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Commutative Banach Algebras: Shilov's Idempotent Theorem and Applications

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"Man darf nicht das, was uns unwahrscheinlich und unnatürlich erscheint, mit dem verwechseln, was absolut unmöglich ist."

Carl Friedrich Gauß

Introduction

Banach algebras are mathematical objects which play a major role in functional analysis. A Banach algebra is a Banach space $\mathcal{A}$ equipped with a continuous multiplication such that

$$\|xy\| \leq \|x\|\|y\| \quad \text{for all } x, y \in \mathcal{A}. \tag{1}$$

Like Banach spaces, Banach algebras are named after the famous mathematician **Stefan Banach**, although he had never studied Banach algebras. In fact, it was **Israel M. Gelfand** who introduced them under the name "normed rings" in his work *Normierte Ringe., Rec. Math. (Mat. Sbornik)*, in 1941. Many mathematicians would rather speak of "Gelfand algebras", and I tend to agree with them.
The theory of Banach algebras has led to many fundamental results, such as the *Gelfand-Mazur theorem* (Theorem 1.3.18), which states that every complex Banach algebra that is a division algebra is isomorphic to $\mathbb{C}$. The basic notions and definitions as well as classical results will be discussed in **Chapter 1**.
In general, Banach algebra theory is very elegant. But there is more to the axioms than meets the eye. We have an algebraic structure - an algebra - and an analytic structure - a Banach space -, and the two structures are linked by means of the inequality (1), which guarantees that multiplication in Banach algebras is continuous. As it turns out, the relationship between the analytic and the algebraic structure is much more subtle.
As for commutative Banach algebras, they allow for a powerful representation theory, which is known as *Gelfand theory* after its creator. Suppose that $\mathcal{A}$ is a commutative Banach algebra with identity e and let

$$\Delta(\mathcal{A}) = \{\varphi : \mathcal{A} \to \mathbb{C} : \varphi \text{ is nonzero, linear and multiplicative}\}.$$

We can see that $\Delta(\mathcal{A})$ lies in the unit ball of the dual space $\mathcal{A}'$, and restricting the weak* topology (or *Gelfand topology*) of $\mathcal{A}'$ to $\Delta(\mathcal{A})$ turns it into a locally compact Hausdorff space (Theorem 2.3.3). The space $\Delta(\mathcal{A})$ equipped with the Gelfand topology is called the *Gelfand space*. For $x \in \mathcal{A}$ we define $\widehat{x} : \Delta(\mathcal{A}) \to \mathbb{C}$ by $\widehat{x}(\varphi) = \varphi(x)$. Then $\widehat{x}$ is a continuous function, which is called the *Gelfand transform* and the mapping

$$\Gamma_{\mathcal{A}} : \mathcal{A} \to C(\Delta(\mathcal{A})), \quad x \mapsto \widehat{x}$$

the *Gelfand representation*. The remarkable insight of Gelfand was that x is invertible in $\mathcal{A}$ if and only if $\widehat{x}$ is invertible as a continuous function in $\Delta(\mathcal{A})$, that is has no zeros (Theorem 2.3.6).
The Gelfand transform can be thought of as an abstract Fourier transform. Using his representation theory, Gelfand gave a strinkingly short and simple proof of *Wiener's theorem* (Theorem 2.3.20): If f is nonzero and has an absolutely convergent Fourier

expansion, then $1/f$ has such an expansion as well. In **Chapter 2** we will thoroughly study the fundamental properties of the Gelfand transform and Gelfand representation. We will also briefly discuss semisimple Banach algebras since they are needed for later purposes.

To most mathematicians it is a well-known fact that the Gelfand space $\Delta(\mathcal{A})$ is compact if a commutative Banach algebra $\mathcal{A}$ has an identity (Theorem 2.3.4). The obvious question arising in this case is whether $\mathcal{A}$ must have an identity if $\Delta(\mathcal{A})$ is compact. Indeed, this is true, but only for semisimple Banach algebras (Theorem 3.3.1). This converse turns out to be a consequence of *Shilov's idempotent theorem* (Theorem 3.2.1), which was published and proved by **Georgi E. Shilov** in 1954 [Shi]. Shilov's idempotent theorem states that the characteristic function of a compact open subset of $\Delta(\mathcal{A})$ is the Gelfand transform of an idempotent in $\mathcal{A}$. Without a doubt, this theorem is one of the major highlights in commutative Banach algebra theory and is presented in **Chapter 3**. Unfortunately, to this day, it is not known how to prove the idempotent theorem without recourse to the multivariable holomorphic functional calculus.

Finally, in **Chapter 4** we will introduce the notion of a regular Banach algebra. We will observe that the hull-kernel topology and the Gelfand topology on $\Delta(\mathcal{A})$ coincide (Theorem 4.2.6), and present a much simpler proof of Theorem 3.3.1, provided that $\mathcal{A}$ is a semisimple regular commutative Banach algebra (Corollary 4.2.16).

For the sake of completeness, some basic definitions and fundamental theorems in topology and functional analysis are listed in the **Appendix**.

The main focus of this Master thesis will be on general Banach algebra theory, especially on commutative Banach algebras and the development of Shilov's idempotent theorem, followed by several applications that will illustrate the power of this remarkable result. Many notions and ideas are based on the very insightful books of [Kan] and [Lar].

Danksagung

Zuallererst möchte ich mich bei meinen Eltern bedanken, die mich während meines Mathematikstudiums nicht nur finanziell, sondern auch moralisch unterstützt haben. Sie haben immer an mich geglaubt, als ich auf Abwegen war und haben nie daran gezweifelt, dass ich letzten Endes all meine Ziele (auch außerhalb des Studiums) erreichen werde. Liebe Eltern, ich bin euch unendlich dankbar, dass ihr mir so vieles in meinem Leben mitgegeben und ermöglicht habt.

Des Weiteren bin ich glücklich darüber, dass ich in meiner Studienzeit neue Freundschaften knüpfen konnte und mir gleichzeitig alte Freunde erhalten geblieben sind. Ohne sie wären die letzten Jahre nur halb so lustig gewesen!

Zu guter Letzt bedanke ich mich bei meinem Betreuer Prof. Blümlinger, der mir mit wertvollen Ratschlägen zur Seite gestanden ist und für meine Anliegen stets Zeit gehabt hat. Insbesondere werde ich seine Anekdoten in Erinnerung behalten.

Filip Žepinić — Wien, Mai 2017

1. Preliminaries

1.1. Banach algebras

This chapter is dedicated to developing the basic concepts and constructions of Banach algebras. We shall be concerned here with introducing some of the basic definitions needed in the succeeding development and in proving several essentially algebraic results. It is quite possible that some of the definitions and results are already known to the reader, whereas others may seem strange and surprising at first glance.
From now on, we will always consider Banach algebras over the complex number field $\mathbb{C}$, primarily due to two reasons: first, some fundamental theorems (such as Gelfand-Mazur) only work for complex algebras, and second, we will need some tools from complex analysis later on in order to develop the theory of the holomorphic functional calculus. Also, we will provide many examples of Banach algebras, so one can get a better and deeper understanding of the subject.

Definition 1.1.1 A linear space $\mathcal{A}$ over $\mathbb{C}$ is an *algebra* if it is equipped with a binary operation, referred to as multiplication, from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ such that

(i) $x(yz) = (xy)z$

(ii) $x(y + z) = xy + xz; \quad (x + y)z = xz + yz$

(iii) $\lambda(xy) = (\lambda x)y = x(\lambda y)$

for all $x, y, z \in \mathcal{A}$ and every $\lambda \in \mathbb{C}$.

Definition 1.1.2 A normed linear space $(\mathcal{A}, \|.\|)$ over $\mathbb{C}$ is called a *normed algebra* if it is an algebra and the norm is submultiplicative, that is

$$\|xy\| \leq \|x\|\|y\| \quad \text{for all } x, y \in \mathcal{A}.$$

A normed algebra $\mathcal{A}$ is a *Banach algebra* if the normed space $(\mathcal{A}, \|.\|)$ is a Banach space.

- The completion $(\widetilde{\mathcal{A}}, \|.\|)$ of a normed algebra $(\mathcal{A}, \|.\|)$ is a Banach algebra. If $x, y \in \widetilde{\mathcal{A}}$ and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are sequences in $\mathcal{A}$ converging in $\widetilde{\mathcal{A}}$ to x and y, respectively, then it follows that $(x_n y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\widetilde{\mathcal{A}}$, and the product of x and y can be defined to be $xy = \lim\limits_{n\to\infty} x_n y_n$.

- Let $\mathcal{A}$ be a Banach algebra. A *subalgebra* is a linear subspace $\mathcal{B}$ of $\mathcal{A}$ such that $x, y \in \mathcal{B}$ implies $xy \in \mathcal{B}$. If $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$, then $\mathcal{B}$ is complete und hence a Banach algebra (under the same operations and norms as $\mathcal{A}$). We then say $\mathcal{B}$ is a *Banach subalgebra* of $\mathcal{A}$.

- An algebra $\mathcal{A}$ is called *commutative* if

$$xy = yx \quad \text{for all } x, y \in \mathcal{A}.$$

  We do not assume commutativity until Chapter 2.

- $\mathcal{A}$ is called *unital* or an *algebra with identity* if there exists $e \in \mathcal{A}$ such that

$$ex = xe = x \text{ for all } x \in \mathcal{A} \text{ and } \|e\| = 1.$$

  To avoid triviality we shall always assume $\mathcal{A} \neq \{0\}$.

Remark 1.1.3 It is clear that $e \in \mathcal{A}$ is unique. Let $\tilde{e}$ be a unit, then $\tilde{e} = \tilde{e}e = e$. Note that $\|e\| \geq 1$ already follows from submultiplicativity of the norm: $\|e\| = \|ee\| \leq \|e\|\|e\|$.

Lemma 1.1.4 *Let $\mathcal{A}$ be a Banach algebra. Then the inequality $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{A}$ makes the multiplication a continuous operation in $\mathcal{A}$.*

Proof. If $(x_n, y_n) \to (x, y)$ in $\mathcal{A} \times \mathcal{A}$, i.e.

$$\|(x_n, y_n) - (x, y)\| \to 0 \quad \text{as } n \to \infty,$$

then the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are bounded and thus

$$\|x_n y_n - xy\| = \|x_n(y_n - y) + y(x_n - x)\| \leq \|x_n\| \cdot \|y_n - y\| + \|y\| \cdot \|x_n - x\|$$

converges to 0. □

The following theorem of Gelfand asserts that we can weaken one of the requirements in the definition of a Banach algebra and still obtain the same sort of mathematical object.
In fact, we will not require the validity of the norm inequality $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{A}$, in order to obtain a Banach algebra. This weaker hypothesis was the one originally used in [Gel], albeit for the case that $\mathcal{A}$ is without identity.

Theorem 1.1.5 (**Gelfand**) *Let $\mathcal{A}$ be an algebra with identity e and with a norm $\|.\|$ under which it is a Banach space. Then there exists a norm $\|.\|_0$ on $\mathcal{A}$ that is equivalent to $\|.\|$ and for which $\|xy\|_0 \leq \|x\|_0\|y\|_0$ holds for all $x, y \in \mathcal{A}$, thus making A into a Banach algebra.*

Proof. Assign to each $x \in \mathcal{A}$ the left translation operator $L_x : \mathcal{A} \to \mathcal{A}$, defined by

$$L_x(y) = xy, \quad y \in \mathcal{A}.$$

Since multiplication is continuous by the preceding lemma, L_x is continuous. Because of $L_x(e) = xe = x$, the map $x \mapsto L_x$ is an isomorphism of $\mathcal{A}$ into $\mathfrak{B}(\mathcal{A})$, the algebra of all bounded linear operators. Let

$$\|x\|_0 = \|L_x\| = \sup\{\|xy\| : \ \|y\| \leq 1\}, \quad x \in \mathcal{A}.$$

It is easy to verify that $\|.\|_0$ is a norm on $\mathcal{A}$ satisfying

$$\|xy\|_0 = \|L_{xy}\| \overset{(ass.)}{=} \|L_x L_y\| \leq \|L_x\| \|L_y\| = \|x\|_0 \|y\|_0.$$

Now we claim that $(\mathcal{A}, \|.\|_0)$ is complete. But first we note that

$$\|x\| = \|xe\| = \|L_x e\| \leq \|L_x\| \|e\| = \|x\|_0 \|e\| \Rightarrow \|x\|_0 \geq \|e\|^{-1} \|x\|, \quad x \in \mathcal{A}.$$

Hence, if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|.\|_0$, then it is also a Cauchy sequence in $\|.\|$. Therefore we have $x_n \to x$ for some $x \in \mathcal{A}$. It follows that $L_{x_n} \to T$ for some $T \in \mathfrak{B}(\mathcal{A})$, because $(L_{x_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{B}(\mathcal{A})$. By continuity of the product in the first variable, we get $L_{x_n} y \to L_x y$ for each $y \in \mathcal{A}$. So $T = L_x$, which proves the claim.
Since we have established above that $\|.\| \leq \|e\| \|.\|_0$, the closed graph theorem implies that the two norms are equivalent.

□

Remark 1.1.6 In the preceding theorem we can see that there always exists an equivalent norm $\|.\|_0$ on $\mathcal{A}$ such that $\|e\|_0 = 1$. In fact

$$\|e\|_0 = \|L_e\| = \sup_{\|y\|=1} \|L_e(y)\| = 1.$$

Consequently in the following chapters we shall always assume, without loss of generality, that the identity in a unital Banach algebra has norm 1.

Next, we wish to list a number of standard examples of Banach algebras, which can be divided in two main classes: algebras of functions (with pointwise multiplication) and algebras of operators (with composition of operators).[1] We shall not prove all of the assertions made about the following examples since the proof techniques are pretty similar.

Example 1.1.7 The complex numbers $\mathbb{C}$ with the usual algebraic operations and with absolute value as the norm are a commutative Banach algebra with identity. In Section 1.3 we will see that $\mathbb{C}$ is the only complex Banach algebra, which is a division algebra (Gelfand-Mazur).

Example 1.1.8 Let $\mathcal{X}$ be a locally compact Hausdorff space. By $C_b(\mathcal{X}), C_0(\mathcal{X})$, and $C_c(\mathcal{X})$ we denote, respectively, the algebras of all continuous complex-valued functions on $\mathcal{X}$ that are bounded, vanish at infinity, or have compact support. The algebra operations are the usual ones of pointwise addition, multiplication, and scalar multiplication. We can show that with the common supremum norm

$$\|f\|_\infty = \sup_{x\in\mathcal{X}} |f(x)|, \quad f \in C_0(\mathcal{X}),$$

[1] Actually there is another class: group algebras (with convolution product) which will not be covered in this thesis.

the algebras $C_b(\mathcal{X})$ and $C_0(\mathcal{X})$ are commutative Banach algebras, whereas $C_c(\mathcal{X})$ is complete only when $\mathcal{X}$ is compact. If $\mathcal{X}$ is noncompact, then only $C_b(\mathcal{X})$ is unital where the constant function 1 is the identity. Indeed, the pointwise product

$$(fg)(x) = f(x)g(x), \quad f, g \in C_b(\mathcal{X}), x \in \mathcal{X}$$

turns $C_b(\mathcal{X})$ into a commutative Banach algebra. The product is clearly commutative and associative, it is linear in f and g and

$$\|fg\|_\infty = \sup_{x\in\mathcal{X}} |f(x)||g(x)| \leq \sup_{x_1\in\mathcal{X}} |f(x_1)| \cdot \sup_{x_2\in\mathcal{X}} |g(x_2)| = \|f\|_\infty \cdot \|g\|_\infty.$$

Example 1.1.9 Let $\mathcal{X}$ be a Banach space. Then $\mathfrak{B}(\mathcal{X})$, the algebra of all bounded linear operators on $\mathcal{X}$, is a Banach algebra, with respect to the usual operator norm. The identity operator I is its unit element. If $\dim \mathcal{X} = n < \infty$, then $\mathfrak{B}(\mathcal{X})$ is isomorphic to the algebra of all complex n-by-n matrices. If $\dim \mathcal{X} > 1$, then $\mathfrak{B}(\mathcal{X})$ is not commutative (the trivial space $\mathcal{X} = \{0\}$ must be excluded).

Example 1.1.10 Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ (closed unit disc in the plane). The *disc algebra* is the following Banach subalgebra of $C(\overline{\mathbb{D}})$:

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f \text{ is analytic on } \mathbb{D}\}.$$

With the usual pointwise operations and the uniform norm

$$\|f\|_\infty = \sup_{z\in\overline{\mathbb{D}}} |f(z)|, \quad f \in A(\overline{\mathbb{D}}),$$

it is easily verified that the disc algebra is a commutative Banach algebra with identity. We claim that $A(\overline{\mathbb{D}})$ is a unital Banach subalgebra of $C(\overline{\mathbb{D}})$, hence a unital Banach algebra. Clearly, $A(\overline{\mathbb{D}}) \subseteq C(\overline{\mathbb{D}})$, and the unit of $C(\overline{\mathbb{D}})$, the constant function 1, is in $A(\overline{\mathbb{D}})$. Moreover, if $f, g \in A(\overline{\mathbb{D}})$, then

- for $\lambda \in \mathbb{C}$ we have $\lambda f \in C(\overline{\mathbb{D}})$ and $(\lambda f)|_{\mathbb{D}} = \lambda \cdot f|_{\mathbb{D}}$ is analytic, so $\lambda f \in A(\overline{\mathbb{D}})$.
- $f + g \in C(\overline{\mathbb{D}})$ and $(f + g)|_{\mathbb{D}} = f|_{\mathbb{D}} + g|_{\mathbb{D}}$ is the sum of two analytic functions, so is analytic, and therefore $f + g \in A(\overline{\mathbb{D}})$.
- $f \cdot g \in C(\overline{\mathbb{D}})$ and $(f \cdot g)|_{\mathbb{D}} = f|_{\mathbb{D}} \cdot g|_{\mathbb{D}}$ is the product of two analytic functions, so is analytic, and therefore $f \cdot g \in A(\overline{\mathbb{D}})$.
- if $h_n \in A(\overline{\mathbb{D}})$ and $h_n \to h \in C(\overline{\mathbb{D}})$, then $h_n|_{\mathbb{T}}$ is a sequence of analytic functions which converges uniformly to $h|_{\mathbb{T}}$, which is therefore analytic, hence $h \in A(\overline{\mathbb{D}})$. ($\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}) = \{z \in \mathbb{C} : |z| = 1\}$ ... 1-torus)

Thus $A(\overline{\mathbb{D}})$ is a closed unital subalgebra of $C(\overline{\mathbb{D}})$, i.e. a unital Banach subalgebra of $C(\overline{\mathbb{D}})$.

Example 1.1.11 Let $\mathcal{X}$ be a compact subset of $\mathbb{C}$. We introduce three unital closed subalgebras of $(C(\mathcal{X}), \|.\|_\infty)$ as follows.

- The first one, denoted by $A(\mathcal{X})$, is the algebra of all functions $f : \mathcal{X} \to \mathbb{C}$ which are continuous on $\mathcal{X}$ and holomorphic on the interior $\mathcal{X}^\circ$ of $\mathcal{X}$. Obviously, $A(\mathcal{X})$ is complete since the uniform limit of a sequence of holomorphic functions is holomorphic.
- The second one, $P(\mathcal{X})$, is the subalgebra of $C(\mathcal{X})$ consisting of all functions which are uniform limits of polynomial functions on $\mathcal{X}$ where the constant polynomial 1 is the identity.
- And finally, the third one, $R(\mathcal{X})$, is the subalgebra of $C(\mathcal{X})$ of all functions which are uniform limits on $\mathcal{X}$ of rational functions p/q, where p und $q \neq 0$ are polynomials.

Note that we always have $P(\mathcal{X}) \subseteq R(\mathcal{X}) \subseteq A(\mathcal{X})$ and that equality holds at either place can be interpreted as a result in approximation theory.

Example 1.1.12 Let $a, b \in \mathbb{R}$ with $a < b$, and for each $n \in \mathbb{N}$ let $C^n([a,b])$ denote the space of all n-times continuously differentiable complex-valued functions defined on the closed interval $[a,b]$, with the usual convention about one-sided derivatives at the end points of the interval. With pointwise operations and the norm

$$\|f\|_n = \sum_{k=0}^{n} \frac{1}{k!}\|f^{(k)}\|_\infty, \quad f \in C^n([a,b]),$$

where $f^{(k)}$ denotes the kth derivative of f, $C^n([a,b])$ becomes a commutative Banach algebra. To verify this, we first show that the norm is submultiplicative. For $f, g \in C^n([a,b])$,

$$\begin{aligned}
\|fg\|_n &= \sum_{k=0}^{n} \frac{1}{k!}\|(fg)^{(k)}\|_\infty \\
&= \sum_{k=0}^{n} \frac{1}{k!}\left\| \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)} \right\|_\infty \\
&= \sum_{k=0}^{n} \left\| \sum_{j=0}^{k} \frac{1}{j!(k-j)!} f^{(j)} g^{(k-j)} \right\|_\infty \\
&\leq \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{1}{j!}\|f^{(j)}\|_\infty \frac{1}{(k-j)!}\|g^{(k-j)}\|_\infty \\
&\leq \sum_{l=0}^{n} \sum_{j=0}^{n} \frac{1}{j!}\|f^{(j)}\|_\infty \frac{1}{l!}\|g^{(l)}\|_\infty \\
&= \|f\|_n \|g\|_n.
\end{aligned}$$

Next, we claim that $C^n([a,b])$ is complete. To verify this, let $(f_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $C^n([a,b])$. With a simple induction argument we eventually get $f_m \to f$ in $C^n([a,b])$.

Example 1.1.13 Let $(\mathcal{X}, \mathfrak{S}, \mu)$ be a positive measure space, where $\mathcal{X}$ is a Banach space, $\mathfrak{S}$ a σ-algebra and μ a measure on $\mathfrak{S}$. Furthermore let $L^\infty(\mathcal{X}, \mathfrak{S}, \mu)$ denote the family of all equivalence classes of essentially bounded μ-measurable complex-valued functions on $\mathcal{X}$. With the operations of addition, multiplication, and scalar multiplication of equivalence classes obtained via pointwise operations on equivalence class representatives and with the usual essential supremum norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in\mathcal{X}} |f(x)| = \inf_M \{M \geq 0 : \mu(\{x \in \mathcal{X} : |f(x)| > M\}) = 0\}, \quad f \in L^\infty(\mathcal{X}, \mathfrak{S}, \mu)$$

it can be shown that $L^\infty(\mathcal{X}, \mathfrak{S}, \mu)$ is a commutative Banach algebra with identity.

Example 1.1.14 Let $l^1(\mathbb{Z})$ denote the vector space of complex sequences $(a_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ such that

$$\|a\| = \sum_{n\in\mathbb{Z}} |a_n| < \infty,$$

thus making it a Banach space. We define a product $*$ such that if $a = (a_n)_{n\in\mathbb{Z}}$ and $b = (b_n)_{n\in\mathbb{Z}}$ are in $l^1(\mathbb{Z})$ then the nth entry of $a * b$ is

$$(a * b)_n = \sum_{m\in\mathbb{Z}} a_m b_{n-m}.$$

This series is absolutely convergent, and $a * b \in l^1(\mathbb{Z})$ since

$$\begin{aligned}\|a * b\| &= \sum_n |(a * b)_n| = \sum_n \Big| \sum_m a_m b_{n-m} \Big| \\ &\leq \sum_{m,n} |a_m||b_{n-m}| = \sum_m |a_m| \sum_n |b_{m-n}| = \|a\|\|b\| < \infty.\end{aligned}$$

It is easy to verify that $*$ is associative and linear in each variable. Therefore $l^1(\mathbb{Z})$ is a (commutative) Banach algebra with identity $e_0 = (\ldots, 0, 1, 0, \ldots)$. Note that $e = (1, 1, 1, \ldots)$ is not in $l^1(\mathbb{Z})$.

Example 1.1.15 Given $f \in C(\mathbb{T})$ and $n \in \mathbb{Z}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{it} : 0 \leq t \leq 2\pi\}$ denotes the 1-torus, we define the nth Fourier coefficient c_n of f by

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt.$$

Let $W(\mathbb{T})$ denote the space of all absolutely convergent *Fourier series* $\sum_{n\in\mathbb{Z}} c_n(f) e^{int}$, which is often referred to as the *Wiener algebra*. So we have $f \in W(\mathbb{T})$ if and only if $(c_n(f))_{n\in\mathbb{Z}} \in l^1(\mathbb{Z})$ (the space of all sequences whose series is absolutely convergent in $\mathbb{Z}$).

Conversely, for $(c_n)_{n\in\mathbb{Z}} \in l^1(\mathbb{Z})$, define $f \in W(\mathbb{T})$ by

$$f(e^{it}) = \sum_{k\in\mathbb{Z}} c_k e^{ikt}, \quad t \in [0, 2\pi].$$

Then we get for each $n \in \mathbb{Z}$

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k\in\mathbb{Z}} c_k e^{ikt} \right) e^{-int} dt = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} c_k \int_0^{2\pi} e^{it(k-n)} dt = c_n.$$

We claim that $W(\mathbb{T})$ with pointwise multiplication and equipped with the norm

$$\|f\|_{W(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |c_n(f)|$$

is a commutative unital Banach algebra with the constant function 1 as its identity. Indeed, clearly $W(\mathbb{T})$ is a Banach space and isometrically isomorphic to $l^1(\mathbb{Z})$ with the isomorphism given by the Fourier transform.
For $f, g \in W(\mathbb{T})$ and $n \in \mathbb{Z}$ we have

$$\begin{aligned} c_n(fg) &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} \left(\sum_{k\in\mathbb{Z}} c_k(g) e^{-ikt} \right) dt \\ &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} c_k(g) \int_0^{2\pi} f(e^{it}) e^{-it(n-k)} dt = \sum_{k\in\mathbb{Z}} c_k(g) c_{n-k}(f), \end{aligned}$$

which implies

$$\|fg\|_{W(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |c_n(fg)| \leq \sum_{j\in\mathbb{Z}} |c_j(f)| \cdot \sum_{k\in\mathbb{Z}} |c_k(g)| = \|f\|_{W(\mathbb{T})} \cdot \|g\|_{W(\mathbb{T})}.$$

The Wiener algebra $W(\mathbb{T})$ will reappear in Section 2.3 where it will play a major role in Wiener's theorem.

Example 1.1.16 Let

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

be an algebra under the usual addition und multiplication of matrices. We claim that $\mathcal{A}$ is a Banach algebra under the norm

$$\left\| \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \right\| = |\alpha| + |\beta|.$$

We only show completeness, the rest is clear. Suppose that $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ is a Cauchy sequence. Since

$$\|A_n - A_m\| = |\alpha_n - \alpha_m| + |\beta_n - \beta_m| \qquad (m \in \mathbb{N}),$$

we know that (α_n) and (α_m) are Cauchy sequences, hence $\alpha_n \to \alpha$ and $\beta_n \to \beta$ for some α and β. It is then straightforward to see that $A_n \to \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$.

In the next theorem we want to establish a link between a so called derivation on a Banach algera $\mathcal{A}$ and a continuous automorphism on $\mathcal{A}$. But first we will show a slight alteration of the well-known Leibniz rule.

Lemma 1.1.17 (**Leibniz's rule**) *A derivation on an algebra $\mathcal{A}$ is a linear mapping D of $\mathcal{A}$ into $\mathcal{A}$ such that*

$$D(xy) = x(Dy) + (Dx)y, \quad x, y \in \mathcal{A}.$$

Let D be a derivation on $\mathcal{A}$. Then D satisfies (with the convention that $D^0 = I$) the Leibniz rule

$$D^n(xy) = \sum_{j=0}^{n} \binom{n}{j} (D^j x)(D^{n-j} y) \quad \text{for all } x, y \in \mathcal{A}, n \in \mathbb{N}. \tag{1.1}$$

Proof. We use induction on n.

- $\underline{n = 1}$:

$$\begin{aligned} D^1(xy) &= \sum_{j=0}^{1} \binom{1}{j} (D^j x)(D^{1-j} y) = \binom{1}{0} D^0 x D^1 y + \binom{1}{1} D^1 x D^0 y \\ &= x(Dy) + (Dx)y. \end{aligned}$$

- $\underline{n \to n+1}$:

$$\begin{aligned} D^{n+1}(xy) &= D^n(xy)D(xy) \\ &= \sum_{j=0}^{n} \binom{n}{j} \big((D^j x)(D^{n-j} y)\big)\big(D^0 x D^1 y + D^1 x D^0 y\big) \\ &= \sum_{j=0}^{n} \binom{n}{j} \big((D^j x)(D^{n-j+1} y) + (D^{j+1} x)(D^{n-j} y)\big) \\ &= \sum_{j=0}^{n} \binom{n}{j} \big(D^{j+1} x D^{n-j} y + D^j x D^{n+1-j} y\big) \\ &= \binom{n+1}{0} x(D^{n+1} y) + \binom{n+1}{n+1} (D^{n+1} x) y \\ &+ \sum_{j=1}^{n} \left(\binom{n}{j-1} + \binom{n}{j} \right) (D^j x)(D^{n+1-j} y) \\ &= \sum_{j=1}^{n} \binom{n+1}{j} (D^j x)(D^{n+1-j} y) \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} (D^j x)(D^{n+1-j} y). \end{aligned}$$

□

Corollary 1.1.18 *Let D be a derivation on an algebra $\mathcal{A}$. Then the following statements hold.*

(i) $D(x^n) = nx^{n-1}Dx \Longleftrightarrow xDx = (Dx)x,\ n-1 \in \mathbb{N}$.

(ii) *if* $D^2x = 0$, *then* $D^n(x^n) = n!(Dx)^n,\ n \in \mathbb{N}$.

Proof.
(i) follows directly by induction.
(ii) Let $D^2x = 0$. Then, by induction,

$$D(Dx)^n = 0, n \in \mathbb{N}.$$

Assume that $D^{n-1}(x^{n-1}) = (n-1)!(Dx)^{n-1}$. Then it follows that $D^n(x^{n-1}) = 0$. Hence, by the Leibniz rule (1.1), we get

$$\begin{aligned} D^n(x^n) &= D^n(xx^{n-1}) \\ &= xD^n(x^{n-1}) + \binom{n}{1}(Dx)D^{n-1}(x^{n-1}) + \ \dots\ + \binom{n}{n}(D^nx)x^{n-1} \\ &= n(Dx)D^{n-1}(x^{n-1}) \\ &= n(Dx)(n-1)!(Dx)^{n-1} \\ &= n!(Dx)^n. \end{aligned}$$

□

Example 1.1.19 For $y \in \mathcal{A}$, let $d_y : \mathcal{A} \to \mathcal{A}$ be defined by

$$d_y(x) = xy - yx, \quad x \in \mathcal{A}.$$

Then d_y is a derivation on $\mathcal{A}$. Each such derivation is called an *inner derivation.* If $\mathcal{A}$ is a normed algebra, each inner derivation d_y is continuous and $\|d_y\| \leq 2\|y\|$. Obviously, $\mathcal{A}$ is commutative if and only if 0 is the only inner derivation.

Theorem 1.1.20 (**Singer-Wermer**) *Let D be a continuous derivation on a Banach algebra $\mathcal{A}$. Then*

$$\exp D : \mathcal{A} \to \mathcal{A}, \quad (\exp D)(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n x, \quad x \in \mathcal{A},$$

is a continuous automorphism on A.

Proof. It is easy to check that $\exp D$ is bounded:

$$\| \exp D\| = \| \sum_{n=0}^{\infty} \frac{1}{n!} D^n \| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \|D\|^n = \exp \|D\|.$$

Furthermore, $\exp D$ is obviously linear.

And since we have

$$\begin{aligned}
(\exp D)(xy) &= \sum_{n=0}^{\infty} \frac{1}{n!} D^n(xy) \\
&\overset{(1.1)}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (D^j x)(D^{n-j} y) \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{j!}(D^j x) \frac{1}{(n-j)!}(D^{n-j} y) \\
&= \left(\sum_{n=0}^{\infty} \frac{1}{n!} D^n x \right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} D^n y \right) \\
&= (\exp D)(x)(\exp D)(y)
\end{aligned}$$

for all $x, y \in \mathcal{A}$, $\exp D$ is multiplicative. Thus $\exp D$ is an algebra homomorphism. It is easy to see that for any commutative operators S and T

$$\exp(S + T) = \exp(S) \cdot \exp(T).$$

Thus $\exp D$ has an inverse $\exp(-D)$ (which is also continuous) and therefore is an automorphism. $\square$

Next we wish to examine in detail how to embed an algebra without identity into an algebra with identity. Suppose that $\mathcal{A}$ is an algebra without identity and denote by $\mathcal{A}_e$ the set of all pairs $(x, \lambda), x \in \mathcal{A}, \lambda \in \mathbb{C}$, that is a point set $\mathcal{A}_e = \mathcal{A} \times \mathbb{C}$. Then $\mathcal{A}_e$ becomes an algebra if the linear space operations and multiplications are defined by

(*i*) $(x, \lambda) + (y, \mu) = (x + y, \lambda + \mu)$

(*ii*) $\mu(x, \lambda) = (\mu x, \mu\lambda)$

(*iii*) $(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$ for all $x, y \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$.

We will only verify (*iii*) as (*i*) and (*ii*) are trivial. Let $x, y, z \in \mathcal{A}$ and $\lambda, \mu, \nu \in \mathbb{C}$.

- Bilinearity:

$$\begin{aligned}
(\alpha(x, \lambda) + \beta(y, \mu))(z, \nu) &= (\alpha x + \beta y, \alpha\lambda + \beta\mu)(z, \nu) \\
&= (\alpha\nu x + \beta\nu y + \alpha\lambda z + \beta\mu z, \alpha\lambda\nu + \beta\mu\nu) \\
&= (\alpha\nu x + \alpha\lambda z, \alpha\lambda\nu) + (\beta\nu y + \beta\mu z, \beta\mu\nu) \\
&= \alpha(x, \lambda)(z, \nu) + \beta(y, \mu)(z, \nu) \quad \text{for all } \alpha, \beta \in \mathbb{C}.
\end{aligned}$$

- Associativity:

$$\begin{aligned}
((x, \lambda)(y, \mu))(z, \nu) &= (xy + \lambda y + \mu x, \lambda\mu)(z, \nu) \\
&= ((xy + \lambda y + \mu y)z + \lambda\mu z + \nu(xy + \lambda y + \mu y), \lambda\mu\nu) \\
&= (xyz + \lambda yz + \mu yz + \lambda\mu z + \nu xy + \nu\lambda y + \nu\mu y), \lambda\mu\nu) \\
&= (x, \lambda)(yz + \mu z + \nu y, \mu\nu) \\
&= (x, \lambda)((y, \mu)(z, \nu)).
\end{aligned}$$

A simple calculation shows that the element $e = (0, 1)$ is an identity for $\mathcal{A}_e$:

$$\begin{aligned}(x, \lambda)(0, 1) &= (1x + \lambda 0, \lambda 1) = (x, \lambda) \\ &= (\lambda 0 + 1x, \lambda 1) = (0, 1)(x, \lambda)\end{aligned}$$

$\mathcal{A}$ is a subalgebra of $\mathcal{A}_e$ if we identify x with $(x, 0)$: $(.,.)(.,.)$ is clearly well-defined and $(x, 0)(y, 0) = (xy, 0)$. Moreover, the mapping $x \mapsto (x, 0)$ is an algebra isomorphism of $\mathcal{A}$ onto an ideal of codimension 1 in $\mathcal{A}_e$.[2]
Evidently, $\mathcal{A}_e$ is commutative if and only if $\mathcal{A}$ is commutative:

$$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu) = (yx + \mu x + \lambda y, \mu\lambda) = (y, \mu)(x, \lambda).$$

Now suppose that $\mathcal{A}$ is a normed algebra. We introduce a norm on $\mathcal{A}_e$ by

$$\|(x, \lambda)\| = \|x\| + |\lambda|, \quad x \in \mathcal{A},\ \lambda \in \mathbb{C}.$$

This turns $\mathcal{A}_e$ into a normed algebra and a Banach algebra provided that $\mathcal{A}$ is complete. Indeed, for $x, y \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$ we have

$$\begin{aligned}\|(x, \lambda)(y, \mu)\| &= \|(xy + \lambda y + \mu x, \lambda\mu)\| \\ &= \|xy + \lambda y + \mu x\| + |\lambda\mu| \\ &\leq \|x\|\|y\| + |\lambda|\|y\| + |\mu|\|x\| + |\lambda||\mu| \\ &= (\|x\| + |\lambda|)(\|y\| + |\mu|) \\ &= \|(x, \lambda)\| + \|(y, \mu)\|.\end{aligned}$$

The norm of the unit e is one:

$$\|e\| = \|(0, 1)\| = \|0\| + |1| = 1.$$

As $(x, \lambda) = (x, 0) + \lambda(0, 1)$, it is customary to write elements (x, λ) as $x + \lambda e$.
In the succeeding chapters we shall always use the symbol A_e to denote the algebra with identity obtained from an algebra without identity by the previously developed construction. This process is usually referred to as that of *adjoining an identity* to $\mathcal{A}$ and A_e is called the *unitization* of $\mathcal{A}$. It should be noted that even when $\mathcal{A}$ has an identity we can still contruct $\mathcal{A}_e$, however the identity for $\mathcal{A}$ is not the identity for $\mathcal{A}_e$. This observation will be useful at times.
The general utility of $\mathcal{A}_e$ lies in the fact that algebras with identity are often easier to deal with than algebras without identity, and one can often deduce properties of $\mathcal{A}$ by examining a related property in $\mathcal{A}_e$.
If $\mathcal{A}$ lacks an identity , then an approximate identity oftern serves as a good substitute. We proceed by introducing this notion.

Definition 1.1.21 Let $\mathcal{A}$ be a normed algebra. A *left (right) approximate identity* for $\mathcal{A}$ is a net $(e_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{A}$ such that $e_\lambda x \to x$ ($xe_\lambda \to x$) for each $x \in \mathcal{A}$.
An *approximate identity* for $\mathcal{A}$ is a net $(e_\lambda)_{\lambda\in\Lambda}$ which is both a left and a right approximate identity. A (left or right) approximate identity $(e_\lambda)_{\lambda\in\Lambda}$ is *bounded* by $C > 0$ if

$$\|e_\lambda\| \leq C \quad \text{for all } \lambda \in \Lambda.$$

[2] We will discuss ideals and their role in Banach algebra theory in Section 2.1.

Definition 1.1.22 $\mathcal{A}$ has *left (right) approximate units* if, for each $x \in \mathcal{A}$ and $\varepsilon > 0$ there exists $u \in \mathcal{A}$ such that $\|x - ux\| \leq \varepsilon$ ($\|x - xu\| \leq \varepsilon$), and $\mathcal{A}$ has an *approximate unit* if, for each $x \in \mathcal{A}$ and $\varepsilon > 0$, there exists $u \in \mathcal{A}$ such that $\|x - ux\| \leq \varepsilon$ and $\|x - xu\| \leq \varepsilon$. $\mathcal{A}$ has a (left or right) approximate unit *bounded* by $C > 0$, if the element u can be chosen such that $\|u\| \leq C$.

Lemma 1.1.23 *Let $(e_\lambda)_\lambda$ and $(f_\mu)_\mu$ be bounded left and right approximate identities for $\mathcal{A}$, repectively. Then the net*

$$(e_\lambda + f_\mu - f_\mu e_\lambda)_{\lambda,\mu} \quad (\lambda, \mu \in \Lambda),$$

is a bounded approximate identity for $\mathcal{A}$.

Proof. Let $g_{\lambda,\mu} = e_\lambda + f_\mu - f_\mu e_\lambda$. Then we get for any $x \in \mathcal{A}$

$$\begin{aligned}\|g_{\lambda,\mu}x - x\| &= \|(e_\lambda + f_\mu - f_\mu e_\lambda)x - x\| \\ &= \|(e_\lambda x - x) + f_\mu(x - e_\lambda x)\| \\ &\leq (1 + \|f_\mu\|)\|e_\lambda x - x\|,\end{aligned}$$

and similarly

$$\|x g_{\lambda,\mu} - x\| \leq (1 + \|e_\lambda\|)\|x - x f_\mu\|.$$

Hence $(g_{\lambda,\mu})_{\lambda,\mu}$ is an approximate identity for A. Cleary $(g_{\lambda,\mu})_{\lambda,\mu}$ is bounded since

$$\|g_{\lambda,\mu}\| = \|e_\lambda + f_\mu - f_\mu e_\lambda\| \leq \|e_\lambda\| + \|f_\mu\| + \|e_\lambda\|\|f_\mu\|.$$

□

Proposition 1.1.24 *Let $\mathcal{A}$ be a normed algebra and let $C > 0$. Then the following three conditions are equivalent.*

(*i*) *$\mathcal{A}$ has left approximate units bounded in C.*

(*ii*) *Given finitely many elements $x_1, \dots, x_n$ in $\mathcal{A}$ and $\varepsilon > 0$, there exists $u \in \mathcal{A}$ such that $\|u\| \leq C$ and $\|x_j - ux_j\| \leq \varepsilon$ for $j = 1, \dots, n$.*

(*iii*) *$\mathcal{A}$ has a left approximate identity bounded by C.*

Proof. See [Kan], Section 1.1, Proposition 1.1.11, p.7. □

Example 1.1.25 We claim that the Banach space $L^1([0,1])$ with

$$(fg)(t) = \int_0^t f(t-s)g(s)ds, \qquad f, g \in L^1([0,1]),\ t \in [0,1]$$

is a non-unital commutative Banach algebra. For that purpose we proceed as follows. We show that $L^1([0,1])$ has no identity but there exists a net $(e_i)_{i\in I}$ with

$$\lim_{i\in I} e_i f = f, \quad f \in L^1([0,1]), \quad \sup_{i\in I}\|e_i\| < \infty.$$

Let e be an identity. Then

$$1 = (e \cdot 1)(t) = \int_0^t e(s)ds, \quad t \in [0,1]$$

which is impossible for $e \in L^1$.

Let $e_i \geq 0$ with $\int_0^1 e_i \, dt = 1, i \in I$ and $\operatorname{supp}(e_i) \subseteq [0, \frac{1}{i}]$. Then for $f \in C([0,1])$

$$(f - fe_i)(t) = \int_0^{\frac{1}{i}} (f(t) - f(t-s))e_i(s)ds \longrightarrow 0 \quad \text{uniformly in } t.$$

(Recall that $C([0,1])$ is dense in $L^1([0,1])$.)

1.2. Invertibility

A major key in understanding Banach algebras is the notion of invertibility[3]. It is closely linked to the spectrum of a Banach algebra element which we will get acquainted with in the next section.

Definition 1.2.1 Let $\mathcal{A}$ be a complex algebra with identity e. An element $x \in \mathcal{A}$ is called *invertible* if there exists $y \in \mathcal{A}$ such that $xy = yx = e$. Then y is called the *inverse*, denoted x^{-1}, of x. Let $\mathcal{G}(\mathcal{A})$ denote the set of all invertible elements.

Remark 1.2.2 We mention some basic properties of $\mathcal{G}(\mathcal{A})$.

- For $x \in \mathcal{A}$, $0x = x0 = 0 \neq e$ implies $0 \neq \mathcal{G}(\mathcal{A})$, thus $\mathcal{G}(\mathcal{A}) \neq \mathcal{A}$.
- If $x \in \mathcal{A}$ has a left and a right inverse, i.e. $zx = e = xy$ for some $y, z \in \mathcal{A}$, then it follows from

  $$z = ze = z(xy) = (zx)y = ey = y$$

  that $x \in \mathcal{G}(\mathcal{A})$ and $y = z = x^{-1}$. Hence inverses are unique.
- $\mathcal{G}(\mathcal{A})$ is a group, since it has an identity e, an inverse x^{-1} and multiplication is associative.
- For $x, y \in \mathcal{G}(\mathcal{A})$ and $\lambda \in \mathbb{C} \setminus \{0\}$ we have

  $$(xy)^{-1} = y^{-1}x^{-1} \text{ and } \lambda x \in \mathcal{G}(\mathcal{A}) \text{ with } (\lambda x)^{-1} = \frac{1}{\lambda}x^{-1}.$$

Definition 1.2.3 Let $\mathcal{A}$ be an algebra with identity. If each $x \in \mathcal{A}, x \neq 0$, is invertible, then $\mathcal{A}$ is called a *division algebra*.

[3] As Gelfand used to say: "It's all about invertibility!"

Example 1.2.4 If $\mathcal{X}$ is a compact topological space then

$$\mathcal{G}(C(\mathcal{X})) = \{f \in C(\mathcal{X}) : f(x) \neq 0 \text{ for all } x \in \mathcal{X}\}.$$

Indeed, if $f(x) \neq 0$ for all $x \in \mathcal{X}$ then we can define $g : \mathcal{X} \to \mathbb{C}, x \mapsto f(x)^{-1}$. The function g is then continuous with $fg = 1$. Conversely, if $x \in \mathcal{X}$ and $f(x) = 0$ then $fg(x) = f(x)g(x) = 0$ so $fg \neq 1$ for all $g \in C(\mathcal{X})$, thus f is not invertible.

Example 1.2.5 We claim that if $\mathcal{X}$ is a Banach space, then

$$\mathcal{G}(\mathfrak{B}(\mathcal{X})) \subseteq \{T \in \mathfrak{B}(\mathcal{X}) : \ker T = \{0\}\}.$$

If $\ker T \neq \{0\}$, then T is not injective, so cannot be invertible. If $\mathcal{X}$ is finite-dimensional and $\ker T = \{0\}$, then T is surjective, therefore T is an invertible linear map. Since $\mathcal{X}$ is finite-dimensional, the linear map T^{-1} is bounded, so T is invertible in $\mathfrak{B}(\mathcal{X})$. Hence

$$\mathcal{G}(\mathfrak{B}(\mathcal{X})) = \{T \in \mathfrak{B}(\mathcal{X}) : \ker T = \{0\}\}, \text{ if } \dim \mathcal{X} < \infty.$$

On the other hand, if $\mathcal{X}$ is infinite-dimensional then we generally have

$$\mathcal{G}(\mathfrak{B}(\mathcal{X})) \subsetneq \{T \in \mathfrak{B}(\mathcal{X}) : \ker T = \{0\}\}.$$

For example, let $\mathcal{X} = \mathcal{H}$ be an infinite-dimensional Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Consider the operator $T \in \mathfrak{B}(\mathcal{H})$ definied by $Te_n = \frac{1}{n}e_n$ for $n > 0$. It is easy to see that $\ker T = \{0\}$. However, T is not invertible. Indeed, if $S \in \mathfrak{B}(\mathcal{H})$ with $ST = I$ then $Se_n = S(nTe_n) = nSTe_n = ne_n$, and consequently $\|Se_n\| = n \to \infty$. Hence S is not bounded, which is a contradiction.

Example 1.2.6 An element $x \neq 0$ of a Banach algebra $\mathcal{A}$ is called a *topological divisor of zero* if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that

- $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and
- $xx_n \to 0$ as $n \to \infty$.

As it turns out, topological divisors of zero are not invertible. Indeed, let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$ a topological divisor of zero. Suppose that x is invertible. Then there exists $x^{-1} \in \mathcal{A}$ such that $x^{-1}x = e$. Now

$$1 = \|x_n\| = \|ex_n\| = \|x^{-1}xx_n\| \leq \|x^{-1}\|\|\underbrace{xx_n}_{\to 0}\| \to 0,$$

which is impossible.

The converse is not necessarily true, i.e. not every element that is not invertible, is automatically a topological divisor of zero. For more details, see [Zel], Section 14, p.57.

Next, we want to find conditions that guarantee the invertibility of an element of a unital Banach algebra.

Theorem 1.2.7 *Let $\mathcal{A}$ be a Banach algebra with identity e and let $x \in \mathcal{A}$ with $\|x\| < 1$. Then $(e-x) \in \mathcal{G}(\mathcal{A})$ and*

$$(e-x)^{-1} = \sum_{n=0}^{\infty} x^n. \qquad (x^0 := e)$$

Proof. Since $\|x^n\| \leq \|x\|^n$ and $\|x\| < 1$ it follows that

$$\sum_{n=0}^{\infty} \|x^n\| \leq \sum_{n=0}^{\infty} \|x\|^n = \frac{1}{1-\|x\|},$$

hence the series $\sum_{n=0}^{\infty} x^n$ is absolutely convergent. Let $y_N = \sum_{n=0}^{N} x^n$. We observe that

$$y_N(e-x) = (e-x)y_N = (e-x)(1+x+x^2+\ \dots\ +x^N) = e - x^{N+1} \to e \text{ as } N \to \infty.$$

Since multiplication is continuous by Lemma 1.1.4, we get that $y(e-x) = (e-x)y = e$, which shows that $(e-x) \in \mathcal{G}(\mathcal{A})$ and $(e-x)^{-1} = y$. □

The following corollary will turn out to be quite useful.

Corollary 1.2.8 *If $\mathcal{A}$ is a unital Banach algebra, then $\mathcal{G}(\mathcal{A})$ is an open subset of $\mathcal{A}$.*

Proof. Let $x \in \mathcal{G}(\mathcal{A})$ and let $r_x = \|x^{-1}\|^{-1}$. We claim that the open ball $B(x, r_x) = \{y \in \mathcal{A} : \|x-y\| < r_x\}$ is contained in $\mathcal{G}(\mathcal{A})$. If $y \in B(x, r_x)$ then $\|x-y\| < r_x$ and

$$y = (x-(x-y))x^{-1}x = (e-(x-y)x^{-1})x.$$

Since $\|(x-y)x^{-1}\| < r_x\|x^{-1}\| < 1$, the element $e-(x-y)x^{-1}$ is invertible by Theorem 1.2.7. Hence y is the product of two invertible elements, so $y \in \mathcal{G}(\mathcal{A})$. This shows that every element of $\mathcal{G}(\mathcal{A})$ lies in an open ball which is contained in $\mathcal{G}(\mathcal{A})$, hence $\mathcal{G}(\mathcal{A})$ is open.

□

Corollary 1.2.9 *Let $\mathcal{A}$ be a unital Banach algebra. Then the mapping $\psi : \mathcal{G}(\mathcal{A}) \to \mathcal{G}(\mathcal{A})$, $x \mapsto x^{-1}$ is a homeomorphism.*

Proof. Since $(x^{-1})^{-1} = x$, the map ψ is a bijection with $\psi = \psi^{-1}$. So we only need to show that ψ is continuous.

If $x, y \in \mathcal{G}(\mathcal{A})$ with $\|x-y\| < \frac{1}{2}\|x^{-1}\|^{-1}$, then using the triangle inequality and the identity

$$x^{-1} - y^{-1} = x^{-1}(y-x)y^{-1} \tag{1.2}$$

we get

$$\|y^{-1}\| \leq \|x^{-1}-y^{-1}\| + \|x^{-1}\| \leq \|x^{-1}\|\|x-y\|\|y^{-1}\| + \|x^{-1}\| \leq \frac{1}{2}\|y^{-1}\| + \|x^{-1}\|,$$

so $\|y^{-1}\| \le 2\|x^{-1}\|$. Using (1.2) again, we obtain

$$\|\psi(x) - \psi(y)\| = \|x^{-1} - y^{-1}\| \le \|x^{-1}\|\|x - y\|\|y^{-1}\| \le 2\|x^{-1}\|^2\|x - y\|,$$

which shows that ψ is continuous at x.

□

Lemma 1.2.10 *Let $\mathcal{A}$ be a Banach algebra with identity and $(x_n)_{n\in\mathbb{N}} \in \mathcal{G}(\mathcal{A})$ such that*

$$x_n \to x \in \mathcal{A} \setminus \mathcal{G}(\mathcal{A}) \text{ as } n \to \infty.$$

Then

(*i*) $\lim_{n\to\infty} \|x_n^{-1}\| = \infty$.

(*ii*) *there exists* $(y_n)_{n\in\mathbb{N}} \in \mathcal{A}, \|y_n\| = 1$, *such that* $\lim_{n\to\infty} xy_n = \lim_{n\to\infty} y_n x = 0$.

Proof.

(*i*) Suppose that there exists a constant $C > 0$ such that $\|x_n^{-1}\| \le C$ for all n, then $\|x_n^{-1}(x-x_n)\| < 1$ for n sufficiently large, which means that $e+x_n^{-1}(x-x_n) \in \mathcal{G}(\mathcal{A})$ by Theorem 1.2.7 and $x = x_n(e + x_n^{-1}(x - x_n)) \in \mathcal{G}(\mathcal{A})$. Contradiction.

(*ii*) Let $y_n = \frac{1}{\|x_n^{-1}\|}x_n^{-1}$. Then

$$\begin{aligned}\|xy_n\| &= \frac{\|xx_n^{-1}\|}{\|x_n^{-1}\|} \\ &= \frac{\|(x - x_n)x_n^{-1} + e\|}{\|x_n^{-1}\|} \\ &\le \frac{\|(x - x_n)x_n^{-1}\| + \|e\|}{\|x_n^{-1}\|} \\ &\le \|x - x_n\| + \frac{\|e\|}{\|x_n^{-1}\|} \longrightarrow 0\end{aligned}$$

as $n \to \infty$. Hence $xy_n \to 0$. Similarly we can show $y_n x \to 0$.

□

Definition 1.2.11 Let $\mathcal{A}$ be an algebra. An element $x \in \mathcal{A}$ is called *quasi-invertible*, if there exists some $y \in \mathcal{A}$ such that

$$xy + x + y = yx + x + y = 0.$$

Then y, denoted by x_{-1}, is called the *quasi-inverse* of x. Let $\mathcal{Q}(\mathcal{A})$ denote the set of all quasi-invertible elements (note that $0 \in \mathcal{Q}(\mathcal{A})$).

Proposition 1.2.12 *Let $\mathcal{A}$ be an algebra with identity e. Then $(e + x)$ is invertible if and only if x is quasi-invertible.*

Proof. Let $y = \tilde{y} - e$. Then

$$\begin{aligned}(e+x) \in \mathcal{G}(\mathcal{A}) &\Longleftrightarrow \exists\ \tilde{y} : (e+x)\tilde{y} = e = \tilde{y}(e+x) \\ &\Longleftrightarrow \exists\ y : (e+x)(y+e) = e = (y+e)(e+x) \\ &\Longleftrightarrow \exists\ y : y+e+xy+x = e = y+yx+e+x \\ &\Longleftrightarrow \exists\ y : xy+x+y = 0 = yx+x+y \\ &\Longleftrightarrow x \in \mathcal{Q}(\mathcal{A}).\end{aligned}$$

□

Remark 1.2.13 Let $\mathcal{A}$ be a unital algebra and $x \in \mathcal{A}$.

- As inverses, quasi-inverses are unique. Indeed, let $y, z \in \mathcal{A}$ be two quasi-inverses of x. Then
$$(e+y)(e+x) = (e+z)(e+x) = e,$$
so $(e+y) = (e+z)$ by uniqueness of inverses and hence $y = z$.

- It is easily checked that if x is quasi-invertible then
$$(e+x)^{-1} = (e+x_{-1}),$$
since
$$(e+x_{-1})(e+x) = e+x+x_{-1}+x_{-1}x = e+0 = e.$$

Theorem 1.2.14 *Let $\mathcal{A}$ be a Banach algebra with identity e and let $x \in \mathcal{A}$ with $\|x\| < 1$. Then $x \in \mathcal{Q}(\mathcal{A})$.*

Proof. By Theorem 1.2.7, $\|x\| = \|e-(e-x)\| < 1$ implies that $(e-x) \in \mathcal{G}(\mathcal{A})$, hence $x \in \mathcal{Q}(\mathcal{A})$. □

Similarly, one can show that $\mathcal{Q}(\mathcal{A})$ is an open subset of $\mathcal{A}$ (see Corollary 1.2.8) and that the mapping $\mathcal{Q}(\mathcal{A}) \to \mathcal{Q}(\mathcal{A}), x \mapsto x_{-1}$ is continuous (see Corollary 1.2.9).

1.3. Spectrum, resolvent set and spectral radius

In this section we introduce the basic concept of the spectrum of an element of a Banach algebra and establish various important results about spectra. We begin with the definition of a spectrum and then prove some fundamental theorems (such as Gelfand-Mazur), the most important one of these being that the spectrum of a Banach algebra is a nonempty compact subset of $\mathbb{C}$. After that we will provide a proof for the spectral mapping theorem and the spectral radius formula.

Definition 1.3.1 Let $\mathcal{A}$ be a Banach algebra with identity e. For $x \in \mathcal{A}$ the set

$$\sigma_{\mathcal{A}}(x) = \{\lambda \in \mathbb{C} : \lambda e - x \notin \mathcal{G}(\mathcal{A})\}$$

is called the *spectrum* of x in $\mathcal{A}$, and the complement $\rho_{\mathcal{A}}(x) = \mathbb{C} \setminus \sigma_{\mathcal{A}}(x)$ the *resolvent set* of x. When $\mathcal{A}$ does not have an identity, we define $\sigma_{\mathcal{A}}(x)$ and $\rho_{\mathcal{A}}(x)$ by $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{A}_e}(x)$ and $\rho_{\mathcal{A}}(x) = \rho_{\mathcal{A}_e}(x)$.

Theorem 1.3.2 *For a Banach algbera $\mathcal{A}$ and $x \in \mathcal{A}$*

$$\sigma_{\mathcal{A}}(x) \cup \{0\} = \sigma_{\mathcal{A}_e}(x).$$

Proof. Suppose that $\mathcal{A}$ does not have an identity. Then $0 \in \sigma_{\mathcal{A}}(x)$ because otherwise $x^{-1}x \in \mathcal{A}$. Hence we get $\sigma_{\mathcal{A}}(x) \cup \{0\} = \sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{A}_e}(x)$.

Now suppose that $\mathcal{A}$ has an identity u. For $y \in \mathcal{A}$ we can verify that

$$(u - y) \in \mathcal{G}(\mathcal{A}) \Leftrightarrow (e - y) \in \mathcal{G}(\mathcal{A}_e).$$

Indeed, if $(u - y) \in \mathcal{G}(\mathcal{A})$ and $(u - y)^{-1} = z + u, z \in \mathcal{A}$, then $(e - y)(z + e) = e = (z + e)(e - y)$. Conversely, if $(e - y) \in \mathcal{G}(\mathcal{A}_e)$ and $(e - y)^{-1} = z + \mu e, z \in \mathcal{A}, \mu \in \mathbb{C}$, then we get $\mu = 1$ and $(u - y)(z + u) = u = (z + u)(u - y)$.

This implies, for $\lambda \neq 0$ and $x \in \mathcal{A}$, that

$$\lambda u - x = \lambda(u - \frac{1}{\lambda}x) \notin \mathcal{G}(\mathcal{A}) \Leftrightarrow \lambda e - x = \lambda(e - \frac{1}{\lambda}x) \notin \mathcal{G}(\mathcal{A}_e),$$

which is equivalent to $\lambda \in \sigma_{\mathcal{A}}(x) \Leftrightarrow \lambda \in \sigma_{\mathcal{A}_e}(x)$. Thus $\sigma_{\mathcal{A}}(x) \setminus \{0\} = \sigma_{\mathcal{A}_e}(x) \setminus \{0\}$. And since $0 \in \sigma_{\mathcal{A}_e}(x)$ this shows that $\sigma_{\mathcal{A}}(x) \cup \{0\} = \sigma_{\mathcal{A}_e}(x)$.

□

Remark 1.3.3

- In most cases, whenever the algebra $\mathcal{A}$ under consideration is understood, we simply write $\sigma(x)$ and $\rho(x)$ for $x \in \mathcal{A}$.
- Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. For all $\lambda \neq 0$ we have

  $\lambda \in \sigma(x) \Leftrightarrow (\lambda e - x) \notin \mathcal{G}(\mathcal{A}) \Leftrightarrow (e - \frac{x}{\lambda}) \notin \mathcal{G}(\mathcal{A}) \Leftrightarrow \frac{x}{\lambda} \notin \mathcal{Q}(\mathcal{A})$ by Proposition 1.2.12.

Example 1.3.4 We have $\sigma(\lambda e) = \{\lambda\}$ for any $\lambda \in \mathbb{C}$.

Example 1.3.5 Let $\mathcal{X}$ be a compact Hausdorff space. If $f \in C(\mathcal{X})$, then

$$\sigma_{C(\mathcal{X})}(f) = f(\mathcal{X}) = \{f(x) : x \in \mathcal{X}\}.$$

Indeed,

$$\begin{aligned} \lambda \in \sigma_{C(\mathcal{X})}(f) &\iff \lambda e - f \notin \mathcal{G}(C(\mathcal{X})) \\ &\iff (\lambda e - f)(x) = 0 \text{ for some } x \in X \text{ (by Example 1.2.4)} \\ &\iff \lambda = f(x) \text{ for some } x \in \mathcal{X} \\ &\iff \lambda \in f(\mathcal{X}). \end{aligned}$$

Example 1.3.6 Let $\mathcal{X}$ be a locally compact, noncompact Hausdorff space and let $f \in C_0(\mathcal{X})$. We claim that

$$\sigma_{C_0(\mathcal{X})}(f) = f(\mathcal{X}) \cup \{0\}.$$

Since $C_0(\mathcal{X})$ does not have an identity, we have $\{0\} \cup f(\mathcal{X}) \subseteq \sigma_{C_0(\mathcal{X})}(f)$. Conversely, let $\lambda \neq 0$ such that $\lambda \notin f(\mathcal{X})$. Then the function $\lambda - f(x)$ is invertible in

$$C_0(\mathcal{X})_e = C_0(\mathcal{X}) + \mathbb{C} \cdot \mathbb{1}_{\mathcal{X}}.$$

Define the function g by

$$g(x) = \frac{f(x)}{1 - \frac{1}{\lambda} f(x)}, \quad x \in \mathcal{X}.$$

Then g is continuous on $\mathcal{X}$ by hypothesis. Furthermore, because f vanishes at infinity, so does g. Hence $g \in C_0(\mathcal{X})$ and

$$\left(\frac{1}{\lambda} + g(x)\right)(\lambda - f(x)) = \frac{1 + f(x)(\lambda - \frac{1}{\lambda})}{\lambda - f(x)}(\lambda - f(x)) = 1$$

for all $x \in \mathcal{X}$ and $f \in C_0(\mathcal{X})$. This proves that $\sigma_{C_0(\mathcal{X})}(f) = f(\mathcal{X}) \cup \{0\}$.

Example 1.3.7 If $\mathcal{X}$ is a finite-dimensional Banach space and $T \in \mathfrak{B}(\mathcal{X})$, then

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}.$$

Indeed,

$$\begin{aligned} \lambda \in \sigma(T) &\Longleftrightarrow T - \lambda \notin \mathcal{G}(\mathfrak{B}(\mathcal{X})) \\ &\Longleftrightarrow \ker(\lambda I - T) \neq \{0\} \text{ (by Example 1.2.4)} \\ &\Longleftrightarrow (\lambda I - T)(x) = 0 \text{ for some nonzero } x \in \mathcal{X} \\ &\Longleftrightarrow Tx = \lambda x \text{ for some nonzero } x \in \mathcal{X} \\ &\Longleftrightarrow \lambda \text{ is an eigenvalue of } T. \end{aligned}$$

If $\mathcal{X}$ is an infinite-dimensional Banach space, then a similar argument can be made: $\sigma(T)$ contains the eigenvalues of T, but generally this inclusion is strict.

Lemma 1.3.8 *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{G}(\mathcal{A})$. Then*

$$\sigma(x^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(x)\}.$$

Proof. Since x and x^{-1} are invertible, 0 is not in $\sigma(x)$ or $\sigma(x^{-1})$. If $\lambda \in \mathbb{C} \setminus \{0\}$ then $\lambda^{-1} - x^{-1} = \lambda^{-1}(x - \lambda)x^{-1}$ by (1.2). Since $\lambda^{-1}(x - \lambda)$ and x^{-1} commute we have

$$\lambda \in \sigma(x) \Leftrightarrow\ x - \lambda e \notin \mathcal{G}(\mathcal{A}) \Leftrightarrow \lambda^{-1} - x^{-1} \notin \mathcal{G}(\mathcal{A}) \Leftrightarrow \lambda^{-1} \in \sigma(x^{-1}).$$

□

Lemma 1.3.9 *Let $\mathcal{A}$ be a unital Banach algebra and let $x, y \in \mathcal{A}$. Then*

$$\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\}.$$

Proof. Suppose that $(e - yx) \in \mathcal{G}(\mathcal{A})$. Then there exists $z \in \mathcal{A}$ such that

$$z(e - yx) = (e - yx)z = e.$$

- First we verify $z(e - yx) = e$:

$$\begin{aligned}(e + yzx)(e - yx) &= ee + eyzx - eyx - yzxyx \\ &= e + yzx - yx - yzxyx \\ &= e + x(yz - y - yzxy) \\ &= e + x(y(z - zxy) - y) \\ &= e + x(y(z(e - xy)) - y) \\ &= e + y(ye - y) \\ &= e.\end{aligned}$$

- Now we check $(e - yx)z = e$:

$$\begin{aligned}(e - yx)(e + yzx) &= e - yx + yzx - yxyzx \\ &= e + x(-y + yz - yxyz) \\ &= e + x(y(z - xyz) - y) \\ &= e + x(y(z(e - xy)) - y) \\ &= e.\end{aligned}$$

In a similar way, we can prove that $(e - xy) \in \mathcal{G}(\mathcal{A})$. Now let $\lambda \in \mathbb{C} \setminus \{0\}$. Then

$$xy - \lambda e = -\lambda(e - x\lambda^{-1}y) \text{ and } yx - \lambda e = -\lambda(e - y\lambda^{-1}x),$$

i.e. $xy - \lambda e \in \mathcal{G}(\mathcal{A}) \Leftrightarrow yx - \lambda e \in \mathcal{G}(\mathcal{A})$. So we have $\lambda \in \rho(xy) \Leftrightarrow \lambda \in \rho(yx)$ and eventually

$$\begin{aligned}\sigma(xy) \setminus \{0\} &= (\mathbb{C} \setminus \rho(xy)) \setminus \{0\} = \mathbb{C} \setminus (\rho(xy) \cup \{0\}) \\ &= \mathbb{C} \setminus (\rho(yx) \cup \{0\}) = (\mathbb{C} \setminus \rho(yx)) \setminus \{0\} \\ &= \sigma(yx) \setminus \{0\}.\end{aligned}$$

□

Definition 1.3.10 Let $\mathcal{A}$ be a Banach algebra. The *spectral radius* of an element $x \in \mathcal{A}$ is the number

$$r_{\mathcal{A}}(x) = r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Remark 1.3.11 Obviously, $r(\lambda x) = |\lambda| r(x)$ for $\lambda \in \mathbb{C}$. The spectral radius is the radius of the smallest closed circular disc in $\mathbb{C}$, with center at 0, which contains $\sigma(x)$. Clearly, the definition given above makes no sense if $\sigma(x)$ is empty. But this never happens, as we shall see in Theorem 1.3.16.

Example 1.3.12 If $\mathcal{X}$ is a compact Hausdorff space and $f \in C(\mathcal{X})$, then

$$r(f) = \sup\{|\lambda| : \lambda \in \sigma(f)\} = \sup\{|\lambda| : \lambda \in f(\mathcal{X})\} = \|f\|.$$

Example 1.3.13 To see that strict inequality is possible, take $\mathcal{X} = \mathbb{C}^2$ with the usual Hilbert space norm and let $T \in \mathfrak{B}(\mathcal{X})$ be the operator with matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since $\det(T - \lambda I) = \lambda^2$, the only eigenvalue of T is 0 and so $\sigma(T) = \{0\}$ by Example 1.3.4. Thus $r(T) = 0 < 1 = \|T\|$.

The following theorem is one the most fundamental results in the theory of Banach algebras. The proof given here is the standard one involving an application of Liouville's theorem. This use of Liouville's theorem is the first example of how the theory of holomorphic functions of one complex variable enters the study of Banach algebras. An alternative proof can be found in [Kan], Section 1.2, p.12.
Before we state the theorem, we first give a vector-valued version of Liouville.

Theorem 1.3.14 (**Liouville**) *Every bounded entire function is constant. That is, every holomorphic function $f : \mathbb{C} \to \mathbb{C}$ for which there exists a constant $c \in \mathbb{R}$ such that $|f(z)| \leq c$ for all $z \in \mathbb{C}$, is constant.*

Lemma 1.3.15 *Let $\mathcal{X}$ be a Banach space and suppose that $f : \mathbb{C} \to \mathcal{X}$ is an entire function in the sense that $\frac{1}{\mu-\lambda}(f(\mu) - f(\lambda))$ converges in $\mathcal{X}$ as $\mu \to \lambda$, for every $\lambda \in \mathbb{C}$. If f is bounded, then f is constant.*

Proof. Given a continuous linear functional $\vartheta \in \mathcal{X}'$, let $g = \vartheta \circ f : \mathbb{C} \to \mathbb{C}$. Since

$$\frac{1}{\mu - \lambda}(g(\mu) - g(\lambda)) = \vartheta\Big(\frac{1}{\mu - \lambda}(f(\mu) - f(\lambda))\Big) \text{ and } |g(\lambda)| \leq \|g\|\|f(\lambda)\|,$$

the function g is entire and bounded. By Liouville's theorem it is constant, so $\vartheta(f(\lambda)) = \vartheta(f(\mu))$ for all $\vartheta \in \mathcal{X}'$ and $\lambda, \mu \in \mathbb{C}$. By the Hahn-Banach theorem it follows that $f(\lambda) = f(\mu)$ for all $\lambda, \mu \in \mathbb{C}$. Hence f is constant. □

We are now in a position to prove the **fundamental theorem of Banach algebra theory**.

Theorem 1.3.16 *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. Then $\sigma(x)$ is a nonempty compact subset of $\mathbb{C}$ with $\sigma(x) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$.*

Proof. The map $f : \mathbb{C} \to \mathcal{A},\ \lambda \mapsto \lambda e - x$ is continuous and

$$\sigma(x) = \{\lambda \in \mathbb{C} : f(\lambda) \notin \mathcal{G}(\mathcal{A})\} = \mathbb{C} \setminus f^{-1}(\mathcal{G}(\mathcal{A})).$$

Since $\mathcal{G}(\mathcal{A})$ is open by Corollary 1.2.8 and f is continuous, $f^{-1}(\mathcal{G}(\mathcal{A}))$ is open and so its complement $\sigma(x)$ is closed. If $|\lambda| > \|x\|$, then $\|\lambda^{-1}x\| = |\lambda|^{-1}\|x\| < 1$ and hence $\lambda e - x = \lambda(e - \lambda^{-1}x)$ is invertible by Theorem 1.2.7, so $\lambda \notin \sigma(x)$. Thus $\sigma(x)$ is contained in the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$. In particular, $\sigma(x)$ is bounded as well as closed, so $\sigma(x)$ is a compact subset of $\mathbb{C}$.

We next show that $\sigma(x) \neq \emptyset$. If $\sigma(x) = \emptyset$, then the map

$$R : \mathbb{C} \to \mathcal{A},\ \lambda \mapsto (\lambda e - x)^{-1}$$

is well-defined. For $\lambda, \mu \in \mathbb{C}$ we can compute that

$$\begin{aligned}(\lambda e - x)^{-1} &= (\lambda e - x)^{-1}(\mu e - x)(\mu e - x)^{-1} \\ &= (\lambda e - x)^{-1}((\mu - \lambda)e + \lambda e - x)(\mu e - x)^{-1} \\ &= ((\mu - \lambda)(\lambda e - x)^{-1} + e)(\mu e - x)^{-1} \\ &= (\mu - \lambda)(\lambda e - x)^{-1}(\mu e - x)^{-1} + (\mu e - x)^{-1}\end{aligned}$$

and therefore, with $\lambda \neq \mu$,

$$\frac{R(\mu) - R(\lambda)}{\mu - \lambda} = -R(\lambda)R(\mu).$$

Since R is continuous by Corollary 1.2.9, we conclude that R is an entire function with derivate

$$R'(\lambda) = \lim_{\mu \to \lambda} \frac{R(\mu) - R(\lambda)}{\mu - \lambda} = -R(\lambda)^2.$$

Now $\|R(\lambda)\| = \|(\lambda e - x)^{-1}\| = |\lambda|^{-1}\|(e - \lambda^{-1}x)^{-1}\|$ and $e - \lambda^{-1}x \to e$ as $|\lambda| \to \infty$, so by Corollary 1.2.9 we get $(e - \lambda^{-1}x)^{-1} \to e$. Consequently $\|R(\lambda)\| \to 0$ as $|\lambda| \to \infty$. So R is a bounded entire function and therefore constant by Lemma 1.3.15. Since $R(\lambda) \to 0$ as $|\lambda| \to \infty$ we have $R(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. This is a contradiction since $R(\lambda)$ is invertible for any $\lambda \in \mathbb{C}$.

□

Remark 1.3.17 Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. The map

$$R_\lambda(x) : \rho(x) \to \mathcal{A},\ \lambda \mapsto (\lambda e - x)^{-1}$$

is called the *resolvent* of x in $\mathcal{A}$.

The next theorem by *Gelfand-Mazur* is truly one of the most fundamental theorems in the study of (commutative) Banach algebras. Its important role in the investigation of such algebras will become apparent when we delevop Gelfand's representation theory for commutative Banach algebras in Chapter 2. The theorem turns out to be a simple consequence of Theorem 1.3.16 and generalizes Frobenius' classical theorem which states that every finite-dimensional complex division algebra is isomorphic to the complex number field.

Theorem 1.3.18 (**Gelfand-Mazur**) *Let $\mathcal{A}$ be a Banach algebra with identity e, and suppose that every nonzero element x of $\mathcal{A}$ is invertible. Then $\mathcal{A}$ is (isometrically) isomorphic to the field of complex numbers.*

Proof. Let $x \in \mathcal{A}$. Since $\sigma(x) \neq \emptyset$ by Theorem 1.3.16, there exists $\lambda_x \in \mathbb{C}$ such that $\lambda_x e - x \notin \mathcal{G}(\mathcal{A})$. Since $\mathcal{G}(\mathcal{A}) = \mathcal{A} \setminus \{0\}$ by hypothesis, it follows that $\lambda_x e = x$. Then λ_x is unique and the mapping $x \mapsto \lambda_x$ is an isomorphism of $\mathcal{A}$ onto $\mathbb{C}$, which is also an isometry, since $|\lambda_x| = \|\lambda_x e\| = \|x\|$ for every $x \in \mathcal{A}$.

□

Remark 1.3.19 The Gelfand-Mazur theorem particularly asserts that every Banach division algebra is commutative. If we drop our standard assumption that $\|e\| = 1$, then we can only conclude that the isomorphism from $\mathcal{A}$ to $\mathbb{C}$ is continuous.
Mazur's original proof, which depends on the submultilplicativity of the norm, can be found in [Zel] pp.18.

This is good point at which to emphasize the advantages of considering only algebras over $\mathbb{C}$. If $\mathcal{A}$ is a normed division algebra over $\mathbb{R}$, then it may be isomorphic to either the complex numbers, the real numbers, or the quaternions $\mathbb{H}$.

We additionally present two conditions on a Banach algebra $\mathcal{A}$ which ensure that it is isomorphic to $\mathbb{C}$.

Theorem 1.3.20 (**Edwards**) *Let $\mathcal{A}$ be a unital Banach algebra. If*

$$\|x^{-1}\| \leq \|x\|^{-1}, \quad x \in \mathcal{G}(\mathcal{A}),$$

then there exists an isometric isomorphism of $\mathcal{A}$ onto $\mathbb{C}$.

Proof. In view of the Gelfand-Mazur theorem it is sufficient to show that $\mathcal{A}$ is a division algebra, i.e. $\mathcal{G}(\mathcal{A}) = \mathcal{A} \setminus \{0\}$.

Note that $\mathcal{A} \setminus \{0\}$ is connected[4], since any two points of $\mathcal{A}$ lie in a 2-dimensional linear subspace of $\mathcal{A}$, and $\mathbb{C}^2 \setminus \{0\}$ is connected. By Lemma 1.2.8, $\mathcal{G}(\mathcal{A})$ is open in $\mathcal{A}$, and clearly also open in $\mathcal{A} \setminus \{0\}$.

We show that $\mathcal{G}(\mathcal{A})$ is closed in $\mathcal{A}$. To that end let $x_n \to x$ be a convergent sequence in $\mathcal{A} \setminus \{0\}$ with $x_n \in \mathcal{G}(\mathcal{A})$ for all $n \in \mathbb{N}$. Therefore there exists $\varepsilon > 0$ such that $\|x_n\| \geq \varepsilon$ for all $n \in \mathbb{N}$ as well as $\|x\| \geq \varepsilon$. But then we have

$$\|x_n^{-1}\| \leq \frac{1}{\|x_n\|} \leq \frac{1}{\varepsilon}, \quad n \in \mathbb{N}.$$

Hence we obtain

$$\|x_n^{-1} - x_m^{-1}\| \leq \|x_m^{-1}(x_n - x_m)x_n^{-1}\| \leq \|x_m^{-1}\|\|x_n - x_m\|\|x_n^{-1}\| \leq \frac{1}{\varepsilon^2}\|x_n - x_m\|$$

for all $m \in \mathbb{N}$. Since $(x_n)_n$ is a Cauchy sequence, $(x_n^{-1})_n$ is also a Cauchy sequence, and so there exists some $y \in \mathcal{A}$ such that $\lim_{n\to\infty} \|x_n^{-1} - y\| = 0$. But then

$$xy = \lim_{n\to\infty} (x_n) \lim_{n\to\infty} (x_n^{-1}) = \lim_{n\to\infty} (x_n x_n^{-1}) = \lim_{n\to\infty} (e) = e,$$

and similarly $yx = e$, so $y = x^{-1}$, and hence $x \in \mathcal{G}(\mathcal{A})$. Thus we see that $\mathcal{G}(\mathcal{A})$ is an open and closed ("clopen") subset of the connected set $\mathcal{A} \setminus \{0\}$. Hence $\mathcal{G}(\mathcal{A})$ is either empty or $\mathcal{A} \setminus \{0\}$ itself, but it cannot be empty since $e \in \mathcal{G}(\mathcal{A})$, whence $\mathcal{G}(\mathcal{A}) = \mathcal{A} \setminus \{0\}$. □

Corollary 1.3.21 (**Mazur**) *Let $\mathcal{A}$ be a unital Banach algebra. If*

$$\|xy\| = \|x\|\|y\|, \quad x, y \in \mathcal{A},$$

then there exists an isometric isomorphism of $\mathcal{A}$ onto $\mathbb{C}$.

[4] See Definition 2.4.10.

Proof. If $x \in \mathcal{G}(\mathcal{A})$, then $1 = \|e\| = \|xx^{-1}\| = \|x\|\|x^{-1}\|$, whence $\|x^{-1}\| = \|x\|^{-1}$. □

Our next goal is to show the *spectral radius formula* which states that $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ equals $r(x)$. In order to establish this result we first need to prove another theorem, which is of considerable interest in itself - the *spectral mapping theorem for polynomials.* It represents a special case of a more general spectral mapping theorem in the context of the holomorphic functional calculus for commutative Banach algebras (Section 3.1).

Definition 1.3.22 If x is an element of a unital Banach algebra $\mathcal{A}$ and $p \in \mathbb{C}[z]$ is a *complex polynomial*, say $p(z) = \lambda_0 + \lambda_1 z + \ \dots \ + \lambda_n z^n$ where $\lambda_0, \lambda_1, \dots, \lambda_n$ are complex numbers, then we write

$$p(x) = \lambda_0 e + \lambda_1 x + \ \dots \ + \lambda_n x^n.$$

Obviously $p(x)$ is an element of $\mathcal{A}$ whenever $x \in \mathcal{A}$. Consider the mapping

$$\vartheta_x : \mathbb{C}[z] \to \mathcal{A}, \ p \mapsto p(x).$$

It is easliy verified that ϑ_x is an algebra homomorphism. In particular, the range of ϑ_x is a commutative subalgebra of $\mathcal{A}$.

Theorem 1.3.23 (**Polynomial spectral mapping theorem**) *If p is a complex polynomial and x an element of a unital Banach algebra $\mathcal{A}$, then*

$$\sigma(p(x)) = p(\sigma(x)) = \{p(\lambda) : \lambda \in \sigma(x)\}.$$

Proof. If p is constant, say $p = \alpha$, then $p(x) = \alpha e$ and hence $\sigma(p(x)) = \sigma(\alpha e) = \{\alpha\} = p(\sigma(x))$.

So let p be non-constant and suppose that $n = \deg p \geq 1$. For now, fix any $\lambda \in \mathbb{C}$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be the roots of the polynomial $q(z) = \lambda - p(z)$. Since $\mathbb{C}$ is algebraically closed and by using the fundamental theorem of algebra we can write

$$\lambda e - p(x) = (\lambda - p(z))(x) = \alpha(\lambda_1 e - x) \cdot \ \dots \ \cdot (\lambda_n e - x)$$

where $\alpha \in \mathbb{C} \setminus \{0\}$. Therefore $\lambda e - p(x) \in \mathcal{G}(\mathcal{A}) \Leftrightarrow \lambda_i e - x \in \mathcal{G}(\mathcal{A})$ for all $i = 1, \dots, n$. It follows that if $\lambda \in \sigma(p(x))$ then $\lambda_i \in \sigma(x)$ for at least one i and hence $\lambda = p(\lambda_i) \in p(\sigma(x))$. This shows $\sigma(p(x)) \subseteq p(\sigma(x))$.

Conversely, let $\mu \in \sigma(x)$ and put $\lambda = p(\mu)$. Then we get $q(\mu) = p(\mu) - p(\mu) = 0$ and thus $\mu = \lambda_i$ for some i. This means that $\lambda_i \in \sigma(x)$ and consequently $\lambda e - p(x) \notin \mathcal{G}(\mathcal{A})$, whence $\lambda \in \sigma(p(x))$. □

Eventually, we are able to prove the spectral radius formula.

Theorem 1.3.24 (**Spectral radius formula**) *The spectral radius of an element x of a unital Banach algebra $\mathcal{A}$ is given by*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}}.$$

Proof. If $\lambda \in \sigma(x)$ and $n \in \mathbb{N}$, then $\lambda^n \in \sigma(x^n)$ by Theorem 1.3.23. So we have $|\lambda|^n \leq \|x^n\|$ by Theorem 1.3.16. Hence $|\lambda| \leq \|x^n\|^{\frac{1}{n}}$ and $r(x) \leq \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}}$. Now consider the function

$$S : \left\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{r(x)}\right\} \to \mathcal{A},\ \lambda \mapsto (e - \lambda x)^{-1}.$$

We observe that for $|\lambda| < \frac{1}{r(x)}$ we have $r(\lambda x) = |\lambda| r(x) < 1$ by Theorem 1.3.23, so $(e - \lambda x) \in \mathcal{G}(\mathcal{A})$ and $S(\lambda)$ is well-defined. Now, we can make the same argument as in the proof of Theorem 1.3.16 to see that S is holomorphic. By Theorem 1.2.7 we have $S(\lambda) = \sum_{n=0}^{\infty} \lambda^n x^n$ for $|\lambda| < \frac{1}{\|x\|}$. If $\vartheta \in \mathcal{A}'$ with $\|\vartheta\| = 1$, then the complex-valued function $f = \vartheta \circ S$ is given by the power series $f(\lambda) = \sum_{n=0}^{\infty} \vartheta(x^n)\lambda^n$ for $|\lambda| < \frac{1}{\|x\|}$. Moreover, f is holomorphic for $|\lambda| < \frac{1}{r(x)}$, so this series converges to $f(\lambda)$ for $|\lambda| < \frac{1}{r(x)}$. Thus for $R > r(x)$ we get

$$\vartheta(x^n) = \frac{1}{2\pi i} \int_{|\lambda| = \frac{1}{R}} \frac{1}{\lambda^{n+1}} f(\lambda)\, d\lambda,$$

and obtain the estimate

$$|\vartheta(x^n)| \leq \frac{1}{2\pi} \cdot \frac{2\pi}{R} \cdot R^{n+1} \cdot \sup_{|\lambda| = \frac{1}{R}} |\vartheta(S(\lambda))| \leq R^n M(R),$$

where $M(R) = \sup_{|\lambda| = \frac{1}{R}} \|S(\lambda)\|$, which is finite by the continuity of S on the compact set $\left\{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{R}\right\}$. Since $S(\lambda) \neq 0$ for any λ in the domain of S, we have $M(R) > 0$. Hence

$$\limsup_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leq \limsup_{n\in\mathbb{N}} R(M(R))^{\frac{1}{n}} = R,$$

whenever $R > r(x)$. We conclude that

$$r(x) \leq \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leq \liminf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leq \limsup_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leq r(x),$$

which completes the proof. □

Next we want to show an interesting identity.

Corollary 1.3.25 *If $\mathcal{A}$ is a unital Banach algebra and $\mathcal{B}$ is a closed unital subalgebra of $\mathcal{A}$, then*

$$r_{\mathcal{A}}(y) = r_{\mathcal{B}}(y) \quad \text{for all } y \in \mathcal{B}.$$

Proof. The norm of an element of $\mathcal{B}$ is the same whether we measure it in $\mathcal{B}$ or in $\mathcal{A}$. By the spectral radius formula we get $r_{\mathcal{A}}(y) = \lim_n \|y^n\|^{\frac{1}{n}} = r_{\mathcal{B}}(y)$. □

Remark 1.3.26

- If $\mathcal{X}$ is a compact Hausdorff space, then it is evident that $\|f\|_\infty = \lim_n \|f^n\|_\infty^{\frac{1}{n}} = r(f),\ f \in C(\mathcal{X})$. But for arbitrary Banach algebras it is not generally the case that $\|x\| = r(x), x \in \mathcal{A}$.
- Whether an element of $\mathcal{A}$ is or is not invertible in $\mathcal{A}$ is a purely algebraic property. The spectrum and the spectral radius of some $x \in \mathcal{A}$ are thus defined in terms of the algebraic structure of $\mathcal{A}$, regardless of any metric (or topological) considerations. On the other hand, $\lim_n \|x^n\|^{\frac{1}{n}}$ depends obviously on metric properties of $\mathcal{A}$. This is one of the remarkable features of the spectral radius formula. It asserts the equality of certain quantities which arise in entirely different ways. So by "enlarging" the algebra, the spectrum may change, but the spectral radius will not.

Suppose that $\mathcal{A}$ is a Banach algebra with identity e and $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$ which contains e. Given an element x of $\mathcal{B}$, what can be said about the relationship between $\sigma_{\mathcal{A}}(x)$ and $\sigma_{\mathcal{B}}(x)$? The result, Theorem 1.3.28 below, together with its corollary, provides considerable information on this relationship and will be employed several times in Chapter 2.

For future references, we briefly recall the notion of topological boundary.

Definition 1.3.27 For any topological space $\mathcal{X}$ and subset $\mathcal{Y}$ of $\mathcal{X}$, $\mathcal{Y}^\circ$ denotes the *interior* of $\mathcal{Y}$ and $\partial(\mathcal{Y})$ denotes the *topological boundary* of $\mathcal{Y}$; that is

$$\mathcal{Y}^\circ = \mathcal{X} \setminus \overline{(\mathcal{X} \setminus \mathcal{Y})} \text{ and } \partial(\mathcal{Y}) = \overline{\mathcal{Y}} \setminus \mathcal{Y}^\circ.$$

Theorem 1.3.28 (**Shilov**) *Let $\mathcal{A}$ be a Banach algebra with identity e and $\mathcal{B}$ a closed subalgebra of $\mathcal{A}$ containing e. If $x \in \mathcal{B}$, then*

(i) $\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}(x)$ *and*

(ii) $\partial(\sigma_{\mathcal{B}}(x)) \subseteq \partial(\sigma_{\mathcal{A}}(x))$.

Proof.

(i) Clearly, if $\lambda e - x \notin \mathcal{G}(\mathcal{B})$, then $\lambda e - x \notin \mathcal{G}(\mathcal{A})$.

(ii) It suffices to show that $\partial(\sigma_{\mathcal{B}}(x)) \subseteq \sigma_{\mathcal{A}}(x)$, because then

$$\partial(\sigma_{\mathcal{B}}(x)) \subseteq \sigma_{\mathcal{A}}(x) \cap \overline{\rho_{\mathcal{B}}(x)} \subseteq \sigma_{\mathcal{A}}(x) \cap \overline{\rho_{\mathcal{A}}(x)} = \partial(\sigma_{\mathcal{A}}(x)).$$

Let $\lambda \in \partial(\sigma_{\mathcal{B}}(x))$ and set $y = \lambda e - x$. Then $y \notin \mathcal{G}(\mathcal{B})$ and there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\rho_{\mathcal{B}}(x)$ with $\lim_{n\to\infty} \lambda_n = \lambda$. Hence for $y_n = \lambda_n e - x$ we have $y_n \in \mathcal{G}(\mathcal{B})$ and $y_n \to y$. Let $z_n = y_n^{-1}, n \in \mathbb{N}$. Then $\|z_n\| \to \infty$ as $n \to \infty$. Indeed this is true, because otherwise there exists $C > 0$ and a subsequence $(z_{n_k})_k$ such that $\|z_{n_k}\| \leq C$ for all k and thus

$$\|e - z_{n_k} y\| = \|z_{n_k}(y_{n_k} - y)\| \leq C\|y_{n_k} - y\| \to 0 \text{ as } k \to \infty.$$

So it follows that $z_{n_k}y$ is invertible for large k, and hence y is invertible in $\mathcal{B}$, which is a contradiction. Therefore

$$\frac{\|z_n y_n\|}{\|z_n\|} = \frac{\|e\|}{\|z_n\|} = \frac{1}{\|z_n\|} \to 0 \text{ as } n \to \infty$$

and

$$\left| \frac{\|z_n y\| - \|z_n y_n\|}{\|z_n\|} \right| \leq \frac{1}{\|z_n\|} \|z_n(y - y_n)\| \leq \|y - y_n\| \to 0 \text{ as } n \to \infty.$$

Thus the elements $w_n = \|z_n\|^{-1} z_n, n \in \mathbb{N}$, of $\mathcal{A}$ satisfy $\|w_n\| = 1$ and $\|w_n y\| \to 0$ as $n \to \infty$. This implies that y cannot be invertible in $\mathcal{A}$ or else

$$1 = \|w_n\| = \|(w_n y)y^{-1}\| \leq \|w_n y\| \|y^{-1}\| \to 0 \text{ as } n \to \infty.$$

Therefore we have $\lambda \in \sigma_{\mathcal{A}}(x)$. □

The succeeding corollary applies, in particular, when $\sigma_{\mathcal{A}}(x) \subseteq \mathbb{R}$ or when $\sigma_{\mathcal{A}}(x)$ is countable.

Corollary 1.3.29 *Let $\mathcal{A}$ be a Banach algebra with identity e and let $x \in \mathcal{A}$. Then the following conditions are equivalent.*

(i) $\rho_{\mathcal{A}}$ is connected.

(ii) $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{B}}(x)$ for every closed subalgebra $\mathcal{B}$ of $\mathcal{A}$ containing x and e.

Proof. See [Kan], Section 1.2, p.15. □

We close this chapter by mentioning that the spectral radius is subadditive and submultiplicative on commuting elements.

Lemma 1.3.30 *Let $\mathcal{A}$ be a normed algebra and suppose that $x, y \in \mathcal{A}$ are such that $xy = yx$. Then*

(i) $r(xy) \leq r(x)r(y)$ and

(ii) $r(x + y) \leq r(x) + r(y)$.

Proof. (i) is easy to prove: since $(xy)^n = x^n y^n$ for all $n \in \mathbb{N}$, and by applying the spectral radius formula we get

$$r(xy) = \lim_{n\to\infty} \|x^n y^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \cdot \lim_{n\to\infty} \|y^n\|^{\frac{1}{n}} = r(x)r(y).$$

Proving (ii) however, requires more effort and is rather technical. Pick $\alpha > r(x)$ and $\beta > r(y)$, and let $a = x/\alpha$ and $b = y/\beta$. Then $r(a) < 1$ and $r(b) < 1$. Because x and y commute we have

$$\|(x+y)^n\|^{\frac{1}{n}} = \left\| \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right\|^{\frac{1}{n}} \leq \left(\sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \|a^k\| \|b^{n-k}\| \right)^{\frac{1}{n}}.$$

Now let

$$k_n = \arg\max_k \|a^k\| \|b^{n-k}\|,$$

so

$$\|(x+y)^n\|^{\frac{1}{n}} \leq (\alpha+\beta) \|a^{k_n}\|^{\frac{1}{n}} \|b^{n-k_n}\|^{\frac{1}{n}}$$

for all $n \in \mathbb{N}$. Since $0 \leq k_n/n \leq 1$, we can choose a subsequence such that $k_{n_i}/n_i \to \delta$ for some δ as $i \to \infty$. Denote this subsequence by k_n. If $\delta = 0$, then

$$\limsup_{n\to\infty} \|a^{k_n}\|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \|a\|^{\frac{k_n}{n}} \leq 1.$$

If $\delta \neq 0$, then $k_{n_i} \neq 0$ for i big enough and thus

$$\limsup_{n\to\infty} \|a^{k_n}\|^{\frac{1}{n}} = \limsup_{n\to\infty} \left(\|a^{k_n}\|^{\frac{1}{k_n}} \right)^{\frac{k_n}{n}} = r(a)^{\delta} < 1,$$

because $r(a) \leq \|a\| < 1$. Therefore $r(x+y) \leq \alpha+\beta$ and since this holds for all $\alpha > r(x)$ and $\beta > r(y)$, the conclusion follows.

□

Remark 1.3.31 It should be noted that by using the Gelfand homomorphism (Section 2.3), a much simpler proof can be given. Thus, if $\mathcal{A}$ is a commutative algebra, r is an algebra seminorm on $\mathcal{A}$.

2. Gelfand Theory

The fundamental results presented in this chapter are the pioneering work of Gelfand [Gel]. Our focus will be almost entirely on commutative Banach algebras. This restriction is imposed because the main tool we shall utilize in the further study of Banach algebras is the Gelfand representation theory, which is valid only for commutative algebras. The reason for this will be quite obvious.

2.1. Ideals and multiplicative linear functionals

We first start with invastigating the link between maximal ideals of a unital Banach algebra and the structure space $\Delta(\mathcal{A})$ of all multiplicative linear functionals. Algebras in this section are not necessarily commutative.

Recall that if $\mathcal{Y}$ is a subspace of a complex vector space $\mathcal{X}$, then the *quotient vector space* $\mathcal{X}/\mathcal{Y}$ is given by $\mathcal{X}/\mathcal{Y} = \{x + \mathcal{Y} : x \in \mathcal{X}\}$ with

- $\lambda(x + \mathcal{Y}) = \lambda x + \mathcal{Y}$ (scalar multiplication),
- $(x + \mathcal{Y}) + (y + \mathcal{Y}) = (x + y) + \mathcal{Y}$ (vector addidtion), and
- $0 + \mathcal{Y} = \mathcal{Y}$ (zero vector in $\mathcal{X}/\mathcal{Y}$)

for all $x, y \in \mathcal{X}, \lambda \in \mathbb{C}$.

Definition 2.1.1 If $\mathcal{Y}$ is a closed subspace of a Banach space $\mathcal{X}$, then the *quotient Banach space* $\mathcal{X}/\mathcal{Y}$ is the vector space $\mathcal{X}/\mathcal{Y}$ equipped with the *quotient norm*, defined by

$$\|x + \mathcal{Y}\| = \inf_{y \in \mathcal{Y}} \|x + y\|.$$

Proposition 2.1.2 *Let $\mathcal{X}$ be a Banach space and let $\mathcal{Y}$ be a closed vector subspace of $\mathcal{X}$. The quotient norm is a norm on the vector space $\mathcal{X}/\mathcal{Y}$, with respect to which $\mathcal{X}/\mathcal{Y}$ is complete. Hence the quotient Banach space $\mathcal{X}/\mathcal{Y}$ is a Banach space.*

Proof. To see that the quotient norm is a norm, observe that

- $\|x + \mathcal{Y}\| \geq 0$ with equality if and only if $\inf_{y \in \mathcal{Y}} \|x + y\| = 0$, which is equivalent to x being in the closure of $\mathcal{Y}$. Since $\mathcal{Y}$ is closed, this means that $x \in \mathcal{Y}$, so $y + \mathcal{Y} = \mathcal{Y}$, the zero vector of $\mathcal{X}/\mathcal{Y}$.
- For $\lambda \in \mathbb{C} \setminus \{0\}$ we have

$$\begin{aligned} \|\lambda(x + \mathcal{Y})\| &= \inf_{y \in \mathcal{Y}} \|\lambda x + y\| = |\lambda| \inf_{y \in \mathcal{Y}} \|x + \lambda^{-1} y\| \\ &= |\lambda| \inf_{\tilde{y} \in \mathcal{Y}} \|x + \tilde{y}\| = |\lambda| \|x + \mathcal{Y}\|. \end{aligned}$$

- The triangle inequality holds since $\mathcal{Y} = \{s + t : s, t \in \mathcal{Y}\}$ and so for $x, z \in \mathcal{X}$

$$\begin{aligned}
\|(x+\mathcal{Y}) + (z+\mathcal{Y})\| &= \|(x+z)+\mathcal{Y}\| \\
&= \inf_{y\in\mathcal{Y}} \|x+z+y\| \\
&= \inf_{s,t\in\mathcal{Y}} \|x+z+s+t\| \\
&\le \inf_{s\in\mathcal{Y}} \|x+s\| + \inf_{t\in\mathcal{Y}} \|z+t\| \\
&= \|x+\mathcal{Y}\| + \|z+\mathcal{Y}\|.
\end{aligned}$$

It remains to show that $\mathcal{X}/\mathcal{Y}$ is complete in the quotient norm. For any $x \in \mathcal{X}$ we can see that

- if $\varepsilon > 0$ then there exists $y \in \mathcal{Y}$ such that $\|x+y\| < \|x+\mathcal{Y}\| + \varepsilon$ and
- $\|x+\mathcal{Y}\| \le \|x\|$, since $0 \in \mathcal{Y}$.

Let $x_k \in \mathcal{X}$ with $\sum_{k=1}^{\infty} \|x_k + \mathcal{Y}\| < \infty$. From the observation made above it follows that there exist $y_k \in \mathcal{Y}$ with

$$\sum_{k=1}^{\infty} \|x_k + y_k\| < \infty,$$

hence the series $\sum_{k=1}^{\infty} x_k + y_k$ converges in $\mathcal{X}$, say to $z \in \mathcal{X}$. Because

$$\left\| z + \mathcal{Y} - \Big(\sum_{k=1}^{n} x_k + \mathcal{Y} \Big) \right\| = \left\| \Big(z - \sum_{k=1}^{n} x_k + y_k \Big) + \mathcal{Y} \right\| \le \left\| z - \sum_{k=1}^{n} x_k + y_k \right\| \to 0$$

as $n \to \infty$, $\sum_{k=1}^{\infty} x_k + y_k$ converges to $z + \mathcal{Y}$. This shows that every absolutely convergent series in $\mathcal{X}/\mathcal{Y}$ is convergent with respect to the quotient norm, thus $\mathcal{X}/\mathcal{Y}$ is complete. □

Definition 2.1.3

- An *ideal* of a Banach algebra $\mathcal{A}$ is a subalgebra I of $\mathcal{A}$ such that for all $i \in I$ and $x \in \mathcal{A}$ we have $xi \in I$ and $ix \in I$.
- If $I \neq \mathcal{A}$ and $I \neq \{0\}$, then I is called a *proper ideal.*
- A *maximal ideal* of $\mathcal{A}$ is a proper ideal that is not contained in any strictly larger proper ideal of $\mathcal{A}$. We henceforth denote by $\mathrm{Max}(\mathcal{A})$ the set of all maximal ideals of $\mathcal{A}$.

Remark 2.1.4 A Banach algebra $\mathcal{A}$ is called *simple* if $\mathcal{A}^2 \neq \{0\}$ and if the only ideals in $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$.

Theorem 2.1.5 *Let I be a closed ideal of a Banach algebra $\mathcal{A}$ and $\mathcal{A}/I$ a quotient Banach space equipped with the product*

$$(\mathcal{A}/I) \times (\mathcal{A}/I) \to \mathcal{A}/I, \ \ ((x+I),(y+I)) \mapsto (x+I)(y+I) = xy + I \tag{2.1}$$

for all $x, y \in \mathcal{A}$. Then the following conditions hold.

(*i*) *$\mathcal{A}/I$ is a Banach algebra.*

(*ii*) *If $\mathcal{A}$ is commutative, then so is $\mathcal{A}/I$.*

(*iii*) *If $\mathcal{A}$ is unital, then so is $\mathcal{A}/I$ and $e_{\mathcal{A}/I} = e_{\mathcal{A}} + I$.*

Proof. (*i*) The product (2.1) is well-defined: if $x_1 + I = x_2 + I$ and $y_1 + I = y_2 + I$, then $x_1 - x_2 \in I$ and $y_1 - y_2 \in I$. Hence $x_1(y_1 - y_2) + (x_1 - x_2)y_2 \in I$ and so

$$(x_1y_1 + I) - (x_2y_2 + I) = x_1y_1 - x_2y_2 + I = x_1(y_1 - y_2) + (x_1 - x_2)y_2 + I = I,$$

thus $x_1y_1 + I = x_2y_2 + I$. It is easy to verify that the product is linear in each variable. Now let $x, y \in \mathcal{A}$. We have

$$\begin{aligned}
\|x + I\|\|y + I\| &= \inf_{i,j \in I} \|x + i\|\|y + j\| \\
&\geq \inf_{i,j \in I} \|(x + i)(y + j)\| \\
&= \inf_{i,j \in I} \|xy + \underbrace{xj + iy + ij}_{\in I}\| \\
&\geq \inf_{i \in I} \|xy + i\| = \|xy + I\| \\
&= \|(x + I)(y + I)\|,
\end{aligned}$$

and therefore $\mathcal{A}/I$ is a Banach algebra.

(*ii*) If $\mathcal{A}$ is commutative, then $(x + I)(y + I) = xy + I = yx + I = (y + I)(y + I)$ for all $x, y \in \mathcal{A}$, so $\mathcal{A}/I$ is commutative.

(*iii*) Since $(x + I)(e + I) = x + I = (e + I)(x + I)$, the element $e + I$ is an identity for $\mathcal{A}/I$. On the one hand we have

$$\|e + I\| = \inf_{i \in I} \|e + i\| \leq \|e + 0\| = 1,$$

and on the other hand, since I is a proper ideal, $e \notin I$, so $e + I \neq 0$. Hence $\|e + I\| \neq 0$. Moreover,

$$\|e + I\| = \|(e + I)(e + I)\| \leq \|(e + I)\|\|(e + I)\|,$$

and cancelling $\|e + I\|$ gives $\|e + I\| \geq 1$.

□

Lemma 2.1.6 *Let $\mathcal{A}$ be a unital Banach algebra. If $I \neq \{0\}$ is an ideal of $\mathcal{A}$, then I is a proper ideal if and only if $I \cap \mathcal{G}(\mathcal{A}) = \emptyset$.*

Proof. We have $e \in \mathcal{G}(\mathcal{A})$, so if $I \cap \mathcal{G}(\mathcal{A}) = \emptyset$, then $e \notin I$, so $I \neq \mathcal{A}$ and I is a proper ideal. Conversely, if $I \cap \mathcal{G}(\mathcal{A}) \neq \emptyset$, let $i \in I \cap \mathcal{A}$. If $x \in \mathcal{A}$, then $x = (xi^{-1})i \in I$, since I is an ideal and $i \in I$. Therefore $I = \mathcal{A}$. □

Theorem 2.1.7 *Let $\mathcal{A}$ be a unital Banach algebra.*

(*i*) *If I is a proper ideal, then the closure $\overline{I}$ is also a proper ideal of $\mathcal{A}$.*

(*ii*) *Any maximal ideal of* $\mathcal{A}$ *is closed.*

(*iii*) *Every proper ideal is contained in a maximal ideal.*

Proof. (*i*) The closure of a vector subspace of $\mathcal{A}$ is again a vector subspace. If $x \in \mathcal{A}$ and $(i_n)_{n\in\mathbb{N}}$ is a sequence in I converging to $i \in \overline{I}$, then $xi_n \to xi$ and $i_n x \to ix$ as $n \to \infty$. Since each xi_n and $i_n x$ is in I, this shows that xi and ix are in $\overline{I}$, which is therefore an ideal of $\mathcal{A}$. Because I is a proper ideal we have $I \cap \mathcal{G}(\mathcal{A}) = \emptyset$ by Lemma 2.1.6 and since $\mathcal{G}(\mathcal{A})$ is open by Corollary 1.2.8, this shows that $\overline{I} \cap \mathcal{G}(\mathcal{A}) = \emptyset$, so $\overline{I} \neq \mathcal{A}$.

(*ii*) Let M be a maximal ideal. Since $M \subseteq \overline{M}$ and $\overline{M}$ is a proper ideal by (*i*) we must have $M = \overline{M}$, so M is closed.

(*iii*) Let I be a proper ideal and $e \in \mathcal{A}$ an identity. Let $\mathcal{J}$ be the set of all ideals J of $\mathcal{A}$ such that $I \subseteq J$ and $e \notin J$. Then $\mathcal{J}$ is nonempty since $I \in \mathcal{J}$. We order $\mathcal{J}$ by inclusion and show that $\mathcal{J}$ satisfies the hypothesis of Zorn's lemma. Let $\mathcal{L}$ be a totally ordered subset of $\mathcal{J}$ and set

$$J = \bigcup_{L\in\mathcal{L}} L.$$

Then $e \notin J$ and J is an ideal since $\mathcal{L}$ is totally ordered. So $J \in \mathcal{J}$ and J is an upper bound for $\mathcal{L}$. Hence by Zorn's lemma $\mathcal{J}$ has a maximal element M, which is obviously a maximal ideal. □

Remark 2.1.8 If $\mathcal{A}$ is commutative, then every maximal ideal I has codimension 1. Indeed, the Banach algebra $\mathcal{A}/I$ has no invertible elements except 0. Hence by Gelfand-Mazur (Theorem 1.3.18), $\mathcal{A}/I$ is 1-dimensional, thus I has codimension 1.

Definition 2.1.9 Let $\mathcal{A}$ be a Banach algebra. A linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is called *multiplicative* if $\varphi \neq 0$ and

$$\varphi(xy) = \varphi(x)\varphi(y) \text{ for all } x, y \in \mathcal{A}.$$

The *kernel* of φ is the set

$$\ker\varphi = \{x \in \mathcal{A} : \varphi(x) = 0\}.$$

Throughout this thesis, for any Banach algebra $\mathcal{A}$, $\Delta(\mathcal{A})$ denotes the set of all nonzero multiplicative linear functionals on $\mathcal{A}$ and is called the *Gelfand space* or *structure space* of $\mathcal{A}$. Other common names given to $\Delta(\mathcal{A})$ are the *maximal ideal space* or *spectrum* of $\mathcal{A}$.

Example 2.1.10 Let $\mathcal{A} = C(\mathcal{X})$ where $\mathcal{X}$ is a compact Hausdorff space. For each $x \in \mathcal{X}$ define $\psi_x : \mathcal{A} \to \mathbb{C}$ by $\psi_x(f) = f(x)$ for all $f \in \mathcal{A}$. Then ψ_x is multiplicative.

Lemma 2.1.11 *Let* $\mathcal{A}$ *be a Banach algebra with identity* e*. If* $\varphi \in \Delta(\mathcal{A})$*, then* $\varphi(e) = 1$ *and* $I = \ker\varphi$ *is an ideal. If* $\widetilde{\varphi} \in \Delta(\mathcal{A})$ *with* $\ker\widetilde{\varphi} = I$*, then* $\varphi = \widetilde{\varphi}$*. For* $\dim\mathcal{A} > 1$ *it follows that* I *is a maximal ideal.*

Proof. Let φ be a multiplicative linear functional. Since $\varphi \neq 0$ we have $\varphi(x) \neq 0$ for some $x \in \mathcal{A}$ such that

$$\varphi(x) = \varphi(xe) = \varphi(x)\varphi(e).$$

Hence by cancelling $\varphi(x)$ we get $\varphi(e) = 1$.

Let $x \in \mathcal{A}$ and $i \in \ker\varphi$. Then

$$\varphi(xi) = \varphi(ix) = \varphi(x)\varphi(i) = \varphi(x) \cdot 0 = 0,$$

and thus $xi, ix \in \ker\varphi$. Hence $I = \ker\varphi$ is an ideal and since $\varphi \neq 0, I \neq \mathcal{A}$. If $\widetilde{\varphi}$ is another multiplicative linear functional with $\ker\widetilde{\varphi} = I$, then for $x \in \mathcal{A}$ it follows from $\varphi(x) - \varphi(x)\varphi(e) = \varphi(x - \varphi(x)e) = 0$ that $x - \varphi(x)e \in I$ and therefore $\widetilde{\varphi}(x - \varphi(x)e) = 0$. Since $\widetilde{\varphi}(e) = 1$ we get $\varphi(x) = \widetilde{\varphi}(x)$ and therefore uniqueness.

If $\dim\mathcal{A} > 1$, then $\ker\varphi \neq \{0\}$, so I is proper. Since I is a subalgebra of $\mathcal{A}$ with codimension 1, there cannot exist a greater proper ideal, hence $I \in \mathrm{Max}(\mathcal{A})$. □

Lemma 2.1.12 *Let $\mathcal{A}$ be a Banach algebra with unit e. Then $\varphi(x) \neq 0$ for every invertible $x \in \mathcal{A}$.*

Proof. By Lemma 2.1.11 we have $\varphi(e) = 1$. Hence if $x \in \mathcal{A}$ is invertible, then

$$\varphi(x)\varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(e) = 1, \text{ thus } \varphi(x) \neq 0.$$

□

The next Lemma is due to Zelazko [Zel].

Lemma 2.1.13 (**Zelazko**) *Let $\mathcal{A}$ be a real or complex algebra with identity e, and let φ be a linear functional on $\mathcal{A}$ satisfying*

$$\varphi(e) = 1 \text{ and } \varphi(x^2) = \varphi(x)^2 \tag{2.2}$$

for all $x \in \mathcal{A}$. Then φ is multiplicative.

Proof. The proof is pretty straightforward. Utilizing the fact that φ is linear and satisfies (2.2) we have for $x, y \in \mathcal{A}$

$$\begin{aligned}\varphi(x^2) + \varphi(xy + yx) + \varphi(y^2) &= \varphi(x^2 + xy + yx + y^2)\\ &= \varphi((x+y)^2) = (\varphi(x) + \varphi(y))^2\\ &= \varphi(x)^2 + 2\varphi(x)\varphi(y) + \varphi(y)^2\\ &= \varphi(x^2) + 2\varphi(x)\varphi(y) + \varphi(y^2),\end{aligned}$$

and hence $\varphi(xy + yx) = 2\varphi(x)\varphi(y)$ for all $x, y \in \mathcal{A}$. It remains to verify that $\varphi(yx) = \varphi(xy)$. Using the identity

$$(ab - ba)^2 + (ab + ba)^2 = 2(a(bab) + (bab)a),$$

we get

$$\begin{aligned}\varphi(ab - ba)^2 + 4\varphi(a)^2\varphi(b)^2 &= \varphi((ab - ba)^2) + \varphi(ab + ba)^2\\ &= \varphi((ab - ba)^2 + (ab + ba)^2)\\ &= 2\varphi(a(bab) + (bab)a)\\ &= 4\varphi(a)\varphi(bab).\end{aligned}$$

for all $a, b \in \mathcal{A}$. Now set $a = x - \varphi(x)e$, so that $\varphi(a) = \varphi(x) - \varphi(x)\varphi(e) = 0$ and $b = y$. Thus we obtain $\varphi(ay) = \varphi(ya)$ and eventually $\varphi(xy) = \varphi(yx)$.

□

Lemma 2.1.14 *Let φ be a multiplicative linear functional on a unital Banach algebra $\mathcal{A}$. Then φ is continuous and $\|\varphi\| = 1$.*

Proof. A linear functional in a Banach space is continuous if and only if its kernel is closed. By Lemma 2.1.11 we know that $\ker \varphi$ is a maximal ideal which itself is closed by Lemma 2.1.7. Hence φ is continuous.

Since $\varphi(e) = 1$ by Lemma 2.1.11 and $\|e\| = 1$ we have $\|\varphi\| \geq |\varphi(e)| = 1$. Now if $\|\varphi\| > 1$, then there exists $x \in \mathcal{A}$ such that $\|x\| < 1$ and $|\varphi(x)| = 1$. Hence, for $n \in \mathbb{N}, \|x^n\| \leq \|x\|^n \to 0$ as $n \to \infty$ and $|\varphi(x^n)| = |\varphi(x)^n| = 1$, which is a contradiction to the continuity of φ. Thus $\|\varphi\| \leq 1$.

□

Remark 2.1.15 We have $|\varphi(x)| \leq r(x)$. Indeed, if $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ are such that $|\lambda| > r(x)$, then $r(x/\lambda) < 1$ and hence $\lambda e - x = \lambda(e - x/\lambda)$ is invertible in $\mathcal{A}$ by Theorem 1.2.7. This implies $\varphi(x) \neq \lambda$ for all such λ, so $|\varphi(x)| \leq r(x)$.

The following theorem characterizes multiplicative linear functionals on (not necessarily commutative) Banach algebras. It has been established independently by Gleason, Kahane and Zelazko, using analytic tools. For a purely algebraic proof see [Kan], Section 2.1, p.45.

In order to prove the Gleason-Kahane-Zelazko theorem (GKZ), we first need a (rather interesting) lemma from complex analysis.

Lemma 2.1.16 *Suppose that f is an entire function of one complex variable such that*

$$f(0) = 1, \ f'(0) = 0 \quad \textit{and} \quad 0 < |f(\lambda)| < e^{|\lambda|}, \quad \lambda \in \mathbb{C}.$$

Then $f(\lambda) = 1$ for all $\lambda \in \mathbb{C}$.

Proof. Since f has no zero, there is an entire function $g(\lambda)$ such that $f(\lambda) = \exp(g(\lambda))$ (recall that $g(\lambda) = f'(\lambda)/f(\lambda)$, hence one can show that f/e^g is constant). Let $u(\lambda)$ and $v(\lambda)$ be real and imaginary parts of $g(\lambda)$, respectively. By hypothesis, we have $g(0) = 0 = g'(0)$. Also note that $e^{u(\lambda)} = |f(\lambda)| < e^{|\lambda|}$, hence $u(\lambda) < |\lambda|$ for all $\lambda \in \mathbb{C}$.

For $|\lambda| \leq r, r > 0$, we have

$$|g(\lambda)| = |\overline{g(\lambda)}| = |2u(\lambda) - g(\lambda)| \leq |2r - g(\lambda)|.$$

The function

$$g_r(\lambda) = \frac{r^2 g(\lambda)}{\lambda^2(2r - g(\lambda))} \quad \text{for all } 0 < |\lambda| \leq r$$

is holomorphic in $\{\lambda : |\lambda| < 2r\}$, and $|g_r(\lambda)| \leq 1$ if $|\lambda| = r$. By the maximum modulus principle it follows that $|g_r(\lambda)| \leq 1$ for $|\lambda| \leq r$. Now fix λ and let $r \to \infty$. Then

$$|g(\lambda)| \leq \frac{|\lambda^2||2r - g(\lambda)|}{r^2} \to 0.$$

But $g(0) = 0$, thus $f = 1$. □

Theorem 2.1.17 (**Gleason-Kahane-Zelazko**) *Let $\mathcal{A}$ be a unital Banach algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a linear functional. Then φ is multiplicative if and only if $\varphi(e) = 1$ and $\varphi(x) \neq 0$ for every invertible element $x \in \mathcal{A}$.*

Proof. In Lemma 2.1.11 and 2.1.12 we have already shown that if $\varphi \in \Delta(\mathcal{A})$, then $\varphi(e) = 1$ and $\varphi(x) \neq 0$ for all $x \in \mathcal{G}(\mathcal{A})$, the group of all invertible elements in $\mathcal{A}$.

Conversely, assume that $\varphi(e) = 1$ and $\varphi(x) \neq 0$ for all $x \in \mathcal{G}(\mathcal{A})$. Then φ is multiplicative. We prove this in three steps.

Step 1: By Lemma 2.1.14 φ is bounded with $\|\varphi\| = 1$.

Step 2: Let $N = \ker \varphi$. We want to show that $a \in N$ implies $a^2 \in N$.

Let $a \in N$ and $\|a\| \leq 1$. Define $f : \mathbb{C} \to \mathbb{C}$

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{\varphi(a^n)\lambda^n}{n!}, \ \lambda \in \mathbb{C}.$$

As $|\varphi(a^n)| \leq \|a^n\| \leq \|a\|^n \leq 1$ for all $n \in \mathbb{N}, f$ is entire and and satisfies $|f(\lambda)| \leq e^{|\lambda|}$ for all $\lambda \in \mathbb{C}$. Also, $f(0) = \varphi(e) = 1$, and $f'(0) = \varphi(a) = 0$. The continuity of φ implies that

$$f(\lambda) = \varphi\Big(\sum_{n=0}^{\infty} \frac{(\lambda a)^n}{n!}\Big) = \varphi(\exp(\lambda a)).$$

Since $\exp(\lambda a)$ is invertible, $f(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$, hence $|f(\lambda)| > 0$. Now by Lemma 2.1.16, $f(\lambda) = 1$ for all $\lambda \in \mathbb{C}$. This shows that

$$\frac{\varphi(a^0)\lambda^0}{0!} + \frac{\varphi(a^1)\lambda^1}{1!} + \frac{\varphi(a^2)\lambda^2}{2!} + \frac{\varphi(a^3)\lambda^3}{3!} + \ \ldots \ = 1$$

where $a^0 = e$. Thus $\varphi(a^2) = 0 = \varphi(a^3) = \ldots$ by uniqueness of power series, and therefore $a^2 \in N$.

Step 3: Again, let $N = \ker \varphi$. For $x, y \in A$ we have

$$x = a + \varphi(x)e, \qquad y = b + \varphi(y)e,$$

where $a, b \in N$. Then, by applying φ to the product of x and y, we obtain

$$\varphi(xy) = \varphi(ab) + \varphi(a)\varphi(y) + \varphi(x)\varphi(b) + \varphi(x)\varphi(y) = \varphi(ab) + \varphi(x)\varphi(y).$$

Therefore we have to show that $\varphi(ab) = 0$. This is equivalent to the assertion that $ab \in N$ for $a, b \in N$. We have already proved the special case $a^2 \in N$ if $a \in N$, so by substituting $x = y$ we get

$$\varphi(x^2) = \varphi(a^2) + \varphi(x)\varphi(x) = \varphi(x)^2, \ x \in \mathcal{A}.$$

It follows now from Lemma 2.1.13 (Zelazko) that φ is multiplicative. □

Let $\mathcal{A}$ be a commutative Banach algebra with identity. The next theorem forms the basic link between the structure space $\Delta(\mathcal{A})$ and ideals in $\mathcal{A}$.

Theorem 2.1.18 *For a unital commutative Banach algebra $\mathcal{A}$, the mapping*

$$\varphi \mapsto \ker \varphi$$

is a bijection between $\Delta(\mathcal{A})$ and $\mathrm{Max}(\mathcal{A})$, the set of all maximal ideals in $\mathcal{A}$.

Proof. If $\varphi \in \Delta(\mathcal{A})$, then $\ker \varphi$ is a maximal ideal of $\mathcal{A}$ by Lemma 2.1.11. Hence the mapping is well-defined.

The mapping $\varphi \mapsto \ker \varphi$ is injective, since if φ_1 and φ_2 are in $\Delta(\mathcal{A})$ with $\ker \varphi_1 = \ker \varphi_2$, then for any $x \in \mathcal{A}$ we have $x - \varphi_2(x)e \in \ker \varphi_2 = \ker \varphi_1$, so $\varphi_1(x - \varphi_2(x)e) = 0$, hence $\varphi_1(x) = \varphi_2(x)$. Thus $\varphi_1 = \varphi_2$.

We show that the mapping is surjective. Let M be a maximal ideal of $\mathcal{A}$ and let $q : \mathcal{A} \to \mathcal{A}/M,\ x \mapsto x + M$ be the corresponding quotient map. Observe that q is a homomorphism and $\ker q = M$. By Theorem 2.1.5 the map $\psi : \mathbb{C} \to \mathcal{A}/M,\ \lambda \mapsto \lambda e + M$ is an isomorphism. Let $\varphi = \psi^{-1} \circ q : \mathcal{A} \to \mathbb{C}$. Since φ is the composition of two homomorphisms, it is a homomorphism, and $\varphi(e) = \psi^{-1}(q(e)) = \psi^{-1}(e + M) = e$, so $\varphi \neq 0$. Hence $\varphi \in \Delta(\mathcal{A})$. Because ψ is an isomorphism, we have $\ker \varphi = \ker q = M$. This shows that $\varphi \mapsto \ker \varphi$ is a bijection from $\Delta(\mathcal{A})$ onto $\mathrm{Max}(\mathcal{A})$.

□

Lemma 2.1.19 *Let $\mathcal{A}$ be a unital commutative Banach algebra and let $x \in \mathcal{A}$. Then the following assertions are equivalent.*

(*i*) $x \notin \mathcal{G}(\mathcal{A})$.

(*ii*) *$x \in I$ for some proper ideal I of $\mathcal{A}$.*

(*iii*) *$x \in M$ for some maximal ideal M of $\mathcal{A}$.*

Proof.

(*i*) ⇒ (*ii*): If $x \notin \mathcal{G}(\mathcal{A})$, consider the set $I = \{xy : y \in \mathcal{A}\}$. Since $\mathcal{A}$ is commutative by hypothesis, I is an ideal of $\mathcal{A}$, and since $\mathcal{A}$ is unital we have $x = xe \in I$. If $e \in I$, then $xy = 1$ for some $y \in \mathcal{A}$, so $x \in \mathcal{G}(\mathcal{A})$, which is a contradiction. Hence $e \notin I$ and therefore I is a proper ideal.

(*ii*) ⇒ (*iii*): Suppose that $x \in I$ where I is a proper ideal of $\mathcal{A}$. By Lemma 2.1.7 (*iii*), every proper ideal is contained in a maximal ideal, that is $I \subseteq M$ for some maximal ideal M of $\mathcal{A}$, hence $x \in M$.

(*iii*) ⇒ (*i*): If M is a maximal ideal of $\mathcal{A}$, then M is a proper ideal, so $M \cap \mathcal{G}(\mathcal{A}) = \emptyset$ by Lemma 2.1.6. Thus $x \notin \mathcal{G}(\mathcal{A})$ for all $x \in M$. □

Theorem 2.1.20 (**Beurling-Gelfand**) *Let $\mathcal{A}$ be a commutative Banach algebra with identity and $x \in \mathcal{A}$. Then*

(*i*) $\sigma(x) = \{\varphi(x) : \varphi \in \Delta(\mathcal{A})\}$.

(*ii*) *$x \in \mathcal{G}(\mathcal{A})$ if and only if $\varphi(x) \neq 0$ for all $\varphi \in \Delta(\mathcal{A})$.*

(iii) $r(x) = \sup\limits_{\varphi \in \Delta(\mathcal{A})} |\varphi(x)|.$

Proof. *(i)*

$$\begin{aligned} \lambda \in \sigma(x) &\iff \lambda e - x \notin \mathcal{G}(\mathcal{A}) \\ &\iff \varphi(\lambda e - x) = 0 \text{ for some } \varphi \in \Delta(\mathcal{A}) \text{ by Theorem 2.1.17 (GKZ)} \\ &\iff \lambda = \varphi(x) \text{ for some } \varphi \in \Delta(\mathcal{A}), \text{ since } \varphi(\lambda) = \lambda \text{ by Lemma 2.1.14.} \end{aligned}$$

(ii)

$$\begin{aligned} x \in \mathcal{G}(\mathcal{A}) &\iff x \notin M \text{ for all maximal ideals } M \text{ of } \mathcal{A} \text{ by Lemma 2.1.19} \\ &\iff x \notin \ker \varphi \text{ for all } \varphi \in \Delta(\mathcal{A}) \text{ by Theorem 2.1.18} \\ &\iff \varphi(x) \neq 0 \text{ for all } \varphi \in \Delta(\mathcal{A}). \end{aligned}$$

(iii) Follows immediately from *(i)* and the definition of $r(x)$. □

Let $\mathcal{A}, \mathcal{B}$ be Banach algebras. In general, if the map $\varphi : \mathcal{A} \to \mathcal{B}$ is linear and preserves invertible elements, then φ need not be multiplicative. Recall that $\mathcal{A} = \mathcal{M}_n(\mathbb{C}), n \geq 2$, is the set of all complex $n \times n$ matrices with matrix addition and matrix multiplication. Equipped with the Frobenius norm defined by

$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}, \ A \in \mathcal{A},$$

$\mathcal{A}$ is a non-commutative unital Banach algebra.

Example 2.1.21 Let $\mathcal{A} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}$ and $\mathcal{B} = \mathcal{M}_2(\mathbb{C})$. Define $\varphi : \mathcal{A} \to \mathcal{B}$ by

$$\varphi\left(\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \right) = \begin{pmatrix} \alpha & \alpha + \beta \\ 0 & \gamma \end{pmatrix}.$$

It is clear that φ maps invertible elements into invertible elements, but

$$\varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence φ is not multiplicative since

$$\varphi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) = \varphi\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},$$

and

$$\varphi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \varphi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Example 2.1.22 Let $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ with I being the identity matrix. Then the map $\varphi : \mathcal{A} \to \mathcal{A}$ given by

$$\varphi(A) = A^T, \ A \in \mathcal{A},$$

maps I into I and invertible elements into invertible elements. But it is not multiplicative since

$$\varphi(AB) = (AB)^T = B^T A^T = \varphi(B)\varphi(A), \ B \in \mathcal{A}.$$

Remark 2.1.23 Since the Gelfand space $\Delta(\mathcal{A})$ plays a major role throughout commutative Banach algebra theory, it is important to understand how $\Delta(\mathcal{A})$ and $\Delta(\mathcal{A}_e)$ are related. Recall that $\mathcal{A}_e$ denotes the unitization of $\mathcal{A}$.

Because $\psi(e) = 1$ for every $\psi \in \Delta(\mathcal{A}_e)$, each $\varphi \in \Delta(\mathcal{A})$ has a unique extension $\widetilde{\varphi} \in \Delta(\mathcal{A}_e)$ given by

$$\widetilde{\varphi}(x + \lambda e) = \varphi(x) + \lambda, \quad x \in \mathcal{A}, \ \lambda \in \mathbb{C}.$$

Let $\widetilde{\Delta}(\mathcal{A}) = \{\tilde{\varphi} : \varphi \in \Delta(\mathcal{A})\}$. Furthermore, let $\varphi_\infty : \mathcal{A}_e \to \mathbb{C}$ denote the homomorphism with $\ker \varphi_\infty = \mathcal{A}$, that is $\varphi_\infty(x + \lambda e) = \varphi_\infty(x) + \lambda\varphi_\infty(e) = \lambda, \ x \in \mathcal{A}$. Then

$$\Delta(\mathcal{A}_e) = \widetilde{\Delta}(\mathcal{A}) \cup \{\varphi_\infty\}.$$

In fact, if $\psi \in \Delta(\mathcal{A}_e)$ and $\psi \neq \varphi_\infty$, then $\psi|_\mathcal{A} \in \Delta(\mathcal{A})$ and hence $\psi = \widetilde{\psi|_\mathcal{A}}$. Identifying $\Delta(\mathcal{A})$ with $\widetilde{\Delta}(\mathcal{A}) \subseteq \Delta(\mathcal{A}_e)$, we always regard $\Delta(\mathcal{A})$ as a subset of $\Delta(\mathcal{A}_e)$. In this sense

$$\Delta(\mathcal{A}_e) = \Delta(\mathcal{A}) \cup \{\varphi_\infty\}.$$

So far in the development of this chapter there is one obvious and vital question that we have not faced: If $\mathcal{A}$ is a commutative Banach algebra, do there exist any complex homomorphisms of $\mathcal{A}$, that is, is $\Delta(\mathcal{A}) \neq \emptyset$? The answer is no. That is, there exist commutative Banach algebras $\mathcal{A}$ such that the only homomorphism of $\mathcal{A}$ into $\mathbb{C}$ is the zero homomorphism. One rather trivial instance of such a phenomenon is given by the following example.

Example 2.1.24 Let $(\mathcal{A}, \|.\|)$ be a Banach space over $\mathbb{C}$ and define a product on $\mathcal{A}$ by setting $xy = 0$ for all $x, y \in \mathcal{A}$. It is easily verified that with this multiplication $\mathcal{A}$ becomes a commutative Banach algebra without unit. However, $\Delta(\mathcal{A}) = \emptyset$ because if φ is a multiplicative linear functional on $\mathcal{A}$, then

$$\varphi(x)^2 = \varphi(x)\varphi(x) = \varphi(xx) = \varphi(0) = 0,$$

from which it is apparent that $\varphi = 0$. Note that, for any commutative Banach algebra $\mathcal{A}$, $\Delta(\mathcal{A}) = \emptyset$ whenever $r(x) = 0$ for every $x \in \mathcal{A}$.

A less trivial example which shows that nonzero multiplicative linear functionals need not exist, is the following one.

Example 2.1.25 Define a bounded linear operator T on $C([0,1])$ by

$$Tf(t) = \int_0^t f(s)ds, \quad f \in C([0,1]),\ t \in [0,1].$$

Let $\mathcal{A}$ be the norm closure in $\mathfrak{B}(C([0,1]))$ of the set of all polynomials in T of the form

$$\sum_{j=1}^{n} \lambda_j T^j, \quad \lambda_1, \ldots, \lambda_n \in \mathbb{C},\ n \in \mathbb{N}.$$

We show that $\mathcal{A}$ is a commutative Banach algebra without identity such that $\Delta(\mathcal{A}) = \emptyset$.
For $f \in C([0,1])$ we have

$$\begin{aligned} |T^2 f(t)| &= \left| \int_0^t \left(\int_0^s f(u)du \right) ds \right| \\ &\leq \int_0^t \left(\int_0^s |f(u)|du \right) ds \\ &\leq \|f\|_\infty \int_0^t \left(\int_0^s du \right) ds \\ &= \|f\|_\infty \frac{t^2}{2}, \qquad t \in [0,1]. \end{aligned}$$

More generally, a straightforward induction argument reveals that

$$|T^n f(t)| \leq \|f\|_\infty \frac{t^n}{n!} \quad \text{for all } t \in [0,1] \text{ and } n \in \mathbb{N}.$$

Consequently

$$\|T^n f\|_\infty \leq \frac{\|f\|_\infty}{n!},$$

from which we conclude that

$$\|T^n\|^{\frac{1}{n}} \leq \left(\frac{1}{n!} \right)^{\frac{1}{n}} \quad \text{for all } n \in \mathbb{N}.$$

Since $(n!)^{\frac{1}{n}} \to \infty$ as $n \to \infty$, we get $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = 0$. Recall that the spectral radius is continuous, and by Lemma 1.3.30 also subadditive and submultiplicative. Hence it follows that $r(S) = 0$ for all $S \in \mathcal{A}$ and therefore $\Delta(\mathcal{A}) = \emptyset$.

If $\mathcal{A}$ is a commutative Banach algebra with identity, then the Gelfand space $\Delta(\mathcal{A})$ cannot be empty. This is an immediate consequence of the next theorem.

Theorem 2.1.26 *Let $\mathcal{A}$ be a unital commutative Banach algebra. Then $\Delta(\mathcal{A}) \neq \emptyset$.*

Proof. Since $\{0\}$ is a proper ideal in $\mathcal{A}$, by Theorem 2.1.7 (*iii*) it is contained in some maximal ideal, which by Theorem 2.1.18 is the kernel of a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$. □

2.2. Semisimple Banach algebras

In this section we want to introduce so called semisimple Banach algebras. Since we are interested in their role in Gelfand's representation theory, we will only present some fundamental results. For a more detailed approach we refer to [Lar], Chapter 5, pp.129.

Definition 2.2.1 Let $\mathcal{A}$ be a commutative Banach algebra. The *radical* of $\mathcal{A}$, $\mathrm{rad}(\mathcal{A})$, is defined by

$$\mathrm{rad}(\mathcal{A}) = \bigcap\{M : M \in \mathrm{Max}(\mathcal{A})\} = \bigcap\{\ker\varphi : \varphi \in \Delta(\mathcal{A})\},$$

where $\mathrm{rad}(\mathcal{A})$ is understood to be $\mathcal{A}$ if $\Delta(\mathcal{A}) = \emptyset$. Cleary, $\mathrm{rad}(\mathcal{A})$ is a closed ideal of $\mathcal{A}$, since maximal ideals are closed (Theorem 2.1.7 (*ii*)). The algebra $\mathcal{A}$ is called *semisimple* if $\mathrm{rad}(\mathcal{A}) = \{0\}$ and *radical* if $\mathrm{rad}(\mathcal{A}) = \mathcal{A}$.

In Example 2.1.25 we have already seen an example of a radical Banach algebra with nontrivial multiplication. On the other hand, it will follow from Corollary 2.3.8 in the next section that $\mathcal{A}$ is semisimple if and only if for every $x \in \mathcal{A}, r_{\mathcal{A}}(x) = 0$ implies that $x = 0$.[1] Since the spectral radius is subadditive and submultiplicative, this means that $\mathcal{A}$ is semisimple if and only if $r_{\mathcal{A}}$ is an algebra norm on $\mathcal{A}$. Thus $\Delta(\mathcal{A}) \neq \emptyset$.

First we list some examples of semisimple Banach algebras.

Example 2.2.2 Let $\mathcal{X}$ be a compact Hausdorff space. Then $C(\mathcal{X})$ is a semisimple Banach algebra.

Example 2.2.3 Let $\mathcal{A} = C^1([0,1])$ with norm

$$\|f\|_{C^1} = \|f\| + \|f'\|.$$

Then $\mathcal{A}$ is a semisimple commutative Banach algebra. Indeed, it can be easily checked that $\mathcal{A}$ is a commutative Banach algebra with unit. We show semisimplicity of $\mathcal{A}$ by verifying

$$r(f) = \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}} = 0 \Rightarrow f = 0.$$

In fact, $r(f) = 0$ means that $\lambda e - f$ is invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$. If $f(x_0) = \lambda \neq 0$ for some x_0, then $\lambda e - f$ would not be invertible. Hence $f(x) = 0$ for all $x \in [0,1]$.

Example 2.2.4 Let $\mathcal{A}$ be the algebra of all continuously differentiable functions $f : [0,1] \to \mathbb{C}$ with pointwise multiplication and the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Let

$$I = \{f \in \mathcal{A} : f(0) = f'(0) = 0\}.$$

We claim that $\mathcal{A}/I$ is a 2-dimensional algebra which has a 1-dimensional radical. Thus $\mathcal{A}$ is an example of a semisimple commutative Banach algebra which admits a non-semisimple quotient.

[1] This characterization will turn out to be quite useful when we develop the Shilov idempotent theorem in Chapter 3.

In fact, $\Delta(\mathcal{A})$ is homeomorphic and isomorphic to $[0,1]$. For a given $x \in [0,1]$, it is apparent that I is a closed ideal. It is also easy to verify that

$$\varphi : \mathcal{A}/I \to \mathbb{C}^2 \text{ defined by } \varphi(f) = (f(x), f'(x))$$

is an isomorphism, where the multiplication of $\mathbb{C}^2$ is given by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, x_1y_2 + x_2y_1).$$

Clearly, the identity element of $\mathbb{C}^2$ is $(0,1)$ and all non-invertible elements have the form $(0,\lambda), \lambda \in \mathbb{C}$. In fact, those non-invertible elements represent the only maximal ideal in $\mathcal{A}$. Hence the radical is 1-dimensional.

We continue with a number of interesting properties of semisimple Banach algebras.

Theorem 2.2.5 *Let ψ be a homomorphism from a commutative Banach algebra $\mathcal{A}$ into a semisimple commutative Banach algebra $\mathcal{B}$. Then ψ is contiuous.*

Proof. We apply the closed graph theorem. It suffices to show that if $x_n \in \mathcal{A}, n \in \mathbb{N}$, are such that $x_n \to 0$ and $\psi(x_n) \to y$ for some $y \in \mathcal{B}$, then $y = 0$.

Let $\varphi \in \Delta(\mathcal{B})$. Then $\varphi \circ \psi \in \Delta(\mathcal{A}) \cup \{0\}$ and hence both, φ and $\varphi \circ \psi$, are continuous by Lemma 2.1.14. Therefore we have

$$\varphi(y) = \lim_{n\to\infty} \varphi(\psi(x_n)) = \lim_{n\to\infty} (\varphi \circ \psi)(x_n) = 0.$$

Since this holds for all $\varphi \in \Delta(\mathcal{B})$ and $\mathcal{B}$ is semisimple by hypothesis, we eventually get $y = 0$. □

Corollary 2.2.6 *Every automorphism or endomorphism of a semisimple commutative Banach algebra $\mathcal{A}$ is continuous.*

Proof. Follows directly from Theorem 2.2.5. □

Corollary 2.2.7 *On a semisimple commutative Banach algebra all Banach algebra norms are equivalent.*

Proof. Let $\mathcal{A}$ be a semisimple commutative Banach algebra and let $\|.\|_1$ and $\|.\|_2$ be two Banach algebra norms on $\mathcal{A}$. By applying Theorem 2.2.5 with ψ to the identity mappings $(\mathcal{A}, \|.\|_1) \to (\mathcal{A}, \|.\|_2)$ and $(\mathcal{A}, \|.\|_2) \to (\mathcal{A}, \|.\|_1)$, the statement follows. □

We conclude this section by presenting a rather interesting example.

Example 2.2.8 Let $C^\infty([0,1])$ denote the algebra of all infinitely many times differentiable functions on $[0,1]$. We show that $C^\infty([0,1])$ admits no Banach algebra norm.

Suppose there exists a Banach algebra norm $\|.\|$ on $C^\infty([0,1])$. We use the same method as in the corollary above and apply Theorem 2.2.5 to the identity mapping from $C^\infty([0,1])$ into $C([0,1])$. Hence we can observe that there exists $c > 0$ such that

$$\|f\|_\infty \leq c\|f\| \quad \text{for all } f \in C^\infty([0,1]). \tag{2.3}$$

Now, using (2.3), we want to prove that the differentiation mapping $D : f \to f'$ from $C^\infty([0,1])$ into itself is continuous. To achieve that, we once again apply the closed graph theorem. Let $f_n \in C^\infty([0,1]), n \in \mathbb{N}$, be such that

$$\lim_{n\to\infty} \|f_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|f_n' - g\| = 0$$

for some $g \in C^\infty([0,1])$. Then

$$\lim_{n\to\infty} \|f_n\|_\infty = 0 \quad \text{and} \quad \lim_{n\to\infty} \|f_n' - g\|_\infty = 0.$$

For each $a, b \in [0,1]$ we have

$$\begin{aligned}\left|\int_a^b g(t)dt\right| &\leq |f_n(b) - f_n(a)| + \left|\int_a^b (f_n'(t) - g(t))dt\right| \\ &\leq 2\|f_n\|_\infty + |b-a| \cdot \|f_n' - g\|_\infty \to 0 \text{ as } n \to \infty.\end{aligned}$$

So we have $\displaystyle\int_a^b g(t)dt = 0$ and hence $g = 0$ since a and b are arbitrary. Thus D is continuous and there exists $d > 0$ such that

$$\|f'\| \leq d\|f\| \quad \text{for all } f \in C^\infty([0,1]).$$

Now let $f(t) = e^{2dt}, t \in [0,1]$. Because $f'(t) = 2d \cdot e^{2dt} = 2d \cdot f(t)$ we get

$$2d\|f\| = \|f'\| \leq d\|f\|,$$

which is a contradiction. Hence there cannot exist a Banach algebra norm on $C^\infty([0,1])$.

2.3. The Gelfand representation

The purpose of this section is to develop the basic elements of Gelfand's theory for commutative (semisimple) Banach algebras. Associated with any such algebra $\mathcal{A}$ is a locally compact Hausdorff space $\Delta(\mathcal{A})$, the structure space or Gelfand space of $\mathcal{A}$, and a norm-decreasing homomorphism $\Gamma_\mathcal{A}$ from $\mathcal{A}$ into $C_0(\Delta(\mathcal{A}))$. If $\mathcal{A}$ has an identity, $\Delta(\mathcal{A})$ is compact. The converse is true whenever $\Gamma_\mathcal{A}$ is injective ($\mathcal{A}$ is semisimple), a fact that can only be shown later (Section 3.3). This representation of $\mathcal{A}$ as an algebra of functions on a locally compact Hausdorff space is fundamental to any thorough study of commutative Banach algebras.

From here now on all Banach algebras are assumed to be commutative if not mentioned otherwise.

Definition 2.3.1 Let $\mathcal{A}$ be a commutative Banach algebra and $\Delta(\mathcal{A})$ the set of all nonzero (hence surjective) multiplicative linear functionals from $\mathcal{A}$ to $\mathbb{C}$. We endow the Gelfand space $\Delta(\mathcal{A})$ with the weakest topology, i.e. the relative w^*-topology, with respect to which all the functions

$$\Delta(\mathcal{A}) \to \mathbb{C}, \ \varphi \mapsto \varphi(x), \quad x \in \mathcal{A},$$

are continuous. In view of the definition of the w^*-topology, we see that a neighbourhood basis at $\varphi_0 \in \Delta(\mathcal{A})$ consists of sets of the form

$$U(\varphi_0, x_1, \ldots, x_n, \varepsilon) = \{\varphi \in \Delta(\mathcal{A}) : |\varphi(x_k) - \varphi_0(x_k)| < \varepsilon, \ k = 1, \ldots, n\},$$

where $\varepsilon > 0, n \in \mathbb{N}$ and $x_1, \ldots, x_n$ are arbitrary elements of $\mathcal{A}$. This topology on $\Delta(\mathcal{A})$ is called the *Gelfand topology.*

Remark 2.3.2 In Lemma 2.1.14 we have already seen that every $\varphi \in \Delta(\mathcal{A})$ is a bounded linear functional on $\mathcal{A}$. Hence $\Delta(\mathcal{A})$ is contained in the unit ball of $\mathcal{A}'$ (the dual space of $\mathcal{A}$). Evidently the Gelfand topology coincides with the relative w^*-topology of $\mathcal{A}'$ on $\Delta(\mathcal{A})$. When adjoining an identity e to $\mathcal{A}, \Delta(\mathcal{A}_e) = \Delta(\mathcal{A}) \cup \{\varphi_\infty\}$ (Remark 2.1.23) and according to the following theorem, the topology on $\Delta(\mathcal{A})$ is the one induced by $\Delta(\mathcal{A}_e)$.

Theorem 2.3.3 *Let $\mathcal{A}$ be a commutative Banach algebra. Then*

(*i*) $\Delta(\mathcal{A})$ *is a locally compact Hausdorff space.*

(*ii*) $\Delta(\mathcal{A}_e) = \Delta(\mathcal{A}) \cup \{\varphi_\infty\}$ *is the one-point compactification of* $\Delta(\mathcal{A})$.

Proof.
(*i*) First we show that $\Delta(\mathcal{A})$ is a Hausdorff space. Let φ_1 and φ_2 be elements of $\Delta(\mathcal{A})$ with $\varphi_1 \neq \varphi_2$. Then for some $x \in \mathcal{A}$

$$\delta = \frac{1}{2}|\varphi_1(x) - \varphi_2(x)| > 0,$$

and so

$$U(\varphi_1, x, \delta) \cap U(\varphi_2, x, \delta) = \emptyset.$$

Thus $U(\varphi_1, x, \delta)$ and $U(\varphi_2, x, \delta)$ are open disjoint neighbourhoods of $\Delta(\mathcal{A})$.
Next, we want to prove that $\Delta(\mathcal{A})$ is locally compact. For that consider $\Delta(\mathcal{A}_e)$ and $\Delta(\mathcal{A}) \subseteq \Delta(\mathcal{A}_e)$. Let U and U_e denote the basic neighbourhoods in $\Delta(\mathcal{A})$ and $\Delta(\mathcal{A}_e)$, respectively. Then, for $\varphi \in \Delta(\mathcal{A}), \varepsilon > 0$ and a finite subset F of $\mathcal{A}$,

$$U_e(\varphi, F, \varepsilon) = \begin{cases} U(\varphi, F, \varepsilon) \cup \{\varphi_\infty\} & \text{if } |\varphi(x)| < \varepsilon \text{ for all } x \in F, \\ U(\varphi, F, \varepsilon) & \text{otherwise.} \end{cases}$$

Hence the Gelfand topology on $\Delta(\mathcal{A})$ and the relative topology on $\Delta(\mathcal{A})$ induced by the the Gelfand topology on $\Delta(\mathcal{A}_e)$ coincide. Since the singleton $\{\varphi_\infty\}$ is a closed subset of $\Delta(\mathcal{A}_e)$, it follows at once that $\Delta(\mathcal{A})$ is an open subset of $\Delta(\mathcal{A}_e)$ and hence is locally compact.
(*ii*) Let $x \in \mathcal{A}$ and $\varepsilon > 0$. Then

$$\begin{aligned} U_e(\varphi_\infty, x, \varepsilon) &= \{\varphi_\infty\} \cup \{\varphi \in \Delta(\mathcal{A}) : |\varphi(x)| < \varepsilon\} \\ &= \Delta(\mathcal{A}_e) \setminus \{\varphi_e \in \Delta(\mathcal{A}_e) : |\varphi_e(x)| \geq \varepsilon\}. \end{aligned}$$

The sets $\{\varphi_e \in \Delta(\mathcal{A}_e) : |\varphi_e(x)| \geq \varepsilon\}, x \in \mathcal{A}$, are closed in $\Delta(\mathcal{A}_e)$ and hence compact. The complement of a basic neighbourhood of φ_∞ is a finite union of such compact sets. Therefore $\Delta(A_e)$ is the one-point compactification of $\Delta(\mathcal{A})$. □

It turns out that if $\mathcal{A}$ has a unit, then the Hausdorff space is compact.

Theorem 2.3.4 *Let $\mathcal{A}$ be a commutative Banach algebra. If $\mathcal{A}$ has an identity, then $\Delta(\mathcal{A})$ is compact.*

Proof. Before we start with the actual proof let us recall what we have already shown. We know that the w^*-topology is Hausdorff and $\Delta(\mathcal{A})$ is a Hausdorff space (Theorem 2.3.3). By Remark 2.3.2, $\Delta(\mathcal{A})$ is contained in the unit ball of $\mathcal{A}'$, which is compact in the w^*-topology by Banach-Alaoglu. A closed subset of a compact set is compact, so it suffices to show that $\Delta(\mathcal{A})$ is w^*-closed in $\mathcal{A}'$.

Let $x \in \mathcal{A}$. Since the evaluation functional

$$f_x : \mathcal{A}' \to \mathbb{C}, \ \varphi \mapsto \varphi(x)$$

is w^*-continuous, we can conclude that the mapping

$$\mathcal{A}' \to \mathbb{C} \times \mathbb{C} \times \mathbb{C}, \ \varphi \mapsto (\varphi(xy), \varphi(x), \varphi(y))$$

for all $x, y \in \mathcal{A}$, is also w^*-continuous. Obviously the map $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \ (a, b, c) \mapsto a - bc$ is continuous. Hence for $x, y \in \mathcal{A}$, the set

$$M_{x,y} = \{\varphi \in A' : \varphi(xy) - \varphi(x)\varphi(y) = 0\},$$

which is the kernel of the w^*-continuous mapping

$$(f_{xy} - f_x f_y) : \mathcal{A}' \to \mathbb{C}, \ \varphi \mapsto \varphi(xy) - \varphi(x)\varphi(y),$$

is w^*-closed. Thus $\Delta(\mathcal{A}) \cup \{0\} = \bigcap_{x,y \in \mathcal{A}} M_{x,y} = \bigcap_{x,y \in \mathcal{A}} (f_{xy} - f_x f_y)^{-1}(0)$ is w^*-closed. Since $M_1 = \{\varphi \in \mathcal{A}' : \varphi(e) = 1\} = f_1^{-1}(1)$ is w^*-closed, it follows that $\Delta(\mathcal{A}) = M_1 \cap \bigcap_{x,y \in \mathcal{A}} M_{x,y}$ is w^*-closed. Now, by Banach-Alaoglu the closed unit ball of $\mathcal{A}'$ is w^*-compact. And finally, since $\Delta(\mathcal{A})$ is a w^*-closed subset of the unit ball, it is w^*-compact. □

We have already mentioned that the converse - a commutative Banach algebra $\mathcal{A}$ has an identity if $\Delta(\mathcal{A})$ is compact - holds true if $\mathcal{A}$ is semisimple. This fact will be proven later in Theorem 3.3.1 when we apply the Shilov idempotent theorem (Theorem 3.2.1). Interestingly enough, a much simpler proof is available when $\mathcal{A}$ is regular (Corollary 4.2.16).

We continue with the notion of the Gelfand representation.

Definition 2.3.5 Let $\mathcal{A}$ be a commutative Banach algebra. For $x \in \mathcal{A}$, we define the *Gelfand transform* of x by

$$\widehat{x} : \Delta(\mathcal{A}) \to \mathbb{C}, \ \widehat{x}(\varphi) = \varphi(x), \quad \varphi \in \Delta(\mathcal{A}).$$

Since the Gelfand topology is the relative w^*-topology on $\Delta(\mathcal{A})$, it is immediately apparent that $\widehat{x}$ is a continuous function on $\Delta(\mathcal{A})$. The mapping

$$\Gamma_{\mathcal{A}} : \mathcal{A} \to C(\Delta(\mathcal{A})), \quad x \mapsto \widehat{x}$$

is called the *Gelfand representation* or *Gelfand homomorphism* of $\mathcal{A}$. We often denote $\Gamma_{\mathcal{A}}(\mathcal{A})$ by $\widehat{\mathcal{A}}$.

Theorem 2.3.6 (**Gelfand's representation theorem**) *Let $\mathcal{A}$ be a unital commutative Banach algebra.*

(*i*) *The Gelfand representation $x \mapsto \widehat{x}$ is a continuous homomorphism from $\mathcal{A}$ onto $C(\Delta(\mathcal{A}))$ with norm* 1.

(*ii*) $\widehat{x}(\Delta(\mathcal{A})) = \sigma_{\mathcal{A}}(x)$.

Proof.
(*i*) Clearly, $\Gamma_{\mathcal{A}}$ is linear. For $x, y \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$ we have

$$\widehat{xy}(\varphi) = \varphi(xy) = \varphi(x)\varphi(y) = \widehat{x}(\varphi)\widehat{y}(\varphi) \text{ and } \widehat{e}(\varphi) = \varphi(e) = 1.$$

Hence $\Gamma_{\mathcal{A}}$ is an algebra homomorphism. Now because

$$|\widehat{x}(\varphi)| = |\varphi(x)| \leq \|\varphi\|\|x\| = \|x\|,$$

it follows that $\|\hat{\cdot}\| \leq 1$. And since $\widehat{e}(\varphi) = 1$ we get $\|\hat{\cdot}\| \geq 1$, so $\|\hat{\cdot}\| = 1$.

(*ii*) Let $\lambda \in \sigma_{\mathcal{A}}(x)$. Then by Theorem 2.1.17 (GKZ) and Theorem 2.1.18

$$\begin{aligned}
\lambda \in \sigma_{\mathcal{A}}(x) &\iff \lambda e - x \notin \mathcal{G}(\mathcal{A}) \\
&\iff \lambda e - x \text{ is contained in a maximal ideal} \\
&\iff \lambda e - x \text{ is contained in } \ker \varphi \\
&\iff \exists \varphi \in \Delta(\mathcal{A}) : \varphi(\lambda e - x) = 0 \\
&\iff \exists \varphi \in \Delta(\mathcal{A}) : \lambda = \varphi(x) \\
&\iff \lambda \in \widehat{x}(\Delta(\mathcal{A})),
\end{aligned}$$

hence $\widehat{x}(\Delta(\mathcal{A})) = \sigma_{\mathcal{A}}(x)$.

□

Remark 2.3.7

- The Gelfand representation theorem is equivalent to

$$x \in \mathcal{G}(\mathcal{A}) \iff \widehat{x} \in \mathcal{G}(C(\Delta(\mathcal{A}))).$$

  That is, $\Gamma_{\mathcal{A}}$ maps invertible elements of $\mathcal{A}$ into the invertible elements of $C(\Delta(\mathcal{A}))$.

- As it turns out, the theorem holds true for non-unital algebras. If $\mathcal{A}$ has no identity, then

$$\begin{aligned}
\sigma_{\mathcal{A}}(x) \setminus \{0\} = \sigma_{\mathcal{A}_e}(x) \setminus \{0\} &= \widehat{x}(\Delta(\mathcal{A}_e)) \setminus \{0\} \\
&\subseteq \widehat{x}(\Delta(\mathcal{A})) = \widehat{x}(\Delta(\mathcal{A}_e)) = \sigma_{\mathcal{A}_e}(x) \\
&= \sigma_{\mathcal{A}}(x).
\end{aligned}$$

Corollary 2.3.8 *Let $\mathcal{A}$ be a unital commutative Banach algebra and $x \in \mathcal{A}$. Then $\widehat{x} = 0$ if and only if*

$$r_{\mathcal{A}}(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = 0.$$

Proof. Follows immediately from Theorem 2.3.6 (Gelfand's representation theorem) and Theorem 1.3.24 (spectral radius formula). □

Definition 2.3.9 Let F be a nonempty set and $\mathcal{A}$ a unital algebra. A subset E of $\mathcal{A}$ *separates points* of F if for each $s, t \in F$ with $s \neq t$, there exists $f \in E$ with $f(s) \neq f(t)$, and E *strongly separates points* of F if, further, for each $s \in F$, there exists $f \in E$ with $f(s) \neq 0$.

Theorem 2.3.10 *Let $\mathcal{A}$ be a commutative Banach algebra and Γ the Gelfand representation of $\mathcal{A}$.*

(*i*) *Γ maps $\mathcal{A}$ into $C_0(\Delta(\mathcal{A}))$ and is norm decreasing.*

(*ii*) *$\Gamma(\mathcal{A})$ strongly separates the points of $\Delta(\mathcal{A})$.*

(*iii*) *Γ is isometric if and only if $\|x\|^2 = \|x^2\|$ for all $x \in \mathcal{A}$.*

Proof.
(*i*) By Theorem 2.3.3, $\Delta(\mathcal{A}_e)$ is the one-point compactification of $\Delta(\mathcal{A})$. Since $\varphi_\infty : \mathcal{A}_e \to \mathbb{C}$ is a homomorphism with kernel $\mathcal{A}$, we have $\varphi_\infty(x) = \widehat{x}(\varphi_\infty) = 0$ for $x \in \mathcal{A}$, and hence $x \in C_0(\Delta(\mathcal{A}))$. Moreover, by Theorem 2.3.6 (Gelfand's representation theorem) and Theorem 1.3.24 (spectral radius formula) we get

$$\|\widehat{x}\|_{C_0(\Delta(\mathcal{A}))} = \|\widehat{x}\|_\infty = \sup_{\varphi \in \Delta(\mathcal{A})} |\widehat{x}(\varphi)| = \sup\{|\lambda| : \lambda \in \widehat{x}(\Delta(\mathcal{A}))\} = r(x) \leq \|x\|,$$

thus Γ is norm decreasing.
(*ii*) Obviously, $\Gamma(\mathcal{A})$ strongly separates the points of $\Delta(\mathcal{A})$, that is, $\Gamma(\mathcal{A})(\varphi) \neq \{0\}$ for each $\varphi \in \Delta(\mathcal{A})$, and if $\varphi_1, \varphi_2 \in \Delta(\mathcal{A})$ with $\varphi_1 \neq \varphi_2$, then $\widehat{x}(\varphi_1) \neq \widehat{x}(\varphi_2)$ for some $x \in \mathcal{A}$.

(*iii*) If $\|y\|^2 = \|y^2\|$ for all $y \in \mathcal{A}$, then $\|x^{2^n}\| = \|x\|^{2^n}$ for every $x \in \mathcal{A}$ and $n \in \mathbb{N}$. Therefore

$$\|\widehat{x}\|_\infty = r(x) = \lim_{n\to\infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|.$$

Conversely, if Γ is an isometry, then

$$\|x^2\| = \|\widehat{x}^2\|_\infty = \|\widehat{x}\|_\infty^2 = \|x\|^2.$$

□

Remark 2.3.11 The Gelfand representation of a commutative Banach algebra $\mathcal{A}$ need not be injective. However, when it is, then $\mathrm{rad}(\mathcal{A}) = \bigcap_{\varphi \in \Delta(\mathcal{A})} \ker \varphi = \{0\}$, hence $\mathcal{A}$ is semisimple.

Lemma 2.3.12 *Let $\mathcal{A}$ be a commutative Banach algebra and for $x \in \mathcal{A}$ let*

$$r = \inf_{x \neq 0} \frac{\|x^2\|}{\|x\|^2} \quad \text{and} \quad s = \inf_{x \neq 0} \frac{\|\widehat{x}\|_\infty}{\|x\|}.$$

Then $s^2 \leq r \leq s$.

Proof. Since $\|\widehat{x}\|_\infty \geq s\|x\|$,

$$\|x^2\| \geq \|\widehat{x}^2\|_\infty = \|\widehat{x}\|_\infty^2 \geq s^2\|x\|^2$$

for every $x \in \mathcal{A}$. Thus $s^2 \leq r$. It remains to show $r \leq s$. Clearly, $\|x^2\| \geq r\|x\|^2$, so $\|x^4\| \geq r\|x^2\|^2 \geq r^3\|x\|^4$. By induction we eventually get $\|x^{2^k}\| \geq r^{2^k-1}\|x\|^{2^k}, k \in \mathbb{N}$, hence

$$\|x^{2^k}\|^{\frac{1}{2^k}} \geq r^{1-\frac{1}{2^k}}\|x\|.$$

Letting $k \to \infty$ and by the spectral radius formula and Theorem 2.3.10 we get for $x \in \mathcal{A}$

$$\|\widehat{x}\|_\infty = r(x) \geq r\|x\|.$$

It follows immediately that $s \geq r$. □

Theorem 2.3.13 *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. Then $\Gamma(\mathcal{A}) = \widehat{\mathcal{A}} = \{\widehat{x} : x \in \mathcal{A}\}$ is closed in $C(\Delta(\mathcal{A}))$ if and only if there exists a constant $K > 0$ such that $\|x\|^2 \leq K\|x^2\|$ for all $x \in \mathcal{A}$.*

Proof. We use Lemma 2.3.12.

Let $K > 0$ and $\|x\|^2 \leq K\|x^2\|$ for all $x \in \mathcal{A}$. This implies that $r \geq \frac{1}{K} > 0$, thus $s > r > 0$, i.e. $s\|x\| \leq \|\widehat{x}\|_\infty$ for all $x \in \mathcal{A}$. Suppose that $(\widehat{x_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(\Delta(\mathcal{A}))$. Then $(x_n)_n$ is a Cauchy sequence in $\mathcal{A}$. Thus there exists $x \in \mathcal{A}$ such that $x_n \to x$ as $n \to \infty$. Since Γ is continuous, $\widehat{x_n} \to \widehat{x}$ as $n \to \infty$, whence it follows that $\Gamma(\mathcal{A})$ is closed.

Conversely, it suffices to show that $s \neq 0$, then $r \geq s^2 > 0$. The Gelfand transform $x \mapsto \widehat{x}$ is a continuous isomorphism between two Banach spaces, $\mathcal{A}$ and $C(\Delta(\mathcal{A}))$, as $\Gamma(\mathcal{A})$ is closed and $\mathcal{A}$ is semisimple. By the open mapping theorem, Γ^{-1} exists and is continuous, that is, there exists $c > 0$ such that $\|\Gamma^{-1}(\widehat{x})\|_\infty \leq c\|\widehat{x}\|_\infty$ for all $\widehat{x} \in C(\Delta(\mathcal{A}))$. Thus $\|x\| \leq c\|\widehat{x}\|_\infty$ for all $x \in \mathcal{A}$.

□

We continue with a number of examples in order to get a better understanding of the Gelfand representation.

Example 2.3.14 Let $\mathcal{X}$ be a locally compact Hausdorff space. For $x \in \mathcal{X}$, the map

$$\varphi_x : C_0(\mathcal{X}) \to \mathbb{C}, \quad f \mapsto f(x)$$

is a nonzero homomorphism, so $\{\varphi_x : x \in \mathcal{X}\} \subseteq \Delta(C_0(\mathcal{X}))$. We claim that we have equality.

For $x \in \mathcal{X}$, let

$$M_x = \ker \varphi_x = \{f \in C_0(\mathcal{X}) : f(x) = 0\},$$

which is a maximal ideal of $C_0(\mathcal{X})$ by Theorem 2.1.18. Now let I be an ideal of $C_0(\mathcal{X})$. If $I \nsubseteq M_x$ for every $x \in \mathcal{X}$, then for each $x \in \mathcal{X}$ there exists $f_x \in I$ with $f_x \neq 0$. Since I is an ideal, $g_x = |f_x|^2 = \overline{f_x}f_x \in I$, and because g_x is continuous and non-negative with $g_x(x) > 0$, there exists an open set U_x with $x \in U_x$ and $g_x(y) > 0$ for all $y \in U_x$. As x varies over $\mathcal{X}$, the open sets U_x cover $\mathcal{X}$. Since $\mathcal{X}$ is locally compact, there exist $n \in \mathbb{N}$

and $x_1, \ldots, x_n \in \mathcal{X}$ such that $U_{x_1}, \ldots, U_{x_n}$ cover $\mathcal{X}$. Let $g = g_{x_1} + \ \ldots \ + g_{x_n}$. Then $g \in I$ and $g(x) > 0$ for all $x \in \mathcal{X}$, so g is invertible in $C_0(\mathcal{X})$. Hence $I = C_0(\mathcal{X})$.

This shows that every proper ideal I of $C_0(\mathcal{X})$ is contained in M_x for some $x \in \mathcal{X}$. Let $\varphi \in \Delta(C_0(\mathcal{X}))$. Since $\ker\varphi$ is a maximal ideal, we must have $\ker\varphi = M_x$ for some $x \in \mathcal{X}$, so $\varphi = \varphi_x$ by Theorem 2.1.18.

Now consider the map $\Phi : \mathcal{X} \to \Delta(C_0(\mathcal{X}))$, $x \mapsto \varphi_x$ where $\varphi_x(f) = f(x)$ for $f \in C_0(\mathcal{X})$. We have just shown that Φ is surjective. Since $\mathcal{X}$ is locally compact and Hausdorff, $C_0(\mathcal{X})$ separates the points of $\mathcal{X}$ by Urysohn's lemma. Hence if $\varphi_x = \varphi_y$ for $x, y \in \mathcal{X}$, then $f(x) = f(y)$ for all $f \in C_0(\mathcal{X})$, so $x = y$. Therefore Φ is a bijection.

We claim that the map $x \mapsto \varphi_x$ is a homeomorphism. Indeed, given $x \in \mathcal{X}$ and an open neighbourhood V_x of x, by Urysohn's lemma there exists $f \in C_0(\mathcal{X})$ such that $f(x) \neq 0$ and $f|_{\mathcal{X}\setminus V_x} = 0$, and hence V_x contains the Gelfand neighbourhood $\{y \in \mathcal{X} : |\varphi_y(f) - \varphi_x(f)| < |f(x)|\}$ of x.
If $f \in C_0(\mathcal{X})$, then

$$\widehat{f} : \Delta(C_0(\mathcal{X})) \to \mathbb{C},\ \varphi_x \mapsto \varphi_x(f) = f(x), \quad x \in \mathcal{X}.$$

This means that if we identify $\Delta(C_0(\mathcal{X}))$ with $\mathcal{X}$, then $\widehat{f} = f$. Hence the Gelfand homomorphism $\Gamma : C_0(\mathcal{X}) \to C_0(\mathcal{X})$ is the identity mapping.

Example 2.3.15 Consider the matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $M_2(\mathbb{C})$. Then $\mathcal{A} = \operatorname{span}\{I, T\}$ is a unital commutative Banach subalgebra of $M_2(\mathbb{C})$. Indeed, since $\mathcal{A}$ is a finite dimensional vector subspace of the Banach space $M_2(\mathbb{C}) = \mathfrak{B}(\mathbb{C}^2)$, it is closed. Moreover, the unit of $M_2(\mathbb{C})$ is I, and $I \in \mathcal{A}$. Since $T^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$, we have for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$

$$(\alpha I + \beta T)(\gamma I + \delta T) = \alpha\gamma I + (\alpha\delta + \gamma\beta)T \in \mathcal{A}.$$

Hence $\mathcal{A}$ is a unital commutative Banach subalgebra of $M_2(\mathbb{C})$.

Furthermore, we want to examine the Gelfand space $\Delta(\mathcal{A})$. Let $\varphi \in \Delta(\mathcal{A})$. By Lemma 2.1.11, $\varphi(I) = 1$. Since $T^2 = 0$, we have $\varphi(T)^2 = \varphi(T^2) = 0$, so $\varphi(T) = 0$. Hence $\varphi(\gamma I + \delta T) = \varphi(\gamma)\varphi(I) + \varphi(\delta)\varphi(T) = \gamma$ for all $\delta, \gamma \in \mathbb{C}$. This is the unique multiplicative linear functional, that is $\Delta(\mathcal{A}) = \{\varphi\}$.

We claim that $\mathcal{A}$ is not semisimple, i.e. the Gelfand representation of $\mathcal{A}$ is not injective. Indeed, note that we have $\widehat{T}(\varphi) = \varphi(T) = 0$, thus $\widehat{T} = 0$ and $\Gamma_{\mathcal{A}}$ is not injective. Alternatively, we can observe that $\|\widehat{T}\| = r(T) = 0$ by Theorem 2.3.10, so $\widehat{T} = 0$.

Example 2.3.16 Let $\mathcal{A} = C^n([a,b])$, and for each $t \in [0,1]$ define $\varphi_t \in \Delta(\mathcal{A})$ by $\varphi_t(f) = f(t)$. We claim that

$$\Phi : [a,b] \to \Delta(\mathcal{A}), \quad t \mapsto \varphi_t$$

is a homeomorphism.

We proceed in a similar way as in Example 2.3.14. Φ is clearly injective and continuous. Let M be any maximal ideal in $\mathcal{A}$. Then, by Theorem 2.1.18, we find $s \in [a,b]$ such that

$$M = \{f \in \mathcal{A} : f(s) = 0\}.$$

Thus $M = \ker \varphi_s$. Therefore Φ is a a homeomorphism since $[a, b]$ is compact and $\Delta(\mathcal{A})$ is Hausdorff. As in Example 2.3.14, we identify $[a, b]$ with $\Delta(\mathcal{A})$. Hence the Gelfand homomorphism of $\mathcal{A}$ is the identity mapping.

Example 2.3.17 Recall the disc algebra $A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f \text{ is analytic on } \mathbb{D}\}$ (Example 1.1.10). Let $w \in \overline{\mathbb{D}}$. Then $\varphi_w : A(\overline{\mathbb{D}}) \to \mathbb{C},\ f \mapsto f(w)$ is a multiplicative linear functional on $A(\overline{\mathbb{D}})$. We claim that every such functional arises in this way.

Consider the function $z \in A(\overline{\mathbb{D}})$ defined by $z(w) = w, w \in \overline{\mathbb{D}}$. If $\varphi \in \Delta(A(\overline{\mathbb{D}}))$, then $\varphi(e) = 1$ and $|\varphi(z)| \leq \|z\| = 1$, hence $\varphi(z) \in \overline{\mathbb{D}}$. Note that the polynomials $\mathbb{D} \to \mathbb{C}$ form a dense unital subalgebra of $A(\overline{\mathbb{D}})$. Now if $p : \mathbb{D} \to \mathbb{C}$ is a polynomial, then $p = \lambda_0 e + \lambda_1 z + \ \dots\ + \lambda_n z^n$ for constants $\lambda_i, i = 0, \dots, n$. Hence

$$\varphi(p) = \lambda_0 + \lambda_1 \varphi(z) + \ \dots\ + \lambda_n \varphi(z)^n = p(\varphi(z)).$$

Since the polynomials are dense in $A(\overline{\mathbb{D}})$, we have $\varphi(f) = f(\varphi(z))$ for all $f \in A(\overline{\mathbb{D}})$. Thus $\varphi = \varphi_{\varphi(z)}$ and hence $\Delta(A(\overline{\mathbb{D}})) = \{\varphi_w : w \in \overline{\mathbb{D}}\}$. Just as in Example 2.3.14, we can easily verify that the map $w \mapsto \varphi_w$ is a homeomorphism from $\overline{\mathbb{D}}$ onto $\Delta(A(\overline{\mathbb{D}}))$.

Example 2.3.18 Recall the commutative Banach algebra $l^1(\mathbb{Z})$ with product $*$ from Example 1.1.14. We claim that $l^1(\mathbb{N}_0)$ is a closed Banach subalgebra of $l^1(\mathbb{Z})$.

Define

$$\iota : l^1(\mathbb{N}_0) \to l^1(\mathbb{Z}),\ x = (x_n)_{n \in \mathbb{N}_0} \mapsto \iota(x) = \begin{cases} x_n, & n \in \mathbb{N}_0, \\ 0 & \text{otherwise } (\mathbb{Z} \setminus \mathbb{N}_0). \end{cases}$$

Let $x \in l^1(\mathbb{N}_0)$, so $\|x\|_{l^1(\mathbb{N}_0)} = \sum\limits_{n \in \mathbb{N}_0} |x_n| < \infty$. Hence we get

$$\|\iota(x)\|_{l^1(\mathbb{Z})} = \sum_{n \in \mathbb{Z}} |\iota(x)| = \sum_{n \in \mathbb{N}_0} |x_n| + \sum_{n \in \mathbb{Z} \setminus \mathbb{N}_0} 0 = \|x\|_{l^1(\mathbb{N}_0)}.$$

Thus ι is isometric. Next we want to verify that ι is closed under addition, limit operation and multiplication (convolution).

- ι is closed under addition: Let $y = (y_n)_{n \in \mathbb{N}_0}$ and $\lambda > 0$. Then $\iota(x + \lambda y) = \iota(x) + \lambda \iota(y)$.
- ι is closed under limit operation: Let $(x_k)_k$ in $\iota(l^1(\mathbb{N}_0))$ and $y \in l^1(\mathbb{Z})$ such that $x_k \to y$ as $k \to \infty$. Thus

  $$\|x_k - y\|_{l^1(\mathbb{Z})} = \sum_{n \in \mathbb{Z}} |x_{k_n} - y_n| = \sum_{n \in \mathbb{Z} \setminus \mathbb{N}_0} |y_n| + \sum_{n \in \mathbb{N}_0} |x_{k_n} - y_n|.$$

  Clearly, $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m \geq N : \|x_m - y\|_{l^1(\mathbb{Z})} < \varepsilon$. Suppose that $\sum\limits_{n \in \mathbb{Z} \setminus \mathbb{N}_0} |y_n| > 0$. Then

  $$\sum_{n \in \mathbb{Z} \setminus \mathbb{N}_0} |y_n| + \sum_{n \in \mathbb{N}_0} |x_{k_n} - y_n| < \sum_{n \in \mathbb{Z} \setminus \mathbb{N}_0} |y_n|.$$

Therefore $\sum_{n\in\mathbb{N}_0} |x_{k_n} - y_n| < 0$, which is a contradiction. Hence $\sum_{n\in\mathbb{Z}\setminus\mathbb{N}_0} |y_n| = 0$ and so $|y_n| = 0$ for all $n \in \mathbb{Z} \setminus \mathbb{N}_0$. Thus $y \in \iota(l^1(\mathbb{N}_0))$.

- ι is closed under convolution: By applying the Cauchy product we get

$$\iota(xy) = \iota(\Big(\sum_{n\in\mathbb{N}_0} x_n \sum_{n\in\mathbb{N}_0} y_n\Big)) = \iota(\Big(\sum_{k\in\mathbb{N}_0} x_k y_{n-k}\Big)_{n\in\mathbb{N}_0}) = \sum_{k\in\mathbb{Z}} \iota(x)_k \iota(y)_{n-k} = \iota(x)\iota(y),$$

since $\sum_{k\in\mathbb{Z}\setminus\mathbb{N}_0} \iota(x)_k \iota(y)_{n-k} = 0$.

Now we want to determine the structure space of $l^1(\mathbb{N}_0)$. We claim that for every $z \in \mathbb{C}, |z| \leq 1$, the functional $\varphi_z : l^1(\mathbb{N}_0) \to \mathbb{C}$ defined by

$$\varphi_z((a_n)_{n\in\mathbb{N}_0}) = \sum_{n\in\mathbb{N}_0} a_n z^n$$

is in $\Delta(l^1(\mathbb{N}_0))$. Indeed, let $(a_n)_n, (b_n)_n$ be sequences in $l^1(\mathbb{N}_0)$ and $\lambda > 0$. Then

- φ_z is linear:

$$\varphi_z(a_n) + \lambda\varphi_z(b_n) = \sum_{n\in\mathbb{N}_0} a_n z^n + \lambda \sum_{n\in\mathbb{N}_0} b_n z^n = \sum_{n\in\mathbb{N}_0} (a_n + \lambda b_n) z^n = \varphi_z(a_n + \lambda b_n).$$

- φ_z is multiplicative:

$$\varphi_z(a_n b_n) = \sum_{n\in\mathbb{N}_0} \sum_{k\in\mathbb{N}_0} a_k b_{n-k} z^n = \sum_{n\in\mathbb{N}_0} a_n z^n \sum_{n\in\mathbb{N}_0} b_n z^n = \varphi_z(a_n)\varphi_z(b_n).$$

It should be noted that there exists a multiplicative functional on $l^1(\mathbb{N}_0)$ which cannot be extended to a multiplicative functional on $l^1(\mathbb{Z})$. Indeed, every multiplicative linear functional $\varphi \in \Delta(l^1(\mathbb{N}_0))$ can be written as $\varphi_z = \varphi$ for some $z \in \mathbb{C}, |z| < 1$. But for $z \in \mathbb{D}, \varphi_z$ does not have an extension on $l^1(\mathbb{Z})$. This is an important observation because the Hahn-Banach theorem is not valid for multiplicative functionals.

The next example will lead to an interesting application of Banach algebra theory.

Example 2.3.19 Consider the space $l^1(\mathbb{Z})$ and recall that $l^1(\mathbb{Z})$ with product $*$ is a commutative Banach algebra (Example 1.1.14). For $n \in \mathbb{Z}$, let $e_n = (\delta_{mn})_{m\in\mathbb{Z}}$. Then $e_n \in l^1(\mathbb{Z})$, and the linear span of $\{e_n : n \in \mathbb{Z}\}$ is dense in $l^1(\mathbb{Z})$. Moreover, it is easy to check that

$$e_n * e_m = e_{n+m}, \quad n, m \in \mathbb{Z}.$$

In particular, $e_0 * e_m = e_m$, hence e_0 is the unit element for $l^1(\mathbb{Z})$.

Now we want to determine the Gelfand space of $l^1(\mathbb{Z})$. If $\varphi \in \Delta(l^1(\mathbb{Z}))$, then $\varphi(e_0) = 1$. Additionally, $e_n * e_{-n} = e_0$, so $e_n = (e_{-n})^{-1}$ and $|\varphi(e_n)| \leq \|e_n\| = 1$ for each $n \in \mathbb{Z}$. Hence

$$1 \leq |\varphi(e_{-n})|^{-1} = |\varphi(e_{-n})^{-1}| = |\varphi(e_n)| \leq 1.$$

Thus we have equality, particularly $|\varphi(e_1)| = 1$. Recall the 1-torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then $\varphi(e_1) \in \mathbb{T}$. Since $e_n = e_1^n$, we have $\varphi(e_n) = \varphi(e_1)^n$. So for $x \in l^1(\mathbb{Z})$ we get

$$\varphi(x) = \varphi\Big(\sum_{n\in\mathbb{Z}} x_n e_n\Big) = \sum_{n\in\mathbb{Z}} x_n \varphi(e_1)^n.$$

Therefore φ is determined by the complex number $\varphi(e_1) \in \mathbb{T}$.

Conversely, we claim that given any $z \in \mathbb{T}$, there exists a multiplicative linear functional $\varphi_z \in \Delta(l^1(\mathbb{Z}))$ with $\varphi_z(e_1) = z$. For that purpose let $\mathcal{A}_0 = \operatorname{span}\{e_n : n \in \mathbb{Z}\}$, which is a dense subalgebra of $l^1(\mathbb{Z})$. Let $\varphi_0 : \mathcal{A}_0 \to \mathbb{C}$ be the unique linear map such that $\varphi_0(e_n) = z^n$ for all $n \in \mathbb{Z}$. If $x, y \in \mathcal{A}_0$, then

$$\begin{aligned}
\varphi_0(x * y) &= \varphi_0\Big(\sum_{n,m\in\mathbb{Z}} x_m y_{n-m} e_n\Big) = \sum_{n,m\in\mathbb{Z}} x_m y_{n-m} z^n \\
&= \sum_{m,n\in\mathbb{Z}} x_m z^m y_{n-m} z^{n-m} = \sum_{m\in\mathbb{Z}} x_m z^m \Big(\sum_{n\in\mathbb{Z}} y_{n-m} z^{n-m}\Big) \\
&= \varphi_0(x)\varphi_0(y).
\end{aligned}$$

Hence φ_0 is a homomorphism and since

$$|\varphi_0(x)| = \Big|\sum_{n\in\mathbb{Z}} x_n z^n\Big| \le \sum_{n\in\mathbb{Z}} |x_n||z^n| = \|x\|_{l^1(\mathbb{Z})},$$

φ_0 is continuous. Thus φ_0 extends to a continuous linear homomorphism $\varphi_z : l^1(\mathbb{Z}) \to \mathbb{C}$, which is multiplicative on $l^1(\mathbb{Z})$. Clearly, $\varphi_z(e_1) = z$.

This shows that the map $\Phi : \mathbb{T} \to \Delta(l^1(\mathbb{Z})), z \mapsto \varphi_z$ is a bijection. By routine arguments (as in Example 2.3.14) we can show that Φ is a homeomorphism.

In Example 1.1.15 we have seen that the Wiener algebra $W(\mathbb{T})$ is isomorphic to $l^1(\mathbb{Z})$, the isomorphism being given by $f \mapsto (c_n(f))_{n\in\mathbb{Z}}$ where

$$c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}dt, \quad n \in \mathbb{Z}.$$

By the preceding Example 2.3.19 we can identify $\Delta(W(\mathbb{T}))$ with $\mathbb{T}$ as follows. For $z \in \mathbb{T}$ let

$$\varphi_z(f) = \sum_{n\in\mathbb{Z}} c_n(f) z^n, \quad f \in W(\mathbb{T}).$$

Then $z \mapsto \varphi_z$ is a homeomorphism between $\mathbb{T}$ and $\Delta(W(\mathbb{T}))$. By identifying

$$\widehat{f}(z) = \sum_{n\in\mathbb{Z}} c_n(f) z^n = f(z) \quad \text{for all } f \in W(\mathbb{T}),$$

the Gelfand representation of $W(\mathbb{T})$ is the identity mapping. Hence the Gelfand transform $\widehat{f}$ can be viewed as an abstract inverse Fourier transform of f.

As a simple consequence we obtain the following classical result by Wiener whose proof is quite simple yet beautiful. It should be noted that Wiener's original proof was hard Fourier analysis and a good deal more complicated.

Theorem 2.3.20 (**Wiener**) *If $f \in W(\mathbb{T})$ is such that $f(z) \neq 0$ for all $z \in \mathbb{T}$, then $1/f \in W(\mathbb{T})$. That is, if f is nonzero and has an absolutely convergent Fourier series, then $1/f$ has an absolutely convergent Fourier series as well.*

Proof. By the previous considerations, we know that $\Delta(W(\mathbb{T}))$ can be identified with $\mathbb{T}$. Now by Theorem 2.1.20 (*ii*) (Beurling-Gelfand), f is invertible in $W(\mathbb{T})$ if and only if $0 \notin \sigma(f)$. By Theorem 2.3.6 (Gelfand's representation theorem) and Example 2.3.19 we get $\sigma(f) = \widehat{f}(\Delta) = f(\mathbb{T})$. By hypothesis we have $0 \notin f(\mathbb{T})$, so $0 \notin \sigma(f)$ and hence f is invertible in $W(\mathbb{T})$. Thus $1/f \in W(\mathbb{T})$.

□

We continue with a result concerning homomorphisms and isomorphisms between commutative Banach algebras.

Theorem 2.3.21 *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras. If $\mathcal{A}$ and $\mathcal{B}$ are algebraically isomorphic, then $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$ are homeomorphic.*

Proof. Suppose $\Phi : \mathcal{A} \to \mathcal{B}$ is an algebra isomorphism. Let $\Phi' : \Delta(\mathcal{B}) \to \Delta(\mathcal{A})$ denote the dual mapping, i.e.

$$\Phi'(\varphi)(x) = \varphi(\Phi(x)), \quad x \in \mathcal{A},\ \varphi \in \Delta(\mathcal{B}).$$

By routine arguments it can be easily verified that Φ' is a bijection. We claim that Φ' is a homeomorphism. First note that Φ' is continuous provided that all functions

$$\Delta(\mathcal{B}) \to \mathbb{C},\ \varphi \mapsto \Phi'(\varphi)(x), \quad x \in \mathcal{A},$$

are continuous. Indeed, it follows immediately from the definition of Φ' and the definition of the topology on $\Delta(\mathcal{B})$ that such functions are continuous. Similarly, $(\Phi')^{-1}$ is continuous.

□

Remark 2.3.22 If $\mathcal{B}$ is semisimple, then by Theorem 2.2.5 Φ is continuous, and Φ is completely determined by the equation $\widehat{\Phi}(x)(\varphi) = \widehat{x}(\Phi'(\varphi))$ where $\varphi \in \Delta(\mathcal{B}), x \in \mathcal{A}$ and Φ' is the homeomorphism defined in the proof of Theorem 2.3.21.

The converse of Theorem 2.3.21 need not be valid. Generally speaking, the question of precisely which homeomorphisms between $\Delta(\mathcal{B})$ and $\Delta(\mathcal{A})$ induce isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ is a rather intricate one, even in the case that $\mathcal{A} = \mathcal{B}$. For some specific algebras the answers are known, for instance, when $\mathcal{A} = C_0(\mathcal{X})$ and $\mathcal{B} = C_0(\mathcal{Y})$ where $\mathcal{X}$ and $\mathcal{Y}$ are locally compact Hausdorff spaces.

Corollary 2.3.23 *For locally compact Hausdorff spaces $\mathcal{X}$ and $\mathcal{Y}$ the following conditions are equivalent.*

(*i*) *$C_0(\mathcal{X})$ and $C_0(\mathcal{Y})$ are isometrically isomorphic.*

(*ii*) *$C_0(\mathcal{X})$ and $C_0(\mathcal{Y})$ are algebraically isomorphic.*

(*iii*) *$\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic.*

Proof.
$(i) \Rightarrow (ii)$ is trivial and $(ii) \Rightarrow (iii)$ is a consequence of Theorem 2.3.21 and Example 2.3.14. Finally, for $(iii) \Rightarrow (i)$ let $\Phi : \mathcal{X} \to \mathcal{Y}$ be a homeomorphism. Then $f \mapsto f \circ \Phi$ is an isometric algebra isomorphism from $C_0(\mathcal{Y})$ to $C_0(\mathcal{X})$.

□

We conclude this section with a result which can be efficiently used to identify the Gelfand topology.

Proposition 2.3.24 *Let $\mathcal{X}$ be a locally compact Hausdorff space and let $\mathcal{A}$ be a family of functions in $C_0(\mathcal{X})$ which strongly separates the points of $\mathcal{X}$. Then the topology of $\mathcal{X}$ equals the weak topology with respect to the functions $x \mapsto f(x), f \in \mathcal{A}$.*

Proof. The general idea is to utilize the one-point compactification technique. We know that the given topology on $\mathcal{X}$ is stronger than the weak topology. Hence it suffices to show that for $x \in \mathcal{X}$ and an open neighbourhood U of x in $\mathcal{X}$, there exists a set V such that $x \in V \subseteq U$ and V is open in the weak topology. Now let $\widetilde{\mathcal{X}}$ be $\mathcal{X}$ if $\mathcal{X}$ is compact, and let $\widetilde{\mathcal{X}} = X \cup \{\infty\}$ be the one-point compactification of $\mathcal{X}$ if $\mathcal{X}$ is noncompact. Every $f \in C_0(\mathcal{X})$ extends to a continuous function on $\widetilde{\mathcal{X}}$, also denoted f, by setting $f(\infty) = 0$. By hypothesis, $\mathcal{A}$ strongly separates the points of $\mathcal{X}$, so for every $y \in \widetilde{\mathcal{X}} \setminus U$ there exists $f_y \in \mathcal{A}$ such that

$$\varepsilon_y = |f_y(y) - f_y(x)| > 0.$$

Hence for each $y \in \widetilde{\mathcal{X}} \setminus U$,

$$V_y = \{z \in \widetilde{\mathcal{X}} : |f_y(z) - f_y(y)| < \varepsilon_y/2\}$$

is an open neighbourhood of y in $\widetilde{\mathcal{X}}$. Since $\widetilde{\mathcal{X}} \setminus U$ is compact, there exist finitely many $y_1, \ldots, y_n \in \widetilde{\mathcal{X}} \setminus U$ such that

$$\widetilde{\mathcal{X}} \setminus U \subseteq \bigcup_{k=1}^{n} V_{y_k}.$$

Let

$$V = \{z \in \mathcal{X} : |f_{y_k}(z) - f_{y_k}(x)| < \varepsilon_{y_k}/2 \text{ for all } 1 \leq k \leq n\}.$$

Then $x \in V$ und V is contained in U. Indeed, if $z \in V$ and $z \notin U$, then $z \in V_{y_k}$ for some $k \in \mathbb{N}$ and therefore

$$\begin{aligned} |f_{y_k}(x) - f_{y_k}(y_k)| &= |f_{y_k}(x) - f_{y_k}(z) + f_{y_k}(z) - f_{y_k}(y_k)| \\ &\leq |f_{y_k}(x) - f_{y_k}(z)| + |f_{y_k}(z) - f_{y_k}(y_k)| \\ &< 2\frac{\varepsilon_{y_k}}{2} = \varepsilon_{y_k}, \end{aligned}$$

which is a contradiction to the definition of ε_{y_k}. □

2.4. Finitely generated Banach algebras

In this final section we briefly discuss so called finitely generated Banach algebras. Many naturally occuring Banach algebras are generated by finitely many elements (for example $l^1(\mathbb{Z})$). Such algebras admit a satisfying description of their structure spaces. In particular, we are interested in the notion of the joint spectrum as it will be needed later.

Definition 2.4.1 Let $\mathcal{A}$ be a commutative Banach algebra with identity e. A subset E of $\mathcal{A}$ is said to *generate* $\mathcal{A}$ if every closed subalgebra of $\mathcal{A}$ containing E and e coincides with $\mathcal{A}$. Equivalently, the set of all finite linear combinations of elements of the form

$$x_1^{n_1} x_2^{n_2} \cdot \ \ldots \ \cdot x_r^{n_r}, \quad x_i \in E, \ n_i \in \mathbb{N}_0, \ r \in \mathbb{N},$$

is dense in $\mathcal{A}$.
$\mathcal{A}$ is called *finitely generated* if there exists a finite subset of $\mathcal{A}$ that generates $\mathcal{A}$. In particular, an element x of $\mathcal{A}$ is a *generator* for $\mathcal{A}$ if the algebra generated by x equals $\mathcal{A}$, i.e. $\mathcal{A}(x) = \mathcal{A}$. An algebra that has a generator is called *monothetic*.

Definition 2.4.2 Let $\mathcal{A}$ be a commutative Banach algebra with identity and let $x_1, \ldots, x_n \in \mathcal{A}$. Then the *joint spectrum* of $x_1, \ldots, x_n$ is the subset $\sigma_{\mathcal{A}}(x_1, \ldots, x_n)$ of $\mathbb{C}^n$ defined by

$$\sigma_{\mathcal{A}}(x_1, \ldots, x_n) = \{(\varphi(x_1), \ldots, \varphi(x_n)) : \varphi \in \Delta(\mathcal{A})\}.$$

Since $\Delta(\mathcal{A})$ is compact and the mapping

$$\Delta(\mathcal{A}) \to \mathbb{C}^n, \quad \varphi \mapsto (\varphi(x_1), \ldots, \varphi(x_n))$$

is continuous, $\sigma_{\mathcal{A}}(x_1, \ldots, x_n)$ is a compact subset of $\mathbb{C}^n$. Evidently, by Theorem 2.3.6 (Gelfand's representation theorem) the joint spectrum of a single element x reduces to the spectrum $\sigma_{\mathcal{A}}(x)$ of x.

If $x_1, \ldots, x_n$ generate a Banach algebra $\mathcal{A}$, then $\Delta(\mathcal{A})$ is canonically homeomorphic to the joint spectrum of $x_1, \ldots, x_n$, which is a compact subset of $\mathbb{C}^n$. Hence it is an important issue to identify the compact subsets of $\mathbb{C}^n$ arising in this manner of joint spectra.

Lemma 2.4.3 *Let $\mathcal{A}$ be a unital commutative Banach algebra, and suppose that $E \subseteq \mathcal{A}$ generates $\mathcal{A}$. Then the mapping*

$$\Phi : \Delta(\mathcal{A}) \to \prod_{x \in E} \sigma_{\mathcal{A}}(x), \ \varphi \mapsto (\varphi(x))_{x \in E}$$

is a homeomorphism between $\Delta(\mathcal{A})$ and $\Phi(\Delta(\mathcal{A})) \subseteq \prod_{x \in E} \sigma_{\mathcal{A}}(x)$. Particularly, if E is finite, say $E = \{x_1, \ldots, x_n\}$, then we have a homeomorphism

$$\Delta(\mathcal{A}) \to \sigma_{\mathcal{A}}(x_1, \ldots, x_n), \ \varphi \mapsto (\varphi(x_1), \ldots, \varphi(x_n)).$$

Proof. Clearly Φ is surjective, hence we need to show that Φ is injective. Assume that $\varphi_1, \varphi_2 \in \Delta(\mathcal{A})$ are such that $\varphi_1(x) = \varphi_2(x)$ for all $x \in E$. Let $\mathcal{B}$ denote the smallest Banach subalgebra of $\mathcal{A}$ containing E and the identity. Then $\mathcal{B}$ is dense in $\mathcal{A}$, and $\varphi_1(y) = \varphi_2(y)$ for all $y \in \mathcal{B}$. Now, since elements in $\Delta(\mathcal{A})$ are continuous, it follows immediately that $\varphi_1 = \varphi_2$. Thus Φ is injective.

We know that $\prod_{x\in E} \sigma_{\mathcal{A}}(x)$ carries the weak topology with respect to the projections

$$p_y : \prod_{x\in E} \sigma_{\mathcal{A}}(x) \to \sigma_{\mathcal{A}}(y),\ y \in E.$$

Since $p_y \circ \Phi(\varphi) = \varphi(y)$, the functions $p_y \circ \Phi, y \in E$, are continuous. Hence Φ is continuous. Therefore

$$\Phi : \Delta(\mathcal{A}) \to \Phi(\Delta(\mathcal{A})),\ \varphi \mapsto (\varphi(x))_{x\in E}$$

is a continuous bijection between a compact space and a Hausdorff space, and hence is a homeomorphism. □

Example 2.4.4 Let $\mathcal{A}$ denote the disc algebra $A(\overline{\mathbb{D}})$ (Example 1.1.10), and y the element of $\mathcal{A}$ given by $y(z) = z,\ z \in \overline{\mathbb{D}}$. Then y is a generator for $\mathcal{A}$. For given $x \in \mathcal{A}$ and $\varepsilon \in (0, 1)$, let x_ε be defined by

$$x_\varepsilon(z) = x(\varepsilon z),\ z \in \overline{\mathbb{D}}.$$

Then $x_\varepsilon \in \mathcal{A}(y)$ and $\lim_\varepsilon \|x - x_\varepsilon\|_\infty = 0$, so that $x \in A(y)$. Now it follows from Lemma 2.4.3 that the mapping $\varphi \mapsto \varphi(y)$ is a homeomorphism of $\Delta(\mathcal{A})$ onto $\overline{\mathbb{D}}$. Because

$$\varphi(x) = x(\varphi(y)), \quad x \in \mathcal{A},\ \varphi \in \Delta(\mathcal{A}),$$

we get $x(z) = \sum_{n=0}^{\infty} a_n z^n$ with $|z| < 1$, and hence $x_\varepsilon = \sum_{n=0}^{\infty} a_n \varepsilon^n y^n$. Therefore

$$\varphi(x_\varepsilon) = x_\varepsilon(\varphi(y)) = x(\varepsilon\varphi(y)), \quad \varepsilon \in (0, 1).$$

Example 2.4.5 Let $\mathcal{A}$ denote the Wiener algebra $W(\mathbb{T})$ (Example 1.1.15), and let g be the element of $\mathcal{A}$ given by

$$g(t) = e^{it}, \quad t \in [0, 2\pi].$$

Then $g \in \mathcal{G}(\mathcal{A})$ and $g^{-1}(t) = e^{-it},\ t \in [0, 2\pi]$. For $f \in \mathcal{A}$ we have

$$f(t) = \sum_{n\in\mathbb{Z}} c_n(f) e^{int}, \quad t \in [0, 2\pi],$$

where c_n is the nth Fourier coefficient of f with norm

$$\|f\|_{W(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |c_n(f)|.$$

Thus the series

$$f = \sum_{n\in\mathbb{Z}} c_n(f) g^n \tag{2.4}$$

is convergent in norm and hence $\{g, g^{-1}\}$ is a set of generators for $\mathcal{A}$.

Next, we can observe that $\sigma(g) = \mathbb{T} = \{e^{it} : 0 \leq t \leq 2\pi\} = \{z \in \mathbb{C} : |z| = 1\}$. Indeed, since $W(\mathbb{T})$ is an algebra of functions, we have $\mathbb{T} \subseteq \sigma(g)$. Conversely, the inequalities

$$r(g) \leq \|g\|_{W(\mathbb{T})} = 1 \quad \text{and} \quad r(g^{-1}) \leq \|g^{-1}\|_{W(\mathbb{T})} = 1$$

show that

$$|z| \leq 1 \quad \text{and} \quad |z^{-1}| \leq 1$$

whenever $z \in \sigma(g)$. Hence $\sigma(g) \subseteq \mathbb{T}$.

For given $f \in \mathcal{A}$ we apply $\varphi \in \Delta(\mathcal{A})$ to (2.4),

$$\varphi(f) = \sum_{n \in \mathbb{Z}} c_n(f)(\varphi(g))^n = f(t),$$

where $e^{it} = \varphi(g)$ with $t \in [0, 2\pi]$. Therefore $\sigma(f) = \{f(t) : t \in [0, 2\pi]\}$. Moreover, $f \in \mathcal{G}(\mathcal{A})$ if and only if $f(t) \neq 0$ for $t \in [0, 2\pi]$ (Wiener's theorem).

We now aim at characterizing those compact subsets of $\mathbb{C}^n$ which arise in this way as structure spaces of commutative Banach algebras generated by n elements, $n \in \mathbb{N}$ (Theorem 2.4.8). To achieve that, we first need the geometrical notion of polynomial convexity.

Definition 2.4.6 A compact subset K of $\mathbb{C}^n, n \in \mathbb{N}$, is called *polynomially convex* if for every $z \in \mathbb{C}^n \setminus K$ there exists a polynomial p such that

$$p(z) = 1 \quad \text{and} \quad |p(w)| < 1 \quad \text{for all } w \in K.$$

Lemma 2.4.7 *Every compact convex subset K of $\mathbb{C}^n$ is polynomially convex.*

Proof. The proof is pretty straightforward and can be found in [Kan], Lemma 2.3.5, p.61. □

Theorem 2.4.8 *For a compact subset K of $\mathbb{C}^n$ the following conditions are equivalent.*

(i) *There exists a unital commutative Banach algebra $\mathcal{A}$ which is generated by n elements $x_1, \ldots, x_n \in \mathcal{A}$ such that $K = \sigma_{\mathcal{A}}(x_1, \ldots, x_n)$.*

(ii) *K is polynomially convex.*

Proof. See [Kan], Theorem 2.3.6, p.62. □

Remark 2.4.9 Suppose that $K \subseteq \mathbb{C}^n$ is polynomially convex and let $\mathcal{A} = P(K)$ be the algebra of all functions $f : K \to \mathbb{C}$ that are uniform limits of polynomial functions on K. Then $\mathcal{A}$ is generated by the functions

$$f_j(z) = z_j, \quad z = (z_1, \ldots, z_n) \in K, \quad 1 \leq j \leq n.$$

It follows now by Theorem 2.4.8 that $\Delta(P(K)) = K$.

We conclude this chapter with a theorem by Shilov, which states that a compact subset of $\mathbb{C}$ is polynomially convex if and only if its complement is connected.

Definition 2.4.10 A topological space $\mathcal{X}$ is called *connected* if it cannot be divided into two disjoint nonempty open sets. That is, for all nonempty open subsets E, F of $\mathcal{X}$

$$E \cap F = \emptyset \Rightarrow E \cup F \neq \mathcal{X}.$$

Theorem 2.4.11 *A compact subset K of $\mathbb{C}$ is polynomially convex if and only if $\mathbb{C} \setminus K$ is connected.*

Proof. See [Kan], Theorem 2.3.7, p.64. □

Remark 2.4.12 Let $n \in \mathbb{N}$. One can prove (by employing the maximum modulus principle for polynomials of several complex variables) that if $K \subseteq \mathbb{C}^n$ is polynomially convex, then $\mathbb{C}^n \setminus K$ is connected. However, the following example shows that for $n \geq 2$ there exist compact subsets of $\mathbb{C}^n$ which fail to be polynomially convex, even though $\mathbb{C}^n \setminus K$ is connected. To date, the problem of a topological characterization of polynomial convex subsets of $\mathbb{C}^n$ for $n \geq 2$ remains open.

Example 2.4.13 Let $n \geq 2$ and

$$K = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| = 1, 1 \leq j \leq n\}.$$

Now assume that K is polynomially convex. We can find a polynomial p in n variables such that $|p(z)| < 1$ for all $z \in K$ and $p(0, 1, \ldots, 1) = 1$. Let q be a polynomial in one variable defined by

$$q(w) = p(w, 1, \ldots, 1), \quad w \in \mathbb{C}.$$

Then $|q(w)| < 1$ for all $w \in \mathbb{C}$ with $|w| = 1$ and $q(0) = p(0, 1, \ldots, 1) = 1$. This is a contradiction to the maximum modulus principle. Hence K is not polynomially convex. Nevertheless, we claim that $\mathbb{C}^n \setminus K$ is connected. Recall Definition 2.4.10 and let

$$E_j = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| > 1\}$$

and

$$F_j = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1\},$$

$1 \leq j \leq n$. It follows that

$$\mathbb{C}^n \setminus K = \bigcup_{j=1}^{n} (E_j \cup F_j).$$

The sets E_j and F_j are arcwise connected[2] and we have

$$E_j \cap E_k \neq \emptyset,\ F_j \cap F_k \neq \emptyset,\ \text{and, for } j \neq k,\ E_j \cap F_k \neq \emptyset.$$

Thus $\mathbb{C}^n \setminus K$ is connected.

[2] A topological space $\mathcal{X}$ is said to be arcwise connected if any two distinct points can be joined by an arc.

3. Functional Calculus and Shilov's Idempotent Theorem

Our primary objective in this chapter is to establish and prove the Shilov idempotent theorem (Section 3.2), which states that for a commutative Banach algebra $\mathcal{A}$, the characteristic function of a compact open subset of $\Delta(\mathcal{A})$ is the Gelfand transform of an idempotent in $\mathcal{A}$. We then continue with some important applications (Section 3.3) of this powerful theorem such as the converse of Theorem 2.3.4. But before we can do so, we first need a new tool from complex analysis - the so called holomorphic functional calculus (Section 3.1).

Let $\mathcal{A}$ be a commutative Banach algebra. The single-variable holomorphic functional calculus associates with a complex-valued function f, which is defined and holomorphic in a neighbourhood of the spectrum of an element x of $\mathcal{A}$, an element $f(x)$ of $\mathcal{A}$. In order to prove the idempotent theorem we need a generalization - the (weaker) several-variable holomorphic functional calculus.

3.1. The holomorphic functional calculus and applications

We begin with developing the single-variable holomorphic functional calculus which provides an efficient method to construct from a given algebra element new elements with specified properties. Most results in this section will be given without any proof. For a more detailed approach on holomorphic functions and several complex variables we refer, for instance, to [Gun] and [Ran].

Definition 3.1.1 Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. Suppose that U is an open set containing $\sigma_{\mathcal{A}}(x)(\subseteq \mathbb{C})$, and denote by $R(U)$ the *set of all rational functions* on U. That is,

$$f \in R(U) \text{ if and only if } f = (p/q)|_U,$$

where p and q are polynomials with $q(z) \neq 0$ for all $z \in U$. By the polynomial spectral mapping theorem we have $\sigma_{\mathcal{A}}(q(x)) = q(\sigma_{\mathcal{A}}(x))$ and hence $0 \notin \sigma_{\mathcal{A}}(q(x))$. Therefore by Theorem 2.1.20 *(ii)* (Beurling-Gelfand), $q(x)$ is invertible in $\mathcal{A}$. We define $f(x) \in \mathcal{A}$ by

$$f(x) = p(x)q(x)^{-1}.$$

Since $\sigma_{\mathcal{A}}$ is nonempty, U is a nonempty open set and the representation p/q of f is unique apart from common factors of numerator and denominator. Furthermore, polynomials in x and the inverses of such polynomials commute with each other. Thus $f(x)$ is independent of the choice of p and q. Now let

$$R(x) = \bigcup \{R(U) : U \text{ is open and } U \supseteq \sigma_{\mathcal{A}}(x)\}.$$

Then $R(x)$ is an *algebra* and $f(x) \in \mathcal{A}$ is well-defined for every $f \in R(x)$.

Lemma 3.1.2 *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. Then the mapping*

$$R(x) \to \mathcal{A}, \quad f \mapsto f(x)$$

is a homomorphism and satisfies

$$\varphi(f(x)) = f(\varphi(x)) \text{ for all } \varphi \in \Delta(\mathcal{A}) \text{ and } \sigma_{\mathcal{A}}(f(x)) = f(\sigma_{\mathcal{A}}(x)).$$

Proof. We apply the spectral mapping theorem and use the fact that $\varphi(q(x)^{-1}) = 1/\varphi(q(x))$ for all $\varphi \in \Delta(\mathcal{A})$ and polynomials $q \neq 0$. □

Definition 3.1.3 For an open subset U of $\mathbb{C}$ let $H(U)$ denote the *set of all holomorphic functions* on U. Let $x \in \mathcal{A}$ and

$$H(x) = \bigcup \{H(U) : U \text{ is open and } U \supseteq \sigma_{\mathcal{A}}(x)\}.$$

Then, with pointwise operations, $H(x)$ is the *algebra* of all functions that are holomorphic in some neighbourhood of $\sigma_{\mathcal{A}}(x)$.

Next, we wish to extend the homomorphism from $R(x)$ into $\mathcal{A}$ to a homomorphism from $H(x)$ into $\mathcal{A}$. Proofs of the following lemmas can be found, for example, in [Bon], Section 7, pp.31 and in [Kan], Section 3.1, pp.141. We essentially exploit Cauchy's integral formula in one variable to define functions of Banach algebra elements.

Definition 3.1.4 For a rectifiable closed curve $\gamma : [a, b] \to \mathbb{C}$ and $z \in \mathbb{C} \setminus \gamma([a, b])$, the number

$$w(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w - z}$$

denotes the *winding number* of γ relative to the point z.

Lemma 3.1.5 *Let U be an open subset of $\mathbb{C}$ and K a compact subset of U. Then there are closed, piecewise smooth curves $\gamma_1, \ldots, \gamma_n$ in $U \setminus K$ such that for any holomorphic function f on U and $z \in K$,*

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(w)}{w - z} dw.$$

In particular, $\displaystyle\sum_{j=1}^{n} w(\gamma_j, z) = 1.$

Lemma 3.1.6 *If $f : U \to \mathbb{C}$ is a holomorphic function, then for every compact subset K of U and $\varepsilon > 0$ there exists a rational function r, the poles of which are contained in $\mathbb{C} \setminus K$, such that*

$$\|f|_K - r|_K\|_\infty \leq \varepsilon.$$

Lemma 3.1.7 *Let $\mathcal{A}$ be a Banach algebra with identity $e, x \in \mathcal{A}$, and U an open neighbourhood of $\sigma_{\mathcal{A}}(x)$. Suppose that $\gamma_1, \ldots, \gamma_n$ are closed, piecewise smooth curves in $U \setminus \sigma_{\mathcal{A}}(x)$ having the properties of Lemma 3.1.5. Then for any rational function f on U,*

$$f(x) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} f(z)(ze - x)^{-1} dz.$$

Lemma 3.1.8 *Let U be an open neighbourhood of $\sigma_{\mathcal{A}}(x)$ and let f be a holomorphic function on U. Moreover, let $\gamma_1, \ldots, \gamma_n$ and $\delta_1, \ldots, \delta_m$ be systems of closed, piecewise smooth curves in $U \setminus \sigma_{\mathcal{A}}(x)$ with the properties of Lemma 3.1.5. Then*

$$\sum_{j=1}^{n} \int_{\gamma_j} f(z)(ze - x)^{-1} dz = \sum_{k=1}^{m} \int_{\delta_k} f(z)(ze - x)^{-1} dz.$$

Definition 3.1.9 Let $\mathcal{A}$ be a unital Banach algebra. For $x \in \mathcal{A}$ and $f \in H(x)$ we define $f(x) \in \mathcal{A}$ as follows. Suppose that f is a holomorphic function on the open set U containing $\sigma_{\mathcal{A}}(x)$, and choose closed, piecewise smooth curves $\gamma_1, \ldots, \gamma_n$ in $U \setminus \sigma_{\mathcal{A}}(x)$ with the properties of Lemma 3.1.5. Then, define $f(x) \in \mathcal{A}$ by

$$f(x) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} f(z)(ze - x)^{-1} dz.$$

Hence, by Lemma 3.1.8, this definition does not depend on the choice of U and of the curves $\gamma_1, \ldots, \gamma_n$. Also, by Lemma 3.1.7, it extends the definition of $f(x)$ for rational functions f to functions holomorphic in U. The set of mappings

$$H(x) \to \mathcal{A}, \quad f \mapsto f(x), \quad x \in \mathcal{A},$$

is referred to as the *single-variable holomorphic functional calculus.*

The basic properties of the mapping $f \mapsto f(x)$ are listed in the next theorem.

Theorem 3.1.10 (**Single-variable holomorphic functional calculus**) *Let $\mathcal{A}$ be a unital commutative Banach algebra. For $x \in \mathcal{A}$ the following assertions hold.*

(*i*) *The mapping $f \mapsto f(x)$ is a homomorphism from $H(x)$ into $\mathcal{A}$.*

(*ii*) *If f is an entire function and $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

the series being absolutely convergent.

(*iii*) *Suppose that* f *and* $f_n, n \in \mathbb{N}$, *are holomorphic functions on some open set* U *containing* $\sigma_{\mathcal{A}}(x)$ *and that* f_n *converges uniformly to* f *on every compact subset of* U. *Then*

$$\|f_n(x) - f(x)\| \to 0.$$

(*iv*) *For* $f \in H(x)$ *we have* $\varphi(f(x)) = f(\varphi(x))$ *for all* $\varphi \in \Delta(\mathcal{A})$, *and thus*

$$\sigma_{\mathcal{A}}(f(x)) = f(\sigma_{\mathcal{A}}(x)).$$

Proof. (*i*) We show that $f \mapsto f(x)$ is a homomorphism.

- Clearly, $f \mapsto f(x)$ is linear.
- $f \mapsto f(x)$ is multiplicative: Let f and g be holomorphic on an open neighbourhood U of $\sigma_{\mathcal{A}}(x)$ and choose curves $\gamma_1, \ldots, \gamma_m : [0,1] \to U \setminus \sigma_{\mathcal{A}}(x)$ as in Lemma 3.1.5, and let $\Gamma_j = \gamma_j([0,1]), 1 \leq j \leq m$. Then there exist sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ of rational functions, each of which has its poles outside of $\sigma_{\mathcal{A}}(x) \cup (\bigcup_{j=1}^m \Gamma_j)$ such that $f_n \to f$ and $g_n \to g$ uniformly on $\bigcup_{j=1}^\infty \Gamma_j$. Hence $f_n g_n \to fg$ uniformly on $\bigcup_{j=1}^\infty \Gamma_j$. Now by Lemma 3.1.2 we know that for a rational function $r \in R(x)$, the mapping $R(x) \to \mathcal{A}, r \mapsto r(x)$ is a homomorphism. Thus for $x \in \mathcal{A}$ we get

$$\begin{aligned}\|f(x)g(x) - (fg)(x)\| \leq{}& \|f(x) - f_n(x)\| \cdot \|g(x)\| \\ &+ \|f_n(x)\| \cdot \|g(x) - g_n(x)\| \\ &+ \|(f_n g_n)(x) - (fg)(x)\|,\end{aligned}$$

  which converges to 0 as $n \to \infty$. Hence $(fg)(x) = f(x)g(x)$.

(*ii*) Let $R > \|x\|$ and $\gamma(t) = Re^{2\pi it}, t \in [0,1]$. Then γ has the properties of Lemma 3.1.5 and the series $\sum_{k=0}^\infty z^{-(k+1)}x^k$ converges uniformly on $\gamma([0,1])$. Therefore we have

$$\begin{aligned}f(x) &= \frac{1}{2\pi i}\int_\gamma f(z)(ze - x)^{-1}dz \\ &= \sum_{k=0}^\infty \frac{1}{2\pi i}\int_\gamma f(z) \cdot z^{-(k+1)} \cdot x^k dz \\ &= \sum_{k=0}^\infty \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{k+1}} x^k dz \\ &= \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} x^k \qquad \text{(Taylor)} \\ &= \sum_{k=0}^\infty a_k x^k.\end{aligned}$$

(*iii*) follows immediately from the estimate

$$\left\| \int_\gamma g(z)(ze - x)^{-1}dz \right\| \leq L(\gamma) \cdot \left\| g|_{\gamma([0,1])} \right\| \cdot \sup_{z \in \gamma([0,1])} \|(ze - x)^{-1}\|,$$

where $L(\gamma)$ denotes the length of the rectifiable curve γ.
(iv) Let $\varphi \in \Delta(\mathcal{A})$ and $z \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(x), x \in \mathcal{A}$. Then

$$\begin{aligned}1 = \varphi(e) &= \varphi((ze - x)(ze - x)^{-1}) = \varphi(ze - x)\varphi((ze - x)^{-1}) \\ &= (\varphi(z)\varphi(e) - \varphi(x))\varphi((ze - x)^{-1}) = (z - \varphi(x))\varphi((ze - x)^{-1}).\end{aligned}$$

This implies $\varphi((ze - x)^{-1}) = (z - \varphi(x))^{-1}$, and since $\Delta(\mathcal{A})$ is a subset of the dual space A' we have

$$\begin{aligned}\varphi(f(x)) &= \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\gamma_k} \varphi(f(z)(ze - x)^{-1}) dz \\ &= \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\gamma_k} f(z)(z - \varphi(x))^{-1} dz \\ &= f(\varphi(x)).\end{aligned}$$

By Theorem 2.3.6 (Gelfand's representation theorem) we eventually get

$$\sigma_{\mathcal{A}}(f(x)) = \widehat{f(x)}(\Delta(\mathcal{A})) = f(\widehat{x}(\Delta(\mathcal{A}))) = f(\sigma_{\mathcal{A}}(x)).$$

□

In order to prove Shilov's idempotent theorem (Theorem 3.2.1), we need to construct a functional calculus for holomorphic functions in n variables, $n \geq 2$, similarly to the single-variable calculus, but weaker and less explicit.

Definition 3.1.11 A complex-valued function f defined on an open subset U of $\mathbb{C}^n$ is called *holomorphic* in U if each point $w \in U$ has an open neighbourhood $V, w \in V \subseteq U$, such that the function f has a power series expansion

$$\sum_{k_1=0}^{\infty} \cdot \ \ldots \ \cdot \sum_{k_n=0}^{\infty} a_{k_1,\ldots,k_n}(z_1 - w_1)^{k_1} \cdot \ \ldots \ \cdot (z_n - w_n)^{k_n},$$

which converges for all $z \in V$.

Theorem 3.1.12 (**Several-variable holomorphic functional calculus**) *Let $\mathcal{A}$ be a unital commutative Banach algebra and let $x_1, \ldots, x_n \in \mathcal{A}$. Let f be a complex-valued function of n variables which is defined and holomorphic on some open set containing the joint spectrum $\sigma_{\mathcal{A}}(x_1, \ldots, x_n)$ of $x_1, \ldots, x_n$. Then there exists $x \in \mathcal{A}$ such that*

$$\widehat{x}(\varphi) = f(\widehat{x_1}(\varphi), \ldots, \widehat{x_n}(\varphi)) \tag{3.1}$$

for all $\varphi \in \Delta(\mathcal{A})$.

Remark 3.1.13

- (3.1) can be written as

$$\varphi(x) = f(\varphi(x_1), \ldots, \varphi(x_n)) \quad \text{for all } \varphi \in \Delta(\mathcal{A}).$$

- When $\mathcal{A}$ is semisimple, there exists exactly one such element $x \in \mathcal{A}$.

Indeed, suppose there exist two distinct elements $x, y \in \mathcal{A}$ such that

$$\varphi(x) = \varphi(y) \Leftrightarrow \varphi(x) - \varphi(y) = 0 \Leftrightarrow (x - y) \in \ker\varphi \quad \text{for all } \varphi \in \Delta(\mathcal{A}).$$

Now, by Theorem 2.1.18, the mapping $\varphi \mapsto \ker\varphi$ is a bijection between $\Delta(\mathcal{A})$ and $\mathrm{Max}(\mathcal{A})$, the set of all maximal ideals in $\mathcal{A}$. Hence

$$(x - y) \in \bigcap\{M : M \in \mathrm{Max}(\mathcal{A})\} = \bigcap\{\ker\varphi : \varphi \in \Delta(\mathcal{A})\} = \mathrm{rad}(\mathcal{A}).$$

Since $\mathrm{rad}(\mathcal{A}) = \{0\}$ for semisimple algebras, we conclude that $x = y$. Thus $x \in \mathcal{A}$ is unique.

To prove the multi-variable calculus (see Theorem 3.1.18), the following result, which is due to Oka, is employed. Let $\mathbb{D}^n$ denote the closed polydisc in $\mathbb{C}^n$, that is

$$\mathbb{D}^n = \{z \in \mathbb{C}^n : |z_j| \leq 1, j = 1, \dots, n\}.$$

Theorem 3.1.14 (**Oka's extension theorem**) *Let $n, m \in \mathbb{N}$. Let $p_1, \dots, p_m$ be polynomials in n complex variables and let $\pi : \mathbb{C}^n \to \mathbb{C}^{n+m}$ denote the mapping defined by*

$$\pi(z) = (z, p_1(z), \dots, p_m(z)), \quad z \in \mathbb{C}^n.$$

If f is holomorphic on an open neighbourhood of $\pi^{-1}(\mathbb{D}^{n+m})$, then there exists a holomorphic function F, defined on some open neighbourhood of $\mathbb{D}^{n+m}$ such that

$$F(\pi(z)) = f(z) \quad \text{for all } z \in \pi^{-1}(\mathbb{D}^{n+m}).$$

Proof. See [Werm], Chapter 7, pp.38. □

For a proof of the following lemmas and propositions we refer to [Kan], Section 3.1, pp. 149. As in the case $n = 1$, for an open subset U of $\mathbb{C}^n$, $H(U)$ denotes the algebra of holomorphic functions on U.

Proposition 3.1.15 *Let $n, m \in \mathbb{N}$ and $c_j > 0$ for $1 \leq j \leq n + m$, and let $p_1, \dots, p_m$ be polynomials in n variables. Let*

$$D = \{z \in \mathbb{C}^{n+m} : |z_j| \leq c_j, j = 1, \dots, n + m\},$$

and define $\pi : \mathbb{C}^n \to \mathbb{C}^{n+m}$ as above. Then, given f is holomorphic on an open neighbourhood of $\pi^{-1}(D)$, there exists a function F which is holomorphic on an open neighbourhood of D such that

$$F(\pi(z)) = f(z) \quad \text{for all } z \in \pi^{-1}(D).$$

In the following $\mathcal{A}$ is always a commutative Banach algebra with identity e and $\mathcal{A}^n$ denotes the Cartesian product of n copies of $\mathcal{A}$.

Lemma 3.1.16 *Let $x = (x_1, \dots, x_n) \in \mathcal{A}^n$ and let U be an open neighbourhood of $\sigma_{\mathcal{A}}(x)$ in $\mathbb{C}^n$. Then there exists a finitely generated closed subalgebra $\mathcal{B}$ of $\mathcal{A}$ containing $e, x_1, \dots, x_n$ such that $\sigma_{\mathcal{B}}(x) \subseteq U$.*

Lemma 3.1.17 *Let $\{x_1, \dots, x_n\}$ be a set of generators of $\mathcal{A}$ and $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n \setminus \sigma_{\mathcal{A}}(x_1, \dots, x_n)$. Then there exists a polynomial p such that*

$$|p(\lambda_1, \dots, \lambda_n)| > 1 + \|p(x_1, \dots, x_n)\|.$$

Proposition 3.1.18 *Let $x = (x_1, \dots, x_n) \in \mathcal{A}^n$ and let U be an open neighbourhood of $\sigma_{\mathcal{A}}(x)$ in $\mathbb{C}^n$. Then there exist $x_{n+1}, \dots, x_N \in \mathcal{A}$ with the following property. Given $f \in H(U)$, there exists a function F, holomorphic on some open neighbourhood of the polydisc $\{z \in \mathbb{C}^N : |z_j| \le 1 + \|x_j\|, 1 \le j \le N\}$, such that*

$$f(\varphi(x_1), \dots, \varphi(x_n)) = F(\varphi(x_1), \dots, \varphi(x_N))$$

for all $\varphi \in \Delta(\mathcal{A})$.

Now we are in a position to prove the several-variable holomorphic functional calculus.

Proof. (of Theorem 3.1.12)

Let $\mathcal{A}$ be a unital commutative Banach algebra and $x_1, \dots, x_n \in \mathcal{A}$, and let f be holomorphic on some open neighbourhood of $\sigma_{\mathcal{A}}(x_1, \dots, x_n)$. Now by Proposition 3.1.18 there exist $x_{n+1}, \dots, x_N \in \mathcal{A}$ and a function F defined and holomorphic on an open neighbourhood of the polydisc

$$D = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : |z_j| \le 1 + \|x_j\|, 1 \le j \le N\}$$

such that

$$f(\varphi(x_1), \dots, \varphi(x_n)) = F(\varphi(x_1), \dots, \varphi(x_N)) \quad \text{for all } \varphi \in \Delta(\mathcal{A}).$$

The function F admits a power series expansion

$$F(z_1, \dots, z_N) = \sum_{k \in (\mathbb{N}_0)^N} \lambda_k \cdot z_1^{k_1} \cdot \ \dots \ \cdot z_N^{k_N},$$

with $k = (k_1, \dots, k_N)$, which is convergent on a neighbourhood of D and therefore absolutely convergent on D. Hence the series

$$\sum_{k \in (\mathbb{N}_0)^N} |\lambda_k| \cdot \|z_1\|^{k_1} \cdot \ \dots \ \cdot \|z_N\|^{k_N}$$

converges. As a consequence the series $\sum_{k \in (\mathbb{N}_0)^N} \lambda_k \cdot z_1^{k_1} \cdot \ \dots \ \cdot z_N^{k_N}$ converges in norm to an element y of $\mathcal{A}$. Then, for all $\varphi \in \Delta(A)$, we obtain

$$\begin{aligned} \widehat{y}(\varphi) = \varphi(y) &= \sum_{k \in (\mathbb{N}_0)^N} \lambda_k \cdot \varphi(x_1)^{k_1} \cdot \ \dots \ \cdot \varphi(x_N)^{k_N} \\ &= F(\varphi(x_1), \dots, \varphi(x_N)) \\ &= F(\widehat{x_1}(\varphi), \dots, \widehat{x_N}(\varphi)) \\ &= f(\widehat{x_1}(\varphi), \dots, \widehat{x_N}(\varphi)). \end{aligned}$$

□

The multi-variable holomorphic functional calculus was originally proved by Shilov [Shi] under the hypothesis that $\mathcal{A}$ be finitely generated, and was extended to the general case by Arens and Calderon. There are many applications of the (single-variable) calculus, such as concerning

- the structure of the group $\mathcal{G}(\mathcal{A})$ of invertible elements of $\mathcal{A}$,
- approximation theory (Runge's theorem),
- the question of whether compactness of $\Delta(\mathcal{A})$ forces $\mathcal{A}$ to be unital and
- the existence of idempotents.

We won't discuss all of them since only the latter two are of particular interest. The next theorem, which we will need for the succeeding corollary, characterizes the behaviour of the functional calculus with respect to compositions.

Theorem 3.1.19 *Let $\mathcal{A}$ be a commutative Banach algebra and suppose that $\Delta(\mathcal{A})$ is compact. Let $x \in \mathcal{A}$ be such that $\widehat{x}(\varphi) \neq 0$ for all $\varphi \in \Delta(\mathcal{A})$, and let f be a holomorphic function on some open neighbourhood of $\widehat{x}(\Delta(\mathcal{A}))$. Then there exists $y \in \mathcal{A}$ so that $\widehat{y} = f \circ \widehat{x}$.*

Proof. By hypothesis, $\widehat{x}(\Delta(\mathcal{A}))$ is a compact subset of $\mathbb{C} \setminus \{0\}$. Now we choose disjoint open sets U and V in $\mathbb{C}$ such that $\widehat{x}(\Delta(\mathcal{A})) \subseteq U, 0 \in V$, and f is a holomorphic function on U. Next we define $g : U \cup V \to \mathbb{C}$ by $g|_U = f$ and $g|_V = 0$ and let $y = g(x) \in \mathcal{A}$. By the single-variable holomorphic functional calculus (Theorem 3.1.10 (*iv*)) we get

$$\widehat{y}(\varphi) = \varphi(y) = \varphi(g(x)) = g(\varphi(x)) = f(\varphi(x)) = f(\widehat{x}(\varphi)) = f \circ \widehat{x}(\varphi)$$

for all $\varphi(x) \in \Delta(\mathcal{A})$ since $\widehat{x}(\Delta(\mathcal{A})) \subseteq U$. □

Corollary 3.1.20 *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. Suppose that $\Delta(\mathcal{A})$ is compact and that there exists $x \in \mathcal{A}$ such that $\widehat{x}(\varphi) \neq 0$ for all $\varphi \in \Delta(\mathcal{A})$. Then $\mathcal{A}$ has an identity.*

Proof. Let f be the function $f(z) = 1/z$ on $\mathbb{C} \setminus \{0\}$. Since $\widehat{x}(\Delta(\mathcal{A})) \subseteq \mathbb{C} \setminus \{0\}$, there exists $y \in \mathcal{A}$ by Theorem 3.1.19 such that

$$\varphi(y) = f(\varphi(x)) = \frac{1}{\varphi(x)}$$

for all $\varphi \in \Delta(\mathcal{A})$. Then the element $u = xy \in \mathcal{A}$ satisfies

$$\varphi(ua) = \varphi(u)\varphi(a) = \varphi(xy)\varphi(a) = \varphi(x)\varphi(y)\varphi(a) = \varphi(x)(1/\varphi(x))\varphi(a) = \varphi(a)$$

for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. Moreover,

$$\varphi(ua) - \varphi(a) = 0 \Leftrightarrow (ua - a) \in \ker \varphi,$$

and since $\mathcal{A}$ is semisimple (i.e. $\text{rad}(\mathcal{A}) = \{0\}$), it follows immediately that $ua = a$ for all $a \in \mathcal{A}$. Hence u is an identity for $\mathcal{A}$. □

In general it is true that a semisimple commutative Banach algebra with compact structure space is unital. The proof, of course, will require the Shilov idempotent theorem which in turn is based on the several-variable holomorphic functional calculus.

We conclude this section by presenting an application of the holomorphic functional calculus which concerns the existence of idempotents.

Definition 3.1.21 An element x of an algebra $\mathcal{A}$ is called an *idempotent* if it satisfies $x^2 = x$. Note that the identity element e is always an idempotent since $e \cdot e = e$.

Theorem 3.1.22 *Let $\mathcal{A}$ be a commutative Banach algebra $\mathcal{A}$ with identity e. Let $x \in \mathcal{A}$ and suppose that $\sigma(x) = \bigcup_{j=1}^m C_j$, where the sets $C_j, 1 \leq j \leq m$, are nonempty, pairwise disjoint and closed in $\sigma(x)$. Then there exist idempotents $e_1, \dots, e_m$ in $\mathcal{A}$ such that*

$$e = \sum_{j=1}^m e_j, \;\; e_j \neq 0 \quad \text{and } e_j e_k = 0 \text{ for } 1 \leq j, k \leq m, j \neq k.$$

Moreover, each e_j is contained in the closed linear span of all elements of the form $(\lambda e - x)^{-1}, \lambda \in \rho(x)$.

Proof. Since the spectrum $\sigma(x)$ is compact, so are $C_1, \dots, C_m$. Thus there exist pairwise disjoint open subsets $U_1, \dots, U_m$ of $\mathbb{C}$ such that $C_j \subseteq U_j$.

Let $U = \bigcup_{j=1}^m U_j$ and define a function f_j on U for each j by

$$f_j(z) = \begin{cases} 1 & \text{if } z \in U_j, \\ 0 & \text{if } z \in U \setminus U_j. \end{cases}$$

Then the holomorphic functions f_j satisfy $f_j^2 = f_j$ and $\sum_{j=1}^m f_j(z) = 1$ for all $z \in U$. Now, let $e_j = f_j(x) \in \mathcal{A}, 1 \leq j \leq m$. It follows that $e_j^2 = f_j^2 = f_j = e_j, e_j e_k = f_j f_k = 0$ for $j \neq k$ and

$$e = \mathbb{1}_U(x) = \sum_{j=1}^m f_j(x) = \sum_{j=1}^m e_j.$$

Applying the single-variable holomorphic functional calculus (Theorem 3.1.10) we see that $1 \in f_j(\sigma(x)) = \sigma(f_j(x)) = \sigma(e_j)$. Thus $e_j \neq 0$ for each j.

By the definition of $f_j(x)$, this element is a norm limit of finite linear combinations of elements of the form $(\lambda e - x)^{-1}$ with $\lambda \in \rho(x) = \mathbb{C} \setminus \sigma(x)$. □

Corollary 3.1.23 *Let $\mathcal{A}$ and e be as in Theorem 3.1.22. If, for some $x \in \mathcal{A}$, the spectrum $\sigma(x)$ is not connected, then there exists an idempotent $\tilde{e}$ in $\mathcal{A}$ such that $\tilde{e} \neq 0$ and $\tilde{e} \neq e$.*

3.2. Shilov's idempotent theorem

We have finally gathered all the necessary tools for proving one of the most beautiful results in general banach algebra theory - the celebrated Shilov idempotent theorem. In 1954, Shilov devised the functional calculus precisely to prove the idempotent theorem. It remains just possible that after more than 60 years a quite different proof might be found.

Theorem 3.2.1 (**Shilov's idempotent theorem**) *Let $\mathcal{A}$ be a commutative Banach algebra and let C be a compact open subset of $\Delta(\mathcal{A})$. Then there exists an idempotent a in $\mathcal{A}$ such that $\widehat{a}$ equals the characteristic function of C, i.e. $\widehat{a} = \mathbb{1}_C$.*

Before we present the actual proof, we first give an overview of our approach.
Let $\mathcal{A}$ be a unital commutative Banach algebra and suppose that the Gelfand space $\Delta(\mathcal{A})$ is a disjoint union of two non-empty compact subsets U_1 and U_2 of $\Delta(\mathcal{A})$, i.e.

$$\Delta(\mathcal{A}) = U_1 \cup U_2.$$

Then there exists an element a in $\mathcal{A}$ such that

$$\varphi(a) = 0 \text{ for all } \varphi \in U_1 \text{ and } \varphi(a) = 1 \text{ for all } \varphi \in U_2.$$

Next, we want to find elements $a_1, \ldots, a_n \in \mathcal{A}$ whose joint spectrum is likewise a disjoint union of two compact subsets C_1 and C_2, that is

$$\sigma_{\mathcal{A}}(a_1, \ldots, a_n) = C_1 \cup C_2.$$

We show that C_1 and C_2 are indeed disjoint, so we can find disjoint open neighbourhoods W_1 and W_2 of C_1 and C_2 in $\mathbb{C}^n$, respectively.

Furthermore, let $f : W_1 \cup W_2 \to \mathbb{C}$ be a holomorphic function on the neighbourhood $W_1 \cup W_2$ of the joint spectrum $\sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ such that

$$f|_{W_1} = 0 \quad \text{and} \quad f|_{W_2} = 1.$$

Then, by the multi-variable holomorphic functional calculus

$$\begin{aligned} \varphi(a) &= f(\varphi(a_1), \ldots, \varphi(a_n)) \\ \iff \widehat{a}(\varphi) &= f(\widehat{a_1}(\varphi), \ldots, \widehat{a_n}(\varphi)). \end{aligned}$$

for all $\varphi \in \Delta(\mathcal{A})$. It turns out that if $\mathcal{A}$ is semisimple, the element a is already an idempotent. In the next step we drop the semisimplicity of $\mathcal{A}$ and show that there exists a unique idempotent element in $\mathcal{A}$. Finally, by employing the unitization technique, we show that the idempotent theorem holds true even for nonunital Banach algebras.

We start with the actual proof by presenting a very helpful lemma.

Lemma 3.2.2 *Let $\mathcal{A}$ be a commutative Banach algebra and let $\varphi_1, \varphi_2 \in \Delta(\mathcal{A})$ with $\varphi_1 \neq \varphi_2$. Then there exists $x \in \mathcal{A}$ such that*

$$\varphi_1(x) = 1 \quad \text{and} \quad \varphi_2(x) = 0.$$

Proof. The proof is straightforward and is based on the standard argument used in the proof of the classical Stone-Weierstrass theorem.

By Theorem 2.3.10 (*ii*), the set $\widehat{A} = \{\widehat{x} : x \in \mathcal{A}\}$ of Gelfand transforms strongly separates the points of $\Delta(\mathcal{A})$. Hence there exist elements $a_1, a_2 \in \mathcal{A}$ and $b \in \mathcal{A}$ such that

$$\varphi_1(a_1) \neq 0, \quad \varphi_2(a_2) \neq 0 \quad \text{and} \quad \varphi_1(b) \neq \varphi_2(b).$$

Now let

$$c_j = \frac{a_j}{\varphi_j(a_j)}$$

for $j = 1, 2$ and

$$c = c_1 + c_2 - c_1 c_2 \in \mathcal{A}.$$

Since φ_j is linear und multiplicative we get

$$\varphi_j(c) = \varphi_j(c_1 + c_2 - c_1 c_2) = \varphi_j(c_1) + \varphi_j(c_2) - \varphi_j(c_1)\varphi_j(c_2) = 1 + 1 - 1 \cdot 1 = 1$$

for $j = 1, 2$, so $\varphi_1(c) = \varphi_2(c) = 1$. Let

$$x = \frac{b - \varphi_2(b)c}{\varphi_1(b) - \varphi_2(b)} \in \mathcal{A}.$$

Then, by applying φ_1 and φ_2 to x, we have

$$\varphi_1(x) = \frac{\varphi_1(b) - \varphi_2(b)\varphi_1(c)}{\varphi_1(b) - \varphi_2(b)} = 1$$

and

$$\varphi_2(x) = \frac{\varphi_2(b) - \varphi_2(b)\varphi_2(c)}{\varphi_1(b) - \varphi_2(b)} = 0.$$

Thus x has the required properties. □

The several-variable holomorphic functional calculus (Theorem 3.1.12) will be employed in the next result.

Proposition 3.2.3 *Let $\mathcal{A}$ be a unital commutative Banach algebra and let U_1 and U_2 be disjoint open subsets of $\Delta(\mathcal{A})$ such that $\Delta(\mathcal{A}) = U_1 \cup U_2$. Then there exists $x \in \mathcal{A}$ such that*

$$\widehat{x}|_{U_1} = 0 \quad \text{and} \quad \widehat{x}|_{U_2} = 1.$$

Proof. Let $\varphi \in U_1$ and $\psi \in U_2$. By Lemma 3.2.2 there exists $a_{\varphi,\psi} \in \mathcal{A}$ such that

$$\varphi(a_{\varphi,\psi}) = 0 \quad \text{and} \quad \psi(a_{\varphi,\psi}) = 1.$$

Now let

$$V_{\varphi,\psi} = \left\{\alpha \in U_1 : |\alpha(a_{\varphi,\psi})| < \frac{1}{2}\right\} \quad \text{and} \quad W_{\varphi,\psi} = \left\{\beta \in U_2 : |\beta(a_{\varphi,\psi})| > \frac{1}{2}\right\}.$$

Clearly, these sets are open neighbourhoods of φ and ψ, respectively, where $V_{\varphi,\psi} \subseteq U_1$ and $W_{\varphi,\psi} \subseteq U_2$. Now, fix $\psi \in U_2$. By hypothesis, U_1 is an open subset of $\Delta(\mathcal{A})$. Since the structure space $\Delta(\mathcal{A})$ of a unital commutative Banach algebra is compact (Theorem 2.3.4), U_1 is also compact. Hence there exists a finite subset F_ψ of U_1 such that

$$U_1 = \bigcup_{\varphi \in F_\psi} V_{\varphi,\psi}.$$

So, if $\alpha \in U_1$, then $|\alpha(a_{\varphi,\psi})| < \frac{1}{2}$ for at least one $\varphi \in F_\psi$. Now, define an open neighbourhood W_ψ of ψ in U_2 by

$$W_\psi = \bigcap_{\varphi \in F_\psi} W_{\varphi,\psi}.$$

By the same argument made above, U_2 is compact. Thus there exist $\psi_1, \ldots, \psi_n \in U_2$ such that

$$U_2 = \bigcup_{j=1}^{n} W_{\psi_j}.$$

Now, consider the finite subset

$$E = \{a_{\varphi,\psi_j} : 1 \leq j \leq n, \varphi \in F_{\psi_j}\}$$

of $\mathcal{A}$ and enumerate E, for example $E = \{x_1, \ldots, x_k\}$. Let

$$C_j = \{(\varphi(x_1), \ldots, \varphi(x_k)) : \varphi \in U_j\}$$

for $j = 1, 2$. Then C_1 and C_2 are compact since U_1 and U_2 are compact. Because $U_1 \cup U_2 = \Delta(\mathcal{A})$ by premise, we have

$$C_1 \cup C_2 = \{(\varphi(x_1), \ldots, \varphi(x_k)) : \varphi \in U_1 \cup U_2\} = \sigma_{\mathcal{A}}(x_1, \ldots, x_k).$$

Now, assume C_1 and C_2 are not disjoint, i.e. $C_1 \cap C_2 \neq \emptyset$. Then there exist $\alpha \in U_1$ and $\beta \in U_2$ such that

$$\alpha(a_{\varphi,\psi_j}) = \beta(a_{\varphi,\psi_j}) \tag{3.2}$$

for each $1 \leq j \leq n$ and all $\varphi \in F_{\psi_j}$. Now, $\beta \in W_{\psi_j}$ for some j and then $\alpha \in V_{\varphi,\psi_j}$ for some $\varphi \in F_{\psi_j}$. By the definition of W_{ψ_j}, β is in W_{φ,ψ_j} and hence

$$|\alpha(a_{\varphi,\psi_j})| < \frac{1}{2} \quad \text{and} \quad |\beta(a_{\varphi,\psi_j})| > \frac{1}{2},$$

which is a contradiction to (3.2). Therefore C_1 and C_2 are disjoint compact subsets of $\mathbb{C}^k$ and consequently we can find disjoint open neighbourhoods W_1 and W_2 of C_1 and C_2 in $\mathbb{C}^k$, respectively.

Next, define a function $f : W_1 \cup W_2 \to \mathbb{C}$ by

$$f|_{W_1} = 0 \quad \text{and} \quad f|_{W_2} = 1.$$

Then f is holomorphic on the neighbourhood $W_1 \cup W_2$ of $\sigma_{\mathcal{A}}(x_1, \ldots, x_k)$. By the several-variable holomorphic functional calculus (Theorem 3.1.12) there exists $x \in \mathcal{A}$ such that

$$\widehat{x}(\varphi) = f(\varphi(x_1), \ldots, \varphi(x_k))$$

for all $\varphi \in \Delta(\mathcal{A})$. Hence $\widehat{x}(\varphi) = 0$ for all $\varphi \in U_1$ and $\widehat{x}(\varphi) = 1$ for all $\varphi \in U_2$. $\square$

When $\mathcal{A}$ is semisimple, the element x of $\mathcal{A}$ in the preceding proposition is an idempotent because $\widehat{x^2} = \widehat{x}$. Therefore the idempotent theorem has been verified so far for semisimple unital commutative Banach algebras.

Next, we drop the hypothesis of $\mathcal{A}$ being semisimple and show that there exists a unique idempotent element in $\mathcal{A}$.

Lemma 3.2.4 *Let $\mathcal{A}$ be a commutative Banach algebra with identity e and let $b \in \mathcal{A}$ be such that $\widehat{b^2} = \widehat{b}$. Then there exists $a \in \mathcal{A}$ such that $\widehat{a} = \widehat{b}$ and $a^2 = a$.*

Proof. Recall that for any $x \in \mathcal{A}$ the geometric series $\sum_{n=0}^{\infty} x^n$ converges in $\mathcal{A}$ whenever $r(x) < 1$. Let $x = 4(b^2 - b)$. Since $\widehat{b^2} = \widehat{b}$ by hypothesis, it follows that

$$\widehat{x} = 4(\widehat{b^2 - b}) = 4(\widehat{b^2} - \widehat{b}) = 0.$$

Thus $x \in \mathrm{rad}(\mathcal{A})$ and hence $r(x) = 0$. Because $\left|\binom{-1/2}{n}\right| \leq 1$ for all $n \in \mathbb{N}_0$, the series

$$\sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$$

converges in $\mathcal{A}$. Now recall the binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

and the well-known equation

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Therefore it is only logical that we denote the element $\sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$ by $(e+x)^{-1/2}$. We claim that

$$(e+x)^{-1/2}(e+x)^{-1/2}(e+x) = e.$$

Indeed,

$$\begin{aligned}
\left((e+x)^{-1/2}\right)^2 (e+x) &= (e+x) \cdot \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n \cdot \sum_{m=0}^{\infty} \binom{-1/2}{m} x^m \\
&= (e+x) \cdot \sum_{m=0}^{\infty} \Bigg(\underbrace{\sum_{k+l=m} \binom{-1/2}{k} \binom{-1/2}{l}}_{=(-1)^m \text{ for } m \in \mathbb{N}_0} \Bigg) x^m \\
&= e + \sum_{m=1}^{\infty} (-1)^m x^m + \sum_{m=0}^{\infty} (-1)^m x^{m+1} \qquad (x^0 = e) \\
&= e + \sum_{m=0}^{\infty} (-1)^{m+1} x^{m+1} + \sum_{m=0}^{\infty} (-1)^m x^{m+1}
\end{aligned}$$

$$= e + \sum_{m=0}^{\infty} \Big((-1)^{m+1} + (-1)^m \Big) x^{m+1}$$
$$= e.$$

Hence $\big((e+x)^{-1/2}\big)^2 = (e+x)^{-1}$. Now, set

$$a = \left(b - \frac{1}{2}e\right)(e+x)^{-1/2} + \frac{1}{2}e.$$

Then we get

$$\begin{aligned} a(a-e) &= \left(\left(b - \frac{1}{2}e\right)(e+x)^{-1/2} + \frac{1}{2}e\right) \cdot \left(\left(b - \frac{1}{2}e\right)(e+x)^{-1/2} - \frac{1}{2}e\right) \\ &= \left(b - \frac{1}{2}e\right)^2 \cdot \big((e+x)^{-1/2}\big)^2 - \frac{1}{4}e \\ &= \left(b^2 - b + \frac{1}{4}e\right) \cdot (e+x)^{-1} - \frac{1}{4}e \\ &= \left(\frac{1}{4}x + \frac{1}{4}e\right) \cdot (e+x)^{-1} - \frac{1}{4} \\ &= \frac{1}{4} \cdot \frac{e+x}{e+x} - \frac{1}{4} \\ &= 0. \end{aligned}$$

Thus $a(a-e) = 0 \Leftrightarrow a^2 = a$, and hence a is an idempotent. It remains to verify that $\widehat{a} = \widehat{b}$. For that purpose let

$$y = \left(b - \frac{1}{2}e\right) \cdot \sum_{n=1}^{\infty} \binom{-1/2}{n} x^n.$$

Then we obtain

$$\begin{aligned} a &= \left(b - \frac{1}{2}e\right) \cdot (e+x)^{-1/2} + \frac{1}{2}e \\ &= \left(b - \frac{1}{2}e\right) \cdot \left(\sum_{n=0}^{\infty} \binom{-1/2}{n} x^n\right) + \frac{1}{2}e \\ &= \left(b - \frac{1}{2}e\right) \cdot \left(e + \sum_{n=1}^{\infty} \binom{-1/2}{n} x^n\right) + \frac{1}{2}e \\ &= be - \frac{1}{2}ee + y + \frac{1}{2}e \\ &= b + y. \end{aligned}$$

Now, since $x \in \mathrm{rad}(\mathcal{A})$ and $\mathrm{rad}(\mathcal{A})$ is a closed ideal, it follows that $y \in \mathrm{rad}(\mathcal{A})$. Hence $\widehat{y} = 0$ and therefore $\widehat{a} = \widehat{b+y} = \widehat{b} + \widehat{y} = \widehat{b}$. □

Lemma 3.2.4 completes the proof of Shilov's idempotent theorem when $\mathcal{A}$ is unital. Now assume that $\mathcal{A}$ has no identity and consider the unitization $\mathcal{A}_e$ of $\mathcal{A}$. As usual, by embedding $\Delta(\mathcal{A})$ into $\Delta(\mathcal{A}_e) = \Delta(\mathcal{A}) \cup \{\varphi_\infty\}$ (recall that φ_∞ denotes the homomorphism from A_e into $\mathbb{C}$ with kernel $\mathcal{A}$, that is $\varphi_\infty(x + \lambda e) = \varphi_\infty(x) + \lambda\varphi_\infty(e) = \lambda$ for all $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$), C is still an open and closed set. Hence there exists an idempotent u in $\mathcal{A}_e$ such that

$$\widehat{u}|_C = 1 \quad \text{and} \quad \widehat{u}|_{\Delta(\mathcal{A})\setminus C} = 0.$$

Since $\widehat{u}(\varphi_\infty) = \varphi_\infty(u) = 0$, it follows immediately that u is in $\mathcal{A}$.

Remark 3.2.5 The proof of Shilov's idempotent theorem amounted to solving the equation $x^2 - x = 0$ in a commutative Banach algebra $\mathcal{A}$, with the solution having a specified Gelfand transform. There is a considerable extension of this idea - an interesting approach was chosen in [Pal], pp.405, utilizing an *implicit function theorem* which in turn is also based on the heavy machinery of several-variable complex analysis.

3.3. Applications of the idempotent theorem

In this section we present a number of applications of Shilov's idempotent theorem. The first one has been announced several times throughout this thesis. It is the ultimate solution to the question of whether compactness of $\Delta(\mathcal{A})$ forces $\mathcal{A}$ to be unital.

Theorem 3.3.1 *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. If $\Delta(\mathcal{A})$ is compact, then $\mathcal{A}$ has an identity.*

Proof. By premise, $\Delta(\mathcal{A})$ is compact. Thus by the idempotent theorem (Theorem 3.2.1) there exists $e \in \mathcal{A}$ such that $\widehat{e} = 1$ on $\Delta(\mathcal{A})$. Hence we get

$$(\widehat{xe - x})(\varphi) = \widehat{x}(\varphi)\widehat{e}(\varphi) - \widehat{x}(\varphi) = 0$$

for all $x \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. Since $\mathcal{A}$ is semisimple we have $\operatorname{rad}(\mathcal{A}) = \{0\}$ and therefore $xe - x = 0 \Leftrightarrow xe = x$ for all $x \in \mathcal{A}$, whence e is an identity for $\mathcal{A}$. □

We have already mentioned the fact that there is a considerably simpler proof of Theorem 3.3.1 given that $\mathcal{A}$ is a regular Banach algebra. This result will be shown at the end of Chapter 4 in Corollary 4.2.16.

Definition 3.3.2 A topological space $\mathcal{X}$ is called *totally disconnected* if the connected components in $\mathcal{X}$ are the one-point sets.

Remark 3.3.3 If $\mathcal{X}$ is a totally disconnected compact Hausdorff space, then the compact open subsets of $\mathcal{X}$ form a base for its topology (see [Rud], Appendix A, p.395).

Corollary 3.3.4 *Let $\mathcal{A}$ be a commutative Banach algebra and suppose that $\Delta(\mathcal{A})$ is totally disconnected. Then $\widehat{\mathcal{A}} = \{\widehat{a} : a \in \mathcal{A}\}$ is dense in $C_0(\Delta(\mathcal{A}))$.*

Proof. Let $f \in C_0(\Delta(\mathcal{A}))$ and $\varepsilon > 0$. Since f vanishes and infinity and every point of $\Delta(\mathcal{A})$ has a neighbourhood basis of compact open sets, there exists a compact open subset K of $\Delta(\mathcal{A})$ such that

$$|f(\varphi)| < \varepsilon \quad \text{for all } \varphi \in \Delta(\mathcal{A}) \setminus K.$$

Next, we can write K as a disjoint union of compact open sets $E_1, \ldots, E_k$, that is $K = \bigcup_{j=1}^{k} E_j$, such that

$$|f(\varphi) - f(\psi)| < \varepsilon \quad \text{for all } \varphi, \psi \in E_j, 1 \leq j \leq k.$$

Hence there exist $c_1, \ldots, c_k \in \mathbb{C}$ with the property that the function

$$g = \sum_{j=1}^{k} c_j \mathbb{1}_{E_j}$$

satisfies the condition

$$|f(\varphi) - g(\varphi)| < \varepsilon \quad \text{for all } \varphi \in K.$$

Now by Shilov's idempotent theorem (Theorem 3.2.1) there exist $a_j \in \mathcal{A}, 1 \leq j \leq k$, such that

$$\widehat{a}_j = \mathbb{1}_{E_j}.$$

So for $a = \sum_{j=1}^{k} c_j a_j$ we get

$$\widehat{a} = \sum_{j=1}^{k} c_j \widehat{a}_j = \sum_{j=1}^{k} c_j \mathbb{1}_{E_j} = g$$

and hence

$$\|\widehat{a} - f\|_\infty < \varepsilon.$$

Thus $\widehat{\mathcal{A}}$ is dense in $C_0(\Delta(\mathcal{A}))$. □

We now investigate the relation between (not necessarily finite) coverings of $\Delta(\mathcal{A})$ through disjoint open subsets.

Definition 3.3.5 Let $\mathcal{A}$ be a commutative Banach algebra. For a subset M of $\mathcal{A}$, the *hull* of M is defined as

$$\text{hul}(M) = \{\varphi \in \Delta(\mathcal{A}) : \varphi(M) = \{0\}\}.$$

Lemma 3.3.6 *Let $\mathcal{A}$ be a commutative Banach algebra, I a closed ideal of $\mathcal{A}$ and $q : \mathcal{A} \to \mathcal{A}/I$ the quotient homomorphism.*

(i) The map $\varphi \mapsto \varphi \circ q$ is a homeomorphism from $\Delta(\mathcal{A}/I)$ onto $\text{hul}(I)$.

(ii) The map $\varphi \mapsto \varphi|_I$ is a homeomorphism from $\Delta(\mathcal{A}) \setminus \text{hul}(I)$ onto $\Delta(I)$.

Proof. See [Kan], Section 2.2, p.58. □

So by the preceding lemma, for each $y \in \mathcal{A}$ there is a unique continuous function f_y on $\Delta(I)$ such that $\widehat{yx}(\varphi) = f_y(\varphi)\widehat{x}(\varphi)$ for all $\varphi \in \Delta(I)$ and $x \in \mathcal{A}$.

In what follows, Λ always denotes the index set for $\lambda \in \Lambda$.

Theorem 3.3.7 *Let $\mathcal{A}$ be a nonunital commutative Banach algebra and let $\Delta(\mathcal{A}) = \bigcup_{\lambda\in\Lambda} F_\lambda$ be a decomposition of $\Delta(\mathcal{A})$ into open and compact subsets $F_\lambda, \lambda \in \Lambda$. Then there exists a family of closed ideals $I_\lambda, \lambda \in \Lambda$, with the following properties.*

(*i*) $\Delta(I_\lambda) = F_\lambda$ *for each* λ.

(*ii*) $I_\lambda \cap \left(\sum_{\mu\in\Lambda,\mu\neq\lambda} I_\mu \right) \subseteq \mathrm{rad}(\mathcal{A})$ *for each* λ.

(*iii*) $\sum_{\lambda\in\Lambda} I_\lambda$ *is dense in* $\mathcal{A}$ *provided that every proper closed ideal of* $\mathcal{A}$ *is contained in a maximal ideal.*

Proof.

We start by applying Shilov's idempotent theorem (Theorem 3.2.1). For each $\lambda \in \Lambda$ there exists an idempotent u_λ in $\mathcal{A}$ such that

$$\widehat{u}_\lambda|_{F_\lambda} = 1 \quad \text{and} \quad \widehat{u}_\lambda|_{\Delta(\mathcal{A})\setminus F_\lambda} = 0.$$

Let $I_\lambda = Au_\lambda$. Then I_λ is an ideal. We claim that I_λ is closed. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in I_λ and $x_n \to x$ in $\mathcal{A}$ as $n \to \infty$. Then $x_n = x_n u_\lambda \to xu_\lambda$ and therefore $x = xu_\lambda \in I_\lambda$.

(*i*) By Lemma 3.3.6 we have

$$\Delta(I_\lambda) = \Delta(\mathcal{A}) \setminus \mathrm{hul}(I_\lambda) = \{\varphi \in \Delta(\mathcal{A}) : \varphi(x) \neq 0 \text{ for some } x \in I_\lambda\}.$$

If $\varphi(x) \neq 0$ and $x = yu_\lambda, y \in A$, then it follows directly from $\varphi(x) = \varphi(y)\varphi(u_\lambda) \neq 0$ that $\varphi(u_\lambda) \neq 0$. Hence $\Delta(I_\lambda) \subseteq F_\lambda$. Conversely, if $\varphi \in F_\lambda$, then $\varphi(u_\lambda) \neq 0$. Thus $\varphi \notin \mathrm{hul}(I_\lambda)$ and so $F_\lambda \subseteq \Delta(I_\lambda)$.

(*ii*) Fix λ and let $J = \sum_{\substack{\mu\in\Lambda \\ \mu\neq\lambda}} I_\mu$ and $L = I_\lambda \cap J$. We have to verify that $L \subseteq \mathrm{rad}(\mathcal{A})$. That is equivalent to $\mathrm{hul}(L) = \Delta(\mathcal{A})$. We show this identity by contradiction. Suppose that there exists $\varphi \in \Delta(\mathcal{A}) \setminus \mathrm{hul}(L)$ and choose $x \in L$ with $\varphi(x) \neq 0$. Since $x \in I_\lambda, x = xu_\lambda$ and therefore $\varphi(u_\lambda) \neq 0$. Hence $\varphi \in F_\lambda$ by (*i*). We want to show that also $\varphi \in F_\mu$ for some $\mu \in \Lambda, \mu \neq \lambda$. For $x \in J$, and since J is a sum of ideals I_μ, we can write x as a sum

$$x = \sum_{j=1}^{n} c_j x_j$$

for $x_j \in I_{\mu_j}, \mu_1, \ldots, \mu_n \in \Lambda, \mu_j \neq \lambda$ for all j and $c_1, \ldots, c_n \in \mathbb{C}$. Now, since $x_j = x_j u_{\lambda_j}$, we have

$$0 \neq \varphi(x) = \varphi\left(\sum_{j=1}^{n} c_j x_j \right) = \sum_{j=1}^{n} \varphi(c_j)\varphi(x_j) = \sum_{j=1}^{n} c_j \varphi(x_j u_{\lambda_j}) = \sum_{j=1}^{n} c_j \varphi(x_j)\varphi(u_{\lambda_j}),$$

and hence $\varphi(u_{\lambda_j}) \neq 0$ for some j, so that $\varphi \in F_{\lambda_j}$ by (*i*). This contradicts the fact that $F_\lambda \cap F_\mu = \emptyset$ for $\lambda \neq \mu$.

(*iii*) By (*i*) we know that $\Delta(I_\lambda) = F_\lambda$ and so by premise we have $\Delta(\mathcal{A}) = \bigcup_{\lambda\in\Lambda} F_\lambda = \bigcup_{\lambda\in\Lambda} \Delta(I_\lambda)$. Thus no element of $\Delta(\mathcal{A})$ annihilates $\sum_{\lambda\in\Lambda} I_\lambda$ and so it follows that this ideal is dense in $\mathcal{A}$ by hypothesis. □

We continue with a converse of Theorem 3.3.7.

Theorem 3.3.8 *Let $\mathcal{A}$ be a nonunital commutative Banach algebra and let $\{I_\lambda : \lambda \in \Lambda\}$ be a family of unital closed ideals of $\mathcal{A}$ satisfying property (ii) of Theorem 3.3.7 and such that the ideal $\sum_{\lambda\in\Lambda} I_\lambda$ is dense in $\mathcal{A}$. Then $\Delta(\mathcal{A}) = \bigcup_{\lambda\in\Lambda} \Delta(I_\lambda)$ and the sets $\Delta(I_\lambda), \lambda \in \Lambda$, are open and disjoint.*

Proof. Clearly, by Lemma 3.3.6, each $\Delta(I_\lambda) = \Delta(\mathcal{A}) \setminus \text{hul}(I_\lambda)$ is open in $\Delta(\mathcal{A})$. Now let $\lambda, \mu \in \Lambda$, so that $\lambda \neq \mu$, and suppose that there exists $\varphi \in \Delta(I_\lambda) \cap \Delta(I_\mu)$. Choose $x \in I_\lambda$ and $y \in I_\mu$ such that $\varphi(x) \neq 0$ and $\varphi(y) \neq 0$. It follows that $xy \in I_\lambda \cap I_\mu$ and $I_\lambda \cap I_\mu \subseteq \text{rad}(\mathcal{A})$ by hypothesis. But this is a contradiction to $\varphi(\text{rad}(\mathcal{A})) = \{0\}$. Thus $\Delta(I_\lambda) \cap \Delta(I_\mu) = \emptyset$ for $\lambda \neq \mu$.

Finally, we have to check the identity $\Delta(\mathcal{A}) = \bigcup_{\lambda\in\Lambda} \Delta(I_\lambda)$. If $\varphi \in \Delta(\mathcal{A})$ and φ annihilates all I_λ, then $\varphi\big(\sum_{\lambda\in\Lambda} I_\lambda\big) = \{0\}$. On the other hand, since φ is continuous and the ideal $\sum_{\lambda\in\Lambda} I_\lambda$ is dense in $\mathcal{A}$, this is impossible. □

We conclude this chapter by characterizing the link between finite coverings of $\Delta(\mathcal{A})$ and decomposition of $\mathcal{A}$ into the direct sum of ideals. Note that $\bigoplus$ denotes the direct sum.

Theorem 3.3.9 *Let $\mathcal{A}$ be a unital commutative Banach algebra.*

(*i*) *If $\Delta(\mathcal{A})$ is a disjoint union $\Delta(\mathcal{A}) = \bigcup_{j=1}^n F_j$ of open (and closed) subsets F_j, then there exist unital closed ideals $I_1, \dots, I_n$ of $\mathcal{A}$ such that $\mathcal{A} = \bigoplus_{j=1}^n I_j$ and $\Delta(I_j) = F_j$ for $j = 1, \dots, n$.*

(*ii*) *Conversely, if $\mathcal{A}$ is the direct sum of closed ideals $I_1, \dots, I_n$, then the sets $\Delta(I_j)$ are closed and open ("clopen") in $\Delta(\mathcal{A})$ and $\Delta(\mathcal{A})$ is the disjoint union of the sets $\Delta(I_j), 1 \leq j \leq n$.*

Proof. We only consider the case $n = 2$. The final conclusion follows from a straightforward induction argument for both (*i*) and (*ii*).

(*i*) Since $\mathcal{A}$ has an identity, the Gelfand space is compact by Theorem 2.3.4. Hence F_1 and F_2 are compact. By the Shilov idempotent theorem (Theorem 3.2.1) there exists an idempotent $e_1 \in \mathcal{A}$ such that $\widehat{e_1} = \mathbb{1}_{F_1}$. Let e denote the identity element of $\mathcal{A}$ and set $e_2 = e - e_1$. Then, since e is always an idempotent ($ee = e$), e_2 is an idempotent and $\widehat{e_2} = \mathbb{1}_{F_2}$. Let $I_j = e_j\mathcal{A}$ for $j = 1, 2$. Then, by the same argument as in the proof of Theorem 3.3.7, I_1 and I_2 are closed ideals of $\mathcal{A}$ and $\Delta(I_j) = F_j$ for $j = 1, 2$. Now,

$$e_1 + e_2 = e = e^2 = (e_1 + e_2)^2 = e_1^2 + 2e_1e_2 + e_2^2 = e_1 + e_2 + 2e_1e_2.$$

Hence $e_1e_2 = 0$ and therefore, if $x \in I_1 \cap I_2$, then $x = xe_2 = xe_2e_1 = 0$. Thus $I_1 + I_2$ is the direct sum of I_1 and I_2. Eventually, if $x \in \mathcal{A}$, then

$$x = xe = x(e_1 + e_2) = xe_1 + xe_2 \in I_1 + I_2,$$

which completes the proof of (*i*).

(*ii*) As in the proof of Theorem 3.3.8, it follows immediately that $\Delta(\mathcal{A}) = \Delta(I_1) \cup \Delta(I_2)$ and $\Delta(I_1) \cap \Delta(I_2) = \emptyset$. Obviously, $\Delta(I_1)$ and $\Delta(I_2)$ are clopen in $\Delta(\mathcal{A})$. □

As a final result we observe that a similar statement can be made for nonunital Banach algebras.

Corollary 3.3.10 *Let $\mathcal{A}$ be a commutative Banach algebra.*

(*i*) *Suppose that $\Delta(\mathcal{A})$ is a disjoint union $\Delta(\mathcal{A}) = \bigcup_{j=1}^{n} F_j$, where F_1 is closed and $F_2, \dots, F_n$ are compact. Then there exist closed ideals $I_1, \dots, I_n$ of $\mathcal{A}$ such that $\mathcal{A} = \bigoplus_{j=1}^{n} I_j, \Delta(I_j) = F_j$ for $j = 1, \dots, n$ and $I_2, \dots, I_n$ are unital.*

(*ii*) *Conversely, let $I_1, I_2, \dots, I_n$ be closed ideals of $\mathcal{A}$ such that $\mathcal{A} = \bigoplus_{j=1}^{n} I_j$ and $I_2, \dots, I_n$ are unital. Then $\Delta(\mathcal{A})$ is the disjoint union of the closed set $\Delta(I_1)$ and the compact sets $\Delta(I_2), \dots, \Delta(I_n)$.*

Proof.
(*i*) Unlike Theorem 3.3.9, we can assume that $\mathcal{A}$ does not have an identity. Recall Remark 2.1.23 and let $\mathcal{A}_e$ be the algebra obtained by the technique of adjoining an identity e to $\mathcal{A}$. Let $E_1 = F_1 \cup \{\varphi_\infty\}$ and $E_j = F_j$ for $j = 2, \dots, n$. Then

$$\Delta(A_e) = \Delta(\mathcal{A}) \cup \{\varphi_\infty\} = \bigcup_{j=1}^{n} E_j$$

is a disjoint union of open and closed subsets. Hence by Theorem 3.3.9 there exist closed unital ideals $J_1, \dots, J_n$ of $\mathcal{A}_e$ such that $\mathcal{A}_e = \bigoplus_{j=1}^{n} J_j$ and $\Delta(J_j) = E_j$ for $j = 1, \dots, n$.

Clearly, we have $J_j \subseteq \mathcal{A}$ for $j = 2, \dots, n$. Indeed, because otherwise $\varphi_\infty(x) \neq 0$ for some $x \in J_j$ and hence $\varphi_\infty \in \Delta(J_j)$. But this is impossible since $\Delta(J_j) = E_j \subseteq \Delta(\mathcal{A})$. We define closed ideals $I_1, \dots, I_n$ of $\mathcal{A}$ by

$$I_1 = J_1 \cap \mathcal{A} \quad \text{and} \quad I_j = J_j \text{ for } j = 2, \dots, n.$$

We want to verify that $\Delta(I_1) = F_1$. Notice that $I_1 \subseteq J_1$ and therefore $\Delta(I_1) \subseteq \Delta(J_1) = E_1$. Since $\varphi_\infty(I_1) = \{0\}$, we get $\Delta(I_1) \subseteq E_1 \setminus \{\varphi_\infty\} = F_1$. Conversely, let $\varphi \in F_1 \subseteq \Delta(J_1)$ and choose $x \in J_1$ with $\varphi(x) \neq 0$. Since $\varphi \in \Delta(\mathcal{A})$, there exists $y \in \mathcal{A}$ such that $\varphi(y) \neq 0$. Thus $xy = J_1 \cap \mathcal{A} = I_1$ and $\varphi(xy) \neq 0$, whence $\varphi \in \Delta(I_1)$.

Finally, we have to show the equality $\mathcal{A} = \bigoplus_{j=1}^{n} I_j$, that is $\mathcal{A}$ is the direct sum of closed ideals $I_1, \dots, I_n$. Let $x \in \mathcal{A}$. Then there exist elements $x_1 \in J_1, \dots, x_n \in J_n$ such that $x = x_1 + \dots + x_n$. Since $x, x_2, \dots, x_n \in \mathcal{A}$, we have $x_1 \in J_1 \cap \mathcal{A} = I_1$. Therefore $\mathcal{A} = I_1 + \dots + I_n$. By the same argument as in the proof of Theorem 3.3.9 (*i*) this sum is direct.

(*ii*) Same argument as in the proof of Theorem 3.3.9 (*ii*). □

4. Regularity

In this final chapter we adress the concept of regularity, which plays a major role in the study of the ideal structure of a commutative Banach algebra $\mathcal{A}$. We introduce a new topology on $\Delta(\mathcal{A})$, the hull-kernel topology, and observe that it coincides with the Gelfand topology. We are particularly interested in some specific properties of regular commutative Banach algebras, such as normality and the existence of partitions of unity. For further details on this topic we refer to [Lar], Chapter 7. As a final result, we present a much simpler proof of Theorem 3.3.1, provided that $\mathcal{A}$ is regular (Corollary 4.2.16).

4.1. The hull-kernel topology

Let $\mathcal{A}$ be a commutative Banach algebra. Recall Theorem 2.1.18 where we have seen that there is a bijection between $\Delta(\mathcal{A})$, the set of all multiplicative linear functionals of $\mathcal{A}$ onto $\mathbb{C}$, and $\operatorname{Max}(\mathcal{A})$, the set of all maximal ideals in $\mathcal{A}$, given by $\varphi \mapsto \ker\varphi$. This is the way we always identify $\Delta(\mathcal{A})$ and $\operatorname{Max}(\mathcal{A})$.

So far we have only considered the Gelfand topology on $\Delta(\mathcal{A})$. Now we introduce a new topology on $\Delta(\mathcal{A}) = \operatorname{Max}(\mathcal{A})$, the so-called hull-kernel topology, which is much more appropriate for studying the ideal structure of $\mathcal{A}$ and, in general, is weaker than the Gelfand topology.

We begin with a number of preliminary definitions.

Definition 4.1.1 Let $\mathcal{A}$ be a commutative Banach algebra. For $E \subseteq \Delta(\mathcal{A}) = \operatorname{Max}(\mathcal{A})$, the *kernel* of E, denoted by $k(E)$, is defined as

$$k(E) = \{x \in \mathcal{A} : \varphi(x) = 0 \text{ for all } \varphi \in E\} = \bigcap\{M \in \operatorname{Max}(\mathcal{A}) : M \in E\}$$

if $E \neq \emptyset$, whereas $k(\emptyset) = \mathcal{A}$. For $\varphi \in \Delta(\mathcal{A})$ we write $k(\varphi)$ instead of $k(\{\varphi\}) = \ker\varphi$.

If $B \subseteq \mathcal{A}$, then the *hull* $h(B)$ of B is defined by

$$h(B) = \{\varphi \in \Delta(\mathcal{A}) : B \subseteq k(\varphi)\} = \{M \in \operatorname{Max}(\mathcal{A}) : B \subseteq M\}.$$

Also, for $x \in \mathcal{A}$, we simply write $h(x)$ instead of $h(\{x\}) = \operatorname{hul} x$.

Remark 4.1.2 Obviously, $k(E)$ is a closed ideal in $\mathcal{A}$ since maximal ideals are closed (Theorem 2.1.7 (*ii*)) and $h(B)$ is a closed subset of $\Delta(\mathcal{A})$ since the functions $\widehat{x}, x \in \mathcal{A}$, are continuous on $\Delta(\mathcal{A})$.

Next, we list some elementary properties of the formation of hulls and kernels.

Lemma 4.1.3 *Let $\mathcal{A}$ be a commutative Banach algebra. Let B, B_1, and B_2 be subsets of $\mathcal{A}$ and let E, E_1, and E_2 be subsets of $\Delta(\mathcal{A})$. Then* (*i*) $B_1 \subseteq B_2 \Longrightarrow h(B_1) \supseteq h(B_2)$.

(*ii*) $h(\overline{B}) = h(B)$ *and* $\overline{B} \subseteq k(h(B))$.

(*iii*) $h(B) = h(k(h(B)))$.

(*iv*) $E_1 \subseteq E_2 \Longrightarrow k(E_1) \supseteq k(E_2)$.

(*v*) $E \subseteq h(k(E))$ *and* $k(E) = k(h(k(E)))$.

(*vi*) $h(k(E_1 \cup E_2)) = h(k(E_1)) \cup h(k(E_2))$.

Proof. All of the proofs are rather elementary. Clearly, $(i), (ii)$ and (iv) follow directly from the definitions made above.

(iii) If $M \in h(k(h(B)))$, then $M \supseteq k(h(B)) \supseteq B$, whence $M \in h(B)$. Conversely, suppose $\varphi \in \Delta(\mathcal{A})$ is such that $k(\varphi) \notin h(k(h(B)))$. Then there exists some $x \in k(h(B))$ such that $\varphi(x) \neq 0$ and hence $\varphi \notin h(B)$.

(v) Since $E \subseteq \Delta(\mathcal{A})$, we have $E \subseteq h(k(E))$, and therefore by (iv) it follows that $k(E) \supseteq k(h(k(E)))$. Conversely, let $B = k(E)$ in (ii). Thus we get $k(E) \subseteq k(h(k(E)))$.

(vi) Obviously, $h(k(E_1)) \cup h(k(E_2)) \subseteq h(k(E_1) \cap k(E_2)) = h(k(E_1 \cup E_2))$. On the other hand, let

$$\varphi \in h(k(E_1 \cup E_2)) = h(k(E_1) \cap k(E_2)) \subseteq h(k(E_1)k(E_2))$$

and assume that $\varphi \notin h(k(E_2))$. Choose $y \in k(E_2)$ with $\varphi(y) \neq 0$. Then

$$\varphi(x)\varphi(y) = \varphi(xy) = 0$$

for all $x \in k(E_1)$. Thus $\varphi \in h(k(E_1))$. □

We know that $h(B)$ is a closed subset of $\Delta(\mathcal{A})$ (Remark 4.1.2). The idea behind the hull-kernel topology is to use such closed sets as the closed sets in a topology. With this in mind we make the following definition.

Definition 4.1.4 Let $\mathcal{A}$ be a commutative Banach algebra. For $E \subseteq \Delta(\mathcal{A})$ the *hull-kernel closure* $\overline{E}$ of E is defined to be $\overline{E} = h(k(E))$. The correspondence $E \to \overline{E}, E \subseteq \Delta(\mathcal{A})$, is a closure operation, that is, satisfies the following conditions.

(1) $E \subseteq \overline{E}$ and $\overline{\overline{E}} = \overline{E}$.

(2) $\overline{E_1 \cup E_2} = \overline{E}_1 \cup \overline{E}_2$.

This is easily verified: (2) is exactly the property (vi) in Lemma 4.1.3 and (1) follows from $(v) : E \subseteq h(k(E)) = \overline{E}$ and

$$\overline{\overline{E}} = h(k(h(k(E)))) = h(k(E)) = \overline{E}.$$

Thus, there is a unique topology on $\Delta(\mathcal{A})$ such that, for each subset E of $\Delta(\mathcal{A})$, $\overline{E} = h(k(E))$ is the closure of E. This topology is called the *hull-kernel topology (hk-topology).*

Example 4.1.5 Let $\mathcal{X}$ be a locally compact Hausdorff space. Then the *hk*-topology on $\mathcal{X} = \Delta(C_0(\mathcal{X}))$ coincides with the given topology. Let E be a closed subset of $\mathcal{X}$ and $x_0 \in \mathcal{X} \setminus E$. It follows by Urysohn's lemma that there exists $f \in C_0(\mathcal{X})$ such that

$$f(x_0) \neq 0 \quad \text{and} \quad f(x) = 0$$

for all $x \in E$. Hence $E = h(k(E))$ is a *hk*-closed set.

Example 4.1.6 Recall the disc algebra from Example 1.1.10. The *hk*-topology on $\overline{\mathbb{D}} = \Delta(A(\overline{\mathbb{D}}))$ is weaker than the usual topology on $\overline{\mathbb{D}}$ since the set of zeros of a nonzero holomorphic function in a region cannot have an accumulation point within that region. Therefore every *hk*-closed subset of $\overline{\mathbb{D}}$ has an at most countable intersection with the open unit disc $\mathbb{D}$.

The next lemma is similar to Lemma 3.3.6.

Lemma 4.1.7 *Let $\mathcal{A}$ be a commutative Banach algebra, I a closed ideal of $\mathcal{A}$ and let $q : \mathcal{A} \to \mathcal{A}/I$ denote the quotient homomorphism.*

(*i*) *The map $\varphi \mapsto \varphi \circ q$ is a homeomorphism for the hull-kernel topologies between $\Delta(\mathcal{A}/I)$ and the closed subset $h(I)$ of $\Delta(\mathcal{A})$.*

(*ii*) *The map $\varphi \mapsto \varphi|_I$ is a homeomorphism for the hull-kernel topologies between the open subset $\Delta(\mathcal{A}) \setminus h(I)$ of $\Delta(\mathcal{A})$ and $\Delta(I)$.*

Proof. The proof is similar to the one in Lemma 3.3.6, see [Kan], Section 4.1, p.196. □

Lemma 4.1.8 *Let $\mathcal{A}$ be a commutative Banach algebra without identity and let $a \in \mathcal{A}$ be such that $\widehat{a}$ is continuous in the hull-kernel topology on $\Delta(\mathcal{A})$. Then $\widehat{a}$ is also continuous on $\Delta(\mathcal{A}_e)$, with respect to the hull-kernel topology.*

Proof. By Remark 2.1.23, $\Delta(\mathcal{A}_e) = \Delta(\mathcal{A}) \cup \{\varphi_\infty\}$ where each $\varphi \in \Delta(\mathcal{A})$ is identified with its canonical extension $x + \lambda e \mapsto \varphi(x) + \lambda, x \in \mathcal{A}, \lambda \in \mathbb{C}$ and φ_∞ denotes the homomorphism $A_e \to \mathbb{C}$ with kernel $\mathcal{A}$. By h and k we denote the hull and kernel operations with respect to $\mathcal{A}$ and by h_e and k_e those with respect to A_e.

Let E be a subset of $\Delta(\mathcal{A}_e)$. From the definitions of hulls and kernels we get that

$$h_e(k_e(E)) \subseteq h(k(E \cap \Delta(\mathcal{A}))) \cup \{\varphi_\infty\}. \tag{4.1}$$

Furthermore, let $F \neq \emptyset$ be a closed subset of $\mathbb{C}$ and $E = \{\varphi \in \Delta(\mathcal{A}_e) : \varphi(a) \in F\}$. By premise on a we get that $E \cap \Delta(\mathcal{A})$ is *hk*-closed in $\Delta(\mathcal{A})$. In order to show that E is *hk*-closed in $\Delta(\mathcal{A}_e)$, we need to distinguish the two cases $0 \in F$ and $0 \notin F$.

Case 1: If $0 \in F$, then $\varphi_\infty \in E$ and so $h(k(E \cap \Delta(\mathcal{A}))) = E \cap \Delta(\mathcal{A})$. Hence by (4.1) we have

$$h_e(k_e(E)) \subseteq (E \cap \Delta(\mathcal{A})) \cup \{\varphi_\infty\} = E.$$

Thus E is *hk*-closed in $\Delta(\mathcal{A}_e)$.

Case 2: If $0 \notin F$, then $\varphi_\infty \notin E$ and therefore $E \subseteq \Delta(\mathcal{A})$. Let $\delta = \inf\{|\lambda| : \lambda \in F\}$. Then $\delta > 0$ and $|\varphi(a)| \geq \delta$ for all $\varphi \in E$ and $a \in \mathcal{A}$. Note that E is compact and since $E = \Delta(\mathcal{A}/k(E))$ and $\varphi(a) = \widehat{a}(\varphi) \neq 0$ for all $\varphi \in E$, we can apply Theorem 3.1.19: There exists some $b \in \mathcal{A}$ such that $\varphi(b) = 1/\varphi(a)$ for all $\varphi \in E$. Now let

$$x = e - ab \in \mathcal{A}_e.$$

Then x satisfies

$$\varphi_\infty(x) = \varphi_\infty(e - ab) = \varphi_\infty(e) - \varphi_\infty(a)\varphi_\infty(b) = 1$$

and

$$\varphi(x) = \varphi(e - ab) = \varphi(e) - \varphi(a)\varphi(b) = 1 - \varphi(a)\frac{1}{\varphi(a)} = 0,$$

for all $\varphi \in E$. Thus $\varphi_\infty \notin h_e(k_e(E))$ and so

$$h_e(k_e(E)) \subseteq h(k(E \cap \Delta(\mathcal{A}))) = h(k(E)) = E.$$

Hence E is hk-closed in $\Delta(\mathcal{A}_e)$, which completes the proof.

□

Before we present the next lemma, we first need to introduce the notion of *modularity*.

Definition 4.1.9 Let $\mathcal{A}$ be a Banach algebra and I an ideal of $\mathcal{A}$. Then I is called *modular*, if $\mathcal{A}/I$ is unital, that is, there exists $u \in \mathcal{A}$ such that the two sets

$$\mathcal{A}(1-u) = \{x - xu : x \in \mathcal{A}\} \quad \text{and} \quad (1-u)\mathcal{A} = \{x - ux : x \in \mathcal{A}\}$$

are both contained in I. Such an element u is called an *identity modulo I*. The ideal I is called a *maximal modular* ideal if it is modular and also a maximal proper ideal.

Lemma 4.1.10 *Let I be a closed ideal of the commutative Banach algebra $\mathcal{A}$ and let E be a hull-kernel-closed subset of $\Delta(\mathcal{A})$ such that $E \cap h(I) = \emptyset$ and $k(E)$ is modular. Then I contains an identity modulo $k(E)$.*

Proof. By hypothesis, $k(E)$ is modular, that is, $\mathcal{A}/(I + k(E))$ is unital, and because E is hk-closed in $\Delta(\mathcal{A})$, we get

$$h(I + k(E)) = h(I) \cap h(k(E)) = h(I) \cap E = \emptyset.$$

Hence $I + k(E) = \mathcal{A}$. Now let $u \in \mathcal{A}$ be such that $ux - x \in k(E)$ for all $x \in \mathcal{A}$. Then $u = v + y$ for $v \in I$ and $y \in k(E)$. It follows that

$$vx - x = (u - y)x - x = ux - x - yx \in k(E)$$

for all $x \in \mathcal{A}$. Thus v is an identity of I modulo $k(E)$. □

4.2. Regular Banach algebras and applications

In this section we present a powerful result (Theorem 4.2.12), which has some very useful applications such as proving Theorem 3.3.1 without recourse to the Shilov idempotent theorem.

We begin with the basic notion of regularity. Let T be a T_1 topological space, that is, every singleton set in T is closed, and let $\mathcal{F}$ be a family of complex-valued functions on T. Recall from point set topology that $\mathcal{F}$ is called *regular* if for any given closed subset E of T and $t \in T \setminus E$, there exists $f \in \mathcal{F}$ with $f(t) \neq 0$ and $f|_E = 0$. This leads to the following definition.

Definition 4.2.1 A commutative Banach algebra $\mathcal{A}$ is called *regular* if its algebra of Gelfand transforms is regular in the above sense, that is, given any closed subset E of $\Delta(\mathcal{A})$ and $\varphi_0 \in \Delta(\mathcal{A}) \setminus E$, there exists $x \in \mathcal{A}$ such that

$$\varphi_0(x) \neq 0 \quad \text{and} \quad \varphi(x) = 0$$

for all $\varphi \in E$.

Example 4.2.2 For a compact Hausdorff space $\mathcal{X}, C(\mathcal{X})$ is a regular Banach algebra. Indeed, it is a well-known fact that $\mathcal{X}$ is a regular and also a normal[1] topological space. The first assertion is equivalent to saying that for each closed set $E \subseteq \mathcal{X}$ and each point $t \in \mathcal{X} \setminus E$, there exists some $f \in C(\mathcal{X})$ such that

$$0 \leq f(s) \leq 1, \ s \in \mathcal{X}, \ f(t) = 1$$

and

$$f(s) = 0, \ s \in E.$$

The second assertion, combined with Urysohn's lemma, reveals that, if $E_1 \subseteq \mathcal{X}$ and $E_2 \subseteq \mathcal{X}$ are disjoint closed sets, then there exists $f \in C(\mathcal{X})$ such that

$$0 \leq f(s) \leq 1, \ s \in \mathcal{X},$$

$$f(s) = 1, \ s \in E_1,$$

and

$$f(s) = 0, \ s \in E_2.$$

Recall that the Gelfand representation of the commutative Banach algebra $\mathcal{A} = C(\mathcal{X})$ is just $\widehat{\mathcal{A}} = C(\mathcal{X})$. Thus $C(\mathcal{X})$ is regular since the Gelfand homomorphism is just the identity mapping.

Example 4.2.3 $C_0(\mathcal{X})$ is also regular. If $E \subseteq \mathcal{X} = \Delta(C_0(\mathcal{X}))$ is closed, then it is closed in the hk-topology, that is, $E = h(k(E))$ (see Example 4.1.5).

Example 4.2.4 It is also easily seen that for $a, b \in \mathbb{R}, a < b$, and $n \in \mathbb{N}$, $C^n([a, b])$ is regular since, when $\Delta(C^n([a, b]))$ is identified with $[a, b]$, the Gelfand homomorphism is nothing but the identity.

[1] see Appendix Definition A.1.2.

The disc algebra $A(\overline{\mathbb{D}})$ is an example of a non-regular Banach algebra.

Example 4.2.5 Recall Example 4.1.6 with identification $\overline{\mathbb{D}} = \Delta(A(\overline{\mathbb{D}}))$. Then $A(\overline{\mathbb{D}})$ fails to be regular. Indeed, consider the set $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Evidently, $E \subseteq \overline{\mathbb{D}}$, hence $E \cup \{0\}$ is closed in the Gelfand topology. But recalling that the Gelfand transformation on $A(\overline{\mathbb{D}})$ is again the identity mapping, we see that

$$k(E) = \{f \in A(\overline{\mathbb{D}}) : f(1/n) = 0,\ n \in \mathbb{N}\} = \{0\},$$

since the zeros in an open set of a nonconstant analytic function are isolated. Thus

$$\overline{E} = h(k(E)) = h(\{0\}) = \Delta(A(\overline{\mathbb{D}})) = \overline{\mathbb{D}} \neq E \cup \{0\}.$$

The next theorem relates regularity of a commutative Banach algebra $\mathcal{A}$ to properties of the hull-kernel topology on $\Delta(\mathcal{A})$.

Theorem 4.2.6 *For a commutative Banach algebra $\mathcal{A}$, the following conditions are equivalent.*

- (i) *$\mathcal{A}$ is regular.*
- (ii) *The hull-kernel topology and the Gelfand topology on $\Delta(\mathcal{A})$ coincide.*
- (iii) *The hull-kernel topology on $\Delta(\mathcal{A})$ is Hausdorff, and every point in $\Delta(\mathcal{A})$ has a hull-kernel neighbourhood with modular kernel.*

Proof. We only show $(i) \Rightarrow (ii)$, since this implication will turn out to be quite useful later on. For a complete proof we refer to [Kan], Section 4.2, Theorem 4.2.3, p.199.

Suppose that $\mathcal{A}$ is regular and let E be a subset of $\Delta(\mathcal{A})$ that is closed in the Gelfand topology. Then, by definition, for every $\varphi \in \Delta(\mathcal{A}) \setminus E$, there exists $x_\varphi \in \mathcal{A}$ such that

$$\widehat{x_\varphi}|_E = 0 \quad \text{and} \quad \widehat{x_\varphi}(\varphi) \neq 0.$$

It follows that $k(E) \nsubseteq \ker\varphi$ for every $\varphi \in \Delta(\mathcal{A}) \setminus E$, and hence $E = h(k(E))$, which means that E is closed in the hk-topology. Thus the two topologies on $\Delta(\mathcal{A})$ coincide. □

Now recall Proposition 2.3.24 where we have proved that the Gelfand topology on $\Delta(\mathcal{A})$ equals the weak topology with respect to the functions $\widehat{x}, x \in \mathcal{A}$. Therefore the equivalence of (i) and (ii) in Theorem 4.2.6 can be reformulated as follows.

Corollary 4.2.7 *Let $\mathcal{A}$ be a commutative Banach algebra. Then $\mathcal{A}$ is regular if and only if $\widehat{x}$ is hull-kernel continuous on $\Delta(\mathcal{A})$ for each $x \in \mathcal{A}$.*

Remark 4.2.8 In [Lar], Section 7.1, Theorem 7.1.2, p.165, it is claimed that a commutative Banach algebra $\mathcal{A}$ is regular provided that the hk-topology on $\Delta(\mathcal{A})$ is Hausdorff. For a unital algebra this is obviously true. Nevertheless, this strengthening of the implication $(iii) \Rightarrow (i)$ in Theorem 4.2.6 does not seem to be correct (even though we are unaware of a counterexample).

We continue with some hereditary properties of regularity such as adjoining an identity and forming closed ideals and quotients.

Theorem 4.2.9 *Let $\mathcal{A}$ be a commutative Banach algebra.*

(i) Let I be a closed ideal of $\mathcal{A}$. If $\mathcal{A}$ is regular, then so are the algebras I and $\mathcal{A}/I$.

(ii) $\mathcal{A}$ is regular if and only if $\mathcal{A}_e$, the unitization of $\mathcal{A}$, is regular.

Proof.
(i) Since $\mathcal{A}$ is regular, the Gelfand topology coincides with the hk-topology on $\Delta(\mathcal{A})$ by Theorem 4.2.6. Now, by Lemma 4.1.7 (ii), the map $\varphi \mapsto \varphi|_I$ is a homeomorphism for the hk-topologies between the open subset $\Delta(\mathcal{A}) \setminus h(I)$ of $\Delta(\mathcal{A})$ and $\Delta(I)$. Similarly, the same is true of the Gelfand topologies by Lemma 3.3.6 (ii). Hence the Gelfand topology and the hk-topology coincide on $\Delta(I)$. By further application of Theorem 4.2.6 we eventually get that I is regular. Similarly, using Lemma 4.1.7 (i) and Lemma 3.3.6 (i) as well as applying Theorem 4.2.6, it follows that $\mathcal{A}/I$ is regular.

(ii) If A_e is regular, so is $\mathcal{A}$ by (i). On the other hand, suppose that $\mathcal{A}$ is regular. Then, by Corollary 4.2.7, for every $a \in \mathcal{A}$, $\widehat{a}$ is hk-continuous on $\Delta(\mathcal{A})$. Hence, by Lemma 4.1.8, $\widehat{a}$ is also hk-continuous on $\Delta(A_e)$. This of course implies that $\widehat{x}$ is hk-continuous on $\Delta(A_e)$ for each $x \in \mathcal{A}_e$. Thus $\mathcal{A}_e$ is regular by Corollary 4.2.7. □

Remark 4.2.10 One can show that the converse of (i) in the preceding Theorem, that is, $\mathcal{A}$ is regular whenever $\mathcal{A}$ has a closed ideal I such that both I and $\mathcal{A}/I$ are regular, holds. However, this result is much more difficult to prove and involves the existence of a greatest closed regular ideal in a commutative Banach algebra. For more details, see [Kan], Section 4.3, pp.207.

It is also worth mentioning that a closed subalgebra of a regular algebra need not be regular. In fact, $C(\overline{\mathbb{D}})$ is regular, whereas the closed subalgebra $A(\overline{\mathbb{D}})$ is not (see Example 4.2.5).

Lemma 4.2.11 *Let I be an ideal in the regular commutative Banach algebra $\mathcal{A}$. Given any $\varphi_0 \in \Delta(\mathcal{A}) \setminus h(I)$, there exists $u \in I$ such that $\widehat{u} = 1$ in some neighbourhood of φ_0.*

Proof. Since $\mathcal{A}$ is regular by premise, it follows from Theorem 4.2.6 that the hk-topology on $\Delta(\mathcal{A})$ is Hausdorff and φ_0 has a hk-neighbourhood with modular kernel. Hence we can choose a neighbourhood V of φ_0 such that $\overline{V} \cap h(I) = \emptyset$ and $k(V)$ is modular. Now we can apply Lemma 4.1.10 which yields the existence of some $u \in I$ such that $\widehat{u}|_V = 1$. □

The following theorem is one of the most striking results on regular commutative Banach algebras. We will need it later in order to prove some very useful applications.

Theorem 4.2.12 *Let $\mathcal{A}$ be a regular commutative Banach algebra, and suppose that I is an ideal in $\mathcal{A}$ and C is a compact subset of $\Delta(\mathcal{A})$ with $C \cap h(I) = \emptyset$. Then there exists $x \in I$ such that*

$$\widehat{x}|_C = 1 \quad \text{and} \quad \widehat{x} = 0 \text{ on some neighbourhood of } h(I).$$

Proof. We start with the existence of some $y \in I$ with $\widehat{y}|_C = 1$. As C is compact by hypothesis, by Lemma 4.2.11 there exist open subsets V_i of $\Delta(\mathcal{A})$ and $u_i \in I$ for $i = 1, \ldots, k$ such that

$$\widehat{u}_i|_{V_i} = 1 \quad \text{and} \quad C \subseteq \bigcup_{i=1}^{k} V_i.$$

Now, we define elements y_i of $\mathcal{A}$ inductively by $y_1 = u_1$ and

$$y_{i+1} = y_i + u_{i+1} - y_i u_{i+1}, \quad i = 1, \ldots, k-1. \tag{4.2}$$

It will be immediately clear why (4.2) was defined this way. A straightforward induction argument reveals that $y_i \in I$ and

$$\widehat{y}_j|_{\bigcup_{i=1}^{j} V_i} = 1.$$

Indeed, this is true. By applying φ to (4.2) we get

$$\begin{aligned}\varphi(y_{j+1}) &= \varphi(y_j) + \varphi(u_{j+1}) - \varphi(y_j)\varphi(u_{j+1}) \\ &= \begin{cases} 1 + \varphi(u_{j+1}) - \varphi(u_{j+1}) & \text{for } \varphi \in \bigcup_{i=1}^{j} V_i, \\ \varphi(y_j) + 1 - \varphi(y_j) & \text{for } \varphi \in V_{j+1}. \end{cases}\end{aligned}$$

Thus $\widehat{y}_{i+1} = 1$ on $\bigcup_{i=1}^{j+1} V_i$. Now $y = y_k$, and y has the desired properties.

Next, we want to show that $\widehat{x}, x \in I$, vanishes on some neighbourhood of $h(I)$. To that end we choose an open subset V of $\Delta(\mathcal{A})$ with $C \subseteq V$ and $\overline{V} \subseteq \Delta(\mathcal{A}) \setminus h(I)$. We observe that

$$C \cap h(k(\Delta(\mathcal{A}) \setminus V)) = C \cap (\Delta(\mathcal{A}) \setminus V) = \emptyset,$$

since $\Delta(\mathcal{A}) \setminus V$ is hk-closed. Therefore we can argue in a similar way as before. For the ideal $J = k(\Delta(\mathcal{A}) \setminus V)$ we obtain $z \in J$ with $\widehat{z}|_C = 1$. So, by the first part of the proof, there exists $y \in I$ such that $\widehat{y}|_C = 1$. Now let $x = yz \in I$. Then x satisfies

$$\widehat{x}(\varphi) = \varphi(x) = \varphi(yz) = \varphi(y)\varphi(z) = \widehat{y}(\varphi)\widehat{z}(\varphi) = 1 \cdot 1 = 1,$$

for all $\varphi \in C$ and

$$\operatorname{supp} \widehat{x} \subseteq \operatorname{supp} \widehat{z} \subseteq \overline{V} \subseteq \Delta(\mathcal{A}) \setminus h(I).$$

Thus $\widehat{x} = 0$ in a neighbourhood of $h(I)$. □

We conclude this chapter by presenting a series of interesting applications of the preceding theorem. We start with a corollary which characterizes the notion of *normality* of a regular Banach algebra.

Corollary 4.2.13 *Every regular commutative Banach algebra $\mathcal{A}$ is normal in the sense that whenever $E \subseteq \Delta(\mathcal{A})$ is closed, $C \subseteq \Delta(\mathcal{A})$ is compact and $E \cap C = \emptyset$, then there exists $x \in \mathcal{A}$ such that* $\operatorname{supp} \widehat{x} \subseteq \Delta(\mathcal{A}) \setminus E$ *and* $\widehat{x}|_C = 1$.

As it turns out, this corollary is just the case $n = 1$ for the following result. In a regular commutative Banach algebra $\mathcal{A}$ there exist *partitions of unity* on $\Delta(\mathcal{A})$ subordinate to a given finite open cover of a compact set.

Corollary 4.2.14 *Let $\mathcal{A}$ be a regular commutative Banach algebra. Suppose that C is a compact subset of $\Delta(\mathcal{A})$ and $U_1, \ldots, U_n$ are open subsets of $\Delta(\mathcal{A})$ such that $C \subseteq \bigcup_{j=1}^n U_j$. Then there exist $x_1, \ldots, x_n \in \mathcal{A}$ with the following properties.*

(*i*) $(\widehat{x}_1 + \ \ldots \ + \widehat{x}_n)|_C = 1$.

(*ii*) $\widehat{x}_j|_{\Delta(\mathcal{A})\setminus U_j} = 0$ *for each* $j = 1, \ldots, n$.

Proof. We proceed in a similar way as in the proof of Theorem 4.2.12 and choose open subsets V_j of $\Delta(\mathcal{A}), 1 \leq j \leq n$, such that

$$\overline{V_j} \subseteq U_j \quad \text{and} \quad C \subseteq \bigcup_{j=1}^n V_j.$$

Moreover, let

$$I_j = k(\Delta(\mathcal{A}) \setminus V_j),\ j = 1, \ldots, n \quad \text{and} \quad I = I_1 + \ \ldots \ + I_n.$$

Then we get $h(I_j) = h(k(\Delta(\mathcal{A}) \setminus V_j)) = \Delta(\mathcal{A}) \setminus V_j$ and therefore

$$h(I) = h(I_1 + \ \ldots \ + I_n) = \bigcap_{j=1}^n h(I_j) = \bigcap_{j=1}^n (\Delta(\mathcal{A}) \setminus V_j) = \Delta(\mathcal{A}) \setminus \bigcup_{j=1}^n V_j.$$

Hence $h(I) \cap C = \emptyset$. Then, by Theorem 4.2.12, there exists $x \in I$ with $\widehat{x}|_C = 1$. Write x as $x = x_1 + \ \ldots \ + x_n$ where $x_j \in I_j$. Then $x_1, \ldots, x_n$ satisfy (*i*) and (*ii*). □

Corollary 4.2.15 *Let $\mathcal{A}$ be a regular commutative Banach algebra such that its range under the Gelfand homomorphism $\Gamma : \mathcal{A} \to C_0(\Delta(\mathcal{A}))$ is closed under complex conjugation. Suppose that C and E are disjoint closed subsets of $\Delta(\mathcal{A})$ with C compact. Then there exists $x \in \mathcal{A}$ such that*

$$\widehat{x}|_C = 1,\ 0 \leq \widehat{x} \leq 1 \quad \textit{and} \quad \operatorname{supp} \widehat{x} \subseteq \Delta(\mathcal{A}) \setminus E.$$

Proof. By Theorem 4.2.12 there exists $y \in \mathcal{A}$ such that $\widehat{y}|_C = 1$ and by the second part of the proof of said theorem we have $\operatorname{supp} \widehat{y} \subseteq \Delta(\mathcal{A}) \setminus E$. By hypothesis, there exists $z \in \mathcal{A}$ such that $\widehat{z} = \overline{\widehat{y}}$ where $^-$ denotes the complex conjugation. Now let f be an entire function defined by

$$f(w) = \sin^2\left(\frac{\pi}{2} w\right),\ w \in \mathcal{A},$$

and let $x = f(yz)$. It follows from the the single-variable holomorphic calculus (Theorem 3.1.10) that

$$\begin{aligned}
\widehat{x}(\varphi) &= \varphi(x) = \varphi(f(yz)) \\
&= f(\varphi(yz)) = f(\varphi(y)\varphi(z)) \\
&= \sin^2\left(\frac{\pi}{2}\varphi(y)\varphi(z)\right) = \sin^2\left(\frac{\pi}{2}|\varphi(y)|^2\right)
\end{aligned}$$

for all $\varphi \in \Delta(\mathcal{A})$. Hence we obtain

$$\widehat{x}|_C = \sin^2\left(\frac{\pi}{2}\left|\widehat{y}|_C\right|^2\right) = \sin^2\left(\frac{\pi}{2}\cdot 1\right) = 1,\ 0 \le \widehat{x} \le 1$$

and

$$\operatorname{supp}\widehat{x} \subseteq \Delta(\mathcal{A}) \setminus E.$$

□

As a final application we present a much simpler proof of Theorem 3.3.1, provided that $\mathcal{A}$ is regular.

Corollary 4.2.16 *Let $\mathcal{A}$ be a semisimple regular commutative Banach algebra. If $\Delta(\mathcal{A})$ is compact, then $\mathcal{A}$ has an identity.*

Proof. By Theorem 4.2.12 there exists $u \in \mathcal{A}$ such that $\widehat{u}|_{\Delta(\mathcal{A})} = 1$. Hence

$$\widehat{x - ux} = \widehat{x} - \widehat{u}\widehat{x} = 0$$

on $\Delta(\mathcal{A})$ for all $x \in \mathcal{A}$. Since $\mathcal{A}$ is semisimple by hypothesis, we get $x - ux = 0 \Leftrightarrow x = ux$ for all $x \in \mathcal{A}$, whence u is an identity for $\mathcal{A}$. □

Of course, the conclusion of Corollary 4.2.16 holds true without assuming that $\mathcal{A}$ be regular as we have witnessed in Theorem 3.3.1. However, since the proof of Shilov's idempotent theorem is based on the several-variable holomorphic functional calculus and therefore requires much more effort, it appears to be justified to give a simpler proof in the case of a regular semisimple algebra.

A. Topology and Functional Analysis

In this short appendix we will provide and repeat some fundamental notions and theorems of topology and functional analysis which are needed in Banach algebra theory. No proofs will be given since they can be found in several textbooks, such as in [Rud].

A.1. Topology

Definition A.1.1 Let $\mathcal{X}$ be a topological space. Then

- $C(\mathcal{X})$ denotes the set of all continuous complex-valued functions on $\mathcal{X}$.
- $C_b(\mathcal{X})$ denotes the subspace of all bounded functions in $C(\mathcal{X})$.
- $C_c(\mathcal{X})$ denotes the set of all functions in $C(\mathcal{X})$ with compact support

$$\operatorname{supp}(f) = \overline{\{f \in C(\mathcal{X}) : f(x) \neq 0 \text{ for all } x \in \mathcal{X}\}}.$$

- $C_0(\mathcal{X})$ denotes the set of all functions $f \in C(X)$ which vanish at infinity, that is if for each $\varepsilon > 0$ there exists a compact subset K_ε of $\mathcal{X}$ such that $|f(x)| < \varepsilon$ for all $x \in \mathcal{X} \setminus K_\varepsilon$.

Obviously, $C_c(\mathcal{X}) \subseteq C_0(\mathcal{X}) \subseteq C_b(\mathcal{X})$ and all these spaces coincide with $C(\mathcal{X})$ when $\mathcal{X}$ is compact. Also, all these spaces are algebras under pointwise operations.

On $C_b(\mathcal{X})$ we can introduce the supremum norm defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathcal{X}\}.$$

This norm turns $C_b(\mathcal{X})$ and $C_0(\mathcal{X})$ into Banach spaces. If $\mathcal{X}$ is a locally compact Hausdorff space, then $C_c(\mathcal{X})$ is dense in $C_0(\mathcal{X})$.

Definition A.1.2 A topological space $\mathcal{X}$ is called *normal* if it is Hausdorff and for each pair $\{A, B\}$ of disjoint closed subsets of $\mathcal{X}$ there exist open subsets U and V of $\mathcal{X}$ such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$.

Theorem A.1.3 (Urysohn's lemma)

(*i*) *Let $\mathcal{X}$ be a normal topological space, and let A and B be disjoint closed subsets of $\mathcal{X}$. Then there exists a continuous function $f : \mathcal{X} \to [0, 1]$ such that $f|_A = 1$ and $f|_B = 0$.*

(*ii*) *Let $\mathcal{X}$ be a locally compact Hausdorff space, and let C be a compact subset of $\mathcal{X}$ and U an open set containing C. Then there exists $f \in C_c(\mathcal{X})$ with $f|_C = 1$, $0 \leq f(x) \leq 1$ for all $x \in \mathcal{X}$ and $\operatorname{supp}(f) \subseteq U$.*

Theorem A.1.4 (**Tietze's extension theorem**) *A Hausdorff space $\mathcal{X}$ is normal if and only if every real-valued function, which is defined and continuous on a closed subset of $\mathcal{X}$, admits a continuous extension to all of $\mathcal{X}$.*

Definition A.1.5 A family F of complex-valued functions on a topological space $\mathcal{X}$ is said to *strongly separate the points* of $\mathcal{X}$ if for each $x \in \mathcal{X}$, there exists $f \in F$ with $f(x) \neq 0$ and for each $x, y \in \mathcal{X}$ with $x \neq y$, there exists $g \in F$ such that $g(x) \neq g(y)$. The family F is called *self-adjoint* if it contains with a function f the conjugate complex function $\overline{f}$.

Theorem A.1.6 (**Stone-Weierstrass theorem**) *Let $\mathcal{X}$ be a locally compact Hausdorff space, and let $\mathcal{A}$ be a self-adjoint subalgebra of $C_0(\mathcal{X})$. Suppose that $\mathcal{A}$ strongly separates the points of $\mathcal{X}$. Then $\mathcal{A}$ is uniformly dense in $C_0(\mathcal{X})$.*

Definition A.1.7 Let $\mathcal{X}$ and $\mathcal{Y}$ be two topological spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is called a *homeomorphism* if it satisfies the following properties.

- f is a bijection.
- f is continuous.
- f^{-1} is continuous (f is an open mapping).

Theorem A.1.8 (**Arzela-Ascoli**) *Let $\mathcal{X}$ be a locally compact Hausdorff space and $F \subseteq C_0(\mathcal{X})$. Suppose that F satisfies the following two conditions.*

(*i*) *The set $F(x) = \{f(x) : f \in F\}$ is bounded for every $x \in \mathcal{X}$.*

(*ii*) *F is equicontinuous; that is, for each $x \in \mathcal{X}$ and $\varepsilon > 0$, there exists a neighbourhood U of x such that $|f(y) - f(x)| < \varepsilon$ for all $f \in F$ and $y \in U$.*

Then F is relatively compact in $(C_0(\mathcal{X}), \|.\|_\infty)$.

Theorem A.1.9 (**Baire's category theorem**) *Let $\mathcal{X}$ be either a locally compact Hausdorff space or a complete metric space.*

(*i*) *If $\mathcal{X}$ is the union of countably many closed subsets, then one of them contains a nonempty open set.*

(*ii*) *The intersection of a countable collection of dense open subsets of $\mathcal{X}$ is dense in $\mathcal{X}$.*

A compact space C is a *compactification* of a topological space $\mathcal{X}$ if there exists a continuous injective mapping from $\mathcal{X}$ onto a dense subset of C. Let $\mathcal{X}$ be a locally compact Hausdorff space. Then there exists a compact Hausdorff space $\widetilde{\mathcal{X}}$ together with an embedding $\iota : \mathcal{X} \to \widetilde{\mathcal{X}}$ such that $\widetilde{\mathcal{X}} \setminus \iota(\mathcal{X})$ is a singleton. $\widetilde{\mathcal{X}}$ is uniquely determined up to homeomorphisms and is called the *one-point compactification* of $\mathcal{X}$. The space $\widetilde{\mathcal{X}}$ can be constructed as follows. Let $\widetilde{\mathcal{X}} = \mathcal{X} \cup \{\infty\}$ as a set and take the open sets in $\widetilde{\mathcal{X}}$ to be the open sets in $\mathcal{X}$ together with the complements in $\widetilde{\mathcal{X}}$ of the compact subset of $\mathcal{X}$.

Note that each $f \in C_0(\mathcal{X})$ extends to a continuous function on $\widetilde{\mathcal{X}}$, also denoted f, by setting $f(\infty) = 0$.

Let $\mathcal{X}$ be a compact space and $\mathcal{Y}$ a Hausdorff space. If f is a continuous and injective mapping from $\mathcal{X}$ into $\mathcal{Y}$, then f is a homeomorphism from $\mathcal{X}$ onto its range $f(\mathcal{X})$.

Proposition A.1.10 *If f is a continuous open map of a locally compact Hausdorff space $\mathcal{X}$ onto a Hausdorff space $\mathcal{Y}$ and if K is a compact subset of $\mathcal{Y}$, then there exists a compact subset C of $\mathcal{X}$ such that $f(C) = K$.*

Proposition A.1.11 *Let $\mathcal{X}$ be a locally compact Hausdorff space. A subset $\mathcal{Y}$ of $\mathcal{X}$ is locally compact (in the induced topology) if and only if there exist a closed subset A of $\mathcal{X}$ and an open subset B of $\mathcal{X}$ such that $\mathcal{Y} = A \cap B$. In particular, a dense subset of $\mathcal{X}$ is locally compact if and only if it is open in $\mathcal{X}$.*

A *closure operation* on a set $\mathcal{X}$ is an assignment $A \to \overline{A}$ from $\mathcal{P}(\mathcal{X})$, the collection of all subsets of $\mathcal{X}$, to itself such that

$$\overline{\emptyset} = \emptyset, \ A \subseteq \overline{A} = \overline{\overline{A}} \text{ and } \overline{A \cup B} = \overline{A} \cup \overline{B} \quad \text{for all } A, B \subseteq \mathcal{X}.$$

If such a closure operation is given, there exists a unique topology on $\mathcal{X}$ such that for each $A \subseteq \mathcal{X}$, $\overline{A}$ equals the closure of A in $\mathcal{X}$ with respect to this topology.

Lemma A.1.12 (**Zorn**) *Suppose a partially ordered set $\mathcal{X}$ has the property that every chain has an upper bound in $\mathcal{X}$. Then the set $\mathcal{X}$ contains at least one maximal element.*

A.2. Functional analysis

Definition A.2.1 For a normed space E, let E' denote the *dual space* of E; that is $E' = \mathfrak{B}(E, \mathbb{C})$, the vector space of all continuous linear functionals on E. Thus E' is a Banach space when equipped with the norm

$$\|f\| = \sup\{|f(x)| : x \in E, \|x\| \leq 1\}, \quad f \in E'.$$

The space E embeds isometrically into the *second dual space* E'' as follows. For each $x \in E$, define $\widehat{x} : E' \to \mathbb{C}$ by $\widehat{x}(f) = f(x)$ for $f \in E'$. Then $\widehat{x} \in E''$, and it is a consequence of the Hahn-Banach theorem (see below) that $\|\widehat{x}\| = \|x\|$.

Definition A.2.2 The *weak topology* $\sigma(E, E')$ on E is the coarsest topology with respect to which all the functionals $f \in E'$ are continuous on E. Similarly, the *weak*-topology* (or *w^*-topology*) $\sigma(E', E)$ is the coarsest topology on E' with respect to which all the linear functionals $\widehat{x}$ on $E', x \in E$, are continuous. Thus a neighbourhood basis of $f_0 \in E'$ in the w^*-topology is formed by the sets

$$U(f_0, F, \varepsilon) = \{f \in E' : |f(x) - f_0(x)| < \varepsilon \text{ for all } x \in F\},$$

where $\varepsilon > 0$ and F is any finite subset of E.

We now collect some fundamental results about dual spaces and bounded linear operators.

Theorem A.2.3 (**Hahn-Banach**) *Let E be a normed space and F a (not necessarily closed) linear subspace of E. If f is a bounded linear functional on F, then there exists $g \in E'$ such that $g(x) = f(x)$ for all $x \in F$ and $\|g\| = \|f\|$.*

Corollary A.2.4 *If F is a linear subspace of E and x an element of E, which is not contained in the closure of F, then there exists $g \in E'$ such that $g|_F = \{0\}$ and $g(x) \neq 0$.*

Theorem A.2.5 (**Banach-Alaoglu**) *Let E be a normed space. Then the unit ball*

$$E'_1 = \{f \in E' : \|f\| \leq 1\}$$

of E' is w^*-compact.

However, E'_1 is compact in the norm topology only if E is finite-dimensional.

Corollary A.2.6 *If M is a w^*-closed linear subspace of E' and $f \in E' \setminus M$, then there exists $x \in E$ such that $f(x) \neq 0$ but $g(x) = 0$ for all $g \in M$.*

Theorem A.2.7 (**Closed graph theorem**) *Let E and F be Banach spaces, and let $T : E \to F$ be a linear map. Then the following conditions on T are equivalent.*

(*i*) *T is continuous.*

(*ii*) *The graph $G_T = \{(x, Tx) : x \in E\}$ of T is closed in $E \times F$.*

(*iii*) *If $x_n \to 0$ in E and $Tx_n \to y$ in F, then $y = 0$.*

Theorem A.2.8 (**Open mapping theorem**) *Let E and F be Banach spaces, and let $T : E \to F$ be a continuous linear mapping. If T is surjective, then T is open. In particular, if $T \in \mathfrak{B}(E, F)$ is bijective, then $T^{-1} \in \mathfrak{B}(F, E)$.*

Corollary A.2.9 *If a vector space E is a Banach space with respect to two norms, say $\|.\|_1$ and $\|.\|_2$, and if there is a constant c such that $\|x\|_2 \leq c\|x\|_1$ for all $x \in E$, then the two norms are equivalent, that is, there is a constant d such that $\|x\|_1 \leq d\|x\|_2$ for all $x \in E$.*

Theorem A.2.10 (**Uniform boundedness principle**) *Let E be a Banach space, F a normed space, and $\{T_\lambda : \lambda \in \Lambda\}$ a family of continuous linear maps from E into F. Suppose that $\{T_\lambda x : \lambda \in \Lambda\}$ is bounded in F for each $x \in E$. Then there exists a constant $C \geq 0$ such that $\|T_\lambda\| \leq C$ for all $\lambda \in \Lambda$.*

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