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Berechenbare Transformationen von Strukturklassen

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Computable Transformations of Classes of Structures

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Erklärung zur Verfassung der Arbeit

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Wien, 20. August 2015

Dino Rossegger

Kurzfassung

Die Arbeit befasst sich mit berechenbaren Transformationen von Strukturklassen. Wir geben einen Überblick über die vorhandenen berechenbaren Transformationen und deren Relationen. Der Fokus liegt hier auf effektiver Bi-Interpretierbarkeit. Wir zeigen, dass die Klassen der Graphen und partiellen Ordnungen bi-interpretierbar sind und befassen uns mit einem neuen Resultat über die Äquivalenz der effektiven Bi-Interpretierbarkeit von Klassen und berechenbaren Funktoren. Weiters untersuchen wir zwei kürzlich erforschte berechenbarkeitstheoretische Eigenschaften von Strukturen, Theorie spektra und Σ_n-spectra im Kontext der effektiven Bi-Interpretierbarkeit. Wir zeigen, dass jedes mögliche Σ_1- und Σ_2-Spektrum in den Klassen der partiellen Ordnungen und Graphen existiert.

Abstract

The work reviews different notions of computable transformations on elementary classes of structures. We review the most prominent computable transformations, effective reducibility, computable embeddings, Turing computable embeddings, effective bi-interpretability and computable functors, discussing the different ideas behind them and their relations. Recent results on the equivalence of effective bi-interpretability and computable functors are discussed in detail and we prove that graphs and partial orders are effectively bi-interpretable and therefore share many computability theoretic properties such as degree spectra and computable dimension. At last the two recently examined notions of theory spectra and Σ_n-spectra and their relation in context of computable transformations are examined. We show that any existing Σ_1- and Σ_2-spectrum can be found in the classes of graphs and partial orders.

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CHAPTER 1

Introduction

1.1 Motivation

Computable model theory is the study of computability theoretic properties of mathematical structures. It extends computability theory, which deals with sets to structures made of a universe, functions and relations. When a computability theoretic property of structures is found one often wants to know whether there is a structure possessing that property in a well studied mathematical class such as graphs, partial orders or lattices. The common way to prove that such a structure exists in a class is to prove it directly for the class, which is not very productive as the constructions and ideas used in these proofs often can not be used for other classes. Another way is to use what we call computable, or effective transformations. These transformations take a structure in some class and produce a structure in the target class preserving many computability theoretic properties depending on the strength of the transformation. This method is reusable, when a new computability theoretic property is studied one only has to show that the transformations preserve the property to prove that there is a structure in the target class possessing the property.

Hirschfeldt, Khoussainov, Shore and Slinko [Hir+02] gave effective transformations from arbitrary structures to graphs, and from graphs to partial orderings, lattices, rings, integral domains, commutative semigroups and 2-step nilpotent groups showing that for many computability theoretic properties such as degree spectra, computable dimension and computable categoricity this classes are complete, i.e. if any structure possessing such a property is found there is a structure in those classes also having this property. Since then research interest in effective transformations has risen and many different notions were studied. Recently Montalbán [Mon14] tried to formally capture the interpretations given by Hirschfeldt, Khoussainov, Shore and Slinko using the notion of effective bi-interpretability. He showed that several computability theoretic properties are preserved by effective bi-interpretations. However for many computability theoretic properties, such as for instance theory spectra and Σ_n-spectra, the question whether they are preserved or not is still open. In a recent paper Harrison-Trainor, Melnikov, Miller and Montalbán [Har+15] showed that the notions of effective bi-interpretability and computable functors is equivalent. This is an interesting result as it establishes a connection between the syntactic notion of effective bi-interpretability and the semantic notion of computable functors.

1.2 Problem statement

The interpretations of structures in structures of another language or class are a well known concept in model theory and have been heavily used in computable model theory to show that structures in different classes possess certain properties. Many authors suggested their own interpretations adding computability theoretic constraints on the relations used in the interpretations to show that their property of interest can also be found in well known mathematical classes. For many of these interpretations it is unclear how "strong" they are, i.e. do they also preserve other properties than those they were intended for? Recently the idea of these interpretations has been formalized in the notion of effective interpretations. It was shown that effective interpretations preserve many computability theoretic properties of structures, however it is still unclear for many interpretations whether they are effective.

For some computability theoretic properties, like theory- and Σ_n-spectra it is also not known whether they are preserved by effective interpretations in general. While an interpretation was used to show that for any structure there is a graph with the same theory spectrum a similar result for Σ_n-spectra has not been obtained yet.

1.2.1 Goals

The project had the following goals.

- Review whether existing interpretations are effective interpretations.
- Give an overview of the different notions of computable transformations and their relation.
- Check whether the notions of Σ_1- and Σ_2-spectra are preserved by effective interpretations.

1.3 Methodical approach

The project can be divided in three parts.

1. *Literature research.* At first an extensive literature research on the different computable transformations and interpretations available was done. This includes the study of already given proofs and working out of proofs only sketched or without further reference. In the thesis, proofs which are not easily accessible are usually stated.

2. *Results about effective interpretations.* After the literature research the proofs of existing interpretations to be effective interpretations were done.

3. *Results about* Σ_1*- and* Σ_2*-spectra.* This was the last part of the project since it required deep understanding of the matter obtained in the first two parts.

1.4 Structure of the work

First, in Chapter 2, we give the necessary computability theoretic background. In Chapter 3 we recall the model theoretic definitions of structures and introduce the most important computability theoretic properties of structures. In Chapter 4 different notions of effective transformations are discussed. We take a detailed look on effective bi-interpretability and show for the properties discussed in Chapter 3 that these are preserved by effective bi-interpretability. We furthermore show that graphs and partial orders are effectively bi-interpretable based on the interpretations given in [Hir+02]. The recent results about the equality of computable functors and effective bi-interpretability are discussed too. Chapter 5 looks at an alternative definition of spectra allowing different equivalence relations, we prove that spectra under Σ_1- equivalence are preserved by effective bi-interpretability and show that the effective interpretation of arbitrary structures in graphs preserves spectra under Σ_2-equivalence. We furthermore review a result that theory spectra are preserved by an effective interpretation slightly different than ours. In Chapter 6 we conclude the work and show potential possibilities for future research and open questions.

CHAPTER 2

Computability theoretic background

To analyse computability theoretic properties of structures some advanced computability theoretic notions are needed. We assume that the reader is familiar with basic computability theoretic notions including computable, computably enumerable sets and Turing machines with oracles. We will stick to the notation used in [Coo03] and also refer to this book as a source for basic notions.

Before we start with different notions of relative computability we would like to recall a basic, but very useful operation, the *computable join* $\oplus$ defined as

$$A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}.$$

It is easy to see that given two computable sets, the computable join of these retains computability, as we can easily extract information from it using the computable function

$$\oplus^{-1}(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$$

2.1 The Turing universe and the arithmetical hierarchy

Definition 2.1. A set A is computable relative to a set B, or B-computable, if there is a functional Φ_i^B computing A using B as oracle and we write $A \leq_T B$ to denote that A is *Turing reducible* to B.

We say that a structure A is B-c.e. if there is a functional Φ_i^B enumerating A. Using this notation we can define an equivalence relation $A \equiv_T B$ iff $A \leq_T B$ and $B \leq_T A$, we say that A is Turing equivalent to B. The *Turing degree* of a structure A is then

$$deg(A) = \{X \subseteq \omega \mid X \equiv_T A\}.$$

If we talk about degrees without taking reference on the structures in it, we use bold letters, like $\mathbf{d}$ to denote degrees. The structure formed by Turing degrees is often referred to as the Turing universe $\mathcal{D}$. It is easy to see that the set of computable functions form a degree, denoted as $\mathbf{0}$. The degree $\mathbf{0}$ is the least degree in the Turing universe as for any $X \in \mathbf{0}$ $X \leq_T Y$ where Y is a structure of arbitrary degree.

Until now we have only talked about the degree of sets, but how about the degree of functions? We can define the degree of a function f using its graph $Graph_f = \{(x, y) \mid f(x) = y\}$ as

$$deg(f) = deg(Graph_f).$$

An important concept is the jump of a set.

Definition 2.2. The jump A' of A is

$$A' = \left\{(x, y) \mid x \in W_y^A\right\},$$

where W_y^A denotes the halting set of Φ_y^A.

If we want to denote the n^{th} jump of a set we write $A^{(n)}$. Jumps have interesting properties.

Theorem 2.1. *Let $A, B \subseteq \omega$, then*
(1) A' is A-c.e.
(2) $A \leq_T A'$
(3) $A' \not\leq_T A$
(4) $B \equiv_T A$ iff $B' \equiv_T A'$

For a proof see [Coo03, p.150]. It follows directly from Theorem 2.1 that $\mathbf{0}'$, the jump of the degree of computable sets, is the degree of c.e. sets. Any set in $\mathbf{0}'$ is definable by an existentially quantified computable relation, i.e. for any set $S \in \mathbf{0}'$ $\overline{x} \in S \Leftrightarrow \exists \overline{y} R(\overline{x}, \overline{y})$, where R is computable.

Example 2.1. Recall the halting set $W_i = \{\overline{x} \mid \varphi_i(\overline{x}) \downarrow\}$. The halting set is c.e. and therefore $deg(W_i) = \mathbf{0}'$. We have that

$$x \in W_i \Leftrightarrow \exists s \varphi_{i,s}(x) \downarrow,$$

where $\varphi_{i,s}$ is the computation of φ_i after s steps.

Indeed this holds not only for $\mathbf{0}'$, but in general leading to the following definition of the arithmetical hierarchy.

Definition 2.3. $\Sigma_0^0, \Pi_0^0, \Delta_0^0 = \mathbf{0}$. And for $n \geq 0$:
(1) $\Sigma_{n+1}^0 =$ all relations of the form $\exists \overline{y} R(\overline{x}, \overline{y})$ with $R \in \Pi_n^0$,
(2) $\Pi_{n+1}^0 =$ all relations of the form $\forall \overline{y} R(\overline{x}, \overline{y})$ with $R \in \Sigma_n^0$,
(3) $\Delta_{n+1}^0 = \Sigma_{n+1}^0 \cap \Pi_{n+1}^0$.
R is *arithmetical* if $R \in \bigcup_{n \geq 0}(\Sigma_n^0 \cup \Pi_n^0)$.

It is possible to relativize this definition and the following theorems to arbitrary sets, we then write $\Sigma_n^{0,A}, \Pi_n^{0,A}, \Delta_n^{0,A}$, denoting that the relation R is A-computable. The 0 in the superscript means that we allow quantification over variables, if the context is clear we often leave it and write Σ_n^A, Π_n^A and Δ_n^A.

We say that A is Σ_n-*complete* if $A \in \Sigma_n$ and $X \leq_T A$ for $X \in \Sigma_n$. Δ_n and Π_n-complete are similarly defined. This notation has some useful properties.

Theorem 2.2 (Post's Theorem). *Let $A \subseteq \omega$, then:*
(1) $\mathbf{0}^{(n+1)}$ is Σ_{n+1}-complete.
(2) $A \in \Sigma_{n+1} \Leftrightarrow A$ is c.e. in $\mathbf{0}^n$.
(3) $A \in \Delta_{n+1} \Leftrightarrow A \leq_T \mathbf{0}^n$.

A detailed proof of Theorem 2.2 can be found in [Coo03, p.157].

2.2 Enumeration reducibility

Apart from Turing reducibility there are also other possibilities to model relative computability. One notion we will need later on is enumeration reducibility. Informally we say that A is enumeration reducible to B, $A \leq_e B$ when we can computably enumerate the members of A by enumerating the members of B without any restriction on the order in which B is enumerated. The formal definition is as follows.

Definition 2.4. An *enumeration operator* Ψ is a c.e. set - where for any set $B \subseteq \omega$

$$n \in \Psi^B \Leftrightarrow \exists \text{ a finite } D \subseteq B \text{ such that } (n, D) \in \Psi$$

So $\Psi^B = \{n \mid (n, D) \in \Psi \text{ for some finite } D \subseteq B\}$.
A is then *enumeration reducible* to B, $A \leq_e B$, if $A = \Psi^B$ for some Ψ.

Since Ψ is c.e. there is a functional enumerating it and hence we can computably approximate it. We write Ψ_s to denote the set Ψ_s after s steps of enumeration. Observe that Ψ_s is now computable. Proposition 2.3 follows.

Proposition 2.3. *If $A \leq_e B$, then A is B-c.e.*

Every set A Turing reducible to B is also enumeration reducible to B, as can be seen in Proposition 2.4.

Proposition 2.4. *If $A \leq_T B$, then $A \leq_e B$.*

Proof. Assume $A \leq_T B$, then there is an i such that $\Phi_i^B = \chi_A$. Recall from the definition of oracle Turing machines that Φ_i^B uses only finite information of B to decide wether $x \in A$. Hence if $\Phi_i^B(x) = 1$, then there is a finite subset D of B, such that $\Phi_i^D(x) = 1$. Hence we can construct Ψ by

$$(n, D) \in \Psi \Leftrightarrow \exists s \Phi_{i,s}^D(n) = 1.$$

It follows that Ψ is Σ_1 and therefore c.e. and hence we can reduce A to B by enumeration reducibility using enumeration operator Ψ. □

However enumeration reducibility is weaker than Turing reducibility, as can be seen in Proposition 2.5.

Proposition 2.5. *There is a A, B such that $A \leq_e B$ but $A \not\leq_T B$.*

Proof. Recall the definition of $K_0 = \{x \mid x \in W_x\}$. We give a counter example by showing that $\chi_{K_0} \leq_e S_{K_0 \oplus \overline{K_0}}$ but $\chi_{K_0} \not\leq_T S_{K_0 \oplus \overline{K_0}}$ where χ_{K_0} is the characteristic function of K_0 and $S_{K_0 \oplus \overline{K_0}}$ is defined as

$$S_{K_0 \oplus \overline{K_0}}(x) = \begin{cases} 1 & \text{if } 2x \in K_0 \oplus \overline{K_0} \text{ or } 2x+1 \in K_0 \oplus \overline{K_0} \\ undefined & \text{otherwise} \end{cases}.$$

Observe that $S_{K_0 \oplus \overline{K_0}}$ is equivalent to the constant function returning 1 and is therefore computable. Assume that $\chi_{K_0} \leq_T S_{K_0 \oplus \overline{K_0}}$, then χ_{K_0} is computable, since it is Turing reducible to a computable set. It follows that $\chi_{K_0} \not\leq_T S_{K_0 \oplus \overline{K_0}}$.

However we can construct an enumeration operator Ψ

$$\Psi = \{(n, 1), \{(2n, 1)\} \mid n \in \omega\} \cup \{(n, 0), \{(2n+1, 1)\} \mid n \in \omega\}.$$

Clearly Ψ is c.e., $Graph_{\chi_{K_0}} = \Psi^{Graph_{S_{K_0 \oplus \overline{K_0}}}}$ and therefore $\chi_{K_0} \leq_e S_{K_0 \oplus \overline{K_0}}$. □

2.3 Computable infinitary formulas

Informally, a computable infinitary formula is an infinite disjunction or conjunction over c.e. sets of formulas. We only consider infinitary formulas in conjunctive or disjunctive normal form with finitely many free variables. Computable infinitary formulas are categorized like their finite counterparts by $\Sigma_\alpha^c, \Pi_\alpha^c, \Delta_\alpha^c$ for computable ordinal $\alpha \geq 0$. We will only need computable infinitary formulas up to Σ_1^c. We give a slightly informal definition of computable infinitary formula.

Definition 2.5. The computable Σ_0^c and Π_0^c formulas are the finitary open formulas, and for computable f and $\alpha > 0$:

(1) a Σ_α^c formula $\varphi(\overline{x})$ is of the form

$$\bigvee_{i\in\omega} \exists \overline{u} \psi_{f(i)}(\overline{u}, \overline{x}' \subseteq \overline{x})$$

where ψ_i is computable Π_β for some $\beta < \alpha$.

(2) a Π_α^c formula $\varphi(\overline{x})$ is of the form

$$\bigwedge_{i\in\omega} \forall \overline{u} \psi_{f(i)}(\overline{u}, \overline{x}' \subseteq \overline{x})$$

where ψ_i is computable Σ_β for some $\beta < \alpha$.

(3) $\Delta_\alpha^c = \Sigma_\alpha^c \cap \Pi_\alpha^c$.

The definition is fine for the first level of the hierarchy $\Sigma_1^c, \Pi_1^c, \Delta_1^c$, since computable infinitary formulas on the first level are made of computable finitary formulas and these formulas have computable codes, computed by $f(i)$ in the definition. However formulas on higher levels are infinite disjunctions or conjunctions of computable infinitary formulas, which do not have codes and hence this definition is inprecise for those formulas unless we assign codes to them. Since we only need computable infinitary formulas up to the first level and the formal definition is rather involved we will not give it here, the interested reader can find it in [AK00, Chapter 7].

The most important property of computable infinitary formulas is that their complexity matches the complexity of their interpretation, if the formula is interpreted in a computable structure. A formal statement can be seen in Theorem 2.6.

Theorem 2.6. *For computable structure $\mathcal{A}$, if $\varphi(\overline{x})$ is a Σ_α^c formula, then $\varphi^\mathcal{A}$ is Σ_α^0, and if $\varphi(\overline{x})$ is a Π_α^c formula, then $\varphi^\mathcal{A}$ is Π_α^0 where $\varphi^\mathcal{A}$ is the interpretation of φ in $\mathcal{A}$.*

The complete proof of this theorem can be found in [AK00, Theorem 7.5]. It goes by induction on α. More on structures and interpretations in Chapter 3.

CHAPTER 3

Computability of structures and structure classes

3.1 Model theoretic background

Before we go into detail about effective transformations we introduce the necessary model theoretic definitions and look at some of the computability theoretic properties which will be discussed later with respect to transformations. We mostly use notations as given in [Mar02].

Definition 3.1. A language $\mathcal{L}$ is given by

(i) a set of function symbols $\mathcal{F}$ together with their arity a_f for each $f \in \mathcal{F}$,

(ii) a set of relation symbols $\mathcal{R}$ together with their arity a_R for each $R \in \mathcal{R}$.

To avoid confusion we use capital letters or strings starting with capital letters to denote relation symbols and small letters or strings starting with small letters to denote functions. In many standard textbooks an additional set is used to denote constants, however we use function symbols with arity 0 to denote constants. Two examples of languages are

(i) the language of rings $\mathcal{L}_r = \{+, \cdot, 0, 1\}$ where $+, \cdot$ are binary functions and $0, 1$ are constant symbols,

(ii) the language of graphs $\mathcal{L}_g = \{E\}$ where E is a binary relation symbol.

Definition 3.2. A structure $\mathcal{S}$ is given by

(i) a non-empty set S called universe or domain of $\mathcal{S}$,

(ii) a set of functions,

(iii) a set of relations.

We usually write structures as $\mathcal{S} = (S, R_1^{\mathcal{S}}, \cdots, R_n^{\mathcal{S}}, f_1^{\mathcal{S}}, \cdots, f_m^{\mathcal{S}})$ where function symbols start with lower case letters and relation symbols with upper case letters. We write the name of the structure in the superscript, as in $R^{\mathcal{S}}$, to make clear that we talk about the interpretation of the symbol. In this work we only deal with structures having domain $S \subseteq \omega$.

If we can interpret any symbol of a language $\mathcal{L}$ in a structure $\mathcal{S}$, then we call $\mathcal{S}$ a $\mathcal{L}$-structure. For example the structure $\mathcal{N} = (\mathbb{Z}, +, \cdot, 0, 1)$, where $+, \cdot$ are interpreted as addition and multiplication respectively and $0, 1$ as the numbers 0 and 1, is a $\mathcal{L}_r$-structure. However the structure $\mathcal{N}' = (\mathbb{Z}, +, 0, 1)$ with the same interpretation for $+, 0, 1$ as $\mathcal{N}$ is not.

Recall the definition of a sentence[1]. A *theory* $\mathcal{T}$ is then a set of sentences. We say that a structure $\mathcal{S}$ is a model of a theory $\mathcal{T}$,

$$\mathcal{S} \models \mathcal{T} \text{ iff } \mathcal{S} \models \varphi, \ \forall \varphi \in \mathcal{T}.$$

We can now define elementary classes of structures.

Definition 3.3. An elementary class $\mathcal{K}$ is given as

$$\mathcal{K} = \{\mathcal{S} \mid \mathcal{S} \models \mathcal{T}\}$$

for some fixed theory $\mathcal{T}$.

There are different ways to construct theories and their classes. One way is to look at the full theory $Th(\mathcal{S})$ of a structure $\mathcal{S}$. The class defined by this theory is then the set of exactly those structures elementary equivalent to $\mathcal{S}$. Another typical way to define theories is to look at properties of structures and defining them as sentences of the theory. Using this approach we get classes of structures possessing the same properties. Below we give examples of classes we will need later on.

Example 3.1. *Graphs*
The following sentences describe the class of non trivial irreflexive graphs.

$$\begin{array}{ll} \exists x, y & E(x, y) \\ \forall x & \neg E(x, x). \end{array}$$

To get the class of symmetric graphs one needs to add the sentence

$$\forall x, y \quad E(x, y) \rightarrow E(y, x).$$

Example 3.2. *Partial Orders*
To capture partial orderings, sentences describing reflexivity, antisymmetry and transitivity are needed.

$$\begin{array}{ll} \forall x & x \leq x \\ \forall x, y & x \leq y \wedge y \leq x \rightarrow x = y \\ \forall x, y, z & x \leq y \wedge y \leq z \rightarrow x \leq z \end{array}$$

[1] a sentence is a closed logic formula

For many properties it is not necessary to look at a particular structure and sufficient to look at its isomorphism type.

Definition 3.4. The isomorphism type $Iso(\mathcal{A})$ of structure $\mathcal{A}$ is

$$Iso(\mathcal{A}) = \left\{\tilde{A} : \tilde{A} \cong \mathcal{A} \text{ and } \tilde{A} \subseteq \omega\right\}.$$

Another important definition is that of definable sets, i.e. sets definable by a formula. The formal definition is as follows.

Definition 3.5. Let $\mathcal{M} = (M, \dots)$ be an $\mathcal{L}$-structure. We say that $X \subseteq M^n$ is definable if and only if there is an $\mathcal{L}$-formula $\varphi(v_1, \dots, v_n, w_1, \dots, w_m)$ and $\bar{b} \in M^m$ such that $X = \left\{\bar{a} \in M^n \mid \mathcal{M} \models \varphi(\bar{a}, \bar{b})\right\}$. We say that $\varphi(\bar{v}, \bar{b})$ *defines* X. X is *A-definable*, or *definable in* A if there is a formula $\psi(\bar{v}, w_1, \dots, w_l)$ and $\bar{b} \in A^l$ such that $\psi(\bar{v}, \bar{b})$ defines X.

Consider the following example.

Example 3.3. Let $\mathcal{N} = (\omega, +, \cdot)$, the set of natural numbers together with additon and multiplication. The set $X = \{x \in \omega \mid x \text{ is even}\}$ is definable by the formula

$$\varphi(x) = \exists y\ x = 2 \cdot y$$

since x is even if and only if $\mathcal{N} \models \varphi(x)$. Since φ does not have any other open variable then x, X is $\emptyset$-definable.

3.2 Computability theoretic properties of structures

We only consider computable languages, i.e. they are countable and we can computably list all their symbols and arities. We also assume that they are relational, i.e. they do not have function symbols. This can be done without loss of generality since we can transform any $\mathcal{L}$-structure $\mathcal{S} = (S, f_1^{\mathcal{S}}, \dots, f_n^{\mathcal{S}}, R_1^{\mathcal{S}}, \dots, R_m^{\mathcal{S}})$ into a relational $\mathcal{L}_{\mathcal{R}}$-structure, where $\mathcal{L}_{\mathcal{R}}$ is a relational language, $\mathcal{S}' = (S, R_1^{\mathcal{S}'}, \dots, R_{n+m}^{\mathcal{S}'})$ by setting

$$R_1^{\mathcal{S}'}, \dots, R_m^{\mathcal{S}'} = R_1^{\mathcal{S}}, \dots, R_m^{\mathcal{S}} \text{ and } R_{m+1}^{\mathcal{S}'}, \dots, R_{m+n}^{\mathcal{S}'} = Graph_{f_1^{\mathcal{S}}}, \dots, Graph_{f_n^{\mathcal{S}'}}$$

and since $deg(f) = deg(Graph_f)$, $\mathcal{S}'$ has exactly the same computability theoretic properties as $\mathcal{S}$. We only consider structures with computable domain subset of the natural numbers.

The *atomic diagram* of $\mathcal{S}$, denoted as $D(\mathcal{S})$, is the set of quantifier free sentences of $\mathcal{S}$ expanded by a constant symbol for each $s \in S$. A slightly more formal but equivalent definition using computable joins is often very useful.

$$D(\mathcal{S}) = A \oplus \bigoplus_{1,\dots,m} R_m^{\mathcal{S}}$$

Using this notation it is possible to identify a structure with its atomic diagram. This is often used to make notation easier, e.g. given a functional $f : D(\mathcal{A}) \to D(\mathcal{B})$ transforming the atomic diagram of structures $\mathcal{A}$ in class K into the atomic diagram of structures $\mathcal{B}$ in class K' we often abuse notation and write $f(\mathcal{A}) = \mathcal{B}$ instead of $f(D(\mathcal{A})) = D(\mathcal{B})$ to improve readability.

The *n-quantifier diagram* of $\mathcal{S}$ is the set of sentences up to n-quantifier alternations of S expanded by a constant symbol for each $s \in S$, and the *elementary diagram* of $\mathcal{S}$ is the full first order theory of $\mathcal{S}$ expanded by a constant symbol for each $s \in S$. We can now define what it means for a structure to be (relatively) computable.

Definition 3.6. A structure $\mathcal{S}$ is **d**-*computable* if S is computable and $D(\mathcal{S})$ is **d**-computable. If the n-quantifier diagram is computable then $\mathcal{S}$ is *n-decidable*, while if the full elementary diagram of $\mathcal{S}$ is computable then $\mathcal{S}$ is *decidable*.

It is easy to see that the atomic diagram of a structure is **d**-computable if and only if all relations and functions are **d**-computable. Computable but undecidable structures do exist and occur naturally, a famous example is given in example 3.4.

Example 3.4. Let $\mathcal{N} = (\omega; +, \cdot, ', 0, 1, =)$ be the standard model, or intended interpretation as it is often called in literature, of Peano arithmetic. Here $+, \cdot$ and $=$ are interpreted as usual and $'$ is the successor function, i.e. for a variable x, $x' = x + 1$. $\mathcal{N}$ is computable since $\omega, +, \cdot, '$ and $=$ are all computable but it is not decidable as was shown by Gödel in his proof of the incompleteness theorem.

If $\mathcal{S}$ is isomorphic to a **d**-computable structure $\tilde{\mathcal{S}}$, then $\tilde{\mathcal{S}}$ is a **d**-computable presentation of $\mathcal{S}$. If there exists a **d**-computable presentation of a structure $\mathcal{S}$ then we say that $\mathcal{S}$ is **d**-computably presentable.

Definition 3.7. The *degree* of a structure $\mathcal{S}$ is

$$deg(\mathcal{S}) = deg(D(\mathcal{S})).$$

While this definition of degrees of structures is quite natural and obvious there are also other interesting computability theoretic properties of structures not properly captured by it. Indeed a structure $\mathcal{S}$ might have isomorphic presentations which are of different degree. Therefore the degree of a structure does not tell us sufficiently enough about the information content of structures. To get a better idea about the computability of structures we therefore need further notions.

3.2.1 Degree spectra

A natural way to analyze the information content of a structure is to look at its isomorphism type, i.e. those structures isomorphic to it. One property to study here are the degrees of the structures in an isomorphism type. This is captured by the degree spectrum of a structure.

Definition 3.8. The *degree spectrum* of a structure $\mathcal{A}$ is

$$DgSp(\mathcal{A}) = \{deg(D(\mathcal{B})) \mid \mathcal{B} \cong \mathcal{A}\}.$$

An interesting property to look at when analyzing the computability of structures is the minimal degree in a degree spectrum.

Definition 3.9. If the degree spectrum of a structure $\mathcal{A}$ has a least element, then this element is called the *degree of isomorphism type of* $\mathcal{A}$.

Indeed not every isomorphism type has a degree. Richter showed in [Ric81] that structures without computable presentation which satisfy the effective extendability condition do not have a degree of isomorphism type. A structure $\mathcal{A}$ satisfies the *effective extendability condition* if for every finite structure $\mathcal{M}$ isomorphic to a substructure of $\mathcal{A}$, and every embedding f of $\mathcal{M}$ into $\mathcal{A}$, there is an algorithm that determines whether a given finite structure $\mathcal{F}$ extending $\mathcal{M}$ can be embedded into $\mathcal{A}$ by an embedding extending f [FHM14]. Results about structures satisfying the effective extendability condition were obtained in [Ric81] and more recently in [Khi04], namely that every structure in the classes of linear orders, trees, partially ordered sets, and abelian p-groups satisfies the effective extendability condition. It follows that any structure of this classes which does not have a computable presentation has no degree of its isomorphism type. That there are structures without computable presentation follows for example directly from a theorem shown independently by Slaman and Wehner.

Theorem 3.1 ([Sla98], [Weh98])**.** *There is a structure $\mathcal{A}$ that has presentations of every degree except* $\mathbf{0}$.

A fundamental result about degree spectra is due to Knight.

Theorem 3.2 ([Kni86])**.** *The degree spectrum of any structure is either a singleton or upwards closed.*

Indeed it was shown in [Kni86] that the degree spectrum of any nontrivial structure is closed upwards. An upwards closed degree spectrum is illustrated in figure 3.1.

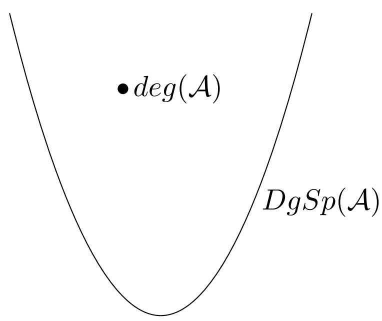

Figure 3.1: The upwards closed degree spectrum of some structure $\mathcal{A}$

3.2.2 Computable dimension

Another property which gives information about the information content of structures is the *computable dimension* of a structure. Given a computable presentable structure we look at the degree of the isomorphisms generating its computable presentations. We define the computable dimension of a structure.

Definition 3.10. Given a degree $\mathbf{d}$, the $\mathbf{d}$-*computable dimension* of a computably presentable structure $\mathcal{S}$ is the number of computable presentations of $\mathcal{S}$ up to $\mathbf{d}$-computable isomorphism. If $\mathcal{S}$ has $\mathbf{d}$-computable dimension 1 then it is $\mathbf{d}$-*computably categorical.*

If we look at the $\mathbf{0}$-computable dimension of a structure, we usually omit the $\mathbf{0}$ and analogously if a structure is $\mathbf{0}$-computably categorical we say that it is computably categorical. In other words a structure $\mathcal{S}$ is computably categorical iff for any two computable presentations $\tilde{\mathcal{S}}, \hat{\mathcal{S}}$ there is a computable function f inducing the isomorphism between $\tilde{\mathcal{S}}, \hat{\mathcal{S}}$. An interesting category of structures are structures with computable dimension 1 or ω. Indeed all structures of many natural classes have either computable dimension 1 or ω. As an example it was shown by Nurtazin that all decidable structures have computable dimension 1 or ω and later Goncharov expanded this result to 1-decidable structures. This result has also been proven for the following algebraic classes by various authors.

Theorem 3.3 ([Gon81], [Gon82], [Gon97], [GD80], [MN79], [Rem81], [Nur74])**.** *All structures in each of the following classes have computable dimension* 1 *or* ω*: algebraically closed fields, real closed fields, Abelian groups, linear orderings, Boolean algebras, and* Δ^0_2*-categorical structures.*

A natural question is wether there exist structures with finite computable dimension greater than 1. This was answered positively in an early paper by Goncharov.

Theorem 3.4 ([Gon80])**.** *For each* $n > 0$ *there is a computable structure with computable dimension* n*.*

An intuitive follow up question is wether there exist structures with finite computable dimension greater than 1 in any well-known class of structures. Indeed several classes which possess this property have been found. The proofs were done by coding families of computably enumerable sets with a finite number of computable enumerations into the structures.

Theorem 3.5 ([GMR89], [Gon80], [Kud96])**.** *For each* $n > 0$ *there are structures with computable dimension* n *in each of the following classes: graphs, lattices, partial orderings,* 2*-step nilpotent groups, and integral domains.*

3.2.3 R.i.c.e and r.i. computable relations

Another option to measure the complexity of a structure is to look at the relations which are computable in the structure. In other words, given a structure $\mathcal{A}$ and a relation R

we ask wether we can compute R with $\mathcal{A}$ as oracle. This question is reflected in the following definitions.

Definition 3.11. We say that a relation R is relatively intrinsically computable, r.i. computable, in $\mathcal{A}$ if for any presentation $\tilde{\mathcal{A}}$ of $\mathcal{A}$ we can compute R in $D(\tilde{\mathcal{A}})$.
A relation R is uniformly relatively intrinsically computable in $\mathcal{A}$ if there is a functional Φ such that for any presentation $\tilde{\mathcal{A}}$

$$\Phi^{D(\tilde{\mathcal{A}})} = \chi_R.$$

Definition 3.12. We say that a relation R is relatively intrinsically computably enumerable, r.i.c.e., in $\mathcal{A}$ if for any presentation $\tilde{\mathcal{A}}$ of $\mathcal{A}$ we can enumerate R in $D(\tilde{\mathcal{A}})$.
A relation R is uniformly relatively intrinsically computably enumerable in $\mathcal{A}$ if there is a functional Φ such that for any presentation $\tilde{\mathcal{A}}$

$$\Phi^{D(\tilde{\mathcal{A}})} = S_R$$

where S_R is the semi characteristic function of R.

We furthermore say that a relation R is co-r.i.c.e. in $\mathcal{A}$ if its complement is r.i.c.e. in $\mathcal{A}$. The following important theorem, which establishes a connection between interpretation and syntactic definition is due to Ash, Knight, Manasse and Slaman.

Theorem 3.6 ([Ash+89])**.** *The following statements are equivalent*
(1) R is (uniformly) r.i.c.e. in $\mathcal{A}$.
(2) R is Σ_1^c-definable (without parameters) in the language of $\mathcal{A}$.

The same holds for co-r.i.c.e. relations, i.e. a relation R is co-r.i.c.e. if and only if its complement is Σ_1^c-definable, and relatively intrinsically computable relations, i.e. a relation R is relatively intrinsically computable if and only if itself and its complement are Σ_1^c-definable, leading to Corollary 3.7.

Corollary 3.7. *The following statements are equivalent*
(1) R is (uniformly) r.i. computable in $\mathcal{A}$.
(2) R is Δ_1^c-definable (without parameters) in the language of $\mathcal{A}$.

Uniformly relatively intrinsically computably relations and uniformly r.i.c.e. relations are strongly connected as can be seen in Theorem 3.8.

Theorem 3.8. *For any structure $\mathcal{A} = \left\{A, R_0^{\mathcal{A}}, \ldots, R_n^{\mathcal{A}}\right\}$, there is a structure $\mathcal{B} = \left\{B, R_0^{\mathcal{B}}, \ldots, R_n^{\mathcal{B}}\right\}$ and equivalence relation $\sim$ such that $\mathcal{A} \cong \mathcal{B}/\sim$ [2]. Furthermore $R_i^{\mathcal{A}}$ is uniformly r.i.c.e. iff $R_i^{\mathcal{B}}$ is uniformly r.i. computable.*

[2] where $\mathcal{B}/\sim$ is the collapse of $\mathcal{B}$ under $\sim$, i.e. $\mathcal{B}/\sim = \left\{B/\sim, R_0^{\mathcal{B}}/\sim, \ldots, R_n^{\mathcal{B}}/\sim\right\}$.

Proof. Assume $R_i^{\mathcal{A}}$ to be uniformly r.i.c.e. in $\mathcal{A}$, then $R_i^{\mathcal{A}}$ is Σ_1^c-definable in $\mathcal{A}$, i.e.

$$\overline{x} \in R_i^{\mathcal{A}} \Leftrightarrow \bigvee_{i\in\omega} \exists \overline{s} \varphi_{f(i)}(\overline{x}, \overline{s}).$$

Define B as the set of tuples of the form $(\overline{x}, \overline{s}, i)$ and let $(overlinex, \overline{s}, i) \sim (\overline{y}, \overline{t}, j)$ iff $\overline{x} = \overline{y}$. We can define $R_i^{\mathcal{B}}$ as

$$(\overline{x}, \overline{s}, i) \in R_i^{\mathcal{B}} \Leftrightarrow \varphi_{f(i)}(\overline{x}, \overline{s}).$$

$R_i^{\mathcal{B}}$ is obviously Δ_1^c-definable without parameters in $\mathcal{B}$ and therefore uniformly r.i. computable. Furthermore it is easy to see that

$$\overline{x} \in R_i^{\mathcal{A}} \Leftrightarrow \overline{x} \in R_i^{\mathcal{B}} / \sim .$$

□

CHAPTER 4

Effective transformations

If a computability theoretic property about structures, like the properties reviewed in Chapter 3, is found, one often wants to know whether this property can also be found in natural classes of structures such as fields, lattices or graphs. One way to approach this question is to prove the property for any of the classes one is interested in, which is tedious and not reusable. A better way would be to define transformations of structures and show that these transformations preserve the properties.

A transformation of structures is generally a functional f taking a structure $\mathcal{A}$ as input and outputting a structure $\mathcal{B}$. Depending on the strength or effectiveness of the transformation f different computability theoretic properties are preserved. Having transformations of structures we say that a class K is transformable to a class K' if every structure $\mathcal{A} \in K$ is transformable to a structure $\mathcal{B} \in K'$. A class K is uniformly transformable to a class K' if we can fix a functional f transforming any structure $\mathcal{A} \in K$ to some structure $\mathcal{B} \in K'$. We say that K is X-reducible to K' where X stands for the type of transformation we used. A class K' is X-complete, or on top for $\mathcal{X}$-reducibility if any class K X-reduces to K'.

The idea of these transformations was first captured by the notion of *Borel reducibility* by Friedman and Stanley [FS89].

Definition 4.1. A class of structures K is *Borel reducible* to a class K', and we write $K \leq_B K'$, if there is a Borel function $f : 2^\omega \to 2^\omega$ that maps presentations of structures in K to structures in K' and preserves isomorphism. That is, for all $\mathcal{A} \in K$, $f(D(\mathcal{A})) = D(\mathcal{B})$ for some $\mathcal{B} \in K'$, and if $\tilde{\mathcal{A}} \in K$ with $f(D(\tilde{\mathcal{A}})) = \tilde{\mathcal{B}}$ for some $\tilde{\mathcal{B}} \in K'$, then

$$\mathcal{A} \cong \tilde{\mathcal{A}} \Leftrightarrow \mathcal{B} \cong \tilde{\mathcal{B}}.$$

Observe that Borel reducibility is not effective, since we do not have any requirements on the computability of f or the reduced structures. In recent years different ideas of effectivizing this notion have been investigated. Often a more general version of Borel reducibility and its effectivizations is studied, allowing arbitrary equivalence relations

E instead of only isomorphisms. We will only consider the restricted version of the reducibilities considering only isomorphism. We will focus on two very strong notions of effective transformations in the sense that they preserve many computability theoretic properties, *effective bi-interpretability* and *computable functors*, and will also review other effectivizations investigated recently.

4.1 Effective bi-interpretability

Before we look at the effective version of interpretability we define the usual model theoretic notion of interpretability similar to the definition in [Mar02, Definition 1.3.9].

Definition 4.2. We say that an $\mathcal{L}_0$-structure $\mathcal{A}$ is *interpretable* in an $\mathcal{L}$-structure $\mathcal{B}$ if there is a definable $X \subseteq \mathcal{B}^n$, a definable equivalence relation $\sim$ on X, and for each symbol of $\mathcal{L}_0$ we can find definable $\sim$-invariant sets on X (where "definable" means definable in $\mathcal{L}$) such that $X/\sim$ with the induced structure is isomorphic to B.

We can now define the effectivization of interpretability similar to the one given in [Mon14, Definition 5.1].

Definition 4.3. We say that a structure $\mathcal{A} = (A, P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \cdots)$ is *effectively interpretable* in $\mathcal{B}$ if there exists a uniformly r.i.c.e. set (in $\mathcal{B}$) $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$ and a uniformly r.i. computable sequence of relations $(\sim, R_0, R_1, \cdots)$ such that

(i) $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \subseteq B^{<\omega}$,
(ii) $\sim$ is an equivalence relation on $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$,
(iii) $R_i \subseteq (B^{<\omega})^{a_{R_i}}$ is closed under $\sim$ within $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$,

and there exists a function $f_{\mathcal{A}}^{\mathcal{B}} : \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$, the *effective interpretation* of $\mathcal{A}$ in $\mathcal{B}$, which induces an isomorphism:

$$(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}/\sim, R_0/\sim, R_1/\sim, \cdots) \cong (A, P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \cdots)$$

Recall the connection between uniformly r.i.c.e sets and Σ_1^c-definability as shown in Theorem 3.6. Any uniformly r.i.c.e. set is Σ_1^c-definable without parameters. The same holds for Δ_1^c-definability and uniformly r.i. sets as established in Corollary 3.7. This equivalence is heavily used in proofs of effective interpretability of a structure in another structure. In the definition given in [Har+15] all relations, including $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$, were required to be Δ_1^c-definable without parameters. However it follows from Theorem 3.8 that this definitions are equivalent. Both definitions have their uses, it is easier to show that a particular interpretation is effective using Definition 4.3, however for most general proofs the definition given in [Har+15] is easier to handle. We will use the latter definition in our proof of Theorem 4.1. We can now look at the notion of effective bi-interpretability.

Definition 4.4. Two structures $\mathcal{A}$ and $\mathcal{B}$ are *effectively bi-interpretable* if there are effective interpretations of one in the other such that the compositions

$$f_{\mathcal{B}}^{\mathcal{A}} \circ \tilde{f}_{\mathcal{A}}^{\mathcal{B}} : \mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})} \to \mathcal{B} \quad \text{and} \quad f_{\mathcal{A}}^{\mathcal{B}} \circ \tilde{f}_{\mathcal{B}}^{\mathcal{A}} : \mathcal{D}om_{\mathcal{A}}^{(\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}})} \to \mathcal{A}$$

are uniformly r.i. computable in $\mathcal{B}$ and $\mathcal{A}$ respectively. (Here $\tilde{f}^{\mathcal{B}}_{\mathcal{A}} : (\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}})^{<\omega} \to \mathcal{A}^{<\omega}$ is the obvious extension of $f^{\mathcal{B}}_{\mathcal{A}} : \mathcal{D}om^{\mathcal{B}}_{\mathcal{A}} \to \mathcal{A}$ mapping $\mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}})}$ to $\mathcal{D}om^{\mathcal{A}}_{\mathcal{B}}$.).

By substituting from the definitions one sees that for two structures $\mathcal{A}, \mathcal{B}$ to be bi-interpretable, $\mathcal{A}$ has to be interpretable in $\mathcal{A}^{<\omega^{<\omega}}$ and $\mathcal{B}$ has to be interpretable in $\mathcal{B}^{<\omega^{<\omega}}$ with the embeddings being r.i. computable in $\mathcal{A}$ and $\mathcal{B}$ resepectively.

When two structures are effectively bi-interpretable then they possess the same computability theoretic properties, in Theorem 4.1 the properties preserved by effective bi-interpretations can be seen.

Theorem 4.1 ([Mon14]). *Let $\mathcal{A}$ and $\mathcal{B}$ be effectively bi-interpretable. Then*

(1) $\mathcal{A}$ and $\mathcal{B}$ have the same degree spectrum.
(2) $\mathcal{A}$ is computably categorical if and only if $\mathcal{B}$ is.
(3) $\mathcal{A}$ and $\mathcal{B}$ have the same computable dimension.
(4) $\mathcal{A}$ is rigid if and only if $\mathcal{B}$ is.
(5) $\mathcal{A}$ and $\mathcal{B}$ have the same Scott rank.
(6) For every $\overline{a} \in \mathcal{A}^{<\omega}$, there is a $\overline{b} \in \mathcal{B}^{<\omega}$ such that $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$ have the same computable dimension, and vice-versa.
(7) For every $R \subseteq \mathcal{A}^{<\omega}$, there is a $Q \subseteq \mathcal{B}^{<\omega}$ which has the same degree spectrum, and vice-versa.
(8) $\mathcal{A}$ has the c.e. extendibility condition if and only if $\mathcal{B}$ does.
(9) The index sets of $\mathcal{A}$ and $\mathcal{B}$ are Turing equivalent, assuming $\mathcal{A}$ and $\mathcal{B}$ are infinite structures.
(10) The jumps of $\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable.

Before we prove some of the properties we make the following crucial observation.

Proposition 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be effectively bi-interpretable, then $deg(\mathcal{A}) = deg(\mathcal{B})$.*

Proof. Recall that $\mathcal{A}$ is coded as a subset of $\mathcal{B}^{<\omega}$ and $\mathcal{B}$ is coded as a subset of $\mathcal{A}^{<\omega}$. Since $\mathcal{A}^{<\omega}$, $\mathcal{B}^{<\omega}$ consist of finite sets of elements of $\mathcal{A}, \mathcal{B}$ respectively, we can compute $\mathcal{A}^{<\omega}, \mathcal{B}^{<\omega}$ given $\mathcal{A}, \mathcal{B}$. It follows that $deg(\mathcal{A}) = deg(\mathcal{A}^{<\omega})$ and $deg(\mathcal{B}) = deg(\mathcal{B}^{<\omega})$. Since the sequences of relations and all functions are uniformly r.i. computable in $\mathcal{A}^{<\omega}$ and $\mathcal{B}^{<\omega}$ respectively, we get that $deg(\mathcal{A}) = deg(\mathcal{B}^{<\omega}) = deg(\mathcal{B})$. $\square$

Observe that in the above proof we use the definition that $\mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}$ is uniformly r.i. computable instead of uniformly r.i.c.e., we can do this as both definitions are equivalent. We can now prove the properties $(1) - (3)$, exactly those properties we have introduced in Chapter 3. For proofs of the other properties see [Mon14].

Proof of (1). In this and all subsequent proofs let $\mathcal{A}$ be coded in $\mathcal{B}^{<\omega}$, hence $A = \mathcal{D}om^{\mathcal{B}}_{\mathcal{A}}$ and let $\tilde{\mathcal{B}} \in Iso(\mathcal{B})$ be coded in $\mathcal{A}^{<\omega}$, i.e. $\tilde{B} = \mathcal{D}om_{\mathcal{A}}^{(\mathcal{D}om^{\mathcal{A}}_{\tilde{\mathcal{B}}})}$. Recall that by Proposition 4.2 $deg(\mathcal{A}) = deg(\mathcal{B})$. Now let $\hat{\mathcal{B}} \cong \mathcal{B}$ by the function h and then obviously also $\hat{\mathcal{B}} \cong \tilde{\mathcal{B}}$. Hence we can code $\hat{\mathcal{B}}$ into $\hat{\mathcal{A}}^{<\omega}$ and $\hat{\mathcal{A}} \cong \mathcal{A}$ by the function

$$f^{\mathcal{B}}_{\mathcal{A}} \circ \tilde{f}^{\mathcal{A}}_{\mathcal{B}} \circ h \circ f^{\hat{\mathcal{A}}-1}_{\hat{\mathcal{B}}}$$

Furthermore $deg(\hat{\mathcal{B}}) = deg(\hat{\mathcal{A}}^{<\omega}) = deg(\hat{\mathcal{A}})$. It follows that $DgSp(\mathcal{A}) = DgSp(\mathcal{B})$. $\square$

Proof of (2). ($\Rightarrow$) Let $\mathcal{A}_1, \mathcal{A}_2$ be computable presentations of $\mathcal{A}$ coded into $\mathcal{B}_1^{<\omega}$, $\mathcal{B}_2^{<\omega}$ and $\mathcal{B}_1, \mathcal{B}_2 \in Iso(\mathcal{B})$ are presentations of $\mathcal{B}$. By the argument about degrees of codings from above we have that $\mathcal{B}_1, \mathcal{B}_2$ are computable. Let $\mathcal{A}_1, \mathcal{A}_2$ be computably isomorphic by the function h. Then $\mathcal{B}_1, \mathcal{B}_2$ are isomorphic by the composition

$$f^{\mathcal{A}_2}_{\mathcal{B}_2} \circ \tilde{f}^{\mathcal{B}_2}_{\mathcal{A}_2} \circ f^{\mathcal{B}_2}_{\mathcal{A}_2} \circ h \circ (f^{\mathcal{A}_1}_{\mathcal{B}_1} \circ \tilde{f}^{\mathcal{B}_1}_{\mathcal{A}_1})^{-1}.$$

$\square$

Proof of (3). As a corollary from the proof of *(2)* we have that two computable presentations of $\mathcal{A}$ are computably isomorphic iff their corresponding structures in $\mathcal{B}$ are computably isomorphic. Hence if $\mathcal{A}$ has k non computably isomorphic presentations, so has $\mathcal{B}$ and vice versa. It follows that $\mathcal{A}$ and $\mathcal{B}$ have the same computable dimension. $\square$

It is also possible to define a transformation between classes of structures based on effective bi-interpretability. Observe that an interpretation is given by a list of Δ_1^c-definable sequence of relations. And since we can write any Δ_1^c-definable relation as the Δ_1^c formula defining the relation, we can view the interpretation as a list of Δ_1^c formulas. With this we can define reducibility between classes.

Definition 4.5. A class K is *reducible to K' via effective bi-interpretability* if there are Δ_1^c formulas such that for every $\mathcal{A} \in K$, there is a $\mathcal{B} \in K'$ such that $\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable using those formulas. A class K is *complete for effective bi-interpretability*, or *ei-complete*, if for every computable language $\mathcal{L}$, the class of $\mathcal{L}$-structures is reducible to K via effective bi-interpretability.

4.1.1 Complete classes

More than a decade before the definition of effective bi-interpretability Hirschfeldt, Khoussainov, Shore and Slinko [Hir+02] defined a slightly weaker notion of completeness, as in Definition 4.6.

Definition 4.6. A theory T is *complete with respect to degree spectra of nontrivial structures, effective dimensions, expansion by constants, and degree spectra of relations* if for every nontrivial countable structure $\mathcal{B}$ there is a nontrivial $\mathcal{A} \models T$ with the following properties.

(1) $DgSp(\mathcal{A}) = DgSp(\mathcal{B})$.
(2) If $\mathcal{B}$ is computably presentable then the following hold
   a) For any degree $\mathbf{d}$, $\mathcal{A}$ has the same $\mathbf{d}$-computable dimension as $\mathcal{B}$.
   b) If $x \in B$ then there exists an $a \in A$ such that $(\mathcal{A}, a)$ has the same computable dimension as $(\mathcal{B}, x)$.

c) If $S \subseteq B$ then there exists a $U \subseteq A$ such that $DgSp_{\mathcal{A}}(U) = DgSp_{\mathcal{B}}(S)$ and if S is intrinsically c.e. then so is U.

We will refer to classes being complete with respect to degree spectra of nontrivial structures, effective dimensions, expansion by constants, and degree spectra of relations as HKSS-complete classes. By giving interpretations between structures they showed that the following theories are HKSS-complete.

Theorem 4.3 ([Hir+02])**.** *Let T be any of the following theories: symmetric, irreflexive graphs, partial orderings, lattices, rings (with zero-divisors), integral domains of arbitrary characteristic, commutative semigroups and 2-step nilpotent groups. Then T is complete with respect to degree spectra of nontrivial structures, effective dimensions, expansion by constants, and degree spectra of relations.*

The authors gave an interpretation of arbitrary structures in asymmetric graphs. To prove the results for symmetric irreflexive graphs, partial orderings and lattices interpretations of asymmetric graphs in these theories were given. To show that rings, integral domains, commutative semigroups and 2-step nilpotent groups are HKSS-complete a finite number of constant symbols needs to be added to their domain.

We will review the interpretations of arbitrary classes in graphs and graphs in partial orders. Instead of giving the original proofs that these interpretations are sufficiently strong to preserve HKSS-completeness, we will show that they are effective interpretations. To do this we first show that any arbitrary structure is effectively interpretable in asymmetric graphs, then we give an interpretation of asymmetric graphs in symmetric irreflexive graphs and at last we show that the class of symmetric irreflexive graphs is effectively interpretable in partial orders, thus we not only proof that the interpretations are effective we also obtain that graphs and partial orders are effectively bi-interpretable and complete for effective bi-interpretability.

In the original proofs the relations given were only required to be r.i. computable and not uniformly r.i. computable. Because of this the effective interpretation of asymmetric graphs in symmetric irreflexive graphs had to be modified. One can show that all interpretations given by Hirschfeldt, Khoussainov, Shore and Slinko are effective interpretations. Hence we obtain Theorem 4.4.

Theorem 4.4. *Symmetric, irreflexive graphs, partial orderings, lattices, rings (with zero-divisors), integral domains of arbitrary characteristic, commutative semigroups and 2-step nilpotent groups are complete for effective bi-interpretability.*

Recently Miller, Park, Poonen, Schoutens and Schlapentokh showed that fields are complete for effective bi-interpretability, expanding the list in Theorem 4.4.

Theorem 4.5 ([Mil+15])**.** *Fields are complete for effective bi-interpretability.*

4.1.2 Completeness of graphs and partial orders

We will show that graphs and partial orders are complete for effective bi-interpretability by giving three interpretations. An interpretation of arbitrary structures into countable graphs, showing that countable graphs are complete, an interpretation of countable graphs into symmetric irreflexive graphs which shows that countable graphs and symmetric irreflexive graphs are effectively bi-interpretable and hence symmetric irreflexive graphs are also complete and finally an interpretation of symmetric irreflexive graphs in partial orders, showing that this class is also complete.

Effective interpretation of arbitrary classes in graphs

We give an effective interpretation of arbitrary classes in countable graphs, that is asymmetric graphs with potential self loops. Given structure $\mathcal{A} = (A, P_0, P_1, \dots, P_n)$ where P_i has arity a_i we construct the corresponding countable graph $\mathcal{G} = (G, E)$ by the following rules.

1. A vertex $\alpha \in G$ with $E(\alpha, \alpha)$,
2. for every $i \in A$ a vertex $\alpha_i \in G$ and $E(\alpha, \alpha_i)$,
3. for every relation P_i and each tuple $(u_1, \dots, u_{a_i}) \in A^{a_i}$ such that $\mathcal{A} \models P(u_1, \dots, u_{a_i})$ vertices $\beta_1, \dots, \beta_{a_i}$, edges $E(\beta_j, \beta_{j+1})$ for $1 \leq j < a_i$, an edge $E(\alpha_i, \beta_j)$ if $u_j = i$ and a cycle of length $2i + 2$ starting at β_1,
4. and for every relation P_i and each tuple $(u_1, \dots, u_{a_i}) \in A^{a_i}$ such that $\mathcal{A} \not\models P(u_1, \dots, u_{a_i})$ vertices $\beta_1, \dots, \beta_{a_i}$, edges $E(\beta_j, \beta_{j+1})$ for $1 \leq j < a_i$, an edge $E(\alpha_i, \beta_j)$ if $u_j = i$ and a cycle of length $2i + 3$ starting at β_1.

In Figure 4.1 one can see the graph corresponding to a structure consisting of two relations $P_0 = 1, \dots$ and $P_1 = \{(1, 3), (2, 3)\}$.

Proposition 4.6. *There is an effective interpretation of $\mathcal{A}$ in $\mathcal{G}$.*

Proof. Let $\mathcal{D}om_{\mathcal{A}}^{\mathcal{G}} = \{\alpha_i \mid i \in A\}$. It is easy to see that $\mathcal{D}om_{\mathcal{A}}^{\mathcal{G}}$ is Σ_1^c-definable without parameters and hence also uniformly r.i.c.e. since

$$x \in \mathcal{D}om_{\mathcal{A}}^{\mathcal{G}} \Leftrightarrow \exists y \; E(y, y) \wedge E(y, x).$$

Define $\sim$ as equality, which is trivially uniformly r.i. computable, and also all the relations we are about to define are $\sim$-invariant since any set is equality invariant. For each P_i with arity a_i we define R_i as

$$\begin{aligned}(x_1, \dots, x_{a_i}) \in R_i \Leftrightarrow \; & \mathcal{D}om_{\mathcal{A}}^{\mathcal{G}}(x_1) \wedge \dots \wedge \mathcal{D}om_{\mathcal{A}}^{\mathcal{G}}(x_{a_i}) \\ & \wedge \exists y_1, \dots, y_{a_i} (E(x_1, y_1) \wedge \dots \wedge E(x_{a_i}, y_{a_i}) \\ & \wedge E(y_1, y_2) \wedge \dots \wedge E(y_{a_i - 1}, y_{a_i}) \\ & \wedge \exists z_1, \dots, z_{2i+1} \; (E(y_1, z_1) \wedge E(z_1, z_2) \wedge \dots \wedge E(z_{2i+1}, y_1))\}\end{aligned}$$

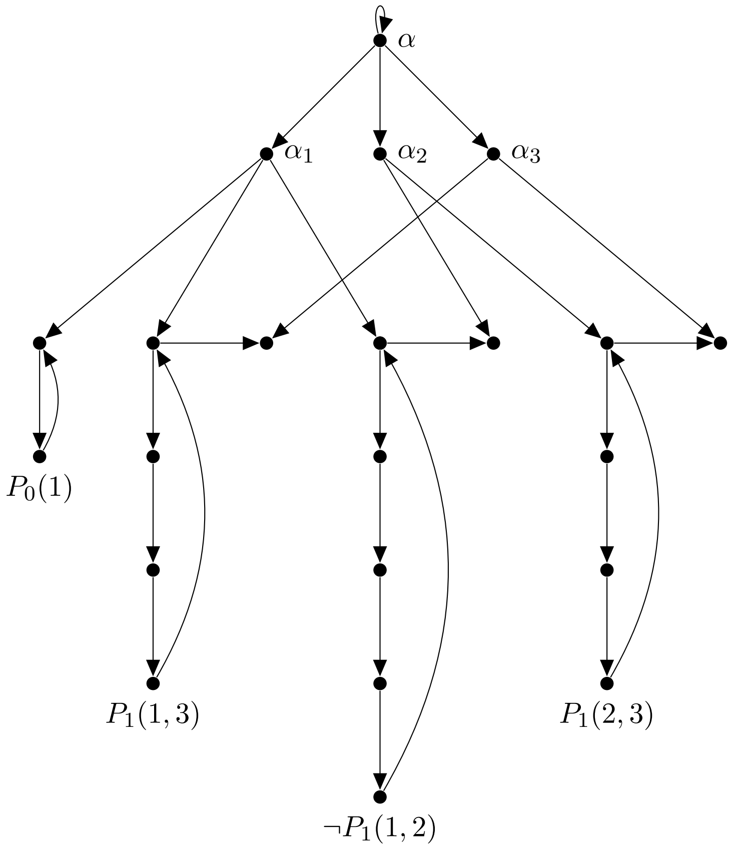

Figure 4.1: The partial graph $\mathcal{G}$ of structure $\mathcal{A}$ with two relations P_0, P_1 and $\mathcal{A} \models P_0(1), \neg P_1(1,2), P_1(1,3), P_1(2,3)$

Since $\mathcal{D}om^{\mathcal{G}}_{\mathcal{A}}$ is Σ^c_1-definable without parameters and a_i is finite, R_i is Σ^c_1-definable. To show that it is uniformly r.i. computable we define the set Q_i such that $(x_1, \ldots, x_{a_i}) \in Q_i \Leftrightarrow (x_1, \ldots, x_{a_i}) \notin P_i$. It is definable by the following formula.

$$\begin{aligned}(x_1, \ldots, x_{a_i}) \in Q_i \Leftrightarrow \quad & \mathcal{D}om^{\mathcal{G}}_{\mathcal{A}}(x_1) \wedge \cdots \wedge \mathcal{D}om^{\mathcal{G}}_{\mathcal{A}}(x_{a_i}) \\ & \wedge \exists y_1, \ldots, y_{a_i} (E(x_1, y_1) \wedge \cdots \wedge E(x_{a_i}, y_{a_i}) \\ & \wedge E(y_1, y_2) \wedge \cdots \wedge E(y_{a_i - 1}, y_{a_i}) \\ & \wedge \exists z_1, \ldots, z_{2i+2} \ (E(y_1, z_1) \wedge E(z_1, z_2) \wedge \cdots \wedge E(z_{2i+2}, y_1))\end{aligned}$$

By the same argument as for R_i, Q_i is Σ^c_1-definable without parameters. Furthermore observe that Q_i is the co-relation of R_i on $\mathcal{D}om^{\mathcal{G}}_{\mathcal{A}}$ and hence R_i is uniformly r.i. computable. We define $f^{\mathcal{G}}_{\mathcal{A}}(i) = \alpha_i$, it follows by construction of $\mathcal{D}om^{\mathcal{G}}_{\mathcal{A}}$ that $f^{\mathcal{G}}_{\mathcal{A}} : \mathcal{D}om^{\mathcal{G}}_{\mathcal{A}} \xrightarrow[onto]{1-1} A$ and that $R_i(x_1, \ldots, x_{a_i}) \Leftrightarrow P_i(f^{\mathcal{G}}_{\mathcal{A}}(x_1), \ldots, f^{\mathcal{G}}_{\mathcal{A}}(x_{a_i}))$. Hence $f^{\mathcal{G}}_{\mathcal{A}}$ is an effective interpretation of $\mathcal{A}$ in $\mathcal{G}$. $\square$

Bi-interpretability of graphs and symmetric irreflexive graphs

Given an arbitrary asymmetric graph $\mathcal{D} = (D, E)$ we construct a symmetric irreflexive graph $\mathcal{G}_{\mathcal{D}} = (G, F)$ satisfying the following conditions.

1. $G = \{a, \widetilde{a}, b, \alpha_1, \alpha_2, \alpha_3\} \cup \{v_i, x_i, \widetilde{x}_i \mid i \in D\}$,
2. $F(a, \widetilde{a}), F(\widetilde{a}, a)$ and $F(b, \widetilde{x}_i), F(\widetilde{x}_i, b)$,
3. $F(\widetilde{a}, \alpha_1), F(\widetilde{a}, \alpha_2), F(\widetilde{a}, \alpha_3), F(\alpha_1, \alpha_2), F(\alpha_2, \alpha_3), F(\alpha_1, \alpha_3)$ and their symmetries,
4. $F(a, v_i), F(x_i, v_i), F(\widetilde{x}_i, v_i)$ and $F(v_i, a), F(v_i, x_i), F(v_i, \widetilde{x}_i)$ for each $i \in D$,
5. if $(i, j) \in E$, then $(x_i, \widetilde{x}_j), (\widetilde{x}_j, x_i) \in F$.

An example can be seen in Figure 4.2. The straight lines code the edge relation and the dashed lines code the edges needed to make the sequence of relations we are about to construct Δ^c_1-definable without parameters.

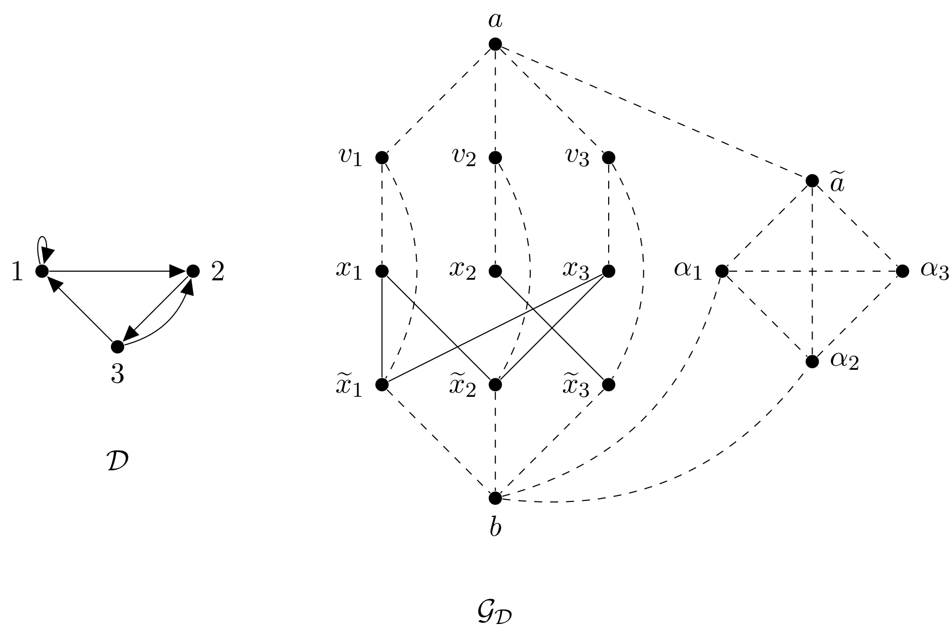

Figure 4.2: Coding of an asymmetric graph $\mathcal{D}$ in a symmetric, irreflexive graph $\mathcal{G}_{\mathcal{D}}$

We modified the original interpretation given in [Hir+02] by adding the complete subgraph on the four vertices $\widetilde{a}, \alpha_1, \alpha_2, \alpha_3$ and connecting it to b and a. The original interpretation only had a vertex $\widetilde{a}$ which was not connected to b. Because we need uniformity we had to add the complete subgraph as otherwise $\mathcal{D}om^{\mathcal{G}_{\mathcal{D}}}_{\mathcal{D}}$ would only be r.i.c.e but not uniform. We first prove that the complete subgraph on $\widetilde{a}, \alpha_1, \alpha_2, \alpha_3$ is the only complete subgraph consisting of 4 or more vertices.

Lemma 4.7. *The subgraph $\widetilde{a}, \alpha_1, \alpha_2, \alpha_3$ of $\mathcal{G}_{\mathcal{D}}$ is the only complete subgraph on 4 vertices in $\mathcal{G}_{\mathcal{D}}$.*

Proof. First observe that in every complete graph on 4 vertices there is a cycle of length 4. By construction of the interpretation the only cycles of length 4 not involving $\widetilde{a}, \alpha_1, \alpha_2, \alpha_3$ are of the form $x_i - \widetilde{x}_j - b - \widetilde{x}_k - x_i$ for any $i, j, k \in G$, but by construction $\neg F(\widetilde{x}_j, \widetilde{x}_k), \neg F(\widetilde{x}_k, \widetilde{x}_j)$ and hence cycles of these form can not build a complete subgraph on 4 vertices. As there is no other possibility to construct a cycle of length 4 the lemma follows. □

Theorem 4.8. *There is an effective interpretation of $\mathcal{D}$ in $\mathcal{G}_{\mathcal{D}}$.*

Proof. Let $\mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}} = \{v_i \mid i \in D\}$. We can define $\mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}$ by the following formula

$$\begin{aligned} x \in \mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}} \Leftrightarrow \exists y_a, y_{\tilde{a}}, y_{\alpha_1}, y_{\alpha_2}, y_{\alpha_3}\ & F(x, y_a) \wedge F(y_a, y_{\tilde{a}}) \wedge x \neq y_{\tilde{a}} \\ & \wedge \neg F(y_a, y_{\alpha_1}) \wedge \neg F(y_a, y_{\alpha_2}) \wedge \neg F(y_a, y_{\alpha_3}) \wedge F(y_{\tilde{a}}, y_{\alpha_1}) \\ & \wedge F(y_{\tilde{a}}, y_{\alpha_2}) \wedge F(y_{\tilde{a}}, y_{\alpha_3}) \wedge F(y_{\alpha_1}, y_{\alpha_2}) \wedge F(y_{\alpha_2}, y_{\alpha_3}) \wedge F(y_{\alpha_1}, y_{\alpha_3}). \end{aligned}$$

The first part of the formula identifies the vertex a by stating that a is the only vertex which is connected to exactly one vertex in the complete subgraph, this is true by construction. The second part of the formula verifies that $y_{\tilde{a}}, y_{\alpha_1}, y_{\alpha_2}, y_{\alpha_3}$ build a complete subgraph. It is easy to see now that the formula defines $\mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}$. As it is also Σ_1^c, $\mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}$ is Σ_1^c-definable without parameters and therefore uniformly r.i.c.e. We let $\sim$ again be equality which is trivially uniformly r.i. computable and define $R(x, y)$ as

$$(x, y) \in R \Leftrightarrow \mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}(x) \wedge \mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}(y) \wedge \exists u, v(F(x, u) \wedge F(y, v) \wedge F(u, v) \wedge F(v, b)).$$

It is easy to see from the construction of the interpretation that the formula defines R. To show that it is Σ_1^c it remains to show that $x = b$ is Σ_1^c-definable. Indeed we can define $x = b$ as

$$\begin{aligned} x = b \Leftrightarrow \exists y_{\tilde{a}}, y_{\alpha_1}, y_{\alpha_2}, y_{\alpha_3} & F(x, y_{\alpha_1}) \wedge F(x, y_{\alpha_2}) \wedge \neg F(x, y_{alpha_3}) \wedge \neg F(x, y_{\tilde{a}}) \\ & \wedge F(y_{\tilde{a}}, y_{\alpha_1}) \wedge F(y_{\tilde{a}}, y_{\alpha_2}) \wedge F(y_{\tilde{a}}, y_{\alpha_3}) \wedge F(y_{\alpha_1}, y_{\alpha_2}) \wedge F(y_{\alpha_2}, y_{\alpha_3}) \\ & \wedge F(y_{\alpha_1}, y_{\alpha_3}). \end{aligned}$$

As in the above formula the first part makes sure that x is connected to exactly 2 vertices in the complete subgraph and the second part verifies that $y_{\tilde{a}}, y_{\alpha_1}, y_{\alpha_2}, y_{\alpha_3}$ induce a complete subgraph. As this is the only complete subgraph and b is the only vertex connected to it by 2 vertices the formula defines $x = b$. It is now easy to see that R is Σ_1^c-definable without parameters. To see that it is uniformly r.i. computable we define the relation Q such that $(x, y) \in Q \Leftrightarrow (x, y) \notin E$. Q is definable in $\mathcal{G}_{\mathcal{D}}$ by

$$(x, y) \in Q \Leftrightarrow \mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}(x) \wedge \mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}(y) \wedge \exists u, v(F(x, u) \wedge F(y, v) \wedge \neg F(u, v) \wedge F(v, b)).$$

The formula is clearly Σ_1^c-definable since $x = b$ is Σ_1^c-definable. By construction of the interpretation we have that $(i, j) \notin E \Leftrightarrow (x_i, \widetilde{x}_j) \notin F$ which is reflected by the formula and therefore it indeed defines Q. Observe that Q is the co-relation of R on $\mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}$, hence R is uniformly r.i. computable. It is also $\sim$-invariant since any set is invariant under equality.

Let $f_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}(v_i) = i$, then $f_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}} : \mathcal{D}om_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}} \xrightarrow[onto]{1-1} D$. We get that $E(x,y) \Leftrightarrow R(f_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}(x), f_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}(x))$ from the arguments above and hence $f_{\mathcal{D}}^{\mathcal{G}_{\mathcal{D}}}$ is an effective interpretation of $\mathcal{D}$ in $\mathcal{G}_{\mathcal{D}}$. $\square$

Using Theorem 4.6 and Theorem 4.8 we get that the classes of asymmetric graphs and irreflexive graphs are effectively bi-interpretable.

Corollary 4.9. *Asymmetric graphs and irreflexive symmetric graphs are effectively bi-interpretable.*

As another corollary, since any structure is reducible to asymmetric countable graphs, we get Corollary 4.10.

Corollary 4.10. *The class of irreflexive symmetric graphs is complete for effective bi-interpretability.*

Bi-interpretability of symmetric irreflexive graphs and partial orders

Let $\mathcal{G}$ be a structure in the language of symmetric, irreflexive, countable graphs. We define the partial ordering $\mathcal{P}_{\mathcal{G}} = (P_{\mathcal{G}}, \prec)$ as in [Hir+02].

1. $P_{\mathcal{G}} = \{a, b\} \cup \{c_i \mid i \in G\} \cup \{d_{i,j} \mid i < j \wedge i, j \in G\}$
2. The relation $\prec$ is the smallest transitive relation on P satisfying the following conditions.
   (a) $a \prec c_i \prec b$ for all $i \in G$,
   (b) if $i, j \in G$, $i < j$ and $E(i,j)$ then $d_{i,j} \prec c_i, c_j$,
   (c) if $i, j \in G$, $i < j$, and $\neg E(i,j)$, then $c_i, c_j \prec d_{i,j}$.

Theorem 4.11. *There is an effective interpretation of $\mathcal{G}$ in $\mathcal{P}_{\mathcal{G}}$.*

Proof. Let $\mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}} = \{x \in P_{\mathcal{G}} \mid a \prec x \prec b\}$, then by definition $\mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}} = \{c_i \mid i \in G\}$ Furthermore let

$$R(x,y) = \Big\{(x,y) \mid x \neq y \wedge \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x) \wedge \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y) \wedge \exists z(z \neq a \wedge z \prec x \wedge z \prec y)\Big\}.$$

Define $\sim$ as equality and observe that $\mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}/\sim = \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}$ and $R/\sim = R$. We first show that this is an interpretation. Consider $f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(c_i) = i$ for $c_i \in P_g$. It follows immediately that $f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}} : \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}} \xrightarrow[onto]{1-1} G$. To show that $f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}$ is an interpretation it remains to show that $R(x,y) \Leftrightarrow E(f(x), f(y))$. To do this we look at the definition of R

$$
\begin{aligned}
&R(x,y) = \\
&\quad \Big\{(x,y) \mid x \neq y \wedge \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x) \wedge \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y) \wedge \exists z(z \neq a \wedge z \prec x \wedge z \prec y)\Big\} \\
&\Leftrightarrow \Big\{(x,y) \mid f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x) \neq f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y) \wedge G(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x)) \wedge G(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y)) \\
&\qquad \wedge \exists z(z \neq a \wedge z \prec x \wedge z \prec y)\Big\}
\end{aligned}
$$

by applying modus ponens on 2.(a) and $\exists z(z \neq a \wedge z \prec x \wedge z \prec y)$ we get

$$
\begin{aligned}
&\Leftrightarrow \Big\{(x,y) \mid f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x) \neq f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y) \wedge G(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x)) \wedge G(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y)) \\
&\qquad \wedge \exists z(z \neq a \wedge z \prec x \wedge z \prec y \wedge z \neq b)\Big\}
\end{aligned}
$$

Figure 4.3 shows a partial Hasse diagram of some value c_i such that $E(i,j), \neg E(i,k)$ in $\mathcal{G}$. Observe that by construction $c_i \prec c_j$ for any i, j and since $z \neq b$ and $z \neq a$, $z = d_{xy}$

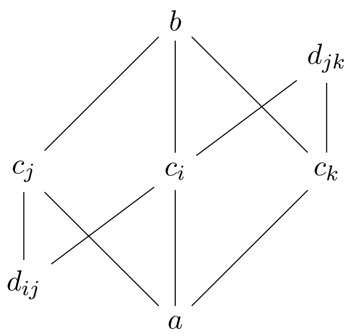

Figure 4.3: The partial Hasse diagram of some $\mathcal{P}_{\mathcal{G}}$ with $E(i,j), \neg E(i,k)$

or $z = d_{yx}$ depending on whether $x < y$. Hence we get

$$
\begin{aligned}
&\Leftrightarrow \Big\{(x,y) \mid f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x) \neq f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y) \wedge G(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x)) \wedge G(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y)) \\
&\qquad \wedge (d_{xy} \prec x \wedge d_{xy} \prec y \vee d_{yx} \prec x \wedge d_{yx} \prec y)\Big\}
\end{aligned}
$$

and by substituting from 2.(a)

$$
\begin{aligned}
&\Leftrightarrow \Big\{(x,y) \mid f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x) \neq f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y) \wedge G(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x)) \wedge G(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y)) \\
&\qquad \wedge (E(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x), f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y)) \vee E(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y), f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x)))\Big\}
\end{aligned}
$$

Recall the axioms of symmetric, irreflexive graphs from Example 3.1. By symmetry we get that

$$
\Leftrightarrow \Big\{(x,y) \mid f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x) \neq f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y) \wedge G(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x)) \wedge G(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y)) \wedge E(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x), f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y))\Big\}
$$

and by irreflexivity

$$E(x,y) \to x \neq y \wedge G(x) \wedge G(y)$$

and hence

$$R(x,y) \Leftrightarrow \left\{(x,y) \mid E(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x), f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y))\right\} \Leftrightarrow E(f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x), f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(y)).$$

The result that f is an interpretation follows. It remains to show that $\mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}$ is uniformly r.i.c.e. and that $\sim$ and R are uniformly r.i. computable. $\sim$ is obviously Δ_1^c-definable without parameters and hence uniformly r.i. computable. By definiton the elements c_i are the only elements that have an upper and a lower element in the order (see Figure 4.3). Hence

$$x \in \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}} \Leftrightarrow \exists u\ x \prec u \wedge \exists v\ v \prec u.$$

It follows that $\mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}$ is Σ_1^c-definable without parameters and therefore uniformly r.i.c.e. Let $R_\exists = \exists z(z \neq a \wedge z \prec x \wedge z \prec y)$. Looking at R one immediately sees that everything but $R_\exists$ is trivially Σ_1^c-definable without parameters. Observe that

$$z \neq a \Leftrightarrow \exists u\ \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(u) \wedge z \not\prec u.$$

We then get

$$R_\exists \Leftrightarrow \exists z\ (\exists u(\mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(u) \wedge z \not\prec u) \wedge z \prec x \wedge z \prec y).$$

Since $\mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}$ is Σ_1^c-definable without parameters and no quantifier alternation is added we get that $R_\exists$ is Σ_1^c definable without paramters and hence R is Σ_1^c-definable without parameters.

To show that R is Δ_1^c-definable without parameters we define the relation Q as

$$(x,y) \in Q \Leftrightarrow \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(x) \wedge \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}} y \wedge (x \neq y \wedge \exists z(z \neq b \wedge x \prec z \wedge y \prec z) \vee x = y)$$

With an analogous argument as for $(x,y) \in R$ one can see that $(x,y) \in Q \Leftrightarrow (x,y) \notin E(x,y)$. The inequivalence $z \neq b$ is definable by the Σ_1^c-formula $\exists u \mathcal{D}om_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}(u) \wedge u \not\prec z$. It follows that Q is Σ_1^c-definable without parameters and therefore R is Δ_1^c-definable without parameters and hence uniformly r.i. computable. It is also $\sim$-invariant since any set is invariant under equality.

We conclude that $f_{\mathcal{G}}^{\mathcal{P}_{\mathcal{G}}}$ is an effective interpretation of $\mathcal{G}$ in $\mathcal{P}_{\mathcal{G}}$. $\square$

By combining Corollary 4.10 and Theorem 4.11 we get the following two corollaries.

Corollary 4.12. *The classes of symmetric irreflexive graphs and partial orders are effectively bi-interpretable.*

Corollary 4.13. *The class of partial orders is complete for effective bi-interpretability.*

4.2 Computable functors

If we look at the degree spectrum (see Definition 3.8) of a structure, a natural way to define a reducibility between two structures $\mathcal{A}$ and $\mathcal{B}$ is to say that $\mathcal{A}$ is reducible to $\mathcal{B}$ if $DgSp(\mathcal{A}) \subseteq DgSp(\mathcal{B})$, or in other words if every presentation of $\mathcal{A}$ can be computed by a presentation of $\mathcal{B}$. This reducibility is known as *Muchnik-reducibility* [Muc63] and we say $\mathcal{A}$ is *Muchnik-reducible* to $\mathcal{B}$. We can also consider the uniform version of this reducibility, *Medvedev-reducibility* [Med55] where we say that a struture $\mathcal{A}$ is *Medvedev-reducible* to a structure $\mathcal{B}$ if there is a Turing functional Φ such that $\Phi^{\tilde{\mathcal{B}}} \to \tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$ are presentations of $\mathcal{A}$ and $\mathcal{B}$ respectively. It is easy to see that the notions of Medvedev-reducibility and effective interpretability are strongly connected, yielding Lemma 4.14.

Lemma 4.14. *If $\mathcal{A}$ is effectively interpretable in $\mathcal{B}$, then $\mathcal{A}$ is also Medvedev reducible to $\mathcal{B}$.*

Proof. From Proposition 4.2 we get that for any $\mathcal{A}$ such that $\mathcal{A}$ is effectively interpretable in $\mathcal{B}$, $deg(\mathcal{A}) = deg(\mathcal{B})$. Let $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$ and let h induce the isomorphism. Furthermore let $\tilde{\mathcal{B}}$ be the structure in which $\mathcal{A}$ is coded. Then $\tilde{\mathcal{B}} \cong \mathcal{B}$ by the isomorphism $f_{\tilde{\mathcal{A}}}^{\tilde{\mathcal{B}}} \circ f_{\mathcal{A}}^{\mathcal{B}-1}$. It follows that $Iso(\mathcal{A}) \subseteq Iso(\mathcal{B})$ and that together with the equivalence of degrees gives $DgSp(\mathcal{A}) \subseteq DgSp(\mathcal{B})$ and hence $\mathcal{A}$ is Medvedev reducible to $\mathcal{B}$. $\square$

However it was shown in [Kal09] by giving counterexamples that the reverse of the implication does not hold, hence Lemma 4.15.

Lemma 4.15. *If $\mathcal{A}$ is Medvedev reducible to $\mathcal{B}$, then $\mathcal{A}$ is not necessarily effectively interpretable in $\mathcal{B}$.*

If we want to achieve equivalence we need to strengthen the notion of Medvedev reducibility. To do this recall the notion of the isomorphism class of a structure $\mathcal{A}$, $Iso(\mathcal{A})$ from Definition 3.4. We can view $Iso(\mathcal{A})$ as a category, having the presentations of $\mathcal{A}$ as its objects and the isomorphisms between them as its morphisms. Now we can define the notion of a computable functor, similar to the definition in [Har+15].

Definition 4.7. A *functor from $\mathcal{A}$ to $\mathcal{B}$* is a functor from $Iso(\mathcal{A})$ to $Iso(\mathcal{B})$, that is, a map F that assigns to each presentation $\tilde{\mathcal{A}}$ in $Iso(\mathcal{A})$ a structure $F(\tilde{\mathcal{A}})$ in $Iso(\mathcal{B})$, and assigns to each morphism $f : \tilde{\mathcal{A}} \to \hat{\mathcal{A}}$ in $Iso(\mathcal{A})$ a morphism $F(f) : F(\tilde{\mathcal{A}}) \to F(\hat{\mathcal{A}})$ in $Iso(\mathcal{B})$ so that the two properties hold below:

(1) $F(id_{\tilde{\mathcal{A}}}) = id_{F(\tilde{\mathcal{A}})}$ for every $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$, and
(2) $F(f \circ g) = F(f) \circ F(g)$ for all morphisms f, g in $Iso(\mathcal{A})$.

A functor $F : Iso(\mathcal{A}) \to Iso(\mathcal{B})$ is *computable* if there exist two computable operators Φ and Φ_* such that

(1) for every $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$, $\Phi^{D(\tilde{\mathcal{A}})}$ is the atomic diagram of $F(\tilde{\mathcal{A}}) \in Iso(\mathcal{B})$,
(2) for every morphism $f : \tilde{\mathcal{A}} \to \hat{\mathcal{A}}$ in $Iso(\mathcal{A})$, $\Phi_*^{D(\tilde{\mathcal{A}}) \oplus f \oplus D(\hat{\mathcal{A}})} = F(f)$.

We often identify a computable functor with its pair (Φ, Φ_*) of Turing operators witnessing its computability.

In the definition *id* stands for the identity isomorphism. Although this definition might seem very complex at first, observe that Φ without Φ_* gives Medvedev reducibility from $\mathcal{A}$ to $\mathcal{B}$. This definition is now equivalent to effective interpretability, see Theorem 4.16.

Theorem 4.16. *Let $\mathcal{A}$ and $\mathcal{B}$ be countable structures. Then $\mathcal{A}$ is effectively interpretable in $\mathcal{B}$ if and only if there exists a computable functor from $\mathcal{B}$ to $\mathcal{A}$.*

The proof that there exists a computable functor from $\mathcal{B}$ to $\mathcal{A}$ if there is an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ is rather straightforward. It is easy to see, and also well known in model theory, that given an interpretation of one structure in another one can construct a functor. Indeed we have used compositions of isomorphisms between structures and their interpretations and showed that those are computable in the proofs of Theorem 4.1, this already suggests that an effective interpretation gives rise to a computable functor.

The other direction, there exists an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ if there is a computable functor from $\mathcal{B}$ to $\mathcal{A}$, is however not so easy to see, since creating an effective interpretation out of a computable functor is not straightforward. For a complete proof of Theorem 4.16 see [Har+15, Section 2].

Harrisson-Trainor, Melnikov, Miller and Montalbán not only showed that the existence of a computable functor given an effective interpretation and vice versa but also a correspondence between those. They showed that given a computable functor F the computable functor $\mathcal{I}_F$ induced by the corresponding interpretation and F are *effectively isomorphic* as defined in Definition 4.8. This is reflected in Proposition 4.17, the proof can be seen in [Har+15, Section 3].

Definition 4.8. A functor $F : Iso(\mathcal{B}) \to Iso(\mathcal{A})$ is *effectively isomorphic* to a functor $G : Iso(\mathcal{B}) \to Iso(\mathcal{A})$ if there is a computable Turing functional Λ such that for every $\tilde{\mathcal{B}} \in Iso(\mathcal{B})$, $\Lambda^{\tilde{\mathcal{B}}}$ is an isomorphism from $F(\tilde{\mathcal{B}})$ to $G(\tilde{\mathcal{B}})$, and the following diagram commutes for every $\tilde{\mathcal{B}}, \hat{\mathcal{B}} \in Iso(\mathcal{B})$ and every morphism $h : \tilde{\mathcal{B}} \to \hat{\mathcal{B}}$:

$$\begin{tikzcd} F(\tilde{\mathcal{B}}) \arrow[r, "\Lambda^{\tilde{\mathcal{B}}}"] \arrow[d, "F(h)"'] & G(\tilde{\mathcal{B}}) \arrow[d, "G(h)"] \\ F(\hat{\mathcal{B}}) \arrow[r, "\Lambda^{\hat{\mathcal{B}}}"] & G(\hat{\mathcal{B}}) \end{tikzcd}$$

Proposition 4.17. *Let $F : Iso(\mathcal{B}) \to Iso(\mathcal{A})$ be a computable functor, then F and $\mathcal{I}_F$ are effectively isomorphic.*

To create an equivalence between effective bi-transformability and computable functors we need one last concept, that of *pseudo-inverse* functors. Let F and G be functors such that $F \circ G$ and $G \circ F$ are effectively isomorphic to the identity functor. Let $\Lambda_{\mathcal{A}}$ be the

Turing functional witnessing the effective isomorphism between $G \circ F$ and the identity functor, i.e. for any $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$, $\Lambda_{\mathcal{A}}^{\tilde{\mathcal{A}}} : \tilde{\mathcal{A}} \to G(F(\tilde{\mathcal{A}}))$. Similarly we can define $\Lambda_{\mathcal{B}}$, then there is a map $\Lambda_{\mathcal{B}}^{F(\tilde{\mathcal{A}})} : F(\tilde{\mathcal{A}}) \to F(G(F(\tilde{\mathcal{A}})))$. If these two maps, and also the similarly defined maps for $Iso(\mathcal{B})$ agree for every $\tilde{\mathcal{A}} \in Iso(\mathcal{A})$ and $\tilde{\mathcal{B}} \in Iso(\mathcal{B})$ respectively, then we say that F and G are *pseudo-inverses*.

We can now define what it means for structures to be computably bi-transformable and state Theorem 4.18 witnessing the equivalence with effective bi-interpretability, for a proof see [Har+15, Section 4].

Definition 4.9. Two structures $\mathcal{A}$ and $\mathcal{B}$ with domain ω are *computably bi-transformable* if there exist computable functors $F : Iso(\mathcal{A}) \to Iso(\mathcal{B})$ and $G : Iso(\mathcal{B}) \to Iso(\mathcal{A})$ which are pseudo-inverses.

Theorem 4.18. *Let $\mathcal{A}$ and $\mathcal{B}$ be countable structures. Then $\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable iff $\mathcal{A}$ and $\mathcal{B}$ are computably bi-transformable.*

As for effective bi-interpretability we can define computable bi-transformations on classes of structures.

Definition 4.10. A class K is *uniformly transformally reducible* in a class K' if there exists a subclass $K'' \subseteq K'$ and computable functors $F : K \to K''$ and $G : K'' \to K$ and F, G are pseudo-inverses. We say that a class is *complete for computable functors* if for every computable language $\mathcal{L}$, the class of $\mathcal{L}$-structures is uniformly transformally reduces to it.

Theorem 4.19, witnessing the equivalence of effective bi-interpretability and computable bi-transformations, follows directly from Theorem 4.18

Theorem 4.19. *A class K is reducible via effective bi-interpretability to a class K' iff it is uniformly transformably reducible to K'.*

As a corollary we obtain that the classes complete for effective bi-interpretability are also complete for computable functors. This equivalence is quite helpful as it establishes an equivalence between the syntactic approach of effective interpretability and the semantic approach of computable functors. It is also of practical interest. For instance Theorem 4.5, the completeness of fields for effective bi-interpretability, has been proven by showing that fields are complete for computable functors.

4.3 Other effectivizations

4.3.1 Effective reducibility

Since it is possible to identify structures with their atomic diagram and the atomic diagram can be coded as a set using computable joins we can view a structure $\mathcal{S}$ as the set coding its atomic diagram $D(\mathcal{S})$. Obviously the set $D(\mathcal{S})$ is computable iff $\mathcal{S}$ is a computable structure. From basic computability theory it follows that $D(S)$ is computable iff its characteristic function is computable, i.e. is φ_i with some index i. We can now define the index set of a class and effective reducibility between two classes.

Definition 4.11. The index set $I(K)$ of a class K is the set of indices of computable structures in K.

Definition 4.12. A class K is *effectively reducible* to a class K' if there is computable $1-1$ function $f : I(K) \to I(K')$ and if $\mathcal{A}, \tilde{\mathcal{A}} \in K$ are computable and isomorphic, i.e. $\mathcal{A} \cong \tilde{\mathcal{A}}$, then $f(\mathcal{A}) \cong f(\tilde{\mathcal{A}})$. We say that $K \leq_e K'$ iff K is effectively reducible to K'.

It was shown in [Fok+12] that linear orderings, fields, trees, p-groups and torsion-free abelian groups are $\leq_e$-complete. Fokina and Friedman [FF09] originally considered both computable reducibility and hyperarithmetical reducibility, where f is required to be hyperarithmetical instead of computable. However Montalbán [Mon14, Theorem 1.6] showed that these two reducibilities coincide for complete classes. This result implies that the isomorphism problem, defined as

$$E(K) = \{(i, j) \mid i, j \text{ are indices of isomorphic computable structures in } K\}$$

for this classes must be Σ^1_1-complete. Hence classes without Σ^1_1-complete index isomorphism set, such as $\mathbb{Q}$-vector spaces, equivalence structures and torsion-free abelian groups can not be $\leq_e$-complete since their isomorphism problem is known to be hyperarithmetical. No examples of classes without hyperarithmetic isomorphism problem which are not $\leq_e$-complete are known, although it has been shown independently by Becker [Bec13] and Knight and Montalbán [unpublished] that such classes exist under the assumption that Vaughts conjecture fails.

4.3.2 Computable embeddings

Computable embeddings are another approach to effectivize the notion of Borel embeddings, first investigated in [Cal+04]. As the above notions it induces a partial order on the classes of structures. To define the notion of computable embeddings we first need the notion of computable transformations.

Definition 4.13 ([Cal+04])**.** Let K and K' be classes of structures, and let Ψ be a c.e. set of pairs (α, φ) where α is a subset of the atomic diagram of a finite structure for the language of K, and φ is an atomic sentence, or the negation of one in the language of K'. We say that Ψ is a *computable transformation from K to K'* if for all $\mathcal{A} \in K$, $\Psi(D(\mathcal{A}))$ has the form $D(\mathcal{B})$, for some $\mathcal{B} \in K'$. We may write $\Psi(\mathcal{A}) = \mathcal{B}$ (identifying the structures with their atomic diagram).

Observe that this notion is essentialy enumeration reducibility (see 2.4) on classes of structures. Hence the output structure of $\Psi(\mathcal{A})$ depends only on $\mathcal{A}$ and not on the order in which $\mathcal{A}$ is processed by Ψ. An important property of computable transformations is that they preserve substructure.

Proposition 4.20 (Preservation of substructure)**.** *Let $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{B} = \Psi(\mathcal{A})$, $B' = \Psi(\mathcal{A}')$, then $\mathcal{B}' \subseteq \mathcal{B}$.*

Proof. Let $\varphi \in D(\mathcal{B}')$, hence by Definition 4.13 there is a subset $\alpha \subseteq D(\mathcal{A}')$ such that $(\alpha, \varphi) \in \Psi$ and since $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{A}' \subseteq \mathcal{A}$, $\varphi \in D(\mathcal{B})$ and $\alpha \subseteq D(\mathcal{A})$. □

To get an effective version of Borel embeddings, isomorphism between structures needs to be preserved by the transformations. If this is added to the definition of computable transformations, computable embeddings are obtained.

Definition 4.14. A *computable embedding* of a class K in K' is a computable transformation Ψ from K to K' preserving isomorphism, i.e. for all $\mathcal{A}, \tilde{\mathcal{A}} \in K$, $\mathcal{A} \cong \tilde{\mathcal{A}}'$ iff $\Psi(\mathcal{A}) \cong \Psi(\tilde{\mathcal{A}}')$. If K is computably embeddable in K' we write $K \leq_c K'$.

Several results on the structure induced by $\leq_c$ have been obtained. It has been shown that the class of infinite graphs is $\leq_c$-complete and that the classes of finite primite fields FPF, finite linear orderings FLO and $\mathbb{Q}$-vector spaces FVS are strictly increasing under $\leq_c$, i.e. $FPF <_c FLO <_c FVS$. This results have all been obtained in [Cal+04]. A general result on the structure of $\leq_c$, that there inifinitely many antichains, and hence $\leq_c$ is not a linear order, has also been obtained in the same paper. It was later shown by Knight [unpublished] that $\leq_c$ is not a lattice, hence there are pairs of classes which neither have meet nor join. A strong tool to prove results about computable embeddings is Proposition 4.20 together with the substructure property. A class K has the *substructure property* if no $\mathcal{A} \in K$ is isomorphic to a substructure of $\tilde{\mathcal{A}} \in K$ unless $\mathcal{A} \cong \tilde{\mathcal{A}}$. So, for instance to show that K_1 does not embed into K_2 and it is known that K_2 has the substructure property it is sufficient to show that K_1 does not.

4.3.3 Turing computable embeddings

The notion of Turing computable embeddings was first defined in [Cal04] but not further investigated until [KMB07]. The difference to computable embeddings is that it uses Turing reducibility instead of enumeration reducibility. The formal definition is as follows.

Definition 4.15. A *Turing computable embedding* of K into K' is an operator $\Phi = \varphi_i$ such that

(1) for each $\mathcal{A} \in K$ there exists $\mathcal{B} \in K$ such that $\varphi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}$,

(2) and Φ preserves isomorphism, i.e. for $\mathcal{A}, \tilde{\mathcal{A}}$, $\mathcal{A} \cong \tilde{\mathcal{A}}$ implies $\Phi(\mathcal{A}) \cong \Phi(\tilde{\mathcal{A}})$.

We write $K \leq_{tc} K'$ iff there is a Turing computable embedding from K to K'.

Interestingly in contrast to enumeration and Turing reducibility on sets as shown in Chapter 2 here the converse holds, namely that $\leq_c$ implies $\leq_{tc}$, see [KMB07] and $\leq_{tc}$ does not imply $\leq_c$, see [Kal09]. The fact that $\leq_{tc}$ does not imply $\leq_c$ was shown constructively by giving counterexamples. This also becomes apparent when we consider that Turing computable embeddings do not preserve the substructure property, since it might still be that structure A has the substructure property and B does not but

A is Turing computably embeddable in B. The two reducibilities agree however on all interesting mathematical classes, in particular on the classes investigated in [Cal04] about which we talked about briefly in Section 4.3.2.

To prove non embeddability results for computable embeddings the substructure property is an important and powerfull tool, but since this property is not preserved by Turing computable embeddings we need something else to show these kind of results. A theorem which makes it possible to show many non embeddability results is the pull back theorem proposed by Knight.

Theorem 4.21 ([KMB07]). *If $K \leq_{tc} K'$ via Φ, then for any computable infinitary sentence φ in the language of K', we can find a computable infinitary sentence φ^* such that for all $\mathcal{A} \in K$, $\Phi(\mathcal{A}) \models \varphi$ iff $\mathcal{A} \models \varphi^*$. Moreover, if φ is computable Σ_α, or computable Π_α , for $\alpha \geq 1$, then so is φ^*.*

Although it has not been shown, it is commonly believed that if two structures are effectively bi-interpretable, then they are Turing computably bi-embeddable. This is suggested by the fact that given two effectively bi-interpretable structures $\mathcal{A}$ and $\mathcal{B}$ and a computable infinitary sentence φ such that $\mathcal{A} \models \varphi$ we can replace each relation in φ in the language of $\mathcal{A}$ by the corresponding relation in the coding in $\mathcal{B}$ which are all Δ^c_1 computable. Hence the complexity of the formula does not increase and Theorem 4.21 does hold also for effective bi-interpretability and computable functors.

CHAPTER 5

Degree spectra of theories and effective bi-interpretability

In Definition 3.8 we defined the degree spectrum of a structure as the set of Turing degrees of its presentations. But this is not the only notion of degree spectra which is studied. In [AM15] Andrews and Miller defined the degree spectrum of a theory T as the set of Turing degrees of all countable models of T. The authors gave several examples of degree spectra of theories including a non-degenerate union of two cones. This example entails that not every degree spectrum of a theory is a spectrum of a structure since it is known that no non-degenerate union of two cones is a degree spectrum of a structure. They also proved that the collection of non-hyperarithmetical degrees is not the spectrum of a theory and since there are structures with its degree spectrum consisting of all non-hyperarithmetical degrees [Kni86, Theorem 4.1] therefore also the converse, that not every spectrum of a structure is a spectrum of a theory, holds. This is also suggested by the fact that the isomorphism class and elementary class of an infinite structure do not coincide.

Fokina, Semukhin and Turetsky [FST15] suggested the following generalization of the definition of degree spectra.

Definition 5.1. The *degree spectrum* of a structure $\mathcal{A}$ *under the equivalence relation* E is

$$DgSp(\mathcal{A}, E) = \{\mathbf{d} \mid \text{there exists a } \mathbf{d}\text{-computable structure } \mathcal{B}, \text{ such that } \mathcal{A}E\mathcal{B}\}.$$

Under this definition the degree spectrum of a structure $\mathcal{A}$ under isomorphism as defined in Definition 3.8 is $DgSp(\mathcal{A}, \cong)$ and the degree spectrum of the equivalence class of a structure, the degree spectrum of the theory of $\mathcal{A}$ is $Dgsp(\mathcal{A}, \equiv)$, the degree spectrum of $\mathcal{A}$ under elementary equivalence.

Andrews and Miller [AM15][Proposition 2.2.] showed the following proposition about spectras of theories, we reformulated it with respect to Definition 5.1.

Proposition 5.1. *Given arbitrary structure $\mathcal{A}$ with $DgSp(\mathcal{A}, \equiv)$, there is a structure $\mathcal{G}_{\mathcal{A}}$ in the theory of graphs such that $DgSp(\mathcal{G}_{\mathcal{A}}, \equiv) = DgSp(\mathcal{A}, \equiv)$.*

They showed the proposition by giving a translation of arbitrary structures into graphs. It can be shown that this translation is an effective interpretation using the same methods we used in our proof of Proposition 4.6.

5.1 Σ_n-spectra

Instead of considering the full theory one can also consider only its Σ_n fragment, that is sentences of the form $\exists \overline{x}_1 \forall \overline{x}_2 \dots Q\overline{x}_n\ \varphi$ where φ is an atomic open formula and Q is either $\forall$ or $\exists$ depending on whether n is even or odd. We define the equivalence relation $\equiv_{\Sigma_n}$ where $\mathcal{A} \equiv_{\Sigma_n} \mathcal{B}$ iff the Σ_n fragments of their theories coincide. The Σ_n-spectrum of a structure $\mathcal{A}$ is then $DgSp(\mathcal{A}, \equiv_{\Sigma_n})$.

Fokina, Semukhin and Turetsky [FST15] studied Σ_n-spectra and made the following crucial observation.

Theorem 5.2. *There is a structure $\mathcal{A}$ with $DgSp(\mathcal{A}, \equiv) \neq DgSp(\mathcal{A}, \equiv_{\Sigma_k})$ for any $k \in \omega$.*

They showed this by constructing a spectrum which is not a Σ_k-spectrum for any $k \in \omega$ and any structure but the theory spectrum of a tree. The authors also looked at Σ_1- and Σ_2-spectra and showed that no Σ_1-spectrum can be a non-degenerate union of two cones but that there is a structure with a Σ_2-spectrum equal to the union of two non-degenerate cones. They used the following proposition about Σ_1-equivalent structures.

Proposition 5.3. *Two structures $\mathcal{A}$ and $\mathcal{B}$ are Σ_1-equivalent iff they have the same finite substructures (in finite sublanguages).*

Using this proposition we can prove the following theorem.

Theorem 5.4. *For any structure $\mathcal{A}$ there is a structure $\mathcal{G}_{\mathcal{A}}$ in the class of graphs such that $DgSp(\mathcal{A}, \equiv_{\Sigma_1}) = DgSp(\mathcal{G}_{\mathcal{A}}, \equiv_{\Sigma_1})$.*

Proof. We prove this theorem by showing that the effective interpretation of arbitrary structures in graphs given in Section 4.1.2 preserves Σ_1-spectra. Let $\mathcal{A}$ be interpreted in $\mathcal{G}_{\mathcal{A}}$. To show that the effective interpretation preserves Σ_1-spectra one has to show that for all structures $\mathcal{B} \equiv_{\Sigma_1} \mathcal{A}$, there is a graph $\mathcal{G}$ such that $\mathcal{G} \equiv_{\Sigma_1} \mathcal{G}_{\mathcal{A}}$ and $deg(\mathcal{B}) = deg(\mathcal{G})$, and vice versa.

First observe that any relation R_i in $\mathcal{G}_{\mathcal{A}}$ is Δ_0-definable, since both it and its complement in $\mathcal{D}om_{\mathcal{A}}^{\mathcal{G}_{\mathcal{A}}}$ are Σ_1-definable. Also the interpretation of the domain, $\mathcal{D}om_{\mathcal{A}}^{\mathcal{G}_{\mathcal{A}}}$, which we showed to be Σ_1-definable in the proof of Proposition 4.6, is Δ_0-definable since its coset can be defined by

$$x \notin \mathcal{D}om_{\mathcal{A}}^{\mathcal{G}_{\mathcal{A}}} \Leftrightarrow E(x,x) \lor \exists y\ E(y,y) \land \neg E(y,x).$$

It follows that every finite substructure $\mathcal{A}_0$ of $\mathcal{A}$ is interpreted in a finite substructure $\mathcal{G}_{\mathcal{A}_0}$ of $\mathcal{G}_{\mathcal{A}}$.
($\Leftarrow$) Let $\mathcal{A}$ be interpreted in $\mathcal{G}_{\mathcal{A}}$. We show that if $\mathcal{G}_{\mathcal{A}} \equiv_{\Sigma_1} \mathcal{G}$, then $\mathcal{G}$ interprets a structure $\mathcal{B}$ such that $\mathcal{A} \equiv_{\Sigma_1} \mathcal{B}$. The fact that $deg(\mathcal{B}) = deg(\mathcal{G})$ follows from Proposition 4.2. Assume $\mathcal{G}_{\mathcal{A}} \equiv_{\Sigma_1} \mathcal{G}$. Notice that the defining formulas of $\mathcal{D}om_{\mathcal{A}}^{\mathcal{G}_{\mathcal{A}}}$ and all R_i we used in the interpretation of $\mathcal{A}$ in $\mathcal{G}_A$ also define relations in $\mathcal{G}$. Pulling back this relations through the effective interpretation of arbitrary structures in graphs we get a structure $\mathcal{B}$ which is interpretable in $\mathcal{G}$. Now let $\mathcal{A}$ be a model of the Σ_1-sentence φ, then we can translate φ into a sentence φ' in the language of graphs by replacing any relation symbol P_i by the defining formula of the corresponding relation R_i in the interpretation. Then $\mathcal{G}_{\mathcal{A}} \models \varphi'$ and furthermore φ' will still be a Σ_1-sentence since all relations of the interpretaion are Δ_0-definable. Since $\mathcal{G}_{\mathcal{A}} \equiv_{\Sigma_1} \mathcal{G}$, $\mathcal{G}$ is also a model of φ' and since $\mathcal{G}$ effectively interprets $\mathcal{B}$, $\mathcal{B}$ is a model of φ. Since by the same argument for any Σ_1-sentence ψ such that $\mathcal{B} \models \psi$, also $\mathcal{A} \models \psi$, it follows that $\mathcal{A} \equiv_{\Sigma_1} \mathcal{B}$.
($\Rightarrow$) Let $\mathcal{B}$ be interpreted in $\mathcal{G}_{\mathcal{B}}$, we show that if $\mathcal{A} \equiv_{\Sigma_1} \mathcal{B}$, then $\mathcal{G}_{\mathcal{A}} \equiv_{\Sigma_1} \mathcal{G}_{\mathcal{B}}$. The fact that $deg(\mathcal{B}) = deg(\mathcal{G}_{\mathcal{B}})$ follows from Proposition 4.2. Assume that $\mathcal{A} \equiv_{\Sigma_1} \mathcal{B}$, but $\mathcal{G}_{\mathcal{A}} \not\equiv_{\Sigma_1} \mathcal{G}_{\mathcal{B}}$. Then by Proposition 5.3 wlog there is a finite substructure $\mathcal{G}'_{\mathcal{A}}$ in $\mathcal{G}_{\mathcal{A}}$ such that $\mathcal{G}'_{\mathcal{A}} \not\cong \mathcal{G}'_{\mathcal{B}}$ for any finite $\mathcal{G}'_{\mathcal{B}} \subseteq \mathcal{G}_{\mathcal{B}}$. It might be the case that $\mathcal{G}'_{\mathcal{A}}$ does not interpret a finite substructure of $\mathcal{A}$, but then we can extend it to a finite substructure $\mathcal{G}_{\mathcal{A}_0} \subseteq \mathcal{G}_{\mathcal{A}}$ that interprets a finite substructure $\mathcal{A}_0 \subseteq \mathcal{A}$, for an example see Figure 5.1.

It is easy to see that since $\mathcal{G}'_{\mathcal{A}}$ is not isomorphic to any finite substructure of $\mathcal{G}_{\mathcal{B}}$, the same holds for $\mathcal{G}_{\mathcal{A}_0}$. Hence if we pull back $\mathcal{G}_{\mathcal{A}_0}$ to the finite substructure $\mathcal{A}_0$ it is interpreting, $\mathcal{A}_0$ can not be isomorphic to any finite substructure $\mathcal{B}_0 \subseteq B$ since otherwise the interpretation of $\mathcal{B}_0$ as a graph, $\mathcal{G}_{\mathcal{B}_0}$ would be isomorphic to $\mathcal{G}_{\mathcal{A}_0}$. This contradicts our assumption that $\mathcal{A} \equiv_{\Sigma_1} \mathcal{B}$. It follows that the interpretation of arbitrary structures in graphs preserves Σ_1-spectra and hence the theorem holds. $\square$

For Σ_2-spectra one can make the following observation.

Proposition 5.5. *$\mathcal{A} \equiv_{\Sigma_2} \mathcal{B}$ iff for every finite substructure $\mathcal{A}_0 \subseteq \mathcal{A}$ (in finite sublanguages) and for all $n \in \omega$ there is a finite $\mathcal{B}_0 \subseteq \mathcal{B}$ such that the finite extensions of size n for $\mathcal{A}_0$ and $\mathcal{B}_0$ coincide.*

Proof. For any structure $\mathcal{S}$, let $\varphi_{D(\mathcal{S})}(\overline{x})$ be the conjunction of atomic open formulas of $\mathcal{S}$. Since we deal with finite languages, if $\mathcal{S}$ is finite, then $\varphi_{D(\mathcal{S})}(\overline{x})$ is a finite formula.
($\Rightarrow$) Assume $\mathcal{A} \equiv_{\Sigma_2} \mathcal{B}$. Observe that for any finite $\mathcal{A}_0$, $\mathcal{A} \models \exists\overline{x}\varphi_{D(\mathcal{A}_0)}(\overline{x})$. Consider extensions $\mathcal{A}_1, \mathcal{A}_2, \ldots$ of size k of $\mathcal{A}_0$. Let $|\overline{y}| = k$, then

$$\begin{aligned}\mathcal{A} \models \exists\overline{x}\ \forall\overline{y}\ &\varphi_{D(\mathcal{A}_0)}(\overline{x}) \wedge (\varphi_{D(\mathcal{A}_1)}(\overline{x},\overline{y}) \vee \varphi_{D(\mathcal{A}_2)}(\overline{x},\overline{y}) \vee \ldots) \wedge \exists\overline{z_1}\ \varphi_{D(\mathcal{A}_1)}(\overline{x},\overline{z_1})\\ &\wedge \exists\overline{z_2}\ \varphi_{D(\mathcal{A}_2)}(\overline{x},\overline{z_2}) \wedge \ldots\end{aligned}$$

Using this notation the disjunction $(\varphi_{D(\mathcal{A}_1)}(\overline{x},\overline{y}) \vee \varphi_{D(\mathcal{A}_2)}(\overline{x},\overline{y}) \vee \ldots)$ and the conjunction $\exists\overline{z_1}\varphi_{D(\mathcal{A}_1)}(\overline{x},\overline{z_1}) \wedge \exists\overline{z_2}\varphi_{D(\mathcal{A}_2)}(\overline{x},\overline{z_2}) \wedge \ldots$ appear to be possibly infinite. This is not the case however since in a finite language using finitely many variables one can only write

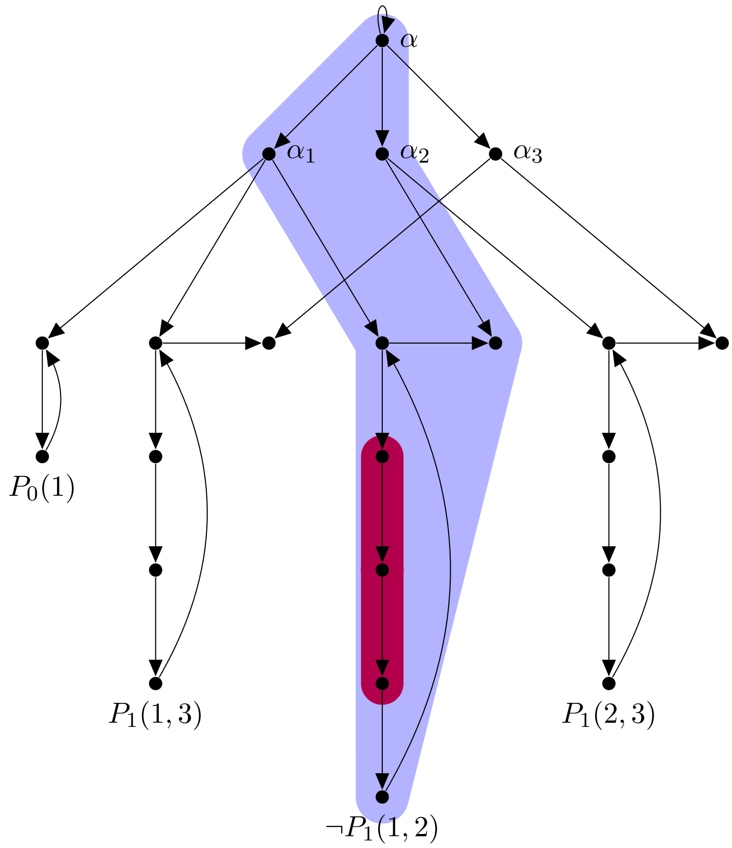

Figure 5.1: A finite substructure of $\mathcal{G}_{\mathcal{A}}$, marked red, and its extension to the interpretation of a substructure of $\mathcal{A}$, marked blue.

finitely many non-elementary equivalent formulas. We therefore may assume wlog that the formulas are finite. Then since $\mathcal{A} \equiv_{\Sigma_2} \mathcal{B}$ also

$$\mathcal{B} \models \exists \overline{x}\ \forall \overline{y}\ \varphi_{D(\mathcal{B}_0)}(\overline{x}) \wedge (\varphi_{D(\mathcal{B}_1)}(\overline{x}, \overline{y}) \vee \varphi_{D(\mathcal{B}_2)}(\overline{x}, \overline{y}) \vee \dots) \wedge \exists \overline{z_1}\ \varphi_{D(\mathcal{B}_1)}(\overline{x}, \overline{z_1})$$
$$\wedge \exists \overline{z_2}\ \varphi_{D(\mathcal{B}_2)}(\overline{x}, \overline{z_2}) \wedge \dots$$

It follows from the subformula $\exists \overline{x}\ \varphi_{D(\mathcal{B}_0)}(\overline{x})$ that there exists a substructure $\mathcal{B}_0 \cong \mathcal{A}_0$ and from the rest of the formula that for any finite extension $\mathcal{A}_i$ of $\mathcal{A}_0$ of size k there is a finite extension $\mathcal{B}_i$ of $\mathcal{B}_0$ of size k such that $\mathcal{A}_i$ is isomorphic to $\mathcal{B}_i$ and moreover, every finite extension $\mathcal{B}_j$ of $\mathcal{B}_0$ is isomorphic to an extension $\mathcal{A}_h$ of $\mathcal{A}_0$.
($\Leftarrow$) Assume the contradiction, i.e. for every finite substructure $\mathcal{A}_0 \subseteq \mathcal{A}$ and for all $n \in \omega$ there is a finite $\mathcal{B}_0 \subseteq \mathcal{B}$ such that the finite extensions of size n for $\mathcal{A}_0$ and $\mathcal{B}_0$ coincide and $\mathcal{A} \not\equiv_{\Sigma_2} \mathcal{B}$. Then there exists an atomic open formula φ such that wlog

$$\mathcal{A} \models \exists \overline{x}\ \forall \overline{y}\ \varphi(\overline{x}, \overline{y}) \text{ and } \mathcal{B} \not\models \exists \overline{x}\ \forall \overline{y}\ \varphi(\overline{x}, \overline{y}).$$

Since $\mathcal{A} \models \exists \overline{x}\ \forall \overline{y}\ \varphi(\overline{x}, \overline{y})$ there are elements $\overline{a} \in A$ such that $\mathcal{A} \models \forall \overline{y}\ \varphi(\overline{a}, \overline{y})$. Let the substructure induced by $\overline{a}$ be $\mathcal{A}_0$ and take an arbitrary extension $\mathcal{A}_i$ of $\mathcal{A}_0$ of size $|y|$.

Then, since $\mathcal{A}$ is a model of the formula, $\mathcal{A}_i \models \exists \overline{x}\ \forall \overline{y}\ \varphi(\overline{x}, \overline{y})$ and since for $\mathcal{A}_0$ there is a finite $\mathcal{B}_0 \subseteq \mathcal{B}$ such that their finite extensions coincide, we can construct an extension $\mathcal{B}_i$ of $\mathcal{B}_0$ such that $\mathcal{B}_i \cong \mathcal{A}_i$. This implies that $\mathcal{B}_i \models \exists \overline{x}\ \forall \overline{y}\ \varphi(\overline{x}, \overline{y})$. Observe that the number of non isomorphic extensions of $\mathcal{A}_0$ and $\mathcal{B}_0$ of size $|y|$ coincides. Hence if there is any finite extension $\mathcal{B}_j$ of $\mathcal{B}_0$ not covered by our construction it is isomorphic to an extension $\mathcal{B}_i$ covered by our construction and therefore $\mathcal{B}_j \models \exists \overline{x}\ \forall \overline{y}\ \varphi(\overline{x}, \overline{y})$. It follows that all extensions of size $|y|$ of $\mathcal{B}_0$ model the formula. If $\mathcal{B}$ is not a model of $\exists \overline{x}\ \forall \overline{y}\ \varphi(\overline{x}, \overline{y})$, then $\mathcal{B}$ is a model of $\forall \overline{x}\ \exists \overline{y}\ \neg\varphi(\overline{x}, \overline{y})$ and therefore it must be the case that for all finite substructures of size $|x|$, there is an extension of size $|y|$ such that the extension is a model of $\neg\varphi(\overline{x}, \overline{y})$. Hence this must also hold for $\mathcal{B}_0$. However this can not be the case since all extensions of $\mathcal{B}_0$ of size $|y|$ are a model of $\varphi(\overline{x}, \overline{y})$. Therefore $\mathcal{B} \models \exists \overline{x}\ \forall \overline{y}\ \varphi(\overline{x}, \overline{y})$, a contradiction to our assumption. $\square$

Using this proposition we can show a result similar to Theorem 5.4 for Σ_1-spectra for Σ_2-spectra.

Theorem 5.6. *For any structure $\mathcal{A}$ there is a structure $\mathcal{G}_{\mathcal{A}}$ in the class of graphs such that $DgSp(\mathcal{A}, \equiv_{\Sigma_2}) = DgSp(\mathcal{G}_{\mathcal{A}}, \equiv_{\Sigma_2})$.*

Proof. To show that the theorem holds we again show that the interpretation of arbitrary structures in graphs given in Section 4.1.2 preserves Σ_2-sepctra.

Let $\mathcal{A}$ be interpreted in $\mathcal{G}_{\mathcal{A}}$. To show that the interpretation preserves Σ_2-spectra one has to show that for all structures $\mathcal{B} \equiv_{\Sigma_2} \mathcal{A}$ there is a graph $\mathcal{G}$ such that $\mathcal{G} \equiv_{\Sigma_2} \mathcal{G}_{\mathcal{A}}$ and $deg(\mathcal{G}) = deg(\mathcal{B})$, and vice versa.

($\Leftarrow$) Let $\mathcal{A}$ be interpreted in $\mathcal{G}_{\mathcal{A}}$. We show that if $\mathcal{G}_{\mathcal{A}} \equiv_{\Sigma_2} \mathcal{G}$, then $\mathcal{G}$ interprets a structure $\mathcal{B}$ such that $\mathcal{A} \equiv_{\Sigma_2} \mathcal{B}$. The fact that $deg(\mathcal{B}) = deg(\mathcal{G}_{\mathcal{B}})$ follows from Proposition 4.2. Assume that $\mathcal{G}_{\mathcal{A}} \equiv_{\Sigma_2} \mathcal{G}$. Notice that the defining formulas of $\mathcal{D}om_{\mathcal{A}}^{\mathcal{G}_{\mathcal{A}}}$ and all R_i we used in the interpretation of $\mathcal{A}$ in $\mathcal{G}_A$ also define relations in $\mathcal{G}$. Pulling back these relations through the effective interpretation of arbitrary structures in graphs we get a structure $\mathcal{B}$ which is interpretable in $\mathcal{G}$. Now let $\mathcal{A}$ be a model of the Σ_2-sentence φ, then we can translate φ into a sentence φ' in the language of graphs by replacing any relation symbol P_i by the defining formula of the corresponding relation R_i in the interpretation. Then $\mathcal{G}_{\mathcal{A}} \models \varphi'$ and furthermore φ' will still be a Σ_2-sentence since all relations of the interpretaion are Δ_0-definable. Since $\mathcal{G}_{\mathcal{A}} \equiv_{\Sigma_2} \mathcal{G}$, $\mathcal{G}$ is also a model of φ' and since $\mathcal{G}$ effectively interprets $\mathcal{B}$, $\mathcal{B}$ is a model of φ. Since by the same argument for any Σ_2-sentence ψ such that $\mathcal{B} \models \psi$, also $\mathcal{A} \models \psi$, it follows that $\mathcal{A} \equiv_{\Sigma_2} \mathcal{B}$.

($\Rightarrow$) Let $\mathcal{B}$ be interpreted in $\mathcal{G}_{\mathcal{B}}$, we show that if $\mathcal{A} \equiv_{\Sigma_2} \mathcal{B}$, then $\mathcal{G}_{\mathcal{A}} \equiv_{\Sigma_2} \mathcal{G}_{\mathcal{B}}$. It then follows from Proposition 4.2 that $deg(\mathcal{A}) = deg(\mathcal{G}_{\mathcal{A}})$.

Assume the contrary, i.e. that $\mathcal{A} \equiv_{\Sigma_2} \mathcal{B}$ but $\mathcal{G}_{\mathcal{A}} \not\equiv_{\Sigma_2} \mathcal{G}_{\mathcal{B}}$. It follows wlog from Proposition 5.5 that there is a finite extension $\mathcal{G}_{\mathcal{A}_1}$ of a finite substructure $\mathcal{G}_{\mathcal{A}_0} \subseteq \mathcal{G}_{\mathcal{A}}$ which is not isomorphic to any finite extension $\mathcal{G}_{\mathcal{B}_1}$ of a finite substructure $\mathcal{G}_{\mathcal{B}_0} \subseteq \mathcal{G}_{\mathcal{B}}$. If $\mathcal{G}_{\mathcal{A}_0}$ does not interpret a substructure $\mathcal{A}_0$ of $\mathcal{A}$ we can extend it to a finite substructure $\mathcal{G}'_{\mathcal{A}_0}$ that interprets a substructure $\mathcal{A}_0 \subseteq \mathcal{A}$ as we did in our proof of Theorem 5.4, see Figure 5.1 for an example. Likewise, if any extension $\mathcal{G}_{\mathcal{A}_1}$ of $\mathcal{G}_{\mathcal{A}_0}$ does not interpret a substructure of $\mathcal{A}$ we can extend it in the same manner to $\mathcal{G}'_{\mathcal{A}_1}$.

If $\mathcal{G}_{\mathcal{A}_1} \subseteq \mathcal{G}'_{\mathcal{A}_0}$, then there is no finite substructure $\mathcal{G}_{\mathcal{B}_0} \subseteq \mathcal{G}_{\mathcal{B}}$ such that $\mathcal{G}_{\mathcal{B}_0} \cong \mathcal{G}'_{\mathcal{A}_0}$. We can pull back $\mathcal{G}'_{\mathcal{A}_0}$ to the finite substructure $\mathcal{A}_0$ it is interpreting and then for any finite substructure $\mathcal{B}_0 \subseteq \mathcal{B}$ $\mathcal{B}_0 \ncong \mathcal{A}_0$ since otherwise $\mathcal{G}_{\mathcal{B}_0}$, the interpretation of $\mathcal{B}_0$ in $\mathcal{B}$, is isomorphic to $\mathcal{G}'_{\mathcal{A}_0}$, a contradiction.

If $\mathcal{G}_{\mathcal{A}_1} \nsubseteq \mathcal{G}'_{\mathcal{A}_0}$, then there may be a finite substructure $\mathcal{G}_{\mathcal{B}_0} \subseteq \mathcal{G}_{\mathcal{B}}$ such that $\mathcal{G}_{\mathcal{B}_0} \cong \mathcal{G}'_{\mathcal{A}_0}$ (if there is none we can proceed as in the above case). If there is such a finite substructure $\mathcal{G}_{\mathcal{B}_0}$ we can pull back $\mathcal{G}'_{\mathcal{A}_1},\ \mathcal{G}'_{\mathcal{A}_0},\ \mathcal{G}_{\mathcal{B}_0}$ to their respective preimages $\mathcal{A}_1,\ \mathcal{A}_0,\ \mathcal{B}_0$. Then $\mathcal{A}_1$ is not isomorphic to any $\mathcal{B}_1$ extending $\mathcal{B}_0$ since otherwise $\mathcal{G}_{\mathcal{B}_1}$, the interpretation of $\mathcal{B}_1$ in $\mathcal{G}_{\mathcal{B}}$, which would be a finite extension of $\mathcal{G}_{\mathcal{B}_0}$ would be isomorphic to $\mathcal{G}'_{\mathcal{A}_1}$, a contradiction.

We conclude that the interpretation of arbitrary structures in graphs preserves Σ_2-spectra and therefore the theorem holds. $\square$

For both Σ_1- and Σ_2-spectra it is unclear whether effective bi-interpretability in general preserves them. The proof methods we used in the proofs of Theorem 5.6 and Theorem 5.4 can not be used to show the general result since effective interpretations allow computably infinitary formulas. Using the same methods as in the above proofs one can however show that also the rest of the interpretations given in Section 4.1.2 preserve Σ_1- and Σ_2-spectra, thus we obtain Corollary 5.7.

Corollary 5.7. *For any structure $\mathcal{A}$ there is a structure $\mathcal{P}$ in the class of partial orders such that $DgSp(\mathcal{A}, \equiv_{\Sigma_1}) = DgSp(\mathcal{P}, \equiv_{\Sigma_1})$ and $DgSp(\mathcal{A}, \equiv_{\Sigma_2}) = DgSp(\mathcal{P}, \equiv_{\Sigma_2})$.*

CHAPTER 6

Conclusion

In Chapter 4 we reviewed different kinds of transformations between classes of structures. These transformations use different approaches, on the one hand there are the notions of effective bi-interpretability and effective interpretability which are syntactic notions using the basic idea of interpretability well known in model theory and applying restrictions on the allowed interpretations to achieve the preservation of desired computability theoretic properties.

These originated in the practical need for tools to prove properties for well known classes, as in [Hir00] or [AM15] and only years after these interpretations first came up it was tried to capture them formally with the definition of effective bi-interpretability. It was shown that such interpretations are very strong in the sense that they preserve many computability theoretic properties but for some properties such as Σ_n-spectra and theory spectra it is unkown whether these are preserved in general or only by some interpretations. If it is true that these properties are not preserved in general, this would justify the need for even stronger notions.

On the other hand there are the notions of computable functors, effective reducibility, Turing computable- and computable embeddings. These notions use a more semantical approach and were defined with a different idea in mind. While effective interpretations and effective bi-interpretability are more practically motivated, the above notions originated either with the idea to effectivize already well known notions (such as Borel reducibility for effective reducibility) or to use well known computability theoretic reducibilities like Turing reducibility and enumeration reducibility on structures. Research on these transformations focused on the structure induced by them and while there are some results on properties they preserve, those were mostly motivated by the need of tools to show non-embeddability of classes.

The result presented in [Har+15], that computable functors and effective bi-interpretability are equivalent, is therefore quite important and also surprising as it establishes an equivalence between the syntactic notion of effective bi-interpretability, where we use the elements of the structure to create an equivalence between classes, and the notion of computable functors, which has a more meta view on structures, looking primarily on their isomorphism classes.

6.1 Future work

There is still a lot of research potential on computable transformations. Not much is known on the relation between computable functors and the other "semantic" transformations. While it is believed that the existence of a computable functor of a class K to K' implies that K is Turing computably embeddable in K' this has not been shown yet.

Another possible research topic are the computability theoretic properties preserved by the different computable transformations. Proving that effective bi-interpretability preserves a property saves a lot of work since it is then not needed to prove a property for all classes seperately. As more and more computability theoretic properties of structures are studied, this proof strategy may become more important. It is for instance still unclear whether effective bi-interpretability preserves theory spectra and Σ_n-spectra for arbitrary n.

The structure induced by effective interpretability is also not well known. While there is a range of complete classes known, and it has been shown for some classes that graphs do not embed in them and hence they are not complete, we do not know of any result on the relation between them.

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