FAKULTÄT FÜR !NFORMATIK
Faculty of Informatics

Überblick über Unabhängigkeitsresultate in der Mengenlehre

DIPLOMARBEIT

zur Erlangung des akademischen Grades

Diplom-Ingenieur

im Rahmen des Studiums

Computational Intelligence

eingereicht von

Thomas Baumhauer
Matrikelnummer 0927499

an der Fakultät für Informatik
ausgeführt am Institut für Diskrete Mathematik und Geometrie
der Technischen Universität Wien

Betreuung: Ao.Univ.-Prof. Martin Goldstern

Wien, 1. Oktober 2015

Thomas Baumhauer	Martin Goldstern

FAKULTÄT FÜR !NFORMATIK
Faculty of Informatics

Survey of Independence Results in Set Theory

DIPLOMA THESIS

submitted in partial fulfillment of the requirements for the degree of

Diplom-Ingenieur

in

Computational Intelligence

by

Thomas Baumhauer
Student Number 0927499

to the Faculty of Informatics
carried out at the Institute of Discrete Mathematics and Geometry
at the Vienna University of Technology

Advisor: Ao.Univ.-Prof. Martin Goldstern

Erklärung zur Verfassung der Arbeit

Hiermit erkläre ich, dass ich diese Arbeit selbständig verfasst habe, dass ich die verwendeten Quellen und Hilfsmittel vollständig angegeben habe und dass ich die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken oder dem Internet im Wortlaut oder dem Sinn nach entnommen sind, auf jeden Fall unter Angabe der Quelle als Entlehnung kenntlich gemacht habe.

1. Oktober 2015

____

Thomas Baumhauer

Abstract

In the first chapter we show that if ZFC is consistent then so is ZFC plus its relativization to a countable, transitive set. To do this we prove a theorem that shows that every formula is reflected in a countable transitive set and then use a compactness argument. This chapter mostly follows [9].

In the second chapter we develop Cohen's forcing method using Boolean algebras, closely following [6]. We show how partial orders can be embedded into Boolean algebras, define Boolean-valued models of set theory and show how Boolean-valued models can be turned into regular two-valued models by factoring with a generic filter. Finally we prove the forcing theorem.

In the following chapters we prove the independence of various statements from ZFC, using the forcing method. Among them are the continuum hypothesis, the Suslin-hypothesis and the Diamond-principle. Furthermore we show various implications between these statements. The proofs in these chapters are collected from [6][7][8][9][10].

We then show how to iterated the forcing method a transfinite number of times, following [6]. Finally we employ this method of iterated forcing to prove the consistency of Martin's axiom, following [6] and [7].

Kurzfassung

Im ersten Kapitel zeigen wir, dass, wenn ZFC widerspruchsfrei ist, auch ZFC plus ZFC relativiert auf eine abzählbare, transitive Menge widerspruchsfrei ist. Um das zu beweisen zeigen wir, dass jede Formel in einer abzählbaren, transitiven Menge reflektiert wird und verwenden ein Kompaktheitsargument. Wir folgen in diesem Kapitel [9].

Im zweiten Kapitel entwickeln wir die Forcing-Methode von Cohen. Wir benutzen dazu Boolsche Algebren und folgen dabei [6]. Wir zeigen wie man Halbordnungen in Boolsche Algebren einbettet, definieren Modelle von ZFC mit Boolschen Wahrheitswerten und zeigen wie aus solchen Modellen, durch Ausfaktorisieren nach einem generischen Filter, gewöhnliche Modelle mit binären Wahrheitswerten werden. Schließlich beweisen wir das Forcing-Theorem.

In den folgenden Kapiteln zeigen wir die Unabhängigkeit verschiedener Aussagen von ZFC. Darunter sind die Kontinuumshypothese, die Suslin-Hypothese und das Karo-Prinzip. Die Beweise in diesen Kapiteln sind aus [6][7][8][9][10] gesammelt.

Weiters zeigen wir, wie man die Forcing-Methode transfinit wiederholt. Dabei folgen wir [6]. Schließlich benutzen wir die wiederholte Forcing-Methode um die Widerspruchsfreiheit des Martinschen Axioms zu beweisen, wobei wir [6] und [7] folgen.

Contents

Chapter 0

Introduction

0.1 Historical Overview

In 1878 Georg Cantor formulated the following question: Is there a set of cardinality strictly greater than the cardinality of the set of natural numbers but strictly less than the cardinality of the set of real numbers? Cantor believed that no such set exists but failed to prove it. His conjecture became known as the *continuum hypothesis* (CH) and in 1900 appeared as the first entry to David Hilbert's famous list of open problems.

The second problem on Hilbert's list was the proof that arithmetic is consistent. In 1931 Kurt Gödel answered this question by showing that any sufficiently powerful first order theory is incomplete (i.e. there are statements that the theory neither proves nor refutes) and in particular does not prove its own consistency.

While in arithmetic such independent statements are often of somewhat artificial character they tend to occur very naturally in set theory. Cantor's continuum hypothesis turned out to be one of them. In 1940 Gödel showed that the continuum hypothesis is consistent by constructing a model, the so-called *constructible universe*, in which the continuum hypothesis holds. In 1963 Paul Cohen applied his newly developed technique of *forcing* to show the consistency of the failure of the continuum hypothesis, thus proving the continuum hypothesis is in fact independent of set theory.

In this thesis we will only work with Cohen's forcing technique which can also be used to construct a model in which the continuum hypothesis holds.

0.2 The Axioms of Zermelo-Fraenkel

We are going to use the *ZFC* formulation of set theory. ZFC is a theory of first order logic. The language of ZFC consists of a single two-ary predicate symbol $\in$. Before we introduce the axioms of ZFC let us introduce some abbreviations to make notation less cumbersome. All free variables of all formulas in this chapter are to be understood as universally quantified.

The symbol $\subseteq$ is an abbreviation for

$$x \subseteq y \quad \leftrightarrow \quad \forall z(z \in x \to z \in y).$$

The symbol $\emptyset$ is an abbreviation for

$$x = \emptyset \quad \leftrightarrow \quad \forall y(y \notin x).$$

Note that $\emptyset$ is well-defined. It is unique because of the axiom of extensionality and its existence can be proved using the separation scheme. Finally $s(x)$ is an abbreviation for

$$y = s(x) \quad \leftrightarrow \quad \forall z(z \in y \leftrightarrow z \in x \vee z = x).$$

Note that even though we only list nine (i.e. finitely many) axioms of ZFC, two of these axioms are really axiom schemes that state an axiom for each of the countably many formulas in the language ZFC.

Axiom 1. Extensionality.

$$\forall z(z \in x \leftrightarrow z \in y) \to x = y$$

The axiom of extensionality states that two sets are equal if and only if they contain the same elements.

Axiom 2. Pairing.

$$\exists z(x \in z \wedge y \in z)$$

The axiom of pairing states that for any two sets there exists a set that contains both of them. Together with separation this means there exists a set that contains exactly both of them, i.e. for sets x, y the set $\{x, y\}$ exists.

Axiom 3. Separation Scheme.

For each formula ϕ in the language of ZFC in which y does not occur free.

$$\exists y \forall z(z \in y \leftrightarrow z \in x \wedge \phi(z, p))$$

The separation scheme states that for a property ϕ that can be formulated in ZFC and any set x there exists the set

$$\{z \in x : \phi(z)\}.$$

Axiom 4. Union.

$$\exists z \forall y \forall u (u \in y \wedge y \in x \to u \in x)$$

The axiom of union states that for every x the set $\bigcup_{y \in x} y$ exists.

Axiom 5. Power Set.

$$\exists y \forall z (z \subseteq x \to z \in y)$$

The power set axiom states that for every set there exists a set that contains all its subsets as elements. Again combined with separation this means there exists a set that contains exactly its subsets.

Axiom 6. Infinity.

$$\exists x (\emptyset \in x \wedge \forall y (y \in x \to s(y) \in x))$$

The axiom of infinity states that there exists a set that contains all natural numbers.

Axiom 7. Collection Scheme.

For any formula ϕ in the language of ZFC in which Y does not occur free.

$$\forall X \exists Y (\forall x \in X)((\exists y)\phi(x, y) \to (\exists y \in Y)\phi(x, y))$$

Over the other axioms of ZFC this is equivalent to the statement that for any function the image of that function is a set.

Axiom 8. Regularity.

$$x \neq \emptyset \to \exists y (y \in x \wedge \neg \exists z (z \in x \wedge z \in y))$$

The axiom of regularity implies that there is no sequence $x_0, x_1, x_2, \ldots$ such that $x_{n+1} \in x_n$ for all $n \in \omega$. In particular there exists no sequence $x_0, x_1, \ldots x_n$ such that $x_{k+1} \in x_k$ for all $k < n$ and $x_0 \in x_n$.

Axiom 9. Choice. Every set can be well-ordered.

Chapter 1

A Countable Transitive Model

1.1 Relativization

Definition 1.1. Let M be a set and σ be a formula in the language of ZFC. We define the *relativization* $\sigma|_M$ of σ to M inductively as[1]

$$\begin{aligned}
(x \in y)|_M &\leftrightarrow x \in y, \\
(x = y)|_M &\leftrightarrow x = y, \\
(\neg\phi)|_M &\leftrightarrow \neg(\phi|_M), \\
(\phi \vee \psi)|_M &\leftrightarrow \phi|_M \vee \psi|_M, \\
(\phi \wedge \psi)|_M &\leftrightarrow \phi|_M \wedge \psi|_M, \\
(\exists x\phi)|_M &\leftrightarrow \exists x \in M : \phi|_M, \\
(\forall x\phi)|_M &\leftrightarrow \forall x \in M : \phi|_M.
\end{aligned} \tag{1.1}$$

Instead of $\phi|_M$ we sometimes write $M \models \phi$.

Definition 1.2. Let M, N be sets. We say that a function $f : M \to N$ is

[1]There are two logical connectives, $\vee$ and $\neg$. Other connectives are abbreviations:

$$\phi \wedge \psi = \neg(\neg\phi \vee \neg\psi)$$

$$\phi \to \psi = \neg\phi \vee \psi$$

$$\phi \leftrightarrow \psi = (\phi \to \psi) \wedge (\psi \to \phi).$$

There is one quantifier $\exists$ and

$$\forall x\phi(x) = \neg\exists x\neg\phi.$$

This is useful when we want to verify statements inductively on the structure of a formula. However if it makes things more clear we may still treat the abbreviated connectives as if they actually exist.

an $\in$-*isomorphism* if f is bijective and for all $x, y \in M$

$$x \in y \quad \leftrightarrow \quad f(x) \in f(y).$$

Lemma 1.3. *Let M, N be sets and let $f : M \to N$ be an $\in$-isomorphism. Then for all formulas ϕ and for all $a_1, \ldots, a_n \in M$*

$$\phi(a_1, \ldots, a_n)|_M \quad \leftrightarrow \quad \phi(f(a_1), \ldots, f(a_n))|_N.$$

Proof. Easy induction on the structure of ϕ. □

Definition 1.4. A set M is called extensional if

$$\forall x, y \in M : \Big(x \neq y \to (\exists t \in M : t \in x \leftrightarrow t \notin y)\Big).$$

This is simply the axiom of extensionality relativized to M. Informally this means that sets $x \neq y$ in M can be "separated" by an element of M, so they also look different from the perspective of M.

Definition 1.5. A set M is called transitive if

$$\forall x : x \in M \to x \subseteq M.$$

An equivalent formulation of transitivity that explains the name is

$$\forall x, y : (y \in x \wedge x \in M) \to y \in M.$$

Definition 1.6. For every set x we can define its *rank* $\rho(x)$ by

$$\rho(x) = \sup\{\rho(y) + 1 : y \in x\}$$

The axiom of regularity is crucial for this definition to work.

Theorem 1.7 (Mostowski collapse)**.** *Let M be an extensional set. Then there exists a transitive set N and an $\in$-isomorphism $f : M \to N$.*

Proof. Inductively by rank we define for $x \in M$

$$f(x) = \{f(y) : y \in x \cap M\}$$

and let[2] $N = f[X]$.

We convince ourselves that N is transitive. If $a \in N$ and $b \in a$ then there exists $x \in M$ such that $f(x) = a$. Now $a = \{f(y) : y \in x \cap M\}$ and therefore there exists $y \in M$ such that $b = f(y)$. Thus $b \in N$.

Assume that f is not injective. Then there exists $x \in M$ such that there is some $y \in M$ with $y \neq x$ and $f(x) = f(y)$. Let x be an element with minimal rank that has this property. Now there are two cases.

[2] Let $f : M \to N$ be a function and $P \subseteq M$ be a subset of M. Then $f[P] = \{f(m) : m \in P\}$ denotes the image of P under f.

1. There is $s \in M$ such that $s \in x$ and $s \notin y$. Since $f(x) = f(y)$ and $f(s) \in f(x)$ there must be some $t \in y \cap M$ such that $f(s) = f(t)$ and of course $s \neq t$. This contradicts the minimality of x.

2. There is $t \in M$ such that $t \notin x$ and $t \in y$. Analogously we find $s \in x \cap M$ such that $f(s) = f(t)$, again contradicting the minimality of x.

Therefore f must be injective. This settles the non trivial direction of

$$x \in y \quad \leftrightarrow \quad f(x) \in f(y)$$

and thus f is indeed an $\in$-isomorphism. □

1.2 Reflexion

Definition 1.8. Let $\mathrm{C}(X)$ be the formula "X is a countable set" and let $\mathrm{CT}(X)$ be the formula "X is a countable, transitive set".

Lemma 1.9. *Let $\phi_0, \ldots, \phi_n$ be formulas in the language of ZFC. Then there exists a countable set M such that for every $i \leq n$ and all $a_1, \ldots a_{n_i} \in M$.*

$$\exists x : \phi_i(x, a_1, \ldots, a_{n_i}) \quad \rightarrow \quad \exists x \in M : \phi_i(x, a_1, \ldots, a_{n_i}).$$

Proof. For every $i \leq n$ we define a Skolem function f_{ϕ_i} such that

$$\exists x : \phi_i(x, a_1, \ldots, a_{n_i}) \quad \rightarrow \quad \phi_i(f_{\phi_i}(a_1, \ldots, a_{n_i}), a_1, \ldots, a_{n_i})$$

and $f_{\phi_i}(a_1, \ldots, a_{n_i}) = a_1$ if no such x exists. We remark that for this we used the axiom of choice.

Now we define M inductively.

1. $M_0 = \{\emptyset\}$.

2. $M_{k+1} = M_k \cup \bigcup_{i \leq n} f_{\phi_i}[M_k^{n_i}]$

3. $M = \bigcup_{k<\omega} M_k$.

It is clear that if M_k is countable so is M_{k+1} Thus M is the countable union of countable sets and therefore M is countable.

If $a_1, \ldots, a_{n_i} \in M$ then there exists $k < \omega$ such that $a_1, \ldots, a_{n_i} \in M_k$. Thus if $\exists x \phi_i(a_1, \ldots, a_{n_i})$ holds then, because there exists $x = f_{\phi_i}(a_1, \ldots, a_{n_i}) \in M_{k+1} \subseteq M$ such that $\phi(x, a_1, \ldots, a_{n_i})$ holds, we have $\exists x \in M : \phi_i(x, a_1, \ldots, a_{n_i})$. □

Remark 1.10. To be more precise lemma 1.9 is really a lemma-scheme that states that for formulas $\phi_0, \dots, \phi_n$ in the language of ZFC it holds that

$$\mathrm{ZFC} \vdash \exists M : \Big(C(M) \wedge \bigwedge_{i \leq n} \forall a_1, \dots a_{n_i} : \big(\exists x : \phi_i(x, \vec{a}) \rightarrow \exists x \in M : \phi_i(x, \vec{a})\big)\Big).$$

Here $\bigwedge_{i \leq n}$ is to be understood as an abbreviation for copying the same formula for every $\phi_0, \dots, \phi_n$.

Theorem 1.11 (Reflection principle)**.** *Let σ be a sentence in the language of ZFC. Then there exists a countable, transitive set M such that*

$$\sigma|_M \quad \leftrightarrow \quad \sigma.$$

Proof. Let τ be the axiom of regularity and let $\phi_1, ..\phi_n$ be a list of formulas such that $\phi_1 = \sigma$, $\phi_2 = \tau$ and for every $i \leq n$ the following holds:

1. If $\phi_i = \neg\psi$ then there is $j \leq n$ such that $\phi_j = \psi$
2. If $\phi_i = \psi \vee \chi$ then there are $j, k \leq n$ such that $\phi_j = \psi$ and $\phi_k = \chi$.
3. If $\phi_i = (\forall x)\psi$ then there is $j \leq n$ such that $\phi_j = \psi$.

In other words the list $\phi_1, \dots, \phi_n$ contains σ, τ and all their subformulas.

Now let M be the countable set obtained by the application of lemma 1.9 to $\neg\phi_i, \dots, \neg\phi_n$. We are going to show that for all $\phi_i, i \leq n$ it holds that

$$\phi_i(a_1, \dots, a_m)|_M \quad \leftrightarrow \quad \phi_i(a_1, \dots, a_m)$$

for all $a_1, \dots, a_m \in M$ by induction on the complexity of ϕ_i.

If ϕ_i is atomic then the equivalence is trivial. Likewise if $\phi_i = \neg\psi$ or $\phi_i = \psi \vee \chi$ then we have already proven the equivalence of for ψ and χ and again the equivalence follows trivially for ϕ_i.

Now let $\phi_i = (\forall x)\psi(x, a_1, \dots, a_m)$. Clearly if ϕ_i holds so does $(\forall x \in M)\psi(x, a_1, \dots, a_m)$ and by induction hypothesis $\phi_i|_M = \psi|_M(x, a_1, \dots, a_m)$ holds. Conversely if ϕ fails then there exists x such that $\neg\psi(x, a_1, \dots, a_m)$ holds and by lemma 1.9 $x \in M$. Thus $(\forall x \in M)\psi(x, a_1, \dots, a_m)$ fails and and by induction hypothesis $\phi_i|_M = \psi|_M(x, a_1, \dots, a_m)$ fails.

Now because $\phi_1 = \sigma$ it holds that $\sigma|_M \leftrightarrow \sigma$ and because $\phi_2 = \tau$ the set M is extensional and we can apply lemma 1.7. □

Remark 1.12. Again reflection principle 1.11 is really theorem-scheme that states that for any sentence σ in the language of ZFC and its corresponding sentence $\sigma|_M$ it holds that

$$\mathrm{ZFC} \vdash \exists M : (\mathrm{CT}(M) \wedge \sigma|_M \leftrightarrow \sigma).$$

Definition 1.13. We define a theory *ZFC*$^+$. The language of ZFC$^+$ is the language of ZFC with a constant symbol M added. Every axiom ϕ of ZFC$^+$ satisfies either

1. ϕ is an axiom of ZFC or
2. $\phi = \mathrm{CT}(M)$
3. $\phi = \psi|_M$ and ψ is an axiom of ZFC.

Theorem 1.14. *Let σ be formula in the language of ZFC. Then if ZFC$^+ \vdash \sigma$ already ZFC $\vdash \sigma$.*

Proof. Assume that we can prove σ from ZFC$^+$. This proof only uses a finite number of axioms of the form $\phi_i|_M$ where ϕ_i is some axiom of ZFC. Let $\phi = \phi_1 \wedge \cdots \wedge \phi_n$ and observe that $\phi|_M = \phi_1|_M \wedge \cdots \wedge \phi_n|_M$. Therefore we have

$$\mathrm{ZFC} \vdash (\mathrm{CT}(M) \wedge \phi|_M) \to \sigma. \tag{1.2}$$

On the other hand by lemma 1.11 we know that

$$\mathrm{ZFC} \vdash \exists N : \mathrm{CT}(N) \wedge (\phi \leftrightarrow \phi|_N)$$

and because ϕ is a theorem of ZFC we get

$$\mathrm{ZFC} \vdash \exists N : \mathrm{CT}(N) \wedge \phi|_N. \tag{1.3}$$

Combining 1.2 and 1.3 we arrive at

$$\mathrm{ZFC} \vdash \sigma.$$

□

Corollary 1.15. *If ZFC is consistent so is ZFC$^+$.*

Proof. Choose $\sigma = \perp$. □

Remark 1.16. The axioms of ZFC$^+$ state that M is a countable, transitive set that satisfies the axioms of ZFC. Thus in some sense M acts as a countable, transitive model of ZFC. However we need to be careful here. What we state is in fact a scheme

$$\text{For all axioms } \phi \text{ of ZFC} : \mathrm{ZFC}^+ \vdash \big(M \models \phi\big)$$

in the meta-theory which is not to be confused with the statement

$$\mathrm{ZFC}^+ \vdash \big(\forall \phi \in \mathrm{ZFC} : M \models \phi\big).$$

The latter statement implies $\mathrm{ZFC}^+ \vdash \mathrm{Con(ZFC)}$ and therefore by 1.14 $\mathrm{ZFC} \vdash \mathrm{Con(ZFC)}$. Assuming that ZFC is consistent this contradicts Gödel's second incompleteness theorem.

In the following we will simply say "Let M be a countable transitive model...". What we actually mean by this is that we work in ZFC^+.

It remains to justify why "pretending" that M is a countable transitive model is good enough to give us consistency results.

Theorem 1.17. *Let σ be a formula in the language of ZFC. Assume that in ZFC^+ we can construct*[3] *a set $M[G]$ such that:*

1. *$ZFC^+ \vdash \big(M[G] \models \sigma\big)$.*
2. *For every axiom ϕ of ZFC it holds that $ZFC^+ \vdash \big(M[G] \models \phi\big)$*

Then if ZFC is consistent so is ZFC + σ.

Proof. Assume that we can prove $\perp$ in ZFC + σ. Again this prove only uses finitely many axioms ϕ_i of ZFC and let again $\phi = \phi_1 \wedge \cdots \wedge \phi_n$. Note that $\psi = \big((\phi \wedge \sigma) \to \perp\big)$ is a theorem in predicate logic and thus $\psi|_{M[G]} = \big((\phi|_{M[G]} \wedge \sigma|_{M[G]}) \to \perp\big)$ is valid too. Since we assume $\psi|_{M[G]}$ and $\sigma|_{M[G]}$ this implies that ZFC^+ is inconsistent which in turn by the previous lemma 1.15 implies that ZFC is inconsistent. This contradicts our assumption that ZFC is consistent and therefore ZFC + σ must also be consistent. $\square$

1.3 Absoluteness

Definition 1.18. Let σ be a formula in the language of ZFC. We say that σ is *absolute* for a set M if for all $x_1, \ldots, x_n \in M$

$$\sigma|_M(x_1, \ldots, x_n) \quad \leftrightarrow \quad \sigma(x_1, \ldots, x_n).$$

Again to be precise this means

$$\mathrm{ZFC} \vdash \Big(\phi \text{ is absolute for } M \quad \leftrightarrow \quad \forall \vec{x} \in M : \big(\phi|_M(\vec{x}) \leftrightarrow \phi(\vec{x})\big)\Big).$$

Definition 1.19. Let σ be a formula in the language of ZFC. We say that σ is a Δ_0*-formula* if it is of one of the following forms.

1. σ is atomic.

[3] For a given forcing $\mathbb{P} \in M$ we can explicitly define an M-generic filter G on $\mathbb{P}$ and the set $M[G]$.

2. $\sigma = \neg\phi$ and ϕ is a Δ_0 formula.
3. $\sigma = \phi * \psi$ and ϕ, ψ are Δ_0-formulas and $* \in \{\wedge, \vee, \to\}$.
4. $\sigma(X) = \big((\exists x \in X)\phi(x)\big)(X)$ and ϕ is a Δ_0 formula.
5. $\sigma(X) = \big((\forall x \in X)\phi(x)\big)(X)$ and ϕ is a Δ_0 formula.

A formula is called Σ_1-formula if it is of the form $\exists x\phi$ where ϕ is a Δ_0-formula.

Theorem 1.20. *Let M be a transitive set and let σ be a Δ_0-formula. Then σ is absolute for M.*

Proof. By induction on the structure of σ. For atomic formulas the theorem is trivial and for conjunction, disjunction and negation the induction step is trivial.

Consider $\sigma = \big((\exists x \in X)\phi(x)\big)(X)$. Then for any $X \in M$ we have, because M is transitive and thus also $X \subseteq M$,

$$\sigma(X)|_M \leftrightarrow (\exists x \in M \cap X)\phi|_M(x) \leftrightarrow (\exists x \in M \cap X)\phi(x) \leftrightarrow \sigma(X)$$

using the induction hypothesis for the second equivalence. The quantifier $\forall$ is treated analogously. □

Lemma 1.21. *"x is an ordinal" is a Δ_0-formula.*

Proof. Using the axiom of regularity we can prove that "x is a an ordinal" is equivalent to "x is transitive and linearly ordered (by $\in$)". The formula "x is transitive" is

$$\forall y \in x : \forall z \in y : z \in x$$

and the formula "x is trichotomic" is

$$\forall y \in x, \forall z \in x : y \in z \vee y = z \vee z \in y.$$

□

This means that for any countable transitive model M of ZFC x is an ordinal in M if x is "really" an ordinal.

Lemma 1.22. *"x is the ordinal ω" is a Δ_0-formula.*

Proof. By the previous lemma 1.21 being an ordinal is a Δ_0-property. The formula "x is a limit ordinal" is

$$\forall y \in x : \exists z \in x : y \in z.$$

The formula "x is the smallest limit ordinal" is

$$\forall y \in x : \text{“}x \text{ is not a limit ordinal”}.$$

Thus we have shown that being the smallest limit ordinal is a Δ_0-property. □

Corollary 1.23. *If M is a countable transitive model of ZFC then $\omega \in M$ and $n \in M$ for all $n \in \omega$.*

Lemma 1.24. *The formula “x can be well-ordered” is a Σ_1-property.*

Proof. The formula "x can be well-ordered" is equivalent to

$$\exists \alpha \Big(\exists e \in x^2 : \exists f \in x \times \alpha : \alpha \text{ is an ordinal } \wedge f \text{ is an isomorphism } (x, e) \to (\alpha, \in) \Big).$$

□

Chapter 2

The Forcing Method

2.1 Forcings, Dense Sets, Generic Filters

In this chapter let M denote a countable, transitive model of ZFC.

Definition 2.1. Any partially ordered set $(\mathbb{P}, \leq) \in M$ is called a *forcing notion* for M.

Let $p, q \in \mathbb{P}$.

1. If $q \leq p$ we say that q *extends* p.
2. If p and q have a common extension we say that p and q are *compatible* and write $p \parallel q$.
3. If p and q are not compatible we call them *incompatible* and write $p \perp q$.

Definition 2.2. We call $D \subseteq \mathbb{P}$ *dense* if $\forall p \in \mathbb{P} : \exists d \in D : d \leq p$.

Definition 2.3. We call $A \subseteq \mathbb{P}$ an *antichain* if

$$\forall p, q \in A : p \neq q \to p \perp q,$$

i.e. the elements of A are pairwise incompatible. We say that an antichain A is *maximal* if no $A' \supset A$ is an antichain.[1]

Definition 2.4. A subset $F \subseteq \mathbb{P}$ is called a *filter* if

(i) F is nonempty (*non-triviality*).

[1] As it is commonly done with every other relation we denote by the symbol $\subseteq$ reflexive and by the symbol $\subset$ irreflexive set inclusion. I.e. for sets M and N, if N is a subset of M we write $N \subseteq M$ and if N is a proper subset of M we write $N \subset M$.

(ii) Any $p, q \in F$ are compatible (*directedness*).

(iii) If $q \in F$ and $q \leq p$ then also $p \in F$ (*upward stability*).

A filter $F \subseteq \mathbb{P}$ is called *generic* over M or simply M-generic if it intersects every dense subset $D \subseteq \mathbb{P}$ with $D \in M$.

Likewise, if $\mathcal{D}$ is a family of dense sets $D \subseteq \mathbb{P}$ we say that a filter F is $\mathcal{D}$-*generic* if it F intersects every $D \in \mathcal{D}$.

Lemma 2.5. *Let $\mathbb{P}$ be a forcing notion and let $\mathcal{D} = \{D_n \subseteq \mathbb{P} : n < \omega\}$ be a family of countably many dense sets. Then there exists a $\mathcal{D}$-generic filter $F \subseteq \mathbb{P}$.*

Proof. We find $d_0 \in D_0$. Then because D_0 is dense we find $d_1 \leq d_0$ such that $d_1 \in D_1$ and because D_2 is dense we find $d_2 \leq d_1$ such that $d_2 \in D_2$ and so on. Now let

$$F = \{p \in \mathbb{P} : (\exists n < \omega)\ p \geq d_n\}.$$

□

Definition 2.6. If B is a Boolean algebra[2] a subset $U \subseteq B$ is called an *ultrafilter* if U is a filter on $(B, \leq)$ and for all $b \in B$

$$b \in U \vee (\neg b) \in U.$$

Lemma 2.7. *For a filter $F \subseteq \mathbb{P}$ the following are equivalent:*

(i) F is generic over M.

(ii) F intersects every maximal antichain $A \subseteq \mathbb{P}$ with $A \in M$.

Proof. Let G be generic and let A be a maximal antichain. We claim that the set

$$D_A = \bigcup_{a \in A} \{p : p \leq a\}$$

is dense. Assume the contrary. Then for some $p \in \mathbb{P}$ there exists no $d \in D_A$ with $d \leq p$. Then for all $a \in A$ it holds that $p \perp a$ because otherwise there exists $r \leq p, r \leq a$ and $r \in D_A$. Thus $A \cup \{p\}$ is an antichain that extends A contradicting the maximality of A.

[2] In Boolean algebras we write $\wedge$ for meet and $\vee$ for join but still $\prod$ and $\sum$ for meeting and joining over a collection of elements. The complement of an element is denoted by $\neg$. We define the abbreviations

$$a \setminus b = a \wedge \neg b$$
$$a \to b = \neg a \vee b.$$

Hence D_A must be dense and by genericity of G there exists $p \in G \cap D_A$ and by upward-stability $a \in G \cap D_A$ for some $a \in A$.

To prove the other direction we claim that every dense set D contains a maximal antichain A. Let $A \subseteq D$ be a maximal antichain in D. Assume that A is not a maximal antichain in $\mathbb{P}$. Then there exists $p \notin A$ and $p \perp a$ for all $a \in A$. Because D is dense there exists $d \in D$ with $d \leq p$ and $d \parallel a$ for some $a \in A$. But this implies $p \parallel a$.

Thus D contains an antichain that is maximal in $\mathbb{P}$ and if F intersects every maximal antichain it also intersects D. □

2.2 Separative Forcings and Boolean Algebras

Definition 2.8. A partially ordered set $(\mathbb{P}, \leq)$ is called *separative* if

$$\forall p, q \in \mathbb{P} : \Big(p \not\leq q \rightarrow \exists r \leq p : r \perp q\Big),$$

i.e. either $p \leq q$ or there exists $r \leq p$ that is incompatible with q.

Let B be a Boolean algebra and let B^+ denote its nonzero elements. We call a subset $D \subseteq B$ *dense* (in B) if it is dense in $(B^+, \leq)$.

We can characterize separative, partially ordered sets by the following theorem.

Theorem 2.9. *Separative partially ordered sets are up to isomorphism exactly the dense subsets of complete Boolean algebras.*

The theorem follows by combining the following two lemmas.

Lemma 2.10. *Let B be a Boolean algebra. Then the following holds:*

(i) Then $(B^+, \leq)$ is separative.

(ii) If D is a dense subset of B then $(D, \leq)$ is a separative partial order.

Proof.

(i) Let $p, q \in B^+$ and $p \not\leq q$. We have to find $r \leq p$ with $r \perp q$. Choose $r = p \setminus q$.

(ii) For $p, q \in D$ with $p \not\leq q$ there exists $r \in B^+$ such that $r \leq p$ and $r \perp q$ because by the previous lemma B^+ is separative. By density of D there exists $d \leq r$ with $d \in D$. Hence also $d \leq p$ and $d \perp q$.

□

Lemma 2.11. *Let* $(\mathbb{P}, \leq)$ *be a separative partially ordered set. Then there exists a complete Boolean algebra* B *such that:*

(i) $\mathbb{P} \subseteq B$.

(ii) $\leq$ *agrees with the partial ordering of* B.

(iii) $\mathbb{P}'$ *is dense in* B.

B *is unique up to isomorphism.*

To be precise we are really talking about an isomorphic copy $\mathbb{Q}$ of $\mathbb{P}$ but the isomorphism is so canonic that we do not distinguish between the two.

The construction of B is analogous to the construction of the completion of a Boolean algebra using Dedekind cuts.

Proof. We call a set $U \subseteq \mathbb{P}$ a cut if

$$\forall p \in U : \forall q \in \mathbb{P} : q \leq p \rightarrow q \in U$$

and for all $p \in P$ let U_p denote the cut $\{q : q \leq p\}$.

We remark that it also makes sense to say that cuts are open sets because they form a basis for a topology.

We say a cut U is regular if for all $p \notin U$ there exists $r \leq p$ such that $U \cap U_r = \emptyset$. For any $q \in \mathbb{P}$ the cut U_q is regular. Assume $p \notin U_q$. Then $p \not\leq q$ and thus because $\mathbb{P}$ is separative there exists $r \leq p$, $r \perp q$ and clearly $U_q \cap U_r = \emptyset$.

Let B be the set of all regular cuts of $\mathbb{P}$. We claim that $(B, \subseteq)$ is a complete Boolean algebra that densely contains $\mathbb{P}$. The intersection of a family $U_i, i \in I$ of regular cuts is a regular cut. If $p \notin \bigcap_{i \in I} U_i$ then there exists $i_0 \in I$ such that $p \notin U_{i_0}$ and because U_{i_0} is regular there exists $r \leq p$ such that $U_{i_0} \cap U_r = \emptyset$. Now since $U_{i_0} \subseteq \bigcap_{i \in I} U_i$ also $\bigcap_{i \in I} U_i \cap U_r = \emptyset$. Therefore every cut U is contained in a minimal regular cut.

For any cut V we define $\overline{V}$ to be the minimal regular cut such that $V \subseteq \overline{V}$.

Now for $u, v \in B$ we define

$$u \wedge v = u \cap v$$

$$u \vee v = \overline{u \cup v}$$

$$\neg u = \{p : U_p \cap u = \emptyset\}$$

and take $\emptyset$ as the zero and $\mathbb{P}$ as the one of B. The definition of $\neg u$ works because if $p \notin \neg u$ then there is some $r \leq p$ with $r \in u$ and thus $U_r \subseteq u$. This means that for all $q \in U_r$ it holds that $U_q \cap u = U_q \neq \emptyset$ and thus $q \notin \neg u$, i.e. $U_r \cap u = \emptyset$. Thus $\neg u$ is regular.

To see that the least regular cut containing u and $\neg u$ is in fact $\mathbb{P}$ assume there exists $p \notin v = \overline{u \cup \neg u}$. Since v is regular there must exist some $r \leq p$ such that $U_r \cap v = \emptyset$. However if $p \notin v$ then in particular $p \notin \neg u$ so with the same argument as before $U_r \subseteq u$ and thus $U_r \cap v \neq \emptyset$. Therefore for all $p \in \mathbb{P}$ it must hold that $p \in v$.

Apart from this it is straight forward to check that B is indeed a complete Boolean algebra and that the embedding $p \mapsto U_p$ is an isomorphism. Finally for any $u \in B$ we can take an arbitrary $p \in u$ and it holds that $U_p \subseteq u$. Therefore the embedding of $\mathbb{P}$ in B is dense.

Again the argument that B is unique is analogous to the argument that the completion of a Boolean algebra is unique. Let C be a complete Boolean algebra with $\mathbb{P} \subseteq C$ dense in C. Define a function $\phi : B \to C$ by

$$\phi(b) = \sum_C \{p \in \mathbb{P} : p \leq_B b\}.$$

It is easy to see that ϕ is a homomorphism.

To see that ϕ is injective consider $b_1 \neq b_2$ and let without loss of generality be $b_1 \setminus b_2 \neq 0$. Because $\mathbb{P}$ is dense there exists some $p \in \mathbb{P}$ such that $p \leq b_1 \setminus b_2$ and obviously $p \not\leq b_2$. Thus $\phi(b_1) \neq \phi(b_2)$.

To see surjectivity consider $c \in C$, let $U = \{p \in \mathbb{P} : p \leq_C c\}$ and let $b = \sum_B U$. Clearly $\phi(b) \leq c$. Assume that $\phi(b) < c$ and consider $p \in \mathbb{P}$ with $p \leq c \setminus \phi(b)$. Then $p \not\leq \phi(b)$ but also $p \in U$ thus $p \leq \phi(b)$. Thus $\phi(b) = c$. □

If our partially ordered set is not separative we can embed it in a separative partially ordered set.

Lemma 2.12. *Let* $(\mathbb{P}, \leq)$ *be a partially ordered set. Then there exists a separative partially ordered set* $(\mathbb{Q}, \leq)$ *and a map* $\phi : \mathbb{P} \to \mathbb{Q}$ *such that for all* $p, q \in \mathbb{P}$ *the following holds:*

(i) $p \leq q \to \phi(p) \leq \phi(q)$.

(ii) $p \parallel q \leftrightarrow \phi(p) \parallel \phi(q)$.

We say that $\mathbb{Q}$ *is the* separative quotient *of* $\mathbb{P}$. *The separative quotient is unique up to isomorphism.*

Proof. We define the equivalence relation $\sim$ on $\mathbb{P}$ as follows:

$$p \sim q \leftrightarrow \forall r\Big(r \parallel p \leftrightarrow r \parallel q\Big)$$

Define $\mathbb{Q} = \mathbb{P}/_\sim$,

$$[p] \leq [q] \leftrightarrow (\forall r \leq p)\ r \parallel q$$

and $\phi : p \mapsto [p]$.

$\mathbb{Q}$ is indeed separative. Assume $[p] \not\leq [q]$, i.e. $\exists r : r \leq p \wedge r \perp q$. Clearly $r \leq p$ implies $[r] \leq [p]$ and $r \perp q$ implies $[r] \perp [q]$. To see the second implication assume $r \perp q$ and $[s] < [r], [s] < [q]$ for some $s \in \mathbb{P}$. Then since $s \leq s$ it holds that $s \parallel r$, i.e. there is some $t \in \mathbb{P}$ such that $t \leq r$ and in particular $t \leq s$ thus $t \parallel q$ and therefore $r \parallel q$.

It is easy to see that in a separative partial order $p \leq q$ if and only if for every r with $r \parallel p$ also $p \parallel q$. Thus the separative quotient is unique. □

Combining the previous lemmas we get the following result.

Theorem 2.13. *For every partially ordered set $(\mathbb{P}, \leq)$ there is a complete Boolean algebra B and an embedding $\pi : \mathbb{P} \to B^+$ such that for all $p, q \in \mathbb{P}$:*

1. $p \leq q \to \pi(p) \leq \pi(q)$.
2. $p \perp q \leftrightarrow \pi(p) \wedge \pi(q) = 0$.
3. *The image $\pi[P]$ of $\mathbb{P}$ under π is dense in B.*

B is unique up to isomorphism and we refer to it as $B(\mathbb{P})$.

Proof. The statements (i) and (iii) follow directly from 2.11 and 2.12. For (ii) observe for $b, c \in B^+$ that $b \perp_{B^+} c \leftrightarrow b \wedge c = 0$.

Uniqueness of B follows again from 2.11 and 2.12. □

This result is useful because it turns out that the generic extensions of M are determined by $B(\mathbb{P})$, based on the following observations.

Lemma 2.14. *In M let $(\mathbb{P}, \leq)$ be a partially ordered set, $(\mathbb{Q}, \leq)$ its separative quotient, and $\phi : \mathbb{P} \to \mathbb{Q}$ an embedding as in 2.12. Then it holds that:*

(i) If $G \subseteq \mathbb{P}$ is an M-generic filter then so is $\phi[G] \subseteq \mathbb{Q}$.

(ii) If $H \subseteq \mathbb{Q}$ is an M-generic filter then so is $\phi^{-1}[H] \subseteq \mathbb{P}$.

Again in M let $(\mathbb{P}, \leq)$ be a partially ordered set and $D \subseteq \mathbb{P}$ be a dense subset of $\mathbb{P}$.

(i) If $G \subseteq \mathbb{P}$ is an M-generic filter the so is $G \cap D \subseteq D$.

(ii) If $H \subseteq D$ is an M-generic filter then so is $\{p \in \mathbb{P} : (\exists d \in D)\ d \leq p\} \subseteq \mathbb{P}$. □

Now if we have $\mathbb{P}$, $B(\mathbb{P})$ and $\pi : \mathbb{P} \to B(\mathbb{P})$ and some generic filter $G \in \mathbb{P}$ then we can define $H = \{b \in B : (\exists p \in G)\ p \leq b\}$ and conversely if $H \in B$ is a generic filter we can define $G = \pi^{-1}[H \cap \pi[\mathbb{P}]]$. In both cases G is generic if and only if H is generic and since we can define one from the other $M[G] = M[H]$ as we will see later.

The following lemma characterizes M-generic filters on B^+.

Lemma 2.15. *Let B be a complete Boolean algebra in M. Then the following are equivalent:*

(i) $G \subseteq B^+$ is generic over M.

(ii) $G \subseteq B$ is an ultrafilter and

$$\forall X \subseteq G, X \in M : \prod X \in G$$

Note that B may not be complete outside of M.

Proof. $(i) \to (ii)$: First observe that for any $b \in B$ the set $\{b, \neg b\}$ is a maximal antichain. By lemma 2.7 a generic filter meets every antichain. Thus G is an ultrafilter in B.

Now take a set $X \subseteq G$ with $X \in M$. Let

$$D = \{b \in B^+ : (\forall x \in X)\ b \leq x\} \cup \{b \in B^+ : (\exists x \in X) b \leq \neg x\}.$$

Clearly D is dense and $D \in M$. Since G is generic there is some $b \in D \cap G$ and b must be contained in the first part of D because it must be compatible with every element of the filter G and thus in particular with all $x \in X$. Therefore $b \leq x$ for all $x \in X$ and by upward-stability of G we get $\prod X \in G$.

$(ii) \to (i)$: Let G be an ultrafilter with the property in (ii). Assume there is a dense set $D \subseteq B^+$, $D \in M$ and $G \cap D = \emptyset$. Because G is an ultrafilter we have $\neg D = \{\neg d : d \in D\} \subseteq G$. If $\prod \neg D = 0$ we get the contradiction $0 \in G$. If $\prod \neg D = b > 0$ for some $b \in B$ then because D is dense there exists some $d \in D$ with $d < b$ which is also a contradiction.

Hence G meets every dense set $D \in M$. □

Therefore we say that $G \subseteq B$ is a generic ultrafilter over M if it satisfies the condition in (ii).

Lemma 2.16. *Let B be a complete Boolean algebra in M and let G be an M-generic filter on B. Let $X \subseteq B$ be a set such that $X \in M$ and let $x^* = \sum X$. Then it holds that*

$$x^* \in G \quad \leftrightarrow \quad \exists x \in X : x \in G.$$

Proof. We work in M. We define a maximal antichain $W \subseteq \{b : b \leq x^*\}$ by

$$W = \{ \prod_{x \in X} \bar{x} : (\forall x \in X)\ \bar{x} \in \{x, \neg x\}, (\exists x_0 \in X)\ \bar{x}_0 = x_0\} \setminus \{0\}.$$

Now if $x^* \in G$ then by lemma 2.7 there exists $w \in W$ such that $w \in G$ and because for some $x_0 \in X$ it holds $w \leq x_0$ and therefore $x_0 \in G$. □

2.3 Boolean-Valued Models

Definition 2.17. A *Boolean-valued model* of set theory is a triple $\mathfrak{A} = (A, B, \|\cdot\|)$ that consists of a universe A, a Boolean algebra B and a function $\|\cdot\|$ that maps all atomic expressions to values in B, written

$$\|x \in y\|, \quad \|x = y\|$$

and $\|\cdot\|$ satisfies the following sanity conditions:

(i) $\|x = x\| = 1$

(ii) $\|x = y\| = \|y = x\|$

(iii) $\|x = y\| \wedge \|y = z\| \leq \|x = z\|$.

(iv) $\|x \in y\| \wedge \|x = s\| \wedge \|y = t\| \leq \|s \in t\|$.

For all formulas $\phi(x_1, \dots, x_n)$ we can now define

$$\|\phi(a_1, \dots, a_n)\|, \quad a_1, \dots, a_n \in A$$

recursively as follows:

1. $\|(\phi \vee \psi)(a_1, \dots a_n)\| = \|\phi(a_1, \dots a_n)\| \vee \|\psi(a_1, \dots a_n)\|$.
2. $\|(\phi \wedge \psi)(a_1, \dots a_n)\| = \|\phi(a_1, \dots a_n)\| \wedge \|\psi(a_1, \dots a_n)\|$.
3. $\|(\phi \to \psi)(a_1, \dots a_n)\| = \|(\neg\phi \vee \psi)(a_1, \dots a_n)\|$.
4. $\|\neg\phi(a_1, \dots a_n)\| = \neg\|\phi(a_1, \dots a_n)\|$.

5. $\|\forall x\phi(x, a_1, \dots a_n)\| = \prod_{a \in A} \|\phi(a, a_1, \dots a_n)\|$.
6. $\|\exists x\phi(x, a_1, \dots a_n)\| = \sum_{a \in A} \|\phi(a, a_1, \dots a_n)\|$.

We call $\phi(a_1, \dots, a_n)$ valid in $\mathfrak{A}$ if $\|\phi(a_1, \dots, a_n)\| = 1$.

It is easy to see that an implication $\phi \to \psi$ is valid iff and only if $\|\phi\| \leq \|\psi\|$. Using this inequality one can verify by a simple calculation that every axiom of predicate calculus is valid in $\mathfrak{A}$. Likewise if ϕ and $\phi \to \psi$ are valid the inequality forces ψ to be valid as well. Therefore anything that can be derived in predicate calculus is valid in $\mathfrak{A}$. Furthermore if ϕ and ψ are equivalent in predicate calculus then $\|\phi\| = \|\psi\|$, again by the above inequality.

Now consider a Boolean-valued model $\mathfrak{A}$ in which every axiom of ZFC is valid and let ϕ be a statement in the language of set theory. Then $\|\phi\| > 0$ implies that ϕ is consistent with ZFC, otherwise $\neg\phi$ would be valid in $\mathfrak{A}$ and therefore $\|\phi\| = \neg\|\neg\phi\| = 0$.

If $\mathfrak{A}$ is a Boolean-valued model of ZFC it seems natural to attempt to turn it into a regular, two-valued model of ZFC by having an ultrafilter $U \subseteq B$ decide which $b \in B$ should evaluate to true.

Definition 2.18. Let $\mathfrak{A} = (A, B, \|\cdot\|)$ be a Boolean-valued model and let $U \subseteq B$ be an ultrafilter.

We define an equivalence relation $\equiv$ on A:

$$x \equiv y \leftrightarrow \|x = y\| \in U.$$

It follows from the sanity conditions for $\|\cdot\|$ that $\equiv$ is indeed an equivalence relation: 2.17 (i) implies reflexivity, (ii) implies symmetry and (iii) together with the upward-stability of U implies transitivity.

Now we define a relation ϵ on $A/_{\equiv}$:

$$[x] \ \epsilon \ [y] \leftrightarrow \|x \in y\| \in U.$$

The relation ϵ is well definied because it follows from 2.17 (iv) that it does not depend on the representatives we choose for $[x]$ and $[y]$.

Finally define

$$\mathfrak{A}/U = (A/_{\equiv}, \epsilon).$$

Our hope is that if $\mathfrak{A}$ is a Boolean-valued model of ZFC then $\mathfrak{A}/U$ is going to be a two-valued model of ZFC. However there is a technical requirement for this.

Definition 2.19. A Boolean-valued model $\mathfrak{A}$ is called *full* if for all formulas $\phi(x, x_1, \dots, x_n)$ and all $a_1, \dots, a_n \in A$ there is an $a \in A$ such that

$$\|\exists x\phi(x, a_1, \dots, a_n)\| = \|\phi(a, a_1, \dots, a_n)\|.$$

Theorem 2.20. *Let $\mathfrak{A} = (A, B\|\cdot\|)$ be a full, Boolean-valued model and let U be an ultrafilter on B. Then for all formulas $\phi(x_1, \ldots x_n)$ and all $a_1, \ldots a_n \in A$ it holds that*

$$\mathfrak{A}/U \models \phi([a_1], \ldots, [a_n]) \quad \leftrightarrow \quad \|\phi(a_1, \ldots, a_n)\| \in U.$$

Proof. We prove the theorem by induction on the structure of ϕ.

If ϕ is atomic then the theorem holds by the definition of $\mathfrak{A}/U$.

For negation we have the following chain of equivalences, using the induction hypothesis, the fact that U is an ultrafilter and the definition of $\|\cdot\|$:

$$\mathfrak{A}/U \models \neg\phi \quad \Leftrightarrow \quad \mathfrak{A}/U \not\models \phi \quad \Leftrightarrow \quad \|\phi\| \notin U \quad \Leftrightarrow$$
$$\Leftrightarrow \quad \neg\|\phi\| \in U \quad \Leftrightarrow \quad \|\neg\phi\| \in U.$$

For disjunction we have

$$\mathfrak{A}/U \models \phi \vee \psi \quad \overset{wlog}{\Rightarrow} \quad \mathfrak{A}/U \models \phi \quad \Rightarrow \quad \|\phi\| \in U \quad \Rightarrow \quad \|\phi \vee \psi\| \in U$$

and the last implication follows because $\phi \to \phi \vee \psi$ implies $\|\phi\| \leq \|\phi \vee \psi\|$. Contrary we have

$$\mathfrak{A}/U \not\models \phi\vee\psi \quad \Rightarrow \quad \mathfrak{A}/U \models \neg\phi\wedge\neg\psi \quad \Rightarrow \quad \mathfrak{A}/U \models \neg\phi \wedge \mathfrak{A}/U \models \neg\psi \quad \Rightarrow$$
$$\Rightarrow \quad \|\phi\| \notin U \wedge \|\psi\| \notin U \quad \Rightarrow \quad \|\phi\| \vee \|\psi\| \notin U \quad \Rightarrow \quad \|\phi \vee \psi\| \notin U.$$

Finally consider the formula $\exists x\psi(x)$. Because $\mathfrak{A}$ is full we can find $a \in A$ with $\|\phi(a)\| = \|\exists\phi(x)\|$. Combined with the fact that for any $b \in A$ it holds that $\phi(b) \to \exists x\phi(x)$ we get

$$\|\exists x\phi(x)\| \in U \quad \leftrightarrow \quad (\exists a \in A)\ \|\phi(a)\| \in U$$

which is by induction hypothesis equivalent to

$$\mathfrak{A}/U \models \exists x\phi(x).$$

□

2.4 The class V^B

In this section B be a complete Boolean algebra. Starting from the class V we define the class V^B of "Boolean-valued sets".

1. $V_0^B = \emptyset$

2. $V^B_{\alpha+1}$ is the set of all functions x that map $\mathrm{dom}(x) \subseteq V^B_\alpha$ to the Boolean algebra B.

3. $V^B_\lambda = \bigcup_{\alpha<\lambda} V^B_\alpha$ if λ is a limit ordinal.

4. $V^B = \bigcup_{\alpha \in ord} V^B_\alpha$.

Sometimes when it is more convenient we will treat elements $y \notin \mathrm{dom}(x)$ as if $x(y) = 0$.

Just as in V we can define a rank function ρ for $x \in V^B$.

$$\rho(x) = \min\{\alpha \in ord : x \in V^B_\alpha\}.$$

We give a mutually recursive definition of $\|\cdot\|$ on the atoms of V^B.

1. $\|x \in y\| = \sum_{t \in \mathrm{dom} y} \big(y(t) \wedge \|x = t\|\big)$

2. $\|x \subseteq y\| = \prod_{t \in \mathrm{dom} x} \big(x(t) \to \|t \in y\|\big)$

3. $\|x = y\| = \|x \subseteq y\| \wedge \|y \subseteq x\|$

Lemma 2.21. *The function $\|\cdot\|$ satisfies the sanity conditions from 2.17.*

Proof. First we show that for all $x \in V^B$

$$\|x = x\| = 1$$

by induction on the rank of x. Clearly it is enough to show that $\|x \subseteq x\| = 1$ which means by definition that $(x(t) \to \|t \in x\|) = 1$ for all $t \in \mathrm{dom}(x)$ or equivalently $x(t) \leq \|t \in x\|$. By induction hypothesis $\|t = t\| = 1$ and therefore by definition of $\|t \in x\|$ it holds that $\|t = t\| \wedge x(t) = x(t) \leq \|t \in x\|$.

The condition

$$\|x = y\| = \|y = x\|$$

is trivially satisfied by symmetric definition of $\|x = y\|$.

The remaining two conditions follow by verifying the following. For all $x, y, z \in V^B$:

1. $\|x = y\| \wedge \|y = z\| \leq \|x = y\|$,

2. $\|x \in y\| \wedge \|x = z\| \leq \|z \in y\|$,

3. $\|y \in x\| \wedge \|x = z\| \leq \|y \in z\|$.

Again we prove this by induction on the ranks of x, y, z.

1. As before it is enough to show that $\|x \subseteq y\| \wedge \|y = z\| \leq \|x \subseteq z\|$. Thus by the definition of $\|x \subseteq z\|$ we want to show that for all $t \in \mathrm{dom}(x)$ it holds that

$$(x(t) \to \|t \in y\|) \wedge \|y = z\| \leq (x(t) \to \|t \in z\|).$$

Now by induction hypothesis $\|t \in y\| \wedge \|y = z\| \leq \|t \in z\|$. Therefore $(\neg x(t) \vee \|t \in y\|) \wedge \|y = z\| = (\|y = z\| \wedge \neg x(t)) \vee (\|t \in y\| \wedge \|y = z\|) \leq \neg x(t) \vee \|t \in z\|$.

2. By induction hypothesis we have $\|x = z\| \wedge \|x = t\| \leq \|z = t\|$ for all $t \in \mathrm{dom}(y)$ and thus

$$\|x = z\| \wedge \|x = t\| \wedge y(t) \leq \|z = t\| \wedge y(t).$$

Summing up over all $t \in \mathrm{dom}(y)$ we get

$$\|x = z\| \wedge \sum_{t \in \mathrm{dom}(y)} \big(\|x = t\| \wedge y(t)\big) \leq \sum_{t \in \mathrm{dom}(y)} \big(\|z = t\| \wedge y(t)\big),$$

i.e. $\|x = z\| \wedge \|x \in y\| \leq \|z \in y\|$.

3. For any $t \in \mathrm{dom}(x)$ we have by the definition of $\|x = z\|$ that $\|x = z\| \leq \neg x(t) \vee \|t \in z\|$. Thus $x(t) \wedge \|x = z\| \leq \|t \in z\|$ and

$$\|y = t\| \wedge x(t) \wedge \|x = z\| \leq \|y = t\| \wedge \|t \in z\|.$$

Now by induction hypothesis $\|y = t\| \wedge \|t \in z\| \leq \|y \in z\|$ and therefore

$$\|y = t\| \wedge x(t) \wedge \|x = z\| \leq \|y \in z\|.$$

Summing up over $t \in \mathrm{dom}(x)$ we get

$$\sum_{t \in \mathrm{dom}(x)} \big(\|y = t\| \wedge x(t)\big) \wedge \|x = z\| \leq \|y \in z\|,$$

i.e. $\|y \in x\| \wedge \|x = z\| \leq \|y \in z\|$. □

Thus V^B is a Boolean-valued model, except for the fact it is not a set. We want to show that every axiom of ZFC is valid in V^B.

Before we do this however we are going to show that V^B is full. We are going to need the following technical lemma.

Lemma 2.22. *Let $W \subseteq B$ be an antichain and let $\{a_u : u \in W\}$ be a family of elements of V^B. Then there exists $a \in V^B$ such that for all $u \in W$ the inequality $u \leq \|a = a_u\|$ holds.*

Proof. We define a as follows. Let $\mathrm{dom}(a) = \bigcup_{u \in W} \mathrm{dom}(a_u)$ and for $t \in \mathrm{dom}(a)$ let $a(t) = \sum_{u \in W}(u \wedge a_u(t))$. Now since W is an antichain we have $u \wedge a(t) = u \wedge a_u(t)$ for all $u \in W, t \in \mathrm{dom}(a)$. This means $u \leq (a(t) \to a_u(t))$ and $u \leq (a_u(t) \to a(t))$ and therefore $u \leq \|a = a_u\|$. □

Lemma 2.23. *V^B is full.*

Proof. We have to show that for every formula $\phi(x, \dots)$ there exists $a \in V^B$ such that

$$\|\phi(a, \dots)\| = \|\exists x \phi(x, \dots)\|.$$

Clearly since for $a \in V^B$ it holds that $\phi(a, \dots) \to \exists x \phi(x, \dots)$ the $\leq$ direction holds for every a. Therefore we need to find $a \in V^B$ such that $\geq$ holds. Let $u_0 = \|\exists x \phi(x, \dots)\|$ and let

$$D = \{u \in B : \exists a_u : u \leq \|\phi(a_u, \dots)\|\}$$

It is easy to see from the definition of $\|\cdot\|$ for $\exists$ that D is in dense below u_0. Let $W \subseteq D$ be a maximal antichain in D. Clearly $\sum_{u \in W} u \leq u_0$ and by lemma 2.22 there exists $a \in V^B$ such that $u \leq \|a = a_u\|$ for all $u \in W$. Therefore $u \leq \|\phi(a, \dots)\|$ for every $u \in W$ and thus it holds that $u_0 \leq \|\phi(a, \dots)\|$. □

The following lemma will be useful for calculations later on.

Lemma 2.24.

$$\|\exists x \in y : \phi(x)\| = \sum_{x \in dom(y)} \Big(\|\phi(x)\| \wedge y(x)\Big)$$

$$\|\forall x \in y : \phi(x)\| = \prod_{x \in dom(y)} \Big(y(x) \to \|\phi(x)\|\Big)$$

Proof.

$$\begin{aligned}
\|\exists x \in y : \phi(x)\| = & \\
&= \|\exists x\big(\phi(x) \wedge x \in y\big)\| = \\
&= \sum_{a \in V^B} \|\phi(a) \wedge a \in y\| = \\
&= \sum_{a \in \mathrm{dom}(y)} \Big(\|\phi(a)\| \wedge \|a \in y\|\Big) = \\
&= \sum_{a \in \mathrm{dom}(y)} \Big(\|\phi(a)\| \wedge \sum_{b \in \mathrm{dom}(y)} \big(\|b = a\| \wedge y(b)\big)\Big) = \\
&= \sum_{x \in \mathrm{dom}(y)} \Big(\|\phi(x)\| \wedge y(x)\Big).
\end{aligned}$$

The second formula can be verified by a similar calculation. □

For ever set $x \in V$ we can find a canonical copy $\check{x}$ in our Boolean-valued model V^B.

Definition 2.25. We define inductively:

1. $\check{\emptyset} = \emptyset$.
2. $\check{x} = \{\check{y} : y \in x\} \times \{1\}$, i.e. $\check{x}$ is a function with $\mathrm{dom}(x) = \{\check{y} : y \in x\}$ and $\check{x}(\check{y}) = 1$ for all $y \in x$.

Furthermore for $\sigma, \tau \in V^B$ we define

$$\mathrm{up}(\sigma, \tau) = \{(\sigma, 1), (\tau, 1)\},$$
$$\mathrm{op}(\sigma, \tau) = \mathrm{up}(\mathrm{up}(\sigma, \sigma), \mathrm{up}(\sigma, \tau)).$$

Lemma 2.26. *If $\phi(x_1, \ldots, x_n)$ is a Δ_0-formula then*

$$\phi(x_1, \ldots, x_n) \quad \leftrightarrow \quad \|\phi(\check{x}_1, \ldots, \check{x}_n)\| = 1.$$

Proof. By induction on the structure of ϕ. For atomic formulas we show that

$$\begin{aligned} x \in y \quad &\leftrightarrow \quad \|\check{x} \in \check{y}\| = 1 \\ x \subseteq y \quad &\leftrightarrow \quad \|\check{x} \subseteq \check{y}\| = 1 \end{aligned}$$

by mutual induction on the ranks of x, y:

$$\|\check{x} \in \check{y}\| = \sum_{\check{t} \in \mathrm{dom}(\check{y})} \big(\|\check{t} = \check{x}\| \wedge \check{y}(\check{t})\big) = \sum_{\check{t} \in \mathrm{dom}(\check{y})} \|\check{t} = \check{x}\|.$$

$$\|\check{x} \subseteq \check{y}\| = \prod_{\check{t} \in \mathrm{dom}(\check{x})} \big(\check{x}(\check{t}) \to \|\check{t} \in \check{y}\|\big) = \prod_{\check{t} \in \mathrm{dom}(\check{x})} \check{x}(\check{t}).$$

For conjunction we have

$$\phi(x) \wedge \psi(y) \quad \Leftrightarrow \quad \|\phi(\check{x})\| = 1 \wedge \|\psi(\check{y})\| = 1 \quad \Leftrightarrow \quad \|\phi(\check{x}) \wedge \psi(\check{y})\| = 1.$$

For negation we have

$$\neg\phi(x) \quad \Leftrightarrow \quad \|\phi(\check{x})\| = 0 \quad \Leftrightarrow \quad \neg\|\phi(\check{x})\| = 1 \quad \Leftrightarrow \quad \|\neg\phi(\check{x})\| = 1.$$

Finally for the existential quantifier we have, using 2.24,

$$\begin{aligned} &\|\exists a \in \check{y} : \phi(a, \check{x})\| = \|\sum_{a \in \mathrm{dom}(\check{y})} \Big(\check{y}(a) \wedge \phi(a, \check{x})\Big)\| = 1 \\ \Leftrightarrow \quad &\exists a \in y : \|\phi(\check{a}, \check{y})\| = 1 \\ \Leftrightarrow \quad &\exists a \in y : \phi(a, x). \end{aligned} \tag{2.1}$$

□

Corollary 2.27. *If ϕ is a Σ_1-formula then $\phi(x_1, \dots, x_n)$ implies $\|\phi(\check{x}_1, \dots, \check{x}_n)\| = 1$.*

The absoluteness of Δ_0 formulas implies, because being an ordinal is a Δ_0-property by 1.21, that the construction of V^B "preserves" the ordinals that occur in V. The following lemma shows that V and V^B have in fact "the same" ordinals, i.e. no new ordinals appear in V^B.

Lemma 2.28. *For $x \in V^B$ it holds that*

$$\|x \text{ is an ordinal}\| = \sum_{\alpha \in ord} \|x = \check{\alpha}\|.$$

Proof. By 1.21 and 2.26 it holds that

$$\sum_{\alpha \in ord} \|x = \check{\alpha}\| \leq \|x \text{ is an ordinal}\|.$$

For the other direction note that for any ordinal γ it holds that

$$\|x \in \check{\gamma}\| = \sum_{\alpha < \gamma} \|x = \check{\alpha}\|$$

and furthermore for any ordinal α it holds that

$$\|x \text{ is an ordinal}\| \leq \|x \in \check{\alpha}\| \vee \|x = \check{\alpha}\| \vee \|\check{\alpha} \in x\|.$$

We observe that $A = \{\alpha : \|\check{\alpha} \in x\| > 0\}$ is a set because $\|\check{\alpha} \in x\| > 0$ implies that there exists $t \in \text{dom}(x)$ such that $\|\check{\alpha} = t\| > 0$. It is easy to see by induction that this implies $\rho(t) \geq \rho(\alpha) = \alpha$. This means that no such $t \in \text{dom}(x)$ exists for $\alpha > \sup_{t \in \text{dom}(x)} \rho(t)$.

Now let $\alpha^* = \sup A + 1$ and let $\gamma = \alpha^* + 1$. Then

$$\|x \text{ is an ordinal}\| \leq \|x \in \check{\alpha}^*\| \vee \|x = \check{\alpha}^*\| = \|x \in \check{\gamma}\|$$

and it follows from what we proved above that

$$\|x \text{ is an ordinal}\| \leq \sum_{\alpha < \gamma} \|x = \check{\alpha}\|.$$

□

Now we can show that V^B is indeed a model of ZFC.

Theorem 2.29. *Every axiom of ZFC is valid in V^B.*

Extensionality. First we observe that for $a, b, c \in B$ if $a \leq b$ then $(b \to c) \leq (a \to c)$. Clearly for $t, X, Y \in V^B$ we have $X(t) \leq \|t \in X\|$ and therefore $(\|t \in X\| \to \|t \in Y\|) \leq (X(t) \to \|t \in Y\|)$. Hence

$$\prod_{t \in V^B} (\|t \in X\| \to \|t \in Y\|) \leq \prod_{t \in V^B} (X(t) \to \|t \in Y\|)$$

and the left side of the inequality is equal to $\|\forall t : t \in X \to t \in Y\|$ and the left side is equal to $\|X \subseteq Y\|$.

Doing the same with the roles of X, Y swapped yields that V^B is extensional.

Pairing. For $x, y \in V^B$ let $z \in V^B$ be such that $\mathrm{dom}(z) = \{x, y\}$ and $z(x) = z(y) = 1$. Clearly $\|x \in z \wedge y \in z\| = 1$.

Separation Scheme. For $X \in V^B$ and a formula ϕ we find $Y \in V^B$ such that

$$\mathrm{dom}(Y) = \mathrm{dom}(X), \quad Y(t) = X(t) \wedge \|\phi(t)\|$$

for all $t \in \mathrm{dom}(X)$. Then $\|x \in Y\| = \|x \in X\| \wedge \|\phi(x)\|$ for all $x \in V^B$, utilizing that $\|x = t\| \wedge \phi(t) = \|x = t\| \wedge \phi(x)$ for all $t \in V^B$ (and in particular all $t \in \mathrm{dom}(Y)$).

Union. For $X \in V^B$ we find $Y \in V^B$ such that

$$\mathrm{dom}(Y) = \bigcup \{\mathrm{dom}(x) : x \in \mathrm{dom}(X)\}, \quad Y(y) = 1$$

for all $y \in \mathrm{dom}(Y)$.

Power Set. For $X \in V^B$ we find $Y \in V^B$ such that

$$\mathrm{dom}(Y) = \{z \in V^B : \mathrm{dom}(z) = \mathrm{dom}(X) \wedge \forall t \in \mathrm{dom}(x) : z(t) \leq X(t)\}$$

and $Y(u) = 1$ forall $u \in \mathrm{dom}(Y)$.

Infinity. By lemma 2.26 we have $\|$"$\breve{\omega}$ is the smallest limit ordinal"$\| = 1$.

Collection Scheme. For $X \in V^B$ and a formula ϕ we find $Y \in V^B$ such that

$$\mathrm{dom}(Y) = \bigcup \{S_x : x \in X\}$$

where $S_x \subseteq V^B$ is a set such that

$$\sum_{y \in V^B} \|\phi(x, y)\| = \sum_{y \in S_x} \|\phi(x, y)\|.$$

Regularity. Let $X \in V^B$. Assume that regularity is not valid for X, i.e.

$$\|\exists x : (x \in X) \wedge \forall y \in X : \exists z \in y : z \in X\| = b > 0.$$

Now let $y \in V^B$ be such that $\|y \in X\| \wedge b > 0$ let y be an element of minimal rank with this property. Then because $\|y \in X\| \wedge b \leq \|\exists z \in y : z \in X\|$ there exists (using fullness) $z \in \mathrm{dom}(y)$ with $\|z \in X\| \wedge \|y \in X\| \wedge b > 0$. But the rank of z is smaller than the rank of y and therefore we have found a contradiction.

Choice. We showed in 1.24 that being well-ordered is a Σ_1-property. Therefore for every $Y \in V$ it holds by corollary 2.27 that

$$\|\check{Y} \text{ can be well-ordered}\| = 1$$

For any $X \in V^B$ we find a set $Y \in V$ and $f \in V^B$ such that

$$\|f \text{ is a function on } \check{Y} \wedge X \subseteq \mathrm{ran}(f)\| = 1$$

We define $Y = \mathrm{dom}(X)$ and

$$\mathrm{dom}(f) = \{\mathrm{op}(\check{y}, y) : y \in Y\}, \quad f(t) = 1$$

for all $t \in \mathrm{dom}(f)$. In other words a well-order on $\mathrm{dom}(X)$ induces a well-order on X. □

Definition 2.30. There is a canonical name Γ for a generic ultrafilter which is a Boolean-valued function with domain

$$\mathrm{dom}(\Gamma) = \{\check{b} : b \in B\}$$

and values

$$\Gamma(\check{b}) = b, \forall b \in B.$$

Indeed it holds that

$$\|\Gamma \text{ is a } V\text{-generic ultrafilter on } B\| = 1.$$

To verify this first note that for $b \in B$ we have

$$\|\check{b} \in \Gamma\| = \sum_{\check{u} \in \mathrm{dom}(\Gamma)} (\|\check{b} = \check{u}\| \wedge \Gamma(\check{b})). = b$$

Using this it is easy to see the following holds:

1. $\|\check{0} \in \Gamma\| = 0$.
2. Let $b, c \in B, b \leq c$. Then $\|\check{b} \in \Gamma\| \leq \|\check{c} \in \Gamma\|$.

3. Let $b, c \in B$ and $d = b \wedge c$. Then $\|\check{b} \in \Gamma\| \wedge \|\check{c} \in \Gamma\| \leq \|\check{d} \in \Gamma\|$.

Therefore Γ is indeed an ultrafilter. Furthermore it holds that:

$$\text{For every } X \subseteq B, P = \prod X: \quad \|\check{X} \subseteq \Gamma \rightarrow \check{P} \in \Gamma\| = 1.$$

This follows from

$$\|\check{X} \subseteq \Gamma\| = \prod_{\check{u} \in \mathrm{dom}(\check{X})} (\check{X}(\check{u}) \rightarrow \|\check{u} \in \Gamma\|) = \prod_{u \in X} u = \|\check{P} \in \Gamma\|.$$

2.5 The Forcing Theorem

As before let M be a countable transitive model of ZFC.

Definition 2.31. Let $(\mathbb{P}, \leq) \in M$ be a forcing and let $B = B(\mathbb{P})$ be the complete Boolean algebra constructed in M as in theorem 2.13 with the embedding $\pi : \mathbb{P} \rightarrow B$.

Then $M^{\mathbb{P}} = M^{B(\mathbb{P})}$ denotes the Boolean-valued model as defined in the previous section, constructed inside of M. We are going to refer to the elements of $M^{\mathbb{P}}$ as $\mathbb{P}$-*names*.

The forcing relation $\Vdash_{\mathbb{P}}$ is defined as

$$p \Vdash_{\mathbb{P}} \phi(\dot{a}_1, \ldots, \dot{a}_n) \quad \leftrightarrow \quad \pi(p) \leq \|\phi(\dot{a}_1, \ldots, \dot{a}_n)\|^M$$

for $\mathbb{P}$-names $\dot{a}_1, \ldots \dot{a}_n \in M^{\mathbb{P}}$.

If $p \Vdash_{\mathbb{P}} \phi(\dot{a}_1, \ldots, \dot{a}_n)$ holds we say that "p forces $\phi(\dot{a}_1, \ldots, \dot{a}_n)$". If every $p \in \mathbb{P}$ forces $\phi(\dot{a}_1, \ldots, \dot{a}_n)$ we write $\Vdash_{\mathbb{P}} \phi(\dot{a}_1, \ldots, \dot{a}_n)$. If it is clear which forcing notion is used we simply write $\Vdash$ instead of $\Vdash_{\mathbb{P}}$.

Lemma 2.32. *Let $\mathbb{P} \in M$ be a forcing notion. Then there exists an M-generic filter G.*

Proof. Because M is countable there are only countably many dense subsets of $\mathbb{P}$ in M. By lemma 2.5 there exists an M-generic filter. □

Definition 2.33. Let again B be a complete Boolean algebra in M and let $G \subseteq B$ be an M-generic ultrafilter. For $x \in M^B$ we define the evaluation x^G of x under G inductively as

1. $\emptyset^G = \emptyset$.
2. $x^G = \{y^G : x(y) \in G\}$.

The generic extension $M[G]$ of M is defined as

$$M[G] = \{x^G : x \in M^B\}.$$

Theorem 2.34. *$M[G]$ is isomorphic to M^B/G as defined in 2.18 with the canonical isomorphism that maps $[x]_\sim$ to x^G.*

Proof. It holds for all $x, y \in M^B$:

1. $\|x = y\| \in G \quad \leftrightarrow \quad x^G = y^G$
2. $\|x \in y\| \in G \quad \leftrightarrow \quad x^G \in y^G$

We verify by alternating induction on the ranks of x, y:

$$\begin{aligned} \|x \in y\| \in G &\Leftrightarrow \exists t \in \mathrm{dom}(y)\Big(y(t) \in G \wedge \|x = t\| \in G\Big) \\ &\Leftrightarrow \exists t\Big(y(t) \in G \wedge x^G = t^G\Big) \\ &\Leftrightarrow x^G \in \{t^G : y(t) \in G\} \\ &\Leftrightarrow x^G \in y^G \end{aligned}$$

$$\begin{aligned} \|x \subseteq y\| \in G &\Leftrightarrow \forall t \in \mathrm{dom}(x)\Big(x(t) \in G \rightarrow \|t \in y\| \in G\Big) \\ &\Leftrightarrow \forall t\Big(x(t) \in G \rightarrow t^G \in y^G\Big) \\ &\Leftrightarrow \{t^G : x(t) \in G\} \subseteq y^G \\ &\Leftrightarrow x^G \subseteq y^G \end{aligned}$$

The non-trivial direction of the first equivalences follows from 2.15 and 2.16 respectively, using the genericity of G. □

Thus we can restate theorem 2.20 as follows.

Theorem 2.35. *Let G be an M-generic ultrafilter on B. Then for all $x_1, \ldots x_n \in M^B$ and all formulas ϕ it holds that*

$$M[G] \models \phi(x_1^G, \ldots, x_n^G) \quad \leftrightarrow \quad \|\phi(x_1, \ldots, x_n)\| \in G.$$

□

It is also possible to verify this theorem directly by induction, using the M-genericity of G.

Corollary 2.36. *If M is a model of ZFC then so is $M[G]$.*

Proof. In 2.29 we showed that every axiom σ of ZFC is valid in M^B, i.e. $\|\sigma\| = 1 \in G$. □

Furthermore we can easily verify the following using what we have already shown.

Corollary 2.37.

1. *$M \subseteq M[G]$*
2. *M and $M[G]$ have the same ordinals.*
3. *$G \in M[G]$*
4. *If $N \supseteq M$ is a transitive model of ZFC then $G \in N$ implies $M[G] \subseteq N$.*

Proof.

1. It can easily be checked by induction that for all $x \in M$ it holds that $\check{x}^G = x$.
2. Follows immediately from 2.28.
3. $\Gamma^G = G$.
4. If $N \subseteq M$ is a transitive model containing G then the construction of $M[G]$ can be done inside of N.

□

Theorem 2.38. *Let $(\mathbb{P}, \leq) \in M$ be a forcing notion and let $G \subseteq \mathbb{P}$ be an M-generic filter. For a formula ϕ and $\dot{a}_1, \ldots, \dot{a}_n \in M^{P(B)}$ it holds that*

$$M[G] \models \phi(\dot{a}_1^G, \ldots, \dot{a}_n^G) \quad \leftrightarrow \quad \exists p \in G : p \Vdash \phi(\dot{a}_1, \ldots, \dot{a}_n).$$

Proof. For a forcing $(\mathbb{P}, \leq) \in M$, its completion $B = B(P)$ with the embedding $\pi : \mathbb{P} \to B$ and an M-generic filter $G \subseteq \mathbb{P}$ we define for $x \in M^B$.

1. $\emptyset^G = \emptyset$
2. $x^G = \{y^G : (\exists p \in G)\ \pi(p) \leq x(y)\}$

We recall that G induces a canonical filter $H = \{b \in B : (\exists p \in G)\ \pi(p) \leq b\}$ on B and it is easy to check that $x^G = x^H$ for all $x \in M^B$. □

Chapter 3

The Continuum Hypothesis

3.1 Notes on Absoluteness

Again let M be a countable transitive model of ZFC. Definitions that make sense in V can be relativized to M. For example there are sets $\omega^M, \mathcal{P}(\omega)^M, \aleph_1^M \in M$ such that

1. $M \models$ "ω^M is the set of the natural numbers."
2. $M \models$ "$\mathcal{P}(\omega)^M$ is the powerset of the natural numbers."
3. $M \models$ "$\aleph_1^M$ is the first uncountable cardinal."

We say that "ω^M plays the role of of ω in M", and so on. We have already seen that "x is the set of natural numbers" is an absolute property and therefore $\omega^M = \omega$.

Clearly since M is countable every set in M, even those that play the role of uncountable sets such as $\aleph_1^M$, are countable (from the outside perspective). However M does not contain a bijection between ω^M and $\aleph_1^M$ and therefore thinks that $\aleph_1^M$ is uncountable. However every set that plays the role of a cardinal also plays the role of an ordinal and this is an absolute property. Thus if κ plays the role of a cardinal in M then it is (from the outside) a countable ordinal.

We have shown that M and a generic extension $M[G]$ contain the same ordinals. However it may be the case that ordinals that played the role of a cardinal in M do not play the role of a cardinal in $M[G]$. For example we might force with partial bijections between ω and $\aleph_1^M$. Then $\aleph_1^M$ is countable in $M[G]$ and therefore cannot play the role of the smallest uncountable ordinal anymore. In other words there may be new bijections in $M[G]$ that did not exists in M that cause cardinals in M to become “smaller”. If this happens to a cardinal κ we say that κ collapsed in $M[G]$.

On the other hand new cardinals cannot appear. If κ plays the role of a cardinal in $M[G]$ then it must have played the role of a cardinal in M. Because if κ was not a cardinal in M then there existed a bijection in M between κ and an ordinal smaller than κ and this bijection still exists in $M[G]$.

3.2 The consistency of CH

Definition 3.1. The *continuum hypothesis* is the statement $2^{\aleph_0} = \aleph_1$.

Starting from M we want to construct a bijection G between $\mathcal{P}(\omega)$ and $\aleph_1$ such that in the model $M[G]$ the continuum hypothesis holds. This means, since we are working in M, we are really constructing a bijection between $\mathcal{P}(\omega)^M$ and $\aleph_1^M$. However what we want is a bijection G between $\mathcal{P}(\omega)^{M[G]}$ and $\aleph_1^{M[G]}$.

This is actually a problem as we will see in the following attempt of forcing CH. Consider, in M, the forcing $(\mathbb{P}, \supseteq)$ of all finite partial bijections between $\mathcal{P}(\omega)$ and $\aleph_1$ and let $G \subseteq \mathbb{P}$ be an M-generic filter.

First of all it is easy to see that $f = \bigcup G$ is a bijection between $\mathcal{P}(\omega)^M$ and $\aleph_1^M$ in $M[G]$. It follows from the directedness of G that f is a function and f is total because for every $A \in \mathcal{P}(\omega)$ the set $D_A = \{p \in \mathbb{P} : A \in \text{dom}(p)\}$ is dense.

However the following lemma shows that f is in fact not a bijection between $\mathcal{P}(\omega)^{M[G]}$ and $\aleph_1^{M[G]}$ in $M[G]$.

Lemma 3.2. *In M let $(\mathbb{P}, \supseteq)$ be the set of all finite partial bijections between $\mathcal{P}(\omega)$ and $\aleph_1$. Then there appear new subsets of ω in $M[G]$, i.e. $\mathcal{P}(\omega)^M \subset \mathcal{P}(\omega)^{M[G]}$.*

Proof. Clearly since ω is absolute $\mathcal{P}(\omega)^M \subseteq \mathcal{P}(\omega)^{M[G]}$. We use $f = \bigcup G$ to define a set $A \subseteq \omega$ by

$$n \in A \quad \leftrightarrow \quad f(n) < \omega.$$

Let $B \subseteq \omega$ be a subset of ω in M. Then the set

$$D_B = \{p \in \mathbb{P} : \exists n \in \text{dom}(p) : (p(n) \geq \omega \wedge n \in B) \vee (p(n) < \omega \wedge n \notin B)\}$$

is dense. This is easy to see because $\text{dom}(p)$ is finite for all $p \in \mathbb{P}$ so we can always add n with this property. Now any generic filter G must contain some $p \in D_B$ and therefore there exists $n \in \omega$ such that

$$n \in A \quad \leftrightarrow \quad f(n) < \omega \quad \leftrightarrow \quad n \notin B,$$

i.e. $A \neq B$ and since B was an arbitrary subset of ω in M it follows that $A \notin M$. □

The above lemma shows that things may go wrong, even if we do not do something "obviously" wrong such as forcing with bijections between ω and $\aleph_1^M$. The following condition ensures that $\aleph_1$ is preserved in the generic extension.

Definition 3.3. A forcing $(\mathbb{P}, \leq))$ is called *σ-closed* if every sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ has a lower bound $p \in \mathbb{P}$.

Lemma 3.4. *Let $(\mathbb{P}, \leq) \in M$ be an σ-closed forcing and let G be an M-generic filter on $\mathbb{P}$. Let $X \in M$ be a set and let $f : \omega \to X$ be a function in $M[G]$. Then f is already contained in M.*

Proof. Let $\dot{f}$ be a $\mathbb{P}$-name and let $p \in \mathbb{P}$ such that

$$p \Vdash f \text{ is a function from } \omega \text{ to } \check{X}$$

and let G be an M-generic filter containing p. For $n < \omega$ let $x_n \in X$ denote the image of n under f in $M[G]$, i.e.

$$M[G] \models f(n) = x_n.$$

Therefore there is $p_0 \in G$ such that $p_0 \Vdash \dot{f}(\check{0}) = \check{x}_0$ and by directedness of G we can assume that $p_0 \leq p$. Now there is $p_1 \in G$ such that $p_1 \Vdash \dot{f}(\check{1}) = \check{x}_1$ and again we may assume $p_1 \leq p_0$. Continuing this way we construct a sequence $p_0 \geq p_1 \geq p_2 \geq \dots$. Let $q \in \mathbb{P}$ be a lower bound of this sequence. Clearly $q \Vdash \dot{f}(\check{n}) = \check{x}_n$ for ever $n < \omega$. Now working in M we can define a function $g : \omega \to X$ by

$$g(n) = x \leftrightarrow q \Vdash \dot{f}(\check{n}) = \check{x}.$$

Of course $g = f$ and therefore $f \in M$. □

Corollary 3.5. *Let $(\mathbb{P}, \leq) \in M$ be an σ-closed forcing and let G be an M-generic filter on $\mathbb{P}$. Then $\aleph_1^M = \aleph_1^{M[G]}$.*

Proof. Assume that $M[G] \models$ "$\aleph_1^M$ is countable". Then there is a surjective function $f : \omega \to \aleph_1^M$ in $M[G]$. However by the above lemma f would already be in M and therefore $M \models$ "$\aleph_1^M$ is countable", a contradiction. □

Theorem 3.6. *Let M be a countable transitive model. Then there is a generic extension $M[G]$ of M such that*

$$M[G] \models 2^{\aleph_0} = \aleph_1$$

Proof. In M let P be the set of all functions p such that

1. $\mathrm{dom}(p)$ is a countable subset of $2^{\aleph_0}$.
2. $\mathrm{ran}(p)$ is a countable subset of $\aleph_1$.
3. p is a bijection.

Clearly $(\mathbb{P}, \supseteq)$ is σ-closed and thus $\aleph_1^M = \aleph_1^{M[G]}$. Furthermore any $f \in 2^{\aleph_0}$ is a function $\omega \to 2$ and therefore if $f \in M[G]$ then $f \in M$. Thus also $(2^{\aleph_0})^M = (2^{\aleph_0})^{M[G]}$.

Now if G is an M-generic filter on $\mathbb{P}$ then $f = \bigcup G$ is a bijection between $(2^{\aleph_0})^M$ and $\aleph_1^M = \aleph_1^{M[G]}$. Indeed f is total because for every $x \in 2^{\aleph_0}$ the set $\{p \in \mathbb{P} : x \in \mathrm{dom}(p)\}$ is dense and likewise f is surjective because for every $x \in \aleph_1$ the set $\{p \in \mathbb{P} : x \in \mathrm{ran}(p)\}$ is dense. □

3.3 The consistency of ¬CH

Definition 3.7. We say that a forcing $(\mathbb{P}, \leq)$ has the *countable chain condition* (c.c.c.) if every antichain $W \subseteq \mathbb{P}$ is at most countable.

Lemma 3.8. *Let $(\mathbb{P}, \leq)$ be a c.c.c. forcing. Let $X, Y \in M$ and let $f : X \to Y$ be a function in $M[G]$. Then there is a function $g : X \to \mathcal{P}(Y)$ in M that "approximates" f in M in the sense that f maps every $x \in X$ to some $y \in g(x)$ and for all $x \in X$ the set $g(x)$ is countable in M.*

Proof. There exist some $p \in G$, $\dot{f} \in M^{B(P)}$ such that

$$p \Vdash \dot{f} \text{ is a function from } \check{X} \text{ to } \check{Y}.$$

Now for $x \in X$ let

$$g(x) = \{y \in Y : (\exists q \leq p)\ q \Vdash \dot{f}(\check{x}) = \check{y}\}.$$

Clearly f can only map x to some $y \in g(x)$. To see that for all $x \in X$ the set $g(x)$ is countable fix for every $y \in g(x)$ a $q_y \in \mathbb{P}$ such that $q_y \Vdash \dot{f}(\check{x}) = \check{y}$. Then since q_y forces different values for $f(x)$ for every $y \in g(x)$ the set $\{q_y : y \in g(x)\}$ is an antichain and thus countable. □

With the help of this lemma we can show that c.c.c. forcings preserve cardinals.

Lemma 3.9. *Let $(\mathbb{P}, \leq)$ be a c.c.c. forcing. Then if α is a cardinal in M it is still a cardinal in $M[G]$.*

Proof. We have shown that M and $M[G]$ have the same ordinals, in particular if α is an ordinal in M it is also an ordinal in $M[G]$. Thus if α is a cardinal in M but not in $M[G]$ then in $M[G]$ there exists an ordinal $\beta < \alpha$ and a surjective function $f : \beta \to \alpha$. Now let $g : \beta \to \mathcal{P}(\alpha)$ a function in M as in lemma 3.8. Then it holds that

$$M \models |\alpha| = |\bigcup_{\gamma<\beta} g(\gamma)| \leq |\beta| \cdot \aleph_0 = |\beta|.$$

This contradicts the fact that α is a cardinal in M. □

To prove that the forcing we are going to use to show the the consistency of ¬CH has the countable chain condition we need the following lemma.

Lemma 3.10 (Δ-Lemma)**.** *Let $\mathcal{C}$ be an uncountable family of finite sets. Then there exists an uncountable set $\mathcal{D} \subseteq \mathcal{C}$ and a set D such that for all $X, Y \in \mathcal{D}$*

$$X \neq Y \to X \cap Y = D.$$

Proof. First note that because $\mathcal{C}$ is uncountable there exists a natural number $n < \omega$ such that $|X| = n$ for uncountably many $X \in \mathcal{C}$. Therefore we can assume without loss of generality that $|X| = n$ for all $X \in \mathcal{C}$. We now prove the lemma by induction on n. The case $n = 1$ is trivial. If $n > 1$ then we have two cases.

1. If there is some a such that $a \in X$ for uncountably many $X \in \mathcal{C}$ then we can assume without loss of generality that $a \in X$ for all $X \in \mathcal{C}$. Now we can apply the lemma to $\{X \setminus \{a\} : X \in \mathcal{C}\}$.

2. Otherwise every a is contained in at most countably many $X \in \mathcal{C}$ and we construct a family $\mathcal{D} = \{X_\alpha : \alpha < \omega_1\}$ of disjoint sets by inductively. For $\alpha < \omega_1$ we can easily find set $X_\alpha \in X$ that is disjoint from every $X_\beta, \beta < \alpha$ because α is countable and every X_β only intersects countably many sets.

□

Theorem 3.11. *Let M be a countable transitive model. Then there is a generic extension $M[G]$ of M such that*

$$M[G] \models 2^{\aleph_0} > \aleph_1.$$

Proof. In M let $\mathbb{P}$ be the set of all functions p such that

1. $\mathrm{dom}(p)$ is a finite subset of $\omega \times \aleph_2$.
2. $\mathrm{ran}(p) \subseteq 2$.

First we show that $(\mathbb{P}, \supseteq)$ satisfies the countable chain condition in M.

Let W be an uncountable subset of $\mathbb{P}$ and consider $\mathcal{C} = \{\mathrm{dom}(p) : p \in W\}$. Clearly, since for every finite $X \subseteq \omega \times \aleph_2$ the set of functions with domain X and values in 2 is also finite, $\mathcal{C}$ must be uncountable. By the Δ-lemma 3.10 there exists an uncountable $\mathcal{D} \subseteq \mathcal{C}$ and a set D such that $X \cap Y = D$ for all $X, Y \in \mathcal{D}$. Now let $V = \{p \in W : \mathrm{dom}(p) \in \mathcal{D}\}$. Again, because there are only finitely many function from D into 2, there must be uncountably many $p \in V$ with equal $p \restriction D$. Now if for $p, q \in V \subseteq W$ we have $p \restriction D = q \restriction D$ then p and q are compatible and W is not an antichain.

Therefore we know by lemma 3.9 that forcing with $\mathbb{P}$ preserves cardinals and in particular $\aleph_1^M = \aleph_1^{M[G]}$ and $\aleph_2^M = \aleph_2^{M[G]}$.

Let G be an M-generic filter on $\mathbb{P}$ and $f = \bigcup G$. As before it is easy to see that f is a total function from $\omega \times \aleph_2$ into 2. For $\alpha < \aleph_2$ we define functions $f_\alpha : \omega \to 2$ by

$$f_\alpha(n) = f(n, \alpha)$$

We show that for $\alpha \neq \beta$ also $f_\alpha \neq f_\beta$. This follows from the fact that the set

$$D_{\alpha,\beta} = \{p \in \mathbb{P} : (\exists n < \omega) p(n, \alpha) \neq p(n, \beta)\}$$

is dense in $\mathbb{P}$.

Thus the set $M[G]$ contains $\aleph_2$ distinct reals and

$$M[G] \models 2^{\aleph_0} > \aleph_1.$$

□

Chapter 4

The Suslin Hypothesis

4.1 Trees

Definition 4.1. We call a partial order $(T, \leq)$ a *plant* if for all $x \in T$ the set

$$T_{<x} = \{y : y < x\}$$

is well-ordered by $\leq$.

Similarly we define

$$T_{\geq x} = \{y : y \geq x\}$$

for every $x \in T$ and it easy to see that $T_{\geq x}$ is a plant.

The *height* $h(x)$ of $x \in T$ is the unique ordinal number α such that $T_{<x} \cong \alpha$.

For an ordinal α let

$$T_\alpha = \{x \in T : h(x) = \alpha\}$$

be the αth *level* of T and we define the height $h(T)$ of T as the least ordinal number α such that $T_\alpha = \emptyset$.

A maximal totally ordered subset $B \subseteq T$ is called a *branch* and again we define the height $h(B)$ of B as the unique ordinal number α such that $B \cong \alpha$.

Definition 4.2. A plant $(T, \leq)$ is called a *tree* if:

1. T has a least element r such that $x \geq r$ for all $x \in T$.
2. If α is a limit ordinal then for all $x, y \in T_\alpha$

$$x \neq y \to T_{<x} \neq T_{<y}.$$

Definition 4.3. Let $(T, \leq)$ be a tree. We call a set $A \subseteq T$ an *antichain* if for every $x \in X$ there is $a \in A$ such that $a \leq x$ or $x \leq a$.

We say that A is *bounded* if there is some $\alpha < h(T)$ such that $h(a) \leq \alpha$ for all $a \in A$. This condition is trivial if $h(T)$ is a successor ordinal.

Note that the definition of antichains on a trees contradicts our previous definition of antichains on general partially ordered sets. However if $A \subseteq T$ is an antichain in the tree sense then A is an antichain in the general sense on $(T, \geq)$.

Definition 4.4. A tree $(T, \leq)$ is called a *Suslin tree* if:

1. T has height ω_1.
2. Every antichain $A \subseteq T$ is at most countable.
3. Every branch B has height less than ω_1.
4. For all $x \in T$ and all ordinals α with $h(x) < \alpha < \omega_1$ there is $y \geq x$ such that $h(y) = \alpha$.

Definition 4.5. The *Suslin hypothesis* SH states that there exists no Suslin tree.

4.2 The consistency of $\neg$SH

Theorem 4.6. *Let M be a countable transitive model. Then there exists a generic extension $M[G]$ of M such that there is a Suslin tree in $M[G]$.*

In this section let $\mathbb{P}^+$ be the set of all sets T such that for some $\alpha \leq \omega_1$:

1. $T \subseteq \omega^{<\alpha}$
2. If $x : \beta \to \omega$ is in T then $x \restriction \gamma \in T$ for every $\gamma < \beta$.
3. If $x : \beta \to \omega$ is in T and $\beta + 1 < \omega_1$ then $x^\frown n \in T$ for every $n < \omega$.
4. If $x : \beta \to \omega$ is in T then for every $\beta < \gamma < \alpha$ there is $y : \gamma \to \omega$ in T such that $y \restriction \beta = x$.
5. $T \cap \omega^\beta$ is at most countable for every $\beta < \alpha$.

It is easy to see that every $T \in \mathbb{P}^+$, ordered by set inclusion, is a tree of height less or equal ω_1. Let $\mathbb{P}$ be the set of all $T \in \mathbb{P}^+$ of height less than ω_1. For $S, T \in \mathbb{P}^+$ we define

$$S \leq T \leftrightarrow (\exists \alpha \leq h(S))\ T = \{x \restriction \alpha : x \in S\}.$$

Let G be an M-generic filter on $(\mathbb{P}, \leq)$ and let $\mathcal{T} = \bigcup G$. We are going to show that $\mathcal{T}$ is a Suslin tree in $M[G]$.

First observe that for two compatible $S, T \in \mathbb{P}$ either $S \leq T$ or $T \leq S$. Thus, because $\mathcal{T}$ is the union of pairwise compatible trees, it is easy to see that $\mathcal{T}$ is a itself tree and every level of $\mathcal{T}$ is at most countable.

Likewise it is easy to see that for a sequence $T_0 \geq T_1 \geq T_2 \geq \dots$ the union $T = \bigcup_{n<\omega} T_n$ lies in $\mathbb{P}$, i.e. $\mathbb{P}$ is σ-closed and $\aleph_1$ is preserved in $M[G]$.

First of all we are going to verify that $\mathcal{T}$ has height $\aleph_1$. We claim that for every $\alpha < \omega_1$ the set

$$D_\alpha = \{T \in \mathbb{P} : h(T) > \alpha\}$$

is dense in $\mathbb{P}$ and thus $h(\mathcal{T}) > \alpha$. It suffices to show that for every $T \in \mathbb{P}$ there is $S \in \mathbb{P}$ such that $h(S) = h(T) + 1$ and $S \leq T$. Then we can use induction, taking unions at limit steps, to find $S' \leq T$ such that $h(S') > \alpha$.

If $h(T) = \beta + 1$ finding S is trivial. If $h(T) = \beta$ is a limit ordinal take for every $x \in T$ a branch B_x such that $x \in B$ and $h(B_x) = \beta$. This is possible by condition 4. Let $b_x = \bigcup B_x$ and let $S = T \cup \{b_x : x \in T\}$. Clearly since T is countable so is S and therefore $S \in \mathbb{P}$.

It remains to show that every antichain in $\mathcal{T}$ is at most countable. Before we do this let us establish the following facts about antichains.

Lemma 4.7. *Let $T \in \mathbb{P}$ and let $A \subseteq T$ be a bounded maximal antichain. Then for every $S \in \mathbb{P}^+$ with $S \leq T$ it holds that A is a maximal antichain in S (and of course A is still bounded in S).*

Proof. Let $x \in S$ be arbitrary and let $\alpha < h(T)$ be such that $h(a) \leq \alpha$ for all $a \in A$. Then because $s \restriction \alpha \in S$ there exists $a \in A$ such that $a \subseteq s \restriction \alpha$. This implies $a \subseteq s$ and therefore A is an antichain. □

Remark 4.8. Note that the condition that A is bounded is necessary. Consider $T = \omega^{<\omega}$,

$$A = \{\underbrace{0^\frown 0^\frown \dots {}^\frown 0}_{n}{}^\frown m : n, m \in \omega, m > 0\}$$

and

$$S = \omega^{<\omega} \cup \{x \in \omega^\omega : x(i) = 0 \text{ for all but finitely many } i \in \omega\}.$$

Then A is a maximal antichain in T but in S the set A is no longer maximal because $x \equiv 0$ is in S.

Lemma 4.9. *Let $T \in \mathbb{P}$ such that $h(T) = \alpha$ is a limit ordinal and let $A \subseteq T$ be a maximal antichain in T. Then there exists $S \leq T$ of height $\alpha + 1$ such that A is a maximal antichain in S (and in particular A is bounded in S),* *Proof.* We construct S as follows. For $x \in T$ there exists $a_x \in A$ such that $a_x \subseteq x$ or $x \subseteq a_x$. Let B_x be a branch of height α (such branch exists because of condition 4.) such that $x \in B_x$ and $a_x \in B_x$ and let $b_x = \bigcup B_x$. Now let $S = T \cup \{b_x : x \in T\}$. Clearly A remains an antichain in S and because T is countable so is S, thus $S \in \mathbb{P}$. □

We are now ready to complete the proof that $\mathcal{T}$ is a a Suslin tree. Because every antichain is contained in a maximal antichain it is enough to show that every maximal antichain is countable.

Let A be a maximal antichain in $\mathcal{T}$. Then there exists a $\mathbb{P}$-name $\dot{A}$ for A, a $\mathbb{P}$-name $\dot{\mathcal{T}}$ for $\mathcal{T}$ and $T \in G$ such that

$$T \Vdash \dot{A} \text{ is a maximal antichain in } \dot{\mathcal{T}}$$

We claim that the set

$$\{S \leq T : \text{there is a bounded maximal antichain in } B \in S \text{ with } S \Vdash \check{B} \subseteq \dot{A}\}$$

is dense below T. In this case there is $S \in G$ such that $B \subseteq A$ is a bounded maximal antichain in S. Because $\mathcal{T} \leq S$ by lemma 4.7 B remains a maximal antichain in $\mathcal{T}$. Thus $B = A$ and because $B \subseteq S$ is countable so is A.

Let $T_0 \leq T$ be arbitrary and of course

$$T_0 \Vdash \dot{A} \text{ is a maximal antichain and } \dot{\mathcal{T}} \leq \check{T}_0.$$

Therefore for any $x \in T$ there exists $T_0' \leq T_0$ and $a_x \in T_0'$ such that $a_x \subseteq x$ or $x \subseteq a_x$ and

$$T_0' \Vdash a_x \in \dot{A}.$$

Because T is countable and $\mathbb{P}$ is σ-closed we can find $T_0' \leq T_0$ that forces this condition for every $x \in T$. Let $T_1 = T_0'$. We construct a sequence $T_0 \geq T_1 \geq T_2, \ldots$ inductively such that for every $n \in \omega$ it holds that for all $x \in T_n$ there is $a_x \in T_{n+1}$ such that $a_x \subseteq x$ or $x \subseteq a_x$ and

$$T_{n+1} \Vdash a \in a_x \in \dot{A}.$$

We find a lowerbound $T_\omega = \bigcup_{n<\omega} T_n$ for this sequence and let $B = \{a_x : x \in T_\omega\}$. Of course B is a maximal antichain in T_ω and

$$T_\omega \Vdash B \subseteq \dot{A}.$$

Now we apply lemma 4.9 to T_ω and get $S \leq \omega$ such that B is bounded in S. Clearly $S \leq T_0$ and

$$S \Vdash B \subseteq \dot{A}$$

and therefore $S \in D$. Thus D is dense. □

Chapter 5

The Diamond Principle

5.1 Club Sets, Stationary Sets

Definition 5.1. A set $C \subseteq \omega_1$ is *closed* if for every countable $X \subseteq C$ it holds that its supremum $\bigcup X \in C$.

A set $C \subseteq \omega_1$ is *unbounded* if for every $\alpha \in \omega_1$ there exists $b > \alpha$ such that $\beta \in C$.

A set $C \subseteq \omega_1$ is *club* set if it is closed and unbounded.

A set $S \subseteq \omega$ is *stationary* if for every club set $C \subseteq \omega_1$ it holds that $C \cap S \neq \emptyset$.

Definition 5.2. The *diamond principle* $\diamondsuit$ states that there exists sequence $\{A_\alpha \subseteq \alpha : \alpha < \omega_1\}$ such that for all sets $A \subseteq \omega_1$ the set $\{\alpha : A \cap \alpha = A_\alpha\}$ is stationary.

This is equivalent to the statement that there exists a sequence of function $\{(h_\alpha : \alpha \to 2) : \alpha < \omega_1\}$ such that for all functions $f : \omega_1 \to 2$ the set $\{\alpha : f \restriction \alpha = h_\alpha\}$ is stationary.

Sequences $\{A_\alpha : \alpha < \omega_1\}$ or $\{h_\alpha : \alpha < \omega_1\}$ that verify the diamond principle are called $\diamondsuit$*-sequence.*

5.2 The consistency of $\diamondsuit$

Theorem 5.3. *Let M be a countable transitive model. Then there exists a generic extension $M[G]$ of M such that there is a $\diamondsuit$-sequence in $M[G]$.*

Proof. We are going to construct a $\diamondsuit$-sequence by forcing.

Let $\mathbb{P}$ be the set of all p such that $p = \{h_\beta : \beta < \alpha\}$ for some $\alpha < \omega_1$ and h_β is a function $\beta \to 2$ for every $\beta < \alpha$. We call α the height of p. For $p, q \in \mathbb{P}$ let $p \leq q$ if $p \supseteq q$.

Clearly, because the supremum of countable many countable ordinals is countable, $\mathbb{P}$ is σ-closed and thus preserves ω_1.

Let G be an M-generic filter on $\mathbb{P}$ and let $\Gamma = \bigcup G$. Obviously for every $\alpha < \omega$ the set $\{p \in \mathbb{P} : p \text{ has height greater than } \alpha\}$ is dense and therefore $\Gamma = \{h_\alpha : \alpha < \omega_1\}$ with h_α being a function $\alpha \to 2$ for every $\alpha < \omega_1$.

We may hope this construction is going to work because, working in M, if f is a function $\omega_1 \to 2$ and C is a club (and in particular unbounded) the set of $p \in \mathbb{P}$ with $f \upharpoonright \alpha = p$ for some $\alpha \in C$ is dense in $\mathbb{P}$. But of course we need this for all club sets in $M[G]$.

Let $f : \omega_1 \to 2$ be a function in $M[G]$, let C be a club set in $M[G]$, let $\dot{f}$ and $\dot{C}$ be $\mathbb{P}$-names for f and C respectively and let $\dot{\Gamma}$ be a $\mathbb{P}$-name for Γ.

First note that for every $p = \{h_\beta : \beta < \alpha\} \in \mathbb{P}$ of height α

$$p \Vdash \dot{\Gamma}(\check{\beta}) = \check{h}_\beta$$

for every $\beta < \alpha$.

We find $p \in G$ such that

$$p \Vdash \dot{f} \text{ is a function } \check{\omega}_1 \to \check{2} \text{ and } \dot{C} \text{ is a club subset of } \check{\omega}_1$$

and claim that for all $q \leq p$ of height α there exists $\beta > \alpha$, a function $g : \alpha \to 2$ in M and $q' \leq q$ of height greater or equal β such that

$$q' \Vdash \check{\beta} \in \dot{C} \wedge \dot{f} \upharpoonright \check{\alpha} = \check{g}.$$

Indeed if G' is an arbitrary M-generic ideal that contains q then for every $\alpha < \omega_1$, because $q \Vdash$ "$\dot{C}$ is unbounded", there exists $q_1 \in G'$ such that $q_1 \Vdash \check{\beta} \in \dot{C}$ for some $\beta > \alpha$. Let $g = \dot{f}^{G'} \upharpoonright \alpha$. Then by lemma 3.4 we have $g \in M$. Therefore there is some $q_2 \in G'$ such that $q_2 \Vdash \dot{f} \upharpoonright \check{\alpha} = \check{g}$. By directedness of G' there is $q' \in G$ such that $q' \leq q, q' \leq q_1, q' \leq q_2$ and again by directedness we may assume that q has height at least β.

Repeating this step and fixing $q \leq p$ we can inductively construct a sequence $q = p_0 \geq p_1 \geq p_2 \geq \ldots$ such that for each p_n of height α_n there exists $\beta_n > \alpha_n$ and $g_n : \alpha \to 2$ such that

$$p_{n+1} \Vdash \check{\beta}_n \in \dot{C} \wedge \dot{f} \upharpoonright \check{\alpha} = \check{g}_n.$$

Utilizing the σ-closedness of $\mathbb{P}$ we find

$$p^* = \bigcup_{n<\omega} p_n$$

of height

$$\alpha^* = \sup_{n<\omega} \alpha_n = \sup_{n<\omega} \beta_n.$$

Clearly $p^* \Vdash \check{\beta}_n \in \dot{C}$ and $p^* \Vdash \dot{f} \restriction \check{\alpha} = \check{g}_n$ for every $n < \omega$. Furthermore because $p^* \Vdash$ "$\dot{C}$ is closed" it follows that $p^* \Vdash \alpha^* \in \dot{C}$.

Likewise let

$$g^* = \bigcup_{n<\omega} g_n$$

and of course g^* is a function $\alpha^* \to 2$ and let $u = p^* \cup g^*$. Clearly

$$u \Vdash \dot{f} \restriction \check{\alpha}^* = \check{g}^* \wedge \alpha^* \in \dot{C} \wedge \dot{\Gamma}(\check{\alpha}^*) = \check{g}^*$$

and of course $u \leq q \leq p$. Finally because q was arbitrary the set of all u with this property for some α^*, g^* is dense below p and therefore G must contain one of them. $\square$

5.3 $\diamondsuit \to$ CH

Theorem 5.4. *The diamond principle implies the continuum hypothesis.*

Proof. Let $\{A_\alpha : \alpha < \omega_1\}$ be a $\diamondsuit$-sequence and let $A \subseteq \omega$ be arbitrary. Then the set $S = \{\alpha : A \cap \alpha = A_\alpha\}$ is stationary and clearly there exists $\alpha > \omega$ with $\alpha \in S$. To see this simply consider the club set $\omega_1 \setminus \omega$. Now because $\alpha \in S$ we have $A \cap \alpha = A_\alpha$ and because $\alpha > \omega$ we have $A \cap \alpha = A$. Thus we have found $\alpha < \omega_1$ such that $A_\alpha = A$ and because A was arbitrary we have shown that every subset of ω occurs in the $\diamondsuit$-sequence. $\square$

5.4 $\diamondsuit \to \neg$SH

Lemma 5.5. *Let $T \subseteq \omega_1^{<\omega^1}$ be a tree and let $A \subseteq T$ be a maximal antichain. Then the set*

$$C = \{\alpha : A \cap T_{<\alpha} \textit{ is a maximal antichain in } T_{<\alpha}\}$$

is a club set.

Proof. Let $\alpha_0 < \alpha_1 < \alpha_2 < \dots$ be a sequence in C and let $\alpha = \sup_{n<\omega} \alpha_n$. For every $x \in T_{<\alpha}$ there is $n \in \omega$ such that $x \in T_{<\alpha_n}$ and there is some $a \in A \cap T_{<\alpha_n}$ with $a \subseteq x$ or $x \subseteq a$. Because $A \cap T_{<\alpha_n} \subseteq A \cap T_{<\alpha}$ it follows that $A \cap T_{<\alpha}$ is a maximal antichain in $T_{<\alpha}$. Thus $\alpha \in C$ and C is closed.

Let $\alpha_0 < \omega_1$ be arbitrary. Then for every $x \in T_{<\alpha_0}$ there exists $a_x \in A$ such that $a_x \subseteq x$ or $x \subseteq a_x$. Let $\alpha_1 \geq \alpha_0$ be such that $\{a_x : x \in T_{<\alpha_0}\} \subseteq$

$T_{<\alpha_1}$. Then let $\alpha_2 \geq \alpha_1$ be such that $\{a_x : x \in T_{<\alpha_1}\} \subseteq T_{<\alpha_2}$ and so on. Let $\alpha = \sup_{n<\omega} \alpha_n$. If $x \in T_{<\alpha}$ then there is $n \in \omega$ such that $x \in T_{<\alpha_n}$ and $a_x \in A \cap T_{<\alpha_{n+1}} \subseteq A \cap T_{<\alpha}$ with $a_x \subseteq x$ or $x \subseteq a_x$. Therefore $A \cap T_{<\alpha}$ is a maximal antichain in $T_{<\alpha}$ and $\alpha \in C$. Of course $\alpha_0 \leq \alpha$ and thus C is unbounded. □

Theorem 5.6. *The diamond principle implies that there exists a Suslin tree.*

Let $\{A_\alpha : \alpha < \omega_1\}$ be a $\diamondsuit$-sequence. We construct a tree $T = \omega_1$ that consists of exactly the countable ordinal numbers. We take the ordinals in order such that every time we add an element to the tree we take the least ordinal we have not used yet.

We start by defining $T_0 = \{0\}$.

If $\alpha = \beta + 1$ is a successor ordinal then we construct T_α by adding countably many successors above every element in T_β. It is clear that T_α is countable.

If α is a limit and we have defined $T_{<\alpha}$ there are two cases.

1. If A_α is a maximal antichain in $T_{<\alpha}$ for every element $x \in T$ there exists $a_x \in A_\alpha$ with $a_x \subseteq x$ or $x \subseteq a_x$. Now let B_x be a branch of height α in $T_{<\alpha}$ such that $x \in B_x$ and $a_x \in B_x$ and add b_x to T_α such that $y \leq b_x$ for all $y \in B_x$.

2. Otherwise simply do the same construction as in the first case but with and arbitrary branch containing x for every $x \in T_{<\alpha}$.

In both cases it is clear that because $T_{<\alpha}$ was countable so is T_α. We continue with this construction until we have constructed a tree $T = \bigcup_{\alpha<\omega_1} T_\alpha$ of height ω_1.

Assume that the exists a branch $T \subseteq B$ of height ω_1. Then $\{\alpha + 1 : \alpha \in B\}$ is an uncountable antichain. Therefore to verify that T is a Suslin tree we only need to show that T does not contain an uncountable maximal antichain.

Let A be a maximal antichain in T and let C be as in lemma 5.5. First note that $D = \{\alpha \in C : T_{<\alpha} = \alpha\}$ is a club set. Closure is obvious. For unboundedness let $\alpha_0 < \omega_1$ be arbitrary. Choose $\alpha_1 \geq \alpha_0$ such that $T_{\alpha_0} \subseteq \alpha_1$, then choose $\alpha_2 \geq \alpha_1$ such that $T_{\alpha_1} \subseteq \alpha_2$ and so on. Let $\alpha = \sup_{n<\omega} \alpha_n$, i.e. $\alpha = \bigcup_{n<\omega} \alpha_n = \bigcup_{n<\omega} T_{<\alpha_n} = T_{<\alpha}$.

Thus by $\diamondsuit$ there exists $\alpha < \omega_1$ such that $A \cap \alpha = A_\alpha$, $A \cap T_{<\alpha}$ is a maximal antichain in $T_{<\alpha}$ and $T_{<\alpha} = \alpha$. Because $A \cap T_{<\alpha} = A \cap \alpha = A_\alpha$ it follows that A_α is a maximal antichain in $T < \alpha$. Now by construction every element in T_α lies above some $a \in A_\alpha$ and thus A_α is also a maximal antichain T. Therefore $A_\alpha = A$ and because $A_\alpha \subseteq \alpha$ it follows that A is countable.

Chapter 6

Iterated Forcing

6.1 Two-step forcing

Definition 6.1. Again let M be a countable transitive model of ZFC. Let $\mathbb{P}$ be a forcing notion in M and $\dot{\mathbb{Q}}$ be a name for a partial order in $M^{\mathbb{P}}$.

We define

$$\mathbb{P} * \dot{\mathbb{Q}} = \{(p, \dot{q}) : p \in P, \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}\}.$$

and for $(p_1, \dot{q}_1), (p_2, \dot{q}_2) \in \mathbb{P} * \dot{\mathbb{Q}}$ let

$$(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2) \quad \leftrightarrow \quad (p_1 \leq p_2) \wedge (p_1 \Vdash \dot{q}_1 \leq \dot{q}_2).$$

Remark 6.2. Note that $\mathbb{P} * \dot{\mathbb{Q}}$ will typically be a proper class, not a set. To see this let $p, q \in \mathbb{P}$ be incompatible and consider the names

$$\sigma_\alpha = \{(\check{\alpha}, p)\}$$

$$\tau_\alpha = \{(\sigma_\alpha, q), (\check{0}, 1)\}$$

where α is an ordinal number. A simple calculation shows that

$$\Vdash_{\mathbb{P}} \tau_\alpha = \{\check{0}\}.$$

Now if $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing with $\Vdash_{\mathbb{P}} \{\check{0}\} \in \dot{\mathbb{Q}}$ it follows that $\{\tau_\alpha : \alpha \in ord\} \subseteq \mathbb{P} * \dot{\mathbb{Q}}$.

We solve this problem with the following lemma.

Lemma 6.3. *There exists a* set Z *of* $\mathbb{P}$*-names such that for all* $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ *there is* $\dot{q}' \in Z$ *with* $\Vdash_{\mathbb{P}} \dot{q}' = \dot{q}$*.*

Proof. Given $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ we construct $\dot{q}'$ as follows. We define the dense set

$$X = \{p \in \mathbb{P} : (\exists \dot{x} \in \mathrm{dom}(\dot{\mathbb{Q}}))\ p \Vdash \dot{q} = \dot{x}\}$$

and choose a maximal antichain $W \subseteq X$. Now let

$$\dot{q}' = \{(\dot{x}, w) : w \in W, \dot{x} \in \mathrm{dom}(\dot{\mathbb{Q}}), w \Vdash \dot{q} = \dot{x}\}.$$

It is clear that $\Vdash_{\mathbb{P}} \dot{q} = \dot{q}'$ and the names $\dot{r}$ with $\mathrm{dom}(r) \subseteq \mathrm{dom}(\dot{\mathbb{Q}})$ form a set. □

Lemma 6.4.

1. *Let G be an M-generic filter on $\mathbb{P}$, let $\mathbb{Q} = \dot{\mathbb{Q}}^G$ and let H be an $M[G]$-generic filter on $\mathbb{Q}$. Then*

$$G * H = \{(p, \dot{q}) : p \in G \wedge \dot{q}^G \in H\}$$

is a M-generic filter on $\mathbb{P} * \dot{\mathbb{Q}}$ and

$$M[G][H] = M[G * H].$$

2. *Conversely let K be an M-generic filter on $\mathbb{P} * \dot{\mathbb{Q}}$. Let*

$$G = \{p \in \mathbb{P} : (\exists \dot{q} \in \dot{\mathbb{Q}})\ (p, \dot{q}) \in K\}$$
$$H = \{\dot{q}^G : (\exists p \in \mathbb{P})\ (p, \dot{q}) \in K\}.$$

Then G is an M-generic filter on $\mathbb{P}$ and H is an $M[G]$ generic filter on $\mathbb{Q} = \dot{\mathbb{Q}}^G$.

Proof. 1. Let $D \in M$ be a dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$. In $M[G]$ let

$$D' = \{\dot{q}^G : (\exists p \in G)\ (p, \dot{q}) \in D\}.$$

It is easy to see that D' is dense in $\mathbb{Q}$ because for every $\dot{q} \in \dot{\mathbb{Q}}$ the set (in M)

$$D_{\dot{q}} = \{p \in \mathbb{P} : (\exists \dot{q}' \in \dot{\mathbb{Q}})\ p \Vdash \dot{q}' \leq \dot{q} \wedge (p, \dot{q}') \in D\}$$

is a dense subset of $\mathbb{P}$ (because D is dense). Thus there exists $q \in D' \cap H$ (and in particular $q \in H$) such that for some $p \in G$ and a name $\dot{q}$ with $\dot{q}^G = q$ it holds that $(p, \dot{q}) \in D$. Therefore $D \cap (G * H) \neq \emptyset$ and $G * H$ is M-generic.

For any model N it holds that $G \in N \wedge H \in N$ if and only if $G * H \in N$. Thus $M[G][H] = M[G * H]$.

2. Let $D \in M$ be a dense subset of $\mathbb{P}$. Then $D' = \{(p, \dot{q}) : p \in D, \dot{q} \in \dot{\mathbb{Q}}, p \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$ is a dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$. Thus $D \cap G \neq \emptyset$ and G is M-generic.

Let $D \in M[G]$ be a dense subset of $\mathbb{Q}$ and let $\dot{D}$ be a name such that $\dot{D}^G = D$. Then $D' = \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} : p \Vdash \dot{q} \in \dot{D}\}$ is a dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$. Thus $D \cap H \neq \emptyset$ and H is $M[G]$-generic. □

Theorem 6.5. *If $\mathbb{P}$ satisfies the countable chain condition in M and $\dot{\mathbb{Q}}$ satisfies the countable chain condition in $M^{\mathbb{P}}$ then $\mathbb{P}*\dot{\mathbb{Q}}$ satisfies the countable chain condition in M.*

Proof. Assume there exists an uncountable antichain $W = \{(p_\alpha, \dot{q}_\alpha), \alpha < \aleph_1\} \subseteq \mathbb{P} * \dot{\mathbb{Q}}$. Let G be an arbitrary M-generic filter and let $\dot{Z}$ be a $\mathbb{P}$-name for the set $\{\alpha : p_\alpha \in G\}$, i.e. $\mathrm{dom}(\dot{Z}) = \check{\aleph}_1$ and $\dot{Z}(\alpha) = p_\alpha$. Because W is an antichain it holds that

$$\forall \alpha \neq \beta < \aleph_1 : \quad p_\alpha \perp p_\beta \quad \vee \quad (p_\alpha \wedge p_\beta) \Vdash \dot{q}_\alpha \perp \dot{q}_\beta.$$

For all $\alpha, \beta \in Z = \dot{Z}^G$ it holds that $p_\alpha \parallel p_\beta$ because G is a filter and therefore $q_\alpha \perp q_\beta$ in $M[G]$. Now because $\mathbb{Q}$ satisfies the countable chain condition in $M[G]$ it holds that $\Vdash_{\mathbb{P}} |\dot{Z}| < \aleph_1$. The intuition now is that because $\mathbb{P}$ satisfies the countable chain condition that if Z is countable in $M[G]$ it must already be countable in M. However Z may not exist in M. Therefore we argue as follows.

Let $X = \{p_n : n < \omega\} \subseteq \mathbb{P}$ be a maximal antichain. Then for every $p_n \in X$ there exists $\alpha_n < \aleph_1$ such that $p_n \Vdash \dot{Z} \subseteq \check{\alpha}_n$. Let $\alpha = \sup_{n<\omega} \alpha_n$ and of course $\alpha < \aleph_0$. Then $\Vdash_{\mathbb{P}} \dot{Z} \subseteq \check{\alpha}$ and because $\mathbb{P}$ satisfies the countable chain condition also $\Vdash_{\mathbb{P}} \check{\alpha} < \aleph_1$. This contradicts the fact that $p_\alpha \Vdash \check{\alpha} \in \dot{Z}$. $\square$

Theorem 6.6. *If $\mathbb{P} * \dot{\mathbb{Q}}$ satisfies the countable chain condition then*

1. *$\mathbb{P}$ satisfies the countable chain condition.*
2. *$\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ satisfies the countable chain condition.*

Proof.

1. $\mathbb{P}$ satisfies the countable chain condition because every antichain in $\mathbb{P}$ induces an antichain in $\mathbb{P} * \dot{\mathbb{Q}}$.

2. Let $\dot{W} \in M^B$ and $p \in \mathbb{P}$ such that

$$p \Vdash \dot{W} \text{ is an uncountable subset of } \dot{\mathbb{Q}}.$$

Now let $\dot{f} \in M^B$ be such that

$$p \Vdash \dot{f} \text{ is a bijection between } \aleph_1 \text{ and } \dot{W}.$$

This means that for every $\alpha < \aleph_1$

$$p \Vdash (\exists \dot{x} \in \dot{W})\ \dot{f}(\check{\alpha}) = \dot{x}$$

and because M^B is full there exists $\dot{q}_\alpha \in \dot{\mathbb{Q}}$ such that

$$p \Vdash \dot{q}_\alpha \in \dot{W} \wedge \dot{f}(\check{\alpha}) = \dot{q}_\alpha.$$

We let $X = \{(p, \dot{q}_\alpha) : \alpha < \aleph_0\}$ and because $p \Vdash \dot{q}_\alpha \neq q_\beta$ for $\alpha \neq \beta$ we know that X is uncountable. Now because $\mathbb{P} * \dot{\mathbb{Q}}$ does not contain an uncountable antichain X cannot be an antichain and therefore there exists $\alpha, \beta < \aleph_0$ such that $\Vdash_B (\dot{q}_\alpha \parallel \dot{q}_\beta)$. and thus

$$p \Vdash \dot{W} \text{ is not an antichain.}$$

□

Corollary 6.7. *If $\mathbb{P}$ and $\mathbb{Q}$ satisfy the countable chain condition then $\mathbb{P} \times \mathbb{Q}$ satisfies the countable chain condition if and only if*

$$\Vdash_{\mathbb{P}} \check{\mathbb{Q}} \textit{ satisfies the countable chain condition.}$$

Remark 6.8. Note that the condition ($\Vdash_{\mathbb{P}} \check{\mathbb{Q}}$ satisfies the countable chain condition) is necessary. It is not enough that $\check{\mathbb{Q}}$ is the name of a forcing that satisfies the countable chain condition in the ground model. Consider for example forcing with a Suslin tree $T \in M$. It is easy to see that a generic filter on T corresponds to a branch of height ω_1 and therefore T is no longer a Suslin tree in $M[G]$ (see 7.11 for details) and in particular no longer satisfies the countable chain condition. Thus $T \times T$ does not satisfies the countable chain condition in M.

6.2 Iterated Forcing with Finite Support

Let λ be an ordinal. We are going to define a sequence of forcing notions $\{\mathbb{P}_\alpha : \alpha < \lambda\}$ such that for every successor ordinal $\alpha + 1$ it holds that $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ for some $\dot{\mathbb{Q}}_\alpha \in M^{\mathbb{P}_\alpha}$. For limit ordinals α we are going to define a limit of $\{\mathbb{P}_\beta : \beta < \alpha\}$ in a way described below. For $\alpha < \lambda$ the symbol $\leq_\alpha$ denotes the partial order of $\mathbb{P}_\alpha$ and $\Vdash_\alpha$ denotes the forcing relation induced by $\mathbb{P}_\alpha$.

Definition 6.9. We call a forcing $\mathbb{P}$ *rooted* if it has a greatest element. As with Boolean algebras we denote the greatest element by 1.

Definition 6.10. Let $\alpha \geq 1$. We say that a set $\mathbb{P}_\alpha$ of sequences of length α is an *iteration* with *finite support* (of length α) if the following holds.

1. If $\alpha = 1$:
   There exists a rooted forcing notion $\mathbb{Q}_0$ such that $\mathbb{P}_1 = \{(p(0)) : p(0) \in \mathbb{Q}_0\}$ and
   $$(p(0)) \leq_1 (q(0)) \quad \leftrightarrow \quad p(0) \leq_{\mathbb{Q}_0} q(0).$$

2. If $\alpha = \beta + 1$:
The set $\mathbb{P}_\beta = \mathbb{P}_\alpha \restriction \beta = \{p \restriction \beta : p \in \mathbb{P}_\alpha\}$ is an iteration with finite support of length β and there exists a $\mathbb{P}_\beta$- name $\dot{\mathbb{Q}}_\beta \in M^{\mathbb{P}_\beta}$ for a rooted forcing such that

$$p \in \mathbb{P}_\alpha \quad \leftrightarrow \quad \Vdash_\beta p(\beta) \in \dot{\mathbb{Q}}_\beta$$

$$p \leq_\alpha q \quad \leftrightarrow \quad p \restriction \beta \leq_\beta q \restriction \beta \quad \wedge \quad p \restriction \beta \Vdash_\beta p(\beta) \leq_{\dot{\mathbb{Q}}_\beta} q(\beta).$$

3. If α is a limit ordinal:
For all $\beta < \alpha$ the set $\mathbb{P}_\beta = \mathbb{P}_\alpha \restriction \beta = \{p \restriction \beta : p \in \mathbb{P}_\alpha\}$ is an iteration with finite support of length β and

$$p \in \mathbb{P}_\alpha \quad \leftrightarrow \quad \forall \beta < \alpha : p \in \mathbb{P}_\beta \quad \wedge \quad |\{\beta < \alpha : p(\beta) \neq \dot{1}_{\mathbb{Q}_\beta}\}| \leq \aleph_0.$$

$$p \leq_\alpha q \quad \leftrightarrow \quad \forall \beta < \alpha : p \restriction \beta \leq_\beta q \restriction \beta.$$

For $p \in \mathbb{P}_\alpha$ we define $\operatorname{supp}(p) = \{\beta < \alpha : p(\beta) \neq \dot{1}_{\mathbb{Q}_\beta}\}$ and say $\operatorname{supp}(p)$ is the support of p.

Since $\mathbb{P}_\alpha$ is uniquely determined by $\{\dot{\mathbb{Q}}_\beta, \beta < \alpha\}$ we say that $\mathbb{P}_\alpha$ is an iteration of $\{\dot{\mathbb{Q}}_\beta, \beta < \alpha\}$ and it is clear that for all $\beta < \alpha$ it holds that $\mathbb{P}_{\beta+1} \cong \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$.

Theorem 6.11. *Let $\mathbb{P}_\alpha$ be is an iteration of finite support of $\{\dot{\mathbb{Q}}_\beta, \beta < \alpha\}$ such that for all $\beta < \alpha$ it holds that*

$$\Vdash_\beta \dot{\mathbb{Q}}_\beta \text{ satisfies the countable chain condition.}$$

Then $\mathbb{P}_\alpha$ satisfies the countable chain condition.

Proof. We prove the theorem by induction. If α is a successor ordinal then the theorem follows from 6.5.

Thus let α be a limit ordinal and let $W \subseteq \mathbb{P}_\alpha$ be an uncountable set. Let $\mathcal{C} = \{\operatorname{supp}(p) : p \in W\}$. Then by lemma 3.10 there is an uncountable $\mathcal{D} \subseteq \mathcal{C}$ and a set d such that for $s, t \in \mathcal{D}$

$$s \neq t \rightarrow s \cap t \ = d.$$

Because α is a limit there exists $\beta < \alpha$ such that $d \subseteq \beta$. By induction hypothesis $\mathbb{P}_\beta$ satisfies the countable chain condition so there are $p, q \in W$ such that $p \restriction \beta \parallel_{\mathbb{P}_\beta} q \restriction \beta$. Now because $(\operatorname{supp}(p) \setminus \beta) \cap (\operatorname{supp}(q) \setminus \beta) = \emptyset$ it is easy to see by induction that $p \parallel_{\mathbb{P}_\alpha} q$.

Indeed if $u \leq p \restriction \beta, u \leq q \restriction \beta$ and without loss of generality $\Vdash_\beta p(\beta) = 1$. Then $u \Vdash_\beta q(\beta) \leq p(\beta)$ and of course also $u \Vdash_\beta q(\beta) \leq q(\beta)$. Thus $u^\frown q(\beta) \leq p \restriction (\beta + 1), u^\frown q(\beta) \leq q \restriction (\beta + 1)$, i.e. $p \restriction (\beta + 1) \parallel q \restriction (\beta + 1)$. Limit steps are trivial. Therefore W is not an antichain. $\square$

Chapter 7

Martin's Axiom

Definition 7.1. *Martin's axiom* MA is the statement that if $(\mathbb{P}, \leq)$ is a partial order that satisfies the countable chain condition and $\mathcal{D}$ is a family of less than $2^{\aleph_0}$ dense sets then there exists a filter $G \subseteq P$ that intersects every $D \in \mathcal{D}$, i.e. there exists a $\mathcal{D}$-generic filter G.

7.1 CH $\rightarrow$ MA

Theorem 7.2. *The continuum hypothesis implies Martin's axiom.*

Proof. Clearly if CH holds then every family $\mathcal{D}$ of less than $2^{\aleph_0}$ families is countable and therefore a $\mathcal{D}$-generic filter exists by lemma 2.5. $\square$

7.2 The consistency of MA $\wedge$ $\neg$CH

Definition 7.3. In this section a $\mathbb{P}$-name τ will be a relation such that

$$\forall(\sigma, p) \in \tau : \sigma \text{ is a } \mathbb{P}\text{-name} \wedge p \in \mathbb{P},$$

i.e. τ is not necessarily a function and $\operatorname{ran}(\tau)$ is a subset of $\mathbb{P}$, not $B(\mathbb{P})$. Just as in theorem 2.38 it is easy to see that this approach yields the same generic extension $M[G]$ as the old definition.

Again we refer to the set of all $\mathbb{P}$-names (of the new definition) as $M^{\mathbb{P}}$.

Definition 7.4. Let $\tau \in M^{\mathbb{P}}$. A *nice name* for a subset of τ is a $\mathbb{P}$-name of the form

$$\bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \operatorname{dom}(\tau), A_\sigma \text{ is an antichain in } \mathbb{P}\}.$$

Lemma 7.5. *Let $\tau \in M^{\mathbb{P}}$ and let $M \models \kappa = |\mathbb{P}|$, $M \models \lambda = |dom(\tau)|$. If $\mathbb{P}$ satisfies the countable chain condition and κ and λ are infinite then there exist no more than κ^λ nice names for subsets of τ.*

Proof. Because $\mathbb{P}$ is ccc there are at most $\kappa^{\aleph_0}$ antichains and therefore there are at most $(\kappa^{\aleph_0})^\lambda = \kappa^{\aleph_0 \cdot \lambda} = \kappa^\lambda$ nice names. □

Lemma 7.6. *Let $\tau, \mu \in M^{\mathbb{P}}$. Then there is a nice name $\theta \in M^{\mathbb{P}}$ for a subset of τ such that*

$$\Vdash \mu \subseteq \tau \to \mu = \theta.$$

Proof. We work in M. Choose $\theta = \bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \mathrm{dom}(\tau)\}$ such that each A_σ is a maximal antichain in the set $\{p \in \mathbb{P} : p \Vdash \sigma \in \mu\}$.

Let G be an arbitrary M-generic filter on $\mathbb{P}$. We want to show that $M[G] \models \mu \subseteq \tau \to \mu = \theta$. We assume that $\mu^G \subseteq \tau^G$ and show $\mu^G = \theta^G$.

$\theta^G \subseteq \mu^G$: Fix $x \in \theta^G$. Then there exists $(\sigma, p) \in \theta$ with $p \in G$ such that $x = \sigma^G$. By definition of θ it holds that $p \Vdash \sigma \in \mu$ and thus $x \in \mu^G$.

$\mu^G \subseteq \theta^G$: Assume there exists $x \in \mu^G \setminus \theta^G$. Then because $\mu^G \subseteq \tau^G$ it holds that $x \in \tau^G$ and again $x = \sigma^G$ for some $(\sigma, p) \in \tau$. Therefore there exists $q \in G$ such that $q \Vdash \sigma \in \mu \wedge \sigma \notin \theta$ and $q \perp p$ for all $p \in A_\sigma$ because $(\sigma, p) \in \theta$ implies that $p \Vdash \sigma \in \theta$. Thus q contradicts the maximality of A_σ. □

Lemma 7.7. *In M let $\mathbb{P}$ satisfy the countable chain condition, let κ, λ be infinite cardinals with $|\mathbb{P}| = \kappa$. Let G be an M-generic filter on $\mathbb{P}$. Then*

$$M[G] \models 2^\lambda \leq \kappa^\lambda.$$

Proof. In M the name $\check{\lambda}$ has cardinality λ and so we can use lemma 7.5 to get an enumeration $\{\theta_\alpha : \alpha < \kappa^\lambda\}$ of the nice names for subsets of the subsets of $\check{\lambda}$. Now let σ be the name $\{(\mathrm{op}(\check{\alpha}, \theta_\alpha), 1) : \alpha < \kappa^\lambda\}$.

In M^G it holds that σ^G is a function on κ^λ with $\sigma^G(\alpha) = \theta_\alpha^G$ for all $\alpha < \kappa^\lambda$. If s is a subset of λ in $M[G]$ then by lemma 7.6 $\sigma^G(\alpha) = s$ for some $\alpha < \kappa^\lambda$. Thus λ has at most κ^λ subsets in $M[G]$. □

Lemma 7.8. *Martin's axiom is equivalent to its restriction to partial orders of size less than $2^{\aleph_0}$.*

Proof. Let $(\mathbb{P}, \leq)$ be a partial order and let $\mathcal{D}$ be a family of fewer than $2^{\aleph_0}$ dense subsets of $\mathbb{P}$. For each $D \in \mathcal{D}$ let W_D be a maximal antichain in D. Because W_D is countable for every D there exists a set $\mathbb{Q} \subseteq \mathbb{P}$ of size less than $2^{\aleph_0}$ such that $W_D \subseteq \mathbb{Q}$ for every D and if $p, q \in \mathbb{Q}$ are compatible (in $\mathbb{P}$) then there is $r \in \mathbb{Q}$ such that $r \leq p, r \leq q$ as a witness of compatibility. Thus

W_D remains a maximal antichain in $\mathbb{Q}$ for each D and $\mathbb{Q}$ retains countable chain condition.

We define $\mathbb{Q}$-dense sets $E_D = \{q \in \mathbb{Q} : (\exists w \in W_D)\ q \leq w\}$. Let H be a be a $\{E_D : D \in \mathcal{D}\}$-generic filter. Then using lemma 2.7 it is easy to see that H induces a G generic filter on $\mathbb{P}$. □

Remark 7.9. It is clear that we may restrict ourselves to only consider partial orders on cardinal numbers.

Theorem 7.10. *Let M be a countable transitive model. Then there exists a forcing notion $\mathbb{P}$ that satisfies the countable chain condition such that the generic extension $M[G]$ (with $G \subseteq \mathbb{P}$) satisfies Martin's axiom and $2^{\aleph_0} = \aleph_2$.*

Proof. In 3.6 we showed that we may assume that

$$M \models 2^{\aleph_0} = \aleph_1.$$

Completely analogous to 3.6 we can force with bijections between $2^{\aleph_1}$ and $\aleph_2$ of size $\aleph_1$. Thus we may additionally assume that

$$M \models 2^{\aleph_1} = \aleph_2.$$

Informally our plan is as follows: Using lemma 7.8 we want to make sure that every ccc partial order on $\aleph_1$ has a generic set. Thus we try to simply add those generic sets by iteratively forcing with every ccc partial order on $\aleph_1$. There are a number of issues with this plan we need to pay attention to. A partial order that was ccc in the ground model may no longer be ccc in a generic extension. In this case we skip this partial order to make sure that $\aleph_1$ does not collapse, making the work we already did irrelevant (see lemma 6.11). Furthermore new ccc partial orders may appear and new dense subsets of ccc partial orders we have already forced with may appear. We will use a bookkeeping method to make sure we treat every ccc partial order we have to.

Let $f : \aleph_2 \to \aleph_2 \times \aleph_2$ be a surjective map such that $f(\alpha) = (\eta, \gamma)$ implies that $\eta \leq \alpha$.

We construct a finite support iteration $\mathbb{P}$ of length $\aleph_2$ from a sequence $\{\dot{\mathbb{Q}}_\alpha : \alpha < \aleph_2\}$. It will hold that $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is a ccc partial order of $\check{\aleph}_1$ for every $\alpha < \aleph_2$. Thus $\mathbb{P}_\alpha$ is ccc for every $\alpha \leq \aleph_2$ by lemma 6.11 and by lemma 3.9 every $\mathbb{P}_\alpha$ preserves cardinals. In particular $\aleph_1$ does not collapse.

Of course for ever $\alpha < \aleph_2$ the name $\dot{\mathbb{Q}}_\alpha$ is really a triple $(\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)$. We will work with $\dot{Q}_\alpha = \check{\aleph}_1$ (see lemma 7.8). Hence for all $\alpha < \aleph_2$ it holds that $|\dot{\mathbb{Q}}_\alpha| = \aleph_1$ and thus $|\mathbb{P}_\alpha| = \aleph_1$ for $\alpha < \aleph_2$ and $|\mathbb{P}| = \aleph_2$.

We define the sequence $\{\dot{\mathbb{Q}}_\alpha : \alpha < \aleph_2\}$ inductively. Assume we have defined $\mathbb{P}_\beta$ and let $\alpha = \beta + 1$ (the definition of iterated forcing tells us what to do at limit steps). Firstly let $\{\dot{\leq}_{\alpha\gamma} : \gamma < \aleph_2\}$ be an enumeration of all nice $\mathbb{P}_\beta$-names for subsets of $\aleph_1 \times \aleph_1$. This is possible because there are $|\mathbb{P}_\beta|^{|\aleph_1\times\aleph_1|} = \aleph_1^{\aleph_1} \leq (2^{\aleph_1})^{\aleph_1} = 2^{\aleph_1\times\aleph_1} = 2^{\aleph_1} = \aleph_2$ such names by lemma 7.5 and our assumption that $M \models 2^{\aleph_1} = \aleph_2$.

Secondly let $(\eta,\gamma) = f(\alpha)$ and let $\dot{\mathbb{Q}}_\alpha$ be the triple $(\check{\aleph}_1, \dot{\leq}_\alpha, \check{0})$ where $\dot{\leq}_\alpha$ be a name such that for every $p \in \mathbb{P}_\beta$ it holds that ($p \Vdash (\check{\aleph}_1, \dot{\leq}_{\eta\gamma}, \check{0})$ is a ccc partial order with maximal element $\check{0}$) implies ($p \Vdash \dot{\leq}_\alpha = \dot{\leq}_{\eta\gamma}$) and ($p \Vdash (\check{\aleph}_1, \dot{\leq}_{\eta\gamma}, \check{0})$ is *not* a ccc partial order with maximal element $\check{0}$) implies ($p \Vdash \dot{\leq}_\alpha$ is the linear order on $\check{\aleph}_1$ defined by reverse inclusion). Defining such $\dot{\leq}_\alpha$ is possible using maximal antichains in the sets $\{p \in \mathbb{P}_\beta : p \Vdash (\check{\aleph}_1, \dot{\leq}_{\eta\gamma}, \check{0})$ is a ccc partial order with maximal element $\check{0}\}$ and $\{p \in \mathbb{P}_\beta : p \Vdash (\check{\aleph}_1, \dot{\leq}_{\eta\gamma}, \check{0})$ is *not* a ccc partial order with maximal element $\check{0}\}$ and lemma 2.22.

We claim that if R is a subset of $\aleph_1$ in $M[G]$ where G is a generic filter on $\mathbb{P}$ then there exists $\alpha < \aleph_2$ such that $R \in M[G_\alpha]$ where G_α is a generic filter on $\mathbb{P}_\alpha$. By lemma 7.6 R has a nice $\mathbb{P}$-name $\dot{R}$, i.e. a name where $\beta \in R$ is decided by a countable antichain W_β for every $\beta < \aleph_1$. Because $|\mathbb{P}| = \aleph_2$ is regular it holds for $\alpha = \sup(\bigcup_{\beta<\aleph_1}\bigcup_{w\in W_\beta} \text{supp}(w))$ that $\alpha < \aleph_2$. Clearly there is a canonical $\mathbb{P}_\alpha$-name $\dot{S}$ that corresponds to $\dot{R}$ such that $\dot{S}^{G_\alpha} = \dot{R}^G = R$ and thus $R \in M[G_\alpha]$.

Now let $\mathbb{Q}$ be a ccc partial order of $\aleph_1$ in $M[G]$ and and let $\mathcal{D}$ be a family of $\aleph_1$ $\mathbb{Q}$-dense subset of $\aleph_1$. Then by the above for every $D \in \mathcal{D}$ there is $\alpha_D < \aleph_2$ such that $D \in M[G_{\alpha_D}]$ and because $\mathbb{Q}$ corresponds to a subset of $\aleph_1 \times \aleph_1 \cong \aleph_1$ there exists $\alpha_\mathbb{Q}$ such that $\mathbb{Q} \in M[G_{\alpha_\mathbb{Q}}]$. Let $\eta = \sup(\{\alpha_D : D \in \mathcal{D}\} \cup \{\alpha_\mathbb{Q}\})$ and again because $\aleph_2$ is regular $\eta < \aleph_2$. Now because a name $\dot{\mathbb{Q}} = \leq_{\eta\gamma_0}$ for $\mathbb{Q}$ appears in $\{\dot{\leq}_{\eta\gamma} : \gamma < \aleph_2\}$ and because there exits $\alpha \geq \eta$ such that $f(\alpha) = (\eta, \gamma_0)$ it holds that $\dot{\mathbb{Q}} = \dot{\mathbb{Q}}_\alpha$ for some $\alpha < \aleph_2$. Thus G_α is a $\mathcal{D}$-generic filter and clearly $G_\alpha \in M[G]$.

Thus every ccc partial order of size $\aleph_1$ in $M[G]$ and every family of at most $\aleph_1$ dense subsets of this partial order in $M[G]$ has a generic filter in $M[G]$. It remains to verify the size of the continuum in $M[G]$. Using $|\mathbb{P}| = \aleph_2$, lemma 7.7 and our assumption $M \models 2^{\aleph_0} = \aleph_1$ we can bound the continuum in $M[G]$ by $2^{\aleph_0} \leq \aleph_2^{\aleph_0} = \aleph_2 \cdot \aleph_1^{\aleph_0} = \aleph_2 \cdot \aleph_1 \cdot \aleph_0^{\aleph_0} = \aleph_2$.

To see that $2^{\aleph_0} \geq \aleph_2$ consider the forcing $\mathbb{Q} = (2^{<\omega}, \supseteq)$. $\mathbb{Q}$ is countable and thus ccc and can be embedded into a forcing that lives on $\aleph_1$ (simply take countably many elements to represent $\mathbb{Q}$ and put the rest above). Therefore there exists a generic filter for every family of $\aleph_1$ dense subset of $\mathbb{Q}$ by what we showed above. Consider $\mathcal{D} = \{D_x, x \in 2^\omega\}$ where $D_x = \{q \in \mathbb{Q} : q \not\subseteq x\}$. It is easy to see that D_x is in fact dense for every $x \in 2^\omega$ and that $D_x \neq D_y$ for $x \neq y$. Thus $|\mathcal{D}| = 2^{\aleph_0}$. Assume that G is a $\mathcal{D}$-generic filter on $\mathbb{Q}$. Clearly $x = \bigcup G$ is a branch, i.e. $x \in 2^\omega$ is a new real, and $G \cap D_x = \emptyset$, contracting the genericity of G. Thus $2^{\aleph_0} > \aleph_1$.

□

7.3 (MA $\wedge$ $\neg$CH) $\rightarrow$ SH

Theorem 7.11. *If Martin's axiom holds and $2^{\aleph_0} > \aleph_1$ then there exists no Suslin tree.*

Proof. Let $(T, \leq)$ be a Suslin tree and consider the partial order $(T, \geq)$. Clearly $(T, \geq)$ satisfies the countable chain condition. For every $\alpha < \omega_1$ the set $D_\alpha = \{x \in T : h(x) > \alpha\}$ is dense in $(T, \geq)$. Let $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ and let G be a $\mathcal{D}$-generic filter. It is clear that G is a branch and because G is $\mathcal{D}$-generic G contains an element at every level $\alpha < \omega_1$, i.e. G is a branch of height ω_1. This contradicts our assumption that $(T, \leq)$ is a Suslin tree. □

Bibliography

[1] Georg Cantor: *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die Reine und Angewandte Mathematik 84, S. 242–258., 1878.

[2] David Hilbert: *Mathematische Probleme*, Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen,Mathematisch-Physikalische Klasse. Heft 3, 1900

[3] Kurt Gödel: *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I.* Monatshefte für Mathematik und Physik, 38, S. 173–198, 1931.

[4] Kurt Gödel: *The Consistency of the Continuum-Hypothesis*, Princeton University Press, 1940.

[5] Paul Cohen: *The independence of the continuum hypothesis, I, II*, Proceedings of the National Academy of Sciences of the United States of America 50, 1143-1148, 1963; 51, 105-110, 1964.

[6] Thomas Jech: *Set Theory,* The Third Millennium Edition, Springer-Verlag 2002

[7] Kenneth Kunen: *Set Theory,* Studies in Logic 34. London: College Publications, 2011.

[8] Tomek Bartoszyński, Haim Judah: *Set Theory: On the Structure of the Real Line*, A K Peters, Wellesley, Massachusetts, 1995.

[9] Nik Weaver: *Forcing for Mathematicians*, World Scientific Publishing Co Pte Ltd, 2014.

[10] Juris Steprāns: *Combinatorial consequences of adding Cohen reals*, Israel Mathematical Conference Proceedings: Set Theory of the Reals, Haim Judah, Editor, A publication of Bar-Ilan University, 1993

Index