Die approbierte Originalversion dieser Diplom-/Masterarbeit ist in der Hauptbibliothek der Technischen Universität Wien aufgestellt und zugänglich.
http://www.ub.tuwien.ac.at

The approved original version of this diploma or master thesis is available at the main library of the Vienna University of Technology.
http://www.ub.tuwien.ac.at/eng

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Unterschrift des Betreuers

DIPLOMARBEIT

Monoids, Automata and Related Constructions in Monoidal Categories

ausgeführt am

Institut für Analysis und Scientific Computing
der Technischen Universität Wien

unter Anleitung von

ao. Univ. Prof. Dr. Wolfgang Herfort

durch

Manuel Eder
Karmeliterhofgasse 2/6
1150 Wien

Datum

Unterschrift (Student)

This work is licensed under the Creative Commons Attribution 4.0 International License. To view a copy of this license, visit http://creativecommons.org/licenses/by/4.0/.

Abstract

We collect the concepts and tools necessary to describe monoids, monoid acts and automata in some abstract category **C**.

We give the details of a proof of the coherence theorem for monoidal categories suggested in [Mac98] and develop a visual calculus for equational reasoning in monoidal categories that simplifies calculations in comparison to the naïve approach, bringing them close to the ease with which such calculations are carried out in the category **Set** of sets.

We sum up and give proofs of results that describe limits, colimits and free objects in the various categories of monoids and monoid acts internal to our abstract category **C**. In fact the results we get describe these structures in any category corresponding to what would in **Set** be the category of a variety of algebras for which the universally quantified variables in the equations describing that variety are in the same order on both sides of every equation (this restriction leads to a more "economical" description than can be given in the general case). This description works in terms of limits and colimits in the ambient category **C**.

Finally we relate the category of automata to the category of biacts via a functor from the former to the latter and present the construction of the tensor product of biacts over some monoid, showing that our functor takes serial composition of automata to the tensor product of biacts.

Contents

1 Introduction

1.1 Overview

The process which lead to the creation of this paper was born from the desire to understand the structure of limits, colimits and other categorical structures in the various categories of monoid acts and automata — with the hope that such an understanding might be beneficial in developing a theory of connected automata which maneuver each other into specific states — expecting that one might be able to say something about the effect that the (co-)limit process has on this property of manouvering each other into some specific state. This could have had applications in the development of algorithms which generate solvers for pathfinding problems in automata which are (co-)limits of other automata from solvers for these other automata.

We will not develop such a theory in this paper and it is also not clear to the author at this point whether this is possible or not. Nonetheless, in the process of trying to understand these categories of automata/actions (the two are closely related in a number of possible ways) some things have come to light which seemed worth writing down.

Monoids, acts and algebraic objects in general, although usually presented as *sets* with some additional structure can be described in a more general context. In fact very little additional structure is needed in a category to be able to describe what for example a monoid is.

Remember that a monoid in the classical sense is a set M together with an associative binary operation $(\cdot) : M \times M \to M$ and a specified element $1 \in M$ which acts as both a left and right unit of $(\cdot)$. That is, $\langle M, (\cdot), 1 \rangle$ form a monoid if

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad \text{and} \qquad 1 \cdot a = a \cdot 1 = a \tag{1}$$

for all $a, b, c \in M$.

These equations for a monoid can also be expressed as the commutativity of the following diagrams

$$\begin{tikzcd}
M \times M \arrow[d, "(\cdot)"] & M \times M \times M \arrow[l, "(\cdot)\times 1_M"'] \arrow[d, "1_M \times (\cdot)"] \\
M & M \times M \arrow[l, "(\cdot)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\{*\} \times M \arrow[r, "\eta \times 1_M"] \arrow[dr, "\lambda_M"'] & M \times M \arrow[d, "(\cdot)"] & M \times \{*\} \arrow[l, "1_M \times \eta"'] \arrow[dl, "\rho_M"] \\
& M &
\end{tikzcd}$$

(2)

Here η is the map which maps the element of the one point set $\{*\}$ to the element $1 \in M$, and we will write 1_M for the identity map on M. ρ_M and λ_M are the isomorphisms $\rho_M(m, *) := m$ and $\lambda_M(*, m) := m$.

As is also done in for example [Mac98] we will transport the concept of a monoid (and of other algebraic structures) from the category **Set** to some other category **C** by taking the diagrams in (2) and interpreting them in the category **C**. One can already guess what will be required of the category **C** for these diagrams to make sense — we will need some functor that takes the role of the product and for this functor there should be an object of the category which is a kind of "identity" (the one point set in **Set**). Also, it makes sense to require that this functor be “associative”. (All of this will be made precise in the next part.)

It turns out that very little about the description which is usually given of limits and colimits in the category of monoids and other algebraic categories actually depends on the ambient category being **Set**. (For a good explicit account of the structure of limits and colimits in categories of acts in the case where the ambient category is **Set** see [KKM00].)

In this paper we work out what this description looks like in an arbitrary ambient category with the appropriate structures to support talking about algebraic objects. A lot of what is familiar carries over. Still, some work has to be done because many things — like for example associativity of the product up to isomorphism and “in the right way” — that are so obvious (or maybe just familiar) in the category of sets that one does not usually even mention them, one cannot take for granted in an abstract category described only by axioms.

Sections 2 and 3 are dedicated to working out the details of what one has to think about when describing algebraic structures in arbitrary categories.
In section 2 we give the definition of a monoidal category (taken from [Mac98]) which will be the structure inside which all other considerations in this paper take place and we give the definitions of the various algebraic structures we will be working with.
In section 3 we present a proof of the coherence theorem for monoidal categories (a fleshed out version of a proof suggested in an exercise in [Mac98]) — a theorem which essentially says that the definition of a monoidal category makes sense and models what we were hoping it would model — and develop a visual calculus which is useful for calculations in monoidal categories and reduces the cognitive burden of carrying out these calculations (and allowed the author to draw pretty pictures under the pretext of doing mathematics).

In section 4 we take a moment to lay out how the following parts 5, 6 and 7 will combine to give a picture of limits, colimits and free objects in categories of algebras internal to monoidal categories. As an application of the calculus developed in part 3 and also as a preliminary to and a justification for considering some of the structures we talk about in parts 6 and 7 we give a proof of the generalized associative law for monoids in monoidal categories. This proof — although not very surprising or complicated — is interesting because it hints at the nature of the connection between monoids and monoidal categories. Ignoring some of the technical details one could even say that we reuse the same proof that was used for the coherence theorem. We take a moment to discuss this connection, albeit in informal terms.

In section 5 we state and prove a number of general theorems about limits/colimits. These can be found elsewhere but it seemed useful to include them here for the reader not familiar with them as we will make heavy use of them later.

Section 6 is dedicated to monads and the structure of limits and colimits in the category of Eilenberg-Moore algebras for some monad. A lot can be said about this topic and we focus on what is important for us in this context. We recount a theorem about colimits in categories of Eilenberg-Moore algebras due to [Lin69] — a theorem which gives colimits in terms of free objects and corresponds very closely to the usual explicit description of for example the coproduct of groups in the category of sets.

Free objects in the categories $\mathbf{Mon_C}$, $\mathbf{Act_C}$ and $\mathbf{BiAct_C}$ of monoids, acts

and biacts respectively (and many others), which are considered internal to some category $\mathbf{C}$, are constructed in section 7. These are interesting in their own right, but they also consitute the final piece of the puzzle in the description of colimits in these categories.
The construction we give is inspired by and can in fact be seen as a direct generalization of the construction of the free monoid in the general setting of an arbitrary ambient category, as it can be found in for example [Mac98, section VII.3]. Although the general case is a little unwieldier than the special case of monoids (but not unreasonably so) it has the benfit of making it clearer *why* the free monoid can be constructed in this manner. Remember that although free objects in categories of algebras in **Set** can always be described as an appropriate factor set of the set of all trees which can be constructed from operator symbols and generating elements, the free monoid has a more concise description which does not require taking any factor sets. The construction we give shares this property of requiring only coproducts and no coequalizers (provided the product of the ambient category gets along well enough with the coproduct of the same).

In section 8 we describe one way of representing automata as biacts. We describe a bifunctor on the category of automata which we call serial composition and which intuitively corresponds to serial composition of automata in the category of sets. We also construct the tensor product of biacts and show that the functor giving the representation of automata as biacts takes the operation of serial composition to the operation of taking the tensor product. In this section we will again be making heavy use of the visual calculus developed in section 3.

Section 9 consists of remarks and ideas for future work. We hint at some of the results that one would expect to hold for the operations of serial composition and tensor product and which could probably be shown with reasonable effort. We discuss how some of the properties of these two operations, which are starting to come into view, suggest a reworking in more general terms of parts of the theory that we have seen in this paper.

1.2 On notation

We use boldface to denote categories, so **Set** is the category of sets and $\mathbf{C}$ will be the default name for an arbitrary category (often with additional structure) relative to which our considerations take place. $\mathbf{I}$ and $\mathbf{J}$ will be used to denote categories that conceptually play the role of index categories — for example categories that appear as the domain of a functor whose limit we are interested in.

We use lower case letters a, b, c, d from the beginning of the latin alphabet to denote objects of a category and lower-case letters f, g, h, k to denote morphisms of a category. For some categories which play a special role we will use a different typeface for the objects and morphisms.

We write $\mathrm{Obj}(\mathbf{C})$ for the set/class of objects of the category $\mathbf{C}$ and $\mathrm{Arr}(\mathbf{C})$ for the set/class of arrows of the category $\mathbf{C}$. Often we will write $a \in \mathbf{C}$ to mean $a \in \mathrm{Obj}(\mathbf{C})$ and f in $\mathbf{C}$ to mean $f \in \mathrm{Arr}(\mathbf{C})$.

For f in $\mathbf{C}$ we use $s(f)$ to denote the source of the arrow f and $t(f)$ to denote the target of f.

For $a, b \in \mathbf{C}$ we write $f : a \to b$ when we mean that f is an arrow of $\mathbf{C}$ and that $s(f) = a$ and $t(f) = b$.

When $f : a \to b$ and $g : b \to c$ we write $g{\circ}f$ for the composite "first f then g".
With regard to directions this is the traditional convention which is compatible with writing the argument that a function is applied to to the right of that function. To lessen the cognitive burden when switching between reading diagrams and reading formulas we will usually draw arrows in diagrams from the right to the left (and from top to bottom). This way the order of arrows in the picture

$$c \xleftarrow{\quad g \quad} b \xleftarrow{\quad f \quad} a$$

is the same order that we get when writing out the composite $g{\circ}f$.
We use 1_a to denote the identity morphism on a.

We use $\hom_{\mathbf{C}}(a,b)$ or $\hom(a,b)$ for the hom-set of arrows $f : a \to b$ in $\mathbf{C}$.

We use upper case letters for functors (upper case letters are also used for sets). $F : \mathbf{C} \to \mathbf{C}'$ means that F is a functor from the category $\mathbf{C}$ to the category $\mathbf{C}'$. If additionally $G : \mathbf{C}' \to \mathbf{C}''$ then we write $G \circ F$ for their composite.

We use parentheses to denote application of functors (and also of functions). $F(a)$ is the object of $\mathbf{C}'$ which is the result of applying F to the object $a \in \mathbf{C}$ and $F(f)$ is the morphism of $\mathbf{C}'$ which is the result of applying F to the morphism f in $\mathbf{C}$.
In addition we of course use parentheses for disambiguation.

We usually use greek letters to denote natural transformations. We write $\sigma : F \dot{\to} G : \mathbf{C} \to \mathbf{C}'$ when we mean that σ is a natural transformation from the functor F to the functor G and that F and G are functors from $\mathbf{C}$ to $\mathbf{C}'$. When the domain and codomain of F and G are clear we may omit the second part to write only $\sigma : F \dot{\to} G$.
We use a subscript to denote the value of the natural transformation at a certain object $a \in \mathbf{C}$ as in e.g. $\sigma_a : F(a) \to G(a)$.
If $\sigma : F \dot{\to} G : \mathbf{C} \to \mathbf{C}'$ and $\tau : G \dot{\to} H : \mathbf{C} \to \mathbf{C}'$, then we write $\tau \bullet \sigma$ for the vertical composite "first σ then τ" — that is $\tau \bullet \sigma : F \dot{\to} H : \mathbf{C} \to \mathbf{C}'$ and $(\tau \bullet \sigma)_a = \tau_a {\circ} \sigma_a$.
We use the same symbol as for functor composition for the "horizontal" composite of natural transformations — that is, if $\sigma : F \dot{\to} G : \mathbf{C} \to \mathbf{C}'$ and $\tau : F' \dot{\to} G' : \mathbf{C}' \to \mathbf{C}''$, then $\tau \circ \sigma : F' \circ F \dot{\to} G' \circ G : \mathbf{C} \to \mathbf{C}''$ is defined by

$$\begin{aligned}(\tau \circ \sigma)_a &= \tau_{G(a)} {\circ} (F'(\sigma_a)) \\ &= (G'(\sigma_a)) {\circ} \tau_{F(a)} \ .\end{aligned}$$

As is customary we will use $\tau \circ F$ as an abbreviation for $\tau \circ 1_F$ and $F' \circ \sigma$ as an abbreviation for $1_{F'} \circ \sigma$ — that is

$$\begin{aligned}(\tau \circ F)_a &= \tau_{F(a)} \\ (F' \circ \sigma)_a &= F'(\sigma_a) \ .\end{aligned}$$

$\mathrm{Nat}(F, G)$ is the set of natural transformations from F to G.

We use angle brackets to denote tuples/lists, as in $\langle a, b, c \rangle$.

We use $A \times B$ to denote the product of sets A and B. We use $\mathbf{C} \times \mathbf{C}'$ to denote the product of categories $\mathbf{C}$ and $\mathbf{C}'$ and if $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{D}'$ then $F \times G : \mathbf{C} \times \mathbf{C}' \to \mathbf{D} \times \mathbf{D}'$ denotes the product-functor of F and G — that is, the functor which sends an object $\langle a, a' \rangle \in \mathbf{C} \times \mathbf{C}'$ which consists of objects $a \in \mathbf{C}$ and $a' \in \mathbf{C}'$ to the object $\langle F(a), G(a') \rangle$ of $\mathbf{D} \times \mathbf{D}'$, and likewise for morphisms.

We will usually silently identify the categories $(\mathbf{C} \times \mathbf{C}') \times \mathbf{C}''$ and $\mathbf{C} \times (\mathbf{C}' \times \mathbf{C}'')$ — writing just $\mathbf{C} \times \mathbf{C}' \times \mathbf{C}''$ for this category. In a similar vein, if $F : \mathbf{C} \times \mathbf{C}' \to \mathbf{D}$ then we will usually write $F(a, a')$ instead of $F(\langle a, a' \rangle)$ for the result of applying F to the object $\langle a, a' \rangle \in \mathbf{C} \times \mathbf{C}'$ and likewise for morphisms.

Some of the structures we describe consist of multiple levels of nested lists. There we do try to be exact and keep e.g. $\langle a, b, \langle c, d \rangle \rangle$ distinct from $\langle a, b, c, d \rangle$ to avoid confusion.

For tuples we will — in addition to ellipsis-notation $\langle F(a_1), \ldots, F(a_n) \rangle$ — use the notation $\langle F(a_i) \rangle_{i=1}^{n}$ to mean the same thing. For space reasons we will also sometimes omit the bounds, as in $\langle F(a_i) \rangle_i$. We will sometimes also have an index run "in the other direction", so e.g. $\langle \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_i} \rangle_{i=k}^{1}$ means

$$\Big\langle \langle \mathfrak{f}_{k,1}, \ldots, \mathfrak{f}_{k,n_k} \rangle, \big\langle \mathfrak{f}_{k-1,1}, \ldots, \mathfrak{f}_{k-1,n_{k-1}} \big\rangle, \ldots, \langle \mathfrak{f}_{1,1}, \ldots, \mathfrak{f}_{1,n_1} \rangle \Big\rangle \ .$$

$\langle a_{i,j} \rangle_{j=1}^{n_i}{}_{i=1}^{k}$ means the list

$$\langle a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}, a_{2,1}, \ldots, a_{2,n_2}, \ldots, a_{k,1}, \ldots, a_{k,n_k} \rangle$$

(no nesting). A single dot "$\cdot$" denotes concatenation of tuples/lists, e.g.

$$\langle a_{1,j} \rangle_{j=1}^{n_1} \cdot \langle a_{2,j} \rangle_{j=1}^{n_2} = \langle a_{i,j} \rangle_{j=1}^{n_i}{}_{i=1}^{2} \ .$$

$\ell(l)$ denotes the length of a tuple

$$\ell(\langle a_1, \ldots, a_n \rangle) := n \ .$$

If $\star$ is an associative binary operator, then we will use the notation

$$\star_{i=1}^{n} (f_i) := f_1 \star f_2 \star \cdots \star f_n \ .$$

For the product (of categories or sets) we use the more traditional

$$\prod_{i=1}^{n} \mathbf{C}_i := \mathbf{C}_1 \times \cdots \times \mathbf{C}_n \ .$$

For a set X we will use X^* to denote the set of all sequences with letters in X, that is

$$X^* := \dot{\cup}_{i=0}^{\infty} \left(\prod_{j=1}^{i} X \right) \ .$$

2 A first look at monoids and monoidal categories

2.1 Monoidal categories

In this section we give the definition of a monoidal category. We will later develop some of the theory of monoidal categories. For additional background and context the reader can also refer to [Mac98, chapter VII].

The tuple $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$, where $\mathbf{C}$ is a category, $\boxtimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a bifunctor and $\alpha_{a,b,c} : a \boxtimes (b \boxtimes c) \to (a \boxtimes b) \boxtimes c$, $\lambda_a : e \boxtimes a \to a$, $\rho_a : a \boxtimes e \to a$ are natural isomorphisms, is called a monoidal category if the diagrams below commute for all $a, b, c, d \in \mathbf{C}$.

$$\begin{tikzcd}
((a \boxtimes b) \boxtimes c) \boxtimes d & (a \boxtimes b) \boxtimes (c \boxtimes d) \arrow[l, "\alpha_{a \boxtimes b, c, d}"'] & a \boxtimes (b \boxtimes (c \boxtimes d)) \arrow[l, "\alpha_{a,b,c \boxtimes d}"'] \arrow[d, "1_a \boxtimes \alpha_{b,c,d}"] \\
(a \boxtimes (b \boxtimes c)) \boxtimes d \arrow[u, "\alpha_{a,b,c} \boxtimes 1_d"] & & a \boxtimes ((b \boxtimes c) \boxtimes d) \arrow[ll, "\alpha_{a, b \boxtimes c, d}"]
\end{tikzcd} \tag{3}$$

$$\begin{tikzcd}
(a \boxtimes e) \boxtimes b \arrow[dr, "\rho_a \boxtimes 1_b"'] & & a \boxtimes (e \boxtimes b) \arrow[ll, "\alpha_{a,e,b}"'] \arrow[dl, "1_a \boxtimes \lambda_b"] \\
& a \boxtimes b &
\end{tikzcd} \tag{4}$$

A monoidal category where α, λ and ρ are identities (and therefore $a \boxtimes (b \boxtimes c) = (a \boxtimes b) \boxtimes c$ and $e \boxtimes a = a \boxtimes e = a$) is called a strict monoidal category. $\boxtimes$ is called the tensor product or just the product of the monoidal category and we will call α, λ and ρ the (basic) structural transformations of the monoidal category.

If the selection of diagrams here seems a little arbitrary that is because it is. What we actually think of when we talk about a monoidal category is a category where all diagrams involving only α, λ and ρ commute. We shall soon see that the diagrams (3) and (4) imply just that.

Note that the definition of a monoidal category is symmetric. Specifically, if we define $a \boxtimes' b := b \boxtimes a$, $\alpha'_{a,b,c} := \alpha^{-1}_{c,b,a}$, $\lambda'_a := \rho_a$ and $\rho'_a := \lambda_a$, then $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ is a monoidal category if and only if $\langle \mathbf{C}', \boxtimes', e', \alpha', \lambda', \rho' \rangle$ is a monoidal category. This will often come in useful in situations where there is a "left" and a "right" version of some structure — allowing us to treat only one case, because the other case can be seen as the first case in the category $\langle \mathbf{C}', \boxtimes', e', \alpha', \lambda', \rho' \rangle$ as defined above, which we will call the *monoidally opposite category* of $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$.

Now we have the concepts to be able to define what we mean in general by a monoid and by other algebraic structures.

2.2 Monoids in a monoidal category

Definition 2.1. A monoid in a monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ is a tuple $\langle m, \mu, \eta \rangle$ consisting of an object $m \in \mathbf{C}$ and of two morphisms $\mu : m \boxtimes m \to m$

and $\eta : e \to m$ such that the two diagrams below commute.

$$\begin{tikzcd}
m \boxtimes m \arrow[d, "\mu"'] & (m \boxtimes m) \boxtimes m \arrow[l, "\mu \boxtimes 1_m"'] & m \boxtimes (m \boxtimes m) \arrow[l, "\alpha_{m,m,m}"'] \arrow[d, "1_m \boxtimes \mu"] \\
m & & m \boxtimes m \arrow[ll, "\mu"]
\end{tikzcd} \tag{5}$$

$$\begin{tikzcd}
e \boxtimes m \arrow[r, "\eta \boxtimes 1_m"] \arrow[dr, "\lambda_m"'] & m \boxtimes m \arrow[d, "\mu"] & m \boxtimes e \arrow[l, "1_m \boxtimes \eta"'] \arrow[dl, "\rho_m"] \\
& m &
\end{tikzcd} \tag{6}$$

As is the case in **Set**, the diagram (5) implies a general associative law. We will prove this fact a little later when we have developed the appropriate tools to easily deal with calculations in monoidal categories.

Note that the definition of a monoid is also symmetric. $\langle m, \mu, \eta \rangle$ is a monoid in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ if and only if it is a monoid in the monoidally opposite category.

Definition 2.2. The category $\mathbf{Mon}_{\langle \mathbf{C}, \boxtimes, \mathbf{e}, \alpha, \lambda, \rho \rangle}$ (or $\mathbf{Mon}_{\mathbf{C}}$) of monoids in the monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ has objects all monoids in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$. An arrow $g : \langle m, \mu, \eta \rangle \to \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$ is an arrow g in $\mathbf{C}$ such that

$$\begin{tikzcd}
\dot{m} \boxtimes \dot{m} \arrow[d, "\dot{\mu}"] & m \boxtimes m \arrow[l, "g \boxtimes g"] \arrow[d, "\mu"] \\
\dot{m} & m \arrow[l, "g"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& e \arrow[dl, "\dot{\eta}"'] \arrow[d, "\eta"] \\
\dot{m} & m \arrow[l, "g"]
\end{tikzcd} \tag{7}$$

commute.

Remark 2.3. This is not a completely formal definition. When we say that an arrow $g : \langle m, \mu, \eta \rangle \to \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$ is an arrow g in $\mathbf{C}$ with certain properties, we mean this as a shortcut for saying that an arrow g of $\mathbf{Mon}_{\mathbf{C}}$ is given by its source $\langle m, \mu, \eta \rangle$, its target $\langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$ and an arrow $\dot{g} : m \to \dot{m}$ with those properties; and that moreover composition of arrows of $\mathbf{Mon}_{\mathbf{C}}$ is given by composition of the underlying arrows in $\mathbf{C}$ and the identity arrow on $\langle m, \mu, \eta \rangle$ is the arrow with underlying arrow 1_m in $\mathbf{C}$.

We will use similar language in defining our other categories of algebraic structures, in the conviction that the reader will be able to fill in the details of a formal definition themselves.

2.3 Left acts, right acts and biacts

Definition 2.4. A *left action* of the monoid $\langle m, \mu, \eta \rangle$ in the monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ on the object $a \in \mathbf{C}$ is a morphism $h : m \boxtimes a \to a$ of $\mathbf{C}$ such that the following diagrams commute.

$$\begin{tikzcd} m \boxtimes a \arrow[d, "h"'] & (m \boxtimes m) \boxtimes a \arrow[l, "\mu \boxtimes 1_a"'] & m \boxtimes (m \boxtimes a) \arrow[l, "\alpha_{m,m,a}"'] \arrow[d, "1_m \boxtimes h"] \\ a & & m \boxtimes a \arrow[ll, "h"] \end{tikzcd} \qquad \begin{tikzcd} m \boxtimes a \arrow[d, "h"'] & e \boxtimes a \arrow[l, "\eta \boxtimes 1_a"'] \arrow[dl, "\lambda_a"] \\ a & \end{tikzcd} \tag{8}$$

We call the pair $\langle h, a\rangle$ a left $\langle m, \mu, \eta\rangle$-act, or just a *left (monoid) act*, when $\langle m, \mu, \eta\rangle$ is clear or not relevant. We can define a right action $\langle b, k\rangle$ ($b \in \mathbf{C}$, $k : b \boxtimes m \to b$) of the monoid $\langle m, \mu, \eta\rangle$ in an analogous manner. This will amount to the same thing as saying that a right action of $\langle m, \mu, \eta\rangle$ in the category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ is a left action of $\langle m, \mu, \eta\rangle$ in the monoidally opposite category of $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$.

As a mnemonic device we will write the morphism of a left action on the left hand side of the object and the morphism of a right action on the right hand side of the object in the tuple.

Definition 2.5. The category $\langle m, \mu, \eta\rangle-\mathbf{Act}$ of left $\langle m, \mu, \eta\rangle$-acts has objects all left $\langle m, \mu, \eta\rangle$-acts and if $\langle h, a\rangle, \langle h', a'\rangle$ are left $\langle m, \mu, \eta\rangle$-acts then $f : \langle h, a\rangle \to \langle h', a'\rangle$ is a morphism of left $\langle m, \mu, \eta\rangle$-acts if $f : a \to a'$ and

$$\begin{tikzcd} m \boxtimes a' \arrow[d, "h'"'] & m \boxtimes a \arrow[l, "1_m \boxtimes f"'] \arrow[d, "h"] \\ a' & a \arrow[l, "f"] \end{tikzcd} \tag{9}$$

commutes.

The category $\mathbf{Act}-\langle m, \mu, \eta\rangle$ of right $\langle m, \mu, \eta\rangle$-acts is defined in an analogous manner.

Definition 2.6. Let $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ be a monoidal category. The category $\mathbf{Act}_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle}$ (or $\mathbf{Act}_{\mathbf{C}}$ for short) of left acts in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ is the category whose objects are tuples $\langle \langle m, \mu, \eta\rangle, \langle h, a\rangle\rangle$ such that $\langle m, \mu, \eta\rangle$ is a monoid in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ and $\langle h, a\rangle$ is a left $\langle m, \mu, \eta\rangle$-act and in which a morphism $\langle g, f\rangle : \langle \langle m, \mu, \eta\rangle, \langle h, a\rangle\rangle \to \langle \langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle, \langle h', a'\rangle\rangle$ consists of two arrows $g : m \to \dot{m}$ and $f : a \to a'$ such that g is a morphism of monoids and such that the following diagram commutes.

$$\begin{tikzcd} \dot{m} \boxtimes a' \arrow[d, "h'"] & m \boxtimes a \arrow[l, "g \boxtimes f"] \arrow[d, "h"] \\ a' & a \arrow[l, "f"] \end{tikzcd} \tag{10}$$

The category **ʇɔA**$_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle}$ of right monoid acts in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ is defined in an analogous manner.

We will also study the case where we have both a left and a right action on an object $a \in \mathbf{C}$. In this case some nice-to-have properties will depend on these actions being compatible in a certain sense.

Definition 2.7. When $\langle m, \mu, \eta\rangle$ and $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$ are monoids in a monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$, $a \in \mathbf{C}$, $h : m \boxtimes a \to a$ and $k : a \boxtimes \dot{m} \to a$, then we say that $\langle h, a, k\rangle$ is an $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-biact if $\langle h, a\rangle$ is a left $\langle m, \mu, \eta\rangle$-act, $\langle a, k\rangle$ is a right $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-act and the diagram

$$\begin{tikzcd} m \boxtimes a \arrow[dd, "h"] & m \boxtimes (a \boxtimes \dot{m}) \arrow[l, "1_m \boxtimes k"'] \arrow[d, "\alpha_{m,a,\dot{m}}"] \\ & (m \boxtimes a) \boxtimes \dot{m} \arrow[d, "h \boxtimes 1_{\dot{m}}"] \\ a & a \boxtimes \dot{m} \arrow[l, "k"] \end{tikzcd} \tag{11}$$

commutes.

Comparing (11) with (9) one might be tempted to say that an $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-biact is just a right $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-act in the category of left $\langle m, \mu, \eta\rangle$-acts. Or replacing (11) with the equivalent (because α is an isomorphism) diagram

$$\begin{tikzcd} a \boxtimes \dot{m} \arrow[dd, "k"] & (m \boxtimes a) \boxtimes \dot{m} \arrow[l, "h \boxtimes 1_{\dot{m}}"'] \arrow[d, "\alpha^{-1}_{m,a,\dot{m}}"] \\ & m \boxtimes (a \boxtimes \dot{m}) \arrow[d, "1_m \boxtimes k"] \\ a & m \boxtimes a \arrow[l, "h"] \end{tikzcd}$$

that an $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-biact is a left $\langle m, \mu, \eta\rangle$-act in the category of right $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-acts.

But with our current definitions this statement will not make sense in a lot of cases because it would mean that we have a monoidal product on the category of left $\langle m, \mu, \eta\rangle$-acts and that $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$ is itself a $\langle m, \mu, \eta\rangle$-act. This may not necessarily be the case and it is also not really what we mean. We will see in section 8.2 how to adapt our definitions so that we can say that an $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-biact is a left $\langle m, \mu, \eta\rangle$-act in the category of right $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-acts.

Definition 2.8. Let $\langle m, \mu, \eta\rangle$ and $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$ be monoids in a monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$. The category $\langle m, \mu, \eta\rangle{-}\mathbf{Act}{-}\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$ of $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-biacts has objects all $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-biacts and a morphism $f : \langle h, a, k\rangle \to \langle h', b, k'\rangle$ of $\langle m, \mu, \eta\rangle{-}\mathbf{Act}{-}\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$ is a morphism $f : a \to b$ of $\mathbf{C}$ which is both a morphism of left acts and a morphism of right acts.

Definition 2.9. Let $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ be a monoidal category. The category $\mathbf{BiAct}_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle}$ (or $\mathbf{BiAct}_{\mathbf{C}}$) of biacts in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ has objects all tuples $\langle\, \langle m, \mu, \eta\rangle, \langle h, a, k\rangle, \langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle\rangle$ such that $\langle m, \mu, \eta\rangle$, $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$ are monoids and $\langle h, a, k\rangle$ is an $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-biact. A morphism

$$\langle g, f, \dot{g}\rangle \colon \langle\, \langle m, \mu, \eta\rangle, \langle h, a, k\rangle, \langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle\rangle \to \langle\, \langle m', \mu', \eta'\rangle, \langle h', b, k'\rangle, \langle \dot{m}', \dot{\mu}', \dot{\eta}'\rangle\rangle$$

of $\mathbf{BiAct}_{\mathbf{C}}$ consists of two morphisms of monoids $g : \langle m, \mu, \eta\rangle \to \langle m', \mu', \eta'\rangle$, $\dot{g} : \langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle \to \langle \dot{m}', \dot{\mu}', \dot{\eta}'\rangle$ and a morphism $f : a \to b$ of $\mathbf{C}$ which combines with

g and $\dot{g}$ to yield a morphism $\langle g, f\rangle : \langle\, \langle m, \mu, \eta\rangle, \langle h, a\rangle\rangle \to \langle\, \langle m', \mu', \eta'\rangle, \langle h', b\rangle\rangle$ of $\mathbf{Act_C}$ and a morphism $\langle f, \dot{g}\rangle : \langle\, \langle a, k\rangle, \langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle\rangle \to \langle\, \langle b, k'\rangle, \langle \dot{m}', \dot{\mu}', \dot{\eta}'\rangle\rangle$ of $\mathbf{ʇɔA}_{\mathbf{C}}$ respectively.

2.4 Automata

Definition 2.10. Let $\langle m, \mu, \eta\rangle$ and $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$ be monoids in a monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$. An $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-automaton is a tuple $\langle a, \delta\rangle$ where $a \in \mathbf{C}$ and $\delta : m \boxtimes a \to a \boxtimes \dot{m}$ such that the following diagrams commute.

$$\begin{tikzcd}
m \boxtimes a \arrow[ddd, "\delta"] & (m \boxtimes m) \boxtimes a \arrow[l, "\mu\boxtimes 1_a"'] & m \boxtimes (m \boxtimes a) \arrow[l, "\alpha_{m,m,a}"'] \arrow[d, "1_m\boxtimes\delta"] \\
& & m \boxtimes (a \boxtimes \dot{m}) \arrow[d, "\alpha_{m,a,\dot{m}}"] \\
& & (m \boxtimes a) \boxtimes \dot{m} \arrow[d, "\delta\boxtimes 1_{\dot{m}}"] \\
a \boxtimes \dot{m} & a \boxtimes (\dot{m} \boxtimes \dot{m}) \arrow[l, "1_a\boxtimes\dot{\mu}"] & (a \boxtimes \dot{m}) \boxtimes \dot{m} \arrow[l, "\alpha^{-1}_{a,\dot{m},\dot{m}}"]
\end{tikzcd} \tag{12}$$

$$\begin{tikzcd}
m \boxtimes a \arrow[dd, "\delta"] & e \boxtimes a \arrow[l, "\eta\boxtimes 1_a"'] \arrow[d, "\lambda_a"'] \\
& a \\
a \boxtimes \dot{m} & a \boxtimes e \arrow[l, "1_a\boxtimes\dot{\eta}"] \arrow[u, "\rho_a"']
\end{tikzcd} \tag{13}$$

[Deu71] gives a definition of which the above definition is a generalization.

Definition 2.11. Let $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ be a monoidal category and let $\langle m, \mu, \eta\rangle$ and $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$ be monoids in that monoidal category. The category

$$\langle m, \mu, \eta\rangle{-}\mathbf{Aut}{-}\,\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$$

of $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-Automata is the category in which the objects are the $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-Automata and in which a morphism $f : \langle a, \delta\rangle \to \langle b, \vartheta\rangle$ is a morphism f of $\mathbf{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
m \boxtimes b \arrow[d, "\vartheta"] & m \boxtimes a \arrow[l, "1_m\boxtimes f"'] \arrow[d, "\delta"] \\
b \boxtimes \dot{m} & a \boxtimes \dot{m} \arrow[l, "f\boxtimes 1_{\dot{m}}"]
\end{tikzcd} \tag{14}$$

3 Coherence

In this section we try to develop a feeling for the algebraic properties of a monoidal category. To this end we will be comparing different monoidal categories. To do this we need the notion of a monoidal functor.

3.1 Monoidal functors

Definition 3.1. A *monoidal functor* is a tuple

$$\langle R, R_2, R_0\rangle : \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle \to \langle \mathbf{C}', \boxtimes', e', \alpha', \lambda', \rho'\rangle$$

where $R : \mathbf{C} \to \mathbf{C}'$ is a functor, $R_0 : e' \to R(e)$ is a morphism in $\mathbf{C}'$ and $R_{2;a,b} : R(a) \boxtimes' R(b) \to R(a \boxtimes b)$ is a natural transformation, such that the following diagrams commute.

$$\begin{tikzcd}
(R(a) \boxtimes' R(b)) \boxtimes' R(c) \arrow[d, "R_{2;a,b} \boxtimes' 1_{R(c)}"'] & & R(a) \boxtimes' (R(b) \boxtimes' R(c)) \arrow[ll, "\alpha'_{R(a),R(b),R(c)}"'] \arrow[d, "1_{R(a)} \boxtimes' R_{2;b,c}"] \\
R(a \boxtimes b) \boxtimes' R(c) \arrow[d, "R_{2;a\boxtimes b,c}"'] & & R(a) \boxtimes' R(b \boxtimes c) \arrow[d, "R_{2;a,b\boxtimes c}"] \\
R((a \boxtimes b) \boxtimes c) & & R(a \boxtimes (b \boxtimes c)) \arrow[ll, "R(\alpha_{a,b,c})"]
\end{tikzcd} \tag{15}$$

$$\begin{tikzcd}
R(a) \arrow[dd, "1_{R(a)}"] & e' \boxtimes' R(a) \arrow[l, "\lambda'_{R(a)}"'] \arrow[d, "R_0 \boxtimes' 1_{R(a)}"'] \\
& R(e) \boxtimes' R(a) \arrow[d, "R_{2;e,a}"'] \\
R(a) & R(e \boxtimes a) \arrow[l, "R(\lambda_a)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
R(a) \arrow[dd, "1_{R(a)}"] & R(a) \boxtimes' e' \arrow[l, "\rho'_{R(a)}"'] \arrow[d, "1_{R(a)} \boxtimes' R_0"'] \\
& R(a) \boxtimes' R(e) \arrow[d, "R_{2;a,e}"'] \\
R(a) & R(a \boxtimes e) \arrow[l, "R(\rho_a)"]
\end{tikzcd} \tag{16}$$

A monoidal functor is called strong if both R_0 and R_2 are isomorphisms.
The composite of two monoidal functors

$$\langle R, R_2, R_0\rangle : \mathbf{C} \to \mathbf{C}' \text{ and } \langle R', R'_2, R'_0\rangle : \mathbf{C}' \to \mathbf{C}''$$

is defined as

$$\begin{aligned}&\langle R', R'_2, R'_0\rangle \circ \langle R, R_2, R_0\rangle := \\ &\qquad \left\langle R' \circ R, \left\langle R'(R_{2;a,b}) \circ R'_{2;R(a),R(b)} \right\rangle_{a,b\in\mathbf{C}}, R'(R_0) \circ R'_0 \right\rangle\end{aligned} \tag{17}$$

The reader is invited to draw the diagrams necessary to prove that this is indeed a monoidal functor.

The diagrams in (15) and (16) all share a common shape. Horizontally we use structural transformations of the monoidal categories — those of the target category at the top and those of the source category with R applied at the bottom, while vertically we use R_0 and instances of R_2 and 1 to, informally speaking, shift all instances of the tensor product inside the functor R.

When R is a monoidal functor commutativity in fact holds for any diagram of such type. To be able to formulate this precisely we need to be able to talk about iterated products.

Definition 3.2. The set of *tensor words* is defined recursively as the (least) set which contains

- the symbol e_0,
- the symbol $(\_)$ and
- for any two tensor words v and w the tensor word $v\Box w$.

In short, tensor words are the free algebra generated by the two constants e_0 and $(\_)$ and the binary operation $\Box$ with no relations imposed.

We assign to e_0 the *length* 0, to $(\_)$ length 1 and to $v\Box w$ the sum of the lengths of v and w. We write $\ell(v)$ for the length of the tensor word v.

We introduce a family of "canonical" tensor words $v^{(n)}$ which are recursively defined by

$$\begin{aligned} v^{(0)} &:= e_0 \\ v^{(n+1)} &:= (\_)\Box v^{(n)} \ . \end{aligned}$$

Clearly $\ell\left(v^{(n)}\right) = n$.

Definition 3.3. For any tensor word v of length n and any monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ we define a functor

$$v_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} : \prod_{i=1}^{n} \mathbf{C} \to \mathbf{C}$$

(where we identify the empty product with the category $\mathbf{1}$ with exactly one object and one identity arrow for that object) by defining

- $e_{0\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} : \mathbf{1} \to \mathbf{C}$ as the functor which sends the object of $\mathbf{1}$ to e and the arrow of $\mathbf{1}$ to 1_e,
- $(\_)_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} : \mathbf{C} \to \mathbf{C}$ as the identity functor on $\mathbf{C}$ and
- $(v\Box w)_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} : \prod_{i=1}^{n} \mathbf{C} \to \mathbf{C}$ as the composite

$$\begin{aligned} &\boxtimes \circ \left(v_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} \times w_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}\right) : \\ &\qquad \prod_{i=1}^{n} \mathbf{C} = \left(\prod_{i=1}^{n_v} \mathbf{C}\right) \times \left(\prod_{i=1}^{n_w} \mathbf{C}\right) \to \mathbf{C} \times \mathbf{C} \to \mathbf{C} \ . \end{aligned}$$

Intuitively $v_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}(a_1, \ldots, a_n)$ consecutively inserts a_1 to a_n into the blanks ($\_$) in v and replaces e_0 by e and $\Box$ by $\boxtimes$. When it is clear from the context which monoidal category we are talking about, we will often abbreviate $v_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}$ to $v_{\mathbf{C}}$ or omit the subscript completely, writing $v(a_1, \ldots, a_n)$ for $v_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}(a_1, \ldots, a_n)$.

In analogy to the functor defined above we write $w\left(\langle v_i \rangle_{i=1}^{\ell(w)}\right)$ for the tensor word in which we have substituted v_i into the i-th blank in w.

Definition 3.4. For a monoidal functor $\langle R, R_2, R_0 \rangle$ and a tensor word v we recursively define the natural transformation

$$R_{v, a_1, \ldots, a_n} : v_{\langle \mathbf{C}', \boxtimes', e', \alpha', \lambda', \rho' \rangle}(R(a_1), \ldots, R(a_n)) \to R\left(v_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}(a_1, \ldots, a_n)\right)$$

by setting

$$\begin{aligned} R_{e_0} &:= R_0 \\ R_{(\_)} &:= 1 \\ R_{v \Box w, a_1, \ldots, a_{n_v}, b_1, \ldots, b_{n_w}} &:= \\ &R_{2; R(v(a_1, \ldots, a_{n_v})), R(w(b_1, \ldots, b_{n_w}))} \circ \left(R_{v, a_1, \ldots, a_{n_v}} \boxtimes' R_{w, b_1, \ldots, b_{n_w}}\right) \ . \end{aligned} \tag{18}$$

The morphisms thus defined do indeed constitute natural transformations because

$$\begin{aligned} &1 : 1_{\mathbf{C}} \dot{\to} 1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C} \ , \\ &R_0 : e' \dot{\to} R(e) : \mathbf{1} \to \mathbf{C} \text{ and} \\ &R_2 : \boxtimes' \circ (R \times R) \dot{\to} R \circ \boxtimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}' \end{aligned}$$

are all natural transformations, as is the pairing $\langle \sigma, \tau \rangle : F \times G \dot{\to} F' \times G' : C \times D \to C' \times D'$ of natural transformations for $\sigma : F \dot{\to} F' : C \to C'$ and $\tau : G \dot{\to} G' : D \to D'$. The image of a natural transformation under a functor (that is the horizontal composite of a functor and a natural transformation) and the vertical composite of natural transformations are also natural and therefore the last equation in (18) also defines a natural transformation.

Conceptually speaking R_v shifts all monoidal structures (that is tensor product and unit) inside the functor application.

We now define a structure whose main purpose is to enable us to make exact the two claims we have made informally that "any diagram of such-and-such shape commutes".

Definition 3.5. We define the set of *formal structural transformations* and at the same time we define for any element of this set a "source" and a "target", both of which are tensor words. In the listing below we write $\beta_0 : v \to w$ to indicate that β_0 is a formal structural transformation and that its source is v and its target is w. The set of formal structural transformations is recursively defined as the least set which contains

- for any tensor word v, the symbol $1_{0;v} : v \to v$ (called the identity on v);
- for tensor words u, v, w, the symbol $\alpha_{0;u,v,w} : u \Box (v \Box w) \to (u \Box v) \Box w$;

- for any tensor word v, the symbol $\lambda_{0;v} : e_0 \Box v \to v$;
- for any tensor word v, the symbol $\rho_{0;v} : v \Box e_0 \to v$;
- for any formal structural transformation $\beta_0 : v \to w$, a formal structural transformation ${\beta_0}^{-1} : w \to v$;
- for any two formal structural transformations $\beta_0 : v \to w$ and $\dot{\beta}_0 : \dot{v} \to \dot{w}$, a formal structural transformation $\beta_0 \Box \dot{\beta}_0 : v \Box \dot{v} \to w \Box \dot{w}$; and
- for any two formal structural transformations $\beta_0 : u \to v$ and $\dot{\beta}_0 : v \to w$, a formal structural transformation $\dot{\beta}_0 \circ \beta_0 : u \to w$.

Note that for any formal structural transformation $\beta_0 : v \to w$ we have $\ell(v) = \ell(w)$. Set $\ell(\beta_0) := \ell(v) = \ell(w)$.

For any monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ we recursively define a map which sends any formal structural tranformation $\beta_0 : v \to w$ to a natural isomorphism $\beta_{0;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} : v_{\mathbf{C}} \dot{\to} w_{\mathbf{C}} : \prod_{i=1}^{n} \mathbf{C} \to \mathbf{C}$ (where $n = \ell(\beta_0)$), by setting

$$
\begin{aligned}
1_{0;v;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} &:= 1_{v_{\mathbf{C}}} \\
\alpha_{0;u,v,w;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} &:= \alpha \circ (u_{\mathbf{C}} \times v_{\mathbf{C}} \times w_{\mathbf{C}}) \\
\lambda_{0;v;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} &:= \lambda \circ v_{\mathbf{C}} \\
\rho_{0;v;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} &:= \rho \circ v_{\mathbf{C}} \\
\left({\beta_0}^{-1}\right)_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} &:= {\beta_{0;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}}^{-1} \\
\left(\beta_0 \Box \dot{\beta}_0\right)_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} &:= \boxtimes \circ \left(\beta_{0;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} \times \dot{\beta}_{0;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}\right) \\
\left(\dot{\beta}_0 \circ \beta_0\right)_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} &:= \dot{\beta}_{0;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle} \bullet \beta_{0;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}
\end{aligned}
$$

We may abbreviate $\beta_{0;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}$ to $\beta_{0;\mathbf{C}}$. We will also write β for $\beta_{0;\mathbf{C}}$ and β' for $\beta_{0;\mathbf{C}'}$, etc. We call a natural isomorphism $\beta = \beta_{0;\mathbf{C}}$ which is the image of some formal structural transformation $\beta_0 : v \to w$ under the mapping just defined a structural transformation from v to w (in $\mathbf{C}$).

Again, intuitively this operation sends formal structural transformations β_0 to structural transformations β of $\mathbf{C}$ by replacing all placeholders by "the real thing" — that is α_0 is replaced by α, λ_0 is replaced by λ, ρ_0 is replaced by ρ, formal inverses are replaced by real inverses, formal composites by real complosites and the symbol $\Box$ is replaced by $\boxtimes$.

We could take a factor set of the set of formal structural transformations, so as to make the formal composition associative, make the formal identity an identity of composition, make the formal inverses real inverses with regard to composition and the formal identities and turn $\Box$ into a bifunctor on the resulting category (which has as objects the tensor words). Because all of these relations hold in a monoidal category, our operation $\beta_{0;\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}$ would still be well-defined as a map from the factor set — giving a functor from this category to any monoidal category. For our considerations this is not really important though, so we do not work out the details here.

Now we can formulate precisely what we already hinted at earlier in this section.

Lemma 3.6. *For any two tensor words v, w and any formal structural transformation $\beta_0 : v \to w$ between these tensor words, the diagram below commutes.*

$$\begin{tikzcd}
w\left(R(a_1),\ldots,R(a_n)\right) \arrow[d, "R_{w,a_1,\ldots,a_n}"] & v\left(R(a_1),\ldots,R(a_n)\right) \arrow[l, "\beta'_{R(a_1),\ldots,R(a_n)}"'] \arrow[d, "R_{v,a_1,\ldots,a_n}"] \\
R(w\left(a_1,\ldots,a_n\right)) & R(v\left(a_1,\ldots,a_n\right)) \arrow[l, "R(\beta_{a_1,\ldots,a_n})"]
\end{tikzcd} \tag{19}$$

Proof. We proceed by induction over the structure of our formal structural transformations. When β_0 is $\alpha_{0;u,v,w}$, $\lambda_{0;u}$ or $\rho_{0;u}$ commutativity of (19) follows from one of the diagrams in (15) or (16) with a replaced by $u\left(a_1,\ldots,a_{n_u}\right)$ etc., combined with naturality of α', λ' or ρ'. For example for the case of α_0 we get the diagram in (20) (on page 18).

The case of the inverses is also clear, as is the case where β_0 is an identity. If β_0 is a composite of structural transformations for which the corresponding diagrams are already known to commute, then writing the diagrams side by side horizontally we see that the diagram for β_0 also commutes.

So the only case we still need to check is the case when β_0 is a tensor product $\dot{\beta}_0\Box\ddot{\beta}_0$ of structural transformations $\dot{\beta}_0 : \dot{v} \to \dot{w}$ and $\ddot{\beta}_0 : \ddot{v} \to \ddot{w}$ already known to make their respective diagrams commute. In this case (19) is of a shape as in (21).

$$\begin{tikzcd}
 & \dot{v}\left(\langle R(a_i)\rangle_i\right) \boxtimes' \ddot{v}\left(\langle R(b_j)\rangle_j\right) \arrow[dl, "\dot{\beta}'_{\langle R(a_i)\rangle_i}\boxtimes'\ddot{\beta}'_{\langle R(b_j)\rangle_j}"] \arrow[d, "R_{\dot{v},\langle a_i\rangle_i}\boxtimes' R_{\ddot{v},\langle b_j\rangle_j}"] \\
\dot{w}\left(\langle R(a_i)\rangle_i\right) \boxtimes' \ddot{w}\left(\langle R(b_j)\rangle_j\right) \arrow[d, "R_{\dot{w},\langle a_i\rangle_i}\boxtimes' R_{\ddot{w},\langle b_j\rangle_j}"'] & R(\dot{v}\left(\langle a_i\rangle_i\right)) \boxtimes' R(\ddot{v}\left(\langle b_j\rangle_j\right)) \arrow[dl, "R\left(\dot{\beta}_{\langle a_i\rangle_i}\right)\boxtimes' R\left(\ddot{\beta}_{\langle b_j\rangle_j}\right)"] \arrow[d, "R_{2;\dot{v}(\langle a_i\rangle_i),\ddot{v}(\langle b_j\rangle_j)}"] \\
R(\dot{w}\left(\langle a_i\rangle_i\right)) \boxtimes' R(\ddot{w}\left(\langle b_j\rangle_j\right)) \arrow[d, "R_{2;\dot{w}(\langle a_i\rangle_i),\ddot{w}(\langle b_j\rangle_j)}"'] & R(\dot{v}\left(\langle a_i\rangle_i\right) \boxtimes \ddot{v}\left(\langle b_j\rangle_j\right)) \arrow[dl, "R\left(\dot{\beta}_{\langle a_i\rangle_i}\boxtimes\ddot{\beta}_{\langle b_j\rangle_j}\right)"] \\
R(\dot{w}\left(\langle a_i\rangle_i\right) \boxtimes \ddot{w}\left(\langle b_j\rangle_j\right)) &
\end{tikzcd}$$

(21)

In this diagram the upper square commutes because it is the image under the bifunctor $\boxtimes'$ of two instances of the diagram (19), one for $\dot{\beta}_0$ and one for $\ddot{\beta}_0$, which by the induction hypothesis we already know to commute. The lower square commutes by naturality of R_2. $\square$

This lemma will also form part of the proof of a theorem that we already stated informally in section 2.1 and which we can now formulate precisely.

$$
\begin{tikzcd}[column sep=tiny]
 & u\left(\langle R(a_i)\rangle_i\right) \boxtimes' \Big(v\left(\langle R(b_j)\rangle_j\right) \boxtimes' w\left(\langle R(c_k)\rangle_k\right)\Big) \arrow[dl, "\alpha'_{u(\langle R(a_i)\rangle_i),v(\langle R(b_j)\rangle_j),w(\langle R(c_k)\rangle_k)}"'] \arrow[dd, "R_{u,\langle a_i\rangle_i} \boxtimes' \left(R_{v,\langle b_j\rangle_j} \boxtimes' R_{w,\langle c_k\rangle_k}\right)"] \\
\Big(u\left(\langle R(a_i)\rangle_i\right) \boxtimes' v\left(\langle R(b_j)\rangle_j\right)\Big) \boxtimes' w\left(\langle R(c_k)\rangle_k\right) \arrow[dd, "\left(R_{u,\langle a_i\rangle_i} \boxtimes' R_{v,\langle b_j\rangle_j}\right) \boxtimes' R_{w,\langle c_k\rangle_k}"'] & \\
 & R(u\left(\langle a_i\rangle_i\right)) \boxtimes' \Big(R(v\left(\langle b_j\rangle_j\right)) \boxtimes' R(w\left(\langle c_k\rangle_k\right))\Big) \arrow[dl, "\alpha'_{R(u(\langle a_i\rangle_i)),R(v(\langle b_j\rangle_j)),R(w(\langle c_k\rangle_k))}"'] \arrow[dd, "1_{R(u(\langle a_i\rangle_i))} \boxtimes' R_{2;v(\langle b_j\rangle_j),w(\langle c_k\rangle_k)}"] \\
\Big(R(u\left(\langle a_i\rangle_i\right)) \boxtimes' R(v\left(\langle b_j\rangle_j\right))\Big) \boxtimes' R(w\left(\langle c_k\rangle_k\right)) \arrow[dd, "R_{2;u(\langle a_i\rangle_i),v(\langle b_j\rangle_j)} \boxtimes' 1_{R(w(\langle c_k\rangle_k))}"'] & \\
 & R(u\left(\langle a_i\rangle_i\right)) \boxtimes' R\Big(v\left(\langle b_j\rangle_j\right) \boxtimes w\left(\langle c_k\rangle_k\right)\Big) \arrow[dd, "R_{2;u(\langle a_i\rangle_i),v(\langle b_j\rangle_j) \boxtimes w(\langle c_k\rangle_k)}"] \\
R\Big(u\left(\langle a_i\rangle_i\right) \boxtimes v\left(\langle b_j\rangle_j\right)\Big) \boxtimes' R(w\left(\langle c_k\rangle_k\right)) \arrow[dd, "R_{2;u(\langle a_i\rangle_i) \boxtimes v(\langle b_j\rangle_j),w(\langle c_k\rangle_k)}"'] & \\
 & R\Big(u\left(\langle a_i\rangle_i\right) \boxtimes \Big(v\left(\langle b_j\rangle_j\right) \boxtimes w\left(\langle c_k\rangle_k\right)\Big)\Big) \arrow[dl, "R\left(\alpha_{u(\langle a_i\rangle_i),v(\langle b_j\rangle_j),w(\langle c_k\rangle_k)}\right)"] \\
R\Big(\Big(u\left(\langle a_i\rangle_i\right) \boxtimes v\left(\langle b_j\rangle_j\right)\Big) \boxtimes w\left(\langle c_k\rangle_k\right)\Big) &
\end{tikzcd}
\tag{20}
$$

3.2 The coherence theorem for monoidal categories

Theorem 3.7 (coherence theorem for monoidal categories)**.** *In any monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ we have that for any two tensor words v, w of the same length n and any two formal structural transformations $\beta_0 : v \to w$, $\dot{\beta}_0 : v \to w$ between these two tensor words*

$$\beta_{a_1,\cdots,a_n} = \dot{\beta}_{a_1,\cdots,a_n} \ .$$

Proof. Our proof follows the idea suggested in the exercises section of [Mac98, section XI.3]. [Mac98, section VII] also has a more elementary proof.

We will show that any structural transformation between two tensor words v, w of the same length n is equal to a canonical structural transformation obtained by connecting both tensor words to a specific “simple” tensor word of length n (namely $v^{(n)}$) and then tracing one path forwards and one backwards. Formally we will package a part of this procedure by defining a monoidal functor comparing the base category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ to (a subcategory of) the category $\mathbf{C}^{\mathbf{C}}$ of endofunctors on $\mathbf{C}$ and natural transformations between them and then use the previous lemma.

Observe first that $\left\langle \mathbf{C}^{\mathbf{C}}, \circ, 1_{\mathbf{C}}, 1, 1, 1 \right\rangle$ forms a strict monoidal category. Now define

$$\begin{aligned}
\mathcal{S} &: \mathbf{C} \to \mathbf{C}^{\mathbf{C}} \\
\mathcal{S}(a) &: \mathbf{C} \to \mathbf{C} \\
(\mathcal{S}(a))(b) &= a \boxtimes b \\
(\mathcal{S}(a))(g) &= 1_a \boxtimes g \\
\mathcal{S}(f) &: \mathcal{S}(a) \dot{\to} \mathcal{S}(a') : \mathbf{C} \to \mathbf{C} \\
(\mathcal{S}(f))_b &= f \boxtimes 1_b \\[1ex]
\mathcal{S}_2 &: (\circ) \circ (\mathcal{S} \times \mathcal{S}) \dot{\to} \mathcal{S} \circ (\boxtimes) : \mathbf{C} \times \mathbf{C} \to \mathbf{C}^{\mathbf{C}} \\
\mathcal{S}_{2;a,b} &: (\mathcal{S}(a)) \circ (\mathcal{S}(b)) \dot{\to} \mathcal{S}(a \boxtimes b) : \mathbf{C} \to \mathbf{C} \\
(\mathcal{S}_{2;a,b})_c &: a \boxtimes (b \boxtimes c) \to (a \boxtimes b) \boxtimes c \\
(\mathcal{S}_{2;a,b})_c &= \alpha_{a,b,c} \\[1ex]
\mathcal{S}_0 &: 1_{\mathbf{C}} \to \mathcal{S}(e) \ , \\
&\quad \text{which is a morphism of } \mathbf{C}^{\mathbf{C}} \\
\mathcal{S}_0 &: 1_{\mathbf{C}} \dot{\to} \mathcal{S}(e) : \mathbf{C} \to \mathbf{C} \\
(\mathcal{S}_0)_a &: a \to e \boxtimes a \\
(\mathcal{S}_0)_a &= \lambda_a^{-1} \\[1ex]
&(\text{for } a, b, c, a' \in \mathbf{C} \ ; \ f : a \to a' \text{ and } g \text{ in } \mathbf{C})
\end{aligned} \tag{22}$$

Observation 3.8.

$$(\mathcal{S}(a_1) \circ \cdots \circ \mathcal{S}(a_n))(e) = v^{(n)}(\langle a_i \rangle_{i=1}^n)$$

Lemma 3.9. *$\langle \mathcal{S}, \mathcal{S}_2, \mathcal{S}_0 \rangle$ as defined above is a monoidal functor.*

Proof. The diagram (15) becomes

$$\begin{tikzcd}
(\mathcal{S}(a)\circ\mathcal{S}(b)\circ\mathcal{S}(c))(d) \arrow[d, "(\mathcal{S}_{2;a,b}\circ 1_{\mathcal{S}(c)})_d"'] & (\mathcal{S}(a)\circ\mathcal{S}(b)\circ\mathcal{S}(c))(d) \arrow[l, "1"'] \arrow[d, "(1_{\mathcal{S}(a)}\circ\mathcal{S}_{2;b,c})_d"] \\
(\mathcal{S}(a\boxtimes b)\circ\mathcal{S}(c))(d) \arrow[d, "(\mathcal{S}_{2;a\boxtimes b,c})_d"'] & (\mathcal{S}(a)\circ\mathcal{S}(b\boxtimes c))(d) \arrow[d, "(\mathcal{S}_{2;a,b\boxtimes c})_d"] \\
(\mathcal{S}((a\boxtimes b)\boxtimes c))(d) & (\mathcal{S}(a\boxtimes(b\boxtimes c)))(d) \arrow[l, "(\mathcal{S}(\alpha_{a,b,c}))_d"]
\end{tikzcd},$$

which turns into

$$\begin{tikzcd}
a\boxtimes(b\boxtimes(c\boxtimes d)) \arrow[d, "\alpha_{a,b,c\boxtimes d}"'] & a\boxtimes(b\boxtimes(c\boxtimes d)) \arrow[l, "1"'] \arrow[d, "1_a\boxtimes\alpha_{b,c,d}"] \\
(a\boxtimes b)\boxtimes(c\boxtimes d) \arrow[d, "\alpha_{a\boxtimes b,c,d}"'] & a\boxtimes((b\boxtimes c)\boxtimes d) \arrow[d, "\alpha_{a,b\boxtimes c,d}"] \\
((a\boxtimes b)\boxtimes c)\boxtimes d & (a\boxtimes(b\boxtimes c))\boxtimes d \arrow[l, "\alpha_{a,b,c}\boxtimes 1_d"]
\end{tikzcd}$$

when we expand the definitions, and that is just a slightly distorted version of diagram (3) which is commutative by the definition of a monoidal category. The diagrams in (16) become

$$\begin{tikzcd}
(\mathcal{S}(a))(b) \arrow[dd, "1"] & (1_{\mathbf{C}}\circ\mathcal{S}(a))(b) \arrow[l, "1"'] \arrow[d, "(\mathcal{S}_0\circ 1_{\mathcal{S}(a)})_b"'] \\
 & (\mathcal{S}(e)\circ\mathcal{S}(a))(b) \arrow[d, "(\mathcal{S}_{2;e,a})_b"'] \\
(\mathcal{S}(a))(b) & (\mathcal{S}(e\boxtimes a))(b) \arrow[l, "(\mathcal{S}(\lambda_a))_b"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{S}(a))(b) \arrow[dd, "1"] & (\mathcal{S}(a)\circ 1_{\mathbf{C}})(b) \arrow[l, "1"'] \arrow[d, "(1_{\mathcal{S}(a)}\circ\mathcal{S}_0)_b"'] \\
 & (\mathcal{S}(a)\circ\mathcal{S}(e))(b) \arrow[d, "(\mathcal{S}_{2;a,e})_b"'] \\
(\mathcal{S}(a))(b) & (\mathcal{S}(a\boxtimes e))(b) \arrow[l, "(\mathcal{S}(\rho_a))_b"]
\end{tikzcd}$$

where again by expanding the definitions we get

$$\begin{tikzcd}
a\boxtimes b \arrow[dd, "1"] & a\boxtimes b \arrow[l, "1"'] \arrow[d, "\lambda^{-1}_{a\boxtimes b}"'] \\
 & e\boxtimes(a\boxtimes b) \arrow[d, "\alpha_{e,a,b}"'] \\
a\boxtimes b & (e\boxtimes a)\boxtimes b \arrow[l, "\lambda_a\boxtimes 1_b"]
\end{tikzcd}
\qquad
\begin{tikzcd}
a\boxtimes b \arrow[dd, "1"] & a\boxtimes b \arrow[l, "1"'] \arrow[d, "1_a\boxtimes\lambda^{-1}_b"'] \\
 & a\boxtimes(e\boxtimes b) \arrow[d, "\alpha_{a,e,b}"'] \\
a\boxtimes b & (a\boxtimes e)\boxtimes b \arrow[l, "\rho_a\boxtimes 1_b"]
\end{tikzcd}.$$

The right hand side diagram is (again a slightly distorted version of) diagram (4) from the definition of a monoidal category. So the only diagram whose commutativity we still need to prove is the left hand side diagram above, which, if we clean it up a little bit, looks like the one below.

$$\begin{tikzcd}
(e \boxtimes a) \boxtimes b \arrow[dr, "\lambda_a \boxtimes 1_b"'] & & e \boxtimes (a \boxtimes b) \arrow[ll, "\alpha_{a,e,b}"'] \arrow[dl, "\lambda_{a \boxtimes b}"] \\
& a \boxtimes b &
\end{tikzcd} \tag{23}$$

To see why this diagram commutes in any monoidal category consider the drawing below.

$$\begin{tikzcd}
((e \boxtimes e) \boxtimes a) \boxtimes b \arrow[dr, "(\rho_e \boxtimes 1_a) \boxtimes 1_b"'] & & (e \boxtimes e) \boxtimes (a \boxtimes b) \arrow[ll, "\alpha_{e \boxtimes e,a,b}"'] \arrow[dr, "\begin{matrix}\rho_e \boxtimes 1_{a \boxtimes b} = \\ \rho_e \boxtimes (1_a \boxtimes 1_b)\end{matrix}"'] & & e \boxtimes (e \boxtimes (a \boxtimes b)) \arrow[ll, "\alpha_{e,e,a \boxtimes b}"'] \arrow[dl, "1_e \boxtimes \lambda_{a \boxtimes b}"] \arrow[dd, "1_e \boxtimes \alpha_{e,a,b}"] \\
& (e \boxtimes a) \boxtimes b & & e \boxtimes (a \boxtimes b) \arrow[ll, "\alpha_{e,a,b}"] & \\
(e \boxtimes (e \boxtimes a)) \boxtimes b \arrow[uu, "\alpha_{e,e,a} \boxtimes 1_b"] \arrow[ur, "(1_e \boxtimes \lambda_a) \boxtimes 1_b"] & & & & e \boxtimes ((e \boxtimes a) \boxtimes b) \arrow[llll, "\alpha_{e,e \boxtimes a,b}"] \arrow[ul, "1_e \boxtimes (\lambda_a \boxtimes 1_b)"']
\end{tikzcd}$$

The triangle on the right hand side is the same as (23), only with the functor $e \boxtimes -$ applied; but $e \boxtimes -$ is naturally isomorphic by λ to $1_{\mathbf{C}}$, so if we can prove the right hand side triangle commutative we know that (23) commutes. To that end we inspect the other parts of the diagram.

The outermost pentagon is (3) from the definition of a monoidal category. The triangle in the upper right corner is an instance of (4) as is the triangle on the left. The parallelogram at the top commutes because α is natural. The same holds for the trapezium at the bottom. This concludes the proof that $\langle \mathcal{S}, \mathcal{S}_2, \mathcal{S}_0 \rangle$ is a monoidal functor.. $\square$

Now we can put all this together and finish the proof of the coherence theorem. Have a look at the following diagram — where obeservation 3.8 shows that the uppermost part makes sense, 3.6 shows that the middle part commutes and naturality of ρ shows that the lower part commutes.

$$\begin{tikzcd}
& a_1 \boxtimes (a_2 \boxtimes (a_3 \boxtimes \ldots (a_n \boxtimes e) \ldots)) \arrow[dl, equal] \arrow[dr, equal] & \\
(w (\mathcal{S}(a_1), \ldots, \mathcal{S}(a_n)))(e) \arrow[rr, equal] \arrow[d, "(\mathcal{S}_{w,a_1,\ldots,a_n})_e"] & & (v (\mathcal{S}(a_1), \ldots, \mathcal{S}(a_n)))(e) \arrow[d, "(\mathcal{S}_{v,a_1,\ldots,a_n})_e"] \\
w(a_1, \ldots, a_n) \boxtimes e \arrow[d, "\rho_{w(a_1,\ldots,a_n)}"] & & v(a_1, \ldots, a_n) \boxtimes e \arrow[ll, "\beta_{a_1,\ldots,a_n} \boxtimes 1_e"] \arrow[d, "\rho_{v(a_1,\ldots,a_n)}"] \\
w(a_1, \ldots, a_n) & & v(a_1, \ldots, a_n) \arrow[ll, "\beta_{a_1,\ldots,a_n}"]
\end{tikzcd} \tag{24}$$

This means that we get that for any formal structural transformation $\beta_0 : v \to w$

$$\beta_{a_1,\ldots,a_n} = \rho_{w(a_1,\ldots,a_n)} \circ (\mathcal{S}_{w,a_1,\ldots,a_n})_e \circ (\mathcal{S}_{v,a_1,\ldots,a_n})_e^{-1} \circ \rho^{-1}_{v(a_1,\ldots,a_n)} \,. \tag{25}$$

□

Definition 3.10. We write $\mathrm{can}_{\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle,w\leftarrow v}$ for the structural transformation from v to w in $\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle$. By the coherence theorem this is well-defined. (Note that for example the right hand side in (25) is a structural transformation from v to w, so such a structural transformation exists.)

Usually the monoidal category $\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle$ will be clear from the context and we will omit it and write only $\mathrm{can}_{w\leftarrow v}$.

Observation 3.11. *For any formal structural transformation β_0 of length n and any tuple $\langle v_i\rangle_{i=1}^n$ of tensor words there is a formal structural transformation $\dot\beta_{0;\mathbf{C}}$ of length $\sum_{i=1}^n \ell(v_i)$ such that $\beta_{0;\mathbf{C};\left\langle v_{i;\mathbf{C}}\left(\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}\right)\right\rangle_{i=1}^n} = \dot\beta_{0;\mathbf{C};\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}{}_{i=1}^n}$ for all tuples of objects $a_{i,j} \in \mathbf{C}$.*

Proof. This is easily seen by structural induction. For example for $\alpha_{0;u,v,w}$ choose $\alpha_{0;u\left(\langle v_i\rangle_{i=1}^{\ell(u)}\right),v\left(\langle v_i\rangle_{i=\ell(u)+1}^{\ell(u)+\ell(v)}\right),w\left(\langle v_i\rangle_{i=\ell(u)+\ell(v)+1}^{\ell(u)+\ell(v)+\ell(w)}\right)}$ and for $\beta_0\Box\dot\beta_0$ find $\ddot\beta_0$ and $\dddot\beta_0$ such that

$$\begin{aligned}
\beta_{0;\mathbf{C};\left\langle v_{i;\mathbf{C}}\left(\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}\right)\right\rangle_{i=1}^{\ell(\beta_0)}} &= \ddot\beta_{0;\mathbf{C};\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}{}_{i=1}^{\ell(\beta_0)}} \qquad \text{and}\\
\dot\beta_{0;\mathbf{C};\left\langle v_{i;\mathbf{C}}\left(\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}\right)\right\rangle_{i=\ell(\beta_0)+1}^{\ell\left(\beta_0\Box\dot\beta_0\right)}} &= \dddot\beta_{0;\mathbf{C};\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}{}_{i=\ell(\beta_0)+1}^{\ell\left(\beta_0\Box\dot\beta_0\right)}}
\end{aligned}$$

and take their product $\ddot\beta_0\Box\dddot\beta_0$. For $\dot\beta_0\circ\beta_0$ of lenght n find $\ddot\beta_0$ and $\dddot\beta_0$ such that

$$\begin{aligned}
\beta_{0;\mathbf{C};\left\langle v_{i;\mathbf{C}}\left(\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}\right)\right\rangle_{i=1}^{n}} &= \ddot\beta_{0;\mathbf{C};\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}{}_{i=1}^{n}} \qquad \text{and}\\
\dot\beta_{0;\mathbf{C};\left\langle v_{i;\mathbf{C}}\left(\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}\right)\right\rangle_{i=1}^{n}} &= \dddot\beta_{0;\mathbf{C};\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}{}_{i=1}^{n}}
\end{aligned}$$

and use $\dddot\beta_0\circ\ddot\beta_0$. The rest are even simpler. □

This strengthens the statement of the coherence theorem a little further.

Corollary 3.12.

$$\mathrm{can}_{w\leftarrow v;\left\langle v_{i;\mathbf{C}}\left(\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}\right)\right\rangle_{i=1}^{n}} = \mathrm{can}_{w\left(\langle v_i\rangle_{i=1}^n\right)\leftarrow v\left(\langle v_i\rangle_{i=1}^n\right);\langle a_{i,j}\rangle_{j=1}^{\ell(v_i)}{}_{i=1}^{n}}$$

for $\ell(v) = \ell(w) = n$.

3.3 Calculations in monoidal categories

The coherence theorem already makes dealing with monoidal categories quite a bit easier. In the sequel we will try to figure out some of the properties of the structures structures presented in section 2. In part this will boild down to deriving other equations (or the commutativity of other diagrams) from the diagrams we have taken as axioms. Ideally when doing so we would like to be able to just omit all structural transformations and do our equational reasoning with these simplified diagrams — essentially working in a strict monoidal category — knowing that the diagrams we really *mean* also commute. To see how this is possible we begin by describing the prototypical strict monoidal category.

Definition 3.13 (Free strict monoidal category with signed arrow-atoms)**.** Let X and A be sets which we will call the set of *object-atoms* and *non-trivial arrow-atoms* respectively. Call the set $A\dot{\cup}X$ the set of *arrow-atoms*. Denote by X^* the set of sequences in letters taken from the set X (that is, the free monoid over X).

Let $\mathfrak{s} : A \to X^*$ and $\mathfrak{t} : A \to X^*$ be functions. We think of $\mathfrak{s}$ and $\mathfrak{t}$ as the functions giving the source and target — or the *signature* — of the non-trivial arrow-atoms.

Based on these data we are going to define a category $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ which we will call the *free strict monoidal category with signed arrow-atoms*. All of $X, A, \mathfrak{s}, \mathfrak{t}$ together will be called the *generators* of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$. When $X, A, \mathfrak{s}, \mathfrak{t}$ are clear from the context we will sometimes use just $\mathcal{E}$ to denote $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$. The set of objects of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ will be X^*. The set of arrows of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ will be a little more complex.

Define $\mathscr{s} : A\dot{\cup}X \to X^*$ by $\mathscr{s}\big|_A := \mathfrak{s}$ and $\mathscr{s}\big|_X(\mathfrak{a}) := \langle\mathfrak{a}\rangle$. Similarly define $\mathscr{t} : A\dot{\cup}X \to X^*$ by $\mathscr{t}\big|_A := \mathfrak{t}$ and $\mathscr{t}\big|_X(\mathfrak{a}) := \langle\mathfrak{a}\rangle$.

Define a set $\mathrm{pArr}(X, A, \mathfrak{s}, \mathfrak{t}) \subset \left((A\dot{\cup}X)^*\right)^*$ (which will serve as an auxiliary structure for the definition of the set of arrows of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$) as follows. The elements of $\mathrm{pArr}(X, A, \mathfrak{s}, \mathfrak{t})$ are nonempty lists of (possibly empty) lists of arrow-atoms

$$\langle\langle\mathfrak{f}_{i,j}\rangle_{j=1}^{n_i}\rangle_{i=k}^{1}$$

with the property that

$$\cdot_{j=1}^{n_i}\left(\mathscr{t}(\mathfrak{f}_{i,j})\right) = \cdot_{j=1}^{n_{i+1}}\left(\mathscr{s}(\mathfrak{f}_{i+1,j})\right) \qquad \text{for all } i \in \{1, \ldots, k-1\}. \tag{26}$$

(Where the dot denotes concatenation.)

So, $\langle\rangle$ is not in $\mathrm{pArr}(X, A, \mathfrak{s}, \mathfrak{t})$ while for example $\langle\,\langle\rangle, \langle\rangle, \langle\rangle\rangle$ is.

We call $\mathrm{pArr}(X, A, \mathfrak{s}, \mathfrak{t})$ the set of *pre-arrows*.

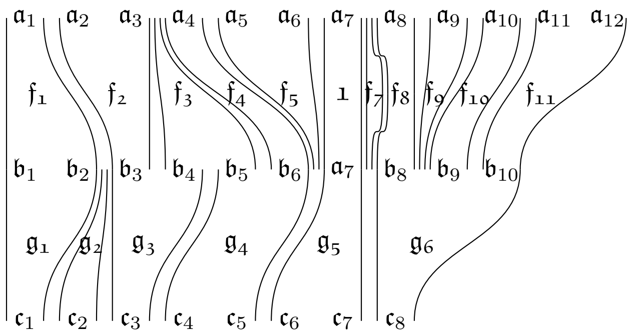
Define functions $s : \mathrm{pArr}(X, A, \mathfrak{s}, \mathfrak{t}) \to X^*$ and $t : \mathrm{pArr}(X, A, \mathfrak{s}, \mathfrak{t}) \to X^*$ by setting

$$\begin{aligned} s\left(\langle\langle\mathfrak{f}_{i,j}\rangle_{j=1}^{n_i}\rangle_{i=k}^{1}\right) &:= \cdot_{j=1}^{n_1}\left(\mathscr{s}(\mathfrak{f}_{1,j})\right) \\ t\left(\langle\langle\mathfrak{f}_{i,j}\rangle_{j=1}^{n_i}\rangle_{i=k}^{1}\right) &:= \cdot_{j=1}^{n_k}\left(\mathscr{s}(\mathfrak{f}_{k,j})\right)\ . \end{aligned}$$

We can represent this structure pictorially. For example the element

$$\langle\,\langle\mathfrak{g}_1, \ldots, \mathfrak{g}_6\rangle, \langle\mathfrak{f}_1, \ldots, \mathfrak{f}_{11}\rangle\rangle$$

of $\mathrm{pArr}(X, A, \mathfrak{s}, \mathfrak{t})$ would be represented in the way shown below.



(27)

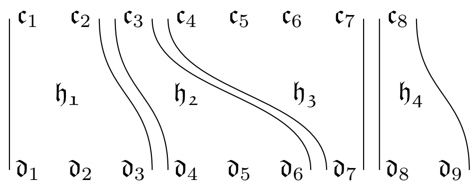
The picture shows not only the arrow-atoms but also their sources and targets. In the top row we see the sources of the arrow-atoms $\mathfrak{f}_i \in A$. In the place where one would expect $\mathfrak{f}_6$ there's a "$\mathtt{1}$" — this is the arrow-atom corresponding to the element $\mathfrak{a}_7 \in A$. The middle row gives the targets of the basic arrows $\mathfrak{f}_i$ which are at the same time the sources of the basic arrows $\mathfrak{g}_j$ and the bottom row gives the targets of $\mathfrak{g}_j$.

Note that there is only one picture to represent any of the tuples

$$\begin{gathered} \langle \, \langle\rangle\rangle \\ \langle \, \langle\rangle, \langle\rangle\rangle \\ \langle \, \langle\rangle, \langle\rangle, \langle\rangle\rangle \\ \vdots \end{gathered}$$

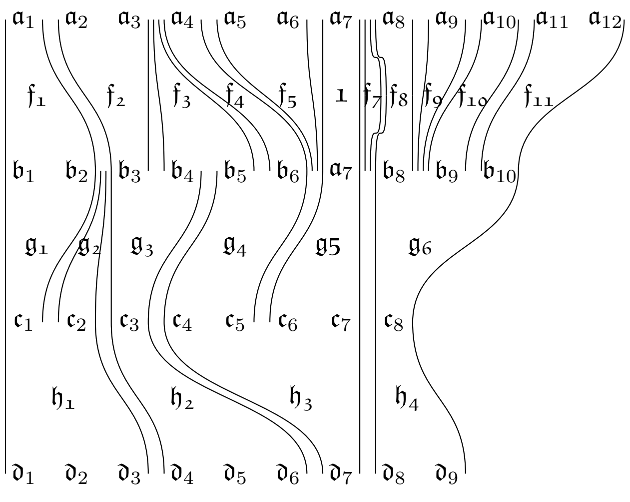
(namely the empty picture). This won't pose any practical problem because in the structure we are about to introduce we will identify all of these.

We introduce a *composition* $\circ$ and a *product* $\boxdot$ on pArr $(X, A, \mathfrak{s}, \mathfrak{t})$ by vertical and horizontal juxtaposition respectively. For composition to be defined the source of the first arrow has to match the target of the second arrow. So, for example, the composite $\langle \, \langle \mathfrak{h}_1, \ldots, \mathfrak{h}_4\rangle\rangle \circ \langle \, \langle \mathfrak{g}_1, \ldots, \mathfrak{g}_6\rangle, \langle \mathfrak{f}_1, \ldots, \mathfrak{f}_{11}\rangle\rangle$ of



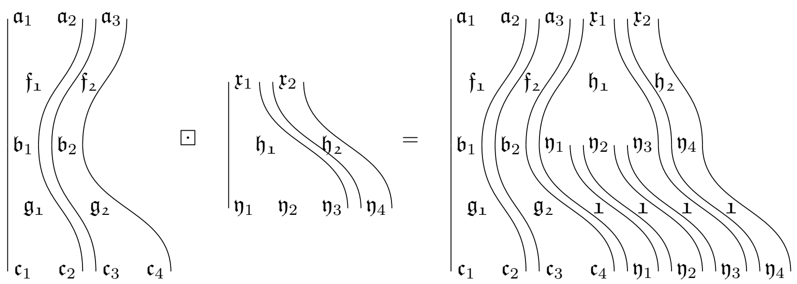
(28)

and (27) is

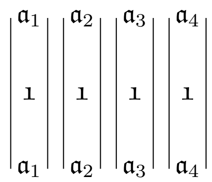


(29)

The product is always defined. If the heights of the two factors don't match, then we pad one of them with identities to get it to the same height as the other. So for example



This structure is not yet a category, let alone a monoidal category. Clearly the composition and the product are associative and we have an identity for the product (the tuple containing only the empty list $\langle\,\langle\rangle\rangle$), but we do not yet have an identity for composition. Arrows of the form



will serve that purpose. So we will be taking the factor set of $\mathrm{pArr}\,(X, A, \mathfrak{s}, \mathfrak{t})$ by a relation $\eqsim$. This relation should relate for any list h the lists $h \cdot \langle s\,(h)\rangle$, $\langle t\,(h)\rangle \cdot h$ and h to each other. To make sure that our product is still well-defined we have to extend this relation a little further. For any element

$$\Big\langle \langle \mathfrak{f}_{k,1}, \ldots, \mathfrak{f}_{k,n_k}\rangle, \ldots, \langle \mathfrak{f}_{i,1}, \ldots, \mathfrak{f}_{i,n_i}\rangle, \Big\langle \mathfrak{f}_{i-1,1}, \ldots, \mathfrak{f}_{i-1,n_{i-1}}\Big\rangle, \ldots, \langle \mathfrak{f}_{1,1}, \ldots, \mathfrak{f}_{1,n_1}\rangle\Big\rangle$$

of $\mathrm{pArr}\,(X, A, \mathfrak{s}, \mathfrak{t})$, if $\Big\langle \mathfrak{f}_{i-1,1}, \ldots, \mathfrak{f}_{i-1,n_{i-1}}\Big\rangle$ can be split into three parts

$$\langle \mathfrak{f}_{i-1,1}, \ldots, \mathfrak{f}_{i-1,l}\rangle\ ,\ \langle \mathfrak{f}_{i-1,l+1}, \ldots, \mathfrak{f}_{i-1,m}\rangle \text{ and } \Big\langle \mathfrak{f}_{i-1,m+1}, \ldots, \mathfrak{f}_{i-1,n_{i-1}}\Big\rangle$$

such that

$$\begin{aligned} t\,(\mathfrak{f}_{i-1,1}) \cdot \ldots \cdot t\,(\mathfrak{f}_{i-1,l}) &= s\,(\mathfrak{f}_{i,1}) \cdot \ldots \cdot s\,(\mathfrak{f}_{i,j-1}) \\ \langle \mathfrak{f}_{i-1,l+1}, \ldots, \mathfrak{f}_{i-1,m}\rangle &= s\,(\mathfrak{f}_{i,j}) \\ t\,(\mathfrak{f}_{i-1,m+1}) \cdot \ldots \cdot t\left(\mathfrak{f}_{i-1,n_{i-1}}\right) &= s\,(\mathfrak{f}_{i,j+1}) \cdot \ldots \cdot s\,(\mathfrak{f}_{i,n_i}) \end{aligned}$$

then we relate

$$\begin{aligned} \Big\langle\ & \langle \mathfrak{f}_{k,1}, \ldots, \mathfrak{f}_{k,n_k}\rangle, \ldots, \langle \mathfrak{f}_{i,1}, \ldots, \mathfrak{f}_{i,n_i}\rangle, \\ & \qquad\qquad\qquad \Big\langle \mathfrak{f}_{i-1,1}, \ldots, \mathfrak{f}_{i-1,n_{i-1}}\Big\rangle, \ldots, \langle \mathfrak{f}_{1,1}, \ldots, \mathfrak{f}_{1,n_1}\rangle\Big\rangle \end{aligned}$$

and

$$\begin{aligned} \Big\langle\ & \langle \mathfrak{f}_{k,1}, \ldots, \mathfrak{f}_{k,n_k}\rangle, \ldots, \langle \mathfrak{f}_{i,1}, \ldots, \mathfrak{f}_{i,j-1}\rangle \cdot t\,(\mathfrak{f}_{i,j}) \cdot \langle \mathfrak{f}_{i,j+1}, \ldots, \mathfrak{f}_{i,n_i}\rangle, \\ & \quad \Big\langle \mathfrak{f}_{i-1,1}, \ldots, \mathfrak{f}_{i-1,l}, \mathfrak{f}_{i,j}, \mathfrak{f}_{i-1,m+1}, \ldots, \mathfrak{f}_{i-1,n_{i-1}}\Big\rangle, \ldots, \langle \mathfrak{f}_{1,1}, \ldots, \mathfrak{f}_{1,n_1}\rangle\Big\rangle\ . \end{aligned}$$

When passing from the former to the latter or the other way round we will say that we (vertically) *shifted* a component in the representation.

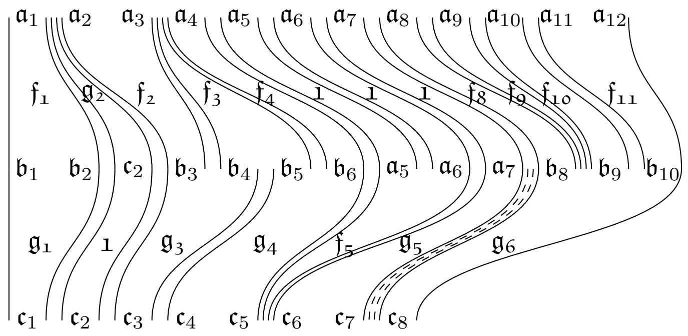
The relation $\approx$ is now the reflexive, symmetric transitive closure of the basic relations just described.

Define the set of arrows of $\mathcal{E}\left(X, A, \mathfrak{s}, \mathfrak{t}\right)$ by

$$\mathrm{Arr}\left(\mathcal{E}\left(X, A, \mathfrak{s}, \mathfrak{t}\right)\right) := \mathrm{pArr}\left(X, A, \mathfrak{s}, \mathfrak{t}\right) / \approx \; .$$

Call the projection $\pi_{\approx, X, A, \mathfrak{s}, \mathfrak{t}} : \mathrm{pArr}\left(X, A, \mathfrak{s}, \mathfrak{t}\right) \to \mathrm{Arr}\left(\mathcal{E}\left(X, A, \mathfrak{s}, \mathfrak{t}\right)\right)$ or just $\pi_{\approx}$, when $X, A, \mathfrak{s}, \mathfrak{t}$ are clear from the context.

In $\mathrm{Arr}\left(\mathcal{E}\left(X, A, \mathfrak{s}, \mathfrak{t}\right)\right)$ we have that for example (27) is equal to



where $\mathfrak{f}_7$ is indicated in a dashed style. Now the relation $\approx$ is compatible with the composition, the product, s and t and therefore these operations can be defined on the set $\mathrm{Arr}(\mathcal{E})$. This makes $\mathcal{E}$ a category. The identity arrow for any object $a \in \mathrm{Obj}(\mathcal{E})$ is just $\langle a \rangle$. $\boxdot$ is a bifunctor on $\mathcal{E}$ (compatibility with composition can be seen by vertically shifting rows of identities on one side or the other if necessary, as allowed by $\approx$). This bifunctor is associative and has as two-sided identity the empty list (which is equal in $\mathrm{pArr}\left(X, A, \mathfrak{s}, \mathfrak{t}\right) / \approx$ to any finite list with all elements empty lists).

Remark 3.14. We will use the kind of pictures introduced above to denote pre-arrows, elements of $\mathrm{Arr}(\mathcal{E})$ and later also equivalence classes of arrows of $\mathcal{E}$ — which will themselves be arrows of another category. It should be clear from the context which we mean.

In a similar manner to the result of the coherence theorem, the main requirement for composites of products of arrows in monoidal categories to behave in the simple way we expect them to behave, is that it is clear how the sources and targets of these arrows arise as $v\left(\langle a_i \rangle_{i=1}^n\right)$ and which iterated products of arrows we are taking. We will now define structures which allow us to capture this idea more precisely.

Definition 3.15. For $X, A, \mathfrak{s}, \mathfrak{t}$ as in definition 3.13 call a pair of functions v_s, v_t from the set of non-trivial arrow-atoms A to the set of tensor words which satisfies

$$\begin{aligned} \ell(v_s(\mathfrak{f})) &= \ell(\mathfrak{s}(\mathfrak{f})) \\ \ell(v_t(\mathfrak{f})) &= \ell(\mathfrak{t}(\mathfrak{f})) \end{aligned} \tag{30}$$

a *varnishing* of the non-trivial arrow atoms A. We will adopt the convention of extending v_s and v_t to the set $A \dot{\cup} X$ (and using the same symbols to denote these maps) by setting

$$\begin{aligned} v_s(\mathfrak{a}) &:= (\_) \\ v_t(\mathfrak{a}) &:= (\_) \end{aligned} \qquad \text{for all } \mathfrak{a} \in X. \tag{31}$$

Definition 3.16. Let $X, A, \mathfrak{s}, \mathfrak{t}$ be generators as in definition 3.13 and let v_s, v_t be a varnishing as in definition 3.15. On the set of structures of the form

$$\left\langle v_e, \langle\langle v_i, \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_i} \rangle\rangle_{i=k}^{1}, v_b \right\rangle \tag{32}$$

where v_e, v_b, v_i are tensor words and $\mathfrak{f}_{i,j}$ are arrow-atoms define a function

$$\operatorname{ptr}\left(\left\langle v_e, \langle\langle v_i, \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_k} \rangle\rangle_{i=k}^{1}, v_b \right\rangle\right) := \langle\langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_k} \rangle_{i=k}^{1} \tag{33}$$

(that is, we simply forget about the tensor words and keep only the arrow atoms). Define a *path-shape* or sometimes just *path* to be any structure p of the form (32) for which $\operatorname{ptr}(p)$ lies in $\operatorname{pArr}(X, A, \mathfrak{s}, \mathfrak{t})$ and for which $\ell(v_e) = \sum_{j=1}^{n_k} \ell(\mathfrak{t}(\mathfrak{f}_{k,j}))$, $\ell(v_b) = \sum_{j=1}^{n_1} \ell(\mathfrak{s}(\mathfrak{f}_{1,j}))$ and $\ell(v_i) = n_i$. That is, a path-shape is any such structure for which the concatenated sources and targets match (as in (26)) and for which the length of v_e matches the length of the target of the ptr and the length of v_b matches the length of the source of the ptr. We call v_b the source tensor word of the path and v_e the target tensor word of the path. We use $\mathbb{P}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t)$ to denote the set of all path-shapes for generators $X, A, \mathfrak{s}, \mathfrak{t}$ and a varnishing v_s, v_t. Define a function $\operatorname{tr} : \mathbb{P}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t) \to \mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ by

$$\operatorname{tr} := \pi_{\diamond} \circ \operatorname{ptr} \ . \tag{34}$$

Call $\operatorname{tr}(p)$ the *trace* of the path p.

An *equation-shape* is a pair $\langle p, q \rangle$ of path shapes — let's say $p = \langle v_e, \hat{p}, v_b \rangle$ and $q = \langle w_e, \hat{q}, w_b \rangle$ — which satisfies

$$v_e = w_e \qquad\qquad \text{and} \qquad\qquad v_b = w_b \ . \tag{35}$$

Definition 3.17. Let X be a set of object-atoms and let $\mathbf{C}$ be a category (in our context $\mathbf{C}$ will always be a monoidal category). A *realization of the object-atoms X in the category $\mathbf{C}$* is any function

$$\mathbb{o} : X \to \operatorname{Obj}(\mathbf{C})$$

Note that this is the same thing as a functor from X to $\mathbf{C}$ when we interpret X as a discrete category and therefore also the same thing as an object of the functor category $\mathbf{C}^X$. We will adopt the convention of writing the same symbol $\mathbb{o}$ for the *lifted* functor $\mathbb{o} : X^n \to \mathbf{C}^n$ (for any natural number n) which sends a tuple $\langle \mathfrak{a}_i \rangle_{i=1}^n \in X^n$ to $\langle \mathbb{o}(\mathfrak{a}_i) \rangle_{i=1}^n$. This will shorten the notation and there should usually be no danger of confusion as it should be clear from the context whether something is a list of elements of X or a single element of X.

Let a monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$, generators $X, A, \mathfrak{s}, \mathfrak{t}$ (definition 3.13), a varnishing v_s, v_t (definition 3.15) and a realization $\mathbb{o}$ of X in $\mathbf{C}$ be given. Relative to these data a *valid realization of the (non-trivial) arrow-atoms A* is a function

$$\mathbb{m} : A \to \operatorname{Arr}(\mathbf{C})$$

which satisfies

$$\begin{aligned} s(\mathbb{m}(\mathfrak{f})) &= v_s(\mathfrak{f})(\mathbb{o}(\mathfrak{s}(\mathfrak{f}))) \ \text{ and} \\ t(\mathbb{m}(\mathfrak{f})) &= v_t(\mathfrak{f})(\mathbb{o}(\mathfrak{t}(\mathfrak{f}))) \ . \end{aligned} \tag{36}$$

We adopt the convention of extending any specified valid realization $\mathbb{m}$ of arrow-atoms to the set $A\dot{\cup}X$ (and again using the same symbol for that extension) by setting

$$\mathbb{m}(\mathfrak{a}) := 1_{\mathbb{o}(\mathfrak{a})} \text{ for all } \mathfrak{a} \in X. \tag{37}$$

With the convention set down in equation (31) of definition (3.15) we can still say that the extended maps satisfy equation (36).

We call the pair $\langle \mathbb{o}, \mathbb{m} \rangle$ a (valid) *realization* (of $X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t$).

Definition 3.18. Let $X, A, \mathfrak{s}, \mathfrak{t}$ be generators, v_s, v_t be a varnishing and $\langle \mathbb{o}, \mathbb{m} \rangle$ be a realization of atoms in a monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$. For any path-shape

$$p = \left\langle v_e, \langle \langle v_i, \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_i} \rangle \rangle_{i=k}^{1}, v_b \right\rangle$$

define the *evaluation* of p (in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$) induced by $\langle \mathbb{o}, \mathbb{m} \rangle$ as

$$\begin{aligned}\mathrm{ev}_{\langle \mathbb{o},\mathbb{m} \rangle}(p) := \mathrm{can}_{v_e \leftarrow v_k\left(\langle v_t(\mathfrak{f}_{k,j}) \rangle_{j=1}^{n_k}\right); \mathbb{o}(t(\mathrm{tr}(p)))} \circ v_k\left(\langle \mathbb{m}(\mathfrak{f}_{k,j}) \rangle_{j=1}^{n_k}\right) \circ \\ \left(\circ_{i=k-1}^{1} \left(\mathrm{can}_{v_{i+1}\left(\langle v_s(\mathfrak{f}_{i+1,j}) \rangle_{j=1}^{n_{i+1}}\right) \leftarrow v_i\left(\langle v_t(\mathfrak{f}_{i,j}) \rangle_{j=1}^{n_i}\right); \mathbb{o}\left(\cdot_{i=1}^{n_i}(\not{t}(\mathfrak{f}_i i))\right)} \circ v_i\left(\langle \mathbb{m}(\mathfrak{f}_{i,j}) \rangle_{j=1}^{n_i}\right) \right) \right) \\ \circ \mathrm{can}_{v_1\left(\langle v_s(\mathfrak{f}_{1,j}) \rangle_{j=1}^{n_1}\right) \leftarrow v_b; \mathbb{o}(s(\mathrm{tr}(p)))} \quad .\end{aligned}$$

This is well-defined because we required that target and source of consecutive parts of a path match and that v_e and v_b have the right length. The idea is that v_b and v_e specify the shape of the source and target of the evaluated path while the tensor words v_i specify which of the possible iterated tensor products to use when evaluating a row of the path. We intersperse the composite with structural transformations to make sources and targets match.

Definition 3.19. We say that a realization of atoms $\langle \mathbb{o}, \mathbb{m} \rangle$ satisfies an equation-shape $\langle p, q \rangle$ if

$$\mathrm{ev}_{\langle \mathbb{o},\mathbb{m} \rangle}(p) = \mathrm{ev}_{\langle \mathbb{o},\mathbb{m} \rangle}(q) \quad .$$

Remark 3.20. Whenever we draw diagrams in monoidal categories, what we are writing down are basically path-shapes. The reason for all these very formal definitions is that we will now show a few facts about path-shapes in general — basically stating that they behave as one would expect — so that we can later forget about them and reason about diagrams without having to repeat the same type of argument over and over again.

Lemma 3.21. *Let $X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \langle \mathbb{o}, \mathbb{m} \rangle, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be as in definition 3.18. If $p = \langle v_e, \hat{p}, v_b \rangle$ and $q = \langle w_e, \hat{q}, w_b \rangle$ are path-shapes such that $v_b = w_e$ then*

$$\mathrm{ev}_{\langle \mathbb{o},\mathbb{m} \rangle}(p) \circ \mathrm{ev}_{\langle \mathbb{o},\mathbb{m} \rangle}(q) = \langle v_e, \hat{p} \cdot \hat{q}, w_b \rangle$$

Proof. This is clear from the definition of evaluation of paths and from the coherence theorem. ☐

Lemma 3.22. *Let $X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \langle \mathbb{o}, \mathbb{m} \rangle, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be as in definition 3.18. If $p = \langle v_e, \hat{p}, v_b \rangle$ and $q = \langle w_e, \hat{q}, w_b \rangle$ are path-shapes such that $\mathrm{ptr}(p) = \mathrm{ptr}(q) =: f$ then*

$$\mathrm{can}_{w_e \leftarrow v_e; \mathbb{o}(t(f))} \circ \mathrm{ev}_{\langle \mathbb{o},\mathbb{m} \rangle}(p) = \mathrm{ev}_{\langle \mathbb{o},\mathbb{m} \rangle}(q) \circ \mathrm{can}_{w_b \leftarrow v_b; \mathbb{o}(s(f))} \quad .$$

Proof. Formally this works by induction but the reader may find the following informal argument more convincing. Say $f = \langle\langle \mathfrak{f}_{i,j}\rangle_{j=1}^{n_i}\rangle_{i=k}^{1}$ and therefore

$$\hat{p} = \langle\langle v_i, \langle \mathfrak{f}_{i,j}\rangle_{j=1}^{n_i}\rangle\rangle_{i=k}^{1} \text{ and } \hat{q} = \langle\langle w_i, \langle \mathfrak{f}_{i,j}\rangle_{j=1}^{n_i}\rangle\rangle_{i=k}^{1} \ .$$

Draw a diagram with the two arrows $\mathrm{ev}_{\langle \mathbb{o},\mathbb{m}\rangle}(p)$ and $\mathrm{ev}_{\langle \mathbb{o},\mathbb{m}\rangle}(q)$ and connect the nodes in the diagram with structural transformations. At the source the diagram looks something like this.

$$\begin{tikzcd}
w_b\left(\mathbb{o}\left(\mathscr{s}\left(f\right)\right)\right) \arrow[d] & v_b\left(\mathbb{o}\left(\mathscr{s}\left(f\right)\right)\right) \arrow[l, "\mathbf{can}_{w_b \leftarrow v_b;\mathbb{o}(\mathscr{s}(f))}"'] \arrow[d] \\
w_1\left(\langle v_s\left(\mathfrak{f}_{1,j}\right)\left(\mathbb{o}\left(\mathscr{s}\left(\mathfrak{f}_{1,j}\right)\right)\right)\rangle_{j=1}^{n_1}\right) \arrow[d, "w_1\left(\langle \mathbb{m}(\mathfrak{f}_{1,j})\rangle_{j=1}^{n_1}\right)"'] & v_1\left(\langle v_s\left(\mathfrak{f}_{1,j}\right)\left(\mathbb{o}\left(\mathscr{s}\left(\mathfrak{f}_{1,j}\right)\right)\right)\rangle_{j=1}^{n_1}\right) \arrow[l, "\mathbf{can}_{w_1 \leftarrow v_1;\ldots}"] \arrow[d, "v_1\left(\langle \mathbb{m}(\mathfrak{f}_{1,j})\rangle_{j=1}^{n_1}\right)"] \\
{} & {}
\end{tikzcd}$$

A middle segment looks like

$$\begin{tikzcd}
w_i\left(\langle v_s\left(\mathfrak{f}_{i,j}\right)\left(\mathbb{o}\left(\mathscr{s}\left(\mathfrak{f}_{i,j}\right)\right)\right)\rangle_{j=1}^{n_i}\right) \arrow[d, "w_i\left(\langle \mathbb{m}(\mathfrak{f}_{i,j})\rangle_{j=1}^{n_i}\right)"'] & v_i\left(\langle v_s\left(\mathfrak{f}_{i,j}\right)\left(\mathbb{o}\left(\mathscr{s}\left(\mathfrak{f}_{i,j}\right)\right)\right)\rangle_{j=1}^{n_i}\right) \arrow[l, "\mathbf{can}_{w_i \leftarrow v_i;\ldots}"'] \arrow[d, "v_i\left(\langle \mathbb{m}(\mathfrak{f}_{i,j})\rangle_{j=1}^{n_i}\right)"] \\
w_i\left(\langle v_t\left(\mathfrak{f}_{i,j}\right)\left(\mathbb{o}\left(\mathscr{t}\left(\mathfrak{f}_{i,j}\right)\right)\right)\rangle_{j=1}^{n_i}\right) \arrow[d] & v_i\left(\langle v_t\left(\mathfrak{f}_{i,j}\right)\left(\mathbb{o}\left(\mathscr{t}\left(\mathfrak{f}_{i,j}\right)\right)\right)\rangle_{j=1}^{n_i}\right) \arrow[l, "\mathbf{can}_{w_i \leftarrow v_i;\ldots}"] \arrow[d] \\
w_{i+1}\left(\langle v_s\left(\mathfrak{f}_{i+1,j}\right)\left(\mathbb{o}\left(\mathscr{s}\left(\mathfrak{f}_{i+1,j}\right)\right)\right)\rangle_{j=1}^{n_{i+1}}\right) & v_{i+1}\left(\langle v_s\left(\mathfrak{f}_{i+1,j}\right)\left(\mathbb{o}\left(\mathscr{s}\left(\mathfrak{f}_{i+1,j}\right)\right)\right)\rangle_{j=1}^{n_{i+1}}\right) \arrow[l]
\end{tikzcd}$$

and at the target end it looks like the picture below.

$$\begin{tikzcd}
{} \arrow[d, "w_k\left(\langle \mathbb{m}(\mathfrak{f}_{k,j})\rangle_{j=1}^{n_k}\right)"'] & {} \arrow[d, "v_k\left(\langle \mathbb{m}(\mathfrak{f}_{k,j})\rangle_{j=1}^{n_k}\right)"] \\
w_k\left(\langle v_s\left(\mathfrak{f}_{k,j}\right)\left(\mathbb{o}\left(\mathscr{t}\left(\mathfrak{f}_{k,j}\right)\right)\right)\rangle_{j=1}^{n_k}\right) \arrow[d] & v_k\left(\langle v_s\left(\mathfrak{f}_{k,j}\right)\left(\mathbb{o}\left(\mathscr{t}\left(\mathfrak{f}_{k,j}\right)\right)\right)\rangle_{j=1}^{n_k}\right) \arrow[l, "\mathbf{can}_{w_k \leftarrow v_k;\ldots}"'] \arrow[d] \\
w_e\left(\mathbb{o}\left(\mathscr{t}\left(f\right)\right)\right) & v_e\left(\mathbb{o}\left(\mathscr{t}\left(f\right)\right)\right) \arrow[l, "\mathbf{can}_{w_e \leftarrow v_e;\mathbb{o}(\mathscr{t}(f))}"]
\end{tikzcd}$$

All the unmarked arrows are structural transformations. Considering for example the lower arrow marked with $\mathbf{can}_{w_i \leftarrow v_i;\ldots}$ in the middle segment note that by observation 3.11 the arrows

$$\mathbf{can}_{w_i \leftarrow v_i;\langle v_t(\mathfrak{f}_{i,j})(\mathbb{o}(\mathscr{t}(\mathfrak{f}_{i,j})))\rangle_{j=1}^{n_i}} \quad \text{and} \quad \mathbf{can}_{w_i\left(\langle v_t(\mathfrak{f}_{i,j})\rangle_{j=1}^{n_i}\right) \leftarrow v_i\left(\langle v_t(\mathfrak{f}_{i,j})\rangle_{j=1}^{n_i}\right);\mathscr{t}\left(\langle\langle \mathfrak{f}_{i,j}\rangle_{j=1}^{n_i}\rangle\right)}$$

are equal. Using the first representation (and the corresponding representation for the upper arrow marked with $\mathbf{can}_{w_i \leftarrow v_i;\ldots}$ in the middle segment) we see that the upper square in the middle segment commutes by naturality of the structural transformations and using the second representation we see that the lower square commutes by the coherence theorem. The remaining parts can be seen to commute in an analogous manner. $\square$

Corollary 3.23. *Let $X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \langle \mathbb{o},\mathbb{m}\rangle, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ be as in definition 3.18. If $p = \langle v_e, \hat{p}, v_b\rangle$ and $q = \langle w_e, \hat{q}, w_b\rangle$ are path-shapes such that $v_e = w_e$, $v_b = w_b$ and $\mathrm{ptr}(p) = \mathrm{ptr}(q)$ then $\mathrm{ev}_{\langle \mathbb{o},\mathbb{m}\rangle}(p) = \mathrm{ev}_{\langle \mathbb{o},\mathbb{m}\rangle}(q)$.*

Corollary 3.24. *If* $\langle \mathbb{o}, \mathbb{m} \rangle$ *is a realization of atoms and* $\langle p, q \rangle$ *and* $\langle \dot{p}, \dot{q} \rangle$ *are equation-shapes such that*

$$\operatorname{ptr}(p) = \operatorname{ptr}(\dot{p}) \qquad \text{and} \qquad \operatorname{ptr}(q) = \operatorname{ptr}(\dot{q})$$

then $\langle \mathbb{o}, \mathbb{m} \rangle$ *satisfies* $\langle p, q \rangle$ *if and only if* $\langle \mathbb{o}, \mathbb{m} \rangle$ *satisfies* $\langle \dot{p}, \dot{q} \rangle$*.*

Proof. Because $\operatorname{ptr}(p) = \operatorname{ptr}(\dot{p})$ and because structural transformations are isomorphisms we can use lemma 3.22 to express $\operatorname{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}(p)$ in terms of $\operatorname{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}(\dot{p})$. The same works for q and $\dot{q}$. Moreover the structural transformations sandwiching the evaluated paths are the same in both cases. ☐

Definition 3.25. Let $X, A, \mathfrak{s}, \mathfrak{t}$ be generators. We call a pair $\langle p, q \rangle$ of arrows of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ with $s(p) = s(q)$ and $t(p) = t(q)$ an equation in $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$.

Define a function Tr from the set of all sets of equation-shapes to the set of all sets of equations in $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ by setting

$$\operatorname{Tr}(D) := \{\langle \operatorname{tr}(p), \operatorname{tr}(q) \rangle | \langle p, q \rangle \in D\} \tag{38}$$

for any set D of equation-shapes.

Let E be a set of equations in $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$. We will construct a new strict monoidal category with the same objects as $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ but with all pairs of arrows $\langle p, q \rangle \in E$ identified. Of course if we just identify these arrows we don't get a monoidal category. So we define a relation $\sim_E$ as the closure of E under reflexivity, symmetry, transitivity, composition and product. That is $\sim_E$ is the least (under ordering by inclusion) relation on the set $\operatorname{Arr}(\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t}))$ which contains E and for which for any three morphisms f, g, h in $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ we have

1. $f \sim_E f$
2. $f \sim_E g$ implies $g \sim_E f$
3. $f \sim_E g$ and $g \sim_E h$ implies $f \sim_E h$
4. $f \sim_E g$ implies both $(f \boxdot h) \sim_E (g \boxdot h)$ and $(h \boxdot f) \sim_E (h \boxdot g)$
5. $f \sim_E g$ implies $(f \circ h) \sim_E (g \circ h)$ if both composites are defined; and $f \sim_E g$ implies $(h \circ f) \sim_E (h \circ g)$ if these composites are defined.

That such a relation exists can be seen by either of the two standard arguments, that is either "from the outside" by observing that the properties 1-5 are preserved by intersection of relations and taking the intersection of all relations containing E and satisfying these properties or "from the inside" by constructing the relation $\sim_E$ as a union $\cup_{i=0}^{\infty} \sim_{Ei}$ with index set the natural numbers where $\sim_{E0} = E$ and for each $n \in \mathbb{N}$ the relation $\sim_{En}$ arises from $\sim_{En-1}$ by adding all pairs of arrows generated from the pairs in $\sim_{En-1}$ by one of the rules 1-5. Because all rules only talk about a finite number of arrows the resulting relation is really closed under properties 1-5 and clearly any relation closed under properties 1-5 has to relate at least the arrows related by $\cup_{i=1}^{\infty} \sim_{Ei}$.

Note that the conditions 1-5 are equivalent to conditions 1-3 together with

4a. $f \sim_E g$ and $h \sim_E k$ implies both $(f \boxdot h) \sim_E (g \boxdot k)$ and $(h \boxdot f) \sim_E (k \boxdot g)$

5a. $f \sim_E g$ and $h \sim_E k$ implies $(f \circ h) \sim_E (g \circ k)$ if both composites are defined; $f \sim_E g$ and $h \sim_E k$ implies $(h \circ f) \sim_E (k \circ g)$ if these composites are defined.

Lemma 3.26. *$f \sim_E g$ implies $s(f) = s(g)$ and $t(f) = t(g)$.*

Proof. This is true for the generating set E by the definition 3.25 of an equation in $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ and clearly this property is preserved under 1-5 above. □

Remark 3.27. This also means that modifying 5 above to only require one of the composites to be defined does not change anything.

We will need the following later.

Lemma 3.28 (explicit description of $\sim_E$)**.** *Let*

$$E' := E \cup \left\{ \langle q, p \rangle \middle| \langle p, q \rangle \in E \right\} \quad .$$

Then $f \sim_E g$ if and only if there exists a natural number n and sequences

$$\langle p_1, q_1 \rangle, \ldots, \langle p_n, q_n \rangle \ , \ h_1, \ldots, h_n \ , \ \dot{h}_1, \ldots, \dot{h}_n \ , \ k_1, \ldots, k_n \ , \ \dot{k}_1, \ldots, \dot{k}_n$$

with $\langle p_i, q_i \rangle \in E'$ and $h_i, \dot{h}_i, k_i, \dot{k}_i \in \mathrm{Arr}(\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t}))$ for all $i \in \{1, \ldots, n\}$ such that

$$\begin{aligned} f &= k_1 \circ \left(h_1 \boxdot p_1 \boxdot \dot{h}_1 \right) \circ \dot{k}_1 \\ k_i \circ \left(h_i \boxdot q_i \boxdot \dot{h}_i \right) \circ \dot{k}_i &= k_{i+1} \circ \left(h_{i+1} \boxdot p_{i+1} \boxdot \dot{h}_{i+1} \right) \circ \dot{k}_{i+1} \\ &\qquad \text{for all } i \in \{1, \ldots, n-1\} \\ k_n \circ \left(h_n \boxdot q_n \boxdot \dot{h}_n \right) \circ \dot{k}_n &= g \quad . \end{aligned} \tag{39}$$

We also allow n to be zero and interpret this case to mean that $f = g$.

Proof. Clearly if such a sequence exists then $\sim_E$ relates f and g. Call the relation which relates f and g if and only if these sequences satisfying (39) exist $\sim'_E$. We need to show that $\sim'_E$ contains E and is closed under 1-5 above. If $\langle f, g \rangle \in E$ then choose $n = 1$, $p_1 = f$, $q_1 = g$, $h_1 = \dot{h}_1 = \pi_{\diamond}(\langle \langle \rangle \rangle)$, $k_1 = 1_{t(f)} = 1_{t(g)}$ and $\dot{k}_1 = 1_{s(f)} = 1_{s(g)}$ (these are equal by the previous lemma). Closure:

1, reflexivity: Empty chain.

2, symmetry: Reverse the direction of the chain.

3, transitivity: Concatenate the chains.

4, closure under product: Assume f, g, h are arrows of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ and

$$f, g \ , \ \langle p_1, q_1 \rangle, \ldots, \langle p_n, q_n \rangle \ , \ h_1, \ldots, h_n \ , \ \dot{h}_1, \ldots, \dot{h}_n \ , \ k_1, \ldots, k_n \ , \ \dot{k}_1, \ldots, \dot{k}_n$$

satisfy equation (39). Then

$$\begin{aligned} h \boxdot f = h \boxdot \left(k_1 \circ \left(h_1 \boxdot p_1 \boxdot \dot{h}_1 \right) \circ \dot{k}_1 \right) = \\ \left(1_{t(h)} \boxdot k_1 \right) \circ \left((h \boxdot h_1) \boxdot p_1 \boxdot \dot{h}_1 \right) \circ \left(1_{s(h)} \boxdot k_1 \right) \end{aligned}$$

$$\left(1_{t(\hbar)} \boxdot k_i\right) \circ \left(\left(\hbar \boxdot \hbar_i\right) \boxdot q_i \boxdot \dot{\hbar}_i\right) \circ \left(1_{s(\hbar)} \boxdot k_i\right) =$$
$$\hbar \boxdot \left(k_i \circ \left(\hbar_i \boxdot q_i \boxdot \dot{\hbar}_i\right) \circ \dot{k}_i\right) =$$
$$\hbar \boxdot \left(k_{i+1} \circ \left(\hbar_{i+1} \boxdot p_{i+1} \boxdot \dot{\hbar}_{i+1}\right) \circ \dot{k}_{i+1}\right) =$$
$$\left(1_{t(\hbar)} \boxdot k_{i+1}\right) \circ \left(\left(\hbar \boxdot \hbar_{i+1}\right) \boxdot p_{i+1} \boxdot \dot{\hbar}_{i+1}\right) \circ \left(1_{s(\hbar)} \boxdot k_{i+1}\right)$$
$$\text{for all } i \in \{1,\ldots,n-1\}$$

$$\left(1_{t(\hbar)} \boxdot k_n\right) \circ \left(\left(\hbar \boxdot \hbar_n\right) \boxdot q_n \boxdot \dot{\hbar}_n\right) \circ \left(1_{s(\hbar)} \boxdot k_n\right) =$$
$$\hbar \boxdot \left(k_n \circ \left(\hbar_n \boxdot q_n \boxdot \dot{\hbar}_n\right) \circ \dot{k}_n\right) = \hbar \boxdot g \ .$$

Multiplication on the right works in the same way.

5, closure under composition: If f, g are connected by a chain as before and $s(k) = t(f) = t(g)$, then by lemma 3.26 we can compose k with each of the k_i. This new sequence connects $k \circ f$ and $k \circ g$. The same works on the other side.

□

Definition 3.29. Let $X, A, \mathfrak{s}, \mathfrak{t}$ be generators and let E be a set of equations in $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$. The strict monoidal category $\left\langle \tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t}), \boxdot, \langle\rangle \right\rangle$ and the strict monoidal functor $\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}$ are defined by

- $\mathrm{Obj}\left(\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})\right) = \mathrm{Obj}(\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t}))$, $\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(a) = a$ for all $a \in \mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$.
- $\mathrm{Arr}\left(\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})\right) = \mathrm{Arr}(\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t}))/\sim_E$ where $\sim_E$ is the relation discussed in detail above. For all morphisms f of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ the arrow $\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(f)$ of $\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})$ is the equivalence class of f under $\sim_E$.
- 1_a in $\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})$ is the equivalence class of 1_a in $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$.
- $s\left(\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(f)\right) := s(f)$ and $t\left(\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(f)\right) := t(f)$
- The composite of two arrows $\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(f)$ and $\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(g)$ is
$$\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(f) \circ \tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(g) := \tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(f \circ g) \ .$$
- $\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(f) \boxdot \tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(g) := \tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(f \boxdot g)$.

When $X, A, \mathfrak{s}, \mathfrak{t}$ are clear from the context we will abbreviate $\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})$ to $\tilde{\mathcal{E}}_E$ and $\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}$ to $\tilde{\pi}_E$. We also define $\tilde{\pi}'_E := \tilde{\pi}_E \circ \pi_{\approx}$.

Lemma 3.30. *$\left\langle \tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t}), \boxdot, \langle\rangle \right\rangle$ and $\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}$ are well-defined. The structure $\left\langle \tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t}), \boxdot, \langle\rangle \right\rangle$ is really a strict monoidal category and the maps denoted by $\tilde{\pi}_{E,X,A,\mathfrak{s},\mathfrak{t}}$ really constitute a strict monoidal functor.*

Proof. Source and target are well-defined by lemma 3.26. Composition is well-defined by property 5a. of the relation $\sim_E$. The bifunctor $\boxdot$ is well-defined by property 4a. of the relation $\sim_E$. Properties such as $s(1_a) = a$ and associativity of composition and $\boxdot$ are inherited from $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$. $\tilde{\varpi}_{E,X,A,\mathfrak{s},\mathfrak{t}}$ is a strict monoidal functor by the very definition of the operations on $\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})$. □

Lemma 3.31. *Let $X, A, \mathfrak{s}, \mathfrak{t}$ be generators and let E be a set of equations in $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$. If $\langle H, H_2, H_0\rangle : \langle \mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t}), \boxdot, \langle\rangle\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ is a strong monoidal functor and $H(p) = H(q)$ for all $\langle p, q\rangle \in E$ then there is a unique strong monoidal functor $\langle L, L_2, L_0\rangle : \left\langle \tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t}), \boxdot, \langle\rangle\right\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ such that*

$$\langle L, L_2, L_0\rangle \circ \tilde{\varpi}_{E,X,A,\mathfrak{s},\mathfrak{t}} = \langle H, H_2, H_0\rangle \ . \tag{40}$$

Proof. By the definition 3.1 of composition of monoidal functors and because $\tilde{\varpi}_{E,X,A,\mathfrak{s},\mathfrak{t}}$ is a strict monoidal functor equation (40) above implies that $L_{2;a,b} = L(1) \circ L_{2;\tilde{\varpi}_E(a),\tilde{\varpi}_E(b)} = H_{2;a,b}$ and $L_0 = L(1) \circ L_0 = H_0$. It also implies

$$L\big(\tilde{\varpi}_E(f)\big) = H(f) \ . \tag{41}$$

So for any equivalence-class $\tilde{\varpi}_{E,X,A,\mathfrak{s},\mathfrak{t}}(f)$ of arrows of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ (which is the same thing as an arrow of $\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})$) there is at most one possible value for L. We still have to show that using (41) as the definition of L is well-defined, that is we have to show that $f \sim_E g$ implies $H(f) = H(g)$. This is true by hypothesis for any pair $\langle p, q\rangle \in E$. We need to check that the relation "images of f and g under H are equal and $f \sim_E g$" satisfies closure properties 1-5 which define the relation $\sim_E$. 1-3 are obvious. 5 is satisfied because H is a functor. It remains to check 4. So let f, g, h be morphisms of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ such that $H(f) = H(g)$ and $f \sim_E g$. By lemma 3.26 we have $s(f) = s(g) =: a$ and $t(f) = t(g) =: b$.

$$\begin{tikzcd}
H(a \boxdot s(h)) \arrow[d, "H(f \boxdot h)"'] & H(a) \boxtimes H(s(h)) \arrow[l, "H_{2;a,s(h)}"'] \arrow[r, "H_{2;a,s(h)}"] \arrow[d, "H(f)\boxtimes H(h)"', "=H(g)\boxtimes H(h)"] & H(a \boxdot s(h)) \arrow[d, "H(g \boxdot h)"] \\
H(b \boxdot t(h)) & H(b) \boxtimes H(t(h)) \arrow[l, "H_{2;b,t(h)}"] \arrow[r, "H_{2;b,t(h)}"'] & H(b \boxdot t(h))
\end{tikzcd}$$

H_2 is an isomorphism because $\langle H, H_2, H_0\rangle$ is strong and therefore

$$H\big(f \boxdot h\big) = H\big(g \boxdot h\big) \ .$$

Closure under product from the left is shown in the exact same way. □

Remark 3.32. This means that $\left\langle \tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t}), \tilde{\varpi}_{E,X,A,\mathfrak{s},\mathfrak{t}}\right\rangle$ is the coequalizer of a suitable pair of strict monoidal functors in the category of strong monoidal functors.

For any realization of atoms we will now construct a specific strong monoidal functor whose value on any arrow f of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ is given by evaluating a specific path whose trace is f. In conjunction with what we already know this will make it clear that if $\langle \mathbb{o}, \mathbb{m} \rangle$ is a realization of atoms which satisfies some set of equation-shapes D and we set $E := \mathrm{Tr}(D)$ then for any two paths $p = \langle v_e, \hat{p}, v_b \rangle$ and $q = \langle w_e, \hat{q}, w_b \rangle$ we have that in generalisation of lemma 3.22 the equality $\tilde{\mathbb{\pi}}_{E,X,A,\mathfrak{s},\mathfrak{t}}(\mathrm{tr}(p)) = \tilde{\mathbb{\pi}}_{E,X,A,\mathfrak{s},\mathfrak{t}}(\mathrm{tr}(p))$ implies $\mathrm{can}_{w_e \leftarrow v_e ; \mathbb{o}(\mathscr{t}(f))} \circ \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}(p) = \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}(q) \circ \mathrm{can}_{w_b \leftarrow v_b ; \mathbb{o}(\mathscr{s}(f))}$.

Let $X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be as in definition 3.18.

We will define a function $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}$ which sends realizations $\langle \mathbb{o}, \mathbb{m} \rangle$ to monoidal functors $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}(\langle \mathbb{o}, \mathbb{m} \rangle)$.

Write $\langle H, H_2, H_0 \rangle = K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}(\langle \mathbb{o}, \mathbb{m} \rangle)$.

We define

$$H\left(\langle \mathfrak{a}_i \rangle_{i=1}^{n}\right) := v^{(n)}\left(\langle \mathbb{o}(\mathfrak{a}_i) \rangle_{i=1}^{n}\right) \ . \tag{42}$$

On single-row arrows $\pi_{\bumpeq}(\langle \langle \mathfrak{f}_i \rangle_{i=1}^{n} \rangle)$ of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ with $\mathscr{s}(\mathfrak{f}_i) = \langle \mathfrak{a}_{i,j} \rangle_{j=1}^{l_i}$ and $\mathscr{t}(\mathfrak{f}_i) = \langle \mathfrak{b}_{i,j} \rangle_{j=1}^{l_i}$ we define H as the composite

$$\begin{tikzcd}
\Big(\mathcal{S}(\mathbb{o}(\mathfrak{a}_{1,1})) \circ \ldots \circ \mathcal{S}(\mathbb{o}(\mathfrak{a}_{1,l_1})) \circ \ldots \circ \mathcal{S}(\mathbb{o}(\mathfrak{a}_{n,1})) \circ \cdots \circ \mathcal{S}(\mathbb{o}(\mathfrak{a}_{n,l_n}))\Big)(e) \arrow[d, "\left(\mathcal{S}_{v_s(\mathfrak{f}_1), \mathbb{o}(\mathscr{s}(\mathfrak{f}_1))} \circ \cdots \circ \mathcal{S}_{v_s(\mathfrak{f}_n), \mathbb{o}(\mathscr{s}(\mathfrak{f}_n))}\right)_e"] \arrow[dddd, bend right, "H(\langle\langle \mathfrak{f}_1, \ldots, \mathfrak{f}_n \rangle\rangle)"'] \\
\Bigg(\mathcal{S}\Big(v_s(\mathfrak{f}_1)(\mathbb{o}(\mathfrak{a}_{1,1}), \ldots, \mathbb{o}(\mathfrak{a}_{1,l_1}))\Big) \circ \ldots \circ \mathcal{S}\Big(v_s(\mathfrak{f}_n)(\mathbb{o}(\mathfrak{a}_{n,1}), \ldots, \mathbb{o}(\mathfrak{a}_{n,l_n}))\Big)\Bigg)(e) \arrow[d, "\left(\mathcal{S}(\mathbb{m}(\mathfrak{f}_1)) \circ \ldots \circ \mathcal{S}(\mathbb{m}(\mathfrak{f}_n))\right)_e"] \\
\Bigg(\mathcal{S}\Big(v_t(\mathfrak{f}_1)(\mathbb{o}(\mathfrak{b}_{1,1}), \ldots, \mathbb{o}(\mathfrak{b}_{1,m_1}))\Big) \circ \ldots \circ \mathcal{S}\Big(v_t(\mathfrak{f}_n)(\mathbb{o}(\mathfrak{b}_{n,1}), \ldots, \mathbb{o}(\mathfrak{b}_{n,m_n}))\Big)\Bigg)(e) \\
\Big(\mathcal{S}(\mathbb{o}(\mathfrak{b}_{1,1})) \circ \ldots \circ \mathcal{S}(\mathbb{o}(\mathfrak{b}_{1,m_1})) \circ \cdots \circ \mathcal{S}(\mathbb{o}(\mathfrak{b}_{n,1})) \circ \ldots \circ \mathcal{S}(\mathbb{o}(\mathfrak{a}_{n,m_n}))\Big)(e) \arrow[u, "\left(\mathcal{S}_{v_t(\mathfrak{f}_1), \mathbb{o}(\mathscr{t}(\mathfrak{f}_1))} \circ \cdots \circ \mathcal{S}_{v_t(\mathfrak{f}_n), \mathbb{o}(\mathscr{t}(\mathfrak{f}_n))}\right)_e"]
\end{tikzcd}$$

(43)

The upper and lower arrows are the (unique by the coherence theorem) structural transformations.

For any arrow $\pi_{\bumpeq}(\langle \langle \mathfrak{f}_{n,1}, \ldots, \mathfrak{f}_{n,k_n} \rangle, \ldots, \langle \mathfrak{f}_{1,1}, \ldots, \mathfrak{f}_{1,k_1} \rangle \rangle)$ of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ we define

$$\begin{aligned} &H\left(\pi_{\bumpeq}(\langle \langle \mathfrak{f}_{n,1}, \ldots, \mathfrak{f}_{n,k_n} \rangle, \ldots, \langle \mathfrak{f}_{1,1}, \ldots, \mathfrak{f}_{1,k_1} \rangle \rangle)\right) := \\ &\qquad H\left(\pi_{\bumpeq}(\langle \mathfrak{f}_{n,1}, \ldots, \mathfrak{f}_{n,k_n} \rangle)\right) \circ \cdots \circ H\left(\pi_{\bumpeq}(\langle \mathfrak{f}_{1,1}, \ldots, \mathfrak{f}_{1,k_1} \rangle)\right) \ . \end{aligned} \tag{44}$$

Observation 3.33. *If* $f = \left\langle \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{k_i} \right\rangle_{i=n}^{1}$ *then*

$$\begin{aligned} &K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}(\langle \mathbb{o}, \mathbb{m} \rangle)(f) = \\ &\qquad \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(\left\langle v^{(\ell(\mathscr{t}(f)))}, \left\langle \left\langle v^{(k_i)}, \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{k_i} \right\rangle \right\rangle_{i=n}^{1}, v^{(\ell(\mathscr{s}(f)))} \right\rangle\right) \ . \end{aligned} \tag{45}$$

By its very definition H preserves composition. If all of the $\mathfrak{f}_i$ in (43) are identities (that is, elements of X), then the middle arrow is an identity and by the convention set down in definition 3.15 we have that $v_s(\mathfrak{f}_i) = v_t(\mathfrak{f}_i)$.

Therefore the composite in (44) is an identity. Now we still need to show that H is well-defined — that is, that the definition given is compatible with the vertical shifting of components of representations of arrows of $\mathcal{E}$. Because of the way in which we have defined H, it suffices to consider an arrow with two rows. So let $\left\langle \left\langle \mathfrak{g}_1, \ldots, \mathfrak{g}_{n_\mathfrak{g}} \right\rangle, \left\langle \mathfrak{f}_1, \ldots, \mathfrak{f}_{n_\mathfrak{f}} \right\rangle \right\rangle$ be an element of $\mathrm{pArr}\,(X, A, \mathfrak{s}, \mathfrak{t})$. When vertical shifting is allowed we can partition each of the lists

$$\left\langle \mathfrak{f}_1, \ldots, \mathfrak{f}_{n_\mathfrak{f}} \right\rangle,\ \left\langle \mathfrak{g}_1, \ldots, \mathfrak{g}_{n_\mathfrak{g}} \right\rangle,\ s\left(\left\langle \left\langle \mathfrak{f}_1, \ldots, \mathfrak{f}_{n_\mathfrak{f}} \right\rangle \right\rangle\right),$$
$$t\left(\left\langle \left\langle \mathfrak{f}_1, \ldots, \mathfrak{f}_{n_\mathfrak{f}} \right\rangle \right\rangle\right) = s\left(\left\langle \left\langle \mathfrak{g}_1, \ldots, \mathfrak{g}_{n_\mathfrak{g}} \right\rangle \right\rangle\right) \text{ and } t\left(\left\langle \left\langle \mathfrak{g}_1, \ldots, \mathfrak{g}_{n_\mathfrak{g}} \right\rangle \right\rangle\right)$$

into three parts, such that the sources and targets of arrows in the first parts are in the first parts of the object lists, and analogously for the second and third parts. Because the horizontal composition of natural transformations is an associative bifunctor we can under these circumstances write the composite

$$\begin{aligned}&\left(\mathcal{S}^{-1}_{v_t(\mathfrak{g}_1)} \circ \cdots \circ \mathcal{S}^{-1}_{v_t\left(\mathfrak{g}_{n_\mathfrak{g}}\right)}\right) \bullet \left(\mathcal{S}\left(\mathfrak{m}(\mathfrak{g}_1)\right) \circ \ldots \circ \mathcal{S}\left(\mathfrak{m}\left(\mathfrak{g}_{n_\mathfrak{g}}\right)\right)\right) \bullet \\ &\left(\mathcal{S}_{v_s(\mathfrak{g}_1)} \circ \cdots \circ \mathcal{S}_{v_s\left(\mathfrak{g}_{n_\mathfrak{g}}\right)}\right) \bullet \left(\mathcal{S}^{-1}_{v_t(\mathfrak{f}_1)} \circ \cdots \circ \mathcal{S}^{-1}_{v_t\left(\mathfrak{f}_{n_\mathfrak{f}}\right)}\right) \bullet \\ &\left(\mathcal{S}\left(\mathfrak{m}(\mathfrak{f}_1)\right) \circ \ldots \circ \mathcal{S}\left(\mathfrak{m}\left(\mathfrak{f}_{n_\mathfrak{f}}\right)\right)\right) \bullet \left(\mathcal{S}_{v_s(\mathfrak{f}_1)} \circ \cdots \circ \mathcal{S}_{v_s\left(\mathfrak{f}_{n_\mathfrak{f}}\right)}\right)\end{aligned} \tag{46}$$

(which is just two chained instances of (43) with evaluation at e omitted) as

$$\begin{aligned}&\Bigg(\left(\mathcal{S}^{-1}_{v_t(\mathfrak{g}_1)} \circ \cdots \circ \mathcal{S}^{-1}_{v_t(\mathfrak{g}_k)}\right) \bullet \left(\mathcal{S}\left(\mathfrak{m}(\mathfrak{g}_1)\right) \circ \ldots \circ \mathcal{S}\left(\mathfrak{m}(\mathfrak{g}_k)\right)\right) \bullet \\ &\left(\mathcal{S}_{v_s(\mathfrak{g}_1)} \circ \cdots \circ \mathcal{S}_{v_s(\mathfrak{g}_k)}\right) \bullet \left(\mathcal{S}^{-1}_{v_t(\mathfrak{f}_1)} \circ \cdots \circ \mathcal{S}^{-1}_{v_t(\mathfrak{f}_i)}\right) \bullet \\ &\left(\mathcal{S}\left(\mathfrak{m}(\mathfrak{f}_1)\right) \circ \ldots \circ \mathcal{S}\left(\mathfrak{m}(\mathfrak{f}_i)\right)\right) \bullet \left(\mathcal{S}_{v_s(\mathfrak{f}_1)} \circ \cdots \circ \mathcal{S}_{v_s(\mathfrak{f}_i)}\right)\Bigg) \circ \\ &\Bigg(\left(\mathcal{S}^{-1}_{v_t(\mathfrak{g}_{k+1})} \circ \cdots \circ \mathcal{S}^{-1}_{v_t(\mathfrak{g}_l)}\right) \bullet \left(\mathcal{S}\left(\mathfrak{m}(\mathfrak{g}_{k+1})\right) \circ \ldots \circ \mathcal{S}\left(\mathfrak{m}(\mathfrak{g}_l)\right)\right) \bullet \\ &\left(\mathcal{S}_{v_s(\mathfrak{g}_{k+1})} \circ \cdots \circ \mathcal{S}_{v_s(\mathfrak{g}_l)}\right) \bullet \left(\mathcal{S}^{-1}_{v_t(\mathfrak{f}_{i+1})} \circ \cdots \circ \mathcal{S}^{-1}_{v_t(\mathfrak{f}_j)}\right) \bullet \\ &\left(\mathcal{S}\left(\mathfrak{m}(\mathfrak{f}_{i+1})\right) \circ \ldots \circ \mathcal{S}\left(\mathfrak{m}(\mathfrak{f}_j)\right)\right) \bullet \left(\mathcal{S}_{v_s(\mathfrak{f}_{i+1})} \circ \cdots \circ \mathcal{S}_{v_s(\mathfrak{f}_j)}\right)\Bigg) \circ \\ &\Bigg(\left(\mathcal{S}^{-1}_{v_t(\mathfrak{g}_{l+1})} \circ \cdots \circ \mathcal{S}^{-1}_{v_t\left(\mathfrak{g}_{n_g}\right)}\right) \bullet \left(\mathcal{S}\left(\mathfrak{m}(\mathfrak{g}_{l+1})\right) \circ \ldots \circ \mathcal{S}\left(\mathfrak{m}\left(\mathfrak{g}_{n_g}\right)\right)\right) \bullet \\ &\left(\mathcal{S}_{v_s(\mathfrak{g}_{l+1})} \circ \cdots \circ \mathcal{S}_{v_s\left(\mathfrak{g}_{n_g}\right)}\right) \bullet \left(\mathcal{S}^{-1}_{v_t(\mathfrak{f}_{j+1})} \circ \cdots \circ \mathcal{S}^{-1}_{v_t\left(\mathfrak{f}_{n_f}\right)}\right) \bullet \\ &\left(\mathcal{S}\left(\mathfrak{m}(\mathfrak{f}_{j+1})\right) \circ \ldots \circ \mathcal{S}\left(\mathfrak{m}\left(\mathfrak{f}_{n_f}\right)\right)\right) \bullet \left(\mathcal{S}_{v_s(\mathfrak{f}_{j+1})} \circ \cdots \circ \mathcal{S}_{v_s\left(\mathfrak{f}_{n_f}\right)}\right)\Bigg)\end{aligned} \tag{47}$$

where i is the first splitting point for the $\mathfrak{f}$'s, j is the second splitting point for the $\mathfrak{f}$'s, k is the first splitting point for the $\mathfrak{g}$'s and l is the second splitting point for the $\mathfrak{g}$'s. When all of

$$\mathfrak{f}_{i+1}, \ldots, \mathfrak{f}_j$$

are identities (that is, elements of X), then

$$\Big(\mathcal{S}(\mathbb{m}(\mathfrak{f}_{i+1})) \circ \ldots \circ \mathcal{S}(\mathbb{m}(\mathfrak{f}_j))\Big)$$

is also an identity and by convention

$$(\_) = v_s(\mathfrak{f}_{i+1}) = v_t(\mathfrak{f}_{i+1}), v_s(\mathfrak{f}_{i+2}) = v_t(\mathfrak{f}_{i+2}), \quad \ldots, \quad v_s(\mathfrak{f}_j) = v_t(\mathfrak{f}_j) \ .$$

Therefore

$$\Big(\mathcal{S}^{-1}_{v_t(\mathfrak{f}_{i+1})} \circ \cdots \circ \mathcal{S}^{-1}_{v_t(\mathfrak{f}_j)}\Big) \bullet \\ \Big(\mathcal{S}(\mathbb{m}(\mathfrak{f}_{i+1})) \circ \ldots \circ \mathcal{S}(\mathbb{m}(\mathfrak{f}_j))\Big) \bullet \Big(\mathcal{S}_{v_s(\mathfrak{f}_{i+1})} \circ \cdots \circ \mathcal{S}_{v_s(\mathfrak{f}_j)}\Big) = 1 \ .$$

By adding an identity

$$\mathcal{S}(\mathbb{m}(\pi_1(\ell(\mathfrak{g}_{k+1})))) \circ \cdots \circ \mathcal{S}\Big(\mathbb{m}\Big(\pi_{\ell(\ell(\mathfrak{g}_{k+1}))}(\ell(\mathfrak{g}_{k+1}))\Big)\Big) \circ \cdots \\ \cdots \circ \mathcal{S}(\mathbb{m}(\pi_1(\ell(\mathfrak{g}_l)))) \circ \cdots \circ \mathcal{S}\Big(\mathbb{m}\Big(\pi_{\ell(\ell(\mathfrak{g}_l))}(\ell(\mathfrak{g}_l))\Big)\Big)$$

on the other side of the middle expression and retracing our steps backwards with the shifted representation $\Big\langle \Big\langle \mathfrak{g}'_1, \ldots, \mathfrak{g}'_{n_{\mathfrak{g}'}} \Big\rangle, \Big\langle \mathfrak{f}'_1, \ldots, \mathfrak{f}'_{n_{\mathfrak{f}'}} \Big\rangle\Big\rangle$ given by

$$\begin{aligned} \langle \mathfrak{f}'_1, \ldots, \mathfrak{f}'_i \rangle &= \langle \mathfrak{f}_1, \ldots, \mathfrak{f}_i \rangle \\ \Big\langle \mathfrak{f}'_{i+1}, \ldots, \mathfrak{f}'_{i+(l-k)} \Big\rangle &= \langle \mathfrak{g}_k, \ldots, \mathfrak{g}_l \rangle \\ \Big\langle \mathfrak{f}'_{i+(l-k)+1}, \ldots, \mathfrak{f}'_{n_{\mathfrak{f}'}} \Big\rangle &= \big\langle \mathfrak{f}_{j+1}, \ldots, \mathfrak{f}_{n_{\mathfrak{f}}} \big\rangle \\ \langle \mathfrak{g}'_1, \ldots, \mathfrak{g}'_k \rangle &= \langle \mathfrak{g}_1, \ldots, \mathfrak{g}_k \rangle \\ \Big\langle \mathfrak{g}'_{k+1}, \ldots, \mathfrak{g}'_{k+\ell(t(\langle\langle \mathfrak{g}_{k+1}, \ldots, \mathfrak{g}_l \rangle\rangle))} \Big\rangle &= t(\langle\langle \mathfrak{g}_{k+1}, \ldots, \mathfrak{g}_l \rangle\rangle) \\ \Big\langle \mathfrak{g}'_{k+\ell(t(\langle\langle \mathfrak{g}_{k+1}, \ldots, \mathfrak{g}_l \rangle\rangle))+1}, \ldots, \mathfrak{g}'_{n_{\mathfrak{g}'}} \Big\rangle &= \big\langle \mathfrak{g}_{l+1}, \ldots, \mathfrak{g}_{n_{\mathfrak{g}}} \big\rangle \end{aligned}$$

we can bring (47) back into a form of type (46) and we see that H is compatible with the relation $\Leftrightarrow$. This concludes our proof that H is a functor.

It is also a strong monoidal functor with $H_0 = 1_e$ and $H_{2;\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_m \rangle, \langle \mathfrak{b}_1, \ldots, \mathfrak{b}_n \rangle}$: $H(\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_m \rangle) \boxtimes H(\langle \mathfrak{b}_1, \ldots, \mathfrak{b}_n \rangle) \to H(\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b}_1, \ldots, \mathfrak{b}_n \rangle)$ the structural transformation $\mathrm{can}_{v^{(m+n)} \leftarrow v^{(m)} \square v^{(n)}; \mathbb{o}(\langle \mathfrak{a}_i \rangle_{i=1}^m \cdot \langle \mathfrak{b}_j \rangle_{j=1}^n)}$. Naturality of H_2 follows from lemma 3.22 because — using bifunctoriality of $\boxtimes$ — we see that both $H(f) \boxtimes H(g)$ and $H(f \boxdot g)$ are the result of evaluating a path, the ptr of which is given by some representation of $f \boxdot g$ in both cases (and we can choose the same representation in both cases).

The equations that must hold for a monoidal functor hold by the coherence theorem.

In summary we have

Definition 3.34. Let $X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be as in definition 3.18 and let $\langle \mathbb{o}, \mathbb{m} \rangle$ be a realization of atoms. The strong monoidal functor

$$K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)$$

which in the scope of this definition we write as $\langle H, H_2, H_0 \rangle$ is defined by

$$\begin{aligned} H\left(\langle \mathfrak{a}_i \rangle_{i=1}^{n}\right) &:= v^{(n)}\left(\langle \mathbb{o}(\mathfrak{a}_i) \rangle_{i=1}^{n}\right) \\ H\left(f\right) &:= \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(\left\langle v^{(\ell(t(f)))}, \left\langle \left\langle v^{(k_i)}, \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{k_i} \right\rangle \right\rangle_{i=n}^{1}, v^{(\ell(s(f)))} \right\rangle\right) \\ &\qquad \text{(where } f = \left\langle \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{k_i} \right\rangle_{i=n}^{1}) \qquad (48) \\ H_0 &:= 1_e \\ H_{2;\langle \mathfrak{a}_i \rangle_{i=1}^{m}, \langle \mathfrak{b}_j \rangle_{j=1}^{n}} &:= \mathrm{can}_{v^{(m+n)} \leftarrow v^{(m)} \Box v^{(n)}; \mathbb{o}\left(\langle \mathfrak{a}_i \rangle_{i=1}^{m} \cdot \langle \mathfrak{b}_j \rangle_{j=1}^{n}\right)} \quad . \qquad (49) \end{aligned}$$

We use only the first equation above to define a functor $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}} : \mathbf{C}^X \to \mathbf{C}^{X^*}$, where we interpret both X and the set of X^* of sequences in letters taken from X as discrete categories. So $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}\left(\mathbb{o}\right) : X^* \to \mathbf{C}$.

When clear from the context we will omit (some of) the subscripts for K. As we have already done in observation 3.33, we will sometimes write $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)\left(f\right)$ or $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)\left(a\right)$ when we mean that we are applying the functor component H of $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)$ to an object or a morphism of $\mathcal{E}\left(X, A, \mathfrak{s}, \mathfrak{t}\right)$. It should be clear from the context when we use such shortcut.

As promised we get a theorem which significantly simplifies calculations in monoidal categories.

Theorem 3.35 (Don't Worry Theorem)**.** *Let $X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be as in definition 3.18, let D be a set of equation-shapes and let $\langle \mathbb{o}, \mathbb{m} \rangle$ be a realization of atoms which satisfies all equation-shapes in D. Set $E := \mathrm{Tr}(D)$. If $p = \langle v_e, \hat{p}, v_b \rangle$ and $q = \langle w_e, \hat{q}, w_b \rangle$ are path-shapes and*

$$\tilde{\mathbb{m}}_{E,X,A,\mathfrak{s},\mathfrak{t}}(\mathrm{tr}(p)) = \tilde{\mathbb{m}}_{E,X,A,\mathfrak{s},\mathfrak{t}}(\mathrm{tr}(q))$$

then with the notation $p := \mathrm{tr}(p)$, $q := \mathrm{tr}(q)$, $a := s\left(p\right) = s\left(q\right)$ and $b := t\left(p\right) = t\left(q\right)$ we have

$$\mathrm{can}_{w_e \leftarrow v_e; \mathbb{o}(b)} \circ \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(p\right) = \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(q\right) \circ \mathrm{can}_{w_b \leftarrow v_b; \mathbb{o}(a)} \quad .$$

Proof. The real work has already been done. If $\mathrm{ptr}(p) = \left\langle \langle \mathfrak{p}_{i,j} \rangle_{j=1}^{k_i} \right\rangle_{i=n}^{1}$ and $\mathrm{ptr}(q) = \left\langle \langle \mathfrak{q}_{i,j} \rangle_{j=1}^{k_i} \right\rangle_{i=n}^{1}$ then by the coherence theorem and by lemma 3.22

$$\begin{aligned} &\mathrm{can}_{w_e \leftarrow v_e; \mathbb{o}(b)} \circ \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(p\right) = \mathrm{can}_{w_e \leftarrow v^{(\ell(b))}; \mathbb{o}(b)} \circ \mathrm{can}_{v^{(\ell(b))} \leftarrow v_e; \mathbb{o}(b)} \circ \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(p\right) = \\ &\quad \mathrm{can}_{w_e \leftarrow v^{(\ell(b))}; \mathbb{o}(b)} \circ \\ &\quad \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(\left\langle v^{(\ell(b))}, \left\langle \left\langle v^{(k_i)}, \langle \mathfrak{p}_{i,j} \rangle_{j=1}^{k_i} \right\rangle \right\rangle_{i=n}^{1}, v^{(\ell(a))} \right\rangle\right) \circ \mathrm{can}_{v^{(\ell(a))} \leftarrow v_b; \mathbb{o}(a)} = \end{aligned}$$

— by definition 3.34 —

$$\mathrm{can}_{w_e \leftarrow v^{(\ell(b))}; \mathbb{o}(b)} \circ K\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)\left(p\right) \circ \mathrm{can}_{v^{(\ell(a))} \leftarrow v_b; \mathbb{o}(a)} \quad .$$

By lemma 3.31 and because $\langle \mathbb{o}, \mathbb{m} \rangle$ satisfies the equations in D there is a functor L such that this is equal to

$$\mathrm{can}_{w_e \leftarrow v^{(\ell(\mathcal{b}))};\mathbb{o}(\mathcal{b})} \circ L\Big(\tilde{\mathbb{m}}_E\big(p\big)\Big) \circ \mathrm{can}_{v^{(\ell(\mathcal{a}))} \leftarrow v_b;\mathbb{o}(\mathcal{a})}$$

which by hypothesis of this theorem is equal to

$$\mathrm{can}_{w_e \leftarrow v^{(\ell(\mathcal{b}))};\mathbb{o}(\mathcal{b})} \circ L\Big(\tilde{\mathbb{m}}_E\big(q\big)\Big) \circ \mathrm{can}_{v^{(\ell(\mathcal{a}))} \leftarrow v_b;\mathbb{o}(\mathcal{a})} =$$

— going in the reverse direction —

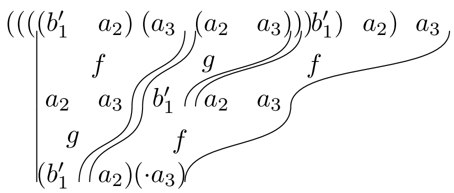
$$\mathrm{can}_{w_e \leftarrow v^{(\ell(\mathcal{b}))};\mathbb{o}(\mathcal{b})} \circ K\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)\big(q\big) \circ \mathrm{can}_{v^{(\ell(\mathcal{a}))} \leftarrow v_b;\mathbb{o}(\mathcal{a})} =$$
$$\mathrm{can}_{w_e \leftarrow v^{(\ell(\mathcal{b}))};\mathbb{o}(\mathcal{b})} \circ \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(\left\langle v^{(\ell(\mathcal{b}))}, \left\langle\left\langle v^{(k_i)}, \langle \mathfrak{q}_{i,j} \rangle_{j=1}^{k_i} \right\rangle\right\rangle_{i=n}^{1}, v^{(\ell(\mathcal{a}))} \right\rangle\right) \circ$$
$$\mathrm{can}_{v^{(\ell(\mathcal{a}))} \leftarrow v_b;\mathbb{o}(\mathcal{a})} =$$
$$\mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(q\right) \circ \mathrm{can}_{w_b \leftarrow v^{(\ell(\mathcal{a}))};\mathbb{o}(\mathcal{a})} \circ \mathrm{can}_{v^{(\ell(\mathcal{a}))} \leftarrow v_b;\mathbb{o}(\mathcal{a})} = \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(q\right) \circ \mathrm{can}_{w_b \leftarrow v_b;\mathbb{o}(\mathcal{a})} \quad .$$

□

Corollary 3.36. *If in theorem 3.35 we have $v_e = w_e$ and $v_b = w_b$, then*

$$\mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(p\right) = \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}\left(q\right) \quad .$$

We will use the Don't Worry Theorem mainly to simplify calculations. Let us introduce at this point a kind of shorthand notation. When we are talking about a collection of arrows $f, g, \ldots$ in $\mathbf{C}$ where $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ is a monoidal category and the domains and codomains of all these arrows have been given in the form $s(f) = v_s(f)\left(\langle a_i \rangle_{i=1}^m\right)$, $t(f) = v_t(f)\left(\langle a'_i \rangle_{i=1}^{m'}\right)$, $s(g) = v_s(g)\left(\langle b_i \rangle_{i=1}^n\right)$, $t(g) = v_t(g)\left(\langle b'_i \rangle_{i=1}^{n'}\right), \ldots$ for some tensor words $v_s(f), v_t(f), v_s(g), v_t(g), \ldots$ then we will use expressions like for example



to denote the morphism of $\mathbf{C}$ which we get if we define generators $X, A, \mathfrak{s}, \mathfrak{t}$, a varnishing v_s, v_t and a realization $\langle \mathbb{o}, \mathbb{m} \rangle$ such that

- the set X contains all objects of $a_i, a'_i, b_i, b'_i, \ldots \in \mathbf{C}$ which appear as components in the specification of sources and targets,
- the set A contains $f, g, \ldots$,
- v_s and v_t are the functions that we already implicitly alluded to when we said that the source and target of $f, g, \ldots$ have to be given in a specific way,
- $\mathbb{o}$ sends elements of X to themselves and
- $\mathbb{m}$ sends elements of A to themselves

and then evaluate some path whose trace is given by the picture and whose source tensor word is that which is indicated by the parentheses in the top row of the picture, while its target tensor word is indicated by the parentheses in the bottom row (a dot "$\cdot$" denotes the unit e of the tensor product) — by corollary 3.23 (or the previous corollary) it doesn't matter which of the possible paths we choose.

(If the reader prefers they can also think of the elements of the set A as being composed of not only a morphism but also the specification of source and target in terms of a tensor word and elements of X. If we want to be very formal this might be necessary because in theory we could have two different names, say f and g — one name associated with a certain specification of source and target and the other with another — such that f and g happen to coincide as morphisms of $\mathbf{C}$ (and therefore somehow their source and target happen to coincide although they are not "formally" equal). We try not to be overly concerned with these kind of matters. The reader can also just think of the set A as containing the *names* which we have chosen in the text for the arrows.)

Sometimes we will also use the more concise notation

$$\begin{aligned}(g \boxdot f) \circ (f \boxdot g \boxdot f) : ((((b_1' \boxtimes a_2) \boxtimes (a_3 \boxtimes (a_2 \boxtimes a_3))) \boxtimes b_1') \boxtimes a_2) \boxtimes a_3 \\ \to (b_1' \boxtimes a_2) \boxtimes (e \boxtimes a_3)\end{aligned}$$

to denote the same arrow as that in the picture.

We will also use these kind of pictures to do calculations when we know that the morphisms $f, g, \ldots$ satisfy some equations. In calculations we will only add parentheses to the source and target when for some reason we want to be explicit about which path we are talking about. Usually though we will omit them because by corollary 3.24 the choice does not make any difference for the truth or falsehood of the equations we are writing down (as long as we make the same choice on both sides of the equation — which we always implicitly assume that we do).

In the next part, in the proof of theorem 4.2 (the main part of which is contained in the proof of lemma 4.3) we will see quite a few of these kind of calculations. There the construction of the generators and the varnishing is spelled out explicitly, so as to give one clear example of how this is done. Usually though we will leave this part implicit and we will assume that the reader can supply the details themselves.

4 Interlude

4.1 Outlook

Observation 4.1. *A monoid $\langle m, \mu, \eta \rangle$ in a monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ is the same thing as a valid realization of atoms for object-atoms $X_{Mon} := \{\mathfrak{m}\}$, non-trivial arrow atoms*

$$\langle A_{Mon}, \mathfrak{s}_{Mon}, \mathfrak{t}_{Mon} \rangle := \{\boldsymbol{\mu} : \langle \mathfrak{m}, \mathfrak{m} \rangle \to \langle \mathfrak{m} \rangle, \boldsymbol{\eta} : \langle\rangle \to \langle \mathfrak{m} \rangle\}$$

(here we deviate a little from the notation used earlier by directly specifying source and target of arrow atoms, but it should be clear what the intended meaning in terms of the earlier notation is) and the varnishing

$$v_{s;Mon}(\boldsymbol{\mu}) := (\_)\Box(\_), v_{t;Mon}(\boldsymbol{\mu}) := (\_), v_{s;Mon}(\boldsymbol{\eta}) := e_0, v_{t;Mon}(\boldsymbol{\eta}) := (\_)$$

which satisfies the set of equation-shapes

$$\begin{aligned}
&D_{Mon} := \\
&\Bigg\{ \bigg\langle \Big\langle (_), \big\langle \langle (_), \langle \boldsymbol{\mu} \rangle \rangle, \langle (_)\Box(_), \langle \boldsymbol{\mu}, \mathfrak{m} \rangle \rangle \big\rangle, (_)\Box\left((_)\Box(_)\right) \Big\rangle, \\
&\qquad\qquad \Big\langle (_), \big\langle \langle (_), \langle \boldsymbol{\mu} \rangle \rangle, \langle (_)\Box(_), \langle \mathfrak{m}, \boldsymbol{\mu} \rangle \rangle \big\rangle, (_)\Box\left((_)\Box(_)\right) \Big\rangle \bigg\rangle, \\
&\bigg\langle \Big\langle (_), \big\langle \langle (_), \langle \boldsymbol{\mu} \rangle \rangle, \langle (_)\Box(_), \langle \mathfrak{m}, \boldsymbol{\eta} \rangle \rangle \big\rangle, (_)\Box e_0 \Big\rangle, \\
&\qquad\qquad \Big\langle (_), \big\langle \langle (_), \langle \mathfrak{m} \rangle \rangle \big\rangle, (_)\Box e_0 \Big\rangle \bigg\rangle, \\
&\bigg\langle \Big\langle (_), \big\langle \langle (_), \langle \boldsymbol{\mu} \rangle \rangle, \langle (_)\Box(_), \langle \boldsymbol{\eta}, \mathfrak{m} \rangle \rangle \big\rangle, e_0\Box(_) \Big\rangle, \\
&\qquad\qquad \Big\langle (_), \big\langle \langle (_), \langle \mathfrak{m} \rangle \rangle \big\rangle, e_0\Box(_) \Big\rangle \bigg\rangle \Bigg\}.
\end{aligned}$$

The underlying object m of the monoid corresponds to the single value of $\mathbb{o}$. μ corresponds to the value of $\mathbb{m}$ at $\boldsymbol{\mu}$ and η corresponds to the value of $\mathbb{m}$ at $\boldsymbol{\eta}$.

The categories of algebras $\mathbf{Act}_{\mathbf{C}}$, **ɟɔA**$_{\mathbf{C}}$ and $\mathbf{BiAct}_{\mathbf{C}}$ defined in section 2 can be described in a completely analogous fashion.

The main result of this section will be a theorem which says that a generalized law of associativity holds for monoids in monoidal categories. The proof of this theorem makes use of the characterization of monoids given above and serves as an illustration for how one can use the visual calculus developed in section 3.

At the same time this result gives concrete meaning to the theory we will develop in section 7. There we will figure out how a characterization of all the morphisms f in $\tilde{\mathcal{E}}_E$, for which $t(f) = \langle \mathfrak{a} \rangle$ (for some $\mathfrak{a} \in X$) leads to a characterization of the free objects in the category of valid realizations which satisfy some set of equation-shapes D such that $\mathrm{Tr}(D) = E$. (This category is defined (in the obvious way) in section 7.)

A characterization of the morphisms

$$f \text{ in } \tilde{\mathcal{E}}_{X_{Mon}}\left(A_{Mon}, \mathfrak{s}_{Mon}, \mathfrak{t}_{Mon}, \mathrm{Tr}\left(D_{Mon}\right)\right) \text{ such that } t\big(f\big) = \langle \mathfrak{m} \rangle$$

is just what the generalized law of associativity gives (see lemma 4.3 below). We do not execute the proofs here but after seeing the proof for the generalized law of associativity for monoids it should be an easy thing for the reader to find similar characterizations for the categories $\tilde{\mathcal{E}}_E$ which give rise to the categories $\mathbf{Act_C}$, $\mathbf{\text{ʇɔ}A_C}$ and $\mathbf{BiAct_C}$ introduced in section 2.

Free objects in the categories $\langle m, \mu, \eta\rangle-\mathbf{Act}$, $\mathbf{Act}-\langle m, \mu, \eta\rangle$ and $\langle m, \mu, \eta\rangle$ $-\mathbf{Act}-\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$ are even simpler. The underlying objects of the free objects on some object $a \in \mathbf{C}$ are given simply by $m \boxtimes a$, $a \boxtimes m$ and $m \boxtimes a \boxtimes \dot{m}$ respectively and the action is given by the multiplication of the monoids. We will describe this in a little more detail in section 6 and in section 8.

When we have free objects in some algebraic category we can think of that category as a category of Eilenberg-Moore algebras. We describe these categories and some of their theory in section 6.

So the roadmap is as follows: In this section we prove the generalized law of associativity for monoids in monoidal categories. In section 5 we work out some properties of colimits that we will need later. Before the end of section 6 we will know the structure of free objects, limits and colimits in the categories $\langle m, \mu, \eta\rangle-\mathbf{Act}$ and $\mathbf{Act}-\langle m, \mu, \eta\rangle$. Section 7 completes the picture for $\mathbf{Mon_C}$, $\mathbf{Act_C}$, $\mathbf{\text{ʇɔ}A_C}$ and $\mathbf{BiAct_C}$. In the beginning of section 8 we supply the details for the remaining case of $\langle m, \mu, \eta\rangle-\mathbf{Act}-\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$. This last case is not really that much more complicated than the case of $\langle m, \mu, \eta\rangle-\mathbf{Act}$ and $\mathbf{Act}-\langle m, \mu, \eta\rangle$ and if the reader wishes he may skip ahead and read (the beginning of) section 8 after section 6.

4.2 Generalized associativity for monoids

Theorem 4.2 (generalized law of associativity for monoids in monoidal categories)**.** *Let $X_{Mon}, A_{Mon}, \mathfrak{s}_{Mon}, \mathfrak{t}_{Mon}, v_{s;Mon}, v_{t;Mon}$ and D_{Mon} be as in observation 4.1. Let $p = \langle v_e, \hat{p}, v_b\rangle$ and $q = \langle w_e, \hat{q}, w_b\rangle$ be path-shapes. If $v_b = w_b$, $v_e = w_e$ and $\ell(v_e) = \ell(w_e) = 1$ then for any valid realization $\langle \mathbb{o}, \mathbb{m}\rangle$ of $X_{Mon}, A_{Mon}, \mathfrak{s}_{Mon}, \mathfrak{t}_{Mon}, v_{s;Mon}, v_{t;Mon}$ which satisfies D_{Mon} we have*

$$\mathrm{ev}_{\langle \mathbb{o}, \mathbb{m}\rangle}(p) = \mathrm{ev}_{\langle \mathbb{o}, \mathbb{m}\rangle}(q) \quad .$$

Proof. This will follow from the next lemma and corollary 3.36 of the Don't Worry Theorem. □

Lemma 4.3. *Let $X_{Mon}, A_{Mon}, \mathfrak{s}_{Mon}, \mathfrak{t}_{Mon}$ and D_{Mon} be as in observation 4.1. Set*

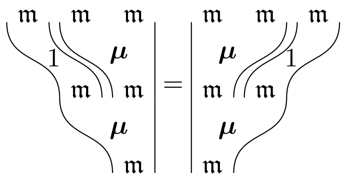
$$\begin{aligned} E_{Mon} &:= \mathrm{Tr}\left(D_{Mon}\right) \\ &= \left\{ \Big\langle \left\langle \langle \boldsymbol{\mu}\rangle, \langle \boldsymbol{\mu}, \mathfrak{m}\rangle\right\rangle, \left\langle \langle \boldsymbol{\mu}\rangle, \langle \mathfrak{m}, \boldsymbol{\mu}\rangle\right\rangle \Big\rangle, \right. \\ &\qquad \Big\langle \left\langle \langle \boldsymbol{\mu}\rangle, \langle \mathfrak{m}, \boldsymbol{\eta}\rangle\right\rangle, \left\langle \langle \mathfrak{m}\rangle\right\rangle \Big\rangle, \\ &\qquad \left. \Big\langle \left\langle \langle \boldsymbol{\mu}\rangle, \langle \boldsymbol{\eta}, \mathfrak{m}\rangle\right\rangle, \left\langle \langle \mathfrak{m}\rangle\right\rangle \Big\rangle \right\} \ . \end{aligned}$$

Then for any two morphisms $\hbar, \dot{\hbar}$ in $\tilde{\mathcal{E}}_{X_{Mon}}(A_{Mon}, \mathfrak{s}_{Mon}, \mathfrak{t}_{Mon}, E_{Mon})$ if $s(\hbar) = s\left(\dot{\hbar}\right)$ and $t(\hbar) = t\left(\dot{\hbar}\right) = \langle \mathfrak{m} \rangle$ then

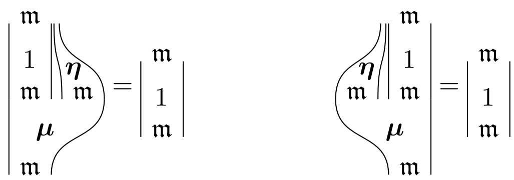
$$\hbar = \dot{\hbar} \ .$$

Proof. We use this opportunity to show how one can use the visual calculus presented in the previous chapter to do calculations in some category $\tilde{\mathcal{E}}_E$. Note that even though the pictures we draw are in one-to-one correspondence to pre-arrows and not to arrows of $\tilde{\mathcal{E}}_{E_{Mon}}$ when writing equations like we will do below the sign "=" is of course intended to refer to the equivalence-classes of these pre-arrows — that is to arrows of $\tilde{\mathcal{E}}_{E_{Mon}}$.

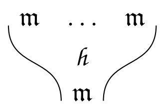
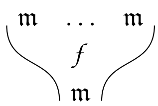
In $\tilde{\mathcal{E}}_{X_{Mon}}(A_{Mon}, \mathfrak{s}_{Mon}, \mathfrak{t}_{Mon}, D_{Mon})$ we have:



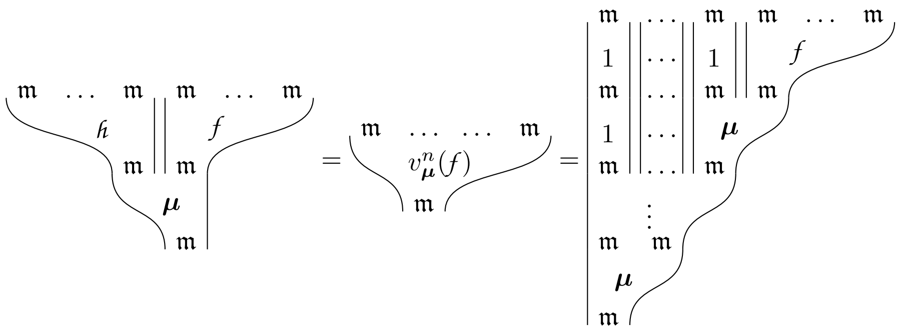
(50)



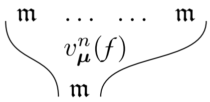
(51)

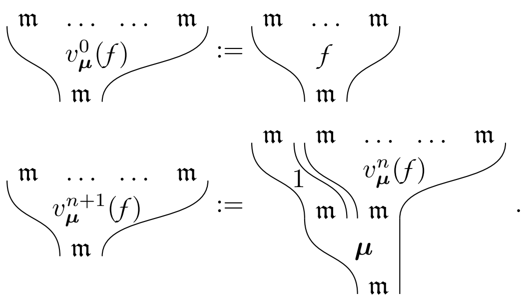
We will proceed by proving that for any  and any 

in $\tilde{\mathcal{E}}_{X_{Mon}}(A_{Mon}, \mathfrak{s}_{Mon}, \mathfrak{t}_{Mon}, D_{Mon})$ with $\ell(s(\hbar)) = n$ we have



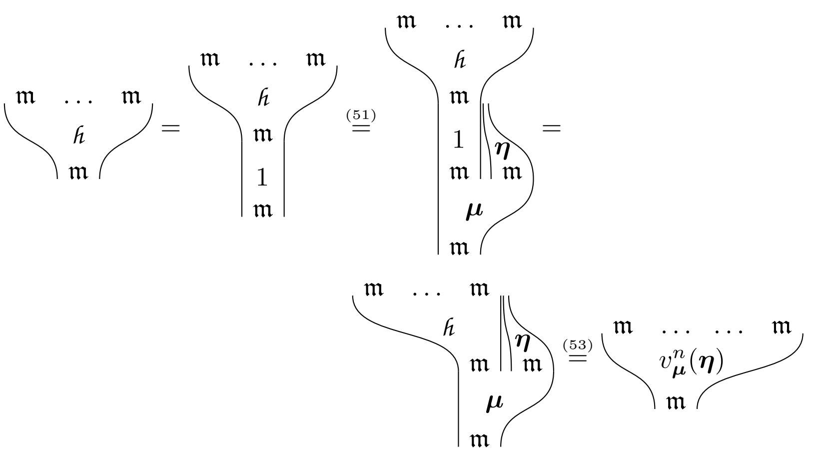
(52)

where  is defined by



(53)

From this it will follow that

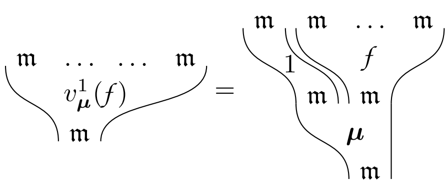


(54)

Any arrow of $\tilde{\mathcal{E}}_{X_{Mon}}(A_{Mon}, \mathfrak{s}_{Mon}, \mathfrak{t}_{Mon}, D_{Mon})$ is the image of some pre-arrow under $\tilde{\mathfrak{m}}_{E_{Mon}} \circ \pi_{\diamond}$. To prove (52) we proceed by induction over the number of rows in such a pre-arrow.

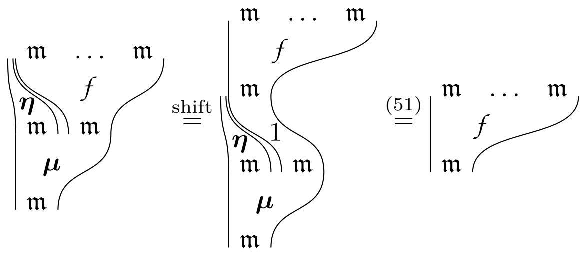
If there is only one row in the pre-arrow mapped to $\hbar$ then we have the following cases.

- $\hbar = 1_{\mathfrak{m}}$:

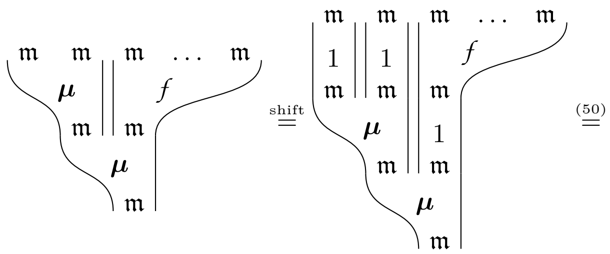
  

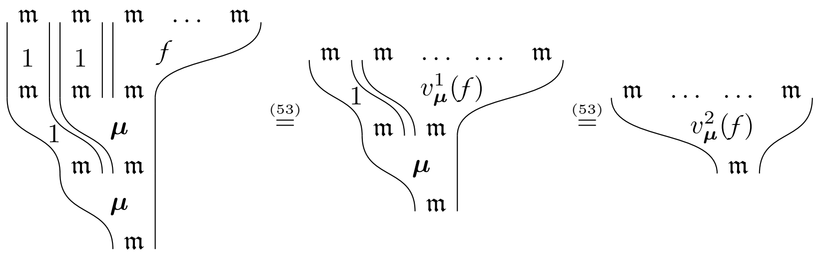
  by definition of $v^1_{\boldsymbol{\mu}}(f)$.

- $\hbar = \boldsymbol{\eta}$:

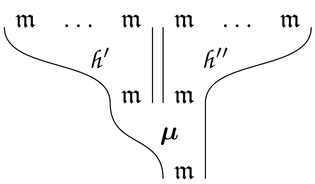


- $\hbar = \boldsymbol{\mu}$:



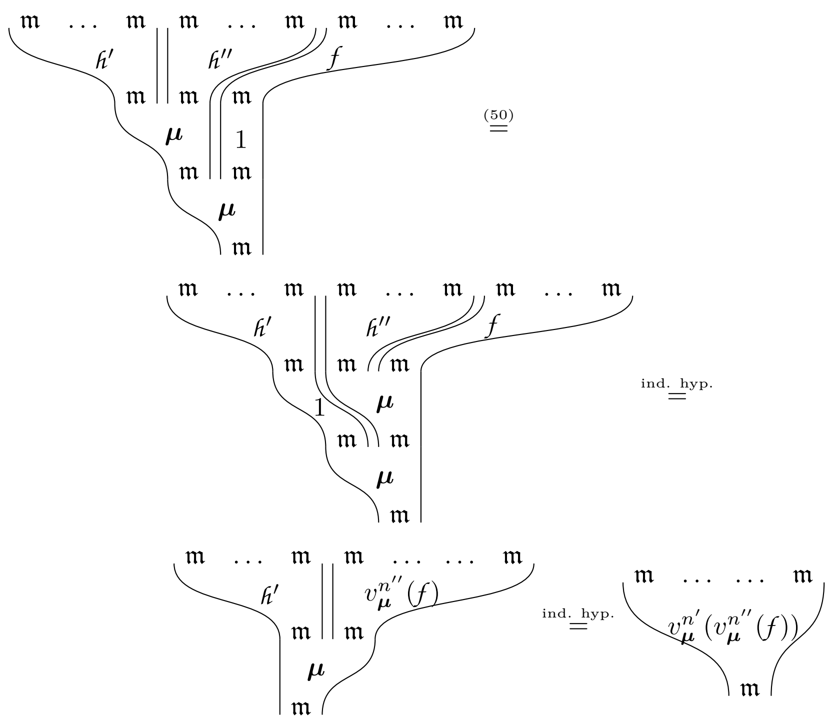


Now assume the statement is true for pre-arrows with $n \geq 1$ rows. Because the target has to be a single $\mathfrak{m}$ and because nothing can precede a lone $\boldsymbol{\eta}$, the last row has to be either a single $1_{\mathfrak{m}}$ or a single $\boldsymbol{\mu}$. Clearly for the first case the statement is true by the induction hypothesis because we can simply omit $1_{\mathfrak{m}}$. In the second case we have

$\hbar =$  (55)

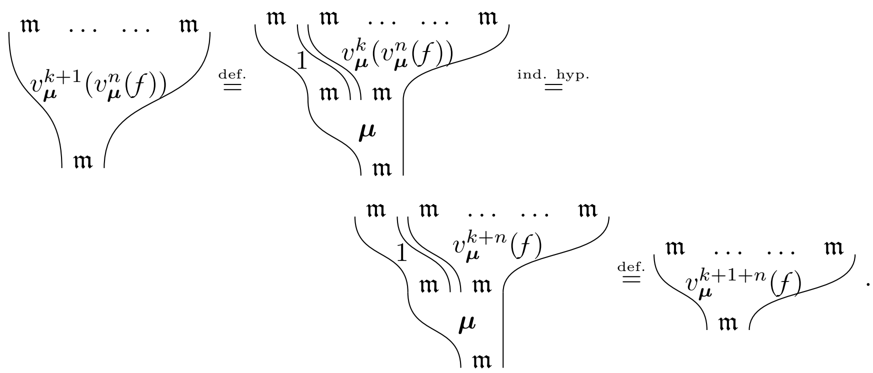
where by induction hypothesis (52) is true for $\hbar'$ and $\hbar''$. Call $\ell(\hbar') = n'$ and

$\ell(\hbar'') = n''$. We calculate



(56)

It remains to show that $v_{\boldsymbol{\mu}}^{m}(v_{\boldsymbol{\mu}}^{n}(f)) = v_{\boldsymbol{\mu}}^{m+n}(f)$. This is done by induction on m. For $m = 0$ the statement is true by the definition of $v_{\boldsymbol{\mu}}^{0}(-)$. Assume the statement is true for $m = k$. For $m = k + 1$ we get



$\square$

If the reader experienced a feeling of déjà-vu while looking at the proof above this is not surprising. The proof draws on the same ideas that we used to prove the coherence theorem for monoidal categories. If we specialize (21) (part of the proof of lemma 3.6 about monoidal functors) to the case which we used in the proof of the coherence theorem, then we can recognize that $R_{\ddot{w}}$ (which is $\mathcal{S}_{\ddot{w}}$ in this specific case) stands as a token for the idea (which is an induction hypothesis there) that for any tensor word v the functors $\left(\circ_{i=1}^{n''}\left(\mathcal{S}\left(-\right)\right)\right)\left(v\left(-,\ldots,-\right)\right) = - \boxtimes \left(\cdots \boxtimes \left(- \boxtimes \left(- \boxtimes v(-,\ldots,-)\right)\right)\right)$ and $\ddot{w}\left(-,\ldots,-\right) \boxtimes v\left(-,\ldots,-\right)$ are closely related. A similar idea is also an induction hypothesis in the preceding proof and we use this induction hypothesis in the second step in (56). Similarly $\mathcal{S}_{\dot{w}}$ represents the idea that for any v the functors $\left(\circ_{i=1}^{n'}\left(\mathcal{S}\left(-\right)\right)\right)\left(v\left(-,\ldots,-\right)\right) = - \boxtimes \left(\cdots \boxtimes \left(- \boxtimes \left(- \boxtimes v(-,\ldots,-)\right)\right)\right)$ and $\dot{w}\left(-,\ldots,-\right) \boxtimes w\left(-,\ldots,-\right)$ are closely related. A similar idea is used in the last step in (56). $\mathcal{S}_2$ — which is an instance of α — is a testament to associativity. The corresponding idea is used in the first step of (56). An analogue to the final step in the coherence theorem (which is recorded in diagram 24) appeared as (54) in the proof of the previous theorem.

Of course the proofs differ when it comes to the details. Much of the complexity we had to deal with in the coherence theorem is not present here because we are only concerned with equality and not “a compatible kind of natural isomorphism”. On the other hand in our proof of the coherence theorem the placeholder which is called f in the proof above was never explicit because we were free to use an exponential object — the functor category — there. This was not possible here and so instead of the *point-free* style used in the proof of the coherence theorem we were forced to name the placeholder.

This comparison should make it clear though that our use of the functor category was purely a matter of convenience. The proof could have been executed equally well without ever using an exponential object by making the placeholder explicit as we did here. This leads at least the author to expect that substituting an arbitrary bicategory in place of the bicategory of categories would allow one to derive results about monoidal objects in such a bicategory that are very similar to the results we got for monoidal categories. At this point we run into a kind of regress though because in this setting 1-cells of this bicategory take the role of functors and to talk about an analogon for monoidal categories we need an analogon for bifunctors — that would be 1-cells whose source is somehow composed of two objects — this seems to boil down to requiring a monoidal product on this bicategory. So we would need to develop a theory of monoidal bicategories. The author is not versed enough in higher order category theory to be able to easily tell whether there is a natural end to what looks like it will lead to an explosion of levels and concepts.

5 Some properties of (co)limits

When figuring out the structure of colimits and free objects in the category of monoids internal to some category and the category of monoid actions we will need a few facts about iterated colimits and preservation of colimits under multifunctors that we state and prove here. The corresponding statements for limits are of course also true because we will only be using general properties of (sometimes monoidal) categories and functors. (Note that $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ is a monoidal category if and only if $\left\langle \mathbf{C}^{op}, \boxtimes^{op}, e, \left(\alpha^{-1}\right)^{op}, \left(\lambda^{-1}\right)^{op}, \left(\rho^{-1}\right)^{op} \right\rangle$ — where $\mathbf{C}^{op}$ is the (normal) opposite category — is a monoidal category.) We will primarily be needing the versions for colimits and therefore we state them in that form.

5.1 Iterated colimits

The statements of the first two lemmata can (with a little loss of detail) be summarized as

$$\mathrm{Colim}_{\langle i,j \rangle \in \mathbf{I} \times \mathbf{J}} (L(i,j)) = \mathrm{Colim}_{i \in \mathbf{I}} (\mathrm{Colim}_{j \in \mathbf{J}} (L(i,j))) \ .$$

We give an elementary proof. [Mac98, section IX.8] has a more sophisticated proof.

Lemma 5.1 (iterated colimit to colimit of bifunctor)**.** *Let $L : \mathbf{I} \times \mathbf{J} \to \mathbf{C}$ be a functor and let $\left\langle L'(i), \langle \tau_{i;j} : L(i,j) \to L'(i) \rangle_{j \in \mathbf{J}} \right\rangle$ be a colimit of the functor $L(i,-)$ for all $i \in \mathbf{I}$. For all $u : i \to i'$ in $\mathbf{I}$ let $L'(u)$ be such that the diagram below commutes for all $j \in \mathbf{J}$.*

$$\begin{tikzcd} L'(i) \arrow[d, "L'(u)"] & L(i,j) \arrow[l, "\tau_{i;j}"'] \arrow[d, "L(u,1_j)"] \\ L'(i') & L(i',j) \arrow[l, "\tau_{i';j}"'] \end{tikzcd} \tag{57}$$

By the colimit property of each τ_i and because L is a functor this implies that $L' : \mathbf{I} \to \mathbf{C}$ is a functor. (If this does not seem clear see for example [Mac98, section V.3.] for some elaboration.) Moreover the arrow function of this functor is uniquely determined once $L'(i)$ and τ_i have been fixed for all $i \in \mathbf{I}$. If $\langle l', \tau' : L' \dot{\to} \Delta(l') \rangle$ is a colimit of L', then

$$\left\langle l', \langle \tau'_i \circ \tau_{i;j} : L(i,j) \to l' \rangle_{\langle i,j \rangle \in \mathbf{I} \times \mathbf{J}} \right\rangle$$

is a colimit of L.

Proof. First of all we have to show that $\langle \tau'_i \circ \tau_{i;j} \rangle_{\langle i,j \rangle \in \mathbf{I} \times \mathbf{J}}$ form a natural transformation.

$$\begin{tikzcd} & & L(i,j) \arrow[dl, "\tau_{i;j}"'] \arrow[d, "L(u,1_j)"] \arrow[dd, bend left=60, "L(u,v)"] \\ & L'(i) \arrow[dl, "\tau'_i"'] \arrow[d, "L'(u)"] & L(i',j) \arrow[dl, "\tau_{i',j}"'] \arrow[d, "L(1_{i'},v)"] \\ l' & L'(i') \arrow[l, "\tau'_{i'}"] & L(i',j') \arrow[l, "\tau_{i',j'}"] \end{tikzcd}$$

The parallelogram commutes by hypothesis of the theorem and the two small triangles commute by naturality of $\tau_{i'}$ and τ'.

Now assume $\dot{\tau} : L\dot{\to}\Delta(a) : \mathbf{I} \times \mathbf{J} \to \mathbf{C}$ is a cone to some object $a \in \mathbf{C}$. Because $L'(i)$ are colimits we get for every $i \in \mathbf{I}$ a unique $\dot{\tau}'_i : L'(i) \to a$ such that $\dot{\tau}_{i,j} = \dot{\tau}'_i \circ \tau_{i;j}$. Have another look at the diagram above, replacing τ' by $\dot{\tau}'$ and l' by a. Then by assumption the big outer triangle commutes. In the right part of the diagram nothing has changed and so it still commutes. This implies $\dot{\tau}'_i \circ \tau_{i;j} = \dot{\tau}'_{i'} \circ L'(u) \circ \tau_{i;j}$, which by the colimit property of τ_i means that $\dot{\tau}'_i = \dot{\tau}'_{i'} \circ L'(u)$ and therefore $\dot{\tau}' : L'\dot{\to}\Delta(a)$ is natural. By assumption τ' is a colimit and therefore we get a unique $f : l' \to a$ such that $\dot{\tau}'_i = f \circ \tau'_i$ and therefore $\dot{\tau}_{i,j} = \dot{\tau}'_i \circ \tau_{i;j} = f \circ \tau'_i \circ \tau_{i;j}$.

Assume $g : l' \to a$ is another morphism such that $\dot{\tau}'_i \circ \tau_{i;j} = \dot{\tau}_{i,j} = g \circ \tau'_i \circ \tau_{i;j}$. Then because τ_i is a colimiting cone we get $\dot{\tau}'_i = g \circ \tau'_i$ for each $i \in \mathbf{I}$. By the colimit property of τ' this means $g = f$. $\square$

Lemma 5.2 (colimit of bifunctor to iterated colimit)**.** *Let*

$$\begin{aligned} L &: \mathbf{I} \times \mathbf{J} \to \mathbf{C} \\ L' &: \mathbf{I} \qquad\ \to \mathbf{C} \end{aligned}$$

be functors and for all $i \in \mathbf{I}$ let

$$\tau_i : L(i,-) \dot{\to} \Delta(L'(i)) : \mathbf{J} \to \mathbf{C}$$

be a natural transformation such that $\langle L'(i), \tau_i \rangle$ is a colimit of $L(i,-)$ and such that for all morphisms $u : i \to i'$ of $\mathbf{I}$ and for all $j \in \mathbf{J}$ the diagram (57) *commutes.*

Moreover let

$$\langle l, \dot{\tau} : L\dot{\to}\Delta(l) \rangle$$

be a colimit of L. Then for each $i \in \mathbf{I}$ the natural transformation

$$\dot{\tau}_i : L(i,-) \dot{\to} \Delta(l) : \mathbf{J} \to \mathbf{C}$$

is a cone from $L(i,-)$ and therefore by the colimit property of τ_i we get a unique morphism $\dot{\tau}'_i : L'(i) \to l$ such that $\dot{\tau}_{i,j} = \dot{\tau}'_i \circ \tau_{i,j}$.

These morphisms form a colmiting cone. That is,

$$\langle l, \langle \dot{\tau}'_i \rangle_{i \in \mathbf{I}} \rangle$$

is a colimit of L'.

Proof. We need to show $\langle \dot{\tau}'_i \rangle_{i \in \mathbf{I}}$ natural.

$$\begin{tikzcd}
 & L'(i) \arrow[dl, "\dot{\tau}'_i"'] \arrow[dd, "L'(u)"] & L(i,j) \arrow[l, "\tau_{i,j}"] \arrow[dd, "L(u,1_j)"] \arrow[dll, bend right, "\dot{\tau}_{i,j}"'] \\
l & & \\
 & L'(i') \arrow[ul, "\dot{\tau}'_{i'}"] & L(i',j) \arrow[l, "\tau_{i',j}"'] \arrow[ull, bend left, "\dot{\tau}_{i',j}"]
\end{tikzcd} \tag{58}$$

The outer triangle commutes by naturality of $\dot{\tau}$ and the square is just (57). Therefore we get $\dot{\tau}'_i \circ \tau_{i,j} = \dot{\tau}'_{i'} \circ L'(u) \circ \tau_{i,j}$ which by the colmit property of τ_i implies $\dot{\tau}'_i = \dot{\tau}'_{i'} \circ L'(u)$.

Let $\sigma' : L' \dot{\to} \Delta(a) : \mathbf{I} \to \mathbf{C}$ be another cone from L'. Note that the requirements for τ stated above mean that it is a natural transformation

$$\tau : L \dot{\to} L' \circ P : \mathbf{I} \times \mathbf{J} \to \mathbf{C}$$

where $P : \mathbf{I} \times \mathbf{J} \to \mathbf{I}$ is the projection. Set

$$\sigma := (\sigma' \circ P) \bullet \tau : L \dot{\to} \Delta(a) : \mathbf{I} \times \mathbf{J} \to \mathbf{C} \ ,$$

and therefore $\sigma_{i,j} = \sigma'_i \circ \tau_{i,j}$. By the colimit property of $\dot{\tau}$ we get a unique $f : l \to a$ such that $\sigma_{i,j} = f \circ \dot{\tau}_{i,j}$. This gives

$$\sigma'_i \circ \tau_{i,j} = \sigma_{i,j} = f \circ \dot{\tau}_{i,j} = f \circ \dot{\tau}'_i \circ \tau_{i,j} \ ,$$

which by the colimit property of τ implies $\sigma'_i = f \circ \dot{\tau}'_i$. If for $g : l \to a$ we also have $\sigma'_i = g \circ \dot{\tau}'_i$, then

$$g \circ \dot{\tau}_{i,j} = g \circ \dot{\tau}'_i \circ \tau_{i,j} = \sigma'_i \circ \tau_{i,j} = f \circ \dot{\tau}'_i \circ \tau_{i,j} = f \circ \dot{\tau}_{i,j} \ ,$$

and by the colimit property of $\dot{\tau}$ this implies $f = g$. $\square$

5.2 Images of colimits under multifunctors

Lemma 5.3. *If $F : \prod_{k=1}^n \mathbf{C}_k \to \mathbf{C}$ is a multifunctor and $L_k : \mathbf{I}_k \to \mathbf{C}_k$ are functors such that for every $k \in \{1, \ldots, n\}$ and for every tuple $\langle a_{k'} \rangle_{\substack{k'=1 \\ k' \neq k}}^n$ with $a_{k'} \in \mathbf{C}_{k'}$ the functor*

$$F(a_1, \ldots, a_{k-1}, -, a_{k+1}, \ldots, a_n) : \mathbf{C}_k \to \mathbf{C}$$

preserves colimits of L_k (we will also say that F preserves colimits of L_k in the k-th variable) and moreover $\langle l_k, \sigma_k : L_k \dot{\to} \Delta(l_k) : \mathbf{I}_k \to \mathbf{C}_k \rangle$ is a colimit of L_k for every $k \in \{1, \ldots, n\}$, then

$$\left\langle F(\langle l_k \rangle_{k=1}^n), \langle F(\langle \sigma_{k;i_k} \rangle_{k=1}^n) : F(\langle L_k(i_k) \rangle_{k=1}^n) \dot{\to} F(\langle l_k \rangle_{k=1}^n) \rangle_{\langle i_k \rangle_{k=1}^n \in \prod_{k=1}^n \mathbf{I}_k} \right\rangle$$

is a colimit of

$$F \circ \left(\prod_{k=1}^n L_k \right) : \prod_{k=1}^n \mathbf{I}_k \to \prod_{k=1}^n \mathbf{C}_k \to \mathbf{C} \ .$$

Proof. We proceed by induction over the number of categories n.

The case $n = 1$ is trivial.

Assume the statement of the theorem is true for $n = m-1$. We apply lemma

5.1 with

$$
\begin{aligned}
\mathbf{I} &= \mathbf{I}_1 \\
\mathbf{J} &= \prod_{k=2}^{m} \mathbf{I}_k \\
L &= F \circ \left(\prod_{k=1}^{m} L_k \right) : \prod_{k=1}^{m} \mathbf{I}_k \to \prod_{k=1}^{m} \mathbf{C}_k \to \mathbf{C} \\
L' &= F\left(L_1\left(-\right), \langle l_k \rangle_{k=2}^{m}\right) : \mathbf{I}_1 \to \mathbf{C}_1 \to \mathbf{C} \\
\tau_{i_1; \langle i_k \rangle_{k=2}^{m}} &= F\left(1_{L_1(i_1)}, \langle \sigma_{k; i_k} \rangle_{k=2}^{m}\right) \\
\tau'_{i_1} &= F\left(\sigma_{1; i_1}, \langle 1_{l_k} \rangle_{k=2}^{m}\right) \ .
\end{aligned}
$$

Indeed, by the induction hypothesis applied to the functor $F\left(L_1\left(i_1\right), -, \ldots, -\right)$ of $m-1$ variables, $\langle L'\left(i_1\right), \tau_{i_1} \rangle$ is a colimit of $L\left(i_1, -\right)$. The diagram (57) turns into

$$
\begin{tikzcd}
F\left(L_1\left(i_1\right), \langle l_k \rangle_{k=2}^{m}\right) \arrow[d, "F\left(L_1(u_1), \langle 1_{l_k} \rangle_{k=2}^{m}\right)"] & F\left(L_1\left(i_1\right), \langle L_k\left(i_k\right) \rangle_{k=2}^{m}\right) \arrow[l, bend right, "F\left(1_{L_1(i_1)}, \langle \sigma_{k; i_k} \rangle_{k=2}^{m}\right)"'] \arrow[d, "F\left(L_1(u_1), \langle L_k\left(1_{i_k}\right) \rangle_{k=2}^{m}\right)"] \\
F\left(L_1\left(i'_1\right), \langle l_k \rangle_{k=2}^{m}\right) & F\left(L_1\left(i'_1\right), \langle L_k\left(i_k\right) \rangle_{k=2}^{m}\right) \arrow[l, bend left, "F\left(1_{L_1\left(i'_1\right)}, \langle \sigma_{k; i_k} \rangle_{k=2}^{m}\right)"]
\end{tikzcd} ,
$$

which commutes because F is a multifunctor. Because F preserves colimits in the first variable, $\langle F\left(l_1, \langle l_k \rangle_{k=2}^{m}\right), \tau' \rangle$ is a colimit of L'. By lemma 5.1

$$
\begin{aligned}
&\Big\langle F\left(\langle l_k \rangle_{k=1}^{m}\right), \\
&\quad \left\langle F\left(\sigma_{1; i_1}, \langle 1_{l_k} \rangle_{k=2}^{m}\right) \circ F\left(1_{L_1(i_1)}, \langle \sigma_{k; i_k} \rangle_{k=2}^{m}\right) \right\rangle_{\langle i_1, \langle i_k \rangle_{k=2}^{m} \rangle \in \mathbf{I}_1 \times \prod_{k=2}^{m} \mathbf{I}_k} = \\
&\qquad\qquad \langle F\left(\langle \sigma_{k; i_k} \rangle_{k=1}^{m}\right) \rangle_{\langle i_k \rangle_{k=1}^{m} \in \prod_{k=1}^{m} \mathbf{I}_k} \Big\rangle
\end{aligned}
$$

is a colimit of $F \circ \left(\prod_{k=1}^{m} L_k\right)$. ☐

The author first learned about the following concept and it's uses from the article [ReflCoeq] on the nLab.

Definition 5.4. A *reflexive pair* is a pair $f, g : a \to b$ of morphisms which has a common right inverse; that is, there is an arrow $h : b \to a$ such that $f \circ h = g \circ h = 1_b$.

A *reflexive coequalizer* is a coequalizer of a reflexive pair.

Definition 5.5. Call the category depicted below $\mathbf{J_r}$.

$$
\begin{tikzcd}
b & \\
b \arrow[u, "1_b"] \arrow[d, "1_b"'] \arrow[r, "h"] & a \arrow[ul, "f"'] \arrow[dl, "g"] \\
b &
\end{tikzcd}
\tag{59}
$$

That is, we mean the category generated by the arrows f, g, h where h is a common right inverse for f and g. Call the subcategory of $\mathbf{J_r}$ which has only the arrows f and g (and of course the identities) $\mathbf{J_c}$, and call the inclusion $K_r : \mathbf{J_c} \to \mathbf{J_r}$.

Lemma 5.6. *A reflexive coequalizer is the same thing as a colimit of a functor from the category* $\mathbf{J_r}$.

Proof. Note first that a cone $\tau : L \dot{\to} \Delta(a) : \mathbf{J_r} \to \mathbf{C}$ from a functor from $\mathbf{J_r}$ is the same thing as a cone $\tau' : L \circ K_r \dot{\to} \Delta(a) : \mathbf{J_c} \to \mathbf{C}$. Obviously the restriction of a cone τ to the category $\mathbf{J_c}$ is still a cone. If on the other hand we have a cone $\tau' : L \circ K_r \dot{\to} \Delta(a) : \mathbf{J_c} \to \mathbf{C}$, then what we have to show is that τ' also commutes with $L(h)$, that is we have to show that $\tau'_a \circ L(h) = \tau'_b$. But this follows from the commutativity with f (or g for that matter) and because h is a right inverse for f.

$$\tau'_a \circ L(h) = \tau'_b \circ L(f) \circ L(h) = \tau'_b \circ L(1_b) = \tau'_b$$

Now if we have a colimit $\langle l, \tau : L \dot{\to} \Delta(l) \rangle$ of a functor $L : \mathbf{J_r} \to \mathbf{C}$ then obviously $L(f) \circ L(h) = L(f) \circ L(h) = 1_{L(b)}$ and therefore $L(f)$ and $L(g)$ form a reflexive pair. A cone from $L \circ K_r$ (which is the same thing as an arrow k from $L(b)$) is also a cone from L and therefore factors uniquely through τ. Therefore τ_b is a coequalizer of $L(f)$ and $L(g)$.

Now assume we have a coequalizer of a reflexive pair $L'(f), L'(g)$. Then because $L'(f), L'(g)$ have a common right inverse h' we can extend L' to $\mathbf{J_r}$ by setting $L'(h) = h'$. By the above argument the coequalizer also forms a cone from the extended functor. Any other cone from $\mathbf{J_r}$ is also a cone from $\mathbf{J_c}$ and therefore factors through the coequalizer. Therefore the coequalizer also gives a colimit of the extended functor. ☐

Definition 5.7. Let $\Delta_{\mathbf{J_r}} : \mathbf{J_r} \to \prod_{i=1}^{n} \mathbf{J_r}$ be the functor with

$$\begin{aligned} \Delta_{\mathbf{J_r}}(j) &:= \langle j \rangle_{i=1}^{n} && \text{for objects } j \in \mathbf{J_r} \\ \Delta_{\mathbf{J_r}}(u) &:= \langle u \rangle_{i=1}^{n} && \text{for morphisms } u \text{ of } \mathbf{J_r}. \end{aligned}$$

That is, $\Delta_{\mathbf{J_r}}$ is just the usual diagonal functor when we regard $\prod_{i=1}^{n} \mathbf{J_r}$ as a functor category $\mathbf{J_r}^{\{1,\ldots,n\}}$.

Lemma 5.8. *If $L_i : \mathbf{J_r} \to \mathbf{C}_i$, $i \in \{1, \ldots, n\}$ are functors with colimits $\langle l_i, \tau_i : L_i \dot{\to} \Delta(l_i) \rangle$ (that is, for each $i \in \{1, \ldots, n\}$ the arrow $\tau_{i;b}$ is the reflexive coequalizer of $L_i(f)$ and $L_i(g)$) and $F : \prod_{i=1}^{n} \mathbf{C}_i \to \mathbf{C}$ is a multifunctor that preserves colimits of L_i in the i-th variable for all $i \in \{1, \ldots, n\}$ (as defined in lemma 5.3), then*

$$\left\langle F(\langle l_i \rangle_{i=1}^{n}), \langle F(\langle \tau_{i;j} \rangle_{i=1}^{n}) : F(\langle L_i(j) \rangle_{i=1}^{n}) \to F(\langle l_i \rangle_{i=1}^{n}) \rangle_{j \in \mathbf{J_r}} \right\rangle$$

is a colimit of $F \circ (\prod_{i=1}^{n} L_i) \circ \Delta_{\mathbf{J_r}}$.

In other words $F(\langle \tau_{i;b} \rangle_{i=1}^{n})$ is a reflexive coequalizer of the pair

$$\langle F(\langle L_i(f) \rangle_{i=1}^{n}), F(\langle L_i(g) \rangle_{i=1}^{n}) \rangle$$

(whose common right inverse is the arrow $F(\langle L_i(h) \rangle_{i=1}^{n})$).

Proof. We will first show, that for any functor $F' : \prod_{i=1}^{n} \mathbf{J_r} \to \mathbf{C}$ and any $c \in \mathbf{C}$ the map $- \circ \Delta_{\mathbf{J_r}} : \mathrm{Nat}\,(F', \Delta\,(c)) \to \mathrm{Nat}\,(F' \circ \Delta_{\mathbf{J_r}}, \Delta\,(c))$ (note that the symbol Δ is used in two different meanings here — first as the diagonl functor from $\prod_{i=1}^{n} \mathbf{J_r}$ and then as the diagonal functor from $\mathbf{J_r}$) is a bijection. We do this by constructing the inverse map. (We assume that the reader is familiar with the simpler fact that a coequalizer can be thought of as either a single arrow which yields equal results when composed with either one of a pair of arrows or as a cone from the category $\mathbf{J_c}$.)

So assume $\tau' : F' \circ \Delta_{\mathbf{J_r}} \dot{\to} \Delta\,(c) : \mathbf{J_r} \to \mathbf{C}$. Let $P : \mathbf{J_r} \times \mathbf{J_r} \to \mathbf{J_r}$ be the projection on the first component and $Q : \mathbf{J_r} \times \mathbf{J_r} \to \mathbf{J_r}$ be the projection on the second component. For any two tuples $\langle P_i \rangle_{i=1}^{n}$, $\langle Q_i \rangle_{i=1}^{n}$ where $P_i, Q_i \in \{P, Q\}$ by the structure of $\mathbf{J_r}$ and because

$$\tau'_b \circ F'\,(\langle f \rangle_{i=1}^{n}) = \tau'_a = \tau'_b \circ F'\,(\langle g \rangle_{i=1}^{n})$$

(by naturality of τ') we get that

$$\begin{aligned}
&\tau'_b \circ F'\,(\langle P_i\,(Q_i\,(f, g)\,, 1_b) \rangle_{i=1}^{n}) = \\
&\tau'_b \circ F'\,(\langle P_i\,(Q_i\,(f, f \circ h \circ g)\,, f \circ h) \rangle_{i=1}^{n}) = \\
&\tau'_b \circ F'\,(\langle f \circ P_i\,(Q_i\,(1_a, h \circ g)\,, h) \rangle_{i=1}^{n}) = \\
&\tau'_b \circ F'\,(\langle f \rangle_{i=1}^{n}) \circ F'\,(\langle P_i\,(Q_i\,(1_a, h \circ g)\,, h) \rangle_{i=1}^{n}) = \\
&\tau'_b \circ F'\,(\langle g \rangle_{i=1}^{n}) \circ F'\,(\langle P_i\,(Q_i\,(1_a, h \circ g)\,, h) \rangle_{i=1}^{n}) = \\
&\tau'_b \circ F'\,(\langle g \circ P_i\,(Q_i\,(1_a, h \circ g)\,, h) \rangle_{i=1}^{n}) = \\
&\tau'_b \circ F'\,(\langle P_i\,(Q_i\,(g, g \circ h \circ g)\,, g \circ h) \rangle_{i=1}^{n}) = \\
&\tau'_b \circ F'\,(\langle P_i\,(g, 1_b) \rangle_{i=1}^{n})
\end{aligned}$$

Knowing this we can define $\dot{\tau}' : F' \dot{\to} \Delta\,(c) : \prod_{i=1}^{n} \mathbf{J_r} \to \mathbf{C}$ by

$$\dot{\tau}'_{\langle P_i(a,b) \rangle_{i=1}^{n}} := \tau'_b \circ F'\,(\langle P_i\,(Q_i\,(f, g)\,, 1_b) \rangle_{i=1}^{n}) \tag{60}$$

where by the above calculation the choice of Q_i does not make any difference for the result. For any arrow $\langle P_i\,(Q'_i\,(h, 1_a)\,, Q''_i\,(Q_i\,(f, g)\,, 1_b)) \rangle_{i=1}^{n}$ (again let each Q'_i and each Q''_i be either one of the projections) of $\prod_{i=1}^{n} \mathbf{J_r}$ (note that every arrow of $\prod_{i=1}^{n} \mathbf{J_r}$ can be written in this way) we calculate

$$\begin{aligned}
&\dot{\tau}'_{\langle P_i(a,b) \rangle_{i=1}^{n}} \circ F'\,(\langle P_i\,(Q'_i\,(h, 1_a)\,, Q''_i\,(Q_i\,(f, g)\,, 1_b)) \rangle_{i=1}^{n}) = \\
&\quad \tau'_b \circ F'\,(\langle P_i\,(Q_i\,(f, g)\,, 1_b) \rangle_{i=1}^{n}) \circ F'\,(\langle P_i\,(Q'_i\,(h, 1_a)\,, Q''_i\,(Q_i\,(f, g)\,, 1_b)) \rangle_{i=1}^{n}) = \\
&\quad \tau'_b \circ F'\,(\langle P_i\,(Q_i\,(f, g) \circ Q'_i\,(h, 1_a)\,, Q''_i\,(Q_i\,(f, g)\,, 1_b)) \rangle_{i=1}^{n}) = \\
&\quad \tau'_b \circ F'\,(\langle P_i\,(Q'_i\,(1_b, Q_i\,(f, g)), Q''_i\,(Q_i\,(f, g)\,, 1_b)) \rangle_{i=1}^{n}) = \\
&\quad \tau'_b \circ F'\,(\langle P'_i\,(Q_i\,(f, g)\,, 1_b) \rangle_{i=1}^{n}) \qquad\qquad = \quad \dot{\tau}'_{\langle P'_i(a,b) \rangle_{i=1}^{n}}
\end{aligned}$$

where

$$P'_i := \begin{cases} P & \text{if } (P_i = P \wedge Q'_i = Q) \vee (P_i = Q \wedge Q''_i = P)\ , \\ Q & \text{otherwise.} \end{cases} \tag{61}$$

Therefore $\dot{\tau}'$ is a cone from F'. The assignment $\tau' \mapsto \dot{\tau}'$ is clearly a right inverse for $- \circ K_r$ and it is left inverse for $- \circ K_r$ because for any cone $\ddot{\tau}'$ from F' we must have

$$\ddot{\tau}'_{\langle P_i(a,b) \rangle_{i=1}^{n}} := \ddot{\tau}'_{\langle b \rangle_{i=1}^{n}} \circ F'\,(\langle P_i\,(Q_i\,(f, g)\,, 1_b) \rangle_{i=1}^{n})$$

and therefore the image of $\ddot{\tau}' \circ K_r$ under the assignment just described is $\ddot{\tau}'$.

Now we prove the lemma. Let $\tau' : F \circ (\prod_{i=1}^n L_i) \circ \Delta_{\mathbf{J_r}} \dot{\to} \Delta(c) : \mathbf{J_r} \to \mathbf{C}$ be a cone. Setting $F' := F \circ (\prod_{i=1}^n L_i)$ in the above argument we get that τ' can be extended to a cone $\dot{\tau}' : F \circ (\prod_{i=1}^n L_i) \dot{\to} \Delta(c) : \prod_{i=1}^n \mathbf{J_r} \to \mathbf{C}$. By lemma 5.3

$$\left\langle F(\langle l_i \rangle_{i=1}^n), \langle F(\langle \tau_{i;j_i} \rangle_{i=1}^n) : F(\langle L_i(j_i) \rangle_{i=1}^n) \to F(\langle l_i \rangle_{i=1}^n) \rangle_{\langle j_i \rangle_{i=1}^n \in \prod_{i=1}^n \mathbf{J_r}} \right\rangle$$

is a colimit of $F \circ (\prod_{i=1}^n L_i)$ and therefore $\dot{\tau}'$ uniquely factors through

$$\langle F(\langle \tau_{i;j_i} \rangle_{i=1}^n) \rangle_{\langle j_i \rangle_{i=1}^n \in \prod_{i=1}^n \mathbf{J_r}} .$$

Among other things this means that there is an arrow $k : F(\langle l_i \rangle_{i=1}^n) \to c$ such that

$$\tau'_b = \dot{\tau}'_{\langle b \rangle_{i=1}^n} = k \circ F(\langle \tau_{i;b} \rangle_{i=1}^n)$$

If for k' we also have

$$\tau'_b = k' \circ F(\langle \tau_{i;b} \rangle_{i=1}^n)$$

then by the above description of the cone $\langle F(\langle \tau_{i;j_i} \rangle_{i=1}^n) \rangle_{\langle j_i \rangle_{i=1}^n \in \prod_{i=1}^n \mathbf{J_r}}$ from $F' = F \circ \prod_{i=1}^n L_i$ in terms of $F(\langle \tau_{i;b} \rangle_{i=1}^n)$ we get that

$$k' \circ F(\langle \tau_{i;j_i} \rangle_{i=1}^n) = k \circ F(\langle \tau_{i;j_i} \rangle_{i=1}^n)$$

for any tuple $\langle j_i \rangle_{i=1}^n \in \prod_{i=1}^n \mathbf{J_r}$. By the colimit property of

$$\langle F(\langle \tau_{i;j_i} \rangle_{i=1}^n) \rangle_{\langle j_i \rangle_{i=1}^n \in \prod_{i=1}^n \mathbf{J_r}}$$

this means that $k = k'$ and therefore $F(\langle \tau_{i;b} \rangle_{i=1}^n)$ is a coequalizer of the pair

$$\langle F(\langle L_i(f) \rangle_{i=1}^n), F(\langle L_i(g) \rangle_{i=1}^n) \rangle .$$

$\square$

6 Monads and algebraic structures

A monoid in the strict monoidal category $\mathbf{C}^{\mathbf{C}}$ of endofunctors on a category $\mathbf{C}$ and natural transformations between them has a special name — it is called a monad. Such monads are interesting in their own right because they can be constructed from adjoint pairs and in many cases a lot of the structure of the underlying categories can be reconstructed from the monad. For more information see [Mac98, chapter VI]. We mention them here because monoid actions and in many cases also monoids themselves can be viewed as so-called Eilenberg-Moore algebras of a suitable monad.

Definition 6.1. The category $\mathbf{C}^T$ of Eilenberg-Moore algebras of a monad $\langle T, \mu, \eta\rangle$ (where $T : \mathbf{C} \to \mathbf{C}$, $\mu : T \circ T \dot{\to} T : \mathbf{C} \to \mathbf{C}$, $\eta : 1_{\mathbf{C}} \dot{\to} T : \mathbf{C} \to \mathbf{C}$) has as objects pairs $\langle a, f\rangle$ where $a \in \mathrm{Obj}(\mathbf{C})$ and $f : T(a) \to a$ such that

$$\begin{tikzcd} T(a) \arrow[d, "f"] & (T\circ T)(a) \arrow[l, "\mu_a"'] \arrow[d, "T(f)"] \\ a & T(a) \arrow[l, "f"] \end{tikzcd} \qquad \begin{tikzcd} T(a) \arrow[d, "f"] & a \arrow[l, "\eta_a"'] \arrow[dl, "1_a"] \\ a & \end{tikzcd} \tag{62}$$

commute. If $\langle a, f\rangle, \langle b, g\rangle \in \mathrm{Obj}\left(\mathbf{C}^T\right)$ then a morphism $h : \langle a, f\rangle \to \langle b, g\rangle$ of these Eilenberg-Moore algebras is an arrow $h : a \to b$ of $\mathbf{C}$ such that

$$\begin{tikzcd} T(b) \arrow[d, "g"] & T(a) \arrow[l, "T(h)"'] \arrow[d, "f"] \\ b & a \arrow[l, "h"] \end{tikzcd} \tag{63}$$

commutes.

The category of monoid acts is the canonical example of a category of Eilenberg-Moore algebras. For some monoid $\langle m, \mu, \eta\rangle$ in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ set $\dot{T} := \mathcal{S}(m) = m \boxtimes -$ (here $\langle \mathcal{S}, \mathcal{S}_2, \mathcal{S}_0\rangle$ is the monoidal functor defined in (22)). We can turn $\dot{T}$ into a monad by setting $\dot{\mu} := \mathcal{S}(\mu) \bullet \mathcal{S}_{2;m,m}$ and $\dot{\eta} := \mathcal{S}(\eta) \bullet \mathcal{S}_0$, explicitly

$$\begin{aligned} \dot{\mu} &: \dot{T} \circ \dot{T} \dot{\to} \dot{T} : \mathbf{C} \to \mathbf{C} \\ \dot{\mu}_a &: m \boxtimes (m \boxtimes a) \to m \boxtimes a \\ \dot{\mu}_a &= (\mu \boxtimes 1_a) \circ \alpha_{m,m,a} \\ \dot{\eta} &: 1_{\mathbf{C}} \dot{\to} \dot{T} : \mathbf{C} \to \mathbf{C} \\ \dot{\eta}_a &: a \to m \boxtimes a \\ \dot{\eta}_a &= (\eta \boxtimes 1_a) \circ \lambda_a^{-1} . \end{aligned}$$

Substituting in the definitions it becomes clear that

Observation 6.2. *A left action of the monoid $\langle m, \mu, \eta\rangle$ is just an Eilenberg-Moore algebra for the monad $\left\langle \dot{T}, \dot{\mu}, \dot{\eta}\right\rangle$. The category of left $\langle m, \mu, \eta\rangle$-acts is the category of these Eilenberg-Moore algebras.*

This justifies spending some effort on figuring out the structure of limits and colimits in categories of Eilenberg-Moore algebras.

Definition 6.3. The functor $G^T : \mathbf{C}^T \to \mathbf{C}$ which sends $\langle a, f \rangle \in \mathrm{Obj}\left(\mathbf{C}^T\right)$ to $a \in \mathbf{C}$ and $h : \langle a, f \rangle \to \langle b, g \rangle$ to $h : a \to b$ is called the forgetful functor of the Eilenberg-Moore category of $\langle T, \mu, \eta \rangle$.

Lemma 6.4. *G^T has a left adjoint $F^T : \mathbf{C} \to \mathbf{C}^T$ which sends an object $a \in \mathbf{C}$ to $\langle T(a), \mu_a \rangle$ and $h : a \to b$ to $T(h) : \langle T(a), \mu_a \rangle \to \langle T(b), \mu_b \rangle$. The unit η^T of this adjunction is η, while the counit $\varepsilon^T : F^T \circ G^T \dot{\to} 1_{\mathbf{C}^T} : \mathbf{C}^T \to \mathbf{C}^T$ is given by $\varepsilon^T_{\langle a,f \rangle} = f$.*

Proof. The condition (63) that $T(h)$ be a morphism of Eilenberg-Moore algebras is satisfied by naturality of μ. The condition that $\varepsilon^T_{\langle a,f \rangle}$ be a morphism of algebras is just the left hand square in (62) and naturality of ε^T is (63). One way of saying that $\langle F, G, \eta, \varepsilon \rangle$ is an adjunction is to say that the identities $(\varepsilon \circ F) \bullet (F \circ \eta) = 1_F$ and $(G \circ \varepsilon) \bullet (\eta \circ G) = 1_G$ hold. (For a detailed account of different ways of characterizing adjunctions see for example [Mac98, chapter IV].) In our case the first of these identities is one of the identities (6) satisfied by the monoid $\langle T, \mu, \eta \rangle$ and the second is just the right hand triangle in (62). $\square$

The following results about limits and colimits in categories of Eilenberg-Moore algebras are either stated or hinted at in [Mac98, chapter VI]. See specifically section VI.2 and VI.7.

6.1 Limits

Lemma 6.5. *G^T creates limits.*

Proof. Let $L : \mathbf{I} \to \mathbf{C}^T$ be a functor and $\left\langle l, \tau : \Delta(l) \dot{\to} G^T \circ L : \mathbf{I} \to \mathbf{C} \right\rangle$ be a limit of $G^T \circ L$. We want to find an arrow $f : T(l) \to l$ which turns l into an object of $\mathbf{C}^T$ such that the components τ_i of τ are algebra morphisms. This means that

$$\begin{tikzcd} T \circ G^T \circ L \arrow[d, "G^T \circ \varepsilon \circ L"'] & \Delta(T(l)) = T \circ (\Delta(l)) \arrow[l, "T \circ \tau"'] \arrow[d, "\Delta(f)"] \\ G^T \circ L & \Delta(l) \arrow[l, "\tau"] \end{tikzcd} \tag{64}$$

commutes. But by the limit property of $\langle l, \tau \rangle$ there is a unique f fulfilling this requirement. We need to check that f turns $\langle l, f \rangle$ into an algebra.

$$\begin{tikzcd} T \circ G^T \circ L \arrow[ddd, "G^T \circ \varepsilon \circ L"'] & & & T \circ T \circ G^T \circ L \arrow[lll, "\mu \circ L"'] \arrow[ddd, "T \circ G^T \circ \varepsilon \circ L"] \\ & \Delta(T(l)) \arrow[ul, "T \circ \tau"'] \arrow[d, "\Delta(f)"'] & \Delta(T(T(l))) \arrow[l, "\Delta(\mu_l)"', "= \mu \circ (\Delta(l))"] \arrow[ur, "T \circ T \circ \tau"] \arrow[d, "\Delta(T(f))"] & \\ & \Delta(l) \arrow[dl, "\tau"'] & \Delta(T(l)) \arrow[l, "\Delta(f)"] \arrow[dr, "T \circ \tau"] & \\ G^T \circ L & & & T \circ G^T \circ L \arrow[lll, "G^T \circ \varepsilon \circ L"] \end{tikzcd}$$

In the diagram above the outermost quadrilateral commutes because $L(i)$ are algebras, the left, right, and lower quadrilateral commute by the definition of f and the upper quadrilateral commutes by naturality of μ. This implies $\tau \bullet \Delta(f) \bullet \Delta(\mu_l) = \tau \bullet \Delta(f) \bullet \Delta(T(f))$. By the limit property we get that $f \circ \mu_l = f \circ T(f)$.

$$\begin{tikzcd}
T \circ G^T \circ L \arrow[dddd, "G^T \circ \varepsilon \circ L"'] & & & G^T \circ L \arrow[lll, "\eta \circ G^T \circ L"'] \arrow[ddddlll, "1"] \\
& \Delta(T(l)) \arrow[ul, "T \circ \tau"'] \arrow[dd, "\Delta(f)"] & \Delta(l) \arrow[l, "\Delta(\eta_l)"', "=\eta \circ \Delta(l)"] \arrow[ur, "\tau"] \arrow[ddl, "1"] & \\
& & & \\
& \Delta(l) \arrow[dl, "\tau"'] & & \\
G^T \circ L & & &
\end{tikzcd}$$

Again, the outermost triangle commutes because $L(i)$ are algebras, while the outer quadrilaterals commute either trivially, by the definition of f or by naturality of η. This implies $\tau \bullet \Delta(f) \bullet \Delta(\eta_l) = \tau \bullet 1$, which by the limit property implies $f \circ \eta_l = 1_l$. $\square$

6.2 Colimits

Lemma 6.6. *If $L : \mathbf{I} \to \mathbf{C}^T$ is a functor and both T and $T \circ T$ preserve colimits of $G^T \circ L : \mathbf{I} \to \mathbf{C}$, then G^T creates colimits for L.*

Proof. The proof is very similar to the previous one. We are given a colimit $\langle l, \tau : G^T \circ L \dot{\to} \Delta(l) : \mathbf{I} \to \mathbf{C} \rangle$ of $G^T \circ L$ and want to find a unique arrow $h : T(l) \to l$ that turns $\langle l, h \rangle$ into an algebra and the components of τ into morphisms of algebras, that is we want the diagram below to commute.

$$\begin{tikzcd}
\Delta(T(l)) = T \circ (\Delta(l)) \arrow[d, "\Delta(h)"'] & T \circ G^T \circ L \arrow[l, "T \circ \tau"'] \arrow[d, "G^T \circ \varepsilon \circ L"] \\
\Delta(l) & G^T \circ L \arrow[l, "\tau"]
\end{tikzcd}$$

By hypothesis of the lemma T preserves colimits of $G^T \circ L$ and therefore there is a unique h satisfying this requirement. We need to check that $\langle l, h \rangle$ is an algebra. Consider the diagram below.

$$\begin{tikzcd}
T \circ G^T \circ L \arrow[dr, "T \circ \tau"] \arrow[ddd, "G^T \circ \varepsilon \circ L"'] & & & T \circ T \circ G^T \circ L \arrow[lll, "\mu \circ L"'] \arrow[dl, "T \circ T \circ \tau"'] \arrow[ddd, "T \circ G^T \circ \varepsilon \circ L"] \\
& \Delta(T(l)) \arrow[d, "\Delta(h)"] & \Delta(T(T(l))) \arrow[l, "\Delta(\mu_l)"', "=\mu \circ (\Delta(l))"] \arrow[d, "\Delta(T(h))"'] & \\
& \Delta(l) & \Delta(T(l)) \arrow[l, "\Delta(h)"] & \\
G^T \circ L \arrow[ur, "\tau"'] & & & T \circ G^T \circ L \arrow[lll, "G^T \circ \varepsilon \circ L"] \arrow[ul, "T \circ \tau"]
\end{tikzcd}$$

Again the topmost quadrilateral commutes by naturality of μ and all other flanking quadrilaterals commute by the definition of h. The outer square commutes because $L(i)$ are algebras. This implies that $\Delta(h \circ \mu_l) \bullet (T \circ T \circ \tau) = \Delta(h \circ T(h)) \bullet (T \circ T \circ \tau)$. Because $T \circ T$ preserves colimits of $G^T \circ L$ we get that $T \circ T \circ \tau$ is a colimiting cone and therefore $h \circ \mu_l = h \circ T(h)$.

$$\begin{tikzcd}
T \circ G^T \circ L \arrow[dr, "T \circ \tau"] \arrow[ddd, "G^T \circ \varepsilon \circ L"] & & & G^T \circ L \arrow[lll, "\eta \circ G^T \circ L"'] \arrow[dl, "\tau"'] \arrow[dddlll, "1"] \\
& \Delta(T(l)) \arrow[d, "\Delta(h)"] & \Delta(l) \arrow[l, "\Delta(\eta_l)"', "=\eta \circ \Delta(l)"] \arrow[dl, "1"] & \\
& \Delta(l) & & \\
G^T \circ L \arrow[ur, "\tau"] & & &
\end{tikzcd}$$

Again, from commutativity of the outer triangle, naturality of η, the definition of h and the colimit property of τ one derives commutativity of the inner triangle. □

Remark 6.7. Note that if a functor F preserves any one colimit of a functor L then it preserves all colimits of L. This is because all colimits are isomorphic, the colimiting cones are mapped onto each other under composition with the isomorphisms, images of isomorphic objects under a functor remain isomorphic and if any one object from a class of isomorphic objects is a colimit of a functor then all other objects are colimits as well — with the colimiting cones again given by composition with the isomorphisms.

Therefore even though the above lemma could have been stated in such a way as to only require T and $T \circ T$ to preserve a specific colimit of $G^T \circ L$ and only claim creation of a specific colimit (the proof would have remained the same) this extra precision is not really required as something more general than this can easily be recovered from the seemingly weaker version of the lemma given above.

Lemma 6.5 resolves the question about the structure of limits in Eilenberg-Moore categories and lemma 6.6 gives a corresponding result for colimits — but only in some cases. These cases include the case of the category $\langle m, \mu, \eta\rangle{-}\mathbf{Act}$ (and of course also $\mathbf{Act}{-}\,\langle m, \mu, \eta\rangle$) when $m\boxtimes-$ (or $-\boxtimes m$ for the category $\mathbf{Act}{-}\,\langle m, \mu, \eta\rangle$) preserves colimits. This seems like a reasonable enough assumption because this is the case in many categories which are relevant in practice. For example, any monoidal category which has an internal hom-functor — that is, a right adjoint of the functor $-\boxtimes m$ — satisfies this condition (see the next lemma) (strictly speaking this only works for the case of $\mathbf{Act}{-}\,\langle m, \mu, \eta\rangle$ but in many cases the monoidal product is also symmetric and then it also works for $\langle m, \mu, \eta\rangle{-}\mathbf{Act}$).

On the other hand, in many practical situations T does not preserve all colimits. For example we will see in section 7 that the category of monoids in some monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ can also be viewed as a category of Eilenberg-Moore algebras under relatively mild assumptions about the structure of $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ — but the monad in question does not preserve colimits in any of the typical examples. In some of these cases (which, as we will see, include the cases we are interested in in the various categories of monoids and monoid actions) one can still describe colimits in $\mathbf{C}^T$ in terms of colimits in $\mathbf{C}$.

This is what we set out to do now. The strategy will be — roughly — to describe general colimits as iterated colimits of two distinct specific kinds, both of which exist and can be described under relatively general assumptions.

First of all remember that

Lemma 6.8. *Any functor $F : \mathbf{X} \to \mathbf{A}$ which is left adjoint to some functor $G : \mathbf{A} \to \mathbf{X}$ preserves colimits.*

Proof. This fact and various ways of proving it can be found in [Mac98, section V.5]. We give another slight variation.

Let

$$\hom_{\mathbf{A}}(F(x), a) \overset{\varphi}{\cong} \hom_{\mathbf{X}}(x, G(a))$$

be an adjunction. When $\tau : L \dot{\to} \Delta(l) : \mathbf{I} \to \mathbf{X}$ is a cone, then saying that $\langle l, \tau : L \dot{\to} \Delta(l) : \mathbf{I} \to \mathbf{X}\rangle$ is a colimit is the same thing as saying that the map

$$\begin{aligned} \varphi'_x &: \hom_{\mathbf{X}}(l, x) \to \hom_{\mathbf{X}^{\mathbf{I}}}(L, \Delta(x)) \\ \varphi'_x(f) &= \Delta(f) \bullet \tau \end{aligned}$$

is a bijection for all $x \in \mathbf{X}$. So to show that F preserves colimits we want to deduce from this that

$$\begin{aligned} \dot{\varphi}'_a &: \hom_{\mathbf{A}}(F(l), a) \to \hom_{\mathbf{X}}(F \circ L, \Delta(a)) \\ \dot{\varphi}'_a(g) &= \Delta(g) \bullet (F \circ \tau) \end{aligned}$$

is a bijection for all $a \in \mathbf{A}$. By naturality of φ we have

$$\left\langle \varphi_{L(i),a}(g \circ F(\tau_i)) \right\rangle_{i \in \mathbf{I}} = \langle \varphi_{x,a}(g) \circ \tau_i \rangle_{i \in \mathbf{I}} = \Delta(\varphi(g)) \bullet \tau = \varphi'_{G(a)}(\varphi(g))$$

and therefore

$$\dot{\varphi}'_a(g) = \left\langle \varphi^{-1}_{L(i),a} \right\rangle_{i \in \mathbf{I}} \left(\varphi'_{G(a)}(\varphi(g)) \right)$$

$(\left\langle \varphi^{-1}_{L(i),a} \right\rangle_{i\in\mathbf{I}} : \hom_{\mathbf{X}^{\mathbf{I}}}(L, G\circ(\Delta(a))) \to \hom_{\mathbf{A}^{\mathbf{I}}}(F\circ L, \Delta(a))$ is the map which sends cones to cones by applying $\varphi^{-1}_{L(i),a}$ to the individual components (to be strict, we did not show that this map really takes cones to cones but we also do not need this property — we only need to know that it is bijective)) which makes it plainly visible that $\dot{\varphi}'_a$ as a composite of bijections is also bijective. $\square$

This means that we know the structure of colimits in $\mathbf{C}^T$ of functors which factor as $F^T \circ L$.

To show that in any category $\mathbf{C}^T$ of Eilenberg-Moore algebras we definitely have one more type of coequalizer we need the concept of a split fork/split coequalizer (this is different from the concept of a reflexive coequalizer introduced earlier). For more on these see [Mac98, VI.6 and VI.7].

Definition 6.9. A *split fork* consists of five arrows $\langle f, g, e, h, k\rangle$ with sources and targets related as shown below and such that the diagram below commutes.

$$\begin{tikzcd}
b \arrow[d, "e"] & a \arrow[l, "f"] \arrow[d, "g"] & b \arrow[l, "k"] \arrow[d, "e"] \arrow[ll, bend right, "1_b"'] \\
c & b \arrow[l, "e"'] & c \arrow[l, "h"'] \arrow[ll, bend left, "1_c"]
\end{tikzcd} \tag{65}$$

Definition 6.10. We call a limit or colimit *absolute* if it is preserved by *every* functor.

Lemma 6.11. *In every split fork* (65) *the arrow e is an absolute coequalizer of f and g.*

Proof. Note first that if $\langle f, g, e, h, k\rangle$ is a split fork and F is any functor, then $\langle F(f), F(g), F(e), F(h), F(k)\rangle$ is also a split fork. So we just need to show that in any split fork (65) the arrow e is a coequalizer of f and g.

Let $r : b \to d$ be such that $r\circ f = r\circ g$. Combining this with what we know for the other arrows we get the diagram below.

$$\begin{tikzcd}
b \arrow[d, "r"] & a \arrow[l, "f"] \arrow[d, "g"] & b \arrow[l, "k"] \arrow[d, "e"] \arrow[ll, bend right, "1_b"'] \\
d & b \arrow[l, "r"'] & c \arrow[l, "h"'] \arrow[ll, bend left, "r'"]
\end{tikzcd}$$

As indicated set $r' := r\circ h$. In the diagram we can see that

$$r'\circ e = r\circ h\circ e = r\circ g\circ k = r\circ f\circ k = r\circ 1_b = r \ .$$

If for r'' we also have $r''\circ e = r$ then

$$r'' = r''\circ e\circ h = r\circ h = r'$$

$\square$

Lemma 6.12. *In any category* $\mathbf{C}^T$ *of Eilenberg-Moore algebras for every object* $a = \langle a', h' \rangle \in \mathbf{C}^T$ *the arrow* ε^T_a *is the coequalizer of the reflexive pair* $\varepsilon^T_{(F^T \circ G^T)(a)}$ *and* $F^T\left(G^T\left(\varepsilon^T_a\right)\right)$*, whose common right inverse is* $F^T\left(\eta^T_{G^T(a)}\right)$*.*

Proof. First we show that $F^T\left(\eta^T_{G^T(a)}\right)$ is a common right inverse for $\varepsilon^T_{(F^T \circ G^T)(a)}$ and $F^T\left(G^T\left(\varepsilon^T_a\right)\right)$. Because G^T is faithful we can calculate in $\mathbf{C}$.

$$
\begin{aligned}
G^T\left(F^T\left(\eta^T_{G^T(a)}\right)\right) &= T(\eta_{a'}) \\
G^T\left(\varepsilon^T_{(F^T \circ G^T)(a)}\right) &= \mu_{a'} \\
G^T\left(F^T\left(G^T\left(\varepsilon^T_a\right)\right)\right) &= T(h')
\end{aligned}
$$

$\mu_{a'} \circ T(\eta_{a'}) = 1_{T(a')}$ because $\langle T, \mu, \eta \rangle$ is a monad and $T(h') \circ T(\eta_{a'}) = 1_{T(a')}$ because a is an Eilenberg-Moore algebra. Next we will show that ε^T_a, $\varepsilon^T_{(F^T \circ G^T)(a)}$ and $F^T\left(G^T\left(\varepsilon^T_a\right)\right)$ become part of a split fork when G^T is applied to them. With the notation of definition 6.9 we set

$$
\begin{aligned}
e &= G^T\left(\varepsilon^T_a\right) = h' \\
f &= G^T\left(\varepsilon^T_{(F^T \circ G^T)(a)}\right) = \mu_{a'} \\
g &= G^T\left(F^T\left(G^T\left(\varepsilon^T_a\right)\right)\right) = T(h') \\
k &= \eta^T_{T(a')} \\
h &= \eta^T_{a'}
\end{aligned}
$$

The diagram in (65) now looks like the one below.

$$
\begin{tikzcd}
T(a') \arrow[d, "h'"] & T(T(a')) \arrow[l, "\mu_{a'}"'] \arrow[d, "T(h')"] & T(a') \arrow[l, "\eta^T_{T(a')}"'] \arrow[d, "h'"] \arrow[ll, bend right, "1_{T(a')}"'] \\
a' & T(a') \arrow[l, "h'"] & a' \arrow[l, "\eta^T_{a'}"] \arrow[ll, bend left, "1_{a'}"]
\end{tikzcd}
$$

The left hand square and the squished bottom "triangle" commute because a is an algebra, the right hand square commutes by naturality of η^T and the upper squished "triangle" commutes because $\langle T, \mu, \eta \rangle$ is a monad. Therefore h' is an absolute coequalizer of $\mu_{a'}$ and $T(h')$. By lemma 6.6 and remark 6.7 this means that G^T creates coequalizers for the pair $\left\langle \varepsilon^T_{(F^T \circ G^T)(a)}, F^T\left(G^T\left(\varepsilon^T_a\right)\right) \right\rangle$. But $G^T\left(\varepsilon^T_a\right) = h'$ which in turn is a coequalizer of

$$
\left\langle G^T\left(\varepsilon^T_{(F^T \circ G^T)(a)}\right), G^T\left(F^T\left(G^T\left(\varepsilon^T_a\right)\right)\right) \right\rangle
$$

and therefore ε^T_a is a coequalizer of $\left\langle \varepsilon^T_{(F^T \circ G^T)(a)}, F^T\left(G^T\left(\varepsilon^T_a\right)\right) \right\rangle$. $\square$

The author learned about the following theorem from the nLab which attributes it to [Lin69].

Theorem 6.13. *Assume that $\mathbf{C}^T$ is the category of Eilenberg-Moore algebras of some monad $\langle T, \mu, \eta\rangle$, that $L : \mathbf{I} \to \mathbf{C}^T$ is a functor and that the colimits of*

$$F^T \circ G^T \circ L \quad \text{and} \quad F^T \circ G^T \circ F^T \circ G^T \circ L$$

exist in $\mathbf{C}^T$.

By lemma 6.8 this is definitely the case if the colimits of $G^T \circ L$ and $G^T \circ F^T \circ G^T \circ L$ exist in $\mathbf{C}$.

Assume further that $\mathbf{C}^T$ has coequalizers of reflexive pairs (it will become evident in the proof, that for every colimit that we are interested in, only the coequalizer of one specific pair is required).

By lemma 6.6 this is the case if $\mathbf{C}$ has reflexive coequalizers which are preserved by T. Under these assumptions a colimit $\langle l, \tau\rangle$ of L exists in $\mathbf{C}^T$.

This colimit is related to the colimits $\langle l_b, \sigma_b\rangle$ of $F^T \circ G^T \circ L$ and $\langle l_a, \sigma_a\rangle$ of $F^T \circ G^T \circ F^T \circ G^T \circ L$ in the way depicted below. In this diagram e is the coequalizer of $F'(f)$ and $F'(g)$. The arrows $F'(f)$, $F'(g)$ and τ_i are uniquely determined by the requirement that the diagram below commute for all $i \in \mathbf{I}$.

$$\begin{tikzcd}
 & & F^T\big(G^T(L(i))\big) \arrow[ddll, "\varepsilon^T_{L(i)}"'] \arrow[d, "\sigma_{b;i}"] & & \\
 & & l_b \arrow[dl, "e"'] & & \\
L(i) \arrow[r, dashed, "\tau_i"] & l & & l_a \arrow[ul, dashed, "F'(f)"'] \arrow[dl, dashed, "F'(g)"] & F^T\Big(G^T\big(F^T\big(G^T(L(i))\big)\big)\Big) \arrow[l, "\sigma_{a;i}"] \arrow[uull, "\varepsilon^T_{F^T(G^T(L(i)))}"'] \arrow[ddll, "F^T\left(G^T\left(\varepsilon^T_{L(i)}\right)\right)"] \\
 & & l_b \arrow[ul, "e"] & & \\
 & & F^T\big(G^T(L(i))\big) \arrow[uull, "\varepsilon^T_{L(i)}"] \arrow[u, "\sigma_{b;i}"'] & &
\end{tikzcd} \tag{66}$$

Proof. This may be obvious to the reader but — justifying the sideclaim in the theorem — note first that if T preserves reflexive coequalizers, then any power of T also preserves reflexive coequalizers, because T maps a reflexive coequalizer to a reflexive coequalizer which is then again mapped to a reflexive coequalizer under T, etc.

From lemma 6.12 we know that the colimit we are hoping to find can be seen as an iterated colimit. Our plan is to use our knowledge from both lemma 5.1 and lemma 5.2 about iterated colimits to reverse the order of taking colimits so that we end up only with colimits that we know to exist.

Define a functor $G : \mathbf{J_r} \to \left(\mathbf{C}^T\right)^{\left(\mathbf{C}^T\right)}$ by setting

$$\begin{aligned}
G(b) &:= F^T \circ G^T \\
G(a) &:= F^T \circ G^T \circ F^T \circ G^T \\
G(f) &:= \varepsilon^T \circ F^T \circ G^T \\
G(g) &:= F^T \circ G^T \circ \varepsilon^T \\
G(h) &:= F^T \circ \eta^T \circ G^T
\end{aligned}$$

and by extending to composites in $\mathbf{J_r}$ by composition of the images of the generating arrows specified here. To show that G is indeed a functor we need to check that $G(f) \bullet G(h) = G(g) \bullet G(h) = 1_{G(b)}$. This is true by lemma 6.12. Now define a functor $F : \mathbf{J_r} \times \mathbf{I} \to \mathbf{C}^T$ as the composite

$$\mathbf{C}^T \xleftarrow[E]{} \left(\mathbf{C}^T\right)^{\left(\mathbf{C}^T\right)} \times \mathbf{C}^T \xleftarrow[G \times L]{} \mathbf{J_r} \times \mathbf{I}$$

where E is the "evaluation functor" — that is, the counit of the adjunction between product and exponential in the (meta-)category of categories. Explicitly this means that for objects $i \in \mathbf{I}$ and morphisms $u : i \to i'$ in $\mathbf{I}$ the functor F is defined by

$$\begin{aligned}
F(b,i) &= F^T\big(G^T(L(i))\big) \\
F(1_b,u) &= F^T\big(G^T(L(u))\big) \\
F(a,i) &= F^T\Big(G^T\Big(F^T\big(G^T(L(i))\big)\Big)\Big) \\
F(1_a,u) &= F^T\Big(G^T\Big(F^T\big(G^T(L(u))\big)\Big)\Big) \\
F(f,u) &= \varepsilon^T_{F^T(G^T(L(i')))} \circ F^T\Big(G^T\Big(F^T\big(G^T(L(u))\big)\Big)\Big) \\
&= F^T\big(G^T(L(u))\big) \circ \varepsilon^T_{F^T(G^T(L(i)))} \\
F(g,u) &= F^T\Big(G^T\Big(\varepsilon^T_{L(i')}\Big)\Big) \circ F^T\Big(G^T\Big(F^T\big(G^T(L(u))\big)\Big)\Big) \\
&= F^T\big(G^T(L(u))\big) \circ F^T\Big(G^T\Big(\varepsilon^T_{L(i)}\Big)\Big) \\
F(h,u) &= F^T\Big(\eta^T_{G^T(L(i'))}\Big) \circ F^T\big(G^T(L(u))\big) \\
&= F^T\Big(G^T\Big(F^T\big(G^T(L(u))\big)\Big)\Big) \circ F^T\Big(\eta^T_{G^T(L(i))}\Big) \quad .
\end{aligned}$$

By hypothesis of the theorem the colimits $\left\langle l_b, \sigma_b : F(b,-) \dot{\to} \Delta(l_b) : \mathbf{I} \to \mathbf{C}^T \right\rangle$ of $F(b,-) = F^T \circ G^T \circ L$ and $\left\langle l_a, \sigma_a : F(a,-) \dot{\to} \Delta(l_a) : \mathbf{I} \to \mathbf{C}^T \right\rangle$ of $F(a,-) = F^T \circ G^T \circ F^T \circ G^T \circ L$ exist.

By the remark in lemma 5.1 there is a unique functor $F' : \mathbf{J_r} \to \mathbf{C}^T$ such that $F'(b) = l_b$, $F'(a) = l_a$ and that a diagram of the shape as in (57) commutes for all arrows $v : j \to j'$ in $\mathbf{J_r}$ and all objects $i \in \mathbf{I}$ (in our case u is replaced by v, i and j are reversed, L is replaced by F, L' is replaced by F' and τ is replaced by σ). This is equivalent to requiring commutativity of these kind of diagrams for a set of arrows which generate the index category, so in our case it means that the upper right and the lower right quadrilaterals in (66) commute for all $i \in \mathbf{I}$, and that the diagram below commutes for all $i \in \mathbf{I}$.

$$\begin{tikzcd}
F^T\big(G^T(L(i))\big) \arrow[r, "F^T\left(\eta^T_{G^T(L(i))}\right)"] \arrow[d, "\sigma_{b;i}"] & F^T\Big(G^T\Big(F^T\big(G^T(L(i))\big)\Big)\Big) \arrow[d, "\sigma_{a;i}"] \\
l_b \arrow[r, dashed, "F'(h)"] & l_a
\end{tikzcd}$$

By hypothesis of the theorem a colimit $\left\langle l, \sigma' : F' \dot{\to} \Delta(l) : \mathbf{J_r} \to \mathbf{C}^T \right\rangle$ of F' (that is a reflexive coequalizer $e = \sigma'_b : l_b \to l$ of $F'(f)$ and $F'(g)$) exists and by lemma 5.1 this means that F has a colimit

$$\left\langle l, \left\langle \sigma'_j \circ \sigma_{j;i} \right\rangle_{\langle j,i \rangle \in \mathbf{J_r} \times \mathbf{I}} \right\rangle$$

Now we reverse the roles of $\mathbf{I}$ and $\mathbf{J_r}$ and apply lemma 5.2. That is for the functor L in that lemma we use F; for the category $\mathbf{I}$ from the lemma we use our $\mathbf{I}$; for $\mathbf{J}$ from the lemma we use $\mathbf{J_r}$. The reader may have noticed that the arguments for F are in reverse order to that in the lemma. A moments reflection will make it clear that this does not change anything about the validity of the lemma. For L' in the lemma we use our L and $\tau_{i;b}$ (or perhaps we should more appropriately say $\tau_{b;i}$, because of the reversal of argument order) in the lemma is given by our $\varepsilon^T_{L(i)}$. The second component $\tau_{i;a}$ (or $\tau_{a;i}$) is given by $\varepsilon^T_{L(i)} \circ \varepsilon^T_{F^T(G^T(L(i)))} = \varepsilon^T_{L(i)} \circ F^T\left(G^T\left(\varepsilon^T_{L(i)}\right)\right)$. The commutativity condition (57) is satisfied because $\varepsilon^T \circ L$ and $\left(\varepsilon^T \circ L\right) \bullet \left(\varepsilon^T \circ F^T \circ G^T \circ L\right) = \left(\varepsilon^T \circ L\right) \bullet \left(F^T \circ G^T \circ \varepsilon^T \circ L\right)$ are natural. The condition that our $L(i)$ be the required colimit is what is stated in lemma 6.12.

We see that all conditions of lemma 5.2 are met and therefore the arrows τ_i shown in (66) which are uniquely determined by the requirement that the outer left quadrilaterals in that diagram commute ($\tau_i \circ \varepsilon^T_{L(i)} = e \circ \sigma_{b;i}$) form a colimiting cone for the functor L. $\square$

7 Free algebras

7.1 A cousin of the functor category

We are now going to describe a category which is closely related to the the functor category $\mathbf{C}^{\mathbf{I}}$. This category is interesting to us because it in turn is closely related (although in a different way) to many of the categories of algebras we are discussing. The objects are going to be monoidal functors. The morphisms will be natural transformations which are compatible with the relevant structures. These definitions can also be found in [Mac98, p. XI.2].

Definition 7.1. Let $\left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho} \right\rangle$ and $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be monoidal categories and let $\langle R, R_2, R_0 \rangle, \langle R', R'_2, R'_0 \rangle : \left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho} \right\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be monoidal functors. A *monoidal natural transformation*

$$\tau : \langle R, R_2, R_0 \rangle \dot{\to} \langle R', R'_2, R'_0 \rangle : \left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho} \right\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$$

is a natural transformation

$$\tau : R \dot{\to} R' : \mathbf{I} \to \mathbf{C}$$

such that the diagrams (67) and (68) commute for all $a, b \in \mathbf{I}$.

$$\begin{tikzcd} R'(a) \boxtimes R'(b) \arrow[d, "R'_{2;a,b}"] & R(a) \boxtimes R(b) \arrow[l, "\tau_a \boxtimes \tau_b"'] \arrow[d, "R_{2;a,b}"] \\ R'(a \boxdot b) & R(a \boxdot b) \arrow[l, "\tau_{a \boxdot b}"'] \end{tikzcd} \tag{67}$$

$$\begin{tikzcd} e \arrow[d, "R'_0"] & e \arrow[l, "1_e"'] \arrow[d, "R_0"] \\ R'(\dot{e}) & R(\dot{e}) \arrow[l, "\tau_{\dot{e}}"'] \end{tikzcd} \tag{68}$$

Lemma 7.2. *For any monoidal natural transformation*

$$\tau : \langle R, R_2, R_0 \rangle \dot{\to} \langle R', R'_2, R'_0 \rangle : \left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho} \right\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle \ ,$$

any tensor word v and any $a_i \in \mathbf{I}$ the diagram (69) *commutes.*

$$\begin{tikzcd} v_{\mathbf{C}}\left(\langle R'(a_i) \rangle_{i=1}^{\ell(v)} \right) \arrow[d, "R'_{v, \langle a_i \rangle_{i=1}^{\ell(v)}}"] & v_{\mathbf{C}}\left(\langle R(a_i) \rangle_{i=1}^{\ell(v)} \right) \arrow[l, "v_{\mathbf{C}}\left(\langle \tau_{a_i} \rangle_{i=1}^{\ell(v)} \right)"'] \arrow[d, "R_{v, \langle a_i \rangle_{i=1}^{\ell(v)}}"] \\ R'\left(v_{\mathbf{I}}\left(\langle a_i \rangle_{i=1}^{\ell(v)} \right) \right) & R\left(v_{\mathbf{I}}\left(\langle a_i \rangle_{i=1}^{\ell(v)} \right) \right) \arrow[l, "\tau_{v_{\mathbf{I}}\left(\langle a_i \rangle_{i=1}^{\ell(v)} \right)}"'] \end{tikzcd} \tag{69}$$

Proof. Again we proceed by induction over the structure of the tensor word v. The case $v = (\_)$ is obvious. The case $v = e_0$ is (68). For $v \Box w$ we get the diagram below.

$$
\begin{tikzcd}
 & v\left(\langle R(a_i)\rangle_{i=1}^{\ell(v)}\right) \boxtimes w\left(\langle R(b_j)\rangle_{j=1}^{\ell(w)}\right) \arrow[dl, "v\left(\langle \tau_{a_i}\rangle_{i=1}^{\ell(v)}\right)\boxtimes w\left(\langle \tau_{b_j}\rangle_{j=1}^{\ell(w)}\right)"'] \arrow[dd, "R_{v,\langle a_i\rangle_{i=1}^{\ell(v)}}\boxtimes R_{w,\langle b_j\rangle_{j=1}^{\ell(w)}}"] \\
v\left(\langle R'(a_i)\rangle_{i=1}^{\ell(v)}\right) \boxtimes w\left(\langle R'(b_j)\rangle_{j=1}^{\ell(w)}\right) \arrow[dd, "R'_{v,\langle a_i\rangle_{i=1}^{\ell(v)}}\boxtimes R'_{w,\langle b_j\rangle_{j=1}^{\ell(w)}}"'] & \\
 & R\left(v\left(\langle a_i\rangle_{i=1}^{\ell(v)}\right)\right) \boxtimes R\left(w\left(\langle b_j\rangle_{j=1}^{\ell(w)}\right)\right) \arrow[dl, "\tau_{v\left(\langle a_i\rangle_{i=1}^{\ell(v)}\right)}\boxtimes \tau_{w\left(\langle b_j\rangle_{j=1}^{\ell(w)}\right)}"'] \arrow[dd, "R_{2;v\left(\langle a_i\rangle_{i=1}^{\ell(v)}\right),w\left(\langle b_j\rangle_{j=1}^{\ell(w)}\right)}"] \\
R'\left(v\left(\langle a_i\rangle_{i=1}^{\ell(v)}\right)\right) \boxtimes R'\left(w\left(\langle b_j\rangle_{j=1}^{\ell(w)}\right)\right) \arrow[dd, "R'_{2;v\left(\langle a_i\rangle_{i=1}^{\ell(v)}\right),w\left(\langle b_j\rangle_{j=1}^{\ell(w)}\right)}"'] & \\
 & R\left(v\left(\langle a_i\rangle_{i=1}^{\ell(v)}\right) \boxdot w\left(\langle b_j\rangle_{j=1}^{\ell(w)}\right)\right) \arrow[dl, "\tau_{v\left(\langle a_i\rangle_{i=1}^{\ell(v)}\right)\boxdot w\left(\langle b_j\rangle_{j=1}^{\ell(w)}\right)}"'] \\
R'\left(v\left(\langle a_i\rangle_{i=1}^{\ell(v)}\right) \boxdot w\left(\langle b_j\rangle_{j=1}^{\ell(w)}\right)\right) &
\end{tikzcd}
$$

The upper square commutes by induction hypothesis for v and w and because $\boxtimes$ is a bifunctor. The lower square is (67) for $v\left(\langle a_i\rangle_{i=1}^{\ell(v)}\right)$ and $w\left(\langle b_j\rangle_{j=1}^{\ell(w)}\right)$. $\square$

Lemma 7.3. *The vertical composite $\sigma{\bullet}\tau$ of monoidal natural transformations*

$$\tau : \langle R, R_2, R_0\rangle \dot{\to} \langle R', R_2', R_0'\rangle : \left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho}\right\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$$

and

$$\sigma : \langle R', R_2', R_0'\rangle \dot{\to} \langle R'', R_2'', R_0''\rangle : \left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho}\right\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$$

is a monoidal natural transformation

$$\sigma{\bullet}\tau : \langle R, R_2, R_0\rangle \dot{\to} \langle R'', R_2'', R_0''\rangle : \left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho}\right\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle \ .$$

The identity natural transformation 1_R is a monoidal natural transformation

$$1_{\langle R, R_2, R_0\rangle} : \langle R, R_2, R_0\rangle \dot{\to} \langle R, R_2, R_0\rangle : \left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho}\right\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle \ .$$

Proof. Write the instances of (67) for σ and τ side by side. Do the same for (68). The second claim is obvious. $\square$

This justifies the following

Definition 7.4. The category

$$\mathbf{MFun}\left(\left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho}\right\rangle, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle\right)$$

(or $\mathbf{MFun}(\mathbf{I}, \mathbf{C})$ for short when the monoidal structures on $\mathbf{I}$ and $\mathbf{C}$ are clear from the context) is the category which has objects monoidal functors

$$\langle R, R_2, R_0 \rangle : \left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho} \right\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$$

and morphisms monoidal natural transformations

$$\tau : \langle R, R_2, R_0 \rangle \dot{\to} \langle R', R'_2, R'_0 \rangle : \left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho} \right\rangle \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle \ .$$

Composition of morphisms if given by vertical composition $\bullet$ of natural transformations. $1_{\langle R, R_2, R_0 \rangle}$ is given by 1_R.

The category

$$\mathbf{StrgMFun}\left(\left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho} \right\rangle, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right)$$

is the full subcategory of $\mathbf{MFun}\left(\left\langle \mathbf{I}, \boxdot, \dot{e}, \dot{\alpha}, \dot{\lambda}, \dot{\rho} \right\rangle, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right)$ which has objects all strong monoidal functors.

7.2 The category of valid realizations

Definition 7.5. Let $X, A, \mathfrak{s}, \mathfrak{t}$ be generators (definition 3.13) and let v_s, v_t be a varnishing (definition 3.15). Further let D be a set of equation-shapes (definition 3.16) and let $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be a monoidal category. The category

$$\mathbf{Real}\left(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right)$$

(which we will abbreviate to $\mathbf{Real}(D, \mathbf{C})$ when the rest is clear from the context) has as objects all valid realizations $\langle \mathbb{o}, \mathbb{m} \rangle$ of the above atoms with the above varnishing which satisfy all equation-shapes in D. In this context we also call the set D the set of *axioms* for the category $\mathbf{Real}\left(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right)$. An arrow $\mathbb{T} : \langle \mathbb{o}, \mathbb{m} \rangle \dot{\to} \langle \mathbb{o}', \mathbb{m}' \rangle$ of $\mathbf{Real}\left(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right)$ is a natural transformation $\mathbb{T} : \mathbb{o} \dot{\to} \mathbb{o}' : X \to \mathbf{C}$ (where we regard $\mathbb{o}$ and $\mathbb{o}'$ as functors from the discrete category X to the category $\mathbf{C}$) which satisfies

$$\begin{tikzcd}
v_s\left(\mathbb{o}'\left(\mathfrak{s}\left(\mathfrak{f}\right)\right)\right) \arrow[d, "\mathbb{m}'(\mathfrak{f})"] & & v_s\left(\mathbb{o}\left(\mathfrak{s}\left(\mathfrak{f}\right)\right)\right) \arrow[ll, "v_s\left(\left\langle \mathbb{T}_{\pi_i(\mathfrak{s}(\mathfrak{f}))} \right\rangle_{i=1}^{\ell(\mathfrak{s}(\mathfrak{f}))}\right)"'] \arrow[d, "\mathbb{m}(\mathfrak{f})"] \\
v_t\left(\mathbb{o}'\left(\mathfrak{t}\left(\mathfrak{f}\right)\right)\right) & & v_t\left(\mathbb{o}\left(\mathfrak{t}\left(\mathfrak{f}\right)\right)\right) \arrow[ll, "v_s\left(\left\langle \mathbb{T}_{\pi_i(\mathfrak{t}(\mathfrak{f}))} \right\rangle_{i=1}^{\ell(\mathfrak{t}(\mathfrak{f}))}\right)"']
\end{tikzcd} \quad . \tag{70}$$

($\pi_i\left(\mathfrak{s}\left(\mathfrak{f}\right)\right)$ is the i-th component of $\mathfrak{s}\left(\mathfrak{f}\right)$.) The composition of arrows is given by vertical composition of natural transformations and the identity natural transformation is the identity morphism.

Observation 7.6. *The category of monoids in* $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ *is the category*

$$\mathbf{Real}\left(X_{Mon}, A_{Mon}, \mathfrak{s}_{Mon}, \mathfrak{t}_{Mon}, v_{s;Mon}, v_{t;Mon}, D_{Mon}, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right) \ .$$

Observation 7.7. *If* $D \subseteq D'$ *then*

$$\mathbf{Real}\left(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D', \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right)$$

is a full subcategory of

$$\mathbf{Real}\left(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right) \ .$$

Definition 7.8. Again, let $X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be as above. In addition to the function

$$K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}} : \operatorname{Obj}(\mathbf{Real}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \emptyset, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle)) \to \\ \operatorname{Obj}(\mathbf{MFun}(\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t}), \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle))$$

defined in 3.34 we define for any two $\mathbb{o}, \mathbb{o}' \in \mathbf{C}^X$ a function

$$K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}} : \hom_{\mathbf{C}^X}(\mathbb{o}, \mathbb{o}') \to \hom_{\mathbf{C}^{(X^*)}}(K(\mathbb{o}), K(\mathbb{o}'))$$

of the same name by setting

$$(K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}(\mathbb{T}))_{\langle \mathfrak{a}_i \rangle_{i=1}^n} := v^{(n)}(\langle \mathbb{T}_{\mathfrak{a}_i} \rangle_{i=1}^n) \tag{71}$$

for any $\mathbb{T} : \mathbb{o} \dot{\to} \mathbb{o}' : X \to \mathbf{C}$.

Lemma 7.9. *A valid realization $\langle \mathbb{o}, \mathbb{m} \rangle$ (which is the same thing as an object of*

$$\mathbf{Real}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \emptyset, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle) \;\;)$$

satisfies a set of equations-shapes D (which means the same thing as saying that $\langle \mathbb{o}, \mathbb{m} \rangle$ is also an object of

$$\mathbf{Real}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle) \;\;)$$

if and only if $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}(\langle \mathbb{o}, \mathbb{m} \rangle)$ factors through $\tilde{\mathbb{m}}_{E,X,A,\mathfrak{s},\mathfrak{t}}$ (with

$$E = \{ \langle \operatorname{tr}(p), \operatorname{tr}(q) \rangle | \, \langle p, q \rangle \in D \} \;\;).$$

If

$$\langle \mathbb{o}, \mathbb{m} \rangle, \langle \mathbb{o}', \mathbb{m}' \rangle \in \mathbf{Real}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle)$$

then a natural transformation $\mathbb{T} : \mathbb{o} \dot{\to} \mathbb{o}' : X \to \mathbf{C}$ is a morphism

$$\mathbb{T} : \langle \mathbb{o}, \mathbb{m} \rangle \dot{\to} \langle \mathbb{o}', \mathbb{m}' \rangle$$

of

$$\mathbf{Real}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle)$$

if and only if $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}(\mathbb{T})$ defined in (71) is a monoidal natural transformation

$$K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}(\mathbb{T}) : K(\langle \mathbb{o}, \mathbb{m} \rangle) \dot{\to} K(\langle \mathbb{o}', \mathbb{m}' \rangle) : \mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t}) \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle.$$

Proof. If $K(\langle \mathbb{o}, \mathbb{m} \rangle)$ factors through $\tilde{\mathbb{m}}_E$ then

$$K(\langle \mathbb{o}, \mathbb{m} \rangle)(\operatorname{tr}(p)) = K(\langle \mathbb{o}, \mathbb{m} \rangle)(\operatorname{tr}(q)) \text{ for all } \langle p, q \rangle \in D \; .$$

By observation 3.33 the arrow $K(\langle \mathbb{o}, \mathbb{m} \rangle)(\operatorname{tr}(p))$ is the result of the evaluation of a path whose ptr is equal to that of p, and similarly for q. By corallary 3.24 this implies $\operatorname{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}(p) = \operatorname{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}(q)$. If on the other hand $\langle \mathbb{o}, \mathbb{m} \rangle$ satisfies the set of equation-shapes D then for any $\langle p, q \rangle \in D$ we have $\operatorname{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}(p) = \operatorname{ev}_{\langle \mathbb{o}, \mathbb{m} \rangle}(q)$. By the same argument as before this implies $K(\langle \mathbb{o}, \mathbb{m} \rangle)(\operatorname{tr}(p)) = K(\langle \mathbb{o}, \mathbb{m} \rangle)(\operatorname{tr}(q))$ for all $\langle p, q \rangle \in D$. By lemma 3.31 this implies that $K(\langle \mathbb{o}, \mathbb{m} \rangle)$ factors through $\tilde{\mathbb{m}}_E$.

For the second part note first that by interpreting the arrows in the diagram (69) (the monoidality-condition for natural transformations) as evaluations of paths with equal traces (for appropriate generators, varnishings, etc.) we see that this diagram commutes in any case. So we only need to think about naturality. We also do this by choosing appropriate generators and varnishings. More explicitly, consider

- the set of object-atoms $\dot{X} := (\{0\} \times X) \dot{\cup} (\{1\} \times X)$,
- the set of non-trivial arrow-atoms $\dot{A} := (\{0\} \times A) \dot{\cup} (\{1\} \times A) \dot{\cup} X$ (so there will be three copies of X (with different meaning) in the set of arrow-atoms),
- the signature $\dot{\mathfrak{s}}, \dot{\mathfrak{t}} : \dot{A} \to \dot{X}^*$ given by

$$\begin{aligned} \dot{\mathfrak{s}}(\langle 0, \mathfrak{f}\rangle) &:= \langle\langle 0, \pi_i(\mathfrak{s}(\mathfrak{f}))\rangle\rangle_{i=1}^{\ell(\mathfrak{s}(\mathfrak{f}))} & \dot{\mathfrak{t}}(\langle 0, \mathfrak{f}\rangle) &:= \langle\langle 0, \pi_i(\mathfrak{t}(\mathfrak{f}))\rangle\rangle_{i=1}^{\ell(\mathfrak{t}(\mathfrak{f}))} \\ \dot{\mathfrak{s}}(\langle 1, \mathfrak{f}\rangle) &:= \langle\langle 1, \pi_i(\mathfrak{s}(\mathfrak{f}))\rangle\rangle_{i=1}^{\ell(\mathfrak{s}(\mathfrak{f}))} & \dot{\mathfrak{t}}(\langle 1, \mathfrak{f}\rangle) &:= \langle\langle 1, \pi_i(\mathfrak{t}(\mathfrak{f}))\rangle\rangle_{i=1}^{\ell(\mathfrak{t}(\mathfrak{f}))} \end{aligned}$$

$$\text{for } \mathfrak{f} \in A$$

$$\dot{\mathfrak{s}}(\mathfrak{a}) := \langle\,\langle\,\langle 0, \mathfrak{a}\rangle\rangle\rangle \qquad\qquad \dot{\mathfrak{t}}(\mathfrak{a}) := \langle\,\langle\,\langle 1, \mathfrak{a}\rangle\rangle\rangle$$

$$\text{for } \mathfrak{a} \in X \ ,$$

- the varnishing $\dot{v}_s, \dot{v}_t$ given by

$$\begin{aligned} \dot{v}_s(\langle 0, \mathfrak{f}\rangle) = \dot{v}_s(\langle 1, \mathfrak{f}\rangle) &= v_s(\mathfrak{f}) \\ \dot{v}_t(\langle 0, \mathfrak{f}\rangle) = \dot{v}_t(\langle 1, \mathfrak{f}\rangle) &= v_t(\mathfrak{f}) \quad \text{for } \mathfrak{f} \in A \\ \dot{v}_s(\mathfrak{a}) = \dot{v}_t(\mathfrak{a}) &= (\_) \qquad \text{for } \mathfrak{a} \in X \ , \end{aligned}$$

- the realization $\langle \dot{\mathbb{o}}, \dot{\mathbb{m}}\rangle$ given by

$$\begin{aligned} \dot{\mathbb{o}}(\langle 0, \mathfrak{a}\rangle) &:= \mathbb{o}(\mathfrak{a}) \\ \dot{\mathbb{o}}(\langle 1, \mathfrak{a}\rangle) &:= \mathbb{o}'(\mathfrak{a}) \quad \text{for } \mathfrak{a} \in X \\ \dot{\mathbb{m}}(\langle 0, \mathfrak{f}\rangle) &:= \mathbb{m}(\mathfrak{f}) \\ \dot{\mathbb{m}}(\langle 1, \mathfrak{f}\rangle) &:= \mathbb{m}'(\mathfrak{f}) \quad \text{for } \mathfrak{f} \in A \\ \dot{\mathbb{m}}(\mathfrak{a}) &:= \mathbb{T}_{\mathfrak{a}} \qquad \text{for } \mathfrak{a} \in X \ , \end{aligned}$$

- the set of equation-shapes

$$\begin{aligned} \dot{D} := \Bigg\{ \Bigg\langle & \Big\langle v_t(\mathfrak{f})\,, \Big\langle\,\langle v_t(\mathfrak{f})\,, \mathfrak{t}(\mathfrak{f})\rangle, \langle (\_), \langle\,\langle 0, \mathfrak{f}\rangle\rangle\rangle \Big\rangle, v_s(\mathfrak{f}) \Big\rangle, \\ & \Big\langle v_t(\mathfrak{f})\,, \Big\langle\,\langle (\_), \langle\,\langle 1, \mathfrak{f}\rangle\rangle\rangle, \langle v_s(\mathfrak{f})\,, \mathfrak{s}(\mathfrak{f})\rangle \Big\rangle, v_s(\mathfrak{f}) \Big\rangle \Bigg\rangle \Bigg| \mathfrak{f} \in A \Bigg\} \end{aligned}$$

and

- the trace-set

$$\dot{E} := \Big\{ \langle\, \mathrm{tr}(p)\,, \mathrm{tr}(q)\rangle \Big| \, \langle p, q\rangle \in \dot{D} \Big\} \ .$$

Define functors

$$S : \mathcal{E}\left(X, A, \mathfrak{s}, \mathfrak{t}\right) \to \mathcal{E}\left(\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}\right)$$
$$S\left(\langle \mathfrak{a}_i \rangle_{i=1}^{n}\right) := \langle\langle 0, \mathfrak{a}_i \rangle\rangle_{i=1}^{n}$$
$$S\big(\pi_{\Bumpeq, X, A, \mathfrak{s}, \mathfrak{t}}\big(\langle\langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_k}\rangle_{i=k}^{1}\big)\big) := \pi_{\Bumpeq, \dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}}\left(\langle\langle\langle 0, \mathfrak{f}_{i,j} \rangle\rangle_{j=1}^{n_k}\rangle_{i=k}^{1}\right)$$

$$T : \mathcal{E}\left(X, A, \mathfrak{s}, \mathfrak{t}\right) \to \mathcal{E}\left(\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}\right)$$
$$T\left(\langle \mathfrak{a}_i \rangle_{i=1}^{n}\right) := \langle\langle 1, \mathfrak{a}_i \rangle\rangle_{i=1}^{n}$$
$$T\big(\pi_{\Bumpeq, X, A, \mathfrak{s}, \mathfrak{t}}\big(\langle\langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_k}\rangle_{i=k}^{1}\big)\big) := \pi_{\Bumpeq, \dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}}\left(\langle\langle\langle 1, \mathfrak{f}_{i,j} \rangle\rangle_{j=1}^{n_k}\rangle_{i=k}^{1}\right)$$

(where $\mathfrak{a}_i \in X$, $\mathfrak{f}_{i,j} \in A \dot{\cup} X$).

Using the closure properties of the relation $\sim_{\dot{E}}$ we get that

$$\tilde{\mathbb{\pi}}_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{E}}\left(\pi_{\Bumpeq, \dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}}\left(\langle t\left(f\right)\rangle\!\rangle \circ S\left(f\right)\right) = \tilde{\mathbb{\pi}}_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{E}}\left(T\left(f\right) \circ \pi_{\Bumpeq, \dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}}\left(\langle s\left(f\right)\rangle\!\rangle\right)\right).$$

(The elements in $\dot{E}$ are exactly those that stand for the above equation where f is an arrow consisting of a single non-trivial arrow-atom.) Now note that

$$K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\mathbb{\tau}\right)_a = K_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{v_s}, \dot{v_t}, \mathbf{C}}\left(\langle \dot{\mathbb{o}}, \dot{\mathbb{m}} \rangle\right)\left(\pi_{\Bumpeq, \dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}}\left(\langle a\rangle\!\rangle\right)\right)$$
$$K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)\left(f\right) = K_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{v_s}, \dot{v_t}, \mathbf{C}}\left(\langle \dot{\mathbb{o}}, \dot{\mathbb{m}} \rangle\right)\left(S\left(f\right)\right)$$
$$K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)\left(f\right) = K_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{v_s}, \dot{v_t}, \mathbf{C}}\left(\langle \dot{\mathbb{o}}, \dot{\mathbb{m}} \rangle\right)\left(T\left(f\right)\right)$$

and therefore

$$K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\mathbb{\tau}\right)_{t(f)} \circ K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)\left(f\right) = K_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{v_s}, \dot{v_t}, \mathbf{C}}\left(\langle \dot{\mathbb{o}}, \dot{\mathbb{m}} \rangle\right)\left(\pi_{\Bumpeq, \dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}}\left(\langle t\left(f\right)\rangle\right) \circ S\left(f\right)\right)$$

and

$$K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)\left(f\right) \circ K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\mathbb{\tau}\right)_{s(f)} = K_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{v_s}, \dot{v_t}, \mathbf{C}}\left(\langle \dot{\mathbb{o}}, \dot{\mathbb{m}} \rangle\right)\left(T\left(f\right) \circ \pi_{\Bumpeq, \dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}}\left(\langle s\left(f\right)\rangle\right)\right) .$$

This reduces the second part of the theorem to what we have already proved in the first part. Indeed $K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\mathbb{\tau}\right)$ is natural if and only if

$$K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\mathbb{\tau}\right)_{t(f)} \circ K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)\left(f\right) = K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)\left(f\right) \circ K_{X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \mathbf{C}}\left(\mathbb{\tau}\right)_{s(f)}$$

which is the case if and only if

$$K_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{v_s}, \dot{v_t}, \mathbf{C}}\left(\langle \dot{\mathbb{o}}, \dot{\mathbb{m}} \rangle\right)\left(\pi_{\Bumpeq, \dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}}\left(\langle t\left(f\right)\rangle\right) \circ S\left(f\right)\right) = K_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{v_s}, \dot{v_t}, \mathbf{C}}\left(\langle \dot{\mathbb{o}}, \dot{\mathbb{m}} \rangle\right)\left(T\left(f\right) \circ \pi_{\Bumpeq, \dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}}\left(\langle s\left(f\right)\rangle\right)\right)$$

which in turn is the case if and only if $K_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{v_s}, \dot{v_t}, \mathbf{C}}\left(\langle \dot{\mathbb{o}}, \dot{\mathbb{m}} \rangle\right)$ factors through $\tilde{\mathbb{\pi}}_{\dot{X}, \dot{A}, \dot{\mathfrak{s}}, \dot{\mathfrak{t}}, \dot{E}}$. By the first part of the theorem this is equivalent to $\langle \dot{\mathbb{o}}, \dot{\mathbb{m}} \rangle$ satisfying the equations in $\dot{D}$, which by the construction of $\langle \dot{\mathbb{o}}, \dot{\mathbb{m}} \rangle$ and $\dot{D}$ is equivalent to $\mathbb{\tau}$ being a morphism $\mathbb{\tau} : \langle \mathbb{o}, \mathbb{m} \rangle \dot{\to} \langle \mathbb{o}', \mathbb{m}' \rangle$. $\square$

The previous lemma allows us to define a functor mapping realizations satisfying a set of axioms to monoidal functors from a category $\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})$.

Definition 7.10. Let $X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be as before. For any set D of equation-shapes we define a functor

$$\begin{aligned} K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\mathbf{C}} : \mathbf{Real}\left(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right) \to& \\ \mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_{\mathrm{Tr}(D)}(X, A, \mathfrak{s}, \mathfrak{t}), \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right)& \end{aligned}$$

which sends any realization $\langle \mathbb{o}, \mathbb{m} \rangle$ to the unique strong monoidal functor

$$\langle L, L_2, L_0 \rangle$$

for which

$$K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}\left(\langle \mathbb{o}, \mathbb{m} \rangle\right) = \langle L, L_2, L_0 \rangle \circ \tilde{\mathbb{m}}_{E,X,A,\mathfrak{s},\mathfrak{t}}$$

and which sends $\mathbb{t} : \langle \mathbb{o}, \mathbb{m} \rangle \dot{\to} \langle \mathbb{o}', \mathbb{m}' \rangle$ to the monoidal natural transformation

$$K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}\left(\mathbb{t}\right) \ .$$

Lemma 7.11. *$K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\mathbf{C}}$ is an equivalence of categories.*

Proof. Again, set $E := \mathrm{Tr}(D)$. We show that $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\mathbf{C}}$ is full and faithful and that every strong monoidal functor

$$\langle L, L_2, L_0 \rangle \in \mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_{\mathrm{Tr}(D)}(X, A, \mathfrak{s}, \mathfrak{t}), \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right)$$

is isomorphic to $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\mathbf{C}}\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)$ for some

$$\langle \mathbb{o}, \mathbb{m} \rangle \in \mathbf{Real}\left(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right) \ .$$

For more information on the different ways of expressing that a functor is an equivalence of categories see [Mac98, section IV.4].

To see that $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\mathbf{C}}$ is full consider any

$$\tau : K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\mathbf{C}}\left(\langle \mathbb{o}, \mathbb{m} \rangle\right) \dot{\to} K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\mathbf{C}}\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right) \ .$$

By lemma 7.2 we get that $\tau_{\langle \mathfrak{a}_i \rangle_{i=1}^n}$ can be expressed as

$$\begin{tikzcd}[column sep=huge] v^{(n)}\left(\left\langle v^{(1)}\left(\mathbb{o}'\left(\mathfrak{a}_i\right)\right)\right\rangle_{i=1}^n\right) \arrow[d, "\mathrm{can}_{v^{(n)} \leftarrow v^{(n)}\left(\left\langle v^{(1)} \right\rangle_{i=1}^n\right);\left\langle \mathbb{o}'(\mathfrak{a}_i)\right\rangle_{i=1}^n}"'] & v^{(n)}\left(\left\langle v^{(1)}\left(\mathbb{o}\left(\mathfrak{a}_i\right)\right)\right\rangle_{i=1}^n\right) \arrow[l, "v^{(n)}\left(\left\langle \tau_{\langle \mathfrak{a}_i \rangle} \right\rangle_{i=1}^n\right)"'] \\ K_D\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)\left(\langle \mathfrak{a}_i \rangle_{i=1}^n\right) & K_D\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)\left(\langle \mathfrak{a}_i \rangle_{i=1}^n\right) \arrow[l, "\tau_{\langle \mathfrak{a}_i \rangle_{i=1}^n}"] \arrow[u, "\mathrm{can}_{v^{(n)}\left(\left\langle v^{(1)} \right\rangle_{i=1}^n\right) \leftarrow v^{(n)};\left\langle \mathbb{o}(\mathfrak{a}_i)\right\rangle_{i=1}^n}"'] \end{tikzcd}$$

which again by an equality-of-traces style argument means that

$$\begin{aligned} \tau_{\langle \mathfrak{a}_i \rangle_{i=1}^n} &= v^{(n)}\left(\left\langle \mathrm{can}_{(\_) \leftarrow v^{(1)};\mathbb{o}'(\mathfrak{a}_i)} \circ \tau_{\langle \mathfrak{a}_i \rangle} \circ \mathrm{can}_{v^{(1)} \leftarrow (\_);\mathbb{o}(\mathfrak{a}_i)} \right\rangle_{i=1}^n\right) \\ &= K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\mathbf{C}}\left(\left\langle \mathrm{can}_{(\_) \leftarrow v^{(1)};\mathbb{o}'(\mathfrak{a})} \circ \tau_{\langle \mathfrak{a} \rangle} \circ \mathrm{can}_{v^{(1)} \leftarrow (\_);\mathbb{o}(\mathfrak{a})} \right\rangle_{\mathfrak{a} \in \tilde{\mathcal{E}}_E(X,A,\mathfrak{s},\mathfrak{t})}\right) \end{aligned}$$

(where we can write the D in the subscript of K because of lemma 7.9). Similarly we see that $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\mathbf{C}}$ is faithful because for any

$$\mathbb{t} : \langle \mathbb{o}, \mathbb{m} \rangle \dot{\to} \langle \mathbb{o}', \mathbb{m}' \rangle$$

and any $\mathfrak{a} \in X$ we have

$$\begin{tikzcd}
\mathbb{o}'(\mathfrak{a}) & & \mathbb{o}(\mathfrak{a}) \arrow[ll, "\mathbb{t}_{\mathfrak{a}}"'] \arrow[d, "\mathrm{can}_{v^{(1)} \leftarrow (_);\mathbb{o}(\mathfrak{a})}"] \\
v^{(1)}(\mathbb{o}'(\mathfrak{a})) \arrow[u, "\mathrm{can}_{(_) \leftarrow v^{(1)};\mathbb{o}'(\mathfrak{a})}"] & & v^{(1)}(\mathbb{o}(\mathfrak{a})) \arrow[ll, "K_D(\mathbb{t})_{\langle \mathfrak{a} \rangle}"']
\end{tikzcd}.$$

Now let $\langle L, L_2, L_0 \rangle \in \mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t}), \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right)$. Define

$$\mathbb{o}(\mathfrak{a}) := L(\langle \mathfrak{a} \rangle)$$

for all $\mathfrak{a} \in \tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})$ and define $\mathbb{m}(\mathfrak{f})$ by

$$\begin{tikzcd}
v_t(\mathfrak{f})\left(\left\langle L\left(\langle \pi_i(\mathfrak{t}(\mathfrak{f})) \rangle\right) \right\rangle_{i=1}^{\ell(\mathfrak{t}(\mathfrak{f}))}\right) \arrow[d, "L_{v_t(\mathfrak{f}), \langle \langle \pi_i(\mathfrak{t}(\mathfrak{f})) \rangle \rangle_{i=1}^{\ell(v_t(\mathfrak{f}))}}"] & v_s(\mathfrak{f})\left(\left\langle L\left(\langle \pi_i(\mathfrak{s}(\mathfrak{f})) \rangle\right) \right\rangle_{i=1}^{\ell(\mathfrak{s}(\mathfrak{f}))}\right) \arrow[l, "\mathbb{m}(\mathfrak{f})"'] \arrow[d, "L_{v_s(\mathfrak{f}), \langle \langle \pi_i(\mathfrak{s}(\mathfrak{f})) \rangle \rangle_{i=1}^{\ell(v_s(\mathfrak{f}))}}"] \\
L(\mathfrak{s}(\mathfrak{f})) & L(\mathfrak{s}(\mathfrak{f})) \arrow[l, "L(\tilde{\mathbb{w}}'_E(\langle \langle \mathfrak{f} \rangle \rangle))"']
\end{tikzcd}$$

for any $\mathfrak{f}$ in A. The diagram above uniquely determines $\mathbb{m}(\mathfrak{f})$ because $\langle L, L_2, L_0 \rangle$ is strong.

Now consider diagram (72), in which $f = \pi_{\diamond}(\langle \langle \mathfrak{f}_i \rangle_{i=1}^n \rangle)$. The upper quadrilateral commutes by the definition of $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}(\langle \mathbb{o}, \mathbb{m} \rangle)$, the right and left quadrilaterals commute by lemma 3.6, the middle square commutes by the definition of $\mathbb{m}(\mathfrak{f}_i)$ just given and the lower quadrilateral commutes by naturality of $L_{v^{(n)}}$. Therefore the outer square — which generalizes the defining square for $\mathbb{m}(\mathfrak{f})$ — commutes. By placing these kind of squares next to each other it becomes clear that for any f in $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ the following square commutes.

$$\begin{tikzcd}
K(\langle \mathbb{o}, \mathbb{m} \rangle)(t(f)) \arrow[d, "L_{v^{(\ell(t(f)))}, \langle \langle \pi_i(t(f)) \rangle \rangle_{i=1}^{\ell(t(f))}}"] & K(\langle \mathbb{o}, \mathbb{m} \rangle)(s(f)) \arrow[l, "K(\langle \mathbb{o}, \mathbb{m} \rangle)(f)"'] \arrow[d, "L_{v^{(\ell(s(f)))}, \langle \langle \pi_i(s(f)) \rangle \rangle_{i=1}^{\ell(s(f))}}"] \\
L(t(f)) & L(s(f)) \arrow[l, "L(\tilde{\mathbb{w}}_E(f))"']
\end{tikzcd}$$

From this we see that $K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}$ factors through $\tilde{\mathbb{w}}_{E,X,A,\mathfrak{s},\mathfrak{t}}$ and by lemma 7.9 this implies that $\langle \mathbb{o}, \mathbb{m} \rangle$ satisfies the set of equations D. We still have to show that $\left\langle L_{v^{(\ell(a))}, \langle \langle \pi_i(a) \rangle \rangle_{i=1}^{\ell(a)}} \right\rangle_{a \in \tilde{\mathcal{E}}_E(X,A,\mathfrak{s},\mathfrak{t})}$ is a monoidal natural isomorphism.

$$
\begin{tikzcd}
K\left(\langle \mathbb{o},\mathbb{m}\rangle\right)\left(t\left(f\right)\right) \arrow[d, equal] & & & K\left(\langle \mathbb{o},\mathbb{m}\rangle\right)\left(s\left(f\right)\right) \arrow[lll, "K(\langle \mathbb{o},\mathbb{m}\rangle)(f)"'] \arrow[d, equal] \\
v^{(\ell(t(f)))}\left(\left\langle L\left(\langle \pi_j\left(\mathscr{t}(\mathfrak{f}_i)\right)\rangle\right)\right\rangle_{j=1}^{\ell(\mathscr{t}(\mathfrak{f}_i))}{}_{i=1}^{n}\right) \arrow[dr, "\mathrm{can}_{v^{(n)}\left(\langle v_t(\mathfrak{f}_i)\rangle_{i=1}^{n}\right)\leftarrow v^{(\ell(t(f)))};\left\langle L\left(\langle \pi_j(\mathscr{t}(\mathfrak{f}_i))\rangle\right)\right\rangle_{j=1}^{\ell(\mathscr{t}(\mathfrak{f}_i))}{}_{i=1}^{n}}"] \arrow[ddd, bend right, "L_{v^{(\ell(t(f)))},\langle\langle \pi_j(t(f))\rangle\rangle_{j=1}^{\ell(t(f))}}"'] & & & v^{(\ell(s(f)))}\left(\left\langle L\left(\langle \pi_j\left(\mathscr{s}(\mathfrak{f}_i)\right)\rangle\right)\right\rangle_{j=1}^{\ell(\mathscr{s}(\mathfrak{f}_i))}{}_{i=1}^{n}\right) \arrow[dl, "\mathrm{can}_{v^{(n)}\left(\langle v_s(\mathfrak{f}_i)\rangle_{i=1}^{n}\right)\leftarrow v^{(\ell(s(f)))};\left\langle L\left(\langle \pi_j(\mathscr{s}(\mathfrak{f}_i))\rangle\right)\right\rangle_{j=1}^{\ell(\mathscr{s}(\mathfrak{f}_i))}{}_{i=1}^{n}}"'] \arrow[ddd, bend left, "L_{v^{(\ell(s(f)))},\langle\langle \pi_j(s(f))\rangle\rangle_{j=1}^{\ell(s(f))}}"] \\
 & v^{(n)}\left(\left\langle v_t(\mathfrak{f}_i)\left(\left\langle L\left(\langle \pi_j\left(\mathscr{t}(\mathfrak{f}_i)\right)\rangle\right)\right\rangle_{j=1}^{\ell(\mathscr{t}(\mathfrak{f}_i))}\right)\right\rangle_{i=1}^{n}\right) \arrow[d, "v^{(n)}\left(\left\langle L_{v_t(\mathfrak{f}_i),\langle\langle \pi_j(\mathscr{t}(\mathfrak{f}_i))\rangle\rangle_{j=1}^{\ell(\mathscr{t}(\mathfrak{f}_i))}}\right\rangle_{i=1}^{n}\right)"] & v^{(n)}\left(\left\langle v_s(\mathfrak{f}_i)\left(\left\langle L\left(\langle \pi_j\left(\mathscr{s}(\mathfrak{f}_i)\right)\rangle\right)\right\rangle_{j=1}^{\ell(\mathscr{s}(\mathfrak{f}_i))}\right)\right\rangle_{i=1}^{n}\right) \arrow[l, "v^{(n)}(\langle \mathbb{m}(\mathfrak{f}_i)\rangle_{i=1}^{n})"'] \arrow[d, "v^{(n)}\left(\left\langle L_{v_s(\mathfrak{f}_i),\langle\langle \pi_j(\mathscr{s}(\mathfrak{f}_i))\rangle\rangle_{j=1}^{\ell(\mathscr{s}(\mathfrak{f}_i))}}\right\rangle_{i=1}^{n}\right)"'] & \\
 & v^{(n)}\left(\langle L(\mathscr{t}(\mathfrak{f}_i))\rangle_{i=1}^{n}\right) \arrow[dl, "L_{v^{(n)},\langle \mathscr{t}(\mathfrak{f}_i)\rangle_{i=1}^{n}}"] & v^{(n)}\left(\langle L(\mathscr{s}(\mathfrak{f}_i))\rangle_{i=1}^{n}\right) \arrow[l, "v^{(n)}\left(\left\langle L\left(\tilde{\mathbb{w}}'_E(\langle\langle \mathfrak{f}_i\rangle\rangle)\right)\right\rangle_{i=1}^{n}\right)"'] \arrow[dr, "L_{v^{(n)},\langle \mathscr{s}(\mathfrak{f}_i)\rangle_{i=1}^{n}}"'] & \\
L\left(t\left(f\right)\right) & & & L\left(s\left(f\right)\right) \arrow[lll, "L\left(\tilde{\mathbb{w}}_E\left(f\right)\right)"']
\end{tikzcd}
\tag{72}
$$

Naturality is just the previous diagram. It is also clearly invertible because $\langle L, L_2, L_0\rangle$ is strong. It remains to check monoidality. We will do this directly by showing that (69) commutes because that's just as easy in this case. So let $\mathfrak{a}_{i,j} \in \tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})$ and call $\langle \mathfrak{a}_{i,j}\rangle_{j=1}^{k_i} =: \mathscr{a}_i$.

$$\begin{tikzcd}
v\left(\langle L(\mathscr{a}_i)\rangle_{i=1}^{n}\right) \arrow[d, "L_{v,\langle \mathscr{a}_i\rangle_{i=1}^{n}}"] & & v\left(\langle K(\langle \mathbb{o}, \mathbb{m}\rangle)(\mathscr{a}_i)\rangle_{i=1}^{n}\right) \arrow[ll, "v\left(\left\langle L_{v(k_i),\langle\langle \mathfrak{a}_{i,j}\rangle\rangle_{j=1}^{k_i}}\right\rangle_{i=1}^{n}\right)"'] \arrow[d, "\mathrm{can}_{v\left(\sum_{i=1}^{n} k_i\right) \leftarrow v\left(\left\langle v(k_i)\right\rangle_{i=1}^{n}\right); \langle L(\langle \mathfrak{a}_{i,j}\rangle)\rangle_{j=1}^{k_i}{}_{i=1}^{n}}"] \\
L\left(\langle \mathfrak{a}_{i,j}\rangle_{j=1}^{k_i}{}_{i=1}^{n}\right) & & K(\langle \mathbb{o}, \mathbb{m}\rangle)\left(\langle \mathfrak{a}_{i,j}\rangle_{j=1}^{k_i}{}_{i=1}^{n}\right) \arrow[ll, "L_{v\left(\sum_{i=1}^{n} k_i\right),\langle\langle \mathfrak{a}_{i,j}\rangle\rangle_{j=1}^{k_i}{}_{i=1}^{n}}"]
\end{tikzcd}$$

(73)

In diagram (73) we see what diagram (69) looks like in our case. This again commutes by lemma 3.6. $\square$

7.3 Free objects

Definition 7.12. Define a *forgetful functor*

$$\mathbb{G}_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle} : \\ \mathbf{Real}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle) \to \mathbf{C}^X$$

from the category $\mathbf{Real}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle)$ which sends any realization $\langle \mathbb{o}, \mathbb{m}\rangle$ to $\mathbb{o}$ and a morphism $\mathbb{T} : \langle \mathbb{o}, \mathbb{m}\rangle \dot{\to} \langle \mathbb{o}', \mathbb{m}'\rangle$ to $\mathbb{T} : \mathbb{o} \dot{\to} \mathbb{o}' : X \to \mathbf{C}$.

Define another forgetful functor

$$G_{X,A,\mathfrak{s},\mathfrak{t},E,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle} : \\ \mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t}), \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle\right) \to \mathbf{C}^X$$

from $\mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t}), \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle\right)$ as

$$G_{X,A,\mathfrak{s},\mathfrak{t},E,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle} := - \circ \mathrm{inj}_{X,A,\mathfrak{s},\mathfrak{t}}$$

where the functor $\mathrm{inj}_{X,A,\mathfrak{s},\mathfrak{t}} : X \to \tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})$ is the injection from the discrete category X into $\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t})$. More explicitly this means

$$\left(G_{X,A,\mathfrak{s},\mathfrak{t},E,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle}(\langle L, L_2, L_0\rangle)\right)(\mathfrak{a}) = L(\mathfrak{a})$$
$$\left(G_{X,A,\mathfrak{s},\mathfrak{t},E,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle}(\tau)\right)_{\mathfrak{a}} = \tau_{\mathfrak{a}} \ .$$

We will abbreviate

$$\mathbb{G}_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle} \text{ to } \mathbb{G}$$

and

$$G_{X,A,\mathfrak{s},\mathfrak{t},E,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle} \text{ to } G$$

when it is clear from the context what we mean.

Lemma 7.13. *If* $E = \mathrm{Tr}(D)$ *then*

$$G_{X,A,\mathfrak{s},\mathfrak{t},E,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle} \circ K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}} \cong \mathbb{G}_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle} \ .$$

Explicitly the the map which sends $\langle \mathbb{o}, \mathbb{m}\rangle$ *to the natural transformation*

$$\mathrm{can}_{(\_)\leftarrow v^{(1)}} \circ \mathbb{o} : v^{(1)} \circ \mathbb{o} \dot{\to} \mathbb{o} : X \to \mathbf{C}$$

is a natural isomorphism from $G \circ K$ *to* $\mathbb{G}$*. We will call this natural transformation* $\mathbb{p}_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle}$ *or* $\mathbb{p}$ *for short.*

Proof. Naturality of $\mathbb{p}$ means that

$$\begin{tikzcd}
(G \circ K)\left(\langle \mathbb{o}', \mathbb{m}'\rangle\right) \arrow[d, "\mathrm{can}_{(_)\leftarrow v^{(1)}} \circ \mathbb{o}'"] & & (G \circ K)\left(\langle \mathbb{o}, \mathbb{m}\rangle\right) \arrow[ll, "(G\circ K)(\mathbb{t})"'] \arrow[d, "\mathrm{can}_{(_)\leftarrow v^{(1)}} \circ \mathbb{o}"] \\
\mathbb{G}\left(\langle \mathbb{o}', \mathbb{m}'\rangle\right) & & \mathbb{G}\left(\langle \mathbb{o}, \mathbb{m}\rangle\right) \arrow[ll, "\mathbb{G}(\mathbb{t})"']
\end{tikzcd}$$

commutes for all $\mathbb{t} : \langle \mathbb{o}, \mathbb{m}\rangle \dot{\to} \langle \mathbb{o}', \mathbb{m}'\rangle$ in $\mathbf{Real}\,(D, \mathbf{C})$. This means that for every $\mathfrak{a} \in X$

$$\begin{tikzcd}
v^{(1)}\left(\mathbb{o}'(\mathfrak{a})\right) \arrow[d, "\mathrm{can}_{(_)\leftarrow v^{(1)};\mathbb{o}'(\mathfrak{a})}"] & & v^{(1)}\left(\mathbb{o}(\mathfrak{a})\right) \arrow[ll, "v^{(1)}(\mathbb{t}_{\mathfrak{a}})"'] \arrow[d, "\mathrm{can}_{(_)\leftarrow v^{(1)};\mathbb{o}(\mathfrak{a})}"] \\
\mathbb{o}'(\mathfrak{a}) & & \mathbb{o}(\mathfrak{a}) \arrow[ll, "\mathbb{t}_{\mathfrak{a}}"']
\end{tikzcd}$$

commutes, which is true because $\mathrm{can}_{(\_)\leftarrow v^{(1)}}$ is natural.

Because every component of $\mathbb{p}$ is an isomorphism $\mathbb{p}$ is also an isomorphism. □

The next theorem will describe free objects in the categories $\mathbf{Real}\,(D, \mathbf{C})$ and $\mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E, \mathbf{C}\right)$.

Perhaps it is in order to say a few words about how we are going to construct our free objects before we commence the more formal part. The reader may be familiar with the construction of free objects in varieties of algebras. There one constructs the free object on a set of generators by first forming the set of all valid combinations of operators and letters from the set of generators and then identifying all elements which necessarily have to be equal as a consequence of the equations that all algebras from the variety in question are supposed to satisfy. In category-theoretical parlance one would say that the constructed object is a coequalizer of a coproduct of products (of different length) of the set of generators with itself.

On the other hand there is the free monoid on a set of generators which may of course also be constructed in the manner described above but this is not the most “economical” way of doing it. Instead, the reader may be aware that the underlying set of the free monoid is simply the set of all finite-length strings in letters taken from the set of generators. In category-theoretical language we have a coproduct of products — coequalizers are not required.

The construction of free objects we are going to give is close to the second case. It can in fact be seen as a direct generalization of the construction of the free monoid (which can be found for example in [Mac98, section VII.3]) — one which the author thinks shines some light on *why* the free monoid can be constructed in this way.

Note however that the construction we give does not cover all types of varieties usually considered. More specifically, because our ambient category only has a monoidal structure but not a symmetric monoidal structure we are not able to express in our set D of equation-shapes equations like for example the equation which would in the ambient category set be written as $a \cdot b = b \cdot a$. This means that anything "abelian" is not covered by our construction. This will be obvious once we have given the actual construction because for example free abelian groups simply do not have the kind of structure which our construction creates.

On the other hand the restriction of a single base object which is often imposed in the classical setting will not appear in our premises because it is simply not necessary and it would not really make the construction any simpler.

Theorem 7.14. *Let $X, A, \mathfrak{s}, \mathfrak{t}$ be generators, v_s, v_t a varnishing, D a set of equations shapes, $E := \mathrm{Tr}(D)$ and $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ a monoidal category.*

Assume that $\mathbf{C}$ has coproducts of sufficiently large sets of objects of $\mathbf{C}$. More specifically we require $\mathbf{C}$ to have a coproduct for any family of objects of $\mathbf{C}$ indexed by a set $\left\{ f \text{ in } \tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t}) \middle| t(f) = \langle \mathfrak{a} \rangle \right\}$ for any $\mathfrak{a} \in X$. Assume further that for any $a \in \mathbf{C}$ both of the functors $a \boxtimes -$ and $- \boxtimes a$ preserve coproducts.

If

1. *$\ell(\mathfrak{t}(\mathfrak{f})) = 1$ for all $\mathfrak{f} \in A$ and*
2. *$\ell(t(p)) = \ell(t(q)) = 1$ for all $\langle p, q \rangle \in E$*

then $\mathbb{G}_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho \rangle}$ has a left adjoint

$$\mathbb{F}_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho \rangle} : \mathbf{C}^X \to \mathbf{Real}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle)$$

and

$$F_{X,A,\mathfrak{s},\mathfrak{t},E,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho \rangle} := K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\mathbf{C}} \circ \mathbb{F}_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,D,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho \rangle} : \mathbf{C}^X \to \mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E(X, A, \mathfrak{s}, \mathfrak{t}), \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle\right)$$

is left adjoint to $G_{X,A,\mathfrak{s},\mathfrak{t},E,\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho \rangle}$.

Proof. We will assume that the reader is familiar with the different possible ways of defining an adjunction. For a general account see [Mac98, chapter IV].

Although it would suffice to find a left adjoint to either $\mathbb{G}$ or G (because then the equivalence of categories would give the other) we will instead work with both of the categories $\mathbf{Real}(D, \mathbf{C})$ and $\mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E, \mathbf{C}\right)$ because it seems to the author that parts of the argument are more naturally executed in one and others fit better in the other. More specifically we will begin by defining the object function $\mathbb{F}_O$ of $\mathbb{F}$. Then we will chain this with K to get the object function F_O of F.

We will then define what will become the units of the two adjunctions. The unit $\mathbb{\eta}$ of the adjunction between $\mathbb{G}$ and $\mathbb{F}$ is composed of morphisms $\mathbb{\eta}_{\dot{\mathbb{o}}} : \dot{\mathbb{o}} \dot{\to} \mathbb{G}(\mathbb{F}_O(\dot{\mathbb{o}}))$ of the category $\mathbf{C}^X$ and because of the way in which F_O and $\mathbb{F}_O$ are related the unit of the adjunction between G and F is composed of maps $\eta_{\dot{\mathbb{o}}} : \dot{\mathbb{o}} \dot{\to} G(F_O(\dot{\mathbb{o}})) = G(K(\mathbb{F}_O(\dot{\mathbb{o}})))$. Using the natural isomorphism $\mathbb{p}$ from lemma 7.13 we can define $\eta_{\dot{\mathbb{o}}}$ in terms of $\mathbb{\eta}_{\dot{\mathbb{o}}}$ as

$$\eta_{\dot{\mathbb{o}}} = \mathbb{p}^{-1}_{\mathbb{F}(\dot{\mathbb{o}})} \bullet \mathbb{\eta}_{\dot{\mathbb{o}}} \ . \tag{74}$$

We will then show that $F_O(\dot{\mathbb{o}}) \in \mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E, \mathbf{C}\right)$ together with η satisfies the required universal property. This means that in diagram (75) for any $\left\langle \dot{L}, \dot{L}_2, \dot{L}_0 \right\rangle \in \mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E, \mathbf{C}\right)$ and any $\mathbb{\sigma} : \dot{\mathbb{o}} \dot{\to} G\left(\left\langle \dot{L}, \dot{L}_2, \dot{L}_0 \right\rangle\right)$ there is a unique morphism $\tau : K(\mathbb{F}_O(\dot{\mathbb{o}})) \dot{\to} \left\langle \dot{L}, \dot{L}_2, \dot{L}_0 \right\rangle$ of $\mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E, \mathbf{C}\right)$ which makes the outer "triangle" commute. When $\left\langle \dot{L}, \dot{L}_2, \dot{L}_0 \right\rangle = K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)$ then because K is full there is a morphism $\mathbb{\tau} : \mathbb{F}_O(\dot{\mathbb{o}}) \dot{\to} \langle \mathbb{o}', \mathbb{m}' \rangle$ with $K(\mathbb{\tau}) = \tau$. If $\mathbb{\sigma}$ and $\mathbb{\sigma}'$ are related in the way indicated in the diagram then that means that because $\mathbb{p}$ is natural $\mathbb{\sigma}'$ factors through $\mathbb{\eta}_{\dot{\mathbb{o}}}$ as $\mathbb{\sigma}' = \mathbb{G}(\mathbb{\tau}) \bullet \mathbb{\eta}_{\dot{\mathbb{o}}}$. If $\mathbb{\tau}'$ is another morphism of $\mathbf{Real}(D, \mathbf{C})$ for which $\mathbb{\sigma}' = \mathbb{G}(\mathbb{\tau}') \bullet \mathbb{\eta}_{\dot{\mathbb{o}}}$, then again by naturality of $\mathbb{p}^{-1}$ we have $\mathbb{\sigma} = \mathbb{p}^{-1}_{\langle \mathbb{o}', \mathbb{m}' \rangle} \bullet \mathbb{\sigma}' = G(K(\mathbb{\tau}')) \bullet \mathbb{p}^{-1}_{\mathbb{F}_O(\dot{\mathbb{o}})} \bullet \mathbb{\eta}_{\dot{\mathbb{o}}} = G(K(\mathbb{\tau}')) \bullet \eta_{\dot{\mathbb{o}}}$, which by uniqueness of τ implies $K(\mathbb{\tau}') = \tau$ and by faithfulness of K this implies $\mathbb{\tau}' = \mathbb{\tau}$. Therefore we will know at this point that $\mathbb{\eta}_{\dot{\mathbb{o}}}$ is universal from $\dot{\mathbb{o}}$ to $\mathbb{G}$.

$$\begin{tikzcd}
 & & \dot{\mathbb{o}} \arrow[dl, "\mathbb{\sigma}'"'] \arrow[d, "\mathbb{\eta}_{\dot{\mathbb{o}}}"] \arrow[dlll, bend right, "\mathbb{\sigma}"'] \arrow[dd, bend left, "\eta_{\dot{\mathbb{o}}}"] \\
 & \mathbb{G}(\langle \mathbb{o}', \mathbb{m}' \rangle) \arrow[d, "\mathbb{p}^{-1}_{\langle \mathbb{o}', \mathbb{m}' \rangle}"] & \mathbb{G}(\mathbb{F}_O(\dot{\mathbb{o}})) \arrow[l, dashed, "\mathbb{G}(\mathbb{\tau})"] \arrow[d, "\mathbb{p}^{-1}_{\mathbb{F}_O(\dot{\mathbb{o}})}"] \\
G\left(\left\langle \dot{L}, \dot{L}_2, \dot{L}_0 \right\rangle\right) \arrow[r, phantom, "="] & G(K(\langle \mathbb{o}', \mathbb{m}' \rangle)) & G(K(\mathbb{F}_O(\dot{\mathbb{o}}))) \arrow[l, dashed, "G(K(\mathbb{\tau}))"'] \arrow[ll, dashed, bend left, "G(\tau)"]
\end{tikzcd} \tag{75}$$

In this situation there is a unique arrow function for F such that η becomes a natural transformation $\eta : 1_{\mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E, \mathbf{C}\right)} \dot{\to} G \circ F$ and this arrow function makes F left adjoint to G and η the unit of that adjunction. Similarly there is a unique arrow function for $\mathbb{F}$ such that $\mathbb{\eta}$ becomes a natural transformation $\mathbb{\eta} : 1_{\mathbf{Real}(D, \mathbf{C})} \dot{\to} \mathbb{G} \circ \mathbb{F}$ and this arrow function makes $\mathbb{F}$ left adjoint to $\mathbb{G}$ and $\mathbb{\eta}$ the unit of that adjunction. Looking at diagram (76) and remembering that

$\mathbb{p}^{-1}$ is natural we see that $F = K \circ \mathbb{F}$.

$$\begin{tikzcd}
\dot{\mathbb{o}}' \arrow[d, "\mathbb{\eta}_{\dot{\mathbb{o}}'}"] \arrow[ddd, bend right=60, "\eta_{\dot{\mathbb{o}}'}"'] & & \dot{\mathbb{o}} \arrow[ll, "\dot{\mathbb{t}}"'] \arrow[d, "\mathbb{\eta}_{\dot{\mathbb{o}}}"'] \arrow[ddd, bend left=60, "\eta_{\dot{\mathbb{o}}}"] \\
\mathbb{G}(\mathbb{F}(\dot{\mathbb{o}}')) \arrow[d, "\mathbb{p}^{-1}_{\mathbb{F}(\dot{\mathbb{o}}')}"] & & \mathbb{G}(\mathbb{F}(\dot{\mathbb{o}})) \arrow[ll, "\mathbb{G}(\mathbb{F}(\dot{\mathbb{t}}))"'] \arrow[d, "\mathbb{p}^{-1}_{\mathbb{F}(\dot{\mathbb{o}})}"'] \\
G(K(\mathbb{F}(\dot{\mathbb{o}}'))) \arrow[d, equal] & & G(K(\mathbb{F}(\dot{\mathbb{o}}))) \arrow[ll, "G(K(\mathbb{F}(\dot{\mathbb{t}})))"'] \arrow[d, equal] \\
G(F(\dot{\mathbb{o}}')) & & G(F(\dot{\mathbb{o}})) \arrow[ll, "G(F(\dot{\mathbb{t}}))"']
\end{tikzcd} \tag{76}$$

Now we still have to define $\mathbb{F}_O$ and $\mathbb{\eta}$.

So let $\dot{\mathbb{o}} \in \mathbf{C}^X$ and call $\mathbb{F}_O(\dot{\mathbb{o}}) := \langle \mathbb{o}, \mathbb{m} \rangle$. The realization of object-atoms $\mathbb{o}$ is given by (remember the convention set down in definition 3.17 of also using the symbol $\dot{\mathbb{o}}$ to denote the lifted functor)

$$\mathbb{o}(\mathfrak{a}) := \coprod_{\substack{\hbar \text{ in } \tilde{\mathcal{E}}_E \\ t(\hbar) = \langle \mathfrak{a} \rangle}} v^{(\ell(s(\hbar)))}\left(\dot{\mathbb{o}}\left(s\left(\hbar\right)\right)\right) . \tag{77}$$

This coproduct exists by hypothesis of the theorem. For every $\hbar$ in $\tilde{\mathcal{E}}_E$ we have an injection $\mathbb{i}_{\dot{\mathbb{o}};\hbar} : v^{(\ell(s(\hbar)))}\left(\dot{\mathbb{o}}\left(s\left(\hbar\right)\right)\right) \to \mathbb{o}(\mathfrak{a})$. Usually, when $\dot{\mathbb{o}}$ is clear from the context we will omit it in the subscript.

From our hypotheses it follows that for any tensor word v the induced functor $v_{\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle}$ preserves coproducts in every variable seperately. Indeed any functor $v\left(b_1, \ldots, b_{i-1}, -, b_{i+1}, \ldots, b_n\right)$ can be seen as a composite of functors, each of which has either the form $a \boxtimes -$ or $- \boxtimes a$ where a is $w\left(\langle b_{i_j} \rangle_{j=1}^k\right)$ for an appropriate tensor word w and an appropriate selection of indizes $i_1, \ldots, i_k$. Using lemma 5.3 we get that

$$\left\langle v\left(\langle \mathbb{i}_{\hbar_i} \rangle_{i=1}^n\right) : v\left(\left\langle v^{(\ell(s(\hbar_i)))}\left(\dot{\mathbb{o}}\left(s\left(\hbar_i\right)\right)\right)\right\rangle_{i=1}^n\right) \to v\left(\langle \mathbb{o}(\mathfrak{a}_i) \rangle_{i=1}^n\right) \right\rangle_{\substack{\langle \hbar_i \rangle_{i=1}^n \text{ in } \prod_{i=1}^n \tilde{\mathcal{E}}_E \\ t(\hbar_i) = \langle \mathfrak{a}_i \rangle}}$$

forms a colimiting cone for all $\langle \mathfrak{a}_i \rangle_{i=1}^n \in \tilde{\mathcal{E}}_E$.

We want to show now that in analogy to (77) we have

$$K\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)(a) = \coprod_{\substack{\hbar \text{ in } \tilde{\mathcal{E}}_E \\ t(\hbar) = a}} v^{(\ell(s(\hbar)))}\left(\dot{\mathbb{o}}\left(s\left(\hbar\right)\right)\right) . \tag{78}$$

(Remember (definition 3.34) that the object function of $K\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)$ only depends on $\mathbb{o}$ so we can talk about $K\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)(a)$ even though we haven't defined $\mathbb{m}$ yet.) This is where the conditions on the targets of arrow-atoms and of paths in E come into play.

Lemma 7.15. *Let $X, A, \mathfrak{s}, \mathfrak{t}$ be generators and let E be a set of equations in $\mathcal{E}\left(X, A, \mathfrak{s}, \mathfrak{t}\right)$. If conditions 1 and 2 in the statement of the parent theorem are* *met then for all* $\langle \mathfrak{a}_i \rangle_{i=1}^n \in \tilde{\mathcal{E}}_E$ *the function*

$$\kappa_{\langle \mathfrak{a}_i \rangle_{i=1}^n} : \left\{ \langle \mathscr{h}_i \rangle_{i=1}^n \text{ in } \prod_{i=1}^n \tilde{\mathcal{E}}_E \middle| t(\mathscr{h}_i) = \langle \mathfrak{a}_i \rangle \;\; \forall i \in \{1, \dots, n\} \right\} \to$$
$$\left\{ \mathscr{h} \text{ in } \tilde{\mathcal{E}}_E \middle| t(\mathscr{h}) = \langle \mathfrak{a}_i \rangle_{i=1}^n \right\}$$
$$\kappa_{\langle \mathfrak{a}_i \rangle_{i=1}^n} \left(\langle \mathscr{h}_i \rangle_{i=1}^n \right) := \boxdot_{i=1}^n \left(\mathscr{h}_i \right)$$

is a bijection.

Proof. Replacing $\tilde{\mathcal{E}}_E$ by $\mathrm{pArr}\,(X, A, \mathfrak{s}, \mathfrak{t})$ everywhere above we can define a function $\kappa''_{\langle \mathfrak{a}_i \rangle_{i=1}^n}$ which operates on tuples of pre-arrows. We will show that $\kappa''_{\langle \mathfrak{a}_i \rangle_{i=1}^n}$ is bijective.

Next we will show that two pre-arrows are related by $\Bumpeq$ if and only if the elements of the tuples which are the images under ${\kappa''}^{-1}_{\langle \mathfrak{a}_i \rangle_{i=1}^n}$ of these two pre-arrows are pairwise related by $\Bumpeq$. From this it will be clear that $\kappa_{\langle \mathfrak{a}_i \rangle_{i=1}^n}$ is a bijection when E is the empty set, that is, when we replace $\tilde{\mathcal{E}}_E$ by $\mathcal{E}\,(X, A, \mathfrak{s}, \mathfrak{t})$ everywhere above. Call this function $\kappa'_{\langle \mathfrak{a}_i \rangle_{i=1}^n}$.

Then we show that any two arrows of $\mathcal{E}\,(X, A, \mathfrak{s}, \mathfrak{t})$ are related by $\sim_E$ if and only if the elements of the tuples which are the images under ${\kappa'}^{-1}_{\langle \mathfrak{a}_i \rangle_{i=1}^n}$ of these two arrows of $\mathcal{E}\,(X, A, \mathfrak{s}, \mathfrak{t})$ are pairwise related by $\sim_E$. This will show the statement of the lemma to be true.

So let $\langle \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_i} \rangle_{i=k}^1$ be a pre-arrow with $\mathscr{t}\left(\langle \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_i} \rangle_{i=k}^1 \right) = \langle \mathfrak{a}_i \rangle_{i=1}^n$. From condition 1 it is clear that n_k must be equal to n. For each of the arrow-atoms $\mathfrak{f}_{k,1}, \dots, \mathfrak{f}_{k,n}$ we can find all arrow-atoms vertically preceding it by the following procedure.

Because there are no arrow-atoms whose target has length bigger than 1, for every index $j \in \{1, \dots, n_{k-1}\}$ we can find an index $j' \in \{1, \dots, n_k\}$ such that the section of $\mathscr{t}\left(\left\langle \langle \mathfrak{f}_{k-1,l} \rangle_{l=1}^{n_{k-1}} \right\rangle \right) = \mathscr{s}(\langle \langle \mathfrak{f}_l \rangle_{l=1}^{n_k} \rangle)$ corresponding to $\mathscr{t}(\mathfrak{f}_{k-1,j})$ is a subsection of the section corresponding to $\mathscr{s}(\mathfrak{f}_{k,j'})$. Because there are no arrow-atoms whose target has length 0 there is exactly one index j' with this property. We say that we group the arrow-atom $\mathfrak{f}_{k-1,j}$ with the arrow-atom $\mathfrak{f}_{k,j'}$ (or really the index-pair $\langle k-1, j \rangle$ with the index-pair $\langle k, j' \rangle$). Continuing this strategy we can find all (index-pairs of) arrow-atoms which precede one of the arrow-atoms (at one of the index-pairs) previously grouped with (the index-pair of) the arrow-atom $\mathfrak{f}_{k,j'}$ — thereby splitting the pre-arrow $\langle \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_i} \rangle_{i=k}^1$ into n parts, each of which is itself a pre-arrow, in such a way that the horizontal juxtaposition of these n parts gives $\langle \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_i} \rangle_{i=k}^1$. In our visual calculus this procedure corresponds to starting at the bottom and tracing the outlines of the arrow-atoms until we arrive at the top — this is possible precisely because we required the length of the target of every arrow-atom to be 1. From the description of the process by which these parts were found it is clear that there can be no other choice of pre-arrows whose horizontal juxtaposition yields $\langle \langle \mathfrak{f}_{i,j} \rangle_{j=1}^{n_i} \rangle_{i=k}^1$. Therefore $\kappa''_{\langle \mathfrak{a}_i \rangle_{i=1}^n}$ is bijective.

$\Bumpeq$ was defined in such a way as to be compatible with horizontal juxtaposition — so clearly if $\mathscr{f}_i, \mathscr{g}_i$ are pre-arrows with $\mathscr{f}_i \Bumpeq \mathscr{g}_i$ and $\mathscr{t}\left(\mathscr{f}_i\right) = \mathscr{t}\left(\mathscr{g}_i\right) = \mathfrak{a}_i$ for all $i \in \{1, \dots, n\}$ then $\kappa''_{\langle \mathfrak{a}_i \rangle_{i=1}^n} \left(\langle \mathscr{f}_i \rangle_{i=1}^n \right) \Bumpeq \kappa''_{\langle \mathfrak{a}_i \rangle_{i=1}^n} \left(\langle \mathscr{g}_i \rangle_{i=1}^n \right)$.

Now let $\mathscr{f}$ and $\mathscr{g}$ be pre-arrows. If $\mathscr{f}$ and $\mathscr{g}$ differ only by a row of identities then the same is true for their components — that is their preimages under

$\kappa''_{\langle \mathfrak{a}_i \rangle_{i=1}^n}$. If f is the result of shifting up a (single-letter) identity in g then there is a component of f which is the result of shifting up this same identity in the corresponding component of g while the other components of f and g are equal. This concludes the second part and shows that $\kappa'_{\langle \mathfrak{a}_i \rangle_{i=1}^n}$ is a bijection.

Again, $\sim_E$ was defined in such a way as to be compatible with monoidal product, so if f_i, g_i are arrows of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ with $f_i \sim_E g_i$ and $t(f_i) = t(g_i) = \mathfrak{a}_i$ for all $i \in \{1, \ldots, n\}$ then $\kappa'_{\langle \mathfrak{a}_i \rangle_{i=1}^n}(\langle f_i \rangle_{i=1}^n) \sim_E \kappa'_{\langle \mathfrak{a}_i \rangle_{i=1}^n}(\langle g_i \rangle_{i=1}^n)$.

So let $f = \kappa'_{\langle \mathfrak{a}_j \rangle_{j=1}^n}(\langle f_j \rangle_{j=1}^n)$ and $g = \kappa'_{\langle \mathfrak{a}_j \rangle_{j=1}^n}(\langle g_j \rangle_{j=1}^n)$ be arrows of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ such that $f \sim_E g$ (by lemma 3.26 any two arrows which are related by $\sim_E$ have an equal target so $f \sim_E g$ already implies that $t(f)$ and $t(g)$ can be written in the form indicated with the same tuple $\langle \mathfrak{a}_j \rangle_{j=1}^n$).

By lemma 3.28 there exists a natural number l and sequences

$$\left\langle \left\langle p^{(i)}, q^{(i)} \right\rangle \right\rangle_{i=1}^l , \left\langle h^{(i)} \right\rangle_{i=1}^l , \left\langle \dot{h}^{(i)} \right\rangle_{i=1}^l , \left\langle k^{(i)} \right\rangle_{i=1}^l , \left\langle \dot{k}^{(i)} \right\rangle_{i=1}^l$$

with

$$\left\langle p^{(i)}, q^{(i)} \right\rangle \in E \cup \{\langle q, p \rangle | \langle p, q \rangle \in E\}$$

and

$$h^{(i)}, \dot{h}^{(i)}, k^{(i)}, \dot{k}^{(i)} \in \mathrm{Arr}(\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})) \text{ for all } i \in \{1, \ldots, l\}$$

such that

$$\begin{aligned} f &= k^{(1)} \circ \left(h^{(1)} \boxdot p^{(1)} \boxdot \dot{h}^{(1)} \right) \circ \dot{k}^{(1)} \\ k^{(i)} \circ \left(h^{(i)} \boxdot q^{(i)} \boxdot \dot{h}^{(i)} \right) \circ \dot{k}^{(i)} &= k^{(i+1)} \circ \left(h^{(i+1)} \boxdot p^{(i+1)} \boxdot \dot{h}^{(i+1)} \right) \circ \dot{k}^{(i+1)} \\ &\qquad \text{for all } i \in \{1, \ldots, l\} \\ k^{(l)} \circ \left(h^{(l)} \boxdot q^{(l)} \boxdot \dot{h}^{(l)} \right) \circ \dot{k}^{(l)} &= g \ . \end{aligned} \tag{79}$$

Call $k^{(i)} \circ \left(h^{(i)} \boxdot p^{(i)} \boxdot \dot{h}^{(i)} \right) \circ \dot{k}^{(i)} =: f^{(i)}$. Then $f^{(1)} = f$, $f^{(l)} = g$ and

$$f^{(i+1)} = k^{(i+1)} \circ \left(h^{(i+1)} \boxdot p^{(i+1)} \boxdot \dot{h}^{(i+1)} \right) \circ \dot{k}^{(i+1)} = k^{(i)} \circ \left(h^{(i)} \boxdot q^{(i)} \boxdot \dot{h}^{(i)} \right) \circ \dot{k}^{(i)}$$

is obtained from $f^{(i)}$ by swapping out a $p^{(i)}$ contained somewhere in the middle of that arrow for a $q^{(i)}$.

Now we can split $f^{(i)}$ into $f_1^{(i)}, \ldots, f_n^{(i)}$. This splitting corresponds to splitting each of $k^{(i)}$, $h^{(i)} \boxdot p^{(i)} \boxdot \dot{h}^{(i)}$ and $\dot{k}^{(i)}$ into n parts which we call $k_j^{(i)}$, $\dot{p}_j^{(i)}$ and $\dot{k}_j^{(i)}$ respectively. Because $p^{(i)}$ has a target with length 1 there is an index j_i such that $\dot{p}_{j_i}^{(i)} = h_{j_i}^{(i)} \boxdot p^{(i)} \boxdot \dot{h}_{j_i}^{(i)}$ for an $h_{j_i}^{(i)}$ which forms a (possibly empty) right hand flank of $h^{(i)}$ and an $\dot{h}_{j_i}^{(i)}$ which forms a (possibly empty) left hand flank of $\dot{h}^{(i)}$. For all $j \neq j_i$ the arrow $p_j^{(i)}$ is a part of either $h^{(i)}$ or $\dot{h}^{(i)}$. Splitting $f^{(i+1)} = k^{(i)} \circ \left(h^{(i)} \boxdot q^{(i)} \boxdot \dot{h}^{(i)} \right) \circ \dot{k}^{(i)}$ in a similar manner corresponds to splitting each of $k^{(i)}$, $h^{(i)} \boxdot q^{(i)} \boxdot \dot{h}^{(i)}$ and $\dot{k}^{(i)}$ into n parts. Because $q^{(i)}$ has the same shape as $p^{(i)}$ the parts that $k^{(i)}$ and $\dot{k}^{(i)}$ split into are exactly the same as before and we can call them by the same names as before. Call the parts of $h^{(i)} \boxdot q^{(i)} \boxdot \dot{h}^{(i)}$ by the names $\dot{q}_j^{(i)}$. Again $q^{(i)}$ is completely contained in one of these parts and because $q^{(i)}$ has the same shape as $p^{(i)}$ and everything else has remained the same as before this part is the part $\dot{q}_{j_i}^{(i)}$ with the same index j_i as before. For this part we have $\dot{q}_{j_i}^{(i)} = h_{j_i}^{(i)} \boxdot q^{(i)} \boxdot \dot{h}_{j_i}^{(i)}$ with $h_{j_i}^{(i)}$ and $\dot{h}_{j_i}^{(i)}$ as defined above.

$$\text{For all indizes } j \neq j_i \text{ we have } \dot{q}_j^{(i)} = \dot{p}_j^{(i)} . \tag{80}$$

For each $j \in \{1, \dots, n\}$ we can now form a chain which uses the sequences

$$\left\langle \left\langle p^{(i_{j;k})}, q^{(i_{j;k})} \right\rangle \right\rangle_{k=1}^{m_j}, \left\langle h_j^{(i_{j;k})} \right\rangle_{k=1}^{m_j}, \left\langle \dot{h}_j^{(i_{j;k})} \right\rangle_{k=1}^{m_j}, \left\langle k_j^{(i_{j;k})} \right\rangle_{k=1}^{m_j}, \left\langle \dot{k}_j^{(i_{j;k})} \right\rangle_{k=1}^{m_j}$$

where $i_{j;1}, \dots, i_{j;m_j}$ are those indizes i for which $j_i = j$. By the discussion above and because

$$\begin{gathered}
k_j^{(i_{j;k})} \circ \left(h_j^{(i_{j;k})} \boxdot q_j^{(i_{j;k})} \boxdot \dot{h}_j^{(i_{j;k})} \right) \circ \dot{k}_j^{(i_{j;k})} = k_j^{(i_{j;k})} \circ \dot{q}^{(i_{j;k})} \circ \dot{k}_j^{(i_{j;k})} \overset{\substack{\text{follows from (79)} \\ \text{and } \kappa'_{\langle \mathfrak{a}_j \rangle_{j=1}^n} \text{ bij.}}}{=} \\
k_j^{(i_{j;k}+1)} \circ \dot{p}^{(i_{j;k}+1)} \circ \dot{k}_j^{(i_{j;k}+1)} \overset{(80)}{=} k_j^{(i_{j;k}+1)} \circ \dot{q}^{(i_{j;k}+1)} \circ \dot{k}_j^{(i_{j;k}+1)} \overset{(79)}{=} \\
k_j^{(i_{j;k}+2)} \circ \dot{p}^{(i_{j;k}+2)} \circ \dot{k}_j^{(i_{j;k}+2)} = \\
\dots = \\
k_j^{(i_{j;k+1}-1)} \circ \dot{q}^{(i_{j;k+1}-1)} \circ \dot{k}_j^{(i_{j;k+1}-1)} \overset{(79)}{=} \\
k_j^{(i_{j;k+1})} \circ \dot{p}^{(i_{j;k+1})} \circ \dot{k}_j^{(i_{j;k+1})} = k_j^{(i_{j;k+1})} \circ \left(h_j^{(i_{j;k+1})} \boxdot p_j^{(i_{j;k+1})} \boxdot \dot{h}_j^{(i_{j;k+1})} \right) \circ \dot{k}_j^{(i_{j;k+1})}
\end{gathered}$$

this chain (which may be empty) connects f_j to g_j in the manner described in lemma 3.28.

The reader is encouraged to draw a picture of the situation. For the author this was very helpful in developing some intuition for what is happening here. □

We have already seen that $\langle v(\langle \mathbb{1}_{h_i} \rangle_{i=1}^n) \rangle_{\substack{\langle h_i \rangle_{i=1}^n \text{ in } \prod_{i=1}^n \tilde{\mathcal{E}}_E \\ t(h_i) = \langle \mathfrak{a}_i \rangle}}$ forms a colimiting cone to $v(\langle \mathbb{o}(\mathfrak{a}_i) \rangle_{i=1}^n)$. Clearly this is still true if we prepend an isomorphism to every one of these maps. So the maps

$$\begin{aligned}
&\iota_{\dot{\mathbb{o}};v \leftarrow u_{\langle h_i \rangle_{i=1}^n};\langle h_i \rangle_{i=1}^n} := \\
&\qquad v(\langle \mathbb{1}_{\dot{\mathbb{o}};h_i} \rangle_{i=1}^n) \circ \mathrm{can}_{v(\langle v^{(\ell(s(h_i)))} \rangle_{i=1}^n) \leftarrow u_{\langle h_i \rangle_{i=1}^n}; \langle \dot{\mathbb{o}}(\pi_j(s(h_i))) \rangle_{j=1}^{\ell(s(h_i))}\,{}_{i=1}^{n}} : \\
&\qquad\qquad u_{\langle h_i \rangle_{i=1}^n} \left(\left\langle \dot{\mathbb{o}}(\pi_j(s(h_i))) \right\rangle_{j=1}^{\ell(s(h_i))}\,{}_{i=1}^{n} \right) \to v(\langle \mathbb{o}(\mathfrak{a}_i) \rangle_{i=1}^n)
\end{aligned}$$

form a colimiting cone when $\langle h_i \rangle_{i=1}^n$ ranges over

$$\left\{ \langle h'_i \rangle_{i=1}^n \text{ in } \prod_{i=1}^n \tilde{\mathcal{E}}_E \,\middle|\, t(h'_i) = \langle \mathfrak{a}_i \rangle \right\} .$$

$u_{\langle h_i \rangle_{i=1}^n}$ is a tensor word of appropriate length.

By the previous lemma we can just as well let $\kappa_{\langle \mathfrak{a}_i \rangle_{i=1}^n} \left(\langle \hbar_i \rangle_{i=1}^n \right)$ range over

$$\left\{ \hbar \text{ in } \tilde{\mathcal{E}}_E \middle| t(\hbar) = \langle \mathfrak{a}_i \rangle_{i=1}^n \right\} \ .$$

Because $s\left(\kappa_{\langle \mathfrak{a}_i \rangle_{i=1}^n} \left(\langle \hbar_i \rangle_{i=1}^n \right)\right) = \cdot_{i=1}^n \left(s\left(\hbar_i\right) \right)$ this means that if we define

$$\iota_{\dot{\mathbb{o}};v \leftarrow u_\hbar;\hbar} := \iota_{\dot{\mathbb{o}};v \leftarrow u_\hbar;\kappa^{-1}_{t(\hbar)}(\hbar)}$$

then

$$\left\langle v\left(\mathbb{o}\left(a\right)\right), \left\langle \iota_{\dot{\mathbb{o}};v \leftarrow u_\hbar;\hbar} : u_\hbar\left(\dot{\mathbb{o}}\left(s\left(\hbar\right)\right)\right) \to v\left(\mathbb{o}\left(a\right)\right) \right\rangle_{\substack{\hbar \text{ in } \tilde{\mathcal{E}}_E \\ t(\hbar)=a}} \right\rangle$$

is a colimit. Specifically, if we set

$$\iota_{\dot{\mathbb{o}};u_\hbar;\hbar} := \iota_{\dot{\mathbb{o}};v^{(\ell(t(\hbar)))} \leftarrow u_\hbar;\hbar}$$

then

$$\left\langle K\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)(a), \left\langle \iota_{\dot{\mathbb{o}};u_\hbar;\hbar} : u_\hbar\left(\dot{\mathbb{o}}\left(s\left(\hbar\right)\right)\right) \to K\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)(a) \right\rangle_{\substack{\hbar \text{ in } \tilde{\mathcal{E}}_E \\ t(\hbar)=a}} \right\rangle$$

is a colimit. Set

$$\iota_{\dot{\mathbb{o}};\hbar} := \iota_{\dot{\mathbb{o}};v^{(\ell(s(\hbar)))};\hbar} = \iota_{\dot{\mathbb{o}};v^{(\ell(t(\hbar)))} \leftarrow v^{(\ell(s(\hbar)))};\hbar} \ .$$

Again in all of the arrows $\iota_{\ldots}$ we omit $\dot{\mathbb{o}}$ in the subscript when it is clear from the context. Now we define $\mathbb{m}(\mathfrak{f})$ by requiring that the diagram in (81) commutes for all $\hbar$ in $\tilde{\mathcal{E}}_E$ for which $t(\hbar) = \mathfrak{s}(\mathfrak{f})$.

$$\begin{tikzcd}
v_s(\mathfrak{f})(\mathbb{o}(\mathfrak{s}(\mathfrak{f}))) \arrow[d, "\mathbb{m}(\mathfrak{f})"] & & v^{(\ell(s(\hbar)))}\left(\dot{\mathbb{o}}\left(s\left(\hbar\right)\right)\right) \arrow[ll, "\iota_{v_s(\mathfrak{f}) \leftarrow v^{(\ell(s(\hbar)))};\hbar}"'] \arrow[d, equal] \\
v_t(\mathfrak{f})(\mathbb{o}(\mathfrak{t}(\mathfrak{f}))) & & v^{\left(\ell\left(s\left(\tilde{\mathbb{\pi}}'_E(\langle\langle \mathfrak{f} \rangle\rangle) \circ \hbar\right)\right)\right)}\left(\dot{\mathbb{o}}\left(s\left(\tilde{\mathbb{\pi}}'_E\left(\langle \langle \mathfrak{f} \rangle\rangle\right) \circ \hbar\right)\right)\right) \arrow[ll, "\iota_{v_t(\mathfrak{f}) \leftarrow v^{\left(\ell\left(s\left(\tilde{\mathbb{\pi}}'_E(\langle\langle \mathfrak{f} \rangle\rangle) \circ \hbar\right)\right)\right)};\tilde{\mathbb{\pi}}'_E(\langle\langle \mathfrak{f} \rangle\rangle) \circ \hbar}"]
\end{tikzcd}$$

(81)

Note that this diagram also uniquely determines $\mathbb{m}(\mathfrak{a})$ for any $\mathfrak{a} \in X$ — moreover it does so in a manner consistent with the convention laid down in equation (37) from definition 3.17 (which just says that $\mathbb{m}(\mathfrak{a})$ is the appropriate identity).

We will show that $K\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)$ is determined by a diagram which is very similar to (81). Let $\mathfrak{f}_i \in A \dot{\cup} X$, $\hbar_i$ in $\tilde{\mathcal{E}}_E$ with $t(\hbar_i) = \mathscr{s}(\mathfrak{f}_i)$. Now set $f := \pi_{\Bumpeq}\left(\langle \langle \mathfrak{f}_i \rangle_{i=1}^n \rangle\right)$, $\hbar := \boxdot_{i=1}^n \left(\hbar_i\right)$ and $\langle H, H_2, H_0 \rangle := K\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)$. Consider the diagram in (82) (in which all unlabelled arrows stand for the appropriate structural transformation). Commutativity of the inner square follows from the definition of $\mathbb{m}(\mathfrak{f}_i)$ and from functoriality of $v^{(n)}{}_{\mathbf{C}}$. The right hand quadrilateral is trivially commutative and for the upper and lower quadrilaterals commutativity follows from a now already familiar equality-of-traces style argument and because $\kappa^{-1}_{t(\hbar)}\left(\hbar\right) = \cdot_{i=1}^n \left(\kappa^{-1}_{t(\hbar_i)}\left(\hbar_i\right)\right)$. The left hand quadrilateral is commutative by the definition of $K\left(\langle \mathbb{o}, \mathbb{m} \rangle\right)$. Therefore the outer quadrilateral commutes.

$$
\begin{tikzcd}
H\left(s\left(f\right)\right) \arrow[dr] \arrow[ddd, bend right, "H(f)"'] & & & v^{(\ell(s(\hbar)))}\left(\dot{\mathbb{o}}\left(s\left(\hbar\right)\right)\right) \arrow[lll, "\iota_{\hbar}"'] \arrow[dl] \arrow[ddd, bend left, "1"] \\
& v^{(n)}\left(\langle v_s\left(\mathfrak{f}_i\right)\left(\mathbb{o}\left(\mathscr{s}\left(\mathfrak{f}_i\right)\right)\right)\rangle_{i=1}^{n}\right) \arrow[d, "v^{(n)}\left(\langle \mathbb{m}(\mathfrak{f}_i)\rangle_{i=1}^{n}\right)"] & v^{(n)}\left(\left\langle v^{(\ell(s(\hbar_i)))}\left(\dot{\mathbb{o}}\left(s\left(\hbar_i\right)\right)\right)\right\rangle_{i=1}^{n}\right) \arrow[l, bend right, "v^{(n)}\left(\left\langle \iota_{v_s(\mathfrak{f}_i)\leftarrow v^{(\ell(s(\hbar_i)))};\hbar_i}\right\rangle_{i=1}^{n}\right)"'] \arrow[d, equal] & \\
& v^{(n)}\left(\langle v_t\left(\mathfrak{f}_i\right)\left(\mathbb{o}\left(\mathscr{t}\left(\mathfrak{f}_i\right)\right)\right)\rangle_{i=1}^{n}\right) \arrow[dl] & v^{(n)}\left(\left\langle v^{(\ell(s(\hbar_i)))}\left(\dot{\mathbb{o}}\left(s\left(\hbar_i\right)\right)\right)\right\rangle_{i=1}^{n}\right) \arrow[l, bend left, "v^{(n)}\left(\left\langle \iota_{v_t(\mathfrak{f}_i)\leftarrow v^{\left(\ell\left(s\left(\tilde{\varpi}'_E(\langle\langle\mathfrak{f}_i\rangle\rangle)\circ\hbar_i\right)\right)\right)};\tilde{\varpi}'_E(\langle\langle\mathfrak{f}_i\rangle\rangle)\circ\hbar_i}\right\rangle_{i=1}^{n}\right)"] & \\
H\left(t\left(f\right)\right) & & & v^{\left(\ell\left(s\left(\tilde{\varpi}_E(f)\circ\hbar\right)\right)\right)}\left(\dot{\mathbb{o}}\left(s\left(\tilde{\varpi}_E\left(f\right)\circ\hbar\right)\right)\right) \arrow[lll, "\iota_{\tilde{\varpi}_E(f)\circ\hbar}"] \arrow[ul]
\end{tikzcd}
$$

(82)

Define $L(a) := H(a)$ for all $a \in \tilde{\mathcal{E}}_E$ and for any $f \in \tilde{\mathcal{E}}_E$ define $L\left(f\right)$ by requiring that the diagram in (83) commutes for all $\hbar$ in $\tilde{\mathcal{E}}_E$ with $t(\hbar) = s\left(f\right)$.

$$
\begin{tikzcd}
L\left(s\left(f\right)\right) \arrow[d, "L(f)"] & v^{(\ell(s(\hbar)))}\left(\dot{\mathbb{o}}\left(s\left(\hbar\right)\right)\right) \arrow[l, "\iota_{\hbar}"'] \arrow[d, equal] \\
L\left(t\left(f\right)\right) & v^{\left(\ell\left(s\left(f\circ\hbar\right)\right)\right)}\left(\dot{\mathbb{o}}\left(s\left(f\circ\hbar\right)\right)\right) \arrow[l, "\iota_{f\circ\hbar}"]
\end{tikzcd}
\tag{83}
$$

This uniquely determines f because these $\iota_{\hbar}$ form a colimiting cone.

To show that $L = H \circ \tilde{\varpi}_E$ we show that L is a functor.

$$
\begin{tikzcd}
L\left(s\left(g\right)\right) \arrow[d, "L(g)"] & v^{(\ell(s(\hbar)))}\left(\dot{\mathbb{o}}\left(s\left(\hbar\right)\right)\right) \arrow[l, "\iota_{\hbar}"'] \arrow[d, equal] \\
L\left(t\left(g\right)\right) = L\left(s\left(f\right)\right) \arrow[d, "L(f)"] & v^{\left(\ell\left(s\left(g\circ\hbar\right)\right)\right)}\left(\dot{\mathbb{o}}\left(s\left(g\circ\hbar\right)\right)\right) \arrow[l, "\iota_{g\circ\hbar}"] \arrow[d, equal] \\
L\left(t\left(f\right)\right) & v^{\left(\ell\left(s\left(f\circ g\circ\hbar\right)\right)\right)}\left(\dot{\mathbb{o}}\left(s\left(f\circ g\circ\hbar\right)\right)\right) \arrow[l, "\iota_{f\circ g\circ\hbar}"]
\end{tikzcd}
\tag{84}
$$

For any $f, g, \hbar$ in $\tilde{\varpi}_E$ with $s\left(f\right) = t\left(g\right)$ and $s\left(g\right) = t\left(\hbar\right)$ the upper and lower square in the diagram above commute by definition — therefore the outer square commutes.

$$
\begin{tikzcd}
L\left(s\left(\mathscr{g}\right)\right) \arrow[d, "L\left(f\circ\mathscr{g}\right)"] & & v^{\left(\ell\left(s\left(\mathscr{h}\right)\right)\right)}\left(\dot{\phi}\left(s\left(\mathscr{h}\right)\right)\right) \arrow[ll, "\iota_{\mathscr{h}}"'] \arrow[d, equal] \\
L\left(t\left(f\right)\right) & & v^{\left(\ell\left(s\left(f\circ\mathscr{g}\circ\mathscr{h}\right)\right)\right)}\left(\dot{\phi}\left(s\left(f\circ\mathscr{g}\circ\mathscr{h}\right)\right)\right) \arrow[ll, "\iota_{f\circ\mathscr{g}\circ\mathscr{h}}"]
\end{tikzcd}
$$

also commutes and because the $\iota_{\mathscr{h}}$s form a colimiting cone this means that $L\left(f\circ\mathscr{g}\right) = L\left(f\right)\circ L\left(\mathscr{g}\right)$. The same type of argument shows that L applied to any identity is an identity.

By the argument following diagram (82) we know that for single-row arrows f in $\mathcal{E}\left(X, A, \mathfrak{s}, \mathfrak{t}\right)$ we have $L\left(\tilde{\mathbb{\pi}}_E\left(f\right)\right) = H\left(f\right)$. Because both sides are functors this implies that for any arrow f in $\mathcal{E}\left(X, A, \mathfrak{s}, \mathfrak{t}\right)$

$$L\left(\tilde{\mathbb{\pi}}_E\left(f\right)\right) = H\left(f\right) \ .$$

By lemma 7.9 we see that $\langle \mathbb{o}, \mathbb{m}\rangle$ satisfies the set of equation-shapes D and the functor component L of $\langle L, L_2, L_0\rangle = K_D\left(\mathbb{F}_O\left(\dot{\phi}\right)\right) =: F_O\left(\dot{\phi}\right)$ is defined by diagram (83) ($L_{2;\mathscr{a},\mathscr{b}} = H_{2;\mathscr{a},\mathscr{b}}$ and $L_0 = H_0$).

Now we show that $F_O\left(\dot{\phi}\right)$ really is a free object. For any $\dot{\phi} \in \mathbf{C}^X$ define the natural transformations

$$
\begin{aligned}
\mathbb{\eta}_{\dot{\phi}} \quad &: \dot{\phi} \dot{\rightarrow} \mathbb{G}(\mathbb{F}_O(\dot{\phi})) : X \to \mathbf{C} \\
\left(\mathbb{\eta}_{\dot{\phi}}\right)_{\mathfrak{a}} &:= \iota_{\dot{\phi};(\_)\leftarrow(\_);1_{\langle\mathfrak{a}\rangle}}
\end{aligned} \tag{85}
$$

$$
\begin{aligned}
\eta_{\dot{\phi}} \quad &: \dot{\phi} \dot{\rightarrow} G(F_O(\dot{\phi})) : X \to \mathbf{C} \\
\left(\eta_{\dot{\phi}}\right)_{\mathfrak{a}} &:= \iota_{\dot{\phi};(\_);1_{\langle\mathfrak{a}\rangle}} \ .
\end{aligned} \tag{86}
$$

η will be the unit of the adjunction between F and G.

Clearly $\eta_{\dot{\phi}} = \mathbb{p}^{-1}_{\mathbb{F}(\dot{\phi})} \bullet \mathbb{\eta}_{\dot{\phi}}$.

We want to show that for any $\left\langle \dot{L}, \dot{L}_2, \dot{L}_0 \right\rangle \in \mathbf{StrgMFun}\left(\tilde{\mathcal{E}}_E, \mathbf{C}\right)$ and any $\mathbb{\sigma} : \dot{\phi} \dot{\rightarrow} G\left(\left\langle \dot{L}, \dot{L}_2, \dot{L}_0 \right\rangle\right) : X \to \mathbf{C}$ there is a unique monoidal natural transformation $\tau : F_O\left(\dot{\phi}\right) \dot{\rightarrow} \left\langle \dot{L}, \dot{L}_2, \dot{L}_0 \right\rangle : \tilde{\mathcal{E}}_E \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ such that $G\left(\tau\right) \bullet \eta_{\dot{\phi}} = \mathbb{\sigma}$.

Consider the diagram in (87). In this diagram $\mathscr{a} = \langle \mathfrak{a}_i\rangle_{i=1}^{\ell(\mathscr{a})}$, $\mathscr{b} \in \tilde{\mathcal{E}}_E$ and $\mathscr{h}$ is a morphism of $\tilde{\mathcal{E}}_E$ such that $s\left(\mathscr{h}\right) = \mathscr{a}$. The square marked with 1 commutes by the definition of $L\left(\mathscr{h}\right)$, 2 has to commute if τ is to be natural, 3 commutes by an equality-of-traces style argument, 4 has to commute if τ is to be monoidal and finally if we require that $G\left(\tau\right) \bullet \eta_{\dot{\phi}} = \mathbb{\sigma}$, then 5 also has to commute. So in summary, if τ is to be a monoidal natural transformation $\tau : F_O\left(\dot{\phi}\right) \dot{\rightarrow} \left\langle \dot{L}, \dot{L}_2, \dot{L}_0 \right\rangle : \tilde{\mathcal{E}}_E \to \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ such that $G\left(\tau\right) \bullet \eta_{\dot{\phi}} = \mathbb{\sigma}$, then necessarily the diagram in (88) (which is just the outer part of the previous diagram) has to commute for any $\mathscr{h}$ in $\tilde{\mathcal{E}}_E$ for which $t\left(\mathscr{h}\right) = \mathscr{b}$ and $s\left(\mathscr{h}\right) = \mathscr{a} = \langle \mathfrak{a}_i\rangle_{i=1}^{\ell(\mathscr{a})}$. But because

$$\left\langle \iota_{\mathscr{h}} \right\rangle_{\substack{\mathscr{h} \text{ in } \tilde{\mathcal{E}}_E \\ t(\mathscr{h}) = \mathscr{b}}}$$

$$
\begin{tikzcd}[column sep=large]
v^{(\ell(a))}\left(\left\langle \dot{L}(\langle \mathfrak{a}_i\rangle)\right\rangle_{i=1}^{\ell(a)}\right) \arrow[d, "\dot{L}_{v^{(\ell(a))},\langle\langle\mathfrak{a}_i\rangle\rangle_{i=1}^{\ell(a)}}"] \arrow[dr, phantom, "\textcircled{4}"] & v^{(\ell(a))}\left(\langle L(\langle \mathfrak{a}_i\rangle)\rangle_{i=1}^{\ell(a)}\right) \arrow[l, "v^{(\ell(a))}\left(\left\langle \tau_{\langle\mathfrak{a}_i\rangle}\right\rangle_{i=1}^{\ell(a)}\right)"'] \arrow[d, "L_{v^{(\ell(a))},\langle\langle\mathfrak{a}_i\rangle\rangle_{i=1}^{\ell(a)}}"] & \\
\dot{L}(a) \arrow[d, "\dot{L}(h)"] \arrow[dr, phantom, "\textcircled{2}"] & L(a) \arrow[l, "\tau_a"'] \arrow[d, "L(h)"] \arrow[dr, phantom, "\textcircled{1}"] & v^{(\ell(s(1_a)))}\left(\dot{o}(s(1_a))\right) \arrow[l, "\iota_{1_a}"'] \arrow[ul, "v^{(\ell(a))}\left(\left\langle \iota_{1_{\langle\mathfrak{a}_i\rangle}}\right\rangle_{i=1}^{\ell(a)}\right)"'] \arrow[ul, phantom, bend left, "\textcircled{3}"] \arrow[ull, bend right=60, "v^{(\ell(a))}\left(\langle\varpi_{\mathfrak{a}_i}\rangle_{i=1}^{\ell(a)}\right)"'] \arrow[ull, phantom, bend right=30, "\textcircled{5}"] \arrow[d, equal] \\
\dot{L}(b) & L(b) \arrow[l, "\tau_b"'] & v^{(\ell(s(h\circ 1_a)))}\left(\dot{o}(s(h\circ 1_a))\right) \arrow[l, "\iota_{h\circ 1_a}"']
\end{tikzcd}
$$

(87)

$$
\begin{tikzcd}
v^{(\ell(a))}\left(\left\langle \dot{L}(\langle \mathfrak{a}_i\rangle)\right\rangle_{i=1}^{\ell(a)}\right) \arrow[d, "\dot{L}_{v^{(\ell(a))},\langle\langle\mathfrak{a}_i\rangle\rangle_{i=1}^{\ell(a)}}"] & & \\
\dot{L}(a) \arrow[d, "\dot{L}(h)"] & & \\
\dot{L}(b) & L(b) \arrow[l, "\tau_b"'] & v^{(\ell(a))}\left(\dot{o}(a)\right) \arrow[l, "\iota_h"'] \arrow[uull, "v^{(\ell(a))}\left(\langle\varpi_{\mathfrak{a}_i}\rangle_{i=1}^{\ell(a)}\right)"']
\end{tikzcd}
\tag{88}
$$

$$
\begin{tikzcd}
v^{(\ell(a))}\left(\left\langle \dot{L}(\langle \mathfrak{a}_i\rangle)\right\rangle_{i=1}^{\ell(a)}\right) \arrow[d, "\dot{L}_{v^{(\ell(a))},\langle\langle\mathfrak{a}_i\rangle\rangle_{i=1}^{\ell(a)}}"] & & \\
\dot{L}(a) \arrow[d, "\dot{L}(h)"] \arrow[dd, bend right=40, "\dot{L}(f\circ h)"'] \arrow[r, phantom, "\textcircled{2}"] & \, & \\
\dot{L}(b) \arrow[d, "\dot{L}(f)"] \arrow[dr, phantom, "\textcircled{3}"] & L(b) \arrow[l, "\tau_b"'] \arrow[d, "L(f)"] \arrow[dr, phantom, "\textcircled{1}"] & v^{(\ell(s(h)))}\left(\dot{o}(s(h))\right) \arrow[l, "\iota_h"'] \arrow[uull, "v^{(\ell(a))}\left(\langle\varpi_{\mathfrak{a}_i}\rangle_{i=1}^{\ell(a)}\right)"'] \arrow[d, equal] \\
\dot{L}(c) & L(c) \arrow[l, "\tau_c"'] & v^{(\ell(s(f\circ h)))}\left(\dot{o}(s(f\circ h))\right) \arrow[l, "\iota_{f\circ h}"']
\end{tikzcd}
\tag{89}
$$

is a colimiting cone, this already uniquely determines a morphism $\tau_{\mathscr{b}} : L(\mathscr{b}) \to \dot{L}(\mathscr{b})$ of $\mathbf{C}$ for any $\mathscr{b} \in \tilde{\mathcal{E}}_E$. If we can show that the morphisms thus defined form a monoidal natural transformation, then we have shown that η is indeed universal from $\dot{\phi}$ to G.

In diagram (89) the parts marked with ① and ② commute by definition of $L(f)$ and $\tau_{\mathscr{b}}$ respectively. The two outermost paths from $v^{(\ell(s(\mathscr{h})))}(\dot{\phi}(s(\mathscr{h})))$ to $\dot{L}(c)$ are also equal by definition of τ_c. From this it follows that

$$\dot{L}(f) \circ \tau_{\mathscr{b}} \circ \iota_{\mathscr{h}} = \tau_c \circ L(f) \circ \iota_{\mathscr{h}}$$

and because

$$\langle \iota_{\mathscr{h}} \rangle_{\substack{\mathscr{h} \text{ in } \tilde{\mathcal{E}}_E \\ t(\mathscr{h}) = \mathscr{b}}}$$

is a colimiting cone this implies that ③ commutes. Therefore τ is natural.

Now we show that τ is monoidal. For $\mathscr{b} = \langle\rangle$ it follows from the requirement 1 in the hypotheses of this theorem that the only arrow $\mathscr{h}$ in $\tilde{\mathcal{E}}_E$ with $t(\mathscr{h}) = \langle\rangle$ is the identity $1_{\langle\rangle}$. In this case the diagram in (88) turns into

$$\begin{tikzcd}
v^{(0)}() = e \arrow[d, "\dot{L}_{v^{(0)},} = \dot{L}_0"'] & & \\
\dot{L}(\langle\rangle) \arrow[d, "\dot{L}(1_{\langle\rangle}) = 1_{\dot{L}(\langle\rangle)}"'] & & \\
\dot{L}(\langle\rangle) & L(\langle\rangle) = e \arrow[l, "\tau_{\langle\rangle}"] & v^{(0)}() = e \arrow[l, "\iota_{1_{\langle\rangle}} = 1_e"] \arrow[uull, "v^{(0)}() = 1_e"']
\end{tikzcd} .$$

This means that $\tau_{\langle\rangle} = \dot{L}_0$ and therefore the condition (68) in the definition 7.1 of a monoidal natural transformation is satisfied.

To see that (67) in the definition of a monoidal natural transformation also commutes consider diagram (90). In this diagram $\mathscr{h}_1, \mathscr{h}_2$ are morphism of $\tilde{\mathcal{E}}_E$ with $t(\mathscr{h}_j) = \mathscr{b}_j$ and $s(\mathscr{h}_j) = \mathscr{a}_j = \langle \mathfrak{a}_{j,i} \rangle_{i=1}^{n_j}$; and $\mathscr{h} = \mathscr{h}_1 \boxdot \mathscr{h}_2$.

⓪ is the square whose commutativity we want to show. We know that ⑥ and ① commute by the definition of τ. ⑦ commutes by an equality-of-traces style argument, ② commutes because $\dot{L}_2$ is natural, ③ commutes by the definition of $\dot{L}_{v^{(n_1)} \square v^{(n_2)}, \langle\langle \mathfrak{a}_{j,i} \rangle\rangle_{i=1}^{n_j}{}_{j=1}^{2}}$ and ④ commutes by lemma 3.6. The outermost ovoid consisting of the two paths from $v^{(n_1)}(\dot{\phi}(\mathscr{a}_1)) \boxtimes v^{(n_2)}(\dot{\phi}(\mathscr{a}_2))$ to $v^{(n_1+n_2)}\left(\left\langle \dot{L}(\langle \mathfrak{a}_{j,i} \rangle) \right\rangle_{i=1}^{n_j}{}_{j=1}^{2} \right)$ — call it ⑤ — commutes by an equality-of-traces style argument. Now a diagram chase which uses ① to ⑦ in ascending order of their names shows that

$$\dot{L}_{2;\mathscr{b}_1,\mathscr{b}_2} \circ (\tau_{\mathscr{b}_1} \boxtimes \tau_{\mathscr{b}_2}) \circ (\iota_{\mathscr{h}_1} \boxtimes \iota_{\mathscr{h}_2}) = \tau_{\mathscr{b}_1 \boxdot \mathscr{b}_2} \circ L_{2;\mathscr{b}_1,\mathscr{b}_2} \circ (\iota_{\mathscr{h}_1} \boxtimes \iota_{\mathscr{h}_2}) \ .$$

$\boxtimes$ takes $\left\langle \coprod_{i \in \mathbf{I}} a_i, \coprod_{j \in \mathbf{J}} b_j \right\rangle$ to $\coprod_{\langle i,j \rangle \in \mathbf{I} \times \mathbf{J}} (a_i \boxtimes b_j)$ (remember that this follows from the requirement that $\boxtimes$ preserve coproducts independently in each variable) and therefore $\langle \iota_{\mathscr{h}_1} \boxtimes \iota_{\mathscr{h}_2} \rangle_{\substack{\langle \mathscr{h}_1, \mathscr{h}_2 \rangle \text{ in } \tilde{\mathcal{E}}_E \times \tilde{\mathcal{E}}_E \\ t(\mathscr{h}_i) = \mathscr{b}_i}}$ form a colimiting cone. From this

$$
\begin{tikzcd}[column sep=small]
v^{(n_1)}\left(\left\langle \dot{L}(\langle \mathfrak{a}_{1,i}\rangle)\right\rangle_{i=1}^{n_1}\right)\boxtimes v^{(n_2)}\left(\left\langle \dot{L}(\langle \mathfrak{a}_{2,i}\rangle)\right\rangle_{i=1}^{n_2}\right) \arrow[dr, "\dot{L}_{v^{(n_1)},\left\langle\langle \mathfrak{a}_{1,i}\rangle\right\rangle_{i=1}^{n_1}}\boxtimes \dot{L}_{v^{(n_2)},\left\langle\langle \mathfrak{a}_{2,i}\rangle\right\rangle_{i=1}^{n_2}}"] \arrow[ddddr, "\dot{L}_{v^{(n_1)}\boxdot v^{(n_2)},\langle\langle \mathfrak{a}_{j,i}\rangle\rangle_{i=1}^{n_j}{}_{j=1}^{2}}"'] \arrow[ddddd, bend right, "\mathrm{can}_{v^{(n_1+n_2)}\leftarrow v^{(n_1)}\boxdot v^{(n_2)};\left\langle \dot{L}(\langle \mathfrak{a}_{j,i}\rangle)\right\rangle_{i=1}^{n_j}{}_{j=1}^{2}}"'] & & & & \\
& \dot{L}(a_1)\boxtimes \dot{L}(a_2) \arrow[dr, "\dot{L}(h_1)\boxtimes\dot{L}(h_2)"] \arrow[ddd, "\dot{L}_{2;a_1,a_2}"] & & 1 & \\
& & \dot{L}(b_1)\boxtimes\dot{L}(b_2) \arrow[d, "\dot{L}_{2;b_1,b_2}"] & L(b_1)\boxtimes L(b_2) \arrow[l, "\tau_{b_1}\boxtimes\tau_{b_2}"] \arrow[d, "L_{2;b_1,b_2}"] & v^{(n_1)}(\dot{\mathrm{o}}(a_1))\boxtimes v^{(n_2)}(\dot{\mathrm{o}}(a_2)) \arrow[l, "\iota_{h_1}\boxtimes\iota_{h_2}"] \arrow[d, "\mathrm{can}_{v^{(n_1+n_2)}\leftarrow v^{(n_1)}\boxdot v^{(n_2)};\langle\dot{\mathrm{o}}(\mathfrak{a}_{j,i})\rangle_{i=1}^{n_j}{}_{j=1}^{2}}"] \arrow[llllu, bend right, "v^{(n_1)}\left(\left\langle\sigma_{\mathfrak{a}_{1,i}}\right\rangle_{i=1}^{n_1}\right)\boxtimes v^{(n_2)}\left(\left\langle\sigma_{\mathfrak{a}_{2,i}}\right\rangle_{i=1}^{n_2}\right)"'] \\
4 & 3 \quad 2 & \dot{L}(b_1\boxdot b_2) & L(b_1\boxdot b_2) \arrow[l, "\tau_{b_1\boxdot b_2}"] \arrow[u, phantom, "7"] & v^{(n_1+n_2)}(\dot{\mathrm{o}}(a_1\boxdot a_2)) \arrow[l, "\iota_h"] \arrow[lllldd, bend left, "v^{(n_1+n_2)}\left(\left\langle\sigma_{\mathfrak{a}_{j,i}}\right\rangle_{i=1}^{n_j}{}_{j=1}^{2}\right)"] \\
& \dot{L}(a_1\boxdot a_2) \arrow[ur, "\dot{L}(h)"] & 0 & 6 & \\
v^{(n_1+n_2)}\left(\left\langle \dot{L}(\langle \mathfrak{a}_{j,i}\rangle)\right\rangle_{i=1}^{n_j}{}_{j=1}^{2}\right) \arrow[ur, "\dot{L}_{v^{(n_1+n_2)},\langle\langle \mathfrak{a}_{j,i}\rangle\rangle_{i=1}^{n_j}{}_{j=1}^{2}}"'] & & & &
\end{tikzcd}
\tag{90}
$$

it follows that

$$\dot{L}_{2;\mathcal{b}_1,\mathcal{b}_2} \circ (\tau_{\mathcal{b}_1} \boxtimes \tau_{\mathcal{b}_2}) = \tau_{\mathcal{b}_1 \boxdot \mathcal{b}_2} \circ L_{2;\mathcal{b}_1,\mathcal{b}_2}$$

which concludes the proof that τ is a monoidal natural transformation and also the proof of the theorem. □

Theorem 7.16. *If the requirements of theorem 7.14 are met, then*

$$\mathbf{Real}\,(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle)$$

is isomorphic to the category $\left(\mathbf{C}^X\right)^{\mathbb{T}}$ *of Eilenberg-Moore algebras for the monad* $\langle \mathbb{T}, \mathbb{G} \circ \mathbb{e} \circ \mathbb{F}, \mathbb{n} \rangle$ *where* $\mathbb{T} := \mathbb{G} \circ \mathbb{F} : \mathbf{C}^X \to \mathbf{C}^X$.

Proof. First note that $\langle \mathbb{T}, \mathbb{G} \circ \mathbb{e} \circ \mathbb{F}, \mathbb{n} \rangle$ is indeed a monad. Associativity of $\mathbb{G} \circ \mathbb{e} \circ \mathbb{F}$ follows from naturality of $\mathbb{e}$ while the two identity laws follow from the equations $(\mathbb{e} \circ \mathbb{F}) \bullet (\mathbb{F} \circ \mathbb{n}) = 1_{\mathbb{F}}$ and $(\mathbb{G} \circ \mathbb{e}) \bullet (\mathbb{n} \circ \mathbb{G}) = 1_{\mathbb{G}}$ which hold for any adjunction. All of this works for any adjunction. There is in fact a rich theory surrounding the relationship between adjunctions and their induced monads of which we have only seen glimpses in this paper. The interested reader who is not already familiar with these topics is referred to [Mac98, chapter VI].

Before we dive in let us calculate a little more explicitly some of the entities which are involved here and which we will need.

We start by giving a more explicit description of the arrow $\mathbb{t} : \mathbb{F}\,(\dot{\mathbb{o}}) \dot{\to} \langle \mathbb{o}', \mathbb{m}' \rangle$ induced by a morphism $\mathbb{\sigma}' : \dot{\mathbb{o}} \dot{\to} \mathbb{G}\,(\langle \mathbb{o}', \mathbb{m}' \rangle)$ — that is the unique $\mathbb{t} : \mathbb{F}\,(\dot{\mathbb{o}}) \dot{\to} \langle \mathbb{o}', \mathbb{m}' \rangle$ such that $\mathbb{\sigma}' = \mathbb{G}\,(\mathbb{t}) \bullet \mathbb{n}_{\dot{\mathbb{o}}}$. Consider diagram (91), in which $\mathcal{h} : \mathcal{a} = \langle \mathfrak{a}_i \rangle_{i=1}^{\ell(\mathcal{a})} \to \mathcal{b}$ in $\tilde{\mathcal{E}}_E\,(X, A, \mathfrak{s}, \mathfrak{t})$ and all unlabelled arrows denote the appropriate structural transformations. For the lower part of the diagram we assume that $\mathcal{b} = \langle \mathfrak{b} \rangle$ and the outermost part of the diagram only makes sense when $\mathcal{h} = \tilde{\mathbb{w}}'_E\,(\langle\,\langle \mathfrak{h} \rangle\rangle)$. In this diagram the pentagon consisting of the two parts ① and ② commutes by diagram (75) which relates the induced morphisms for η and $\mathbb{n}$ and by diagram (88) which defines the induced morphism for η. All other parts of the diagram commute because the involved arrows can be seen as evaluations of paths whose traces are equal. From this we see that the part of the diagram with the bigger arrows commutes. The requirement that this part commute for all $\mathcal{h}$ in $\tilde{\mathcal{E}}_E$ for which $t\,(\mathcal{h}) = \langle \mathfrak{b} \rangle$ also uniquely determines $\mathbb{t}$.

As $\mathbb{e}_{\langle \mathbb{o}', \mathbb{m}' \rangle}$ is the morphism induced by $1_{\mathbb{G}(\langle \mathbb{o}', \mathbb{m}' \rangle)}$ we get that the diagram in (92) commutes (where $\mathfrak{h} \in A \dot{\cup} X$ with $\mathscr{s}(\mathfrak{h}) = \mathcal{a}$ and $\mathscr{t}(\mathfrak{h}) = \langle \mathfrak{b} \rangle$)

$$\begin{tikzcd}
 & v_s\,(\mathfrak{h})(\mathbb{o}'(\mathcal{a})) \arrow[dl, "\mathbb{m}'(\mathfrak{h})"'] \arrow[dd, "\iota_{\mathbb{o}';(_)\leftarrow v_s(\mathfrak{h});\tilde{\mathbb{w}}'_E(\langle\langle \mathfrak{h} \rangle\rangle)}"] \\
v_t\,(\mathfrak{h})(\mathbb{o}'(\mathfrak{b})) \arrow[d, "\mathrm{can}_{(_)\leftarrow v_t(\mathfrak{h});\mathbb{o}'(\mathfrak{b})}"] & \\
\mathbb{o}'\,(\mathfrak{b}) & \mathbb{G}(\mathbb{F}(\mathbb{o}'))\,(\mathfrak{b}) \arrow[l, "\left(\mathbb{G}\left(\mathbb{e}_{\langle \mathbb{o}', \mathbb{m}' \rangle}\right)\right)_{\mathfrak{b}})"]
\end{tikzcd} \tag{92}$$

while $K_D\left(\mathbb{e}_{\langle \mathbb{o}', \mathbb{m}' \rangle}\right)$ is defined by the requirement that

$$\begin{tikzcd}
 & K_D\,(\langle \mathbb{o}', \mathbb{m}' \rangle)\,(\mathcal{a}) \arrow[dl, "K_D(\langle \mathbb{o}', \mathbb{m}' \rangle)(\mathcal{h})"'] \arrow[d, "\iota_{\mathbb{o}';\mathcal{h}}"] \\
K_D\,(\langle \mathbb{o}', \mathbb{m}' \rangle)\,(\mathcal{b}) & K_D(\mathbb{F}(\mathbb{o}'))\,(\mathcal{b}) \arrow[l, "\left(K_D\left(\mathbb{e}_{\langle \mathbb{o}', \mathbb{m}' \rangle}\right)\right)_{\mathcal{b}}"]
\end{tikzcd} \quad . \tag{93}$$

$$
\begin{tikzcd}[column sep=large]
 & v_s(\mathfrak{h})(\mathbb{o}'(a)) \arrow[d] \arrow[ddddl, bend right, "\mathbb{m}'(\mathfrak{h})"'] & & & \\
 & v^{(\ell(a))}\left(\left\langle K_D\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)\left(\langle \mathfrak{a}_i \rangle\right)\right\rangle_{i=1}^{\ell(a)}\right) \arrow[d] & & & \\
 & K_D\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(a) \arrow[d, "K_D\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(h)"'] \arrow[drr, phantom, "\textcircled{2}"] & & & \\
 & K_D\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(b) \arrow[d] \arrow[ur, phantom, "\textcircled{1}"] & K_D(\mathbb{F}(\dot{\Phi}))(b) \arrow[l, "(K_D(\pi))_b"] \arrow[d] & v^{(\ell(a))}(\dot{\Phi}(a)) \arrow[l, "\iota_{\dot{\Phi};h}"] \arrow[dl, "\iota_{\dot{\Phi};(\_)\leftarrow v^{(\ell(a))};h}"] \arrow[uull, "v^{(\ell(a))}\left(\left\langle \sigma'_{\mathfrak{a}_i} \right\rangle_{i=1}^{\ell(a)}\right)"'] \arrow[uuull, "v^{(\ell(a))}\left(\left\langle \rho^{-1}_{\langle \mathbb{o}', \mathbb{m}' \rangle;\mathfrak{a}_i} \circ \sigma'_{\mathfrak{a}_i} \right\rangle_{i=1}^{\ell(a)}\right)"'] & v_s(\mathfrak{h})(\dot{\Phi}(a)) \arrow[l] \arrow[dll, bend left, "\iota_{\dot{\Phi};(\_)\leftarrow v_s(\mathfrak{h});\tilde{\pi}'_E(\langle\langle \mathfrak{h} \rangle\rangle)"] \arrow[uuuulll, "v_s(\mathfrak{h})\left(\left\langle \sigma'_{\mathfrak{a}_i} \right\rangle_{i=1}^{\ell(a)}\right)"'] \\
v_t(\mathfrak{h})(\mathbb{o}'(\mathfrak{b})) \arrow[r] & \mathbb{o}'(\mathfrak{b}) & \mathbb{G}(\mathbb{F}(\dot{\Phi}))(\mathfrak{b}) \arrow[l, "(\mathbb{G}(\pi))_{\mathfrak{b}}"] & &
\end{tikzcd}
\tag{91}
$$

commute for all $h : a \to b$ in $\tilde{\mathcal{E}}_E$.

Using the above diagram for $K_D\left(\mathbb{e}_{\mathbb{F}(\dot{\mathbb{o}})}\right)$ and the definition of $K_D\left(\mathbb{F}\left(\dot{\mathbb{o}}\right)\right)\left(g\right)$ (given in (83)) we see that the diagram in (94) commutes for all $h : a \to b,\quad g : b \to c$ in $\tilde{\mathcal{E}}_E$ (by the lemma about iterated colimits the requirement that outer part of this diagram commute for these arrows even uniquely determines $K_D\left(\mathbb{e}_{\mathbb{F}(\dot{\mathbb{o}})}\right)$).

$$
\begin{tikzcd}
 & v^{(\ell(a))}\left(\dot{\mathbb{o}}\left(a\right)\right) \arrow[d, "\iota_{\dot{\mathbb{o}};h}"] \arrow[ddl, "\iota_{\dot{\mathbb{o}};g\circ h}"'] \\
 & K_D\left(\mathbb{F}\left(\dot{\mathbb{o}}\right)\right)\left(b\right) \arrow[d, "\iota_{\mathbb{G}(\mathbb{F}(\dot{\mathbb{o}}));g}"] \arrow[dl, "K_D(\mathbb{F}(\dot{\mathbb{o}}))(g)"] \\
K_D\left(\mathbb{F}\left(\dot{\mathbb{o}}\right)\right)\left(c\right) & K_D(\mathbb{F}(\mathbb{G}\left(\mathbb{F}\left(\dot{\mathbb{o}}\right)\right)))\left(c\right) \arrow[l, "\left(K_D\left(\mathbb{e}_{\mathbb{F}(\dot{\mathbb{o}})}\right)\right)_c"]
\end{tikzcd}
\tag{94}
$$

We calculate the defining equation for $F\left(\dot{\mathbb{t}}\right) = K_D(\mathbb{F}(\dot{\mathbb{t}}))$ (where $\dot{\mathbb{t}} : \dot{\mathbb{o}} \dot{\to} \dot{\mathbb{o}}'$). This is the morphism induced by $\eta_{\dot{\mathbb{o}}'} \bullet \dot{\mathbb{t}}$.

$$
\begin{tikzcd}
v^{(\ell(b))}\left(\left\langle F\left(\dot{\mathbb{o}}'\right)\left(\langle \mathfrak{b}_i\rangle\right)\right\rangle_{i=1}^{\ell(b)}\right) \arrow[d, "\mathrm{can}_{\ldots}"] \arrow[dr, phantom, "①"] & v^{(\ell(b))}\left(\dot{\mathbb{o}}'\left(b\right)\right) \arrow[l, "v^{(\ell(b))}\left(\left\langle \left(\eta_{\dot{\mathbb{o}}'}\right)_{\mathfrak{b}_i}\right\rangle_{i=1}^{\ell(b)}\right)"'] \arrow[dl, "\iota_{\dot{\mathbb{o}}';1_b}"] \arrow[d, "\iota_{\dot{\mathbb{o}}';g\circ 1_b} = \iota_{\dot{\mathbb{o}}';g}"] \arrow[dr, phantom, "⓪"] & v^{(\ell(b))}\left(\dot{\mathbb{o}}\left(b\right)\right) \arrow[l, "v^{(\ell(b))}\left(\left\langle \dot{\mathbb{t}}_{\mathfrak{b}_i}\right\rangle_{i=1}^{\ell(b)}\right)"'] \arrow[d, "\iota_{\dot{\mathbb{o}};g}"] \\
F\left(\dot{\mathbb{o}}'\right)\left(b\right) \arrow[r, "F\left(\dot{\mathbb{o}}'\right)\left(g\right)"'] \arrow[ur, phantom, "②"] & F\left(\dot{\mathbb{o}}'\right)\left(c\right) & F\left(\dot{\mathbb{o}}\right)\left(c\right) \arrow[l, "\left(F\left(\dot{\mathbb{t}}\right)\right)_c"]
\end{tikzcd}
$$

(95)

In diagram (95) (in which, like we do for K, we write $F\left(\dot{\mathbb{o}}\right)\left(c\right)$ and mean that we apply the functor component of $F\left(\dot{\mathbb{o}}\right)$ to c) the triangle ① commutes by an equality-of-traces style argument, ② commutes by the definition of $F\left(\dot{\mathbb{o}}'\right)\left(g\right)$ and the outermost part of the diagram commutes by (88) which describes the induced morphism. From this we glean that ⓪ commutes (and the requirement that it commute for all appropriate g in $\tilde{\mathcal{E}}_E$ uniquely determines $F\left(\dot{\mathbb{t}}\right)$).

Now we define two functors

$$V : \mathbf{Real}\left(D, \mathbf{C}\right) \to \left(\mathbf{C}^X\right)^{\mathbb{T}} \text{ and } W : \left(\mathbf{C}^X\right)^{\mathbb{T}} \to \mathbf{Real}\left(D, \mathbf{C}\right)$$

and show that $W \circ V = 1_{\mathbf{Real}(D,\mathbf{C})}$ and $V \circ W = 1_{\left(\mathbf{C}^X\right)^{\mathbb{T}}}$.

V sends any $\langle \mathbb{o}', \mathbb{m}'\rangle \in \mathbf{Real}\left(D, \mathbf{C}\right)$ to $\left\langle \mathbb{o}', \mathbb{G}\left(\mathbb{e}_{\langle \mathbb{o}', \mathbb{m}'\rangle}\right) : \mathbb{G}(\mathbb{F}(\mathbb{o}')) \to \mathbb{o}'\right\rangle$ and any $\mathbb{t}' : \langle \mathbb{o}', \mathbb{m}'\rangle \dot{\to} \langle \mathbb{o}'', \mathbb{m}''\rangle$ to $\mathbb{t}' : \left\langle \mathbb{o}', \mathbb{G}\left(\mathbb{e}_{\langle \mathbb{o}', \mathbb{m}'\rangle}\right)\right\rangle \dot{\to} \left\langle \mathbb{o}'', \mathbb{G}\left(\mathbb{e}_{\langle \mathbb{o}'', \mathbb{m}''\rangle}\right)\right\rangle$. We have to show that $\mathbb{G}\left(\mathbb{e}_{\langle \mathbb{o}', \mathbb{m}'\rangle}\right)$ satisfies (62) and that $\mathbb{t}'$ satisfies (63) (both from definition 6.1 of Eilenberg-Moore algebras and their morphisms). Commutativity of the left diagram in (62) follows from naturality of $\mathbb{e}$ and the right hand diagram is just the equation $(\mathbb{G} \circ \mathbb{e}) \bullet (\mathbb{n} \circ \mathbb{G}) = 1_{\mathbb{G}}$ which holds for any adjunction. Commutativity of (63) also follows from naturality of $\mathbb{e}$. So V is well-defined. (Note that we could have defined a functor like this for any adjunction — again the reader is referred to [Mac98, chapter VI] for a more thorough discussion.)

W sends any $\langle \mathbb{o}', \mathbb{h}' \rangle \in \left(\mathbf{C}^X\right)^{\mathbb{T}}$ to $\langle \mathbb{o}', \mathbb{m}' \rangle$ where $\mathbb{m}'(\mathscr{h})$ is defined as

$$
\begin{tikzcd}
 & v_s(\mathfrak{h})(\mathbb{o}'(\mathscr{a})) \arrow[dl, "\mathbb{m}'(\mathfrak{h})"'] \arrow[dd, "\iota_{\mathbb{o}';(\_)\leftarrow v_s(\mathfrak{h});\bar{\mathbb{w}}'_E(\langle\langle \mathfrak{h} \rangle\rangle)}"] \\
v_t(\mathfrak{h})(\mathbb{o}'(\mathfrak{b})) & \\
\mathbb{o}'(\mathfrak{b}) \arrow[u, "\mathrm{can}_{v_t(\mathfrak{h})\leftarrow(\_);\mathbb{o}'(\mathfrak{b})}"] & \mathbb{G}(\mathbb{F}(\mathbb{o}'))(\mathfrak{b}) \arrow[l, "\mathbb{h}'_{\mathfrak{b}}"]
\end{tikzcd}
\tag{96}
$$

$(\mathfrak{s}(\mathscr{h}) = \mathscr{a}$ and $\mathfrak{t}(\mathscr{h}) = \mathscr{b})$ and any $\mathbb{t}' : \langle \mathbb{o}', \mathbb{h}' \rangle \dot{\to} \langle \mathbb{o}'', \mathbb{h}'' \rangle$ to $\mathbb{t}' : \langle \mathbb{o}', \mathbb{m}' \rangle \dot{\to} \langle \mathbb{o}'', \mathbb{m}'' \rangle$. To see that W is well-defined we have to show that $\langle \mathbb{o}', \mathbb{m}' \rangle$ thus defined satisfies the set of equation-shapes D and that $\mathbb{t}'$ is really a morphism of realizations.

$$
\begin{tikzcd}
v_s(\mathfrak{h})(\mathbb{o}''(\mathscr{a})) \arrow[d, "\iota_{\mathbb{o}'';(\_)\leftarrow v_s(\mathfrak{h});\bar{\mathbb{w}}'_E(\langle\langle \mathfrak{h} \rangle\rangle)}"] \arrow[ddd, bend right=60, "\mathbb{m}''(\mathfrak{h})"'] & v_s(\mathfrak{h})(\mathbb{o}'(\mathscr{a})) \arrow[l, "v_s(\mathfrak{h})\left(\left\langle \mathbb{t}'_{\mathfrak{a}_i} \right\rangle_{i=1}^{\ell(\mathscr{a})}\right)"'] \arrow[d, "\iota_{\mathbb{o}';(\_)\leftarrow v_s(\mathfrak{h});\bar{\mathbb{w}}'_E(\langle\langle \mathfrak{h} \rangle\rangle)}"'] \arrow[ddd, bend left=60, "\mathbb{m}'(\mathfrak{h})"] \\
\mathbb{G}(\mathbb{F}(\mathbb{o}''))(\mathfrak{b}) \arrow[d, "\mathbb{h}''_{\mathfrak{b}}"] & \mathbb{G}(\mathbb{F}(\mathbb{o}'))(\mathfrak{b}) \arrow[l, "\mathbb{G}(\mathbb{F}(\mathbb{t}'))_{\mathfrak{b}}"] \arrow[d, "\mathbb{h}'_{\mathfrak{b}}"'] \\
\mathbb{o}''(\mathfrak{b}) \arrow[d, "\mathrm{can}_{v_t(\mathfrak{h})\leftarrow(\_);\mathbb{o}''(\mathfrak{b})}"] & \mathbb{o}'(\mathfrak{b}) \arrow[l, "\mathbb{t}'_{\mathfrak{b}}"] \arrow[d, "\mathrm{can}_{v_t(\mathfrak{h})\leftarrow(\_);\mathbb{o}'(\mathfrak{b})}"'] \\
v_t(\mathfrak{h})(\mathbb{o}''(\mathfrak{b})) & v_t(\mathfrak{h})(\mathbb{o}'(\mathfrak{b})) \arrow[l, "v_t(\mathfrak{h})(\mathbb{t}'_{\mathfrak{b}})"]
\end{tikzcd}
\tag{97}
$$

In diagram (97) (in which $\mathfrak{h} \in A$ with $\mathfrak{s}(\mathfrak{h}) = \mathscr{a} = \langle \mathfrak{a}_i \rangle_{i=1}^{\ell(\mathscr{a})}$ and $\mathfrak{t}(\mathfrak{h}) = \langle \mathfrak{b} \rangle$) the two paths in the upper square have traces that are equal to those of the two paths that make up the square $\circlearrowleft$ in (95) for an appropriate selection of generators and for an appropriate choice of $\mathscr{c}$ and $\mathscr{b}$ in (95). Therefore this square commutes. The middle square commutes because $\mathbb{t}'$ is a morphism of algebras and the bottom square commutes by naturality of $\mathrm{can}_{v_t(\mathfrak{h})\leftarrow(\_)}$. This shows that if $\langle \mathbb{o}', \mathbb{m}' \rangle$ and $\langle \mathbb{o}'', \mathbb{m}'' \rangle$ turn out to satisfy the equation-shapes D, then $\mathbb{t}'$ is really a morphism of realizations.

To see that $\mathbb{m}'$ as defined in (96) satisfies the equation-shapes D we will show that $K(\langle \mathbb{o}', \mathbb{m}' \rangle) = K_{X,A,\mathfrak{s},\mathfrak{t},v_s,v_t,\mathbf{C}}(\langle \mathbb{o}', \mathbb{m}' \rangle)$ is defined by a diagram which has a similar shape as that in (96). This diagram is shown in (98).

$$
\begin{tikzcd}
 & K(\langle \mathbb{o}', \mathbb{m}' \rangle)(\mathscr{a}) \arrow[dl, "K(\langle \mathbb{o}', \mathbb{m}' \rangle)(\mathscr{h})"'] \arrow[d, "\iota_{\mathbb{o}';\bar{\mathbb{w}}_E(\mathscr{h})}"] \\
K(\langle \mathbb{o}', \mathbb{m}' \rangle)(\mathscr{b}) & K(\mathbb{F}(\mathbb{o}'))(\mathscr{b}) \arrow[l, "v^{(\ell(\mathscr{b}))}\left(\left\langle \mathbb{h}'_{\mathfrak{b}_i} \right\rangle_{i=1}^{\ell(\mathscr{b})}\right)"]
\end{tikzcd}
\tag{98}
$$

($\mathscr{h} : \mathscr{a} \to \mathscr{b} = \langle \mathfrak{b}_i \rangle_{i=1}^{\ell(\mathscr{b})}$ in $\tilde{\mathcal{E}}_E$.) From this it is then obvious that $K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(\mathscr{h})$ depends only on the equivalence class $\tilde{\mathbb{\pi}}_E(\mathscr{h})$ and therefore that $K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)$ factors through $\tilde{\mathbb{\pi}}_E$ and therfore by lemma 7.9 that $\langle \mathbb{o}', \mathbb{m}' \rangle$ satisfies the equation-shapes in D.

(96) is compatible with our convention of extending $\mathbb{m}'$ to X by sending all elements of X to the appropriate identities because $\mathbb{h}'$ satisfies $\mathbb{h}' \bullet \mathbb{\eta}_{\mathbb{o}'} = 1_{\mathbb{o}'}$. By applying $v^{(\ell(\mathscr{b}))}$ to multiple instances of (96) we see that (98) commutes for arrows $\mathscr{h} = \pi_{\approx}\left(\langle \langle \mathfrak{h}_i \rangle_{i=1}^{n} \rangle\right) \in \mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ which consist of a single row of arrow-atoms — this is shown in the diagram below, in which $\mathscr{s}(\mathfrak{h}_i) = \mathscr{a}_i$, $\mathscr{t}(\mathfrak{h}_i) = \langle \mathfrak{b}_i \rangle$ and $\mathscr{a} = \boxdot_{i=1}^{n}(\mathscr{a}_i)$, $\mathscr{b} = \langle \mathfrak{b}_i \rangle_{i=1}^{n}$.

$$
\begin{tikzcd}
 & K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(\mathscr{a}) \arrow[d, "\mathrm{can}_{\ldots}"] \arrow[dddl, bend right, "K(\langle \mathbb{o}', \mathbb{m}' \rangle)(\mathscr{h})"'] \arrow[ddd, bend left, "\iota_{\mathbb{o}'; \tilde{\mathbb{\pi}}_E(\mathscr{h})}"] \\
 & v^{(n)}\left(\langle v_s(\mathfrak{h}_i)(\mathbb{o}'(\mathscr{a}_i)) \rangle_{i=1}^{n}\right) \arrow[dl, "v^{(n)}(\langle \mathbb{m}'(\mathfrak{h}_i) \rangle_{i=1}^{n})"'] \arrow[dd, "v^{(n)}\left(\left\langle \iota_{\mathbb{o}'; (-) \leftarrow v_s(\mathfrak{h}_i); \tilde{\mathbb{\pi}}'_E(\langle \langle \mathfrak{h}_i \rangle \rangle)} \right\rangle_{i=1}^{n}\right)"] \\
v^{(n)}\left(\langle v_t(\mathfrak{h}_i)(\mathbb{o}'(\mathfrak{b}_i)) \rangle_{i=1}^{n}\right) \arrow[d, "\mathrm{can}_{\ldots}"] & \\
v^{(n)}\left(\langle \mathbb{o}'(\mathfrak{b}_i) \rangle_{i=1}^{n}\right) \arrow[d, equal] & v^{(n)}\left(\langle \mathbb{G}(\mathbb{F}(\mathbb{o}'))(\mathfrak{b}_i) \rangle_{i=1}^{n}\right) \arrow[l, "v^{(n)}\left(\left\langle \mathbb{h}'_{\mathfrak{b}_i} \right\rangle_{i=1}^{n}\right)"] \arrow[d, equal] \\
K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(\mathscr{b}) & K\left(\mathbb{F}(\mathbb{o}')\right)(\mathscr{b})
\end{tikzcd}
$$

Now we proceed by induction over the number of rows in a representation of $\mathscr{h}$. So assume that $\mathscr{h}$ and $\mathscr{g}$ are arrows of $\mathcal{E}(X, A, \mathfrak{s}, \mathfrak{t})$ with $t(\mathscr{g}) = \mathscr{c} = \langle \mathfrak{c}_i \rangle_{i=1}^{\ell(\mathscr{c})}$, $s(\mathscr{g}) = \mathscr{b} = \langle \mathfrak{b}_i \rangle_{i=1}^{\ell(\mathscr{b})} = t(\mathscr{h})$ and $s(\mathscr{h}) = \mathscr{a}$ such that (98) commutes for $\mathscr{h}$ and $\mathscr{g}$. Have a look at the diagram in (99). In this diagram the square marked with 1 is just the image under $v^{(\ell(\mathscr{c}))}\mathbf{C}$ of multiple instances of the left hand diagram in (62) which must hold for any Eilenberg-Moore algebra, and therefore 1 commutes. 2 is the square 0 from (95), so also commutes and commutativity of 3 is what we calculated in (94). The "triangles" marked with i commute by the induction hypothesis. In combination this shows that the outermost ovoid commutes and therefore completes the inductive step.

So now we know that both V and W are well-defined.

Comparing diagrams (92) and (96) we see that $W \circ V$ is the identity functor on objects. Comparing (98) and (93) we see that $\mathbb{h}'$ and $\mathbb{G}\left(\mathbb{\epsilon}_{W(\langle \mathbb{o}', \mathbb{h}' \rangle)}\right)$ satisfy the same equations. These equations uniquely determine them because $\langle \iota_{\mathbb{o}'; \mathscr{h}} \rangle_{\substack{\mathscr{h} \text{ in } \tilde{\mathcal{E}}_E \\ t(\mathscr{h}) = \mathscr{b}}}$ forms a colimiting cone to $\left(K\left(\mathbb{F}(\mathbb{o}')\right)\right)(\mathscr{b})$. Therefore $\mathbb{h}'$ and $\mathbb{G}\left(\mathbb{\epsilon}_{W(\langle \mathbb{o}', \mathbb{h}' \rangle)}\right)$ have to be equal. This shows that $V \circ W$ is the identity on objects. That both $W \circ V$ and $V \circ W$ are the identity on arrows is now trivially clear from the definition of W and V. $\square$

$$
\begin{tikzcd}
 & & & K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(a) \arrow[dl, "\iota_{\mathbb{o}';\bar{\mathbb{w}}_E(h)}"'] \arrow[dllldd, bend right, "\iota_{\mathbb{o}';\bar{\mathbb{w}}_E(g \circ h)}"'] \arrow[ddddl, bend left, "K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(h)"] \arrow[dddddlll, bend left=80, "K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(g \circ h)"] \\
 & & K(\mathbb{F}(\mathbb{o}'))(b) \arrow[dl, "\iota_{\mathbb{G}(\mathbb{F}(\mathbb{o}'));\bar{\mathbb{w}}_E(g)}"'] \arrow[ddd, "v^{(\ell(b))}\left(\left\langle \mathbb{h}'_{\mathfrak{b}_i} \right\rangle_{i=1}^{\ell(b)}\right)"] \arrow[r, phantom, "\mathrm{i}"] & \\
K\left(\mathbb{F}\left(\mathbb{o}'\right)\right)(c) \arrow[d, equal] & K\left(\mathbb{F}(\mathbb{G}(\mathbb{F}(\mathbb{o}')))\right)(c) \arrow[l, "\left(K\left(\mathbb{e}_{\mathbb{F}(\mathbb{o}')}\right)\right)_c"'] \arrow[d, equal] \arrow[ur, phantom, "3"] & & \\
v^{(\ell(c))}\left(\langle \mathbb{T}\left(\mathbb{o}'\right)(\mathfrak{c}_i) \rangle_{i=1}^{\ell(c)}\right) \arrow[d, "v^{(\ell(c))}\left(\left\langle \mathbb{h}'_{\mathfrak{c}_i} \right\rangle_{i=1}^{\ell(c)}\right)"] \arrow[dr, phantom, "1"] & v^{(\ell(c))}\left(\langle \mathbb{T}(\mathbb{T}(\mathbb{o}'))(\mathfrak{c}_i) \rangle_{i=1}^{\ell(c)}\right) \arrow[l, "v^{(\ell(c))}\left(\left\langle \left(\mathbb{G}\left(\mathbb{e}_{\mathbb{F}(\mathbb{o}')}\right)\right)_{\mathfrak{c}_i} \right\rangle_{i=1}^{\ell(c)}\right)"'] \arrow[d, "v^{(\ell(c))}\left(\left\langle \left(\mathbb{T}(\mathbb{h}')\right)_{\mathfrak{c}_i} \right\rangle_{i=1}^{\ell(c)}\right)"', "=\left(K\left(\mathbb{F}\left(\mathbb{h}'\right)\right)\right)_c"] \arrow[r, phantom, "2"] & & \\
v^{(\ell(c))}\left(\langle \mathbb{o}'(\mathfrak{c}_i) \rangle_{i=1}^{\ell(c)}\right) \arrow[d, equal] & v^{(\ell(c))}\left(\langle \mathbb{T}\left(\mathbb{o}'\right)(\mathfrak{c}_i) \rangle_{i=1}^{\ell(c)}\right) \arrow[l, "v^{(\ell(c))}\left(\left\langle \mathbb{h}'_{\mathfrak{c}_i} \right\rangle_{i=1}^{\ell(c)}\right)"] \arrow[d, equal] & K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(b) \arrow[dl, "\iota_{\mathbb{o}';\bar{\mathbb{w}}_E(g)}"] \arrow[dll, bend left, "K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(g)"] & \\
K\left(\langle \mathbb{o}', \mathbb{m}' \rangle\right)(c) & K\left(\mathbb{F}\left(\mathbb{o}'\right)\right)(c) \arrow[l, phantom, "\mathrm{i}"] & &
\end{tikzcd}
\tag{99}
$$

7.4 Putting it all together

As remarked at the beginning of section 4, each of the categories $\mathbf{Mon}_{\mathbf{C}}$, $\mathbf{Act}_{\mathbf{C}}$, **ʇɔA**$_{\mathbf{C}}$ and $\mathbf{BiAct}_{\mathbf{C}}$ can be regarded as a category of realizations satisfying some set D of equation-shapes, described by the diagrams given in section 2 as axioms for this category.

Now we know that these categories can in turn be regarded as categories of Eilenberg-Moore algebras (provided the necessary coproducts exist in $\mathbf{C}$ and that $a \boxtimes -$ and $- \boxtimes a$ preserve coproducts).

We immediately see (lemma 6.5) that the forgetful functor creates limits.

For this we do not really need the coproducts in $\mathbf{C}$ though — for any specific category of realizations satisfying some set of equation-shapes it is an easy thing to construct a proof which is very similar to the proof of lemma 6.5, which shows that the forgetful functor creates limits — independently from whether free objects exist or not. The only thing we really need are the conditions 1 and 2 from the statement of theorem 7.14. If we want to prove that the forgetful functor creates limits for an arbitrary category of realizations, then going through the theory of Eilenberg-Moore algebras makes things a little simpler, because it saves us from the technicality of having to deal with diagrams whose shape is not given explicitly.

But for colimits it's not as simple, because for example in the category of monoids in **Set** the forgetful functor does *not* create colimits. This is where the theory we have developed really becomes relevant.

There is one thing we still need to check though. We need to check that under appropriate assumptions the functor $\mathbb{T}$ just constructed satisfies the conditions of theorem 6.13 — specifically we want to show that

Lemma 7.17. *If the requirements of theorem 7.14 are met and for all $a \in \mathbf{C}$ the functors $a \boxtimes -$ and $- \boxtimes a$ preserve not only the coproducts required in theorem 7.14 but also reflexive coequalizers, then the functor $\mathbb{T}$ from theorem 7.16 preserves reflexive coequalizers.*

Note that if $\mathbf{C}$ has reflexive coequalizers, then the hypotheses above imply that $a \boxtimes -$ and $- \boxtimes a$ preserve all colimits of functors from categories which are small enough (because arbitrary colimits can be constructed from coproducts and reflexive coequalizers).

Proof. Building on diagram (95) we see that for $\dot{\mathbb{t}} : \dot{\mathbb{o}} \dot{\to} \dot{\mathbb{o}}'$ in $\mathbf{C}^X$ the value $\mathbb{T}(\dot{\mathbb{t}}) = \mathbb{G}(\mathbb{F}(\dot{\mathbb{t}}))$ of the arrow function of $\mathbb{T}$ at $\dot{\mathbb{t}}$ is given by the requirement that the diagram below commute for all $\mathcal{g}$ in $\tilde{\mathcal{E}}_E$ with $t\left(\mathcal{g}\right) = \langle \mathfrak{c} \rangle$ (where $s\left(\mathcal{g}\right) = \mathcal{b} = \langle \mathfrak{b} \rangle_{i=1}^{\ell(\mathcal{b})}$).

$$\begin{tikzcd}
v^{(\ell(\mathcal{b}))}\left(\dot{\mathbb{o}}'\left(\mathcal{b}\right)\right) \arrow[d, "\iota_{\dot{\mathbb{o}}';(_)\leftarrow v^{(\ell(\mathcal{b}))};\mathcal{g}}"'] & & v^{(\ell(\mathcal{b}))}\left(\dot{\mathbb{o}}\left(\mathcal{b}\right)\right) \arrow[ll, "v^{(\ell(\mathcal{b}))}\left(\langle \dot{\mathbb{t}}_{\mathfrak{b}_i} \rangle_{i=1}^{\ell(\mathcal{b})}\right)"'] \arrow[d, "\iota_{\dot{\mathbb{o}};(_)\leftarrow v^{(\ell(\mathcal{b}))};\mathcal{g}}"] \\
\left(\mathbb{G} \circ \mathbb{F}\right)\left(\dot{\mathbb{o}}'\right)\left(\mathfrak{c}\right) & & \left(\mathbb{G} \circ \mathbb{F}\right)\left(\dot{\mathbb{o}}\right)\left(\mathfrak{c}\right) \arrow[ll, "\left(\left(\mathbb{G}\circ\mathbb{F}\right)\left(\dot{\mathbb{t}}\right)\right)_{\mathfrak{c}}"]
\end{tikzcd} \tag{100}$$

This means that not only the object function of $\mathbb{T}$ but also the arrow function of $\mathbb{T}$ is given by a coproduct. More specifically, $\mathbb{T}$ can be written as the composite below.

$$\begin{array}{c}
\mathbf{C}^X \\
\Big\downarrow - \circ \varpi \\
\mathbf{C}^I \\
\wr\| \\
\displaystyle\prod_{\mathfrak{c} \in X} \prod_{\substack{g \text{ in } \tilde{\mathcal{E}}_E \\ t(g) = \langle \mathfrak{c} \rangle}} \prod_{i=1}^{\ell(s(g))} \mathbf{C} \\
\Big\downarrow \displaystyle\prod_{\mathfrak{c} \in X} \prod_{\substack{g \text{ in } \tilde{\mathcal{E}}_E \\ t(g) = \langle \mathfrak{c} \rangle}} v_{\mathbf{C}}^{(\ell(s(g)))} \\
\displaystyle\prod_{\mathfrak{c} \in X} \prod_{\substack{g \text{ in } \tilde{\mathcal{E}}_E \\ t(g) = \langle \mathfrak{c} \rangle}} \mathbf{C} \\
\Big\downarrow \displaystyle\prod_{\mathfrak{c} \in X} \left(\coprod\nolimits^{\mathbf{C}}_{J_{\mathfrak{a}}} \right) \\
\displaystyle\prod_{\mathfrak{c} \in X} \mathbf{C} \\
\wr\| \\
\mathbf{C}^X
\end{array}$$

The product which appears as the third category from the top is isomorphic to a functor category $\mathbf{C}^I$, where I is the discrete category

$$I := \left\{ \langle \mathfrak{a}, g, i \rangle \in X \times \mathrm{Arr}\left(\tilde{\mathcal{E}}_E\right) \times \mathbb{N} \middle| t(g) = \langle \mathfrak{a} \rangle \text{ and } 1 \leq i \leq \ell(s(g)) \right\} .$$

The functor $- \circ \varpi$ is the functor which works on objects and morphisms of $\mathbf{C}^X$ (remember that these are functors and natural transformations respectively) by prepending the functor $\varpi : I \to X$ (which is really only a function) given by

$$\varpi(\langle \mathfrak{a}, g, i \rangle) := \pi_i (s(g))$$

(and thereby giving a functor / natural transformation from I to $\mathbf{C}$). The functor $- \circ \varpi$ preserves colimits because colimits in functor categories can be calculated pointwise (see for example [Mac98, section V.3]).

We know from lemma 5.8 that under our hypotheses $v_{\mathbf{C}}^{(n)}$ preserves reflexive coequalizers. Again, because colimits in functor categories can be calculated pointwise we get that $\prod_{\mathfrak{c} \in X} \prod_{\substack{g \text{ in } \tilde{\mathcal{E}}_E \\ t(g) = \langle \mathfrak{c} \rangle}} v_{\mathbf{C}}^{(\ell(s(g)))}$ preserves reflexive coequalizers.

The functor which we called $\coprod^{\mathbf{C}}_J : \prod_{j \in J} \mathbf{C} \to \mathbf{C}$ above is the functor which sends a collection of objects indexed by the set J to their coproduct (this functor is usually not unique but any one will do). $J_{\mathfrak{a}}$ (for $\mathfrak{a} \in X$) is the set $\left\{ g \in \mathrm{Arr}\left(\tilde{\mathcal{E}}_E\right) \middle| t(g) = \langle \mathfrak{a} \rangle \right\}$. Using first lemma 5.1 with $\mathbf{I} = J_{\mathfrak{a}}$ and $\mathbf{J} = \mathbf{J_r}$ and then lemma 5.2 with $\mathbf{I} = \mathbf{J_r}$ and $\mathbf{J} = J_{\mathfrak{a}}$ we see that $\coprod^{\mathbf{C}}_{J_{\mathfrak{a}}}$ preserves reflexive coequalizers. Using that colimits can be calculated pointwise one more time we get that $\prod_{\mathfrak{c}\in X} \coprod_{J_{\mathfrak{a}}}^{\mathbf{C}}$ preserves reflexive coequalizers. And of course a composite of functors which preserve reflexive coequalizers also preserves reflexive coequalizers. $\square$

Corollary 7.18. *If* $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ *is a cocomplete monoidal category and both* $a \boxtimes -$ *and* $- \boxtimes a$ *preserve colimits for all* $a \in \mathbf{C}$ *and if*

1. $\ell(\mathfrak{t}(\mathfrak{f})) = 1$ *for all* $\mathfrak{f} \in A$ *and*
2. $\ell(t(p)) = \ell(t(q)) = 1$ *for all* $\langle p, q \rangle \in \mathrm{Tr}(D)$,

then

$$\mathbf{Real}(X, A, \mathfrak{s}, \mathfrak{t}, v_s, v_t, D, \langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle)$$

is cocomplete.

8 The tensor product over a monoid and serial composition of automata

In this chapter we define the tensor product of biacts over a monoid and the operation of serial composition for automata. We define a functor which encodes automata in biacts and show that under this encoding serial composition runs parallel to taking the tensor product over a monoid.

8.1 Supporting structures for acts on the level of categories

Earlier we mentioned that the category of biacts can be arrived at in an iterated fashion by taking the category of right acts in the category of left acts in some monoidal category, or the other way round — provided that we have used appropriate definitions for right and left acts.

As mentioned earlier, if we want this to work out, then clearly the monoid which acts and the object being acted on need to belong to different categories, because the object being acted on has some additional structure — namely the action from the other side — while the monoid generally doesn't. This means we need to think about bifunctors $\mathbin{\scriptstyle\boxtimes} : \mathbf{C} \times \mathbf{D} \to \mathbf{D}$ where $\mathbf{C}$ contains the monoid $\langle m, \mu, \eta \rangle$ and $\mathbf{D}$ contains the object a being acted on. To be able to say that $\langle m, \mu, \eta \rangle$ is a monoid, $\mathbf{C}$ has to be a monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$. It also makes sense to require some kind of associativity between $\boxtimes$ and $\mathbin{\scriptstyle\boxtimes}$. This leads to the following concept.

Definition 8.1. Let $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be a monoidal category, let $\mathbf{D}$ be a category and let $\mathbin{\scriptstyle\boxtimes} : \mathbf{C} \times \mathbf{D} \to \mathbf{D}$ be a bifunctor. Further let $\hat{\alpha}_{a,b,d} : a \mathbin{\scriptstyle\boxtimes} (b \mathbin{\scriptstyle\boxtimes} d) \to (a \boxtimes b) \mathbin{\scriptstyle\boxtimes} d$ and $\hat{\lambda}_d : e \mathbin{\scriptstyle\boxtimes} d \to d$ (where $a, b \in \mathbf{C}$ and $d \in \mathbf{D}$) be natural transformations such that the following diagrams commute for all $a, b, c \in \mathbf{C}$, $d \in \mathbf{D}$.

$$\begin{tikzcd}
((a \boxtimes b) \boxtimes c) \mathbin{\scriptstyle\boxtimes} d & (a \boxtimes b) \mathbin{\scriptstyle\boxtimes} (c \mathbin{\scriptstyle\boxtimes} d) \arrow[l, "\hat{\alpha}_{a\boxtimes b,c,d}"'] & a \mathbin{\scriptstyle\boxtimes} (b \mathbin{\scriptstyle\boxtimes} (c \mathbin{\scriptstyle\boxtimes} d)) \arrow[l, "\hat{\alpha}_{a,b,c\mathbin{\scriptstyle\boxtimes} d}"'] \arrow[d, "1_a \mathbin{\scriptstyle\boxtimes} \hat{\alpha}_{b,c,d}"] \\
(a \boxtimes (b \boxtimes c)) \mathbin{\scriptstyle\boxtimes} d \arrow[u, "\alpha_{a,b,c} \mathbin{\scriptstyle\boxtimes} 1_d"] & & a \mathbin{\scriptstyle\boxtimes} ((b \boxtimes c) \mathbin{\scriptstyle\boxtimes} d) \arrow[ll, "\hat{\alpha}_{a,b\boxtimes c,d}"]
\end{tikzcd} \tag{101}$$

$$\begin{tikzcd}
(a \boxtimes e) \mathbin{\scriptstyle\boxtimes} d \arrow[dr, "\rho_a \mathbin{\scriptstyle\boxtimes} 1_d"'] & & a \mathbin{\scriptstyle\boxtimes} (e \mathbin{\scriptstyle\boxtimes} d) \arrow[ll, "\hat{\alpha}_{a,e,d}"'] \arrow[dl, "1_a \mathbin{\scriptstyle\boxtimes} \hat{\lambda}_d"] \\
& a \mathbin{\scriptstyle\boxtimes} d &
\end{tikzcd} \tag{102}$$

Then we say that $\left\langle \mathbin{\scriptstyle\boxtimes}, \hat{\alpha}, \hat{\lambda} \right\rangle$ is a left (tensor) action of the monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ on $\mathbf{D}$.

Again there is also the symmetric concept of a right action of a monoidal category, which is the same thing as a left action of the monoidally opposite category.

8.2 Acts revisited

We can now generalize definitions 2.4 and 2.5.

Definition 8.2. Let $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be a monoidal category with an action $\left\langle \boxtimes, \hat{\alpha}, \hat{\lambda} \right\rangle$ on a category $\mathbf{D}$. Let $\langle m, \mu, \eta \rangle$ be a monoid in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ and let $a \in \mathbf{D}$. A left action of $\langle m, \mu, \eta \rangle$ on a is a morphism $h : m \boxtimes a \to a$ of $\mathbf{D}$ such that the following diagrams commute.

$$\begin{tikzcd}
m \boxtimes a \arrow[d, "h"'] & (m \boxtimes m) \boxtimes a \arrow[l, "\mu \boxtimes 1_a"'] & m \boxtimes (m \boxtimes a) \arrow[l, "\hat{\alpha}_{m,m,a}"'] \arrow[d, "1_m \boxtimes h"] \\
a & & m \boxtimes a \arrow[ll, "h"]
\end{tikzcd}
\qquad
\begin{tikzcd}
m \boxtimes a \arrow[d, "h"'] & e \boxtimes a \arrow[l, "\eta \boxtimes 1_a"'] \arrow[dl, "\hat{\lambda}_a"] \\
a &
\end{tikzcd} \tag{103}$$

Again we call the pair $\langle h, a \rangle$ a left $\langle m, \mu, \eta \rangle$-act in $\mathbf{D}$, or just a *left (monoid) act*. Flipping the sides gives the definition of a right $\langle m, \mu, \eta \rangle$-act.

Definition 8.3. The category $\langle m, \mu, \eta \rangle - \mathbf{Act}_{\mathbf{D}}$ of left $\langle m, \mu, \eta \rangle$-acts in $\mathbf{D}$ has objects all left $\langle m, \mu, \eta \rangle$-acts in $\mathbf{D}$ and if $\langle h, a \rangle, \langle h', a' \rangle$ are left $\langle m, \mu, \eta \rangle$-acts in $\mathbf{D}$ then $f : \langle h, a \rangle \to \langle h', a' \rangle$ is a morphism of left $\langle m, \mu, \eta \rangle$-acts if $f : a \to a'$ and

$$\begin{tikzcd}
m \boxtimes a' \arrow[d, "h'"] & m \boxtimes a \arrow[l, "1_m \boxtimes f"] \arrow[d, "h"] \\
a' & a \arrow[l, "f"]
\end{tikzcd} \tag{104}$$

commutes.

Definition 8.4. Let $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ be a monoidal category. Let $\langle m, \mu, \eta \rangle$, $\langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$ be monoids in $\mathbf{C}$. We extend the monoidal product $\boxtimes$ to a functor

$$\overset{m\dot{m}}{\boxtimes} : \langle m, \mu, \eta \rangle - \mathbf{Act} \times \mathbf{Act} - \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle \to \langle m, \mu, \eta \rangle - \mathbf{Act} - \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$$

by the definitions

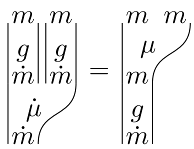
$$\begin{aligned}
\langle h, a \rangle \overset{m\dot{m}}{\boxtimes} \langle b, k \rangle &:= \left\langle \begin{array}{ccc} m & (a & b) \\ & h & 1 \\ & a & b \end{array}, a \boxtimes b, \begin{array}{ccc} (a & b) & m \\ 1 & k & \\ a & b & \end{array} \right\rangle \\
&\quad \text{for } \langle h, a \rangle \in \langle m, \mu, \eta \rangle - \mathbf{Act}\ ,\ \langle b, k \rangle \in \mathbf{Act} - \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle \\
f \overset{m\dot{m}}{\boxtimes} g &:= \begin{cases} f \boxtimes g : \left\langle \begin{array}{ccc} m & (a & b) \\ & h & 1 \\ & a & b \end{array}, a \boxtimes b, \begin{array}{ccc} (a & b) & \dot{m} \\ 1 & k & \\ a & b & \end{array} \right\rangle \\ \qquad \to \left\langle \begin{array}{ccc} m & (a' & b') \\ & h' & 1 \\ & a' & b' \end{array}, a' \boxtimes b', \begin{array}{ccc} (a' & b') & \dot{m} \\ 1 & k' & \\ a' & b' & \end{array} \right\rangle \end{cases} \\
&\quad \text{for morphisms } f : \langle h, a \rangle \to \langle h', a' \rangle \text{ in } \langle m, \mu, \eta \rangle - \mathbf{Act} \text{ and} \\
&\qquad\qquad g : \langle b, k \rangle \to \langle b', k' \rangle \text{ in } \mathbf{Act} - \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle\ .
\end{aligned} \tag{105}$$

(Remember the notation introduced at the end of chapter 3 on coherence.)

Lemma 8.5. *This is well-defined.*

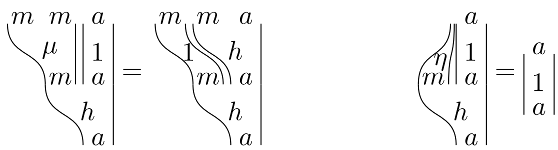
For reference let us first restate the defining diagrams for our algebraic structures in the visual calculus developed in chapter 3. For monoids we have already done this in (50) and (51) of section 4.2. Here's the rest.

Definition 2.2. $g : \langle m, \mu, \eta\rangle \to \langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$ is a morphism of monoids if (diagram (7)):



(106)

Definition 2.4. $\langle h, a\rangle$ is a left $\langle m, \mu, \eta\rangle$-act if (diagram (8)):

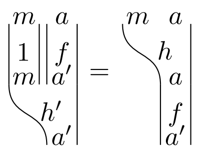


(107)

$\langle b, k\rangle$ is a right $\langle m, \mu, \eta\rangle$-act if:

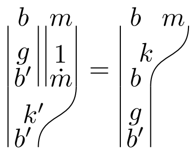
(108)

Definition 2.5. $f : \langle h, a\rangle \to \langle h', a'\rangle$ is a morphism of left $\langle m, \mu, \eta\rangle$-acts if (diagram (9)):



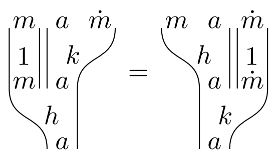
(109)

$g : \langle b, k\rangle \to \langle b', k'\rangle$ is a morphism of right $\langle m, \mu, \eta\rangle$-acts if:



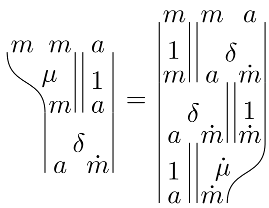
(110)

Definition 2.7. $\langle h, a, k\rangle$ is a $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-biact if $\langle h, a\rangle$ is a left $\langle m, \mu, \eta\rangle$-act, $\langle a, k\rangle$ is a right $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-act and if (diagram (11)):

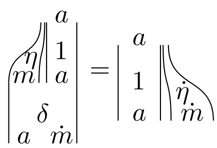


(111)

Definition 2.10. $\langle a, \delta\rangle$ is an $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-automaton if (diagrams (12) and (13)):

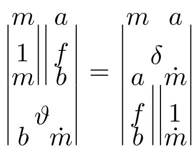


(112)



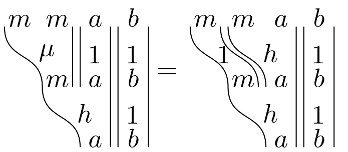
(113)

Definition 2.11. $f : \langle a, \delta\rangle \to \langle b, \vartheta\rangle$ is a morphism of $\langle m, \mu, \eta\rangle$-$\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-automata if (diagram (14)):

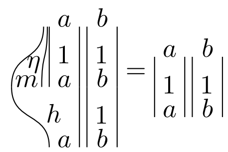


(114)

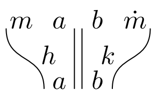
Proof of lemma 8.5. We have to show that $\langle h, a\rangle \overset{m\dot{m}}{\boxtimes} \langle b, k\rangle$ is a biact, that is, that



and

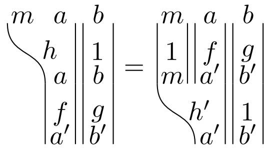


— which obviously follows from (107) —, that the symmetric equations hold for the right action — which is also clear — and that the actions commute — which is also clear because after a shift both ways of composing the actions give the picture



—;

and we have to show that $f \boxtimes g$ commutes with both actions if f is a morphism of left $\langle m, \mu, \eta\rangle$-acts and g is a morphism of right $\langle \dot{m}, \dot{\mu}, \dot{\eta}\rangle$-acts.



(115)

by (109) and a shift. The other side is again symmetric. $\square$

The following will sometimes allow us to skip a few case distinctions.

Observation 8.6. *Let $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ be a monoidal category.*
$\mathbf{e} := \langle e, \lambda_e = \rho_e, 1_e\rangle$ *is a monoid and*

$$\begin{gathered}\mathbf{e}{-}\mathbf{Act} \cong \mathbf{Act}{-}\mathbf{e} \cong \mathbf{e}{-}\mathbf{Act}{-}\mathbf{e} \cong \mathbf{C}\\ \langle m, \mu, \eta\rangle{-}\mathbf{Act}{-}\mathbf{e} \cong \langle m, \mu, \eta\rangle{-}\mathbf{Act}\\ \mathbf{e}{-}\mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle \cong \mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle\end{gathered}$$

where $\cong$ denotes isomorphism of categories.

This means that definition 8.4 also gives functors

$$\begin{array}{cccccc}
\overset{\dot m}{\boxtimes} & : & \mathbf{C} & \times\mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle & \to & \mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle\\
 & & \wr\Vert & \Vert & & \wr\Vert\\
\overset{\mathbf{e}\dot m}{\bowtie} & : & \mathbf{e}{-}\mathbf{Act} & \times\mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle & \to & \mathbf{e}{-}\mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle\\
 & & & & & \\
\overset{m}{\boxtimes} & : & \mathbf{Act}{-}\left\langle m, \mu, \eta\right\rangle\times & \mathbf{C} & \to & \langle m, \mu, \eta\rangle{-}\mathbf{Act}\\
 & & \Vert & \wr\Vert & & \wr\Vert\\
\overset{m\mathbf{e}}{\bowtie} & : & \mathbf{Act}{-}\left\langle m, \mu, \eta\right\rangle\times & \mathbf{e}{-}\mathbf{Act} & \to & \langle m, \mu, \eta\rangle{-}\mathbf{Act}{-}\mathbf{e}
\end{array}$$

by composition with these isomorphisms in the appropriate places.

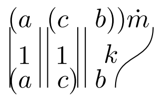
Lemma 8.7. $\left\langle \overset{\dot m}{\boxtimes}, \overset{\dot m}{\hat\alpha}, \overset{\dot m}{\tilde\lambda}\right\rangle$ *is a left tensor action of $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ on $\mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle$, where $\overset{\dot m}{\hat\alpha}$ is just α regarded as a morphism of $\mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle$ and $\overset{\dot m}{\tilde\lambda}$ is just λ regarded as a morphism of $\mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle$. This means that we are also stating that α and λ can be regarded as morphisms of $\mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle$.*

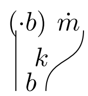
If $\langle h, a\rangle \in \langle m, \mu, \eta\rangle{-}\mathbf{Act}$, $\langle b, k\rangle \in \mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle$, then $\alpha_{a,c,b}$ can also be regarded as a morphism

$$\overset{m\dot m}{\hat\alpha}_{\langle h,a\rangle,c,\langle b,k\rangle} : \langle h, a\rangle \overset{m\dot m}{\bowtie} \left(c \overset{\dot m}{\boxtimes} \langle b, k\rangle\right) \to \left(\langle h, a\rangle \overset{m}{\boxtimes} c\right) \overset{m\dot m}{\bowtie} \langle b, k\rangle$$

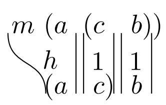
of biacts. $\overset{m\dot m}{\hat\alpha}$ is natural.

Proof. In our calculus both composites of $\alpha_{a,c,b}$ with the right action give the left hand picture below and both composites of lambda with the right action give the right hand picture below.





Both composites of $\alpha_{a,c,b}$ with the left action give the picture below.



The forgetful functors from the categories $\mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle$ and $\langle m, \mu, \eta\rangle{-}\mathbf{Act}{-}\left\langle \dot{m}, \dot{\mu}, \dot{\eta}\right\rangle$ to $\mathbf{C}$ are obviously faithful and therefore commutativity of the diagrams for α and λ implies commutativity of the diagrams (101) and (102) for $\overset{\dot m}{\hat\alpha}$ and $\overset{\dot m}{\tilde\lambda}$, and naturality of α and λ implies naturality of $\overset{\dot m}{\hat\alpha}$, $\overset{m\dot m}{\hat\alpha}$ and $\overset{\dot m}{\tilde\lambda}$. $\square$

Lemma 8.8.

$$\begin{aligned}\langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\,\langle\dot m,\dot\mu,\dot\eta\rangle &\cong \langle m,\mu,\eta\rangle{-}\mathbf{Act}_{\mathbf{Act}-\langle\dot m,\dot\mu,\dot\eta\rangle} \\ &\cong {}_{\langle m,\mu,\eta\rangle-\mathbf{Act}}\mathbf{Act}{-}\,\langle\dot m,\dot\mu,\dot\eta\rangle\end{aligned} \tag{116}$$

If $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ is a monoidal category and $\langle m,\mu,\eta\rangle, \langle\dot m,\dot\mu,\dot\eta\rangle$ are monoids in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ then the category of $\langle m,\mu,\eta\rangle$-$\langle\dot m,\dot\mu,\dot\eta\rangle$-biacts is isomorphic to the category of left $\langle m,\mu,\eta\rangle$-acts in the category of right $\langle\dot m,\dot\mu,\dot\eta\rangle$-acts in $\mathbf{C}$ (where the action of $\mathbf{C}$ on $\mathbf{Act}-\langle\dot m,\dot\mu,\dot\eta\rangle$ is the one from the previous lemma) and also to the category of right $\langle\dot m,\dot\mu,\dot\eta\rangle$-acts in the category of left $\langle m,\mu,\eta\rangle$-acts in $\mathbf{C}$.

Proof. Just expand the definitions. □

In the sequel we will silently identify all of the categories in (116) and assume that the reader can mentally insert the appropriate isomorphisms in the appropriate places.

Note that observation 6.2 is still true in its essence if instead of just a monoidal category we have an action $\left\langle \boxtimes, \hat\alpha, \hat\lambda\right\rangle$ of a monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$ on some other category $\mathbf{D}$. In this case the monad $\left\langle \dot T, \dot\mu, \dot\eta\right\rangle$ is given by

$$\begin{aligned}
\dot T &: \mathbf{D} \to \mathbf{D}\\
\dot T &:= m \boxtimes -\\
\dot\mu &: \dot T \circ \dot T \dot\to \dot T : \mathbf{D} \to \mathbf{D}\\
\dot\mu_d &: m \boxtimes (m \boxtimes d) \to m \boxtimes d\\
\dot\mu_d &= (\mu \boxtimes 1_d) \circ \hat\alpha_{m,m,d}\\
\dot\eta &: 1_{\mathbf{D}} \dot\to \dot T : \mathbf{D} \to \mathbf{D}\\
\dot\eta_d &: d \to m \boxtimes d\\
\dot\eta_d &= (\eta \boxtimes 1_d) \circ \hat\lambda_d^{-1}\ .
\end{aligned}$$

Of course this works symmetrically for the other side.

So with the notation

$$\begin{aligned}
{}_mT &:= m \boxtimes - : \mathbf{C} \to \mathbf{C}\\
T_{\dot m} &:= - \boxtimes \dot m : \mathbf{C} \to \mathbf{C}\\
{}_m\overset{\dot m}{T} &:= m \overset{\dot m}{\boxtimes} - \ : \mathbf{Act}{-}\,\langle\dot m,\dot\mu,\dot\eta\rangle \to \mathbf{Act}{-}\,\langle\dot m,\dot\mu,\dot\eta\rangle\\
\overset{m}{T}_{\dot m} &:= - \overset{m}{\boxtimes} \dot m \ : \langle m,\mu,\eta\rangle{-}\mathbf{Act} \to \langle m,\mu,\eta\rangle{-}\mathbf{Act}
\end{aligned} \tag{117}$$

we can say that if $\langle m,\mu,\eta\rangle$ and $\langle\dot m,\dot\mu,\dot\eta\rangle$ are monoids in $\mathbf{C}$, then for all of the above functors there is a monad which they are a part of and

Observation 8.9.

$$\langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\,\langle\dot m,\dot\mu,\dot\eta\rangle \cong \left(\mathbf{C}^{T_{\dot m}}\right)^{{}_m\overset{\dot m}{T}} \cong \left(\mathbf{C}^{{}_mT}\right)^{\overset{m}{T}_{\dot m}}\ .$$

Let us name the forgetful functors.

$$\begin{array}{llllll}
{}_mG := G^{{}_mT} & : \langle m,\mu,\eta\rangle{-}\mathbf{Act} & & \to & \mathbf{C} & \\
G_{\dot m} := G^{T_{\dot m}} & : & \mathbf{Act}{-}\left\langle \dot m,\dot\mu,\dot\eta\right\rangle & \to & \mathbf{C} & \\
{}^mG_{\dot m} := G^{{}^mT_{\dot m}} & : \langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\left\langle \dot m,\dot\mu,\dot\eta\right\rangle & & \to & \langle m,\mu,\eta\rangle{-}\mathbf{Act} & \\
{}_mG^{\dot m} := G^{{}_mT^{\dot m}} & : \langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\left\langle \dot m,\dot\mu,\dot\eta\right\rangle & & \to & \mathbf{Act}{-}\left\langle \dot m,\dot\mu,\dot\eta\right\rangle & \\
{}_mG_{\dot m} & : \langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\left\langle \dot m,\dot\mu,\dot\eta\right\rangle & & \to & \mathbf{C} &
\end{array} \tag{118}$$

Observation 8.10.

$$\begin{aligned}
G_{\dot m}\circ {}_mG^{\dot m} &= {}_mG\circ {}^mG_{\dot m} = {}_mG_{\dot m}\\
G_{\dot m}\circ {}_mT^{\dot m} &= {}_mT\circ G_{\dot m}\\
{}_mG\circ {}^mT_{\dot m} &= T_{\dot m}\circ {}_mG
\end{aligned}$$

Now we can combine all this with lemma 6.6.

Lemma 8.11. *If $L:\mathbf{I}\to\langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\left\langle \dot m,\dot\mu,\dot\eta\right\rangle$ is a functor, ${}_mG_{\dot m}\circ L$ has a colimit and both ${}_mT$ and $T_{\dot m}$ preserve colimits of $T\circ{}_mG_{\dot m}\circ L$ for all T which are a — possibly empty — composite of ${}_mT$s and $T_{\dot m}$s, then ${}_mG^{\dot m}$ and ${}^mG_{\dot m}$ create colimits of L.*

Proof. We show that ${}_mG^{\dot m}$ creates colimits for L. The proof for ${}^mG_{\dot m}$ is symmetric.

Lemma 6.6 shows that in this situation $G_{\dot m}$ creates colimits of ${}_mG^{\dot m}\circ L$ and therefore if ${}_mG_{\dot m}\circ L = G_{\dot m}\circ{}_mG^{\dot m}\circ L$ has a colimit, then ${}_mG^{\dot m}\circ L$ has a colimit $\langle l,\tau\rangle$ and this colimit is preserved by $G_{\dot m}$.

Because ${}_mT$ preserves colimits of $G_{\dot m}\circ{}_mG^{\dot m}\circ L$ and because ${}_mT\circ G_{\dot m} = G_{\dot m}\circ{}_mT^{\dot m}$ we get that $\left\langle \left(G_{\dot m}\circ{}_mT^{\dot m}\right)(l), G_{\dot m}\circ{}_mT^{\dot m}\circ\tau\right\rangle$ is a colimit of $G_{\dot m}\circ{}_mT^{\dot m}\circ{}_mG^{\dot m}\circ L$.

Because $T_{\dot m}$ and $T_{\dot m}\circ T_{\dot m}$ preserve colimits of $G_{\dot m}\circ{}_mT^{\dot m}\circ{}_mG^{\dot m}\circ L = {}_mT\circ G_{\dot m}\circ{}_mG^{\dot m}\circ L$ we get from lemma 6.6 that $G_{\dot m}$ creates colimits of ${}_mT^{\dot m}\circ{}_mG^{\dot m}\circ L$ and therefore there is a unique cone $\langle l',\tau'\rangle$ which is mapped to $\left\langle \left(G_{\dot m}\circ{}_mT^{\dot m}\right)(l), G_{\dot m}\circ{}_mT^{\dot m}\circ\tau\right\rangle$ under $G_{\dot m}$ and this cone is a colimit of ${}_mT^{\dot m}\circ{}_mG^{\dot m}\circ L$. But the cone $\left\langle {}_mT^{\dot m}(l), {}_mT^{\dot m}\circ\tau\right\rangle$ is mapped to $\left\langle \left(G_{\dot m}\circ{}_mT^{\dot m}\right)(l), G_{\dot m}\circ{}_mT^{\dot m}\circ\tau\right\rangle$ under $G_{\dot m}$. Therefore $\left\langle {}_mT^{\dot m}(l), {}_mT^{\dot m}\circ\tau\right\rangle$ is a colimit of ${}_mT^{\dot m}\circ{}_mG^{\dot m}\circ L$ — that is ${}_mT^{\dot m}$ preserves the colimit $\langle l,\tau\rangle$.

Substituting ${}_mT^{\dot m}\circ{}_mT^{\dot m}$ for ${}_mT^{\dot m}$ and ${}_mT\circ{}_mT$ for ${}_mT$ everywhere above we see that ${}_mT^{\dot m}\circ{}_mT^{\dot m}$ also preserves this colimit. (Where we use that $G_{\dot m}\circ{}_mT^{\dot m}\circ{}_mT^{\dot m} = {}_mT\circ G_{\dot m}\circ{}_mT^{\dot m} = {}_mT\circ{}_mT\circ G_{\dot m}$.)

By remark 6.7 ${}_mT^{\dot m}$ and ${}_mT^{\dot m}\circ{}_mT^{\dot m}$ preserve all colimits of ${}_mG^{\dot m}\circ L$ and again by lemma 6.6 we get that ${}_mG^{\dot m}$ creates colimits of L. □

Corollary 8.12. *If $m\boxtimes -$ and $-\boxtimes\dot m$ preserve colimits of $T\circ{}_mG_{\dot m}\circ L$ (for all T as in the lemma above), then ${}_mG_{\dot m}$ creates colimits of L.*

Proof. This is just lemma 6.6 combined with the previous lemma. □

We can now define the tensor product of acts over some monoid.

Definition 8.13. Let $\langle m,\mu,\eta\rangle, \langle \dot m,\dot\mu,\dot\eta\rangle, \langle \ddot m,\ddot\mu,\ddot\eta\rangle$ be monoids in a monoidal category $\langle \mathbf{C},\boxtimes,e,\alpha,\lambda,\rho\rangle$.

Let $\langle h : m\boxtimes a \to a,\ \ a,\ \ \dot k : a\boxtimes \dot m \to a\rangle$ be a $\langle m,\mu,\eta\rangle$-$\langle \dot m,\dot\mu,\dot\eta\rangle$-biact and let $\langle \dot h : \dot m\boxtimes b \to b,\ \ b,\ \ \ddot k : b\boxtimes \ddot m \to b\rangle$ be a $\langle \dot m,\dot\mu,\dot\eta\rangle$-$\langle \ddot m,\ddot\mu,\ddot\eta\rangle$-biact.

The *tensor product*

$$\langle h,a,\dot k\rangle {}^{m}\textcircled{\dot m}{}^{\ddot m} \langle \dot h,b,\ddot k\rangle$$

of $\langle h,a,\dot k\rangle$ *and* $\langle \dot h,b,\ddot k\rangle$ *over* $\langle \dot m,\dot\mu,\dot\eta\rangle$ is an object of $\langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\langle \ddot m,\ddot\mu,\ddot\eta\rangle$ and is defined as the coequalizer of the two arrows $\left(\dot k \overset{m\ddot m}{\boxtimes} 1_{\langle b,\ddot k\rangle}\right) \circ \overset{m\ddot m}{\alpha}_{\langle h,a\rangle,\dot m,\langle b,\ddot k\rangle}$ and $1_{\langle h,a\rangle} \overset{m\ddot m}{\boxtimes} \dot h$ (shown in (119)), if that coequalizer exists. By corollary 8.12 this is definitely the case if the base category has coequalizers and if $m\boxtimes -$ and $-\boxtimes \dot m$ preserve coequalizers.

$$\begin{tikzcd}
\left(\langle h,a\rangle \overset{m}{\boxtimes} \dot m\right) \overset{m\ddot m}{\boxtimes} \langle b,\ddot k\rangle \arrow[d, "\dot k \overset{m\ddot m}{\boxtimes} 1_{\langle b,\ddot k\rangle}"] & \langle h,a\rangle \overset{m\ddot m}{\boxtimes} \left(\dot m \overset{\ddot m}{\boxtimes} \langle b,\ddot k\rangle\right) \arrow[l, "\overset{m\ddot m}{\alpha}_{\langle h,a\rangle,\dot m,\langle b,\ddot k\rangle}"'] \arrow[d, "1_{\langle h,a\rangle} \overset{m\ddot m}{\boxtimes} \dot h"] \\
\langle h,a\rangle \overset{m\ddot m}{\boxtimes} \langle b,\ddot k\rangle & \langle h,a\rangle \overset{m\ddot m}{\boxtimes} \langle b,\ddot k\rangle
\end{tikzcd} \tag{119}$$

Lemma 8.14. *When the tensor product over a monoid, as defined above, exists everywhere, then it is a functor*

$$\begin{aligned} {}^{m}\textcircled{\dot m}{}^{\ddot m} : \langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\langle \dot m,\dot\mu,\dot\eta\rangle \times \langle \dot m,\dot\mu,\dot\eta\rangle{-}\mathbf{Act}{-}\langle \ddot m,\ddot\mu,\ddot\eta\rangle \to \\ \langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\langle \ddot m,\ddot\mu,\ddot\eta\rangle \end{aligned} \tag{120}$$

Or, to be more exact, there is a unique way of extending the definition above to a functor in such a way that the collection of coequalizers turns into a natural transformation from $\overset{m\ddot m}{\boxtimes} \circ \left(\overset{m}{G}_{\dot m} \times {}_{\dot m}\overset{\ddot m}{G}\right)$ *to that functor.*

Proof. We show that the tensor product over a monoid as defined in 8.13 is a limit with a parameter — that is, that there is a trifunctor

$$\begin{aligned} L : J_t \times \langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\langle \dot m,\dot\mu,\dot\eta\rangle \times \langle \dot m,\dot\mu,\dot\eta\rangle{-}\mathbf{Act}{-}\langle \ddot m,\ddot\mu,\ddot\eta\rangle \to \\ \langle m,\mu,\eta\rangle{-}\mathbf{Act}{-}\langle \ddot m,\ddot\mu,\ddot\eta\rangle \end{aligned}$$

such that $\langle h,a,\dot k\rangle {}^{m}\textcircled{\dot m}{}^{\ddot m} \langle \dot h,b,\ddot k\rangle$ is the colimit of $L\left(-,\langle h,a,\dot k\rangle,\langle \dot h,b,\ddot k\rangle\right)$. From this the statement of the theorem follows. (As mentioned in lemma 5.1, if this last part does not seem clear to to reader they can find this discussed in [Mac98, section V.3.].)

Definition 8.15. The category J_t is the category depicted below.

$$\begin{tikzcd}
\mathscr{a}' \arrow[d, "\mathscr{f}"] & \mathscr{a} \arrow[l, "\mathscr{h}"] \arrow[d, "\mathscr{g}"] \\
\mathscr{b} & \mathscr{b}
\end{tikzcd} \tag{121}$$

The category J_t' is the category depicted above sans the arrow $\mathscr{h}$.

We define two functors

$$L_l : J_t' \to \langle m,\mu,\eta\rangle\text{–}\mathbf{Act}^{\langle m,\mu,\eta\rangle\text{–}\mathbf{Act}\text{–}\langle \dot m,\dot\mu,\dot\eta\rangle}$$

$$L_r : J_t' \to \mathbf{Act}\text{–}\,\langle \ddot m,\ddot\mu,\ddot\eta\rangle^{\langle \dot m,\dot\mu,\dot\eta\rangle\text{–}\mathbf{Act}\text{–}\langle \ddot m,\ddot\mu,\ddot\eta\rangle}$$

$$\begin{aligned}
L_l(\mathscr{a}) &:= \overset{m}{G}_{\dot m} & L_r(\mathscr{a}) &:= {}_{\dot m}\overset{\ddot m}{T} \circ {}_{\dot m}\overset{\ddot m}{G} \\
L_l(\mathscr{a}') &:= \overset{m}{T}_{\dot m} \circ \overset{m}{G}_{\dot m} & L_r(\mathscr{a}') &:= {}_{\dot m}\overset{\ddot m}{G} \\
L_l(\mathscr{b}) &:= \overset{m}{G}_{\dot m} & L_r(\mathscr{b}) &:= {}_{\dot m}\overset{\ddot m}{G} \\
L_l(\mathscr{f}) &:= \overset{m}{G}_{\dot m} \circ \overset{m}{\varepsilon}_{\dot m} & L_r(\mathscr{f}) &:= 1_{{}_{\dot m}\overset{\ddot m}{G}} \\
L_l(\mathscr{g}) &:= 1_{\overset{m}{G}_{\dot m}} & L_r(\mathscr{g}) &:= {}_{\dot m}\overset{\ddot m}{G} \circ {}_{\dot m}\overset{\ddot m}{\varepsilon}
\end{aligned} \tag{122}$$

where $\overset{m}{\varepsilon}_{\dot m}$ is the counit of the adjunction of which $\overset{m}{G}_{\dot m}$ is the right component and ${}_{\dot m}\overset{\ddot m}{\varepsilon}$ is the counit of the adjunction of which ${}_{\dot m}\overset{\ddot m}{G}$ is the right component — that is $\overset{m}{G}_{\dot m}\left(\overset{m}{\varepsilon}_{\dot m}\right)$ sends any $\langle m,\mu,\eta\rangle$-$\langle \dot m,\dot\mu,\dot\eta\rangle$-biact to its right action considered as a morphism of left $\langle m,\mu,\eta\rangle$-acts and ${}_{\dot m}\overset{\ddot m}{G}\left({}_{\dot m}\overset{\ddot m}{\varepsilon}\right)$ sends any $\langle \dot m,\dot\mu,\dot\eta\rangle$-$\langle \ddot m,\ddot\mu,\ddot\eta\rangle$-biact to its left action considered as a morphism of right $\langle \ddot m,\ddot\mu,\ddot\eta\rangle$-acts.

Define L' as the composite

$$\begin{array}{c}
J_t' \times \langle m,\mu,\eta\rangle\text{–}\mathbf{Act}\text{–}\,\langle \dot m,\dot\mu,\dot\eta\rangle \times \langle \dot m,\dot\mu,\dot\eta\rangle\text{–}\mathbf{Act}\text{–}\,\langle \ddot m,\ddot\mu,\ddot\eta\rangle \\
\Big\downarrow {\scriptstyle \Delta_{J_t'}\times 1\times 1} \\
J_t' \times J_t' \times \langle m,\mu,\eta\rangle\text{–}\mathbf{Act}\text{–}\,\langle \dot m,\dot\mu,\dot\eta\rangle \times \langle \dot m,\dot\mu,\dot\eta\rangle\text{–}\mathbf{Act}\text{–}\,\langle \ddot m,\ddot\mu,\ddot\eta\rangle \\
\Big\downarrow \text{shuffle} \\
J_t' \times \langle m,\mu,\eta\rangle\text{–}\mathbf{Act}\text{–}\,\langle \dot m,\dot\mu,\dot\eta\rangle \times J_t' \times \langle \dot m,\dot\mu,\dot\eta\rangle\text{–}\mathbf{Act}\text{–}\,\langle \ddot m,\ddot\mu,\ddot\eta\rangle \\
\Big\downarrow {\scriptstyle L_l\times 1\times L_r\times 1} \\
\langle m,\mu,\eta\rangle\text{–}\mathbf{Act}^{\langle m,\mu,\eta\rangle\text{–}\mathbf{Act}\text{–}\langle \dot m,\dot\mu,\dot\eta\rangle} \times \langle m,\mu,\eta\rangle\text{–}\mathbf{Act}\text{–}\,\langle \dot m,\dot\mu,\dot\eta\rangle \times \\
\mathbf{Act}\text{–}\,\langle \ddot m,\ddot\mu,\ddot\eta\rangle^{\langle \dot m,\dot\mu,\dot\eta\rangle\text{–}\mathbf{Act}\text{–}\langle \ddot m,\ddot\mu,\ddot\eta\rangle} \times \langle \dot m,\dot\mu,\dot\eta\rangle\text{–}\mathbf{Act}\text{–}\,\langle \ddot m,\ddot\mu,\ddot\eta\rangle \\
\Big\downarrow {\scriptstyle E_{\langle m,\mu,\eta\rangle\text{–}\mathbf{Act}^{\langle m,\mu,\eta\rangle\text{–}\mathbf{Act}\text{–}\langle \dot m,\dot\mu,\dot\eta\rangle}} \times E_{\langle \ddot m,\ddot\mu,\ddot\eta\rangle\text{–}\mathbf{Act}^{\langle \dot m,\dot\mu,\dot\eta\rangle\text{–}\mathbf{Act}\text{–}\langle \ddot m,\ddot\mu,\ddot\eta\rangle}}} \\
\langle m,\mu,\eta\rangle\text{–}\mathbf{Act} \times \mathbf{Act}\text{–}\,\langle \ddot m,\ddot\mu,\ddot\eta\rangle \\
\Big\downarrow {\scriptstyle \overset{m\ddot m}{\bowtie}} \\
\langle m,\mu,\eta\rangle\text{–}\mathbf{Act}\text{–}\,\langle \ddot m,\ddot\mu,\ddot\eta\rangle
\end{array} \tag{123}$$

where $\Delta_{J_t'} : J_t' \to J_t' \times J_t'$ is the functor $f \mapsto \langle f, f\rangle$; the second functor (labelled "shuffle") is the obvious functor which reorders the components of its argument and which has the specified source and target; and what we have called $E_{\langle m,\mu,\eta\rangle-\mathbf{Act}^{\langle m,\mu,\eta\rangle-\mathbf{Act}-\langle \dot m,\dot\mu,\dot\eta\rangle}}$ and $E_{\langle \ddot m,\ddot\mu,\ddot\eta\rangle-\mathbf{Act}^{\langle \dot m,\dot\mu,\dot\eta\rangle-\mathbf{Act}-\langle \ddot m,\ddot\mu,\ddot\eta\rangle}}$ are the evaluation functors.

Define

$$
\begin{aligned}
L : J_t \times \langle m,\mu,\eta\rangle-\mathbf{Act}-\langle \dot m,\dot\mu,\dot\eta\rangle \times \langle \dot m,\dot\mu,\dot\eta\rangle-\mathbf{Act}-\langle \ddot m,\ddot\mu,\ddot\eta\rangle \to& \\
\langle m,\mu,\eta\rangle-\mathbf{Act}-\langle \ddot m,\ddot\mu,\ddot\eta\rangle&
\end{aligned}
$$

as the functor which is equal to L' on

$$J_t \times \langle m,\mu,\eta\rangle-\mathbf{Act}-\langle \dot m,\dot\mu,\dot\eta\rangle \times \langle \dot m,\dot\mu,\dot\eta\rangle-\mathbf{Act}-\langle \ddot m,\ddot\mu,\ddot\eta\rangle$$

and which is defined on the arrows

$$\langle \mathscr{h}, f, g\rangle : \langle \mathscr{a}, \langle h, a, \dot k\rangle, \langle \dot h, b, \ddot k\rangle\rangle \to \langle \mathscr{a}', \langle h', a', \dot k'\rangle, \langle \dot h', b', \ddot k'\rangle\rangle$$

as

$$
\begin{aligned}
L(\mathscr{h}, f, g) &: \langle h, a\rangle \overset{m\ddot m}{⧖} \left(\dot m \overset{\ddot m}{\boxtimes} \langle b, \ddot k\rangle\right) \to \left(\langle h', a'\rangle \overset{m}{\boxtimes} \dot m\right) \overset{m\ddot m}{⧖} \langle b', \ddot k'\rangle \\
L(\mathscr{h}, f, g) &:= \left(\left(\overset{m}{G}_{\dot m}(f) \overset{m}{\boxtimes} 1_{\dot m}\right) \overset{m\ddot m}{⧖} {}_{\dot m}\overset{\ddot m}{G}(g)\right) \circ \overset{m\ddot m}{\hat\alpha}_{\langle h,a\rangle,\dot m,\langle b,\ddot k\rangle} \\
&= \overset{m\ddot m}{\hat\alpha}_{\langle h',a'\rangle,\dot m,\langle b',\ddot k'\rangle} \circ \left(\overset{m}{G}_{\dot m}(f) \overset{m\ddot m}{⧖} \left(1_{\dot m} \overset{\ddot m}{\boxtimes} {}_{\dot m}\overset{\ddot m}{G}(g)\right)\right)
\end{aligned}
$$

(where $\overset{m\ddot m}{\hat\alpha}$ is the natural transformation from lemma 8.7). Because of the structure of J_t there is exactly one functor which satisfies these requirements. $\square$

Definition 8.16. Let $\langle m,\mu,\eta\rangle$, $\langle \dot m,\dot\mu,\dot\eta\rangle$, $\langle \ddot m,\ddot\mu,\ddot\eta\rangle$ be monoids in some monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$. Define a functor

$$
\begin{aligned}
\overset{m\ \ddot m}{\underset{\dot m}{\supset}} : \langle m,\mu,\eta\rangle-\mathbf{Aut}-\langle \dot m,\dot\mu,\dot\eta\rangle \times \langle \dot m,\dot\mu,\dot\eta\rangle-\mathbf{Aut}-\langle \ddot m,\ddot\mu,\ddot\eta\rangle \to& \\
\langle m,\mu,\eta\rangle-\mathbf{Aut}-\langle \ddot m,\ddot\mu,\ddot\eta\rangle& \qquad (124)
\end{aligned}
$$

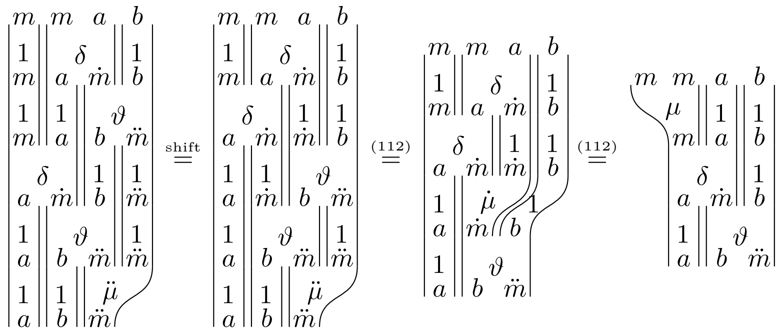
$$
\langle a, \delta\rangle \overset{m\ \ddot m}{\underset{\dot m}{\supset}} \langle b, \vartheta\rangle := \left\langle a \boxtimes b, \left|\begin{array}{ccc} m\ (a & & b) \\ & \delta & 1 \\ a & \dot m & b \\ 1 & & \vartheta \\ (a & b) & \ddot m \end{array}\right| \right\rangle
$$

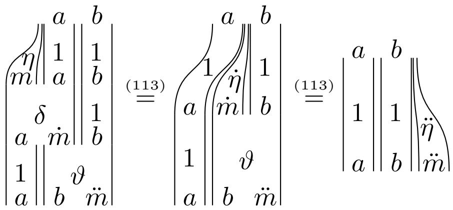
$$f \overset{m\ \ddot m}{\underset{\dot m}{\supset}} g := f \boxtimes g \text{ for } f : \langle a, \delta\rangle \to \langle a', \delta'\rangle \text{ and } g : \langle b, \vartheta\rangle \to \langle b', \vartheta'\rangle\ .$$

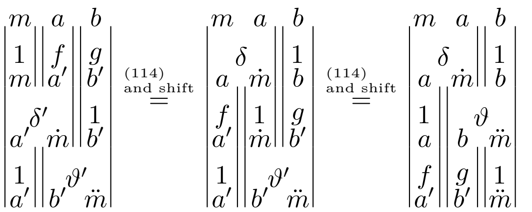
We call $\langle a, \delta\rangle \overset{m\ \ddot m}{\underset{\dot m}{\supset}} \langle b, \vartheta\rangle$ the serial composition of $\langle a, \delta\rangle$ and $\langle b, \vartheta\rangle$.

Lemma 8.17. *This is really a functor.*

Proof. We have to check that the structure defined is an automaton and that $f \boxtimes g$ is a morphism of automata. Compatibility with composition of morphisms and with identities is then obvious from the definitions.







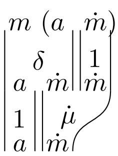
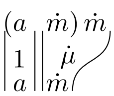
$\square$

We can use biacts to represent automata.

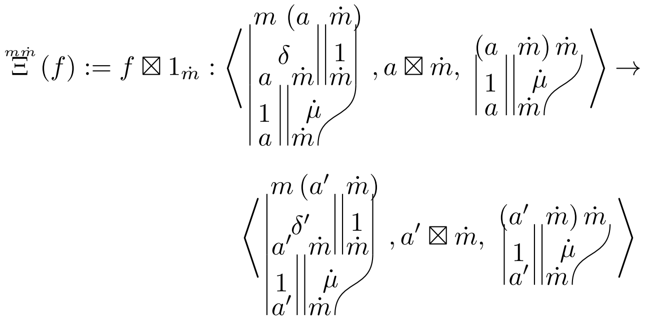
Definition 8.18. Let $\langle m, \mu, \eta\rangle, \langle \dot m, \dot\mu, \dot\eta\rangle$ be monoids in a monoidal category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho\rangle$. Define a functor

$$\overset{m\dot m}{\Xi} : \langle m, \mu, \eta\rangle - \mathbf{Aut} - \langle \dot m, \dot\mu, \dot\eta\rangle \to \langle m, \mu, \eta\rangle - \mathbf{Act} - \langle \dot m, \dot\mu, \dot\eta\rangle$$

by

$\overset{m\dot m}{\Xi}(\langle a, \delta\rangle) := \Big\langle$  $, a \boxtimes \dot m,$  $\Big\rangle$

for $\langle a, \delta\rangle \in \langle m, \mu, \eta\rangle - \mathbf{Aut} - \langle \dot m, \dot\mu, \dot\eta\rangle$

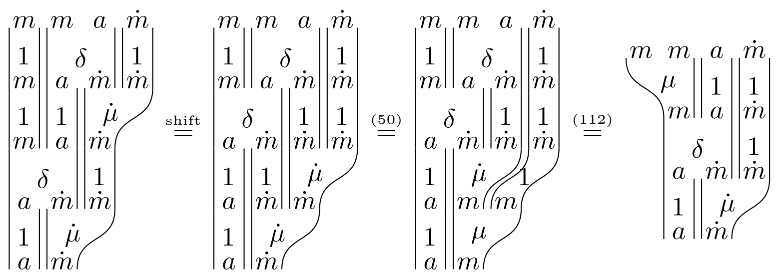


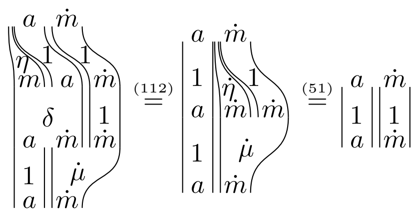
for $f : \langle a, \delta \rangle \to \langle a', \delta' \rangle$ in $\langle m, \mu, \eta \rangle$–**Aut**– $\langle \dot m, \dot\mu, \dot\eta \rangle$

Lemma 8.19. *This makes sense.*

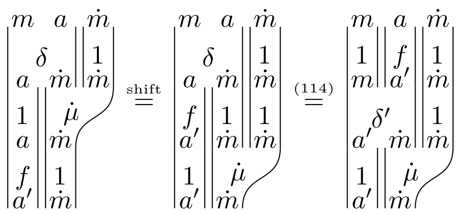
Proof. We need to show that $\overset{m\dot m}{\Xi}(\langle a, \delta \rangle)$ is a biact and that $\overset{m\dot m}{\Xi}(f)$ is a morphism of biacts. Compatibility with composition and identities is again obvious from the definitions.

Left action:

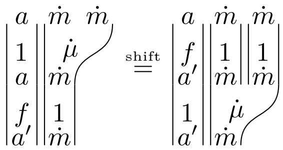




That $(1 \boxdot \dot\mu) : (a \boxtimes \dot m) \boxtimes \dot m \to a \boxtimes \dot m$ is a right action follows directly from the unit and associativity axioms for the monoid $\langle \dot m, \dot\mu, \dot\eta \rangle$. $\overset{m\dot m}{\Xi}(f)$ is a morphism of left acts:



$\overset{m\dot m}{\Xi}(f)$ is a morphism of right acts:



$\square$

Theorem 8.20. *The functors*

$$\left(\overset{m\dot m}{\Xi}(-)\right) {}^{m}\!\odot_{\dot m}^{\ddot m} \left(\overset{\dot m\ddot m}{\Xi}(-)\right)$$

and

$$\overset{m\ddot m}{\Xi}\left(- \;{}^{m}\!\supset_{\dot m}^{\ddot m}\; -\right)$$

are naturally isomorphic.

The tensor product $\left(\overset{m\dot m}{\Xi}(\langle a,\delta\rangle)\right) {}^{m}\!\odot_{\dot m}^{\ddot m} \left(\overset{\dot m\ddot m}{\Xi}(\langle b,\vartheta\rangle)\right)$ *over* $\langle \dot m,\dot\mu,\dot\eta\rangle$ *exists for all*

$$\langle a,\delta\rangle \in \langle m,\mu,\eta\rangle\text{–}\mathbf{Aut}\text{–}\langle \dot m,\dot\mu,\dot\eta\rangle, \quad \langle b,\vartheta\rangle \in \langle \dot m,\dot\mu,\dot\eta\rangle\text{–}\mathbf{Aut}\text{–}\langle \ddot m,\ddot\mu,\ddot\eta\rangle\ .$$

Proof. We prove this by showing that for every pair of objects

$$\langle a,\delta\rangle \in \langle m,\mu,\eta\rangle\text{–}\mathbf{Aut}\text{–}\langle \dot m,\dot\mu,\dot\eta\rangle, \langle b,\vartheta\rangle \in \langle \dot m,\dot\mu,\dot\eta\rangle\text{–}\mathbf{Aut}\text{–}\langle \ddot m,\ddot\mu,\ddot\eta\rangle$$

there is a colimiting cone from $L\left(-,\overset{m\dot m}{\Xi}(\langle a,\delta\rangle),\overset{\dot m\ddot m}{\Xi}(\langle b,\vartheta\rangle)\right)$ (L is defined in the proof of lemma 8.14) to $\overset{m\ddot m}{\Xi}\left(\langle a,\delta\rangle \;{}^{m}\!\supset_{\dot m}^{\ddot m}\; \langle b,\vartheta\rangle\right)$ and that the collection of these colimiting cones is a natural transformation ξ' from L to the functor

$$\langle j,\langle a,\delta\rangle,\langle b,\vartheta\rangle\rangle \mapsto \overset{m\ddot m}{\Xi}\left(\langle a,\delta\rangle \;{}^{m}\!\supset_{\dot m}^{\ddot m}\; \langle b,\vartheta\rangle\right)\ .$$

By the usual type of argument making use of the universal properties, this implies that the objects $\left(\overset{m\dot m}{\Xi}(\langle a,\delta\rangle)\right) {}^{m}\!\odot_{\dot m}^{\ddot m} \left(\overset{\dot m\ddot m}{\Xi}(\langle b,\vartheta\rangle)\right)$ and $\overset{m\ddot m}{\Xi}\left(\langle a,\delta\rangle \;{}^{m}\!\supset_{\dot m}^{\ddot m}\; \langle b,\vartheta\rangle\right)$ are isomorphic and that these isomorphisms form a natural transformation. (And clearly the tensor product exists because we have found at least one coequalizer of the required kind.)

Because of the shape of the category J_t this means finding for all

$$\langle a,\delta\rangle \in \langle m,\mu,\eta\rangle\text{–}\mathbf{Aut}\text{–}\langle \dot m,\dot\mu,\dot\eta\rangle, \langle b,\vartheta\rangle \in \langle \dot m,\dot\mu,\dot\eta\rangle\text{–}\mathbf{Aut}\text{–}\langle \ddot m,\ddot\mu,\ddot\eta\rangle$$

a morphism

$$\begin{aligned}\xi_{\langle a,\delta\rangle,\langle b,\vartheta\rangle} = {\xi'}_{\langle a,\delta\rangle,\langle b,\vartheta\rangle;\mathscr{l}} : \left({}^{m}G_{\dot m}\left(\overset{m\dot m}{\Xi}(\langle a,\delta\rangle)\right)\right) \overset{m\ddot m}{\bowtie} \left({}_{\dot m}G^{\ddot m}\left(\overset{\dot m\ddot m}{\Xi}(\langle b,\vartheta\rangle)\right)\right) \to \\ \overset{m\ddot m}{\Xi}\left(\langle a,\delta\rangle \;{}^{m}\!\supset_{\dot m}^{\ddot m}\; \langle b,\vartheta\rangle\right)\end{aligned}$$

of biacts which is the coequalizer of the two paths in (119) when we substitute the appropriate actions from definition 8.18 and for which the diagram in (125)

$$\begin{tikzcd}
 & \left(\overset{m}{G}_{\dot m}\left(\overset{m\dot m}{\Xi}\left(\langle a,\delta\rangle\right)\right)\right)\overset{m\dot m\ddot m}{\boxtimes}\left({}_{\dot m}\overset{\ddot m}{G}\left(\overset{\dot m\ddot m}{\Xi}\left(\langle b,\vartheta\rangle\right)\right)\right) \arrow[dl, "\left(\overset{m}{G}_{\dot m}\left(\overset{m\dot m}{\Xi}(f)\right)\right)\overset{m\dot m\ddot m}{\boxtimes}\left({}_{\dot m}\overset{\ddot m}{G}\left(\overset{\dot m\ddot m}{\Xi}(g)\right)\right)"'] \arrow[d, "\xi_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}"] \\
\left(\overset{m}{G}_{\dot m}\left(\overset{m\dot m}{\Xi}\left(\langle a',\delta'\rangle\right)\right)\right)\overset{m\dot m\ddot m}{\boxtimes}\left({}_{\dot m}\overset{\ddot m}{G}\left(\overset{\dot m\ddot m}{\Xi}\left(\langle b',\vartheta'\rangle\right)\right)\right) \arrow[d, "\xi_{\langle a',\delta'\rangle,\langle b',\vartheta'\rangle}"'] & \overset{m\ddot m}{\Xi}\left(\langle a,\delta\rangle\overset{m\;\ddot m}{\underset{\dot m}{\mathcal{D}}}\langle b,\vartheta\rangle\right) \arrow[dl, "\overset{m\ddot m}{\Xi}\left(f\overset{m\;\ddot m}{\underset{\dot m}{\mathcal{D}}}g\right)"] \\
\overset{m\ddot m}{\Xi}\left(\langle a',\delta'\rangle\overset{m\;\ddot m}{\underset{\dot m}{\mathcal{D}}}\langle b',\vartheta'\rangle\right) &
\end{tikzcd}$$

(125)

commutes. Using that all the other arrows in the colimiting cone are composites of $\xi_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}$ with arrows $L\left(f,1_{\overset{m\dot m}{\Xi}(\langle a,\delta\rangle)},1_{\overset{\dot m\ddot m}{\Xi}(\langle b,\vartheta\rangle)}\right)$ (for f in J_t) and that L is a functor, one easily sees that this implies naturality of ξ'.

To show that $\xi_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}$ is a coequalizer we will show that

$$\underset{m\;\ddot m}{G}\left(\xi_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}\right)$$

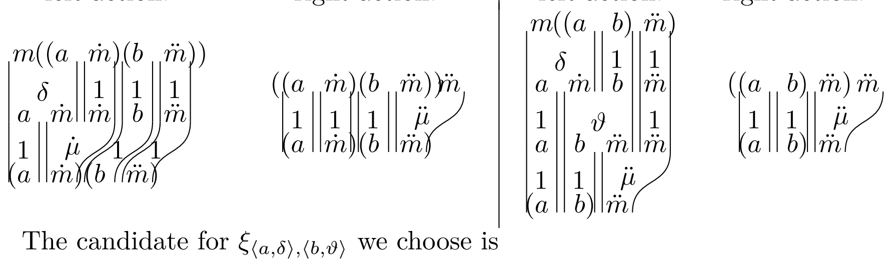
is a split coequalizer and therefore an absolute coequalizer. Then it follows from corollary 8.12 that $\xi_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}$ is a coequalizer.

So we will need the actions on $\left(\overset{m}{G}_{\dot m}\left(\overset{m\dot m}{\Xi}\left(\langle a,\delta\rangle\right)\right)\right)\overset{m\dot m\ddot m}{\boxtimes}\left({}_{\dot m}\overset{\ddot m}{G}\left(\overset{\dot m\ddot m}{\Xi}\left(\langle b,\vartheta\rangle\right)\right)\right)$ and $\overset{m\ddot m}{\Xi}\left(\langle a,\delta\rangle\overset{m\;\ddot m}{\underset{\dot m}{\mathcal{D}}}\langle b,\vartheta\rangle\right)$ as well as a candidate for $\xi_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}$.

Substituting in the definitions we get:

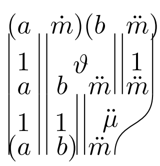
$\left(\overset{m}{G}_{\dot m}\left(\overset{m\dot m}{\Xi}\left(\langle a,\delta\rangle\right)\right)\right)\overset{m\dot m\ddot m}{\boxtimes}\left({}_{\dot m}\overset{\ddot m}{G}\left(\overset{\dot m\ddot m}{\Xi}\left(\langle b,\vartheta\rangle\right)\right)\right)$: | $\overset{m\ddot m}{\Xi}\left(\langle a,\delta\rangle\overset{m\;\ddot m}{\underset{\dot m}{\mathcal{D}}}\langle b,\vartheta\rangle\right)$:

left action: right action: | left action: right action:



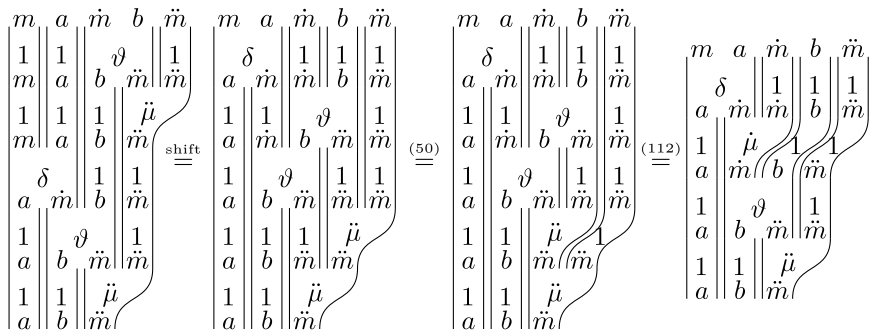
The candidate for $\xi_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}$ we choose is

$$\underset{m\;\ddot m}{G}\left(\xi_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}\right)\overset{\text{to be seen}}{=}\hat{\xi}_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}:=$$

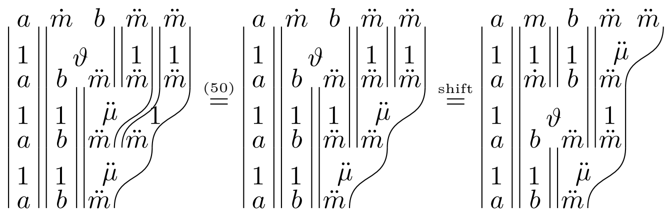


. (126)

We need to show that $\hat{\xi}_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}$ is compatible with the actions from the left.



And that it is compatible with the actions from the right.

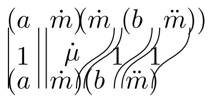


Now we show that $\hat{\xi}_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}$ is part of a split fork in such a way that lemma 6.11 will show that $\hat{\xi}_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}$ is an absolute coequalizer. Using the names from definition 6.9 we have.

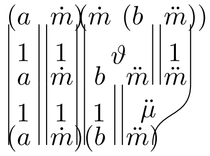
$$e = \hat{\xi}_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}$$

$$f = \underset{m\,\ddot{m}}{G}\left(\left(\overset{m}{\underset{\dot{m}}{G}}\left(\overset{m}{\varepsilon}_{\dot{m}}\overset{m\dot{m}}{\Xi}(\langle a,\delta\rangle)\right)\overset{m\ddot{m}}{\boxtimes}1\ \underset{\dot{m}}{\overset{\ddot{m}}{G}}\left(\overset{\dot{m}\ddot{m}}{\Xi}(\langle b,\vartheta\rangle)\right)\right)\circ\hat{\alpha}^{m\ddot{m}}_{m\dot{m}}{}_{\overset{m\dot{m}}{\Xi}(\langle a,\delta\rangle),\dot{m},\overset{\dot{m}\ddot{m}}{\Xi}(\langle b,\vartheta\rangle)}\right)$$

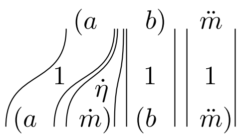
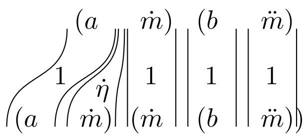
$$=$$



$$g = \underset{m\,\ddot{m}}{G}\left(1_{\overset{m}{\underset{\dot{m}}{G}}\left(\overset{m\dot{m}}{\Xi}(\langle a,\delta\rangle)\right)}\overset{m\ddot{m}}{\boxtimes}\underset{m}{\overset{\ddot{m}}{G}}\left(\underset{m}{\overset{\dot{m}}{\varepsilon}}\overset{\dot{m}\ddot{m}}{\Xi}(\langle b,\vartheta\rangle)\right)\right) =$$

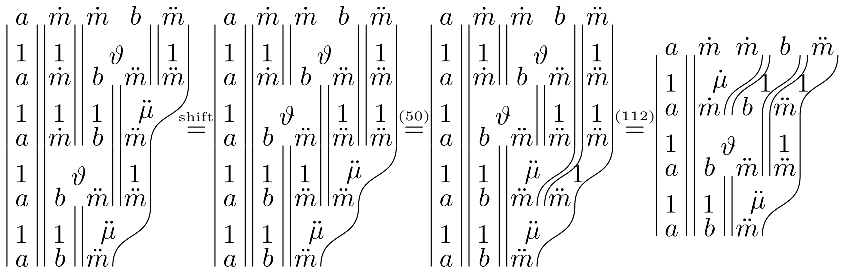


Up to permutation of f and g it was clear what f and g had to be because we want to show that e is a coequalizer of the two arrows above — it will turn out that this choice of permutation is the one that works. For the remaining arrows we choose:

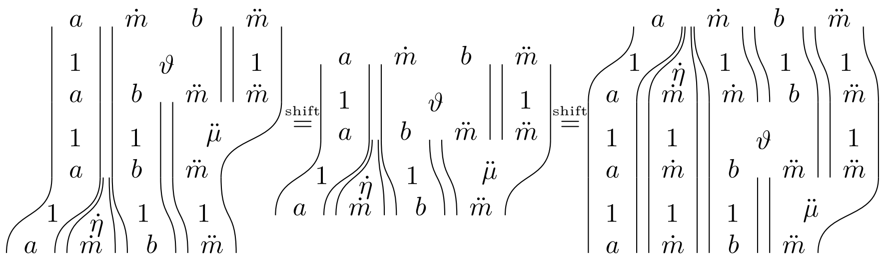
$h :=$  $k :=$ 

We need to check that all of the equations in definition 6.9 hold.

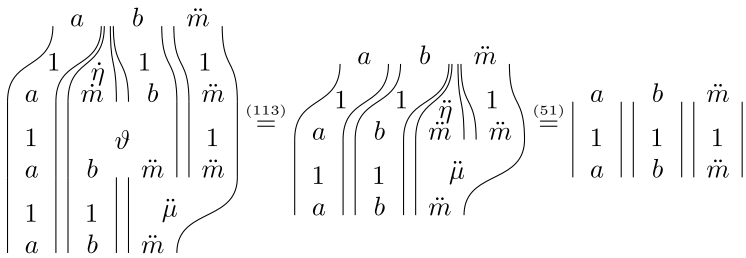
- $e \circ g = e \circ f$:



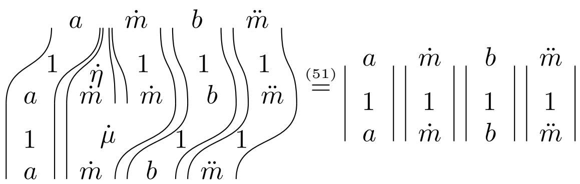
- $h \circ e = g \circ k$:



- $e \circ h = 1$:



- $f \circ k = 1$:



We still have to show that (125) commutes. So let $f : \langle a, \delta \rangle \to \langle a', \delta' \rangle$ and

$g : \langle b, \vartheta \rangle \to \langle b', \vartheta' \rangle$ be morphisms of automata. We have

$$\underset{m\ \ddot{m}}{G}\left(\xi_{\langle a',\delta'\rangle,\langle b',\vartheta'\rangle}\circ\left(\left(\overset{m}{\underset{\dot{m}}{G}}\left(\overset{m\dot{m}}{\Xi}(f)\right)\right)\overset{m\ddot{m}}{\bowtie}\left(\underset{\dot{m}}{\overset{\ddot{m}}{G}}\left(\overset{\dot{m}\ddot{m}}{\Xi}(g)\right)\right)\right)\right) =$$

$$\begin{array}{|c|c|c|c|}
a & \dot{m} & b & \ddot{m} \\
\genfrac{}{}{0pt}{}{f}{a'} & \genfrac{}{}{0pt}{}{1}{\dot{m}} & \genfrac{}{}{0pt}{}{g}{b'} & \genfrac{}{}{0pt}{}{1}{\ddot{m}} \\
\genfrac{}{}{0pt}{}{1}{a'} & \multicolumn{2}{c|}{\genfrac{}{}{0pt}{}{\vartheta'}{b'\ \ \ddot{m}}} & \genfrac{}{}{0pt}{}{1}{\ddot{m}} \\
\genfrac{}{}{0pt}{}{1}{a'} & \genfrac{}{}{0pt}{}{1}{b'} & \multicolumn{2}{c}{\genfrac{}{}{0pt}{}{\ddot{\mu}}{\ddot{m}}}
\end{array}
\overset{\substack{(114)\\ \text{for } g}}{=}
\begin{array}{|c|c|c|c|}
a & \dot{m} & b & \ddot{m} \\
\genfrac{}{}{0pt}{}{f}{a'} & \multicolumn{2}{c|}{\genfrac{}{}{0pt}{}{\vartheta}{b\ \ \ddot{m}}} & \genfrac{}{}{0pt}{}{1}{\ddot{m}} \\
\genfrac{}{}{0pt}{}{1}{a'} & \genfrac{}{}{0pt}{}{g}{b'} & \genfrac{}{}{0pt}{}{1}{\ddot{m}} & \genfrac{}{}{0pt}{}{1}{\ddot{m}} \\
\genfrac{}{}{0pt}{}{1}{a'} & \genfrac{}{}{0pt}{}{1}{b'} & \multicolumn{2}{c}{\genfrac{}{}{0pt}{}{\ddot{\mu}}{\ddot{m}}}
\end{array}
\overset{\text{shift}}{=}
\begin{array}{|c|c|c|c|}
a & \dot{m} & b & \ddot{m} \\
\genfrac{}{}{0pt}{}{1}{a} & \multicolumn{2}{c|}{\genfrac{}{}{0pt}{}{\vartheta}{b\ \ \ddot{m}}} & \genfrac{}{}{0pt}{}{1}{\ddot{m}} \\
\genfrac{}{}{0pt}{}{1}{a} & \genfrac{}{}{0pt}{}{1}{b} & \multicolumn{2}{c}{\genfrac{}{}{0pt}{}{\ddot{\mu}}{\ddot{m}}} \\
\genfrac{}{}{0pt}{}{f}{a'} & \genfrac{}{}{0pt}{}{g}{b'} & \genfrac{}{}{0pt}{}{1}{\ddot{m}} &
\end{array}$$

$$= \underset{m\ \ddot{m}}{G}\left(\overset{m\ddot{m}}{\Xi}\left(f \overset{m\quad\ddot{m}}{\dot{m}} g\right)\circ\xi_{\langle a,\delta\rangle,\langle b,\vartheta\rangle}\right) .$$

$\square$

9 Where are we now?

Now seems like a good time to let the narrative rest for a while, take a step back and consider where we have come. We started out with the concept of a monoidal category and saw how one can describe algebraic structures in a monoidal category.

The coherence theorem for monoidal categories reassured us that the concept of a monoidal category is not unnecessarily complex and behaves roughly like a generalisation of what we have become used to in the category of sets. We developed a calculus for reasoning about morphisms in monoidal categories.

With these tools in hand we were able to set out to tackle the problem of describing limits, colimits and free objects of algebraic categories in a monoidal category. For the act categories $\langle m, \mu, \eta \rangle - \mathbf{Act}$, $\mathbf{Act} - \langle m, \mu, \eta \rangle$ and $\langle m, \mu, \eta \rangle - \mathbf{Act} - \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$, where we regard the monoid as fixed, this was relatively simple. In these categories both, limits and colimits, are the same as in the base category (under the assumption that the monoidal product preserves colimits on either side). For the other algebraic categories we found that limits are in essence the same as in the base category and that colimits can be described through free objects, which in turn can be constructed from coproducts of the base category.

Now we have arrived at a point where we have constructed from the base category $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$ two new classes of categories: the categories $\langle m, \mu, \eta \rangle - \mathbf{Act} - \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$ and $\langle m, \mu, \eta \rangle - \mathbf{Aut} - \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$, for both of which we have bifunctors which somehow combine two objects into one object. We even know that these bifunctors are compatible in a way.
Now the obvious question is: "Are these bifunctors themselves products of some monoidal category?"

Although this is the obvious question it turns out that it is not quite the right question — for it only makes sense for the categories $\langle m, \mu, \eta \rangle - \mathbf{Act} - \langle m, \mu, \eta \rangle$ and $\langle m, \mu, \eta \rangle - \mathbf{Aut} - \langle m, \mu, \eta \rangle$ — that is, when the monoids on both sides are the same. This is because a monoidal product is a functor which has the product of two categories which are the same as its source and also maps into that same category. What we have here is more general.

The author thinks it very likely that both of the collections $\langle m, \mu, \eta \rangle - \mathbf{Act} - \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$ and $\langle m, \mu, \eta \rangle - \mathbf{Aut} - \langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$ (where $\langle m, \mu, \eta \rangle$ and $\langle \dot{m}, \dot{\mu}, \dot{\eta} \rangle$ are monoids in $\langle \mathbf{C}, \boxtimes, e, \alpha, \lambda, \rho \rangle$) fit the definition of a bicategory. What we would have to show is that serial composition and tensor product are associative up to isomporphism and that diagrams similar to the diagrams (3), (4) for monoidal categories commute. For serial composition this is almost immediately obvious; for the tensor product it is a little more complicated but the author thinks it extremely likely that a proof which works by interchanging colimits is possible. Some ingenuity is probably necessary because the naïve approach of simply calculating explicitly all the necessary diagrams — albeit definitely doable in theory — would result in *very* large diagrams and such a proof would be neither pleasant to write nor read.

So provided this works out we have a situation in which we started with a monoidal category and ended up with two new bicategories. Of course one can now ask: "Could we have started with a bicategory?" The author thinks that this is very likely. We would have to extend the theory of coherence and calculations to this more general setting but at least on first inspection it seems like this should be possible in a fairly straightforward way. This does add a little extra complexity though because there are more entities that have to be distinguished.

This puts us in a situation where we can iterate this construction. Whether this makes any sense from an interpretation point of view and whether it has any applications is still to be explored.

References

[Deu71] Peter Deussen. *Halbgruppen und Automaten.* Springer, 1971.

[KKM00] Mati Kilp, Ulrich Knauer, and Alexander V. Mikhalev. *Monoids, Acts And Categories.* De Gruyter Expositions in Mathematics Series. Walter De Gruyter, 2000.

[Lin08] Fred E. J. Linton. “Coequalizers in categories of algebras”. In: *Seminar on Triples and Categorical Homology Theory.* Reprints in Theory and Applications of Categories. Ed. by Beno Eckmann and Myles Tierney. 18. 2008, pp. 61–72. URL: `www.tac.mta.ca/tac/reprints/articles/18/tr18.pdf`. Repr. of “Coequalizers in categories of algebras”. In: *Seminar on Triples and Categorical Homology Theory.* Lecture Notes in Mathematics. Ed. by Beno Eckmann and Myles Tierney. Vol. 80. 1969, pp. 75–90. Reprinted in: [Lin08].

[Lin69] Fred E. J. Linton. “Coequalizers in categories of algebras”. In: *Seminar on Triples and Categorical Homology Theory.* Lecture Notes in Mathematics. Ed. by Beno Eckmann and Myles Tierney. Vol. 80. 1969, pp. 75–90. Reprinted in: [Lin08].

[Mac98] Saunders Mac Lane. *Categories for the Working Mathematician.* 2nd ed. Graduate Texts in Mathematics. Springer, 1998.

[ReflCoeq] Todd Trimble et al. *reflexive coequalizer.* Version 9. URL: `http://ncatlab.org/nlab/show/reflexive+coequalizer` (visited on 02/28/2013).