L^2(0,T;(L^2(\Omega))^2), \\ \sqrt{r_0}v &\in L^2(0,T;(L^2(\Omega))^2), \qquad &r_1^{1/3} h^{1/3} v \in L^3(0,T;(L^3(\Omega))^2), \\ \sqrt{\kappa}\nabla^2 h &\in L^2(0,T;(L^2(\Omega))^4). \end{aligned} \tag{2.41}$$

Without loss of generality, we will assume that $\tilde{f} = 0$ since everything holds also true for a regular enough $\tilde{f}$. We take the initial data as

$$\begin{aligned} &h_0 \in L^2(\Omega), \quad h_0|v_0|^2 = \frac{|m_0|^2}{h_0} \in L^1(\Omega), \quad \sqrt{\kappa}\nabla h_0 \in (L^2(\Omega))^2, \\ &\nabla\sqrt{h_0} \in (L^2(\Omega))^2, \quad -r_0 \log_- h_0 \in L^1(\Omega), \end{aligned} \tag{2.42}$$

where $m_0 = 0$ on $h^{-1}(\{0\})$ and $\log_- g = \log\min(g, 1)$.

Theorem 2.18. *Let $(h_n, v_n)_{n\in\mathbb{N}}$ be a sequence of weak solutions of system (2.36) - (2.37) with initial data*

$$h_n(x,0) = h_0^n, \qquad (h_n v_n)(x,0) = m_0^n,$$

where h_0^n, v_0^n are such that

$$h_0^n \to h_0 \text{ in } L^1(\Omega), \quad h_0^n v_0^n \to h_0 v_0 \text{ in } L^1(\Omega), \tag{2.43}$$

and satisfying (2.32) and (2.42) and let h_0, m_0 also satisfy (2.32) and (2.42) and assume that either $r_1 > 0$ or $\kappa > 0$. Then $(h_n, v_n)_{n\in\mathbb{N}}$ converges strongly to a weak solution of system (2.36) - (2.37) with initial conditions (2.38). In particular, h_n converges strongly in $L^2(0,T;L^2(\Omega))$ and $\rho_n v_n$ converges strongly in $L^{4/3}(0,T;(L^{4/3}(\Omega))^2)$.

Proof. The classical energy estimate (2.39), together with the bounds on the initial data, yields the following uniform bounds on $(h_n, v_n)_{n\in\mathbb{N}}$:

$$\begin{aligned}
\left|\left|\sqrt{h_n}v_n\right|\right|_{L^\infty(0,T;(L^2(\Omega))^2)} &\le C, & \left|\left|\sqrt{h_n}D(v_n)\right|\right|_{L^2(0,T;(L^2(\Omega))^4)} &\le C,\\
\sqrt{\kappa}\,||\nabla h_n||_{L^\infty(0,T;(L^2(\Omega))^2)} &\le C, & \left|\left|\frac{h_n}{Fr}\right|\right|_{L^\infty(0,T;L^2(\Omega))} &\le C,\\
\sqrt{r_0}\,||v_n||_{L^2(0,T;(L^2(\Omega))^2)} &\le C, & r_1^{1/3}\left|\left|h_n^{1/3}v_n\right|\right|_{L^3(0,T;(L^3(\Omega))^2)} &\le C,
\end{aligned} \tag{2.44}$$

for some constant $C > 0$. The bounds on the initial data (2.42) yield

$$\int_\Omega \left(\frac{1}{2}h_0|v_0 + 2\nu\nabla\log h_0|^2 + \frac{h_0^2}{2Fr^2} + \frac{\kappa}{2}|\nabla h_0|^2 \,\mathrm{d}x\right) \le C,$$

since, $\sqrt{h_0}v_0 \in L^2(\Omega)$, $h_0 \in L^2(\Omega)$ and $\nabla h_0 \in L^2(\Omega)$. But also

$$h_0|\nabla\log h_0|^2 = 4|\nabla\sqrt{h_0}|^2 \in L^1(\Omega), \tag{2.45}$$

and

$$2h_0v_0\cdot\nabla\log h_0 = 2v_0\cdot\nabla h_0 = v_0\sqrt{h_0}\nabla\sqrt{h_0} \in L^1(\Omega),$$

by Hölder's inequality (Theorem A.31), since $\sqrt{h_0}v_0$ and $\nabla\sqrt{h_0} \in L^2(\Omega)$.

Since also $h_0 \in L^2(\Omega)$,

$$-r_0 \int_\Omega \log_+ h_0 \ \mathrm{d}x \le C,$$

where $\log_+ g = \log \max(g, 1)$. Additionally,

$$-r_0 \log_- h_0 \in L^1(\Omega),$$

by assumption on the initial data (2.42). Therefore, also

$$-r_0 \int_\Omega \log h_0 \ \mathrm{d}x \le C.$$

So the BD entropy estimate (2.40) integrated with respect to t yields the additional a priori bounds (using equality (2.45)):

$$\begin{aligned} &\left|\left|\nabla\sqrt{h_n}\right|\right|_{L^\infty(0,T;(L^2(\Omega))^2)} \le C, &&\left|\left|\frac{\nabla h_n}{Fr}\right|\right|_{L^2(0,T;(L^2(\Omega))^2)} \le C, \\ &\sqrt{\kappa}\left|\left|\nabla^2 h_n\right|\right|_{L^2(0,T;(L^2(\Omega))^4)} \le C, &&\left|\left|\sqrt{h_n}\nabla v_n\right|\right|_{L^2(0,T;(L^2(\Omega))^4)} \end{aligned} \tag{2.46}$$

The momentum $h_n v_n$ is bounded in $L^\infty(0,T;(L^1(\Omega))^2)$ by Hölder's inequality (Theorem A.31), since both, $\sqrt{h_n}v_n$ and h_n, are bounded in $L^\infty(0,T;(L^2(\Omega))^2)$. Therefore, by the mass balance equation (2.36), $\partial_t h_n$ is bounded in $L^\infty(0,T;W^{-1,1}(\Omega))$ (the definition of Sobolev spaces with negative exponents is given in Definition A.10). Since $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$(see Definition A.8) by Sobolev's embedding theorem (Theorem A.9), it follows that $W^{-1,1}(\Omega) \hookrightarrow H^{-2}(\Omega)$ (again by Definition A.10), therefore, $\partial_t h_n$ is bounded in $L^\infty(0,T;H^{-2}(\Omega))$. Combined with the uniform $L^2(0,T;H^1(\Omega))$ bound on h_n, Aubin's Lemma (Lemma A.12) gives the strong convergence, up to a subsequence, of h_n to h in $L^2(0,T;L^2(\Omega))$, since $H^1 \hookrightarrow\hookrightarrow L^2 \hookrightarrow H^{-2}$, see Definition A.10. In the capillary case, when $\kappa \neq 0$, we even have $h_n \in L^\infty(0,T;H^1(\Omega))$ and, thus, compactness in $C([0,T];L^2(\Omega))$.

Since $v_n \in L^2(0,T;(L^2(\Omega))^2)$ is bounded it converges weakly to some $v \in L^2(0,T;(L^2(\Omega))^2)$, up to a subsequence, by Theorem A.15.

In the next step, Bresch and Desjardins, [7], used Fourier projection to cut off high frequency parts of $\sqrt{h_n}v_n$. They later proposed a simpler proof in [5] using an additional estimate on $h_n|v_n|^2 \ln(1+|v_n|^2)$, which allows for vanishing drag terms, see Remark 2.20. We will give a slightly different proof using the same idea with the

Fourier projector, which was proposed by N. Zamponi:

For any $k \in \mathbb{N}$, the k^{th} Fourier projector P_k is defined on $L^2(\Omega)$ as follows: if $\sum_{\ell\in\mathbb{Z}^2} c_\ell \exp(i\ell\cdot x)$ denotes the Fourier decomposition of $f \in L^2(\Omega)$, then $P_k f$ is given by the low frequency part $\sum_{|\ell|\le k} c_\ell \exp(i\ell\cdot x)$. The following classical estimate holds,

$$||f - P_k f||_{L^2(\Omega)} \le \frac{C_p}{k^{2(1-1/p)}}||\nabla f||_{L^p(\Omega)},\ \forall\, p\in(1,2),\ [7]. \tag{2.47}$$

We introduce $\beta \in C^\infty(\mathbb{R})$ such that $0 \le \beta \le 1$,

$$\beta(s) = \begin{cases} 0 & \text{for} \quad s \le 1, \\ 1 & \text{for} \quad s \ge 2. \end{cases}$$

Furthermore, we denote $\beta_\alpha(\cdot) := \beta(\cdot/\alpha)$, for some positive number α. We have the following estimate for any α,

$$\begin{aligned} &\left|\left|\sqrt{h_n}v_n - P_k(\sqrt{h_n}v_n)\right|\right|_{L^2(0,T;(L^2(\Omega))^2)} \\ &\quad\le \left|\left|\sqrt{h_n}v_n - \sqrt{h_n}\beta_\alpha(h_n)v_n\right|\right|_{L^2(0,T;(L^2(\Omega))^2)} \\ &\qquad+ \left|\left|\sqrt{h_n}\beta_\alpha(h_n)v_n - P_k(\sqrt{h_n}\beta_\alpha(h_n)v_n)\right|\right|_{L^2(0,T;(L^2(\Omega))^2)} \\ &\qquad+ \left|\left|P_k(\sqrt{h_n}v_n) - P_k(\sqrt{h_n}\beta_\alpha(h_n)v_n)\right|\right|_{L^2(0,T;(L^2(\Omega))^2)} \\ &\quad\le C\sqrt{\alpha}||v_n||_{L^2(0,T;(L^2(\Omega))^2)} \\ &\qquad+ \frac{C_p}{k^{1/3}}\left|\left|\nabla(\sqrt{h_n}\beta_\alpha(h_n)v_n)\right|\right|_{L^2(0,T;(L^{6/5}(\Omega))^4)}, \end{aligned} \tag{2.48}$$

by (2.47) and since

$$|\sqrt{h_n}v_n - \sqrt{h_n}\beta_\alpha(h_n)v_n| \le C\sqrt{\alpha}\ v_n, \tag{2.49}$$

$$|P_k(\sqrt{h_n}v_n) - P_k(\sqrt{h_n}\beta_\alpha(h_n)v_n)| \le C\sqrt{\alpha}\ v_n, \tag{2.50}$$

by the definition of β_α. In the case when $r_1 > 0$, we have

$$
\begin{aligned}
&||\nabla(\sqrt{h_n}\beta_\alpha(h_n)v_n)||_{L^2(0,T;(L^{6/5}(\Omega))^4)} \\
&\quad = ||\sqrt{h_n}\beta_\alpha(h_n)\nabla v_n + \nabla\sqrt{h_n}\beta_\alpha(h_n)v_n + 2\nabla\sqrt{h_n}\beta'_\alpha(h_n)h_n v_n||_{L^2(0,T;(L^{6/5}(\Omega))^4)} \\
&\quad \le ||\sqrt{h_n}\beta_\alpha(h_n)\nabla v_n||_{L^2(0,T;(L^{6/5}(\Omega))^4)} \\
&\qquad + \left|\left|(2\beta'_\alpha(h_n)h_n^{2/3} + \beta_\alpha h_n^{-1/3}) \cdot h_n^{1/3} v_n \nabla\sqrt{h_n}\right|\right|_{L^2(0,T;(L^{6/5}(\Omega))^1)} \\
&\quad \le C||\beta_\alpha(h_n)\sqrt{h_n}\nabla v_n||_{L^2(0,T;(L^2(\Omega))^4)} + \left|\left|h_n^{1/3} v_n \nabla\sqrt{h_n}\right|\right|_{L^2(0,T;(L^{6/5}(\Omega))^1)} \\
&\qquad \cdot \left|\left|2\beta'_\alpha(h_n)h_n^{2/3} + \beta_\alpha(h_n)h_n^{-1/3}\right|\right|_{L^\infty((0,T)\times\Omega)},
\end{aligned}
$$

since, by the definition of $\beta_\alpha(h_n)$ and by (2.46), $\beta_\alpha\sqrt{h_n}\nabla v_n \in L^2(0,T;(L^2(\Omega))^4)$ and $2\beta'_\alpha(h_n)h_n^{2/3} + \beta_\alpha h_n^{-1/3} \in L^\infty((0,T)\times\Omega)$, furthermore,

$$
\begin{aligned}
&\le C\left|\left|\beta_\alpha(h_n)\sqrt{h_n}\nabla v_n\right|\right|_{L^2(0,T;(L^2(\Omega))^4)} + \left|\left|h_n^{1/3} v_n\right|\right|_{L^3(0,T;(L^3(\Omega))^2)} \\
&\quad \cdot \left|\left|\nabla\sqrt{h_n}\right|\right|_{L^6(0,T;(L^2(\Omega))^2)} \cdot \left|\left|2\beta'_\alpha(h_n)h_n^{2/3} + \beta_\alpha(h_n)h_n^{-1/3}\right|\right|_{L^\infty((0,T)\times\Omega)},
\end{aligned}
$$

by the generalized Hölder inequality (Remark A.33), thus, since $\nabla\sqrt{h_n} \in L^\infty(0,T;(L^2(\Omega))^2)$ by (2.46),

$$
\begin{aligned}
&\le C\left|\left|\beta_\alpha(h_n)\sqrt{h_n}\nabla v_n\right|\right|_{L^2(0,T;(L^2(\Omega))^4)} + C\left|\left|h_n^{1/3} v_n\right|\right|_{L^3(0,T;(L^3(\Omega))^2)} \\
&\quad \cdot \left|\left|\nabla\sqrt{h_n}\right|\right|_{L^\infty(0,T;(L^2(\Omega))^2)} \cdot \left|\left|2\beta'_\alpha(h_n)h_n^{2/3} + \beta_\alpha(h_n)h_n^{-1/3}\right|\right|_{L^\infty((0,T)\times\Omega)}.
\end{aligned}
$$

In the capillary case $\kappa > 0$, we see that

$$
\begin{aligned}
&||\nabla(\sqrt{h_n}\beta_\alpha(h_n)v_n)||_{L^2(0,T;(L^{6/5}(\Omega))^4)} \\
&\quad = ||\sqrt{h_n}\beta_\alpha(h_n)\nabla v_n + \nabla\sqrt{h_n}\beta_\alpha(h_n)v_n + 2\nabla\sqrt{h_n}\beta'_\alpha(h_n)h_n v_n||_{L^2(0,T;(L^{6/5}(\Omega))^4)} \\
&\quad \le ||\sqrt{h_n}\beta_\alpha(h_n)\nabla v_n||_{L^2(0,T;(L^{6/5}(\Omega))^4)} \\
&\qquad + \left|\left|(\beta'_\alpha(h_n) + \frac{\beta_\alpha}{2h_n}) \cdot 2h_n v_n \nabla\sqrt{h_n}\right|\right|_{L^2(0,T;(L^{6/5}(\Omega))^1)}
\end{aligned}
$$

$$\leq C \left|\left|\beta_\alpha(h_n)\sqrt{h_n}\nabla v_n\right|\right|_{L^2(0,T;(L^2(\Omega))^4)} + \left|\left|2h_n v_n \nabla\sqrt{h_n}\right|\right|_{L^\infty(0,T;(L^{6/5}(\Omega))^1)}$$
$$\cdot \left|\left|\beta'_\alpha(h_n) + \frac{\beta_\alpha}{2h_n}\right|\right|_{L^\infty((0,T)\times\Omega)},$$

again, since $\beta_\alpha(h_n)\sqrt{h_n}\nabla v_n \in L^2(0,T;(L^2(\Omega))^4)$ by (2.46) and $2\beta'_\alpha(h_n) + \beta_\alpha/2h_n \in L^\infty((0,T)\times\Omega)$ by the definition of β_α, additionally, with $2h_n v_n\nabla\sqrt{h_n} = \sqrt{h_n}v_n\nabla h_n$

$$\leq C \left|\left|\beta_\alpha(h_n)\sqrt{h_n}\nabla v_n\right|\right|_{L^2(0,T;(L^2(\Omega))^4)}$$
$$+ C\left|\left|\sqrt{h_n}v_n\nabla h_n\right|\right|_{L^\infty(0,T;(L^{6/5}(\Omega))^1)} \cdot \left|\left|\beta'_\alpha(h_n) + \frac{\beta_\alpha}{2h_n}\right|\right|_{L^\infty((0,T)\times\Omega)},$$
$$\leq C \left|\left|\beta_\alpha(h_n)\sqrt{h_n}\nabla v_n\right|\right|_{L^2(0,T;(L^2(\Omega))^4)}$$
$$+ C\left|\left|\sqrt{h_n}v_n\right|\right|_{L^\infty(0,T;(L^2(\Omega))^2)} \cdot ||\nabla h_n||_{L^2(0,T;(L^3(\Omega))^2)}$$
$$\cdot \left|\left|\beta'_\alpha(h_n) + \frac{\beta_\alpha}{2h_n}\right|\right|_{L^\infty((0,T)\times\Omega)},$$

again by the generalized Hölder inequality (Remark (A.33) and, finally, we use the Gagliardo-Nirenberg inequality (Theorem A.28) on the norm of ∇h_n (the precise derivation was done in Remark A.29),

$$\leq C \left|\left|\beta_\alpha(h_n)\sqrt{h_n}\nabla v_n\right|\right|_{L^2(0,T;(L^2(\Omega))^4)} + C\left|\left|\sqrt{h_n}v_n\right|\right|_{L^\infty(0,T;(L^2(\Omega))^2)}$$
$$\cdot \left(||\nabla h_n||^{2/3}_{L^\infty(0,T;(L^2(\Omega))^2)} \cdot \left|\left|\nabla^2 h_n\right|\right|^{1/3}_{L^2(0,T;(L^2(\Omega))^4)} + C\,||\nabla h_n||_{L^\infty(0,T;(L^2(\Omega))^2)}\right)$$
$$\cdot \left|\left|\beta'_\alpha(h_n) + \frac{\beta_\alpha(h_n)}{2h_n}\right|\right|_{L^\infty((0,T)\times\Omega)}.$$

Thus, the left hand side of equation (2.48) is uniformly bounded by $C\sqrt{\alpha}+C_\alpha\cdot k^{-1/3}$, by the a priori bounds (2.44) and (2.46) in both cases. Therefore, the high frequency part of $\sqrt{h_n}v_n$ is arbitrarily small in $L^2(0,T;(L^2(\Omega))^2)$ uniformly in n, for large enough k.
Since $\sqrt{h_n}v_n \in L^\infty(0,T;(L^2(\Omega))^2)$ is bounded, we have weak convergence, up to a subsequence of $\sqrt{h_n}v_n$ to some $\sqrt{h}v$ in $L^2(0,T;L^2(\Omega)) \supset L^\infty(0,T;L^2(\Omega))$, by The-

orem A.15. Let $\epsilon > 0$, we want to prove the strong convergence in $L^2(0,T;L^2(\Omega))$:

$$||\sqrt{h_n}v_n - \sqrt{h}v||_{L^2(0,T;L^2(\Omega))} \leq \left|\left|\sqrt{h_n}v_n - P_k(\sqrt{h_n}v_n)\right|\right|_{L^2(0,T;L^2(\Omega))} + \left|\left|\sqrt{h}v - P_k(\sqrt{h}v)\right|\right|_{L^2(0,T;L^2(\Omega))} + \left|\left|P_k(\sqrt{h_n}v_n) - P_k(\sqrt{h}v)\right|\right|_{L^2(0,T;L^2(\Omega))}. \tag{2.51}$$

Since the high frequency part of $\sqrt{h_n}v_n$ is arbitrarily small uniformly in n for some large enough k, there exists a $k = k(\epsilon)$ such that

$$\left|\left|\sqrt{h_n}v_n - P_k(\sqrt{h_n}v_n)\right|\right|_{L^2(0,T;L^2(\Omega))} < \frac{\epsilon}{3},\ n \geq 1, \quad \left|\left|\sqrt{h}v - P_k(\sqrt{h}v)\right|\right|_{L^2(0,T;L^2(\Omega))} < \frac{\epsilon}{3}.$$

We use the definition of P_k, to see that

$$||P_k(\sqrt{h_n}v_n) - P_k(\sqrt{h}v)||^2_{L^2(0,T;L^2(\Omega))} = ||P_k(\sqrt{h_n}v_n - \sqrt{h}v)||^2_{L^2(0,T;L^2(\Omega))}$$
$$= \left|\left| C \sum_{|\ell|\leq k} \underbrace{\int_\Omega e^{-i\ell\cdot x}(\sqrt{h_n}v_n - \sqrt{h}v)\ \mathrm{d}x}_{\longrightarrow\ 0\ (n\to\infty)}\ e^{i\ell\cdot x} \right|\right|^2_{L^2(0,T;L^2(\Omega))}$$

since $\sqrt{h_n}v_n$ converges weakly in $L^2(0,T;L^2(\Omega))$. In other words, we can find an index $n_\epsilon \geq 1$ such that

$$\left|\left|P_k(\sqrt{h_n}v_n) - P_k(\sqrt{h}v)\right|\right|^2_{L^2(0,T;L^2(\Omega))} < \frac{\epsilon}{3} \qquad \forall\ n > n_\epsilon.$$

Therefore, inequality (2.51) yields

$$||\sqrt{h_n}v_n - \sqrt{h}v||_{L^2(0,T;L^2(\Omega))} < \epsilon, \quad n > n_\epsilon,$$

i.e. the strong convergence of $\sqrt{h_n}v_n$ to $\sqrt{h}v$ in $L^2(0,T;L^2(\Omega))$.

This, combined with the strong convergence of the h_n in $L^2(0,T;L^2(\Omega))$, the weak convergence of v_n in $L^2(0,T;L^2(\Omega))$ and the weak convergence, up to a subsequence, of $h_n^{1/3}v_n$ in $L^3(0,T;L^3(\Omega))$, enables us to pass to the limit in the terms $h_nv_n, h_nv_n \otimes v_n, h_n^2, r_0v_n$ and $r_1h_n|v_n|v_n$ by Theorem A.14.

Next we will show the convergence of $h_nD(v_n)$ in $\mathcal{D}'((0,T)\times\Omega)$. We calculate

$$h_n \nabla v_n = \nabla(h_n v_n) - 2\sqrt{h_n} v_n \nabla \sqrt{h_n},$$

therefore,

$$\int_0^T \int_\Omega h_n \nabla v_n \phi \, \mathrm{d}x \, \mathrm{d}t = -\int_0^T \int_\Omega h_n v_n \nabla \phi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega 2\sqrt{h_n} v_n \nabla \sqrt{h_n} \phi \, \mathrm{d}x \, \mathrm{d}t.$$

Since $\sqrt{h_n}$ converges strongly in $L^4(0,T;L^4(\Omega))$ and $\sqrt{h_n}v_n$ converges strongly in $L^2(0,T;(L^2(\Omega))^2)$, $h_n v_n$ converges strongly in $L^{4/3}(0,T;(L^{4/3}(\Omega))^2)$, by Hölder's inequality (Theorem A.31). Therefore, the first term on the right-hand side converges to

$$-\int_0^T \int_\Omega h v \nabla \phi \, \mathrm{d}x \, \mathrm{d}t,$$

The sequence $\sqrt{h_n}v_n$ converges strongly in $L^2(0,T;(L^2(\Omega))^2)$ and $\nabla\sqrt{h_n}$ is uniformly bounded and, therefore, converges weakly in $L^2(0,T;(L^2(\Omega))^2)$, up to a subsequence. Therefore, $2\sqrt{h_n}v_n\nabla\sqrt{h_n}$ converges weakly to $2\sqrt{h}v\nabla\sqrt{h}$ in $L^2(0,T;L^2(\Omega))$, by Theorem A.14. Thus, the second term is convergent, too. A similar argument holds for $h\nabla v^T$.

We rewrite the nonlinear term $\kappa h_n \nabla \Delta h_n$:

$$\kappa h_n \nabla \Delta h_n = \kappa \nabla \left(\Delta \frac{h_n^2}{2} - \frac{|\nabla h_n|^2}{2} \right) - \kappa \mathrm{div}(\nabla h_n \otimes \nabla h_n).$$

The strong convergence of ∇h_n in $L^2(0,T;(L^2(\Omega))^2)$ suffices to pass to the limit in the nonlinear terms. Since $\partial_t h_n \in L^2(0,T;H^{-2}(\Omega))$, $\partial_t \nabla h_n \in L^2(0,T;(H^{-3}(\Omega))^2)$ and also $\nabla h_n \in L^2(0,T;(H^1(\Omega))^2)$ by (2.46). We can apply Aubin's Lemma (Lemma A.12) to $H^1 \hookrightarrow\hookrightarrow L^2 \hookrightarrow H^{-3}$ and obtain the desired strong convergence.

Since ∇h_n and h_n converge strongly in $L^2(0,T;(L^2(\Omega))^2)$, the pressure term $h_n \nabla h_n$ converges strongly in $L^1(0,T;L^1(\Omega))$ by Theorem A.14.

We have shown the strong convergence of $(h_n, v_n)_{n\in\mathbb{N}}$ to (h_n, v_n), i.e. the stability of weak solutions of system (2.36)-(2.37). $\square$

2.3.1. Construction of approximate sequences of weak solutions to the shallow water equations

It is important to know how to construct such uniformly bounded sequences of solutions with the desired qualities. We will recall the most important steps in the construction of such a sequence which was shown by Bresch and Desjardins in [9].

We look at a slightly perturbed system which preserves the BD entropy. Three smoothing terms are added to the right hand side of the momentum equation (2.37). Two of them are associated with a small parameter $\epsilon > 0$, they mollify the height function and keep it away from zero (then we have existence of global weak solutions for this mollified system). The third term is a parabolic term in v_ϵ, scaled by a positive parameter $\eta > 0$. The modified system is globally well posed for fixed ϵ and η. By taking $\eta \to 0$, we have to make sure that the bounds on the approximate solutions, which are required in order to prove the stability results, are fulfilled uniformly in ϵ. Then, using the stability results given in the proof, the global existence of weak solutions is obtained by taking $\epsilon \to 0$.

Let $\epsilon > 0$ be fixed and $\eta > 0$. We consider the system

$$
\begin{aligned}
&\partial_t h_{\epsilon\eta} + \operatorname{div}(h_{\epsilon\eta} v_{\epsilon\eta}) = 0, \\
\partial_t(h_{\epsilon\eta} v_{\epsilon\eta}) &+ \operatorname{div}(h_{\epsilon\eta} v_{\epsilon\eta} \otimes v_{\epsilon\eta}) + \frac{\nabla h_{\epsilon\eta}^2}{2Fr^2} + r_0 v_{\epsilon\eta} + r_1 |v_{\epsilon\eta}| v_{\epsilon\eta} - \kappa h_{\epsilon\eta} \nabla \Delta h_{\epsilon\eta} \\
&= \operatorname{div}(2\nu h_{\epsilon\eta} D(v_{\epsilon\eta})) + \epsilon h_{\epsilon\eta} \nabla \Delta^{2s+1} h_{\epsilon\eta} - \epsilon \nabla p(h_{\epsilon\eta}) - \eta \Delta^2 v_{\epsilon\eta}, \\
&h_{\epsilon\eta}(0,\cdot) = h_{0,\epsilon,\eta}, \qquad q_{\epsilon\eta}(0,\cdot) = q_{0,\epsilon,\eta},
\end{aligned}
$$

for some large enough integer s and an additional nondecreasing pressure term $p(h)$, which we will choose such that h remains spatially regular enough. W.l.o.g. we set $\nu = 1$. The initial conditions satisfy

$$
h_{0,\epsilon,\eta},\ q_{0,\epsilon,\eta} \in C^\infty(\Omega), \qquad h_{0,\epsilon,\eta} \geq \epsilon > 0 \text{ a.e. in } \Omega.
$$

The classical energy estimate for this system is given by:

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega \left(h_{\epsilon\eta} \frac{|v_{\epsilon\eta}|^2}{2} + \frac{h_{\epsilon\eta}^2}{2Fr^2} + \frac{\epsilon}{2} \left|\nabla^{2s+1} h_{\epsilon\eta}\right|^2 + \frac{\kappa}{2} |\nabla h_{\epsilon\eta}|^2 + \epsilon h_{\epsilon\eta} \Pi(h_{\epsilon\eta}) \right) \mathrm{d}x \\
&\quad + \int_\Omega (2h_{\epsilon\eta} |D(v_{\epsilon\eta})|^2 + \eta |\Delta v_{\epsilon\eta}|^2 + r_0 |v_{\epsilon\eta}|^2 + r_1 h_{\epsilon\eta} |v_{\epsilon\eta}|^3) \,\mathrm{d}x = 0.
\end{aligned}
$$

The system also preserves the BD structure (again $\phi = \log h_{\epsilon\eta}$):

$$\frac{d}{dt}\int_\Omega\Bigg(h_{\epsilon\eta}\frac{|v_{\epsilon\eta}+2\nabla\log h_{\epsilon\eta}|^2}{2}+\frac{h_{\epsilon\eta}^2}{2Fr^2}+\frac{\epsilon}{2}\left|\nabla^{2s+1}h_{\epsilon\eta}\right|^2$$
$$+\frac{\kappa}{2}\left|\nabla h_{\epsilon\eta}\right|^2+\epsilon h_{\epsilon\eta}\Pi(h_{\epsilon\eta})-2r_0\log h_{\epsilon\eta}\Bigg)\,\mathrm{d}x+\int_\Omega\Bigg(2h_{\epsilon\eta}|A(v_{\epsilon\eta})|^2+\eta|\Delta v_{\epsilon\eta}|^2$$
$$+\frac{2|\nabla h_{\epsilon\eta}|^2}{Fr^2}+\epsilon\left|\Delta^{2s+1}h_{\epsilon\eta}\right|^2+2\epsilon p'(h_{\epsilon\eta})\left|\nabla\sqrt{h_{\epsilon\eta}}\right|^2+\kappa\left|\Delta h_{\epsilon\eta}\right|^2+r_0|v_{\epsilon\eta}|^2$$
$$+r_1h_{\epsilon\eta}|v_{\epsilon\eta}|^3\Bigg)\,\mathrm{d}x=-2\eta\int_\Omega\Delta v_{\epsilon\eta}\cdot\nabla\Delta\log h_{\epsilon\eta}\,\mathrm{d}x-2\int_\Omega r_1|v_{\epsilon\eta}|v_{\epsilon\eta}\cdot\nabla h_{\epsilon\eta}\,\mathrm{d}x.$$

These two energy estimates give regularity on $h_{\epsilon\eta}$ and $v_{\epsilon\eta}$. As a matter of fact, the additional pressure may be taken as $\rho(h)=-h^{-3}$. Then the internal energy has the form $\Pi(h)=h^{-4}/4$. In that case we can take $s\geq 2$ and the energy estimates yield that the height is bounded and bounded away from zero all the time, [9].
Also, the following estimates hold:

$$\begin{aligned}
\nabla\sqrt{h}&\in L^\infty(0,T;(L^2(\Omega))^2), & \sqrt{h}v&\in L^\infty(0,T;(L^2(\Omega))^2),\\
\sqrt{h}\nabla v&\in L^2(0,T;(L^2(\Omega))^4), & \nabla h&\in L^2(0,T;(L^2(\Omega))^2),\\
\sqrt{r_0}v&\in L^2(0,T;(L^2(\Omega))^2), & r_1^{1/3}h^{1/3}v&\in L^3(0,T;(L^3(\Omega))^2),\\
\sqrt{\kappa}\nabla^2h&\in L^2(0,T;(L^2(\Omega))^4), & \sqrt{\eta}v&\in L^2(0,T;(L^2(\Omega))^3),\\
\sqrt{\epsilon}\Delta^{s+1}h&\in L^2(0,T;L^2(\Omega)), & \sqrt{\epsilon}\nabla^{2s+1}h&\in L^\infty(0,T;(L^2(\Omega))^3),\\
\frac{\epsilon}{h^3}&\in L^\infty(0,T;L^1(\Omega)), & \sqrt{\epsilon}\nabla h^{-3/2}&\in L^2(0,T;(L^2(\Omega))^3).
\end{aligned}\tag{2.52}$$

For given ϵ and η the mollified system has global in time unique solutions since the height function is bounded away from zero all the time. The construction of solutions of this system, which satisfy the energy estimates, is done by a Galerkin method and by applying existence results for ordinary differential equations, see [26]. Then we let η go to zero while keeping ϵ fixed. The positive lower bounds on $(h_{\epsilon\eta})_{\eta>0}$ hold uniformly in η. Furthermore, $(v_{\epsilon\eta})_{\eta>0}$ converges strongly to some limit velocity v_ϵ in $L^2((0,T)\times\Omega)$, [9].

Remark 2.19. The drag terms give additional information on (h,v). The laminar friction term yields the a priori estimate $||v||_{L^2(0,T;(L^2(\Omega))^2)}\leq C$. It takes care of possible concentrations on the vacuum set:

$$||v||_{L^2(0,T;(L^2(\Omega))^2)}=\left|\left|\frac{m}{h}\right|\right|_{L^2(0,T;(L^2(\Omega))^2)}\leq C.$$

The turbulent friction term is necessary to prove the stability if the capillary coefficient $\kappa = 0$.

Remark 2.20. With a newer estimate on the boundedness of $h|v|^2 \ln(1 + |v|^2)$ in $L^\infty(0, T; L^1(\Omega))$ given by A. Vasseur and A. Mellet in [27] (see Lemma 3.14), we can prove the stability of weak solutions without drag terms. The proof follows exactly the steps given in the proof of Theorem 2.18, but the strong convergence of $\sqrt{h_n} v_n$ is proven by Lemma 3.18.

Remark 2.21. Note that in dimensions greater than two, building a sequence of approximate solution with the desired qualities without the presence of drag terms is still an open problem. If one succeeds to construct such a sequence without drag terms in dimension three, one will have proven the global existence of weak solution of the friction shallow water equations without drag terms in dimension three, by Remark 2.20.

2.4. Energy estimates for Quantum Hydrodynamic models

Another Korteweg-type equation is given by the Quantum-Navier-Stokes model. We will rewrite the balance equations entirely in terms of the new effective velocity variable $v + \nabla\phi(\rho)$.

Let Ω be the whole space $\mathbb{R}^d$. We consider the barotropic Euler equations

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{2.53}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p = \rho f + 2\operatorname{div}(\mu(\rho) D(v)) + \nabla(\lambda(\rho)\operatorname{div}\, v) + \operatorname{div}\, K, \tag{2.54}$$

$$\rho(.,0) = \rho_0, \quad u(.,0) = u_0, \qquad \text{in } \Omega, \tag{2.55}$$

where we assume the Korteweg-type stress tensor K to be of the form
$K = \mu(\rho)\nabla^2\phi(\rho) = \mu(\rho)\phi''(\rho)|\nabla\rho|^2 + \mu(\rho)\phi'(\rho)\nabla^2\rho$. This expression is obtained from the general form of the Korteweg tensor (1.24) by setting $\alpha = \mu(\rho)\phi''(\rho), \beta = \gamma = 0$ and $\delta = \mu(\rho)\phi'(\rho)$.

As will be proven in the next theorem, rewriting the balance equations entirely in terms of the new effective velocity variable $w = v + \nabla\phi(\rho)$ eliminates the third order capillary terms.
We have to make different assumptions on the relation between μ and λ than those we used in Section 2.1.

Theorem 2.22 (Viscous Euler Formulation, [22]).
Let (ρ, v) be a smooth solution to the barotropic Euler equations (2.53)-(2.55) and let

$$\Psi(\rho) = \mu(\rho), \text{ i.e. } \rho\phi'(\rho) = \mu'(\rho), \quad \text{and} \quad \lambda(\rho) = \rho\mu'(\rho) - \mu(\rho). \tag{2.56}$$

Then (ρ, w) with $w = v + \nabla\phi(\rho)$ is a smooth solution to the viscous Euler system

$$\partial_t\rho + \operatorname{div}(\rho w) = \Delta\mu(\rho), \tag{2.57}$$

$$\partial_t(\rho w) + \operatorname{div}(\rho w \otimes w) + \nabla p = \rho f + \Delta(\mu(\rho)w) \quad \text{in } \Omega,\ t > 0, \tag{2.58}$$

$$\rho(.,0) = \rho_0, \quad w(.,0) = u_0 + \nabla\phi(\rho_0) \quad \text{in } \Omega. \tag{2.59}$$

Moreover, if (ρ, w) is a smooth solution to system (2.57)-(2.59) then (ρ, v) with $v = w - \nabla\phi(\rho)$ solves (2.53)-(2.55). Furthermore, the following energy identity holds:

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho|w|^2 + \Pi(\rho)\right)\, dx + \int_\Omega \left(\mu(\rho)|\nabla w|^2 + \phi'(\rho)p'(\rho)|\nabla\rho|^2\right)\, dx = \int_\Omega \rho f \cdot w\, dx. \tag{2.60}$$

Before we give the proof of Theorem 2.22, we mention the differences of the energy identity above to the ones derived in the former sections.

Remark 2.23. The relation between μ and λ differs from the relation given in Section 2.1,

$$\mu(\rho) = \mu_1\Psi(\rho), \qquad \lambda = 2\mu_1(\rho\Psi'(\rho) - \Psi(\rho)). \tag{2.61}$$

Also the treated capillary stress tensor is a different one.

Remark 2.24. In Theorem 2.22, Jüngel has shown that the BD entropy is a mathematical entropy for a class of capillary Navier-Stokes equations, which was not yet covered in the works of Bresch and Desjardins, since the treated Korteweg tensor is a different one.

Remark 2.25. The right hand side of the momentum equation written in terms of v contains third order derivatives of the function $\Psi(\rho)$, while the right hand side of the momentum equation written in terms of w only contains second order derivatives of the function $\mu(\rho) = \Psi(\rho)$.

Remark 2.26. With the relation between μ and λ and the capillary tensor given in Section 2.1, we can rewrite the system (2.1) - (2.2) partially in terms of w.

We assume that $\mu_1 = \frac{1}{2}$ in equation (2.61). Therefore, $\Psi(\rho) = 2\mu(\rho)$ and $\rho\phi(\rho)' = 2\mu'(\rho)$, furthermore $\lambda = 2(\rho\mu'(\rho) - \mu(\rho))$. We can rewrite the equations (2.1) and (2.2) with $w := v + \nabla\phi$ and Remark 2.7 as:

$$\begin{aligned} \partial_t\rho + \operatorname{div}(\rho v) &= 0, \\ \partial_t(\rho w) + \operatorname{div}(\rho v \otimes w) + \nabla p(\rho) &= \rho f + 2\operatorname{div}(\mu(\rho)A(v)) + \rho\nabla(\Psi'(\rho)\Delta\Psi(\rho)). \end{aligned} \tag{2.62}$$

Proof of the assertion made in Remark 2.26. We will first discuss $\partial_t(\rho\nabla\phi)$:

$$\partial_t\nabla\phi(\rho) = \nabla(\phi'(\rho)\partial_t\rho) = -\nabla(\phi'(\rho)\operatorname{div}(\rho v)),$$

thus,

$$\begin{aligned} \partial_t(\rho\nabla\phi(\rho)) &= -\nabla\phi(\rho)\operatorname{div}(\rho v) - \rho\nabla(\phi'(\rho)\operatorname{div}(\rho v)) \\ &= -\nabla(\rho\phi'(\rho)\operatorname{div}(\rho v)) = -\nabla(2\mu'(\rho)\operatorname{div}(\rho v)) \\ &= -2\nabla(\rho\mu'(\rho)\operatorname{div} v) - \underbrace{\nabla(2\mu'(\rho)v\operatorname{div}\rho)}_{=-\nabla(2v\operatorname{div}\mu(\rho))} \\ &= -2\nabla(\rho\mu'(\rho)\operatorname{div} v) - \nabla(\operatorname{div}(2\mu(\rho)v)) + \nabla(2\mu(\rho)\operatorname{div} v) \\ &= -\nabla(\operatorname{div}(2\mu(\rho)v)) + \nabla(2(\mu(\rho) - \rho\mu'(\rho))\operatorname{div} v). \end{aligned}$$

We used $\rho\phi'(\rho) = 2\mu'(\rho)$. Additionally,

$$\begin{aligned} \operatorname{div}(\rho v \otimes \nabla\phi(\rho)) &= \operatorname{div}(\rho v \otimes \phi'(\rho)\nabla\rho) = \operatorname{div}(\rho\phi'(\rho)v \otimes \nabla\rho) \\ &= \operatorname{div}(2\mu'(\rho)v \otimes \nabla\rho) = \operatorname{div}(v \otimes 2\nabla\mu(\rho)) \\ &= \nabla(\operatorname{div}(2\mu(\rho)v)) - \operatorname{div}(2\mu(\rho)\nabla v^T). \end{aligned}$$

Therefore, we obtain, with $\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho))$,

$$\begin{aligned} &\partial_t(\rho w) + \operatorname{div}(\rho v \otimes w) \\ &\quad = \partial_t(\rho v) + \partial_t(\rho\nabla\phi(\rho)) + \operatorname{div}(\rho v \otimes v) + \operatorname{div}(\rho v \otimes \nabla\phi(\rho)) = \\ &\qquad = -\nabla p + \rho f + 2\operatorname{div}(\mu(\rho)D(v)) + \nabla(\lambda(\rho)\operatorname{div} v) + \rho\nabla(\Psi'(\rho)\Delta\Psi(\rho)) \\ &\qquad\quad + \nabla(\operatorname{div}(2\mu(\rho)v)) - \operatorname{div}(2\mu(\rho)\nabla v^T) - \nabla(\operatorname{div}(2\mu(\rho)v)) - \nabla(\lambda(\rho)\operatorname{div} v). \end{aligned}$$

Together with

$$
\begin{aligned}
2\mathrm{div}(\mu(\rho)D(v)) - &2\mathrm{div}(\mu(\rho)\nabla v^T) \\
&= \mathrm{div}(\mu(\rho)(\nabla v + \nabla v^T - 2\nabla v^T)) = 2\mathrm{div}(\mu(\rho)A(v)),
\end{aligned}
$$

this proofs the assumption. □

Remark 2.27. Note that the capillary tensor cannot be eliminated in this formulation.

Proof of Theorem 2.22. Equation (2.57) follows directly from

$$
\begin{aligned}
\partial_t\rho + \mathrm{div}(\rho w) &= \partial_t\rho + \mathrm{div}(\rho(v + \nabla\phi(\rho))) = \partial_t\rho + \mathrm{div}(\rho v) + \mathrm{div}(\rho\nabla\phi(\rho)) \\
&= \mathrm{div}(\rho\phi'(\rho)\nabla\rho) = \mathrm{div}(\mu'(\rho)\nabla\rho) = \Delta\mu(\rho).
\end{aligned}
$$

In order to prove the momentum equation we first give some helpful results:
Similar to the the proof of Remark 2.26, with the different relation $\mu'(\rho) = \rho\phi'(\rho)$, we can show that

$$
\partial_t(\rho\nabla\phi(\rho)) = -\nabla(\mathrm{div}(\mu(\rho)v)) + \nabla((\mu(\rho) - \rho\mu'(\rho))\mathrm{div}\ v).
$$

Additionally,

$$
\begin{aligned}
&\mathrm{div}(\rho\nabla\phi(\rho) \otimes \nabla\phi(\rho)) = \mathrm{div}(\nabla\mu(\rho) \otimes \nabla\phi(\rho)) = \Delta(\mu(\rho)\nabla\phi(\rho)) - \mathrm{div}(\mu(\rho)\nabla^2\phi(\rho)), \\
&\mathrm{div}(\rho\nabla\phi(\rho) \otimes v + \rho v \otimes \nabla\phi(\rho)) = \Delta(\mu(\rho)v) - 2\mathrm{div}(\mu(\rho)D(v)) + \nabla\mathrm{div}(\mu(\rho)v).
\end{aligned}
$$

Thus, using the system of equations (2.53)-(2.54) alongside all equations above, we can show that

$$
\begin{aligned}
&\partial_t(\rho w) + \mathrm{div}\ (\rho w \otimes w) \\
&\quad = \partial_t(\rho v) + \partial_t(\rho\nabla\phi(\rho)) + \mathrm{div}(\rho v \otimes v) \\
&\qquad\quad + \mathrm{div}(\rho\nabla\phi(\rho) \otimes \nabla\phi(\rho)) + \mathrm{div}(\rho\nabla\phi(\rho) \otimes v + \rho v \otimes \nabla\phi(\rho)) \\
&\quad = (-\nabla p + \rho f + 2\mathrm{div}(\mu(\rho)D(v)) + \nabla(\lambda(\rho)\mathrm{div}\ v) + \mathrm{div}(\mu\nabla^2\phi(\rho))) - \nabla(\mathrm{div}(\mu(\rho)v)) \\
&\qquad\quad + \nabla((\mu(\rho) - \rho\mu'(\rho))\mathrm{div}\ v) + \Delta(\mu(\rho)\nabla\phi(\rho)) - \mathrm{div}(\mu(\rho)\nabla^2\phi(\rho)) \\
&\qquad\quad + \Delta(\mu(\rho)v) - 2\mathrm{div}(\mu(\rho)D(v)) + \nabla(\mathrm{div}(\mu(\rho)v)) \\
&\qquad = \Delta(\mu(\rho)w) - \nabla p + \rho f,
\end{aligned}
$$

since $\lambda(\rho) = (\rho\mu'(\rho) - \mu(\rho))$ by definition.

Finally, we want to proof the energy equality (2.60). Firstly, we differentiate the energy with respect to the time t,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \left(\frac{\rho}{2}|w|^2 + \Pi(\rho)\right)\,\mathrm{d}x &= \int_\Omega \partial_t\rho\frac{1}{2}|w|^2 + \Pi'(\rho)\partial_t\rho + \rho\partial_t w\cdot w\,\mathrm{d}x \\
&= \int_\Omega \partial_t\rho\frac{1}{2}|w|^2 + \Pi'(\rho)\partial_t\rho + \partial_t(\rho w)w - \partial_t\rho|w|^2\,\mathrm{d}x \\
&= \int_\Omega \left(\partial_t\rho\left(-\frac{1}{2}|w|^2 + \Pi'(\rho)\right) + \partial_t(\rho w)\cdot w\right)\,\mathrm{d}x.
\end{aligned}$$

Then we integrate by parts and use the mass balance equation equation (2.57) and the definition of Π, $\rho\Pi''(\rho) = p'(\rho)$, to obtain the following identities,

$$\int_\Omega \mathrm{div}(\rho w)\frac{1}{2}|w|^2\,\mathrm{d}x = \int_\Omega \mathrm{div}(\rho w\otimes w)\cdot w\,\mathrm{d}x,$$

similar to equation (2.8). Additionally,

$$\begin{aligned}
&-\int_\Omega \mathrm{div}(\rho w)\Pi'\,\mathrm{d}x = \int_\Omega \rho w\cdot\nabla\Pi'\,\mathrm{d}x = \int_\Omega \rho w\cdot\nabla\rho\Pi''\,\mathrm{d}x = \int_\Omega w\cdot\nabla p(\rho)\,\mathrm{d}x, \\
&-\int_\Omega \Delta\mu\frac{1}{2}|w|^2\,\mathrm{d}x = \int_\Omega \nabla\mu\cdot\nabla\left(\frac{1}{2}|w|^2\right)\,\mathrm{d}x = \int_\Omega \nabla\mu\nabla w\cdot w\,\mathrm{d}x \\
&\qquad = \int_\Omega \nabla(\mu w)\cdot\nabla w - \mu\,|\nabla w|^2\,\mathrm{d}x = -\int_\Omega \Delta(\mu w)w\,\mathrm{d}x - \int_\Omega \mu\,|\nabla w|^2\,\mathrm{d}x, \\
&\int_\Omega \Delta\mu\Pi'\,\mathrm{d}x = -\int_\Omega \nabla\mu\cdot\nabla\Pi'\,\mathrm{d}x = -\int_\Omega \mu'\Pi''|\nabla\rho|^2\,\mathrm{d}x.
\end{aligned}$$

Using these identities and the balance of mass and momentum equations, (2.57) and (2.58), the energy estimate reads

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(\frac{\rho}{2}|w|^2 + \Pi(\rho)\right)\,\mathrm{d}x = \int_\Omega \left(\partial_t\rho\left(-\frac{1}{2}|w|^2 + \Pi'(\rho)\right) + \partial_t(\rho w)\cdot w\right)\,\mathrm{d}x \\
&= \int_\Omega \left(\mathrm{div}(\rho w)\frac{1}{2}|w|^2 - \mathrm{div}(\rho w)\Pi'(\rho) - \Delta\mu\frac{1}{2}|w|^2 + \Delta\mu\Pi'(\rho) + \partial_t(\rho w)\cdot w\right)\,\mathrm{d}x \\
&= \int_\Omega \left(\mathrm{div}(\rho w\otimes w)\cdot w + \nabla p\cdot w - \Delta(\mu w)w - \mu|\nabla w|^2 - \mu'\Pi''|\nabla\rho|^2 + \partial_t(\rho w)\cdot w\right)\,\mathrm{d}x \\
&= \int_\Omega \left(-\mu(\rho)|\nabla w|^2 - \mu'\Pi''(\rho)|\nabla\rho|^2 + \rho f\cdot w\right)\,\mathrm{d}x.
\end{aligned}$$

By the definitions of $\mu(\rho)$ and $\Pi(\rho)$ we get $\mu'\Pi'' = \rho\phi'\Pi'' = \phi'p'$, which proves the claim. □

2.4.1. The Quantum Navier-Stokes model

A quantum fluid model is given by

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho v) &= 0 \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p - 2\epsilon^2 \rho \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) &= \rho f + 2\nu \operatorname{div}(\rho D(v)). \end{aligned} \tag{2.63}$$

The viscosity is given by $\mu(\rho) = \nu\rho$ and $\epsilon > 0$ is the scaled Planck constant, [18]. This system can be obtained from system (2.53)-(2.54) by choosing $\phi(\rho) = (\epsilon^2/\nu)\log\rho$. For $\nu = 0$, it is called quantum hydrodynamic model which is employed in semiconductor simulations.

Jüngel used the formulation of the Euler equations entirely in terms of w and the arising energy identity to prove the global existence of weak solutions for the one dimensional quantum Navier-Stokes model with $\epsilon = \nu$:

Theorem 2.28 ([22]). *Let d=1 and $\rho_0 \in W^{1,\infty}(\mathbb{R}), v_0 \in L^\infty(\mathbb{R})$ such that $\rho_0 \geq \delta > 0$ in $\mathbb{R}$. Assume that $\mu(\rho) = \rho$ for $\rho \geq 0, \mu'(\rho) = \rho\phi'(\rho), \lambda(\rho) = \rho\mu'(\rho) - \mu(\rho), \epsilon = \nu$, and $f = 0$. Then there exists a smooth bounded solution (ρ, v) to (2.53)-(2.55) satisfying $\rho(x,t) \geq c(\delta, t) > 0$ for $(x,t) \in \mathbb{R} \times [0,\infty)$.*

3. Stability of weak solutions to the barotropic compressible Navier-Stokes equations

A. Mellet and A. Vasseur showed, in their paper *On the barotropic compressible Navier-Stokes equations*, [27], that the entropy discovered by D. Bresch and B. Desjardins is not only useful to prove the stability of solutions of Navier-Stokes equations with Korteweg terms, but it can also be used to prove the stability of weak solutions of a system that does not contain any capillary terms. We will discuss these stability results in this chapter.

We assume that $\Omega = \mathbb{R}^d$ or $\Omega = T^d$, for $d = 2$ or $d = 3$, is a domain with periodic boundary conditions. We consider the Cauchy problem for the following system of isentropic compressible Navier-Stokes equations with positive initial conditions:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \tag{3.1}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla \rho^\gamma - \operatorname{div}(2\mu(\rho)D(v)) - \nabla(\lambda(\rho)\operatorname{div}\, v) = 0 \tag{3.2}$$

$$\rho(0,x) = \rho_0 \geq 0, \qquad \rho v(0,x) = m_0. \tag{3.3}$$

The density $\rho(x,t)$ and the velocity $v(x,t)$ are both functions of the spatial variables x and the time t. Throughout this chapter we will use the notation $\rho := \rho(x,t)$ and $v := v(x,t)$. The pressure is given by $p(\rho) = \rho^\gamma$ for any $\gamma > 1$ and $D(v)$ is the symmetric part of the velocity gradient $D(v) = 1/2(\nabla v + \nabla v^T)$. The viscosity coefficients $\mu(\rho), \lambda(\rho) \in C^2(0,\infty)$ are once again assumed to fulfill

$$\lambda(\rho) = 2\rho\mu'(\rho) - 2\mu(\rho). \tag{3.4}$$

Furthermore, we have to make additional assumptions on $\mu(\rho)$ and $\lambda(\rho)$. We assume that there exists a positive constant $\nu \in (0,1)$ such that

$$\mu'(\rho) \geq \nu, \qquad \mu(0) \geq 0, \tag{3.5}$$

$$|\lambda'(\rho)| \leq \frac{1}{\nu}\mu'(\rho), \tag{3.6}$$

$$\nu\mu(\rho) \leq 2\mu(\rho) + d\lambda(\rho) \leq \frac{1}{\nu}\mu(\rho). \tag{3.7}$$

If $d = 3$ and $\gamma \geq 3$ we need the additional requirement,

$$\liminf_{\rho\to\infty} \frac{\mu(\rho)}{\rho^{\gamma/3+\epsilon}} > 0, \tag{3.8}$$

for some $\epsilon > 0$.

Remark 3.1. In particular, $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$ satisfy the requirements (3.4) - (3.7). In the two-dimensional case, for $\gamma = 2$, we recover the Saint-Venant equations. Therefore, we can apply the stability results given in this section to the shallow water system (2.36)-(2.37), see Remark 2.20.

Remark 3.2. The estimates in (3.7) yield $|\lambda(\rho)| \le C_\nu \mu(\rho),\ C_\nu > 0,\ \forall \rho > 0$. Together with (3.4) condition (3.7) implies

$$\frac{d-1+\nu}{d\rho} \le \frac{\mu'(\rho)}{\mu(\rho)} \le \frac{d-1+1/\nu}{d\rho},\ \forall \rho > 0, \tag{3.9}$$

thus, for some $C > 0$,

$$\begin{cases} C\rho^{(d-1)/d+\nu/d} \le \mu(\rho) \le C\rho^{(d-1)/d+1/(d\nu)}, & \rho \ge 1, \\ C\rho^{(d-1)/d+1/(d\nu)} \le \mu(\rho) \le C\rho^{(d-1)/d+\nu/d}, & \rho \le 1. \end{cases} \tag{3.10}$$

In particular, $\mu(0) = 0$.

We treat systems with zero capillarity, i.e. the Korteweg tensor $K = 0$. The viscosity coefficients $\mu(\rho), \lambda(\rho) \in C^2(0,\infty)$ are regular enough to derive the energy estimates (2.6) and (2.7) for smooth solutions of system (3.1)-(3.2) and they are also assumed to satisfy the relation for the BD entropy (3.4), with $\mu_1 = 1/2$ in (2.3). Thus the energy estimates associated with system (3.1)-(3.2), are given by:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho v^2 \ \mathrm{d}x + 2\int_\Omega \mu(\rho) D(v)^2 \ \mathrm{d}x + \int_\Omega \lambda(\rho)(\mathrm{div}\ v)^2 \ \mathrm{d}x + \frac{d}{dt}\int_\Omega \frac{1}{\gamma-1}\rho^\gamma \ \mathrm{d}x \le 0, \tag{3.11}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|v+\nabla\phi|^2 \ \mathrm{d}x + \int_\Omega \nabla\phi(\rho)\cdot\nabla\rho^\gamma \ \mathrm{d}x \\ &\qquad + \frac{d}{dt}\int_\Omega \frac{1}{\gamma-1}\rho^\gamma \ \mathrm{d}x + \int_\Omega 2\mu(\rho)|A(v)|^2 \ \mathrm{d}x = 0, \end{aligned} \tag{3.12}$$

with $A(v) = \frac{1}{2}(\nabla v - \nabla v^T)$ and since

$$\Pi(\rho) = \frac{1}{\gamma-1}\rho^\gamma.$$

If $\mu(\rho)$ and $\lambda(\rho)$ satisfy $2\mu(\rho) + d\lambda(\rho) \geq 0$, and the initial energy is bounded, i.e.,

$$\frac{1}{2}\int_\Omega \rho_0 v_0^2 + 2\frac{1}{\gamma - 1}\rho_0^\gamma \;\; \mathrm{d}x < +\infty,$$
$$\frac{1}{2}\int_\Omega \rho_0 |\nabla\phi(\rho_0)|^2 \;\mathrm{d}x < +\infty,$$

integrating the estimate (3.11) with respect to the time yields the following a priori bounds on smooth solutions (ρ, v) of system (3.1)-(3.2):

$$||\sqrt{\rho}v||_{L^\infty(0,T;(L^2(\Omega))^d)} \leq C, \tag{3.13}$$
$$||\rho||_{L^\infty(0,T;L^\gamma(\Omega))} \leq C, \tag{3.14}$$
$$||\sqrt{\mu(\rho)}D(v)||_{L^2(0,T;(L^2(\Omega))^{d\times d})} \leq C. \tag{3.15}$$

The BD entropy estimate (3.12) gives the additional bounds,

$$||\sqrt{\rho}\nabla\phi(\rho)||_{L^\infty(0,T;(L^2(\Omega))^d)} \leq C,$$
$$||\nabla\phi \cdot \nabla\rho^\gamma||_{L^1(0,T;L^1(\Omega))} \leq C.$$

Additionally, since $\mu(\rho)$ is assumed to be increasing with respect to ρ and, therefore, also $\phi' = 2\mu'(\rho)/\rho \geq 0$,

$$\frac{1}{2}||\sqrt{\rho}\nabla\phi(\rho)||_{L^\infty(0,T;(L^2(\Omega))^d)} = \frac{1}{2}||\sqrt{\rho}\phi'\nabla\rho||_{L^\infty(0,T;(L^2(\Omega))^d)}$$
$$= \frac{1}{2}\left|\left|\frac{1}{\sqrt{\rho}}\rho\phi'\nabla\rho\right|\right|_{L^\infty(0,T;(L^2(\Omega))^d)} = 2||\mu'(\rho)\nabla\sqrt{\rho}||_{L^\infty(0,T;(L^2(\Omega))^d)} \leq C,$$

and also

$$||\nabla\phi\cdot\nabla\rho^\gamma||_{L^1(0,T;L^1(\Omega))} = ||\phi'\nabla\rho\cdot\nabla\rho^\gamma||_{L^1(0,T;L^1(\Omega))} = ||\gamma\phi'\rho^{\gamma-1}(\nabla\rho)^2||_{L^1(0,T;L^1(\Omega))}$$
$$= ||\gamma\mu'\rho^{\gamma-2}(\nabla\rho)^2||_{L^1(0,T;L^1(\Omega))} = C||\sqrt{\mu'(\rho)\rho^{\gamma-2}}\nabla\rho||_{L^2(0,T;(L^2(\Omega))^d)} \leq C.$$

So, in particular, we have

$$||\mu'(\rho)\nabla\sqrt{\rho}||_{L^\infty(0,T;(L^2(\Omega))^d)} \leq C, \tag{3.16}$$
$$||\sqrt{\mu'(\rho)\rho^{\gamma-2}}\nabla\rho||_{L^2(0,T;(L^2(\Omega))^d)} \leq C. \tag{3.17}$$

Furthermore,

$$||\sqrt{\mu(\rho)}\nabla v||_{L^2(0,T;(L^2(\Omega))^{d\times d})} \leq C, \tag{3.18}$$

since we have this estimate on both, the symmetric and the antisymmetric part of ∇v by (3.11) and (3.12).

Finally, if we integrate the mass balance equation (3.1) over Ω and use Reynold's Transport Theorem (Theorem 1.2) for smooth solutions of system (3.1)-(3.2),

$$0 = \int_\Omega \partial_t \rho + \operatorname{div}(\rho v)\ \mathrm{d}x = \frac{d}{dt}\int_\Omega \rho\ \mathrm{d}x,$$

we obtain the natural L^1 estimate,

$$||\rho||_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{3.19}$$

if we assume that the density is bounded initially, $\rho_0 \in L^\infty(0,T;L^1(\Omega))$. This L^1 estimate is already covered by the classical energy estimate, since (3.14) automatically yields (3.19).

Remark 3.3. Under assumption (3.5) the two estimates (3.16) and (3.17) give additional control on ρ and ρ^γ. This will be enough to prove the stability of weak solutions.

Remark 3.4. The conditions allow for viscosity coefficients to vanish on the vacuum set. In fact, we gain some regularity on the density on the vacuum set while we lose some regularity on the velocity.

One of the key tools of the proof will be the following Lemma.

Lemma 3.5. *Assume that (3.7) holds, then smooth solutions of system (3.1)-(3.3) satisfy the following inequality for any $\delta \in (0,2)$:*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \rho\frac{1+|v|^2}{2}\ln(1+|v|^2)\ \ dx + \frac{\nu}{2}\int_\Omega \mu(\rho)(1+\ln(1+|v|^2))|D(v)|^2\ \ dx \\
&\quad \leq C\left(\int_\Omega \left(\frac{\rho^{2\gamma-\delta/2}}{\mu(\rho)}\right)^{2/(2-\delta)}\ dx\right)^{(2-\delta)/2}\left(\int_\Omega \rho(2+\ln(1+|v|^2))^{2/\delta}\ \ dx\right)^{\delta/2} \\
&\quad + C\int_\Omega \mu(\rho)|\nabla v|^2\ \ dx.
\end{aligned} \tag{3.20}$$

Proof. We mulitply the momentum equation (3.1) by $(1 + \ln(1 + |v|^2)) \cdot v$,

$$\begin{aligned}\partial_t(\rho v)(1 + \ln(1 + |v|^2)) \cdot v + \operatorname{div}(\rho v \otimes v)(1 + \ln(1 + |v|^2)) \cdot v \\ + \nabla \rho^\gamma (1 + \ln(1 + |v|^2)) \cdot v - \operatorname{div}(2\mu(\rho) D(v))(1 + \ln(1 + |v|^2)) \cdot v \\ - \nabla(\lambda(\rho) \operatorname{div} v)(1 + \ln(1 + |v|^2)) \cdot v = 0.\end{aligned} \tag{3.21}$$

Then, we expand the terms $\partial_t \rho v$, $\operatorname{div}(\rho v \otimes v)$ and use the mass balance equation together with the identities

$$\begin{aligned}\partial_t v \cdot (1 + \ln(1 + |v|^2))v &= \partial_t \left(\frac{1 + |v|^2}{2} \ln(1 + |v|^2) \right), \\ \nabla v \cdot (1 + \ln(1 + |v|^2))v &= \nabla \left(\frac{1 + |v|^2}{2} \ln(1 + |v|^2) \right).\end{aligned}$$

This gives

$$\begin{aligned}&(\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v)) \cdot (1 + \ln(1 + |v|^2)) \cdot v = \\ &\quad \rho \partial_t v \cdot (1 + \ln(1 + |v|^2))v - \operatorname{div}(\rho v) \cdot v \cdot (1 + \ln(1 + |v|^2))v \\ &\qquad + \operatorname{div}(\rho v) \cdot v \cdot (1 + \ln(1 + |v|^2))v + \rho v \cdot \nabla v \cdot (1 + \ln(1 + |v|^2))v \\ &\qquad\quad = \rho \partial_t \left(\frac{1 + |v|^2}{2} \ln(1 + |v|^2) \right) + \rho v \cdot \nabla \left(\frac{1 + |v|^2}{2} \ln(1 + |v|^2) \right).\end{aligned}$$

We insert this equality into equation (3.21) and integrate over Ω. By integrating by parts in the diffusion terms,

$$\begin{aligned}&- \int_\Omega \operatorname{div}(2\mu(\rho) D(v)) \cdot (1 + \ln(1 + |v|^2) \cdot v \, \mathrm{d}x = \\ &\quad \int_\Omega 2\mu(\rho)(1 + \ln(1 + |v|^2) D(v) : \nabla v \, \mathrm{d}x + \int_\Omega 2\mu(\rho) \frac{2}{1 + |v|^2} (\nabla v \; v) \cdot (D(v) \; v) \, \mathrm{d}x, \\ &- \int_\Omega \nabla(\lambda(\rho) \operatorname{div} v)(1 + \ln(1 + |v|^2)) \cdot v \, \mathrm{d}x = \\ &\quad \int_\Omega \lambda(\rho)(1 + \ln(1 + |v|^2)(\operatorname{div} v)^2 \, \mathrm{d}x + \int_\Omega \lambda(\rho) \operatorname{div} v \frac{2}{1 + |v|^2} (\nabla v \; v) \cdot v \, \mathrm{d}x,\end{aligned}$$

we obtain, with $D(v) : \nabla v = |D(v)|^2$, since $D(v)$ is symmetric,

$$
\begin{aligned}
&\int_\Omega \rho\partial_t \left(\frac{1+|v|^2}{2}\ln(1+|v|^2)\right)\ \mathrm{d}x + \int_\Omega \rho v\cdot\nabla\left(\frac{1+|v|^2}{2}\ln(1+|v|^2)\right)\ \mathrm{d}x \\
&\quad+\int_\Omega 2\mu(\rho)(1+\ln(1+|v|^2)|D(v)|^2\ \mathrm{d}x + \int_\Omega 2\mu(\rho)\frac{2}{1+|v|^2}(\nabla v\ v)\cdot(D(v)\ v)\ \mathrm{d}x \\
&\quad+\int_\Omega \lambda(\rho)(1+\ln(1+|v|^2)(\mathrm{div}\ v)^2\ \mathrm{d}x + \int_\Omega \lambda(\rho)\mathrm{div}\ v\frac{2}{1+|v|^2}(\nabla v\ v)\cdot v\ \mathrm{d}x \\
&\quad+\int_\Omega (1+\ln(1+|v|^2)v\cdot\nabla\rho^\gamma\ \mathrm{d}x = 0,
\end{aligned}
$$

We use the general identities,

$$
(\mathrm{div}\ v)^2 = \sum_i\sum_j \partial_i v_i\partial_j v_j \le \sum_i\sum_j \frac{1}{2}(\partial_i v_i^2 + \partial_j v_j^2) \le d|D(v)|^2,
$$

condition (3.7) and Remark 3.2 to obtain:

$$
\begin{aligned}
&\int_\Omega \rho\partial_t \left(\frac{1+|v|^2}{2}\ln(1+|v|^2)\right)\ \mathrm{d}x + \int_\Omega \rho v\cdot\nabla\left(\frac{1+|v|^2}{2}\ln(1+|v|^2)\right)\ \mathrm{d}x \\
&\quad+\nu\int_\Omega \mu(\rho)(1+\ln(1+|v|^2))|D(v)|^2\ \mathrm{d}x \\
&\qquad\le -\int_\Omega (1+\ln(1+|v|^2))v\cdot\nabla\rho^\gamma\ \mathrm{d}x + C\int_\Omega \mu(\rho)|\nabla v|^2\ \mathrm{d}x.
\end{aligned} \tag{3.22}
$$

By multiplying the mass balance equation by $\frac{1+|v|^2}{2}\ln(1+|v|^2)$ and integrating by parts, we obtain

$$
\int_\Omega \frac{1+|v|^2}{2}\ln(1+|v|^2)\partial_t\rho\ \mathrm{d}x - \int_\Omega \rho v\cdot\nabla\left(\frac{1+|v|^2}{2}\ln(1+|v|^2)\right)\ \mathrm{d}x = 0.
$$

Adding this equality to inequality (3.22) gives

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \rho\frac{1+|v|^2}{2}\ln(1+|v|^2)\ \mathrm{d}x + \nu\int_\Omega \mu(\rho)(1+\ln(1+|v|^2))|D(v)|^2\ \mathrm{d}x \\
&\qquad\le -\int_\Omega (1+\ln(1+|v|^2))v\cdot\nabla\rho^\gamma\ \mathrm{d}x + C\int_\Omega \mu(\rho)|\nabla v|^2\ \mathrm{d}x.
\end{aligned} \tag{3.23}
$$

We need to bound the right hand side of inequality (3.23). Therefore, we integrate by parts in the first term on the right hand side and use

Hölder's inequality (Theorem A.31) in the second step,

$$\begin{aligned}
&\left|\int_\Omega (1+\ln(1+|v|^2))v\cdot\nabla\rho^\gamma \,\mathrm{d}x\right| \qquad (3.24)\\
&\quad\le \left|\int_\Omega \frac{2}{1+|v|^2}(\nabla v\; v)\cdot v\;\rho^\gamma \,\mathrm{d}x\right| + \left|\int_\Omega (1+\ln(1+|v|^2))(\mathrm{div}\; v)\rho^\gamma \,\mathrm{d}x\right|\\
&\quad\le \left|\int_\Omega 2\;\nabla v\;\rho^\gamma \,\mathrm{d}x\right| + \left|\int_\Omega (1+\ln(1+|v|^2))(\mathrm{div}\; v)\rho^\gamma \,\mathrm{d}x\right|\\
&\quad\le 2\left(\int_\Omega \mu(\rho)|\nabla v|^2 \,\mathrm{d}x\right)^{1/2}\left(\int_\Omega \frac{\rho^{2\gamma}}{\mu(\rho)} \,\mathrm{d}x\right)^{1/2} + \left|\int_\Omega (1+\ln(1+|v|^2))(\mathrm{div}\; v)\rho^\gamma \,\mathrm{d}x\right|.
\end{aligned}$$

The last term on the right-hand side of this equation can be equally bounded by Hölder's inequality in the first step and by using the general inequality

$$2\sqrt{f\cdot g}\le f+g, \qquad \forall\; f,g\in[0,\infty), \tag{3.25}$$

together with (3.7) and Remark 3.2 in the second step:

$$\begin{aligned}
&\left|\int_\Omega (1+\ln(1+|v|^2))(\mathrm{div}\; v)\rho^\gamma \,\mathrm{d}x\right|\\
&\quad\le \left(\int_\Omega (1+\ln(1+|v|^2))\mu(\rho)(\mathrm{div}\; v)^2 \,\mathrm{d}x\right)^{1/2}\left(\int_\Omega (1+\ln(1+|v|^2))\frac{\rho^{2\gamma}}{\mu(\rho)} \,\mathrm{d}x\right)^{1/2}\\
&\quad\le \frac{\nu}{2}\int_\Omega (1+\ln(1+|v|^2))\mu(\rho)|D(v)|^2 \,\mathrm{d}x + C_\nu\int_\Omega (1+\ln(1+|v|^2))\frac{\rho^{2\gamma}}{\mu(\rho)} \,\mathrm{d}x.
\end{aligned} \tag{3.26}$$

Inserting the inequality (3.26) into inequality (3.24) while using the general inequality (3.25) for the nonnegative terms,

$$2\left(\int_\Omega \mu(\rho)|\nabla v|^2 \,\mathrm{d}x\right)^{1/2}\left(\int_\Omega \frac{\rho^{2\gamma}}{\mu(\rho)} \,\mathrm{d}x\right)^{1/2},$$

we obtain

$$\begin{aligned}
\left|\int_\Omega (1+\ln(1+|v|^2))v\cdot\nabla\rho^\gamma \,\mathrm{d}x\right| &\le C\int_\Omega \mu(\rho)|\nabla v|^2 \,\mathrm{d}x\\
&+\frac{\nu}{2}\int_\Omega (1+\ln(1+|v|^2))\mu(\rho)|D(v)|^2 \,\mathrm{d}x + C_\nu\int_\Omega (2+\ln(1+|v|^2))\frac{\rho^{2\gamma}}{\mu(\rho)} \,\mathrm{d}x.
\end{aligned}$$

By Hölder's inequality (Theorem A.31), the last term satisfies

$$
\begin{aligned}
&\int_\Omega (2+\ln(1+|v|^2))\frac{\rho^{2\gamma}}{\mu(\rho)}\ \mathrm{d}x \\
&\quad\le \left(\int_\Omega \left(\frac{\rho^{2\gamma-\delta/2}}{\mu(\rho)}\right)^{2/(2-\delta)}\ \mathrm{d}x\right)^{(2-\delta)/2}\left(\int_\Omega \rho(2+\ln(1+|v|^2))^{2/\delta}\ \mathrm{d}x\right)^{\delta/2},
\end{aligned}
$$

for any $\delta \in (0,2)$, which proves the Lemma. □

Definition 3.6 (Weak solutions)**.** We say that (ρ, v) is a weak solution of system (3.1)-(3.3) on $[0,T]\times\Omega$ if

$$
\begin{aligned}
&\rho \in L^\infty(0,T;L^1(\Omega)\cap L^\gamma(\Omega)),\\
&\sqrt{\rho}\in L^\infty(0,T;H^1(\Omega)),\\
&\sqrt{\rho}v \in L^\infty(0,T;(L^2(\Omega))^d),\\
&\mu(\rho)D(v)\in L^2(0,T;(W^{-1,1}_{loc}(\Omega))^{d\times d}),\\
&\lambda(\rho)\mathrm{div}\ v \in L^2(0,T;W^{-1,1}_{loc}(\Omega)),
\end{aligned}
$$

with $\rho \ge 0$ and $(\rho, \sqrt{\rho}v)$ are solving

$$
\begin{aligned}
\partial_t\rho + \mathrm{div}(\sqrt{\rho}\sqrt{\rho}v) &= 0 \\
\rho(0,x) &= \rho_0(x)
\end{aligned}
\quad \text{in } \mathcal{D}'((0,T)\times\Omega),
$$

and the equality

$$
\begin{aligned}
&\int_\Omega m_0\cdot\phi(0,\cdot)\ \ \mathrm{d}x + \int_0^T\int_\Omega \sqrt{\rho}(\sqrt{\rho}v)\partial_t\phi + \sqrt{\rho}v\otimes\sqrt{\rho}v : \nabla\phi\ \ \mathrm{d}x\ \mathrm{d}t \\
&\quad + \int_0^T\int_\Omega \rho^\gamma \mathrm{div}\phi\ \ \mathrm{d}x\ \mathrm{d}t - \langle 2\mu(\rho)D(v),\nabla\phi\rangle - \langle\lambda(\rho)\mathrm{div}\ v, \mathrm{div}\phi\rangle = 0,
\end{aligned}
$$

holds for all smooth test functions with compact support $\phi \in \mathcal{D}((0,T)\times\Omega)$, such that $\phi(T,\cdot)=0$.

Remark 3.7. The diffusion terms make sense when we write them in index notation, while we use Einstein's summing convention (Definition A.26), as

$$\langle 2\mu(\rho)D(v), \nabla\phi\rangle$$

$$= -\int_0^T \int_\Omega \frac{\mu(\rho)}{\sqrt{\rho}}(\sqrt{\rho}v_j)\partial_{ii}\phi_j \quad \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega (\sqrt{\rho}v_j)2\mu'(\rho)\partial_i\sqrt{\rho}\partial_i\phi_j \quad \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^T \int_\Omega \frac{\mu(\rho)}{\sqrt{\rho}}(\sqrt{\rho}v_i)\partial_{ji}\phi_j \quad \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega (\sqrt{\rho}v_i)2\mu'(\rho)\partial_j\sqrt{\rho}\partial_i\phi_j \quad \mathrm{d}x \ \mathrm{d}t,$$

and

$$\langle \lambda(\rho)\mathrm{div}\ v, \mathrm{div}\phi\rangle$$

$$= -\int_0^T \int_\Omega \frac{\lambda(\rho)}{\sqrt{\rho}}(\sqrt{\rho}v_i)\partial_{ij}\phi_j \quad \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega (\sqrt{\rho}v_j)2\lambda'(\rho)\partial_i\sqrt{\rho}\partial_j\phi_i \quad \mathrm{d}x \ \mathrm{d}t.$$

Theorem 3.8. *Assume that $\gamma > 1$ and $\mu(\rho), \lambda(\rho) \in C^2(0,\infty)$ satisfy the conditions (3.4) - (3.7), and (3.8) if $\gamma \geq 0$ and $d = 3$.*
Let $(\rho_n, v_n)_{n\in\mathbb{N}}$ be a sequence of weak solutions of system (3.1)-(3.2) satisfying the entropy inequalities (3.11) and (3.12) together with the inequality (3.20), with initial data

$$\rho_n(x,0) = \rho_0^n(x) \quad \textit{and} \quad \rho_n v_n(x,0) = \rho_0^n(x)v_0^n(x) =: m_0^n(x), \tag{3.27}$$

where ρ_0^n, v_0^n are such that

$$\rho_0^n \geq 0, \quad \rho_0^n \to \rho_0 \ \textit{in} \ L^1(\Omega), \quad \rho_0^n v_0^n \to \rho_0 v_0 \ \textit{in} \ L^1(\Omega), \tag{3.28}$$

and satisfy

$$\int_\Omega \rho_0^n \frac{|v_0^n|^2}{2} + \frac{1}{\gamma-1}\rho_0^{n\gamma} \quad dx < C, \qquad \int_\Omega (\rho_0^n)^{-1}|\nabla\mu(\rho_0^n)|^2 \quad dx < C, \tag{3.29}$$

and

$$\int_\Omega \rho_0^n \frac{1+|v_0^n|^2}{2}\ln(1+|v_0^n|^2) \quad dx < C, \tag{3.30}$$

for a constant $C > 0$ independent of n. Then, up to a subsequence, $(\rho_n, \sqrt{\rho_n} v_n)_{n\in\mathbb{N}}$ converges strongly to a weak solution of system (3.1)-(3.3), which satisfies (3.11), (3.12) and (3.20).
In particular, ρ_n converges strongly in $C^0(0,T; L^{3/2}_{loc}(\Omega))$, $\sqrt{\rho_n} v_n$ converges strongly in $L^2(0,T; L^2_{loc}(\Omega))$ and the momentum $\rho_n v_n$ converges strongly in $L^1(0,T; L^1_{loc}(\Omega))$ for any $T > 0$.

Remark 3.9. For the sake of simplicity we will dismiss the superscripts $(L^2(\Omega))^{d\times d}$ and $(L^2(\Omega))^d$ in the following and will just write $L^2(\Omega)$ and proceed similarly with all other spaces.

Remark 3.10. To prove the stability of weak solutions we have to pass to the limit in the term $\rho_n v_n \otimes v_n$ which requires the strong convergence of $\sqrt{\rho_n} v_n$. Thus, we have to show the compactness of $\sqrt{\rho_n} v_n$ in $L^2((0,T) \times \Omega)$. This is achieved by having a better estimate on $\rho_n v_n^2$ than the $L^\infty(0,T; L^1(\Omega))$ bound, namely a $L^\infty(0,T; L \log L(\Omega))$ bound, which will be a result of Lemma 3.5.

Remark 3.11. It is an important open problem to construct a sequence of approximate weak solutions $(\rho_n, v_n)_{n\in\mathbb{N}}$ that fulfills all the requirements made in Theorem 3.8 for the barotropic Navier-Stokes equations in 3 dimensions, [5]. Anyone who succeeds in constructing such a sequence will have proven the global existence of weak solutions.

Proof of Theorem 3.8. If the initial data (ρ_0^n, v_0^n) satisfies the assumptions (3.28) - (3.30) we have, in particular,

$$
\begin{aligned}
&\rho_0^n \in L^1 \cap L^\gamma, \qquad \rho_0^n \geq 0 \text{ a.e.},\\
&\rho_0^n |v_0^n|^2 \in L^1(\Omega),\\
&\sqrt{\rho_0^n}\nabla\phi(\rho_0^n) = 2\nabla\mu(\rho_0^n)/\sqrt{\rho_0^n} \in L^2(\Omega),\\
&\int_\Omega \rho_0^n \frac{|v_0^n|^2}{2} \ln(1 + |v_0^n|^2) \ \mathrm{d}x < C.
\end{aligned}
$$

Since we assumed that the sequence of weak solutions $(\rho_n, v_n)_{n\in\mathbb{N}}$ satisfies the energy inequalities (3.11) and (3.12), we conclude in the exact same fashion as before that

$$||\sqrt{\rho_n} v_n||_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{3.31}$$
$$||\rho_n||_{L^\infty(0,T;L^1\cap L^\gamma(\Omega))} \leq C, \tag{3.32}$$
$$||\sqrt{\mu(\rho_n)}\nabla v_n||_{L^2(0,T;(L^2(\Omega))^{d\times d})} \leq C, \tag{3.33}$$
$$||\mu'(\rho_n)\nabla\sqrt{\rho_n}||_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{3.34}$$
$$||\sqrt{\mu'(\rho_n)\rho_n^{\gamma-2}}\nabla\rho_n||_{L^2(0,T;L^2(\Omega))} \leq C. \tag{3.35}$$

With hypothesis (3.5) and since

$$\begin{aligned}\sqrt{\mu'(\rho_n)\rho_n^{\gamma-2}}\nabla\rho_n &= \sqrt{\mu'(\rho_n)}\rho^{(\gamma-2)/2}\nabla\rho_n \\ &= \sqrt{\mu'(\rho_n)}\nabla\rho_n^{\gamma/2},\end{aligned}$$

we obtain from the estimates (3.33) - (3.35),

$$||\sqrt{\rho_n}\nabla v_n||_{L^2(0,T;L^2(\Omega))} \leq C, \tag{3.36}$$

$$||\nabla\sqrt{\rho_n}||_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{3.37}$$

$$||\nabla\rho_n^{\gamma/2}||_{L^2(0,T;L^2(\Omega))} \leq C, \tag{3.38}$$

We will recall here the proof given in [27], which was nicely presented in single steps.

Step 1. Convergence of $\sqrt{\rho_n}$

Lemma 3.12. *The following estimates hold true:*

$$\sqrt{\rho_n} \in L^\infty(0,T;H^1(\Omega)), \tag{3.39}$$

$$\partial_t\sqrt{\rho_n} \in L^2(0,T;H^{-1}(\Omega)). \tag{3.40}$$

Thus, $\sqrt{\rho_n}$ converges, up to a subsequence, almost everywhere and strongly in $L^2(0,T;L^2_{loc}(\Omega))$:

$$\sqrt{\rho_n} \to \sqrt{\rho} \qquad \textit{a.e. and strongly in } L^2_{loc}((0,T)\times\Omega). \tag{3.41}$$

Additionally, ρ_n converges to ρ in $C^0(0,T;L^{3/2}_{loc}(\Omega))$.

Proof. Estimate (3.32), together with estimate (3.37), yields $\sqrt{\rho_n} \in L^\infty(0,T;H^1(\Omega))$. In oder to proof the second estimate, we compute

$$\begin{aligned}\partial_t\sqrt{\rho_n} &= \frac{1}{2}\frac{1}{\sqrt{\rho_n}}\,\partial_t\rho_n = -\frac{1}{2}\frac{1}{\sqrt{\rho_n}}\mathrm{div}(v_n\rho_n) = -\frac{1}{2}\sqrt{\rho_n}\ \mathrm{div}\ v_n - \frac{1}{2}\frac{1}{\sqrt{\rho_n}}v_n\cdot\nabla\rho_n = \\ &= -\frac{1}{2}\sqrt{\rho_n}\ \mathrm{div}\ v_n - v_n\cdot\nabla\sqrt{\rho_n} = \frac{1}{2}\sqrt{\rho_n}\ \mathrm{div}\ v_n - \mathrm{div}(\sqrt{\rho_n}v_n),\end{aligned} \tag{3.42}$$

using the mass balance equation (3.1). Since $\sqrt{\rho_n}v_n \in L^\infty(0,T;L^2(\Omega))$ by (3.31), $\mathrm{div}(\sqrt{\rho_n}v_n) \in L^\infty(0,T;H^{-1}(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))$, by the definition of negative Sobolev spaces (Definition A.10). Also $\sqrt{\rho_n}\nabla v_n \in L^2(0,T;L^2(\Omega)) \hookrightarrow L^2(0,T;H^{-1}(\Omega))$ by (3.36) and by Definition A.10. Thus, it follows from equation (3.42) that

$\partial_t\sqrt{\rho_n} \in L^2(0,T;H^{-1}(\Omega))$.
Since $\sqrt{\rho_n} \in L^\infty(0,T;H^1(\Omega))$ by (3.39), it is certainly bounded in $L^2(0,T;H^1(\Omega))$. We have the embeddings $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, see Definition A.8, by Sobolev's embedding Theorem (Theorem A.9) and by the definition of $H^{-1}(\Omega)$. Note that, in the case $\Omega = \mathbb{R}^d$, we have these embeddings for every compact $K \subset \Omega$. Therefore, we can apply Aubin's Lemma (Lemma A.12) and obtain the strong convergence of $\sqrt{\rho_{n_k}}$ in $L^2_{loc}((0,T)\times\Omega)$. By Theorem A.21, we have almost everywhere convergence of a subsequence.

Sobolev embedding (Theorem A.9) and (3.39) imply that $\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^q(\Omega))$ $\forall q \in [2,+\infty)$ if $d=2$ and $\forall\, q \in [2,6]$ if $d=3$. So, in either case $\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^6(\Omega))$ and ρ_n is bounded in $L^\infty(0,T;L^3(\Omega))$. Therefore,

$$\rho_n v_n = \sqrt{\rho_n}\sqrt{\rho_n}v_n \in L^\infty(0,T;L^{3/2}(\Omega)), \tag{3.43}$$

and similarly,

$$\nabla\rho_n = 2\sqrt{\rho_n}\nabla\sqrt{\rho_n} \in L^\infty(0,T;L^{3/2}(\Omega)), \tag{3.44}$$

by Hölder's inequality (Theorem A.31), since both, $\sqrt{\rho_n}v_n$ and $\nabla\sqrt{\rho_n}$, are bounded in $L^\infty(0,T;L^2(\Omega))$ by (3.31) and (3.37). So $\rho_n \in L^\infty(0,T;H^{3/2}(\Omega))$. The mass balance equation (3.1) gives $\partial_t\rho_n \in L^\infty(0,T;W^{-1,3/2}(\Omega))$, by estimate (3.43). We have the embeddings $H^{3/2}(\Omega) \hookrightarrow\hookrightarrow L^{3/2}(\Omega) \hookrightarrow W^{-1,3/2}(\Omega)$ by Sobolev's embedding theorem (again, this is true for every compact $K \subset \Omega = \mathbb{R}^d$). Thus, Aubin's Lemma yields the compactness, up to a subsequence, of ρ_n in $C([0,T];L^{3/2}_{loc})$.

□

Step 2. Convergence of the pressure ρ_n^γ

Lemma 3.13. *The pressure ρ_n^γ is bounded in*

$$\begin{cases} L^{5/3}((0,T)\times\Omega) & \text{if } d=3,\\ L^r((0,T)\times\Omega),\ \forall\, r\in[1,2) & \text{if } d=2. \end{cases}$$

In particular ρ_n^γ converges strongly to ρ^γ in $L^1_{loc}((0,T)\times\Omega)$.

Proof. The estimates (3.32) and (3.38) yield $\rho_n^{\gamma/2} \in L^2(0,T;H^1(\Omega))$. Thus, exactly as in the proof of Lemma 3.12, we have that $\rho_n^{\gamma/2} \in L^2(0,T;L^q(\Omega))$ for all $q \in [2,\infty)$ when $d=2$. Then, again with (3.32), $\rho_n^\gamma \in L^1(0,T;L^p(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ $\forall\, p \in [1,\infty)$. Thus, ρ_n^γ is bounded in $L^r((0,T)\times\Omega)$ for all $r\in[1,2)$, by Lyapunov's inequality (Corollary A.32).

If $d = 3$ we obtain $\rho_n^{\gamma/2} \in L^2(0,T;L^6(\Omega))$, or equivalently, $\rho_n^\gamma \in L^1(0,T;L^3(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$. We can use Lyapunov's inequality to obtain:

$$||\rho_n^\gamma||_{L^{5/3}(0,T;L^{5/3}(\Omega))} \leq ||\rho_n^\gamma||^{2/5}_{L^\infty(0,T;L^1(\Omega))} ||\rho_n^\gamma||^{3/5}_{L^1(0,T;L^3(\Omega))}. \tag{3.45}$$

Hence, ρ_n^γ is bounded in $L^{5/3}((0,T)\times\Omega)$. We make use of de la Vallée-Poussin's Theorem (Theorem A.18) with $G(t) := t^{5/3}$ and obtain that ρ_n^γ is uniformly integrable. Afterwards we apply Vitali's Theorem (Theorem A.19) on a bounded subspace of Ω and obtain the strong convergence of ρ_n^γ in $L^1_{loc}((0,T)\times\Omega)$, since we already know that ρ_n^γ converges almost everywhere to ρ^γ, by Lemma 3.12.

□

Step 3. Bounds for $\sqrt{\rho_n} v_n$

Lemma 3.14. *If $d = 2$ and $\gamma > 1$ or if $d = 3$ and either $\gamma < 3$ or $\gamma \geq 3$ and (3.8) holds, then $\rho_n |v_n|^2 \ln(1 + |v_n|^2)$ is bounded in $L^\infty(0,T;L^1(\Omega))$.*

Remark 3.15. Using the bounds of $\rho_n|v_n|^2$ in $L^\infty(0,T;L\log L(\Omega))$ we will only have to prove the convergence almost everywhere to obtain strong convergence of $\sqrt{\rho_n} v_n$ in $L^2_{loc}((0,T)\times\Omega)$.

Proof of Lemma 3.14. By Lemma 3.5, we have, for any $\delta \in (0,2)$:

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \rho_n \frac{1+|v_n|^2}{2}\ln(1+|v_n|^2)\ \mathrm{d}x + \frac{\nu}{2}\int_\Omega \mu(\rho_n)(1+\ln(1+|v_n|^2))|D(v_n)|^2\ \mathrm{d}x \\ &\quad\leq C\left(\int_\Omega \left(\frac{\rho_n^{2\gamma-\delta/2}}{\mu(\rho_n)}\right)^{2/(2-\delta)}\ \mathrm{d}x\right)^{(2-\delta)/2} \left(\int_\Omega \rho_n(2+\ln(1+|v_n|^2))^{2/\delta}\ \mathrm{d}x\right)^{\delta/2} \\ &\qquad + C\int_\Omega \mu(\rho_n)|\nabla v_n|^2\ \mathrm{d}x. \end{aligned}$$

Using estimate (3.33), $\mu(\rho_n) \geq 0$, $\rho_n \in L^\infty(0,T;L^1(\Omega))$ and $\rho_n v_n^2 \in L^\infty(0,T;L^1(\Omega))$ this inequality gives:

$$\frac{d}{dt}\int_\Omega \rho_n \frac{1+|v_n|^2}{2}\ln(1+|v_n|^2)\ \mathrm{d}x \leq C\left(\int_\Omega \left(\frac{\rho_n^{2\gamma-\delta/2}}{\mu(\rho_n)}\right)^{2/(2-\delta)}\ \mathrm{d}x\right)^{(2-\delta)/2} + C. \tag{3.46}$$

It follows from (3.5) that $\mu(\rho) \geq \nu\rho$, thus,

$$\frac{d}{dt}\int_\Omega \rho_n \frac{1+|v_n|^2}{2}\ln(1+|v_n|^2)\ \mathrm{d}x \le C\left(\int_\Omega \left(\rho_n^{2\gamma-1-\delta/2}\right)^{2/(2-\delta)}\ \mathrm{d}x\right)^{(2-\delta)/2} + C. \tag{3.47}$$

By Lemma 3.13, ρ_n^γ is bounded in $L^1((0,T)\times\Omega)$ for every $\gamma > 1$, in dimension $d = 2$, so, in particular, $\rho_n^{2\gamma-1-\delta/2}$ is bounded in $L^1((0,T)\times\Omega)$ in dimension $d = 2$. Furthermore, Lemma 3.13 gives the boundedness of $\rho_n^{5/3\gamma}$ in $L^1((0,T)\times\Omega)$ when $d = 3$. Thus, if we require that

$$2\gamma - 1 < \frac{5}{3}\gamma \Longleftrightarrow \gamma < 3,$$

then $\rho_n^{2\gamma-1-\delta/2}$ is bounded in $L^1((0,T)\times\Omega)$ also in dimension $d = 3$. Therefore, we had to make the requirement $\gamma < 3$ in the statement of the Lemma. In either case, the right hand side of inequality (3.47) is bounded for small δ, thus,

$$\frac{d}{dt}\int_\Omega \rho_n \frac{1+|v_n|^2}{2}\ln(1+|v_n|^2)\ \mathrm{d}x \le C,$$

in dimension 2 and 3. Which, together with assumption (3.30), gives the boundedness in $L^\infty(0,T;L^1(\Omega))$.
When $d = 3$ and $\gamma \ge 3$, we need hypothesis (3.8) to show that the right-hand side of (3.46) is bounded. For large ρ_n, in the limes $\rho_n \to \infty$, we have,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \rho_n \frac{1+|v_n|^2}{2}\ln(1+|v_n|^2)\ \mathrm{d}x &\le C\left(\int_\Omega \left(\frac{\rho_n^{2\gamma-\delta/2}}{\mu(\rho_n)}\right)^{2/(2-\delta)}\ \mathrm{d}x\right)^{(2-\delta)/2} + C \\
&\le C\left(\int_\Omega \left(\frac{\rho_n^{2\gamma-\delta/2}}{\rho_n^{\gamma/3+\epsilon}}\right)^{2/(2-\delta)}\ \mathrm{d}x\right)^{(2-\delta)/2} + C \\
&= C\left(\int_\Omega \left(\rho_n^{2\gamma-\gamma/3-\epsilon-\delta/2}\right)^{2/(2-\delta)}\ \mathrm{d}x\right)^{(2-\delta)/2} + C \\
&\le C\left(\int_\Omega \left(\rho_n^{5/3\gamma}\right)^{2/(2-\delta)}\ \mathrm{d}x\right)^{(2-\delta)/2} + C \quad < \infty,
\end{aligned}$$

with $\epsilon > 0$ and δ small, by Lemma 3.13. □

Step 4. Convergence of the momentum $\rho_n v_n$

Firstly, we will prove the following Lemma, which will be important in the proof of the compactness of the momentum $\rho_n v_n$.

Lemma 3.16. *The functions $\mu(\rho_n)/\sqrt{\rho_n}$ and $\lambda(\rho_n)/\sqrt{\rho_n}$ are bounded in $L^\infty(0,T;L^6(\Omega))$.*

Proof. We only prove the result for $\mu(\rho_n)/\sqrt{\rho_n}$, since, with inequality (3.6) and Remark 3.2,

$$|\lambda(\rho_n)| \leq C\mu(\rho_n), \qquad \text{and} \qquad |\lambda'(\rho_n)| \leq C\mu'(\rho_n)$$

the proof is similar for $\lambda(\rho_n)/\sqrt{\rho_n}$.

In view of inequality (3.10),

$$\frac{\mu(\rho_n)}{\sqrt{\rho_n}} \leq C\rho_n^\nu \leq C, \qquad \text{if } \rho_n \leq 1,$$

in either dimension. Thus, it is sufficient to control $\mu(\rho_n)/\sqrt{\rho_n}$ for large ρ_n.

We split the problem into two steps, in the first step, we will treat the problem in dimension two, in the second, we will consider it in dimension three.

Step 1. Dimension d = 2

Again, the boundedness of $\sqrt{\rho_n}$ in $L^\infty(0,T;H^1(\Omega))$ implies that ρ_n is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p \in [1,+\infty)$. With (3.10), we have

$$\frac{\mu(\rho_n)}{\sqrt{\rho_n}} \leq \begin{cases} C\rho_n^{\frac{1}{2\nu}} & \text{if} \quad \rho \geq 1, \\ C\rho_n^{\frac{\nu}{2}} & \text{if} \quad \rho \leq 1. \end{cases}$$

Therefore, also $\mu(\rho_n)/\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p \in [1,+\infty)$.

Step 2. Dimension d = 3

We need to show that $\mu(\rho_n)/\sqrt{\rho_n} \in L^\infty(0,T;H^1(\Omega))$. Then Sobolev embedding (Theorem A.9) implies that $\mu(\rho_n)/\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^6(\Omega))$. Firstly,

$$\nabla\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right) = 2\mu'(\rho_n)\nabla\sqrt{\rho_n} - \frac{\mu(\rho_n)}{2\rho_n^{3/2}}\nabla\rho_n. \tag{3.48}$$

The conditions (3.4) and (3.7) yield

$$2\mu'(\rho)\rho = 2\mu(\rho) + \lambda(\rho) \geq \frac{2\mu(\rho) + 3\lambda(\rho)}{3} \geq \frac{\nu}{3}\mu(\rho),$$

thus,

$$\begin{aligned}
\left|\nabla\left(\frac{\mu(\rho)}{\sqrt{\rho}}\right)\right| &\leq C\left|\mu'(\rho)\nabla\sqrt{\rho}\right| + \left|\frac{\mu(\rho)}{2\rho^{3/2}}\nabla\rho\right| \\
&\leq C\left|\mu'(\rho)\nabla\sqrt{\rho}\right| + C_\nu\left|\frac{2\mu'(\rho)\rho}{2\rho^{3/2}}\nabla\rho\right| \\
&\leq C\left|\mu'(\rho)\nabla\sqrt{\rho}\right| + C_\nu\left|2\mu'(\rho)\nabla\sqrt{\rho}\right| \leq C_\nu|\mu'(\rho)\nabla\sqrt{\rho}|\,.
\end{aligned}$$

Therefore, estimate (3.34) yields:

$$\left\|\nabla\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right)\right\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \tag{3.49}$$

Again, (3.10) gives

$$\frac{\mu(\rho)}{\sqrt{\rho}} \leq \begin{cases} C\rho^{\frac{1}{6}+\frac{1}{3\nu}} & \text{if} \quad \rho \geq 1, \\ C\rho^{\frac{1}{6}+\frac{\nu}{3}} & \text{if} \quad \rho \leq 1. \end{cases} \tag{3.50}$$

So there exists a constant $s \leq 1$, such that $\rho^{\frac{s}{6}+\frac{s}{3\nu}} \leq \rho^{1/2} \in L^\infty(0,T;L^2(\Omega))$, thus

$$\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right)^s \in L^\infty(0,T;L^2(\Omega)). \tag{3.51}$$

Moreover,

$$\left|\nabla\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right)^s \mathbb{1}_{\mu(\rho_n)/\sqrt{\rho_n}\geq 1}\right| = \left|\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right)^{s-1}\nabla\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right)\mathbb{1}_{\mu(\rho_n)/\sqrt{\rho_n}\geq 1}\right| \leq \left|\nabla\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right)\right|,$$

since $s-1 \leq 0$. By inequality (3.5), $\mu(\rho) \geq \nu\rho$, we see that $\mu(\rho) \geq \sqrt{\rho}$ if and only if $\sqrt{\rho} = \frac{1}{\nu} \geq 1$, therefore, the set $\{\rho_n | \frac{\mu(\rho_n)}{\sqrt{\rho_n}} \geq 1\}$ coincides with the set $\{\rho_n | \rho_n \geq 1\}$. Thus, $(\mu(\rho_n)/\sqrt{\rho_n})^s \mathbb{1}_{\rho_n \geq 1}$ is bounded in $L^\infty(0,T;H^1(\Omega))$ by (3.49) and (3.51). Sobolev embedding implies that

$$\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right)^{s_1} \mathbb{1}_{\rho_n \geq 1} \in L^\infty(0,T;L^2(\Omega)),$$

for all $s_1 \in (s, 3s)$. As long as $3s \leq 1$ we can apply the same argument to $3s$ and will eventually obtain

$$\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right) \mathbb{1}_{\rho_n \geq 1} \in L^\infty(0,T;L^2(\Omega)).$$

Together with (3.49),

$$\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right) \mathbb{1}_{\rho_n \geq 1} \in L^\infty(0,T;H^1(\Omega)),$$

and Sobolev embedding gives $(\mu(\rho_n)/\sqrt{\rho_n})\mathbb{1}_{\rho_n \geq 1} \in L^\infty(0,T;L^6(\Omega))$. □

Remark 3.17. In particular, if $\mu(\rho_n) = \nu\rho_n$, Lemma 3.16 follows directly from Lemma 3.12, since we have $\mu(\rho_n)/\sqrt{\rho_n} = \nu\sqrt{\rho_n}$.

We are now able to prove the compactness of the momentum.

Lemma 3.18. *Up to a subsequence, the momentum $m_n = \rho_n v_n$ converges strongly to $m(x,t)$ in $L^2(0,T;L^p_{loc}(\Omega))$ for all $p \in [1, 3/2)$. In particular, up to a subsequence, m_n converges almost everywhere to m.*

Remark 3.19. We can already define $v(x,t) = m(x,t)/\rho(x,t)$ outside the vacuum set, but we do not yet know if $m(x,t)$ is zero on the vacuum set.

Proof of Lemma 3.18. As shown in Lemma 3.12, $\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^6(\Omega))$ in both dimensions. Thus, by Hölder's Inequality (Theorem A.31),

$$\rho_n v_n = \sqrt{\rho_n}\sqrt{\rho_n} v_n \in L^\infty(0,T;L^q(\Omega)) \quad \text{for all } q \in [1, 3/2], \tag{3.52}$$

since $\sqrt{\rho_n} v_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$ by estimate (3.31). We rewrite the i^{th} component of the gradient of $\rho_n v_n$ in index notation:

$$\partial_i(\rho_n u_{n_j}) = \rho_n \partial_i v_{n_j} + v_{n_j}\partial_i \rho_n = \sqrt{\rho_n}\sqrt{\rho_n}\partial_i v_{n_j} + v_{n_j} \cdot 2\sqrt{\rho_n}\partial_i\sqrt{\rho_n}$$

Using the estimates (3.36), (3.37) and (3.31), the second term is bounded in $L^\infty(0,T;L^1(\Omega))$, and the first term is bounded in $L^2(0,T;L^q(\Omega))$ $\forall q \in [1,3/2]$, again by Hölder's Inequality. Thus, we have

$$\nabla(\rho_n v_n) \text{ is bounded in } \; L^2(0,T;L^1(\Omega)).$$

In particular, with estimate (3.52),

$$\rho_n v_n \text{ is bounded in } \; L^2(0,T;W^{1,1}(\Omega)).$$

In order to use Aubin's Lemma (Lemma A.12), we have to show that

$$\partial_t(\rho_n v_n) \text{ is bounded in } \; L^\infty(0,T;W^{-2,4/3}(\Omega)).$$

Then we obtain compactness of the sequence $\rho_n v_n$ in $L^2(0,T;L^p_{loc}(\Omega))$ for all $p \in [1,3/2)$, since $W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, $\forall\, q \in [1,3/2)$, by Sobolev's embedding Theorem (Theorem A.9). If Ω is the whole space, the compact embedding holds true for every compact subset $K \subset \Omega$. But also $L^{\tilde{q}}(\Omega) \hookrightarrow W^{-2,4/3}(\Omega)$ $\forall\, \tilde{q} \in [1,\infty)$, since $W^{\tilde{q}+2}(\Omega) \hookrightarrow W^{0,4/3}(\Omega)$ by Sobolev's embedding theorem, see the definition of Sobolev spaces with negative exponents as spaces of distributional derivatives (Definition A.10).

We will use the momentum equation (3.2). With (3.31) and (3.32) we obtain:

$$\operatorname{div}(\sqrt{\rho_n}v_n \otimes \sqrt{\rho_n}v_n) \in L^\infty(0,T;W^{-1,1}(\Omega)), \tag{3.53}$$

$$\nabla\rho_n^\gamma \in L^\infty(0,T;W^{-1,1}(\Omega)). \tag{3.54}$$

We have to show that the diffusion terms $\operatorname{div}(2\mu(\rho_n)D(v_n))$ and $\nabla(\lambda(\rho_n)\operatorname{div} v_n)$ are bounded in $L^\infty(0,T;W^{-2,4/3}(\Omega))$. Therefore, we rewrite the term

$$
\begin{aligned}
\mu(\rho_n)\nabla v_n &= \nabla(\mu(\rho_n)v_n) - v_n\nabla\mu(\rho_n) \\
&= \nabla\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n}v_n\right) - \sqrt{\rho_n}v_n\frac{\nabla\mu(\rho_n)}{\sqrt{\rho_n}} \\
&= \nabla\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n}v_n\right) - 2\sqrt{\rho_n}v_n\mu'(\rho_n)\nabla\sqrt{\rho_n},
\end{aligned}
$$

and proceed similarly with the terms involving $\lambda(\rho_n)$ and ∇v_n^T. The second term is bounded in $L^\infty(0,T;L^1(\Omega))$ due to Lemma 3.14 and estimate (3.34), by Hölder's inequality (Theorem A.31). The first term is bounded in $L^\infty(0,T;W^{-1,3/2}(\Omega))$ due to estimate (3.31), Lemma 3.14 and Hölder's inequality.
Thus, both terms contributing to $\mu(\rho_n)\nabla v_n$ are bounded in $L^\infty(0,T;W^{-1,4/3}(\Omega))$, since $L^1(\Omega) \hookrightarrow W^{-1,4/3}(\Omega)$ (because $W^{1,1}(\Omega) \hookrightarrow W^{0,4/3}(\Omega)$ by Sobolev's embedding theorem) and also $W^{-1,3/2}(\Omega) \hookrightarrow W^{-1,4/3}(\Omega)$ (since $W^{0,3/2}(\Omega) \hookrightarrow W^{0,4/3}(\Omega)$). Similarly, we conclude that $\mu(\rho_n)D(v_n)$ and $\lambda(\rho_n)\mathrm{div}v_n$ are bounded in $L^\infty(0,T;W^{-1,4/3}(\Omega))$. Since $W^{-1,1}(\Omega) \hookrightarrow W^{-2,4/3}(\Omega)$ (Sobolev's embedding theorem yields $W^{1,1}(\Omega) \hookrightarrow W^{0,4/3}(\Omega)$), also the terms (3.53) and (3.54) are bounded in $L^\infty(0,T;W^{-2,4/3}(\Omega))$. Therefore, the momentum equation (3.2) gives $\partial_t(\rho_n v_n) \in L^\infty(0,T;W^{-2,4/3}(\Omega))$. Application of Aubin's Lemma to $W^{1,1}_{loc}(\Omega) \hookrightarrow\hookrightarrow L^p_{loc}(\Omega \hookrightarrow W^{-2,4/3}_{loc}(\Omega)$ gives the strong convergence of $\rho_n v_n = m_n$ to m in $L^2(0,T;L^p_{loc}(\Omega))$ $\forall p \in [1,3/2)$. In particular, by Theorem A.21, we have a subsequence which converges a.e. to m.

□

Step 5. Convergence of $\sqrt{\rho_n}v_n$

Lemma 3.20. *The sequence $\sqrt{\rho_n}v_n$ converges strongly in $L^2_{loc}((0,T)\times\Omega)$ to $m/\sqrt{\rho}$, which is defined to be zero when $m = 0$.*
In particular, we have $m(x,t) = 0$ a.e. on $\{\rho(x,t) = 0\}$ and there exists a $v(x,t)$ such that $m(x,t) = \rho(x,t)v(x,t)$ and

$$
\begin{aligned}
&\rho_n v_n \to \rho v \quad \textit{strongly in } L^2(0,T;L^p_{loc}(\Omega)),\ \forall p \in [1,3/2), \\
&\sqrt{\rho_n}v_n \to \sqrt{\rho}v \quad \textit{strongly in } L^2_{loc}((0,T)\times\Omega).
\end{aligned}
$$

Remark 3.21. Note that v is not uniquely defined on the vacuum set.

Proof. Since $m_n/\sqrt{\rho_n} := \sqrt{\rho_n}v_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$ uniformly in n by estimate (3.31), Fatou's Lemma (Lemma A.23) yields

$$
\int_\Omega \liminf_{n\to\infty}\frac{m_n^2}{\rho_n}\ \mathrm{d}x < \infty.
$$

Thus, $m(x,t) = 0$ a.e. on $\{\rho(x,t) = 0\}$. This allows us to define the limit velocity $v(x,t)$ by $v(x,t) = m(x,t)/\rho(x,t)$ when $\rho(x,t) \neq 0$ and by $v(x,t) = 0$ when $\rho(x,t) = 0$. In particular, we have

$$m(x,t) = v(x,t)\rho(x,t),$$

and

$$\int_\Omega \frac{m^2}{\rho} \, \mathrm{d}x = \int_\Omega \rho |v|^2 \, \mathrm{d}x < \infty.$$

By Fatou's lemma and Lemma 3.14,

$$\begin{aligned}\int_\Omega \rho |v|^2 \ln(1+|v|^2) \, \mathrm{d}x &\leq \int_\Omega \liminf_{n\to\infty} \rho_n |v_n|^2 \ln(1+|v_n|^2) \, \mathrm{d}x \\ &\leq \liminf_{n\to\infty} \int_\Omega \rho_n |v_n|^2 \ln(1+|v_n|^2) \, \mathrm{d}x,\end{aligned}$$

also $\rho|v|^2 \ln(1+|v|^2)$ is in $L^\infty(0,T;L^1(\Omega))$.
Lemma 3.18 gives the strong convergence of $\rho_n v_n$ to $m(x,t)$ in $L^2(0,T;L^p_{loc}(\Omega))$ for all $p \in [1,3/2)$. It remains to prove the strong convergence of $\sqrt{\rho_n} v_n$ to $\sqrt{\rho} v$ in $L^2_{loc}((0,T)\times\Omega)$.

Since m_n and ρ_n converge almost everywhere, by Lemma 3.18 and Lemma 3.12, $\sqrt{\rho_n} v_n$ converges almost everywhere to $\sqrt{\rho} v$ in $\{\rho(x,t) \neq 0\}$. In particular, $\sqrt{\rho_n} v_n \mathbb{1}_{|v_n|\leq M}$ converges almost everywhere in $\{\rho(x,t) \neq 0\}$ to $\sqrt{\rho} v \mathbb{1}_{|v|\leq M}$, for some constant $M > 0$. In $\{\rho(x,t) = 0\}$, we have $\sqrt{\rho_n} v_n \mathbb{1}_{|v_n|\leq M} \leq M\sqrt{\rho_n} \to 0$ almost everywhere.

We have

$$\begin{aligned}\int_0^T \int_\Omega |\sqrt{\rho_n} v_n - \sqrt{\rho} v|^2 \, \mathrm{d}x \, \mathrm{d}t &\leq \int_0^T \int_\Omega |\sqrt{\rho_n} v_n \mathbb{1}_{|v_n|\leq M} - \sqrt{\rho} v \mathbb{1}_{|v|\leq M}|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + 2\int_0^T \int_\Omega |\sqrt{\rho_n} v_n \mathbb{1}_{|v_n|\geq M}|^2 \, \mathrm{d}x \, \mathrm{d}t + 2\int_0^T \int_\Omega |\sqrt{\rho} v \mathbb{1}_{|v|\geq M}|^2 \, \mathrm{d}x \, \mathrm{d}t.\end{aligned} \tag{3.55}$$

Since $|\sqrt{\rho_n} v_n \mathbb{1}_{|v_n|\leq M}| \leq \sqrt{\rho_n} M$, where $\sqrt{\rho_n} \in L^\infty(0,T;L^2(\Omega))$ is integrable for every n, we can apply the Dominated Convergence Theorem (Theorem A.20) and the convergence almost everywhere of $\sqrt{\rho_n} v_n \mathbb{1}_{|v_n|\leq M}$ gives:

$$\int_0^T \int_\Omega |\sqrt{\rho_n} v_n \mathbb{1}_{|v_n|\le M} - \sqrt{\rho} v \mathbb{1}_{|v|\le M}|^2 \ \mathrm{d}x \ \mathrm{d}t \to 0.$$

Furthermore, we have

$$\int_0^T \int_\Omega |\sqrt{\rho_n} v_n \mathbb{1}_{|v_n|\ge M}|^2 \ \mathrm{d}x \ \mathrm{d}t \le \frac{1}{\ln(1+M^2)} \int_0^T \int_\Omega \rho_n v_n^2 \ln(1+|v_n|^2) \ \mathrm{d}x \ \mathrm{d}t,$$

and

$$\int_0^T \int_\Omega |\sqrt{\rho} v \mathbb{1}_{|v|\ge M}|^2 \ \mathrm{d}x \ \mathrm{d}t \le \frac{1}{\ln(1+M^2)} \int_0^T \int_\Omega \rho v^2 \ln(1+|v|^2) \ \mathrm{d}x \ \mathrm{d}t,$$

which applied to the inequality (3.55) yields:

$$\limsup_{n\to\infty} \int_0^T \int_\Omega |\sqrt{\rho_n} v_n - \sqrt{\rho} v|^2 \ \mathrm{d}x \ \mathrm{d}t \le \frac{C}{\ln(1+M^2)}, \qquad M > 0.$$

The strong convergence in $L^2_{loc}((0,T)\times\Omega)$ follows by taking $M \to \infty$. □

Step 6. Convergence of the diffusion terms

Lemma 3.22. *We have*

$$\begin{aligned} \mu(\rho_n)\nabla v_n &\to \mu(\rho)\nabla v \ \ in \ \mathcal{D}'((0,T)\times\Omega), \\ \mu(\rho_n)\nabla v_n^T &\to \mu(\rho)\nabla v^T \ \ in \ \mathcal{D}'((0,T)\times\Omega), \\ \lambda(\rho_n) div \ v_n &\to \lambda(\rho) div \ v \ \ in \ \mathcal{D}'((0,T)\times\Omega). \end{aligned}$$

Proof. Let $\psi \in \mathcal{D}((0,T)\times\Omega)$ be a test function, then

$$\begin{aligned} \int_0^T \int_\Omega \mu(\rho_n)\nabla v_n \psi \ \mathrm{d}x \ \mathrm{d}t &= -\int_0^T \int_\Omega \mu(\rho_n) v_n \nabla\psi \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \nabla\mu(\rho_n) v_n \psi \ \mathrm{d}x \ \mathrm{d}t \\ &= -\int_0^T \int_\Omega \frac{\mu(\rho_n)}{\sqrt{\rho_n}} \sqrt{\rho_n} v_n \nabla\psi \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \nabla\rho_n \frac{\mu'(\rho_n)}{\sqrt{\rho_n}} \sqrt{\rho_n} v_n \psi \ \mathrm{d}x \ \mathrm{d}t \end{aligned}$$

As shown in Lemma 3.16, $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty(0,T;L^6(\Omega))$. Since, by estimate (3.50), there exists a constant $s \le 1$ such that

$$\left(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right)^s \leq C\sqrt{\rho_n},$$

$\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is a continuous function of $\sqrt{\rho_n}$ and, therefore, converges almost everywhere to $\frac{\mu(\rho)}{\sqrt{\rho}}$, which is defined to be zero on the vacuum set. We can apply de la Vallée-Poussin's Theorem (Theorem A.18), with $G(t) := t^3$, then the family $\{|\frac{\mu(\rho_n)}{\sqrt{\rho_n}}|^2\}_{n\in\mathbb{N}}$ is uniformly integrable. Thus, Vitali's Theorem (Theorem A.19) gives the strong convergence in $L^2_{loc}((0,T)\times\Omega)$. Also $\sqrt{\rho_n}v_n$ converges strongly in $L^2_{loc}((0,T)\times\Omega)$ by Lemma 3.20. Since the scalar product of two strongly convergent sequences converges strongly to the product of the limits, the first term converges to

$$\int_0^T\int_\Omega \frac{\mu(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n}v_n\nabla\psi \;\; \mathrm{d}x\mathrm{d}t.$$

In order to prove the convergence of the second term, we define

$$\nabla F(\rho_n) := \frac{\mu'(\rho_n)}{\sqrt{\rho_n}}\nabla\rho_n,$$

with $F' := \mu'(\rho_n)/\sqrt{\rho_n} = \frac{1}{2}\sqrt{\rho_n}\phi'(\rho_n)$, as defined in the equations (2.3) and (2.4). Then,

$$\int_\Omega |\nabla F(\rho_n)|^2 \;\; \mathrm{d}x = \int_\Omega \frac{1}{4}\rho_n|\nabla\phi(\rho_n)|^2 \;\; \mathrm{d}x \leq C,$$

by estimate (3.16). Inequality (3.10) yields

$$\mu'(\rho_n) \leq C\rho_n^{-1/2+\nu/3},$$

when $\rho_n \leq 1$ in either dimension. Thus, $F(\rho_n) \leq C\rho_n^{\nu/3}$. And for large $\rho_n \geq 1$, we see, exactly as in the proof of Lemma 3.16, that

$$(F(\rho_n))^s \leq \rho_n^{1/2},$$

for some $s \leq 1$ and that $F(\rho_n) \in L^\infty(0,T;L^6(\Omega))$. Also, $F(\rho_n)$ is a continuous function and, thus, $F(\rho_n)$ converges almost everywhere to $F(\rho)$. We can apply de la

Vallée-Poussin's Theorem with $G(t) := t^3$ and Vitali's Theorem to $F(\rho_n)$ to obtain the strong converge of $F(\rho_n)$ to $F(\rho)$ in $L^2_{loc}((0,T) \times \Omega)$.

Since, in particular, $\nabla F(\rho_n) = 2\mu'(\rho_n)\nabla\sqrt{\rho_n} \in L^2((0,T) \times \Omega)$ by (3.34), it follows that:

$$\nabla F(\rho_n) \rightharpoonup \nabla F(\rho) \quad \text{weakly in} \quad L^2_{loc}((0,T) \times \Omega).$$

Since the product of a strongly, $\sqrt{\rho_n} v_n$ converges strongly in $L^2_{loc}((0,T) \times \Omega)$ by Lemma 3.20, and a weakly convergent sequence converges weakly to the product of the limits, see Theorem A.14,

$$\int_0^T \int_\Omega \nabla \rho_n \frac{\mu'(\rho_n)}{\sqrt{\rho_n}} \sqrt{\rho_n} v_n \psi \, \mathrm{d}x \, \mathrm{d}t$$

is convergent, too. This yields the convergence of $\mu(\rho_n)\nabla v_n$ to $\mu(\rho)\nabla v$ in $D'((0,T) \times \Omega)$.

A similar argument holds for $\mu(\rho_n)\nabla v^T$ and $\lambda(\rho_n)\mathrm{div} v_n$, since $|\lambda(\rho_n)| \leq C\mu(\rho_n)$ and $|\lambda'(\rho_n)| \leq C\mu'(\rho_n)$, by Remark . □

We have proven the stability of weak solutions for the barotropic compressible Navier-Stokes equations with a pressure law $p(\rho) = \rho^\gamma, \gamma > 1$, in dimension 2 and 3, since we have shown the desired convergence of the sequence of weak solutions $(\rho_n, \sqrt{\rho_n} v_n)_{n \in \mathbb{N}}$ to $(\rho, \sqrt{\rho} v)$. □

A. Appendix

For the convenience of the reader, we recall some definitions and theorems which are used in this thesis.

Definition A.1. The topological dual space of a Banach space X is defined by $X' := \{f : X \to \mathbb{R} \mid \text{ f is linear and continuous}\}$. We write the duality product $\langle f, u \rangle_{X'}$ instead of $f(u)$.
The weak topology on X is the initial topology with respect to X'.

Definition A.2. We denote by $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ the space of all functions with compact support in Ω that have continuous derivatives of all orders and call functions $\phi \in \mathcal{D}(\Omega)$ test functions.

Definition A.3. Suppose $f \in L^1_{loc}(\Omega)$, and $\alpha = (\alpha_1, ..., \alpha_d)$ is a multi-index. We say $g \in L^1_{loc}(\Omega)$ is the α^{th}- weak derivative of f, $D^\alpha f = g$, provided that

$$\int_\Omega f D^\alpha \phi \ \ \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g \phi \ \ \mathrm{d}x,$$

for all test functions $\phi \in \mathcal{D}(\Omega)$.

Definition A.4. The space of distributions is defined as the dual space of $\mathcal{D}(\Omega)$, which we will denote by $\mathcal{D}'(\Omega)$. A regular distribution $T(\cdot)$ is a distribution which is generated by a locally integrable function $f \in L^1_{loc}(\Omega)$. It has a representation

$$T_f(\phi) = \int_\Omega f(x)\phi(x) \ \ \mathrm{d}x.$$

We denote the distributions by the duality product,

$$\langle T, \phi \rangle := T(\phi).$$

The strong convergence in $\mathcal{D}'(\Omega)$ is defined as follows: A sequence $(T_n)_{n\in\mathbb{N}}(\cdot)$ of distributions converges to a $T \in \mathcal{D}'(\Omega)$, if for every test function $\phi \in \mathcal{D}$:

$$\lim_{n\to\infty} \langle T_n, \phi \rangle = \langle T, \phi \rangle.$$

Definition A.5. Let $1 \leq p \leq \infty$ and k be a nonnegative integer. The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $f : \Omega \to \mathbb{R}$ such that for each multi-index $\alpha = (\alpha_1, ..., \alpha_d)$, with $|\alpha| \leq k$, $D^\alpha f$ exists in the weak (or distributional) sense and belongs to $L^p(\Omega)$.
We usually write $H^k(\Omega)$ for $W^{k,2}(\Omega)$ and $H^0(\Omega) = L^2(\Omega)$.
We define the norm on $W^{k,p}(\Omega)$ as follows:

$$||f||_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum\limits_{|\alpha| \leq k} \int_\Omega |D^\alpha f|^p \ \mathrm{d}x \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sum\limits_{|\alpha| \leq k} \ \text{ess } \sup_\Omega |D^\alpha f| & \text{if } p = \infty. \end{cases}$$

The space $W^{k,p}(\Omega)$ is a Banach space for every $k \in \mathbb{N}_0, 1 \leq p \leq \infty$ and a Hilbert space for $k \in \mathbb{N}_0, p = 2$, [14]. The scalar product is then given by

$$(u, v)_{H^k} = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)_{L^2}.$$

Remark A.6. Actually, the Sobolev spaces contain equivalence classes of functions which coincide outside of Lebesgue sets of measure zero.

Definition A.7. We define the Sobolev space $W_0^{k,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $|| \cdot ||_{W^{k,p}(\Omega)}$.

Definition A.8. Let X, Y be Banach spaces, $X \subset Y$. We say X is compactly embedded in Y, provided that

- the embedding is continuous, i.e. $||x||_Y \leq C||x||_X \ \forall x \in X$, and,
- each bounded sequence in X is precompact in Y.

The notations $X \hookrightarrow Y$ and $X \hookrightarrow\hookrightarrow Y$ stand for continuous and compact embedding, respectively.

Theorem A.9 (Sobolev embedding theorem, [2])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega \in C^1$ (it suffices to have a Lipschitz boundary) or the whole space $\mathbb{R}^d$ and $1 \leq p, q < \infty, k, m \in \mathbb{N}_0$ with $m > k$.*
The embedding $W^{m,p}(\Omega) \hookrightarrow W^{k,q}(\Omega)$ is continuous if

$$m - \frac{d}{p} \geq k - \frac{d}{q}.$$

The embedding is compact in the case when Ω is not the whole space if the inequality is strict.
Both propositions hold also true for $W_0^{m,p}(\Omega)$ for any bounded domain Ω.

We want to extend the natural property, that for a function in $W^{k,p}(\Omega)$, $k \geq 1$, its distributional derivative is in $W^{k-1,p}(\Omega)$. Therefore, we define:

Definition A.10. For a function $f \in W^{k,p}(\Omega), k \in \mathbb{Z}$, we define the space $W^{k-1,p}(\Omega)$ as the space consisting of all distributional derivatives of f of order one.

If we regard the space $W^{k,p}(\Omega), k \in \mathbb{N}_0$ as a closed subspace of the Cartesian product of $L^p(\Omega)$ spaces, i.e. let $N = \sum_{0\leq|\alpha|\leq k} 1$ and let $L^p_N(\Omega) := \prod_{j=1}^{N} L^p(\Omega)$, with the norm of $v = (v_1, ..., v_N) \in L^p_N(\Omega)$ given by

$$||v||_{L^p_N(\Omega)} := \begin{cases} \left(\sum\limits_{j=1}^{N} ||v_j||^p_{L^p}\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\limits_{1\leq j\leq N} ||v_j||_{L^\infty} & \text{if } p = \infty, \end{cases}$$

then $W^{k,p}(\Omega)$ is a closed subspace of L^p_N, [1]. We can generally define the space $W^{k-m,p}(\Omega)$ for $k, m \in \mathbb{N}_0$, by:

$$\begin{aligned} W^{k-m,p}(\Omega) := \{T \in \mathcal{D}'(\Omega) \mid \\ T(\phi) = \sum_{0\leq|\alpha|\leq m} (-1)^{|\alpha|} \int_\Omega v_\alpha D^\alpha \phi \ \ \mathrm{d}x, \ \forall \ \phi \in \mathcal{D}(\Omega), v \in L^p_N\}, \end{aligned}$$

In particular, the space $W^{-k,p}(\Omega)$ is isometrically isomorphic to the dual space of $W^{k,q}_0(\Omega)$, $k \in \mathbb{N}_0, 1 \leq q < \infty$, where q is the conjugate index of p, $\frac{1}{p} + \frac{1}{q} = 1$, [1]. The space $W^{-k,p}(\Omega)$ is a Banach space and the embedding $W^{k,p}(\Omega) \hookrightarrow W^{k-m,p}(\Omega)$ is continuous.

For parabolic differential equations we need to define Sobolev spaces in space and time.

Definition A.11. Let B be a Banach space and $T > 0$.

- The space $C^k([0,T]; B)$ is the set of all functions $u : [0,T] \to B$, whose first k derivatives exist and are continuous. The norm is given by:

$$||u||_{C^k([0,T];B)} = \sum_{i=0}^{k} \max_{0\leq t\leq T} ||D^i u(t)||_B.$$

- The space $L^p(0,T;B(\Omega))$ is the set of all (equivalence classes of) measurable functions $u:(0,T)\to B$ such that:

$$||u||_{L^p(0,T;B(\Omega))} := \begin{cases} \left(\int_0^T ||u(t)||_B^p \,\mathrm{d}t\right)^{1/p} & < \infty, \quad \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{0<t<T} ||u(t)||_B & < \infty, \quad \text{if } p = \infty. \end{cases}$$

These spaces are Banach spaces. If H is a Hilbert space, then $L^2(0,T;H(\Omega))$ is also a Hilbert space with respect to the scalar product

$$(u,v)_{L^2(0,T;H(\Omega))} = \int_0^T (u(t),v(t))_H \,\mathrm{d}t,$$

for $u,v \in L^2(0,T;H(\Omega))$.

Lemma A.12 (Aubin's Lemma, [4], [30]). *Let X,B,Y be Banach spaces with $X \subseteq B \subseteq Y$. Let U be a set of functions such that U is bounded in $L^p(0,T;X(\Omega))$ and $\partial_t U = \{\partial_t u \mid u \in U\}$ is bounded in $L^r(0,T;Y(\Omega))$, for $1 \le p,r \le +\infty$, and*

$$X \hookrightarrow\hookrightarrow B, \quad B \hookrightarrow Y.$$

If $p < +\infty$, then the embedding of U in $L^p(0,T;B(\Omega))$ is compact. If $p = +\infty$ and $r > 1$, the embedding of U in $C([0,T];B)$ is compact.

Definition A.13. Strong convergence of a sequence in a Banach space X, denoted $u_n \to u$, is defined as the convergence in the topology on X. In particular, if the topology on X is the norm topology, we have

$$u_n \to u \quad \text{if} \quad ||u_n - u||_X \to 0,\ n \to \infty.$$

The weak convergence of a sequence in X is the convergence in the weak topology on X. We write

$$u_n \rightharpoonup u \quad \text{for } n \to \infty.$$

Equivalently, we can say $(u_n)_{n\in\mathbb{N}} \subset X$ converges weakly to $u \in X$, if for all $f \in X'$,

$$\langle f, u_n \rangle_{X'} \to \langle f, u \rangle_{X'} \quad \text{for } n \to \infty.$$

If H is a Hilbert space, the duality product is the scalar product on H.

Theorem A.14 ([28]). *The product of two strongly convergent sequences converges strongly to the product of the limits. The product of a weakly and a strongly convergent sequence converges weakly to the product of the limits.*

Theorem A.15 ([33]). *Every bounded sequence in a Hilbert space contains a weakly convergent subsequence.*

Definition A.16. A sequence of functions f_n converges almost everywhere to a function f in a measurable space X if $f_n(x) \to f(x)$ for all $x \in Y$, where Y is a measurable subspace of X such that $\lambda(X \setminus Y) = 0$, where λ is the Lebesgue measure.

Definition A.17. A collection of random variables $\{X_\alpha\}_{\alpha \in A}$ is called uniformly integrable if

$$\lim_{M\to\infty} \sup_{\alpha\in A} \mathbb{E}[|X_\alpha| \mathbb{1}_{|X_\alpha|>M}] = 0,$$

where $\mathbb{E}(X)$ is the mean value of X in A,

$$\mathbb{E}(X) = \int_A X \, \mathrm{d}\lambda.$$

Theorem A.18 (De la Vallée-Poussin theorem, [15]). *A family of random variables $\{X_\alpha\}_{\alpha\in A}$ is uniformly integrable if and only if there exists some increasing non-negative function $G(t)$, such that*

$$\lim_{t\to\infty} \frac{G(t)}{t} = \infty,$$

and $\mathbb{E}[G(|X_\alpha|)] \leq C$, for some constant C and for all $\alpha \in A$.

Theorem A.19 (Vitali's theorem, [16]). *Let $(X, \mathcal{F}, \mu)$ be a positive measure space and $\{f_n\}_{n\in\mathbb{N}} \in L^p(X)$. If*

$$\begin{aligned}
&\mu(X) < \infty, \\
&\{|f_n|^p\}_{n\in\mathbb{N}} \textit{ is uniformly integrable}, \\
&f_n(x) \to f(x) \textit{ a.e. as } n \to \infty, \textit{ and} \\
&|f(x)| < \infty \textit{ a.e.},
\end{aligned}$$

then $f \in L^p(X)$ and f_n converges strongly to f in $L^p(X)$.

Theorem A.20 (Dominated convergence theorem, [17]). *Let $p \in [1, \infty)$ and f_n be a sequence of functions in $L^p(\Omega)$. If $f_n \to f$ a.e. in Ω for some measurable function f and there exists a nonnegative function $g \in L^p(\Omega)$ such that $|f_n(x)| \leq g(x)$ a.e. in Ω, then $f \in L^p(\Omega)$ and $f_n \to f$ in $L^p(\Omega)$.*

The partial converse is also true:

Theorem A.21 ([11]). *Let f_n be a sequence of functions in $L^p(\Omega)$ and $f \in L^p(\Omega)$ such that $f_n \to f$ in $L^p(\Omega)$. Then there exists a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e. in Ω and $|f_{n_k}(x)| \leq g(x)\ \forall\ k \in \mathbb{N}$ a.e. in Ω with $g \in L^p(\Omega)$.*

Theorem A.22 (Gauß' Theorem or Divergence Theorem, [3]). *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with $\partial\Omega \in C^1$ and outer normal vector n, which is defined on $\partial\Omega$. Let $f \in C^1(\overline{\Omega}; \mathbb{R}^d)$ be a vector valued function. Then*

$$\int_\Omega div\ f\ dx = \int_{\partial\Omega} f \cdot n\ ds.$$

Lemma A.23 (Fatou's Lemma [17]). *For every sequence of nonnegative, measurable functions $f_n : \Omega \to \mathbb{R}$ it holds:*

$$\int_\Omega \liminf_{n\to\infty} f_n(x)\ dx\ \leq\ \liminf_{n\to\infty} \int_\Omega f_n(x)\ dx.$$

Theorem A.24 (Localization theorem). *Let $f : \Omega \to \mathbb{R}$ be a vector valued function, which is continuous on the open set $\Omega \subset \mathbb{R}^d$ and let B be an arbitrary subset of Ω, then*

$$\int\limits_B f(x)\ dx = 0,\ \forall\ B \subset \Omega,$$

implies that $f(x) = 0,\ \forall\ x \in \Omega$.

Definition A.25. The Levi-Civita symbol, or Epsilon tensor, represents a collection of numbers. It has indices $1, ..., d$, for the space dimension d. It is defined by its antisymmetric properties:

- $\epsilon_{12...d} = 1$,
- $\epsilon_{ij...u...v...} = -\ \epsilon_{ij...v...u...}$.

The second property yields that the Epsilon tensor is zero if two indices are equal,

$$\epsilon_{ij...v...v...} = 0.$$

Using the antisymmetric property of the tensor, we can easily rewrite the i^{th} component of the outer product of two vectors $v, u \in \mathbb{R}^3$ in terms of ϵ,

$$(v \times u)_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} v_j u_k.$$

Definition A.26 (Einstein's summing convention)**.** In any expression in which an index appears twice in a term, the expression is assumed to be summed over all possible values of this index.

Example A.27. The i^{th} component of the outer product can be written as

$$(v \times u)_i = \epsilon_{ijk} v_j u_k,$$

by Einstein's summing convention.

Theorem A.28 (Gagliardo-Nirenberg inequality)**.** *Let u belong to $L^q(\mathbb{R}^d)$ and its derivatives of order m, $D^m u$, belong to L^r, $1 \leq q, r \leq \infty$. The following inequality holds for the derivatives $D^j u$, $0 \leq j < m$,*

$$\left|\left|D^j u\right|\right|_{L^p} \leq C \left|\left|D^m u\right|\right|_{L^r}^{\alpha} \left|\left|u\right|\right|_{L^q}^{1-\alpha},$$

where

$$\frac{1}{p} = \frac{j}{d} + \alpha\left(\frac{1}{r} - \frac{m}{d}\right) + (1-\alpha)\frac{1}{q},$$

for all α in the interval

$$\frac{j}{m} \leq \alpha \leq 1.$$

For a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary (i.e. $\partial\Omega \in C^1$) the result is true if we add the term $C\left|\left|u\right|\right|_{L^{\tilde{q}}(\Omega)}, \tilde{q} > 0$ to the right-hand side of the inequality.

Remark A.29. This remark is used to prove the inequality claimed in the proof of Theorem 2.18, namely

$$||\nabla h_n||_{L^2(0,T;(L^3(\Omega))^2)} \\ \leq ||\nabla h_n||^{2/3}_{L^\infty(0,T;(L^2(\Omega))^2)} \cdot \left|\left|\nabla^2 h_n\right|\right|^{1/3}_{L^2(0,T;(L^2(\Omega))^4)} + C\,||\nabla h_n||_{L^\infty(0,T;(L^2(\Omega))^2)}\,.$$

We will apply the Gagliardo-Nirenberg inequality (Theorem A.28) to $u(x) = \nabla h_n$. We have $\nabla h_n \in L^\infty(0,T;(L^2(\Omega))^2)$ and $\nabla^2 h_n \in L^2(0,T;(L^2(\Omega))^4)$, therefore, the dimension $d = 2$, $q = 2$, $r = 2$ and $m = 1$. We want to bound ∇h_n, thus, we set $j = 0$. Then $0 = j < m = 1$. The constant α must be in the interval,

$$0 \leq \alpha \leq 1,$$

we will take $\alpha = 1/3$. Therefore,

$$\frac{1}{p} = 0 + \frac{1}{3}\left(\frac{1}{2} - \frac{1}{2}\right) + \frac{2}{3}\cdot\frac{1}{2},$$

yields $p = 3$. The Gagliardo-Nirenberg inequality reads as

$$||\nabla h_n||_{(L^3(\Omega))^2} \leq ||\nabla h_n||^{2/3}_{(L^2(\Omega))^2} \cdot \left|\left|\nabla^2 h_n\right|\right|^{1/3}_{(L^2(\Omega))^4} + C\,||\nabla h_n||_{(L^2(\Omega))^2}\,.$$

The last term is necessary since we have a bounded domain $\Omega \subset \mathbb{R}^d$ (which is assumed to have a smooth boundary), where we chose $\tilde{q} = 2$.

If we now use the inequality derived above for the hyperbolic Sobolev space $L^2(0,T;(L^3(\Omega))^2)$, we obtain:

$$||\nabla h_n||_{L^2(0,T;(L^3(\Omega))^2)} \\ \leq ||\nabla h_n||^{2/3}_{L^\infty(0,T;(L^2(\Omega))^2)} \cdot \left|\left|\nabla^2 h_n\right|\right|^{1/3}_{L^2(0,T;(L^2(\Omega))^4)} + C\,||\nabla h_n||_{L^\infty(0,T;(L^2(\Omega))^2)}\,.$$

Definition A.30. The indicator function of a subset B of a set X is defined as

$$\mathbb{1}_B(x) := \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases}$$

Theorem A.31 (Hölder's Inequality, [14])**.** *Assume* $1 \leq p, q \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Let* $u \in L^p(\Omega), v \in L^q(\Omega)$*, then*

$$||uv||_{L^1(\Omega)} \leq ||u||_{L^p(\Omega)} ||v||_{L^q(\Omega)}.$$

Corollary A.32 (Lyapunov's Inequality, [14])**.** *Assume* $1 \leq s \leq r \leq t \leq \infty$ *and*

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}.$$

Suppose $u \in L^s(\Omega) \cap L^t(\Omega)$*. Then* $u \in L^r(\Omega)$ *and*

$$||u||_{L^r(\Omega)} \leq ||u||^{\theta}_{L^s(\Omega)} ||u||^{1-\theta}_{L^t(\Omega)}.$$

Remark A.33. Hölder's inequality can be generalized for p, q, r with

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$

Then it holds for $u \in L^q, v \in L^r$ that

$$||uv||_{L^p(\Omega)} = |||uv|^p||^{1/p}_{L^1(\Omega)} \leq |||u|^p||^{1/p}_{L^{q/p}(\Omega)} |||v|^p||^{1/p}_{L^{r/p}(\Omega)} = ||u||_{L^q(\Omega)} ||v||_{L^r(\Omega)}.$$

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