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Diffusive approximation of the Liouville equation

ausgeführt von

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Datum

Unterschrift

Ich erkläre hiermit an Eides statt, die folgende Arbeit selbstständig angefertigt zu haben. Die aus fremden Quellen übernommenen Ideen, Argumente und Aussagen sind als solche durch eine Quellenreferenz bzw. bei allgemein bekannten Resultaten durch den entsprechenden Namen kenntlich gemacht.

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Ort, Datum Unterschrift

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Contents

Introduction

The Liouville equation (1) is a linear partial differential equation and occurs in many different fields of physics.

$$f_t + v\, f_x - V'(x)\, f_v = 0 \tag{1}$$

In the field of kinetic equations it describes the behavior of a phase space density f under the influence of a potential $V(x)$ and could be considered as prototype for more advanced (nonlinear) partial differential equations as for example the Kramer-Fokker-Planck equation or the Boltzmann equation and Vlasov equation. Heuristically a phase space density f describes the evolution of an ensemble of particles. Considering the Liouville equation these particles cannot interact (e.g. through collisions). The other examples mentioned allow such interactions, which makes their treatment rather delicate in general.

The goal of this work is to investigate the following stationary problem on the phase plane

$$\begin{aligned} v\, f_x - V'(x)\, f_v &= 0 \quad (x,v) \in [0,1] \times \mathbb{R} \\ f|_{\{x=0 \wedge v>0\}} &= g_+ \\ f|_{\{x=1 \wedge v<0\}} &= g_- \end{aligned} \tag{2}$$

using a vanishing viscosity method. Constructing a solution using the method of characteristics is rather straight forward, but such a solution is not uniquely determined by the boundary conditions (see Chapter 1). Therefore we introduce an artificial diffusion term εf_{vv} which acts only on the velocity variable. This yields

$$v\, f_x^\varepsilon - V'(x)\, f_v^\varepsilon = \varepsilon f_{vv}^\varepsilon$$

which is a parabolic-elliptic degenerated [Hil70] partial differential equation.

Some similar turning point problems

$$v\,f_x - \varepsilon f_{vv} = 0$$
$$f|_{\{x=0 \wedge v>0\}} = g_+$$
$$f|_{\{x=1 \wedge v<0\}} = g_-$$

were considered quite some years ago in [BG68] and [Bea79]. Unfortunately the approaches of both references strongly rely on the symmetry (self adjointness) of the second order derivative and is therefore not directly applicable to the non symmetric differential operator $-V'(x)\partial_v - \varepsilon f_{vv}$. Another approach given in [GVdMP87] generalizes the results to accretive differential operators, but the problem is posed on a different domain. Using the notion of accretive operators in Krein space finally allows to solve (2) for linear potentials $V(x) = ax+b$ as described in [Ćur00] (see Chapter 3).

As already pointed out in [Deg86], each of these methods restricts itself more or less directly to bounded domains (in v direction). To overcome this problem, one would have to find suitable weighted spaces (for example weighted Sobolev spaces [KO84]) that allow Poincare type inequalities on unbounded domains. Unfortunately it is not straight forward to construct those spaces, since the weight has to be compatible to the equation (see [Vil09]).

Fortunately the function space framework presented in [BG68] allows us to prove existence, uniqueness and regularity of the solution to (2). At least on bounded domains we can establish all results modifying the proofs given in [BG68] and [Bea79].

Chapter 1

The Liouville equation

1.1 Hamiltonian dynamical systems and flows

The Liouville equation naturally occurs in Hamiltonian dynamics, where it is an expression derived from the continuity equation and the volume preserving property of a Hamiltonian (or more general divergence free) flow in phase space. Therefore we will shortly repeat some very basic concepts concerning flows.

Definition 1.1. A map $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n : (t, y) \mapsto \phi_t(y)$ which is differentiable with respect to t is called a complete flow if and only if it fulfills the following prerequisites:

1. $\phi_0(y) = y \quad \forall y \in \mathbb{R}^n$
2. $\phi_s \circ \phi_t = \phi_{s+t} \quad \forall s, t \in \mathbb{R}$

The flow naturally defines a vector field F by

$$F(y) = \frac{d}{dt}\phi_t(y)\Big|_{t=0}$$

and satisfies [Mei07, Lemma 4.1] the initial value problem

$$\begin{aligned} \frac{d}{dt}\phi_t(y_0) &= F \circ \phi_t(y_0) \\ \phi_0(y_0) &= y_0. \end{aligned}$$

Another very fundamental fact [Mei07, Lemma 4.2] enables us to interpret solutions of a system of differential equations as a flow.

Lemma 1.2. *Let $U \subseteq \mathbb{R}^n$ be open and let $F \in C^1(U, \mathbb{R}^n)$ such that there exist a solution to*

$$y'(t) = F(y(t))$$
$$y(0) = y_0$$

which stays in U i.e. $y(t, y_0) \in U$ for every $y_0 \in U$ and all $t \in \mathbb{R}$. Then $\phi_t(y_0) := y(t, y_0)$ is a complete flow.

Obviously to be a complete flow on some bounded region U is a very strong property since it tells a lot about the long time behavior. In many cases the solutions will not stay in the region. Therefore we have to relax the concept of a complete flow to a flow which is defined (with appropriate parametrization) for $t \in [0, T]$ or $t \in [0, T)$ and $U \subseteq \mathbb{R}$. Hereby the flow leaves the region U at

$$t = T := \inf\{t \in \mathbb{R}^+ : \phi_t(U) \not\subset U\}.$$

Lemma 1.3. *The Jacobian of the flow $D\phi_t(x)$ satisfies the matrix differential equation.*

$$\frac{d}{dt} D\phi_t(x) = DF(\phi_t(x)) D\phi_t(x), \quad D\phi_0(x) = I$$

Therefore it is a fundamental matrix of:

$$\dot{y}(t) = A(t)y(t)$$

Proof. [Mei07, Chapter 7.2, Page 250] □

Theorem 1.4 (Abel)**.** *Let $\Phi(t, t_0)$ be the fundamental matrix of the linear system*

$$\dot{y}(t) = A(t)y(t)$$

then a generalized Abel identity holds.

$$\det(\Phi(t, t_0)) = \exp \int_{t_0}^{t} \mathrm{tr} A(t)\, dt$$

Proof. [Mei07, Theorem 2.11] □

Corollary 1.5. *Let ϕ_t be the flow of the vector field $F \in C^1$ then the following identity holds.*

$$\det(D\phi_t(x)) = \exp \int_{0}^{t} \mathrm{tr} DF(\phi_t(x))\, dt$$

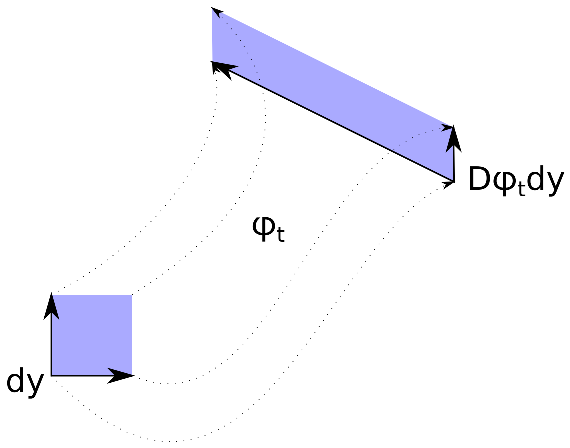

Figure 1.1: The image of an infinitesimal volume under a flow ϕ_t.

Corollary 1.6. *Let ϕ_t be a flow on $U \subseteq \mathbb{R}^n$ with associated vector field F and let $U_0 \subseteq \Omega$, $U_t := \phi_t(U_0) \subseteq U$ be bounded regions (see Corollary 1.1), then a flow is volume preserving i.e.*

$$\int_{U_0} dy = \int_{U_t} dy, \quad \forall t \in [0, T]$$

if only if

$$\text{div } F = \sum_{i=0}^{n} \partial_{y_i} F_i = \text{tr} DF = 0.$$

Proof. Using

$$\int_{U_t} dy = \int_{U_0} |\det D\phi_t(y)| dy$$

and the previous results, we directly get the statement. □

Now that we mentioned some basic properties of flows, we will consider systems of differential equations with special symplectic structure.

Definition 1.7. Let $U \subseteq \mathbb{R}^{2n}$ be open and let $H : U \to \mathbb{R}$. Let furthermore

$$\mathcal{J} = \begin{pmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{pmatrix}.$$

A system of differential equations in the form

$$\dot{y}(t) = \mathcal{J} \nabla H(y(t)) \tag{1.1}$$

is called Hamiltonian system with Hamiltonian H.

Remark 1.8. Hamiltonians often occur as energy functional of mechanical models and therefore it is very common to interpret the Hamiltonian as a functional $H(x,v) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, where $x \in \mathbb{R}^n$ denotes a (generalized) coordinate and $v \in \mathbb{R}^n$ a (generalized) velocity. With this notion we can reformulate (1.1) by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} \nabla_v H(x(t), v(t)) \\ -\nabla_x H(x(t), v(t)) \end{pmatrix}$$

Remark 1.9. Using the implicit function theorem we can find orbits (the graphs of the solutions) of $H(x(t), v(t)) = \mathit{const.}$ for every Hamiltonian system with $H \in C^1$. The nullclines of the system are exactly those points where the implicit function theorem fails.

Further investigating the volume preserving property we can easily conclude the following lemma for Hamiltonian flows.

Lemma 1.10. *Let ϕ_t be the Hamiltonian flow associated with $\mathcal{J}\nabla H \in C^2$, then ϕ_t is volume preserving and if H is not explicitly time dependent, then $H(\phi_t(x))$ is constant.*

Proof. Let $F = \mathcal{J}\nabla H$. Using Corollary 1.6 the following identity proves the first result.

$$0 = \nabla_x \nabla_v H(x,v) - \nabla_v \nabla_x H(x,v) = \mathrm{tr} D\mathcal{J}\nabla H = \mathrm{tr} DF$$

To conclude the second one we just differentiate $H(\phi_t(x))$ and get

$$\frac{d}{dt} H \circ \phi_t(x) = \nabla H(x)^T \dot{\phi}_t(x) = \nabla H(x)^T \mathcal{J} \nabla H(x) = 0$$

where the last term is zero due to the antisymmetric structure of $\mathcal{J}$. □

Theorem 1.11. *Let $U \subset \mathbb{R}^{2n}$ be a region in space and velocity i.e. $y = (x,v) \in U$ and let $t \in [0,T)$ be the time variable. Let $f : U \times [0,T) \to \mathbb{R}^+ : (x,v,t)^T \mapsto f(x,v,t)$ be a density function defined on a subset U of the phase space of a Hamiltonian system induced by $H \in C^2(U)$. H is supposed to have no critical points in U. Furthermore f and the vector field $F = \mathcal{J}\nabla H$ should satisfy the continuity equation on U.[1] Then the density function along trajectories is constant i.e. the Liouville equation holds.*

$$\frac{d}{dt} f(\phi_t(y), t) = \partial_t f(\phi_t(y), t) + \nabla_v H(\phi_t(y))^T \nabla_x f(\phi_t(y), t) - \nabla_x H(\phi_t(y), t)^T \nabla_v f(\phi_t(y), y) = 0$$

[1] Actually this is a very natural condition in many cases, especially as long as our flow describes a quantity that is conserved e.g. particles, fluids, etc.

Proof. Starting from the continuity equation

$$\frac{\partial}{\partial t} f(\phi_t(y), t) + \mathrm{div}(f(\phi_t(y), t) F(\phi_t(y))) = 0$$

we can use $F = \mathcal{J}\nabla H$. Now we apply the product rule and use the volume preserving property of the Hamiltonian vector field $\mathrm{div}\mathcal{J}\nabla H = 0$. This directly results in the required Liouville equation. □

From another point of view we can interpret the Liouville equation as partial differential equation with respect to f. This perfectly works on the given domain U because $\phi_t : U \to U$ is bijective due to the lack of fixed points. Since we will concentrate on stationary problems from now on we will do so, but only consider the time independent version.

$$\nabla_v H^T \nabla_x f - \nabla_x H^T \nabla_v f = 0 \tag{1.2}$$

Although in some cases the resulting first order linear partial differential equation is simple enough to calculate explicit solutions (which we will do in the following) we cannot provide those in general. Nevertheless the explict solutions will give us some insights about the kind of difficulties we have to deal with.

As last step we will need boundary conditions of some kind. The following definition will give us the possibility to impose boundary conditions that lead to a well posed problem.

Definition 1.12. Let $U \subset \mathbb{R}^n$ be a region with C^1 boundary and let $\nu(y) : \partial U \to \mathbb{S}^{n-1}$ be the inner normal vector of ∂U. Furthermore let $F : C^1(U)$ be a vector field, then we denote the inflow boundary by $\Gamma_{in} := \left\{ y \in \partial U : F(y)^T \nu(x, v) > 0 \right\}$ and the outflow boundary by $\Gamma_{out} := \left\{ y \in \partial U : F(y)^T \nu(x, v) < 0 \right\}$.

As the name suggests the inflow/outflow boundary is the part of the boundary where the trajectories of the Hamiltonian system run into/out of the domain and this leads to the idea of constructing solutions from such trajectories with initial values at the inflow boundary.

1.2 Method of characteristics

In the following we will write a (real valued) first order partial differential equation with Dirichlet boundary condition $u = g$ on $\Gamma \subseteq \partial U$ as

$$G(Du, u, y) = 0 \quad \forall y \in U$$
$$u(y) = g(y) \quad \forall y \in \Gamma$$

where $G : \mathbb{R}^n \times \mathbb{R} \times \bar{U} \to \mathbb{R} : (p, z, y) \to G(p, z, y)$, $U \subseteq \mathbb{R}^n$ is open and $u : \bar{U} \to \mathbb{R}$. Such an equation can be solved by the method of characteristics, whose idea is to reduce the partial differential equation to ordinary differential equations along paths through the domain. These paths are called the characteristics $y(s)$, with values $z(s) := u(y(s))$ and gradient $p(s) := \nabla u(y(s))$. The characteristic equations defining these are

$$\dot{p}(s) = -\nabla_y G(p(s), z(s), y(s)) - \nabla_z G(p(s), z(s), y(s))^T p(s) \tag{1.3}$$

$$\dot{z}(s) = \nabla_p G(p(s), z(s), y(s))^T p(s) \tag{1.4}$$

$$\dot{y}(s) = \nabla_p G(p(s), z(s), y(s)) \tag{1.5}$$

For motivation of those equations and details see [Eva10].

Definition 1.13. Let $G \in C^1(F : \mathbb{R}^n \times \mathbb{R} \times \bar{U}, \mathbb{R})$ then the triple (p_0, z_0, y_0) satisfies the noncharacteristic condition if and only if

$$G_p(p_0, z_0, y_0) \neq 0$$

Theorem 1.14 (local existence [Eva10])**.** *Let (p_0, z_0, y_0) satisfy the noncharacteristic condition then there exists an open $V \subset U$ and $u \in C^2(V)$ such that*

$$\begin{aligned} F(Du(y), u(y), y) = 0 \quad \forall y \in V \\ u(y) = g(y) \quad \forall y \in \Gamma \cap V \end{aligned}$$

From (1.5) we instantly see that the direction $\dot{y}(s)$ of the characteristic $y(s)$ at a point s depends only on $\nabla_p G(p, z, y)$. In the concrete example of the stationary Liouville equation one can observe that the direction just depends on the Hamiltonian i.e.

$$\begin{aligned} G(p, z, y) &= p^T \mathcal{J} \nabla H(y) = -p_x \nabla_v H(x, v) + p_y \nabla_x H(x, v) \\ G_p(p, z, y) &= \mathcal{J} \nabla H(y) = F(y) \end{aligned}$$

So, on one hand, it is obvious that we can use Definition 1.12 to define inflow and outflow boundaries for characteristics which satisfy the noncharacteristic condition and on the other hand we can conclude that solutions along the characteristics are constant by

$$\dot{z}(s) = G_p(p(s), z(s), y(s))^T p(s) = G(p(s), z(s), y(s)) = 0.$$

Actually we also see that in the case of the Liouville equation, the concepts of characteristics and the trajectories of the underlying Hamiltonian systems coincide because we could use the H invariant orbits to locally construct trajectories and moreover a flow.

Lemma 1.15. *Let U be a region in $\mathbb{R}^{2n}$, let $\psi(\cdot) := \begin{pmatrix} x(\cdot) \\ v(\cdot) \end{pmatrix} : [0,\tau] \to U$ be a solution to*

$$\begin{pmatrix} \dot{x}(s) \\ \dot{v}(s) \end{pmatrix} = \begin{pmatrix} \nabla_v H(x(s), v(s)) \\ -\nabla_x H(x(s), v(s)) \end{pmatrix}$$

and let $f : U \to \mathbb{R}$ be constant along ψ i.e. $f(x(s), v(s)) = const.$ for all $s \in [0, \tau]$ then f satisfies (1.2).

Proof.

$$\begin{aligned} 0 =& \frac{d}{ds}\left(f \circ \psi\right)(s) = \nabla f(\psi(s))^T \, \dot{\psi}(s) + \frac{\partial f}{\partial s}(s) = \nabla f(\psi(s))^T \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \nabla H(\psi(s)) \\ =& \nabla_v H(\psi(s)) \nabla_x f(\psi(s)) - \nabla_x H(\psi(s)) \nabla_v f(\psi(s)) \end{aligned}$$

□

1.3 Properties of the solution

We now will consider a very special kind of Hamiltonian system, the one dimensional harmonic oscillator[2] with quadratic potential $V(x)$.

$$H : \begin{cases} \mathbb{R}^2 \to \mathbb{R} \\ (x, v)^T \mapsto v^2 + V(x) \end{cases}$$

We are mainly interested in the problem induced by the Liouville equation on a slab $\Omega = [0,1] \times \mathbb{R}$. Here $\Gamma_+ := \{(0, v) : v > 0\}$ and $\Gamma_- := \{(1, v) : v < 0\}$ denote the inflow boundary $\Gamma_{in} = \Gamma_+ \cup \Gamma_-$, where we will impose boundary conditions.

$$\begin{aligned} v\, f_x - V'(x)\, f_v &= 0 \\ f|_{\Gamma_+} &= g_+ \\ f|_{\Gamma_-} &= g_- \end{aligned} \tag{1.6}$$

Since Theorem 1.14 provides us with a solution that is locally C^2, the question arises which kind of regularity we can expect on Ω and if the solution to the boundary value problem is unique. The following example shows that on one hand the solution is far from being unique and that

[2] Can also be written as $x' - V'(x) = 0$

we cannot even expect the solution to be continuous on Ω. Let therefore $V(x) = (x-\mu)^2$ be a quadratic potential. We can solve (1.6) explicitly and see that the characteristics are circles with center at $(0,\mu)$. If we set $g_+ \equiv 1$ and $g_- \equiv -1$ we can observe (in Equation 1.3) that flow areas of both boundaries touch each other, which leads to a discontinuity. Furthermore there is a region where the characteristics never run through a boundary condition and therefore we don't know how to choose the values in this region. Actually in this region every function would satisfy the equation and the boundary conditions, as long as it is constant along circles with center $(0,\mu)$. So our solution is not unique.

Obviously we cannot do much about the regularity, but we can try to pick a particular solution. For this reason we will try to find a viscosity solution, which is the limit of solutions of a viscous version of (1.6).

We choose to add a small diffusivity in velocity direction and therefore get (for $\varepsilon > 0$) a viscous Liouville equation.

$$v\, f_x - V'(x)\, f_v = \varepsilon f_{vv} \tag{1.7}$$

$$f|_{\Gamma_+} = g_+ \tag{1.8}$$

$$f|_{\Gamma_-} = g_- \tag{1.9}$$

Although we expect the solutions to be regularized, the preceding equation is not trivial to treat on bounded or unbounded domains. (1.6) is also known as the stationary linear Vlasov equation [Deg86].

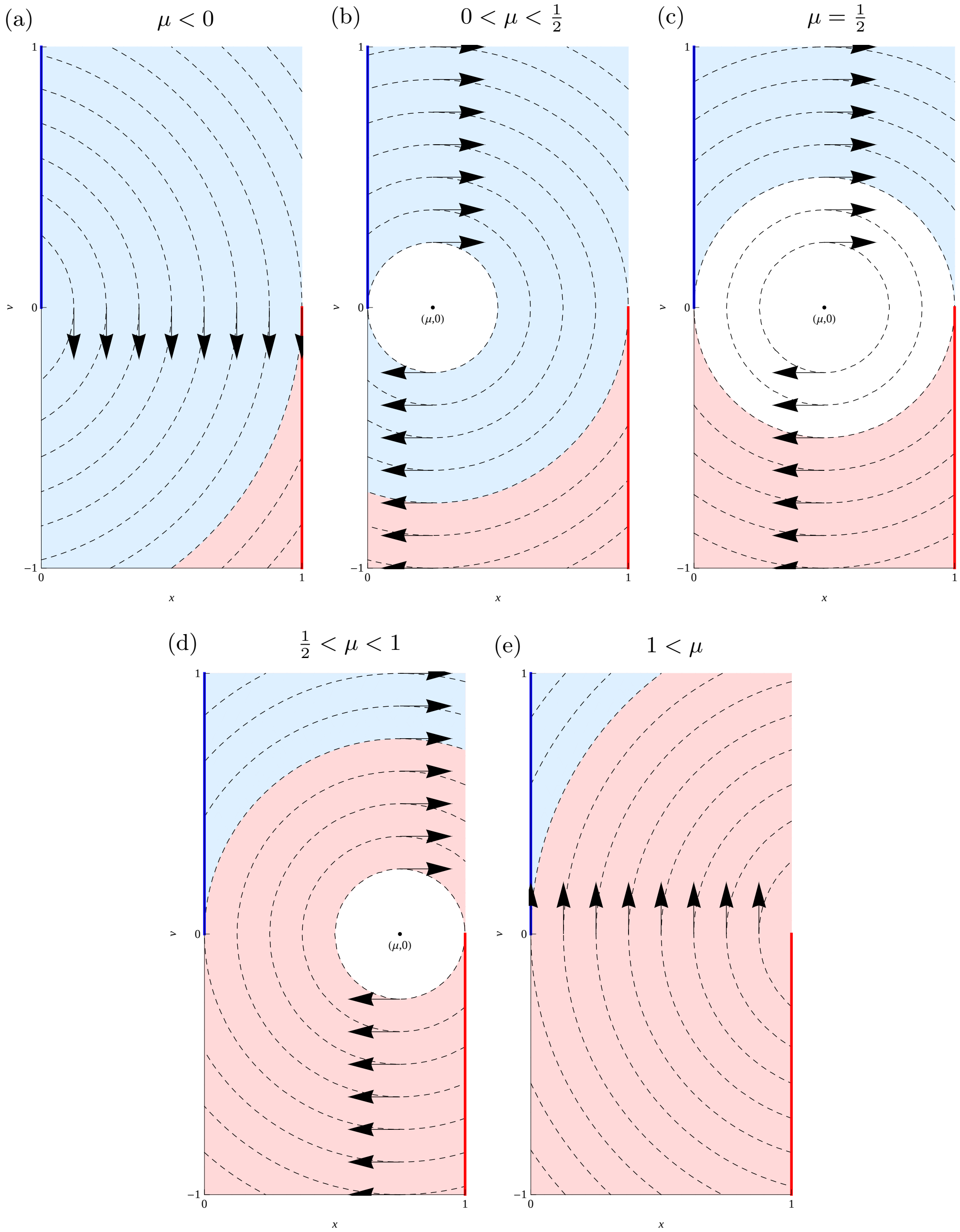

Figure 1.2: The dynamics corresponding to $H(x, v) = v^2 + (x - \mu)^2$. The blue boundary represents Γ_+ and the red one Γ_-. The areas reached by characteristics starting at the respective boundaries are colored accordingly. Characteristics in the white area are closed and do not reach the boundary and so the solution's value along these is not well defined.

Chapter 2

Velocity diffusive approximation on bounded domains

Suppose we have given boundary data g_+ and g_- with compact support and for example a quadratic potential $V(x) = (x-\mu)^2 + \nu$ then we can choose $a < 0 < b$ such that the solution of the Liouville equation, denoted by f has compact support $\operatorname{supp} f \subset \Omega_0 := [0,1] \times [a,b]$ by construction (see Theorem 1.14 and Figure 2.1).

Due to the degenerated parabolic-elliptic structure (see Definition A.8) we will see that on unbounded slab domains it is rather difficult to prove existence of a solution to (1.7). Mainly this fails because of the Poincare inequality does not hold, but since f vanishes outside Ω_0 for compactly supported boundary values (the cases we will treat), we will approximate f by

$$f_{\Omega_0,\varepsilon}(x,v) := \begin{cases} f_\varepsilon(x,v) & (x,v)^T \in \Omega_0 \\ 0 & (x,v)^T \in \Omega \setminus \Omega_0 \end{cases} \tag{2.1}$$

where f_ε is a solution to (1.7) on Ω_0 and additionally solves homogeneous Dirichlet boundary conditions on $\Gamma_0 = \partial\Omega_0 \setminus \partial\Omega$ (see Figure 2.1).

In the following we therefore restrict ourselves to the problem given by compactly supported g_- and g_+ and (1.7) on bounded domains.

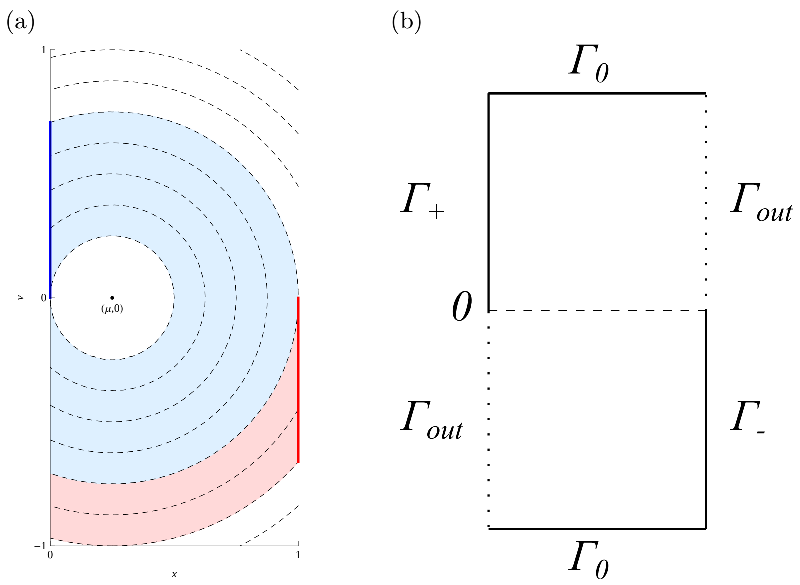

Figure 2.1: In (a) we can see that if the boundary data have compact support (the blue and the red line) then the solution has compact support too. In the illustrated example the support of the solutions stays in the compact set $[0,1] \times [-1,1]$. In (b) the notation for the boundary are illustrated. Hereby the bold lines mark the boundary, on which boundary conditions are imposed. (Equation 1.8 on Γ_+, Equation 1.9 on Γ_- and the homogeneous Dirichlet boundary condition on Γ_0.

2.1 A weighted L_2 space

To treat the multiplication of v onto the derivative in the parabolic variable we will use a weighted version of L^2. Choosing v as a weight would deliver us a Krein space, see Chapter 3 instead of a Hilbert space. Fortunately the closely related Hilbert space $L^2_{|v|}(J)$ enables us to use some modified results from the theory of Lebesgue-Bochner spaces and we therefore do not need a Krein space approach.

Definition 2.1. Let $w : \Omega \to \mathbb{R}_0^+$ be a Lebesgue measurable function, then we introduce a real valued weighted L^p space by:

$$L^p(\Omega, w) := \left\{ f \text{ measurable on } \Omega : \|f\|_{L^p(\Omega,w)} := \sqrt[p]{\int_\Omega |f(x)|^p \, w(x) \, dx} < \infty \right\} \tag{2.2}$$

Furthermore we define $L^2_{|v|}(\Omega) := L^2(\Omega, |id|)$.

Remark 2.2. Since $|v|dx$ is a σ-finite Borel measure (on the real line) $L^2_{|v|}(\Omega)$ is a Hilbert space (see for example [RS72], II.1 example 4) with inner product

$$(f,g)_{L^2_{|v|}(\Omega)} = \left(\mathrm{id}_{|v|} f, g\right)_{L^2(\Omega)}. \tag{2.3}$$

On bounded domains Ω the inclusion $L^2(\Omega) \subset L^2_{|v|}(\Omega)$ holds. Hereby the boundedness of Ω is necessary since in the unbounded case we could easily construct a counter example by $f(x) := \frac{1}{x}$ on the interval $[1, \infty)$.

$f \in L^2([1,\infty))$ since

$$\|f\|^2_{L^2([1,\infty))} = \int_1^\infty \frac{1}{x^2}\, dx = 1,$$

but due to

$$\|f\|^2_{L^2_{|v|}([1,\infty))} = \int_1^\infty \frac{1}{x^2}|x|\, dx = \infty$$

f is not in $L^2_{|v|}([1,\infty))$.

Lemma 2.3. *Let $K \subset \mathbb{R}^n$ be compact such that there exist c_1 and c_2 with $0 < c_1 < w(x) < c_2 \quad \forall x \in K$. Then $\|\cdot\|_{L^p(K)} \cong \|\cdot\|_{L^p(K,w)}$*

Proof.

$$\begin{aligned}
(c_1\, \lambda^n(K))^{\frac{1}{p}}\, \|f\|_{L^p(K)} &= \left(c_1\, \lambda^n(K) \int_K |f|^p\, dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_K |f|^p\, w dx\right)^{\frac{1}{p}} = \|f\|_{L^p(K,w)} \\
&\leq \left(c_2\, \lambda^n(K) \int_K |f|^p\, dx\right)^{\frac{1}{p}} = (c_1\, \lambda^n(K))^{\frac{1}{p}}\, \|f\|_{L^p(K,w)}
\end{aligned}$$

□

Theorem 2.4. *The infinitely often differentiable functions with compact support, denoted by $C_0^\infty(\mathbb{R}^\pm)$ are dense in $L^2_{|v|}(\mathbb{R}^\pm)$.*

Proof. We can approximate $f \in L^2_{|v|}(\mathbb{R}^+)$ by $f_n := \mathbb{1}_{[\frac{1}{n},n]} f$. This sequence converges to f, pointwise almost everywhere, since with $n_0 := \lceil \max\left(x, \frac{1}{x}\right) \rceil$ we see:

$$\forall x \in \mathbb{R}^+\ \forall \varepsilon > 0\ \exists n_0 : |f - f_n| < \varepsilon \quad a.e. \quad \forall n \geq n_0$$

Since each f_n is measurable and absolutely dominated by f we can apply the dominated convergence theorem and get

$$\forall \varepsilon > 0\ \exists n_0 : \|f_n - f\|_{L^2_{|v|}(\mathbb{R})} < \varepsilon \quad \forall n \geq n_0$$

Every f_n is in $L^2_{|v|}\left(\left[\frac{1}{n}, n\right]\right)$ and by lemma 2.3 also in $L^2\left(\left[\frac{1}{n}, n\right]\right)$.Therefore we can approximate f_n by a sequence $(f_{n,m})_{m\in\mathbb{N}} \subset C_c^\infty\left(\left[\frac{1}{n}, n\right]\right)$ for each $n \in \mathbb{N}$ as classical L^2 function [Wer05]. Using lemma 2.3 again we get convergence in $L^2_{|v|}\left(\left[\frac{1}{n}, n\right]\right)$ for all $n \in \mathbb{N}$ i.e.

$$\forall n \in \mathbb{N} \quad \forall \varepsilon > 0\ \exists m_0 : \|f_{n,m} - f_n\|_{L^2_{|v|}([\frac{1}{n},n])} < \varepsilon \quad \forall m \geq m_0$$

To approximate $f \in L^2_{|v|}(\mathbb{R}^+)$ we use a diagonal sequence defined by $\tilde{f}_n := f_{n,n} \in C_c^\infty\left(\left[\frac{1}{n}, n\right]\right) \subset C_c^\infty(\mathbb{R}^+)$. Then for all ε there exists a n_0 such that the following estimate holds.

$$\begin{aligned}
\|f - \tilde{f}_n\|_{L^2_{|v|}(\mathbb{R}^+)} =& \|f - f_n + f_n - f_{n,n}\|_{L^2_{|v|}(\mathbb{R}^+)} \\
\leq& \|f - f_n\|_{L^2_{|v|}(\mathbb{R}^+)} + \|f_n - f_{n,n}\|_{L^2_{|v|}(\mathbb{R}^+)} \\
=& \|f - f_n\|_{L^2_{|v|}(\mathbb{R}^+)} + \|f_n - f_{n,n}\|_{L^2_{|v|}([\frac{1}{n}, n])} < \varepsilon \quad n \geq n_0
\end{aligned}$$

Therefore we find a sequence of smooth and compactly supported functions that converge in $L^2_{|v|}(\mathbb{R}^+)$ against f. For the negative real axis we use exactly the same method of proof. □

Corollary 2.5. *Let $J \subseteq \mathbb{R}$. The functions $C^\infty_{0,0}(J) := \{f \in C^\infty_0(J) : f(0) = 0\}$ are dense in $L^2_{|v|}(J)$.*

2.2 A modified Sobolev-Bochner space

Let us use the real valued, reflexive and separable Hilbert space $L^2(I, H^1_0(J))$ (see Theorem B.10) and its dual space $L^2(I, H^{-1}(J))$ equipped with the dual norm (see [Bea79], p.4)

$$\|f\|_{L^2(I,H^{-1}(J))} := \sup_{\|g\|_{L^2(I,H^1_0(J))}=1} |\langle f, g\rangle|.$$

We always consider $x \in I := [0,1]$ to be the Lebesgue-Bochner space variable and $v \in J := [a,b]$ with $a < 0 < b$ to be the Sobolev space variable. Furthermore let $J_- := [a,0]$ and $J_+ := [0,b]$. The evaluation of a function $f \in L^2(0,T,H^1_0(J))$ at a point (x,v) is denoted by $f(x,v) := f(x)(v)$ (which makes sense since $f(x) \in H^1_0(J)$ *a.e.* in $[0,1]$ and therefore there exist a $\hat{f} \in C(J)$ with $\hat{f} = f$ *a.e.*). The weak derivatives f_x, f_v with respect to x and v are thought to be in the sense of Definition B.17 and $H^1_0(J)$ respectively. To be more specific we use $(f_x)(x,v) := f'(x)(v)$ and $(f_v)(x,v) := (f(x))'(v)$.

Lemma 2.6. *Let $J \subset \mathbb{R}$ be a bounded interval. Let $|f|_{L^2(I,H^1_0(J))} := \|f_v\|_{L^2(I,L^2(J))}$, then $|\cdot|_{L^2(I,H^1_0(J))}$ and $\|\cdot\|_{L^2(I,H^1_0(J))}$ are equivalent on $L^2(I,H^1_0(J))$ i.e. there exist $\alpha, \beta > 0$ such that*

$$\alpha |f|^2_{L^2(I,H^1_0(J))} \leq \|f\|^2_{L^2(I,H^1_0(J))} \leq \beta |f|^2_{L^2(I,H^1_0(J))} \quad \forall f \in L^2(I,H^1_0(J))$$

Proof. The first estimate is trivial since we just add a positive term.

$$|f|^2_{L^2(I,H^1_0(J))} = |f_v|^2_{L^2(I,L^2(J))} \leq \|f_v\|^2_{L^2(I,L^2(J))} + \|f\|^2_{L^2(I,L^2(J))} = \|f\|^2_{L^2(I,H^1_0(J))}$$

The other direction can be obtained by the Poincaré inequality given by Lemma B.21. □

Lemma 2.7. *Let $J := [a, b]$ and let $\Phi := C^1([0,1], H_0^1(J))$ be the continuously differentiable functions with images in $H_0^1(J)$. Furthermore define*

$$\|\phi\|_\Phi := \|\phi\|_{L^2(I,H_0^1(J))} + \|\phi(0)\|_{L^2_{|v|}(J^+)} + \|\phi(1)\|_{L^2_{|v|}(J^-)}. \tag{2.4}$$

Then $(\Phi, \|\cdot\|_\Phi)$ is a normed space which is continuously embedded in $L^2(I, H_0^1(J))$ i.e.

$$\|\phi\|_{L^2(I,H_0^1(J))} \le c\|\phi\|_\Phi \quad \forall \phi \in \Phi$$

and $c = 1$.

Proof. $\|\phi\|_{L^2(I,H_0^1(J))}$ is a norm and the other two terms in (2.4) are seminorms. Therefore their sum is a norm. Since seminorms are non negative we get

$$\|\phi\|_{L^2(I,H_0^1(J))} \le \|\phi\|_{L^2(I,H_0^1(J))} + \|\phi(0)\|_{L^2_{|v|}(J^+)} + \|\phi(1)\|_{L^2_{|v|}(J^-)} = \|\phi\|_\Phi \quad \forall \phi \in \Phi.$$

□

We also consider the subspace $\Phi_0 \subset \Phi$ whose elements $\phi \in \Phi_0$ satisfy $\phi|_{\Gamma_{out}} = 0$.

Next we introduce

$$\mathcal{B} := \left\{ f \in L^2(I, H_0^1(J)) : \mathrm{id}_v f_x \in L^2(I, H^{-1}(J)) \right\}$$

equipped with the norm

$$\|f\|_{\mathcal{B}}^2 := \|f\|^2_{L^2(I,H_0^1(J))} + \|\mathrm{id}_v f_x\|^2_{L^2(I,H^{-1}(J))}.$$

Before we go into more details with this Sobolev-Bochner space we need a technical lemma.

Lemma 2.8 ([Bea79])**.** *Let $f \in C^1(I, H_0^1(J))$ and let there exist $x_0 \in I$ such that $f(x_0) = 0$. Then the following inequality holds.*

$$\sup_{x \in I} \|f(x)\|^2_{L^2_{|v|}(J)} \le 2\, \|\mathrm{id}_v f_x\|_{L^2(I,L^2(J))}\, \|f\|_{L^2(I,L^2(J))}$$

Proof. We use the Cauchy-Schwarz inequality, the differentiability of the parameter integral on J and the definition of the $L^2(J)$ norm to conclude

$$\begin{aligned}
\frac{d}{dx}\, \|f(x)\|^2_{L^2_{|v|}} \le& \left| \frac{d}{dx} \left(f(x), f(x)\right)_{L^2_{|v|}(J)} \right| = \left| \frac{d}{dx} \left(|\mathrm{id}_v| f(x), f(x)\right)_{L^2(J)} \right| \\
=& 2 \left| \left(|\mathrm{id}_v| f_x(x), f(x)\right)_{L^2(J)} \right| \\
\le& 2\, \||\mathrm{id}_v| f_x(x)\|_{L^2(J)}\, \|f(x)\|_{L^2(J)} = 2\, \|\mathrm{id}_v f_x(x)\|_{L^2(J)}\, \|f(x)\|_{L^2(J)}
\end{aligned}$$

Integrating this inequality from x_0 to x and using the Cauchy-Schwarz inequality yield the required statement.

$$
\begin{aligned}
\|f(x)\|^2_{L^2_{|v|}} = \|f(x)\|^2_{L^2_{|v|}} - \|f(x_0)\|^2_{L^2_{|v|}} \leq & 2\int_{x_0}^{x} \|\mathrm{id}_v f_s(s)\|_{L^2(J)} \|f(s)\|_{L^2(J)}\, ds \\
\leq & 2\int_{x_0}^{x} \|\mathrm{id}_v f_s(s)\|^2_{L^2(J)}\, ds \int_{x_0}^{x} \|f(s)\|^2_{L^2(J)}\, ds \\
\leq & 2\int_{I} \|\mathrm{id}_v f_s(s)\|^2_{L^2(J)}\, ds \int_{I} \|f(s)\|^2_{L^2(J)}\, ds
\end{aligned}
$$

□

The following lemma delivers most of the technical preliminaries needed for the further investigation and is of the same structure as Theorem B.20.

Lemma 2.9 ([Bea79],[Rou13]). *Let $f \in \mathcal{B}$ and let $I = [0,1]$.*

1. *$C^1(I, H^1_0(J)) = \Phi$ is densely embedded in $\mathcal{B}$.*
   More specifically there is a sequence $(f_n)_{n\in\mathbb{N}} \subset C^1(I, H^1_0(J))$ converging to f i.e.

$$
\lim_{n\to\infty} \|f - f_n\|_{L^2(I,H^1_0(J))} = 0 \tag{2.5}
$$

$$
\lim_{n\to\infty} \left\| \mathrm{id}_v f_x - \mathrm{id}_v \frac{d}{dx} f_n \right\|_{L^2(I,H^{-1}(J))} = 0 \tag{2.6}
$$

2. *$\mathcal{B}$ is embedded in $C(I, L^2_{|v|}(J))$ and the previous sequence $(f_n)_{n\in\mathbb{N}} \subset C^1(I, H^1_0(J))$ additionally converges to f the sense of*

$$
\lim_{n\in\mathbb{N}} \|f - f_n\|_{C(I,L^2_{|v|}(J))} = 0.
$$

3. *The derivative of the norm satisfies*

$$
\begin{aligned}
&\frac{d}{dx}\left(f(x), f(x)\right)_{L^2_v(J)} \\
&= \langle f(x), \mathrm{id}_v\, f_x(x)\rangle_{H^1_0(J)\times H^{-1}(J)} + \langle \mathrm{id}_v\, f_x(x), f(x)\rangle_{H^{-1}(J)\times H^1_0(J)}
\end{aligned}
\qquad a.e. \text{in I.} \tag{2.7}
$$

4. *Let f and $g \in \mathcal{B}$ then integration by parts yields*

$$
\begin{aligned}
&\langle \mathrm{id}_v\, f_x, g\rangle_{L^2(I,H^{-1}(J))\times L^2(I,H^1_0(J))} + \langle f, \mathrm{id}_v\, g_x\rangle_{L^2(I,H^{-1}(J))\times L^2(I,H^1_0(J))} \\
&= \int_J v\, f(1,v) g(1,v)\, dv - \int_J v\, f(0,v) g(0,v)\, dv.
\end{aligned} \tag{2.8}
$$

If $f(x)$ and $g(x)$ are real valued functions for a.e $x \in I$ we additionally get

$$
\langle g, \mathrm{id}_v\, f_x\rangle_{L^2(I,H^1_0(J))\times L^2(I,H^{-1}(J))} = \langle \mathrm{id}_v\, f_x, g\rangle_{L^2(I,H^{-1}(J))\times L^2(I,H^1_0(J))}\,.
$$

Proof. 1. Let

$$\rho_\varepsilon := \begin{cases} \frac{1}{N\varepsilon} e^{\frac{x^2}{x^2-\varepsilon^2}} & x \in (-\varepsilon, \varepsilon) \\ 0 & x \in \mathbb{R} \setminus (-\varepsilon, \varepsilon) \end{cases}$$

where $N := \int_{(-\varepsilon,\varepsilon)} e^{\frac{s^2}{s^2-\varepsilon^2}}\, ds$. Using the smooth function $s \mapsto \rho_\varepsilon(x + \xi_\varepsilon(x) - s)$ with $\xi_\varepsilon(x) := \varepsilon(1-2x)$ and $K := \operatorname{supp} \rho_\varepsilon = [-\varepsilon + x + \xi_\varepsilon(x), +\varepsilon + x + \xi_\varepsilon(x)]$ we define

$$f_\varepsilon(x) := \int_I \rho_\varepsilon(x + \xi_\varepsilon - s) f(s)\, ds$$

which is in $C^1(I, H_0^1(J))$ since we can apply Proposition B.12. Using Lemma B.15, the transformation rule for Lebesgue integrals and the Chauchy-Schwarz inequality we get with $\hat{K} = [\xi_\varepsilon(x) - \varepsilon, \xi_\varepsilon(x) + \varepsilon]$

$$\begin{aligned}
\|f_\varepsilon - f\|^2_{L^2(I,H_0^1(J))} &= \int_I \left\| \int_K \rho_\varepsilon(x+\xi_\varepsilon - s) f(s)\, ds - \int_K \rho_\varepsilon(x + \xi_\varepsilon - s) f(x)\, ds \right\|^2_{H_0^1(J)} dx \\
&\leq \int_I \left[\int_{\hat{K}} \rho_\xi(\xi_\varepsilon(x) - s) \|f(x+s) - f(x)\|_{H_0^1(J)}\, ds \right]^2 dx \\
&\leq \int_{[-\varepsilon,\varepsilon]} \rho_\varepsilon(s)^2\, ds \int_I \int_{\hat{K}} \|f(x+s) - f(x)\|^2_{H_0^1(J)}\, ds\, dx \\
&\leq \frac{C}{\varepsilon} \int_I \int_{\hat{K}} \|f(x+s) - f(x)\|^2_{H_0^1(J)}\, ds\, dx \\
&\leq 2C \sup_{|s|<2\varepsilon} \int_I \|(\iota f)(x+s) - (\iota f)(x)\|^2_{H_0^1(J)}\, dx \\
&= 2C \sup_{|s|<2\varepsilon} \|(\iota f)_s - \iota f\|^2_{L^2(I,H_0^1(J))}
\end{aligned}$$

In the last inequality we use

$$(\iota f)(x) := \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I \end{cases}$$

as introduced in Lemma B.16. Due to the uniform continuity of $s \mapsto (\iota f)_s$ provided by Lemma B.16 we finally conclude (2.5) by $\lim_{\varepsilon\to 0} \|f_\varepsilon - f\|_{L^2(I,H_0^1(J))}$.

By the same strategy we first estimate

$$\begin{aligned}
&\left\| (1-2\varepsilon)^{-1} \mathrm{id}_v \frac{d}{dx} f_\varepsilon - \mathrm{id}_v f_x \right\|^2_{L^2(I,H^{-1}(J))} \\
&= \int_I \left\| \int_K \rho'_\varepsilon(x + \xi_\varepsilon - s) \mathrm{id}_v f(s)\, ds - \int_K \rho_\varepsilon(x+\xi_\varepsilon - s) \mathrm{id}_v f_x(x)\, ds \right\|^2_{H^{-1}(J)} dx \\
&= \int_I \left\| \int_K \rho_\varepsilon(x + \xi_\varepsilon - s) \mathrm{id}_v f_x(s)\, ds - \int_K \rho_\varepsilon(x+\xi_\varepsilon - s) \mathrm{id}_v f_x(x)\, ds \right\|^2_{H^{-1}(J)} dx \\
&\leq 2C \sup_{|s|<\varepsilon} \|\mathrm{id}_v\left((\iota f_x)(\cdot + s) - \iota f_x\right)\|^2_{L^2(I,H^{-1}(J))}
\end{aligned}$$

and subsequently use

$$\begin{aligned}
&\|\mathrm{id}_v \frac{d}{dx} f_\varepsilon - \mathrm{id}_v f_x\|_{L^2(I,H^{-1}(J))} \\
&\leq \|\mathrm{id}_v \frac{d}{dx} f_\varepsilon - (1-2\varepsilon)^{-1}\mathrm{id}_v \frac{d}{dx} f_\varepsilon\|_{L^2(I,H^{-1}(J))} + \|(1-2\varepsilon)^{-1}\mathrm{id}_v \frac{d}{dx} f_\varepsilon - \mathrm{id}_v f_x\|_{L^2(I,H^{-1}(J))} \\
&\leq \frac{2\varepsilon}{1-2\varepsilon} \|\mathrm{id}_v \frac{d}{dx} f_\varepsilon\|_{L^2(I,H^{-1}(J))} + \sqrt{2C} \sup_{|s|<\varepsilon} \|\mathrm{id}_v \left((\iota f_x)(\cdot + s) - \iota f_x\right)\|_{L^2(I,H^{-1}(J))}
\end{aligned}$$

and Lemma B.16 to conclude convergence in the sense of (2.6).

2. Let $(f_n)_{n\in\mathbb{N}} \subset C^1(I, H_0^1(J))$ be the sequence constructed in the previous point. The convergence in $L^2(I, H_0^1(J))$ implies the existence of a subsequence f_j that converges *a.e.* in I. Now fix $x_0 \in I$ such that in particular $\lim_{j\to\infty} \|f_j(x_0) - f(x_0)\|_{L^2(J)}$. Interpreting $f_{j,0}(x) := f_j(x_0)$ as element of $L^2_{|v|}(J)$ which is constant in x, we estimate

$$\begin{aligned}
\sup_{x\in I} \|f_j(x) - f_k(x)\|^2_{L^2_{|v|}(J)} \leq &\sup_{x\in I} \|(f_j(x) - f_{j,0}(x)) - (f_k(x) - f_{k,0}(x))\|^2_{L^2_{|v|}(J)} \\
&+ \sup_{x\in I} \|f_{j,0}(x) - f_{k,0}(x)\|^2_{L^2_{|v|}(J)}
\end{aligned}$$

and on v-bounded domains J the following inequality is trivial to prove.

$$\|f\|_{L^2_{|v|}(J)} \leq C(J)\|f\|_{L^2(J)}. \tag{2.9}$$

This and Lemma 2.8 applied to $(f_k(x) - f_{k,0}(x)) - (f_j(x) - f_{j,0}(x))$ (it obviously attends zero at x_0) lead to

$$\begin{aligned}
\sup_{x\in I} \|f_j(x) - f_k(x)\|^2_{L^2_{|v|}(J)} \leq &2\|\mathrm{id}_v(f_j - f_k)_x\|_{L^2(I,L^2(J))}\|(f_k - f_{j,0}) - (f_j - f_{j,0})\|^2_{L^2(I,L^2(J))} \\
&+ C(J)^2\|f_j(x_0) - f_k(x_0)\|_{L^2(J)}.
\end{aligned}$$

Since $f_k - f_j$ is convergent in $L^2(I, L^2(J)) \subset L^2(I, H_0^1(J))$ and therefore a Cauchy sequence in $L^2(I, L^2(J))$, we deduce that $f_k - f_j$ is a Cauchy sequence in $C(I, L^2_{|v|}(J))$ by the last estimate. $f_k \in C^1(I, L^2(J))$ and convergence a.e. in $L^2(J)$ therefore implies pointwise convergence of $\frac{d}{dx} f_k$ in $L^2(J)$. Finally, the pointwise Cauchy property in $L^2_{|v|}(J)$ follows by (2.9) and since $C(I, L^2_{|v|}(J))$ is complete the Cauchy sequence converges to a limit in $C(I, L^2_{|v|}(J))$. In metric spaces the pointwise limit and the uniform limit (if it exists) are the same. We proved that $f \in C(I, L^2_{|v|}(J))$ with $\lim_{j\to\infty} \|f_j - f\|_{C(I,L^2_{|v|})} = 0$.

3. Using the sequence $(f_n)_{n\in\mathbb{N}}$ of the first point, which is converging to f in the sense of (2.5) and (2.6) every f_n is continuously differentiable in x and therefore we can apply the

product rule for the L^2 product.

$$\int_{[x_0,x_1]} \left\langle \mathrm{id}_v \frac{d}{dx} f_n(x), f_n(x) \right\rangle_{H^{-1}(J)\times H_0^1(J)} + \left\langle f_n(x), \mathrm{id}_v \frac{d}{dx} f_n(x) \right\rangle_{H_0^1(J)\times H^{-1}(J)} dx$$
$$= \int_{[x_0,x_1]} \left(\mathrm{id}_v \frac{d}{dx} f_n(x), f_n(x) \right)_{L^2(J)} + \left(f_n(x), \mathrm{id}_v \frac{d}{dx} f_n(x) \right)_{L^2(J)} dx$$
$$= \int_{[x_0,x_1]} \frac{d}{dx} \left(\mathrm{id}_v f_n(x), f_n(x)\right)_{L^2(J)} dx = \left(\mathrm{id}_v f_n(x_1), f_n(x_1)\right)_{L^2(J)} - \left(\mathrm{id}_v f_n(x_0), f_n(x_0)\right)_{L^2(J)}$$

As previously stated f_n also converge in $C(I, L^2_{|v|}(J))$ and this implies the convergence of the right hand side. The left hand side converges because

$$\left| \int_{[x_0,x_1]} \left\langle \mathrm{id}_v \frac{d}{dx} f_n(x), f_n(x) \right\rangle_{H^{-1}(J)\times H_0^1(J)} - \langle \mathrm{id}_v f_x(x), f(x) \rangle_{H^{-1}(J)\times H_0^1(J)} \, dx \right|$$
$$\leq \left| \left\langle \mathrm{id}_v \frac{d}{dx} f_n(x), f_n(x) - f(x) \right\rangle \right| + \left| \left\langle \mathrm{id}_v (\frac{d}{dx} f_n(x) - f_x(x)), f(x) \right\rangle \right|$$
$$\leq \left\| \mathrm{id}_v \frac{d}{dx} f_n \right\|_{L^2(H^{-1}(J))} \|f_n - f\|_{L^2(H_0^1(J))} + \left\| \mathrm{id}_v f_x - \mathrm{id}_v \frac{d}{dx} f_n \right\|_{L^2(I,H^{-1}(J))} \|f\|_{L^2(I,H_0^1(J))},$$

where the duality products without index act on $L^2([x_0,x_1], H^{-1}(J)) \times L^2([x_0,x_1], H_0^1(J))$.

Now we can differentiate almost everywhere in I with respect to x_1 since

$$x_1 \mapsto \int_{[x_0,x_1]} \left\langle \mathrm{id}_v \frac{d}{dx} f_n(x), f_n(x) \right\rangle_{H^{-1}(J)\times H_0^1(J)} + \left\langle \mathrm{id}_v \frac{d}{dx} f_n(x), f_n(x) \right\rangle_{H^{-1}(J)\times H_0^1(J)} dx$$

is absolutely continuous. The fundamental theorem of calculus for the Lebesgue integral delivers the statement.

4. The same proof as in the third point delivers the statement. Hereby we have to chose $x_0 := 0$ and $x_1 := 1$ and skip the differentiation.

□

2.3 Existence

In this chapter we will introduce the methods of [BG68], and [Bea79]. As long as we restrict ourselves to bounded domains in v, (1.7) is perfectly tractable with those methods.

We introduce our domain as finite slab $\Omega := I^\circ \times J^\circ$, where $I := [0,1]$ and $J := [a,b]$ are closed bounded intervals as in the previous section with $J^+ := [0,a]$ and $J^- := [0,b]$. Now we also consider the real valued spaces introduced in Chapter 2.1 and Chapter 2.2 and recall the inhomogeneous version of (1.7)

$$vf_x(x,v) - V'(x)f_v(x,v) - \varepsilon f_{vv}(x,v) = h(x,v) \tag{2.10}$$

$$f|_{\Gamma_+} = g_+ \tag{2.11}$$

$$f|_{\Gamma_-} = g_- \tag{2.12}$$

for arbitrary $V \in C^1(I)$ and $g := \chi_{J^+} g_+ + \chi_{J^-} g_-$.

For smooth functions f and h we can introduce a weak formulation of (2.10)-(2.12) by multiplying with a smooth function $\phi \in \Phi_0$ and integration by parts.

$$\begin{aligned}
&\int_I \int_J v f \phi_x - V'(x)\, f\phi_v - \varepsilon\, f_v \phi_v \, dv\, dx \\
&= \int_0^b v g^+(v)\phi(0,v)dv - \int_a^0 v g_-(v)\phi(1,v)dv - \langle h, \phi\rangle_{L^2(I,H^{-1}(J))\times L^2(I,H_0^1(J))} \quad \forall \phi \in \Phi_0.
\end{aligned}$$

Starting from this distributional formulation we extend our function space and define bilinear map $B : L^2(I, H_0^1(J)) \times \Phi_0 \to \mathbb{R}$ and linear map $L : \Phi_0 \to \mathbb{R}$ by

$$\begin{aligned}
B(f,\phi) &:= \int_I -\left(\mathrm{id}_v f(x), \phi_x(x)\right)_{L^2(J)} + \left(V'(x)\, f(x), \phi_v(x)\right)_{L^2(J)} + \varepsilon \left(f_v, \phi_v\right)_{L^2(J)} \, dx \\
L(\phi) &:= -\int_{J_+} v g^+(v)\phi(0,v)dv + \int_{J_-} v g_-(v)\phi(1,v)dv + \langle h, \phi\rangle_{L^2(I,H^{-1}(J))\times L^2(I,H_0^1(J))}
\end{aligned}$$

So we want to find $f \in L^2(I, H_0^1(J))$ satisfying,

$$B(f,\phi) = L(\phi) \quad \forall \phi \in \Phi_0 \tag{2.13}$$

Now two questions arise. First of all we obviously want to know if there exist such a solution $f \in L^2(I, H_0^1(J))$ or maybe even $f \in W^{2,2}(I, H_0^1(J), H^{-1}(J))$. The other one concerns the boundary conditions. Since a L^2 function does not allow point evaluation in general, we have to clarify if and in which sense the boundary conditions (2.11)-(2.12) are fulfilled by a solution to (2.13).

Lemma 2.10. *For every $\phi \in \Phi$ we get*

$$\left(\phi_v(x), \phi(x)\right)_{L^2(J)} = 0 \quad \forall x \in I.$$

More generally

$$\mathrm{Re}\ \left(\phi_v(x), \phi(x)\right)_{L^2(J)} = 0 \quad x \in I$$

holds for a complex valued version of Φ.

Proof. Integration by parts yields

$$(\phi_v(x), \phi(x))_{L^2(J)} = -(\phi(x), \phi_v(x))_{L^2(J)}$$

and since

$$2(\phi_v(x), \phi(x))_{L^2(J)} = 2\mathrm{Re}(\phi_v(x), \phi(x))_{L^2(J)} = (\phi_v(x), \phi(x))_{L^2(J)} + (\phi(x), \phi_v(x))_{L^2(J)} = 0$$

we proved the statement. □

Theorem 2.11. *Let $h \in L^2(I, H^{-1}(J))$. If $g \in L^2_{|v|}(J)$ then problem (2.13) has at least one solution $f \in L^2(I, H^1_0(J))$ such that*

$$B(f, \phi) = L(\phi) \quad \forall \phi \in \Phi_0.$$

Proof. We would like to use Corollary A.14 and therefore have to check its requirements.

As we showed in Chapter 2.1 and Theorem B.10, Φ_0 is a normed space equipped with the norm $\|\cdot\|_\Phi$ and $L^2(I, H^1_0(J))$ a Hilbert space equipped with the inner product $(\cdot,\cdot)_{L^2(I,H^1_0(J))}$. Furthermore Φ_0 is continuously embedded in $L^2(I, H^1_0(J))$ (see Lemma 2.7). $B(\cdot,\cdot)$ obviously is a bilinear map. For the induced linear map $B(\cdot, \phi)$ we prove continuity on $L^2(I, H^1_0(J))$ for every $\phi \in \Phi_0$. Therefore we estimate

$$\begin{aligned}|B(f,\phi)| \leq& \int_I |(\mathrm{id}_v f(x), \phi_x(x))_{L^2(J)}| + |\left(V'(x)\, f(x), \phi_v(x)\right)_{L^2(J)}| + \varepsilon |(f_v(x), \phi_v(x))_{L^2(J)}|\, dx \\ \leq& \left(\max_{v\in J} |v|\, \|\phi_x\|_{L^2(I,L^2(J))} + (\varepsilon + \max_{x\in I} |V'(x)|)\, \|\phi_v\|_{L^2(I,L^2(J))}\right) \|f\|_{L^2(I,H^1_0(J))}\end{aligned}$$

using the Cauchy-Schwarz inequality. Again using the Cauchy-Schwarz inequality delivers the continuity of the linear map $L(\cdot) : \Phi_0 \to \mathbb{R}$. So proving coercivity on Φ_0 remains. Applying the identities of Lemma 2.10 and Lemma 2.9 we directly conclude

$$\begin{aligned}B(\phi,\phi) =& \mathrm{Re}\, B(\phi,\phi) = \frac{B(\phi,\phi) + \overline{B(\phi,\phi)}}{2} \\ =& \varepsilon |\phi|_{L^2(I,H^1_0(J))} + \frac{1}{2}\int_{J^+} v\, \phi(0,v)^2\, dv - \frac{1}{2}\int_{J^-} v\, \phi(1,v)^2\, dv.\end{aligned}$$

Now Lemma 2.6 and (2.4) yield

$$B(\phi,\phi) = \varepsilon |\phi|^2_{L^2(I,H^1_0(J))} + \frac{1}{2}\int_0^b v\, \phi(0,v)^2\, dv - \frac{1}{2}\int_a^0 v\, \phi(1,v)^2\, dv \geq C(J,\varepsilon)\|\phi\|^2_\Phi.$$

with $C(J, \varepsilon) > 0$. □

Corollary 2.12. *Let $f \in L^2(I, H_0^1(J))$ be the solution provided by Theorem 2.11. Then* $\mathrm{id}_v f \in L^2(I, H^{-1}(J))$ *and fulfills*

$$\mathrm{id}_v f_x = V'\, f_v + \varepsilon\, f_{vv} + h$$

in the sense of $L^2(I, H^{-1}(J))$.

Proof. A solution $f \in L^2(I, H_0^1(J))$ of Theorem 2.11 satisfies $B(f, \phi) = L(\phi) \quad \forall \phi \in \Phi_0$ and therefore in particular for every $\phi(x) := \zeta(x)\eta$ with arbitrary $\zeta \in C_0^\infty(I)$ and $\eta \in H_0^1(J)$. Using this special test function we conclude

$$\int_I (\mathrm{id}_v f(x), \eta)_{L^2(J)}\, \zeta_x(x)\, dx = \int_I \Big(\big(V'(x)\, f(x) + \varepsilon\, f_v(x), \eta_v\big)_{L^2(J)} - \langle h(x), \eta\rangle_{H^{-1}(J)\times H_0^1(J)} \Big)\, \zeta(x)\, dx \tag{2.14}$$

for every $\eta \in H_0^1(J)$ and every $\zeta \in C_0^\infty(I)$.

Now we can use the weak derivative (in the sense of $L^2(J)$) $f_{vv}(x) \in H^{-1}(J)$ for almost every $x \in I$ defined by

$$\langle f_{vv}(x), \psi\rangle_{H^{-1}\times H_0^1} = -\,(f_v(x), \psi_v) \quad \forall \psi \in C_0^\infty(J).$$

Since $C_0^\infty(J)$ is dense in $H_0^1(J)$ we get

$$\langle f_{vv}(x), \eta\rangle_{H^{-1}\times H_0^1} = -\,(f_v(x), \eta_v) \quad \forall \eta \in H_0^1(J) \tag{2.15}$$

By

$$\begin{aligned}
\|f_{vv}\|^2_{L^2(I,H^{-1}(J))} &= \int_I \|f_{vv}(x)\|^2_{H^{-1}(J)}\, dx = \int_I \sup_{\substack{\eta\in H_0^1(J)\\ \|\eta\|_{H_0^1(J)}\le 1}} |\,\langle f_{vv}(x), \eta\rangle\,|^2_{H^{-1}(J)\times H_0^1(J)}\, dx \\
&= \int_I \sup_{\substack{\eta\in H_0^1(J)\\ \|\eta\|_{H_0^1(J)}\le 1}} |\,(f_v(x), \eta_v)_{L^2(J)}\,|^2\, dx \\
&\le \int_I \|f(x)\|^2_{H_0^1(J)}\, dx < \infty
\end{aligned}$$

we conclude $f_{vv} \in L^2(I, H^{-1}(J))$. If we furthermore identify $V'(x) f_v(x) \in L^2(J)$ with an element of $H^{-1}(J)$ we know (by the same density argument for the weak derivative)

$$\big\langle V'(x)\, f_v(x), \eta\big\rangle_{H^{-1}(J)\times H_0^1(J)} = -\,\big(V'(x) f(x), \eta_v\big)_{L^2(J)} \quad \forall \eta \in H_0^1(J) \tag{2.16}$$

for almost every $x \in I$. With this identity we can conclude $V'(x) f_v(x) \in L^2(I, H^{-1}(J))$ as just before for f_{vv}.

Finally applying (2.15) and (2.16) to (2.14) yields

$$\int_I (\mathrm{id}_v f(x), \eta)_{L^2(J)} \, \zeta_x(x) \, dx = -\int_I \Big(\big\langle V'(x) \, f_v(x) + \varepsilon \, f_{vv}(x) + h(x), \eta \big\rangle_{H^{-1}(J) \times H_0^1(J)} \Big) \, \zeta(x) \, dx. \tag{2.17}$$

We also observe that $x \mapsto w(x) := V'(x) \, f_v(x) + \varepsilon \, f_{vv}(x) + h(x) \in L^2(I, H^{-1}(J)) \subset L^1(I, H^{-1}(J))$ as all the terms are. So by Lemma B.18 and (2.17), w is the unique weak x-derivative of $\mathrm{id}_v f(x)$ and therefore corresponds with $\mathrm{id}_v f_x$ almost everywhere (with respect to x). So $\mathrm{id}_v f_x \in L^2(I, H^{-1}(J))$ and

$$\mathrm{id}_v f_x = V' \, f_v + \varepsilon \, f_{vv} + h$$

in the sense of $L^2(I, H^{-1}(J))$. □

2.4 Regularity

Lemma 2.13. *Let $P : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega) : f \mapsto v \, f_x - V'(x) \, f_v - \varepsilon f_{vv}$ and let V', $h \in C^\infty(\Omega)$ then every $f \in \mathcal{D}'(\Omega)$ satisfying $Pf = h$ is in $C^\infty(\Omega)$.*

Proof. We test for Hörmander's condition Theorem A.5 and will use $X_0 := v \, \partial_x - V'(x) \, \partial_v$ and $X_1 := i\sqrt{\varepsilon}\partial_v$.

$$\begin{aligned}
[X_0, X_1] &= (X_0 X_1 - X_1 X_0) \\
&= i\sqrt{\varepsilon}(v \, \partial_{xv} - V'(x) \, \partial_{vv} - \partial_x - v \, \partial_{vx} + V'(x) \, \partial_{vv}) \\
&= -i\sqrt{\varepsilon}\partial_x
\end{aligned}$$

Since X_1 and $[X_0, X_1]$ are linearly independent and do not depend on the parameters x and v we get linear independence of X_1 and $[X_0, X_1]$ for every $(x, v) \in \Omega$. Using Theorem A.5 we get the required regularity result. □

Theorem 2.14 (Regularity on the interior)**.** *Let $h \in C^\infty(\Omega) \cap L^2(\Omega) \subset L^2(I, H^{-1}(J))$. Then every solution f provided by Theorem 2.11 is smooth too i.e. $f \in C^\infty(\Omega)$. Furthermore f is a solution to* (2.10) *in sense of distributions and therefore a classical solution on Ω.*

Proof. This proof will be done in three steps.

1. We will show that $f \in L^2(I, H_0^1(J))$ delivers us a distributional solution to (1.7). Therefore define a distribution $\Lambda \in \mathcal{D}'(\Omega)$ by

$$\Lambda : \mathcal{D}(\Omega) \to \mathbb{R} : \phi \mapsto \int_I (f(x), \phi(x, \cdot)) \, dx$$

Since $f \in L^2(I, H_0^1(J))$, Λ actually is continuous. The distributional derivative $D_v\Lambda$ is defined by $\langle D_v\Lambda, \phi\rangle = -\langle \Lambda, \phi_v\rangle \quad \forall \phi \in \mathcal{D}(\Omega)$. By the definition of the distributional derivative and the calculus of C^∞ and distributions we also get $\langle D_x \mathrm{id}_v \Lambda, \phi\rangle = -\langle \Lambda, \mathrm{id}_v \phi_x\rangle$. Collecting the new identities, applying some calculus again and using (2.13) yield

$$\left\langle \mathrm{id}_v D_x\Lambda - V' \circ \mathrm{id}_x\, D_v\Lambda - \varepsilon\, D_{vv}\Lambda, \phi \right\rangle = \langle h, \phi\rangle \quad \forall \in \phi\mathcal{D}(\Omega) \subset \Phi_0. \tag{2.18}$$

Hereby we can interpret h as distribution on Ω since Fubini's theorem gives us

$$\int_\Omega h(x,v)\phi(x,v)\, dv\, dx = \int_I \int_J h(x,v)\phi(x,v)\, dx\, dv.$$

2. Next we apply the Hypoellipticity result Lemma 2.13 and therefore get $\Lambda \in C^\infty(\Omega)$.

3. Finally we have to prove that this implies $f \in C^\infty(\Omega)$. Due to the smoothness we can write Λ as regular distribution i.e. there is $\lambda \in C^\infty(\Omega) \cap L^1_{loc}(\Omega)$ satisfying $\langle \Lambda, \phi\rangle = \int_\Omega \lambda(x,v)\phi(x,v)\, dv\, dx$. Now we can apply Fubini's Theorem (on the compact support of ϕ) which leads together with the definition of Λ to

$$\begin{aligned}\int_\Omega \lambda(x,v)\phi(x,v)\, dv\, dx &= \int_I \int_J \lambda(x,v)\phi(x,v)\, dv\, dx \\ &= \langle \Lambda, \phi\rangle = \int_I \int_J f(x,v)\phi(x,v)\, dv\, dx \quad \forall \phi \in \mathcal{D}(\Omega)\end{aligned}$$

So λ and f coincide almost everywhere and therefore there is a $C^\infty(\Omega)$ representative in the $L^2(I, H_0^1(J))$ equivalence class of f.

□

Corollary 2.15. *Let $f \in \mathcal{B}$ be a solution to* (2.13) *Then*

$$\|f(x)\|_{L^2_{|v|}(J)} \le \|g\|_{L^2_{|v|}(J)} \quad \forall x \in I.$$

Proof. Let $x \in I$ be arbitrary. Using (1.7) and (2.7) of Lemma 2.9 we can apply the definition of the weak derivative which yields

$$\frac{d}{dx}\|f(x)\|^2_{L^2_v(J)} = -\varepsilon \|f_v(x)\|^2_{L^2(J)} \le 0 \quad a.e.$$

Integration yields

$$\|f(0)\|_{L^2_v(J)} \ge \|f(x)\|_{L^2_v(J)} \ge \|f(1)\|_{L^2_v(J)}$$

and subsequently

$$\|g_+\|_{L^2_{|v|}(J^+)} - \|f(0)\|_{L^2_{|v|}(J^-)} \geq \|f(x)\|_{L^2_{|v|}(J^+)} - \|f(x)\|_{L^2_{|v|}(J^-)} \geq \|f(1)\|_{L^2_{|v|}(J^+)} - \|g_-\|_{L^2_{|v|}(J^-)}.$$

Now we can write $f(x) = \chi_{J^-} f(x) + \chi J^+ f(x)$ as linear combination of linear independently functions and get for each of them:

$$\begin{aligned}\|g_+\|_{L^2_{|v|}(J^+)} \geq & \|f(x)\|_{L^2_{|v|}(J^+)} \geq \|f(1)\|_{L^2_{|v|}(J^+)} \\ \|f(0)\|_{L^2_{|v|}(J^-)} \leq & \|f(x)\|_{L^2_{|v|}(J^-)} \leq \|g_-\|_{L^2_{|v|}(J^-)}\end{aligned}$$

The statement is now proved by $\|f\|_{L^2_{|v|}(I)} = \|f\|_{L^2_{|v|}(J^-)} + \|f\|_{L^2_{|v|}(J^+)}$. □

2.5 Trace operators and boundary conditions

Proposition 2.16 ([Lio61]). *The trace operators*

$$\begin{aligned}\mathrm{tr}_a &: L^2(I, H^1(J)) \to L^2(I) : f \mapsto f(\cdot)(a) \\ \mathrm{tr}_b &: L^2(I, H^1(J)) \to L^2(I) : f \mapsto f(\cdot)(b)\end{aligned}$$

are continuous and $\mathrm{tr}_a f = 0$, $\mathrm{tr}_b f = 0$ *for every* $f \in L^2(I, H^1_0(J))$.

Proposition 2.17. *The trace operators*

$$\begin{aligned}\mathrm{tr}_{\Gamma_+} &: C(I, L^2_{|v|}(J)) \to L^2_{|v|}(J^+) : f \mapsto \chi_{J^+} f(0) \\ \mathrm{tr}_{\Gamma_-} &: C(I, L^2_{|v|}(J)) \to L^2_{|v|}(J^-) : f \mapsto \chi_{J^-} f(1)\end{aligned}$$

are continuous.

Proof. The proof is a trivial consequence of the fact that the point evaluation for continuous function is continuous concerning the uniform norm.

$$\begin{aligned}\|f(0)\|_{L^2_{|v|}(J^+)} &\leq \max_{x\in I} \|f(x)\|_{L^2_{|v|}(J)} \\ \|f(1)\|_{L^2_{|v|}(J^+)} &\leq \max_{x\in I} \|f(x)\|_{L^2_{|v|}(J)}\end{aligned}$$

□

Theorem 2.18. *A solution $f \in \mathcal{B}$ to* (2.13) *provided by Theorem 2.11 fulfills the boundary conditions* (2.11) *and* (2.12) *such that*

$$
\begin{aligned}
\operatorname{tr}_a f =&0 & \operatorname{tr}_b f =&0\\
\operatorname{tr}_{\Gamma_+} f =&g_+ & \operatorname{tr}_{\Gamma_-} f =&g_-
\end{aligned}
$$

are satisfied.

Proof. The first both trace conditions are satisfied by Proposition 2.16. For the second pair of conditions let $f \in \mathcal{B}$ be a solution to (2.13) then by Lemma 2.9 we can conclude $f \in C(I, L^2_{|v|}(J))$ and furthermore there is a sequence $(f_n)_{n\in\mathbb{N}} \subset C^1(I, H^1_0(J))$ converging to f in the sense of $\mathcal{B}$. Therefore we define

$$
h_n := \operatorname{id}_v (f_n)_x - V' \circ \operatorname{id}_x (f_n)_v - \varepsilon (f_n)_{vv} \in L^2(I, H^{-1}(J))
$$

which is well defined as we have seen in the proof Corollary 2.12. We also define

$$
g_n := \chi_{J^-} f_n(1) + \chi_{J^+} f_n(0) \in L^2_{|v|}(J).
$$

Due to (2.5) and (2.6) we can deduce:

$$
\lim_{n\to\infty} \|\operatorname{id}_v (f_n)_x - V' \circ \operatorname{id}_x (f_n)_v - \varepsilon (f_n)_{vv} - \operatorname{id}_v f_x + V' \circ \operatorname{id}_x f_v + \varepsilon f_{vv}\|_{L^2(I,H^{-1}(J))} = 0
$$

or equivalently

$$
\lim_{n\to\infty} \|h_n - h\|_{L^2(I,H^{-1}(J))} = 0.
$$

Using the preceding convergence results and the fact that f is a solution to $B(f,\phi) = L(\phi)$ for every $\phi \in \Phi_0$ we can deduce:

$$
\begin{aligned}
&(g, \phi(0) - \phi(1))_{L^2_{|v|}(J)} \qquad (2.19)\\
&= \left\langle f, V' \circ \operatorname{id}_x \phi_v - \varepsilon \phi_{vv} - \operatorname{id}_v \phi_x \right\rangle_{L^2(I,H^1_0(J))\times L^2(I,H^{-1}(J))} - \langle h, \phi \rangle_{L^2(I,H^{-1}(J))\times L^2(I,H^1_0(J))}\\
&= \lim_{n\to\infty} \left\langle f_n, V' \circ \operatorname{id}_x \phi_v - \varepsilon \phi_{vv} - \operatorname{id}_v \phi_x \right\rangle_{L^2(I,H^1_0(J))\times L^2(I,H^{-1}(J))} - \langle h_n, \phi \rangle_{L^2(I,H^{-1}(J))\times L^2(I,H^1_0(J))}.
\end{aligned}
$$

As $f_n \in C^1(I, H^1_0(J))$ we furthermore observe for every $\phi \in \Phi_0$

$$
\begin{aligned}
&\left\langle f_n, V' \circ \operatorname{id}_x \phi_v - \operatorname{id}_v \phi_x \right\rangle_{L^2(I,H^1_0(J))\times L^2(I,H^{-1}(J))}\\
&= \left(f_n, V' \circ \operatorname{id}_x \phi_v - \operatorname{id}_v \phi_x \right)_{L^2(I,L^2(J))} \qquad (2.20)\\
&= (g_n, \phi(0) - \phi(1))_{L^2_{|v|}(J)} - \left(V' \circ \operatorname{id}_x (f_n)_v - \operatorname{id}_v (f_n)_x, \phi \right)_{L^2(I,L^2(J))}.
\end{aligned}
$$

Using the definition of the second weak derivative in $H_0^1(J)$ (a $H^{-1}(J)$ function) as in the proof of Corollary 2.12 we conclude

$$\begin{aligned}\langle f_n, \phi_{vv}\rangle_{L^2(I,H_0^1(J))\times L^2(I,H^{-1}(J))} &= ((f_n)_v, \phi_v)_{L^2(I,L^2(J))}\\ &= \langle (f_n)_{vv}, \phi\rangle_{L^2(I,H^{-1}(J))\times L^2(I,H_0^1(J))}\,. \end{aligned} \tag{2.21}$$

Putting (2.20) and (2.21) together yields

$$\big\langle f_n, V'\circ \mathrm{id}_x\,\phi_v - \varepsilon\phi_{vv} - \mathrm{id}_v\,\phi_x\big\rangle - \langle h_n, \phi\rangle = (g_n, \phi(0)-\phi(1))_{L^2_{|v|}(J)} \quad \forall \phi\in\Phi_0.$$

which, together with (2.19) finally leads to

$$\begin{aligned}(g, \phi(0)-\phi(1))_{L^2_{|v|}(J)} &= \lim_{n\to\infty}(g_n, \phi(0)-\phi(1))_{L^2_{|v|}(J)}\\ &= (\chi_{J^-}f(1) + \chi_{J^+}f(0), \phi(0)-\phi(1))_{L^2_{|v|}(J)} \quad \forall\phi\in\Phi_0.\end{aligned}$$

Hereby the last equality follows by Lemma 2.9, b).

For every $\psi\in H_0^1(J)$ we yield $\chi_{J^\pm}\psi\in H_0^1(J)$ and we can find a linear combination $l_\psi : x\mapsto x\,\chi_{J^-}\psi + (1-x)\,\chi_{J^+}\psi \in C^1(I, H_0^1(J))$ such that $l_\psi(0)-l_\psi(1)=\psi$. Although $H_0^1(J)$ is per se too small as test function set, we can use the density of $C^\infty_{0,0}(J) = \{f\in C_0^\infty(J) : f(0)=0\}\subset H_0^1(J)\subset L^2_{|v|}(J)$ provided by Corollary 2.5. Therefore we finally proved the statement by

$$\|g-\chi_{J^-}f(1)+\chi_{J^+}f(0)\|_{L^2_{|v|}(J)} = 0$$

or equivalently

$$\|g-\chi_{J^-}f(1)\|_{L^2_{|v|}(J)} + \|\chi_{J^+}f(0)\|_{L^2_{|v|}(J)} = 0.$$

□

2.6 Uniqueness of the solution

Theorem 2.19. *The solution provided by Theorem 2.11 and Corollary 2.12 is unique i.e. there is exactly one $f\in\mathcal{B}$ that satisfies* (2.10) *in the sense of $L^2(I,H^{-1}(J))$ and the boundary conditions* (2.11)-(2.12) *in the sense of the traces of Theorem 2.18.*

Proof. For readability we will skip the indices of the duality products. Unless otherwise stated they all act on $L^2(I,H^{-1}(J))\times L^2(I,H_0^1(J))$. Assume there exist solutions $f_1, f_2\in\mathcal{B}$ that fulfill

(2.13) and the boundary conditions (2.11)-(2.12) in the sense of tr_{Γ_1} and tr_{Γ_2}. Then $f := f_1 - f_2$ is a solution in $\mathcal{B}$ to

$$\begin{aligned} v f_x(x,v) - V'(x) f_v(x,v) - \varepsilon f_{vv}(x,v) &= 0 \\ \mathrm{tr}_{\Gamma_+} f &= 0 \\ \mathrm{tr}_{\Gamma_-} f &= 0 \end{aligned}$$

Due to (2.8) we instantly see

$$\begin{aligned} &2\mathrm{Re}\,\langle \mathrm{id}_v\, f_x, f\rangle \\ &= \langle f, \mathrm{id}_v\, f_x\rangle + \langle f, \mathrm{id}_v\, f_x\rangle \\ &= \int_J v\, f(v,1)^2\, dv - \int_J v\, f(v,0)^2\, dv \\ &= \int_{J^+} |v| f(v,1)^2 dv + \int_{J^-} |v| f(v,0)^2 dv \geq 0 \end{aligned}$$

By Lemma 2.9 f can be approximated by a sequence $(f_n)_{n\in\mathbb{N}} \subset C^1(I, H_0^1(J))$. Define

$$\begin{aligned} h_n &:= \mathrm{id}_v\,(f_n)_x - V' \circ \mathrm{id}_x (f_n)_v - \varepsilon (f_n)_{vv} \in L^2(I, H^{-1}(J)) \\ g_n &:= \chi_{J^-} f_n(1) + \chi_{J^+} f_n(0) \in L^2_{|v|}(J). \end{aligned}$$

The definition of the weak derivative on $H_0^1(J)$ and integration by parts yield

$$\mathrm{Re}\,\langle h_n, f_n\rangle = \int_I 2\varepsilon\, ((f_n(x))_v, (f_n(x))_v)_{L^2(J)}\, dx + (\mathrm{id}_v\, f_n(1), f_n(1))_{L^2(J)} - (\mathrm{id}_v\, f_n(0), f_n(0))_{L^2(J)}$$

which directly leads to

$$\begin{aligned} &2\varepsilon \int_I ((f_n(x))_v, (f_n(x))_v)_{L^2(J)}\, dx \\ &= \mathrm{Re}\,\langle h_n, f_n\rangle - (\mathrm{id}_v\, f_n(1), f_n(1))_{L^2(J)} + (\mathrm{id}_v\, f_n(0), f_n(0))_{L^2(J)} \\ &\leq \mathrm{Re}\,\langle h_n, f_n\rangle + (g_n, g_n)_{L^2_{|v|}(J)}\,. \end{aligned}$$

Since $h_n \to h = 0$ in $L^2(I, H^{-1}(J))$, $g_n \to g = 0$ in $L^2_{|v|}(J)$ and $f_n \to f \in L^2(I, H_0^1(J))$ the last inequality implies

$$2\varepsilon \|f_v\|^2_{L^2(I,L^2(J))} = 2\varepsilon \lim_{n\to\infty} \|(f_n)_v\|^2_{L^2(I,L^2(J))} \leq \lim_{n\to\infty} \mathrm{Re}\,\langle h_n, f_n\rangle + (g_n, g_n)_{L^2_{|v|}(J)} = 0.$$

Due to Lemma 2.6 and the fact that $\|\cdot\|_{L^2(I,H_0^1(J))}$ is a norm, we can conclude $f = 0$. Therefore the solutions f_1 and f_2 coincide as elements of $L^2(I, H_0^1(J))$. $\square$

2.7 Maximum principle

With the definitions of Chapter A.2 we can state the following.

Lemma 2.20. *Let $V(x) \in C^2(\overline{\Omega})$ then the propagation set of the differential operator*

$$(Pf)(x,v) := v\partial_x f(x,v) - V'(x)\partial_v f(x,v) - \varepsilon\partial_{vv} f(x,v)$$

in Ω is $S(p,\Omega) = \Omega$ for every $p \in \Omega$.

Proof. Let $\phi_t((x_0,v_0))$ be the flow of X_0 i.e. $t \mapsto \phi_t((x_0,v_0))$ is a drift trajectory and let $\psi_t((x_0,v_0))$ be the flow of $(X_1,X_2)^T$ i.e. $t \mapsto \psi_t((x_0,v_0))$ is a diffusion trajectory. First of all we observe that the diffusion trajectories are exactly all curves satisfying $\{(x,v) \in \Omega : x = c\}$ with $c \in I^\circ$. So they are just the "vertical" lines through the domain i.e. for the trajectories we get

$$\psi_t((x,v)) = (x_0, \xi(t))$$

where $\xi(t)$ can take values in $\mathbb{R}$.

The drift trajectories are given by the C^1 vector field

$$X_0 := \begin{pmatrix} v \\ -V'(x) \end{pmatrix}$$

and have positive first coordinate everywhere in $\Omega \setminus \{(x,v) \in \Omega : v = 0\}$. Let $(x_0,v_0) \in \Omega$ with $v_0 > 0$ and let $U_\varepsilon((x_0,v_0))$ be an open ball around (x_0,v_0) with fixed radius $0 < \varepsilon < v_0$. Since there are no critical points of X_0 in $U_\varepsilon((x_0,v_0))$ we can apply the implicit function theorem. This delivers us the fact that for every drift trajectory $\phi_t((x_0,v_0))$ starting in (x_0,v_0) with $(x_1,v_1) = \phi_{t_1}((x_0,v_0))$ we get $x_1 - x_0 > c$ with $c > 0$. Hereby the difference is positive (even on the compact ball $K_\varepsilon((x_0,v_0))$) since the first component of X_0 is larger or equal than $v_0 - \varepsilon$ as long as we stay in $K((x_0,v_0))$.

Using a diffusion trajectory $\psi_t((x_1,v_1))$ we now can build a chain $(x_0,v_0) \to (x_1,v_1) \to (x_2,v_2) = (x_1,v_2)$. Connecting such chains together we can connect (x_0,v_0) and every point $(x_n,v_n) \in U_\varepsilon((x_0,v_0))$ by a chain of diffusion and drift trajectories as long as $x_n \geq x_0$. Since we stay in $K_\varepsilon((x_0,v_0))$ the amount of drift trajectories needed is at most $\frac{x_1-x_0}{c}$ and because we can arbitrarily change v coordinate by a diffusion trajectory we need at most as much diffusion trajectories as drift trajectories. This implies that the chain is finite.

If $v_0 < 0$ we can do the same as long as $x_n \leq x$. Since $\overline{\Omega}$ is compact we can cover in particular Ω by a finite collection of overlapping open sets. Therefore we can use the preceding arguments to connect each point in the upper half plane sector of Ω with any point right hand sided by a finite chain. We can do the same on the lower half plane sector of Ω with left hand sided points. Since every point in $\{(x, v) \in \Omega : v = 0\}$ is hit by a diffusion trajectory we can connect those to every point in Ω too. This finally enables us furthermore to connect every pair of points in the Ω by a finite chain of diffusion and drift trajectories since we can switch between the upper and the lower part and therefore also between the left and right direction. $\square$

Theorem 2.21. *Let $h \in C^\infty(\Omega)$ and $h(x, v) \geq 0 \quad \forall (x, v) \in \Omega$. Let $f \in \mathcal{B}$ be solution to $Pf = h$ then f attains its maximum at the boundary $\partial\Omega$. If furthermore $h = 0$ then the solution attains its minimum at the boundary as well.*

Proof. Suppose, on the contrary, its maximum $M = f(p)$ for some $p \in \Omega$. The propagation set $S(p, \Omega) = \Omega$ and Lemma A.12 yields $f = M$ on $\overline{\Omega}$, which is in contradiction to the assumption.This implies the maximum to be at the boundary.

In the case of $h = 0$, $-f$ is a solution too and we can apply the already proved maximum principle. $\square$

Chapter 3

Abstract kinetic equation in Krein space

In this chapter we try to treat the problem given by (1.7) in an more abstract way. Due to the special structure it is convenient to state the problem in a Krein space.

In [Ćur00] an abstract treatment in Krein space was given which suits to Everything Theorem 3.10 Due to the form of the treated domain [Ćur00, Theorem 3.4] seems to be a convenient result. Unfortunately as in most references only autonomous equations are studied and indeed the results e.g. presented in [Paz92, Chapter 5, Theorem 3.1] and [Paz92, Chapter 7, Theorem 2.3] suggest that the nonautonomous Cauchy problem has stricter and more involved necessary conditions. So we aim to interpret the previous results considering

$$Tf'(x) = A(x)f + Df \quad x \in [0,1] \tag{3.1}$$

$$g = P_+f(0) + P_-f(1) \tag{3.2}$$

where $T := \mathrm{id}_v$, $A(x) = A := c\partial_v$ and $D = \varepsilon\partial_{vv}$. (The potential $V(x)$ has to be linear)

3.1 Krein spaces

Introductions to Krein spaces can be found in [Lan], [Wor08] or [AI89].

Definition 3.1. Let $\mathcal{K}$ be a vector space equipped with a hermitian sesquilinear map

$$[\cdot,\cdot] : \mathcal{K} \times \mathcal{K} \to \mathbb{C}$$

$$\begin{aligned}[\alpha x + \beta y, z] =& \alpha\,[x,z] + \beta\,[y,z] \\ [x,y] =& \overline{[y,x]}.\end{aligned}$$

and let $A \in B(\mathcal{K})$ a bounded operator.

1. A subspace $X \subset \mathcal{K}$ is called A-invariant if and only if $Ax \in X \quad \forall x \in X$
2. A subspace $X \subset \mathcal{K}$ is called maximal invariant if X is A-invariant and for every A-invariant subspace Y is already included in X.
3. A subspace $X \subset \mathcal{K}_+ := \{x \in \mathcal{K} : [Ax, x] > 0\}$ is called positive.
4. A subspace $X \subset \mathcal{K}_- := \{x \in \mathcal{K} : [Ax, x] < 0\}$ is called negative.

Theorem 3.2. *Let $(\mathcal{K}, (\cdot,\cdot))$ be a complex Hilbert space and let $[\cdot,\cdot]$ be a hermitian sesquilinear map. Then $(\mathcal{K}, [\cdot,\cdot])$ is a Krein space if there is a self adjoint Gram operator $G \in B(\mathcal{K})$ with bounded inverse satisfying:*

$$[x,y] = (Gx, y) \quad \forall x, y \in \mathcal{K} \tag{3.3}$$

In particular we call G a fundamental symmetry if $G = G^*$ and $G^2 = \mathrm{id}$. [Lan]

Remark 3.3. As pointed out in the more general Definition 2.1.1 of [Wor08] Krein spaces are equipped with the same topology as their Hilbert space counterparts. In particular L^2_v is a Krein space and its Hilbert space counterpart is $L^2_{|v|}$ [Lan].

3.2 Contractive C_0-semigroups

Definition 3.4. Let $(X, \|\cdot\|)$ be a Banach space. We call a family of bounded operators $\{S(t)\}_{t\in\mathbb{R}_0^+}$ a strongly continuous or C_0-semigroup if and only if

1. $S(0) = \mathrm{id}$
2. $S(t+s) = S(t)S(s) \quad \forall t, s \geq 0$
3. $\lim_{t\to 0} \|S(t)x - x\| = 0 \quad \forall x \in X$

The infinitesimal generator of $S(t)$ is given by

$$Ax := \lim_{t\downarrow 0} \frac{S(t)x - x}{t} \quad \forall x \in \mathrm{dom}\, A$$

Theorem 3.5. *Let $(X, \|\cdot\|)$ be a Banach space and let $A \in L(X)$ be the possibly unbounded generator of a strongly continuous semigroup. Then for every $x \in \operatorname{dom} A$ the following Cauchy problem is uniquely solved by $u(t) := S(t)x$ in the classical sense.*

$$
\begin{aligned}
u'(t) =& Au(t) \quad 0 \leq t \\
u(0) =& x
\end{aligned}
$$

[Paz92, Theorem 4.1.3]

Theorem 3.6 (Hille-Yoshida [Paz92])**.** *Let $(X, \|\cdot\|)$ be a Banach space. A linear operator $A \in L(X)$ generates a strongly continuous semigroup satisfying $\|S(t)\| \leq Me^{\omega t}$ if and only if*

1. *A is closed and $\overline{\operatorname{dom} A} = X$.*

2. *There exist $\omega \in \mathbb{R}$ and $M > 0$ such that*

$$
(\omega, +\infty) \subset \rho(A)
$$

where $\rho(A)$ is the resolvent set of A and the estimate

$$
\|(A - \lambda \operatorname{id})^{-n}\| \leq \frac{M}{(\lambda - \omega)^n} \quad \forall \lambda > \omega
$$

holds for every $n \in \mathbb{N}$.

If a semigroup satisfies the above stated estimate with $M = 1$ and $\omega = 0$ then we call it a semigroup of contractions. The following theorem gives a characterization for the generators of those.

Definition 3.7. Let $(X, \|\cdot\|)$ be a Banach space and X' its dual space.

$$
F(x) := \{x' \in X' : \langle x', x \rangle = \|x\|^2 = \|x'\|^2\}
$$

A linear operator $A \in L(X)$ is called dissipative if and only if for every $x \in \operatorname{dom} A$ there exists $x' \in X'$ such that

$$
\operatorname{Re} \langle Ax, x' \rangle \leq 0.
$$

If, furthermore, there is no proper extension of A it is called maximal dissipative. This is true if and only if there exists $\lambda > 0$

$$
(A - \lambda \mathrm{id}) \operatorname{dom} A = X. \tag{3.4}
$$

In the case of a Hilbert space obviously $F(x) = \{x\}$ after identification.

Theorem 3.8 (Lumer-Phillips [Paz92]). *Let $A \in L(X)$ be closed and densely defined. Then A is maximal dissipative if and only if A is the generator of a semigroup of contractions.*

Although we can introduce the notion of dissipativity in Krein spaces, this property doesn't characterize the generators of a semigroups of contractions in Krein spaces. Nevertheless we have the following statement.

Lemma 3.9. *Let $(H, (\cdot, \cdot))$ be a Hilbert space and let J be a fundamental symmetry of the Krein space $(H, (J\cdot, \cdot))$. $A \in L(H)$ is (maximal) J-dissipative in $(H, (J\cdot, \cdot))$ if and only if JA and AJ are (maximal) dissipative in $(H, (\cdot, \cdot))$. [AI89, II.2, 2.3]*

Theorem 3.10 ([Ćur00, Theorem 3.4]). *Let $A \in L(H)$ be maximal dissipative in a Krein space $(H, [\cdot, \cdot])$, which allows a fundamental symmetry J commuting with A. Then*

$$\begin{aligned} u'(t) =& Au(t) \\ g =& P_+ u(0) + P_- u(1) \end{aligned}$$

has a unique solution. If furthermore $J = P_+ - P_-$ then the solution has the form

$$u(t) = (e^{-tAP_+} P_+ + e^{(1-t)AP_-} P_-)g.$$

3.3 Statement of the problem

Let H be a Hilbert space. We now assume an arbitrary $T \in B(H)$ to be selfadjoint with $\ker T = 0$.

Furthermore we assume that P_+ is the projection onto the maximal T-invariant positive subspace of H and P_- the orthogonal projection onto the maximal invariant negative subspace. The positive part of T is given by

$$|T| := TP_+ - TP_- = P_+ T - P_- T.$$

We now can introduce the completion of H with respect to $(y, z)_T := \left(|T|^{\frac{1}{2}} y, |T|^{\frac{1}{2}} z\right) = (|T|y, z)$ and denote it with H_T. Equipped with $(\cdot, \cdot)_T$, H_T is a Hilbert space and since $J := P_+ - P_-$ fulfills $J^2 = \mathrm{id}$ and $J = J^*$, $(H_T, (J\cdot, \cdot)_T)$ is a Krein space. Hereby P_+ and P_- are now the extensions to H_T.

Finally we can state our problem. Find $u \in C^1([0,1], H_T)$ such that

$$Tf'(x) = A(x)f + Df \quad x \in [0,1] \tag{3.5}$$

$$g = P_+ f(0) + P_- f(1) \tag{3.6}$$

is satisfied.

The whole Krein space construction aims to prove $T^{-1}(A(x)u + Du)$ is an infinitesimal generator of a strongly continuous semigroup. Given a solution to

$$u'(x) = T^{-1}(A(x)f + Df) \quad t \in [0,T] \tag{3.7}$$

$$g = P_+ u(0) + P_- u(1) \tag{3.8}$$

we could construct a solution to (3.5) by simply applying T. Concerning the dissipativity it should be noted, that

$$\begin{aligned} \operatorname{Re}\left[JT^{-1}(A(x)f + Df), f\right] &= \operatorname{Re}\left(JJT^{-1}(A(x)f + Df), f\right)_T \\ = \operatorname{Re}\left(T^{-1}(A(x)f + Df), f\right)_T &= \operatorname{Re}\left((A(x)f + Df), f\right) \end{aligned} \quad \forall f \in H_T \tag{3.9}$$

Going back to the concrete example with $T := \mathrm{id}_v$, $A(x) := V'(x)\,\partial_v$ and $D := \varepsilon\partial_{vv}$ we will stick to the autonomous case $V'(x) = c$ for $c \in \mathbb{R}$ and use the weighted L^2 Hilbert space $H_T := L^2_{|v|}$ equipped with $(|v|\cdot, \cdot)_{L^2}$ and the corresponding Krein space H_T is equipped with $(v\cdot, \cdot)$. Obviously $J := P_+ - P_-$ is a fundamental symmetry. Furthermore the problem was given with $P_+ := \chi_{J^-}$ and $P_- := \chi_{J^+}$.

Again, as in the previous chapters, we will treat the case on v-bounded domains i.e. $J = [a, b]$ with $a < 0 < b$ to preserve coercivity. Using $\operatorname{dom} A := C^2_{0,0}(J)$ (see Chapter 2.1) which is a dense subset of $L^2_v(J)$ or equivalently $L^2_{|v|}(J)$ we see $JA = AJ$ on H_T by integration by parts.

As in the proof of Theorem 2.11 we can conclude that $A(x) + D$ is dissipative with respect to the standard L^2 and with (3.9) we therefore get J-dissipativity. Since A is densely defined A is closable [Paz92, 1.4, Theorem 4.5]. To get maximality we would have to prove (3.4) for $\overline{A}$. Fortunately this is equivalent to

$$\overline{(A - \lambda \mathrm{id})\operatorname{dom} A} = H_T$$

and therefore we only have to solve an ordinary differential equation in $\operatorname{dom} A$. For existence of a solution to an autonomous version of Equation 3.1 we then could apply Theorem 3.10.

Appendix A

Some results on partial differential operators

A.1 Hypoellipticity

Definition A.1. A linear differential operator of order k is a mapping of the form

$$P : \begin{cases} \mathcal{D}(\Omega) \to \mathcal{D}(\Omega) \\ \phi \mapsto \sum_{|\alpha| \leq k} a_\alpha \, \partial_\alpha \phi \end{cases}$$

where α is a multiindex and ∂_α the corresponding partial distributional derivative with coefficient function $a_\alpha \in C^\infty(\Omega)$.

Definition A.2. The principal symbol of a linear differential operator is given by $p(x, \xi) := \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$.

Definition A.3. Let Ω be an open subset of $\mathbb{R}^n$ and P a differential operator defined on $\mathcal{D}'(\Omega)$ then we call P hypoelliptic if $Pu \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$.

Definition A.4 (Lie bracket)**.** For all $f \in C^\infty$ and linear first order homogeneous differential operators X, Y with smooth coefficient functions, the Lie bracket is given by

$$[X, Y](f) := X(Y(f)) - Y(X(f)) \tag{A.1}$$

and therefore $[X, Y](f) \in C^\infty(\Omega)$.

Theorem A.5 (Hörmander Condition [Hör67]). *Let Ω be an open subset of $\mathbb{R}^n$, $c \in C^\infty(\Omega)$ and $X_0, \dots, X_r$ be first order differential operators. Let*

$$P = \sum_{i=1}^{r} X_i^2 + X_0 + c \tag{A.2}$$

be a second order differential operator. P is hypoelliptic if n of the following operators (iterated Lie brackets)

$$X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \dots$$

are linearly independent for every point in Ω.

Theorem A.6. *Let P be a linear differential operator as above. If $p(x, \xi) \in \mathbb{R}$, $\forall x \in \Omega$ and there exist $\xi \neq 0$ and $x \in \Omega$ such that*

$$p(x, \xi) = 0$$
$$\exists j : \frac{\partial p(x, \xi)}{\partial \xi_j} \neq 0,$$

then P is not hypoelliptic. [Hör67]

Corollary A.7. *A second order linear differential operator with real principal part needs to be a semi-definite form to be hypoelliptic. [Hör67]*

A.2 Maximum principle for degenerate problems [Hil70]

Definition A.8. Let $P : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ be a differential operator. We call P degenerate elliptic-parabolic if and only if P is of the form

$$P : f \mapsto \nabla^T A \nabla f + b^T \nabla f$$

with positive semidefinite $A \in C^2(\Omega, \mathbb{R}^n \times \mathbb{R}^n)$ i.e. $\xi^T A(x) \xi^T \geq 0 \quad \forall \xi \in \mathbb{R}^n$ and $b \in C^1(\Omega, \mathbb{R})$.

Let $X_k : \Omega \to \mathbb{R}^n$ be a Lipschitz continuous vector field, then there is unique solution $x \in C^1(\Omega)$ satisfying

$$\dot{x}(t) = X_k(x(t))$$
$$x(0) = p.$$

The trajectory $\{x \in \Omega : \exists t \in \mathbb{R} : x = x(t)\}$ of this solution can be split up into curves $\Gamma_{[t_1,t_2]} := \{x \in \Omega : \exists t \in [t_1, t_2] : x = x(t)\}$ parametrized on any subinterval of $[t_1, t_2]$. This parametrization is denoted by $\Gamma(t)$.

Definition A.9. Let $X_k := A\, e_k$. Γ is called diffusion trajectory in ω if

$$x(t) \in \omega \quad \forall t \in [t_1, t_2]$$
$$X_k(x(t)) \neq 0 \quad \forall t \in [t_1, t_2]$$

.

Definition A.10. Let $X_0 := b - \sum_{j=0}^{n} \partial_{x_j} A e_j$. Γ is called drift trajectory in ω if

$$x(t) \in \omega \quad \forall t \in [t_1, t_2]$$
$$X_0(x(t)) \neq 0 \quad \forall t \in [t_1, t_2]$$

and for $p = x(t_0)$ the trajectory points into the same direction as $X_0(p)$ i.e.

$$\frac{\dot{\Gamma}(t)}{\|\dot{\Gamma}(t)\|} = \frac{X_0(p)}{\|X_0(p)\|}$$

Definition A.11. Let ω a domain and let $p \in \omega$. $S(p, \omega)$, called the propagation set, is the union of p and the set of all points that is reached by a finite chain of diffusion or drift trajectories.

Lemma A.12 (interior principle [Hil70, Theorem 1])**.** *Let P be degenerate elliptic-parabolic and let $f \in C^2$ such that*

$$(Pf)(x) \geq 0 \quad \forall x \in S(p, \omega).$$

Suppose

$$\sup_{x \in S(p,\omega)} f(x) \leq M := f(p) < \infty$$

then $f(x) = M$ for all $x \in \overline{S(p, \omega)}$.

A.3 Lions' theorem and inf-sup condition

Theorem A.13 (Lions [Sho13, III.2 Theorem 2.1 and Corollary 2.1])**.** *Let $(H, (\cdot\,, \cdot))$ be a Hilbert space, let $(F, \|\cdot\|_F)$ a normed space and let $B : H \times F \to \mathbb{C}$ a bilinear map. Suppose $B(\cdot, \phi)$ is continuous i.e. $B(\cdot, \phi) \in H'$ for every $\phi \in \Phi$. Then the following statements are equivalent.*

a) $\inf_{\|\phi\|_F = 1} \sup_{\|u\|_H \leq 1} |B(u, \phi)| \geq \alpha > 0$

b) For all $f \in F'$ there exists $u \in H$ satisfying

$$B(u, \phi) = f(\phi) \quad \forall \phi \in F$$

A solution u to b) furthermore can be bounded by $\|u\|_H \leq \frac{1}{\alpha}\,\|f\|_{F'}$

Corollary A.14. *Let F be continuously embedded in H, i.e. there exists a constant c such that $\|\phi\|_H \leq c\|\phi\|_F$ for all $\phi \in F$. Let B be bilinear and $B(\cdot,\phi) \in H'$ for every fixed $\phi \in F$. Suppose B is coercive on F, i.e. there is $\alpha_F > 0$ such that*

$$B(\phi,\phi) \geq \alpha_F \|\phi\|_F^2 > 0 \quad \forall \phi \in F,$$

then Theorem A.13 still holds with coercivity constant $\alpha = \frac{\alpha_F}{c}$.

[Sho13, III.2 Corollary 2.3]

Appendix B

Function spaces

B.1 Gelfand triples

Let V, H be topological vector spaces, then the embedding $i : (V, \mathcal{T}_V) \to (H, \mathcal{T}_H)$ is continuous if and only $\mathcal{T}_V$ is finer than $\mathcal{T}_H$ i.e. $\mathcal{T}_H \subset \mathcal{T}_V$. Suppose V, H are Banach spaces and $\mathcal{T}_V$,$\mathcal{T}_H$ the norm induced topologies, then the embedding is continuous if and only if there is a positive constant c such that

$$\|v\|_H \leq c\|v\|_V \quad \forall v \in V.$$

Definition B.1 (Gelfand Triple [Rou13])**.** Let V be a reflexiv space which is densely included in a Hilbert space H i.e. $V \subset H \subset \overline{V}^H$. Furthermore let V be continuously embedded in H. We call (V, H, V') a Gelfand triple. If V is a Hilbert space we also call (V, H, V') a Hilbert triple or rugged Hilbert spaces.

Remark B.2. A Gelfand triple satisfies

$$\langle h', h \rangle_{V' \times V} = \left(h', h\right)_H \quad \forall h', h \in H. \tag{B.1}$$

Since we can identify $H = H'$ and the continuous embedding also provides $H = H' \subset V'$. Finally we note that H' is dense in V' and $i' : H' \to V'$ is injective and continuous. Actually the proof for this relies on the Hahn-Banach theorem and the reflexivity of V.

By definition of the adjoint operator we get:

$$(x, y)_H = \langle x, y \rangle_{H' \times H} = \langle x, iy \rangle_{H' \times H} = \langle i'x, y \rangle_{V' \times V} = \langle x, y \rangle_{V' \times V} \quad \forall x \in H, y \in V$$

Mostly we will identify the elements automatically i.e. if we use $i(y) \in H$ for any $y \in V$ we will also write $y := i(y) \in H$. [Rou13]

Example B.3. Let Ω be an open set. The triple of the classical test functions $\mathcal{D}(\Omega)$, the quadratic integrable functions $L^2(\Omega)$ and the distributions $\mathcal{D}(\Omega)'$ is a Gelfand triple. See for example [Rud91, Chapter 6].

Example B.4. Let Ω be a bounded region. Let $H_0^1(\Omega)$ be the well known Sobolev space and $H^{-1}(\Omega)$ its continuous dual space. An important example of a Hilbert triple is

$$(H_0^1(\Omega), L^2(\Omega), H^{-1}(\Omega)).$$

The density of $H_0^1(\Omega)$ in $L^2(\Omega)$ follows from the density of $C_0^\infty(\Omega)$ and the continuity is obvious from the definitions of the norms.

B.2 Lebesgue-Bochner spaces

In this section we will shortly state some basic definitions and properties concerning Banach space valued integration. This is done a little more extensive in [Eva10, Appendix E.5] although the proofs for the theorems are given in [Yos80, Chapter V.4 and V.5] except stated otherwise.

Let $(X, \|\cdot\|_X)$ be a real Banach space. Let I be a compact interval.

Definition B.5. The space of all continuous functions $f : I \to X$ is denoted by $C(I, X)$ and equipped with the (obviously well defined) norm $\|f\|_{C(I,X)} := \max_{t \in I} \|f(t)\|_X$.

Definition B.6. A function $f : I \to X$ is called strongly measureable if and only if there exists a sequence of simple functions $s_k(t) = \sum_{i=0}^k \chi_{E_i}(t) x_i$, where $(x_i)_{i=0,\dots,k} \subset X$ and each E_i is a Lebesgue measureable subsets of I such that

$$\lim_{k\to\infty} \|f(t) - s_k(t)\|_X = 0 \quad a.e.\, t \in I$$

Definition B.7 (Bochner integral)**.** The integral of a simple function

$$s : I \to X : t \mapsto s(t) = \sum_{i=0}^{m} \chi_{E_i}(t) x_i$$

is defined by

$$\int_I s(t)\, dt = \sum_{i=0}^{m} |E_i| x_i$$

Let $f : I \to X$ be strongly measureable such that there exist a sequence of simple functions $(s_k)_{k\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty} \int_I \|f(t) - s_k(t)\|\, dt = 0$$

then we define the Bochner integral by

$$\int_I f(t)\,dt = \lim_{k\to\infty}\int_I s_k(t)\,dt$$

Theorem B.8 (Bochner)**.** *A stronly measureable function $f: I \to X$ is Bochner integrable if and only if $t \mapsto \|f(t)\|_X$ is integrable. In this case the following estimate*

$$\|\int_I f(t)\,dt\|_X \leq \int_I \|f(t)\|_X\,dt$$

and the identity

$$\left\langle x', \int_I f(t)\,dt \right\rangle = \int_I \langle x', f(t)\rangle\,dt$$

hold for every $x' \in X'$

Definition B.9. For every $1 \leq p < \infty$ the Lebesgue-Bochner space $L^p(I,X)$ consists of all strongly measureable $f: I \to X$ satisfying

$$\|f\|_{L^p(I,X)} := \sqrt[p]{\int_I \|f(t)\|_X^p\,dt} < \infty$$

Theorem B.10. *Let $1 \leq p < \infty$ and let X be a separable Banach space.*

1. *$L^p(I,X)$ is a Banach space.*

2. *If X is additionally separable then the dual of $L^p(I,X)$ is $L^q(I,X')$ where X' is the dual of X and q is the Hölder conjugate of p. The duality pairing is given by:*

$$\langle x', x\rangle_{L^p(I,X)\times L^q(I,X')} = \int_I \langle x'(t), x(t)\rangle_{X'\times X}\,dt \quad \forall x \in X, x' \in X'$$

3. *If X is additionally reflexiv, separable and $1 < p < \infty$ then $L^p(I,X)$ is reflexiv too.*

4. *If X is additionally a Hilbert space equipped with a product $(\cdot,\cdot)_X$ then $L^2(I,X)$ is a Hilbert space too. The product is*

$$(x,y)_{L^2(I,X)} = \int_I (x(t),y(t))_X\,dt \quad \forall x,y \in X$$

Proof. [Rou13] □

Proposition B.11 (dominated convergence)**.** *Let X be a Banach space and let $(f_n)_{n\in\mathbb{N}} \subset L^1(I,X)$ a sequence converging to $f: I \to X$ almost everywhere. If there exists $g \in L^1(I,\mathbb{R})$ such that $\|f_n\| \leq g$ for every $n \in \mathbb{N}$ then $f \in L^1(I,X)$ and*

$$\lim_{n\to\infty}\int_I \|f_n(t) - f(t)\|\,dt = 0.$$

Furthermore

$$\lim_{n\to\infty}\int_I f_n(t)\,dt = \int_I f(t)\,dt$$

Proof. Since $\|f_n - f\| \le 2|g|$ almost everwhere, we can apply the dominated convergence theorem for Lebesgue integrals, which delivers the statement. $\square$

Proposition B.12. *Let $f : I \times J \mapsto X$ such that $t \mapsto f(t,x) \in L^1(I,X)$ for every $x \in J$ and there is a $g \in L^1(I,\mathbb{R})$ satisfying*

$$\left|\frac{d}{dx}f(t,x)\right| \le g(t) \quad \forall (t,x) \in I \times J.$$

Then we can differentiate the parameter integral by differentiating the integrand i.e.

$$\frac{d}{dx}\int_I f(t,x)\,dt = \int_I \frac{d}{dx}\phi(t,x)\,dt.$$

Proof. There exist ξ_k between x_k and x_0 such that

$$\left\|\frac{f(t,x_k) - f(t,x_0)}{x_k - x_0}\right\| \le \left\|\frac{d}{dx}f(t,\xi_k)\right\| \le g(t) \quad \forall t \in I.$$

Now we can apply the Proposition B.11 to conclude

$$\begin{aligned}
&\frac{d}{dx}\int_I f(t,x_0)\,dt = \lim_{n\to\infty}\frac{\int_I f(t,x_0)\,dt - \int_I f(t,x_k)\,dt}{x_0 - x_k} \\
&= \lim_{n\to\infty}\int_I \frac{f(t,x_0) - f(t,x_k)}{x_0 - x}\,dt = \int_I \lim_{n\to\infty}\frac{f(t,x_0) - f(t,x_k)}{x_0 - x}\,dt \\
&= \int_I \frac{d}{dx}f(t,x_0)\,dt.
\end{aligned}$$

This proof was adopted from [Kal11]. $\square$

Corollary B.13. *Let $\phi \in C_0^\infty(I \times J)$ then $x \mapsto \int_J \phi(t,x)f(x)\,dx \in C^\infty(I,X)$ for every $f \in L^1(I,X)$.*

Corollary B.14. *Let $f \in L^1(I,X)$ and let $\phi \in C_0^\infty(I^\circ)$. The convolution $f * \phi : x \mapsto \int_I \phi(x - s)\,f(s)\,ds \in C^\infty(I^\circ, X)$.*

Lemma B.15. *Let $f \in C(I \times J,\mathbb{R})$ and $g : J \to X$ strongly measurable. The following identity holds.*

$$\int_J f(t,x)g(t)\,dx = g(t)\int_J f(t,x)\,dx \quad \forall t \in I$$

Hereby the left hand side is a Bochner integral and the right hand side a Lebesgue integral.

Proof. Since $f(t,\cdot) : J \to \mathbb{R}$ is obviously continuous, it is measureable too. For $t \in I$ fixed let $s_k(x) := \sum \alpha_i \chi_{E_i}(x)$ be the approximating sequence that is guaranteed by measurability, i.e. $\lim_{n\in\mathbb{N}} |s_k(x) - f(t,x)|$ for $a.e.\, x \in J$. Now $s_k(x)g(t)$ is a simple function converging to $f(t,x)g(t)$ due to

$$\|s_k(x)g(t) - f(t,x)g(t)\|_X \leq |s_k(x) - f(t,x)|\|g(t)\|$$

By definition the Bochner integral attends

$$\int_I f(x,t)g(t)\,dx = \lim_{n\in\mathbb{N}} \sum_{i=0}^{n} \alpha_i \lambda(E_i) g(t)$$

i.e.

$$\lim_{n\in\mathbb{N}} \| \int_I f(x,t)g(t)\,dx - \sum_{i=0}^{n} \alpha_i \lambda(E_i) g(t)\|_X = 0$$

Again the homogenity of the norm and the definition of the Lebesgue integral deliver us the required result.

$$\begin{aligned} &\lim_{n\in\mathbb{N}} \| \int_I f(x,t)\,dx g(t) - \sum_{i=0}^{n} \alpha_i \lambda(E_i) g(t)\|_X \\ =&\lim_{n\in\mathbb{N}} \|g(t)\|_X \left| \int_I f(x,t)\,dx - \sum_{i=0}^{n} \alpha_i \lambda(E_i) \right| = 0 \end{aligned}$$

Every Banach space is a metric space, in which limits in X are unique. So we proved

$$\int_J f(t,x)g(t)\,dx = g(t) \int_J f(t,x)\,dx \quad \forall t \in I$$

□

Lemma B.16. *Let $f \in L^p(I,X)$ be fixed for $1 \leq p < \infty$. Let*

$$(\iota f)(x) := \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I. \end{cases}$$

be the trivial embedding of $L^p(I,X)$ into $L^p(\mathbb{R},X)$.

Then the mapping

$$s \mapsto (\iota f)_s := (\iota f)(\cdot + s) : I \to L^p(I,X)$$

is uniformly continuous.

Proof. The proof follows the classical strategy and is an adopted version of [Kal11, Korollar 13.3.6].

1. First prove the result for step functions of the type $\chi_{[a,b)}(x)u$ where $u \in X$.

   Let $f_t := \chi_{[a,b)-t}(x)u$ and $f_s := \chi_{[a,b)-s}(x)u$, then

$$\begin{aligned}\|(\iota f)_t - (\iota f)_s\|^p_{L^p(I,X)} &= \int_I \|(\chi_{[a,b)-t}(x) - \chi_{[a,b)-s}(x))u\|^p_X \, dx \\ &= \|u\|^p_X \int_I |\chi_{[a,b)-t}(x) - \chi_{[a,b)-s}(x)| \, dx \le 2\|u\|_X \, |s-t|\end{aligned}$$

   as long as $|t-s|$ is sufficiently small. Actually as long as $|t-s|$ is small enough, the indicator functions overlap and therefore the integral over their difference is at most twice the non overlapping area, which is $|t-s|$.

   Using the estimate we conclude that $(\iota f)_t$ is uniformly continuous.

2. This obviously leads to uniform continuity for simple functions s_k too.

3. Finally we take $f \in L^p(I,X)$ which can be approximated by a sequence of simple functions s_k by definition. The convergence in $L^p(I,X)$ actually means $\lim_{k\to\infty} \|f - s_k\|_{L^p(I,X)}$. The estimate

$$\int_I \|f(x)\|^p_X \, dx = \int_I \|(\iota f)(x)\|^p_X \, dx \ge \int_I \|(\iota f)(x-s)\|^p_X \, dx, \tag{B.2}$$

   is a simple observation for the classical Lebesgue integral. Now we can conclude uniform convergence of $t \mapsto (s_k)_t$ by $\|f - s_k\|_{L^p(I,X)} \ge \|(\iota f)_t - (\iota s_k)_t\|_{L^p(I,X)}$.

   Since $t \mapsto (\iota s_k)_t = t \mapsto (s_k)_t$ on I and the latter is continuous, $t \mapsto (\iota f)_t$ is continous on I as uniform limit of continuous functions.

   Due to $(\iota f)_t - (\iota f)_s = (\iota f - (\iota f)_{s-t})_t$ and the already mentioned fact for translations (B.2) we get $\|\iota f - (\iota f)_{s-t}\|_{L^p(I,X)} \ge \|(\iota f)_t - (\iota f)_s\|_{L^p(I,X)}$.

   Finally $t \mapsto (\iota f)_t$ is continuous at 0 and we therefore proved uniform continuity.

□

B.3 Sobolev-Bochner spaces

In this section we shortly introduce the most important facts about the Sobolev spaces generated by the previous Lebesgue-Bochner spaces. The references are [Rou13] and [Mel14].

Definition B.17. Let X be a Banach space and let $f,\ g \in L^1_{loc}(I,X)$. We call g the weak derivative of f denoted by $f' := g$ if and only if

$$\int_I f(t)\phi'(t)\, dx = -\int_I g(t)\phi(t)\, dt \quad \forall \phi \in C_0^\infty(I)$$

In separabel Banach spaces the weak derivative is unique.

The following characterization of the weak derivative is often handy.

Lemma B.18 ([Mel14]). *Let V be separabel and reflexiv. Let H be a Hilbert space and (V', H, V) the corresponding Gelfand triple. Let $f \in L^1(I, V)$ and $g \in L^1(I, V')$. Then g is the unique weak derivative of f if and only if*

$$\int_I (f(t), h)_H \, \phi'(t) \, dt = -\int_I \langle g(t), h \rangle_{V' \times V} \, \phi(t) \, dt \quad \forall \phi \in C_0^\infty(I) \quad h \in V.$$

Definition B.19. Let (V, H, V') be a Gelfand triple. Then we call

$$W^{p,q}(I, V, V') = \{f \in L^p(I, V) : f' \in L^q(I, V')\}$$

a Sobolev-Bochner space.

For Sobolev-Bochner spaces we get the following very useful theorem.

Theorem B.20 (Calculus for Bochner spaces [Rou13]). *Let (V, H, V') be a Gelfand triple. Furthermore let $q = \frac{p}{p-1}$ be the Hölder conjugate of p. Then the following statements hold:*

1. *$C^1(I, V)$ is densely embedded in $W^{p,q}(I, V, V')$*
2. *$W^{p,q}(I, V, V')$ is continuously embedded in $C(I, H)$.*
3. *The following integration by parts formula holds for every $t_1, t_2 \in I$ and $f, g \in W^{p,q}(I, V, V')$*

$$(f(t_2), g(t_2))_H - (f(t_1), f(t_1))_H = \int_{[t_1, t_2]} \langle f'(t), g(t) \rangle_{V' \times V} + \langle f(t), g'(t) \rangle_{V \times V'} \, dt$$

4. *For the derivative of the norm we get:*

$$\frac{d}{dt} \|f(t)\|_H^2 = 2 \langle f'(t), f(t) \rangle_{V' \times V} \quad a.e.$$

Lemma B.21. *Let Ω be a bounded domain, then for every $f \in L^2(I, H_0^1(\Omega))$ the Poincaré inequality holds i.e.*

$$\|f\|_{L^2(I, H_0^1(\Omega))} \leq C(\Omega) \|\nabla f\|_{L^2(I, L^2(\Omega))}$$

Here ∇f is the weak gradient in the sense of $L^2(\Omega)$.

If $\Omega \subset (-L, L) \times \mathbb{R}^n$ then $C(\Omega) = 2L$.

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