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**MSc Economics**

# Booms, Busts and Informational Cascades in a Sequential Trade Model

A Master's Thesis submitted for the degree of
"Master of Science"

supervised by

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# Affidavit

I, Clemens Possnig

hereby declare

that I am the sole author of the present Master's Thesis,

Booms, Busts and Informational Cascades in a Sequential Trade Model

28 pages, bound, and that I have not used any source or tool other than those referenced or any other illicit aid or tool, and that I have not prior to this date submitted this Master's Thesis as an examination paper in any form in Austria or abroad.

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## Abstract

This paper analyzes a security market with sequential trade and a specialist dealer. I study how heterogeneous Bayes-rational agents with private signals interact if allowed to decide multiple times. I show that in this setting, informational cascades can occur. That is, all agents choose to follow public information over their private signal. I introduce a shortselling constraint on the dealer side, i.e. finite supply of the traded asset. When this constraint binds, cascades can be terminated endogenously in the model.

# 1 Introduction

Within the last two decades, the literature on rational models of herding and cascades has provided intuitive insights into market outcomes that at first glance might seem anomalous. The authors Bikhchandani, Hirshleifer and Welch, from now on BHW, were the first to put forth the notion of informational cascades (Bikhchandani *et al.*, 1992). Banerjee independently introduced similar mechanisms under the label of herd behavior (Banerjee, 1992). The structure of their models is the same: a countable sequence of agents must make a once-in-a-lifetime decision, given some exogenous order. The common result of BHW and Banerjee was that it is possible that the publicly available information on actions of past agents can overrule the incentives given by private signals.

Glosten and Milgrom, henceforth GM, proposed that under endogenous prices, such results would not occur. They introduced a seminal sequential trading model with Bayes-rational agents (Glosten et Milgrom, 1985); (Glosten, 1989). GM concluded that under their price mechanism, full learning would occur in the sense that all private information got would be incorporated in the price. This is often referred to as the "price critique", a notion coined by Chari and Kehoe (Chari et Kehoe, 2004). However, by the end of the 90s, articles emerged pointing towards the other direction: Lee presented a model with endogenous prices and continuous choice in which cascades may indeed occur (Lee, 1998). Many other models followed, adapting the GM framework and incorporating cascades: Avery and Zemsky, Décamps and Lovo, Cipriani and Guarino are only some authors working in that field. They came to similar conclusions: even with endogenous prices, cascades and herding behavior can occur (Avery et Zemsky, 1998); (Decamps et Lovo, 2002); (Cipriani et Guarino, 2008).

All of these models have in common the fact that as soon as a cascade arises, it never ends: see for example "Corollary 1 (Cascades last for ever)" Cipriani et Guarino (2008, pg.14). Although Lee actually makes way for the possibility of ending the cascade, he only uses an exogenously arriving signal, which trivially breaks down the cascade by introducing a large amount of new information (Lee, 1998). Should it not be possible to find a simple, but intuitive way to incorporate a cascade that terminates endogenously?

In reality, we do not observe cascades and herding behavior that goes on forever - there are a lot of constraints one could think of that would impede such events from happening. The first one was already touched: it is not plausible that throughout an infinite amount of time, no new and unexpected amount of information would enter a certain market, say a stock market. As discussed, such an event would likely destroy a cascade, but is unsatisfactory for a theorist since its incorporation is ad hoc.

Another constraint is finite supply: a herd can only endure as long as the action that is being executed again and again is actually feasible. This paper analyzes the implications of relaxing this crucial simplification. The GM framework as well as the model defined by Cipriani and Guarino (2008) might be seen as a basis for the model that I present, however many important changes are introduced. As a motivation and real life example of my model, I study the NYSE. Its structure is indeed nested in the abstract GM scheme. The literature on cascades in trading models was mainly introduced to rationally explain behavior on stock markets that could not be explained by fundamentals. The bubble behavior that led to many of the recent financial crises in the 90s and the Dot-com Bubble in the early 2000s can be interpreted as having arisen due to herding and cascades. My model presents a simple framework that rationally explains the development of a price bubble and its bursting, just as it has been observed multiple times in history.

The structure of the paper is as follows: Sections 2 and 3 present the model and its solution, Section 4 deals with informational cascades in the model, Section 5 shows how finite supply of the asset can affect cascades and Section 6 concludes.

## 2 The Model

The model analyzes the trading behavior of individuals sequentially arriving at the market for a risky asset. Prices will be determined endogenously by a market maker. Contrary to most of the herding and cascades literature, I study not once-in-a-lifetime decisions. Individuals are allowed to reevaluate their position at every point in time at which they are in the market and state new orders. They are endowed with differing preferences for discounting and a private signal on the terminal value of the risky asset. Agents trade two assets, a risky asset with ex post liquidation value $V \in \{\underline{V}, \bar{V}\}$ and a riskless asset, called cash. At each trading round the final value of the risky asset becomes known with probability $\beta$ and at the last period, $T+1$, it becomes known with certainty. Agents are risk neutral and maximize expected utility conditional on the information available at the time of the decision. At every period $t$, traders are allowed to sell, buy or hold out: $z \in \{-1, 0, 1\}$. The discrete choice set will prove a crucial part of the model. Additionally, since the focus of this paper is the description of bubble schemes in sequential trade settings, concentrating the analysis on whether an agent buys or sells is sufficient. At every period in the game traders reevaluate their planned trading schedule and are allowed to change it. As in the GM model, agents act as price takers.

Traders maximize $u(W)$, where $W$ is their final cash holding which depends on the vector $z \in \{-1,0,1\}^{T-t}$, if the trader had entered the market at $t$. They start out with different initial wealth levels $W^i \in \mathbb{Z}$. Different private signals $\theta^i \in \{\underline{\theta}, \bar{\theta}\}$ correlated with the liquidation value of the risky asset such that their precision is $q = Pr(\theta = \bar{\theta} \mid V = \bar{V}) = Pr(\theta = \underline{\theta} \mid V = \underline{V})$ with $0.5 < q < 1$ are assigned to every trader. Furthermore, a factor $k^i \in \{\underline{k}, \bar{k}\}$ denoting a discount factor on the final value of the asset for agent i, distributed symmetrically around 1 is allotted to each agent. Thus, types are fully described by $(W^i, \theta^i, k^i)$. When considering future events, agents take into account all public information $h_t = \{\{z^i_\tau\}^t_{i,\tau=1}, x^M_t\}$, where $x^M_t$ is the risky asset holding of the market maker, plus their private signal. Let $\mu_t = Pr(V = \bar{V} \mid h_t)$ be the public belief on the liquidation value of the asset being the higher one, with $\mu_0 = 0.5$. Agents trade the asset against a single market maker with discount factor $k^M$ normalized to 1 and a shortselling constraint on his asset holding $x^M_t \geq 0\ \forall t$. The game evolves over trading rounds $t = 1, ..., T+1$, where $T$ is the total number of agents (the trader entering last is allowed to reevaluate his position once).

### 2.1 Traders

Traders are indexed by the period in which they entered the market, i.e. $z^i_t$ corresponds to the order posted at period $t$ by the agent who entered at period $i \leq t$. At every trading round $t$, all incumbent traders plus the newly entered trader are allowed to post trading orders, which are executed at the same time. The ordering of agents arriving at the market is exogenously given. Formally, agent $i$'s decision problem at period $t$ is given by:

$$\max_z E^i_t[u(W^i_t + k^i V x^i_t + \sum_{\tau=t}^{T+1} (k^i V - p_\tau) z^i_\tau]$$

$$\text{s.t.: } W^i_{t+1} = W^i_t - p_t z^i_t \quad x^i_{t+1} = x^i_t + z^i_t$$

Where $E^i_t$ is the expectation operator conditional on $i$'s information available at $t$, $W^i_t$ is the agent's cash holding, $x^i_t$ is their risky asset holding, $p_t$ is the price of the risky asset and $z^i_t$ is $i$'s trading decision at $t$. From now on, let $E^i_t \equiv E_t$ as long as the agent taking the expectation is clearly identifiable.

### 2.2 Market Maker

The market maker does not hold a signal or any other additional private information. He posts the price of the risky asset and buys and sells at any trading order arriving at the market. Following a simple rule, he sets the price as the expected liquidation value of the asset conditional on all past trading orders, i.e. conditional on all public information $h_t = \{\{z^i_\tau\}^t_{i,\tau=1}, x^M_t\}$. The risky asset is not available in abundant amounts. The holding $x^M_t$ is therefore accompanied by a shortselling constraint: $x^M_t \geq c \, \forall t$, where $c$ is a non positive constant. Without loss of generality, $c = 0$ for the rest of the paper. This will prohibit the market maker from always being able to post the price as stated in the rule above, which will play an important role in the termination of cascades later in the paper. The market maker's inventory evolves in the following way:

$$x^M_t - Z_t = x^M_{t+1} \geq 0 \qquad Z_t = \sum_{i=1}^{t} z^i_t$$

## 3 Solution

Due to their knowledge of the market maker's inventory, traders don't expect the price to be different from the expected value conditional on public information except if the shortselling constraint is binding. The more traders (buyers) are in the market, the more probable is it that the market maker's constraint will bind. This could lead to complex adjustments in the expectation of future prices, which is however simplified by the assumption of risk neutrality of the traders.

The trader's optimization problem can be defined recursively as:

$t = T + 1$:

$$V_t(\pi^i_t, p_t, W^i_t, x^i_t) = \max_{z_t} E_t[W^i_t + k^i V x^i_t + (k^i V - p_t) z^i_t]$$

Setting $E_t[V] \equiv v$, the optimal choice is

$$z^i_t = \begin{cases} 1 \Leftrightarrow k^i v > p_t \\ 0 \Leftrightarrow k^i v = p_t \\ -1 \Leftrightarrow k^i v < p_t \end{cases}$$

and for $t \leq T$

$$V_t(\pi^i_t, p_t, W^i_t, x^i_t) = \max_{z_t} E_t[\beta(W^i_t + k^i V x^i_t + (k^i V - p_t) z^i_t) + (1-\beta) V_{t+1}(\pi^i_{t+1}, p_{t+1}, W^i_{t+1}, x^i_{t+1})]$$

$$\text{s.t.: } W^i_t - z^i_t p_t = W^i_{t+1} \qquad x^i_t + z^i_t = x^i_{t+1}$$

$$p_{t+1} = E[V \mid h_{t+1}] \quad \pi^i_{t+1} = Pr(V = \bar{V} \mid h_{t+1}, \theta^i)$$

In every period $t$, traders solve their decision problem recursively, conditional on their expectation of future prices. Given an optimal decision for the last period, $T+1$, the value function for period $t = T$ can be written as follows:

$$V_t(\pi^i_t, p_t, W^i_t, x^i_t) = \max_{z_t} E_t[\beta(W^i_t + k^i V x^i_t + (k^i V - p_t) z^i_t) +$$

$$+(1-\beta)(W^i_{t+1} + k^i V x^i_{t+1} + (k^i V - p_{t+1}) z^i_{t+1})]$$

$$= \max_{z_t} E_t[(W^i_t + k^i V x^i_t + (k^i V - p_t) z^i_t) + (1-\beta)(k^i V - p_{t+1}) z^i_{t+1}]$$

For an arbitrary $t$ then, the decision is:

$$V_t(\pi^i_t, p_t, W^i_t, x^i_t) = \max_{z_t} E_t[(W^i_t + k^i V x^i_t + (k^i V - p_t) z^i_t) + \sum_{m=1}^{T+1-t} (1-\beta)^m (k^i V - p_{t+m}) z^i_{t+m}]$$

The additive separability between the trading choice to be executed today against those of the future implies that every trader in the market faces a static decision at any given period, just as in the final period $T+1$. By the linearity of the problem and since traders don't face any shortselling constraints, they do not change their behavior as their cash endowment changes. Thus, an agent's type is fully described by the pair $(k, \theta)$.

## 4 Informational Cascades and Expectation Formation

An informational cascade is defined as a situation in which traders disregard their private signal $\theta^i$ and act on public information $h_t$ only (Bikhchandani *et al.*, 1992).

**Definition 1.** In an informational cascade, $Pr(z_t = y \mid h_t, \theta) = Pr(z_t = y \mid h_t)$ holds for all $y \in \{-1, 0, 1\}$, all signals $\theta$ and all traders.

Agents thus trade independently of their private signal. From this point on, no information can be inferred from an agent's trades and the price remains constant throughout the duration of the cascade. Here it would mean that there is so much public information already accumulated in the market (i.e. in the set of public information) that the additional incentive given by the private signal is less than the incentive to trade given by the subjective discount factor. It is important to observe that this behavior is not herding as defined by Banerjee: in his model, action convergence (all agents end up choosing the same option) is observed for the whole realm of all partaking individuals (Banerjee, 1992). Here, this is not the case, since agents do not only differ by their signal, but also by their discount factor. Still, informational cascades and herding behavior are two sides of the same coin, as already observed by Smith and Sorensen (2000). The cascade leads to all individuals ignoring their signal, which in turn leads to herd behavior for all agents with the same discount factor (but possibly differing signals!). Given Definition 1 and optimal behavior of traders described in the previous section, Lemma 1 can be introduced:

**Lemma 1.** *As long as there is no informational cascade, every trader reveals his signal with his first nonzero trade.*

*Proof.* When there is no cascade, newly entering traders use their private signal as incentive to trade, no matter their discount factor. Thus if at $t$ there is no cascade, it holds that

$$E_t[V \mid h_t, \underline{\theta}] < E_t[V \mid h_t] < E_t[V \mid h_t, \bar{\theta}]$$

Which means that upon observation of this traders action, one can undoubtedly infer his signal. □

This implies that their second action will reveal their discount factor $k$, as it represents their only motive to trade from that time on. Thus, signals and discounting of traders already in the market are common knowledge and will be taken into account by the market maker. Given Definition 1 and the characterization of optimal behavior of traders given in Section 3, the only incentive to trade that remains for agents in a cascade is their individual discount factor. Since the market maker's discount factor is normalized to one with respect to the trader's discount factors, their decision is governed by having either a high or low discount factor. A cascade can therefore equivalently be characterized by two inequalities:

**Definition 2.** An informational cascade is reached if and only if one of the following two inequalities hold:

$$\underline{k} E_t[V \mid h_t, \bar{\theta}] < E_t[V \mid h_t]$$

$$\bar{k}E_t[V \mid h_t, \underline{\theta}] > E_t[V \mid h_t]$$

The set of expected values fulfilling these two inequalities will from now on be called the "cascade region". The model is only meaningful if at the beginning of the game, agents have incentives to trade based on their signal. Thus, some parameter restrictions are needed, which are introduced as Assumption 1:

$$\underline{k} > \frac{(\bar{V} + \underline{V})}{2(q\bar{V} + (1-q)\underline{V})} \quad \bar{k} < \frac{(\bar{V} + \underline{V})}{2((1-q)\bar{V} + q\underline{V})} \tag{1}$$

The assumption is a function of the uninformed average of the liquidation value of the risky asset in comparison to what can be inferred from having obtained a signal. The more precise the signal, the more extreme the allowable discount factors can be, since the incentive to trade on a signal increases in its precision.

### 4.1 Expectation Formation

By Lemma 1 and since there are only two possible private signals, the updating on the public belief given a revealed signal amounts to counting the number of good revealed signals in excess of bad signals or vice versa. Suppose in period $t$, the incoming trader reveals his signal $\bar{\theta}$, so the public belief on the value of the risky asset is updated to $E[V \mid h_t, \bar{\theta}]$. Since there are only two possible values for the risky asset, it suffices to consider $Pr(V = \bar{V} \mid h_t)$ in order to fully characterize the expectation given the information $h_t$. The Bayesian updating scheme given a new signal is then formalized as follows:

$$Pr(V = \bar{V} \mid h_t, \bar{\theta}) = \frac{qPr(V = \bar{V} \mid h_t)}{qPr(V = \bar{V} \mid h_t) + (1 - Pr(V = \bar{V} \mid h_t))(1-q)}$$

$$= \frac{Pr(V = \bar{V} \mid h_t)}{Pr(V = \bar{V} \mid h_t) + \frac{(1-q)}{q}(1 - Pr(V = \bar{V} \mid h_t))}$$

Suppose now that in the next period a buy order reveals another signal $\bar{\theta}$ :

$$Pr(V = \bar{V} \mid h_{t+1}, \bar{\theta}) = \frac{qPr(V = \bar{V} \mid h_t)}{qPr(V = \bar{V} \mid h_t) + (1-q)\frac{(1-q)}{q}(1 - Pr(V = \bar{V} \mid h_t))}$$

$$= \frac{Pr(V = \bar{V} \mid h_t)}{Pr(V = \bar{V} \mid h_t) + (\frac{1-q}{q})^2(1 - Pr(V = \bar{V} \mid h_t))}$$

Iterating this scheme supposing a longer sequence of signals shows that the number of good revealed signals in excess of bad signals will become important in the analysis. I therefore introduce the variable $\nu_t^s$, the net number of revealed signals. It denotes the difference between the number of good and bad signals that have been revealed in the market between period $t$ and $s$, with $s > t$. As long as it is clear by context, I will drop the starting period $t$. By Lemma 1, $\nu^s = \left|\{z_{t+i}^{t+i} \mid z_{t+i}^{t+i} = 1\}_{i=0}^{s-t}\right| - \left|\{z_{t+i}^{t+i} \mid z_{t+i}^{t+i} = -1\}_{i=0}^{s-t}\right|$. The following two lemmata will be integral in the upcoming proof of Proposition 1.

**Lemma 2.** *The updating of the public belief $\mu_s = Pr(V = \bar{V} \mid h_s)$ is a function of $\nu^s$ and can be formulated in terms of the belief at $t < s$ and the integer $\nu^s$:*

$$\mu_s = Pr(V = \bar{V} \mid h_t, \nu^s = m) = \frac{Pr(V = \bar{V} \mid h_t)}{Pr(V = \bar{V} \mid h_t) + (\frac{1-q}{q})^m (1 - Pr(V = \bar{V} \mid h_t))}$$

Where $m$ is positive if there are more good than bad signals, and negative if not.

*Proof.* The updating after a bad signal simply multiplies the likelihood ratio $\frac{1-q}{q}$ in the denominator by its inverse, thus decreasing its exponent by one. To illustrate, suppose by period $t+1$, two good signals have been revealed and let the signal revealed in period $t+2$ be $\underline{\theta}$:

$$Pr(V = \bar{V} \mid h_{t+2}, \underline{\theta}) = \frac{(1-q)Pr(V = \bar{V} \mid h_t)}{(1-q)Pr(V = \bar{V} \mid h_t) + q(\frac{1-q}{q})^2(1 - Pr(V = \bar{V} \mid h_t))}$$

$$= \frac{Pr(V = \bar{V} \mid h_t)}{Pr(V = \bar{V} \mid h_t) + (\frac{1-q}{q})(1 - Pr(V = \bar{V} \mid h_t))}$$

This shows that in the updating process every revealed good signal can be canceled out by a bad signal, which completes the proof. □

Since $\frac{1-q}{q} < 1$, as $m$ increases (decreases) the belief gets more and more concentrated on the higher (lower) value of the asset. The orders of other agents are the only channel which conveys information to the traders other than their own private signal. Since outside the cascade region traders fully reveal their signal by their order, these revealed signals are the only additional information available about the risky asset, which leads to the following remark:

*Remark* 1. From $t = 1$ on, all that governs the public belief are the net revealed signals $\nu$.

Applying the insights of Remark 1 to Definition 2, the cascade region, shows that the characterizing inequalities are actually functions of $\nu^t$. The cascade region only depends on the beliefs of traders and the market maker (i.e. the public belief), and as was stated above, beliefs ultimately only depend on the net revealed signals. Therefore, define $\Phi(\nu) < 0$ and $\Theta(\nu) > 0$ as representing the first and second inequality of Definition 2, respectively. Lemma 3 follows naturally:

**Lemma 3.** *At any trading period $t$, the cascade region is characterized by*

$$\Phi(\nu) < 0 \text{ and } \Theta(\nu) > 0$$

*with $\nu$ denoting the difference between the number of revealed good and bad signals.*

*Proof.* Without loss of generality, I consider period $t = 0$ as the period of reference for $\nu^s$, with $s > 0$. Let $\nu_0^t \equiv \nu^t$ then be the net number of signals revealed from period 0 up until period $t$. First recall that at any $t$, the public belief on the asset having the high liquidation value is $\mu_t = Pr(V = \bar{V} \mid h_t)$. The expected value of the asset can be reformalized as follows:

$$E[V \mid h_t] = \mu_t \bar{V} + (1-\mu_t)\underline{V} = (\bar{V} - \underline{V})\mu_t + \underline{V}$$

Then the cascade region at period $t$ can be rewritten as:

$$\underline{k}(\bar{V} - \underline{V}) \frac{\mu_t}{\mu_t + \frac{(1-q)}{q}(1-\mu_t)} + \underline{k}\underline{V} < (\bar{V} - \underline{V})\mu_t + \underline{V}$$

$$\bar{k}(\bar{V} - \underline{V}) \frac{\mu_t}{\mu_t + \frac{q}{(1-q)}(1-\mu_t)} + \bar{k}\underline{V} > (\bar{V} - \underline{V})\mu_t + \underline{V}$$

Using Lemma 2, the above inequalities can be analyzed as the belief varies due to changes in $\nu^t$. For a fixed $t$, $\nu^t \in \{-t, -t+1, ..., t-1, t\}$. Thus, $-T+1 \leq \nu^t \leq T+1$ holds for all $t$. Keeping these bounds in mind, we can abstract from $t$ when considering the number of revealed net signals. Further, let $\mu_0 \equiv \mu$. Then, for a given $\nu$, the cascade region can be restated as follows:

$$\underline{k}(\bar{V} - \underline{V}) \frac{\mu}{\mu + (\frac{1-q}{q})^{\nu+1}(1-\mu)} + \underline{k}\underline{V} - (\bar{V} - \underline{V}) \frac{\mu}{\mu + (\frac{1-q}{q})^{\nu}(1-\mu)} - \underline{V} < 0$$

$$\bar{k}(\bar{V} - \underline{V}) \frac{\mu}{\mu + (\frac{q}{1-q})^{-\nu+1}(1-\mu)} + \bar{k}\underline{V} - (\bar{V} - \underline{V}) \frac{\mu}{\mu + (\frac{q}{1-q})^{-\nu}(1-\mu)} - \underline{V} > 0$$

Referring to the left hand side of the first inequality as $\Phi(\nu)$ and the left hand side of the second inequality as $\Theta(\nu)$ completes the proof of Lemma 2. □

Under this formulation, Assumption 1 ensures that $\Phi(0) > 0$ and $\Theta(0) < 0$. The two functions thus act as an accounting device that shows at any trading period if the market is in an informational cascade or not. Since all past trades are public information, the knowledge of being in a cascade is as well. Proposition 1 ensures the existence of cascades after a finite number of trading rounds.

**Proposition 1.** *There exists an initial value for the market maker's inventory $x_0^M$, a minimum number of trading periods $\tau$ and a sequence of traders such that for any $T \geq \tau$, after a finite number of trading rounds t, $\Phi(\nu^t) < 0$ and $\Theta(\nu^t) > 0$ hold.*

*Proof.* Using Lemmas 2 and 3, I reformulate the cascade region as a function of $\nu^t$ and find that the underlying inequalities indeed hold for finite values of $\nu^t$. Since all agents update their beliefs in the same way, if one trader enters the cascade region, all do, which will complete the proof. Throughout the proof I assume that the market maker's inventory $x_0^M$ is large enough such that the constraint never becomes binding.

I begin by considering $\Phi(\nu)$, the left hand side of the first inequality. Recall that $\Phi(\nu) < 0$ is a necessary condition describing the cascade region. I prove now that there exist two roots $\bar{r}_1, \underline{r}_1$ such that for all $\nu$ satisfying $\nu > \bar{r}_1$ or $\nu < \underline{r}_1$, $\Phi(\nu) < 0$ holds.

Without loss of generality, let $\mu = 0.5$, as it is an uninformative prior in period $t = 0$. Taking the first derivative of $\Phi$ with respect to $\nu$ yields:

$$\frac{\Phi'}{(\bar{V} - \underline{V})} = -\underline{k}\frac{(\frac{1-q}{q})^{\nu+1} log(\frac{1-q}{q})}{(1+(\frac{1-q}{q})^{\nu+1})^2} + \frac{(\frac{1-q}{q})^{\nu} log(\frac{1-q}{q})}{(1+(\frac{1-q}{q})^{\nu})^2}$$

To find an extremum, set $\Phi' = 0$ and solve for $\nu$.

$$\sqrt{\underline{k}(\frac{1-q}{q})}(1+(\frac{1-q}{q})^{\nu}) = 1+(\frac{1-q}{q})^{\nu+1}$$

$$(\frac{1-q}{q})^{\nu} = \frac{1-\sqrt{\underline{k}(\frac{1-q}{q})}}{\sqrt{\underline{k}(\frac{1-q}{q})} - (\frac{1-q}{q})}$$

$$\nu^{\star} = \frac{log(\frac{1-\sqrt{\underline{k}(\frac{1-q}{q})}}{\sqrt{\underline{k}(\frac{1-q}{q})}-(\frac{1-q}{q})})}{log(\frac{1-q}{q})}$$

$\nu^\star$ exists under mild conditions: $\frac{1-q}{q} < \underline{k} < 1$, turning the argument of the logarithm in the numerator non negative, and the argument of the one in the denominator nonzero. Due to the monotonicity of the logarithm and because $log(\frac{1-q}{q}) < 0$, it holds that $\nu < \nu^\star \implies \Phi'(\nu) > 0$ and $\nu > \nu^\star \implies \Phi'(\nu) < 0$. This shows that the extremum is a unique maximum to $\Phi(.)$. Using that by assumption 1, $\Phi(0) > 0$, this shows that $\nu^\star$ is a positive global maximum to $\Phi$. Next I prove that as $\nu$ approaches its negative and positive limits, $\Phi < 0$ will hold. Solving for the limits:

$$\lim_{\nu\to\infty} \Phi(\nu) = \underline{k} - 1 - \frac{(1-\underline{k})\underline{V}}{(\bar{V}-\underline{V})} < 0 \quad \lim_{\nu\to-\infty} \Phi(\nu) = -\frac{(1-\underline{k})\underline{V}}{(\bar{V}-\underline{V})} < 0$$

As $\Phi$ is continuous, I conclude that the function has two roots $\bar{\nu}, \underline{\nu}$ . To complete the proof of the proposition, it remains to show that the second inequality behaves in a similar way.

Recall that

$$\Theta(\nu) = \bar{k}(\bar{V}-\underline{V})\frac{\mu}{\mu + (\frac{q}{1-q})^{-\nu+1}(1-\mu)} + \bar{k}\underline{V} - (\bar{V}-\underline{V})\frac{\mu}{\mu + (\frac{q}{1-q})^{-\nu}(1-\mu)} - \underline{V}$$

describes the second inequality of the cascade region as a function $\Theta(\nu)$ of the number of net signals. Since $\nu$ describes the difference between revealed good and bad signals and the ratio $\frac{1-q}{q}$ is inverted in $\Theta(\nu)$, we need to use $-\nu$ instead of $\nu$. $\Theta(\nu) < 0$ is a necessary condition for the cascade region to be reached. Again I set $\mu = 0.5$ as it is an uninformative prior in period $t = 0$. The procedure is similar to the one before: I will find an extremum to $\Theta(\nu)$ and, taking limits, show that there are two roots. Setting $\Theta'(\nu) = 0$ and solving for $\nu$:

$$\nu^\star = -\frac{log(\frac{1-\sqrt{\bar{k}(\frac{q}{1-q})}}{\sqrt{\bar{k}(\frac{q}{1-q})}-(\frac{q}{1-q})})}{log(\frac{q}{1-q})}$$

which exists under the assumption $1 < \bar{k} < \frac{q}{1-q}$. Further, we have that $\nu > \nu^\star \implies \Theta'(\nu) < 0$ and $\nu < \nu^\star \implies \Theta'(\nu) > 0$, which shows that the extremum is a unique minimum. Assumption 1 ensures that the minimum yields $\Theta(\nu^\star) < 0$. Considering the limits:

$$\lim_{\nu\to\infty} \Theta(\nu) = \bar{k} - 1 + \frac{(\bar{k}-1)\underline{V}}{(\bar{V}-\underline{V})} > 0 \quad \lim_{\nu\to-\infty} \Phi(\nu) = \frac{(\bar{k}-1)\underline{V}}{(\bar{V}-\underline{V})} > 0$$

This proves by continuity that $\Theta(v)$ has two roots. Of course, the roots of the two inequalities will not necessarily be the same, but since to the left and right of their extrema the two functions are monotone, one only needs to take the larger (smaller) value in order for both inequalities to be satisfied. Specifically, let $\bar{r}_1, \underline{r}_1$ be the roots of $\Phi$, and $\bar{r}_2, \underline{r}_2$ be the roots of $\Theta$. Then for $\bar{v} = max\{\bar{r}_1, \bar{r}_2\}$ and $\underline{v} = min\{\underline{r}_1, \underline{r}_2\}$ it is true that for all $v$ satisfying $v > \bar{v}$ or $v < \underline{v}$, $\Phi(v) < 0$ and $\Theta(v) > 0$ hold. Given that $\bar{v}, \underline{v}$ are both finite and $-T+1 \leq v \leq T+1$ always holds, the minimal number of trading periods $\tau$ needed for the cascade to happen is $\tau = \bar{v} = \left|\{z_i^i \mid z_i^i = 1\}_{i=0}^{\tau}\right|$ or $\tau = \underline{v} = \left|\{z_i^i \mid z_i^i = -1\}_{i=0}^{\tau}\right|$. □

Cipriani and Guarino introduce a model with the same choice set and discount factor structure for their traders, however no dynamic choices or shortselling constraints (Cipriani et Guarino, 2008). They proof the emergence of cascades using the public belief and looking for upper and lower bounds. Here this is done differently, since the actual knowledge of the necessary values of $v$ for a cascade will prove useful in upcoming issues. Since only net revealed signals matter, it could be the case that only high-signal types entered the market until $\bar{v}$ was reached, which amounts to $v$ being equal to the number of high signals revealed in the market. This then is the minimum number of trading periods in order for the cascade region to be reached. Similarly for $\underline{v}$. Proposition 1 shows that in this model, after some finite amount of periods, it can happen that traders will not act on their signal anymore. The probability of such events will depend on the precision of the private signals as they are integral in the emergence of cascades, which is discussed in the next subsection. Before that, it is insightful to consider the market situation as the number of revealed net signals reaches one of the roots. Suppose the root is an integer. Then, at the root, it could happen that a zero-order will be posted, but only a newly arriving agent would do so. This is the case since all incumbent agents have already revealed their signal, thus for them the two functions $\Phi(v)$ and $\Theta(v)$ by definition can not be equal to zero at that point. The zero-order of the newly arrived agent can be caused by him having both a high and a low signal. However, buy-orders (sell-orders) can still be unambiguously associated with a high (low) signal. Thus it could be that as one of the roots is reached, information gets more slowly revealed in the market, but it still does so. Indeed, only if the number of revealed net signals is large (small) enough such that both roots are "passed", the cascade starts.

**Corollary 1.** *Throughout the duration of a cascade, the number of net revealed signals*

$\nu$ *remains constant at*

$$\nu = \begin{cases} \bar{\nu}+1 \vee \underline{\nu}-1 \; if \; \bar{\nu}, \underline{\nu} \in \mathbb{Z} \\ \lceil \bar{\nu} \rceil \vee \lfloor \underline{\nu} \rfloor \; otherwise \end{cases}$$

*Proof.* By the above discussion, at the roots $\bar{\nu}$ and $\underline{\nu}$, signals can still get revealed. However, as soon as one more net signal in the right direction (i.e. a high signal if $\nu = \bar{\nu}$ and a low signal otherwise) is revealed, the cascade starts and the believe cannot be updated any longer. As no more signals are revealed, the number of revealed net signals stays constant. □

## 4.2 Probability of a Cascade

Proposition 1 provided evidence that in this model, and given that the market maker has enough inventory, cascades can happen. In the upcoming formulations, I will again assume a market maker who never faces a binding shortselling constraint. This greatly simplifies the expressions characterizing the probability of a cascade happening, however the argument does generalize. By Corollary 1, the number of net revealed signals remains constant at a certain level during a cascade. For ease of notation, let $\bar{m} \equiv \lceil \bar{\nu} \rceil$ and $\underline{m} \equiv \lfloor \underline{\nu} \rfloor$, abstracting from integer roots. Knowing the roots, expressing the probability of a cascade is a combinatorial problem. One has to consider the amount of possible sequences of signals that traders could reveal such that a cascade is reached. Let the variable $B \in \mathbb{Z}$ denote this amount. Furthermore, since this is just a combinatorial exercise of distributing the right amount of high or low signals, only positive values of $m$ can be considered, and that without loss of generality. Thus, in a slight abuse of notation, throughout this subsection: $\underline{m} = \mid \underline{m} \mid$ holds.

*Remark* 2. A necessary condition for a cascade to be reached at period $t > \bar{m}$ is that the number of revealed low signals is $\frac{t-\bar{m}}{2}$, implying that the number of revealed good ones is $\frac{t+\bar{m}}{2}$. For a cascade at $\underline{m}$, $\frac{t-\underline{m}}{2}$ high signals and $\frac{t+\underline{m}}{2}$ low signals are required. This condition ensures that the difference of the revealed signals results in the required number.

*Remark* 3. Caution has to be taken, since after a cascade is reached, no signals are revealed anymore. This implies that by considering the number of possible combinations of $\frac{t-\bar{m}}{2}$ low signals in a given set of $t$ signals, too many possibilities are included. Lemma 4 elucidates this observation:

**Lemma 4.** *The number of possible sequences of signals $B_t$ leading to a cascade at (and not before) period t is characterized by:*

$$\bar{B}_t = \binom{t-2}{\frac{t-\bar{m}}{2}} - \sum_{l=1}^{\frac{t-\bar{m}}{2}} \binom{t-\bar{m}-2l}{\frac{t-\bar{m}-2(l-1)}{2}} \bar{B}_{\bar{m}+2(l-1)} -$$

$$-\mathbf{1}_{t\geq 2\underline{m}+\bar{m}} \sum_{i=1}^{\lfloor \frac{t+\bar{m}}{2(\bar{m}+\underline{m})} \rfloor} \binom{t-(2i-1)(\bar{m}+\underline{m})}{\frac{t-\bar{m}}{2} - i\underline{m} - (i-1)\bar{m}} + \mathbf{1}_{t\geq 2\underline{m}+3\bar{m}} \sum_{j=1}^{\lfloor \frac{t-\bar{m}}{2(\bar{m}+\underline{m})} \rfloor} \binom{t-2j(\bar{m}+\underline{m})}{\frac{t-\bar{m}}{2} - j(\bar{m}+\underline{m})}$$

*$\forall t \geq \bar{m}$, where $\bar{B}_{\bar{m}} = 1$ and $\lfloor x \rfloor$ gives the largest integer less or equal to x. For a cascade at $\underline{m}$, the expression is similar:*

$$\underline{B}_t = \binom{t-2}{\frac{t-\underline{m}}{2}} - \sum_{l=1}^{\frac{t-\underline{m}}{2}} \binom{t-\underline{m}-2l}{\frac{t-\underline{m}-2(l-1)}{2}} \underline{B}_{\underline{m}+2(l-1)} -$$

$$-\mathbf{1}_{t\geq \underline{m}+2\bar{m}} \sum_{i=1}^{\lfloor \frac{t+\underline{m}}{2(\bar{m}+\underline{m})} \rfloor} \binom{t-(2i-1)(\bar{m}+\underline{m})}{\frac{t-\underline{m}}{2} - (i-1)\underline{m} - i\bar{m}} + \mathbf{1}_{t\geq 3\underline{m}+2\bar{m}} \sum_{j=1}^{\lfloor \frac{t-\underline{m}}{2(\bar{m}+\underline{m})} \rfloor} \binom{t-2j(\bar{m}+\underline{m})}{\frac{t-\underline{m}}{2} - j(\bar{m}+\underline{m})}$$

*$\forall t \geq \underline{m}$, where $\underline{B}_{\underline{m}} = 1$.*

*Proof.* The expressions follow directly from the requirement that in order to reach a cascade at a specific period, no cascade must have been reached in any preceding point in time. The last two signals of a sequence are fixed, since each high signal cancels with a low signal. In order to reach the required level of $\nu$, the difference between revealed high and low signals, the last two signals must be of the kind that exists in excess of the other kind. That is, for a cascade at $\bar{m}$ it is necessary that the last two signals of the sequence be positive and conversely for the cascade at $\underline{m}$.

From now on, only the first expression, i.e. the expression treating a cascade at $\bar{m}$, will be discussed, since all arguments apply symmetrically to the latter one. The expression will be analyzed term by term in order to fully characterize it. To make references clearer, enumerate the four terms in the expression:

$$\binom{t-2}{\frac{t-\bar{m}}{2}} \tag{2}$$

$$\sum_{l=1}^{\frac{t-\bar{m}}{2}} \binom{t-\bar{m}-2l}{\frac{t-\bar{m}-2(l-1)}{2}} \bar{B}_{\bar{m}+2(l-1)} \tag{3}$$

$$\mathbf{1}_{t\geq 2\underline{m}+\bar{m}} \sum_{i=1}^{\lfloor \frac{t+\bar{m}}{2(\bar{m}+\underline{m})} \rfloor} \binom{t-(2i-1)(\bar{m}+\underline{m})}{\frac{t-\bar{m}}{2} - i\underline{m} - (i-1)\bar{m}} \tag{4}$$

$$\mathbf{1}_{t\geq 2\underline{m}+3\bar{m}} \sum_{j=1}^{\lfloor \frac{t-\bar{m}}{2(\bar{m}+\underline{m})} \rfloor} \binom{t-2j(\bar{m}+\underline{m})}{\frac{t-\bar{m}}{2}-j(\bar{m}+\underline{m})} \tag{5}$$

By the above remark, the total number of ways to distribute the required negative signals such that $\nu = \bar{m}$, is given by $\binom{t-2}{\frac{t-\bar{m}}{2}}$ at any $t \geq \bar{m}$. However, this number includes also all combinations of such signals that, for a high enough $t$, could have already reached a cascade at some point $l < t$. This fact is incorporated by (3). It is the number of all cascade events that are incorporated in the total amount of combinations $\binom{t-2}{\frac{t-\bar{m}}{2}}$ but would have led to a cascade at some earlier point in the sequence. To be exact, $\bar{B}_{\bar{m}+2(l-1)}$ gives the number of possible signals that would lead to a cascade at period $\bar{m}+2(l-1) < t-2$, while $\binom{t-\bar{m}-2l}{\frac{t-\bar{m}-2(l-1)}{2}}$ counts all combinations of the remaining signals of the sequence in the time periods from $\bar{m}+2(l-1)$ to $t-2$.

This term only subtracts possible cascades at the same level as the one which is sought, in this case $\bar{m}$. Recall that at $t$, in order to arrive at the right root, $\frac{t-\bar{m}}{2}$ negative signals are needed. For high enough $t$, this number can be larger than $\underline{m}$ - which means that by counting $\binom{t-2}{\frac{t-\bar{m}}{2}}$ , also some sequences with cascade at the low level are incorporated. These need to be subtracted. The argument extends to these subtracted possibilities: for even higher $t$, a lot of positive signals are necessary for a cascade to happen at $\underline{m}$, such that they have to be subtracted as well, which explains the occurrence of (4) and (5) in the expression for $B_t$. These terms arise only at time horizons for $t \geq 2\underline{m}+\bar{m}$ and $t \geq 2\underline{m}+3\bar{m}$, respectively. This is so because as soon as $\frac{t-\bar{m}}{2} = \underline{m}$, there are enough negative signals such that also a cascade at the lower root is possible. Then, the number of positive signals required to arrive at $\underline{m}$ is given by $\frac{t-\bar{m}}{2} - \underline{m}$. If this number is greater or equal to $\bar{m}$, the possibility for a positive cascade has to be subtracted as well, which leads to the two conditions. The form of these two terms is given by a simple relationship. First consider (4) : suppose that $2\underline{m}+\bar{m} \leq t$. That means, that cascades at level $\underline{m}$ are counted in the combinations $\binom{t-2}{\frac{t-\bar{m}}{2}}$, which need to be subtracted. One simply needs to find all such possible "unwanted" cascades. As the number of negative signals available is $\frac{t-\bar{m}}{2}$, the number of positive signals required such that level $\underline{m}$ is reached equals $\frac{t-\bar{m}}{2} - \underline{m}$. Thus, the number of combinations of cascades at the lower root is given by all ways of incorporating the amount of positive signals within the sum of required positive and available negative signals: $\binom{t-\bar{m}+\underline{m}}{\frac{t-\bar{m}}{2}-\underline{m}}$. This is the first term in (4) . Should the number of required positive signals be high enough, one needs to subtract all possible combinations leading to an "unwanted" cascade at the high level from the aforementioned term, which gives the first term of (5) : $\binom{t-2(\bar{m}+\underline{m})}{\frac{t-\bar{m}}{2}-(\bar{m}+\underline{m})}$. Again, if the required negative signals in this case are numerous enough, one would need to subtract another term, leading to the second term of the first sum, and so on. This comes to an end as soon as there are not enough signals anymore to distribute such that an "unwanted" cascade would be reached. Thus, the last term in the sum is given by the one which has $k \in \{0,1,...,m-1\}$ signals to distribute, for $m \in \{\underline{m},\bar{m}\}$. The endpoints of the two sums are characterized by this argument: $\frac{t+\bar{m}}{2(\bar{m}+\underline{m})}$ is the value of $i$ which solves $\frac{t-\bar{m}}{2} - i\underline{m} - (i-1)\bar{m} = 0$. Should this not be an integer, the largest integer smaller than this value applies. It is intuitive to observe that $\lfloor \frac{t+\bar{m}}{2(\bar{m}+\underline{m})} \rfloor \geq 1$ for $t \geq 2\underline{m} + \bar{m}$. A similar argument holds for (5) . □

The proof of Lemma 4 shows that there are certain subtleties linked to the emergence of cascades. A striking one is given in the following Corollary:

**Corollary 2.** *For all $t \geq m$, a cascade can only happen at periods $t$ with the same parity as $m$, where $m \in \{\underline{m},\bar{m}\}$.*

*Proof.* The proof is done for cascades at $\bar{m}$, but applies in a similar argument to $\underline{m}$. For a cascade at period $t$, $\frac{t-\bar{m}}{2}$ low signals are required such that the difference between high and low signals gives the root. However, only if $t$ and $\bar{m}$ have the same parity, will the number of required low signals be divisible by 2, i.e. an integer. Since signals only exist in integer amounts, the result follows. □

Now, using the insights given by Lemma 4, the probability of cascades can be formulated:

**Lemma 5.** *Suppose that the market maker's inventory $x_t^M$ is such that the constraint is never binding. Then at $t = 0$, given any horizon of coming periods $s$, the probability of a cascade at root $m$ is*

$$Pr(cascade \mid s,m) = \frac{1}{2}(q^{\frac{s}{2}}(1-q)^{\frac{s}{2}}\left( (\frac{q}{1-q})^{\frac{m}{2}} + (\frac{1-q}{q})^{\frac{m}{2}}) \sum_{j=0}^{\frac{s-m}{2}} q^{-j}(1-q)^{-j} B_{s-2j} \right)$$

*Where $m \in \{\underline{m},\bar{m}\}$ and $B \in \{\underline{B},\bar{B}\}$ are as defined in Lemma 4.*

*Proof.* Since $B_t$ gives the number of possible combinations of signals leading to a cascade at $t$ and all such terms are mutually exclusive as was shown in the proof for Lemma 4, one simply needs to sum up all possibilities of cascades for all $t \leq s$, and multiply each such term by the respective probability of observing the required amount of high and low signals. Integrating the uninformative prior over the two possible values of the risky asset and rearranging terms yields the result. □

**Proposition 2.** *Suppose that the market maker's inventory $x_t^M$ is such that the constraint is never binding. Then at $t = 0$, given any horizon of coming periods s, the probability of a cascade is*

$$Pr(cascade \mid s) = Pr(cascade \mid s, \bar{m}) + Pr(cascade \mid s, \underline{m})$$

*Where $Pr(cascade \mid s, m)$ is as defined in Lemma 5.*

*Proof.* Due to their structure, $\bar{B}_t$ and $\underline{B}_t$ count mutually exclusive events of cascades at either the high or the low root. Therefore, their occurrence is independent, which leads to the result above. □

A numerical example will provide insight into the magnitude of these probabilities:

**Example 1.** Let $k \in \{0.95, 1.05\}$, $V \in \{1, 5\}$ and $q = 0.8$. It is straightforward to check that these priors abide by the constraints set by Assumption 1 and in the proof of Proposition 1. Then, using a root finding algorithm, one finds that $\bar{m} = 3$ and $\underline{m} = -4$. For a cascade with belief concentrated at the higher value of the risky asset, there have to be 3 more high signals revealed in the market than low ones, and vice versa for the low belief cascade. Using the formula given by Proposition 2, the probability of a cascade can be plotted against an increasing sequence of future horizons $s$. For $s = 3$, which is the minimum number of trading periods needed in order for the cascade to arise with belief concentrated at the high level, the probability is $Pr(cascade \mid \nu = \bar{m}, s = \bar{m}) = 0.26$. I plot the probability that a cascade happens as a function of how long the market will operate:

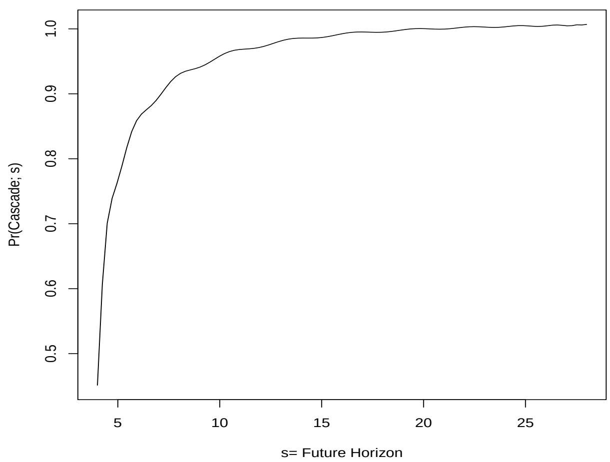


Figure 1: Probability of a Cascade
Figure rendered using R Software: (R Core Team, 2016)

It is shown how with increasing $s$ the probability of a cascade gradually approaches 1. This is an intuitive result due to the fact that as the future horizon increases, the amount of possible sequences of signals increases exponentially. Thus, it becomes more and more unlikely that one does not observe a cascade in this model. The wiggles in the plot are due to the fact that depending on the parity of the roots, cascades at say $\bar{m}$ can only happen at time periods of parity equal to that of $\bar{m}$, as shown in Corollary 2.

## 5 The Market Maker's Constraint

Up until now I was assuming that the market maker had enough inventory to maintain an orderly market. As stated in the Introduction, cascades seemed to be unstoppable once they started, which is a somewhat unsatisfactory outcome. This model allows for the incorporation of finite supply of the asset of interest, a quite realistic extension. Now I will discuss what happens in the case that the market maker's constraint starts to bind. As defined in the beginning, the market maker faces a shortselling constraint at all $t$:

$$x_t^M - Z_t = x_{t+1}^M \geq 0 \quad Z_t = \sum_{i=1}^{t} z_t^i$$

Additionally, recall that the market maker's price setting rule gives that $p_t = E[V \mid h_t]$. However, when the market maker's constraint becomes binding, this is no longer possible. Suppose that at period $t$, after all orders have been satisfied, it holds that $x_t^M = 0$. As the market maker knows the optimal strategies of the traders as well as either the whole type of incumbent traders (if no cascade) or their discount factor (if in a cascade), he can always correctly expect the orders he will face in the coming period, for all traders but the newly arriving one. There is no information as to which type this trader might have. Thus, should the market maker's constraint bind in period $t$, he has to make sure that the incoming trader of period $t+1$ has no incentive to buy, because he will not be able to cater to such an order. The market maker will set $p_{t+1} \equiv p_{t+1}^{bd} = \bar{k}E[V \mid h_t, \bar{\theta}]$. Given this price, no possible type of trader will want to buy. Remark 4 summarizes this observation:

*Remark* 4. Suppose at $t$, $x_t^M = 0$. Then, $p_{t+1}^{bd} = \bar{k}E[V \mid h_t, \bar{\theta}]$

That is, no trader will buy: only traders with high signal and discount factor set $z_t^i = 0$, all others sell. The content of the information conveyed by these orders is not as unambiguous as is the case if the market maker has no binding constraint. For $z_t^i = 0$ the high signal can immediately be concluded. However, nonzero trades are not as easily understandable to the public as before (the public is to be understood as all incumbent traders plus the market maker).

Suppose that the market maker's constraint binds during a cascade. Since the cascade has started, no trader has revealed his private information. All their orders were governed by them having a high or low discount factor, thus this is the only information that the market maker (and also the public) then has on any trader who entered during the cascade. Upon observing the amount of negative orders at $t$, it cannot be discerned by the public which trader is the newly arrived one, which entered during the cascade and which before the cascade. The public will update their belief on the value of the risky asset in a similar, but more complicated way than was shown in Lemma 2. Up until now, at most one new piece of information (signal) was revealed in every trading round. In the current situation, a lot of incumbent traders have not revealed their signal yet, and some information on their signals is revealed by their trades given $p_t^{bd}$. Let us consider the information conveyed by trades of the newly arrived agent. If he set $z_t^t = 0$, his high signal is perfectly revealed. Suppose for a moment that the newly entered trader posted a negative trading order and that the public can observe who he is. Then, all that incumbent agents can infer is that this trader was not of type $(\bar{k}, \bar{\theta})$. Lemma 6 describes how this information will be incorporated in the public's belief:

**Lemma 6.** *Suppose at $t$ the market has been in a cascade and $p_t = p_t^{bd}$. If $z_t^t = -1$, the market maker's belief gets updated in the following way:*

$$Pr(V = \bar{V} \mid h_t, z_t^t = -1) = \frac{\mu_t(\frac{1}{3}q + \frac{2}{3}(1-q))}{\mu_t(\frac{1}{3}q + \frac{2}{3}(1-q)) + (1-\mu_t)(\frac{1}{3} + \frac{1}{3}q)}$$

$$= \frac{\mu_t}{\mu_t + (\frac{1+q}{2-q})(1-\mu_t)}$$

*Proof.* As stated before, if $z_t^t = -1$ there are three types who could have posted the order: $(\bar{k}, \underline{\theta})$, $(\underline{k}, \bar{\theta})$ and $(\underline{k}, \underline{\theta})$. Then we have: $Pr[\theta^t = \bar{\theta}] = \frac{1}{3}$, and $Pr[\theta^t = \underline{\theta}] = \frac{2}{3}$. Thus,

$$Pr(z_t^t = -1 \mid V = \bar{V}) = \frac{1}{3}q + \frac{2}{3}(1-q)$$

Then applying Bayes Rule similarly as in Lemma 2 to the belief $Pr(V = \bar{V} \mid h_t) = \mu_t$ yields the result. □

In order to fully characterize the updating of the market maker's belief given trades at the boundary price, we are left with analyzing what else gets revealed by the orders taken at $t$. It is helpful to list what exactly the market maker knows at $t$, given that the cascade started at period $s < t$:

1. $\left|\{z_t^i \mid z_t^i = 0\}_{i=0}^t\right|$ and $\left|\{z_t^i \mid z_t^i = -1\}_{i=0}^t\right|$, the number of zero and negative orders at $t$.
2. The exact number of zero and negative orders taken by traders who entered before period $s$, since they fully revealed their types $(k, \theta)$.
3. The exact number of high and low discount factors of traders who entered in periods $\tau$, where $s \leq \tau < t$, since these governed their trades during the cascade.

Using this knowledge the market maker can infer the following about the incumbent traders, apart from the newly entered one: He can set aside the identified trades done by agents who entered before $s$, since their information has already been incorporated in his belief. Then for all traders who entered during the cascade and revealed low discount factors it holds that they will have posted negative orders. They are, equally likely to have had high or low signals, so no information can be revealed by their trades. The market maker can thus concentrate his analysis on the traders who entered during the cascade and revealed a high discount factor. For them, every negative order reveals a low signal, and every zero order reveals a high signal. There remains one problem: by reducing the number of orders to those that are of interest for him, the market maker will be left with a number of orders equal to the number of traders who entered during the cascade and revealed a high signal plus one, since the order done by the newly arrived agent cannot be assigned to any group. It is not possible for the market maker to infer the type of the agent who arrived in $t$ by his first trade. Given this argument, Lemma 7 characterizes the updating given a boundary price during a cascade.

The reader might have expected that a bit of new notation will be convenient: Let $n_h$ and $n_l$ be the number of traders who entered the market during the cascade, have high discount factors and posted zero and negative orders, respectively. Specifically, if the cascade started at period $s < t$, then $n_h = \left|\{z_t^i \mid z_t^i = 0, k^i = \bar{k}\}_{i=s}^{t-1}\right|$ and $n_l = \left|\{z_t^i \mid z_t^i = -1, k^i = \bar{k}\}_{i=s}^{t-1}\right|$. Let $\underline{m} = \lfloor \underline{v} \rfloor$, respectively $\bar{m} = \lceil \bar{v} \rceil$ be the level of revealed signals that prevails during a cascade by Corollary 1. Without loss of generality, I abstract from the case where $\bar{v}$ or $\underline{v}$ are integers. The following two Lemmata describe the beliefs of the market maker and the traders upon observing the orders as $p_t = p_t^{bd}$.

**Lemma 7.** *Suppose at $t$, the market has been in a cascade since period $s < t$ and $p_t = p_t^{bd}$. Then, given all trading orders posted in $t$, the market maker's belief is updated in the following way:*

$$Pr(V=\bar{V} \mid h_{t+1}) = \mu^M_{t+1} = \frac{1}{4}\frac{1}{1+(\frac{1-q}{q})^{n_h-n_l+m+1}} + \frac{3}{4}\frac{1}{1+(\frac{1-q}{q})^{n_h-n_l+m+1}(\frac{1+q}{2-q})}$$

*where*

$n_h = \left|\{z^i_t \mid z^i_t = 0, k^i = \bar{k}\}^{t-1}_{i=s}\right|$ , $n_l = \left|\{z^i_t \mid z^i_t = -1, k^i = \bar{k}\}^{t-1}_{i=s}\right|$ *and* $m = \bar{m}$ *if the belief is concentrated at the high value of the asset and* $\underline{m}$ *if it is concentrated at the low level.*

*Proof.* As stated above, the market maker cannot observe which of the orders was posted by the newly arrived agent. He only knows that $Pr(z^t_t = 0) = \frac{1}{4}$ and $Pr(z^t_t = -1) = \frac{3}{4}$ since the agent can have four possible types and only the type $(\bar{k}, \bar{\theta})$ would post a zero-order. Given these two possibilities, he will have two possible updating schemes:

If $z^t_t = 0$, the newly arrived trader's signal is fully revealed, and for all other traders with high discount factor the signal is revealed as well. Recall that during the cascade the belief was stuck at

$$\mu_t = \frac{1}{1+(\frac{1-q}{q})^m}$$

with $m = \bar{m}$ if the belief is concentrated at the high value of the asset and $\underline{m}$ if it is concentrated at the low level. I used the fact that $v = v^t_0$ constitutes all information available to form the expectation, and that at $t = 0$ the public starts out with an uninformative prior $\mu_0 = \frac{1}{2}$. Then using the revealed signals with probability $\frac{1}{4}$ it holds that

$$\mu_{t+1} = \frac{1}{1+(\frac{1-q}{q})^{n_h-n_l+m+1}}$$

where $n_h = \left|\{z^i_t \mid z^i_t = 0, k^i = \bar{k}\}^{t-1}_{i=s}\right|$ and $n_l = \left|\{z^i_t \mid z^i_t = -1, k^i = \bar{k}\}^{t-1}_{i=s}\right|$

If $z^t_t = -1$, the newly arrived trader could have had a high or a low signal, but not with equal probabilities. As for all types with high discount factor who entered during the cascade their signal is revealed, subtracting one (the trade done by the newly arrived trader) from $n_l$ gives exactly the number of revealed low signals. Then, using Lemma 6, with probability $\frac{3}{4}$ it holds that

$$\mu_{t+1} = \frac{1}{1+(\frac{1-q}{q})^{n_h-n_l+m+1}(\frac{1+q}{2-q})}$$

Taking expectations over the type of the newly arrived trader completes the proof. □

Here, traders will not in general hold the same belief as the market maker: except for those who entered before the cascade started traders may still hold private information on their type. Lemma 8 gives a characterization of the updated beliefs of all incumbent traders observing the orders at the boundary price:

**Lemma 8.** *Suppose at t, the market has been in a cascade since period $s < t$ and $p_t = p_t^{bd}$. Then, given all trading orders posted in t, the incumbent trader's beliefs are characterized by the following:*

*Equal to the market maker's belief, for trades with index $\tau < s$:*

$$Pr(V = \bar{V} \mid h_{t+1}^{\tau}) = \mu_{t+1}^{M} = \frac{1}{4}\frac{1}{1+(\frac{1-q}{q})^{n_h-n_l+m}} + \frac{3}{4}\frac{1}{1+(\frac{1-q}{q})^{n_h-n_l+m+1}(\frac{1+q}{2-q})}$$

*For traders with index $s \leq \tau < t$ :*

$$Pr(V = \bar{V} \mid h_{t+1}^{\tau}) = \begin{cases} \mu_{t+1}^{M} & if\ k^{\tau} = \bar{k} \\ \frac{\mu_{t+1}^{M}}{\mu_{t+1}^{M}+(1-\mu_{t+1}^{M})\lambda_{\theta}} & if\ k^{\tau} = \underline{k} \end{cases}$$

*For the trader with index t (the newly arrived trader):*

$$Pr(V = \bar{V} \mid h_{t+1}^{t}) = \begin{cases} \frac{1}{1+(\frac{1-q}{q})^{n_h-n_l+m+1}} & if\ type = (\bar{k},\bar{\theta}) \\ \frac{1}{1+(\frac{1-q}{q})^{n_h-n_l+m}\lambda_{\theta}} & otherwise \end{cases}$$

*Where $n_h = \left|\{z_t^i \mid z_t^i = 0, k^i = \bar{k}\}_{i=s}^{t-1}\right|$ and $n_l = \left|\{z_t^i \mid z_t^i = -1, k^i = \bar{k}\}_{i=s}^{t-1}\right|$ , $\lambda_{\theta} = \frac{1-q}{q}$ if the trader had a high signal, and $\frac{q}{1-q}$ if he had a low signal.*

*Proof.* For $\tau < s$, the proof goes exactly as the proof for Lemma 7. Traders who entered before the cascade started fully revealed their signal. For traders with index $s \leq \tau < t$ and high discount factor, we have shown that their trade fully reveals their signal. Therefore, they also do not hold any informational advantage over the market maker. Nothing can be inferred from traders who arrived during the cascade and revealed a low discount factor, as was stated before Lemma 7. Therefore, their information encompasses the information held by the market maker plus the information they hold via their private signal. By Lemma 2, this information is incorporated in $\lambda_{\theta}$. Finally, the newly arrived trader will, if he has a high signal and a high discount factor, fully reveal his type and hold the same information the market maker does. In any other case, his signal will not be revealed and thus also enters his belief using the term $\lambda_{\theta}$. □

Using the above characterizations of the more complicated updating schemes at a binding constraint for the market maker, Proposition 3 can be proven. It shows that due to the implications of the finite supply of the risky asset, the model carries some intriguing possibilities. At any period of time, if a cascade prevails, it may terminate, given sufficiently low inventory for the market maker.

**Proposition 3.** *Given a cascade, there exists a finite number of newly arriving traders such that the market maker's constraint binds and the number of newly revealed net signals* $\nu$ *is such that* $\Phi(\nu^t) < 0$ *and* $\Theta(\nu^t) > 0$ *both don't hold anymore, so the cascade region is left.*

*Proof.* Suppose that in $t$ the market is in a cascade and the belief is concentrated around $\underline{V}$ (existence was shown in Proposition 1). Further, suppose that $p_t = p_t^{bd}$. Recall from Corollary 1 that this means that $\nu^t = \lfloor \underline{\nu} \rfloor$, where $\underline{\nu}$ is the smaller one of the negative roots of $\Phi(\nu)$ and $\Theta(\nu)$ (and w.l.o.g. abstract from the case where $\underline{\nu}$ is an integer). Furthermore, given the monotonicity of both $\Phi(\nu)$ and $\Theta(\nu)$ around the root, it is implied that for an amount of positive information large enough, the cascade region can be left again. However, as argued in Lemma 7, the updating at the boundary of the market maker's constraint is not as obvious as it was in the usual case. In the now supposed situation, the public belief on the high value of the asset is

$$Pr(V = \bar{V} \mid h_t) = \frac{1}{1 + (\frac{1-q}{q})^{\underline{m}}}$$

With $\underline{m} = \lfloor \underline{\nu} \rfloor$. I exploit the fact that $\nu = \nu_0^t$ constitutes all information available and is used to form the expectation, and that at $t = 0$ the public starts out with an uninformative prior $\mu_0 = \frac{1}{2}$. Given that the belief on the high value of the asset is very low in this situation, we need to find that due to the newly available information the mass allocated to$\bar{V}$ by the belief increases sufficiently such that the cascade region is left. I will show that possibly the market maker's belief after incorporating all information as in Lemma 7 is larger than the belief that prevailed throughout the cascade. First, after some algebra it is possible to reformulate the market maker's belief in the following way:

$$Pr[V = \bar{V} \mid h_{t+1}] = \frac{1}{4}\frac{1}{1 + (\frac{1-q}{q})^{n_h - n_l + \underline{m}}} + \frac{3}{4}\frac{1}{1 + (\frac{1-q}{q})^{n_h - n_l + \underline{m} + 1}(\frac{1+q}{2-q})} =$$

$$= \frac{4 + (\frac{1-q}{q})^{n_h - n_l + \underline{m}}(3 + (\frac{1-q}{q})(\frac{1+q}{2-q}))}{4(1 + (\frac{1-q}{q})^{n_h - n_l + \underline{m}}(1 + (\frac{1-q}{q})(\frac{1+q}{2-q})(1 + (\frac{1-q}{q})^{n_h - n_l + \underline{m}})))}$$

It remains to show that for certain values of $n_h - n_l$, this belief allocates more weight to the high value of the asset than the belief that has prevailed during the cascade:

$$\frac{4+(\frac{1-q}{q})^{n_h-n_l+\underline{m}}(3+(\frac{1-q}{q})(\frac{1+q}{2-q}))}{4(1+(\frac{1-q}{q})^{n_h-n_l+\underline{m}}(1+(\frac{1-q}{q})(\frac{1+q}{2-q})(1+(\frac{1-q}{q})^{n_h-n_l+\underline{m}})))} > \frac{1}{1+(\frac{1-q}{q})^{\underline{m}}}$$

$$\Leftrightarrow (3+(\frac{1-q}{q})(\frac{1+q}{2-q}))((\frac{1-q}{q})^{n_h-n_l+\underline{m}}+(\frac{1-q}{q})^{n_h-n_l+2\underline{m}})+4(\frac{1-q}{q})^{\underline{m}} >$$

$$> 4(\frac{1-q}{q})^{n_h-n_l+\underline{m}}(1+(\frac{1-q}{q})(\frac{1+q}{2-q}))(1+(\frac{1-q}{q})^{n_h-n_l+\underline{m}})$$

$$\Leftrightarrow 4(\frac{1-q}{q})^{-(n_h-n_l)}+(\frac{1-q}{q})^{\underline{m}}(3+(\frac{1-q}{q})(\frac{1+q}{2-q})) >$$

$$> 1+(\frac{1-q}{q})(\frac{1+q}{2-q})(3+4(\frac{1-q}{q})^{n_h-n_l})$$

As $(\frac{1-q}{q}) < 1$, the left hand side increases the larger $n_h - n_l$ gets, while the right hand side decreases with $n_h - n_l$. Thus, for $n_h - n_l$ large enough, the above inequality will hold, which proves the statement for $\nu^t = \lfloor \underline{\nu} \rfloor$. Even more, the above relationship holds just as well for $\nu^t = \lceil \bar{\nu} \rceil$. The belief is concentrated around the high value of the risky asset, thus in order to leave the cascade region, it must be decreased. Using the above argument, for a small enough $n_h - n_l$, the above inequality will be reversed and the cascade region is left. For every trader who enters after this event, their belief will be governed by all that the market maker can infer (this is public knowledge) plus their own signal, and the updating scheme outlined by Lemma 2 applies. This proves that for every trader arriving at least one period after the boundary price was posted, they will trade on their signal and not on their discount factor, so their signal is revealed. □

The insights of Proposition 3 can be used to create some interesting dynamics in this model. Essentially, we observe that this structure allows for a sequence of traders that at first engage in an informational cascade, but due to the shortselling constraint of the market maker something changes: enough information is released into the market such that the cascade ends. An example will illuminate this intuition.

**Example 2.** I reuse the priors from Example 1: $k \in \{0.95, 1.05\}$, $V \in \{1, 5\}$ and $q = 0.8$. Then, $\bar{m} = 3$ and $\underline{m} = -4$. Furthermore, let $x_1^M = 82$. In order to exactly illustrate what happens, assume the following sequence of traders arrive from $t = 1$ to $t = 11$ at the market:

| $t$ | Enter | $p_t$ | $x_t^M$ |
|---|---|---|---|
| 1 | $(\bar{k},\underline{\theta})$ | 3 | 82 |
| 2 | $(\bar{k},\bar{\theta})$ | 1.8 | 83 |
| 3 | $(\bar{k},\underline{\theta})$ | 3 | 83 |
| 4 | $(\bar{k},\underline{\theta})$ | 1.8 | 80 |
| 5 | $(\underline{k},\bar{\theta})$ | 1.24 | 78 |
| 6 | $(\underline{k},\underline{\theta})$ | 1.8 | 75 |
| 7 | $(\underline{k},\bar{\theta})$ | 1.24 | 75 |
| 8 | $(\bar{k},\bar{\theta})$ | 1.8 | 76 |
| 9 | $(\underline{k},\bar{\theta})$ | 3 | 74 |
| 10 | $(\underline{k},\bar{\theta})$ | 4.2 | 71 |
| 11 | $(\bar{k},\bar{\theta})$ | 4.77 | 69 |

Table 1: Sequence of Traders, $1 \leq t \leq 11$

The assumed priors make sure that from $t = 1$ on, the market starts from outside the cascade region. That is, traders reveal their signals by their trading behavior. When checking the sequence, it becomes clear that while the trader arriving at $t = 11$ still reveals his high signal, his trade is the one that sets $\nu_{11} = \bar{m} = 3$. Sufficiently many traders with high signals have arrived in the market such that the resulting positive public belief is overwhelming: the incentive to trade based on private signals is less important than the information already disseminated in the market. From the next period onwards, the market is in a cascade. Starting from an uninformative prior, the price of the risky asset at the first period is $p_1 = 3$. At $t = 12$, the price of the asset has risen to $p_{12} = 4.94$ since the public belief on the high value of the asset has increased to $\mu_{12} = 0.985$. Let the sequence of traders arriving from $t = 12$ be the following:

| $t$ | Enter | $p_t$ | $x_t^M$ |
|---|---|---|---|
| 12 | $(\bar{k},\underline{\theta})$ | 4.94 | 68 |
| 13 | $(\bar{k},\bar{\theta})$ | 4.94 | 66 |
| 14 | $(\bar{k},\underline{\theta})$ | 4.94 | 63 |
| 15 | $(\bar{k},\underline{\theta})$ | 4.94 | 59 |
| 16 | $(\bar{k},\underline{\theta})$ | 4.94 | 54 |
| 17 | $(\bar{k},\bar{\theta})$ | 4.94 | 48 |
| 18 | $(\underline{k},\bar{\theta})$ | 4.94 | 41 |
| 19 | $(\underline{k},\bar{\theta})$ | 4.94 | 35 |
| 20 | $(\underline{k},\underline{\theta})$ | 4.94 | 30 |
| 21 | $(\bar{k},\underline{\theta})$ | 4.94 | 26 |
| 22 | $(\bar{k},\underline{\theta})$ | 4.94 | 21 |
| 23 | $(\bar{k},\underline{\theta})$ | 4.94 | 15 |
| 24 | $(\bar{k},\underline{\theta})$ | 4.94 | 8 |
| 25 | $(\bar{k},\underline{\theta})$ | 5.23 | 0 |
| 26 | any | 1.14 | 17 |

Table 2: Sequence of Traders, $12 \leq t \leq 26$

I am listing the values of $x_t^M$ as well, to check if at some point, the market maker's shortselling constraint becomes binding. Indeed, this happens after the trades of $t = 24$. The market maker will therefore have to set the boundary price at $t = 25$, which amounts to $p_{25}^{bd} = 5.23$. It is higher than the possible high value of the asset because the market maker needs to impede a potential new trader with high discount factor and signal. It is interesting to observe that at this point, the price of the asset cannot be explained by the fundamentals of the risky asset anymore. As outlined by Proposition 3 and the preceding Lemmata, new information is conveyed to the public by the actions taken by incumbent traders at $t = 25$. Only traders with high discount factor and high signal will order zero, while all other traders want to sell. That is, at $t = 25$, the market maker is faced with a cumulative order of $-\sum_{i=1}^{25} z_t^i = 17$, so four zero-orders have been made. Recall that inference can only be made on the signals held by traders who entered during the cascade and have a high discount factor. There are ten traders of this kind in the market. The market maker can assign two of the four zero-orders to traders who fully revealed their type before the cascade had started. Thus, of the ten traders interesting to him due to the possible inference of new information, at least one has made a zero-order, depending on the type of the newly arrived trader. In the notation adopted in Proposition 3 and its preceding Lemmata, $n_h = 2$ and $n_l = 8$. Applying Lemma 7 yields the belief of the market maker given the trades done after having set the boundary price. Due to the overwhelming number of negative signals that have accumulated during the cascade, the market maker decreases his belief drastically: $\mu_{26}^M = 0.034$. This sudden pessimism drops the price to $p_{26} = 1.14$. The following figure shows the development of the price (left axis) and inventory (right axis) as the sequence of traders above enters the market. It is clearly visible how from periods 1 to 11, the price changes as different signals get revealed. The cascade is highlighted by the time points between the two dashed red lines. At $t = 25$ the boundary price is posted, which explains the upward spike in the price sequence. Afterward the price drops as the cascade is terminated and a lot of low signals are revealed.

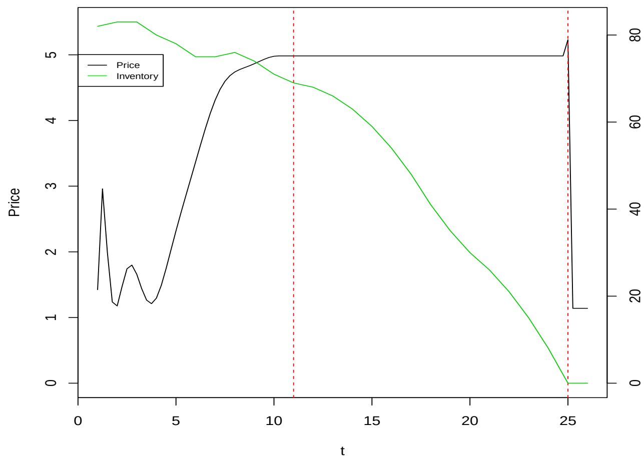


Figure 2: Price and Inventory

Figure rendered using R Software: (R Core Team, 2016)

# 6 Conclusion

In a sequential trading model with risk neutral agents, I show that under endogenous prices, cascades emerge even if traders are allowed to act more than once. The model introduces a shortselling constraint for the market maker which becomes crucial in the termination of cascades. The constraint endogenously defines a point such that a cascade, which under other circumstances would last forever, has to end. This is due to the dissemination not of new information, but of information about traders who are already in the market but have not revealed their private signal yet. Additionally, the probability of cascades in this model is formalized and explored. The assumption of risk neutrality should be seen as a first approximation, opening the field for analysis of similar models under more general specifications of preferences.

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